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2015

ESSAYS IN FINANCIAL ECONOMICS AND ECONOMETRICS

LA SPADA, GABRIELE

Princeton, NJ : Princeton University

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ESSAYS IN FINANCIAL ECONOMICS AND ECONOMETRICS

Gabriele La Spada

A DISSERTATION
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Advisers:

Professor Hyun S. Shin
Professor Markus K. Brunnermeier

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Abstract

Chapter 1 (my job market paper) asks the following question: Do asset managers reach for yield because of competitive pressures in a low rate environment? I propose a tournament model of money market funds (MMFs) to study this issue. I show that funds with different costs of default respond differently to changes in interest rates, and that it is important to distinguish the role of risk-free rates from that of risk premia. An increase in the risk premium leads funds with lower default costs to increase risk-taking, while funds with higher default costs reduce risk-taking. Without changes in the premium, low risk-free rates reduce risk-taking. My empirical analysis shows that these predictions are consistent with the risk-taking of MMFs during the 2006–2008 period.

Chapter 2, co-authored with Fabrizio Lillo and published in Studies in Nonlinear Dynamics and Econometrics (2014), studies the effect of round-off error (or discretization) on stationary Gaussian long-memory process. For large lags, the autocovariance is rescaled by a factor smaller than one, and we compute this factor exactly. Hence, the discretized process has the same Hurst exponent as the underlying one. We show that in presence of round-off error, two common estimators of the Hurst exponent, the local Whittle (LW) estimator and the detrended fluctuation analysis (DFA), are severely negatively biased in finite samples. We derive conditions for consistency and asymptotic normality of the LW estimator applied to discretized processes and compute the asymptotic properties of the DFA for generic long-memory processes that encompass discretized processes.

Chapter 3, co-authored with Fabrizio Lillo, studies the effect of round-off error on integrated Gaussian processes with possibly correlated increments. We derive the variance and kurtosis of the realized increment process in the limit of both "small" and "large" round-off errors, and its autocovariance for large lags. We propose novel estimators for the variance and lag-one autocorrelation of the underlying, unobserved increment process. We also show that for fractionally integrated processes, the realized increments have the same Hurst exponent as the underlying ones, but the LW estimator applied to the realized series is severely negatively biased in medium-sized samples.

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I thank my second adviser, Markus Brunnermeier. Markus had the patience to listen to me and my crazy ideas even before I was a student of his. And the first time we discussed my research projects, I remember, he taught me one of the most important lessons I have learned about research, the question every researcher should ask herself or himself before embarking in an intellectual quest: "Why should people care about what you are doing?" If we believe that knowledge and intellectual production are public goods, and I truly do, their social value is defined and ultimately realized in their communication. Then, Markus' question is not only an advice given to "a student on the market" but becomes an obligation for the profession. A rule that I will keep following.

My job market paper started as a theoretical work but in the end had also an important empirical part. That empirical part would not be there as it is now if it were not for Atif Mian and David Sraer. Atif and David helped me understand the strengths and weaknesses of my work, pointing my attention to the critical empirical issues. Their help has been a terrific training.

Valentin Haddad deserves a special acknowledgement. Valentin read my job market paper several times, gave me feedback and advice on many occasions, motivated and supported me from the very beginning. And he did all of this without being my adviser or recommender, having no obligation to do so. I am profoundly grateful to him for this.

A large part of this thesis is on time series econometrics and greatly benefited from the invaluable advice and kindness of Ulrich Mueller. Even if I have never been a student of his, Ulrich had the generosity to sit with me and discuss my projects multiple times, challenging and advising me, and ultimately making me want to know more econometrics. I am truly thankful to him for that.

I thank Laura Hedden, the Graduate Program Administrator of the Economics Department at Princeton. Always available, always kind, always supportive, Laura helped me so many times that I lost count of it. She has been just incredible.

I thank Fabrizio Lillo, my co-author and friend for many years now. I have learned and keep learning much from him. That, among many other things, is what makes him a great co-author.

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Finally, I thank my family: my parents and my sister, the most wonderful family one could ask for. They are the ultimate owners of this thesis. It is impossible to overestimate the impact of their love and support on my work. It would take too many pages to thank them for all they have done for me over the years, and I am not sure I would be able to do that properly. I will just say this: they have always been and are my anchor, my reference point.

Dulcis in fundo, I thank Federica, my marvelous "partner in crime." Simply put, without her, without her constant support and love, I couldn't even imagine this moment.

To my parents

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Chapter 1

Competition, Reach for Yield, and Money Market Funds

1.1 Introduction

Do money market funds "reach for yield" because of competitive pressures when risk-free rates decrease? Are there differences in the cross-section? What is the proper notion of competitive pressure for money market funds? To answer these questions, I propose a tournament model of money market funds and test its predictions on the period 2006–2008.

"Reach for yield" refers to the tendency to buy riskier assets in order to achieve higher returns. Recently, there has been much debate about asset managers reaching for yield in a low risk-free rate environment, especially in competitive industries. Asset managers are typically compensated with asset-based fees, and it has been widely observed that investors positively respond to fund performance. This induces asset managers to compete among each other over relative performance to attract money flows. The concern is that lower returns on safe assets might exacerbate this risk-taking incentive and lead asset managers to delve into riskier assets.[1] US prime money market funds (MMFs), in particular, are seen as a leading example of asset managers reaching for yield because of competitive forces.[2] Both regulators and academics have lately paid close attention to prime MMFs because of their crucial role in the recent financial

[1] See FSOC (2013), Office of Financial Research (2013), Bernanke (2013), Haldane (2014), and Yellen (2014). Similar concerns have been previously raised by Rajan (2006). On the other hand, by lowering interest rates the Federal Reserve tries to induce risky investment in the real economy.

[2] See Stein (2013): *"[...] A leading example here comes from the money market fund sector, where even small increases in a money fund's yield relative to its competitors can attract large inflows of new assets under management."*

crisis. However, although the possible "reach for yield" of MMFs is central to the agenda of regulators and academics, there is a relative lack of theoretical and empirical literature on the topic.[3]

The two economic forces at work in the MMF industry are: *fund competition* and *risk of "breaking the buck."* To capture these features, I model the industry as a static fund tournament with a continuum of risk-neutral funds that have heterogeneous costs of default. The cost of default in the model represents the cost of "breaking the buck" in the real world. The heterogeneity of default costs captures the real-world heterogeneity of reputational damages to fund sponsors in case their funds default. These damages include outflows from other funds in the same fund family and losses in the franchise value of other parts of a sponsor's business.[4] In terms of methodological contribution, to the best of my knowledge, this is the first paper that solves a tournament model with a continuum of players in a fully analytic way without first-order approximations.

First, I show that the tournament has a unique Nash equilibrium, fund risk-taking strictly decreases with the cost of default, and the equilibrium default probability is strictly positive for (almost) all funds. Funds trade off expected costs of default for the expected gains of outperforming competitors by taking on more risk. The fund with the highest default cost anticipates that, in equilibrium, it will have the lowest expected rank of performance and keeps its default probability equal to zero, regardless of other funds' actions. Funds with slightly lower default costs anticipate its move and keep their default probability slightly above zero to outperform it. This reasoning applies to the other funds in descending order of default costs. This means that, in equilibrium, funds with lower default costs face higher competitive pressure and therefore take on more risk. I show that the fund-specific competitive pressure is uniquely determined by the distribution of default costs in the industry and is independent of asset returns. Importantly, because of competition, the equilibrium default probability is positive for (almost) all funds, *regardless* of the scale of default costs in the industry. This result comes from the tournament nature of the game and would not hold if funds were compensated based on absolute performance.

The equilibrium default probability depends on asset returns only through a tournament version of the standard risk premium, which is exogenously given. This "tournament incentive" represents the risk-taking channel of competition. An increase in the risk premium increases the

[3] To the best of my knowledge, the only other papers on the "reach for yield" of MMFs are the recent works by Chodorow-Reich (2014), and Di Maggio and Kacperczyk (2014). See below for a literature review.

[4] Kacperczyk and Schnabl (2013) introduced this notion of sponsor's reputation concern for MMFs. In my empirical analysis, I use it as proxy for the fund's default cost to map the model to the data. See Section 1.7 for details.

equilibrium default probability of all funds. However, its effect on observable risky investment is heterogeneous in the cross-section. Consider an increase in the riskiness of the risky asset that increases the risk premium. Funds with higher default costs face lower competitive pressure, and the increase in their default probability will be smaller. If the increase in the riskiness of the risky asset is sufficiently large, they will be forced to cut their risky investment to keep their probability of default sufficiently close to zero. Funds with lower default costs, on the other hand, face a higher competitive pressure and are more affected by the increase in the risk premium. If they face sufficiently high competition, they will increase their risky investment. This bifurcation of fund risk-taking in response to changes in the risk premium comes from the heterogeneity of competition in equilibrium.

Importantly, equilibrium default probability does not depend explicitly on the level of the risk-free rate. This is because, absent default, funds only care about relative performance, and if they default, they receive a fixed pay equal to their idiosyncratic default cost. The equilibrium risky investment, however, does depend on the level of the risk-free rate. If the return on safe assets decreases, funds are forced to cut their risky investment to keep the same probability of default. That is, holding the premium constant, a decrease in the risk-free rate decreases the risky investment of all funds. This anti-"reach for yield" effect is stronger for funds with higher default costs, which means that the cross-sectional differential in risky investment decreases with the risk-free rate.

These results show that to understand the risk-taking of MMFs, it is important to distinguish the level of the risk-free rate from the risk premium. Risk premia are key to trigger risk-taking but affect funds with low and high default costs in opposite ways. Low risk-free rates, on the other hand, increasing the buffer of safe assets necessary to maintain a given default probability, reduce risky investment. This effect of risk-free rates on risk-taking is peculiar to MMFs and comes from their distinctive feature of a stable net asset value and consequent risk of "breaking the buck."

In my empirical analysis, I show that these predictions are consistent with the risk-taking behavior of MMFs during the 2006–2008 period. To map the model to the data, I identify fund's cost of default with sponsor's cost of reputational damages.[5] First, I show that the rank of fund performance, not raw performance, determines money flows in the MMF industry, justifying the choice of a tournament model.

Second, I provide evidence that supports the predictions of the model on the level of risky investment in the time series. Figure 1.1 shows that in the period August 2007–August 2008,

[5]Following Kacperczyk and Schnabl (2013), I proxy sponsor's reputation concern with the share of non-money market fund business in the sponsor's total mutual fund business. See Section 1.7 for details.

when the risk premium increased and the risk-free rate decreased, funds with higher default costs decreased their risky investment, while funds with lower default costs increased it, as predicted by the model. These qualitative observations are confirmed by results in Table 1.4,

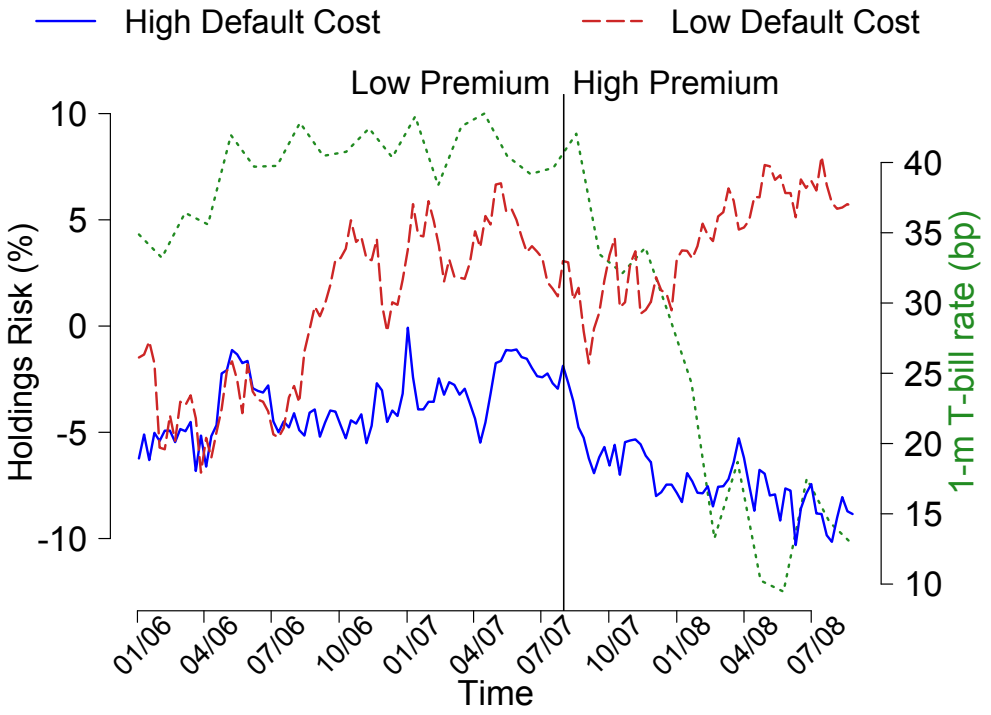


FIGURE 1.1: **MMF risk-taking in the time series: high vs low default costs.** The risk premia available to MMFs increased significantly after July 2007. The solid blue (dashed red) line is the average percentage of risky assets net of the safe assets for funds whose sponsor's reputation concern is consistently above (below) the industry median. The dotted green line is the monthly return on 1-month T-bills. See Section 1.7 for definitions and details.

in which I disentangle the effect of the risk-free rate from that of the risk premium.[6] For funds whose sponsors have reputation concerns above the industry median, an increase of 1% in the risk premium decreases the net share of risky assets by 2 percentage points, portfolio maturity by 2.1 days, and increases the share of safe assets by 1.6 percentage points. On the contrary, after an increase of 1% in the risk premium, funds whose sponsors have reputational concerns below the industry median increase the net share of risky asset by 1.5 percentage points and portfolio maturity by 1.2 days. On the other hand, a decrease of 1% in the 1-month T-bill rate increases the share of safe assets by 10.7 percentage points for funds whose sponsor's reputational concern is above the industry median, and by 8.6 percentage points for funds whose sponsor's reputational concern is below the industry median.

[6]My main proxy for the risk premium is the excess bond premium for financial firms introduced by Gilchrist and Zakrajsek (2012). See Section 1.7 for details.

Finally, I fully exploit the cross-sectional variation in sponsor's reputation concerns to test model's predictions on the cross-sectional risk-taking differential and identify the effects of risk-free rates and risk premia on it. I show that funds with lower default costs always take on more risk, and confirm that the differential increases when either risk-free rates decrease or risk premia increase. In the period January 2006–July 2007, when premia were low and risk-free rates relatively high, the difference in the net share of risky assets between funds in the lowest and highest percentile of default costs was 9 percentage points. The same figure more than doubled in the period August 2007–July 2008, when premia were high and risk-free rates low. In particular, I show that changes in the risk premium are more important for the asset class composition of fund portfolios, while changes in the risk-free rate are more important for portfolio maturity. An increase of 1% in the risk premium increases the difference in risky investment between funds in the lowest and highest percentile of default costs by 6 percentage points, and the difference in the share of safe assets by 4 percentage points. A decrease of 1% in the 1-month T-bill rate increases the difference in portfolio maturity between funds in the lowest and the highest percentile of default costs by 20 days.

The reminder of the paper is organized as follows. The next section reviews the literature. Section 1.2 describes prime MMFs and their institutional setting. Section 1.3 introduces the model. Section 1.4 characterizes and discusses the equilibrium. Section 1.5 studies the response of equilibrium risk-taking to shocks in asset returns. Section 1.6 studies the response of equilibrium risk-taking to shocks in the competitive environment (i.e., in the distribution of default costs). Section 1.7 presents the empirical analysis and tests model's predictions. Section 1.8 concludes. Proofs of the theoretical results and supplementary material are in the appendices.

Related literature

This paper belongs to the recent and growing literature on the risk-taking and systemic importance of MMFs. The most closely related paper is Kacperczyk and Schnabl (2013); hereafter simply KS. KS empirically observe that, in the period August 2007–July 2008, funds whose sponsors have lower reputation concerns took on more risk than funds whose sponsors have higher reputation concerns. This paper extends their work in that: (1) I propose a theoretical model of MMFs that provides predictions on their risk-taking in both the cross-section and the time series; (2) I disentangle the effect of the risk-free rate from that of the risk premium; (3) I show that the rank of performance is the true determinant of money flows to MMFs.

To the best of my knowledge, the only other papers that study the "reach for yield" of MMFs are Chodorow-Reich (2014), and Di Maggio and Kacperczyk (2014). Chodorow-Reich (2014) considers heterogeneity in the MMF industry along the dimension of administrative costs. Di Maggio and Kacperczyk (2014) look at the cross-section of MMFs in terms of affiliation to financial conglomerates.[7] Both papers are empirical. One of the main contributions of my paper is to provide a theoretical model of MMFs to explain how competition and risk-free rates affect their risk-taking.[8]

Parlatore Siritto (2014) is the only other paper that I am aware of that presents a model of MMFs. She proposes a 3-period general equilibrium model that focuses on the effects of the new regulation put forward by the Securities and Exchange Commission (SEC), specifically the transition from a stable NAV (net asset value) to a floating NAV. This paper contributes to that debate by showing that the stable NAV, generating a risk of default and the consequent need for a buffer of safe assets, also generates a channel of monetary policy that reduces risky investment when risk-free rates decrease. Moving to a floating NAV would eliminate that channel, so that the whole industry would take on more risk in a low rate environment.

On a theoretical level, this paper belongs to the literature on fund tournaments. Most of that literature has focused on the relative risk-taking of interim winners and losers in a dynamic context (Goriaev, Palomino, and Prat, 2003; Basak and Makarov, 2012, 2014). In contrast, in this paper heterogeneity comes from the cost of default, which is an intrinsic property of the funds. Under a technical point of view, most theoretical papers on fund tournaments consider tournaments with only two players (winner and loser). Basak and Makarov (2012) consider a tournament with a continuum of funds but assume that a fund's payoff only depends on its performance relative to the average.[9] The methodological contribution of this paper is to develop a technique to solve tournaments with a continuum of players without resorting to first-order approximations.

On a broader level, this paper belongs to the literature on the transmission of monetary policy to financial intermediaries. Borio and Zhu (2012) introduced the term "risk-taking channel" of

[7] This characteristic is related to the notion of sponsor's reputation concerns that I use to map the model to the data. However, it is significantly different from the proxy I use in my empirical analysis. See Section 1.7 for details.

[8] Other empirical papers on MMFs are Baba, McCauley, and Ramaswami (2009), McCabe (2010), Squam Lake Group (2011), Hanson, Scharfstein, and Sunderam (2014), Chernenko and Sunderam (2014), Strahan and Tanyeri (2014), and Schmidt, Timmermann, and Wermers (2014).

[9] Basak and Makarov (2012) also assume that the flow-performance relation is convex. This paper assumes that it is linear in the rank of performance because the focus in on the competitive nature of the MMF environment alone. However, the qualitative results of my model hold for any payoff function that increases with the rank of performance.

monetary policy to denote the impact of monetary policy on the willingness of market participants to take on risk. Most of that literature has focused on banks (Adrian and Shin, 2009; Jiménez *et al.*, 2014; Landier, Sraer, and Thesmar, 2014). This paper contributes to the literature by studying how the level of risk-free rates affects the risk-taking of important non-bank financial institutions such as prime MMFs.

1.2 Prime Money Market Funds: Institutional Features

US prime money market funds (MMFs) are open-ended mutual funds that invest in money market instruments. Prime MMFs are pivotal players in the financial markets. As of the end of 2013, they had roughly $1.5 trillion in assets under management and held approximately 40% of the global outstanding volume of commercial papers. In particular, they are a critical source of short-term financing for financial institutions. As of May 2012, they provided roughly 35% of such funding and 73% of their assets consisted of debt instruments issued by large global banks.[10]

Similarly to other mutual funds, MMFs are paid fees as a fixed percentage of assets under management and are therefore subject to the tournament-like incentives generated by a positive flow-performance relation. On the other hand, contrary to regular mutual funds, MMFs aim to keep the net asset value (NAV) of their assets at $1 per share. They do so by valuing assets at amortized cost and providing daily dividends as securities progress toward their maturity date. Since their deposits are not insured by the government and are daily redeemable, MMFs are subject to the risk of runs. If a fund "breaks the buck," i.e. its NAV drops below $1, it will likely suffer a run, as it happened on September 16, 2008, when Reserve Primary Fund, the oldest money fund, broke the buck because its shares fell to 97 cents after writing off debt issued by Lehman Brothers.

MMFs are regulated under Rule 2a-7 of the Investment Company Act of 1940. This regulation restricts fund holdings to short-term, high-quality debt securities. For example, it limits commercial paper holdings to those that carry either the highest or second-highest rating from at least two of the nationally recognized credit rating agencies. In the period of analysis, January 2006–August 2008, MMFs were not permitted to hold more than 5% of investments in second tier (A2-P2) paper, or to hold more than a 5% exposure to any single issuer (other than the government and agencies), and the weighted average maturity of the portfolio was capped to 90 days. In 2010, after the turmoil generated by the collapse of Reserve Primary Fund, the

[10] See ICI Fact Book (2013) and Hanson, Scharfstein, and Sunderam (2014).

SEC adopted amendments to Rule 2a-7, requiring funds to invest in even higher-quality assets of shorter maturities.[11]

On July 23, 2014, the SEC approved a new set of rules for MMFs.[12] The main pillar of these rules is that institutional prime MMFs will have to sell and redeem shares based on the current market-based value of the securities in their underlying portfolios. That is, they will have to move from a stable NAV to a floating NAV. The goal is to eliminate the risk of runs when the NAV falls below \$1. This new regulation, which will take effect on October 2016, has encountered the strong opposition of the industry.[13]

1.3 A Model of Money Market Funds

The model is a static fund tournament with a continuum of risk neutral funds of measure 1. Funds are indexed by $c \in [\underline{c}, \overline{c}] \subseteq \mathbb{R}_+$, where c represents the idiosyncratic cost of default defined below. c is distributed in the population according to a distribution function F_c, absolutely continuous on $[\underline{c}, \overline{c}]$, with density f_C.

Each fund is endowed with the same amount of initial deposits, $D > 0$. At the end of the tournament, deposits pay a gross interest rate equal to 1 to some outside investor. Funds can invest in two assets: a risk-free asset with deterministic gross return $R_f > 1$, and a risky asset with random gross return R distributed according to a distribution function F_R, absolutely continuous on $[\underline{R}, \overline{R}] \subset \mathbb{R}_+$, with density f_R.

ASSUMPTION 1. $\underline{R} < 1$ and $median(R) > R_f$.

As discussed below, Assumption 1 provides the proper notion of risk premium in a tournament context. Funds can neither short-sell nor borrow.

Let $x_c \in [0, D]$ be the risky investment of fund c. The *ex post* profit of fund c's portfolio is

$$\pi(x_c) = (R - R_f)x_c + (R_f - 1)D$$

[11] SEC Release No. IC-29132. For instance, the weighted average maturity is now capped to 60 days, funds are required to have enhanced reserves of cash and readily liquidated securities to meet redemption requests, and they can invest only 3% (down from 5%) of total assets in second tier securities.

[12] SEC Release No. IC-31166.

[13] ICI, "A Bad Idea: Forcing Money Market Funds to Float Their NAVs" (January 2013), and http://www.preservemoneymarketfunds.org (last visited September 4, 2014).

Hereafter, when it causes no confusion, $\pi(x_c)$ will be simply denoted as π_c and will be referred to as fund c's *performance*. Fund c is said to default, or "*break the buck*," if $\pi_c < 0$. In that case, fund c pays a fixed cost equal to its type.

If a fund does not default, its payoff is proportional to its assets under management (AUM) at the end of the tournament. Conditional on no default, fund c's final AUM are

$$AUM(c) = (Rk(\pi_c) + a)\, D,$$

where $Rk(\pi_c)$ is the rank of fund c's performance at the end of the tournament, and a is the fraction of money flows that does not depend on relative performance. $Rk(\pi_c)$ represents a postive flow-performance relation, a can be regarded as the effect of advertising or the overall attractiveness of the industry. For simplicity a is assumed to be the same for all funds and positive.

Given a profile of *ex post* performance $\pi : [\underline{c}, \overline{c}] \to \mathbb{R}$, the rank of a performance equal to y is

$$Rk(y) := \int_{\{c\,:\,\pi_c < y\}} \mathrm{d}F_c(c) \tag{1.1}$$

That is, the rank of a fund's performance is equal to the measure of funds with worse performance. $Rk(\pi_c) \in [0, 1]$ for all c, $Rk(\pi_c) = 1$ if c has the (strictly) highest performance, and $Rk(\pi_c) = 0$ if c has the lowest performance.[14]

The *ex post* payoff of fund c is

$$\begin{cases} \gamma\,(Rk(\pi_c) + a)\, D & \text{if } \ c \ \text{ does not default} \\ -c & \text{if } \ c \ \text{defaults} \end{cases}$$

where $\gamma \in (0, 1)$ represents the management fee paid by outside investors. The interplay between asset-based fee and positive flow-performance relation generates the fund tournament.

Under a strategy profile $x : [\underline{c}, \overline{c}] \to [0, D]$, the expected payoff of fund c is

$$v_c(x_c, x_{-c}) = \underbrace{\gamma\, D\, \mathbb{E}_R[Rk(\pi_c) + a | \pi_c \geq 0]\mathbb{P}_R(\pi_c \geq 0)}_{\text{expected tournament reward}} - \underbrace{c\, \mathbb{P}_R(\pi_c < 0)}_{\text{expected cost of default}} \tag{1.2}$$

[14] Under this definition of performance rank, the aggregate end-of-the-game AUM coming from the tournament are equal to the initial aggregate deposits divided by 2. Since the model is static, this plays no role, and for notational simplicity I omit the normalization factor 2 in my definition.

where x_{-c} is the risky investment of all funds except c, and $\mathbb{E}_R[\cdot]$ and $\mathbb{P}_R(\cdot)$ are the expected value and probability measure over the risky return R, respectively.

Finally, all information above is common knowledge.

Discussion of model's assumptions

The interest rate on deposits equal to 1 represents the stable NAV of \$1 in the MMF industry. The cost of default c represents sponsor's costs when the NAV of its MMF falls below \$1. These costs include reputational costs, as well as negative spillovers to other parts of sponsor's business.

The safe asset can be regarded as a Treasury bill, while the risky asset can be regarded as a bank obligation, or some other risky fixed-income security. Under Assumption 1 negative realizations of the risky return can trigger a default if the fund is too exposed to the risky asset. The premium on the risky asset is in terms of its median because, in a tournament context, fund payoffs depend only on relative performance. The *ex post* rank of performance is equal to the *ex ante* rank of risky investment when the realized risky return is above the risk free-rate, while it is equal to the *reverse* of the *ex ante* rank of risky investment when the realized risky return is below the risk free-rate. Hence, there is a tournament-driven risk-taking incentive if and only if the realized risky return is more likely to be above the risk-free rate than below it, i.e., $median(R) > R_f$.

The assumption that a fund's payoff is proportional to its AUM is consistent with the fee structure typically used in the MMF industry (ICI Fact Book, 2013). The assumption that a fund's AUM at the end of the tournament depend on fund's net return only via the flow-performance relation is consistent with the common practice in the MMF industry of redistributing dividends to keep the NAV fixed at \$1. The assumption that short-selling and borrowing are not allowed is also consistent with the regulation of MMFs.

The assumption that fund performance is a major determinant of fund flows is supported by a vast empirical literature (Chevalier and Ellison, 1997). In Section 1.7.2, I show empirically that the rank of performance, and not the raw performance, is the main determinant of the flow-performance relation in the MMF industry, which supports the choice of a tournament model.

Contrary to the majority of the positive theoretical literature on fund tournaments (Basak and Makarov, 2012), the above model does not assume a convex flow-performance relation. Although

there is some evidence that the flow-performance relation for MMFs is convex (Christoffersen and Musto, 2002), that risk-taking channel is shut off to focus on the incentives generated only by the tournament nature of fund competition.[15] However, the qualitative predictions of the model are robust to both convex and concave specifications of the flow-performance relation.

In the above model, the flow-performance relation is exogenously given. In mapping the model to the data, this amounts to assume that investors do not take into account funds' costs of default when making their investment decisions. That is, investors do not risk-adjust funds' performance based on sponsors' reputation concerns. In Section 1.7.2, I show that this assumption is satisfied in the data. In Appendix 1.A, I also present a random utility model that rationalizes this assumption and discuss possible ways to formally endogeneize a rank-based flow-performance relation.

Under specification (1.1) the rank of a fund's performance is equal to the measure of funds with strictly lower performance. In Appendix 1.C, I consider the more general specification in which the rank of a fund's performance is equal to the measure of funds with strictly lower performance plus a fraction ($\delta \in [0,1]$) of the funds with the same performance. All theoretical results in the paper are proved under the general specification, deriving conditions on δ for their validity.

The assumption that fund flows also depend on factors that are not related to funds' relative performance, e.g. advertising, has been vastly documented in the empirical literature on mutual funds (Jain and Wu, 2000). The assumption is made mainly for technical reasons as it ensures the existence of an equilibrium without imposing further conditions on the primitives of the model. However, the model can be solved and gives the same results even if that assumption is relaxed ($a = 0$) and substituted with a regularity condition on the distribution of default costs.

Finally, the above model abstracts away from any agency problem that may arise within the fund management company. That is, funds are identified with their sponsors.

1.4 The Nash Equilibrium

This section analytically characterizes the unique Nash equilibrium of the tournament. Before characterizing the equilibrium, let me introduce the following variable:

$$x_0 := \frac{R_f - 1}{R_f - \underline{R}} D \in (0, D)$$

[15] Moreover, Spiegel and Zhang (2013) have recently argued that the empirically observed convexity of the flow-performance relationship in the mutual fund industry is due solely to misspecification of the empirical model.

x_0 is the maximum risky investment such that the probability of default is zero. Given a risky investment $x \in [0, D]$, the probability of default is zero for $x \leq x_0$ and strictly positive for $x > x_0$. Hereafter, I refer to x_0 as the *critical risky investment*. $D - x_0$ is the minimum buffer of safe assets required to fully insure the fund against the risk of default. Importantly, x_0 strictly increases with the risk-free rate. This means that the minimum buffer of safe assets necessary to avoid "breaking the buck" is larger when the risk-free rate is low.

As solution concept, I use the standard definition of Nash equilibrium for games with a continuum of players introduced by Aumann (1964).

Definition 1.1. A risky investment strategy $x : [\underline{c}, \overline{c}] \to [0, D]$ is a Nash equilibrium of the game defined by (1.2) if and only if

$$\upsilon_c(x_c, x_{-c}) \geq \upsilon_c(z, x_{-c}) \qquad \text{for all } \; z \in [0, D]\,,$$

almost everywhere (a.e.) on $[\underline{c}, \overline{c}]$.

Hereafter, for simplicity, I drop the "*a.e.*" notation. All following results are true *a.e.* on $[\underline{c}, \overline{c}]$.[16]

Proposition 1.2. *Any equilibrium risky investment $x(c)$ must be strictly decreasing, differentiable with strictly negative derivative, and $\lim_{c \to \overline{c}} x(c) = x_0$.*

The first part of Proposition 1.2 is the differentiability and strict monotonicity of any equilibrium. This result comes from the fact that the payoff of funds depends on the rank order of their actions.[17] The second part of Proposition 1.2 reveals that any equilibrium must be in the region of positive probability of default, as summarized by the following corollary.

Corollary 1.3. *In equilibrium, the probability of "breaking the buck" is strictly positive for all funds and decreasing in the cost of default.*

In the region of zero probability of default, bounded above by x_0, the expected payoff increases with the risky investment regardless of other players' actions. Hence, each fund has an incentive to invest at least x_0 in the risky asset. The fund with the highest default cost invests exactly x_0 because it anticipates that it will have the lowest rank in expectation and optimally chooses to keep its default probability equal to zero, regardless of what other funds do. The pressure of competition drives all other funds to invest more than x_0 in the attempt to outperform their

[16] Since F_c is assumed absolutely continuous with respect to the Lebesgue measure, "*a.e.*" and "$F_c - a.e.$" coincide.

[17] Similar results are obtained in auction theory (Krishna, 2010).

competitors. The strategic interactions coming from the tournament make (almost) all MMFs not perfectly safe *ex ante*.[18] Corollary 1.3 can be regarded as the most basic form of risk-shifting and is the first main result on MMF risk-taking. Importantly, it holds true regardless of the scale of default costs, i.e., even if all funds have extremely large costs of default. This result would not hold if funds were compensated according their absolute performance (see Appendix 1.B for a detailed comparison).

To explicitly determine the equilibrium, I proceed as follows. Under Assumption 1, the expected rank of a fund's performance increases with the *ex ante* rank of its risky investment. Since any equilibrium is decreasing, the rank of a fund's risky investment is equal to the mass of funds with higher cost of default. That is, given an equilibrium investment profile $x(c)$, the rank of a risky investment y is $1 - F_C(x^{-1}(y))$. Since any equilibrium is differentiable with negative first derivative, I can take the first-order condition of the objective function (1.2) with respect to $x(c)$ and obtain an ordinary differential equation (ODE) in $\mathrm{d}x(c)/\mathrm{d}c$. The ODE, together with the boundary condition given by Proposition 1.2, provides a well-defined Dirichlet problem. Under regularity conditions on the primitives of the model, the Dirichlet problem can be solved exactly, and the solution is unique. Finally, I prove that the unique solution of the Dirichlet problem is indeed the unique equilibrium of the tournament by checking a second-order condition.

Proposition 1.4. *A Nash equilibrium exists if and only if* $\mathbb{E}_C\left[\frac{\gamma D}{\gamma D(F_C(c)+a)+c}\right] \leq \log\left(1+\frac{F_R(1)}{1-2F_R(R_f)}\right)$. *Moreover, if there is a Nash equilibrium, it must be unique.*

The equilibrium default probability is

$$p(c) = 2 \underbrace{q(R_f)}_{\substack{\textit{tournament}\\ \textit{incentive}}} \cdot \underbrace{Q(c)}_{\substack{\textit{incentive}\\ \textit{multiplier}}},$$

and the equilibirum risky investment is

$$x(c) = \frac{R_f - \underline{R}}{R_f - F_R^{-1}(p(c))}\, x_0\,, \tag{1.3}$$

where

$$q(R_f) := 0.5 - F_R(R_f) > 0\,,$$
$$Q(\widetilde{c}) := \exp\left\{\gamma D\, \mathbb{E}_C\left[(\gamma D\,(F_C(c)+a)+c)^{-1}\,|c>\widetilde{c}\right](1-F_C(\widetilde{c}))\right\} - 1,$$

$F_R^{-1}(\cdot)$ *is the quantile function of* R, *and* $\mathbb{E}_C[\cdot]$ *is the expected value over the cost of default.*

[18] Except for the fund with the highest cost of default, which has measure zero.

The equilibrium default probability is uniquely determined by $q(R_f)$ and $Q(c)$. $q(R_f)$ is common to all funds, strictly positive under Assumption 1, and is referred to as the *tournament incentive*. To understand its role as risk-taking incentive, note that the *ex post* rank of fund c's performance, $Rk(\pi_c)$, depends on the *ex ante* rank of fund c's risky investment, $Rk(x_c)$, in the following way

$$Rk(\pi_c) = \begin{cases} Rk(x_c) & \text{if } R > R_f, \text{ i.e. with probability } 1 - F_R(R_f) \\ \\ 1 - Rk(x_c) & \text{if } R < R_f, \text{ i.e. with probability } F_R(R_f) \end{cases}$$

The incentive to increase the default probability by investing in the risky asset increases with the difference between the above probabilities, i.e. $2q(R_f)$. Within the debate on a competition-driven "reach for yield" of MMFs, $q(R_f)$ represents the *incentive to reach for yield*. Larger $q(R_f)$, larger the default probability and risky investment for all funds. $q(R_f)$ is the only risk-taking incentive in the model and, as discussed below, a proxy for the standard risk premium.

$Q(c)$ is fund-specific, positive, strictly decreasing in the cost of default, and goes to zero as c goes to $\overline{c}$. $Q(c)$ is referred to as the *incentive multiplier* because it determines a fund's sensitiveness to the tournament incentive by measuring the competitive pressure faced by the fund in equilibrium. To see this, let us consider the fund with the highest cost of default, $\overline{c}$. As discussed above, $\overline{c}$ anticipates that, in equilibrium, its expected performance will have the lowest rank. Hence, it decides to keep the probability of default equal to zero by investing x_0 in the risky asset, regardless of other players' actions. That is, $\overline{c}$ is not affected by fund competition, and $Q(\overline{c}) = 0$. Funds with slightly lower default costs anticipate that $\overline{c}$ will invest x_0. Hence, in order to outperform $\overline{c}$, they keep a default probability slightly greater than zero by investing a little bit more than x_0 in the risky asset. This reasoning extends to the other funds in descending order of default costs. In other words, since the fund with the highest default cost is insensitive to competition, each fund faces competitive pressure only from the funds with higher default costs. Figure 1.2 shows the equilibrium risky investment and the incentive multiplier as functions of the cost of default.

To gain a deeper insight into how competition works in the MMF tournament, let us take a closer look at $Q(c)$. The competitive pressure faced by an agent in a competitive context depends on: (1) *how many* competitors she has, and (2) *how competitive* her competitors can be. For a fund $\widetilde{c}$ in the MMF tournament, the multiplier, $Q(\widetilde{c})$, captures both effects through

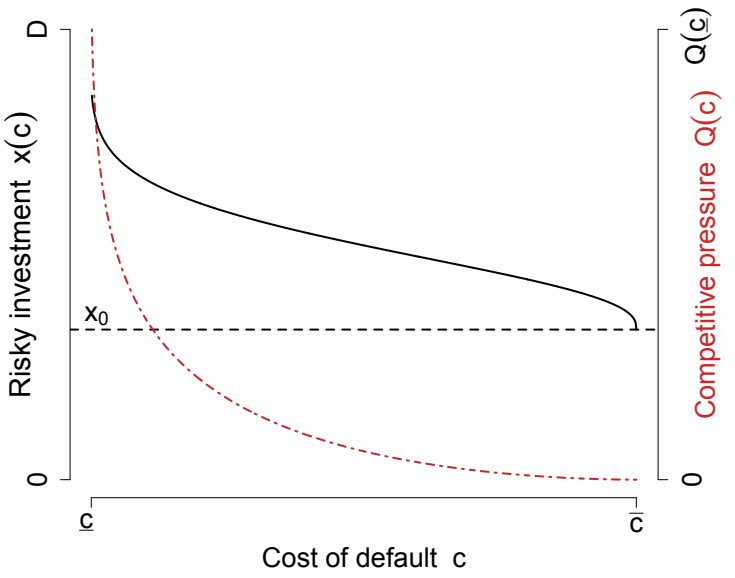


FIGURE 1.2: **Equilibrium risky investment and competitive pressure.** The solid black line (left y-axis) is the equilibrium risky investment, $x(c)$, as function of the default cost. The dashed black line is the maximum risky investment such that the probability of default is zero, x_0. The dot-dashed red line (right y-axis) is the incentive multiplier, $Q(c)$, as function of the default cost. $Q(c)$ measures the competitive pressure faced by each fund in equilibrium.

the interaction term:

$$\mathbb{E}_C\left[\left(\underbrace{\gamma D\left(F_C(c)+a\right)+c}_{\substack{\text{marginal cost}\\ \text{of risky investment}}}\right)^{-1}\middle| c>\widetilde{c}\right]\cdot\underbrace{\left(1-F_C(\widetilde{c})\right)}_{\substack{\text{mass of funds with}\\ \text{higher default costs}}}$$

$1-F_C(\widetilde{c})$ represents the mass of fund $\widetilde{c}$'s competitors, i.e. the mass of funds with higher default costs. $\mathbb{E}_C\left[\left(\gamma D\left(F_C(c)+a\right)+c\right)^{-1}|c>\widetilde{c}\right]$ captures the competitiveness of fund $\widetilde{c}$'s competitors: it is the average of their inverse marginal cost of increasing the default probability by investing in the risky asset. Investing more in the risky asset increases the *ex post* performance rank if the realized risky return is above the risk-free rate, but it *decreases* the *ex post* performance rank if the realized risky return falls below the risk-free rate. Hence, for a competitor c, increasing the probability of default by investing more in the risky asset has both a *direct cost*, its explicit cost of default c, and an *opportunity cost*, the AUM that c will receive at the end of the tournament if it does not default, and the risky return falls below the risk-free rate. This opportunity cost is $\gamma D\left(F_C(c)+a\right)$.[19] The economic intuition is: the lower the competitors' marginal cost of

[19]This is because the equilibrium risky investment decreases with the cost of default. If the risky return falls below the risk-free rate, the *ex post* rank of performance is equal to the mass of funds with lower costs of default.

risky investment is, the more competitive they can be. Figure 1.3 shows the two components of competitive pressure at work.

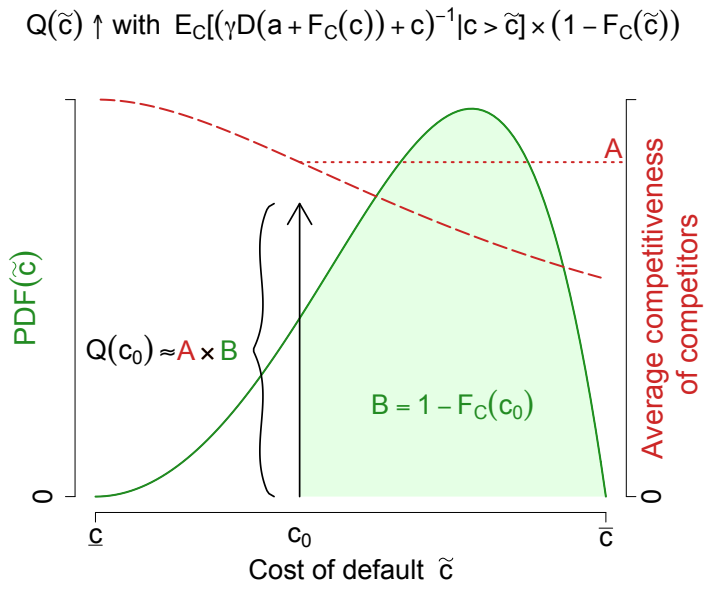


FIGURE 1.3: **Components of competitive pressure.** The solid green line (left y-axis) is the probability density function of the default cost, f_C. The green shaded area is the mass of funds with default costs higher than c_0 and represents the mass of fund c_0's competitors. The long-dashed red line (right y-axis) is the average inverse marginal cost of risky investment for funds with default costs higher than $\widetilde{c}$ as a function of $\widetilde{c}$ and represents the competitiveness of a fund's competitors. The upward arrow is the product of the mass of fund c_0's competitors and their competitiveness (B and A, respectively) and represents the competitive pressure faced by c_0 in equilibrium.

$Q(c)$ shows that competitive pressure (1) is a local property in the MMF tournament, in the sense that it is fund-specific, and (2) is determined only by the distribution of default costs in the industry. Importantly, competitive pressure does not depend on the distribution of returns.

Standard risk premium, approximate equilibrium, and sufficient conditions

The tournament incentive is a spread between risky and safe returns expressed in terms of probabilities. Under mild regularity conditions on the distribution of risky returns, it can be directly related to the standard risk premium.[20]

Lemma 1.5. *Suppose that F_R is twice differentiable at $\mu := \mathbb{E}[R]$ and $|F_R(\mu) - 0.5|$ is small. Then,*

$$q(R_f) \approx f_R(\mu)\,(\mu - R_f) \qquad \text{for small } \mu - R_f > 0.$$

[20] Hereafter, $f \approx g(x)$ for small x means $f = g(x) + o(x)$ in the standard small $o(\cdot)$ notation.

If the distribution of risky returns is sufficiently smooth, with its mean and median being close, $q(R_f)$ is linearly proportional to the standard risk premium, $\mu - R_f$, when the standard risk premium is small. Since the spread on the risky securities available to MMFs is typically very small, the approximation provided by Lemma 1.5 is likely to hold in the data. This relation suggests to proxy the tournament incentive with a measure of risk premium in the empirical analysis.

For some comparative statics I study below, the equilibrium risky investment (1.3) is not easily tractable. However, under mild regularity conditions, it can be written in a more tractable form.

Corollary 1.6. *Suppose that F_R is twice differentiable on $[\underline{R}, 1]$ and $f_R(\underline{R}) > 0$. Then, for small equilibrium default probability, i.e. small $q(R_f)Q(c)$, the equilibrium risky investment is*

$$x(c) \approx \left(1 + 2\frac{q(R_f)}{f_R(\underline{R})(R_f - \underline{R})}Q(c)\right) x_0\,. \tag{1.4}$$

If the distribution of risky returns is sufficiently smooth in its left tail, the equilibrium risky investment is proportional to the tournament incentive, normalized by a measure of tail risk and scaled by the fund-specific multiplier. Approximation (1.4) always holds for funds with higher default costs because they keep the default probability close to zero. It also holds for all funds if the maximum competitive pressure in the industry is sufficiently small. In the following, I will use (1.4) to study how the cross-sectional risk-taking differential reacts to changes in the risk premium and riskiness of the risky asset.

Finally, the following corollary provides two sufficient conditions, with a straightforward economic interpretation, for the existence of the equilibrium.

Corollary 1.7. *The equilibrium exists if*

$$\textit{either} \quad \left(e^{1/a} - 1\right)^{-1} \geq 2\frac{q(R_f)}{F_R(1)}; \qquad \textit{or} \quad \left(e^{\gamma D/\underline{c}} - 1\right)^{-1} \geq 2\frac{q(R_f)}{F_R(1)}$$

The first condition says that the fraction of AUM that do not depend on fund performance must be sufficiently larger than the tournament incentive, normalized by the probability of risky returns falling below the rate on deposits (\$1). The second condition says that the minimum default cost in the industry must be sufficiently larger than the normalized tournament incentive. Both conditions are likely to hold in the MMF industry, where the spread on eligible risky securities is small, and therefore $q(R_f)$ is also small (Lemma 1.5). Moreover, the second condition is likely to hold more generally because the cost of "breaking the buck," as it happened

to Reserve Primary Fund, is arguably very high in absolute terms even for those funds with relatively low default costs.

1.5 Shocks to Asset Returns

This section studies how the equilibrium responds to changes in the risk-free rate and distribution of risky returns. The goal of this section is to characterize the "reach for yield" behavior of MMFs in response to changes in the available investment opportunities.

The equilibrium default probability, $p(c)$, depends on asset returns only via the tournament incentive, $q(R_f) = 0.5 - F_R(R_f)$. It does not depend on the level of the risk-free rate or other parts of the risky return distribution. This is because, absent default, the payoff only depends on relative performance, and in case of default, the payoff is a fund-specific fixed cost that is independent of how much the fund defaulted.[21]

Proposition 1.8. *The equilibrium default probability $p(c)$ increases with the tournament incentive $q(R_f)$ for all funds, except for that with the highest default cost for which it is always zero. The effect of $q(R_f)$ on the equilibrium default probability is stronger for funds with lower default costs.*

Proposition 1.8 follows immediately from the formula for the equilibrium default probability, $p(c) = 2q(R_f)Q(c)$. An increases in $q(R_f)$ increases the equilibrium default probability of all funds, except that with the highest default cost, for which $Q(\overline{c}) = 0$. Since the effect of $q(R_f)$ is weighted by the idiosyncratic multiplier $Q(c)$, it is stronger for funds with lower default costs.

The equilibrium risky investment, on the other hand, does depend on the level of the risk-free rate and the risky return distribution. The distribution of risky returns affects risky investment via $F_R^{-1}(p(c))$. Importantly, only the left tail of the distribution matters. This is because from the no-short-selling constraint $x(c) \leq D$, and hence the equilibrium default probability must be smaller than or equal to the probability that the risky return falls below 1, i.e. $p(c) \leq F_R(1)$. The risk-free rate, on the other hand, affects the equilibrium risky investment both explicitly via its level, R_f, and implicitly via the tournament incentive, $q(R_f)$. In the following section, I study the effects of these variables on fund risky investment both separately and jointly. This allows me to make predictions on the risky investment of MMFs when changes in the risk-free rate, risk premium and riskiness of the risky assets occur simultaneously and to disentangle the different channels.

[21] This feature can be regarded as a Value-at-Risk rule, or a form of limited liability.

1.5.1 Changes to the risk premium, holding the risk-free rate constant

First, I consider the effect of changes in the risk premium, holding the risk-free rate constant. In the real world, the premium on risky assets usually increases with their riskiness. To mimic this scenario, I consider an increase in the tournament incentive accompanied by an increase in the left tail of the return distribution.

I show that under realistic conditions funds with high and low default costs respond in opposite ways to such changes. The increase in the tournament incentive increases the equilibrium default probability of (almost) all funds. On the other hand, for a given default probability, a shift to the left in the return distribution reduces the corresponding amount of risky investment. The heterogeneous competitive pressure in equilibrium determines which effect dominates.

Let $H := \dfrac{F_R(r)}{F_R(1)}$ for all $r \in [\underline{R}, 1]$ be the left tail of F_R, renormalized to 1 so to have a proper distribution function. Suppose there is a stochastic shift from $H^{(1)}$ to $H^{(2)}$, both with support $[\underline{R}, 1]$. In particular, assume that $H^{(1)}$ dominates $H^{(2)}$ in terms of likelihood ratio order, $H^{(1)} \succ_{LRD} H^{(2)}$. Finally, suppose the tournament incentive goes from $q^{(1)}$ to $q^{(2)} > q^{(1)}$.

Proposition 1.9. *Let $H^{(1)} \succ_{LRD} H^{(2)}$ and $q^{(1)} < q^{(2)}$.*

(i) If $\dfrac{q^{(2)}}{q^{(1)}} > (\geq) \sup \dfrac{H^{(2)}}{H^{(1)}}$, all funds (weakly) increase their risky investment.

(ii) If $\dfrac{q^{(2)}}{q^{(1)}} < \sup \dfrac{H^{(2)}}{H^{(1)}}$,

- *(a) funds with relatively high costs of default decrease their risky investment;*
- *(b) funds with relatively low costs of default increase their risky investment if and only if they face sufficiently high competitive pressure.*

Moreover, if $\dfrac{H^{(2)}}{H^{(1)}}$ is decreasing, the cutting point between (a) and (b) is unique.

Part (i) provides a predictable result: if the increase in the tournament incentive is sufficiently larger than the increase in the riskiness of the risky asset,[22] all funds increase their risky investment.

Part (ii) considers the more realistic and interesting scenario when the increase in the probability of low returns and that in the tournament incentive are of comparable sizes.[23] The intuition

[22] I.e., the growth in q is larger than the maximum growth in the left tail of the return distribution.
[23] I.e., $q^{(2)}/q^{(1)} < \sup H^{(2)}/H^{(1)}$.

for this case is as follows. Funds with higher default costs face lower competitive pressure and keep their default probability closer to zero. Therefore, their risky investment is more sensitive to shocks in the probability of very low returns. Under likelihood ratio dominance, the growth in the leftmost part of the left tail of the return distribution is greater than the growth in the tournament incentive. Hence, even though their default probability increases, funds with higher default costs are forced to decrease their risky investment.[24] On the other hand, funds with lower default costs have a larger incentive multiplier, due to larger competitive pressure, and are more sensitive to shocks in the tournament incentive. If competitive pressure on those funds is sufficiently high, the increase in q dominates, and their risky investment increases. In other words, the heterogeneous competitive pressure in equilibrium generates a bifurcation in the response of funds' risky investment to changes in the risk premium.

Importantly, this bifurcation occurs even if we substitute the likelihood ratio dominance in the left tail with a simple decrease in the lowest possible risky return (i.e., $\underline{R}^{(2)} < \underline{R}^{(1)}$), confirming the robustness of the above economic intuition.

These results suggest that the cross-sectional risky investment differential increases with the premium and riskiness of the risky asset. When competitive pressure on funds with lower default costs is high, this intuition is formalized by Proposition 1.9 (ii). To have a formal result for when the competitive pressure on those funds is low, I use the approximate equilibrium (1.4):

$$x_{app}(c) := (1 + 2\widetilde{q}Q(c))\, x_0, \qquad where \quad \widetilde{q} := \frac{q(R_f)}{f_R(\underline{R})(R_f - \underline{R})}$$

As discussed above, this approximation is valid for all funds when the competitive pressure on funds with lower default costs is low. Since $\widetilde{q}$ incorporates both the tournament incentive and the risk of low returns, I differentiate the approximate equilibrium w.r.t. $\widetilde{q}$ to capture the effect of a simultaneous change in both variables.[25]

Corollary 1.10. $\dfrac{\mathrm{d}}{\mathrm{d}\widetilde{q}} \left| \dfrac{\mathrm{d}x_{app}(c)}{\mathrm{d}c} \right| > 0$ *for all* c.

Corollary 1.10 confirms that the cross-sectional risky investment differential increases with the risk premium also when competitive pressure on funds with lower default costs is low.

[24] Except the fund with the highest cost of default, which always invests x_0. Since, by assumption, $H^{(1)}$ and $H^{(2)}$ have the same support, x_0 does not change after the stochastic shift in the tail.

[25] Note that $\mathrm{d}x_{app}/\mathrm{d}\widetilde{q} > 0$ for all c. In Proposition 1.9, this corresponds to (i) vs. (ii) when competitive pressure is low, which is the necessary condition for x_{app} to be a valid approximation of the equilibrium for all funds.

1.5.2 Changes to the risk-free rate, holding the risk premium constant

Here I consider changes in the risk-free rate, holding the tournament incentive $q(R_f)$ constant. Since q is proportional to the standard risk premium, this amounts to assume that the risk premium remains constant when the risk-free rate changes. This exercise can be regarded as a rigid shift of the distribution of risky returns together with the risk-free rate.

Holding the tournament incentive constant, a decrease in the risk-free rate does not change the equilibrium default probability of any fund. On the other hand, it forces all funds to invest more in the safe asset to keep the same probability of default in equilibrium.

Proposition 1.11. *Holding the tournament incentive $q(R_f)$ constant, the equilibrium risky investment strictly increases with the risk-free rate for all funds.*

This effect on equilibrium risky investment is stronger for funds with relatively higher default costs. To see this, let us consider the following scenario. Suppose that the competitive pressure on the fund with the lowest default cost, $\underline{c}$, is sufficiently high so that its equilibrium default probability is exactly equal to $F_R(1)$. That is, $\underline{c}$ fully invests its portfolio in the risky asset, i.e., $x(\underline{c}) = D$. Holding $q(R_f)$ constant, this equilibrium investment is unaffected by changes in the level of the risk-free rate. On the other hand, in equilibrium, the fund with the highest default cost invests exactly x_0, which increases with the risk-free rate. Hence, when the risk-free rate decreases, holding $q(R_f)$ constant, the risky investment differential between funds with the highest and lowest default costs increases. This intuition is summarized by the following corollary.

Corollary 1.12. *Holding $q(R_f)$ constant, if $\underline{R} > 2 - R_f$, the cross-sectional risky investment differential decreases with the risk-free rate. That is,*

$$\frac{\partial}{\partial R_f}\left|\frac{\mathrm{d}x(c)}{\mathrm{d}c}\right| < 0 \quad \textit{for all } c \textit{ if } \underline{R} > 2 - R_f.$$

The partial derivative with respect to R_f indicates that $q(R_f)$ is being held constant. The assumption that the lowest possible return on the risky asset is not too low is likely to hold in the data,[26] since MMFs can only invest in securities of the highest credit quality and very short maturity. Moreover, for the approximate equilibrium (1.4) of Corollary 1.6, the result in Corollary 1.12 holds true without any assumption on $\underline{R}$, further confirming the above economic intuition.

[26] Unless the net risk-free rate is exactly zero, which is ruled out by model's assumption.

1.5.3 Simultaneous changes in the risk premium and risk-free rate

Finally, in the real world, periods of low risk-free rates are often associated with periods of high risk premia. Here I do comparative statics for this scenario.

For simplicity, I hold the distribution of risky returns constant, so that a decrease in the risk-free rate, R_f, mechanically increases the tournament incentive, $q(R_f) = 0.5 - F_R(R_f)$.[27] This corresponds to a mechanical increase in the risk premium (Lemma 1.5). In the following proposition, the symbol of total derivative with respect to R_f indicates exactly this scenario: $q(R_f)$ is allowed to vary with R_f, while the distribution of returns is held constant. That is, $\mathrm{d}x/\mathrm{d}R_f > (<)0$ means that risky investment decreases (increases) when the risk-free rate goes down and the tournament incentive (i.e., the premium) goes up.

ASSUMPTION 2. The reverse hazard rate of the risky return, $\frac{f_R}{F_R}$, is non-increasing on $[\underline{R}, 1)$.

Proposition 1.13. *Under Assumption 2, there exists $c^* \in (\underline{c}, \overline{c})$ s.t.*

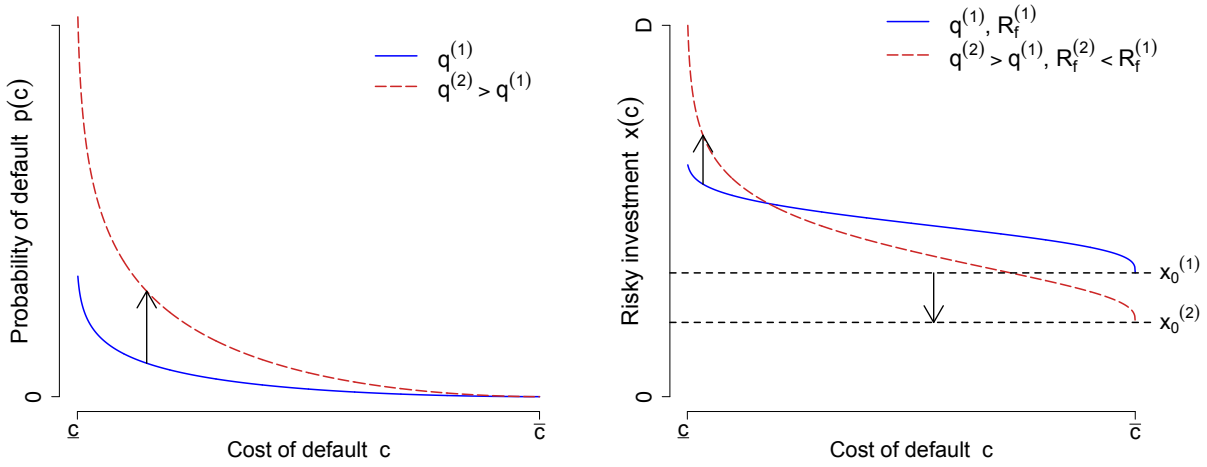
i) $\dfrac{\mathrm{d}x(c)}{\mathrm{d}R_f} > 0$ *for all* $c > c^*$*;*

ii) $\dfrac{\mathrm{d}x(c)}{\mathrm{d}R_f} < 0$ *for all* $c < c^*$ *if and only if the competitive pressure on the funds with the lowest cost of default, $Q(\underline{c})$, is sufficiently high.*

A decrease in the risk-free rate that mechanically increases the tournament incentive also increases the equilibrium default probability of (almost) all funds, with the effect being stronger for funds with lower default costs. On the other hand, holding the default probability constant, a decrease in the risk-free rate decreases the equilibrium risky investment of all funds, with the effect being stronger for funds with higher default costs. The idiosyncratic multiplier, $Q(c)$, determines which effect dominates by measuring the relative importance of fund competition. To see this, take the fund with the highest cost of default, $\overline{c}$. $\overline{c}$ is unaffected by competition ($Q(\overline{c}) = 0$) and always keeps the equilibrium default probability equal to zero by investing exactly x_0. Since x_0 increases with R_f, after a decrease in the risk-free rate, $\overline{c}$ is forced to cut its risky investment to keep its default probability equal to zero, even though $q(R_f)$ increases. On the other hand, funds with relatively low costs of default face a higher competitive pressure, captured by higher $Q(c)$. If competition is sufficiently strong, $Q(c)$ is sufficiently large and the effect on the default probability via $q(R_f)$ dominates. Figure 1.4 qualitatively shows this result.

[27] For simplicity, I do not consider an increase in the left tail of the return distribution because, from the previous section, we already know that its effect goes in the same direction as that of a decrease in R_f.

Finally, note that Assumption 2 is very weak. Many common distributions with support in $\mathbb{R}_+$ satisfy a decreasing reverse hazard rate condition in the left tail, including uniform, log-normal, Beta, chi-squared, and exponential (Shaked and Shanthikumar, 1994). Even more importantly, Assumption 2 is not necessary for part (i) of Proposition 1.13.



(A) Equilibrium default probability (B) Equilibrium risky investment

FIGURE 1.4: **Equilibrium risk-taking when the premium (q) increases and the risk-free rate (R_f) decreases.** Left panel: default probability (unaffected by changes in R_f). Right panel: risky investment.

1.6 Shocks to the competitive environment

For now I have only considered how fund risk-taking responds to shocks in investment opportunities. This section studies how it responds to changes in the competitive environment. As discussed in Section 1.4, fund-specific competitive pressure is uniquely determined by the distribution of default costs. Suppose the distribution of default costs shifts from $F_C^{(1)}$ to $F_C^{(2)}$, both with support $[\underline{c}, \overline{c}]$.

Proposition 1.14. *If $F_C^{(2)} \succ_{LRD} F_C^{(1)}$, there exist $c_* \leq c^* \in (\underline{c}, \overline{c})$ s.t. equilibrium default probability and risky investment decrease for all $c < c_*$ and increase for all $c \in (c^*, \overline{c})$. The equilibrium default probability and risky investment of $\overline{c}$ remain the same: 0 and x_0, respectively.*

The intuition is as follows. The equilibrium risk-taking of fund $\widetilde{c}$ depends on the distribution of default costs only through the incentive multiplier $Q(\widetilde{c})$, which strictly increases with

$$\mathbb{E}_C\left[(\gamma D(F_C(c)+a)+c)^{-1}\,|c>\widetilde{c}\right](1-F_C(\widetilde{c}))=\int_{\widetilde{c}}^{\overline{c}}\frac{f_C(u)\mathrm{d}u}{\gamma D(F_C(u)+a)+u}\,. \tag{1.5}$$

The fund with the highest default cost, $\overline{c}$, is unaffected by shocks to the distribution of default costs because it is unaffected by competition. That is, $Q(\overline{c})=0$ under any F_C. For the other funds with relatively high default costs, the LRD shift increases both the mass of competitors ($f_C^{(2)}(u)>f_C^{(1)}(u)$ in the upper tail) and their competitiveness by lowering the opportunity cost of risky investment ($F_C^{(2)}(u)<F_C^{(1)}(u)$ everywhere). As a result, for large $\widetilde{c}$, the right-hand side of (1.5) increases, increasing the multiplier and so the risk-taking of funds with relatively high default costs. On the other hand, for the fund with the lowest default cost $\underline{c}$, the mass of competitors remains equal to 1, and their average opportunity cost of risky investment does not change.[28] However, the LRD shift decreases the average competitiveness of $\underline{c}$'s competitors by increasing the average cost of default in the industry. Hence, for $\underline{c}$ (and the other funds with relatively low default costs), a shift of F_C to the right decreases the incentive multiplier and so the risk-taking.[29]

Intuitively, if the distribution of default costs shifts to the right, competition becomes relatively stronger for funds with higher costs of default, and relatively weaker for funds with lower costs of default. Again, this result shows that, in the MMF tournament, competitive pressure is not a global property of the industry but a local property of each fund.

Proposition 1.14 suggests that shocks to the competitive landscape might have surprising effects in the aggregate. For example, if the right tail of the distribution of default costs is sufficiently fat, an increase in the fraction of funds with relatively high default costs could increase aggregate risk-taking rather than decrease it. This is because the increase in risk-taking by funds with higher default costs could more than offset the decrease in risk-taking by funds with lower default costs.

Model Predictions: Summary

The model makes the following testable predictions.

[28] Specifically, $\int_{\underline{c}}^{\overline{c}}\frac{f(u)\mathrm{d}u}{\gamma D(F(u)+a)}=\frac{1}{\gamma D}\log\left(1+a^{-1}\right)$ for any CDF F on $[\underline{c},\overline{c}]$.

[29] The assumption of LRD is made only for simplicity. For Proposition 1.14 to hold, it is sufficient to assume a first-order stochastic dominance shift such that the two density functions cross only a finite number of times.

P.1 Funds with lower costs of default always hold more risky assets.

P.2 Holding the risk-free rate constant, an increase in the risk premium:

(a) decreases the risky investment of funds with higher default costs;

(b) increases the risky investment of funds with lower default costs (if and only if they face sufficiently high competitive pressure);

(c) always increases the cross-sectional differential.

P.3 Holding the risk premium constant, a decrease in the risk-free rate:

(a) decreases the risky investment of all funds;

(b) increases the cross-sectional differential.

P.4 When the fraction of funds with relatively high costs of default increases, funds with lower default costs decrease their risk-taking, while funds with higher default costs increase it.

The following empirical analysis provides evidence that supports predictions P.1 to P.3. Testing P.4 is left for future work.

Default Probability

The model also makes predictions on the equilibrium default probability. Specifically, all funds have a strictly positive probability of "breaking the buck," and this probability decreases with fund's cost of default. That is, all MMFs are not perfectly safe *ex ante*. Also, the probability of "breaking the buck" increases with the risk premium, with the effect being stronger for funds with lower default costs. The level of the risk-free rate, on the other hand, does not affect the probability of default.

Since in the data I do not observe MMF portfolio holdings at the security level, I cannot test these predictions in my empirical analysis. However, Brady, Anadu, and Cooper (2012) find that at least 21 MMFs would have broken the buck if they had not received sponsor support between 2007 and 2011. That period was characterized by low risk-free rates and high risk premia for MMFs. Their data suggest that sponsor support was frequent and significant: 78 MMFs (out of a total of 341 MMFs) received sponsor support in 123 instances for a total amount of at least $4.4 billion. This evidence supports model's predictions on the default probability of money market funds.

1.7 Empirical Analysis

As other recent studies on prime MMFs, I focus on institutional funds because they exhibit a stronger flow-performance relation than retail funds (KS; Chernenko and Sunderam, 2014). I consider the period from January 2006 to August 2008 because in this period there were significant variations in both the risk premium and the risk-free rate. I use those variations to identify the differential effects of risk premia and risk-free rates on fund risk-taking highlighted by the model.

Mapping the model to the data

To map the model to the data, I use the notion of sponsor's reputation (or business) concern introduced by KS. The fund's cost of default in the model is the sponsor's cost of possible negative spillovers in the data. The rationale is that sponsors with a larger share of non-MMF business expect to incur in larger costs if the NAV of their MMFs falls below \$1. This is because of possible outflows from other mutual funds managed by the same sponsor or a loss of sponsor's other business due to reputational damages. Following KS, I proxy sponsor's reputation concern using

$$Fund\,Business = \frac{\text{sponsor's mutual fund assets not in institutional prime MMFs}}{\text{sponsor's total mutual fund assets}}$$

Fund Business is the share of sponsor's mutual fund assets that are not in prime institutional MMFs. Another plausible measure of sponsor's reputation concern is affiliation to financial conglomerates (e.g., banks or insurance companies). However, this proxy (*Conglomerate*) is a binary variable, while in my model the cost of default is a continuous variable. *Fund Business* is continuous by construction and is therefore the natural proxy for the model's cost of default.[30]

As discussed above, the tournament incentive $q(R_f)$ in the model is mapped into the risk premium in the data. My main proxy for the risk premium is the excess bond premium for financial firms introduced by Gilchrist and Zakrajsek (2012), hereafter referred to as *GZ Premium*. Since MMFs mainly invest in debt securities issued by financial firms, this is the most appropriate measure of risk premium for MMFs. As robustness check, I also use an index of realized spreads on the risky securities available to MMFs. This index is defined and discussed below.

[30]However, in some regressions, I also use *Conglomerate* as extra control for sponsor's reputation concern.

My main proxy for the risk-free rate is the return on 1-month T-bills. Since in the tournament model the risk-free rate affects fund risk-taking through the return on safe assets (e.g., treasuries), and MMFs are restricted to invest only in short-term securities, this is the appropriate proxy for the model's risk-free rate. As robustness check, I also use the return on 3-month T-bills as secondary proxy for the risk-free rate.

The data set

I construct a data set that maps MMFs to their sponsors. Data on individual MMFs are provided by iMoneyNet. Data on fund sponsors are from the CRSP Mutual Fund Database.

iMoneyNet data are the most comprehensive source of information on MMF holdings and have been used by KS, Chodorow-Reich (2014), and Di Maggio and Kacperczyk (2014). They are at the weekly, share-class level and contain information on yields, assets under management, expense ratio, age, portfolio composition by instrument type, and weighted average maturity. Since my model is at the fund level, I aggregate share classes by fund and compute fund characteristics as the weighted average of the share class values, with assets per share class as weights. Details on the construction of the data set are in Appendix 1.D.

Data on the *GZ Premium* are at the monthly level from Simon Gilchrist's website: `http://people.bu.edu/sgilchri/Data/data.htm`. Data on T-bill rates are from Kenneth French's website (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`) and CRSP.

Sumamry Statistics

Table 1.D.1 in Appendix 1.D shows summary statistics for all institutional prime MMFs as of January 2006. The sample includes 143 funds and 82 sponsors. The average fund size is \$6.3 billion and the average fund age is 11.2 years. The spread is computed as the annualized gross yield (i.e., before expenses) minus the yield of the 1-month Treasury bill. The average spread is 7.5 basis points and the average expense ratio is 35.9 basis points. In terms of assets holdings, funds hold 31.4% in commercial paper, 19.9% in floating-rate notes, 13.4% in repurchase agreements, 13.6% in asset-backed commercial paper, 12.4% in bank obligations, 5.9% in U.S. Treasuries and agency-backed debt, and 3.4% in time deposits. The average family size at the sponsor level is \$73.3 billion, and the average fund business is 74.5%.[31]

[31] These results are very close to those of KS, confirming the consistency of my data set with theirs.

Since my analysis focuses on the different risk-taking of MMFs with high and low default costs, I check that funds whose sponsors have different levels of reputation concern do not significantly differ along other dimensions. Column (2) and (3) of Table 1.D.1 show summary statistics for funds whose sponsors have *Fund Business* above and below the industry median, respectively. I find that both groups are quite similar in terms of observable characteristics, such as spread, expense ratio, maturity, age, and holdings. The main differences are that funds sponsored by firms with higher *Fund Business* are (1) smaller on average, and (2) on average less likely to be part of financial conglomerates. These results suggest that the extent of a sponsor's non-MMF business is not systematically correlated with other observable fund characteristics.

Appendix 1.D also analyzes the distributional properties of *Fund Business*. It shows that there is significant, widespread variation in the cross-section of sponsors' reputation concerns, which supports the validity of a "continuum-of-funds" approach and helps the identification of reputation effects on the cross-sectional risk-taking differential. In various robustness checks, I also show that *Fund Business* does not covary with fund's incurred expenses, which means that my results are not driven by the cross-sectional variation considered by Chodorow-Reich (2014). On the other hand, the cross-sectional correlation between *Fund Business* and affiliation to a financial conglomerate–the heterogeneity considered by Di Maggio and Kacperczyk (2014)– tends to be negative but not in a statistically significant way. These results are omitted for brevity but are available upon request.

1.7.1 Investment Opportunities: Risk Premium vs. Risk-free Rate

Prime MMFs can invest only in U.S treasuries, GSE debt, repurchase agreements, certificate of deposits (i.e., time deposits and bank obligations), floating-rate notes, commercial papers, and asset-backed commercial papers. Among these eligible securities, U.S. treasuries, GSE debt, and repos are the safest ones. Certificates of deposits (CDs), floating-rate notes (FRNs), commercial papers (CPs) and asset-backed commercial papers (ABCPs) have historically been the riskiest ones.

To investigate the time variation in the investment opportunities available to MMFs, I construct an index of spreads on the risky securities available to MMFs. The index contains the 3-month CD rate, 3-month LIBOR (often used as reference rate for FRNs), 3-month AA financial CP rate, and 3-month AA ABCP rate. Data are at the monthly level from FRED. The index is

$$Spread\ Index_t = \left(a_{2006}^{CD}\, r_t^{CD} + a_{2006}^{FRNS}\, r_t^{LIBOR} + a_{2006}^{CP}\, r_t^{CP} + a_{2006}^{ABCP}\, r_t^{ABCP}\right) - GS3M_t \quad (1.6)$$

where r_t^K is the interest rate of category K in month t, and $GS3M_t$ is the 3-month constant maturity rate on T-bills (from FRED). The coefficient a_{2006}^K is the industry average relative weight of category K in the portfolios of institutional prime MMFs' as of January 3, 2006. Weights are held constant as of January 2006 to alleviate possible endogeneity issues. a's are normalized to sum up to 1. Figure 1.5 shows *Spread Index* (red line) from January 2006 to August 2008. Before July 2007, *Spread Index* was consistently below 0.5%, with an average value of 0.3%. From July 2007, *Spread Index* started to rise, reaching 1.4% in August 2007 and a maximum of 1.9% in August 2008, with an average value of 1.4% from August 2007 to August 2008.

Spread Index is an *ex post* measure of investment opportunities, which I use as secondary proxy for the risk premium. Figure 1.5 also shows my primary proxy for the risk premium, *GZ Premium* (blue line). The pattern is similar. Until July 2007, it was negative and relatively flat with an average value of -0.27%. In August 2007, it became positive and started to rise steadily, reaching a maximum of almost 2% in August 2008, with an average value of more than 0.9% from August 2007 to August 2008. The time paths of *Spread Index* and *GZ Premium* indicate that starting from July-August 2007 the premia available to prime MMFs experienced a significant increase. Hereafter, I refer to the period from January 2006 to July 2007 as the *Pre* period, and to the period from August 2007 to August 2008 as the *Post* period.

Figure 1.5 also shows the 1-month T-bill monthly return (green line) over the period of interest. The 1-month T-bill rate was relatively high in the pre period, with an average value of 40 basis points (bps), and decreased substantially in the post period, reaching a minimum of 9 bps in May 2008, with an average value of 22 bps. This evidence says that, in the period of analysis, low risk-free rates were associated with high risk premia, and *vice versa*. Importantly, note that the increase in risk premia started few months earlier than the decrease in the risk-free rate, which helps to identify the differential effect of these variables on fund risk-taking.

Proxies of risk-taking

Since *Spread Index* and *GZ Premium* aggregate risk premia across different instrument types, they do not identify the single, riskiest asset class available to MMFs in the period of analysis. In order to do that, since I do not observe fund portfolios at the individual security level, I run a panel regression with current fund spread on the left-hand side and past holdings by instrument category, together with a set of controls, on the right-hand side. Details on the regression specification are in Appendix 1.E, and results are in Table 1.E.1. I find that bank obligations experienced the largest increase in spread relative to U.S. treasuries in the post

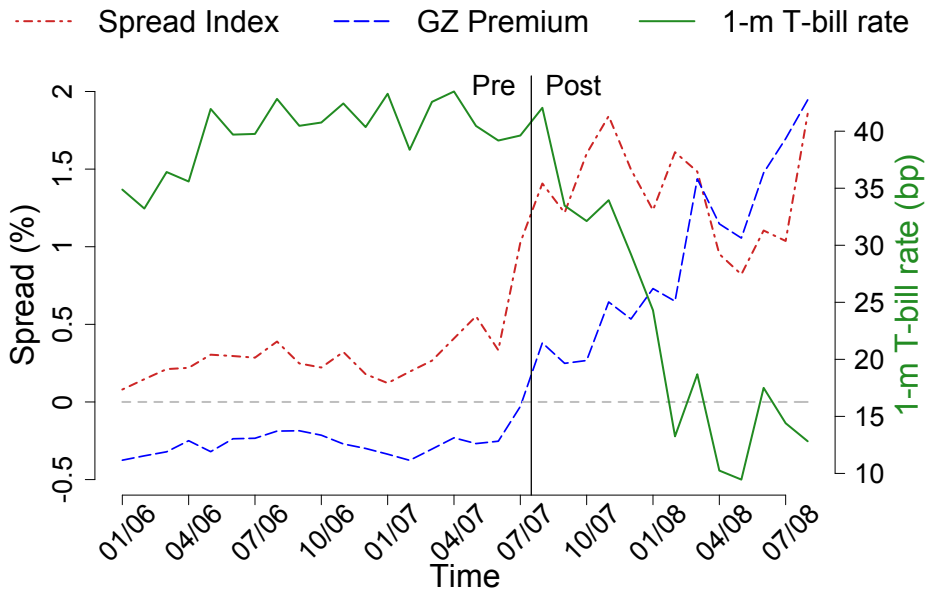


FIGURE 1.5: **Risk Premium vs. Risk-free Rate.** *Spread Index* (dot-dashed line) is an index of spreads on risky securities available to prime MMFs, weighted according to the industry average holdings as of January 3, 2006 (see (1.6)). *GZ Premium* (dashed line) is the the excess bond premium for financial firms from Gilchrsit and Zakrajsek (2012). The solid line, whose scale in on the righ y-axis, represent the monthly return on the 1-month T-bill. Data are monthly.

period and were the riskiest asset class over the whole period. In view of this result, I use the percentage holdings of bank obligations net of U.S. treasuries, GSE debt, and repurchase agreements as a measure of the riskiness of MMF portfolios in terms of asset class composition. I refer to this measure as *Holdings Risk* and use it as main proxy for fund risk-taking in my regressions.[32] In the following, I also use other proxies for fund risk-taking: *Maturity Risk* and *Spread*. *Maturity Risk*, is the weighted average maturity of assets in a fund portfolio. In general, funds with longer portfolio maturities are considered riskier. *Spread*, is fund's gross yield minus the 1-month T-bill rate. In the context of MMFs, the spread is a good measure of risk because there is little scope for managerial skill, so that fund spreads largely reflect fund portfolio risk.[33]

Finally, for robustness purposes, I also use the share of U.S. treasuries, GSE debt, and repos as negative proxy for fund risk-taking. As mentioned above, these asset classes have consistently been the safest instruments available to prime MMFs. In the following, I refer to this (negative) measure of risk-taking as *Safe Holdings*.

[32] KS obtained similar results and used the same measure of risk-taking in their empirical analysis.

[33] A potential problem with using this measure is that it may vary over time even though managers may not actively change the risk profile of their portfolios, only because the yields of individual assets in the portfolio change.

1.7.2 Flow-performance Relationship and Tournament Assumption

This sections analyzes the flow-performance relation in the MMF industry during the period of analysis. In particular, it tests the assumptions that investor money flows are determined by the rank of fund performance (tournament), and not raw performance.

I estimate the sensitivity of fund flows to past performance using the following regression model:

$$Fund\ Flow_{i,t+1} = \alpha_i + \mu_t + \beta Performance_{i,t} + \gamma \cdot X_{it} + \varepsilon_{i,t+1} \tag{1.7}$$

where $Fund\ Flow_{i,t+1}$ is the percentage increase in fund i's size from week t to week $t+1$, adjusted for earned interests and trimmed at the 0.5% level to alleviate the concern of outliers. $Performance_{i,t}$ is a measure of fund i's performance in week t (see below). $X_{i,t}$ is a vector of fund-specific controls that includes the natural logarithm of fund size in millions of dollars ($Log(Fund\ Size)$), fund expenses in basis points ($Expense\ Ratio$), fund age in years (Age), the natural logarithm of the fund family size in billions of dollars ($Log(Family\ Size)$), and the volatility of fund flows ($Flow\ Volatility_{i,t}$), measured as the standard deviation of weekly fund flows over the previous quarter. μ_t denotes week fixed effects, which account for variations in the macroeconomic environment, and α_i denotes fund fixed effects, which account for any unobserved time-invariant fund characteristics within the pre or post periods. The coefficient of interest is β.

In my first specification, $Performance_{i,t}$ is the raw spread: $Spread_{i,t}$ is the annualized gross yield (i.e., before expenses) minus the yield of the 1-month T-bill, in basis points. Columns (1) and (2) of Table 1.1 show the results for this specification. Standard errors are heteroskedasticity-and-autocorrelation (HAC) robust.[34] When raw spreads are used as measure of performance, the flow-performance relation is positive and significant only in the post period.

In the MMF industry, however, cross-sectional spreads are typically very small and a difference of even few basis points can crucially alter fund flows. Hence, raw measures of past performance might not be appropriate to explain investor money flows. In my second specification, $Performance_{i,t}$ is the spread rank: $Spread\ Rank_{i,t}$ is the rank of fund i's spread in week t. The rank is expressed in percentiles normalized over the interval $[0, 1]$, with $Spread\ Rank = 0$ for the worst performance and $Spread\ Rank = 1$ for the best one. Columns (3) and (4) of Table 1.1 show the results.

[34] Hereafter, unless otherwise specified, reported standard errors are always HAC robust, and all panel data models are estimated using the "within" estimator.

When the spread rank is the main explanatory variable, the flow-performance relation is positive and statistically significant in both periods, and the adjusted R^2 slightly increases (by 0.1–0.2%). These results suggest that ranks are more important than raw performance in explaining money flows to MMFs. To further test this hypothesis, I estimate model (1.7) including both measures of performance. Columns (5) and (6) of Table 1.1 show the results.

When both raw spreads and spread ranks are included, the spread rank remains positive and statistically significant in both periods, while the raw spread is not statistically significant in either one. These results indicate that the performance rank, not the raw performance, is the actual determinant of fund flows in the MMF industry.[35] Moving from the lowest to the highest rank of past performance increases subsequent fund flows by 0.92% per week in the pre period and 1.79% per week in the post period. Both effects are economically large because they imply that a fund could increase its annual revenue by 60.9% in the pre period and 151.8% in the post period by moving from the lowest to the highest rank.[36]

As robustness check, I run regression (1.7) using the rank of *Fund Flow* as dependent variable to alleviate the concern of outliers, without resorting to trimming. Results are similar and reported in Table 1.F.1 in Appendix 1.F. As further robustness checks, I also run regression (1.7) trimming the distribution of fund flows at multiples of the interquartile range and/or using only time fixed effects. Results are similar and are available upon request.

In Appendix 1.F, I also show that the flow-performance relation is not explicitly affected by the reputation concerns of fund sponsors. That is, investors do not risk-adjust a fund's yield based on the reputation concerns of its sponsor. This evidence shows that the flow-performance relation can be taken as exogenous in the context of my model. Similar results have also been obtained by KS.

1.7.3 Risk-taking in the Time Series: Who's Reaching For Yield?

This section provides empirical evidence in agreement with model's predictions on the level of MMF risky investment in the time series. The model predicts that an increase in the risk premium increases the risky investment of funds with lower default costs (if they face sufficiently high competition) but decreases the risky investment of funds with higher default costs. On

[35] Massa (1997), and Patel, Zeckhauser, and Hendricks (1994) obtained similar results for equity mutual funds.

[36] An increase equal to the cross-sectional average of the within-fund standard deviation of *Spread Rank*, equal to a shift of 20% across the rank distribution, increases subsequent fund flows by 0.18% per week in the pre period, and 0.36% per week in the post period. These figures imply that a fund, by increasing its rank of 20%, could increase its annual revenue by 10.0% in the pre period and 20.4% in the post period.

the other hand, a decrease in the risk-free rate leads all funds to reduce their risky investment (with the effect being stronger for funds with higher default costs).

In the period from July 2007 to August 2008, there were both a decrease in risk-free rates and an increase in the premia of some securities available to MMFs because of an increase in their riskiness. Figure 1.1 in the Introduction shows the net risky investment (*Holdings Risk*) of funds whose sponsors have reputation concerns (*Fund Business*) consistently below the industry median (dashed red line) and funds whose sponsors have reputation concerns consistently above the industry median (solid blue line). In the context of my model, the first category represents funds with relatively low default costs, and the second one represents funds with relatively high default costs. Figure 1.1 shows that in the period of high risk premia (and low risk-free rates), there is a bifurcation in the risk-taking of MMFs, as predicted by the model: funds with lower default costs do "reach for yield," while funds with higher default costs do the opposite.

To give a more quantitative estimate of the above evidence, I perform the following analysis. I consider a sub-period with relatively high risk-free rates and low risk premia (July-December 2006), and a sub-period with relatively low risk-free rates high risk premia (January-June 2008). The average monthly return on 1-month T-bills was about 41.1 basis points from July to December 2006, and about 15.6 basis points from January to June 2008. The average monthly excess bond premium for financial firms, on the other hand, was about -0.19% from July to December 2006, and about 1.1% from January to June 2008. Table 1.2 reports summary statistics for the 1-month T-bill rate, *GZ Premium*, and *Spread Index* over the two periods. Then, I compare the average change over time in net risk exposure for funds with high *Fund Business* to the average change over time in net risk exposure for funds with low *Fund Business*. For roustness, I do the same for the average change over time in safe holdings. Results are in Table 1.3.

Table 1.3 confirms the qualitative observations of Fig. 1.1. For the funds with high reputation concerns the average change in net risk exposure across the two periods was -3.3 percentage points, and the average change in safe holdings was almost 3.7 percentage points. Both changes are statistically significant at the 1% level. On the contrary, for funds with low reputation concerns the average change in net risk exposure across the two periods is 4.7 percentage points, and the average change in safe holdings is 1.3 percentage points. The change in net risky investment is statistically significant at the 1% level, while the change in safe holdings is not statistically significant.[37] These preliminary results qualitatively confirm the predictions of my model on the risk-taking behavior of MMFs in the time series. This analysis, however, does not

[37]Similar results hold also if we restrict the definition of *Safe Assets* to U.S. Treasuries and GSE debt.

distinguish the effect of decreasing risk-free rates from that of increasing risk premia. In the next section, I try to disentangle the differential effects of risk-free rates and risk premia.

1.7.3.1 Disentangling the risk-free rate from the risk premium

The model predicts that, holding the risk-free rate constant, an increase in the premium and riskiness of the available risky securities increases the risky investment of funds with lower default costs and decrease that of funds with higher default costs (prediction P.2a, b). On the other hand, holding the premium constant, a decrease in the risk-free rate decreases the risky investment of all funds, with the effect being stronger for funds with higher default costs (prediction P.3a). This section aims to disentangle these two channels and shows evidence in support of model's predictions.

Hereafter, I consider only MMFs that remain in the data set throughout the whole period of analysis. There are 122 such funds. On this balanced panel, I consider the following regression:

$$\begin{aligned} Risk_{i,t} =& \alpha_i + \beta_1^H High\,FB_{i,t-1} * \widehat{rp}_t + \beta_2^H High\,FB_{i,t-1} * rf_t + \\ &+ \beta_1^L Low\,FB_{i,t-1} * \widehat{rp}_t + \beta_2^L Low\,FB_{i,t-1} * rf_t + \gamma \cdot X_{i,t-1} + \varepsilon_{i,t} \end{aligned} \tag{1.8}$$

Risk is either *Holdings Risk* or *Maturity Risk* defined as in Section 1.7.1.[38] $\widehat{rp}$ is a proxy for the risk premium: *GZ Premium* in the main specification. rf is the return on 1-month T-bills. Since data on $\widehat{rp}$ are at the monthly level, weekly fund-specific data are averaged over months, and regression (1.8) is run at the monthly level. $High\,FB_{i,t}$ ($Low\,FB_{i,t}$) is a dummy variable that is equal to 1 if fund i's *Fund Business* is above (below) the industry median in month t and zero otherwise. X is the set of fund-specific controls defined in (1.7) with, in addition, sponsor's *Fund Business* and without *Flow Volatility*. All fund-specific right-hand side variables are lagged to alleviate possible endogeneity issues. α_i denotes fund fixed effects, which account for unobserved time-invariant fund characteristics.

β_1^H and β_2^H represent how the risky investment of MMFs with higher default costs (i.e., with higher sponsor's reputation concerns) responds to changes in the risk premium and risk-free rate, respectively. β_1^L and β_2^L represent the corresponding sensitivities for funds with lower default costs. The model predicts: $\beta_1^H < 0 < \beta_1^L$ and $\beta_2^H > \beta_2^L > 0$. For robustness, I also run regression (1.8) with *Safe Holdings* as dependent variable. In that case, the model predicts: $\beta_1^H > 0 > \beta_1^L$ and $\beta_2^H < \beta_2^L < 0$.

[38] Since *Spread* mechanically co-varies with the risk-free rate, I do not use it as measure of risk-taking in (1.8).

Columns (1)–(3) of Table 1.4 show the results. Since *Fund Business* is a fund sponsor attribute, risk-taking within the same sponsor may be correlated across its funds. To address this concern, reported standard errors are HAC and cross-correlation robust. As for the effect of changes in the risk premium, the data confirm the predictions of the model for all measures of risk. For funds whose sponsors have *Fund Business* above the median, an increase of 1% in the risk premium decreases *Holdings Risk* by roughly 2%, increases *Safe Holdings* by 1.6%, and decreases *Maturity Risk* by 2.1 days. These results are statistically significant at the 1% level. On the contrary, funds whose sponsors have fund business below the median increase their *Holdings Risk* by 1.5% and their *Maturity Risk* by 1.2 days. These results are statistically significant at the 10% level.

As for the effect of changes in the risk-free rate, the data confirm the predictions of the model for the risk measures based on asset class holdings. After a decrease of 1% in the 1-month T-bill monthly return, funds whose sponsors have high *Fund Business* decrease their *Holdings Risk* by 3.2 percentage points and increase their safe holdings by 10.7 percentage points, with the result for safe assets being statistically significant at the 1% level. Funds whose sponsors have low *Fund Business* decrease their *Holdings Risk* by only 0.7 percentage points and increase their *Safe Holdings* by 8.6 percentage points, with the result for safe assets being statistically significant at the 10% level. For both categories of funds, the effect of the risk-free rate on safe assets is economically important.

Interestingly, the results for *Maturity Risk* suggest that the shift to safer asset classes is accompanied by a maturity extension. *Ceteris paribus*, after a decrease of 1% in the risk-free rate, both funds with relatively high and low default costs increase their portfolio maturity by roughly 7 days. This apparent disagreement with the model comes from the fact that: (1) longer maturity does not necessarily mean riskier portfolio, especially if shorter-term bank obligations are substituted by longer-term treasuries; (2) the model does not distinguish between maturity risk and holdings risk, and hence it cannot capture the trade-off between these two types of risk. Together with the observations on *Holdings Risk* and *Safe Holdings*, the results for *Maturity Risk* suggest that when risk-free rates decrease, funds do increase their safe holdings, but compensate to this loss of yield by increasing the maturity of their portfolio.[39]

As first robustness check, I run regression (1.8) identifying as funds with high (low) default costs those whose sponsors have *Fund Business* consistently above (below) the median *over the whole period* of analysis. This specification eliminates the concern that the results might be driven by endogenous changes in sponsors' reputation concerns in response to changes in the interest rate environment. Results are in columns (4)–(6) of Table 1.4 and are qualitatively

[39] A similar behavior was qualitatively observed also by Baba, McCauley, and Ramaswami (2009).

similar to those in columns (1)–(3). When the risk premium increases, holding the risk-free rate constant, funds with higher default costs decrease their risky investment, while funds with lower default costs tend to do the opposite (even though not in a statistically significant way). When the risk-free rate decreases, holding the premium constant, all funds reduce risky investment and shift to safer assets, with the effect being statistically more significant for funds with higher default costs.

Another possible concern is that since reputation concerns are a sponsor's characteristic, the cutoff between funds with high and low default costs should be calculated at the sponsor level. To address this issue, I run regression (1.8) using the median value of *Fund Business* in the sponsor population as cross-sectional cutoff. Results are similar and reported in Tables 1.G.6. Finally, I run regression (1.8) using *Spread Index* as proxy for the risk premium, and the 3-month T-bill rate as proxy for the risk-free rate. Results are similar and are available upon request.

1.7.4 Risk Taking in the Cross-section

This section exploits the significant cross-sectional variation in sponsor's reputation concerns to test the predictions of the model on the risky investment of MMFs in the cross-section. My model predicts that funds whose sponsors have lower reputation concerns (i.e., lower default costs) always hold more risky assets (P.1), and that the cross-sectional differential increases when either the risk premium goes up (P.2c), or the risk-free rate goes down (P.3b).

On the balanced panel of MMFs active throughout the whole period of analysis, I estimate the following weekly regression:

$$Risk_{i,t} = \alpha_i + \mu_t + \beta_1 FB\ Rank_{i,t-k} + \beta_2 Post_t * FB\ Rank_{i,t-k} + \gamma \cdot X_{i,t-k} + \varepsilon_{i,t} \tag{1.9}$$

where $FB\ Rank_{i,t}$ is the rank of fund i's *Fund Business* in week t. $FB\ Rank$ is calculated at the fund level (i.e., two funds have the same $FB\ Rank$ if they have the same sponsor) and is expressed in percentiles normalized to $[0, 1]$. $FB\ Rank = 0$ for those funds whose sponsor has the lowest fund business, and $FB\ Rank = 1$ for those funds whose sponsor has the highest fund business. X is the set of fund-specific controls defined in (1.8), with in addition their interaction terms with $Post_t$ and $FB\ Rank$ instead of *Fund Business*. *Post* is an indicator variable equal to 1 for the post period and 0 for the pre period. Both $FB\ Rank$ and X are lagged by k weeks to alleviate endogeneity concerns. For robustness, I run various regression specifications with different values of k, namely, $k = 1, 4, 8$, and 12 (corresponding to 1 week, 1 month, 2, and 3

months). μ_t and α_i denote week and fund fixed effects, respectively. I use three measures of risk (*Risk*) at a weekly frequency: *Holdings Risk*, *Maturity Risk*, and *Spread*, as defined in Section 1.7.1.

The coefficients of interest are β_1 and β_2. β_1 measures the effect of a sponsor's reputation concern on the risky investment of its funds in the pre period. β_2 measures the risk-taking differential between the post and the pre period. In the model, these coefficients correspond to the derivative of equilibrium risky investment w.r.t. the cost of default in different interest rate environments. Since in the post period there was a significant increase in risk premia and decrease in risk-free rates, the model predicts: $\beta_1 < 0$ and $\beta_2 < 0$.

Table 1.5 shows the result for $k = 4$ and 8. Reported standard errors are HAC and cross-correlation robust. For *Holdings Risk* and *Maturity Risk*, both β_1 and β_2 are negative and statistically significant at the 1% level. These results are also economically important. Going from the highest to the lowest rank of *Fund Business* increases the net holdings of risky assets by 8.5 percentage points in the pre period and 17.7 percentage points in the post period, and it increases the portfolio maturity by 6.6 days in the pre period and 14 days in the post period. When *Spread* is the dependent variable, β_2 is negative and statistically significant, while β_1 is negative but insignificant.[40] Results for $k = 1$ and 12 are similar and omitted for brevity.

To address possible concerns about the above measures of risk, I also run regression (1.9) using the share of safe holdings (*Safe Holdings*) as dependent variable. In this case, the model predicts $\beta_1 > 0$ and $\beta_2 > 0$. Columns (7) and (8) of Table 1.5 show the results, which confirm the predictions of the model. Going from the highest to the lowest rank of *Fund Business* decreases the holdings of safe assets by 2.3 percentage points in the pre period and 7.7 percentage points in the post period, with the differential being statistically significant at the 1% level.

Since in the model risky investment is determined by a fund's rank in the distribution of default costs, the rank of sponsor's reputation concern, *FB Rank*, is the natural explanatory variable in regression (1.9). However, as robustness check, I also run regression (1.9) using raw *Fund Business* as main explanatory variable. Results are similar and can be found in Table 1.G.2 in Appendix 1.G.

To further check the robustness of my results, I run Instrumental Variable (IV) regressions in which *Fund Business* is instrumented with its lagged value, separately on the post and the pre period. The results are similar to those reported above and confirm the predictions of the

[40] The lack of statistical significance for *Spread* in the pre period could be due to the fact that: (1) spreads may change without the risk profile of the underlying portfolio changing, (2) the differential spread across eligible instruments was very small in the pre period, making identification more difficult.

model. Details on the regression specification are in Appendix 1.G, and results are in Table 1.G.3.

1.7.4.1 Disentangling the risk-free rate from the risk-premium

This section aims to disentangle the effect of changes in the level of the risk-free rate from the effect of changes in the risk premium on the cross-sectional risky investment differential. I.e., here I test predictions P.2c and P.3b separately. To this aim, I run the following regression:

$$Risk_{i,t} = \alpha_i + \mu_t + \beta_1 FB\ Rank_{i,t-1} * \widehat{rp}_t + \beta_2 FB\ Rank_{i,t-1} * rf_t + \gamma \cdot X_{i,t-1} + \varepsilon_{i,t} \quad (1.10)$$

where $Risk$, $FB\ Rank$, and X are defined as in (1.9), without the interaction terms with $Post$. $\widehat{rp}$ is a proxy for the risk premium: $GZ\ Premium$ in the main specification. Since $GZ\ Premium$ is estimated at the monthly level, regression (1.10) is also at the monthly level. Fund-specific weekly-observed quantities are averaged over months. rf_t is the risk-free rate, proxied by the 1-month T-bill return. Fund-level variables are lagged to alleviate possible endogeneity issues.

The coefficients of interest are β_1 and β_2. In the context of the model, β_1 represents the cross-derivative of fund risky investment w.r.t. the cost of default and the risk premium. β_2 represents the cross-derivative of fund risky investment w.r.t. the cost of default and the risk-free rate. The model predicts: $\beta_1 < 0$ (prediction P.2c) and $\beta_2 > 0$ (prediction P.3b). To check the robustness of my results, I also run regression (1.10) using $Safe\ Holdings$, defined as in (1.9), as dependent variable. In this case the model predicts: $\beta_1 > 0$ and $\beta_2 < 0$.

Results are in Table 1.6. Reported standard errors are HAC and cross-correlation robust. The data confirms the predictions of the model. When only either one of the interaction terms is included in regression (1.10), the sign of the coefficient is consistent with the model and the coefficient is statistically significant at the 1% level for all measures of risk. When both interaction terms are included, the sign of all coefficients is consistent with the model, but statistical significance and economic importance depend on the specific measure of risk.

For $Holdings\ Risk$ and $Safe\ Holdings$, only the interaction term with the risk premium is statistically significant, and it is also economically important. After an increase of 1% in the risk premium, the difference in net risk exposure between funds in the lowest and highest percentile of $Fund\ Business$ increases by 6.2 percentage points. Similarly, the difference in safe assets between funds in the highest and lowest percentile of $Fund\ Business$ increases by 3.6 percentage points. The cross-sectional effect of the risk-free rate is not statistically significant for either $Holdings\ Risk$ or $Safe\ Holdings$, but it is economically significant for $Holdings\ Risk$.

After a decrease of 1% in the risk-free rate, the difference in net risk exposure between funds in the lowest and highest percentile of *Fund Business* increases by more than 4.7 percentage points.

For *Maturity Risk* and *Spread*, only the interaction term with the risk-free rate is statistically significant (at the 1% and 10% level, respectively). Its economic importance is also much greater than that of the interaction term with the risk premium. A decrease of 1% in the 1-month T-bill return increases the difference in portfolio maturity between funds in the lowest and highest percentile of *Fund Business* by more than 20 days. On the other hand, an increase of 1% in the risk premium increases the same differential by only 1.6 days. A decrease of 1% in the 1-month T-bill return increases the spread differential between funds in the lowest and highest percentile of *Fund Business* by almost 15 bps. On the other hand, an increase of 1% in the risk premium increases the same differential by only 0.4 bps.

These results suggest that the risk premium and the risk-free rate affect the risk-taking of MMFs in different ways. Changes in the risk premium are more important in determining the asset class composition of MMF portfolios. Changes in the risk-free rate, on the other hand, are more important in determining the weighted average maturity of MMF portfolios.

As first robustness check, I run regression (1.10) using *Fund Business* instead of its rank as main explanatory variable. Results are similar and are available upon request. As further robustness checks, I run regression (1.10) using the 3-month T-bill rate as proxy for rf, and *Spread Index* as proxy for $\widehat{rp}$. Results are reported in Table 1.G.4 and Table 1.G.5 of Appendix 1.G, respectively. Again, they confirm model's predictions.[41] Finally, I run regression (1.10) interacting also the fund-specific controls with rf and $\widehat{rp}$, and/or lagging all fund-specific variables on the RHS by 2 months. Results are similar and omitted for brevity.

1.8 Conclusions

In this paper, I propose a novel tournament model of money market funds (MMFs) to study whether competitive factors generate "reach for yield" in a low risk-free rate environment. First, the model shows that competitive pressure is heterogeneous in the cross-section: funds with lower default costs face a higher competitive pressure and therefore take on more risk. Second, the model shows that it is important to distinguish low risk-free rates from high risk premia. The risk premium is key to trigger fund risk-taking. However, its effect is heterogeneous in the cross-section. When the premium increases due to an increase in the riskiness of the risky asset,

[41] Not surprisingly, the relative weights of risk premium and risk-free rate change. See Appendix 1.G for details.

funds with lower default costs increase their risky investment because, facing higher competitive pressure, they are more sensitive to the increase in investment opportunities. Funds with higher default costs, on the other hand, decrease they risky investment because, aiming to keep the default probability closer to zero, they are more sensitive to the increase in the probability of negative return realizations. On the other hand, contrary to conventional wisdom, a decrease in the risk-free rate reduces the risky investment of all funds, with the effect being stronger for funds with higher default costs. This is because a decrease in the level of the risk-free rate increases the buffer of safe assets necessary to keep the probability of default at the equilibrium level.

The empirical analysis shows that these predictions are consistent with the risk-taking behavior of MMFs during the 2006–2008 period. When risk premia increased, funds whose sponsors have low reputation concerns increased risk-taking, while funds whose sponsors have high reputation concerns decreased risk-taking. The empirical analysis also confirms the differential role of risk-free rate and spread to explain changes in fund portfolios. Holding the premium constant, when risk-free rates decreased, funds shifted their portfolios toward safer asset classes. Finally, I show that the rank of fund performance, not the raw performance, determines investor money flows in the industry, justifying the modeling assumption of a fund tournament.

These results shed light on the transmission of monetary policy to money market funds. The level of the risk-free rate affects fund risk-taking through the stable NAV and consequent risk of "breaking the buck" typical of money market funds. This channel of monetary policy, peculiar to MMFs, goes in the opposite direction of the conventional "reach for yield" argument and reduces fund risk-taking in a low risk-free rate environment. This result contributes to the recent debate on the systemic importance of MMFs and the new regulation recently approved by the SEC. Under the new regulation, taking effect in October 2016, institutional prime funds will move from a stable NAV to a floating NAV. This institutional change, while possibly eliminating the risk of runs, might actually lead all institutional prime MMFs to take on more risk.

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	$Fund\ Flow_{i,t+1}$					
	(1)	(2)	(3)	(4)	(5)	(6)
	Post	Pre	Post	Pre	Post	Pre
$Spread\ Rank_{i,t}$			1.897***	0.651**	1.792**	0.919**
			(0.371)	(0.276)	(0.740)	(0.431)
$Spread_{i,t}$	0.026***	0.027			0.002	−0.022
	(0.008)	(0.019)			(0.015)	(0.029)
$Log(Fund\ Size)_{i,t}$	−5.488***	−4.163***	−5.575***	−4.181***	−5.576***	−4.186***
	(0.893)	(0.662)	(0.912)	(0.669)	(0.912)	(0.671)
$Expense\ Ratio_{i,t}$	−0.094	−1.275	0.562	−1.241	0.491	−1.071
	(2.418)	(3.366)	(2.470)	(3.339)	(2.511)	(3.383)
$Age_{i,t}$	−0.120	−0.491**	−0.139	−0.498**	−0.137	−0.495**
	(0.132)	(0.233)	(0.133)	(0.233)	(0.132)	(0.232)
$Flow\ Volatility_{i,t}$	−0.037***	2.919**	−0.015***	2.843**	−0.017	2.818**
	(0.010)	(1.369)	(0.004)	(1.384)	(0.015)	(1.392)
$Log(Family\ Size)_{i,t}$	0.843*	−0.009	0.841*	−0.002	0.843*	0.010
	(0.444)	(0.176)	(0.453)	(0.175)	(0.453)	(0.183)
Week fixed effect	Y	Y	Y	Y	Y	Y
Fund fixed effect	Y	Y	Y	Y	Y	Y
Observations	7,387	9,467	7,387	9,467	7,387	9,467
Adj. R^2 (within)	0.030	0.023	0.032	0.024	0.032	0.024
R^2 (overall)	0.079	0.060	0.080	0.061	0.080	0.061

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.1: **Flow-performance relation: performance rank matters more than raw performance.** Columns (1), (3) and (5) cover the period 8/1/2007-8/31/2008 (*post* period). Columns (2), (4) and (6) cover the period 1/1/2006-7/31/2007 (*pre* period). The dependent variable is *Fund Flow*, computed as the percentage change in total net assets from week t to week $t+1$, adjusted for earned interests and trimmed at the 0.5%. Independent variables are the weekly annualized spread from $t-1$ to t, its rank in percentiles normalized to $[0, 1]$, logarithm of fund size, fund expense ratio, fund age, volatility of fund flows based on past 12-week fund flows, and logarithm of fund family size. All regressions are at the weekly level and include week and fund fixed effects. Standard errors are HAC robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

Period	1-month T-bill monthly return (bp)					
	Min	1st Qu.	Median	Mean	3rd Qu.	Max.
Jul-Dec 2006	39.76	40.39	40.61	41.10	42.01	42.83
Jan-Jun 2008	9.46	11.00	15.38	15.58	18.39	24.30
	GZ Premium (%)					
	Min	1st Qu.	Median	Mean	3rd Qu.	Max.
Jul-Dec 2006	−0.300	−0.271	−0.224	−0.232	−0.188	−0.186
Jan-Jun 2008	0.649	0.730	1.102	1.108	1.437	1.476
	Spread Index (%)					
	Min	1st Qu.	Median	Mean	3rd Qu.	Max.
Jul-Dec 2006	0.179	0.222	0.267	0.274	0.322	0.389
Jan-Jun 2008	0.822	0.953	1.170	1.201	1.484	1.609

TABLE 1.2: **Summary Statistics for risk-free rate and risk premium.** *GZ Premium* is the excess bond premium for financial firms from Gilchrist and Zakrajsek (2012). *Spread Index* is the index of spreads on eligible risky securities defined by (1.6).

Fund Business	*Holdings Risk* (%)			*Safe Holdings* (%)		
	Jul-Dec 06	Jan-Jun 08	Δ_t	Jul-Dec 06	Jan-Jun 08	Δ_t
High ($N = 39$)	-4.37^{***}	-7.67^{***}	-3.30^{***}	16.74^{***}	20.40^{***}	3.66^{***}
	(0.14)	(0.24)		(0.11)	(0.25)	
Low ($N = 50$)	0.64	5.37^{***}	4.73^{***}	16.51^{***}	17.87^{***}	1.36
	(0.97)	(0.40)		(0.65)	(0.52)	
$\Delta_t(High) - \Delta_t(Low)$			-8.03^{***}			2.30^{***}
t-stat			−7.40			2.63

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.3: **Risky investment over time: high vs. low default costs.** The sample includes all prime institutional funds continuously active throughout the period from 1/1/2006 to 08/31/2008 and whose sponsor's *Fund Business* is consistently either above (*High*) or below (*Low*) the industry median. *Fund Business* is the share of mutual fund assets other than institutional prime MMFs in sponsor's total mutual fund assets. *Holdings Risk* is the percentage of risky assets (i.e., bank obligations) net of the safe assets (US treasuries, GSE debt, and repos) in fund portfolios. *Safe Holdings* is the percentage of safe assets in fund portfolios. Δ_t is the average change over time. Standard errors of each within-period average (in parentheses) are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

	High (Low) Fund Business at $t-1$			*High (Low) Fund Business* over whole period		
	(1)	(2)	(3)	(4)	(5)	(6)
	Holdings Risk	*Maturity Risk*	*Safe Holdings*	*Holdings Risk*	*Maturity Risk*	*Safe Holdings*
$High\,FB_{i,t-1} * \widehat{rp}_t$	-1.955^{***}	-2.111^{***}	1.584^{***}	-3.416^{***}	-0.596	2.159^{***}
	(0.694)	(0.627)	(0.573)	(0.671)	(0.680)	(0.512)
$Low\,FB_{i,t-1} * \widehat{rp}_t$	1.545^{*}	1.150^{*}	-0.006	1.448	1.078	-0.073
	(0.909)	(0.682)	(0.861)	(1.395)	(1.157)	(1.219)
$High\,FB_{i,t-1} * rf_t$	3.199	-7.642^{*}	-10.662^{***}	8.159^{**}	-0.351	-11.632^{***}
	(4.316)	(4.035)	(3.914)	(3.223)	(3.592)	(3.481)
$Low\,FB_{i,t-1} * rf_t$	0.670	-7.101^{*}	-8.579^{*}	2.609	-9.396	-13.768^{*}
	(5.921)	(4.076)	(5.122)	(8.223)	(6.179)	(7.984)
$Controls_{i,t-1}$	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y
Adj. R^2 (within)	0.062	0.084	0.050	0.074	0.079	0.059
R^2 (overall)	0.757	0.568	0.753	0.759	0.566	0.755
Observations	3,782	3,782	3,782	3,782	3,782	3,782

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.4: **Reach for yield: risk premium vs. risk-free rate.** The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008 ($n = 122$). Data are at the monthly level ($T = 31$). The dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1) and (4), average portfolio maturity (*Maturity Risk*) in columns (2) and (5), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (3) and (6). The risk premium $\widehat{rp}$ is the excess bond premium for financial firms from Gilchrist and Zakrajsek (2012). The risk-free rate rf_t is the return on 1-month T-bills. In columns (1)–(3), *High* (*Low*) $FB_{i,t}$ is a binary variable equal to 1 if fund i's *Fund Business* is above (below) the median value in month t, and 0 otherwise. In columns (4)–(6), *High* (*Low*) $FB_{i,t}$ is a binary variable equal to 1 for all t if fund i's *Fund Business* is above (below) the median value consistently over the whole period, and 0 otherwise. *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. Other independent variables (*Controls*) are fund assets, expense ratio, fund age, fund family size, and *Fund Business*. All regressions include fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

	$Holdings\ Risk_{i,t}$		$Maturity\ Risk_{i,t}$		$Spread_{i,t}$		$Safe\ Holdings_{i,t}$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	k=4	k=8	k=4	k=8	k=4	k=8	k=4	k=8
$FB\ Rank_{i,t-k}$	-8.523^{***}	-7.868^{***}	-6.637^{***}	-5.828^{***}	-1.923	-1.965	2.315	1.664
	(1.805)	(1.732)	(1.434)	(1.596)	(1.995)	(1.969)	(1.591)	(1.644)
$FB\ Rank_{i,t-k} * Post_t$	-9.210^{***}	-8.756^{***}	-7.426^{***}	-7.361^{***}	-4.113^{***}	-4.186^{***}	5.424^{***}	5.120^{***}
	(1.504)	(1.481)	(0.932)	(0.978)	(0.743)	(0.791)	(0.952)	(0.918)
$Controls_{i,t-k}$	Y	Y	Y	Y	Y	Y	Y	Y
$Controls_{i,t-k} * Post_t$	Y	Y	Y	Y	Y	Y	Y	Y
Week Fixed Effects	Y	Y	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y	Y	Y
Observations	16,470	15,982	16,470	15,982	16,470	15,982	16,470	15,982
Adj. R^2 (within)	0.032	0.030	0.049	0.045	0.038	0.034	0.024	0.023
R^2 (overall)	0.759	0.762	0.588	0.591	0.983	0.983	0.756	0.758

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.5: **Cross-sectional risk-taking differential in the *Pre* and *Post* period.** The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008 ($n = 122$). Data are at the weekly level ($T = 139$). The dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1)–(2), average portfolio maturity (*Maturity Risk*) in columns (3)–(4), the weekly annualized fund spread (*Spread*) in column (5)–(6), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (7)–(8). *FB Rank* is the rank in percentiles normalized to $[0, 1]$ of *Fund Business*, where *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. *Post* is an indicator variable equal to 1 for the period from 8/1/2007 to 8/ 31/2008, and 0 otherwise. The other independent variables (*Controls*) are fund assets, expense ratio, fund age, and fund family size. All regressions are at the weekly level and include week and fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

	(1) *Holdings Risk*	(2) *Safe Holdings*	(3) *Holdings Risk*	(4) *Safe Holdings*	(5) *Holdings Risk*	(6) *Safe Holdings*
$FB\,Rank_{i,t-1} * \widehat{rp}_t$	-6.865^{***}	3.688^{***}			-6.189^{***}	3.587^{**}
	(1.019)	(0.787)			(1.971)	(1.714)
$FB\,Rank_{i,t-1} * rf_t$			38.501^{***}	-20.285^{***}	4.718	-0.706
			(7.562)	(4.692)	(11.203)	(9.905)
Adj. R^2 (within)	0.049	0.028	0.046	0.026	0.049	0.028
R^2 (overall)	0.787	0.788	0.787	0.788	0.787	0.788
	Maturity Risk	*Spread*	*Maturity Risk*	*Spread*	*Maturity Risk*	*Spread*
$FB\,Rank_{i,t-1} * \widehat{rp}_t$	-4.568^{***}	-2.568^{***}			-1.637	-0.442
	(0.993)	(0.731)			(1.181)	(1.369)
$FB\,Rank_{i,t-1} * rf_t$			29.404^{***}	17.262^{***}	20.469^{***}	14.852^{*}
			(4.272)	(3.477)	(6.148)	(8.108)
Adj. R^2 (within)	0.031	0.010	0.033	0.011	0.033	0.011
R^2 (overall)	0.617	0.983	0.618	0.983	0.617	0.983
$Controls_{i,t-1}$	Y	Y	Y	Y	Y	Y
Month Fixed Effects	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y
Observations	3,782	3,782	3,782	3,782	3,782	3,782

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.6: **Cross-sectional risk-taking differential: risk premium vs. risk-free rate.** The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008 ($n = 122$). Data are at the monthly level ($T = 31$). In the top panel, the dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1), (3), and (5), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (2), (4), and (6). In the bottom panel, the dependent variables are: average portfolio maturity (*Maturity Risk*) in columns (1), (3), and (5), the weekly annualized fund spread (*Spread*) in column (2), (4), (6). *FB Rank* is the rank in percentiles normalized to $[0, 1]$ of *Fund Business*, where *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. $\widehat{rp}$ is the excess bond premium for financial firms from Gilchrist and Zakrajsek (2012). rf_t is the net return on 1-month T-bills. The other independent variables (*Controls*) are fund assets, expense ratio, fund age, and fund family size. All regressions include month and fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

Appendix 1.A Microfoundation of the tournament

This appendix presents a random utility model of fund investors that rationalizes the rank-based payoff function of the MMF tournament.

The standard theoretical justification for the empirically observed positive relation between money flows and past performance is that investors assume that fund managers have idiosyncratic, unobservable skills and try to infer them from historical data. Higher past performance is perceived as a signal of higher ability and generates money inflows.

Let us assume that there is a continuum of investors. Each investor is associated with a single fund and endowed with a wealth $D > 0$. I refer to the investor associated with fund c as "investor c". I assume that investor c has only two options: she can either put her money into her idiosyncratic fund c or invest in an alternative technology outside the MMF industry. The investor demand for delegated management satisfies the following random utility model:

$$\text{investor } c \text{ invests in } \begin{cases} \text{fund } c & \text{with probability} \quad p = Rk_\pi(c) \\ \text{alternative technology} & \text{with probability} \quad 1-p \end{cases}$$

This model can be motivated by arguing that investors have limited information, or limited capacity of processing information, on the management industry and market structure. Each investor has accumulated some information on a given fund, which she prefers to the others for some idiosyncratic reason. Investor c uses the ex post rank of fund c's performance as an indication of its manager's skill. The acquisition of ex post information on other funds is too costly. Hence, each investor only decides whether to invest in the idiosyncratic fund or in the alternative technology.

There are other ways to endogeneize the rank-based flow-performance relation observed in the data as the outcome of an optimal investment strategy of rational investors. Huang, Wei,and Yan (2007) formally show that rank-based reward functions arise in equilibrium due to information acquisition and participation costs faced by retail investors. Matejka and McKay (2013) show that the logit model (closely related to the above random utility model) is the optimal decision rule for a rationally inattentive agent who is uncertain on the fundamental value of her investment possibilities but faces a cost of acquiring information. In the context of my model, the unobservable, fundamental value of investment opportunities would be a fund's underlying quality, and the logit model would represent the endogenous rank based flow-performance relation. Frankel (2013) shows that ranking is the optimal delegated alignment contract when a principal delegates multiple decisions to an agent, who has private information relevant to each

decision, but the principal is uncertain about the agent's preferences. In the case of mutual fund industry, we can think of the principal as the investor and the agent as a financial adviser. Finally, the normative literature on tournament theory (e.g., Lazaer and Rosen, 1981) shows that a tournament reward structure is optimal for a principal-agent problem in presence of moral hazard.

Appendix 1.B Relative Performance vs. Absolute Performance

This section compares fund risk-taking under the tournament to fund risk-taking under absolute performance compensation. As in the tournament, funds have initial deposits $D > 0$ that pay a gross interest rate equal to 1 to some outside investors. Funds can invest only in a risk-free asset with gross return $R_f > 1$ and a risky asset with a random gross return $R \sim F_R$ on $[\underline{R}, \overline{R}] \subseteq \mathbb{R}_+$. Let us assume that the risky asset has a positive risk premium in the standard sense, i.e. $\mathbb{E}[R] > R_f$, so that the funds have an incentive to take on risk. As in the fund tournament, F_R is assumed to be absolutely continuous, and f_R denotes its density.

Each fund receives a payoff proportional to the raw profit on its portfolio if it does not default, and it pays a fixed cost otherwise. For consistency with the tournament, funds are risk-neutral, the payoff is a linear function of the net profit, and each fund faces an idiosyncratic cost of default, $c \in (0, \infty)$. The payoff of fund c for investing x in the risky asset is

$$\upsilon_c(x) = \gamma \mathbb{E}_R \left[\pi(x) | \pi(x) \geq 0\right] \mathbb{P}_R \left(\pi(x) \geq 0\right) - c\mathbb{P}_R(\pi(x) < 0) \tag{1.11}$$

where $\gamma \in (0, 1)$ can be regarded as a percentage fee on profits, and $\pi(x)$ is fund's profit and is referred to as the absolute performance. Funds maximize (1.11) under no short-selling and no borrowing constraints. Contrary to what happens in the fund tournament, here the distribution of default costs plays no role in the optimization problem. Under absolute performance compensation, there are no strategic interactions among funds.

The maximization of (1.11) is a linear optimization problem on a bounded interval, with control variable $x \in [0, D]$. To simplify the analysis I make the following assumption.

ASSUMPTION 3. (i) $\inf_{r \in (\underline{R}, 1)} f_R(r) > 0$; (ii) $\sup_{r \in (\underline{R}, 1)} f_R(r) < \infty$.

Assumption 3 is weak. Several common distributions with support in $\mathbb{R}_+$ satisfy it, including uniform, exponential, Beta(1, β), Gamma(1, θ), and generalized Pareto. Moreover, for any absolutely continuous distribution F_R that does not satisfy Assumption 3 (i), we can construct another absolutely continuous distribution that is practically identical to F_R and satisfies

Assumption 3 (i) by infinitesimally increasing f_R in an arbitrarily small neighborhood of its infima.[42]

Proposition 1.15. *Let Assumption 3 hold. The optimal risky investment of a fund maximizing the payoff function (1.11), subject to no short-selling and no borrowing constraints, is as follows.*

(i) *If* $\inf_{(\underline{R},1)} \frac{f'_R}{f_R} < \frac{2}{R_f - \underline{R}}$, *there exists a unique* $c_0 \in (0,\infty)$ *s.t.*

$$x(c) = \begin{cases} D & \text{for } c \leq c_0 \\ x_0 & \text{for } c > c_0 \end{cases}$$

In this case the optimal strategy profile is said to be "bang-bang".

(ii) *In general, there exist* $c_1 \leq c_2 \in (0,\infty)$ *s.t.*

$$x(c) \begin{cases} = D & \text{for } c \leq c_1 \\ \in \{\widetilde{x} \in (x_0, D) | \widetilde{x} = h(\widetilde{x}; c) \text{ and } \widetilde{x} < g(\widetilde{x}; c)\} & \text{for } c_1 < c \leq c_2 \\ = x_0 & \text{for } c > c_2 \end{cases}$$

where $h(x;c) = \left(\frac{c\,D\,(R_f - 1)\,f_R(R_0(x))}{\gamma \mathbb{E}[R - R_f | R > R_0(x)] \mathbb{P}(R > R_0(x))}\right)^{1/2}$, $g(x;c) = \frac{(R_f-1)Dc}{2c + \gamma(R_f-1)D} \frac{f'_R(R_0(x))}{f_R(R_0(x))}$, *and* $R_0(x) = R_f - (R_f - 1)D/x$. *Moreover,* $x(c)$ *is strictly decreasing on* (c_1, c_2).

If $\{\widetilde{x} \in (x_0, D) | \widetilde{x} = h(\widetilde{x}; c) \text{ and } \widetilde{x} < g(\widetilde{x}; c)\} = \varnothing$, $c_1 = c_2$ *and the optimal strategy profile is "bang-bang".*

Under no short-selling constraint the maximum achievable profit is bounded from above. If the cost of default is sufficiently high, the expected loss from default dominates the expected gain from absolute performance compensation, the fund invests exactly the maximum risky investment such that the probability of default is zero, x_0. On the other hand, if the cost of default is sufficiently low, the expected gain from absolute performance dominates the expected loss from default, and the fund invests its whole portfolio in the risky asset. These results are in contrast with those of the fund tournament. In that case, due to the strategic nature of the game, the pressure of competition drives all funds to invest more than x_0 in the risky security, so that the equilibrium default probability is strictly positive for all funds, regardless of the scale of default costs. On the other hand, in the tournament, funds with lower default costs do not invest their whole portfolio in the risky asset because, in equilibrium, they do not need that to outperform their competitors.

[42] Since F_R is assumed to be absolutely continuous, the set $\{r \in [\underline{R}, \overline{R}] : f_R(r) = 0\}$ has zero Lebesgue measure, and therefore this modification would have a practically negligible effect on the distributional properties of R.

Corollary 1.16. *If the minimum cost of default in the industry is sufficiently high, equilibrium risk-taking in the fund tournament is strictly greater than optimal risk-taking under absolute performance compensation for all funds. (Except for that with the highest default cost for which optimal risk-taking is the same under the two compensation schemes.)*

In the MMF industry, the cost of default is arguably very high in absolute terms even for those funds with relatively lower default costs. Hence, Corollary 1.16 says that the competitive forces coming from the relative performance evaluation of MMFs by fund investors observed empirically generate more risk-taking than what there would be under absolute performance compensation.

Most predictions of the absolute performance model on the response of funds to shocks in the economic environment are different from the predictions of the tournament. Under absolute performance compensation, funds with (absolutely) high default costs are insensitive to shocks in the risk premium, and funds with (absolutely) low default costs are insensitive to both changes in the risk premium and changes in the risk-free rate.[43] This is in disagreement with the data. Moreover, by construction, the absolute performance model is unable to capture the effect of competitive pressure. Under the absolute performance model, fund risk-taking is insensitive to shifts in the distribution of default costs.

Appendix 1.C Proofs

This appendix proves the theoretical results presented in the main text. Let us consider the following generalization of the *ex post* rank-order of profits

$$Rk_{\pi}(c) := \int_{\{c':\pi_{c'}<\pi_c\}} \mathrm{d}F_c(c') + \delta \int_{\{c':\pi_{c'}=\pi_c\}} \mathrm{d}F_c(c')$$

with $\delta \in [0,1]$. The results presented in the body of the paper are for the special case $\delta = 0$. δ represents the "premium" for pooling coming from investors' money flow. For risk-averse investors δ is arguably close to zero, since risk-averse investors would penalize the uncertainty about unobservable skills when a pool of funds have the same ex post profits.

Let us introduce the following notation: $\Omega := (\underline{c}, \overline{c})$ and $\lambda(\cdot)$ is the measure induced by $F_C(\cdot)$. I.e., for any $C = (c_1, c_2) \subseteq \Omega$ with $c_1 \leq c_2$, $\lambda(C) = F_C(c_2) - F_C(c_1)$. Since F_C is assumed to be

[43] The results for funds with low default costs are due to the assumption of risk-neutrality. If the funds were risk-averse, we should expect the risk-taking of funds with low cost of default to be positively related with the premium and, holding the premium constant, with the risk-free rate.

absolutely continuous with respect to the Lebesgue measure, any set of Lebesgue measure zero has also measure zero under λ. As in the main text, I drop the "a.e." notation for simplicity; the following results on the properties of the equilibrium are to be interpreted as valid almost everywhere.

Lemma 1.17. *For a given strategy profile $x : \Omega \to [0, D]$, the objective function of player c is*

$$\begin{aligned} \upsilon_c(x) &= \gamma D \left\{ a + F_R(R_f) + F_X(x) \left[1 - 2F_R(R_f) \right] \right\} - \left\{ \gamma D \left[a + 1 - F_X(x) \right] + c \right\} F_R\left(R_0(x) \right) + \\ &\quad + \gamma D \left\{ \delta \left[1 - F_R\left(R_0(x) \right) \right] - F_R(R_f) + F_R\left(R_0(x) \right) \right\} \lambda\left(C_x \right) \end{aligned}$$

where $R_0(x) := R_f - (R_f - 1)\frac{D}{x}$ and $C_x := \{c' \in \Omega : x(c') = x\}$.

Proof. Trivial (by substitution). □

Preliminary results

Lemma 1.18. *If there exists a Nash equilibrium $x : \Omega \to [0, D]$ s.t.*

$$x(c) \in \begin{cases} (x_0, D] & \text{for all } c \in C_1 \\ [0, x_0] & \text{for all } c \in \Omega \setminus C_1 \,, \end{cases}$$

then C_1 is connected, $\inf C_1 = \inf \Omega$, and $x(c)$ is weakly decreasing on C_1 for all δ.

Proof. By contradiction, suppose that there exist $c_1 \in \Omega$ and $c_2 \in C_1$ s.t. $c_1 < c_2$ and $x(c_1) < x(c_2)$. For notational simplicity let $x(c_1) = x_1$ and $x(c_2) = x_2$. Then,

$$\begin{aligned} \upsilon_{c_2}(x_1) &= \upsilon_{c_1}(x_1) - (c_2 - c_1) F_R\left(R_0(x_1) \right) \\ &\underset{\text{(by optimality of } \upsilon_{c_1}(x_1))}{\geq} \upsilon_{c_1}(x_2) - (c_2 - c_1) F_R\left(R_0(x_1) \right) \\ &= \upsilon_{c_2}(x_2) + \underbrace{(c_2 - c_1)}_{>0} \underbrace{\left[F_R\left(R_0(x_2) \right) - F_R\left(R_0(x_1) \right) \right]}_{>0} > \upsilon_{c_2}(x_2) \end{aligned}$$

which contradicts optimality of $\upsilon_{c_2}(x_2)$. Thus, $x(c)$ is weakly decreasing on C_1, C_1 is connected and $\inf C_1 = \inf \Omega$ for all δ. □

Lemma 1.19. *If there exists a Nash equilibrium $x : \Omega \to [0, D]$ s.t.*

$$x(c) \begin{cases} = D & \text{for all } c \in C_0 \\ \in (x_0, D) & \text{for all } c \in C_1 \\ \in [0, x_0] & \text{for all } c \in \Omega \setminus \{C_0 \cup C_1\} \,, \end{cases}$$

then, if $\delta < 1 - F_R(R_f)$,

(i) $C_0 \cup C_1 = \Omega$, *i.e.* $x(c) > x_0$ *for all* $c \in \Omega$;

(ii) C_0 *is connected,* C_1 *is connected,* $\inf C_0 = \inf \Omega$, $\sup C_0 = \inf C_1$, *and* $\sup C_1 = \sup \Omega$;

(iii) $x(c)$ *is strictly decreasing on* C_1 *and continuous on the interior of* C_1;

(iv) $\lim_{c \to \sup \Omega} x(c) = x_0$ *if* $\sup \Omega > \gamma D \frac{\delta(1-F_R(1)) - F_R(R_f) - aF_R(1)}{F_R(1)}$.

Proof. (i) Let $C_2 := \Omega \setminus \{C_0 \cup C_1\}$. By contradiction, suppose that C_2 has positive Lebesgue measure, i.e. $\lambda(C_2) > 0$. Let us define

$$\widetilde{C} := \left\{ c : \lambda\left(\left\{ c' \in C_2 : x(c') \geq x(c) \right\}\right) > 0 \right\} \subseteq C_2 \,.$$

$\widetilde{C}$ is the subset of players whose risky investment is weakly smaller than the risky investment of a positive measure subset of players in C_2. By construction $\widetilde{C}$ is a subset of C_2 and has positive measure, i.e. $\lambda\left(\widetilde{C}\right) > 0$. For any $c_b \in \widetilde{C}$ let $x_b := x(c_b)$ and $\lambda_b := \lambda\left(\{c : x(c) = x_b\}\right)$. By construction $\lambda\left(C_2\right) - F_x(x_b) > 0$ and $\lambda\left(C_2\right) - F_x(x_b) - \lambda_b \geq 0$.

For sufficiently small $\varepsilon > 0$,

$$\begin{aligned} \upsilon_{c_b}(x_0 + \varepsilon) =& \gamma D \left\{ a + F_R(R_f) + \left[\lambda(C_2) + g(\varepsilon)\right] \left[1 - 2F_R(R_f)\right] \right\} + \\ & - \left\{ \gamma D \left[a + 1 - \left(\lambda(C_2) + g(\varepsilon)\right) \right] + c_b \right\} F_R\left(R_0(x_0 + \varepsilon)\right) \end{aligned}$$

where $g(\varepsilon) := \lambda\left(\{c \in C_1 : x(c) < x_0 + \varepsilon\}\right) \geq 0$. Then, if $\delta < 1 - F_R(R_f)$, for all $c_b \in \widetilde{C}$ there exists a sufficiently small $\varepsilon > 0$ s.t.

$$\begin{aligned} \upsilon_{c_b}(x_0 + \varepsilon) \geq & \upsilon_{c_b}(x_b) + \gamma D \underbrace{\left[\lambda\left(C_2\right) - F_X(x_b) - \lambda_b\right]}_{\geq 0 \,, \text{ and } > 0 \text{ if } \lambda_b = 0} \left[1 - 2F_R(R_f)\right] + \\ & + \gamma D \lambda_b \underbrace{\left[1 - F_R(R_f) - \delta\right]}_{> 0} - \left\{ \gamma D \left[a + 1 - \lambda\left(C_2\right) \right] + c_b \right\} \underbrace{F_R(R_0(x_0 + \varepsilon))}_{\approx 0 \text{ for small } \varepsilon} \\ > & \upsilon_{c_b}(x_b) \end{aligned}$$

which contradicts optimality of x_b. Note that optimality is violated by a positive measure set of players since $\lambda\left(\widetilde{C}\right) > 0$ by construction.

(ii) Trivial, it follows directly from (i) and Lemma 1.18.

(iii) *Monotonicity*. By contradiction, suppose that there exists a non-empty, non-singleton subset $C_a \subseteq C_1$ s.t. $x(c) = x_a \in (x_0, D)$ for all $c \in C_a$ and $x(c) \neq x_a$ otherwise. Since C_1 is connected and any NE is weakly decreasing on C_1, C_a has positive measure, i.e. $\lambda(C_a) > 0$. Pick $\varepsilon > 0$ and let $g(\varepsilon) := \lambda(\{c : x(c) \in (x_a, x_a + \varepsilon)\}) \geq 0$. Then, if $\delta < 1 - F_R$, for all $c_a \in C_a$ there exists a sufficiently small $\varepsilon > 0$ s.t.

$$\begin{aligned}
\upsilon_{c_a}(x_a + \varepsilon) &= \gamma D\left\{a + F_R(R_f) + \left[F_X(x_a) + \lambda\left(C_a\right) + g(\varepsilon)\right]\left(1 - 2F_R(R_f)\right)\right\} + \\
&\quad - \left\{\gamma D\left[a + 1 - F_X(x_a) - \lambda\left(C_a\right) - g(\varepsilon)\right] + c\right\} F_R\left(R_0(x_a + \varepsilon)\right) \\
&\geq \upsilon_a(x_a) + \gamma D \underbrace{\lambda\left(C_a\right)}_{>0}\underbrace{\left[1 - F_R(R_f) - \delta\left(1 - F_R\left(R_0(x_a)\right)\right)\right]}_{>0 \text{ for all } x_a \in (0,D) \text{ if } \delta < 1 - F_R(R_f)} + \\
&\quad - \left\{\gamma D\left[a + 1 - F_X(x_a) - \lambda\left(C_a\right)\right] + c\right\} \underbrace{\Delta F_R\left(\varepsilon\right)}_{\approx 0} \\
&> \upsilon_a(x_a)
\end{aligned}$$

for any $x_a \in (x_0, D)$, which contradicts optimality of x_a for a positive measure set of players.

(iii) *Continuity*. Let us prove that $x(c)$ is left-continuous. The proof of right-continuity is very similar and thus omitted. We proceed by contradiction. Let $\text{int}\left(C_1\right)$ be the interior of C_1. Since $x(c)$ is strictly decreasing on C_1, let us suppose that there exists $c_1 \in \text{int}\left(C_1\right)$ and $\eta > 0$ s.t. $x(c) > x(c_1) + \eta$ for all $c < c_1$. Pick $\varepsilon > 0$ and let $C_\varepsilon := (c_1 - \varepsilon, c_1)$. For notational simplicity let $x_1 := x(c_1)$, $\Delta F_R(\eta) := F_R\left(R_0(x_1 + \eta)\right) - F_R\left(R_0(x_1)\right) > 0$, and for all $c_\varepsilon \in C_\varepsilon$ let $x_\varepsilon := x(c_\varepsilon)$ and $g(\varepsilon) := \lambda((c_\varepsilon, c_1))$. Note that for ε sufficiently small $C_\varepsilon \subset C_1$ and, since $x(c)$ is strictly decreasing on C_1, $g(\varepsilon) = F_X(x_\varepsilon) - F_X(x_1)$. By absolute continuity of λ, there exists $\varepsilon > 0$ sufficiently small s.t. for all $c_\varepsilon \in C_\varepsilon$

$$\begin{aligned}
\upsilon_{c_\varepsilon}(x_1) &= \gamma D\left\{a + F_R(R_f) + F_X(x_1)\left[1 - 2F_R(R_f)\right]\right\} - \left\{\gamma D\left[a + 1 - F_X(x_1)\right] + c_\varepsilon\right\} F_R\left(R_0(x_1)\right) \\
&= \upsilon_{c_\varepsilon}(x_\varepsilon) - \gamma D \underbrace{g(\varepsilon)}_{\approx 0}\left[1 - 2F_R(R_f) + F_R\left(R_0(x_\varepsilon)\right)\right] + \left\{\gamma D\left[a + 1 - F_X(x_1)\right] + c_\varepsilon\right\} \underbrace{\Delta F_R(\eta)}_{>0} \\
&> \upsilon_{c_\varepsilon}(x_\varepsilon)
\end{aligned}$$

which contradicts the optimality of x_ε for a positive measure set of players.

(iv) First, let us prove that if $\sup\Omega > \gamma D \frac{\delta[1 - F_R(1)] - F_R(R_f) - aF_R(1)}{F_R(1)}$, then $\lim_{c \to \sup\Omega} x(c) < D$. By contradiction, suppose that $\lim_{c \to \sup\Omega} x(c) = D$. Since $x(c)$ is weakly decreasing on $C_0 \cup C_1 = \Omega$,

it must be $x(c) = D$ for all $c \in \Omega$. Under $\sup\Omega > \gamma D \frac{\delta[1-F_R(1)]-F_R(R_f)-aF_R(1)}{F_R(1)}$, by continuity of the payoff function with respect to fund's type c, there exists $\varepsilon > 0$ sufficiently small s.t. for all $c \in (\sup\Omega - \varepsilon, \sup\Omega)$

$$\begin{aligned} v_c(x_0) &= \gamma D\left[a + F_R(R_f)\right] \\ &= v_c(D) + \left\{\gamma D\left[a+1\right] + c\right\} F_R(1) - \gamma D\left\{\delta\left[1 - F_R(1)\right] - F_R(R_f) + F_R(1)\right\} \\ &> v_c(D) \end{aligned}$$

which contradicts optimality of $x(c) = D$ for a positive measure set of players. Therefore, since $x(c) > x_0$ for all $c \in \Omega$, it must be $\lim_{c\to\sup\Omega} x(c) \in [x_0, D)$.

Now, by contradiction, suppose $\lim_{c\to\sup\Omega} x(c) = x_0 + \eta$ with $\eta > 0$. Pick $\varepsilon > 0$ small and let $C_\varepsilon := (\sup\Omega - \varepsilon, \sup\Omega) \subset C_1$. For notational simplicity let $x_\varepsilon := x(c_\varepsilon)$ for all $c_\varepsilon \in C_\varepsilon$. Since $x(c)$ is strictly decreasing on C_1, $F_X(x_\varepsilon) = 1 - F_C(c_\varepsilon)$ for all $c_\varepsilon \in C_\varepsilon$. By continuity of F_C there exists $\varepsilon > 0$ sufficiently small s.t. for all $c_\varepsilon \in C_\varepsilon$

$$\begin{aligned} v_c(x_0) &= \gamma D\left[a + F_R(R_f)\right] \\ &\geq v_c(x(c)) - \underbrace{\left[1 - F_C(c_\varepsilon)\right]}_{\approx 0 \text{ for small } \varepsilon}\left[1 - 2F_R(R_f)\right] + \left\{\gamma D\left[a + F_C(c_\varepsilon)\right] + c\right\}\underbrace{F_R\left(R_0(x_0+\eta)\right)}_{>0} \\ &> v_c(x(c)) \end{aligned}$$

which contradicts optimality of $x(c)$ for a positive measure set of players. □

Intuition If the "premium" for pooling is sufficiently small, "pooling" is not optimal because each player playing the pooling strategy can slightly increase her risky exposure so to increase the expected rank of her profits by outperforming, in expectation, the other pooling players, without significantly increasing the risk of default. It is obvious that this argument does not hold for $x(c) = D$ because of no-short-selling and no-borrowing constraints. On the other hand, the Nash equilibrium cannot have jumps because, under strict monotonicity, the marginal player on the left-hand side of a jump can decrease the risk of default by a finite amount without affecting the expected rank-order of her profits. By continuity of the payoff function with respect to fund's type, this is true for a positive measure set of players.

Proposition 1.20. *If there exists a Nash equilibrium $x : \Omega \to [0, D]$, then, if $\delta < \frac{F_R(R_f)-F_R(1)}{1-F_R(1)} < 1 - F_R(R_f)$,*

(i) $x(c) \in (x_0, D)$ for all $c \in \Omega$;

(ii) $x(c)$ *is strictly decreasing and continuous for all* $c \in \Omega$*;*

(iii) $\lim_{c \to \sup \Omega} x(c) = x_0$ *for any* $\sup \Omega \in \mathbb{R}_{>0}$.

Proof. (i). From Lemma 1.19 $x(c) \in (x_0, D]$ for all $c \in \Omega$. By contradiction, suppose there exists $C_a \subseteq \Omega$ with $\lambda(C_a) > 0$ s.t. $x(c) = D$ for all $c \in C_a$ and $x(c) < D$ otherwise. Pick $\varepsilon > 0$ and let $g(\varepsilon) := \lambda(\{c : x(c) \in (D - \varepsilon, D)\})$ and $\Delta F_R(\varepsilon) := F_R(1) - F_R((R_0(D - \varepsilon))$. Then, if $\delta < \frac{F_R(R_f) - F_R(1)}{1 - F_R(1)}$, by absolute continuity of F_C and F_R there exists a sufficiently small $\varepsilon > 0$ s.t. for all $c_a \in C_a$

$$\begin{aligned}
\upsilon_{c_a}(D - \varepsilon) &= \gamma D \{a + F_R(R_f) + [1 - \lambda(C_a) - g(\varepsilon)][1 - 2F_R(R_f)]\} + \\
&\quad - \{\gamma D [a + \lambda(C_a) + g(\varepsilon)] + c_a\} [F_R(1) - \Delta F_R(\varepsilon)] \\
&= \upsilon_{c_a}(D) + \gamma D \lambda(C_a) \underbrace{[F_R(R_f) - F_R(1) - \delta(1 - F_R(1))]}_{>0} + \\
&\quad + \{\gamma D [a + \lambda(C_a) + g(\varepsilon)] + c_a\} \underbrace{\Delta F_R(\varepsilon)}_{\approx 0} - \gamma D [1 - 2F_R(R_f) + F_R(1)] \underbrace{g(\varepsilon)}_{\approx 0} \\
&> \upsilon_{c_a}(D)
\end{aligned}$$

which contradicts the optimality of x_a for a positive measure set of players since $\lambda(C_a) > 0$.

(ii) It follows directly from (i) and Lemma 1.19.

(iii) Since $x(c) \in (x_0, D)$ is strictly decreasing and continuous on Ω, the proof is the same as the second part of the proof of Lemma 1.19 (iv), and thus it is omitted. □

Proof of results in the main text

Proof of Proposition 1.2. From Proposition 1.20 it follows that, under $\delta = 0$, any NE must be continuous and strictly decreasing. Let $x : \Omega \to (x_0, D)$ be a NE and let $x[\Omega] \subseteq (x_0, D)$ be its image. From strict monotonicity it follows that $F_X(y) = 1 - F_C(x^{-1}(y))$ for all $y \in x[\Omega]$, where $x^{-1}(\cdot)$ is the inverse of $x(\cdot)$. Under any NE the payoff function of fund c for investing $y \in x[\Omega]$ can be written as

$$\upsilon(y, c) = A(x^{-1}(y)) - B(x^{-1}(y), c) G(y)$$

where

$$\begin{aligned} A(x^{-1}(y)) &= \gamma D\left\{a + F_R(R_f) + [1 - 2F_R(R_f)]\left[1 - F_C(x^{-1}(y))\right]\right\} \\ B(x^{-1}(y), c) &= \left\{\gamma D\left[a + F_C(x^{-1}(y))\right] + c\right\} \\ G(y) &= F_R\left(R_0(y)\right) \end{aligned}$$

By optimality of the NE, for any Δc

$$A(c) - B(c,c)G(x(c)) \geq A(c + \Delta c) - B(c + \Delta c, c)G(x(c + \Delta c))$$

and

$$A(c + \Delta c) - B(c + \Delta c, c + \Delta c)G(x(c + \Delta c)) \geq A(c) - B(c, c + \Delta c)G(x(c))$$

Since F_R is absolutely continuous by assumption and $R_0(\cdot)$ is continuously differentiable on (x_0, D), with strictly positive first derivative, by using the Mean Value Theorem on $G(\cdot)$ we can write

$$\frac{[A(c + \Delta c) - A(c)] - [B(c + \Delta c, c) - B(c, c)]\, G(x(c))}{B(c + \Delta c, c)G'(x^*)} \leq x(c + \Delta c) - x(c)$$

and

$$\frac{[A(c + \Delta c) - A(c)] - [B(c + \Delta c, c + \Delta c) - B(c, c + \Delta c)]\, G(x(c))}{B(c + \Delta c, c + \Delta c)G'(x^*)} \geq x(c + \Delta c) - x(c)$$

where $G'(\cdot)$ is the strctly positive first derivative of $G(\cdot)$, and $x^* \in (x(c), x(c + \Delta c))$. Combining the last two inequalities and dividing by $\Delta c > 0$, we obtain

$$\begin{aligned} &\frac{[A(c + \Delta c) - A(c)] - [B(c + \Delta c, c + \Delta c) - B(c, c + \Delta c)]\, G(x(c))}{\Delta c\, B(c + \Delta c, c + \Delta c)G'(x^*)} \geq \frac{x(c + \Delta c) - x(c)}{\Delta c} \\ &\geq \frac{[A(c + \Delta c) - A(c)] - [B(c + \Delta c, c) - B(c, c)]\, G(x(c))}{\Delta c\, B(c + \Delta c, c)G'(x^*)} \end{aligned}$$

Because $x(c)$ is continuous, the left- and right-most terms of this double inequality converge to

$$\frac{A'(c) - B'(c,c)G(x(c))}{B(c,c)G'(x(c))}$$

as $\Delta c \to 0$, where $A'(\cdot)$ is the first derivative of $A(\cdot)$, and $B'(\cdot,\cdot)$ is the first derivative of $B(\cdot,\cdot)$ with respect to the first argument. By plugging the explicit expressions for A, B, and G, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}c} = -\frac{\gamma D f_C(c)\left[2q(R_f) + F_R\left(R_0(x(c))\right)\right] x(c)^2}{\left\{\gamma D\left[a + F_C(c)\right] + c\right\} f_R(R_0(x(c)))(R_f - 1)D} < 0\,.$$

where $q(R_f) := 0.5 - F_R(R_f)$. Therefore, $x(c)$ is continuously differentiable, strictly decreasing with strictly negative first derivative everywhere, and it must satisfy the above ODE. Note that the above ODE is the same ODE we would obtain by taking the first-order condition of the objective function under the assumption that the NE is continuously differentiable with strictly negative first derivative (so that the objective function would be continuously differentiable as well).

The boundary condition follows from Proposition 1.20 with $\delta = 0$. □

Proof of Proposition 1.4. From the proof of Porposition 1.2 we know that any NE is differentiable and must satisfy the following Dirichlet problem

$$\begin{cases} S(x)\mathrm{d}x + \widetilde{Q}(c)\mathrm{d}c = 0 & \text{with } c \in \Omega \text{ and } x \in (x_0, D) \\ \lim\limits_{c \to \sup \Omega} x(c) = x_0 \end{cases}$$

where

$$S(x) = \frac{(R_f - 1)Df_R(R_0(x))x^{-2}}{2q(R_f) + F_R(R_0(x))}$$
$$\widetilde{Q}(c) = \frac{\gamma D f_C(c)}{\gamma D\left[a + F_C(c)\right] + c}$$

$S(x)$ is integrable on (x_0, D), and under $a > 0$ (or $\underline{c} > 0$) $\widetilde{Q}(c)$ is integrable on Ω. By integrating the above separable ODE we obtain

$$\int^c \widetilde{Q}(s)\mathrm{d}s = -\int^x \frac{(R_f - 1)Df_R(R_0(u))u^{-2}}{2q(R_f) + F_R(R_0(u))}\mathrm{d}u + K = -\log\left[2q(R_f) + F_R(R_0(x))\right] + K$$

from which it follows

$$x(c) = \frac{(R_f - 1)D}{R_f - F_R^{-1}\left(\exp\left(-\int_{\underline{c}}^c \widetilde{Q}(s)\mathrm{d}s + K\right) - 2q(R_f)\right)}$$

where F_R^{-1} is the quantile function of R (i.e., the inverse of the cumulative distribution function). By using the boundary condition $\lim\limits_{c \to \overline{c}} x(c) = x_0$ we derive $K = \int_{\underline{c}}^{\overline{c}} \widetilde{Q}(s)\mathrm{d}s + \log\left(2q(R_f)\right)$, and obtain the unique solution of the Dirichlet problem

$$x(c) = \frac{(R_f - 1)D}{R_f - F_R^{-1}\left(2q(R_f)Q(c)\right)}\,. \tag{1.12}$$

where $Q(c) := \exp\left(\int_c^{\overline{c}} \widetilde{Q}(s)\mathrm{d}s\right) - 1$. Therefore, if there exists a NE, it is unique and equal to (1.12).

The next step is to check that $x(c) \in (x_0, D)$ for all c. From the boundary condition and the fact that $x(c)$ is continuous and strictly decreasing it follows that $x(c) > x_0$ for all $c \in \Omega$. It is easy to show that $x(c) < D$ for all $c \in \Omega$ if and only if

$$\int_{\underline{c}}^{\overline{c}} \widetilde{Q}(s)\mathrm{d}s = \mathbb{E}_C\left[\frac{\gamma D}{\gamma D\left[a + F_C(c)\right] + c}\right] < \log\left(1 + \frac{F_R(1)}{2q(R_f)}\right)$$

The last step is to prove that the unique solution of the Dirichlet problem is indeed a NE. Under the strategy profile (1.12) the objective function of each player, $\upsilon_c(y)$ with $y \in [0, D]$, is continuous everywhere and continuously differentiable on $[0, x_0) \cup (x_0, x(\underline{c})) \cup (x(\underline{c}), D]$. It straightforward to show that

$$\frac{\partial \upsilon_c}{\partial y}(y) = \begin{cases} 0 & \text{if } \ 0 \le y < x_0 \\ -\left[\gamma D a + c\right] f_R\left(R_0(y)\right)(R_f - 1)Dy^{-2} < 0 & \text{if } \ x(\underline{c}) < y \le D \end{cases}$$

The first derivative of $\upsilon_c(y)$ on $(x_0, x(\underline{c}))$ is given by

$$\begin{aligned} \frac{\partial \upsilon_c}{\partial y}(y) \quad = \quad & \gamma D\left[2q(R_f) + F_R(R_0(y))\right] f_C(x^{-1}(y)) \left(\left.\frac{\mathrm{d}x}{\mathrm{d}c}\right|_{c=x^{-1}(y)}\right)^{-1} \\ & - \left\{\gamma D\left[a + F_C(x^{-1}(y))\right] + c\right\} f_R\left(R_0(y)\right)(R_f - 1)Dy^{-2} \end{aligned}$$

By substituting the above ODE we obtain

$$\frac{\partial \upsilon_c}{\partial y}(y) = \left(x^{-1}(y) - c\right) f_R\left(R_0(y)\right)(R_f - 1)Dy^{-2} \begin{cases} > 0 & \text{if } \ x_0 < y < x(c) \\ = 0 & \text{if } \ y = x(c) \\ < 0 & \text{if } \ x(c) < y < x(\underline{c}) \end{cases}$$

Since $\upsilon_c(y)$ is continuous everywhere, it follows that, under the strategy profile $x(\cdot)$ given by (1.12), $x(c)$ is a global maximum for all $c \in \Omega$. Therefore, $x(c)$ is the unique NE of the tournament. □

Proof of Lemma 1.5. Trivial, by applying Taylor's theorem on $q(R_f)$ around μ. □

Proof of Corollary 1.6. Trivial, by applying Taylor's theorem on $x^{NE}(c)$ around $\overline{c}$. □

Proof of Corollary 1.7. Trivial. □

Proof of Proposition 1.9. Let $F^{(i)}$ denote the distribution of risky returns, F_R, when the left tail is $H^{(i)}$. Let $x^{(i)}(c)$ be the NE under $H^{(i)}$ and $R_0^{(i)}(c)$ be the critical return, $R_0(x)$, calculated at $x^{(i)}(c)$.

(i) $\left[\dfrac{q^{(2)}}{q^{(1)}} \geq \sup \dfrac{H^{(2)}}{H^{(1)}}\right]$ For every c,

$$q^{(1)}Q(c) = F^{(1)}(R_0^{(1)}(c)) = F_R(1)H^{(1)}(R_0^{(1)}(c)) \geq \frac{q(1)}{q^{(2)}}F_R(1)H^{(2)}(R_0^{(1)}(c)) = \frac{q(1)}{q^{(2)}}F^{(2)}(R_0^{(1)}(c))$$

Hence, $q^{(2)}Q(c) \geq F^{(2)}(R_0^{(1)}(c))$, from which it follows that $R_0^{(2)}(c) \geq R_0^{(1)}(c)$, and therefore $x^{(2)}(c) \geq x^{(1)}(c)$.

(ii) $\left[\dfrac{q^{(2)}}{q^{(1)}} < \sup \dfrac{H^{(2)}}{H^{(1)}}\right]$ First, it is straightforward to prove (by contradiction) that $\frac{q^{(2)}}{q^{(1)}} < \sup \frac{H^{(2)}}{H^{(1)}}$ implies $\frac{q^{(2)}}{q^{(1)}} < \sup \frac{h^{(2)}}{h^{(1)}}$, where $h^{(i)}$ is the density of $H^{(i)}$.[44] Second, since $\frac{q^{(2)}}{q^{(1)}} > 1 > \inf \frac{h^{(2)}}{h^{(1)}}$ and $\frac{h^{(2)}}{h^{(1)}}$ is decreasing by LRD, there exists $r^* \in (\underline{R}, 1)$ s.t.

$$\frac{f^{(2)}}{f^{(1)}}(r) = \frac{h^{(2)}}{h^{(1)}}(r) \begin{cases} \leq q^{(2)}/q^{(1)} & \text{for all } r \geq r^* \\ > q^{(2)}/q^{(1)} & \text{for all } r < r^* \end{cases}$$

Since $R_0^{(1)}(c)$ goes to $\underline{R}$ as $c \to \bar{c}$, is continuous and strictly decreasing, there exists $c^* \in (\underline{c}, \bar{c})$ s.t., for all $c > c^*$,

$$\begin{aligned} F^{(2)}\left(G^{(1)}\left(q^{(1)}Q(c)\right)\right) &= \frac{f^{(2)}}{f^{(1)}}(\widetilde{r})q^{(1)}Q(c) \quad \text{with } \widetilde{r} \in (\underline{R}, r^*) \qquad \text{(by the Mean Value Theorem)} \\ &> q^{(2)}Q(c) \end{aligned}$$

Hence, $> G^{(1)}\left(q^{(1)}Q(c)\right) > G^{(2)}\left(q^{(2)}Q(c)\right)$, and therefore $x^{(1)}(c) > x^{(2)}(c)$ for all $c > c^*$. Note that, if $Q(\underline{c}) = \mathbb{E}_C\left[\left(\gamma D\left(F_C(c) + a\right) + c\right)^{-1}\right] < \dfrac{F^{(1)}(r^*)}{q(1)}$, then $x^{(1)}(c) > x^{(2)}(c)$ for all c.

Now, consider the case when competitive pressure on funds with low cost of default is sufficiently high, i.e., $Q(\underline{c})$ is sufficiently large. By the single crossing property of LRD, there exists r^{**}

[44] Since F_R is assumed to be absolutely continuous, $h^{(i)}$ is strictly positive for all $i = 1, 2$.

s.t. $\frac{F^{(2)}}{F^{(1)}} = \frac{H^{(2)}}{H^{(1)}}$ is monotonically decreasing on $(r^{**}, 1)$. Since $\frac{q^{(2)}}{q^{(1)}} > 1 = \frac{F^{(2)}}{F^{(1)}}(1)$, there exists $Q(\underline{c})$ sufficiently large s.t. $R_0^{(1)}(\underline{c}) \in (r^{**}, 1)$ and

$$\frac{F^{(2)}}{F^{(1)}}(R_0^{(1)}(\underline{c})) < \frac{q^{(2)}}{q^{(1)}}$$

which implies $q^{(2)}Q(c) > F^{(2)}\left(R_0^{(1)}(\underline{c})\right)$, and hence $x^{(2)}(\underline{c}) > x^{(1)}(\underline{c})$. Then, by continuity and monotonicity of the NE, for $Q(\underline{c})$ sufficiently large there exists $c_* \in (\underline{c}, \overline{c})$ s.t. $x^{(2)}(\underline{c}) < x^{(1)}(\underline{c})$. □

Corollary 1.10. Trivial. □

Proof of Proposition 1.11. Trivial, by differentiating equilibrium (1.3) w.r.t R_f holding $q(R_f)$ constant. □

Proof of Proposition 1.13. Let $x(c; R_f)$ be the unique NE (1.3), where the second argument indicates the explicit dependence on the risk-free rate, R_f. Let

$$R_0^{NE}(c) := F_R^{-1}\left(2q(R_f)Q(c)\right) \quad \text{for all } c \in (\underline{c}, \overline{c})$$

be the equilibrium critical threshold of realized risky returns: at the uniqe NE (1.3), fund c "breaks the buck" if and only if $R < R_0^{NE}(c)$. Since $x(c) \in (x_0, D)$ is strictly decreasing and $\lim_{c\to\overline{c}} x(c) = x_0$, $R_0^{NE}(c) \in (\underline{R}, 1)$ is also strictly decreasing and $\lim_{c\to\overline{c}} R_0^{NE}(c) = \underline{R}$. Moreover, since F_R is absolutely continuous with support $(\underline{R}, \overline{R})$, $R_0^{NE}(c)$ is continuous on $(\underline{c}, \overline{c})$.

Since the probability density f_R is strictly positive everywhere on $(\underline{R}, \overline{R})$, the unique NE is differentiable with respect to R_f everywhere on $(\underline{c}, \overline{c})$, and

$$\frac{\mathrm{d}x(c; R_f)}{\mathrm{d}R_f} = \frac{2(R_f - 1)f_R(R_f)}{2q(R_f)\left[R_f - R_0^{NE}(c)\right]^2}\left[\frac{(2q(R_f))(1 - R_0^{NE}(c))}{2(R_f - 1)f_R(R_f)} - \frac{F_R\left(R_0^{NE}(c)\right)}{f_R\left(R_0^{NE}(c)\right)}\right]$$

From Proposition 1.11 and the fact that $Q(\overline{c}) = 0$, it follows that $x(c)$ is strictly increasing with R_f in a neighborhood $I_{\overline{c}}$ of $\overline{c}$. Hence, $\frac{\mathrm{d}x(c;R_f)}{\mathrm{d}R_f}$ must be non-negative on $I_{\overline{c}}$ and strictly positive

on a dense subset of $I_{\overline{c}}$. Therefore,

$$\begin{aligned} &\frac{F_R\left(R_0^{NE}(c)\right)}{f_R\left(R_0^{NE}(c)\right)} \leq \frac{(2q(R_f))(1-R_0^{NE}(c))}{2f_R(R_f)(R_f-1)} && \text{everywhere on } I_{\overline{c}}, \\ \text{and} \quad &\frac{F_R\left(R_0^{NE}(c)\right)}{f_R\left(R_0^{NE}(c)\right)} < \frac{(2q(R_f))(1-R_0^{NE}(c))}{2f_R(R_f)(R_f-1)} && \text{on a dense subset of } I_{\overline{c}}. \end{aligned}$$

Since F_R/f_R is weakly increasing by assumption, the strict inequality holds true everywhere in a sufficiently small neighborhood of $\overline{c}$, i.e. for $R_0^{NE}(c)$ sufficiently close to $\underline{R}$.

If

$$\mathbb{E}_C\left[\frac{\gamma D}{\gamma D\left(a+F_C(c)\right)+c}\right] \leq \lim_{c\to\underline{c}} \log\left(1+F_R\left(1-\frac{2f_R(R_f)(R_f-1)}{2q(R_f)}\frac{F_R\left(R_0^{NE}(c)\right)}{f_R\left(R_0^{NE}(c)\right)}\right)\right),$$

then

$$\lim_{c\to\underline{c}} \frac{F_R\left(R_0^{NE}(c)\right)}{f_R\left(R_0^{NE}(c)\right)} \leq \lim_{c\to\underline{c}} \frac{(2q(R_f))(1-R_0^{NE}(c))}{2f_R(R_f)(R_f-1)}$$

in a neighborhood of $\underline{c}$, and from (weak) monotonicity of F_R/f_R it follows that $\frac{\mathrm{d}x(c;R_f)}{\mathrm{d}R_f} \geq 0$ for all c and $\frac{\mathrm{d}x(c;R_f)}{\mathrm{d}R_f} > 0$ almost everywhere on $(\underline{c},\overline{c})$.

On the other hand, if

$$\mathbb{E}_C\left[\frac{\gamma D}{\gamma D\left(a+F_C(c)\right)+c}\right] > \lim_{c\to\underline{c}} \log\left(1+F_R\left(1-\frac{2f_R(R_f)(R_f-1)}{2q(R_f)}\frac{F_R\left(R_0^{NE}(c)\right)}{f_R\left(R_0^{NE}(c)\right)}\right)\right),$$

then

$$\lim_{c\to\underline{c}} \frac{F_R\left(R_0^{NE}(\underline{c})\right)}{f_R\left(R_0^{NE}(\underline{c})\right)} > \lim_{c\to\underline{c}} \frac{(2q(R_f))(1-R_0^{NE}(c))}{2f_R(R_f)(R_f-1)}$$

and it follows that $\frac{\mathrm{d}x(c;R_f)}{\mathrm{d}R_f} < 0$ in a neighborhood of $\underline{c}$.

Since F_R/f_R is weakly increasing on $(\underline{R},1)$ by assumption and $R_0(c)$ is strictly decreasing, there exists a unique $c^* \in (\underline{c},\overline{c})$ s.t. $\frac{\mathrm{d}x(c;R_f)}{\mathrm{d}R_f} < 0$ for all $c < c^*$ and $\frac{\mathrm{d}x(c;R_f)}{\mathrm{d}R_f} > 0$ for all $c > c^*$. □

Proposition 1.14. If $F_C^{(2)} \succ_{LRD} F_C^{(1)}$, then there exists c^* s.t. $f^{(2)}(c) > f^{(1)}(c)$ for all $c > c^*$. Since $F_C^{(2)}(c) < F_C^{(1)}(c)$ for all c,

$$\int_c^{\overline{c}} \frac{\gamma D f_C^{(2)}(u)\mathrm{d}u}{\gamma D\left[F_C^{(2)}(u)+a\right]+u} > \int_c^{\overline{c}} \frac{\gamma D f_C^{(1)}(u)\mathrm{d}u}{\gamma D\left[F_C^{(1)}(u)+a\right]+u} \qquad \text{for all } c > c^*,$$

from which it follows that $Q^{(2)}(c) > Q^{(1)}(c)$ and therefore $x^{(2)}(c) > x^{(1)}(c)$ for all $c > c^*$

Using integration by parts,

$$\int_{\underline{c}}^{\overline{c}} \frac{\gamma D f_C(u)\mathrm{d}u}{\gamma D\left[F_C(u)+a\right]+u} = \log\left(\frac{\gamma D(a+1)+\overline{c}}{\gamma D\, a+\underline{c}}\right) - \int_{\underline{c}}^{\overline{c}} \frac{\mathrm{d}u}{\gamma D(F_C(u)+a)+u}.$$

Hence, if $F_C^{(2)}(c) < F_C^{(1)}(c)$ everywhere because $F_C^{(2)} \succ_{LRD} F_C^{(1)}$, $Q^{(2)}(\underline{c}) < Q^{(1)}(\underline{c})$. By continuity and monotonicity of the NE, there exists c_* s.t. $Q^{(2)}(c) < Q^{(1)}(c)$ and therefore $x^{(2)}(c) < x^{(1)}(c)$ for all $c < c_*$.

□

Appendix 1.D Data Construction and Summary Statistics

Data Construction

Data on fund characteristics are from iMoneyNet. These data are the most comprehensive source of information on MMFs and are widely used for both academic research and investment decisions. KS check that the iMoneyNet database covers the universe of US MMFs by comparing it to the list of funds registered at the SEC, and Chodorow-Reich (2014) shows that the coverage of iMoneyNet data matches that of the Financial Accounts of the United States.

I focus on prime MMFs over the period from January 2006 to August 2008. Data are at the weekly, share class level. In my sample, there are a total of 830 share classes. I find that 7 of these share classes have some missing data for some week. Almost all missing data come from funds that report monthly for the first few months of their existence and later switch to weekly reporting. Following KS, I use linear interpolation to generate weekly data for these share classes. Since my analysis is at the fund level, I aggregate share classes at the fund level. To identify funds, I use information on the underlying portfolio, which must be the same for all share classes belonging to the same fund. Share classes that have the same portfolio composition in terms of asset classes and the same weighted average maturity identify a unique fund. Over my period of analysis, consisting of 139 weeks, I identify 330 prime MMFs. I double-check the accuracy of my fund identifier by verifying that the assets for all share classes add up to total fund size. The difference between the two exceeds \$100,000 (data are reported in \$100,000 increments) only for 177 fund-week observations out of 35,608, i.e., less than 0.5% of the sample.

To construct fund level characteristics, I follow KS and average share class characteristics using share class assets as weights. Each fund can have both retail share classes, which are available

only to retail investors, and institutional share classes, which are available only to institutional investors. In my empirical analysis, I label a fund as institutional if it has at least one institutional share classes. A fund is labeled as retail if it has no institutional share class. KS use the same convention. Moreover, institutional share classes are typically much larger than retail share classes, which justifies this identification. My empirical analysis focuses on institutional funds because it has been observed that they face a more sensitive flow-performance relation than retail funds. In my sample, I observe 192 institutional funds, for a total of 19,642 institutional fund-week observations. (I observe 149 retail funds, for a total of 15,966 retail fund-week observations.) The institutional funds that remain active throughout the whole period of analysis are 122, for a total of 16,958 observations. The main empirical analysis is restricted to this balanced panel.

I merge the iMoneyNet database with the CRSP Survivorship Bias Free Mutual Fund Database. CRSP data are at the quarterly level. Therefore, share classes in the two data sets are matched at that frequency. (Any within-quarter variation at the sponsor level is assumed to be constant.) To match funds in the iMoneyNet database to sponsors in the CRSP database, I proceed as follows. First, I match share classes by using the NASDAQ ticker. If a share class is matched, I assign to it a sponsor based on the entry `mgmt_cd` in the corresponding CRSP match. If `mgmt_cd` is not available, I use `mgmt_name`. If there is no match in CRSP using the NASDAQ ticker, I use the 9-digit CUSIP number. For some observations neither NASDAQ nor CUSIP have a match in the CRSP database. In those cases, I assign a match based on the other share classes in the same fund for which a match is available. (If share classes from the same fund are assigned to different sponsors in CRSP, I only use the largest share class.) If no other share class in the fund has a valid match in CRSP, I assign a match based on the other share classes in the same fund complex, as indicated by MoneyNet. (Again, if share classes from the same complex are assigned to different sponsors in CRSP, I only use the largest share class.) If no other share class in the complex has a valid match in CRSP, I match share classes by matching the name of the complex as reported by iMoneyNet with the fund name in the CRSP database. Under this algorithm, only 14 funds out of a total of 330 are not matched with a unique sponsor in CRSP. I manually match 13 of these funds with their sponsor in CRSP by using SEC filings in EDGAR, company sources, and press coverage.[45] In this way, I manage to match at least 99.26% of funds every week, corresponding to a coverage of at least 99.94% in terms of asset volume.

[45]The only fund that I did not manage to match with the CRSP database is the Williams Capital Liquid Assets Fund, whose institutional share is WLAXX.

For each sponsor, I use CRSP data to calculate the total amount of its mutual fund assets at a given quarter and use that measure to calculate the proxy for sponsor's reputational concern as described in the main text.

Summary Statistics

Table 1.D.1 provides summary statistics for all institutional prime MMFs as of January 3, 2006. The sample includes 143 funds and 82 sponsors. Column (1) shows summary statistics for all funds, column (2) shows summary statistics for funds whose sponsors have *Fund Business* above the median value of 81.9% as of January 2006, and column (3) shows summary statistics for funds whose sponsors have values below the median. Results are discussed in Section 1.7 of the main text. My findings are close to those of KS, validating the consistency of my data set with theirs.

Distributional properties of *Fund Business*

This section presents a descriptive analysis of the distributional properties of *Fund Business*. It shows that there is significant dispersion in the cross-section of sponsors' reputation concerns, which supports the validity of a "continuum-of-funds" approach and helps the identification of the effect of default cost (i.e., sponsor's reputation concerns) on the cross-sectional risk-taking differential.

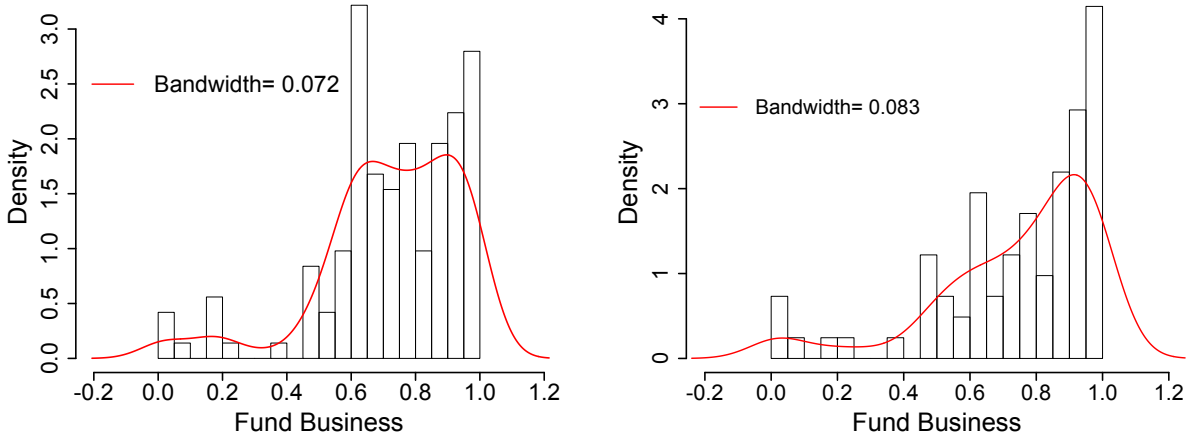
The left panel of Figure 1.6 shows the distribution of *Fund Business* in the population of funds in January 3, 2006. The distribution is widely spread on the interval $[0, 1]$, suggesting that a binary distribution would be a poor approximation of the actual one. The distribution shows some degree of multimodality with a small peak around zero (sponsors specialized in MMFs, e.g. City National Rochdale), and two pronounced peaks around 0.7 and 1 (largest asset managers, e.g PIMCO). The right panel of Figure 1.6 shows the distribution of *Fund Business* in the population of sponsors for the same date. Again, the distribution is widely spread on its support and shows some degree of multimodality, even though to a lesser extent. The comparison of the two distribution suggests that the some sponsors in the mid-range of *Fund Business* offer relatively more MMFs.

Both at the fund level and at the sponsor level, the distribution of *Fund Business* is stable over the period of analysis. Figure 1.7 shows the evolution of the mean and quartiles of *Fund Business* from January 2006 to August 2008. The left panel is at the fund level, and the right panel is at the sponsor level. The results are similar.

				Kacperczyk & Schnabl (2013)		
	(1) All	(2) High FB	(3) Low FB	(4) All	(5) High FB	(6) Low FB
Fund Characteristics						
Spread (bp)	7.54	7.27	7.70	6.93	6.60	7.28
	(6.46)	(6.22)	(6.64)	(6.44)	(7.54)	(5.00)
Expense Ratio (bp)	35.90	34.89	36.53	31.64	32.40	30.81
	(21.79)	(22.81)	(21.23)	(19.10)	(18.43)	(19.90)
Fund Size ($mil)	6,318	4,195	7,645**	4,886	2,981	6,951***
	(10,793)	(8,413)	(11,899)	(8,685)	(4,833)	(11,169)
Maturity (days)	33.27	33.93	32.85	34.32	35.12	33.45
	(10.65)	(10.90)	(10.54)	(11.02)	(12.48)	(9.17)
Age (years)	11.20	11.12	11.25	10.61	10.43	10.81
	(6.84)	(7.36)	(6.53)	(4.75)	(5.53)	(3.75)
Family Size ($bil)	73.3	99.1	47.5*	72.8	97.5	45.9**
	(157.1)	(211.7)	(61.8)	(149.1)	(200.9)	(39.2)
Fund Business	0.745	0.929	0.562***	0.764	0.897	0.619***
	(0.248)	(0.051)	(0.230)	(0.198)	(0.064)	(0.192)
Conglomerate	0.566	0.418	0.659***	0.601	0.558	0.648
	(0.497)	(0.498)	(0.477)	(0.491)	(0.500)	(0.481)
Portfolio Holdings						
U.S. Treasuries & agency	0.059	0.065	0.055	0.060	0.072	0.048
	(0.096)	(0.092)	(0.099)	(0.109)	(0.120)	(0.095)
Repurchase Agreements	0.134	0.128	0.138	0.135	0.142	0.126
	(0.151)	(0.169)	(0.139)	(0.150)	(0.169)	(0.128)
Bank Deposits	0.034	0.016	0.045***	0.032	0.021	0.044**
	(0.058)	(0.034)	(0.067)	(0.057)	(0.039)	(0.069)
Bank Obligations	0.124	0.112	0.132	0.122	0.111	0.135
	(0.127)	(0.116)	(0.133)	(0.126)	(0.120)	(0.132)
Floating-Rate Notes	0.199	0.225	0.183	0.198	0.192	0.204
	(0.164)	(0.184)	(0.149)	(0.162)	(0.168)	(0.156)
Commercial Paper	0.314	0.305	0.319	0.320	0.356	0.280**
	(0.216)	(0.226)	(0.212)	(0.224)	(0.252)	(0.182)
Asset-Backed CP	0.136	0.149	0.128	0.134	0.106	0.164**
	(0.154)	(0.180)	(0.136)	(0.155)	(0.151)	(0.154)
Funds	143	55	88	148	77	71

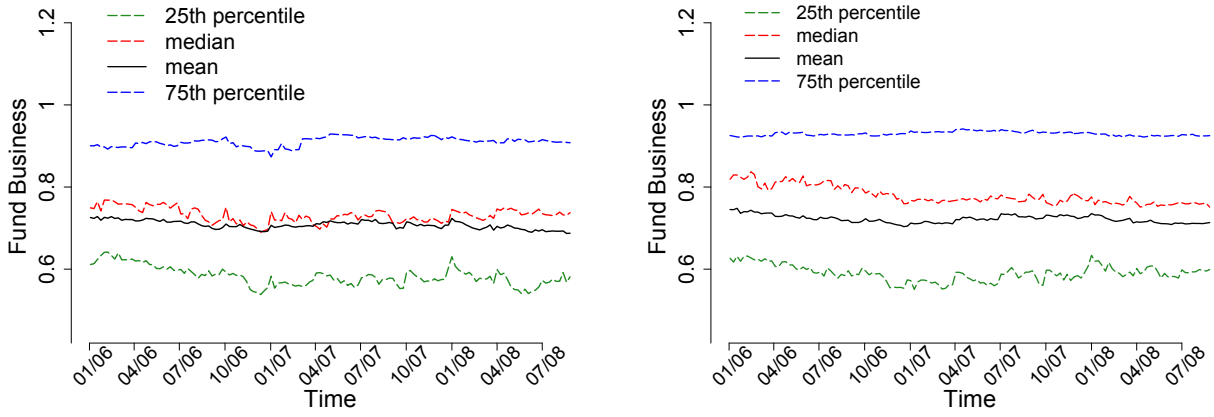
$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.D.1: Summary statistics for all U.S. institutional prime MMFs as of January 3, 2006. *Fund Business* (FB) is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. High (Low) FB includes all funds with *Fund Business* above (below) the median value of *Fund Business* (81.9%). Fund characteristics are spread, expense ratio, fund size, average portfolio maturity, age, family size, and whether the fund sponsor is part of a conglomerate. Holdings are the share of assets invested in Treasuries and agency debt, repurchase agreements, bank deposits, bank obligations, floating-rate notes, commercial paper, and asset-backed commercial paper. Cross-sectional standard deviations of the given characteristics are in parentheses. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.



(A) Fund Population (B) Sponsor Population

FIGURE 1.6: **Distribution of** *Fund Business* **as of January 3, 2006.** The sample is all institutional prime MMFs (left panel), and all sponsors of institutional prime MMFs (right panel). The red line represents the density of *Fund Business* estimated using a Gaussian kernel. The bandwidth is determined according to Silverman's "rule of thumb" (Silverman, 1986, pg. 48, eqn. (3.31)).



(A) Fund Population (B) Sponsor Population

FIGURE 1.7: **Time Evolution of the Summary Statistics of** *Fund Business*.

Appendix 1.E Risk-taking Opportunities: supplementary evidence

This section analyzes the risk-taking opportunities of prime MMFs from January, 2006 to August, 2008. The analysis is at the level of the asset classes in MMF portfolios as reported by iMoneyNet. This is a more granular investigation of the risk-taking opportunities of MMFs

than the one in Section 1.7.1 and allows me to identify the riskiest asset class in the period of interest.

Since I do not directly observe the yield of the individual instruments, following KS I infer the spread of each instrument via regression

$$Spread_{i,t+1} = \alpha_i + \mu_t + \sum_j \beta_j Holdings_{i,j,t} + \gamma \cdot X_{i,t} + \varepsilon_{i,t+1} \tag{1.13}$$

where $Spread_{i,t+1}$ is the gross yield of fund i in week $t+1$ minus the risk-free rate (30-day T-bill weekly return), $Holdings_{i,j,t}$ denotes fund i's fractional holdings of instrument type j in week t, α_i denotes fund fixed effects, and μ_t denotes week fixed effects. The instrument types include repurchase agreements, time deposits, bank obligations (i.e., negotiable deposits), floating-rate notes, commercial papers, and asset-backed commercial papers. The omitted category is Treasuries and GSE debt. X is the set of fund-specific controls defined in (1.7). The coefficients of interest are β_j, which measure the return on money market instrument j in week $t+1$ relative to that of Treasuries and GSE debt. Following KS, I estimate the regression model separately for the post period from August 2007 to August 2008 and the pre period from January 2006 to July 2007. I also estimate the regression on the whole period (January 2006-August 2008).

Table 1.E.1 shows the results. All standard errors are HAC robust. Columns (1) and (2) include only week fixed effects, while Columns (3) and (4) include both week and fund fixed effects.[46] My results are very close to those of KS. In general, all instruments, except *Bank Deposits*, have significantly larger yields in the post period. In particular, *Bank Obligations* show the largest increase in the spread relative to treasuries and GSE debt. When accounting for fund fixed effects, the yield of a fund fully invested in bank obligations would have been 94 basis points higher than the yield of a fund fully invested in Treasury and agency debt. Similar results hold for ABCP and, to a lesser extent, FRNS and CP.

Columns (5) and (6) show the results for the whole period. Qualitatively these results are similar to those for the post period. *Bank Obligations* have the largest yield, followed by ABCP, CP and FRNS. All instruments have positive and statistically significant yields relative to Treasuries and GSE debt securities.

Figure 1.8 shows the spread of eligible risky instruments relative to U.S. Treasuries and GSE debt from January 2006 to August 2008. I run regression (1.13) with two-way fixed effects for every month in the time window. Each point represents the three-month-backward MA of the

[46] Including fund fixed effects alleviates the concern that funds with large holdings of risky assets might also be riskier, along other unobserved dimensions.

coefficients on the instrument type. Before August 2007, the spread between risky instruments and U.S. Treasuries and GSE debt was at most 25 basis points, thus leaving little scope for large variations in the cross-section of funds' yields. After August 2007, the spread between risky instruments, such as bank obligations, and safe instruments, such as U.S. Treasuries, increased to 90 basis points.

In view of the results of this section, bank obligations can be regarded as the riskiest security available to MMFs in the period of analysis, consistently with KS. This justifies the definition of the main risk measure, *Holdings Risk*, as the percentage of assets held in bank obligations net of U.S. treasuries, GSE debt and repurchase agreements.

Appendix 1.F Flow-performance: supplementary evidence

Table 1.F.1 shows the results for the estimation of regression (1.7) using the rank of *Fund Flow* (*Fund Flow Rank*) as dependent variable. The rank is calculated in percentiles that are normalized to $[0, 1]$ (e.g., for a fund-week in the top 98% of the distribution of fund flows, *Fund Flow Rank* $= 0.98$). Using the rank of fund flows is an alternative to trimming to take care of the effect of possible extreme outliers. Results are qualitatively similar to those obtained using *Fund Flow* trimmed at the 0.5%. The past *Spread* is a significant determinant of the fund-flow performance only in the post period. On the other hand, the past *Spread Rank* is statistically significant both in the post and the pre period, and its economic significance is considerably larger in both periods. Moreover, when both are included, only *Spread Rank* is statistically significant.

As further robustness check, I also run regression (1.7) including only time fixed effects, and using trimming conditions based on the interquartile range. Results are similar and are available upon request. This additional empirical evidence confirms that the rank of performance is a better explanatory variable of money flows in the MMF industry than the raw performance and supports the choice of modeling the MMF industry as a pure tournament.

1.F.1 Exogeneity of the flow-performance relationship

Here I test the assumption that the flow-performance relationship can be taken as exogenous in the context of my model. Specifically, I test the hypothesis that the flow-performance relation is not explicitly affected by sponsor's reputation concerns (*Fund Business* and *Conglomerate*). This characteristic may affect the flow-performance relation if investors anticipate the effect of

	$Spread_{i,t+1}$					
	(1) Post	(2) Pre	(3) Post	(4) Pre	(5) Whole Period	(6)
Portfolio Holdings						
$Repurchase\ Agreements_{i,t}$	5.464	17.444***	44.292***	8.410**	17.430**	44.933***
	(11.351)	(3.536)	(14.964)	(4.072)	(8.464)	(12.508)
$Bank\ Deposits_{i,t}$	−7.111	16.376***	13.712	8.481**	5.601	32.814**
	(26.207)	(4.298)	(21.260)	(4.049)	(17.713)	(16.228)
$Bank\ Obligations_{i,t}$	78.037***	15.312***	94.000***	0.291	49.975***	68.781***
	(10.385)	(3.580)	(15.523)	(4.005)	(7.159)	(12.194)
$Floating\text{-}Rate\ Notes_{i,t}$	77.542***	22.005***	72.438***	5.205	51.156***	53.829***
	(8.657)	(3.484)	(17.763)	(4.448)	(6.197)	(12.758)
$Commercial\ Paper_{i,t}$	55.131***	16.518***	75.221***	10.494***	38.350***	53.634***
	(10.734)	(3.490)	(21.158)	(4.051)	(7.351)	(12.675)
$Asset\text{-}Backed\ CP_{i,t}$	74.212***	20.260***	82.188***	10.470***	47.770***	63.007***
	(11.444)	(3.480)	(17.896)	(3.972)	(7.459)	(12.247)
Fund Characteristics						
$Log(Fund\ Size)_{i,t}$	0.772	−0.026	4.055***	0.178	0.460	2.067
	(0.532)	(0.114)	(1.123)	(0.445)	(0.396)	(1.306)
$Expense\ Ratio_{i,t}$	6.133*	−0.543	31.518	8.187*	3.115	10.993
	(3.683)	(0.863)	(26.000)	(4.303)	(1.920)	(11.488)
$Age_{i,t}$	0.045	0.012	−0.292	0.169	−0.022	0.054
	(0.111)	(0.021)	(0.709)	(0.206)	(0.062)	(0.684)
$Log(Family\ Size)_{i,t}$	0.413	0.311***	−2.170	0.755*	0.357	0.554
	(0.547)	(0.118)	(2.464)	(0.431)	(0.313)	(0.927)
Week fixed effect	Y	Y	Y	Y	Y	Y
Fund fixed effect	N	N	Y	Y	N	Y
Observations	7,602	11,599	7,602	11,599	19,335	19,335
Adj. R^2 (within)	0.51	0.25	0.18	0.04	0.29	0.11
R^2 (overlall)	0.95	0.97	0.97	0.98	0.97	0.98

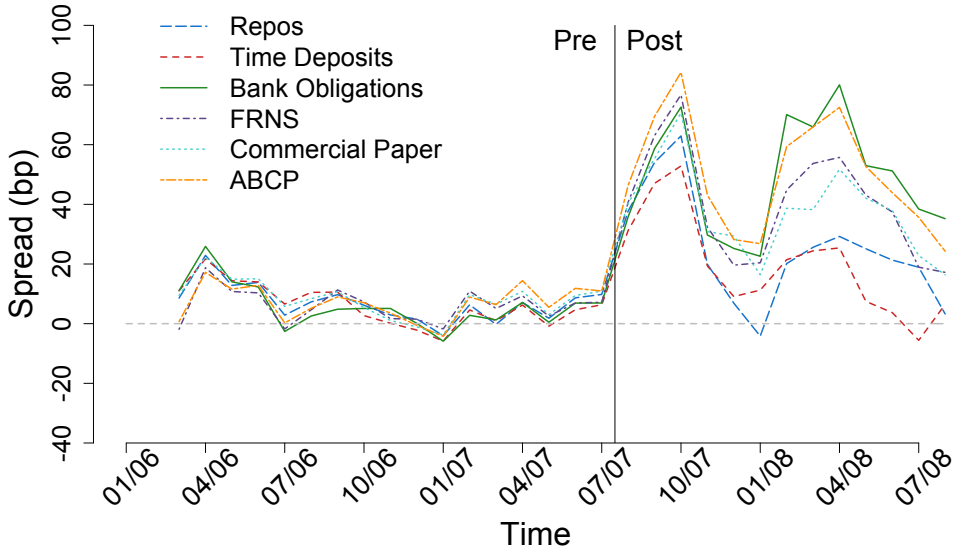
$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.E.1: **Investment opportunities at the security-class level.** The sample is all U.S. institutional prime money market funds. The dependent variable *Spread* is computed as the annualized fund yield minus the Treasury bill rate. Holdings variables are the share of assets invested in repurchase agreements, bank deposits, bank obligations, floating-rate notes, commercial paper (CP), and asset-backed CP (omitted category is U.S. Treasury and GSE debt). Fund characteristics are logarithm of fund size, expense ratio, fund age, and logarithm of fund family size. All regressions are at the weekly level and include week fixed effects. Columns (3) and (4) include also fund fixed effects. Columns (1) and (3) cover the period 8/1/2007-8/31/2008 (post period). Columns (2) and (4) cover the period 1/1/2006-7/31/2007 (pre period). Standard errors are HAC-robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

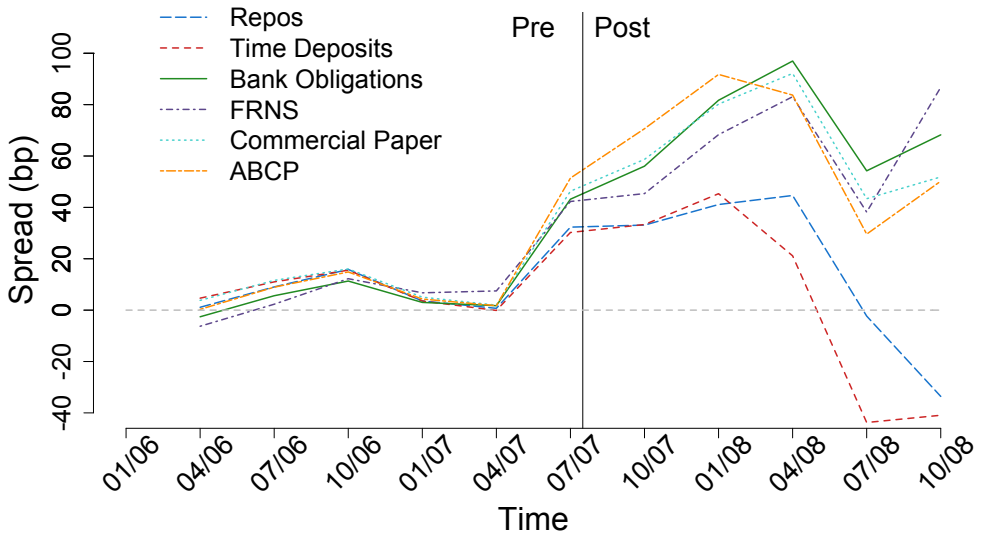
	$Fund\ Flow\ Rank_{i,t+1}$					
	(1)	(2)	(3)	(4)	(5)	(6)
	Post	Pre	Post	Pre	Post	Pre
$Spread\ Rank_{i,t}$			0.118^{***}	0.026^{*}	0.078^{*}	0.049^{**}
			(0.022)	(0.015)	(0.043)	(0.025)
$Spread_{i,t}$	0.002^{***}	0.001			0.001	-0.002
	(0.000)	(0.001)			(0.001)	(0.002)
$Log(Fund\ Size)_{i,t}$	-0.255^{***}	-0.214^{***}	-0.256^{***}	-0.215^{***}	-0.257^{***}	-0.215^{***}
	(0.062)	(0.034)	(0.067)	(0.034)	(0.065)	(0.034)
$Expense\ Ratio_{i,t}$	0.077	-0.108	0.132	-0.111	0.104	-0.096
	(0.155)	(0.172)	(0.166)	(0.171)	(0.162)	(0.172)
$Age_{i,t}$	-0.003	-0.028^{**}	-0.005	-0.028^{**}	-0.004	-0.028^{**}
	(0.007)	(0.012)	(0.007)	(0.012)	(0.007)	(0.012)
$Flow\ Volatility_{i,t}$	-0.004^{***}	0.082	-0.002^{***}	0.079	-0.003^{***}	0.077
	(0.001)	(0.062)	(0.000)	(0.062)	(0.001)	(0.063)
$Log(Family\ Size)_{i,t}$	0.044	0.002	0.042	0.002	0.043	0.003
	(0.027)	(0.011)	(0.028)	(0.010)	(0.027)	(0.011)
Week fixed effect	Y	Y	Y	Y	Y	Y
Fund fixed effect	Y	Y	Y	Y	Y	Y
Observations	7,496	9,625	7,496	9,625	7,496	9,625
Adj. R^2 (within)	0.024	0.020	0.025	0.021	0.025	0.021
R^2 (overall)	0.050	0.043	0.050	0.043	0.050	0.044

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.F.1: **Flow-performance relation: performance rank matters more than raw performance.** Columns (1), (3) and (5) cover the period 8/1/2007-8/31/2008 (*post* period). Columns (2), (4) and (6) cover the period 1/1/2006-7/31/2007 (*pre* period). The dependent variable is *Fund Flow Rank*, which is the rank of *Fund Flow*. *Fund Flow* is the percentage change in total net assets from week t to week $t+1$, adjusted for earned interests. The rank is computed in percentiles normalized to $[0, 1]$. Using the rank of *Fund Flow* alleviates the concern of outliers in the distribution of fund flows. Independent variables are the weekly annualized spread from $t-1$ to t, its rank in percentiles normalized to $[0, 1]$, logarithm of fund size, fund expense ratio, fund age, volatility of fund flows based on past 12-week fund flows, and logarithm of fund family size. All regressions are at the weekly level and include fund and week fixed effects. Standard errors are HAC robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.



(A) Monthly



(B) Quarterly

FIGURE 1.8: **Spread by instrument type.** I implement the panel regression model in Table 1.E.1 with two-way fixed effects over the period from January 2006 to August 2008. In the top chart (*a*), I estimate the regression model every month: each point represents the three-month-backward MA of coefficients on the instrument type. In the bottom chart (*b*), I estimate the regression model every quarter: each point represents the two-quarter-backward MA of coefficients on the instrument type. Each point represents the return relative to the omitted category (Treasuries and GSE debt) measured in basis points.

sponsor's reputation concern on fund's risk taking. I test this hypothesis by estimating the flow-performance relation in equation (1.7) with additional interactions of *Fund Business* and

Conglomerate with *Spread Rank*. I present the results in Table 1.F.2. Standard errors are HAC robust.

For both periods I find that the coefficients on the interaction terms are statistically and economically insignificant for both measures of reputation concern. Hence, conditional on flow performance, there is no effect of sponsor characteristics on fund flows. As first robustness check, I run the same test using *Spread* instead of *Spread Rank* as main explanatory variable, as done by KS. Results are similar and omitted for brevity. As further robustness check, I run the same test using the rank of *Fund Flow* as independent variable to alleviate the concern of outliers. Results are very similar and thus omitted for brevity.

These findings suggest that investors do not risk adjust fund performance based on sponsor characteristics and that the assumption of an exogenous flow-performance relation in the context of my model is satisfied in the data.

Appendix 1.G MMF risk-taking: supplementary evidence

1.G.1 Cross-sectional risk-taking holding sponsor's concerns fixed as of 2006

Here I follow Kacperczyk and Schnabl (2013) and analyze fund risk-taking in the *Post* and *Pre* period by estimating the following regression:

$$\begin{aligned} Risk_{i,t+1} =& \alpha + \mu_t + \beta_1 Reputation\ Concerns_{i,2006} \\ &+ \beta_2 Post_t * Business\ Spillovers_{i,2006} + \gamma \cdot X_{i,2006} + \varepsilon_{i,t+1} \end{aligned} \tag{1.14}$$

where $Reputation\ Concerns_{i,2006}$ is a generic name for either *Fund Business* or *Conglomerate*. *Post* is an indicator variable equal to 1 for the post period and 0 for the pre period. $X_{i,2006}$ is a vector of control variables similar to those in equation (1.9). Both business spillover variables and other controls are measured as of January 2006, which mitigates the concern that fund risk choices are driven by changes in fund characteristics due to investment opportunity changes. Regression model (1.14) also includes week fixed effects (μ_t), which account for any time differences in aggregate fund flows or macroeconomic conditions. As KS, I use three measures of risk (*Risk*): *Spread*, *Holdings Risk*, and *Maturity Risk*.

To be consistent with KS (see Table IV therein), Table 1.G.1 shows the results when both *Fund Business* and *Conglomerate* are included on the RHS of (1.14). Reported standard errors are HAC and cross-correlation robust. My results are qualitatively similar to those of KS.

	$Fund\ Flow_{i,t+1}$					
	(1)	(2)	(3)	(4)	(5)	(6)
	Post	Pre	Post	Pre	Post	Pre
$Spread\ Rank_{i,t}$	3.229***	1.339**	2.156***	0.618*	3.617***	1.368*
	(1.039)	(0.676)	(0.494)	(0.367)	(1.179)	(0.814)
$Fund\ Business_{i,t}$*	−0.019	−0.010			−0.020	−0.010
$Spread\ Rank_{i,t}$	(0.014)	(0.009)			(0.014)	(0.009)
$Conglomerate_{i,t}$*			−0.004	0.001	−0.005	0.000
$Spread\ Rank_{i,t}$			(0.006)	(0.005)	(0.007)	(0.006)
$Log(Fund\ Size)_{i,t}$	−5.656***	−4.229***	−5.578***	−4.182***	−5.664***	−4.230***
	(0.938)	(0.681)	(0.647)	(0.442)	(0.938)	(0.679)
$Expense\ Ratio_{i,t}$	1.114	−1.168	0.714	−1.237	1.333	−1.169
	(2.491)	(3.339)	(2.721)	(1.679)	(2.508)	(3.340)
$Age_{i,t}$	−0.137	−0.488**	−0.138	−0.497***	−0.136	−0.489**
	(0.135)	(0.233)	(0.131)	(0.112)	(0.137)	(0.233)
$Flow\ Volatility_{i,t}$	−0.018***	2.849**	−0.016***	2.850**	−0.020***	2.844**
	(0.005)	(1.391)	(0.006)	(1.289)	(0.005)	(1.392)
$Log(Family\ Size)_{i,t}$	0.952**	0.072	0.846**	−0.003	0.964**	0.073
	(0.427)	(0.205)	(0.378)	(0.186)	(0.425)	(0.208)
Week fixed effect	Y	Y	Y	Y	Y	Y
Fund fixed effect	Y	Y	Y	Y	Y	Y
Observations	7,387	9,467	7,387	9,467	7,387	9,467
Adj. R^2 (within)	0.032	0.024	0.032	0.024	0.032	0.024
R^2 (overlall)	0.081	0.061	0.080	0.061	0.081	0.061

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.F.2: **Flow-performance relation: no explicit effect of sponsor's reputation concerns.** Columns (1), (3) and (5) cover the period 8/1/2007-8/31/2008 (*post* period). Columns (2), (4) and (6) cover the period 1/1/2006-7/31/2007 (*pre* period). The dependent variable is *Fund Flow*, computed as the percentage change in total net assets from week t to week $t+1$, adjusted for earned interests and trimmed at the 0.5%. Independent variables are the weekly spread rank from $t-1$ to t, logarithm of fund size, fund expense ratio, fund age, volatility of fund flows based on past 12-week flows, and logarithm of fund family size. Additional independent variables are the interactions of *Spread Rank* with *Fund Business* and *Conglomerate*. *Fund Business* is the share of mutual fund assets other than institutional prime MMF in sponsor's total mutual fund assets. *Conglomerate* is an indicator variable equal to 1 if the fund sponsor is affiliated with a financial conglomerate and 0 otherwise. All regressions are at the weekly level and include week and fund fixed effects. Standard errors are HAC robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

	$Holdings\ Risk_{i,t+1}$	$Maturity\ Risk_{i,t+1}$	$Spread_{i,t+1}$	$Safe\ Holdings_{i,t+1}$
	(1)	(2)	(3)	(4)
$Fund\ Business_{i,2006} * Post_t$	-14.828^{***}	-12.071^{***}	-9.862^{***}	7.899^{***}
	(1.707)	(1.153)	(1.533)	(1.173)
$Conglomerate_{i,2006} * Post_t$	-2.393^{***}	0.725	-5.296^{***}	2.530^{***}
	(0.425)	(0.500)	(0.490)	(0.387)
$Fund\ Business_{i,2006}$	-11.007^{***}	6.725^{***}	-0.266	3.051^{***}
	(1.120)	(0.747)	(0.239)	(0.933)
$Conglomerate_{i,2006}$	-2.728^{***}	-0.830^{**}	-1.983^{***}	0.032
	(0.228)	(0.336)	(0.160)	(0.150)
$Controls_{i,2006}$	Y	Y	Y	Y
$Controls_{i,2006} * Post_t$	Y	Y	Y	Y
Week fixed effect	Y	Y	Y	Y
Fund fixed effect	N	N	N	N
Observations	16,836	16,836	16,836	16,836
R^2 (overall)	0.181	0.121	0.977	0.114

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.G.1: **Cross-sectional risk-taking differential in the *Pre* and *Post* period.** The sample is all U.S. institutional prime money market funds for the period from 1/1/2006 to 8/31/2008. The dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in column (1), average portfolio maturity (*Maturity Risk*) in column (2), the weekly annualized fund spread (*Spread*) in column (3), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (4). *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. *Conglomerate* is an indicator variable equal to 1 if the fund sponsor is affiliated with a financial conglomerate, and 0 otherwise. *Post* is an indicator variable equal to 1 for the period from 8/1/2007 to 8/31/2008, and 0 otherwise. The other independent variables (*Controls*) are fund assets, expense ratio, fund age, and fund family size as of 1/3/2006. All regressions are at the weekly level and include week fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

Sponsor's reputation concerns are negatively correlated with fund risk-taking in the post period, and their effect is statistically significant at the 1% level for most measures of risk. In both my regressions and those of KS, sponsor's reputation concerns tend to be negatively correlated with *Holdings Risk* also in the pre period, with the effect being statistically significant at the 1% level in my regression but insignificant in that of KS. Moreover, both in my regressions and those of KS, *Conglomerate* is negatively correlated with all measures of risk in the pre period, with its effect being statistically significant at least at the 5% level in my regressions. For robustness, I also run regression (1.14) using $Safe\ Holdings$ (U.S. trasuries + GSE debt + repos) as dependent variable (see column 4). In this case, my model predicts $\beta_2 > 0$ and $\beta_1 > 0$. In the data, I find that for $Fund\ Business$ both $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are positive and statistically significant at the 1% level, in agreement with the model and empirical results of Section 1.7.4.

In regression (1.14), sponsor's reputation concern is instrumented with its value as of January 2006. While this eliminates possible endogenous correlations between sponsor's level of reputation concern and the unobserved regression error, it also excludes all truly exogenous variations in sponsor's reputation concern (e.g., shocks to the sponsor's equity mutual fund business that are orthogonal to the investment opportunities in the money market). For these reasons, in Section 1.7.4 I extend the analysis of KS by using as explanatory variable the lagged value of a sponsor's reputation concern.

1.G.2 Cross-sectional risk-taking: robustness checks

Table 1.G.2 reports the results of regression (1.9) when $Fund\ Business$ is used as main explanatory variable instead of its rank ($FB\ Rank$). Standard errors are HAC and cross-correlation robust. Results are very similar to those reported in Table 1.5 of Section 1.7.4, confirming the validity of my model's predictions.

As further robustness check, I also run the following instrumental variable (IV) regression model separately on the post and the pre period:

$$Risk_{i,t} = \alpha_i + \mu_t + \beta Fund\ Business_{i,t;IV:t-k} + \gamma \cdot X_{i,t-k} + \varepsilon_{i,t} \tag{1.15}$$

where $Risk$ and X are defined as in (1.9), and $Fund\ Business_{i,t;IV:t-k}$ is fund i's measure of sponsor's reputation concern in week t instrumented with sponsor's reputation concern in week $t-k$. $Fund\ Business$ is instrumented with its lagged values in order to alleviate possible endogeneity issues. For the same reason the controls X are lagged.[47] I run several regression

[47] I also run a regression specification in which all time-varying and fund-varying RHS variables are instrumented with their lagged values. The results are very similar and thus omitted.

	$Holdings\ Risk_{i,t}$		$Maturity\ Risk_{i,t}$		$Spread_{i,t}$		$Safe\ Holdings_{i,t}$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	k=4	k=8	k=4	k=8	k=4	k=8	k=4	k=8
$Fund\ Business_{i,t-k}$	-20.891^{***}	-19.473^{***}	-9.238^{***}	-8.217^{***}	-3.019	-3.422	6.278^{***}	4.357^{*}
	(3.438)	(3.337)	(2.118)	(2.174)	(3.590)	(3.231)	(2.425)	(2.598)
$Fund\ Business_{i,t-k} * Post_t$	-11.037^{***}	-10.247^{***}	-7.728^{***}	-7.763^{***}	-3.237^{***}	-3.522^{***}	8.106^{***}	7.510^{***}
	(1.528)	(3.174)	(0.937)	(1.955)	(0.711)	(3.209)	(1.265)	(2.469)
$Controls_{i,t-k}$	Y	Y	Y	Y	Y	Y	Y	Y
$Controls_{i,t-k} * Post_t$	Y	Y	Y	Y	Y	Y	Y	Y
Week Fixed Effects	Y	Y	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y	Y	Y
Observations	16,470	15,982	16,470	15,982	16,470	15,982	16,470	15,982
Adj. R^2 (within)	0.037	0.034	0.042	0.039	0.035	0.031	0.029	0.026
R^2 (overall)	0.761	0.763	0.585	0.589	0.982	0.983	0.757	0.759

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.G.2: **Cross-sectional risk-taking differential in the *Pre* and *Post* period.** The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008. The dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1)–(2), average portfolio maturity (*Maturity Risk*) in columns (3)–(4), the weekly annualized fund spread (*Spread*) in column (5)–(6), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (7)–(8). *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. *Post* is an indicator variable equal to 1 for the period from 8/1/2007 to 8/31/2008, and 0 otherwise. The other independent variables (*Controls*) are fund assets, expense ratio, fund age, and fund family size. All regressions are at the weekly level and include week and fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

specifications with different values of k, namely $k = 1, 4, 8$, and 12 (corresponding to 1 week, 1 month, 2 and 3 months). Table 1.G.3 shows the results for $k = 4$ and 8. Standard errors are HAC and cross-correlation robust. The results for $k = 1$ and 12 are similar and thus omitted for brevity.

The estimation of regression (1.15) confirms my previous results. For both *Holdings Risk* and *Maturity Risk* the effect of (past) *Fund Business* on MMF risk-taking is negative, statistically and economically significant in both periods. Moreover, the effect in the post period tends to be stronger, with the difference being statistically significant for *Holdings Risk*. The effect of *Fund Business* on *Spread* is negative, statistically and economically significant in the post period, while it is not significant in the pre period.[48] The effect of sponsor's reputation concern on *Safe Holdings* is positive and statistically significant in the post period, and positive but not statistically significant in the pre period. These results confirm the findings of Table 1.5 and the predictions (1) and (3) of my model on differential risk-taking in the MMF industry. I also run regression (1.15) using *FB Rank* as main explanatory variable. The results are similar and omitted for brevity.

Finally, as further robustness check, I also run IV regressions in which all RHS variables in (1.9) are instrumented with their lagged values. The results are similar and thus omitted.

1.G.3 Disentangling risk premium from risk-free rate: robustness checks

Table 1.G.4 shows the estimation of regression (1.10) when rf is the 3-month T-bill rate. The data qualitatively confirm the predictions of the model. The interaction of *FB Rank* with the risk premium is is in agreement with model's predictions, statistically significant at least at the 5% level for all measures of risk, and economically important. After an increase of 1% in the GZ excess bond premium the difference in *Holdings Risk* between funds in the lowest and highest percentile of *Fund Business* increases by roughly 6.3 percentage points, the absolute value of the difference in *Safe Holdings* increases by 3.4 percentage points, the difference in portfolio maturity by 4.2 days, and the spread differential increases by 1.8 basis points. The interaction of *FB Rank* with the rate on 3-month T-bills is in agreement with the model but never statistically significant. However, its economic effect is comparable to that of the interaction term with the risk premium. After a decrease of 1% in the 3-month T-bill rate, the difference in *Holdings Risk* between funds in the lowest and highest percentile

[48] When *Spread* is the dependent variable the predictive power of the "within" estimation is very small (0.3%), which means that most of the variation in spread is due to unobserved fund-specific components and time-fixed effects. Moreover, as discussed in the main text, *Spread* does not necessarily reflect active risk-taking by the fund.

	Holdings Risk$_{i,t}$		*Maturity Risk*$_{i,t}$		*Spread*$_{i,t}$		*Safe Holdings*$_{i,t}$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	Post	Pre	Post	Pre	Post	Pre	Post	Pre
					$k = 4$			
Fund Business$_{i,t}$(IV: $t-k$)	-54.030^{***}	-13.159^{***}	-14.867^{***}	-8.856^{**}	-42.688^{***}	3.107	34.512^{***}	2.182
	(12.639)	(4.271)	(5.523)	(3.821)	(7.558)	(2.126)	(8.220)	(2.432)
F-statistics (1st stage IV)	$3.1 \cdot 10^5$	$5.3 \cdot 10^5$	$3.1 \cdot 10^5$	$5.3 \cdot 10^5$	$3.1 \cdot 10^5$	$5.3 \cdot 10^5$	$3.1 \cdot 10^5$	$5.3 \cdot 10^5$
Adj. R^2 (within)	0.018	0.018	0.009	0.011	0.015	0.003	0.023	0.009
R^2 (overall)	0.841	0.823	0.713	0.648	0.974	0.981	0.830	0.824
Observations	6,344	9,638	6,344	9,638	6,344	9,638	6,344	9,638
					$k = 8$			
Fund Business$_{i,t}$(IV: $t-k$)	-69.894^{***}	-21.432^{***}	-14.678^{**}	-18.326^{***}	-62.555^{***}	3.674^{*}	37.269^{***}	1.968
	(10.371)	(5.818)	(6.009)	(5.292)	(13.078)	(2.156)	(5.538)	(2.524)
F-statistics (1st stage IV)	$1.7 \cdot 10^5$	$2.9 \cdot 10^5$	$1.7 \cdot 10^5$	$2.9 \cdot 10^5$	$1.7 \cdot 10^5$	$2.9 \cdot 10^5$	$1.7 \cdot 10^5$	$2.9 \cdot 10^5$
Adj. R^2 (within)	0.028	0.017	0.011	0.009	0.007	0.001	0.032	0.011
R^2 (overall)	0.850	0.827	0.734	0.655	0.977	0.982	0.842	0.831
Observations	5,856	9,150	5,856	9,150	5,856	9,150	5,856	9,150
Controls$_{i,t-k}$	Y	Y	Y	Y	Y	Y	Y	Y
Week Fixed Effects	Y	Y	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y	Y	Y

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.G.3: **Cross-sectional risk-taking differential in the *Pre* and *Post* Period: separate regressions.** The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008. The dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1)–(2), average portfolio maturity (*Maturity Risk*) in columns (3)–(4), the weekly annualized fund spread (*Spread*) in column (5)–(6), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (7)–(8). *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. *Fund Business* in week t is instrumented with its lagged value from week $t-k$ ($k = 4$ or $k = 8$). The other independent variables (*Controls*) are fund assets, expense ratio, fund age, and fund family size. All regressions are at the weekly level and include week and fund fixed effects.

Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

of *Fund Business* increases by 4.4 percentage points, the absolute value of the difference in $Safe\ Holdings$ by 2.2 percentage points, the difference in portfolio maturity by more 2.9 days, and the spread differential by 5.8 basis points.

Table 1.G.5 shows the estimation of regression (1.10) when the risk premium $\widehat{rp}$ is proxied by *Spread Index*, defined by (1.6) in Section 1.7.1, and rf is the 30-day T-bill return. Again, the data confirm the predictions of the model: the sign of all coefficients is in agreement with the model and the coefficients of interest are statistically significant for most measures of risk. In terms of economic importance, my results suggest that the 1-month T-bill rate affects the cross-sectional risk-taking differential more than the risk premium, for both asset-class composition and maturity of fund's portfolio. However, note that *Spread Index* is an *ex post* measure of the risk premium. Hence, these results should be regarded as a qualitative robustness check of the model, not as a quantitative statement on the relative weight of risk premium and and risk-free rate on MMF risk-taking

1.G.4 Risk-taking in the time series: supplementary evidence

Figure 1.9 shows the industry average percentage of risky assets net of safe assets in fund portfolios (*Holdings Risk*), over the period 1/1/2006–12/31/2008. The 1-month T-bill rate is superimposed (green line). The industry as a whole did not significantly "reach for yield" in the second half of 2008, when the risk-free rate was falling and the risk premium increased. This is consistent with the opposite risk-taking behavior of MMFs with low and high default costs in response to increases in the premium and riskiness of the risky assets. If any, there was a more significant "reach for yield" in the pre period, when risk-free rates where relatively high and risk premia relatively low.

Table 1.G.6 shows the results of regression (1.8) using the median value of *Fund Business* in the sponsor population as cutoff between "high" and "low" *Fund Business*. The risk premium is proxied by the Gilchrist and Zakrajsek (2012)'s excess bond premium for financial firms, and the risk-free rate by the 1-month T-bill return. Results are similar to those in Table 1.4 and qualitatively confirm the predictions of the model. When the premium and riskiness on the risky assets increase, funds with higher default costs decrease their risky investment, while funds with lower default costs tend to increase it. Holding the premium constant, both categories of funds increase their safe holdings, and for funds with lower default costs this tends to be accompanied by an extension of portfolio maturity.

	(1) *Holdings Risk*	(2) *Safe Holdings*	(3) *Holdings Risk*	(4) *Safe Holdings*	(5) *Holdings Risk*	(6) *Safe Holdings*
$FB\,Rank_{i,t-1} * \widehat{rp}_t$	-6.865^{***}	3.688^{***}			-6.298^{***}	3.411^{***}
	(1.019)	(0.787)			(1.351)	(1.084)
$FB\,Rank_{i,t-1} * rf_t$			28.287^{***}	-15.088^{***}	4.444	-2.177
			(6.702)	(3.851)	(4.619)	(4.240)
Adj. R^2 (within)	0.038	0.023	0.031	0.019	0.038	0.023
R^2 (overall)	0.785	0.787	0.783	0.786	0.785	0.787
	Maturity Risk	*Spread*	*Maturity Risk*	*Spread*	*Maturity Risk*	*Spread*
$FB\,Rank_{i,t-1} * \widehat{rp}_t$	-4.568^{***}	-2.568^{***}			-4.201^{***}	-1.828^{**}
	(0.993)	(0.731)			(1.181)	(0.917)
$FB\,Rank_{i,t-1} * rf_t$			18.779^{***}	12.724^{***}	2.874	5.803
			(4.305)	(4.063)	(4.609)	(5.674)
Adj. R^2 (within)	0.039	0.034	0.031	0.034	0.039	0.036
R^2 (overall)	0.619	0.984	0.616	0.984	0.619	0.984
$Controls_{i,t-k}$	Y	Y	Y	Y	Y	Y
Month Fixed Effects	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y
Observations	3,782	3,782	3,782	3,782	3,782	3,782

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.G.4: **Cross-sectional risk-taking differential: risk-free rate vs. risk premium.** $\widehat{rp}$ is the excess bond premium for financial firms from Gilchrist and Zakrajsek (2012). rf_t is the return on 3-month T-bills. The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008 ($n = 122$). Data are at the monthly level ($T = 31$). In the top panel, the dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1), (3), and (5), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (2), (4), and (6). In the bottom panel, the dependent variables are: average portfolio maturity (*Maturity Risk*) in columns (1), (3), and (5), the weekly annualized fund spread (*Spread*) in column (2), (4), (6). *FB Rank* is the rank in percentiles normalized to $[0, 1]$ of *Fund Business*, which is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. The other independent variables (*Controls*) are fund assets, expense ratio, fund age, fund family size, and lagged *FB Rank*. All regressions include month and fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

	(1)	(2)	(3)	(4)	(5)	(6)
	Holdings Risk	*Safe Holdings*	*Holdings Risk*	*Safe Holdings*	*Holdings Risk*	*Safe Holdings*
$FB\,Rank_{i,t-1} * \widehat{rp}_t$	-6.176^{***}	3.226^{***}			-2.638^{*}	1.346
	(1.852)	(1.145)			(1.525)	(1.232)
$FB\,Rank_{i,t-1} * rf_t$			38.501^{***}	-20.285^{***}	30.343^{***}	-16.123^{***}
			(7.562)	(4.692)	(7.799)	(5.356)
Adj. R^2 (within)	0.042	0.024	0.046	0.026	0.047	0.026
R^2 (overall)	0.786	0.787	0.787	0.788	0.787	0.788
	Maturity Risk	*Spread*	*Maturity Risk*	*Spread*	*Maturity Risk*	*Spread*
$FB\,Rank_{i,t-1} * \widehat{rp}_t$	-5.095^{***}	-2.101^{***}			-2.606^{***}	-0.138
	(1.123)	(0.576)			(0.730)	(0.937)
$FB\,Rank_{i,t-1} * rf_t$			29.404^{***}	17.262^{***}	21.347^{***}	16.835^{***}
			(4.272)	(3.477)	(3.773)	(5.170)
Adj. R^2 (within)	0.029	0.009	0.033	0.011	0.035	0.011
R^2 (overall)	0.616	0.983	0.617	0.983	0.618	0.983
$Controls_{i,t-k}$	Y	Y	Y	Y	Y	Y
Month Fixed Effects	Y	Y	Y	Y	Y	Y
Fund Fixed Effects	Y	Y	Y	Y	Y	Y
Observations	3,782	3,782	3,782	3,782	3,782	3,782

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.G.5: **Cross-sectional risk-taking differential: risk-free rate vs. risk premium.** $\widehat{rp}$ is *Spread Index*, the index of spreads on typical risky securities available to MMFs defined by (1.6) in the main text. rf_t is the return on 30-day T-bills. The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008 ($n = 122$). Data are at the monthly level ($T = 31$). In the top panel, the dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in columns (1), (3), and (5), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in columns (2), (4), and (6). In the bottom panel, the dependent variables are: average portfolio maturity (*Maturity Risk*) in columns (1), (3), and (5), the weekly annualized fund spread (*Spread*) in column (2), (4), (6). *FB Rank* is the rank in percentiles normalized to $[0, 1]$ of *Fund Business*, which is the share of sponsor's mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. The other independent variables (*Controls*) are fund assets, expense ratio, fund age, fund family size, and lagged *FB Rank*. All regressions include month and fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

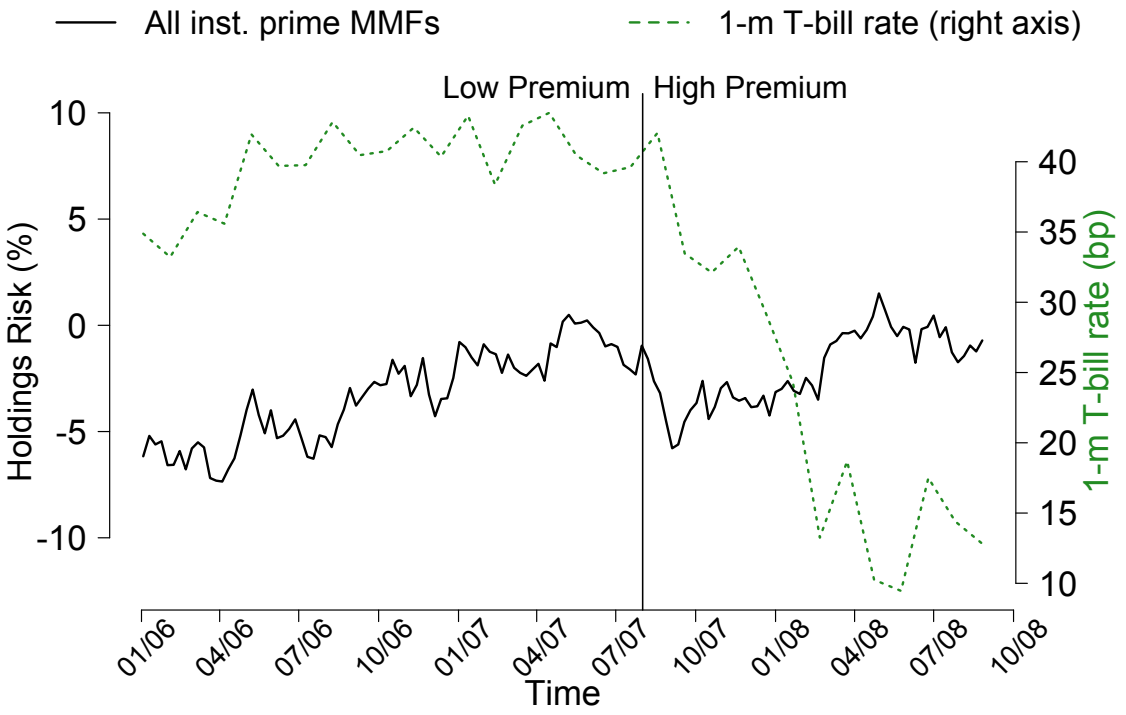


FIGURE 1.9: **Industry average** *Holdings Risk*. The black line represents the industry average percentage of assets held in bank obligations (risky assets) net of U.S. trasuries, GSE debt, and repurchase agreements (safe assets). The green line represents the weekly return on 1-month T-bills (from CRSP). The scale for average holdings is on the left y-axis. The scale for the T-bill rate is on the right y-axis.

As further robustness checks, I also run regression (1.8) using *Spread Index* as proxy for the risk premium and the 3-month T-bill rate as proxy for the risk-free rate. Results are similar and are available upon request.

	(1)	(2)	(3)
	Holding Risk	*Maturity Risk*	*Safe Holdings*
$High\,FB_{i,t-1} * \widehat{rp}_t$	-3.636^{***}	-1.627^{**}	2.531^{***}
	(0.671)	(0.680)	(0.512)
$Low\,FB_{i,t-1} * \widehat{rp}_t$	1.922^{**}	0.297	-0.311
	(0.878)	(0.669)	(0.840)
$High\,FB_{i,t-1} * rf_t$	-0.444	-5.357	-9.397^{**}
	(4.362)	(3.789)	(4.281)
$Low\,FB_{i,t-1} * rf_t$	2.507	-8.558^{**}	-9.843^{*}
	(5.258)	(4.230)	(5.324)
$Controls_{i,t-1}$	Y	Y	Y
Fund Fixed Effects	Y	Y	Y
Adj. R^2 (within)	0.073	0.085	0.058
R^2 (overall)	0.759	0.568	0.755
Observations	3,782	3,782	3,782

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 1.G.6: **Reach for yield: risk premium vs. risk-free rate. Cutoff high vs. low default costs: median reputation concern in sponsor population.** The sample is all U.S. institutional prime money market funds continuously active throughout the period from 1/1/2006 to 8/31/2008 ($n = 122$). Data are at the monthly level ($T = 31$). The dependent variables are: the percentage of risky assets (bank obligations) net of safe assets (US treasuries, GSE debt, and repos) in a fund's portfolio (*Holdings Risk*) in column (1), average portfolio maturity (*Maturity Risk*) in column (2), and the percentage of safe assets in a fund's portfolio (*Safe Holdings*) in column (3). The risk premium $\widehat{rp}$ is the excess bond premium for financial firms from Gilchrist and Zakrajsek (2012). The risk-free rate rf_t is the return on 30-day T-bills. *Low* (*High*) $FB_{i,t}$ is a binary variable equal to 1 if fund i's *Fund Business* is below (above) the median value in the sponsor population in month t, and 0 otherwise. *Fund Business* is the share of mutual fund assets other than institutional prime money market funds in sponsor's total mutual fund assets. Other independent variables (*Controls*) are fund assets, expense ratio, fund age, fund family size, and *Fund Business*. All regressions include fund fixed effects. Standard errors are HAC and cross-correlation robust. ***, **, * represent 1%, 5%, and 10% statistical significance, respectively.

Chapter 2

The Effect of Round-off Error on Long Memory Processes[1]

2.1 Introduction

"All economic data is discrete" (Engle and Russell (2004)). Round-off errors occur whenever a real valued process is observed on a grid of discrete values. A special case of round-off error is obtained by taking the sign of a real valued process. Round-off errors change the properties of the original stochastic process. Few exact results exist on the effect of round-off error on stochastic processes. Delattre and Jacod (1997), for example, proved a central limit theorem on a Brownian motion with round-off error sampled at discrete times. In this paper we study analytically and numerically the effect of round-off error on long memory processes. We will use the term *discretized process* to refer to the process with round-off error and the term *discretization* to refer to the rounding procedure.

Round-off errors can be either due to a limit in the resolution of the observing device or to the fact that an underlying real valued process can manifest itself only as a discrete valued process. When seen as a resolution effect, round-off error can also be considered as a special case of measurement error. One recent strand of econometric literature considers the problem of estimation of a process that can be observed only with some noise due to the measurement process. The typical modeling approach is to consider the measurement error as an additive white noise process uncorrelated with the unobserved latent process (see, for example, Hansen

[1]This chapter is a slightly modified version of a paper co-authored with Fabrizio Lillo and published in *Studies in Nonlinear Dynamics and Econometrics*, 2014; 18(4) 445-482. The paper has been presented at the XIII Workshop on Quantitative Finance, L'Aquila, Italy, January 2012.

and Lunde (2010)). Round-off error can be considered as a different form of measurement error, which is a deterministic function of the underlying latent process and is not uncorrelated with it.

An important example of round-off error due to the fact that the process manifests itself only as a discrete valued process is the dynamics of asset prices. Despite the fact that the price of an asset is a real number, transaction prices (but also quote prices) can assume only values which are multiple of a minimum value called tick size. For transaction by transaction data the tick over price ratio can be large leading to a price dynamics that cannot be even approximated as a real valued process. In the market microstructure literature the round-off error is one of the main sources of disturbance in the estimation of integrated volatility. Several papers have considered how tick size affects the diffusion dynamics of price on short time scales. For example, Gottlieb and Kalay (1985) and Harris (1990) developed microstructure models to investigate the effect of price discretization on return variance and serial correlation. More recent papers consider the effect of price discretization on phase portrait of returns (Szpiro (1998)), on price-dividend relation (Bali and Hite (1998)), and on integrated volatility (Rosenbaum (2009)). See also La Spada *et al.* (2011) and references therein for a recent review of the effect of tick size on the diffusion properties of financial asset prices. Beside market microstructure, discretized processes emerge naturally in discrete choice models, binned data, computer vision, detectors, and digital signal processing. Recently, the effect of discretization has also been studied in the framework of rational inattention (Saint-Paul (2011)). As an example of a financial application of long memory processes when considering the sign of a variable, we mention the work of Lillo and Farmer (2004). There authors empirically show that the sign of the market order flow in a double auction financial market is a long memory process.

Long memory processes are ubiquitous in natural, social, and economic systems (see Beran (1994)). However, a large part of the theory, statistics, and modeling of long memory processes is based on the assumption of normality of the distribution, and very often also on the hypothesis of linearity of the generating mechanism. A major step toward generalization has been the paper by Dittmann and Granger (2002) that investigated the properties of a process obtained from a non-linear transformation of a Gaussian long memory process. Their main result is that when the transformation can be written as a *finite* linear combination of Hermite polynomials, the transformed process has the same or a smaller Hurst exponent of the original process, depending on the Hermite rank of the transformation (see below for a more precise definition of these terms). However, their results "need not hold for transformations with infinite Hermite expansion" (see Dittmann and Granger (2002), Proposition 1). Moreover, recently, several papers have also studied the asymptotic properties of common estimators of the Hurst exponent

when they are applied to non-Gaussian and non-linear time series (e.g., Dalla, Giraitis and Hidalgo (2006); see Section 2.4 for a detailed literature review). The present paper contributes in this direction by presenting several results for a specific, yet important, class of non-linear transformations of a Gaussian long memory process with an *infinite* Hermite expansion, namely the class of discretization transformations.

In this paper we present an in-depth analysis of the properties of a stochastic process obtained from the discretization of a large class of Gaussian long memory processes. We give the asymptotic behavior of the autocovariance and of the spectral density, by computing explicitly the leading term and the order of the second term in a series expansion. We find that the autocovariance and the autocorrelation are asymptotically rescaled by a factor smaller than one, and the Hurst exponent is the same for the continuous and the discretized process. The spectral density is also rescaled for small frequencies by the same scaling factor as the autocovariance. We find an explicit closed form of this scaling factor. It is worth noting that the decrease of the autocorrelation function holds for the discretization of any Gaussian weakly stationary process, either long-memory or short-memory. Our results are consistent with those in Hansen and Lunde (2010) on the estimation of the persistence and the autocorrelation function of a time series measured with error.

We then consider two classic methods to estimate the Hurst exponent, namely the local Whittle (LW) estimator suggested by Künsch (1987) and the Detrended Fluctuation Analysis (DFA) introduced by Peng *et al.* (1994). The LW estimator is a very popular semiparametric method for investigating the long memory properties of a stochastic process. It has been extensively studied in econometric theory and frequently used in empirical work in financial econometrics. However, most of the results in the literature do not hold for the discretized process (see discussion in Section 2.4). Recently, Dalla, Giraitis and Hidalgo (2006) proved consistency and asymptotic normality of the LW estimator for a general class of nonlinear processes, but, as discussed in detail in Section 2.4, we cannot directly apply their results to the class of processes considered in this paper. As we show below, under suitable regularity conditions we are able to extend the results in Dalla *et al.* (2006) to our setting and prove that the LW estimator is consistent and asymptotically normal for Gaussian long memory processes observed with roundoff error. Moreover, we also show that one of the main results of Dalla *et al.* (2006) can be generalized by relaxing some assumptions that are not necessary.

The DFA (Peng *et al.* (1994)) is another very popular semiparametric method for the investigation of long memory properties of generic processes. It was introduced more than fifteen years ago to investigate physiological data, in particular the heartbeat signal. Since its introduction it has been applied to a large variety of systems, including physical, biological, economic, and

technological data. In economics and finance it has been applied for example in Schmitt *et al.* (2000), Lillo and Farmer (2004), Di Matteo *et al.* (2005), Yamasaki *et al.* (2005), and Alfarano and Lux (2007). In a recent paper Bardet and Kammoun (2008) computed explicitly the asymptotic properties of the DFA for the fGn and for a general class of Gaussian weakly stationary long memory processes. Here we generalize their results to a non-Gaussian generic long memory process and we applied them to the discretized process. As a byproduct we show that the order of the error of the root-mean-square fluctuation given by Theorem 4.2 of Bardet and Kammoun (2008) is not correct for a generic Gaussian process. Because of the cancellation of a term the theorem is correct for the fGn and the fARIMA process as claimed in Bardet and Kammoun (2008). By comparing the root-mean-square fluctuation of the discretized process to that of the continuous process we argue that the second-order term induces a negative bias in the estimation of the Hurst exponent.

The paper is organized as follows. In Section 2.2 we define the class of long memory processes we consider in the present paper. In Section 2.3 we derive some analytical results on the distribution, autocovariance, and spectral density of the discretized process. Section 2.4 presents analytical and numerical results on the estimation of the Hurst exponent obtained by using the LW estimator and the DFA. Finally, Section 2.5 concludes.

2.2 Long memory processes

In this paper we are interested in studying the effect of round-off error on a long memory process. There are several possible definitions of a long memory process. The very general definition we are using in the present paper is the following.

Definition 2.1. A discrete time weakly stationary stochastic process $\{X(t)\}_{t\in\mathbb{N}}$ is long memory if its autocovariance function $\gamma(k)$ behaves as

$$\gamma(k) = k^{2H-2}L(k) \qquad for \;\; k \geq 1\,, \tag{2.1}$$

where $H \in (0.5, 1)$ and $L(k)$ is a slowly varying function at infinity[2].

The parameter H is called Hurst exponent or, sometimes, the self-similarity parameter. Under this definition of long memory the autocovariance function does not necessarily decay as a

[2] $L(x)$ is a slowly varying function if $\lim_{x\to\infty} L(tx)/L(x) = 1$, $\forall t > 0$ (see Embrechts *et al.* (1997)). In the definitions above, and for the purposes of this paper, we are considering only positively correlated long-memory processes. Negatively correlated long-memory processes also exist, but the long-memory processes we will consider in the rest of the paper are all positively correlated.

pure power-law. Consider the case $L(k) = \log k$, or any power of the logarithm function. The autocorrelation function of a long memory process is not integrable on the positive real line and, as a consequence, the process does not have a typical time scale.

The class of long memory processes defined above is quite large due to the arbitrariness of the slowly varying function $L(k)$. Some of the results we will present below hold for a more restricted class of long-memory processes characterized by the properties of the slowly varying function in Definition 2.1.

Following Embrechts *et al.* (1997) we denote with $\mathcal{R}_0$ the set of slowly varying functions at infinity. We introduce the following definition

Definition 2.2. We define the set of well behaved slowly varying function as

$$\mathcal{L}(K, I, b_i, \beta_i) \equiv \left\{ l \in \mathcal{R}_0 : \exists\, K > 0 \; s.t. \;\; l(k) = \sum_{i=0}^{I} b_i k^{-\beta_i} \quad \forall\, k > K \right\} \tag{2.2}$$

with $I \leq \infty$, $b_0 > 0$, $b_i \neq 0 \; \forall\, i \geq 1$, $\beta_0 = 0$, and $\beta_i < \beta_{i+1} \; \forall\, i \geq 0$. Moreover, if $I = \infty$, the parameters b_i and β_i are such that the series $\sum_{i=0}^{\infty} b_i k^{-\beta_i}$ converges absolutely $\forall\, k > K$.

Note that all the slowly varying functions which are analytic at infinity are well behaved with $I = \infty$. Hereafter, we abbreviate $\mathcal{L}(K, I, b_i, \beta_i)$ as $\mathcal{L}$.

We are interested here in the discretization of long memory processes characterized by a Gaussian distribution. A stationary Gaussian process is completely characterized by the mean (hereafter assumed to be zero), the variance D and the autocorrelation function $\rho(k) = \gamma(k)/D$. Two classes of stationary Gaussian long memory processes are often considered in the literature. The first one is the fractional Gaussian noise (fGn) (see Mandelbrot and van Ness (1968)), characterized by the autocorrelation function

$$\rho(k) = \frac{1}{2}[(k+1)^{2H} + (k-1)^{2H} - 2k^{2H}]. \tag{2.3}$$

For large k the asymptotic expansion of (2.3) is

$$\rho(k) = \frac{H(2H-1)}{k^{2-2H}} \left(1 + \frac{(2H-2)(2H-3)}{12} \frac{1}{k^2} + \; \dots \right).$$

The second important example is the fARIMA$(0, d, 0)$ process[3], where $d = H - 0.5$, whose autocorrelation is

$$\rho(k) = \frac{\Gamma(3/2 - H)\Gamma(H + k - 1/2)}{\Gamma(H - 1/2)\Gamma(k + 3/2 - H)}, \tag{2.4}$$

where $\Gamma(\cdot)$ is the gamma function. The asymptotic expansion of (2.4) is

$$\rho(k) = \frac{\Gamma(3/2 - H)}{\Gamma(H - 1/2)} \frac{1}{k^{2-2H}} \left(1 - \frac{(4H^3 - 12H^2 + 11H - 3)}{12} \frac{1}{k^2} + \dots\right).$$

Note that both for the fGn and for the fARIMA process the slowly varying function $L(k)$ is analytic at infinity and therefore $L \in \mathcal{L}$ with $I = \infty$. In the following we will present results for the discretization of generic stationary Gaussian long memory processes, and we will consider the fGn or the fARIMA as special cases.

2.3 The discretized process

Given a discrete-time real-valued process $\{X(t)\}_{t\in\mathbb{N}}$ and a grid of points $j\delta$ with $j \in \mathbb{Z}$ and $\delta > 0$, the discretized process at time t is $X_d(t) = round\left(X(t)/\delta\right)\delta$, where $round(x) := \{k \in \mathbb{Z} : -0.5 \leq x - k < 0.5\}$ is the rounding function. The parameter δ sets the level of round-off error. This type of discretization appears, for example, whenever a weakly stationary process is discretized through a binning procedure.

The probability mass function of the discretized process is

$$p_d(x) = \sum_{n=-\infty}^{\infty} q_n \delta_D(x - n\delta), \qquad \text{where} \qquad q_n = \int_{(n-1/2)\delta}^{(n+1/2)\delta} p(x)dx \tag{2.5}$$

$p(x)$ is the probability density function of X_t and $\delta_D(x)$ is the Dirac delta function.

It is useful to introduce the adimensional scaling variable

$$\chi = \frac{D}{\delta^2} \tag{2.6}$$

Since $X(t)$ is Gaussian distributed, the variance D_d of the discretized process can be calculated explicitly. For detailed analytical results on the distributional properties of $X_d(t)$ see Appendix 2.A. Left panel of Figure 2.1 shows the ratio D_d/D as a function of the scaling parameter χ. It is worth noting that this ratio is not monotonic. For small χ the ratio goes to zero because

[3] In the following we consider only fARIMA$(0, d, 0)$. Therefore, when in the following we refer to fARIMA, we mean fARIMA$(0, d, 0)$. For results on more general fARIMA see Section 2.4.3.

for $\delta \gg D$ essentially all the probability mass falls in the bin centered at zero. Finally, the parameter χ sets the fraction q_0 of points which in the discretized process have value zero. It is direct to show that, if $X(t)$ is Gaussian, $q_0 = erf\left[1/2\sqrt{2\chi}\right]$, where $erf[\cdot]$ is the error function. This is clearly a monotonically decreasing function. In the numerical examples below we will use $\chi = 0.1$, 0.25, and 0.5 corresponding to $q_0 = 0.886$, 0.683, and 0.521, respectively.

A different type of discretization that we will consider below is obtained by taking the sign of X_t. Assuming that the distribution function of X_t is absolutely continuous with respect to the Lebesgue measure in a neighborhood of 0, so that the event $X(t) = 0$ has zero probability, this discretization leads to $X_s(t) = sign(X(t)) = \pm 1$ with probability 1. We discuss the sign transformation in more detail in Appendix 2.D.

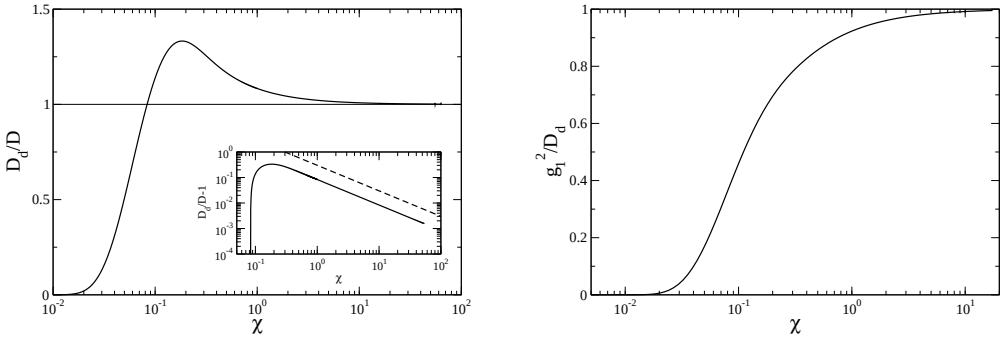


FIGURE 2.1: Left panel. Ratio between the variance D_d of the discretized process and the variance D of the original Gaussian process as a function of the scaling parameter χ. The inset shows the fractional change $D_d/D - 1$ as a function of χ in a log-log scale. The dashed line shows the χ^{-1} behavior. Right panel. Ratio between the squared coefficient g_1^2 (see formula (2.11)) and the variance of the discretized process as a function of the scaling parameter χ. This ratio is equal to the ratio between the autocorrelation function $\rho_d(k)$ of the discretized process and the autocorrelation function $\rho(k)$ for large values of lags k (see Corollary 2.5).

2.3.1 Autocovariance and autocorrelation function

In order to study how the correlation properties change when a stationary Gaussian process is discretized we will make use of a general theory presented, for example, in Beran (1994) and Dittmann and Granger (2002). For the benefit of the reader we recapitulate here this approach.

Without loss of generality, we consider the case of an underlying stationary Gaussian process of unit variance. The starting point is a series expansion of the bivariate Gaussian density function in Hermite polynomials. Hermite polynomials are an orthonormal polynomial system

with Gaussian weight. Specifically, we use the same normalization of Hermite polynomials as in Dittmann and Granger (2002), i.e.,

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = \delta_{nm} \tag{2.7}$$

where δ_{nm} is the Kronecker delta. The expansion of the bivariate Gaussian density function $P(x, y)$ in Hermite polynomials is (see Barrett and Lampard (1955))

$$P(x, y) = P(x)P(y)\left[1 + \sum_{j=1}^{\infty} \rho^j H_j(x)H_j(y)\right] \tag{2.8}$$

where $P(x)$ is the univariate Gaussian density function and ρ is the correlation coefficient between variables x and y.

Following Lemma 1 of Dittmann and Granger (2002), if we transform two Gaussian random variables X and Y with a nonlinear transformation $g(\cdot)$ that can be decomposed in Hermite polynomials

$$g(x) = g_0 + \sum_{j=1}^{\infty} g_j H_j(x) \tag{2.9}$$

the linear covariance and the linear correlation of the transformed variables are

$$Cov[g(X)g(Y)] = \sum_{j=1}^{\infty} g_j^2 \rho^j \qquad Cor[g(X)g(Y)] = \frac{\sum_{j=1}^{\infty} g_j^2 \rho^j}{\sum_{j=1}^{\infty} g_j^2} \tag{2.10}$$

The proof is straightforward. The coefficients g_j in (2.9) are

$$g_j = \int_{-\infty}^{\infty} g(x)H_j(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \tag{2.11}$$

The smallest $j > 0$ such that g_j is non-vanishing is called the Hermite rank of the function $g(x)$. Note that the second equation in (2.10) implies that any non-linear transformation of a bivariate Gaussian distribution decreases the correlation between the variables (the covariance can of course increase or decrease). When X and Y describes the same process at two different times the above equations can be used to compute the autocovariance and autocorrelation properties of the transformed process.

Dittmann and Granger (2002) used the above expansion to study how the long memory properties change as a result of nonlinear transformations that can be written as finite Hermite expansions. (See Proposition 1 of Dittmann and Granger (2002).) They mention that this approach cannot be used if the transformation has an infinite Hermite expansion. As we will

show below, the discretization can be expressed as an infinite sum of Hermite polynomials and therefore we cannot use directly their Proposition 1. In the following we compute explicitly the asymptotic behavior of the autocovariance function and of the spectral density of the discretized process in order to directly obtain its long memory properties.

By using the theory outlined above we compute the asymptotic behavior of the autocovariance $\gamma_d(k)$ and the autocorrelation $\rho_d(k)$ of the discretized process.

Proposition 2.3. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary Gaussian process with autocovariance function given by Definition 2.1. Then the autocovariance function of the discretized process $\{X_d(t)\}_{t\in\mathbb{N}}$ satisfies*

$$\gamma_d(k) = \left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 k^{2H-2}L(k)\left(1 + O\left(k^{4H-4}L^2(k)\right)\right) \qquad as \;\; k \to \infty\,, \tag{2.12}$$

where $\vartheta_a(u, q)$ is the elliptic theta function.

The proof of this and of the other propositions in this section can be found in Appendix 2.B. According to Definition 2.1, this proposition proves that the discretized process is a long memory process with the same Hurst exponent as the original process. Note that asymptotically the autocovariance of the discretized process is proportional to the autocovariance of the original process.

Moreover, as for the original process, the autocovariance function does not necessarily decay as a pure power-law. The second order corrections are nested inside the slowly varying function, and it may happen that the first order term in $O(k^{4H-4}L^2(k))$ dominates the second order term in $k^{2H-2}L(k)$. To compute explicitly the second-order corrections we need to specify a functional form for the slowly varying function $L(k)$. To this end we consider long memory processes whose slowly varying function in the autocovariance is well behaved according to Definition 2.2.

Proposition 2.4. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary Gaussian process with autocovariance function given by Definition 2.1 and $L \in \mathcal{L}$. Then, the autocovariance function of the discretized process $\{X_d(t)\}_{t\in\mathbb{N}}$ satisfies*

$$\gamma_d(k) = \left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 b_0\, k^{2H-2}\left(1 + O\left(k^{-\min(4-4H,\beta_1)}\right)\right) \qquad as \;\; k \to \infty, \tag{2.13}$$

where $\vartheta_a(u, q)$ is the elliptic theta function.

The behavior of the autocorrelation function is trivially obtained from the two above propositions. For example, for the more general case we have the following.

Corollary 2.5. *Under the conditions of Proposition 2.3, the autocorrelation function $\rho_d(k)$ of the discretized process $\{X_d(t)\}_{t\in\mathbb{N}}$ satisfies, as $k \to \infty$,*

$$\rho_d(k) = \left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 \frac{k^{2H-2}L(k)}{D_d}\left(1 + O\left(k^{4H-4}L(k)^2\right)\right) = \frac{g_1^2}{D_d}\rho(k)\left(1 + O\left(k^{4H-4}L(k)^2\right)\right)$$

For large k, $\rho_d(k)/\rho(k) \sim g_1^2/D_d$. Hereafter, the notation $x_k \sim y_k$ means that $x_k/y_k \to 1$ as $k \to \infty$, unless specified otherwise. The right panel of Figure 2.1 shows g_1^2/D_d as a function of χ. We observe that the more severe is the discretization, the larger is the reduction of the autocorrelation function.

We numerically tested our propositions and the error made by considering only the leading term in the asymptotic expansion. We simulated a fGn with unit variance for which $\lim_{k\to\infty} L(k) = H(2H-1)$. Figure 2.2 shows the sample autocovariance of the discretized process with $\chi = 0.1$ for $H = 0.7$ and $H = 0.85$, and for different sample size, namely $n = 2^{10} = 1,024$ and $n = 2^{14} = 16,384$. Before discussing this figure, we remind that the sample autocovariance (and the autocorrelation) is a biased estimator for long memory time series. Hosking (1996) showed that, for a generic long memory process with an autocovariance asymptotically decaying as $\gamma(k) \sim \lambda k^{2H-2}$ with $0.5 < H < 1$ and $\lambda > 0$, the sample autocovariance $\hat{\gamma}(k)$ has an asymptotic bias

$$E[\hat{\gamma}(k)] - \gamma(k) \sim \frac{-\lambda n^{2H-2}}{H(2H-1)} \qquad as \;\; n \to \infty\,, \tag{2.14}$$

where n is the length of the sample time series. Since Hosking's theorem only requires that the process is long memory, we can apply it also to the discretized time series. Figure 2.2 shows a very good agreement between simulations and analytical results with bias correction.

2.3.2 Spectral density

Proposition 2.6. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary Gaussian process with autocovariance function given by Definition 2.1. If L belongs to the Zygmund class[4], the spectral density $\phi_d(\omega)$ of the discretized process $\{X_d(t)\}_{t\in\mathbb{N}}$ satisfies*

$$\lim_{\omega\to 0^+} \phi_d(\omega) \Big/ \left[\left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 c_\phi |\omega|^{1-2H} L(\omega^{-1})\right] = 1 \tag{2.15}$$

where $\vartheta_a(u, q)$ is the elliptic theta function, and $c_\phi = \pi^{-1}\Gamma(2H-1)\sin(\pi H)$.

[4] A positive measurable function f belongs to the Zygmund class if, for every $\alpha > 0$, $x^\alpha f(x)$ is ultimately incrasing, and $x^{-\alpha} f(x)$ is ultimately decreasing (see Zygmund (1959)).

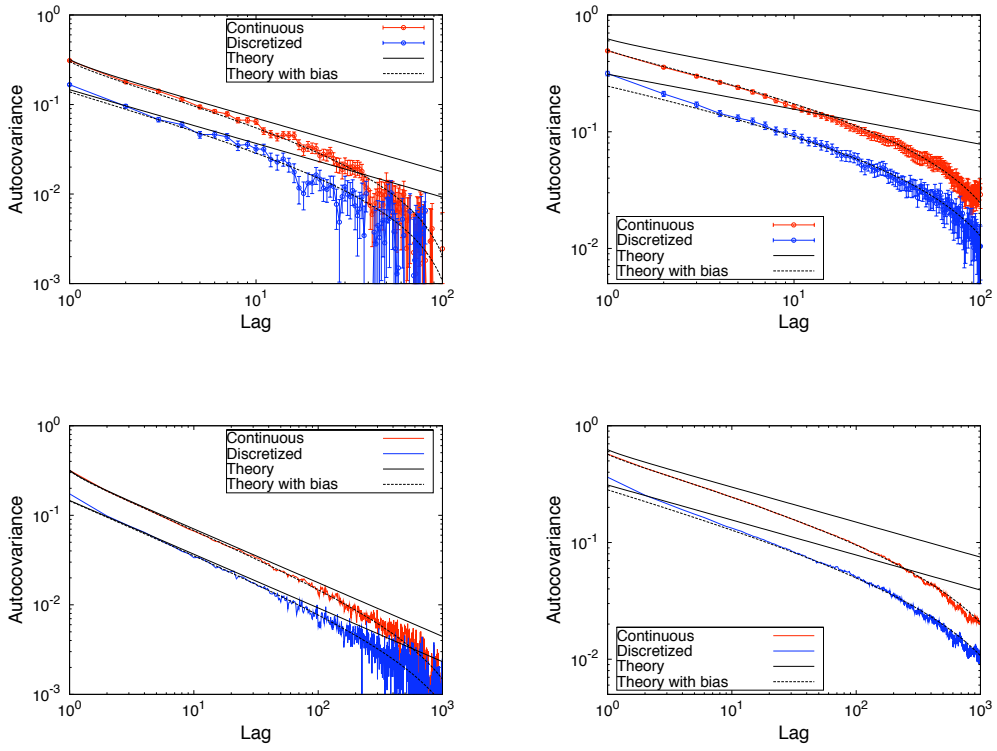


FIGURE 2.2: Sample autocovariance of a numerical simulation of a fGn and its discretization with a scaling parameter $\chi = 0.1$. The time series has length $n = 2^{10}$ (top) and $n = 2^{14}$ (bottom). The Hurst exponent of the fGn is $H = 0.7$ (left) and $H = 0.85$ (right). The figure also shows the asymptotic theoretical autocovariance and the autocovariance corrected for the finite sample bias of (2.14) both for the fGn and for its discretization.

Similarly to the case of the autocovariance, second-order corrections to the spectral density are hidden inside the slowly varying function $L(k)$ and the terms we neglected. To compute these corrections explicitly in terms of powers of $|\omega|$ we need to specify a functional form for the slowly varying function. We therefore consider long memory processes whose slowly varying function in the autocovariance is well behaved according to Definition 2.2. Moreover, we introduce the following assumption

ASSUMPTION 4. In Definition 2.2, either $I < \infty$, or, if $I = \infty$, then

(i) $K = 1$;

(ii) $\sup\{\beta_i\} > 2H - 1$;

(iii) $\sup_i \{\Gamma(2H - 1 - \beta_i)\} < \infty$.

Assumption 4 (i) provides that $L(k) = \sum_{i=0}^{\infty} b_i k^{-\beta_i}$ converges absolutely $\forall\, k \geq 1$. Under this assumption the Fourier transform of $k^{2H-2}L(k)$ can be written as the series of Fourier transforms of the terms $k^{2H-2-\beta_i}$. Assumption 4 (ii) implies that there is only a finite number of terms in $L(k)$ that are not summable over k. Assumption 4 (iii) ensures that we can rearrange an infinite number of terms in a series expansion of polylogarithms that we use below to represent Fourier series (see the proof of Proposition 2.7 and Lemma 2.10).

Then, we can prove the following

Proposition 2.7. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary Gaussian process with autocovariance function given by Definition 2.1 and $L \in \mathcal{L}$. Then, the spectral density $\phi_d(\omega)$ of the discretized process $\{X_d(t)\}_{t\in\mathbb{N}}$ satisfies, as $\omega \to 0^+$,*

$$\phi_d(\omega) = \left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 c_\phi b_0 \,|\omega|^{1-2H} + g(\omega) + o\left(g(\omega)\right) \tag{2.16}$$

where $\vartheta_a(u,q)$ is the elliptic theta function, $c_\phi = \pi^{-1}\Gamma(2H-1)\sin(\pi H)$, and[5]

$$g(\omega) = \begin{cases} c_0 \in \mathbb{R} & if\ H < \min\left(\frac{5}{6}, \frac{1+\beta_1}{2}\right), \\ g_3^2 \left(\frac{b_0}{D}\right)^3 \pi^{-1}\Gamma(6H-5)\sin(3H\pi)|\omega|^{5-6H} & if\ H > \max\left(\frac{5}{6}, \frac{4-\beta_1}{4}\right), \\ \left(\frac{\vartheta_2(0,e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 b_1\pi^{-1}\Gamma(2H-1-\beta_1)\sin\left(\frac{2H-\beta_1}{2}\pi\right)|\omega|^{1-2H+\beta_1} & if\ H \in \left(\frac{1+\beta_1}{2}, \frac{4-\beta_1}{4}\right), \\ \left(\frac{\vartheta_2(0,e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 b_1\pi^{-1}\ln|\omega|^{-1} & if\ H = \frac{1+\beta_1}{2} < \frac{5}{6}, \\ g_3^2\left(\frac{b_0}{D}\right)^3 \pi^{-1}\ln|\omega|^{-1} & if\ H = \frac{5}{6} < \frac{1+\beta_1}{2}, \\ \left(\left(\frac{\vartheta_2(0,e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 b_1 + g_3^2\left(\frac{b_0}{D}\right)^3\right)\pi^{-1}\ln|\omega|^{-1} & if\ H = \frac{1+\beta_1}{2} = \frac{5}{6}, \end{cases} \tag{2.17}$$

where g_3 is given by (2.36).

In addition, under Assumption 4, $\forall\, j \geq 0$ let $\{\widetilde{b}_{j,i}, \widetilde{\beta}_{j,i}\}_{0\leq i\leq I}$ be the real numbers given by the Cauchy product

$$\left(\sum_{i=0}^{I} b_i k^{-\beta_i}\right)^{2j+1} = \sum_{i=0}^{\widetilde{I}_j} \widetilde{b}_{j,i} k^{-\widetilde{\beta}_{j,i}} \qquad \forall\, k \geq 1$$

where $\widetilde{I}_j = \binom{2j+I}{I-1}$ if $I < \infty$, and $\widetilde{I}_j = \infty\ \forall\, j \geq 0$ if $I = \infty$. Then, if $H < \min\left(\frac{5}{6}, \frac{1+\beta_1}{2}\right)$,

$$g(\omega) = \frac{D_d}{2\pi} + \pi^{-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\widetilde{I}_j} \frac{g_{2j+1}^2}{D^{2j+1}}\,\widetilde{b}_{j,i}\,\zeta\left((2j+1)(2-2H) + \widetilde{\beta}_{j,i}\right), \tag{2.18}$$

[5] Note that the third equation in (2.17) may hold only if $\beta_1 < \frac{2}{3}$, so that $\frac{1+\beta_1}{2} < 1 - \frac{\beta_1}{4}$ and the set for H is non-empty.

where D_d is the variance of the discretized process, g_{2j+1} are the Hermite coefficients of the discretization, and $\zeta(\cdot)$ is the Riemann zeta function.

Note that, if we just assume $L \in \mathcal{L}$, we are able to compute exactly only the second-order terms $O\left(|\omega|^{5-6H}\right)$, $O\left(|\omega|^{1-2H+\beta_1}\right)$ and $O\left(\ln|\omega|^{-1}\right)$. However, under Assumption 4 we can calculate exactly also the term $O(1)$ in (2.17), and therefore we have the exact second-order correction for all values of H.

As we have mentioned above, an important case is given by long memory processes whose slowly varying function is analytic at infinity. This class includes the fGn and the fARIMA process. If the slowly varying function is analytic at infinity, then $\beta_1 \in \mathbb{N}$ and therefore we have the following

Corollary 2.8. *If $L \in \mathcal{R}_0$ and L is analytic at infinity, then, as $\omega \to 0^+$,*

$$\phi_d(\omega) = \left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 c_\phi b_0 |\omega|^{1-2H} + \begin{cases} c_0 + o(1) & for \;\; H \in (1/2, 5/6) \\ c_1 \ln|\omega|^{-1} + o\left(\ln|\omega|^{-1}\right) & for \;\; H = 5/6 \\ c_2|\omega|^{5-6H} + o\left(|\omega|^{5-6H}\right) & for \;\; H \in (5/6, 1) \end{cases} \tag{2.19}$$

where $c_1 > 0$ and $c_2 > 0$ are given by the fifth and the second equation of (2.17), respectively.

In addition, under Assumption 4, c_0 is given by (2.18).

The sign of (2.18) depends on the specific autocovariance function (or, equivalently, on the specific spectral density) of the underlying Gaussian process. For example, the fGn satisfies Assumption 4 and in this case we can prove the following

Corollary 2.9. *The spectral density of the discretization of a fGn satisfies (2.19) and the second-order term is strictly positive for all $H \in (0.5, 1)$.*

Unfortunately, the fARIMA process does not satisfy Assumption 4 (i) and we are not able to prove that c_0 satisfies (2.18) and is strictly positive. However, from the first part of Corollary 2.8 we know that the second-order term is strictly positive for $H \geq 5/6$ and our extensive numerical simulations (see below) indicate that it is strictly positive also for $H < 5/6$.

2.3.2.1 The discretization of fGn and fARIMA processes

The fGn and the fARIMA process are often used in modeling Gaussian long memory processes. Starting from the explicit functional form of the spectral density of either of these two processes it is direct to show (see Beran (1994)) that the relative second-order term of the spectral density is $O(\omega^2)$, which is a very small correction to the leading term even for relatively large frequencies. Hence, roughly speaking, on a log-log plot the spectral density of both the fGn and the fARIMA process is close to a straight line even for relatively large frequencies.

On the other hand, the two corollaries above state that for the discretization of a Gaussian long memory process with analytic $L(k)$, as the fGn and the fARIMA, the relative second order correction to the spectral density around 0^+ is always larger than $O\left(|\omega|^{2/3}\right)$, which is a relatively large correction even for small frequencies. On a log-log plot this leads to a significant deviation from the straight-line behavior (leading term) even for relatively small frequencies.

In Figures 2.3 we show examples of the sample periodogram of the discretized fGn ($\chi = 0.1$) for different time series length and Hurst exponent. As expected, while the periodogram of the fGn is very straight in a log-log scale, the periodogram of the discretized process changes its slope for increasing frequencies. For the discretized process the absolute slope of the periodogram decreases for increasing frequencies for all values of H due to the strictly positive second-order term. We ran the same simulations for various fARIMA processes and we obtained similar results.

As we will show below, these observations have important consequences for the Hurst exponent estimators based on the periodogram.

2.4 Estimation of the Hurst exponent

In this section we investigate the estimation of the Hurst exponent from finite time series of the discretized process. We numerically generate time series of discretized long memory processes and we compute the Hurst exponent by using the local Whittle estimator and the Detrended Fluctuation Analysis. Despite the fact that, as we have demonstrated in the previous sections, the Hurst exponent of the discretized process is the same as the one of the original process, we will show here that both methods give estimates of the Hurst exponent which are systematically and significantly negatively biased. These results are consistent with the literature on long-memory signal plus noise processes (Arteche (2004), and Hurvich, Moulines and Soulier (2005))

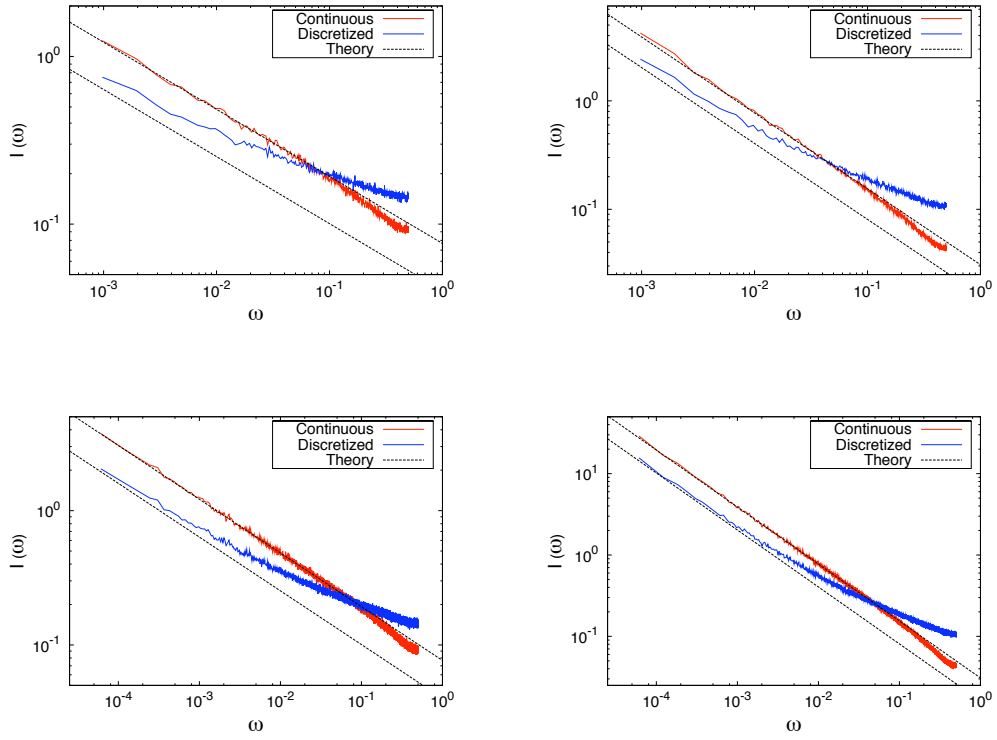


FIGURE 2.3: Sample periodogram of a numerical simulation of a fGn and its discretization with a scaling parameter $\chi = 0.1$. The time series has length $n = 2^{10}$ (top) and $n = 2^{14}$ (bottom). The Hurst exponent of the fGn is $H = 0.7$ (left) and $H = 0.85$ (right). The figure also shows the leading term of the expansion of the spectral density for small ω.

and on general non-linear transformations of long-memory processes (Dalla, Giraitis and Hidalgo (2006)).

2.4.1 Local Whittle estimator

The local Whittle (LW) estimator is a Gaussian semiparametric estimator that works in the frequency domain. It has been suggested by Künsch (1987). Since its introduction it has been extensively studied in econometric theory (see Robinson (1995b), Velasco (1999), Phillips and Shimotsu (2004), Andrews and Sun (2004), Shimotsu and Phillips (2005), Shimotsu and Phillips (2006), Dalla, Giraitis and Hidalgo (2006), Shao and Wu (2007), Abadir, Distaso and Giraitis (2007)), and widely used in theoretical and applied works in financial econometrics to estimate long memory in volatility (see Hurvich and Ray (2003), Arteche (2004) and Hurvich, Moulines and Soulier (2005)).

Let $\{X_t\}_{t\in\mathbb{N}}$ be a weakly stationary long memory process with spectral density $\phi(\omega)$ satisfying

$$\phi(\omega) = c\,|\omega|^{1-2H}\,(1+o(1)) \qquad as\ \ \omega \to 0^{+} \tag{2.20}$$

for some $c > 0$. For a time series $\{X_t\}$, $t = 1, \ldots, n$, define the periodogram

$$I_n(\omega_j) = (2\pi n)^{-1} \left| \sum_{t=1}^{n} X_t \exp^{it\omega_j} \right|$$

Let $\omega_j = 2\pi j/n$, $j = 1, \ldots, n$, be the Fourier frequencies. The LW estimator is defined as the minimizer of the objective function $U_n(h; m)$, i.e.

$$\hat{H} \equiv \hat{H}_n = \underset{h\in[0,1]}{\arg\min}\, U_n(h; m) \tag{2.21}$$

where the local objective function is

$$U_n(h; m) = \log\left(\frac{1}{m}\sum_{j=1}^{m} \omega_j^{2h-1} I_n(\omega_j)\right) - \frac{2h}{m}\sum_{j=1}^{m} \log\omega_j \tag{2.22}$$

$m = m(n)$ is an integer-bandwidth parameter such that

$$m \to \infty, \ \ m = o(n), \ \ \text{as } n \to \infty$$

Note that, in semiparametric models, the spectral density function has property (3.6) and is only locally parameterized around $\omega = 0$ by the parameters H and c. Therefore, contrary to the parametric Whittle estimation, which employs the full spectrum of frequencies, the local Whittle estimator uses only the first m Fourier frequencies.

Under Gaussianity assumption Fox and Taqqu (1986) proved consistency and asymptotic normality of the LW estimator, while Dahlhaus (1989) proved consistency and asymptotic normality of an exact likelihood approach. Giraitis and Surgalis (1990) proved consistency and asymptotic normality of the LW estimator under the assumption of linearity for $H \in (0.5, 1)$. Robinson (1995b) proved consistency and asymptotic normality of the LW estimator under the assumption that the innovations in the Wold representation are a martingale difference sequence with finite fourth moments, for $H \in (0, 1)$. Under a similar set of assumptions Velasco (1999) extended Robinson's results to the non-stationary region and showed that the estimator is consistent for $H \in (0, 1.5)$, and asymptotically normally distributed for $H \in (0, 1.25)$. Velasco also showed that, upon adequate tapering of the observations, the region of consistent estimation of H

may be extended but with corresponding increases in the variance of the limit normal distribution. Hurvich and Chen (2000) also proved consistency and asymptotic normality of a tapered LW estimator for $H \in (0, 2)$. Phillips and Shimotsu (2004) considered a special model of non-stationary fractional integration and extended the results of Velasco (1999) proving that the LW estimator: (i) has a non-normal limit distribution for $H \in [1.25, 1.5)$, (ii) has a mixed normal limit distribution for $H = 1.5$, and (iii) converges to unity in probability for $H > 1.5$. Shimotsu and Phillips (2005, 2006) proposed an exact[6] form of the local Whittle estimator which is based on an exact algebraic manipulation of the Whittle likelihood and does not rely on differencing or tapering. They call this estimator the exact local Whittle (ELW) estimator and show that it is consistent and asymptotically normal when the optimization covers an interval of width less than $9/2$.

From Proposition 2.7 we know that the discretized process satisfies equation (3.6). However, the discretized process is nonlinear, and therefore we cannot apply the results cited above. Recently, several papers have studied the asymptotic properties of the LW estimator for nonlinear processes. For example, Hurvich and Ray (2003), Arteche (2004), and Hurvich, Moulines and Soulier (2005) have studied the asymptotic properties of the LW estimator for signal-plus-noise processes in the presence of long-memory. However, even though the discretized process can be seen as a signal-plus-noise process (see Proof of Theorem 2.11 in Appendix 2.C for more details), we cannot apply their results because they assume that the noise is either a white noise or independent of the signal, which is not the case for the discretized process. Under regularity conditions Dalla, Giraitis and Hidalgo (2006) and Shao and Wu (2007) proved consistency and asymptotic normality of the LW for a very general class of nonlinear processes. Abadir, Distaso and Giraitis (2007) introduced a fully extended local Whittle (FELW) estimator, which is applicable not only for traditional cases but also for nonlinear and non-Gaussian processes, and they proved that it is consistent.

In this paper we follow the approach of Dalla, Giraitis and Hidalgo (2006) (henceforth DGH). The relevant result for our work is their Theorem 4; however, we cannot directly apply it for two reasons. First, DGH define long memory processes in the frequency domain, and therefore they impose assumptions directly on the spectral density. On the contrary, in the present paper we define long memory processes in the time domain (see Definitions 2.1 and 2.2) and therefore we want to impose assumptions on the autocovariance function. For this reason we introduce Lemma 2.10 that makes the connection between the class of processes considered here and those studied in DGH. Second, as we will show below, some assumptions in Theorem 4 of DGH is

[6]The word "exact" is used to distinguish the proposed estimator (which relies on an exact algebraic manipulation) from the conventional local Whittle estimator, which is based on the approximation $I_X(\omega_j) \sim \omega_j^{-2d} I_u(\omega_j)$. Of course, the Whittle likelihood is itself an approximation of the exact likelihood.

too restrictive for the processes considered in this paper. Instead, we can prove consistency and asymptotic normality of the LW estimator for the discretized process under weaker assumptions.

ASSUMPTION 5. In Definition 2.2, $\beta_1 \neq 2H-1$, and either $2H \neq \beta_1 \leq 2$ or $D/2 \neq -\sum_{i:\beta_i \neq 2H-1}^{I} b_i \zeta(2-2H+\beta_i)$, where $\zeta(\cdot)$ is the Riemann zeta function.

Note that if $L(k)$ in Definition 2.2 is analytic at infinity, then $\mathbb{N} \ni \beta_1 \neq 2H-1$, but not necessarily $\beta_1 \leq 2$.

Lemma 2.10. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary Gaussian process with autocovariance given by Definition 2.1 with $L \in \mathcal{L}$. Let Assumptions 4 and 5 hold. Then, as $\omega \to 0^+$, the spectral density satisfies*

$$\phi(\omega) = c_\phi |\omega|^{1-2H} \left(b_0 + c_\beta |\omega|^\beta + o\left(|\omega|^\beta\right) \right), \tag{2.23}$$

where $c_\phi > 0$ is defined as in Proposition 2.6, $c_\beta \neq 0$ and $\beta \in (0,2]$.

In the above lemma β is either β_1 or $2H-1$, depending on the specific functional form of the autocovariance of the underlying Gaussian process (see proof below). The fGn is a special case of Lemma 2.10 with $\beta = \beta_1 = 2$. The fARIMA process does not satisfy the conditions of Lemma 2.10 (specifically, Assumption 4 (i)), but nonetheless its spectral density satisfies (2.23) with $\beta = \beta_1 = 2$

More important, Lemma 2.10 provides conditions under which the underlying Gaussian process $\{X(t)\}_{t\in\mathbb{N}}$ satisfies assumption $T(\alpha_0, \beta)$ in DGH with $\alpha_0 = 2H-1$.

In order to prove asymptotic normality we need some smoothness assumption on the spectral representation of the underlying process. We assume that $\{X(t)\}_{t\in\mathbb{N}}$ is purely non-deterministic, so that it is also linear with finite fourth moments. Therefore, we can write

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$$

where $\sum_{j=0}^{\infty} a_j^2 < \infty$ and ε_t are i.i.d. Gaussian variables with zero mean and unit variance. Let

$$\alpha(\omega) = \sum_{j=0}^{\infty} a_j e^{ij\omega}$$

We introduce the following assumption

ASSUMPTION 6.

$$\frac{d}{d\omega}\alpha(\omega) = O\left(\frac{|\alpha(\omega)|}{\omega}\right) \qquad as \;\; \omega \to 0^+ \tag{2.24}$$

Assumption 6 is the same as Assumption A2′ in Robinson (1995b) and is the standard smoothness assumption used in the literature to prove asymptotic normality of the LW estimator. Unfortunately, we are not able to relax this assumption for the class of long-memory processes considered in this paper.

Following Theorems 3 and 4 in DGH and under the assumptions above we derive the following theorem for a generic measurable transformation of an underlying Gaussian process.

Theorem 2.11. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary purely non-deterministic Gaussian process with autocovariance given by Definition 2.1 and $L \in \mathcal{L}$. Let $Y(t) = g(X(t))\ \forall t \in \mathbb{N}$ for some measurable function g with Hermite rank $j_0 = 1$. Then, as $n \to \infty$, the LW estimator $\widehat{H}^Y_{LW}$ of the process $\{Y(t)\}_{t\in\mathbb{N}}$ satisfies:*

(i) $\widehat{H}^Y_{LW} \xrightarrow{p} H$.

(ii) Let Assumptions 4–6 hold. If $m = o\left(n^{2\beta/(2\beta+1)}\right)$, then

$$\widehat{H}^Y_{LW} - H \quad = \quad O_P\left(m^{-1/2} + \left(\frac{m}{n}\right)^{\beta} + \left(\frac{m}{n}\right)^{r}\right) \tag{2.25}$$

where β is defined as in Lemma 2.10, and

$$r = \begin{cases} H - 1/2 & if \ \ j_1(2-2H) > 1 \\ (j_1 - 1)(1-H) & if \ \ j_1(2-2H) < 1 \\ (j_1-1)/(2j_1) - \varepsilon & if \ \ j_1(2-2H) = 1 \end{cases}$$

for any $\varepsilon > 0$, where $j_1 \geq 2$ is the order of the second non-vanishing Hermite coefficient.

(iii) In addition, if $m = o(n^{2r/(2r+1)})$, then

$$\sqrt{m}\left(\widehat{H}^Y_{LW} - H\right) \xrightarrow{d} N\left(0, \frac{1}{4}\right). \tag{2.26}$$

Note that, contrary to DGH in their Theorem 4 (see *Assumption LM* and equation (50) therein), we do not require $\beta = 2$ in (2.23). As shown in the proof, we do not need this assumption to prove consistency and asymptotic normality of the LW estimator. Moreover, to establish an upper bound on the convergence rate of the LW estimate, i.e. (2.25), we do not need to assume $m \geq n^{\gamma}$ for some $0 < \gamma < 1$ as in DGH (see equations (56) and (58) therein). Our trimming conditions on m are weaker than theirs and our asymptotic results hold true even if $m(n) \to \infty$ more slowly than any power of n.

Theorem 2.11 applies to the discretized process because the rounding is a measurable transformations of the underlying Gaussian process. So, we have the following

Corollary 2.12. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary purely non-deterministic Gaussian process with autocovariance given by Definition 2.1 and $L \in \mathcal{L}$. Then, the LW estimator $\widehat{H}^d_{LW}$ of the discretized process $\{X_d(t)\}_{t\in\mathbb{N}}$ is consistent.*

Moreover, if Assumptions 4–6 hold and $m = o\left(n^{2\beta/(2\beta+1)}\right)$, where β is defined as in Lemma 2.10, then $\widehat{H}^d_{LW}$ satisfies (2.25) with

$$r = \begin{cases} H - 1/2 & if \quad H < 5/6 \\ 2(1-H) & if \quad H > 5/6 \\ 1/3 - \varepsilon & if \quad H = 5/6 \end{cases}$$

for any $\varepsilon > 0$.

In addition, if $m = o(n^{2r/(2r+1)})$, then $\widehat{H}^d_{LW}$ is asymptotically normal and satisfies (2.26).

For fGn and fARIMA processes $\beta = 2$, and therefore $\hat{H}^{LW}_Y - H = O_P\left(m^{-1/2} + \left(\frac{m}{n}\right)^r\right)$, with $r \in (0, 1/3)$. Moreover, if $m = n^\gamma$ with $\gamma > 0.4$, the term $O_P\left((m/n)^r\right)$ will dominate the term $O_P(m^{-1/2})$. As we discuss below, this consideration is important in order to understand the negative finite sample bias we observe in our numerical simulations.

2.4.1.1 Numerical simulations

We numerically generated the Gaussian process as a fGn with unit variance and we considered two values of the Hurst exponent, namely $H = 0.7$ and 0.85. This choice of Hurst exponents is motivated by the need to have values below and above the critical value $H = 5/6$. In order to test the dependence on the time series length we considered series of length $n = 2^{10}$ and $n = 2^{14}$. We then applied four discretization procedures corresponding to discretization parameter $\chi = 0.1, 0.25, 0.5$ and the sign of the process. We performed $L = 10^3$ simulations in order to obtain the statistical properties of the estimators.

Our results are summarized in Tables 2.4.1 and 2.4.2. The tables show the mean, standard error, and three quantiles (2.5%, 50%, and 97.5%) of the estimator $\widehat{H}_{LW}$ defined by (2.21) for four thresholds, namely $m = n^{0.5}$, $n^{0.6}$, $n^{0.7}$ and $n^{0.8}$. We chose these thresholds to be consistent with DGH.

For the fGn the LW estimate increases with the threshold m, with a small negative bias for $m = n^{0.5}$ and a small positive bias for $m = n^{0.8}$. The bias is minimized for values of m between $n^{0.6}$ and $n^{0.7}$, in agreement with the results for fARIMA processes reported in Robinson (1995b) and DGH (see Table I therein). As expected, the absolute bias decreases as we increase the sample size. For $n = 2^{14}$, when $H = 0.7$ the bias is not statistically significant at the 5% level for $m = n^{0.5}$, $n^{0.6}$, $n^{0.7}$; when $H = 0.85$ it is not statistically significant at the 5% level for $m = n^{0.5}$. The standard error obviously decreases with m.

For both the discretized process and the sign process we observe a strong and statistically significant negative bias for all values of n, m, and H. The bias becomes more severe as the discretization becomes coarser, i.e. as δ increases. In general, the absolute bias increases with the threshold[7]. This is because, as m increases, the objective function includes higher-frequency ordinates of the periodogram. From (2.51) in the proof of Theorem 2.11 (see Appendix 2.C) we know that the $O_P\left((m/n)^r\right)$ term in the asymptotic expansion of $\widehat{H}_{LW}$ represents the contribution of the sample estimate, taken with the negative sign, of the higher-order components of the spectral density. From Corollary 2.9 we know that for the discretization of a fGn the second-order term of the spectral density is strictly positive for all H. Therefore, intuitively, the term $O_P\left((m/n)^r\right)$ generates the negative finite-sample bias that we observe here. On the other hand, as expected, the standard error decreases as m increases. The LW estimator with $m = n^{0.5}$ is less biased by the discretization than the estimator with threshold $m = n^{0.8}$, but it has a larger dispersion and is more noisy. These properties are expected from the bias-variance tradeoff. Note that for $n = 2^{10}$ and $m = n^{0.8}$ the 97.5% quantile of $\widehat{H}_{LW}$ is smaller than the true H for $\chi = 0.1$ if $H = 0.7$, and for both $\chi = 0.1$ and the sign process if $H = 0.85$. Even more strikingly, for $n = 2^{14}$ the 97.5% quantile of $\widehat{H}_{LW}$ is smaller than the true H for the sign process and for $\chi = 0.1$, 0.25, both for $H = 0.7$ and for $H = 0.85$.

We also note that the effect of the bias due to the discretization is increasing with the long memory parameter, i.e. the larger is H, the larger is the discretization bias. Finally, note that the bias is a finite size effect.

We ran the same simulations for various fARIMA, and we got similar results.

[7] The only exception is for $H = 0.7$ in small samples ($n = 2^{10}$) and for moderate discretizations ($\chi = 0.25$, 0.5). In this case the LW does change in a statistically significant way across different thresholds.

2.4.2 Detrended Fluctuation Analysis

2.4.2.1 Definition and notation

We now consider the Detrended Fluctuation Analysis (DFA) (see Peng *et al.* (1994)), a method to investigate the properties of a long memory process and to estimate the Hurst exponent. This method was introduced more than fifteen years ago to investigate physiological data, in particular the heartbeat signal. Since its introduction it has been applied to a large variety of systems, including physical, biological, economic, and technological data. Some examples of its application to economics and financial time series can be found in Schmitt *et al.* (2000), Lillo and Farmer (2004), Di Matteo *et al.* (2005), Yamasaki *et al.* (2005), and Alfarano and Lux (2007). The idea is to consider the integrated process and detrend it locally. The scaling of the fluctuations of the residuals as a function of the box size in which the regression is performed gives the estimate of the Hurst exponent. More precisely, let $\{X(t)\}$, $t = 1, \ldots, n$, be a finite sample from a process $\{X(t)\}_{t\in\mathbb{N}}$ and denote the discrete integration of this sample as

$$Y(k) = \sum_{t=1}^{k} X(t)$$

The integrated time series is divided into $[n/m]$ boxes of equal length m, where $[z]$ is the integer part of z. In each box a least squares line is fit to the data (representing the trend in that box). The y-coordinate of the straight line segments is denoted by $\widehat{Y}_m(k)$. Next, one detrends the integrated time series, $Y(k)$, by subtracting the local trend, $\widehat{Y}_m(k)$, in each box. For any given box size m, the root-mean-square fluctuation (or, simply, fluctuation function) of this integrated and detrended time series is calculated by

$$F(m) = \sqrt{\frac{1}{m \cdot [n/m]} \sum_{k=1}^{m\cdot[n/m]} [Y(k) - \widehat{Y}_m(k)]^2}. \tag{2.27}$$

This computation is repeated over all time scales (box sizes) to characterize the relationship between $F(m)$ and the box size m. Typically $F(m)$ increases with box size m. For a long memory process the Hurst exponent is given by the relation $F(m) \propto m^H$. Therefore H is estimated by performing a log-regression of $F(m)$ versus m. The proposers of this method claim that DFA is able to perform well also in the presence of non-stationarities, such as trends, (see Peng *et al.* (1994)), even if some recent results dispute this claim (see Bardet and Kammoun (2008)).

The partial DFA function computed in the j-th window of size m is

$$F_j^2(m) = \frac{1}{m} \sum_{k=m(j-1)+1}^{mj} (Y(k) - \widehat{Y}_m(k))^2$$

for $j \in \{1, \ldots, [n/m]\}$. Then, we can write

$$F^2(m) = \frac{1}{[n/m]} \sum_{j=1}^{[n/m]} F_j^2(m) \tag{2.28}$$

Recently, Bardet and Kammoun (2008) (Lemma 2.2) showed that for a stationary process the series $\{F_j(m)\}_{1 \le j \le [n/m]}$ is a stationary process for all m. This means that in order to study the asymptotic statistical properties of the DFA we can focus on $F_1^2(m)$ only. They then used this result to provide an asymptotic expression (Theorem 4.2) for $\mathbb{E}\left[F_1^2(m)\right]$ for a general class of Gaussian long memory processes. Here we extend this result to non-Gaussian long memory processes (see Theorem 2.13 below). In doing that we show that the order of the correction of the asymptotic expansion given in Bardet and Kammoun (2008) for generic Gaussian processes (Theorem 4.2) is incorrect and we provide the correct order. For the special case of fGn and fARIMA the result provided in Bardet and Kammoun (2008) is correct because the prefactor of the correction term observed for generic long memory processes cancels out exactly.

2.4.2.2 A theorem on the detrended fluctuation analysis of a general long memory process

Here we generalize Theorem 4.2 of Bardet and Kammoun (2008) to a general class of non-Gaussian long memory processes. The discretized process belongs to this class.

Theorem 2.13. *Let $\{X(t)\}_{t \in \mathbb{N}}$ be a weakly stationary long memory process, with zero mean, finite variance, and the following autocovariance function*

$$\gamma(k) = A k^{2H-2} \left(1 + O\left(k^{-\beta}\right)\right) \quad as \;\; k \to \infty\,, \tag{2.29}$$

with $A > 0$, $H \in (0.5, 1)$, and $\beta > 0$. Then, for large m

$$\mathbb{E}\left[F_1^2(m)\right] = A' f(H) m^{2H} \begin{cases} \left(1 + O\left(m^{-\min(2H-1,\beta)}\right)\right) & if \;\; \beta \neq 2H-1 \\ \left(1 + O\left(m^{1-2H} \ln m\right)\right) & if \;\; \beta = 2H-1 \end{cases} \tag{2.30}$$

where $A' = \frac{A}{H(2H-1)}$, and $f(H) = \frac{1-H}{(1+H)(2+H)(1+2H)}$.

First of all, note that the class of processes for which one can apply this theorem is more general than that of Definition 2.2, but clearly less general than that of Definition 2.1. Second, our theorem gives a different order of the correction compared to the one given in Theorem 4.2 of Bardet and Kammoun (2008). In fact, Bardet and Kammoun give $O(m^{-\min(1,\beta)})$, while our theorem gives $O(m^{-\min(2H-1,\beta)})$. In many cases of interest (for example when $\beta > 1$) the leading error term is $O(m^{1-2H})$ rather than $O(m^{-1})$ as stated in Bardet and Kammoun (2008).

For fGn or fARIMA$(0, d, 0)$ we observe a specific behavior. In fact, in these cases the error term is $O(m^{-1})$, instead of $O(m^{-\min(2H-1,\beta)})$, because the prefactor of the terms of order $O(m^{1-2H})$ cancels out exactly, $\beta = 2$ and the next sub-leading term is indeed $O(m^{-1})$. This depends on the specific functional form of the autocovariance of these processes. Therefore, as stated in Bardet and Kammoun (2008) Property 3.1, we have:

Corollary 2.14. *For a fractional Gaussian noise (fGn) with variance D,*

$$\mathbb{E}\left[F_1^2(m)\right] = Df(H)m^{2H}(1 + O(m^{-1}))\,. \tag{2.31}$$

The general Theorem 2.13 gives the asymptotic properties of the fluctuation function for a time series obtained by discretizing or taking the sign of a Gaussian process. Specifically, for discretized processes we have the following

Corollary 2.15. *For the discretization of a Gaussian time series with the properties of Theorem 2.13, for large m*

$$\mathbb{E}\left[F_1^2(m)\right] = \left(\frac{\vartheta_2(0, e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 A' f(H) m^{2H} \begin{cases} \left(1 + O\left(m^{1-2H}\right)\right) & if \;\; H \in (1/2, 5/6) \\ \left(1 + O\left(m^{-2/3}\ln m\right)\right) & if \;\; H = 5/6 \\ \left(1 + O\left(m^{4H-4}\right)\right) & if \;\; H \in (5/6, 1) \end{cases} \tag{2.32}$$

where $\vartheta_a(u, q)$ is the elliptic theta function, and A' and $f(H)$ are defined as in Theorem 2.13.

2.4.2.3 Numerical simulations

We then studied the performance of the DFA on finite time series. As in the numerical analysis of the LW estimator, we considered a fGn with $H = 0.7$ and 0.85 and we generated $L = 10^3$ time series of length $n = 2^{10}$ and 2^{14}. We then applied four discretization procedures: $\chi = 0.1, 0.25, 0.5$, and the sign of the process. Following a standard practice, we consider values of m ranging from $m = 4$ to $m = n/4$.

Figure 2.4 show an example of the root-mean-square fluctuation $F(m)$ as a function of the block size m. As in the case of the LW estimator, a different behavior is observed for the Gaussian process and for its discretization. For the former $F(m)$ is well described by a power law with exponent H over the whole range of investigated block sizes m. On the contrary, for the discretized (or sign) process, $F(m)$ changes significantly slope in the log-log scale, i.e. it has a varying local power law behavior. Only for large box sizes the function $F(m)$ converges to the expected asymptotic behavior from above. This again suggest a significant negative bias in the estimation of H on finite samples, bias that we quantitatively observe in estimation below. In fact, in order to estimate the Hurst exponent through DFA one needs to perform a best fit of F with a power law function. The ambiguity is however the interval of values of m where one performs the fit. To the best of our knowledge there is no rule for selecting optimally such an interval. One expects to obtain a less biased, but noisier, estimate of H by performing the fit in a small region corresponding to large values of the block size m. To investigate this point we estimate the Hurst exponent by performing the fit over a fraction q of the largest values of $\log_{10}[m]$. We consider four values of q, namely $q = 1$ (the whole interval), $q = 0.75$ (the largest three-quarter), $q = 0.5$ (the largest half), and $q = 0.25$ (the largest quartile).

The results of our analysis are summarized in Tables 2.4.3 and 2.4.4. For all q we observe an underestimation of $\widehat{H}$ for the discretized series. This underestimation is always significant in terms of standard errors. Specifically, for short series ($n = 2^{10}$) there is a severe underestimation for the sign and for $\chi = 0.1, 0.25$, both when $H = 0.7$ and when $H = 0.85$. For long series ($n = 2^{14}$) we observe a severe underestimation for the sign and for $\chi = 0.1$, both when $H = 0.7$ and when $H = 0.85$. Strikingly, when $q = 1$ and $n = 2^{14}$, the 97.5% quantile of $\widehat{H}$ is smaller than (or roughly equal to) the true value of H for the sign and for $\chi = 0.1$, both when $H = 0.7$ and when $H = 0.85$. Finally, note that for the discretized process is not always true that the less biased estimator is obtained for small values of q (i.e. large m). In fact, in many cases we observe quite the contrary. For example, in short samples ($n = 2^{10}$) the DFA estimate with $q = 1$ is less biased than that with $q = 0.25$, except when $H = 0.85$ and the discretization is very coarse (sign and $\chi = 0.1$). These results might be due to the shape of the root-mean-square fluctuation function F that changes slowly slope when the block size m varies and converges to its asymptotic value from above (see Figure 2.4). The DFA estimator with small q works better (in terms of bias) in long time series for coarser discretizations and larger H.

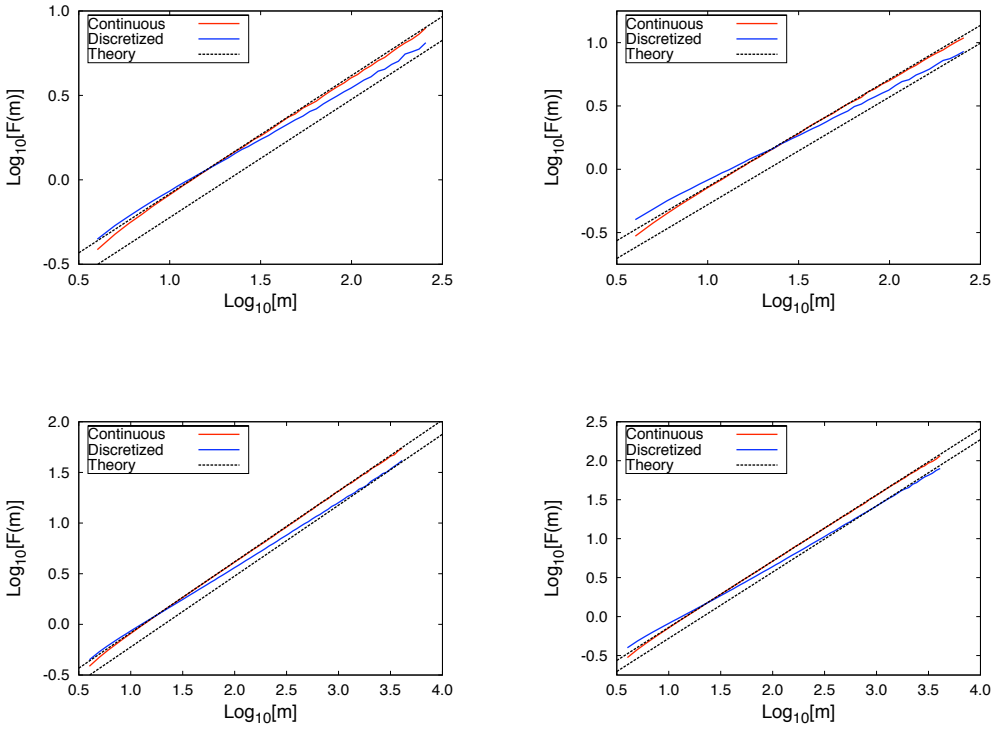


FIGURE 2.4: Root mean square fluctuation $F(m)$ as a function of the block size m of a numerical simulation of a fGn and its discretization with a scaling parameter $\chi = 0.1$. The time series has length $n = 2^{10}$ (top) and $n = 2^{14}$ (bottom). The Hurst exponent of the fGn is $H = 0.7$ (left) and $H = 0.85$ (right). The figure also shows the leading term of the expansion of the $F(m)$ for large m (dashed lines).

2.4.3 Discussion

In conclusion, our analytical considerations and numerical simulations show that both the local Whittle estimator and the DFA consistently give negatively biased estimates of the Hurst exponent H when they are applied to discretized processes. These results are consistent with the literature on long-memory signal plus noise processes (Arteche (2004); Hurvich, Moulines and Soulier (2005)) and on general non-linear transformations of long-memory processes (Dalla, Giraitis and Hidalgo (2006)). Our results show also that the size of the negative bias of $\widehat{H}$ for a discretized process can be significant because the second order correction of the spectral density or of the DFA fluctuation function is large for these processes. The negative bias can be partly overcome by taking small values of m (for the LW estimator) or small intervals for the fit of the root-mean-square fluctuations (for the DFA). However, in both cases the variance of the estimators become large and thus the estimator is not significantly reliable.

In short samples, the LW estimator with $m = n^{0.8}$ and the DFA estimator with $q = 1$ give similar results for $H = 0.7$, while the former is slightly less biased than the latter for $H = 0.85$. In large samples, instead, the DFA estimator with $q = 1$ is less biased than the LW estimator with $m = n^{0.8}$, both for $H = 0.7$ and $H = 0.85$. On the other hand, the LW estimator with $m = n^{0.5}$ performs better (in terms of both variance and bias) than the DFA estimator with $q = 0.25$, both in small and in large samples, and for all values of H. However, it is worth noticing that number of points used in the DFA regression[8] is much smaller than the number of points used for the LW minimization. Therefore, it is not surprising that the variance of the LW estimate is systematically lower than that of the DFA estimate.

Despite the fact that we have presented results only for the fGn and fARIMA$(0, d, 0)$ processes, we have also run extensive Monte Carlo simulations for stationary fARIMA$(1, d, 0)$ and fARIMA$(0, d, 1)$ processes. The results are similar to those obtained for fGn and fARIMA$(0, d, 0)$, with a statistically significant underestimation of the Hurst exponent for the sign process and for coarse discretizations, both for the LW estimation and for the DFA method. However, for stationary fARIMA$(1, d, 0)$ and fARIMA$(0, d, 1)$ the size of the underestimation depends on the high-frequency behavior of the spectral density of the original process. For the sake of brevity, we do not report these results here, but they are available from the authors by request.

2.5 Conclusions

2.5.1 Relation to the measurement error literature

As mentioned in the introduction, round-off error can be considered a form of measurement error, even if the type of measurement error typically considered in the literature is different, being modeled as a white noise uncorrelated with the latent process. On the other hand, the round-off error is a deterministic function of the latent process itself, and it is neither uncorrelated with the latent process nor a white noise.

Recently, in the context of short-memory processes, Hansen and Lunde (2010) have studied the effect of sampling errors on the dynamic properties of an underlying ARMA(p, q) time series. They have proved that the estimates of both the persistence parameter and the autocorrelation function are negatively biased by the measurement error. Our paper contributes in this direction for the case of long-memory processes subject to round-off errors. In fact, we are able to compute exactly the asymptotic scaling factor between the autocorrelation function of the realized process

[8] In our numerical studies, as in the original code of Peng *et al.* (1994), the x-coordinates of the DFA regression are arranged in a geometric series such that the ratio between consecutive box sizes is approximately $2^{1/8}$.

and that of the latent process. This scaling factor is a function only of the adimensional parameter χ, which in turn depends on the grid size (δ) and the variance of the underlying series (D). In most cases of interest the researcher knows the level of discretization. In principle, if the researcher knew also the variance of the underlying process, then she could estimate the autocorrelation function of the latent variable exactly for large lags. The same holds true for the spectral density at small frequencies and the DFA function for large box sizes. In general, however, the variance of the underlying process is not known *ex ante*, and the researcher needs to make inference on it. Even if the problem of the inference on χ (or D) is outside the scope of this paper, it is worth mentioning that by considering the fraction q_0 of points of the discretized process with value zero it is possible to infer the value of the variance of the underlying process. In fact, we showed that $q_0 = erf[\delta/2\sqrt{2D}]$ and therefore the knowledge of the grid size δ and the measurement of q_0 allow to infer D. Finally, note that other methods could be used to estimate D, such as, for example, by comparing the autocovariance of the discretized process with two different values of χ.

In the context of long memory processes, Hurvich, Moulines, and Soulier (2005) considered a process that can be decomposed into the sum of a long memory signal (possibly non-stationary) and a white noise, possibly contemporaneously correlated with the signal. They proved that the observed series has the same Hurst exponent as the underlying signal, and from numerical simulations observed that standard estimators of the Hurst exponent, such as the GPH estimator, suffer from a negative bias. Hurvich, Moulines, and Soulier (2005) proposed a corrected local Whittle estimator, which is unbiased, but has larger asymptotic standard errors. Recently Rossi and Santucci de Magistris (2011) analyzed the effect of measurement errors on the estimation of the Hurst exponent. Specifically, they studied the effect of measurement errors on the estimation of the dynamic properties of the realized variance, as an *ex-post* estimator of the integrated variance. Rossi and Santucci de Magistris (2011) find that discrete frequency sampling and market microstructure noise induce a finite-sample bias in the fractionally integration semiparametric estimates. Based on Hurvich, Moulines, and Soulier (2005) they propose an unbiased local Whittle estimator (with larger asymptotic standard errors) that accounts for the presence of the measurement error. However, the corrected Whittle estimator proposed by Hurvich, Moulines, and Soulier (2005) and Rossi and Santucci de Magistris (2011) does not apply to the round-off error, because the round-off error does not satisfy the standard assumptions on the measurement error. More recently, in a working paper, Corsi and Renò (2011) proposed an indirect inference approach to estimate the long memory parameter of a latent integrated volatility series.

Our paper contributes to this literature by providing the second order term of the spectral

density and of the root-mean-square fluctuation function for the discretized process. A possible extension is to identify analytically and numerically the asymptotic region over which the bias on the long memory estimates due to these second order corrections becomes negligible. From Corollary 2.12 we know the order in probability of the bias of the LW estimator, which can help identifying a suitable bandwidth m for the minimization of the Whittle likelihood function (2.22) in terms of bias reduction. Similarly, one could use our results on the second order correction to the root-mean-square fluctuation to estimate a suitable threshold q for the DFA regression (see Section 2.4.2) in terms of bias reduction. This is outside the scope of the present paper and is left as future work.

2.5.2 Concluding remarks

We have presented an extensive analytical and numerical investigation of the properties of a continuously valued long memory process subject to round-off error. We have shown that the discretized process is also long memory with the same Hurst exponent as the latent process. We have explicitly computed the leading term of the asymptotic expansion of the autocovariance function, of the spectral density for small frequencies, and of the root-mean-square fluctuation for large box sizes. We have shown that the autocovariance, the spectral density and the root-mean-square fluctuation are asymptotically rescaled by a factor smaller than one, and we have computed exactly this scaling factor. More important in all three cases we have computed the order of magnitude the second order correction. This term is important to quantify the bias of the Hurst exponent estimators based on these quantities. An in depth analysis of Hurst exponent estimators, namely the periodogram and the Detrended Fluctuation Analysis, shows that both the estimators are significantly negatively biased as a consequence of the order of magnitude of the second order corrections.

These results can be considered the starting point for several future research directions. A first interesting question is whether the strong negative bias in the estimation of the Hurst exponent observed for the discretization transformation is also observed for other types of (non-linear) transformation of the Gaussian process. Since one often obtains non-Gaussian long memory processes by transforming a fGn or a fARIMA process (Dittman and Granger (2002)), the question of the bias of the estimator of the Hurst exponent is of great interest. A second extension is to study the effect of round-off error on non-stationary processes. The discretization of a non-stationary process is, for example, a more realistic description of the effect of tick size on the price process, and therefore it will have more direct applications to the microstructure literature. The discretization of the price process induces round-off errors in the observed

returns, and therefore also in the realized volatility. From a financial econometrics perspective, it would be interesting to consider also processes with stochastic volatility, i.e. processes where the increments of the process are uncorrelated, but the volatility is significantly correlated or is a long memory process. A very small scale example of this type of analysis has been performed in La Spada *et al.* (2011), where discretization of simulated price process with volatility described by an ARCH process has been considered, finding qualitative results similar to those presented in this paper. Finally, an interesting topic for further research is the extension of our results to continuous time process, and then the study of the combined effect of round-off error and discrete time sampling. We believe that these extensions are potentially of interest in the literature on realized volatility measures, such as the realized variance, that are imperfect estimates of actual volatility.

Since most natural and socio-economic time series are observed on a grid of values, i.e. the measured process is naturally discretized, we believe that our results are potentially of interest in many contexts. The severe bias of the long memory property of a discretized process should warn the scholars investigating processes on the underestimation of the Hurst exponent due to the non-linear transformation induced by round-off errors.

Appendix 2.A Distributional properties

In this appendix we consider the distributional properties of the discretization of a generic stationary Gaussian process.

From (2.5) the m-th moment of the discretized process can be written as $E[X_d^m] = \sum_{n=-\infty}^{\infty} q_n (n\delta)^m$. Since X is Gaussian distributed, the variance D_d of the discretized process can be calculated explicitly. From the above expression with $m = 2$ we obtain

$$D_d = D \sum_{n=-\infty}^{\infty} \frac{n^2}{2\chi} \left[erf\left(\frac{2n+1}{2\sqrt{2\chi}}\right) - erf\left(\frac{2n-1}{2\sqrt{2\chi}}\right)\right] \tag{2.33}$$

Left panel of Figure 2.1 shows the ratio D_d/D as a function of the scaling parameter χ. It is worth noting that this ratio is not monotonic. For small χ the ratio goes to zero because δ is very large relatively to D and essentially all the probability mass falls in the bin centered at zero. In this regime the variance ratio goes to zero as

$$\frac{D_d}{D} \simeq 2\sqrt{\frac{2}{\pi}} \frac{e^{-1/8\chi}}{\sqrt{\chi}}.$$

When $\chi \gg 1$ the ratio tends to one because the effect of discretization becomes irrelevant. In this regime $D_d/D \simeq 1 + 1/(12\chi)$.

Analogously it is possible to calculate the kurtosis $\kappa_d = E[X_d^4]/(E[X_d^2])^2$ of the discretized process. It is direct to show that the kurtosis is

$$\kappa_d = \frac{\sum_{n=1}^{\infty} n^4 \left[erf\left(\frac{2n+1}{2\sqrt{2\chi}}\right) - erf\left(\frac{2n-1}{2\sqrt{2\chi}}\right)\right]}{\left(\sum_{n=1}^{\infty} n^2 \left[erf\left(\frac{2n+1}{2\sqrt{2\chi}}\right) - erf\left(\frac{2n-1}{2\sqrt{2\chi}}\right)\right]\right)^2} \tag{2.34}$$

For small χ the kurtosis diverges as

$$\kappa_d \simeq \sqrt{\frac{\pi}{8\chi}} e^{1/8\chi}$$

because the fourth moment goes to zero slower than the squared second moment. For large χ the kurtosis converges as expected to the Gaussian value 3 as $\kappa_d \simeq 3 - 1/(120\,\chi^2)$. Note that the kurtosis reaches its asymptotic value 3 from below, since it reaches a minimum of roughly 2.982 at $\chi \simeq 0.53$ and then converges to three from below.

Appendix 2.B Proofs for Section 2.3

Proof of Proposition 2.3. The discretized process, $X_d(t)$, is a non-linear transformation of the underlying real-valued process, $X(t)$. Let us denote the discretization transformation with $g(\cdot)$, so that $X_d(t) = g(X(t))$. From (2.10) we know that

$$\gamma_d(k) = \sum_{j=1}^{\infty} g_j^2 \rho^j(k) \qquad \forall\, k = 0, 1, \dots$$

where ρ is the autocorrelation function of the underlying continuous process, and g_j are defined in (2.11).

Since the function $g(x)$ is an odd function $g_j = 0$ for j even, while all the odd coefficients are non-vanishing. Therefore the discretization function has Hermite rank 1 and can be written as an infinite sum of Hermite (odd) polynomials. The generic g_j coefficient is

$$\begin{aligned} g_j &= \frac{1}{\sqrt{2\pi D}} \sum_{n=-\infty}^{\infty} n\delta \int_{(n-1/2)\delta}^{(n+1/2)\delta} H_j\left(\frac{x}{\sqrt{D}}\right) e^{-x^2/2D} dx \\ &= \sqrt{\frac{D}{2\pi}} \sum_{n=-\infty}^{\infty} \frac{n}{\sqrt{\chi}} \int_{(n-1/2)/\sqrt{\chi}}^{(n+1/2)/\sqrt{\chi}} H_j(x) e^{-x^2/2} dx \end{aligned}$$

The first Hermite polynomial is $H_1(x) = x$ and the coefficient g_1 is

$$\frac{g_1}{\sqrt{D}} = \sqrt{\frac{2}{\pi\chi}} \sum_{k=0}^{\infty} \exp\left[-\frac{(2k+1)^2}{8\chi}\right] = \frac{1}{\sqrt{2\pi\chi}} \vartheta_2(0, e^{-1/2\chi}) \tag{2.35}$$

where $\vartheta_a(u, q)$ is the elliptic theta function. For large χ, $g_1 \simeq \sqrt{D}$, while for small χ the coefficient g_1 goes to zero as

$$g_1 \simeq \sqrt{D}\sqrt{\frac{2}{\pi}} \frac{e^{-1/8\chi}}{\sqrt{\chi}}$$

The second non-vanishing coefficient g_3 is given by

$$\frac{g_3}{\sqrt{D}} = -\sqrt{\frac{1}{3\pi\chi}} \sum_{k=0}^{\infty} \exp\left[-\frac{(2k+1)^2}{8\chi}\right] + \frac{1}{\sqrt{48\pi\chi^3}} \sum_{k=0}^{\infty} (2k+1)^2 \exp\left[-\frac{(2k+1)^2}{8\chi}\right] \tag{2.36}$$

In principle one could calculate all the coefficients g_j. Here we want to focus on the case when the correlation coefficient ρ is small (i.e. k is large), so that it suffices to consider the first two

coefficients g_1 and g_3. Therefore, from (2.10) we have

$$\gamma_d(k) \;=\; \frac{g_1^2}{D}\gamma(k)\left(1+O\left(\gamma(k)^2\right)\right) \qquad \text{as } k\to\infty\,. \tag{2.37}$$

If we plug (2.1) and (2.35) into (2.37), we get the result. □

Proof of Proposition 2.4. Under the assumption that $L\in\mathcal{L}$, we can write $\sum_{i=0}^{\infty} b_i k^{-\beta_i} = b_0\left(1+O\left(k^{-\beta_1}\right)\right)$; by substituting this expression into (2.12) we get the result. □

Proof of Corollary 2.5. It follows from the definition of autocorrelation function and Proposition 2.3. □

Proof of Proposition 2.6. From Proposition 2.3 we can write $\gamma_d(k)\sim\left(\frac{\vartheta_2(0,e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2 k^{2H-2}L(k)$, as $k\to\infty$. Then, the proposition follows from Theorem 3.3 (a) in Palma (2007). □

Proof of Proposition 2.7. For the sake of simplicity, we consider the case of an underlying process with unit variance, namely $D=1$. In order to extend the proof to non unit variance we need simply to do the transformation $L(k)\to L(k)/D$. Obviously, $L(k)/D$ is still slowly varying at infinity. Moreover, we consider only the case $I=\infty$, the case $I<\infty$ being trivial.

This proof is divided in two parts: in the first part we prove (2.16) and (2.17) under the assumption that $L\in\mathcal{L}$; in the second part we prove (2.18) under the additional Assumption 4.

First part From Proposition 2.4 and Theorem 2.1 in Beran (1994) we know that the spectral density $\phi_d(\omega)$ exists. Moreover, if $L\in\mathcal{L}$, then $\exists K>0$ s.t. $\forall\, k\geq K$, $L(k)=\sum_{i=0}^{\infty} b_i k^{-\beta_i}$ and the series converges absolutely. Then, from Herglotz's theorem and (2.10) we can write

$$\phi_d(\omega)=\frac{D_d}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{K}\gamma_d(k)\cos(k\omega)+\frac{1}{\pi}\sum_{k>K}^{\infty}\sum_{j=0}^{\infty}g_{2j+1}^2\left(k^{2H-2}\sum_{i=0}^{\infty}b_i k^{-\beta_i}\right)^{2j+1}\cos(k\omega)$$

As $\omega\to 0^+$, the first two terms on the RHS converge to a constant, while the third term diverges.

Since $\sum_i b_i k^{-\beta_i}$ converges absolutely $\forall\, k\geq K$, we can write

$$\sum_{i=0}^{\infty} b_i k^{-\beta_i} = b_0 + b_1 k^{-\beta_i} + b_2 k^{-\beta_2}(1+o(1))$$

Therefore,

$$\begin{aligned}
&\sum_{k>K}^{\infty}\sum_{j=0}^{\infty} g_{2j+1}^2\left(k^{2H-2}\sum_{i=0}^{\infty} b_i k^{-\beta_i}\right)^{2j+1}\cos(k\omega) = \\
&= \sum_{k>K}^{\infty}\left\{g_1^2 k^{2H-2}\left[b_0 + b_1 k^{-\beta_1} + b_2 k^{-\beta_2}(1+o(1))\right] + \right. \\
&\quad \left. + g_3^2 k^{6H-6}\left[b_0^3 + 3b_0^2 b_1 k^{-\beta_1}(1+o(1))\right] + g_5^2 k^{10H-10} b_0^5 (1+o(1))\right\}\cos(k\omega) \\
&= g_1^2 b_0 \sum_{k=1}^{\infty} k^{2H-2}\cos(k\omega) + g_1^2 b_1 \sum_{k=1}^{\infty} k^{2H-2-\beta_1}\cos(k\omega) + O\left(\sum_{k=1}^{\infty} k^{2H-2-\beta_2}\cos(k\omega)\right) + \\
&\quad + g_3^2 b_0^3 \sum_{k=1}^{\infty} k^{6H-6}\cos(k\omega) + O\left(\sum_{k=1}^{\infty} k^{6H-6-\beta_1}\cos(k\omega)\right) + O\left(\sum_{k=1}^{\infty} k^{10H-10}\cos(k\omega)\right) + O(1),
\end{aligned} \tag{2.38}$$

where the term $O(1)$ comes from substituting $\sum_{k>K}^{\infty}$ with $\sum_{k=1}^{\infty}$.

Then, we introduce the following representation of a trigonometric series

$$\sum_{k=1}^{\infty} k^{-s}\cos(k\omega) = \frac{1}{2}\left(Li_s(e^{-i\omega}) + Li_s(e^{i\omega})\right) \tag{2.39}$$

where $Li_s(z) = \sum_{k=1}^{\infty} z^k/k^s$ is the polylogarithm. The series expansion of $Li_s(e^z)$ for small z is (see Gradshteyn and Ryzhik (1980))

$$Li_s(e^z) = \Gamma(1-s)(-z)^{s-1} + \sum_{k=0}^{\infty}\frac{\zeta(s-k)}{k!}z^k, \qquad if \;\; s \notin \mathbb{N} \tag{2.40}$$

$$Li_s(e^z) = \frac{z^{s-1}}{(s-1)!}\left[H_{s-1} - \ln(-z)\right] + \sum_{k=0,\,k\neq s-1}^{\infty}\frac{\zeta(s-k)}{k!}z^k, \qquad if \;\; s \in \mathbb{N} \tag{2.41}$$

where $\zeta(\cdot)$ is the analytic continuation of the Riemann zeta function over the complex plane and H_s is the sth harmonic number.

Finally, we plug (2.40) or (2.41) into (2.39), and then we apply (2.39) into (2.38). By substituting (2.35) for g_1^2 and after some algebraic manipulation we get the result.

Second part Under Assumption 4 (i) the series $L(k) = \sum_{i=0}^{\infty} b_i k^{-\beta_i}$ converges absolutely $\forall\, k \geq 1$. Therefore, by Mertens' theorem[9], for any $j \geq 0$ we can write

$$\left(\sum_{i=0}^{\infty} b_i k^{-\beta_i}\right)^{2j+1} = \sum_{i=0}^{\infty} \widetilde{b}_{j,i} k^{-\widetilde{\beta}_{j,i}} \qquad \forall\, k \geq 1\,,$$

where the series on the RHS is the Cauchy product. Note that $\widetilde{b}_{0,i} = b_i$ and $\widetilde{\beta}_{0,i} = \beta_i\ \forall\, i$, while $\widetilde{b}_{j,0} = b_0^{2j+1}$ and $\widetilde{\beta}_{j,0} = 0\ \forall\, j$. By absolute convergence of the original series the Cauchy product also converges absolutely $\forall\, k \geq 1$.

From Herglotz's theorem and Proposition 2.4 we can write

$$\phi_d(\omega) \;=\; \frac{D_d}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} g_{2j+1}^2 \widetilde{b}_{j,i} k^{-\alpha_{j,i}} \cos(k\omega)\,,$$

where $\alpha_{j,i} \equiv (2j+1)(2-2H) + \widetilde{\beta}_{j,i}$.

Since $\left\{g_{2j+1}^2\right\}$ are bounded above (because $\sum_j g_{2j+1}^2 = D_d < \infty$), Assumption 4 (i) implies that the double series $\sum_j \sum_i g_{2j+1}^2 \widetilde{b}_{j,i} k^{-\alpha_{j,i}}$ converges absolutely $\forall\, k \geq 1$. Moreover, under Assumption 4 (*ii*) there is only a finite number of terms in $L(k)$ (and therefore in $\sum_j \sum_i g_{2j+1}^2 \widetilde{b}_{j,i} k^{-\alpha_{j,i}}$) that are not summable over k. Thus, from Rudin (1976) (Chapter 8, Theorem 8.3) we can invert the order of summation and write

$$\phi_d(\omega) \;=\; \frac{D_d}{2\pi} + \frac{1}{\pi} \sum_{j=0}^{\infty} g_{2j+1}^2 \sum_{i=0}^{\infty} \widetilde{b}_{j,i} \sum_{k=1}^{\infty} k^{-\alpha_{j,i}} \cos(k\omega)$$

In other words, the Fourier transform of the series becomes the series of the Fourier transforms. From this point on the proof is very similar to that of Lemma 2.10 and therefore omitted for brevity. □

Proof of Corollary 2.8. If L is analytic at infinity, then $L(k) = \sum_{n=0}^{\infty} b_n k^{-n}$ for large k for some $\{b_n\}$, and the series converges absolutely within its radius of convergence. Then, $L \in \mathcal{L}$ with $\beta_1 \in \mathbb{N}$, and by applying Proposition 2.7 and (2.35) we get (2.19). It is straightforward to see that $c_2 > 0$ and $c_1 > 0$ for $H > 5/6$. □

Proof of Corollary 2.9. For a fGn $L(k) = \frac{D}{2}\left(\left(1+\frac{1}{k}\right)^{2H} + \left(1-\frac{1}{k}\right)^{2H} - 2\right) k^2$, which is an even analytic function at infinity and whose power series converges absolutely $\forall\, k \geq 1$. Because

[9] See G. H. Hardy, *Divergent Series*, 2nd ed., (1991), Chapter X Theorem 160.

$L(k)$ is analytic and even, $\beta_i = 2i \;\forall\, i \geq 0$ and we can write

$$\Gamma\left(2H-1-\beta_i\right) = (-1)^{2i-1}\frac{\Gamma\left(1-2H\right)\Gamma\left(2H\right)}{\Gamma\left(2(i+1-H)\right)} \longrightarrow 0 \qquad \text{as } i \to \infty\,.$$

Thus, the autocovariance of a fGn satisfies Assumption 4, and therefore from Corollary 2.8 it follows that the spectral density of a discretized fGn satisfies (2.19) with c_0 given by (2.18).

From Corollary 2.8 we already know that the second-order term is strictly positive if $H \geq 5/6$. Hence, we just need to prove that $c_0 > 0$ if $H < 5/6$. Since c_0 is given by (2.18), we can write

$$\begin{aligned} c_0 =& \frac{D_d}{2\pi} + \frac{g_1^2}{\pi}\sum_{i=0}^{\infty}\binom{2H}{2(i+1)}\zeta\left(2(i+1-H)\right) + \\ &+ \sum_{j=1}^{\infty}\sum_{i=0}^{\infty}\frac{g_{2j+1}^2}{\pi D^{2j+1}}\widetilde{b}_{j,i}\,\zeta\left((2j+1)(2-2H)+\widetilde{\beta}_{j,i}\right) \end{aligned} \tag{2.42}$$

where we used the fact that for a fGn $b_i = \binom{2H}{2(i+1)}D$ and $\beta_i = 2i \;\forall\, i \geq 0$. The symbol $\binom{\cdot}{\cdot}$ denotes the generalized binomial coefficient.

First, since $\{b_i\}$ are strictly positive and $(2j+1)(2-2H) > 1 \;\forall\, j \geq 1$ if $H < 5/6$, the third term on the RHS of (2.42) is strictly positive for $H < 5/6$.

Second, from Sinai (1976) and Beran (1994) we know that the spectral density of a fGn satisfies

$$\phi(\omega) = 2c_\phi^*\left(1-\cos\omega\right)\sum_{j=-\infty}^{\infty}\left|2\pi j+\omega\right|^{-2H-1}, \quad \omega \in [-\pi,\pi]\,.$$

where $c_\phi^* = \frac{D}{2\pi}\sin\left(\pi H\right)\Gamma\left(2H+1\right)$. For small ω the behavior of the above spectral density follows by Taylor expansion at zero:

$$\phi(\omega) = c_\phi^*|\omega|^{1-2H} - \frac{1}{12}c_\phi^*|\omega|^{3-2H} + o\left(|\omega|^{3-2H}\right)\,. \tag{2.43}$$

On the other hand, because $L(k)$ is analytic with $\beta_1 = 2$, it satisfies Assumption 5; therefore, the fGn satisfies the conditions of Lemma 2.10. Following the proof of that lemma, after some algebraic manipulations, we get

$$\phi(\omega) = c_\phi^*|\omega|^{1-2H} + c_0^* - \frac{1}{12}c_\phi^*|\omega|^{3-2H} + o\left(|\omega|^{3-2H}\right)\,, \tag{2.44}$$

where $c_0^* = D\pi^{-1}\left(\frac{1}{2} + \sum_{i=0}^{\infty}\binom{2H}{2(i+1)}\zeta\left(2(i+1-H)\right)\right)$. By comparing (2.43) and (2.44) it follows that c_0^* must be equal to zero, and therefore

$$\sum_{i=0}^{\infty}\binom{2H}{2(i+1)}\zeta\left(2(i+1-H)\right) = -\frac{1}{2}. \tag{2.45}$$

Finally, note that from (2.10) we know that $D_d = \sum_{j=0}^{\infty} g_{2j+1}^2$. Hence, by plugging (2.45) into (2.42) we obtain

$$\begin{aligned} c_0 &= \frac{\sum_{j=0}^{\infty} g_{2j+1}^2}{2\pi} - \frac{g_1^2}{2\pi} + \sum_{j=1}^{\infty}\sum_{i=0}^{\infty}\frac{g_{2j+1}^2}{\pi D^{2j+1}}\widetilde{b}_{j,i}\,\zeta\left((2j+1)(2-2H)+\widetilde{\beta}_{j,i}\right) \\ &= \frac{\sum_{j=1}^{\infty} g_{2j+1}^2}{2\pi} + \sum_{j=1}^{\infty}\sum_{i=0}^{\infty}\frac{g_{2j+1}^2}{\pi D^{2j+1}}\widetilde{b}_{j,i}\,\zeta\left((2j+1)(2-2H)+\widetilde{\beta}_{j,i}\right) > 0 \end{aligned}$$

□

Appendix 2.C Proofs for Section 2.4

Proof of Lemma 2.10. From Theorem 2.1 in Beran (1994) we know that the spectral density $\phi(\omega)$ exists and from Hergotz's theorem it is the discrete Fourier transform of the autocovariance $\gamma(k) = k^{2H-2}\sum_i b_i k^{-\beta_i}$.

Let $\alpha_i = 2 - 2H + \beta_i \;\forall\, i \geq 0$. Under Assumption 4 (ii) there is only a finite number of terms in the autocovariance of X that are not summable, and therefore there will be only a finite number of divergent terms in the spectral density. Moreover, under Assumption 4 (i) the series $\sum_{i=0}^{I} b_i k^{-\beta_i}$ converges absolutely $\forall\, k \geq 1$. Therefore, by Rudin (1976) (Theorem 8.3) we can write

$$\phi(\omega) = \frac{D}{2\pi} + \frac{1}{\pi}\sum_{i=0}^{I} b_i \sum_{k=1}^{\infty} k^{-\alpha_i}\cos(k\omega) \tag{2.46}$$

By using the polylogarithm representation (2.39) introduced above, for small ω we can plug (2.40) and (2.41) into (2.46). Let $I_1 = \{i \in \mathbb{N} : \alpha_i \notin \mathbb{N}\}$ and $I_2 = \{i \in \mathbb{N} : \alpha_i \in \mathbb{N}\}$. Then, under

Assumption 4 (iii) we can rearrange the double series in (2.46) in the following way

$$\begin{aligned}\phi(\omega) = \frac{D}{2\pi} \quad &+ \quad \frac{1}{\pi} b_0 \left(\Gamma(2H-1)\sin\left((1-H)\pi\right)|\omega|^{1-2H} + \sum_{j=0}^{\infty}(-1)^j \frac{\zeta(2-2H-2j)}{(2j)!}\omega^{2j} \right) \\ &+ \quad \frac{1}{\pi}\sum_{i\in I_1} b_i \left(\Gamma(1-\alpha_i)\sin\left(\frac{\pi\alpha_i}{2}\right)|\omega|^{\alpha_i-1} + \sum_{j=0}^{\infty}(-1)^j \frac{\zeta(\alpha_i-2j)}{(2j)!}\omega^{2j} \right) \\ &+ \quad \frac{1}{\pi}\sum_{i\in I_2} b_i \left(\frac{|\omega|^{\alpha_i-1}}{(\alpha_i-1)!}\left[\sin\left(\frac{\pi\alpha_i}{2}\right)H_{\alpha_i-1} + \frac{\pi}{2}\cos\left(\frac{\pi\alpha_i}{2}\right) - \sin\left(\frac{\pi\alpha_i}{2}\right)\ln|\omega|\right] + \right. \\ &\qquad \left. + \sum_{j=0,\, j\neq\alpha_i-1}^{\infty}(-1)^j\frac{\zeta(\alpha_i-2j)}{(2j)!}\omega^{2j} \right), \qquad as \;\; \omega\to 0^+, \end{aligned} \tag{2.47}$$

where $\zeta(\cdot)$ is the analytic continuation of the Riemann zeta function over the complex plane and H_s is the sth harmonic number.

Under Assumption 4 we can collect all the terms of the same oder and rearrange (2.47) in powers of ω. Let c_0^* be the term of order $O(1)$ in (2.47); then,

$$c_0^* = \frac{D}{2\pi} + \frac{1}{\pi}\sum_{i:\,\alpha_i\neq 1}^{I} b_i\zeta(\alpha_i) .$$

Under Assumption 5 $\alpha_1 \neq 1$, and therefore, if also $\alpha_1 \neq 2$, we can write

$$\phi(\omega) = c_\phi b_0|\omega|^{1-2H} + c_0^* + c_1^*|\omega|^{1-2H+\beta_1} + o(|\omega|^{\min(0,1-2H+\beta_1)}) \qquad \text{as} \;\; \omega\to 0^+, \tag{2.48}$$

where c_ϕ is defined as in Proposition 2.6 and $c_1^* = \pi^{-1}b_1\Gamma(2H-1-\beta_1)\sin\left(\frac{2H-\beta_1}{2}\pi\right)$.

If $\alpha_1 = 2$, by Assumption 5 $c_0^* \neq 0$ and we can write

$$\phi(\omega) = c_\phi b_0|\omega|^{1-2H} + c_0^* + o(1), \quad as \;\; \omega\to 0^+. \tag{2.49}$$

Putting together (2.48) and (2.49), and noting that if $c_0^* = 0$ in (2.48), then by Assumption 5 $\beta_1 \leq 2$, we get the result

$$\phi(\omega) = c_\phi|\omega|^{1-2H}\left(b_0 + c_\beta|\omega|^\beta + o(|\omega|^\beta)\right), \quad as \;\; \omega\to 0^+,$$

where $c_\beta \neq 0$ and $\beta \in (0,2]$. □

Proof of Theorem 2.11. Following the proof of Theorem 4 in DGH, because $j_0 = 1$ we can write Y_t as a signal plus noise process $Y_t = W_t + Z_t$, where

$$W_t = g_1 H_1(X_t) = g_1 X_t \qquad Z_t = \sum_{j=j_1}^{\infty} g_j H_j(X_t)$$

where j_1 is the second non-vanishing term in the Hermite expansion and $H_j(\cdot)$ is the jth Hermite polynomial.

Part (i) If $L \in \mathcal{L}$, the spectral density of W_t satisfies Assumption A in DGH, i.e.,

$$\phi_w(\omega) = c_w|\omega|^{1-2H}(1 + o(1)) \qquad as \;\; \omega \to 0^+,$$

where $c_w = (g_1^2/D)c_\phi b_0$. Moreover, since X_t is a stationary purely non-deterministic Gaussian process, it is also linear with finite fourth moments. Consequently, $W_t = g_1 X_t$ is also linear with finite fourth moments. Then, we can write

$$W_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$$

where $\sum_{j=0}^{\infty} a_j^2 < \infty$ and ε_t are i.i.d. Gaussian variables with zero mean and unit variance. Let $\alpha(\omega) = \sum_{j=0}^{\infty} a_j e^{ij\omega}$. From Proposition 4 in DGH it follows that W_t satisfies also Assumption B therein.

We show below that the spectral density of Z_t satisfies $\phi_z(\omega) \leq C|\omega|^{1-2H_z}$, as $\omega \to 0^+$, for some $C > 0$ and $H_z \geq 0.5$ such that

$$H > H_z = \begin{cases} 0.5 & if \;\; j_1(2-2H) > 1 \\ 0.5 - j_1(1-H) & if \;\; j_1(2-2H) < 1 \\ \varepsilon & if \;\; j_1(2-2H) = 1 \end{cases} \tag{2.50}$$

for any $\varepsilon \in (0.5, H)$.

Indeed, if $j_1(2-2H) > 1$ from (2.10)

$$\sum_{k=1}^{\infty} \gamma_z(k) = \sum_{k=1}^{\infty} \sum_{j=j_1}^{\infty} g_j^2 (\rho(k))^j \leq C \sum_{k=1}^{\infty} k^{-j_1(2-2H)} < \infty$$

for some $C > 0$. Therefore, $H_z = 0.5 < H$.

If $j_1(2-2H) < 1$, we can prove that

$$\phi_z(\omega) = \frac{g_{j_1}^2}{D^{j_1}} b_0 c_\phi |\omega|^{1-2H_z}(1+o(1)) \leq C|\omega|^{1-2H_z} \qquad as\ \omega \to 0^+\,,$$

for some $C > 0$ and $H_z = H - (j_1 - 1)(1-H) < H \in (0.5, 1)$. Similarly, if $j_1(2-2H) = 1$, we can prove that

$$\phi_z(\omega) = C \ln|\omega|^{-1}(1+o(1)) \leq C|\omega|^{-\varepsilon} \qquad as\ \omega \to 0^+\,,$$

for some $C > 0$ and for any $\varepsilon > 0$. The proof of the above results is a special case of the proof of Proposition 2.7, and thus omitted. The results above prove (2.50).

Since W_t satisfies Assumptions A and B in DGH and the spectral density ϕ_z satisfies the asymptotic conditions above, consistency of $\widehat{H}_{LW}^Y$ follows from Theorem 3 (i) in DGH.

Moreover, if we write the periodogram of Y_t as $I_Y(\omega_j) = I_W(\omega_j) + v_j$, where I_W is the periodogram of the "signal" W_t and v_j is the contribution of the "noise" Z_t at the jth Fourier frequency, it is straightforward to show (see DGH pg. 225–226) that

$$\begin{aligned}\widehat{H}_{LW}^Y - H &= \left(\widehat{H}_{LW}^X - H\right)(1+o_P(1)) - \left(m^{-1}\sum_{j=1}^{m}\left(\log\left(\frac{j}{m}\right)+1\right)\frac{v_j}{c_w\omega_j^{1-2H}}\right)(1+o_P(1)) \\ &\quad + O_P\left(\frac{\log m}{m}\right) \\ &= \left(\widehat{H}_{LW}^X - H\right)(1+o_P(1)) + O_P\left(\left(\frac{m}{n}\right)^{H-H_z} + \frac{\log m}{m}\right)\,, \end{aligned} \tag{2.51}$$

where $\widehat{H}_{LW}^X$ denotes the LW estimator of $\{X_t\}$ if the sequence $\{X_t\}$ were observed.

Note that, roughly speaking, v_j represents the sample estimate of the higher-order terms of the spectral density of Y_t at the jth Fourier frequency. For the discretization of a fGn we know from Corollary 2.9 that the second-order term of the spectral density is strictly positive for all H; therefore, in that case, we expect that the second term on the RHS of the first line of (2.51) will induce a negative finite sample bias on $\widehat{H}_{LW}^Y$. The order of magnitude of this finite sample bias is $O_P\left((m/n)^{H-H_z}\right)$.

Part (ii) Under Assumptions 4 and 5, from Lemma 2.10 it follows that X_t and therefore W_t satisfy Assumption $T(\alpha_0, \beta)$ in DGH, with $\alpha_0 = 2H - 1$ and β defined as in Lemma 2.10. Moreover, under Assumption 6 we can combine the second part of Proposition 5 in DGH with Proposition 3 and Theorem 2 therein, and under the assumption that $m = o\left(m^{2\beta/(2\beta+1)}\right)$ we

can write

$$\widehat{H}^X_{LW} - H = -\left(\frac{m}{n}\right)^{\beta}\left(\frac{c_\beta}{b_0}\right)\frac{B_\beta}{2} - \frac{V_m}{2}(1+o_P(1)) + o_p\left(m^{-1/2} + \left(\frac{m}{n}\right)^{\beta}\right) \tag{2.52}$$

where c_β is defined as in Lemma 2.10, $B_\beta = (2\pi)^\beta\beta/(\beta+1)^2$, and

$$V_m = m^{-1}\sum_{j=1}^{m}\left(\log\left(\frac{j}{m}\right)+1\right)(\eta_j - \mathbb{E}\eta_j)$$

with $\eta_j = I_X(\omega_j)/\phi_x(\omega_j)$.

Let $r = H - H_z$. By plugging (2.52) into (2.51) we obtain

$$\widehat{H}^Y_{LW} - H = -\frac{V_m}{2} - \left(\frac{m}{n}\right)^{\beta}\left(\frac{c_\beta}{b_0}\right)\frac{B_\beta}{2} + O_P\left(\left(\frac{m}{n}\right)^{r} + \frac{\log m}{m}\right) + o_p\left(m^{-1/2} + \left(\frac{m}{n}\right)^{\beta} + V_m\right). \tag{2.53}$$

Moreover, under Assumption 6 and $m = o\left(m^{2\beta/(2\beta+1)}\right)$, by Robinson's (1995) Theorem 2

$$\sqrt{m}V_m \xrightarrow{d} N(0,1)\,, \qquad as\ \ n\to\infty\,. \tag{2.54}$$

Therefore, $V_m = O_P\left(m^{-1/2}\right)$ and from (2.53) follows (2.25).

Part (iii) If $m = o(n^{2r/(2r+1)})$, equation (2.26) follows from applying (2.54) in (2.53). □

Proof of Corollary 2.12. The result of the corollary follows directly from Theorem 2.11 and from noticing that the second non-vaninshing Hermite coefficient for the discretized process is $g_3 \neq 0$, so that $j_1 = 3$. □

For the proof of Theorem 2.13 we need the following lemma. Note that the proofs of the lemmas are at the end of this Appendix.

Lemma 2.16. *Let $m \in \mathbb{N}$, $\alpha < 1$, and*

$$R(\alpha) = -\alpha(\alpha-1)\int_1^m \frac{B_2(\{1-t\})}{2!}t^{\alpha-2}\mathrm{d}t \tag{2.55}$$

$$\widetilde{R}(\alpha) = -\int_1^m \frac{B_2(\{1-t\})}{2!}t^{\alpha-2}\left(\alpha(\alpha-1)\log t - 1 + 2\alpha\right)\mathrm{d}t \tag{2.56}$$

where $B_2(x) = 1/6 - x + x^2$ is the third Bernoulli polynomial and $\{x\}$ represents the fractional part of the real number x. Then, $R(\alpha)$ and $\widetilde{R}(\alpha)$ converge as $m\to\infty$.

Note that $R(\alpha)$ and $\widetilde{R}(\alpha)$ are the remainders of a first order Euler-Maclaurin expansion of the sums $\sum_{k=1}^{m} k^{\alpha}$ and $\sum_{k=1}^{m} k^{\alpha}\log k$, respectively.

For the proof of Theorem 2.13 we need the following lemmas.

Lemma 2.17. *Let $i,k\in\mathbb{N}$, $\alpha>0$, and $\alpha\neq 1,2$. Then, as $i\to\infty$,*

$$\sum_{k=1}^{i}(i-k)k^{-\alpha}=A_0(\alpha)i^{2-\alpha}+A_1(\alpha)\,i+O(1)$$

where $A_0(\alpha)=\frac{1}{(1-\alpha)(2-\alpha)}$ and $A_1(\alpha)=-\frac{(\alpha+2)(\alpha+3)}{12(1-\alpha)}+R(\alpha)$ and $R(\cdot)$ is defined as in (2.55).

Lemma 2.18. *Let $i,k\in\mathbb{N}$. Then, as $i\to\infty$,*

$$\begin{aligned}\sum_{k=1}^{i}(i-k)k^{-1} &= i\ln i+\left(-\frac{5}{12}+R_1\right)i+O(1)\\ \sum_{k=1}^{i}(i-k)k^{-2} &= \left(\frac{5}{3}+R_2\right)i-\ln i+O(1)\end{aligned}$$

where $R_1\equiv R(\alpha=-1)$ and $R_2\equiv R(\alpha=-2)$.

Before proving Theorem 2.13 we prove the following proposition.

Proposition 2.19. *Under the assumptions of Theorem 2.13, let us define $\Sigma_m=Cov(Y(i),Y(j))$, i.e., the covariance matrix of the integrated process $(Y(1),\ldots,Y(m))$. Then,*

(i) if $\beta\neq 2H-1$, then

$$\Sigma_m=\frac{A}{2H(2H-1)}\Big(i^{2H}\left(1+O(i^{-\min(2H-1,\beta)})\right)+j^{2H}\left(1+O(j^{-\min(2H-1,\beta)})\right)+\\ -|i-j|^{2H}\left(1+O(|i-j|^{-\min(2H-1,\beta)})\right)\Big]$$

(ii) if $\beta=2H-1$, then

$$\Sigma_m=\frac{A}{2H(2H-1)}\left[i^{2H}\left(1+O(i^{1-2H}\ln i)\right)+j^{2H}\left(1+O(j^{1-2H}\ln j)\right)+\right.\\ \left.-|i-j|^{2H}\left(1+O(|i-j|^{1-2H}\ln|i-j|)\right)\right]$$

Proof. Under the assumptions on $X(t)$, for $1 \leq i, j, \leq m$ we can write

$$\begin{aligned}\Sigma_m &= Cov(Y(i), Y(j)) = \sum_{k=1}^{i}\sum_{l=1}^{j} Cov(X(k), X(l)) = \sum_{k=1}^{i}(i-k)\,\gamma(k) + \\ &+ \sum_{k=1}^{j}(j-k)\,\gamma(k) - \sum_{k=1}^{|i-j|}(|i-j|-k)\,\gamma(k) + \min(i,j)\,D \end{aligned} \tag{2.57}$$

where D is the variance of the process $X(t)$. By substituting the explicit functional form for $\gamma(k)$ we get

$$\sum_{k=1}^{i}(i-k)\,\gamma(k) \leq A\sum_{k=1}^{i}(i-k)\,k^{2H-2} + M\sum_{k=1}^{i}(i-k)k^{2H-2-\beta}$$

for some $M > 0$ sufficiently large. By Lemma 2.17 we have $A\sum_{k=1}^{i}(i-k)\,k^{2H-2} = A\left(\frac{i^{2H}}{2H(2H-1)} + O(i)\right)$.

Now, we consider the following cases:

(i) $\beta \neq 2H-1$. In this case we have to distinguish two cases.

If $\beta \neq 2H$, we can use Lemma 2.17 and obtain

$$\begin{aligned}\sum_{k=1}^{i}(i-k)\,k^{2H-2-\beta} =& A_0(2H-2-\beta)\,i^{2H-\beta} + A_1(2H-2-\beta)\,i + O(1) \\ =& i^{2H}\,O\left(i^{-\min(2H-1,\beta)}\right)\end{aligned}$$

If $\beta = 2H$, then we can use Lemma 2.18

$$\sum_{k=1}^{i}(i-k)\,k^{2H-2-\beta} = \left(\frac{5}{3} + R_2\right)i - \ln i + O(1) = O\left(i\right)$$

Then, we repeat the same calculation for the second and third term in (2.57). By noting that $\min(i,j)$ is either of order $O(i)$ or $O(j)$ and putting together all the terms, we obtain the result.

(ii) $\beta = 2H-1$. In this case we can use Lemma 2.18

$$\sum_{k=1}^{i}(i-k)\,k^{2H-2-\beta} = \left(i\ln i + \left(-\frac{5}{12} + R_1\right)i + O(1)\right) = i^{2H}O\left(i^{1-2H}\ln i\right)$$

Then, we repeat the same calculation for the second and third term in (2.57). By noting that $\min(i,j)$ is either of order $O(i)$ or $O(j)$ and putting together all the terms, we obtain the result.

□

Proof of Theorem 2.13. First, for $j \in \{1, \dots, [n/m]\}$ let us define the vector:

$$Y^{(j)} = (Y(1 + m(j-1)), \dots, Y(mj))^\top ,$$

where $x^\top$ means the transpose of x.

Then, following Bardet and Kammoun (2008),

$$F_1^2(m) = \frac{1}{m}(Y^{(1)} - P_{E_1}Y^{(1)})^\top (Y^{(1)} - P_{E_1}Y^{(1)}) = \frac{1}{m}(Y^{(1)\top}Y^{(1)} - Y^{(1)\top}P_{E_1}Y^{(1)}) ,$$

where E_1 is the vector subspace of $\mathbb{R}^m$ generated by the two vectors $e_1 = (1, \dots, 1)^\top$ and $e_2 = (1, 2, \dots, m)^\top$, P_{E_1} is the matrix of the orthogonal projection on E_1, and the second equality holds because the projection operator is idempotent. As a consequence,

$$\mathbb{E}\left[F_1^2(m)\right] = \frac{1}{m}\left(\mathrm{Tr}(\Sigma_m) - \mathrm{Tr}(P_{E_1}\Sigma_m)\right)$$

Case (i) If $\beta \neq 2H - 1$, from Proposition 2.19 we get

$$\mathrm{Tr}(\Sigma_m) = \frac{A}{H(2H-1)} m^{2H+1} \left(\int_0^1 x^{2H} \mathrm{d}x + O\left(m^{-\min(2H-1,\beta)}\right) + O\left(m^{-1}\right) \right) ,$$

where the error $O\left(m^{-1}\right)$ comes from approximating the sum with the integral; therefore,

$$\mathrm{Tr}(\Sigma_m) = \frac{Am^{2H+1}}{H(2H-1)} \frac{1}{2H+1} \left(1 + O\left(m^{-\min(2H-1,\beta)}\right)\right) . \tag{2.58}$$

For the term $\mathrm{Tr}(P_{E_1}\Sigma_m)$, we use the following representation of the projection operator

$$P_{E_1} = \frac{2}{m(m-1)} \left((2m+1) - 3(i+j) + 6\frac{i \cdot j}{1+m} \right) . \tag{2.59}$$

Then, using (2.59) and Proposition 2.19 we can write

$$\begin{aligned}\mathrm{Tr}(P_{E_1}\Sigma_m) =& \frac{2Am^{2H+1}m^2}{m(m-1)2H(2H-1)}\left[\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{m^2}\left(\left(2+\frac{1}{m}\right)-3\frac{i+j}{m}+6\frac{ij}{m(m+1)}\right)\times\right.\\ &\times\left(\left(\frac{i}{m}\right)^{2H}\left(1+O\left(i^{-\min(2H-1,\beta)}\right)\right)+\left(\frac{j}{m}\right)^{2H}\left(1+O\left(j^{-\min(2H-1,\beta)}\right)\right)+\right.\\ &\left.\left.-\left(\frac{|i-j|}{m}\right)^{2H}\left(1+O\left(|i-j|^{-\min(2H-1,\beta)}\right)\right)\right)\right].\end{aligned}$$

Approximating sums with integrals, we get

$$\begin{aligned}\mathrm{Tr}(P_{E_1}\Sigma_m) &= \frac{Am^{2H+1}}{H(2H-1)}\left(1+O\left(m^{-\min(2H-1,\beta)}\right)+O\left(m^{-1}\right)\right)\times\\ &\quad\times\int_0^1\int_0^1\left[(2-3(x+y)+6xy)\left(x^{2H}+y^{2H}-|x-y|^{2H}\right)\right]\mathrm{d}x\mathrm{d}y\\ &= \frac{Am^{2H+1}}{H(2H-1)}\frac{1+H(4+H)}{(1+H)(2+H)(1+2H)}\left(1+O\left(m^{-\min(2H-1,\beta)}\right)\right)\end{aligned}\tag{2.60}$$

Putting together (2.58) and (2.60) we obtain

$$\frac{1}{m}\left(\mathrm{Tr}(\Sigma_m)-\mathrm{Tr}(P_{E_1}\Sigma_m)\right)=\frac{A}{H(2H-1)}f(H)m^{2H}\left(1+O\left(m^{-\min(2H-1,\beta)}\right)\right)$$

which is the formula of $\mathbb{E}\left[F_1^2(m)\right]$.

Case (ii) If $\beta = 2H-1$, the proof is exactly the same, except for replacing all the terms $O\left(i^{-\min(2H-1,\beta)}\right)$ with the terms $O\left(i^{1-2H}\ln i\right)$. □

Proof of Corollary 2.14. It follows from the autocovariance of a fractional Gaussian noise (see formula (2.3)). The proof is very similar to the proof of Theorem 2.13, and thus omitted. However, a complete proof can be found in Bardet and Kammoun (2008) (see Proof of Property 3.1. therein). □

Proof of Corollary 2.15. It follows from Proposition 2.4 and Theorem 2.13. □

Proofs of lemmas for Section 2.4

Proof of Lemma 2.16. First we prove that $R(\alpha)$ converges. Because $|B_2(\{1-t\})| \leq B_2(0) = 1/6$ for all t, we have $|R(\alpha)| \leq \frac{|\alpha(\alpha-1)|}{12} \int_1^m t^{-2+\alpha} \mathrm{d}t$, where the integral on the right-hand side converges as $m \to \infty$ because $2 - \alpha > 1$.

Now we prove that $\widetilde{R}(\alpha)$ converges. Pick $\varepsilon > 0$ s.t. $2 - \alpha - \varepsilon > 1$. One can always find such ε because $\alpha < 1$ by assumption. Because $\log t$ is slowly varying, there exists $T > 1$ s.t. $t^{-2+\alpha} \log t < t^{-2+\alpha+\varepsilon}$ for all $t > T$. Because $|B_2(\{1-t\})| \leq 1/6$ for all $t \geq 1$, we can write

$$|\widetilde{R}(\alpha)| < \frac{1}{12} \int_1^T t^{-2+\alpha} \left(\alpha(\alpha-1)\log t - 1 + 2\alpha\right) \mathrm{d}t + \frac{\alpha(\alpha+1)}{12} \int_T^m t^{-2+\alpha+\varepsilon} \mathrm{d}t$$

where the second integral converges as $m \to \infty$ because $2 - \alpha - \varepsilon > 1$. □

Proof of Lemma 2.17. By using Euler-Maclaurin formula up to the first order we obtain

$$i \sum_{k=1}^{i} k^{-\alpha} \quad = \quad i \left(\frac{i^{1-\alpha} - 1}{1-\alpha} + \frac{i^{-\alpha} + 1}{2} + \frac{B_2}{2!}(-\alpha)(i^{-\alpha-1} - 1) + R(\alpha) \right)$$

where $B_2 = 1/6$ is the third Bernoulli number, and $R(\alpha)$ is the remainder of the Euler-Maclaurin expansion given by (2.55). From Lemma 2.16 we know that $R(\alpha)$ converges, as $i \to \infty$. So, we can write

$$i \sum_{k=1}^{i} k^{-\alpha} = \frac{i^{2-\alpha}}{1-\alpha} + A_1(\alpha) i + \frac{i^{1-\alpha}}{2} - \frac{\alpha\, i^{-\alpha}}{12}$$

where $A_1(\cdot)$ is defined as in Lemma 2.17. Note that the second-order term is $O(i)$.

Similarly, we can write

$$\sum_{k=1}^{i} k^{1-\alpha} = \frac{i^{2-\alpha}}{2-\alpha} + \frac{i^{1-\alpha}}{2} + A_1(\alpha - 1) + \frac{(1-\alpha)\, i^{-\alpha}}{12}$$

Note that in this case the second-order term is $O(i^{1-\alpha})$.

Putting together these two terms we have $\sum_{k=1}^{i} (i-k)k^{-\alpha} = A_0(\alpha) i^{2-\alpha} + A_1(\alpha) i + O(1)$. where $A_0(\cdot)$ is defined as Lemma 2.17. Note that the terms of order $i^{1-\alpha}$ cancel out exactly. □

Proof of Lemma 2.18. The proof is similar to the proof of Lemma 2.17, and thus omitted. □

Appendix 2.D Sign Process

Taking the sign of a stochastic process can be thought of as an extreme form of discretization. Hence, to study the asymptotic properties of the sign process we can use the same technique outlined in Section 2.3.1 for general nonlinear transformations of Gaussian processes. By decomposing the sign transformation on the basis of Hermite polynomials we get the following

Proposition 2.20. *Let $\{X(t)\}_{t\in\mathbb{N}}$ be a stationary Gaussian process with autocovariance function given by Definition 2.1. Then, the autocovariance $\gamma_s(k)$ and the autocorrelation $\rho_s(k)$ of the sign process satisfy*

$$\gamma_s(k) = \rho_s(k) = \frac{2}{\pi}\arcsin\rho(k) \tag{2.61}$$

Therefore, also the sign transformation preserves the long memory property and the Hurst exponent. Moreover, if the autocorrelation ρ is small (for example if the lag k is large) we have

$$\gamma_s(k) = \rho_s(k) = \frac{2}{\pi}\left(\rho(k) + \frac{\rho^3(k)}{6}\right) + O(\rho^5(k)). \tag{2.62}$$

This expression has been obtained several times, as, for example, in the context of binary time series (see Keenan (1982)). Note that, trivially, when the discretization is obtained by taking the sign function the variance of the discretized process is $D_s = 1$.

All the results on the discretized process presented above hold true for the sign process as well, with $\left(\frac{\vartheta_2(0,e^{-1/2\chi})}{\sqrt{2\pi\chi}}\right)^2$ replaced by $\frac{2}{\pi D}$ and $g_3 = -\frac{1}{\sqrt{3\pi}}$. Numerical results are available from the authors by request.

Proof of Proposition 2.20. As the discretization, the sign transformation is an odd function, and therefore $g_j = 0$ when j is even. When j is odd the coefficients of the sign function in Hermite polynomials are

$$g_j = 2\int_0^\infty H_j(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = \frac{2^{j+1/2}\Gamma(j+1/2)}{\pi\sqrt{(2j+1)!}}.$$

By inserting these value in (2.10) we obtain the autocorrelation (and autocovariance) function of the sign of a Gaussian process

$$\gamma_s(k) = \rho_s(k) = \sum_{j=1}^{\infty} g_j^2\rho(k)^j = \frac{2}{\pi}\arcsin\rho(k)$$

□

	$m = n^{0.5}$		$m = n^{0.6}$		$m = n^{0.7}$		$m = n^{0.8}$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.4411		0.5097		0.5584		0.5917
Sign	0.6610	0.6606	0.6583	0.6584	0.6578	0.6570	0.6561	0.6563
	(0.0035)	0.8670	(0.0023)	0.7951	(0.0016)	0.7553	(0.0011)	0.7219
		0.4109		0.4828		0.5323		0.5507
$\chi = 0.1$	0.6350	0.6393	0.6308	0.6341	0.6272	0.6287	0.6223	0.6227
	(0.0035)	0.8339	(0.0023)	0.7646	(0.0015)	0.7166	(0.0011)	0.6884
		0.4375		0.5069		0.5708		0.6073
$\chi = 0.25$	0.6710	0.6757	0.6709	0.6726	0.6717	0.6739	0.6726	0.6730
	(0.0036)	0.8899	(0.0024)	0.8053	(0.0016)	0.7644	(0.0010)	0.7350
		0.4625		0.5287		0.5884		0.6266
$\chi = 0.5$	0.6809	0.6855	0.6815	0.6842	0.6861	0.6883	0.6898	0.6907
	(0.0035)	0.8894	(0.0023)	0.8138	(0.0015)	0.7745	(0.0010)	0.7555
		0.4641		0.5406		0.6050		0.6483
Continuous	0.6905	0.6922	0.6935	0.6955	0.7014	0.7034	0.7113	0.7118
	(0.0036)	0.9113	(0.0023)	0.8300	(0.0015)	0.7955	(0.0010)	0.7761
				$n = 2^{14}$				
		0.5773		0.6211		0.6336		0.6409
Sign	0.6814	0.6831	0.6768	0.6763	0.6688	0.6691	0.6606	0.6606
	(0.0016)	0.7772	(0.0009)	0.7331	(0.0005)	0.7019	(0.0003)	0.6816
		0.5594		0.6014		0.6102		0.6106
$\chi = 0.1$	0.6659	0.6691	0.6567	0.6575	0.6447	0.6448	0.6323	0.6327
	(0.0016)	0.7580	(0.0009)	0.7105	(0.0006)	0.6791	(0.0003)	0.6536
		0.5811		0.6267		0.6469		0.6544
$\chi = 0.25$	0.6880	0.6901	0.6849	0.6853	0.6802	0.6798	0.6752	0.6753
	(0.0016)	0.7816	(0.0009)	0.7396	(0.0006)	0.7156	(0.0003)	0.6961
		0.5874		0.6318		0.6561		0.6687
$\chi = 0.5$	0.6928	0.6930	0.6923	0.6933	0.6900	0.6895	0.6887	0.6882
	(0.0016)	0.7897	(0.0009)	0.7469	(0.0005)	0.7247	(0.0003)	0.7094
		0.5941		0.6406		0.6675		0.6831
Continuous	0.6988	0.6993	0.7000	0.7005	0.7008	0.7006	0.7043	0.7044
	(0.0016)	0.7927	(0.0009)	0.7555	(0.0006)	0.7344	(0.0003)	0.7253

TABLE 2.4.1: Estimation of the Hurst exponent of a fGn with $H = 0.7$ and its discretization with the sign function and with $\chi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The estimation of the Hurst exponent is obtained with the Local Whittle estimator by maximizing the objective function (2.21) (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

	$m = n^{0.5}$		$m = n^{0.6}$		$m = n^{0.7}$		$m = n^{0.8}$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.5706		0.6519		0.6872		0.7056
Sign	0.8109	0.8179	0.8018	0.8029	0.7903	0.7924	0.7776	0.7783
	(0.0036)	1.0294	(0.0023)	0.9474	(0.0015)	0.8801	(0.0011)	0.8457
		0.5228		0.5928		0.6281		0.6362
$\chi = 0.1$	0.7671	0.7681	0.7525	0.7530	0.7397	0.7394	0.7268	0.7268
	(0.0038)	1.0013	(0.0026)	0.9072	(0.0018)	0.8495	(0.0015)	0.8122
		0.5797		0.6515		0.7118		0.7312
$\chi = 0.25$	0.8271	0.8354	0.8183	0.8220	0.8092	0.8091	0.7984	0.7993
	(0.0036)	1.0488	(0.0024)	0.9541	(0.0015)	0.9043	(0.0011)	0.8639
		0.6008		0.6737		0.7365		0.7612
$\chi = 0.5$	0.8350	0.8423	0.8338	0.8389	0.8324	0.8332	0.8288	0.8302
	(0.0037)	1.0538	(0.0024)	0.9739	(0.0015)	0.9228	(0.0011)	0.8919
		0.6041		0.6930		0.7612		0.8004
Continuous	0.8468	0.8554	0.8507	0.8538	0.8576	0.8591	0.8694	0.8698
	(0.0036)	1.0568	(0.0024)	0.9952	(0.0015)	0.9484	(0.0011)	0.9338
				$n = 2^{14}$				
		0.7358		0.7632		0.7720		0.7675
Sign	0.8331	0.8353	0.8242	0.8242	0.8077	0.8086	0.7887	0.7886
	(0.0015)	0.9206	(0.0009)	0.8783	(0.0006)	0.8398	(0.0003)	0.8098
		0.7153		0.7350		0.7317		0.7193
$\chi = 0.1$	0.8143	0.8131	0.7971	0.7965	0.7735	0.7727	0.7484	0.7484
	(0.0016)	0.9137	(0.0010)	0.8563	(0.0007)	0.8156	(0.0005)	0.7786
		0.7447		0.7797		0.7898		0.7878
$\chi = 0.25$	0.8425	0.8430	0.8374	0.8379	0.8252	0.8258	0.8092	0.8094
	(0.0015)	0.9338	(0.0009)	0.8900	(0.0006)	0.8584	(0.0003)	0.8302
		0.7486		0.7840		0.8028		0.8117
$\chi = 0.5$	0.8464	0.8478	0.8450	0.8449	0.8382	0.8389	0.8309	0.8308
	(0.0015)	0.9414	(0.0009)	0.8994	(0.0006)	0.8703	(0.0003)	0.8514
		0.7497		0.7916		0.8158		0.8368
Continuous	0.8503	0.8506	0.8518	0.8514	0.8516	0.8519	0.8565	0.8569
	(0.0015)	0.9427	(0.0009)	0.9057	(0.0006)	0.8856	(0.0003)	0.8770

TABLE 2.4.2: Estimation of the Hurst exponent of a fGn with $H = 0.85$ and its discretization with the sign function and with $\chi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The estimation of the Hurst exponent is obtained with the Local Whittle estimator by maximizing the objective function (2.21) (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

	$q = 0.25$		$q = 0.5$		$q = 0.75$		$q = 1$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.3676		0.4771		0.5313		0.5807
Sign	0.6482	0.6430	0.6505	0.6469	0.6493	0.6462	0.6562	0.6547
	(0.0052)	0.9836	(0.0029)	0.8415	(0.0019)	0.7635	(0.0012)	0.7299
		0.3308		0.4416		0.5026		0.5471
$\chi = 0.1$	0.6177	0.6133	0.6168	0.6147	0.6150	0.6147	0.6224	0.6240
	(0.0050)	0.9541	(0.0029)	0.8093	(0.0018)	0.7316	(0.0012)	0.6957
		0.3667		0.4992		0.5538		0.5982
$\chi = 0.25$	0.6610	0.6590	0.6628	0.6611	0.6631	0.6628	0.6717	0.6729
	(0.0051)	0.9773	(0.0029)	0.8537	(0.0019)	0.7770	(0.0012)	0.7448
		0.3632		0.4979		0.5625		0.6145
$\chi = 0.5$	0.6699	0.6605	0.6720	0.6720	0.6749	0.6753	0.6862	0.6861
	(0.0052)	1.0018	(0.0030)	0.8624	(0.0019)	0.7863	(0.0012)	0.7623
		0.3702		0.5098		0.5751		0.6272
Continuous	0.6773	0.6719	0.6840	0.6803	0.6894	0.6880	0.7040	0.7035
	(0.0052)	1.0020	(0.0030)	0.8812	(0.0019)	0.8071	(0.0012)	0.7807
				$n = 2^{14}$				
		0.4577		0.5712		0.6102		0.6295
Sign	0.6643	0.6614	0.6718	0.6723	0.6691	0.6697	0.6664	0.6670
	(0.0035)	0.8934	(0.0016)	0.7691	(0.0009)	0.7242	(0.0006)	0.7010
		0.4584		0.5537		0.5898		0.6043
$\chi = 0.1$	0.6554	0.6519	0.6552	0.6577	0.6480	0.6486	0.6418	0.6425
	(0.0034)	0.8706	(0.0016)	0.7571	(0.0009)	0.7069	(0.0006)	0.6801
		0.4643		0.5745		0.6189		0.6380
$\chi = 0.25$	0.6682	0.6674	0.6777	0.6799	0.6779	0.6791	0.6779	0.6781
	(0.0035)	0.8999	(0.0017)	0.7819	(0.0010)	0.7371	(0.0006)	0.7169
		0.4653		0.5806		0.6250		0.6502
$\chi = 0.5$	0.6714	0.6634	0.6836	0.6849	0.6859	0.6861	0.6885	0.6891
	(0.0035)	0.9043	(0.0017)	0.7870	(0.0010)	0.7442	(0.0006)	0.7268
		0.4663		0.5854		0.6326		0.6616
Continuous	0.6766	0.6756	0.6893	0.6908	0.6941	0.6954	0.7004	0.7015
	(0.0035)	0.9047	(0.0017)	0.7917	(0.0010)	0.7546	(0.0006)	0.7400

TABLE 2.4.3: Estimation of the Hurst exponent of a fGn with $H = 0.7$ and its discretization with the sign function and with $\chi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The estimation of the Hurst exponent is obtained with the Detrended Fluctuation Analysis by performing the least square regression over a q fraction of the largest values of $\log_{10}[m]$ (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

	$q = 0.25$		$q = 0.5$		$q = 0.75$		$q = 1$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.4475		0.5755		0.6327		0.6798
Sign	0.7796	0.7647	0.7782	0.7795	0.7737	0.7735	0.7724	0.7727
	(0.0061)	1.1787	(0.0034)	0.9876	(0.0022)	0.9045	(0.0015)	0.8574
		0.4278		0.5232		0.5866		0.6187
$\chi = 0.1$	0.7345	0.7205	0.7306	0.7288	0.7237	0.7224	0.7220	0.7227
	(0.0059)	1.1161	(0.0033)	0.9401	(0.0023)	0.8700	(0.0016)	0.8298
		0.4557		0.5888		0.6583		0.7038
$\chi = 0.25$	0.7923	0.7749	0.7954	0.7979	0.7929	0.7944	0.7923	0.7931
	(0.0061)	1.1848	(0.0033)	1.0102	(0.0021)	0.9231	(0.0014)	0.8764
		0.4739		0.6140		0.6785		0.7253
$\chi = 0.5$	0.8058	0.7939	0.8106	0.8114	0.8127	0.8147	0.8176	0.8200
	(0.0061)	1.2039	(0.0033)	1.0145	(0.0021)	0.9344	(0.0014)	0.8996
		0.4815		0.6262		0.7008		0.7522
Continuous	0.8140	0.7936	0.8241	0.8248	0.8330	0.8346	0.8485	0.8502
	(0.0062)	1.2178	(0.0033)	1.0272	(0.0021)	0.9580	(0.0014)	0.9341
				$n = 2^{14}$				
		0.5712		0.7131		0.7557		0.7640
Sign	0.8231	0.8201	0.8227	0.8222	0.8150	0.8143	0.8028	0.8025
	(0.0039)	1.0578	(0.0017)	0.9289	(0.0010)	0.8758	(0.0006)	0.8410
		0.5760		0.6924		0.7219		0.7264
$\chi = 0.1$	0.8071	0.8076	0.8018	0.8019	0.7874	0.7869	0.7695	0.7697
	(0.0039)	1.0472	(0.0018)	0.9184	(0.0010)	0.8533	(0.0007)	0.8124
		0.5814		0.7192		0.7657		0.7787
$\chi = 0.25$	0.8271	0.8249	0.8304	0.8299	0.8272	0.8269	0.8185	0.8183
	(0.0039)	1.0749	(0.0018)	0.9410	(0.0010)	0.8891	(0.0006)	0.8565
		0.5871		0.7265		0.7756		0.7929
$\chi = 0.5$	0.8296	0.8301	0.8350	0.8348	0.8357	0.8346	0.8333	0.8329
	(0.0039)	1.0671	(0.0018)	0.9472	(0.0010)	0.8984	(0.0006)	0.8714
		0.5849		0.7303		0.7814		0.8103
Continuous	0.8316	0.8318	0.8388	0.8390	0.8436	0.8428	0.8499	0.8497
	(0.0039)	1.0732	(0.0018)	0.9516	(0.0010)	0.9048	(0.0006)	0.8884

TABLE 2.4.4: Estimation of the Hurst exponent of a fGn with $H = 0.85$ and its discretization with the sign function and with $\chi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The estimation of the Hurst exponent is obtained with the Detrended Fluctuation Analysis by performing the least square regression over a q fraction of the largest values of $\log_{10}[m]$ (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

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Chapter 3

Statistical properties and covariance estimation of integrated processes in the presence of round-off error and serial correlation[1]

3.1 Introduction

Many real phenomena described by a real valued variable are observed with a round-off error. This might be due to a resolution limit of the observing instrument (and in this case round-off can be seen as a measurement error) or to the fact that an underlying real valued process can manifest itself only as a discrete valued process. A typical case of the latter is the presence of a finite tick size for the prices of financial assets.

Round-off errors change the statistical properties of the original stochastic process. Moreover, the effect of rounding is different depending on whether the underlying process is stationary or integrated ($I(1)$). In the former case, recently we studied the the effect of round-off error on long memory processes, considering both analytical properties and finite sample biases of Hurst exponent estimators (La Spada and Lillo, 2014). However, in many cases of interest the rounding acts on the integrated process. Consider again the case of financial asset prices. As it is well known, prices are well described by a unit root stochastic process, and the tick size acts as a round-off error on the price and not on the returns. Therefore, the knowledge of how

[1] This chapter is a slightly modified version of a working paper co-authored with Fabrizio Lillo.

rounding changes the properties of an integrated process is useful in high frequency financial econometrics.

Few exact results exist on the effect of round-off error on integrated stochastic processes. Delattre and Jacod (1997), for example, proved a central limit theorem on a Brownian motion with round-off error sampled at discrete times. In the context of financial time series, Harris (1990) considered the case of Brownian motion and derived expressions for the effect of round-off errors on variance and autocovariance. This paper extends the literature by considering the effect of round-off error on integrated processes with correlated increments. Specifically, we study the statistical properties of the realized increments, i.e. the increments of the underlying process when observed with round-off errors. We study the variance and the covariance, and we consider in detail the limit of small and large round-off errors, while the previous literature focused only on the limit of small round-off errors. We then use these results to construct estimators of both quantities. We show that our estimator of the underlying variance is strongly consistent, while our estimator of the lag-1 correlation converges almost surely to a pseudo-true parameter (coming from a misspecified model), which is close to the true underlying lag-1 correlation if the underlying lag-1 correlation is small. Numerical simulations show that the variance estimator is unbiased, and the bias for the lag-1 correlation estimator is small if the underlying lag-1 correlation is sufficiently small, as predicted. Moreover, numerical simulations indicate that our estimators are also asymptotically normal. Finally, we consider long memory processes and show that the local Whittle (LW) estimator gives biased estimates of the Hurst exponent when applied to realized increments. We believe that our results are potentially of interest in the literature on realized volatility and covariance estimation for high-frequency data in the presence of round-off error and long memory.

The paper is organized as follows. In Section 3.2 we introduce the notation and the problem setting. Section 3.3 presents our main theoretical results for the variance and autocovariance of the process in the presence of round-off errors. Moreover, in this section we introduce the estimators of the variance and lag-1 autocovariance of the underlying process and study their properties. In Section 3.4 we present numerical results testing our theoretical predictions and the performance of the estimators. Moreover, we study numerically the effect of rounding on the estimation with the Whittle estimator of the Hurst exponent of a long memory process. Finally, in Section 3.5 we present our conclusions and propose some applications of our results. For the benefit of the reader we have gathered all the proofs in Appendix A, while in Appendix B we present the numerical results for the Detrended Fluctuation Analysis estimation of the Hurst exponent on long memory processes with rounding.

3.2 Problem setting

Let $\{v_t\}_{t\in\mathbb{Z}}$ be an $I(1)$ Gaussian process. v_t can also be regarded as a continuous time process observed at equally spaced discrete times, as for example a Brownian motion. Let us indicate with $\{r_t\}_{t\in\mathbb{Z}}$ its weakly stationary Gaussian increments, so that

$$v_t = \sum_{s=t_0}^{t} r_t \qquad \text{with } t_0 \in \mathbb{Z}.$$

The process r_t has mean μ, variance σ^2, and autocovariance function γ_s with $s \in \mathbb{N}$. We allow γ_s to satisfy either one of the following two mutually exclusive conditions:

$$\sum_{s=1}^{\infty} |\gamma_s| < \infty \tag{3.1}$$

$$\gamma_s = L(s)s^{2H-2} \text{ for } s \geq 1, \tag{3.2}$$

where $L(\cdot)$ is slowly varying at infinity, and $H \in [0.5, 1)$ is the Hurst exponent. In the following, we refer to processes satisfying (3.1) as *short memory* processes, and processes satisfying (3.2) as *long memory* processes. For example, the Brownian motion and ARIMA models with Gaussian innovations belong to the first class, while the fractional Brownian motion (fBm) and fARIMA models with Gaussian innovations belong to the second class.[2] Hereafter, without loss of generality, we will assume $\mu = 0$.

Suppose that v_t is observed with round-off error $\alpha > 0$, so that the observed process is $v_t^{(\alpha)} = \alpha \cdot round(v_t/\alpha) \;\; \forall\, t \in \mathbb{Z}$, where

$$round(x) = \{k \in \mathbb{Z} : -0.5 \leq x - k < 0.5\}$$

is the rounding function. Hereafter, $v_t^{(\alpha)}$ will be referred to as the *realized integrated process*. Let $r_t^{(\alpha)} = p_t^{(\alpha)} - p_{(t-1)}^{(\alpha)} \;\; \forall\, t \in \mathbb{Z}$. We refer to $r_t^{(\alpha)}$ and r_t as the *realized increments* and the *underlying increments*, respectively.

Finally, we introduce the following adimensional variable

$$\xi = \frac{\sigma^2}{\alpha^2}.$$

[2] A fBm with $H = 0.5$ has by definition *i.i.d.* Gaussian increments and therefore, strictly speaking, belongs to both classes.

The interpretation of the adimensional variable ξ is the following: as $\xi \to 0$, the round-off error (α) becomes more significant relatively to the diffusion of the underlying process; *vice versa*, as $\xi \to \infty$, the round-off error becomes negligible. Hereafter, we will refer to ξ as the *discretization parameter*.

REMARK 1. If the realized process is observed every τ steps, then we can define $\xi = \sigma_\tau^2/\alpha^2$, where $\sigma_\tau^2 := var(\sum_{t=1}^{\tau} r_t)$ is the variance of the underlying increment over τ steps. For a short memory Gaussian process, $\sigma_\tau^2 \approx \sigma^2\tau$ for large τ, for a fBm $\sigma_\tau^2 = \sigma^2\tau^{2H}\ \forall\,\tau$, while for fARIMA $\sigma_\tau^2 \approx \sigma^2\tau^{2H}$ for large τ. In practical applications, τ represents the time scale of interest at which the process is observed.

In this paper, we study the statistical properties of the *realized increments*, and we compare them to those of the *underlying increments*. We give analytical results for both the distributional and dynamical properties of the realized increments. We derive asymptotic behavior in the limit of both large and small round-off errors. We also propose estimators for the variance and correlations of the underlying increment process and show that they are strongly consistent. We test our theoretical results numerically. Finally, we consider the performance of two well known estimators of the Hurst exponent when applied to the realized increments of a long memory underlying process.

3.3 Theoretical Results

3.3.1 Distributional Properties

For any random variable x and real number $\alpha > 0$, let

$$\widetilde{x} = x - round(x/\alpha)\alpha\,,$$
$$\overline{x} = x - \lfloor x/\alpha \rfloor \alpha\,,$$

where $\lfloor x \rfloor$ is the integer part of x. $\widetilde{x}$ and $\overline{x}$ are obviously functions of α, but for notational simplicity we omit to make that dependence explicit. It is straightforward to show that

$$v_t^{(\alpha)} = v_{t-1}^{(\alpha)} + \alpha \cdot \left\lfloor \frac{r_t}{\alpha} \right\rfloor + \alpha \cdot round\left(\frac{\widetilde{v}_{t-1} + \overline{r}_t}{\alpha} \right).$$

The following lemma holds without the assumption of Gaussianity of the underlying process.

Lemma 3.1. *Let $f(x;r_t)$ be the density of v_{t-1} conditional on r_t. For any $N \in \mathbb{N}$, if f is of class $\mathcal{C}^N$ with respect to its first argument, there exists a constant $C_N > 0$ such that, for all $\alpha > 0$ and all Borel sets I in $[0,\alpha]$ of Lebsegue measure $l(I)$,*

$$\left| \frac{\mathbb{P}\left(\overline{v}_{t-1} \in I | r_t\right)}{l(I)} - 1 \right| \leq C_N \alpha^N \int_{\mathbb{R}} \left| \frac{\partial^N f(x;r_t)}{\partial x^N} \right| \mathrm{d}x \,.$$

REMARK 2. Lemma 3.1 gives the error coming from approximating the fractional part of $v_{t-1}|r_t$ with a uniformly distributed random variable independent of r_t. If the density of $v_{t-1}|r_t$ is known, the error term can be derived explicitly. For example, for the Gaussian processes we consider here, it is straightforward to show that for any $N \geq 1$, the relative error term goes to zero as $\xi^{-N/2}(t-t_0)^{-NH}$, as $t_0 \to -\infty$, where $H = 0.5$ if the process is short memory. Hence, if the process started in the infinite past (i.e., $t_0 = -\infty$), $\overline{v}_{t-1}|r_t$ is uniformly distributed over $[0,\alpha)$. Similarly, $\widetilde{v}_{t-1}|r_t$ is uniformly distributed over $[-\alpha/2, \alpha/2)$.

REMARK 3. Lemma 3.1 is a direct application of Lemma 6.1 in Delattre and Jacod (1997). The only difference with respect to their framework is that we allow for correlation of the underlying series. Namely, we work with the density of v_{t-1} conditional on r_t instead of the marginal density of v_{t-1}.

Lemma 3.2. *For any $N \in \mathbb{N}$, as $t_0 \to -\infty$, the probability mass function (pmf) of $r_t^{(\alpha)}$ satisfies*

$$\begin{aligned} p_\alpha(k) = \mathbb{P}\left(r_t^{(\alpha)} = k\alpha\right) = \frac{1}{2}\Bigg[\sqrt{\frac{2\xi}{\pi}} \left(e^{-\frac{(k-1)^2}{2\xi}} - 2e^{-\frac{k^2}{2\xi}} + e^{-\frac{(k+1)^2}{2\xi}} \right) + \\ \left((k-1) erf\left(\frac{k-1}{\sqrt{2\xi}}\right) - 2\,k\, erf\left(\frac{k}{\sqrt{2\xi}}\right) + (k+1) erf\left(\frac{k+1}{\sqrt{2\xi}}\right) \right) \Bigg] + \\ + O\left(\xi^{-N/2} |t_0|^{-NH}\right) \end{aligned}$$

where $erf(\cdot)$ is the error function, and $H = 0.5$ if the underlying process is short memory.

REMARK 4. From Lemma 3.2 it follows that if the underlying process v_t started in the infinite past (i.e., in the limit $t_0 = -\infty$), the distributional properties of the realized increments do not depend on time. The probability that a realized increment is zero is $p_\alpha(0) = \sqrt{2\xi/\pi}(e^{-1/2\xi} - 1) + erf(1/\sqrt{2\xi})$, which is a strictly decreasing function of ξ (see the left panel of Figure 3.1). If the round-off error α is known, the strict monotonicity of $p_\alpha(0)$ can be used to infer the value of the underlying variance σ^2 from the fraction of zero increments in a realized time series.

Henceforth, $a(x) \sim b(x)$ for large (small) x means that $a(x)/b(x) \to 1$ as $x \to \infty$ ($x \to 0$).

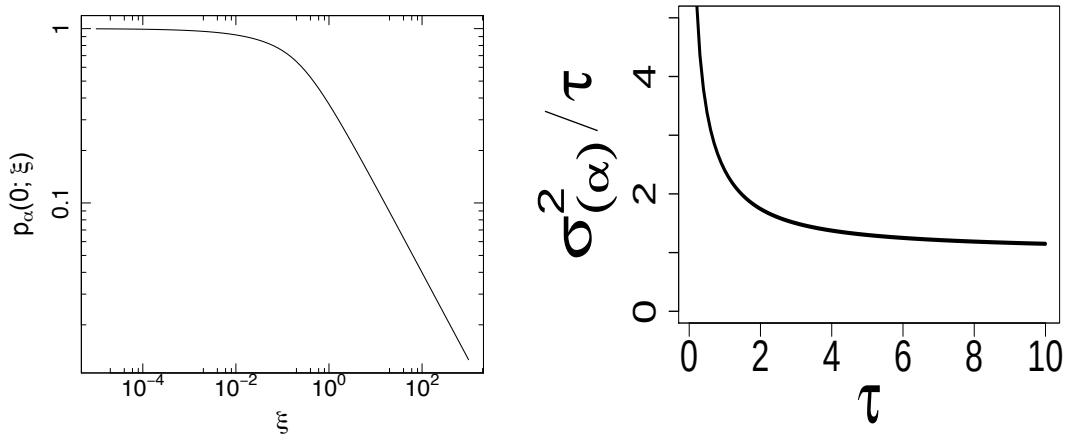


FIGURE 3.1: Left panel: $p_\alpha(0)$ as function of the discretization parameter ξ on a log-log scale. Right panel: rescaled variance, $\sigma^2_{(\alpha)}/\tau$, versus the number of time steps τ of an Gaussian process with *i.i.d.* increments observed with a round-off error α. In financial econometrics this is called the signature plot.

Proposition 3.3. *Suppose the underlying process* $\{v_t\}$ *started in the infinite past (i.e.,* $t_0 = -\infty$*). Let* σ^2_α *and* κ_α *be the variance and the kurtosis of* $\{r_t^{(\alpha)}\}$*, respectively. Then,*

$$\sigma^2_{(\alpha)} = \left(1 + \frac{1}{6\xi} - \frac{1}{\pi^2\xi}\sum_{n\geq 1} n^{-2} e^{-2\pi^2\xi n^2}\right)\sigma^2 \tag{3.3}$$

Moreover, in the limit of large and small ξ*, we have*

$$\sigma^2_\alpha(\xi) \sim \sqrt{\frac{2}{\pi\xi}}\,\sigma^2 \qquad \textit{and} \qquad \kappa_\alpha(\xi) \sim \sqrt{\frac{\pi}{2\xi}} \qquad \textit{for small} \quad \xi;$$
$$\sigma^2_\alpha(\xi) \sim \left(1 + \frac{1}{6\xi}\right)\sigma^2 \quad \textit{and} \qquad \kappa_\alpha(\xi) \sim 3\left(1 - \frac{1}{5\,(1+6\xi)^2}\right) \qquad \textit{for large} \quad \xi;$$

REMARK 5. From Proposition 3.3 it follows that: (1) as $\xi \to 0$, the variance and the kurtosis of the realized increments diverge as $\xi^{-1/2}$; (2) as $\xi \to \infty$, the variance and the kurtosis of the realized increments converge to those of the underlying increments, and the relative error is positive $O\left(\xi^{-1}\right)$ and negative $O\left(\xi^{-2}\right)$, respectively. The limiting behavior of the variance for large ξ is the same as the one obtained by Harris (1990). However, neither Harris (1990) nor Delattre and Jacod (1997) considered the limit of small round-off errors or the kurtosis.

REMARK 6. Let us consider a Wiener process and assume that we observe it with a round-off error. The scaling of the variance with the number τ of time steps at which the process is

observed (see REMARK 1) allows us to use (3.3) to compute how the variance of the realized increments behaves with τ. The right panel of Fig. 3.1 shows the variance divided by τ versus τ. For large τ, the process becomes diffusive, while for small τ the variance per unit time diverges. This is similar to the *signature plot* observed empirically for high frequency financial returns, and it is due to the negative autocorrelation induced by the round-off.

3.3.2 Autocovariance Properties

The following lemma holds without the assumption of Gaussianity of the underlying process.

Lemma 3.4. *Let $f(x_1, x_2; \mathbf{r})$ be the joint density of (v_{t+s-1}, v_{t-1}) conditional on (r_{t+s}, r_t). For any $N \in \mathbb{N}$, if f is of class $\mathcal{C}^N$ on $\mathbb{R}^2$ with respct to its first two arguments, there exists a constant $C_N > 0$ such that, for all $\alpha > 0$ and all Borel sets I_1 and I_2 in $[0, \alpha]$ of Lebsegue measure $l(I_1)$ and $l(I_2)$,*

$$\left| \frac{\mathbb{P}\left(\overline{v}_{t+s-1} \in I_1, \overline{v}_{t-1} \in I_2 | r_{t+s}, r_t\right)}{l(I_1) l(I_2)} - 1 \right| \leq C_N \alpha^{N+2} \sum_{|\beta|=N} \sum_{j_1, j_2 = -\infty}^{\infty} \int_0^1 \left| \partial^{(\beta)} f(\alpha \boldsymbol{j} + u\alpha \mathbf{e}; \mathbf{r}) \right| \mathrm{d}u$$

where $\partial^{(\beta)}$ represents the partial derivative of degree $|\beta|$ with respect to the first two arguments, with β being a 2-dimensional multi-index, $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2$, and $\mathbf{e} = (1, 1) \in \mathbb{Z}^2$.

REMARK 7. Lemma 3.4 gives the error coming from approximating the fractional parts of v_{t-1} and v_{t+s-1} conditional on (r_t, r_{t+s}) with two independent, uniformly distributed random variables, which are also independent of (r_t, r_{t+s}). For the Gaussian processes we consider here, it is relatively straightforward (Lemma 3.12) to show that for any $N \geq 1$, the relative error term goes to zero as $\xi^{-N/2} \left(|t_0|^{-NH} + s^{-NH} \right)$, as $t_0 \to -\infty$ and $s \to \infty$, where $H = 0.5$ if the process is short memory.

By using the series expansion of the bivariate Gaussian density function in Hermite polynomials (Barrett and Lampard, 1955), we can prove the following.

Lemma 3.5. *Let ρ_s be the autocorrelation function of $\{r_t\}$. For any $N \in \mathbb{N}$, as $t_0 \to -\infty$ and for large s, the joint pmf of $(r_t^{(\alpha)}, r_{t+s}^{(\alpha)})$ satisfies*

$$\mathbb{P}\left(r_t^{(\alpha)} = k\alpha, r_{t+s}^{(\alpha)} = l\alpha \right) = p_\alpha(k) p_\alpha(l) + \rho_s p_\alpha^{(1)}(k) p_\alpha^{(1)}(l) + O\left(\rho_s^2\right) + O\left(|t_0|^{-NH} + s^{-NH} \right),$$

where $p_\alpha(k)$ is given by Lemma 3.2,

$$p_\alpha^{(1)}(k) = \frac{\sqrt{\xi}}{2} \left[erf\left(\frac{k-1}{\sqrt{2\xi}} \right) - 2erf\left(\frac{k}{\sqrt{2\xi}} \right) + erf\left(\frac{k+1}{\sqrt{2\xi}} \right) \right],$$

and $H = 0.5$ if the process is short-memory.

REMARK 8. From Lemma 3.2 and 3.5, it follows that $\{r_t^{(\alpha)}\}$ is weakly stationary in the limit $t_0 \to -\infty$, and the error term decays faster than any power law in $|t_0|$, i.e., it is $O(|t_0|^{-NH})$ for any $N \geq 1$. Hereafter, when we say that the underlying process v_t started in the infinite past, it means that we are taking the limit $t_0 \to -\infty$, so that the error term $O(|t_0|^{-NH})$ can be neglected and $\{r_t^{(\alpha)}\}$ can be considered as weakly stationary.

REMARK 9. The error term coming from approximating the joint distribution of the fractional parts of v_{t-1} and v_{t+s-1} conditional on (r_t, r_{t+s}) with two independent uniform distributions goes to zero faster than any power law in s (specifically, $O(s^{-NH})$ for any $N \geq 1$). Hence, as we can see from the next proposition, if the autocovariance of the underlying process decays as a power law in s, the autocovariance of the discretized process is asymptotically equal to that of the underlying one. On the other hand, if the autocovariance of the underlying process goes to zero faster than any power law, the error term might dominate, and the autocovariance of $r_{(\alpha)}$ cannot be directly related to that of r.

Proposition 3.6. *Suppose that the underlying process $\{v_t\}$ started in the infinite past (i.e., $t_0 = -\infty$). Let ρ_s be the autocorrelation function of $\{r_t\}$. For any $N\mathbb{N}$, the autocovariance of $\{r_t^{(\alpha)}\}$ satisfies*

$$cov(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}) = cov(r_t, r_{t+s}) + O(\rho_s^2) + \xi^{-N/2} O(s^{-NH}) \qquad \text{for large } s.$$

Therefore, if $\rho_s \sim s^{-\beta}$ with $\beta > 0$ as $s \to \infty$,

i) $cov(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}) \sim cov(r_t, r_{t+s})$ for large s,

ii) and the autocorrelation of $\{r_t^{(\alpha)}\}$ satisfies, for large s,

$$cor(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}) \sim \sqrt{\frac{\pi\xi}{2}}\rho_s \quad \text{for small } \xi,$$
$$cor(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}) \sim \left(1 - \frac{1}{6\xi}\right)\rho_s \quad \text{for large } \xi.$$

REMARK 10. If the slowly varying function $L(\cdot)$ in (3.2) belongs to the Zygmund class, from Proposition 3.6 it follows that, for small frequencies, the spectral densities of realized increments is (approximately) equal to that of the underlying increments. (See Lemma 3.3 (a) in Palma (2007).)

REMARK 11. From Proposition 3.6 it follows that if the underlying increment process is long-memory, so is the realized increment process, with the same Hurst exponent. Therefore, in

principle, one could use the realized series to estimate the Hurst exponent of the underlying one. However, our numerical simulations (see below) show that very popular and well-known estimators of the Hurst exponent, such as the local Whittle estimator and the Detrended Fluctuation Analysis, are negatively biased in finite samples when applied to the realized increments. This negative bias comes from the short-lag behavior of the autocovariance of the realized increments, or, equivalently, from the high-frequency behavior of their spectral density. Proposition 3.6 provides only an asymptotic result in the limit of large lags. When the number of lags is small, the result of Proposition 3.6 is not valid, and the autocovariance of the realized increments can differ significantly from that of the underlying increments. In fact, from our numerical simulations we observe that the autocovariance of the realized increments is significantly smaller than that of the underlying increments at small lags. (See Figure 3.3.)

The following proposition provides an explicit formula for the joint pmf of $(r_t^{(\alpha)}, r_{t+1}^{(\alpha)})$ in the limit $t_0 \to -\infty$, which can be used to calculate the lag-1 autocovariance of the realized increments.

Proposition 3.7. *Suppose that the underlying process $\{v_t\}$ started in the infinite past (i.e., $t_0 = -\infty$). The joint pmf of $(r_t^{(\alpha)}, r_{t+1}^{(\alpha)})$ satisfies*

$$\mathbb{P}\left(r_t^{(\alpha)} = k\alpha, r_{t+1}^{(\alpha)} = l\alpha\right) = p_\alpha^{(2)}(k,l;\xi) + \rho_1 p_\alpha^{(3)}(k,l;\xi) + O(\rho_1^2), \tag{3.4}$$

where ρ_1 is the one-lag autocorrelation of the underlying increments $\{r_t\}$,

$$\begin{aligned} p_\alpha^{(3)}(k,l;\xi) = \frac{1}{4}\sqrt{\frac{\xi}{\pi}} \Bigg\{ \Bigg[\left(e^{-\frac{(k+l+1)^2}{4\xi}} + e^{-\frac{(k+l-1)^2}{4\xi}} \right) \left(erf\left(\frac{(k-l+1)}{2\sqrt{\xi}} \right) + erf\left(\frac{(l-k+1)}{2\sqrt{\xi}} \right) \right) \Bigg] + \\ - e^{-\frac{(k+l)^2}{4\xi}} \left(erf\left(\frac{(k-l+2)}{2\sqrt{\xi}} \right) + erf\left(\frac{(l-k+2)}{2\sqrt{\xi}} \right) \right) \Bigg\}, \end{aligned}$$

and

$$\begin{aligned} p_\alpha^{(2)}(k,l;\xi) = & \frac{1}{4} \Bigg\{ f(k,l) + f(l,k) + erf\left(\frac{k-1}{\sqrt{2\xi}} \right) \left[l\, erf\left(\frac{l}{\sqrt{2\xi}} \right) - (l+1)\, erf\left(\frac{l+1}{\sqrt{2\xi}} \right) + 1 \right] + \\ & + erf\left(\frac{k}{\sqrt{2\xi}} \right) \left[(l+1)\, erf\left(\frac{l+1}{\sqrt{2\xi}} \right) - (l-1)\, erf\left(\frac{l-1}{\sqrt{2\xi}} \right) - 2 \right] + \\ & + erf\left(\frac{k+1}{\sqrt{2\xi}} \right) \left[(l-1)\, erf\left(\frac{l-1}{\sqrt{2\xi}} \right) - l\, erf\left(\frac{l}{\sqrt{2\xi}} \right) + 1 \right] \Bigg\} + R(k,l), \end{aligned}$$

with

$$f(k,l) = \sqrt{\frac{\xi}{\pi}}\left\{\sqrt{2}\left[e^{-\frac{(k+1)^2}{2\xi}}\left[erf\left(\frac{l}{\sqrt{2\xi}}\right) - erf\left(\frac{l-1}{\sqrt{2\xi}}\right)\right] - e^{-\frac{k^2}{2\xi}}\left[erf\left(\frac{l+1}{\sqrt{2\xi}}\right) - erf\left(\frac{l-1}{\sqrt{2\xi}}\right)\right] + \right.\right.$$
$$+e^{-\frac{(k-1)^2}{2\xi}}\left[erf\left(\frac{l+1}{\sqrt{2\xi}}\right) - erf\left(\frac{l}{\sqrt{2\xi}}\right)\right]\Bigg] + e^{-\frac{(k+l-1)^2}{4\xi}}\left[erf\left(\frac{k-l+1}{2\sqrt{\xi}}\right) + erf\left(\frac{l-k+1}{2\sqrt{\xi}}\right)\right]$$
$$+e^{-\frac{(k+l+1)^2}{4\xi}}\left[erf\left(\frac{k-l+1}{2\sqrt{\xi}}\right) + erf\left(\frac{l-k+1}{2\sqrt{\xi}}\right)\right] +$$
$$\left.-e^{-\frac{(k+l)^2}{4\xi}}\left[erf\left(\frac{k-l+2}{2\sqrt{\xi}}\right) + erf\left(\frac{l-k+2}{2\sqrt{\xi}}\right)\right]\right\}$$

and

$$R(k,l) = \int_{k-1}^{k} \frac{e^{-\frac{x^2}{2\xi}}}{\sqrt{2\pi\xi}}\left((k+l)\Phi\left(\frac{x-(k+l)}{\sqrt{\xi}}\right) - (k+l-1)\Phi\left(\frac{x-(k+l-1)}{\sqrt{\xi}}\right)\right)dx$$
$$+\int_{k}^{k+1} \frac{e^{-\frac{x^2}{2\xi}}}{\sqrt{2\pi\xi}}\left((k+l)\Phi\left(\frac{x-(k+l)}{\sqrt{\xi}}\right) - (k+l+1)\Phi\left(\frac{x-(k+l+1)}{\sqrt{\xi}}\right)\right)dx,$$

$\Phi(\cdot)$ *being the cumulative distribution function (cdf) of the standard Normal distribution.*

Unfortunately, we are not able to derive analytically a closed-form expression for $R(k,l)$, and therefore, for empirical purposes, we have to calculate it numerically.

REMARK 12. If the underlying increment process is long memory, the joint pmf of $(r_t^{(\alpha)}, r_{t+1}^{(\alpha)})$ depends on the Hurst exponent H through ρ_1. Proposition 3.7 provides an asymptotic result for small ρ_1. For a reliable estimation for processes with large one lag autocorrelation one should derive also the third-order term proportional to ρ_1^2, corresponding to the term with Hermite polynomials of degree two in the expansion of the bivariate Gaussian density.

REMARK 13. By taking the limit of small round-off errors ($\xi \to \infty$), we show numerically that when the underlying increments are *i.i.d.*, the lag-1 autocovariance of the realized increments is $-\sigma^2/(12\xi)$. This bias was obtained also by Harris (1990).

3.3.3 Inference on unobservable parameters

3.3.3.1 Inference on underlying variance σ^2

As mentioned above, we can use Lemma 3.2 to infer the unobservable variance of the underlying process.

Proposition 3.8. *Let* $\left\{r_1^{(\alpha)}, \ldots, r_n^{(\alpha)}\right\}$ *be a sample from* $\{r_t^{(\alpha)}\}$. *Let*

$$\widehat{\sigma^2} := \alpha^2 h^{-1}(\widehat{p_0})$$

where $h(x) = \sqrt{\frac{2x}{\pi}}\left(e^{-\frac{1}{2x}} - 1\right) + erf\left(\frac{1}{\sqrt{2x}}\right)$, *and* $\widehat{p_0} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}[r_i^{(\alpha)} = 0]$, *with* $\mathbb{I}[\cdot]$ *being the indicator function. Then,* $\widehat{\sigma^2} \to \sigma^2$ *almost surely as* $n \to \infty$.

Proposition 3.8 says that the variance estimator obtained from Lemma 3.2 using the fraction of realized increments equal to zero is strongly consistent. Note that, according to our definitions (3.1) and (3.2), this result holds for both short-memory and long-memory processes.

REMARK 14. On the other hand, the asymptotic normality of $\widehat{\sigma^2}$ will depend on whether the underlying process is short-memory or long-memory, and we leave the issue for future work. We conjecture that if the underlying process is short-memory, the sequence of indicator functions $\left\{\mathbb{I}\left[r_t^{(\alpha)} = 0\right]\right\}_t$ satisfies Gordin's condition for the $\sqrt{n}$-central limit theorem (CLT), and $\widehat{\sigma^2}$ is asymptotically normal with convergence rate $\sqrt{n}$ by application of the Delta method.

REMARK 15. For the case of an underlying long-memory process, we would probably need to develop an *ad hoc* technique. On the one hand, the existing literature provides functional CLTs for the empirical process of long-memory processes that have a MA representation with *i.i.d.* or *m.d.s.*[3] innovations (see Dehling, Mikosch, and Sorensen, 2003), but $r^{(\alpha)}$ does not satisfy this assumption, and therefore we cannot conclude that $n^{-1/2}\sum_{i=1}^{n} \mathbb{I}[r_i^{(\alpha)} = 0]$ is asymptotically normal. On the other hand, the literature provides CLTs for non-linear, instantaneous functionals of Gaussian or linear long-memory processes, but $r^{(\alpha)}$ (and therefore any of its indicator functions) is a non-linear, *non-instantaneous* functional of the underlying process because, for any t, $r_t^{(\alpha)}$ also depends on the realization of the underlying integrated process at $t-1$, and therefore we cannot apply those results.

3.3.3.2 Inference on underlying correlation

Using Proposition 3.7, we propose a (misspecified) estimator for the unobservable correlation coefficient at lag 1. If we neglect the term $O\left(\rho_1^2\right)$ in Proposition 3.7, we have the misspecified model:

$$\mathbb{P}\left(r_t^{(\alpha)} = 0, r_{t+1}^{(\alpha)} = 0\right) = p_\alpha^{(2)}(0,0;\xi) + \rho_1^* p_\alpha^{(3)}(0,0;\xi). \tag{3.5}$$

where $\rho_1^* = \rho_1(1 + O(\rho_1))$ is the lag-1 correlation coefficient for the misspecified model. If ρ_1 is sufficiently small, we expect (3.5) to be a good approximation of the true equation, and $\rho_1^* \approx \rho_1$.

[3] Martingale difference sequence.

Hence, if we have *a priori* knowledge that the lag-1 autocorrelation of $\{r_t\}$ is relatively small, we can infer ρ_1 by estimating ρ_1^* from (3.5) after plugging in $\widehat{\xi} := \alpha^2\widehat{\sigma^2}$, which is a strongly consistent estimator of ξ (Proposition 3.8).

Proposition 3.9. *Let* $\left\{r_1^{(\alpha)}, \ldots, r_n^{(\alpha)}\right\}$ *be a sample from* $\{r_t^{(\alpha)}\}$. *Let*

$$\widehat{\rho_1} := \frac{\widehat{p_{00}} - p_\alpha^{(2)}(0,0;\widehat{\xi})}{p_\alpha^{(3)}(0,0;\widehat{\xi})},$$

where $\widehat{\xi} := \widehat{\sigma^2}/\alpha^2$, $\widehat{p_{00}} = \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{I}[r_i^{(\alpha)} = 0, r_{i+1}^{(\alpha)} = 0]$, *and* $p_\alpha^{(2)}(0,0;\xi)$ *and* $p_\alpha^{(3)}(0,0;\xi)$ *are defined in Proposition 3.7. Then,* $\widehat{\rho_1} \to \rho_1^* = \rho_1\left(1 + O\left(\rho_1\right)\right)$ *almost surely as* $n \to \infty$.

Morevoer, if $\rho_1 = 0$,

$$\widehat{\beta} := \widehat{p_{00}} - p_\alpha^{(2)}(0,0;\widehat{\xi}) \to 0 \qquad a.s. \qquad as\ n \to \infty.$$

REMARK 16. Again, we leave the proof of asymptotic normality for future work. We expect that if the process is short memory, the Gordin's CLT applies to $\mathbb{I}[r_i^{(\alpha)} = 0, r_{i+1}^{(\alpha)} = 0]$, and again by Delta method we have asymptotic normality of $\widehat{\beta}$ and $\widehat{\rho_1}$ with convergence rate $\sqrt{n}$. On the other hand, if the process is long memory we will have to develop an *ad hoc* technique, and the rate of convergence will probably be lower (see Dehling *et al.*, 2003).

REMARK 17. If the rescaled partial sums of $\mathbb{I}[r_i^{(\alpha)} = 0, r_{i+1}^{(\alpha)} = 0]$ are asymptotically normal, then under the null hypothesis that the underlying process is uncorrelated at lag 1, $\widehat{\beta}$ is asymptotically normal, and we could construct a test for zero correlation under the null $H_0 : \rho_1 = 0$.

REMARK 18. Proposition 3.9 says that $\widehat{\rho_1}$ is a strongly consistent estimator of ρ_1^*, which will be close to the true lag-1 correlation ρ_1 if ρ_1 is sufficiently small because $\rho_1^* = \rho_1\left(1 + O(\rho_1)\right)$. Hence, this estimator is suitable for application to short memory process, but we expect to perform poorly if the underlying process is long memory.

3.4 Numerical results

3.4.1 Distributional Properties and Autocovariance

In this section, we report our numerical results for the distributional properties and autocovariance of the realized increments. We consider three different processes, namely a Brownian motion sampled at constant time intervals (corresponding to an i.i.d. Gaussian process that we label with $H = 0.5$), an ARIMA(1,1,0) process with parameter $\rho_1 = 0.2$, and a fractional

Brownian motion (fBm) with Hurst exponent H sampled at constant time intervals. We show most of the results for $H = 0.7$, but in some cases we will also consider the case $H = 0.85$. The length of the time series is set equal to $n = 2^{14}$. We perform $L = 10^3$ Monte Carlo (MC) simulations in order to obtain the statistical properties of the estimators.

For studying the distributional properties, we show only the case of a fBm with $H = 0.7$, since the distributional properties of the realized series do not depend on the memory parameter. For long memory processes we take into account the finite sample bias given by Hosking (1996). In Figure 3.2 we report our results for the variance and kurtosis of the realized increments. There is a very good agreement between numerical simulations and the theoretical results of Proposition 3.3. In fact, all the points lie on our asymptotic formulas for large and small ξ, for both the variance and for the kurtosis.

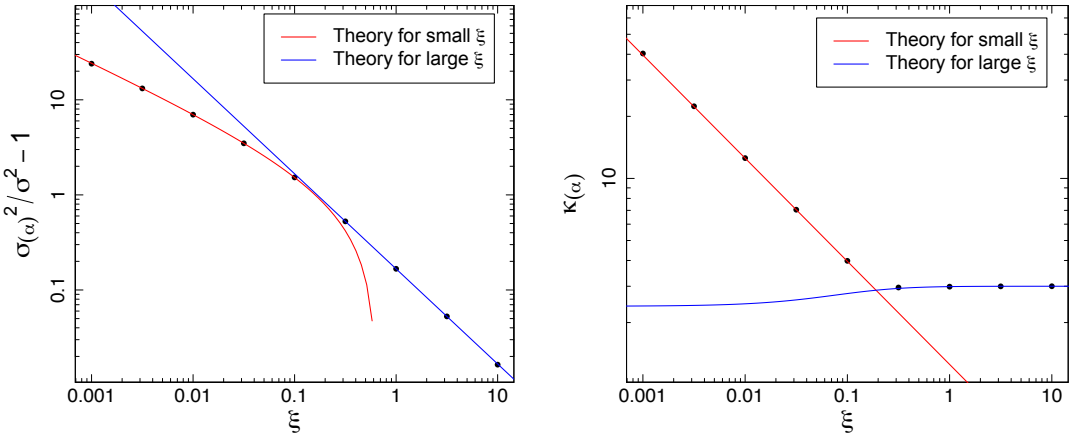


FIGURE 3.2: Variance (left) and Kurtosis (right) of the realized increments, $\{r_t^{(\alpha)}\}$. The time series has length $n = 2^{14}$. The Hurst exponent of the underlying fBm is $H = 0.7$. The figure also shows the theoretical asymptotic behavior given by Proposition 3.3 for small (red line) and for large (blue line) ξ.

Figure 3.3 shows the results for the sample autocovariance. We consider different values of the discretization parameter, namely, $\xi = 0.001$, 0.01, 0.1, and 1, corresponding to probability of a zero realized increment $p_\alpha(0) \approx 0.97$, 0.92, 0.75, and 0.37, respectively. As for the variance, when calculating the autocovariance of the long memory processes, we take into account the finite sample bias given by Hosking (1996). Fig. 3.3 shows that for large lags the autocovariance of the realized increments converges to that of the underlying increments, as expected from Proposition 3.6. However, for small lags the autocovariance of the realized series is significantly smaller than that of the underlying series, and often negative on the first lags. This effect

becomes stronger as the discretization parameter decreases (i.e., as the rounding error becomes relatively more important) and for smaller values of the Hurst exponent.

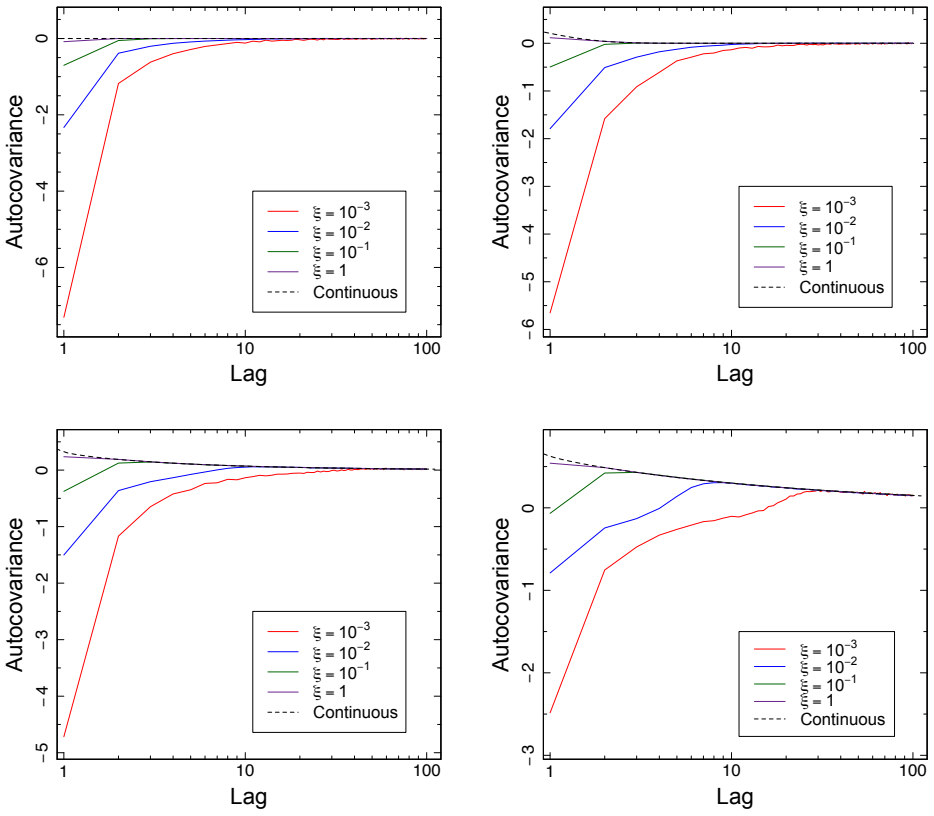


FIGURE 3.3: Sample autocovariance of a simulated time series (black) and realized increments, $\{r_t^{(\alpha)}\}$, of length $n = 2^{14}$ with $\xi = 1$ (violet), 0.1 (green), 0.01 (blue), and 0.001 (red). The simulated processes are a Brownian motion (top left) and ARIMA(1,1,0) process (top right), a fBm with $H = 0.7$ (bottom left) and $H = 0.85$ (bottom right).

Figure 3.4 shows the autocovariance at lag 1 for time series of realized increments. Points correspond to numerical calculations based on Monte Carlo simulations and solid lines correspond to theoretical calculations based on Proposition 3.7. For all the cases, our theoretical calculations are in very good agreement with the numerical simulations on the whole range of ξ considered.

3.4.2 Inference on underlying variance and lag-1 autocorrelation

In this section, we present our numerical results for the estimation of underlying variance and lag-1 correlation from a finite series of realized increments using the estimators proposed in Proposition 3.8 and 3.9. We consider six values of the discretization parameter, namely, $\xi =$ 0.025, 0.05, 0.1, 0.25, 0.5, and 1, corresponding to probability of a zero realized increment

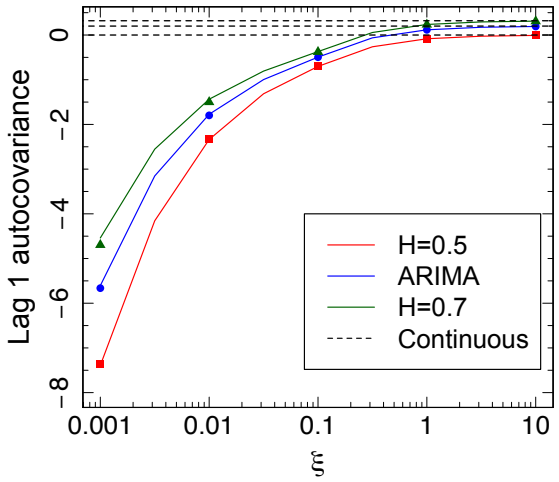
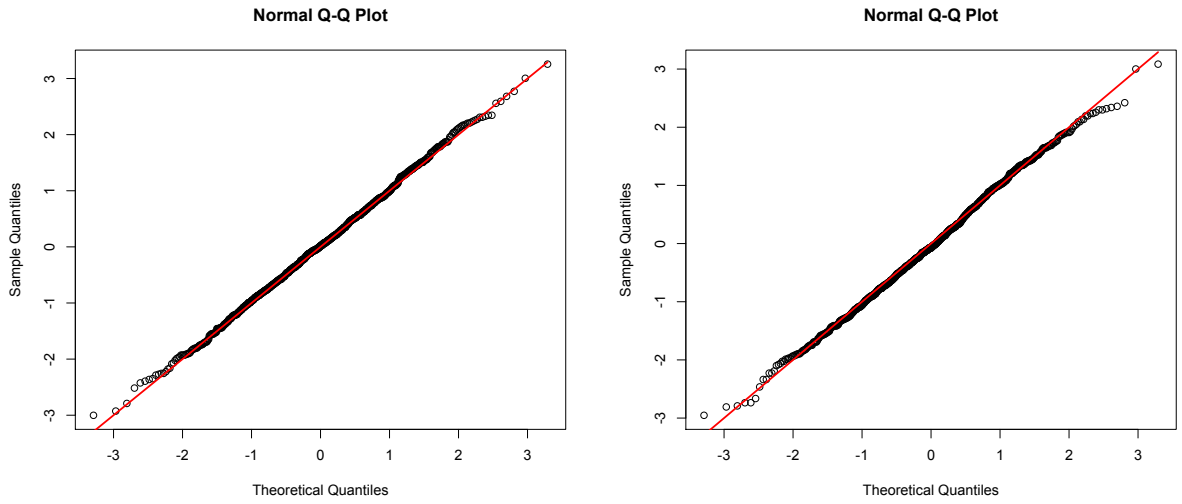


FIGURE 3.4: Lag-one autocovariance of the realized increments for an iid process ($H = 0.5$), an ARIMA(1,1,0) process with $\rho = 0.2$, and a fBm with $H = 0.7$. The time series has length $n = 2^{14}$. Points: numerical calculations based on MC simulations; Solid lines: theoretical calculations based on Proposition 3.7; Dashed lines: lag-one autocovariance of the underlying increments.

$p_\alpha(0) \approx 0.87$, 0.82, 0.75, 0.61, 0.49, and 0.37 respectively. As data generating process for the underlying increment process, we use a Brownian motion and an ARIMA(1,1,0) process with increments described by $r_t = \rho r_{t-1} + \varepsilon_t$ and Gaussian noise ε_t. In both cases, the variance of the underlying increment process is $\sigma^2 = 1$. We consider 4 different values of ρ, namely, $\rho = 0.05$, 0.1, 0.15, and 0.20. Since we know that $\widehat{\rho_1}$ is not suitable when the underlying process is highly persistent (REMARK 18), we do not consider ARIMA models with large ρ or long memory processes. We run $L = 10^3$ MC simulations, and at each MC iteration the underlying time series has length $n = 2^{14}$.

For the estimator of the underlying variance defined in Proposition 3.8, our results are in Table 3.4.1. The results show that for all combinations of the underlying parameters, $\widehat{\sigma^2}$ is unbiased and symmetrically distributed around the true σ^2. To further investigate its distributional properties, Figure 3.5 plots the quantiles of the MC sample $\{\widehat{\sigma^2}_m\}_{m=1,\dots,L}$, normalized by its standard deviation, against the quantiles of a standard Normal distribution. We show our results for $\xi = 0.025$ and two data generating processes: Brownian motion and ARIMA(1,1,0) with $\rho_1 = 0.1$. In both cases, the distribution of the estimator is well approximated by a Normal distribution, suggesting that, when the underlying increment process is short memory, $\widehat{\sigma^2}$ is asymptotically Normal, as we conjecture in REMARK 14. Results for other model specifications are similar and omitted for brevity.



(A) Brownian motion

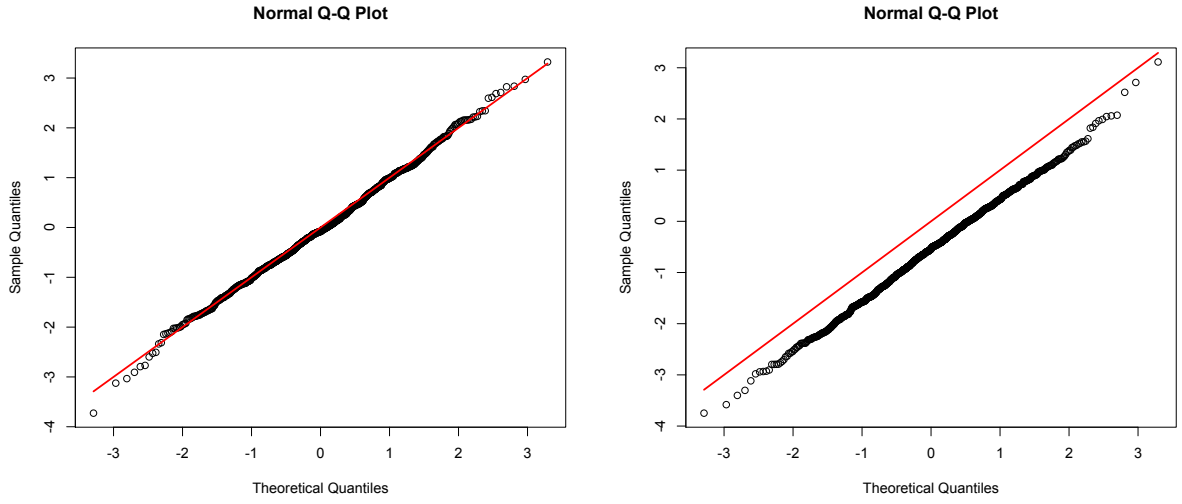
(B) ARIMA(1,1,0) with $\rho_1 = 0.2$

FIGURE 3.5: Normal Q-Q plot for the MC sample $\{\widehat{\sigma^2}_m\}_{m=1,\dots,L}$, normalized by its standard deviation. Sample quantiles of $\widehat{\sigma^2}_m$ on the x-axis. Quantiles of a standard Normal on the y-axis. Left panel: underlying Brownian motion. Right panel: underlying ARIMA(1,1,0) process with $\rho_1 = 0.1$. In both cases, the discretization parameter is $\xi = 0.025$. For each MC iteration, the sample size is $n = 2^{14}$. Number of MC simulations: $L = 10^3$. The red line is the 45-degree line passing through the origin.

For our estimator of the underlying lag-1 correlation defined in Proposition 3.9, results are in Table 3.4.2. The results show that when the underlying process is a Brownian motion, $\widehat{\rho_1}$ is unbiased and symmetrically distributed around the true value $\rho = 0$. Along this dimension, our estimator behaves well also when the true correlation is $\rho_1 = 0.05$, even though the estimator shows a small negative bias for $0.1 \leq \xi \leq 0.5$. As expected, the estimator tends to perform more poorly as ρ_1 increases, and for $\rho_1 \geq 0.1$ it is negatively biased for all values of ξ. This is because, as ρ_1 increases, the weight of the term $O(\rho_1^2)$ in (3.4) increases, and the misspecified model (3.5) is further away from the true model. That is, ρ_1^* is not approximately equal to ρ_1. However, for all $\rho_1 \in [0.1, 0.2]$, when the discretization is very coarse ($\xi = 0.025$), the relative size of this bias is fairly small (i.e., $O(10^{-3})$). More subtle is the dependence of the bias of the estimator with ξ for a fixed value of ρ. The table shows that the bias of the estimator increases when ξ increases, i.e. when the round-off becomes less significant. This is because in the misspecified model (3.5) we neglect the terms $O\left(\rho_1^2\right)$, and for any given ρ_1 those terms increase with ξ.

To further investigate its distributional properties, Figure 3.6 plots the quantiles of the MC sample $\{\widehat{\rho_1}_m\}_{m=1,\dots,L}$, normalized by its standard deviation, against the quantiles of a standard Normal distribution. We show the results for the case of a Brownian motion and an

ARIMA(1,1,0) with $\rho_1 = 0.2$, when $\xi = 0.05$. For both data generating processes, the distribution of $\widehat{\rho_1}$ is well approximated by a Normal distribution, suggesting that if the underlying process is short-memory, $\widehat{\rho_1}$ is asymptotically Normal, as we conjecture in REMARK 16. Also, for the ARIMA(1,1,0) with $\rho_1 = 0.2$, the right panel of Fig. 3.6 clearly shows that the distribution of $\widehat{\rho_1}$ is not centered around ρ_1, but on the left of ρ_1. This corresponds to the negative bias observed in Table 3.4.2 and coming from the fact that $\rho_1^* \not\approx \rho_1$ for relatively large ρ_1.



(A) Brownian motion (B) ARIMA(1,1,0) with $\rho_1 = 0.2$

FIGURE 3.6: Normal Q-Q plot for the MC sample $\{\widehat{\rho_1}_m\}_{m=1,\dots,L}$, normalized by its standard deviation. Sample quantiles of $\widehat{\rho_1}_m$ on the x-axis, and quantiles of a standard Normal on the y-axis. Left panel: underlying Brownian motion process. Right panel: underlying ARIMA(1,1,0) process with $\rho_1 = 0.2$. In both cases, the discretization parameter is $\xi = 0.05$. Number of MC simulations: $L = 10^3$. For each MC iteration, the sample size is $n = 2^{14}$. The red line is the 45-degree line passing through the origin.

3.4.3 Estimation of the Hurst exponent

In this section, we present our numerical results for the estimation of the Hurst exponent of the underlying increments from a finite series of realized increments. We consider three values of the discretization parameter, namely, $\xi = 0.1$, 0.25, and 0.5, corresponding to probability of a zero realized increment $p_\alpha(0) \approx 0.75$, 0.61 and 0.49, respectively. We study the local Whittle estimator, which is one of the most used estimators of the Hurst exponent. In Appendix 3.B, we consider also the Detrended Fluctuation Analysis (DFA) as an alternative estimator. The purpose is to show that the results for the Whittle estimator are common to other Hurst exponent estimators. For a detailed discussion of the asymptotic properties of these estimators

when applied to nonlinear, instantaneous functionals of stationary long-memory processes, see La Spada and Lillo (2014). However, we cannot apply those theoretical results here because the discretization we consider in this paper is a non-linear, non-instantaneous functional of the underlying increment process. We leave the theoretical study of the asymptotic properties of these estimators when applied to the discretized series considered in this paper for future work.

The local Whittle (LW) estimator (see Künsch, 1987) is a Gaussian semiparametric estimator that works in the frequency domain. Let $\{X_t\}_{t\in\mathbb{N}}$ be a weakly stationary long memory process with spectral density $\phi(\omega)$ satisfying

$$\phi(\omega) = c\,|\omega|^{1-2H}\,(1+o(1)) \qquad as\ \ \omega \to 0^{+} \tag{3.6}$$

for some $c > 0$. For a time series $\{X_t\}$, $t = 1, \ldots, n$, define the periodogram

$$I_n(\omega_j) = (2\pi n)^{-1} \left| \sum_{t=1}^{n} X_t \exp^{it\omega_j} \right|$$

Let $\omega_j = 2\pi j/n$, $j = 1, \ldots, n$, be the Fourier frequencies. The LW estimator is defined as the minimizer of the local objective function

$$U_n(h; m) = \log\left(\frac{1}{m} \sum_{j=1}^{m} \omega_j^{2h-1} I_n(\omega_j) \right) - \frac{2h}{m} \sum_{j=1}^{m} \log \omega_j$$

where $m = m(n)$ is an integer-bandwidth parameter such that

$$m \to \infty, \ \ m = o(n), \ \ \text{as } n \to \infty$$

Since from Proposition 3.6 the realized increments satisfy (3.6) with the same H as the underlying increments, to estimate the Hurst exponent of the underlying series we can use the LW estimator on the realized increment series. Note that, in semiparametric models, the spectral density function has property (3.6) and is only locally parameterized around $\omega = 0$ by the parameters H and c. Therefore, contrary to the parametric Whittle estimation, which employs the full spectrum of frequencies, the LW estimator uses only the first m Fourier frequencies. Following a standard procedure, in our numerical simulations we use $m = n^{0.5}$, $n^{0.6}$, $n^{0.7}$ and $n^{0.8}$. (See Dalla, Giraitis and Hidalgo (2006) and La Spada and Lillo (2014) for a detailed discussion of the optimal bandwidth for nonlinear processes.)

Numerical Results Our numerical results for the LW estimator are reported in Tables 3.4.3 and 3.4.4. We studied a fBm with Hurst exponent $H = 0.7$ and $H = 0.85$. Tables show that, in medium-sized samples and for very coarse discretizations ($\xi = 0.1$), the LW estimator is significantly negatively biased for all bandwidths m, for both $H = 0.7$ and $H = 0.85$. However, for $\xi = 0.25$ and 0.5 the bias is significant only when $m = n^{0.7}$ and $n^{0.8}$ in medium-sized samples, and only when $m = n^{0.8}$ in long samples. Importantly, for $m = n^{0.8}$ (which is the optimal bandwidth suggested by Robinson and Henry (1996) for Gaussian processes) the negative bias is statistically significant and severe for all discretizations considered, both in medium-sized and long samples. For $\xi = 0.1$, the RMSE is minimum when $m = n^{0.5} - n^{0.6}$ in both medium-sized and long samples; for $\xi = 0.25$, it is minimum when $m = n^{0.6}$; and for $\xi = 0.5$ and the fBm, it is minimum when $m = n^{0.6} - n^{0.7}$.

We conclude that, when applied to series of realized increments, the LW estimator is statistically significantly negatively biased in medium-sized samples. For coarse discretizations and large bandwidths, the magnitude of the bias is statistically significant and severe also in long samples. These findings indicate that the optimal bandwidth suggested in the literature for Gaussian processes ($m = n^{0.8}$) is not suitable when the underlying integrated process is observed with round-off error. For the realized increments our numerical results suggest an optimal bandwidth, in terms of RMSE minimization, of order $n^{0.6} - n^{0.7}$, in agreement with the results in Dalla *et al.* (2006) for nonlinear transformations of stationary Gaussian processes.

3.5 Conclusions

In this paper we have presented a detailed study of the effect of round-off error on the distributional and autocovariance properties of an integrated process with correlated increments. Our paper generalizes (and makes use) of some previous literature, notably Harris (1990), Delattre and Jacod (1997), and La Spada and Lillo (2014). We derive the variance and kurtosis of the realized increments and provide expansions of the autocovariance both for large lags and at lag 1. Contrary to the previous literature, we also derive the asymptotic behavior of these quantities in the limit of large round-off errors. Building on these results, we propose estimators of the underlying, unobservable variance and lag-1 autocovariance starting from the properties of the realized increments and test them on synthetic data. Finally, we consider the special case an underlying integrated long memory process (specifically, a fractional Brownian motion) and study the estimation of its Hurst exponent. We prove that the Hurst exponent of the realized increments is the same as the one of the underlying process. However, the numerical analysis

of two commonly used estimators of the Hurst exponent shows that there is a strong negative bias in finite samples.

The results and estimators proposed in this paper could be useful in several applications. Considering high frequency financial econometrics, we see at least four interesting problems. First, our estimator for the lag-1 autocorrelation of the underlying increment series could be used to perform tests of high frequency market efficiency in the presence of round-off errors. Efficiency, in fact, requires that $\rho_1 = 0$, and the use of mid-prices in place of transaction prices could help mitigating the bid-ask bounce, while not eliminating the effect of round-off errors. Second, as shown in the right panel of Fig. 3.1, round-off errors alone are capable to explain qualitatively the signature plot of high frequency financial data. It would be interesting to compute which fraction of the signature plot is due to the round-off error, especially for large tick stocks (i.e., stocks with an high tick size over price ratio). Third, with a simple generalization of Proposition 3.7, it should be possible to compute the role of round-off error on the synchronous correlation of *two* separate time series. The decline of correlation for small ξ might be partly responsible of the empirically observed Epps effect, i.e. the decline of the correlation between stock returns when the sampling time declines. Finally, we believe that the machinery developed here is an useful starting point for developing statistical methods to disentangle jumps from rounding effects. Since at high frequency round-off errors become very important, it is crucial and difficult to develop tests able to identify whether a price movement corresponds to a jump of the underlying, unobservable process, or it is due to the discrete nature of the observed price. We plan to investigate these issues in future works.

Appendix 3.A Proofs

Let us first state the following definition and lemma.

Definition 3.10. An n-dimensional multi-index is an n-tuple $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ of non-negative integers (i.e., $\boldsymbol{\beta} \in \mathbb{N}_0^n$). The number $|\boldsymbol{\beta}| := \beta_1 + \ldots + \beta_n$ is called the order of $\boldsymbol{\beta}$.

Lemma 3.11. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* $\mathbf{y} = (y_1, \ldots, y_n) \in [0,1)^n$. *There are universal constants* C_N *such that for all* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{>0}$, *and all Borel functions* k *on* $\mathbb{R}^n$ *and* g *on* $\mathbb{R}^n \times [0,1)^n$ *such that* $h(\mathbf{x}, \mathbf{y}) := k(\mathbf{x})g(\mathbf{x}, \mathbf{y})$ *is of class* C^N $(N \geq 1)$ *with respect to* $\mathbf{x}$, *we*

have

$$\left| \int_{\mathbb{R}^n} k(\mathbf{x}) g\left(\mathbf{x}, \left\{\frac{\mathbf{x}}{\boldsymbol{\alpha}}\right\}\right) \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^n} k(\mathbf{x}) \mathrm{d}\mathbf{x} \int_{[0,1)^n} g(\mathbf{x}, \mathbf{u}) \, \mathrm{d}\mathbf{u} \right| \leq$$
$$\leq C_N \max\{\alpha_1, \ldots, \alpha_n\}^{N+n} \sum_{|\beta|=N} \sum_{j_1,\ldots,j_n=-\infty}^{\infty} \int_0^1 \mathrm{d}t \int_{[0,1)^n} \left| \partial_{\mathbf{x}}^{(\beta)} h(\mathbf{j}\boldsymbol{\alpha} + t\boldsymbol{\alpha}, \mathbf{u}) \right| \mathrm{d}\mathbf{u} \,.$$

where $\{\cdot\}$ *represents the fractional part component-wise,* $\frac{\mathbf{x}}{\boldsymbol{\alpha}} := \left(\frac{x_1}{\alpha_1}, \ldots, \frac{x_n}{\alpha_n}\right)$, $\mathbf{j}\boldsymbol{\alpha} := (j_1\alpha_1, \ldots, j_n\alpha_n)$, *and* $\partial_{\mathbf{x}}^{(\beta)}$ *is a partial derivative of order* $|\boldsymbol{\beta}|$ *with respect to the first* n *arguments, with* β *being a* n*-dimensional multi-index.*

REMARK 19. Lemma 3.11 is a generalization of Lemma 6.1 in Delattre and Jacod (1997). The extension from the 1-dimensional to the n-dimensional case is relatively straightforward when using multi-index notation. The proof is along the same lines as the proof of Lemma 6.1 in Delattre and Jacod (1997) and is thus omitted for brevity. In the following, we use Lemma 3.11 to derive our Lemma 3.1 and 3.4.

Proof of Lemma 3.1. Lemma 3.1 follows from Lemma 3.11 with $n = 1$, $\alpha_1 = \alpha$, $k(x) = f(x; r_t)$, where $f(\cdot; r_t)$ is the density of v_{t-1} conditional on r_t (of class C^N by assumption), and $g(x, y) = \mathbb{I}[y \in I]$ for any Borel set I in $[0, 1]$ of Lebesgue measure $l(I)$. □

REMARK 20. Hereafter, for notational simplicity, $round(x)$ stands for $round(x/\alpha)$.

Proof of Lemma 3.2. We want to calculate $\mathbb{P}\left(r_t^{(\alpha)} = k\alpha\right)$ for large t. First, note that

$$r_t^{(\alpha)} = k\alpha \quad \text{if} \quad \begin{cases} \text{either} & r_t \in [(k-1)\alpha, k\alpha) \ \text{ and } \ round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 \\ \text{or} & r_t \in [k\alpha, (k+1)\alpha) \ \text{ and } \ round\left(\widetilde{v}_{t-1} + \breve{r}_t\right) = 0 \end{cases}$$

Hence, for any $N \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}\left(r_t^{(\alpha)} = k\alpha\right) =& \mathbb{P}\left(r_t \in [(k-1)\alpha, k\alpha), round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1\right) + \\
&+ \mathbb{P}\left(r_t \in [k\alpha, (k+1)\alpha), round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0\right) \\
&= \int_{(k-1)\alpha}^{k\alpha} \mathbb{P}\left(round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 \,|r_t\right) f(r_t)\mathrm{d}r_t + \\
&+ \int_{k\alpha}^{(k+1)\alpha} \mathbb{P}\left(round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 \,|r_t\right) f(r_t)\mathrm{d}r_t \\
&= \int_{(k-1)\alpha}^{k\alpha} \mathbb{P}\left(\frac{\alpha}{2} - \overline{r}_t \leq \widetilde{v}_{t-1} < \frac{\alpha}{2}|r_t\right) f(r_t)\mathrm{d}r_t + \\
&+ \int_{k\alpha}^{(k+1)\alpha} \mathbb{P}\left(-\frac{\alpha}{2} \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} - \overline{r}_t|r_t\right) f(r_t)\mathrm{d}r_t \\
&= \int_{(k-1)\alpha}^{k\alpha} \left(1 + \frac{r_t}{\alpha} - k\right) f(r_t)\mathrm{d}r_t + \int_{k\alpha}^{(k+1)\alpha} \left(1 + k - \frac{r_t}{\alpha}\right) f(r_t)\mathrm{d}r_t + O\left(\xi^{-N/2}|t_0|^{-NH}\right)
\end{aligned}$$

where f is the pdf of r_t (i.e., a Gaussian density with mean 0 and variance σ^2), and the last equality follows from Lemma 3.1. In fact, since the underlying process v_t is Gaussian, $v_{t-1}|r_t$ is also Gaussian with variance proportional to $\sigma^2(t-t_0)^{2H}$, with $H = 0.5$ if r_t is short-memory. Hence, from Lemma 3.1 it follows that, for any $N \geq 1$, the error coming from approximating $\widetilde{v}_{t-1}|r_t$ with a uniformly distributed random variable over $[-\alpha/2, \alpha/2)$ is proportional to $\alpha^N(\sigma(t-t_0)^H)^{-N} = \xi^{-N/2}O\left(|t_0|^{-NH}\right)$. By carrying out the integrals, we obtain the result. □

Proof of Proposition 3.3. The mass distribution function of the realized process $r^{(\alpha)}$, $p_\alpha(\cdot)$, is symmetric around zero. Hence, its expected value is zero, and its variance is

$$\sigma^2_{(\alpha)}) = \mathbb{E}\left[r^2_{(\alpha)}\right] = \sum_{k=-\infty}^{\infty} (k\alpha)^2 p_\alpha(k) = \frac{1}{\xi}\left(\sum_{k=-\infty}^{\infty} k^2 p_\alpha(k)\right)\sigma^2.$$

p_α is continuously differentiable, and $p_\alpha(x)$ and all its derivatives go to zero as $x \to \pm\infty$. Hence, by Euler-MacLaurin formula, we can write

$$\sum_{k=-\infty}^{\infty} k^2 p_\alpha(k) = \int_{-\infty}^{\infty} x^2 p_\alpha(x)\mathrm{d}x - \frac{1}{4!}\int_{-\infty}^{\infty} B_4\left(\{1-t\}\right) p_\alpha^{(4)}(t;\xi)\mathrm{d}t, \tag{3.7}$$

where B_4 is the fourth Bernoulli polynomial, and $p_\alpha^{(4)}$ is the fourth derivative of p_α given by

$$p_\alpha^{(4)}(t;\xi) = \frac{a_1(t;\xi)e^{-\frac{(t+1)^2}{2\xi}} + a_2(t;\xi)e^{-\frac{(t-1)^2}{2\xi}} - 2a_3(t;\xi)e^{-\frac{t^2}{2\xi}}}{\sqrt{2\pi}\xi^{5/2}}$$

with

$$\begin{aligned} a_1(t;\xi) &= t^4 + 2t^3 + (1-9\xi)t^2 - 8\xi t + 12\xi^2 \\ a_2(t;\xi) &= t^4 - 2t^3 + (1-9\xi)t^2 + 8\xi t + 12\xi^2 \\ a_3(t;\xi) &= t^4 - 9\xi t^2 + 12\xi^2 \end{aligned}$$

First,

$$\int_{-\infty}^{\infty} x^2 p_\alpha(x)\mathrm{d}x = \xi + \frac{1}{6}\,. \tag{3.8}$$

Second, $p_\alpha^{(4)}()t;\xi$ is square integrable in t, and its Fourier transform is

$$\int_{-\infty}^{\infty} e^{2\pi i n t} p_{(\alpha)}^{(4)}(t;\xi)\mathrm{d}t = -8\pi^2 n^2 e^{-2\pi^2\xi n^2}.$$

Third, the periodic Bernoulli polynomials have a Fourier series expansion that is absolutely convergent; specifically, $B_{(4)}(\{1-t\}) = \sum_{n\neq 0} b_n e^{2\pi i n t}$, where $b_n = -\frac{3}{2\pi^4 n^4}$ are the Fourier coefficients. Hence, we can write

$$\begin{aligned} \int_{-\infty}^{\infty} B_4\left(\{1-t\}\right) p_{(\alpha)}^{(4)}(t;\xi)\mathrm{d}t &= \sum_{n\neq 0}^{\infty} b_n \int_{-\infty}^{\infty} e^{2\pi i n t} p_{(\alpha)}^{(4)}(t;\xi)\mathrm{d}t \\ &= -8\pi^2 \sum_{n\neq 0} b_n n^2 e^{-2\pi^2\xi n^2} \\ &= \frac{24}{\pi^2}\sum_{n\geq 1} n^{-2} e^{-2\pi^2\xi n^2}. \end{aligned} \tag{3.9}$$

Plugging (3.8) and (3.9) in (3.7), for all $\xi \in (0,\infty)$ we obtain

$$\sigma_{(\alpha)}^2 = \left(1 + \frac{1}{6\xi} - \frac{1}{\pi^2\xi}\sum_{n\geq 1} n^{-2} e^{-2\pi^2\xi n^2}\right)\sigma^2.$$

Since $\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2}$ converges absolutely for all $\xi \in [0,\infty)$, we have $\lim_{\xi\to\infty}\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2} = 0$. That is, $\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2} = o(1)$ as $\xi \to \infty$. Hence, for large ξ (i.e., $\xi \to \infty$), we have the result

$$\sigma^2_{(\alpha)} = \left(1 + \frac{1}{6\xi} + o(\xi^{-1})\right)\sigma^2.$$

For the case of small ξ (i.e., $\xi \to 0^+$), we apply Euler-MacLaurin formula again and write

$$\begin{aligned}\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2} =& \int_1^\infty x^{-2}e^{-2\pi^2\xi x^2}\mathrm{d}x + \frac{2+\pi^2\xi}{3}e^{-2\pi^2\xi} - \int_1^\infty B_2(\{1-t\})\frac{a_4(t;\xi)e^{-2\pi^2\xi t^2}}{t^4}\mathrm{d}t\\ =&\frac{5+\pi^2\xi}{3}e^{-2\pi^2\xi} - 2\sqrt{2\xi}\pi^{3/2}\left[1 - erf\left(\sqrt{2\xi}\pi\right)\right] +\\ &- \int_1^\infty B_2(\{1-t\})\frac{a_4(t;\xi)e^{-2\pi^2\xi t^2}}{t^4}\mathrm{d}t,\end{aligned}$$

where $a_4(t;\xi) = 3 + 6\pi^2\xi t^2 + 8\pi^4\xi^2 t^4$, and $B_2(x)$ is the second Bernoulli polynomial. Therefore, for small ξ,

$$\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2} = \frac{5}{3} - \sqrt{2\xi}\pi^{3/2} + O(\xi) - \int_1^\infty B_2(\{1-t\})\frac{a_4(t;\xi)e^{-2\pi^2\xi t^2}}{t^4}\mathrm{d}t.$$

Since $\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2}$ converges absolutely for all $\xi \in [0,\infty)$, and $\lim_{\xi\to 0} n^{-2}e^{-2\pi^2\xi n^2} = n^{-2}$, we have $\lim_{\xi\to 0}\sum_{n\geq 1} n^{-2}e^{-2\pi^2\xi n^2} = \pi^2/6$. Hence, we must have

$$\int_1^\infty B_2(\{1-t\})\frac{a_4(t;\xi)e^{-2\pi^2\xi t^2}}{t^4}\mathrm{d}t = \frac{5}{3} - \frac{\pi^2}{6} + g(\xi),$$

where $g(\xi) = o(1)$ as $\xi \to 0$. Since the above integrand converges absolutely for all $\xi \in [0,\infty)$,

$$\frac{5}{3} - \frac{\pi^2}{6} = \lim_{\xi\to 0}\int_1^\infty B_2(\{1-t\})\frac{a_4(t;\xi)e^{-2\pi^2\xi t^2}}{t^4}\mathrm{d}t = 3\int_1^\infty \frac{B_2(\{1-t\})}{t^4}\mathrm{d}t.$$

Hence,

$$\begin{aligned}
|g(\xi)| =& \left| \int_1^\infty B_2(\{1-t\}) \frac{a_4(t;\xi) e^{-2\pi^2\xi t^2}}{t^4} \mathrm{d}t - 3\int_1^\infty \frac{B_2(\{1-t\})}{t^4} \mathrm{d}t \right| \\
\leq & B_2(0) \left[3\int_1^\infty \frac{1-e^{-2\pi^2\xi t^2}}{t^4} \mathrm{d}t + 2\pi^2\xi \int_1^\infty \frac{3t^2+4\pi^2\xi t^4}{t^4} e^{-2\pi^2\xi t^2} \mathrm{d}t \right] \\
=& \frac{1}{6}\left[1 + e^{-2\pi^2\xi}\left(10\pi^2\xi - 1\right) - 8\sqrt{2}\pi^{7/2}\xi^{3/2}\left(1 - erf\left(\sqrt{2\xi}\pi\right)\right)\right] \\
=& \frac{\pi^2\xi}{2} + O(\xi^{3/2}) \qquad as \;\; \xi \to 0.
\end{aligned}$$

That is, $g(\xi) = O(\xi)$ as $\xi \to 0$. Therefore, for small ξ, we have the result

$$\begin{aligned}
\sigma^2_{(\alpha)} &= \left(1 + \frac{1}{6\xi} - \frac{1}{\pi^2\xi} \sum_{n\geq 1} n^{-2} e^{-2\pi^2\xi n^2} \right) \sigma^2 \\
&= \left(1 + \frac{1}{6\xi} - \frac{1}{\pi^2\xi}\left[\frac{\pi^2}{6} - \sqrt{2\xi}\pi^{3/2} + O(\xi) \right] \right) \sigma \\
&= \left(\sqrt{\frac{2}{\pi\xi}} + O(1) \right) \sigma.
\end{aligned}$$

For the kurtosis, we have to calculate $\mathbb{E}[r^4_{(\alpha)}]$. The proof is similar and thus omitted for brevity. □

Proof of Lemma 3.4. Lemma 3.4 follows from Lemma 3.11 with $n = 2$, $\alpha_1 = \alpha_2 = \alpha$, $k(\mathbf{x}) = f(\mathbf{x}; r_{t+s}, r_t)$, where $f(\cdot; r_{t+s}, r_t)$ is the joint density of (v_{t+s-1}, v_{t-1}) conditional on (r_{t+s}, r_t) (of class C^N by assumption), and $g(\mathbf{x}, \mathbf{y}) = \mathbb{I}[y_1 \in I_1, y_2 \in I_2]$ for any Borel sets I_1, I_2 in $[0,1]$ of Lebesgue measure $l(I_1)$ and $l(I_2)$, respectively. □

Lemma 3.12. *Let $\{r_t\}_{t\in\mathbb{Z}}$ be a Gaussian process, and $f(\cdot; r_t, r_{t+s})$ be the joint density of (v_{t-1}, v_{t+s-1}) conditional on (r_t, r_{t+s}), where $v_t = \sum_{s=0}^{t} r_s$. For any $\alpha > 0$ and $N \in \mathbb{N}$, let*

$$g(r_t, r_{t+s}) := \alpha^{N+2} \sum_{|\beta|=N} \sum_{j_1,j_2=-\infty}^{\infty} \int_0^1 \partial^{(\beta)} f(\alpha\mathbf{j} + u\alpha\mathbf{e}; r_{t+s}, r_t) \mathrm{d}u \tag{3.10}$$

where β is a multi-index, $\mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2$, and $\mathbf{e} = (1,1)$.

Then, there exists a universal constant C^*_N such that, for large t and s,

$$|g(r_t, r_{t+s})| \leq C^*_N \xi^{-N/2}(t^{-NH} + s^{-NH}) \qquad for\ all\ \ r_t, r_{t+s}.$$

Proof of Lemma 3.12. Let $\mathbf{v} = (v_{t+s-1}, v_{t-1})$ and $\mathbf{r} = (r_{t+s}, r_t)$. Since $\{r_t\}$ is Gaussian with zero mean,

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{r} \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \mathbf{\Sigma}_{vv} & \mathbf{\Sigma}_{vr} \\ \mathbf{\Sigma}_{vr} & \mathbf{\Sigma}_{rr} \end{pmatrix}\right)$$

where $\mathcal{N}$ stands for Normal random variable, $\mathbf{\Sigma}_{vv} = cov(\mathbf{v}, \mathbf{v})$, $\mathbf{\Sigma}_{vr} = cov(\mathbf{v}, \mathbf{r})$, and $\mathbf{\Sigma}_{rr} = cov(\mathbf{r}, \mathbf{r})$. Therefore, f is a bivariate Gaussian density with mean $\boldsymbol{\mu} = \mathbf{\Sigma}_{vr}\mathbf{\Sigma}_{rr}^{-1}\mathbf{r}$ and covariance matrix $\mathbf{\Sigma} = \mathbf{\Sigma}_{vv} - \mathbf{\Sigma}_{vr}\mathbf{\Sigma}_{rr}^{-1}\mathbf{\Sigma}_{vr}$. Note that, for large t and s, the terms in $\mathbf{\Sigma}_{vv}$ are $O\left(t^{2H} + s^{2H} \vee t^{2H} + s^{2H-1} \vee t^{2H}\right)$, those in $\mathbf{\Sigma}_{vr}$ are $O\left(t^{2H-1} + s^{2H-1} \vee t^{2H-1}\right)$, and those in $\mathbf{\Sigma}_{rr}$ are $O\left(1 \vee s^{2H-2}\right)$.

Let $\varphi(\cdot; \boldsymbol{\mu}, \mathbf{\Sigma})$ be the bivariate Gaussian density with mean $\boldsymbol{\mu}$ and variance $\mathbf{\Sigma}$. For any $N \in \mathbb{N}$, it is straightforward to show that

$$\frac{\partial^N}{\partial \mathbf{y}^N}\varphi(\mathbf{y}; \boldsymbol{\mu}, \mathbf{\Sigma}) = \varphi(\mathbf{y}; \boldsymbol{\mu}, \mathbf{\Sigma}) \left(\sum_{k=0}^{\lfloor N/2 \rfloor} b_k \left(\mathbf{\Sigma}^{-1}\left(\mathbf{y} - \boldsymbol{\mu}\right)\right)^{\otimes(N-2k)} \otimes \left(\mathbf{\Sigma}^{-1}\right)^{\otimes k} \right) \tag{3.11}$$

where $b_k \in \mathbb{Z}$ for all k, $\otimes$ represents the Kronecker product, and $\mathbf{A}^{\otimes m}$ is the m-th Kronecker power of the matrix $\mathbf{A}$. For any multi-index β with $|\beta| = N$, the integrand on the right hand-side of (3.10) is a particular element of the tensor of order N given by (3.11) with $\mathbf{y} = \alpha\mathbf{j} + u\alpha\mathbf{e}$.

Each term of the tensor on the RHS of (3.11) can be written as

$$\sum_{k=0}^{\lfloor N/2 \rfloor} b_k \sum_{\text{some pairs } (\gamma_1, \gamma_2)} \omega_{\gamma_1}(\mathbf{\Sigma}^{-1}; N-k)\theta_{\gamma_2}(\mathbf{y}; N-2k)$$

where γ_1 is a 3-dimensional multi-index of order $N-2k$, γ_2 is a 2-dimensional multi-index of order $N-2k$, $\omega_{\gamma_1}(\mathbf{\Sigma}^{-1}; N-k)$ are monomials of order $N-k$ in the unique elements of $\mathbf{\Sigma}^{-1}$, and $\theta_{\gamma_2}(\mathbf{y}; N-2k)$ are monomials of order $N-2k$ in elements of $\mathbf{y} - \boldsymbol{\mu}$. Hence,

$$\begin{aligned} &\sum_{j_1, j_2=-\infty}^{\infty} \int_0^1 \left|\partial^{(\beta)} f(\alpha\mathbf{j} + u\alpha\mathbf{e} | r_{t+s}, r_t)\right| \mathrm{d}u \leq \\ &\leq \sum_{k=0}^{\lfloor N/2 \rfloor} |b_k| \sum_{(\gamma_1, \gamma_2)} |\omega_{\gamma_1}(\mathbf{\Sigma}^{-1}; N-k)| \sum_{j_1, j_2=-\infty}^{\infty} \int_0^1 \varphi\left(\mathbf{y}(u); \boldsymbol{\mu}, \mathbf{\Sigma}\right) \left|\theta_{\gamma_2}(\mathbf{y}(u); N-2k)\right| \mathrm{d}u. \end{aligned} \tag{3.12}$$

For any real function h, $\sum_{j_2=-\infty}^{\infty} \sum_{j_1=-\infty}^{\infty} \int_0^1 h(\alpha\mathbf{j} + u\alpha\mathbf{e})\mathrm{d}u$ is a series of integrals of h over 45-degree lines with intercepts at $\{j_2\alpha\}_{j_2 \in \mathbb{Z}}$. For large t and s, the elements of $\mathbf{\Sigma}$ are large relative to α^2 (i.e., the standard deviations of φ are large relative to the distance between the 45-degree

lines over which integration is carried out), and the operation $\sum_{j_1,j_2=-\infty}^{\infty} \int_0^1 \mathrm{d}u$ in (3.12) is well approximated by a double integral over $\mathbb{R}^2$. Namely, as $t \to \infty$ and $s \to \infty$,

$$\sum_{j_1,j_2=-\infty}^{\infty} \int_0^1 \varphi\left(\mathbf{y}(u); \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \left|\theta_{\gamma_2}(\mathbf{y}(u); N-2k)\right| \mathrm{d}u = \alpha^{-2} \int_{\mathbb{R}^2} \varphi\left(\mathbf{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \left|\theta_{\gamma_2}(\mathbf{y}; N-2k)\right| \mathrm{d}\mathbf{y} + o(1)$$

The proof relies on the Euler-MacLaurin formula and is similar to the derivation of Proposition 3.3. Since the integral of a monomial of order m in the elements of $\mathbf{y} - \boldsymbol{\mu}$ against $\varphi\left(\mathbf{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \mathrm{d}\mathbf{y}$ is a polynomial of order $m/2$ in the elements of $\boldsymbol{\Sigma}$, for all (r_t, r_{t+s}) we have

$$\begin{aligned}
|g(r_t, r_{t+s})| \leq & C_N \alpha^{N+2} \sum_{|\beta|=N} \sum_{j_1,j_2=-\infty}^{\infty} \int_0^1 \left|\partial^\beta f(\alpha\mathbf{j} + u\alpha\mathbf{e} | r_{t+s}, r_t)\right| \mathrm{d}u \\
\leq & C_N \alpha^N \sum_{|\beta|=N} \sum_{k=0}^{\lfloor N/2 \rfloor} |b_k| \sum_{\text{some } (\gamma_1,\gamma_3)} \omega_{\gamma_1}(\boldsymbol{\Sigma}^{-1}; N-k) \left(\eta_{\gamma_3}(\boldsymbol{\Sigma}; N/2-k) + o(1)\right) \\
\leq & C_N^* \left(\frac{\alpha}{\sigma}\right)^N \left(t^{-NH} + s^{-NH}\right) = \xi^{-N/2} O\left(t^{-NH} + s^{-NH}\right) \qquad \text{for large } t \text{ and } s,
\end{aligned}$$

where γ_3 is a 3-dimensional multi-index of order $N/2-k$, $\eta_{\gamma_3}(\boldsymbol{\Sigma}; N/2-k)$ is a polynomial of order $N/2-k$ in the unique elements of $\boldsymbol{\Sigma}$, and the last inequality follows from the structure of $\boldsymbol{\Sigma}$. Note that the upper bound for $|g(r_t, r_{t+s})|$ is uniform in (r_t, r_{t+s}). In fact, the RHS of the second inequality depends on (r_t, r_{t+s}) only through the error term $o(1)$ coming from the Euler-MacLaurin approximation, which is uniformly dominated by the term $O(t^{-NH} + s^{-NH})$ coming from $\omega_{\gamma_1}(\boldsymbol{\Sigma}^{-1}; N-k)\eta_{\gamma_3}(\boldsymbol{\Sigma}; N/2-k)$, which is independent of (r_t, r_{t+s}). □

Proof of Lemma 3.5. We want to calculate $\mathbb{P}\left(r_t^{(\alpha)} = k\alpha, r_{t+s}^{(\alpha)} = l\alpha\right)$ in the limit $t_0 \to -\infty$ and large s. First, note that

$$\begin{aligned}
&\mathbb{I}\left[r_t^{(\alpha)} = k\alpha, r_{t+s}^{(\alpha)} = l\alpha\right] = \\
&\mathbb{I}\left[r_t \in [k\alpha, (k+1)\alpha), r_{t+s} \in [l\alpha, (l+1)\alpha), round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0, round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 0\right] + \\
&\mathbb{I}\left[r_t \in [(k-1)\alpha, k\alpha), r_{t+s} \in [l\alpha, (l+1)\alpha), round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1, round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 0\right] + \\
&\mathbb{I}\left[r_t \in [k\alpha, (k+1)\alpha), r_{t+s} \in [(l-1)\alpha, l\alpha), round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0, round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 1\right] + \\
&\mathbb{I}\left[r_t \in [(k-1)\alpha, k\alpha), r_{t+s} \in [(l-1)\alpha, l\alpha), round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1, round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 1\right]
\end{aligned}$$

Hence, we distinguish four cases.

Case 1: $r_{t+s} \in [l\alpha, (l+1)\alpha)$ and $r_t \in [k\alpha, (k+1)\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_{t+s}, r_t\right) = \\
&\mathbb{P}\left(-\frac{\alpha}{2} \leq \widetilde{v}_{t+s-1} < \frac{\alpha}{2} - \overline{r}_{t+s}, -\frac{\alpha}{2} \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} - \overline{r}_t | r_{t+s}, r_t\right) = \\
&\left(1 + l - \frac{r_{t+s}}{\alpha}\right)\left(1 + k - \frac{r_t}{\alpha}\right)\left(1 + g(r_t, r_{t+s})\right),
\end{aligned}$$

where $g(r_t, r_{t+s})$ is the error coming from approximating the joint distribution of $\widetilde{v}_{t-1}$ and $\widetilde{v}_{t+s-1}$ conditional on (r_t, r_{t+s}) with two independent uniform distributions. Combining Lemma 3.4 and 3.12, for any $N \geq 1$ there exists a universal constant $C_N > 0$ such that, as $t_0 \to -\infty$ and $s \to \infty$,

$$|g(r_t, r_{t+s})| \leq C_N \xi^{-N/2}(|t_0|^{-NH} + s^{-NH}) \qquad \text{for all } r_t, r_{t+s}.$$

Case 2: $r_{t+s} \in [l\alpha, (l+1)\alpha)$ and $r_t \in [(k-1)\alpha, k\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_{t+s}, r_t\right) = \\
&\mathbb{P}\left(-\frac{\alpha}{2} \leq \widetilde{v}_{t+s-1} < \frac{\alpha}{2} - \overline{r}_{t+s}, \frac{\alpha}{2} - \overline{r}_t \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} | r_{t+s}, r_t\right) = \\
&\left(1 + l - \frac{r_{t+s}}{\alpha}\right)\left(1 - k + \frac{r_t}{\alpha}\right)\left(1 + g(r_t, r_{t+s})\right)
\end{aligned}$$

Case 3: $r_{t+s} \in [(l-1)\alpha, l\alpha)$ and $r_t \in [k\alpha, (k+1)\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_{t+s}, r_t\right) = \\
&\mathbb{P}\left(\frac{\alpha}{2} - \overline{r}_{t+s} \leq \widetilde{v}_{t+s-1} < \frac{\alpha}{2}, -\frac{\alpha}{2} \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} - \overline{r}_t | r_{t+s}, r_t\right) = \\
&\left(1 - l + \frac{r_{t+s}}{\alpha}\right)\left(1 + k - \frac{r_t}{\alpha}\right)\left(1 + g(r_t, r_{t+s})\right)
\end{aligned}$$

Case 4: $r_{t+s} \in [(l-1)\alpha, l\alpha)$ and $r_t \in [(k-1)\alpha, k\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_{t+s}, r_t\right) = \\
&\mathbb{P}\left(\frac{\alpha}{2} - \overline{r}_{t+s} \leq \widetilde{v}_{t+s-1} < \frac{\alpha}{2}, \frac{\alpha}{2} - \overline{r}_t \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} | r_{t+s}, r_t\right) = \\
&\left(1 - l + \frac{r_{t+s}}{\alpha}\right)\left(1 - k + \frac{r_t}{\alpha}\right)\left(1 + g(r_t, r_{t+s})\right)
\end{aligned}$$

The joint pmf of $(r_t^{(\alpha)}, r_{t+s}^{(\alpha)})$ is given by

$$\begin{aligned}
&\mathbb{P}\left(r_t^{(\alpha)} = k\alpha, r_{t+s}^{(\alpha)} = l\alpha\right) = \\
&\int_{k\alpha}^{(k+1)\alpha}\int_{l\alpha}^{(l+1)\alpha} \mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_t, r_{t+s}\right) p(r_t, r_{t+s}) \mathrm{d}r_t \mathrm{d}r_{t+s} + \\
&\int_{(k-1)\alpha}^{k\alpha}\int_{l\alpha}^{(l+1)\alpha} \mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_t, r_{t+s}\right) p(r_t, r_{t+s}) \mathrm{d}r_t \mathrm{d}r_{t+s} + \\
&\int_{k\alpha}^{(k+1)\alpha}\int_{(l-1)\alpha}^{l\alpha} \mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_t, r_{t+s}\right) p(r_t, r_{t+s}) \mathrm{d}r_t \mathrm{d}r_{t+s} + \\
&\int_{(k-1)\alpha}^{k\alpha}\int_{(l-1)\alpha}^{l\alpha} \mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_t, r_{t+s}\right) p(r_t, r_{t+s}) \mathrm{d}r_t \mathrm{d}r_{t+s},
\end{aligned} \tag{3.13}$$

where $p(r_t, r_{t+s})$ is the joint Gaussian density of r_t and r_{t+s}. The final step is to use a series expansion of the bivariate Gaussian density in Hermite polynomials. Hermite polynomials are an orthonormal polynomial system with Gaussian weights. Specifically, we use the following normalization of Hermite polynomials,

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \delta_{nm} \tag{3.14}$$

where δ_{nm} is the Kronecker delta. For example, $H_0(x) = 1$, $H_1(x) = x$, and $H_2(x) = x^2 - 1$. The expansion of the bivariate Gaussian density function $p(x, y)$ in Hermite polynomials is

$$p(x, y) = p(x)p(y)\left[1 + \sum_{j=1}^{\infty} \rho^j H_j(x) H_j(y)\right] \tag{3.15}$$

where $p(x)$ is the univariate Gaussian density function and ρ is the correlation coefficient between variables x and y. By using the Hermite polynomial expansion of $p(r_t, r_{t+s})$ up to the second order (i.e., up to the Hermite polynomial of degree one), substituting the above expressions for the terms $\mathbb{P}\left(round\left(\widetilde{v}_{t+s-1} + \overline{r}_{t+s}\right) = m, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = n | r_t, r_{t+s}\right)$ with $m, n \in \{0, 1\}$ and the bound on $|g(r_t, r_{t+s})|$, and carrying out the integrals, we obtain the desired result. □

Proof of Proposition 3.6. From Lemma 3.5, we can write the joint pmf of $\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right)$ as

$$p_\alpha(k, l) = p_0(k)p_0(l) + \rho_s p_1(k)p_1(l) + G(k, l) + H(k, l)$$

where $p_0(\cdot)$, $p_1(\cdot)$ are defined in Lemma 3.2 and 3.5, ρ_s is the autocorrelation function of $\{r_t\}$, $G(k, l)$ contains all the terms of order higher than 2 in the Hermite expansion of (3.13) under

the assumption that $\widetilde{v}_{t-1}$ and $\widetilde{v}_{t+s-1}$ conditional on (r_t, r_{t+s}) are independent and uniformly distributed, and $H(k,l)$ is the error term coming from approximating the joint distribution of $\widetilde{v}_{t-1}$ and $\widetilde{v}_{t+s-1}$ with two independent uniform distributions. Note that $G(k,l) = O(\rho_s^2)$ for large s, and for any $N \geq 1$, $H(k,l) = O(|t_0|^{-NH} + s^{-NH})$ in the limit $t_0 \to -\infty$ and large s.

Then, we can write

$$\begin{aligned} cov\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right) = &\left(\alpha \sum_{k=-\infty}^{\infty} k p_0(k)\right)^2 + \rho_s \left(\alpha \sum_{k=-\infty}^{\infty} k p_1(k)\right)^2 + \\ &\alpha^2 \left(\sum_{k,l} klG(k,l) + \sum_{k,l} klH(k,l)\right) - \left(\mathbb{E}\left[r_t^{(\alpha)}\right]\right)^2 \\ =&\rho_s \left(\alpha \sum_{k=-\infty}^{\infty} k p_1(k)\right)^2 + \alpha^2 \left(\sum_{k,l} klG(k,l) + \sum_{k,l} klH(k,l)\right) \end{aligned}$$

where the last equality follows from the fact that $p_0(\cdot)$ is the marginal density of $r^{(\alpha)}$. Hence,

$$\left| cov\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right) - \rho_s \alpha^2 \left(\sum_{k=-\infty}^{\infty} k p_1(k)\right)^2 \right| \leq \alpha^2 \left(\sum_{k,l} |k||l||G(k,l)| + \sum_{k,l} |k||l||H(k,l)|\right) \quad (3.16)$$

Since $var(|r_t^{\alpha}|) < \infty$ for every t (see Lemma 3.2, its proof, and Proposition 3.3), from Cauchy-Schwartz inequality $\left|cov\left(|r_t^{(\alpha)}|, |r_{t+s}^{(\alpha)}|\right)\right| < \infty$ for all t and s, which implies that $\{klG(k,l)\}$ and $\{klH(k,l)\}$ must be absolutely summable over (k,l).

Since $G(k,l)$ is an absolutely convergent power series in ρ_s with first element of order ρ_s^2 and depends on s only through the powers of ρ_s, and since $\{klG(k,l)\}$ is absolutely summable over (k,l), the first term on the RHS of (3.16) is $O\left(\rho_s^2\right)$ for large s. From the proof of Lemma 3.5 we know that for any $N \geq 1$ there exists a universal constant C_N s.t., as $t_0 \to -\infty$ and large s,

$$|H(k,l)| < C_N \xi^{-N/2} \left(|t_0|^{-NH} + s^{-NH}\right) \left[p_0(k)p_0(l) + \rho_s p_1(k)p_1(l) + G(k,l)\right].$$

Therefore, since $\mathbb{E}\left[|r^{(\alpha)}|\right] < \infty$ for all t, the second term on the RHS of (3.16) is $O\left(|t_0|^{-NH} + s^{-NH}\right)$ as $t_0 \to -\infty$ and for large s.

Now, it is straightforward to see that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} k p_1(k) = \lim_{M\to\infty} 2\sum_{k=1}^{M} k p_1(k) &= \sqrt{\xi} \lim_{M\to\infty} (-M-1)\, erf\left(\frac{M}{\sqrt{2\xi}}\right) + M erf\left(\frac{M+1}{\sqrt{2\xi}}\right) \\ &= -\sqrt{\xi} \end{aligned}$$

Hence, $\rho_s\alpha^2\left(\sum_{k=-\infty}^{\infty} kp_1(k)\right)^2 = \rho_s\sigma^2 = cov(r_t, r_{t+s})$, and we have the desired result:

$$cov\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right) = cov(r_t, r_{t+s}) + O\left(\rho_s^2\right) + \xi^{-N/2}O\left(|t_0|^{-NH} + s^{-NH}\right) \qquad \text{for any } N \geq 1.$$

From the above result, it follows immediately that if $\rho_s \sim s^{-\beta}$ for some $\beta > 0$ as $s \to \infty$, the autocovariance of the realized process is asymptotically equal to that of the underlying one, i.e., $cov\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right) \sim cov(r_t, r_{t+s})$ as $s \to \infty$. On the other hand, if the ρ_s decays faster than any power law, the error term $O\left(s^{-NH}\right)$ might be dominating.

Under the assumption of power law decay of ρ_s, the asymptotic behavior of $cor\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right)$ for small and large ξ follows from the behavior of $var\left(r_t^{(\alpha)}\right)$ in the limit of small and large ξ (Proposition 3.3), and from the fact that $cov\left(r_t^{(\alpha)}, r_{t+s}^{(\alpha)}\right) \sim cov(r_t, r_{t+s})$ for large s. □

Proof of Proposition 3.7. We want to calculate $\mathbb{P}\left(r_t^{(\alpha)} = k\alpha, r_{t+1}^{(\alpha)} = l\alpha\right)$ in the limit $t_0 \to -\infty$. As in the proof of Lemma 3.5, we need to distinguish four cases:

Case 1: $r_t \in [k\alpha, (k+1)\alpha)$ and $r_{t+1} \in [l\alpha, (l+1)\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(-\frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t + \overline{r}_{t+1} < \frac{\alpha}{2}, -\frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t < \frac{\alpha}{2} | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(-\frac{\alpha}{2} \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} - \overline{r}_t - \overline{r}_{t+1} | r_t, r_{t+1}\right) = \\
&\left(1 + k + l - \frac{r_t + r_{t+1}}{\alpha}\right) \mathbb{I}\left[r_t + r_{t+1} \leq \alpha(1 + k + l)\right] \left(1 + g(r_t, r_{t+1})\right)
\end{aligned}$$

Case 2: $r_t \in [(k-1)\alpha, k\alpha)$ and $r_{t+1} \in [l\alpha, (l+1)\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(-\frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t + \overline{r}_{t+1} - \alpha < \frac{\alpha}{2}, \frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t < \frac{3\alpha}{2} | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(\frac{\alpha}{2} - \overline{r}_t \leq \widetilde{v}_{t-1} < \frac{3\alpha}{2} - \overline{r}_t - \overline{r}_{t+1} | r_t, r_{t+1}\right) = \\
&\left(\left(1 + l - \frac{r_{t+1}}{\alpha}\right) \mathbb{I}\left[r_t + r_{t+1} > \alpha(k + l)\right] + \left(1 - k + \frac{r_t}{\alpha}\right) \mathbb{I}\left[r_t + r_{t+1} \leq \alpha(k + l)\right]\right) \left(1 + g(r_t, r_{t+1})\right)
\end{aligned}$$

Case 3: $r_t \in [k\alpha, (k+1)\alpha)$ and $r_{t+1} \in [(l-1)\alpha, l\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(\frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t + \overline{r}_{t+1} < \frac{3\alpha}{2}, -\frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t < \frac{\alpha}{2} | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(\frac{\alpha}{2} - \overline{r}_t - \overline{r}_{t+1} \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} - \overline{r}_t | r_t, r_{t+1}\right) = \\
&\left(\left(1 - l + \frac{r_{t+1}}{\alpha}\right)\mathbb{I}\left[r_t + r_{t+1} \leq \alpha(k+l)\right] + \left(1 + k - \frac{r_t}{\alpha}\right)\mathbb{I}\left[r_t + r_{t+1} > \alpha(k+l)\right]\right)\left(1 + g(r_t, r_{t+1})\right)
\end{aligned}$$

Case 4: $r_t \in [(k-1)\alpha, k\alpha)$ and $r_{t+1} \in [(l-1)\alpha, l\alpha)$. Then,

$$\begin{aligned}
&\mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(\frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t + \overline{r}_{t+1} - \alpha < \frac{3\alpha}{2}, \frac{\alpha}{2} \leq \widetilde{v}_{t-1} + \overline{r}_t < \frac{3\alpha}{2} | r_t, r_{t+1}\right) = \\
&\mathbb{P}\left(\frac{3\alpha}{2} - \overline{r}_t - \overline{r}_{t+1} \leq \widetilde{v}_{t-1} < \frac{\alpha}{2} | r_t, r_{t+1}\right) = \\
&\left(1 - k - l + \frac{r_t + r_{t+1}}{\alpha}\right)\mathbb{I}\left[r_t + r_{t+1} > \alpha(k+l-1)\right]\left(1 + g(r_t, r_{t+1})\right)
\end{aligned}$$

The function $g(r_t, r_{t+1})$ above represents the error coming from approximating the joint distribution of $\widetilde{v}_{t-1}$ and $\widetilde{v}_t$ conditional on (r_t, r_{t+1}) with two independent uniform distributions. From Lemma 3.4 and 3.12, we know that for any $N \in \mathbb{N}$, there exists a universal constant C_N s.t. $|g(r_t, r_{t+1})| < C_N \xi^{-N/2} O(|t_0|^{-NH})$ as $t_0 \to -\infty$.

The joint pmf of $(r_t^{(\alpha)}, r_{t+1}^{(\alpha)})$ is given by

$$\begin{aligned}
&\mathbb{P}\left(r_t^{(\alpha)} = k\alpha, r_{t+1}^{(\alpha)} = l\alpha\right) = \\
= &\int_{k\alpha}^{(k+1)\alpha}\int_{l\alpha}^{(l+1)\alpha} \mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_t, r_{t+1}\right) p(r_t, r_{t+1}) \mathrm{d}r_t \mathrm{d}r_{t+1} \\
+ &\int_{(k-1)\alpha}^{k\alpha}\int_{l\alpha}^{(l+1)\alpha} \mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 0, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_t, r_{t+1}\right) p(r_t, r_{t+1}) \mathrm{d}r_t \mathrm{d}r_{t+1} \\
+ &\int_{k\alpha}^{(k+1)\alpha}\int_{(l-1)\alpha}^{l\alpha} \mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 0 | r_t, r_{t+1}\right) p(r_t, r_{t+1}) \mathrm{d}r_t \mathrm{d}r_{t+1} \\
+ &\int_{(k-1)\alpha}^{k\alpha}\int_{(l-1)\alpha}^{l\alpha} \mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = 1, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = 1 | r_t, r_{t+1}\right) p(r_t, r_{t+1}) \mathrm{d}r_t \mathrm{d}r_{t+1},
\end{aligned}$$

where $p(r_t, r_{t+1})$ is the joint Gaussian density of r_t and r_{t+1}, with correlation coefficient ρ_1. Finally, we use the Hermite polynomial expansion of the bivariate Gaussian density up to the second order (i.e., up to Hermite polynomials of degree one) and substitute the above expressions

for the terms $\mathbb{P}\left(round\left(\widetilde{v}_t + \overline{r}_{t+1}\right) = m, round\left(\widetilde{v}_{t-1} + \overline{r}_t\right) = n | r_t, r_{t+1}\right)$ with $m, n \in \{0, 1\}$ and for the upper bound on $|g(r_t, r_{t+s})|$. By carrying out the integrals and taking the limit $t_0 \to -\infty$, we obtain the result. □

Proof of Proposition 3.8. First, note that

$$cov(\mathbb{I}\left[r_t^{(\alpha)} = 0\right], \mathbb{I}\left[r_{t+s}^{(\alpha)} = 0\right]) = \mathbb{P}\left(r_t^{(\alpha)} = 0, r_{t+s}^{(\alpha)} = 0\right) - \mathbb{P}\left(r_t^{(\alpha)} = 0\right)\mathbb{P}\left(r_{t+s}^{(\alpha)} = 0\right).$$

From Lemma 3.5, it follows that if the process started in the infinite past (i.e. $t_0 = -\infty$),

$$cov(\mathbb{I}\left[r_t^{(\alpha)} = 0\right], \mathbb{I}\left[r_{t+s}^{(\alpha)} = 0\right]) = O\left(\rho_s + s^{-NH}\right)$$

for large s and any $N \geq 1$. Then, the rest of the proof is a direct application of Corollary 4 in Lyons (1988) and the continuous mapping theorem. □

Proof of Proposition 3.9. First, note that for

$$cov(\mathbb{I}\left[r_t^{(\alpha)} = 0, r_{t-1}^{(\alpha)} = 0\right], \mathbb{I}\left[r_{t+s}^{(\alpha)} = 0, r_{t+s-1}^{(\alpha)} = 0\right]) =$$
$$\mathbb{P}\left(r_t^{(\alpha)} = 0, r_{t-1}^{(\alpha)} = 0, r_{t+s}^{(\alpha)} = 0, r_{t+s-1}^{(\alpha)} = 0\right) - \mathbb{P}\left(r_t^{(\alpha)} = 0, r_{t-1}^{(\alpha)} = 0\right)\mathbb{P}\left(r_{t+s}^{(\alpha)} = 0, r_{t+s-1}^{(\alpha)} = 0\right).$$

Then, by using the same argument as in the proof of Lemma 3.5 and the generalized Hermite polynomial expansion of multivariate normal densities (of dimension 4 in this case), it is straightforward to show that if the process started in the infinite past (i.e. $t_0 = -\infty$),

$$cov(\mathbb{I}\left[r_t^{(\alpha)} = 0, r_{t-1}^{(\alpha)} = 0\right], \mathbb{I}\left[r_{t+s}^{(\alpha)} = 0, r_{t+s-1}^{(\alpha)} = 0\right]) = O\left(\rho_s + s^{-NH}\right)$$

for large s and any $N \geq 1$. The rest of the proof is a direct application of Corollary 4 in Lyons (1988) and the continuous mapping theorem. □

Appendix 3.B Detrended Fluctuation Analysis

The Detrended Fluctuation Analysis (DFA) (see Peng *et al.*, 1994) is a semiparametric method that works in the time domain. The idea is to consider the integrated process and detrend it locally. The scaling of the fluctuations of the residuals as a function of the box size in which the regression is performed gives the estimate of the Hurst exponent. More precisely, let $\{X(t)\}$, $t = 1, \ldots, n$, be a finite sample from a process $\{X(t)\}_{t\in\mathbb{N}}$ and denote the discrete integration of

this sample as

$$Y(k) = \sum_{t=1}^{k} X(t)$$

The integrated time series is divided into $[n/m]$ boxes of equal length m, where $[z]$ is the integer part of z. In each box a least squares line is fit to the data (representing the trend in that box). The y-coordinate of the straight line segments is denoted by $\widehat{Y}_m(k)$. Next, one detrends the integrated time series, $Y(k)$, by subtracting the local trend, $\widehat{Y}_m(k)$, in each box. For any given box size m, the root-mean-square fluctuation (or, simply, fluctuation function) of this integrated and detrended time series is calculated by

$$F(m) = \sqrt{\frac{1}{m \cdot [n/m]} \sum_{k=1}^{m \cdot [n/m]} [Y(k) - \widehat{Y}_m(k)]^2}. \tag{3.17}$$

This computation is repeated over all time scales (box sizes) to characterize the relationship between $F(m)$ and the box size m. Typically, $F(m)$ increases with box size m. Recently, La Spada and Lillo (2014), extending and correcting previous results in the literature, proved that $F(m) \sim m^H$ for large m, for a large class of stationary long-memory process. Since the realized increments belong to this class and have the same Hurst exponent as the the underlying increments, we can apply the DFA to the realized series to estimate the Hurst exponent of the underlying one.

H is estimated by performing a log-regression of $F(m)$ versus m. The ambiguity is, however, the interval of values of m where one performs the fit. To the best of our knowledge there is no rule for selecting optimally such an interval. One might expect to obtain a less biased, but noisier, estimate of H by performing the fit in a small region corresponding to large values of the block size m. To investigate this point we estimate the Hurst exponent by performing the fit over a fraction q of the largest values of $\log_{10}[m]$. In the following numerical simulations we consider four values of q, namely, $q = 1$ (the whole interval), $q = 0.75$ (the largest three-quarter), $q = 0.5$ (the largest half), and $q = 0.25$ (the largest quartile). The standard procedure for the DFA is to use $q = 1$.

Numerical Results Our numerical results for the DFA estimator are reported in Tables 3.B.1 and 3.B.2. They show that the DFA estimator applied to the realized increments is significantly and severely negatively biased in both medium-sized and long samples, and for both $H = 0.7$ and $H = 0.85$. The bias is stronger for smaller values of ξ (i.e., coarser discretizations). In medium-sized samples the absolute value of the bias is maximum when $q = 1$ for all the discretizations

considered. In long samples it is maximum when $q = 1$ for coarser discretizations (i.e., $\xi = 0.1$ and 0.25), and when $q = 0.25$ for mild discretization (i.e., $\xi = 0.5$). On the other hand, for the fBm the absolute bias decreases with q in both medium-sized and long samples: it is minimum when $q = 1$ and maximum when $q = 0.25$. For coarse discretizations the root mean square error (RMSE) is minimum for $q \in [0.25, 0.5]$ in medium-sized samples, and for $q \in [0.5, 0.75]$ in long samples. For coarse discretizations the RMSE is maximum when $q = 1$ in both medium-sized and long samples. For mild discretizations the RMSE is minimum when $q = 0.5$ in medium-sized samples, and when $q = 0.75$ in long samples. On the other hand, for the fBm the RMSE decreases with q: it is maximum when $q = 0.25$ and minimum when $q = 1$.

We conclude that, when applied to the realized increments, the DFA is a significantly negatively biased estimator of the Hurst exponent of the underlying increments, especially if the standard bandwidth $q = 1$ is used. For coarse discretizations our numerical simulations suggest to take $q = 0.25$ in medium-sized samples and $q = 0.5$ in long samples, as optimal bandwidth minimizing the RMSE.

	IID		AR(1)							
			$\rho_1 = 0.05$		$\rho_1 = 0.1$		$\rho_1 = 0.15$		$\rho_1 = 0.20$	
	$\text{Mean}(\widehat{\sigma^2} - \sigma^2)$	2.5% 50%	$\text{Mean}(\widehat{\sigma^2} - \sigma^2)$	2.5% 50%	$\text{Mean}(\widehat{\sigma^2} - \sigma^2)$	2.5% 50%	$\text{Mean}(\widehat{\sigma^2} - \sigma^2)$	2.5% 50%	$\text{Mean}(\widehat{\sigma^2} - \sigma^2)$	2.5% 50%
	$\text{SE}(\widehat{\sigma^2})$	97.5%	$\text{SE}(\widehat{\sigma^2})$	97.5%	$\text{SE}(\widehat{\sigma^2})$	97.5%	$\text{SE}(\widehat{\sigma^2})$	97.5%	$\text{SE}(\widehat{\sigma^2})$	97.5%
		-0.109		-0.100		-0.098		-0.088		-0.095
$\xi = 0.025$	$1.75 \cdot 10^{-3}$	0.002	$2.04 \cdot 10^{-3}$	0.000	$-1.50 \cdot 10^{-3}$	-0.005	$2.90 \cdot 10^{-3}$	0.001	$-1.35 \cdot 10^{-3}$	-0.004
	$1.82 \cdot 10^{-3}$	0.114	$1.73 \cdot 10^{-3}$	0.110	$1.68 \cdot 10^{-3}$	0.110	$1.55 \cdot 10^{-3}$	0.100	$1.57 \cdot 10^{-3}$	0.095
		-0.071		-0.075		-0.072		-0.068		-0.074
$\xi = 0.05$	$2.10 \cdot 10^{-3}$	0.003	$-5.52 \cdot 10^{-4}$	0.002	$1.47 \cdot 10^{-3}$	0.001	$6.33 \cdot 10^{-4}$	-0.000	$2.95 \cdot 10^{-3}$	0.004
	$1.26 \cdot 10^{-3}$	0.079	$1.26 \cdot 10^{-3}$	0.075	$1.21 \cdot 10^{-3}$	0.079	$1.17 \cdot 10^{-3}$	0.072	$1.16 \cdot 10^{-3}$	0.073
		-0.055		-0.050		-0.056		-0.051		-0.053
$\xi = 0.1$	$-3.26 \cdot 10^{-5}$	-0.000	$1.20 \cdot 10^{-3}$	0.001	$6.76 \cdot 10^{-4}$	-0.000	$-5.04 \cdot 10^{-4}$	-0.002	$9.95 \cdot 10^{-4}$	-0.001
	$9.05 \cdot 10^{-4}$	0.057	$9.13 \cdot 10^{-4}$	0.056	$9.11 \cdot 10^{-4}$	0.058	$8.42 \cdot 10^{-4}$	0.055	$8.60 \cdot 10^{-4}$	0.055
		-0.043		-0.042		-0.040		-0.042		-0.040
$\xi = 0.25$	$1.08 \cdot 10^{-3}$	0.001	$4.32 \cdot 10^{-4}$	0.001	$6.45 \cdot 10^{-4}$	0.001	$-3.83 \cdot 10^{-4}$	0.000	$9.43 \cdot 10^{-5}$	-0.000
	$7.11 \cdot 10^{-4}$	0.049	$7.08 \cdot 10^{-4}$	0.047	$6.72 \cdot 10^{-4}$	0.044	$6.69 \cdot 10^{-4}$	0.041	$6.71 \cdot 10^{-4}$	0.042
		-0.043		-0.044		-0.046		-0.042		-0.042
$\xi = 0.5$	$3.24 \cdot 10^{-4}$	0.000	$1.42 \cdot 10^{-4}$	-0.001	$4.51 \cdot 10^{-4}$	0.002	$5.75 \cdot 10^{-5}$	-0.001	$7.57 \cdot 10^{-5}$	0.001
	$6.99 \cdot 10^{-4}$	0.047	$7.05 \cdot 10^{-4}$	0.045	$7.14 \cdot 10^{-4}$	0.043	$6.56 \cdot 10^{-4}$	0.040	$6.87 \cdot 10^{-4}$	0.043
		-0.047		-0.046		-0.047		-0.047		-0.045
$\xi = 1$	$-5.98 \cdot 10^{-4}$	-0.001	$6.27 \cdot 10^{-5}$	-0.000	$3.45 \cdot 10^{-5}$	0.000	$9.88 \cdot 10^{-5}$	0.001	$-6.31 \cdot 10^{-4}$	-0.001
	$7.51 \cdot 10^{-4}$	0.047	$7.88 \cdot 10^{-4}$	0.050	$8.04 \cdot 10^{-4}$	0.050	$7.49 \cdot 10^{-4}$	0.046	$7.69 \cdot 10^{-4}$	0.047

TABLE 3.4.1: Estimation of σ^2. Sample size: $n = 2^{14}$. Number of MC iterations: $L = 10^3$.

	IID		AR(1)							
			$\rho_1 = 0.05$		$\rho_1 = 0.1$		$\rho_1 = 0.15$		$\rho_1 = 0.20$	
		2.5%		2.5%		2.5%		2.5%		2.5%
	Mean$(\widehat{\rho_1} - \rho_1)$	50%	Mean$(\widehat{\rho_1} - \rho_1)$	50%	Mean$(\widehat{\rho_1} - \rho_1)$	50%	Mean$(\widehat{\rho_1} - \rho_1)$	50%	Mean$(\widehat{\rho_1} - \rho_1)$	50%
	SE$(\widehat{\rho_1})$	97.5%	SE$(\widehat{\rho_1})$	97.5%	SE$(\widehat{\rho_1})$	97.5%	SE$(\widehat{\rho_1})$	97.5%	SE$(\widehat{\rho_1})$	97.5%
		-0.056		-0.058		-0.056		-0.059		-0.060
$\xi = 0.025$	-1.53 $\cdot 10^{-4}$	0.000	-1.24 $\cdot 10^{-3}$	-0.000	-2.14 $\cdot 10^{-3}$	-0.002	-7.13 $\cdot 10^{-3}$	-0.008	-9.32 $\cdot 10^{-3}$	-0.010
	9.08 $\cdot 10^{-4}$	0.057	8.87 $\cdot 10^{-4}$	0.051	8.67 $\cdot 10^{-4}$	0.049	8.51 $\cdot 10^{-4}$	0.046	8.36 $\cdot 10^{-4}$	0.043
		-0.044		-0.048		-0.049		-0.054		-0.053
$\xi = 0.05$	-8.57 $\cdot 10^{-4}$	-0.002	-3.20 $\cdot 10^{-4}$	0.000	-2.87 $\cdot 10^{-3}$	-0.003	-5.07 $\cdot 10^{-3}$	-0.005	-1.10 $\cdot 10^{-2}$	-0.011
	7.30 $\cdot 10^{-4}$	0.048	7.62 $\cdot 10^{-4}$	0.046	6.91 $\cdot 10^{-4}$	0.041	7.22 $\cdot 10^{-4}$	0.038	7.09 $\cdot 10^{-4}$	0.033
		-0.039		-0.046		-0.045		-0.053		-0.059
$\xi = 0.1$	6.10 $\cdot 10^{-4}$	0.001	-2.48 $\cdot 10^{-3}$	-0.002	-5.14 $\cdot 10^{-3}$	-0.006	-1.06 $\cdot 10^{-2}$	-0.010	-1.92 $\cdot 10^{-2}$	-0.019
	6.67 $\cdot 10^{-4}$	0.041	6.83 $\cdot 10^{-4}$	0.038	6.49 $\cdot 10^{-4}$	0.035	6.68 $\cdot 10^{-4}$	0.028	6.47 $\cdot 10^{-4}$	0.020
		-0.069		-0.075		-0.081		-0.096		-0.115
$\xi = 0.25$	-1.75 $\cdot 10^{-3}$	-0.002	-3.03 $\cdot 10^{-3}$	-0.004	-1.15 $\cdot 10^{-2}$	-0.011	-2.63 $\cdot 10^{-2}$	-0.027	-4.64 $\cdot 10^{-2}$	-0.045
	1.11 $\cdot 10^{-3}$	0.073	1.12 $\cdot 10^{-3}$	0.065	1.14 $\cdot 10^{-3}$	0.057	1.12 $\cdot 10^{-3}$	0.041	1.09 $\cdot 10^{-3}$	0.020
		-0.152		-0.159		-0.173		-0.214		-0.248
$\xi = 0.5$	-5.79 $\cdot 10^{-4}$	-0.001	-4.86 $\cdot 10^{-3}$	-0.006	-2.27 $\cdot 10^{-2}$	-0.026	-5.75 $\cdot 10^{-2}$	-0.055	-1.00 $\cdot 10^{-1}$	-0.101
	2.40 $\cdot 10^{-3}$	0.150	2.44 $\cdot 10^{-3}$	0.147	2.42 $\cdot 10^{-3}$	0.134	2.39 $\cdot 10^{-3}$	0.092	2.42 $\cdot 10^{-3}$	0.050
		-0.385		-0.410		-0.450		-0.513		-0.632
$\xi = 1$	4.19 $\cdot 10^{-4}$	0.001	-7.64 $\cdot 10^{-3}$	-0.008	-4.42 $\cdot 10^{-2}$	-0.046	-1.20 $\cdot 10^{-1}$	-0.116	-2.08 $\cdot 10^{-1}$	-0.206
	6.30 $\cdot 10^{-3}$	0.379	6.20 $\cdot 10^{-3}$	0.369	6.29 $\cdot 10^{-3}$	0.345	6.28 $\cdot 10^{-3}$	0.259	6.63 $\cdot 10^{-3}$	0.196

TABLE 3.4.2: Estimation of ρ_1. Sample size: $n = 2^{14}$. Number of MC iterations: $L = 10^3$.

	$m = n^{0.5}$		$m = n^{0.6}$		$m = n^{0.7}$		$m = n^{0.8}$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.4418		0.5404		0.5527		0.4935
$\xi = 0.1$	0.6839	0.6875	0.6866	0.6893	0.6498	0.6525	0.5603	0.5607
	(0.0035)	0.8908	(0.0023)	0.8167	(0.0015)	0.7369	(0.0011)	0.6234
		0.4492		0.5379		0.5894		0.5691
$\xi = 0.25$	0.6905	0.6906	0.6917	0.6927	0.6844	0.6850	0.6344	0.6352
	(0.0037)	0.9117	(0.0023)	0.8244	(0.0015)	0.7741	(0.0011)	0.6980
		0.4699		0.5483		0.5947		0.6025
$\xi = 0.5$	0.6881	0.6886	0.6949	0.7003	0.6898	0.6910	0.6669	0.6667
	(0.0036)	0.8897	(0.0023)	0.8305	(0.0016)	0.7850	(0.0010)	0.7286
		0.4589		0.5552		0.6016		0.6484
fBm	0.6958	0.7004	0.6990	0.7034	0.7018	0.7031	0.7114	0.7112
	(0.0035)	0.9050	(0.0023)	0.8286	(0.0016)	0.7965	(0.0010)	0.7750
				$n = 2^{14}$				
		0.6032		0.6376		0.6594		0.6236
$\xi = 0.1$	0.7003	0.7029	0.6980	0.6989	0.6927	0.6927	0.6445	0.6444
	(0.0015)	0.7911	(0.0009)	0.7533	(0.0005)	0.7289	(0.0003)	0.6661
		0.5965		0.6384		0.6618		0.6597
$\xi = 0.25$	0.6986	0.6998	0.6987	0.6998	0.6977	0.6980	0.6797	0.6792
	(0.0016)	0.7961	(0.0010)	0.7591	(0.0006)	0.7319	(0.0003)	0.6991
		0.6004		0.6448		0.6652		0.6719
$\xi = 0.5$	0.6970	0.6986	0.6996	0.7007	0.6992	0.6984	0.6924	0.6929
	(0.0015)	0.7844	(0.0009)	0.7532	(0.0005)	0.7331	(0.0003)	0.7127
		0.6055		0.6430		0.6659		0.6853
fBm	0.7010	0.7016	0.6989	0.6995	0.7007	0.7012	0.7050	0.7048
	(0.0015)	0.7934	(0.0009)	0.7518	(0.0005)	0.7328	(0.0003)	0.7258

TABLE 3.4.3: LW estimation of the Hurst exponent of a fBm with $H = 0.7$ and realized increments with $\xi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The LW objective function uses only the first m Fourier frequencies (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

	$m = n^{0.5}$		$m = n^{0.6}$		$m = n^{0.7}$		$m = n^{0.8}$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.5816		0.6918		0.6844		0.5961
$\xi = 0.1$	0.8406	0.8461	0.8375	0.8392	0.7926	0.7940	0.6677	0.6688
	(0.0037)	1.0566	(0.0024)	0.9825	(0.0016)	0.8830	(0.0012)	0.7412
		0.6106		0.6953		0.7450		0.6909
$\xi = 0.25$	0.8438	0.8482	0.8473	0.8493	0.8320	0.8317	0.7571	0.7587
	(0.0034)	1.0490	(0.0023)	0.9913	(0.0015)	0.9258	(0.0010)	0.8214
		0.6372		0.7000		0.7412		0.7383
$\xi = 0.5$	0.8464	0.8463	0.8452	0.8459	0.8419	0.8419	0.8037	0.8050
	(0.0035)	1.0618	(0.0023)	0.9789	(0.0016)	0.9407	(0.0010)	0.8644
		0.6101		0.6987		0.7571		0.8042
fBm	0.8508	0.8529	0.8495	0.8538	0.8529	0.8517	0.8693	0.8703
	(0.0035)	1.0488	(0.0023)	0.9780	(0.0015)	0.9419	(0.0010)	0.9316
				$n = 2^{14}$				
		0.7519		0.7942		0.8106		0.7588
$\xi = 0.1$	0.8519	0.8538	0.8512	0.8512	0.8445	0.8444	0.7815	0.7819
	(0.0015)	0.9443	(0.0009)	0.9050	(0.0005)	0.8775	(0.0004)	0.8038
		0.7655		0.7914		0.8137		0.8028
$\xi = 0.25$	0.8546	0.8545	0.8504	0.8507	0.8492	0.8491	0.8243	0.8241
	(0.0014)	0.9425	(0.0009)	0.9063	(0.0005)	0.8840	(0.0003)	0.8447
		0.7518		0.7934		0.8160		0.8206
$\xi = 0.5$	0.8505	0.8511	0.8513	0.8517	0.8503	0.8516	0.8399	0.8400
	(0.0016)	0.9441	(0.0009)	0.9049	(0.0005)	0.8818	(0.0003)	0.8588
		0.7458		0.7911		0.8162		0.8362
fBm	0.8481	0.8494	0.8518	0.8522	0.8515	0.8512	0.8568	0.8572
	(0.0016)	0.9401	(0.0009)	0.9072	(0.0006)	0.8844	(0.0003)	0.8778

TABLE 3.4.4: LW estimation of the Hurst exponent of a fBm with $H = 0.85$ and realized increments with $\xi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The LW objective function uses only the first m Fourier frequencies (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

	$q = 0.25$		$q = 0.5$		$q = 0.75$		$q = 1$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.3584		0.4694		0.4950		0.4903
$\xi = 0.1$	0.6593	0.6546	0.6468	0.6477	0.6134	0.6143	0.5721	0.5726
	(0.0051)	0.9811	(0.0028)	0.8205	(0.0018)	0.7240	(0.0013)	0.6502
		0.3760		0.4900		0.5447		0.5620
$\xi = 0.25$	0.6786	0.6781	0.6697	0.6709	0.6553	0.6560	0.6338	0.6342
	(0.0051)	1.0001	(0.0028)	0.8373	(0.0018)	0.7649	(0.0012)	0.7116
		0.3747		0.5069		0.5609		0.5903
$\xi = 0.5$	0.6750	0.6658	0.6792	0.6783	0.6738	0.6738	0.6655	0.6641
	(0.0053)	1.0207	(0.0029)	0.8584	(0.0018)	0.7913	(0.0013)	0.7421
		0.3871		0.5096		0.5766		0.6259
fBm	0.6832	0.6762	0.6891	0.6870	0.6938	0.6929	0.7063	0.7047
	(0.0051)	1.0280	(0.0030)	0.8790	(0.0019)	0.8142	(0.0013)	0.7861
				$n = 2^{14}$				
		0.4847		0.5943		0.6226		0.6027
$\xi = 0.1$	0.6830	0.6818	0.6886	0.6895	0.6785	0.6794	0.6413	0.6421
	(0.0034)	0.8959	(0.0016)	0.7831	(0.0009)	0.7308	(0.0006)	0.6756
		0.4767		0.5966		0.6314		0.6331
$\xi = 0.25$	0.6851	0.6831	0.6895	0.6893	0.6871	0.6882	0.6697	0.6704
	(0.0033)	0.9051	(0.0015)	0.7827	(0.0009)	0.7371	(0.0006)	0.7026
		0.4815		0.5909		0.6380		0.6489
$\xi = 0.5$	0.6797	0.6764	0.6880	0.6867	0.6900	0.6900	0.6832	0.6833
	(0.0033)	0.8800	(0.0016)	0.7894	(0.0009)	0.7444	(0.0006)	0.7188
		0.4826		0.5958		0.6386		0.6634
fBm	0.6803	0.6766	0.6890	0.6888	0.6937	0.6948	0.7002	0.7013
	(0.0033)	0.8901	(0.0015)	0.7856	(0.0009)	0.7473	(0.0006)	0.7341

TABLE 3.B.1: DFA estimation of the Hurst exponent of a fBm with $H = 0.7$ and realized increments with $\xi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The least square regression $\log_{10}[F]$ vs. $\log_{10}[m]$ is performed over a q-fraction of the largest values of $\log_{10}[m]$ (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

	$q = 0.25$		$q = 0.5$		$q = 0.75$		$q = 1$	
		2.5%		2.5%		2.5%		2.5%
	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%	Mean($\widehat{H}$)	50%
	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%	(SE($\widehat{H}$))	97.5%
				$n = 2^{10}$				
		0.4568		0.5979		0.6275		0.5934
$\xi = 0.1$	0.8011	0.8023	0.7913	0.7907	0.7485	0.7476	0.6812	0.6809
	(0.0057)	1.1491	(0.0032)	0.9812	(0.0020)	0.8696	(0.0014)	0.7673
		0.4677		0.6257		0.6663		0.6696
$\xi = 0.25$	0.8159	0.8004	0.8162	0.8151	0.7962	0.7939	0.7555	0.7554
	(0.0060)	1.2166	(0.0033)	1.0138	(0.0021)	0.9211	(0.0014)	0.8425
		0.4752		0.6303		0.6867		0.7074
$\xi = 0.5$	0.8192	0.8137	0.8251	0.8268	0.8177	0.8173	0.7962	0.7956
	(0.0058)	1.1924	(0.0032)	1.0148	(0.0020)	0.9425	(0.0014)	0.8845
		0.4824		0.6327		0.7131		0.7656
fBm	0.8177	0.8053	0.8288	0.8275	0.8388	0.8380	0.8529	0.8529
	(0.0058)	1.1810	(0.0031)	1.0223	(0.0021)	0.9706	(0.0014)	0.9439
				$n = 2^{14}$				
		0.6033		0.7291		0.7714		0.7389
$\xi = 0.1$	0.8300	0.8298	0.8372	0.8377	0.8285	0.8282	0.7775	0.7773
	(0.0037)	1.0663	(0.0017)	0.9424	(0.0010)	0.8880	(0.0006)	0.8168
		0.5856		0.7319		0.7767		0.7712
$\xi = 0.25$	0.8269	0.8263	0.8369	0.8359	0.8370	0.8369	0.8107	0.8109
	(0.0037)	1.0530	(0.0017)	0.9498	(0.0010)	0.9013	(0.0007)	0.8539
		0.5921		0.7283		0.7767		0.7849
$\xi = 0.5$	0.8208	0.8244	0.8363	0.8367	0.8397	0.8397	0.8271	0.8275
	(0.0038)	1.0581	(0.0017)	0.9445	(0.0010)	0.8998	(0.0006)	0.8654
		0.6283		0.7361		0.7835		0.8092
fBm	0.8352	0.8319	0.8422	0.8420	0.8452	0.8458	0.8508	0.8507
	(0.0035)	1.0632	(0.0017)	0.9424	(0.0010)	0.9014	(0.0006)	0.8880

TABLE 3.B.2: DFA estimation of the Hurst exponent of a fBm with $H = 0.85$ and realized increments with $\xi = 0.1$, 0.25, and 0.5. The time series has length 2^{10} (top) and 2^{14} (bottom). The least square regression $\log_{10}[F]$ vs. $\log_{10}[m]$ is performed over a q-fraction of the largest values of $\log_{10}[m]$ (see text). The table reports the mean value of the estimator $\widehat{H}$ over 10^3 numerical simulations, together with the standard error (SE) and the 2.5%, 50%, and 97.5% percentile.

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