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# Global Solutions for the gravity water waves system: Infinite depth setting and flat bottom setting

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# Global solutions for the gravity water waves system: Infinite depth setting and Flat bottom setting

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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# Abstract

In this thesis, we study the gravity water waves system in two settings: (i) the water region is two dimensional and there is no bottom, (ii) the water region is three dimensional and the bottom of the water region is flat.

For the $2D$ gravity water waves system in the infinite depth setting, we prove the global existence and the modified scattering for a class of initial data, which can have infinite energy. The initial data is only required to be small above the level $\dot{H}^{1/5} \times \dot{H}^{1/5+1/2}$. No assumption is assumed below this level. As a direct consequence, the momentum condition assumed on the physical velocity in all previous small energy results by Ionescu-Pusateri[25], Alazard-Delort[2] and Ifrim-Tataru [21] is removed.

For the $3D$ gravity water waves system in the flat bottom setting, we prove global existence for suitably small initial data and non-existence of traveling waves below a certain level of smallness, which strongly contrasts the behavior of the solution of the same system in the $2D$ case.

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# Chapter 1

# Introduction

## 1.1 The gravity water waves system

We first describe the problem setting. There is an incompressible irrotational inviscid fluid occupying a time dependent finite depth domain $\Omega(t)$ in $\mathbb{R}^{n+1}$, $n \in \{1, 2\}$, which has a free interface $\Gamma(t)$ and a fixed flat bottom $\Sigma$. The initial height of the interface is very small. Above the domain $\Omega(t)$, it's vacuum. We consider the effect of gravity and ignore the effect of surface tension. We normalize the gravity constant $g$ to be 1.

In the infinite depth setting, we have $\Sigma = \emptyset$. In the Eulerian coordinates, we can describe the domain and the interface as follows,

$$\Omega(t) := \{(x, y) : x \in \mathbb{R}^n, y \leq h(t, x)\}, \quad \Gamma(t) := \{(x, y) : x \in \mathbb{R}^n, y = h(t, x)\}.$$

In the flat bottom setting, we can normalize the depth to be 1. In the Eulerian coordinates, we can describe the domain, the interface and the bottom in Eulerian coordinates as follows,

$$\Omega(t) := \{(x, y) : x \in \mathbb{R}^n, -1 \leq y \leq h(t, x)\},$$

$$\Gamma(t) := \{(x, y) : x \in \mathbb{R}^n, y = h(t, x)\}, \quad \Sigma := \{(x, y) : x \in \mathbb{R}^n, y = -1\}.$$

The evolution of this fluid is described by the Euler equation with boundary conditions as follows,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p - g(0,0,1) & \\ \nabla u = 0,\ \nabla \times u = 0,\ u(0) = u_0 & \\ u \cdot \vec{\mathbf{n}} = 0 & \text{on } \Sigma \\ p = 0 & \text{on } \ \Gamma(t) \\ \partial_t + u \cdot \nabla \text{tangents to } \cup_t \Gamma(t) & \text{on } \ \Gamma(t), \end{cases} \tag{1.1}$$

where $u$ denotes the physical velocity field and $p$ denotes the pressure.

As the velocity field is irrotational, we can represent it in terms of velocity potential $\phi$. Let $\psi$ be the restriction of velocity potential on the boundary $\Gamma(t)$, i.e., $\psi(t,x) := \phi(t,x,\, h(t,x))$. From divergence free condition and boundary conditions, we can derive the following harmonic equation with two boundary conditions: Neumann type on the bottom and Dirichlet type on the interface,

$$(\Delta_x + \partial_y^2)\phi = 0, \quad \frac{\partial \phi}{\partial \vec{\mathbf{n}}}\big|_{\Sigma} = 0, \quad \phi\big|_{\Gamma(t)} = \psi. \tag{1.2}$$

If $\Sigma = \emptyset$, then there is no Neumann type boundary condition.

Hence, we can reduce (see, for example [43]) the motion of fluid to the evolution of height "$h$" and velocity potential on the interface "$\psi$" as follows,

$$\begin{cases} \partial_t h = G(h)\psi, \\ \\ \partial_t \psi = -h - \frac{1}{2}|\nabla \psi|^2 + \dfrac{(G(h)\psi + \nabla h \cdot \nabla \psi)^2}{2(1+|\nabla h|^2)}, \end{cases} \tag{1.3}$$

where $G(h)\psi = \sqrt{1+|\nabla h|^2}\mathcal{N}(h)\psi$ and $\mathcal{N}(h)\psi$ is the Dirichlet-Neumann operator. We remark that although the formulas of equations are same for the infinite depth setting and the flat bottom setting, the behaviors of the Dirichlet-Neumann operator are totally different, which make a huge difference.

For gravity water waves system (1.3), it has the following conservation law,

$$\mathcal{H}(h,\psi) = \int_{\mathbb{R}^2} \frac{1}{2}|h|^2 + \frac{1}{2}\psi G(h)\psi dx. \tag{1.4}$$

## 1.2 Previous results

Without being exhaustive on the progresses made so far on the water waves system, we only mention those representative results on the initial value problem. For the results on the blow up behavior of solutions and "splash singularity", interested readers may refer to [8, 15, 11] and references therein.

On the local theory side, Nalimov [33] and Yosihara [35] considered the small initial data case, Wu [39, 40] considered general initial data in Sobolev spaces, see also the subsequent works by Christodoulou-Lindblad [9], Lannes[30], Lindblad [31], Coutand-Shkoller [10], Shatah-Zeng[34] and Alazard-Burq-Zuily [3, 4]. If the effect of surface tension is also considered, local existence also holds, see Beyer-Gunther [7], Ambrose-Masmoudi [6], Coutand-Shkoller [11], Shatah-Zeng [34] and Alazard-Burq-Zuily [3, 4].

On the long time behavior side, we have several results. For the gravity water waves system in the infinite depth setting. In the $3D$ case, Wu [39] and Germain-Masmoudi-Shatah [16] proved global existence for small initial data. In the $2D$ case, see the work of Wu[40] and the work of Hunter-Ifrim-Tataru [21] for the almost global existence, see the work of Ionescu-Pusateri[25], Alazard-Delort [2], Ifrim-Tataru[21], Wang [36] for the global existence result. For the capillary water waves system in the infinite depth setting. See the work of Germain-Masmoudi-Shatah [17] for the $3D$ case.See the work of Ionescu-Pusateri [28] and Ifrim-Tataru [22] for the $2D$ case. Recenlty, Deng-Ionescu-Pausader-Pusateri [14] proved the global existence for the gravity-capillary water waves system. Therefore, the picture of the small data global solution of the $3D$ water waves system in the infinite depth setting is complete.

For the water waves system in the flat bottom setting. What we know so far about the flat bottom case can be summarized as follows: (i) on the one hand, the local existence holds (with bottom not necessary flat) by the work of Lannes [30] and the works of Alazard-Burq-Zuily [3, 4] and the large time existence holds, which is larger than what predicted by energy estimate, by the work of Alvarez-Samaniego and Lannes in [5]; (ii) on the other hand, the existence of traveling waves for initial data arbitrary small in $L^2$ level, which depends on the dimension and the ratio of the surface tension coefficient and the gravity constant.

The existence of traveling waves makes the problem of global regularity more delicate and more complicated. Traveling waves are more likely to exist in $2D$. In the $2D$ case, the traveling waves exist as long as $g \neq 0$ regardless whether $\sigma$ equals 0 or not (see introduction part of [13] and references therein). Meanwhile, in the $3D$ case, so far the traveling waves are only known to exist when $\sigma/g > 1/3$ (see [13]), i.e., the strong surface tension case.

As a contrast, from the Theorem 1.5.1, we know that there is no traveling waves under certain smallness level when $\sigma/g = 0$, i.e., $\sigma = 0, g = 1$. This result provides a sharp contrast for the fundamental difference of the behavior of small initial data solution between the $3D$ setting and the $2D$ setting.

## 1.3 Notation and Preliminary

For any two numbers $A$ and $B$, we use $A \lesssim B$ and $B \gtrsim A$ to denote $A \leq CB$, where $C$ is an absolute constant. We use $A \sim B$ to denote the case when $A \lesssim B$ and $B \lesssim A$.

For $p \in \mathbb{N}_+$, we use $\Lambda_p(\mathcal{N})$ to denote the $p$-th order terms of the nonlinearity $\mathcal{N}$ if a Taylor expansion of $\mathcal{N}$ is available. Also, we use $\Lambda_{\geq p}[\mathcal{N}]$ to denote the $p$-th and higher orders terms. More precisely, $\Lambda_{\geq p}[\mathcal{N}] := \sum_{q \geq p} \Lambda_q[\mathcal{N}]$. For example, $\Lambda_2[\mathcal{N}]$ denotes the quadratic term of $\mathcal{N}$ and $\Lambda_{\geq 2}[\mathcal{N}]$ denotes the quadratic and higher order terms of $\mathcal{N}$. If there is no special annotation, then Taylor expansions are in terms of $h$ and $\psi$.

For $n \in \{1, 2\}$, we fix an even smooth function $\tilde{\psi} : \mathbb{R}^n \to [0, 1]$ supported in $[-3/2, 3/2]$ and equals to 1 in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define

$$\psi_k(x) := \tilde{\psi}(x/2^k) - \tilde{\psi}(x/2^{k-1}), \quad \psi_{\leq k}(x) := \tilde{\psi}(x/2^k) = \sum_{l \leq k} \psi_l(x), \quad \psi_{\geq k}(x) := 1 - \psi_{\leq k-1}(x),$$

and use $P_k$, $P_{\leq k}$ and $P_{\geq k}$ to denote the projection operators by the Fourier multipliers $\psi_k$, $\psi_{\leq k}$ and $\psi_{\geq k}$ respectively. For a well defined function $f(x)$, we use $f_k(x)$ to abbreviate $P_k f(x)$ and use $f_{k,j}(x)$ to abbreviate $Q_{k,j} f(x)$. Recall that the operator $Q_{k,j}$ is defined in (1.31).

We use both $\widehat{f}(\xi)$ and $\mathcal{F}(f)(\xi)$ to denote the Fourier transform of $f$, which is defined as follows,

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

We use $\mathcal{F}^{-1}(g)$ to denote the inverse Fourier transform of $g(\xi)$. For an integer $k \in \mathbb{Z}$, we use $k_+$ to denote $\max\{k, 0\}$ and use $k_-$ to denote $\min\{k, 0\}$. For two well defined functions $f(x)$ and $g(x)$ and a bilinear form $Q(f, g)$, we use the convention that the symbol $q(\cdot, \cdot)$ of $Q(\cdot, \cdot)$ is defined in the following sense throughout this paper,

$$\mathcal{F}[Q(f, g)](\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta)\widehat{g}(\eta) q(\xi - \eta, \eta) d\eta. \tag{1.5}$$

Very similarly, for a trilinear form $C(f,g,h)$, its symbol $c(\cdot,\cdot,\cdot)$ is defined in the following sense,

$$\mathcal{F}[C(f,g,h)](\xi)=\frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\widehat{f}(\xi-\eta)\widehat{g}(\eta-\sigma)\widehat{h}(\sigma)c(\xi-\eta,\eta-\sigma,\sigma)d\eta d\sigma.$$

**Definition 1.3.1.** *Given $\rho\in\mathbb{N}_+,\rho\geq 0$ and $m\in\mathbb{R}$, we use $\Gamma^m_\rho(\mathbb{R}^n)$ to denote the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^n\times(\mathbb{R}^n/\{0\})$, which are $C^\infty$ with respect to $\xi$ for $\xi\neq 0$. Moreover, they satisfy the following estimate,*

$$\forall|\xi|\geq 1/2,\|\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}}\lesssim_\alpha(1+|\xi|)^{m-|\alpha|},\quad \alpha\in\mathbb{N}^n,$$

*where $W^{\rho,\infty}$ is the usual Sobolev space.*

*Remark* 1.3.2. $\rho$ in above definitions is not necessary an integer, but for our applications, the integer case is sufficient.

**Definition 1.3.3.** *(i) We use notation $\dot{\Gamma}^m_\rho(\mathbb{R}^n)$ denotes the subspace of $\Gamma^m_\rho(\mathbb{R}^n)$ which consists of symbols $a(x,\xi)$ that are homogeneous of degree $m$ in $\xi$.*

*(ii) If $a=\sum_{0\leq j<\rho}a^{(m-j)}$, where $a^{(m-j)}\in\dot{\Gamma}^{m-j}_{\rho-j}(\mathbb{R}^n)$, then we say $a^{(m)}$ is the principal symbol of $a$.*

*(iii) An operator $T$ is said to be of order $m$, $m\in\mathbb{R}$, if for all $\mu\in\mathbb{R}$, it's bounded from $H^\mu(\mathbb{R}^n)$ to $H^{\mu-m}(\mathbb{R}^n)$. We denote the set of all operators of order $m$ as $S^m$.*

For symbol $a\in\Gamma^m_\rho$, we can define its norm as follows,

$$M^m_\rho(a):=\sup_{|\alpha|\leq 2+\rho}\sup_{|\xi|\geq 1/2}\|(1+|\xi|)^{|\alpha|-m}\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}}.$$

For $a,f\in L^2$ and pseudo differential operator $\tilde{a}(x,\xi)$, we define the operator $T_af$ and $T_{\tilde{a}}f$ as follows,

$$T_af=\mathcal{F}^{-1}[\int_{\mathbb{R}^n}\widehat{a}(\xi-\eta)\theta(\xi-\eta,\eta)\widehat{f}(\eta)d\,\eta],$$

$$T_{\tilde{a}}f=\mathcal{F}^{-1}[\int_{\mathbb{R}^n}\mathcal{F}_x(\tilde{a})(\xi-\eta,\eta)\theta(\xi-\eta)\widehat{f}(\eta)d\eta],$$

where the cut-off function is defined as follows,

$$\theta(|\xi-\eta|,\eta)=\begin{cases}1 & \text{when } |\xi-\eta|\leq 2^{-10}|\eta|,\\ 0 & \text{when } |\xi-\eta|\geq 2^{10}|\eta|.\end{cases}$$

We use $\tilde{\theta}(\xi-\eta,\eta)$ to denote cutoff function where two frequencies have comparable size inside the support and define bilinear operator $R_{\mathcal{B}}(\cdot,\cdot)$ by the Fourier multiplier $\tilde{\theta}(\cdot,\cdot)$. More precisely,

$$\tilde{\theta}(\xi-\eta,\eta):=1-\theta(\xi-\eta,\eta)-\theta(\eta,\xi-\eta). \tag{1.6}$$

Therefore, we have the following paraproduct decomposition for two well defined function $a$ and $b$,

$$ab=T_ab+T_ba+R_{\mathcal{B}}(a,b). \tag{1.7}$$

**Lemma 1.3.4.** *For well defined functions $f_1, f_2$, and $f_3$, the following estimate holds for any $k\geq 0$ and $a,b,c$, such that $a\leq s$ $b,c\geq 0$, and $b+c=a$,*

$$\|\big(T_{f_1}T_{f_2}-T_{f_1f_2}\big)f_3\|_{H^s}\lesssim \|f_3\|_{H^{s-a}}\|f_1\|_{\widetilde{W}^b}\|f_2\|_{\widetilde{W}^c}. \tag{1.8}$$

$$\|\big(T_{f_1}T_{f_2}-T_{f_1f_2}\big)f_3\|_{\widetilde{W}^s}\lesssim \|f_1\|_{\widetilde{W}^{s-a}}\|f_2\|_{\widetilde{W}^b}\|f_3\|_{\widetilde{W}^c}, \tag{1.9}$$

$$\|\big(T_{f_1}T_{f_2}-T_{f_1f_2}\big)f_3\|_{W^s}\lesssim \|f_1\|_{W^{s-a}}\|f_2\|_{W^b}\|f_3\|_{W^c}. \tag{1.10}$$

*Proof.* On the Fourier side, we have

$$\mathcal{F}\big[\big(T_{f_1}T_{f_2}-T_{f_1f_2}\big)f_3\big](\xi)=\int_{\mathbb{R}^n\times\mathbb{R}^n}\widehat{f_1}(\xi-\eta)\widehat{f_2}(\eta-\sigma)\widehat{f_3}(\sigma)\big(\theta(\eta-\sigma,\sigma)\theta(\xi-\eta,\eta)-\theta(\xi-\sigma,\sigma)d\eta d\sigma.$$

Note that, except the case when $|\xi-\sigma|\ll|\sigma|\ll|\xi-\eta|\sim|\eta-\sigma|$,

$$\theta(\eta-\sigma,\sigma)\theta(\xi-\eta,\eta)-\theta(\xi-\sigma,\sigma)=0.$$

Therefore,

$$\|\big(T_{f_1}T_{f_2}-T_{f_1f_2}\big)f_3\|_{H^s}\lesssim\big(\sum_{-10\leq k_3-10\leq k\leq k_3+10\leq k_2-10\leq k_1\leq k_2+10}2^{2sk}\|P_{k_1}f_1P_{k_2}f_2P_{k_3}f_3\|_{L^2}^2\big)^{1/2}$$

$$+\|f_1\|_{\widetilde{W}^0}\|f_2\|_{\widetilde{W}^0}\|f_3\|_{L^2}\lesssim\|f_3\|_{H^{s-a}}\|f_1\|_{\widetilde{W}^b}\|f_2\|_{\widetilde{W}^c},$$

where $a \leq s$, $b, c \geq 0$, and $b + c = a$. The estimates (1.9) and (1.10) can be derived very similarly. We omit details here. □

We have the following composition lemma for paradifferential operators. It can be found, for example, in [3, 32].

**Lemma 1.3.5.** *Let $m \in \mathbb{R}$ and $\rho > 0$, if given symbols $a \in \Gamma^m_\rho(\mathbb{R}^n)$ and $b \in \Gamma^{m'}_\rho(\mathbb{R}^n)$, we can define*

$$a \sharp b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial^\alpha_\xi a \partial^\alpha_x b,$$

*then for all $\mu \in \mathbb{R}$, there exists a constant $K$ such that*

$$\|T_a T_b - T_{a\sharp b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M^m_\rho(a) M^{m'}_\rho(b). \tag{1.11}$$

**Lemma 1.3.6.** *Let $m \in \mathbb{R}$ and $\rho > 0$ and let $a \in \Gamma^m_\rho(\mathbb{R}^n)$, if we denote $(T_a)^*$ as the adjoint operator of $T_a$ and denote $\bar{a}$ as the complex conjugate of $a$, then we know that, $(T_a)^* - T_{a^*}$ is of order $m - \rho$, where*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial^\alpha_\xi \partial^\alpha_x \bar{a}.$$

*Moreover, the operator norm of $(T_a)^* - T_{a^*}$ is bounded by $M^m_\rho(a)$.*

*Proof.* See [3][Theorem 3.10]. □

In most applications of Lemma 1.3.6, $m \leq 1$. If we let $\rho = 1$ in above lemma, then it's easy to see $a^* = \bar{a}$, if moreover $a$ is real, then $a^* = \bar{a} = a$.

Define a class of symbol and its associated norms as follows,

$$\mathcal{S}^\infty := \{m : \mathbb{R}^{2n} \text{ or } \mathbb{R}^{3n} \to \mathbb{C}, m \text{ is continuous and } \quad \|\mathcal{F}^{-1}(m)\|_{L^1} < \infty\},$$

$$\|m\|_{\mathcal{S}^\infty} := \|\mathcal{F}^{-1}(m)\|_{L^1}, \quad \|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} := \|m(\xi,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty},$$

$$\|m(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} := \|m(\xi,\eta,\sigma)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}.$$

We have the following lemma on the multilinear estimates,

**Lemma 1.3.7.** *Assume that $m$, $m' \in S^\infty$, $p, q, r, s \in [1, \infty]$ , then the following estimates hold for well defined functions $f(x), g(x)$, and $h(x)$,*

$$\|m \cdot m'\|_{S^\infty} \lesssim \|m\|_{S^\infty} \|m'\|_{S^\infty}, \tag{1.12}$$

$$\Big\|\mathcal{F}^{-1}\big[\int_{\mathbb{R}^n} m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\big]\Big\|_{L^p} \lesssim \|m\|_{\mathcal{S}^\infty}\|f\|_{L^q}\|g\|_{L^r} \quad \text{if } \frac{1}{p}=\frac{1}{q}+\frac{1}{r}, \tag{1.13}$$

$$\Big\|\mathcal{F}^{-1}\big[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m'(\xi,\eta,\sigma)\widehat{f}(\xi-\eta)\widehat{h}(\sigma)\widehat{g}(\eta-\sigma)d\eta d\sigma\big]\Big\|_{L^p} \lesssim \|m'\|_{\mathcal{S}^\infty}\|f\|_{L^q}\|g\|_{L^r}\|h\|_{L^s}, \tag{1.14}$$

*where* $\dfrac{1}{p}=\dfrac{1}{q}+\dfrac{1}{r}+\dfrac{1}{s}$.

*Proof.* The proof is standard, or see [25] for detail. □

To estimate the $\mathcal{S}^\infty_{k,k_1,k_2}$ or the $\mathcal{S}^\infty_{k,k_1,k_2,k_3}$ norms of symbols, we constantly use the following lemma .

**Lemma 1.3.8.** *For $i\in\{2,3\}$, if $f:\mathbb{R}^{in}\to\mathbb{C}$ is a smooth function and $k_1,\cdots,k_i\in\mathbb{Z}$, then the following estimate holds,*

$$\|\int_{\mathbb{R}^{ni}} f(\xi_1,\cdots,\xi_i)\prod_{j=1}^{i} e^{ix_j\cdot\xi_j}\psi_{k_j}(\xi_j)d\xi_1\cdots d\xi_i\|_{L^1_{x_1,\cdots,x_i}} \lesssim \sum_{m=0}^{[in/2]+1}\sum_{j=1}^{i} 2^{mk_j}\|\partial^m_{\xi_j}f\|_{L^\infty}. \tag{1.15}$$

*Proof.* Let's first consider the case when $i=2$. Through scaling, it is sufficient to prove above estimate for the case when $k_1=k_2=0$. From Plancherel theorem, we have the following two estimates,

$$\|\int_{\mathbb{R}^{ni}} f(\xi_1,\xi_2)e^{i(x_1\cdot\xi_1+x_2\cdot\xi_2)}\psi_0(\xi_1)\psi_0(\xi_2)d\xi_1d\xi_2\|_{L^2_{x_1,x_2}} \lesssim \|f(\xi_1,\xi_2)\|_{L^\infty_{\xi_1,\xi_2}},$$

$$\|(|x_1|+|x_2|)^{n+1}\int_{\mathbb{R}^{ni}} f(\xi_1,\xi_2)e^{i(x_1\cdot\xi_1+x_2\cdot\xi_2)}\psi_0(\xi_1)\psi_0(\xi_2)d\xi_1d\xi_2\|_{L^2_{x_1,x_2}} \lesssim \sum_{m=0}^{n+1}\big[\|\partial^m_{\xi_1}f\|_{L^\infty}+\|\partial^m_{\xi_2}f\|_{L^\infty}\big],$$

which are sufficient to finish the proof of (1.15). We can prove the case when $i=3$ very similarly, hence we omit the details here. □

**Lemma 1.3.9.** *For any $k\in\mathbb{Z}$ and $f\in L^2(\mathbb{R}^1)$, we have*

$$\|\widehat{f}(\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim 2^{-k/2}\|f_k\|^{1/2}_{L^2}\big[2^k\|\partial_\xi\widehat{f}(\xi)\psi_k(\xi)\|_{L^2}+\|f_k\|_{L^2}\big]^{1/2}. \tag{1.16}$$

*Proof.* The proof is standard. The desired estimate (1.16) is scale invariant, we only have to prove it for the case when $k=0$, which follows from the Cauchy-Schwartz inequality directly. Or one can find the detailed proof in [27, 28]. □

## 1.4 The main result on the $2D$ gravity water waves system

Before stating our main theorem, we define the main function spaces that will be used constantly. Note that the underlaying space of all function spaces introduced in this section is $\mathbb{R}$.

We define the function spaces as follows,

$$\|f\|_{H^{N,p}} := \big[\sum_{k\in\mathbb{Z}}(2^{Nk}+2^{pk})^2\|P_kf\|_{L^2}^2\big]^{\frac{1}{2}} \approx \|(|\partial_x|^p+|\partial_x|^N)f\|_{L^2}, \quad 0\le p\le N, \tag{1.17}$$

$$\|f\|_{W^{\gamma,b}} := \sum_{k\in\mathbb{Z}}\big(2^{\gamma k_+}+2^{bk_-}\big)\|P_kf\|_{L^\infty}, \quad \|f\|_{W^\gamma} := \sum_{k\in\mathbb{Z}}(2^{N_2k}+1)\|P_kf\|_{L^\infty}, \tag{1.18}$$

$$\|f\|_{\widetilde{W^\gamma}} := \|P_{\le 0}f\|_{L^\infty}+\sum_{k\ge 0}2^{\gamma k}\|P_kf\|_{L^\infty}, \quad \gamma\ge 0, \tag{1.19}$$

$$\mathcal{C}_0 := \{f:\mathbb{R}\to\mathbb{C}\,\text{is continuous and}\,\lim_{|x|\to\infty}|f(x)|=0\}, \quad \|f\|_{\mathcal{C}_0}:=\|f\|_{L^\infty}. \tag{1.20}$$

Note that the function space $\mathcal{C}_0$ is needed as the norm $H^{N,p}$ doesn't define natural spaces of distribution when $p>1/2$. Above function spaces will only be used in the $2D$ gravity water waves in the infinite depth setting.

Our main result is as follows,

**Theorem 1.4.1.** *Let $N_0=8$, $N_1=1$, $N_2=61/20$, $p=1/5$, and $p_0\in(0,10^{-10}]$. Assume that $(h_0,\psi_0)\in H^{N_0+1/2,p}\times(H^{N_0+1/2,1/2+p}\cap\mathcal{C}_0)$ satisfies the following smallness condition,*

$$\|h_0\|_{H^{N_0+1/2,p}}+\|\psi_0\|_{H^{N_0+1/2,1/2+p}}+\|x\partial_xh_0\|_{H^{N_1+1/2,p}}+\|x\partial_x\psi_0\|_{H^{N_1+1/2,1/2+p}}\le\epsilon_0, \tag{1.21}$$

*where $\epsilon_0$ is a sufficiently small constant, then there exists a unique global solution $(h,\psi)$ of the system (1.3) with initial data $(h_0,\psi_0)$. Moreover, the following estimates hold,*

$$\sup_{t\in[0,\infty)}(1+t)^{-p_0}\big[\|h\|_{H^{N_0,p}}+\|\psi\|_{H^{N_0,1/2+p}}+\|Sh\|_{H^{N_1,p}}+\|S\psi\|_{H^{N_1,1/2+p}}\big]\lesssim\epsilon_0, \tag{1.22}$$

$$\sup_{t\in[0,\infty)}(1+t)^{1/2}\|(h,|\nabla|^{\frac{1}{2}}\psi)\|_{W^{N_2}}\lesssim\epsilon_0, \tag{1.23}$$

*where $S:=t\partial_t+2x\partial_x$ is the scaling vector field associated with the system (1.3). Furthermore, the solution possesses modified scattering property as $t\to+\infty$.*

*Remark* 1.4.2. Note that $p=1/5$ in above theorem is not optimal, we can improve it after being more careful about the argument used here. However, given the methods used here, $p$ should strictly

less than 1/4.

*Remark* 1.4.3. The property of modified scattering in the infinite energy setting is same as the small energy setting in [25, 26]. We first describe the modified scattering property and then comment on it. Define $f(t) := h(t) + i|\nabla|^{1/2}\psi(t)$ and

$$G(t,\xi) := \frac{|\xi|^4}{\pi}\int_0^t |\widehat{f}(s,\xi)|^2 \frac{ds}{s+1}, \tag{1.24}$$

then there exist $\omega_\infty$ and $p_1 < p_0$ such that

$$\sup_{t\in[0,\infty)} (1+t)^{p_1} \Big\| |\xi|^{1/4}(1+|\xi|)^{N_2}\big(e^{iG(t,\xi)}\widehat{f}(t,\xi) - e^{-it|\xi|^{1/2}}\omega_\infty(\xi)\big)\Big\|_{L^2_\xi} \lesssim \epsilon_0. \tag{1.25}$$

We can see that, after rotating the profile $\widehat{f}(t,\xi)$ with the angle of $G(t,\xi)$, it approaches to a linear solution. We can also write this modified scattering behavior in the physical space $(t,x)$. Following the argument in [26], one can show that there exists a uniformly bounded function $f_\infty$ such that

$$\Big| f(t,x) - \frac{e^{-it|t/4|x||}}{\sqrt{1+|t|}} f_\infty(\frac{x}{t}) \exp\Big[-\frac{i|f_\infty(x/t)|^2}{64|x/t|^5}\log(1+|t|)\Big]\Big| \lesssim \epsilon_0(1+|t|)^{-1/2-p_1/2}. \tag{1.26}$$

Note that similar formulas have also been derived by Alazard-Delort[2] and Ifrim-Tataru [21].

## 1.5 The main result on the $3D$ gravity water waves system

Before stating our main theorem, we first define the function spaces that will be used and only be used in the $3D$ gravity water waves above a flat bottom setting. Note that the underlaying space of all function spaces introduced is $\mathbb{R}^2$.

Define the $L^\infty$-type space as follows,

$$\|f\|_{W^{\gamma,b}} := \sum_{k\in\mathbb{Z}} (2^{\gamma k} + 2^{bk})\|P_k f\|_{L^\infty}, \quad \|f\|_{W^\gamma} := \|f\|_{W^{\gamma,0}}, \quad 0 \le b \le \gamma. \tag{1.27}$$

$$\|f\|_{\widetilde{W}^\gamma} := \sum_{k\ge 0} 2^{\gamma k}\|P_k f\|_{L^\infty} + \|P_{\le 0} f\|_{L^\infty}. \tag{1.28}$$

Define the $Z$-normed space and the auxiliary space $B_{k,j}$ as follows,

$$\|f\|_Z := \sup_{k\in\mathbb{Z}} \sum_{j\ge \max\{-k,0\}} \|f\|_{B_{k,j}}, \quad \|f\|_{B_{k,j}} := 2^{\alpha k}(1+2^{6k})2^j \|\varphi_j^k(x)\cdot P_k f\|_{L^2}, \quad \alpha = 1/10, \tag{1.29}$$

where the cutoff function $\varphi_j^k(x)$ localizes the physical position in a way depends on the localized frequency. More precisely, it is defined as follows,

$$\varphi_j^k(x) := \begin{cases} \tilde{\psi}_{(-\infty,-k]}(x) & \text{if } k+j=0 \text{ and } k \leq 0, \\ \tilde{\psi}_{(-\infty,0]}(x) & \text{if } j=0 \text{ and } k \geq 0, \\ \tilde{\psi}_j(x) & \text{if } k+j \geq 1 \text{ and } j \geq 1, \end{cases} \tag{1.30}$$

where $\tilde{\psi}(\cdot)$ is a standard bump function. We mention that this type of $Z$–norm was first introduced and used by Ionescu-Pausader in [23] for the context of Euler-Poisson/electron system case.

We use the following notation to include all possible combinations of $k$ and $j$,

$$\mathcal{J} := \{(k,j) \in \mathbb{Z} \times \mathbb{Z}_+ : k+j \geq 0\}.$$

From above definition, it's easy to see the following decomposition holds

$$P_k f = \sum_{j \geq \max\{-k,0\}} Q_{k,j} f \quad f = \sum_{(k,j)\in\mathcal{J}} Q_{k,j} f.$$

where the operator $Q_{k,j}$ is defined as follows,

$$Q_{k,j} f := P_{[k-2,k+2]}[\varphi_j^k(x) \cdot P_k f]. \tag{1.31}$$

Our main result is stated as follows,

**Theorem 1.5.1.** *Let $N_0 = 1000$ and $\delta \in (0, 10^{-9}]$ be fixed and sufficiently small. If $(h_0, \psi_0)$ satisfies the following estimate,*

$$\|h_0\|_{H^{N_0+1/2}} + \|\Lambda\psi\|_{H^{N_0}} + \|(h_0, \Lambda\psi_0)\|_{Z\cap L^1} \leq \epsilon_0 \leq \bar{\epsilon}, \tag{1.32}$$

*for some sufficiently small constant $\bar{\epsilon}$, then there is a unique global solution for the system* (1.3) *with initial data $(h_0, \psi_0)$, moreover, the following estimate holds,*

$$\sup_{t\in[0,\infty)} (1+t)^{-\delta} \|(h, \Lambda\psi)(t)\|_{H^{N_0}} + (1+t)\|(h, \Lambda\psi)(t)\|_{W^{4,2\alpha}} + \|e^{it\Lambda}(h + i\Lambda\psi)(t)\|_Z \lesssim \epsilon_0. \tag{1.33}$$

*Remark* 1.5.2. The top regularity $N_0$ can be improved after being more careful with the argument used in this paper. Since our goal is to show global existence and highlight main difficulties, we do

not pursue the goal of optimizing the required top regularity here.

*Remark* 1.5.3. From (1.33), we can derive that there is no traveling wave below a certain smallness level determined by $\bar{\epsilon}$ in above theorem.

*Remark* 1.5.4. From the improved $Z$-norm estimate part (section 5.2), as a byproduct, we also know that the solution is scattering to a linear solution in a lower regularity Sobolev space, e.g., $H^5(\mathbb{R}^2)$.

# Chapter 2

# The Dirichlet-Neumann operator, Paralinearization, and Symmetrization in the infinite depth setting

In this section, we study the properties of Dirichlet-Neumann operator associated with the $2D$ water waves system in the infinite depth setting. The content of this section is part of the author's paper [36].

The idea of analyzing the Dirichlet-Neumann operator is based on the work of Alazard-Delort [2]. A key observation is that there exists a fixed point type structure inside the velocity potential, which provides a good way to estimate the Dirichlet-Neumann operator in the small data regime. More precisely, we assume that there exist some $T > 0$ and some $\delta > 0$ sufficiently small and the following estimate holds inside the time interval $[0, T]$,

$$\sup_{t\in[0,T]} \|\partial_x h(t)\|_{\widetilde{W^{\gamma'}}} + \|\partial_x h(t)\|_{L^\infty}^{1-2/q} \|\partial_x h(t)\|_{L^2}^{q/2} < \delta, \tag{2.1}$$

where $q \in [2, +\infty)$ is sufficiently large and $\gamma' > n/2 + 1$ is some fixed number and $\widetilde{W}^{\gamma'}$-norm is

defined in (1.28). As a direct consequence of (2.1), the following estimate holds,

$$\sup_{t\in[0,T]} \|\partial_x h\|_{W^{\gamma'}} \lesssim \delta.$$

It is very reasonable to assume (2.1), since the local well posedness of the system (1.3) holds (see [39]). Hence, we know that the solution remains to be small for a certain amount of time if the initial data is very small. For the rest of this section, the time "$t$" is naturally restricted in $[0,T]$ and assumption (2.1) holds.

## 2.1 A fixed point type formulation

We transform the domain $\Omega(t) := \{(x,y) : y \le h(t,x), x \in \mathbb{R}\}$ to the negative half-space through the change of coordinates $(x,y) \longmapsto (x,z) := (x, y-h(t,x))$, $z \in (-\infty, 0]$. Define $\varphi(x,z) = \phi(x, z+h(t,x))$. Hence $\phi(x,y) = \varphi(x, y-h(t,x))$. It is easy to verify the following identities hold,

$$\partial_y^2\phi(x,z+h(t,x)) = \partial_z^2\varphi(x,z), \quad \partial_x^2\phi = \partial_x^2\varphi - 2\partial_x\partial_z\varphi\partial_x h - \partial_z\varphi\partial_x^2 h + \partial_z^2\varphi(\partial_x h)^2. \tag{2.2}$$

Recall (1.2). We know that $\phi$ satisfies a harmonic equation inside $\Omega(t)$. Therefore, from (1.2), the following equation holds,

$$P\varphi := [(1+(\partial_x h)^2)\partial_z^2 + \partial_x^2 - 2\partial_x h\partial_x\partial_z - \partial_x^2 h\partial_z]\varphi = 0, \quad \varphi\big|_{z=0} = \psi. \tag{2.3}$$

They following two terms are very important to the study of Dirichlet-Neumann operator,

$$B(h)\psi = \frac{G(h)\psi + \partial_x h\partial_x\psi}{(1+(\partial_x h)^2)}, \quad V(h)\psi = \partial_x\psi - \partial_x h\partial_x B(h)\psi,$$

which represent the vertical derivative and the horizontal derivative of the velocity potential at the interface respectively. Note that the only nontrivial term is $B(h)\psi$.

In $(x,z)$ coordinates system, we have

$$B(h)\psi = \partial_z\varphi\big|_{z=0}, \quad G(h)\psi = (1+(\partial_x h)^2)\partial_z\varphi\big|_{z=0} - \partial_x h\partial_x\psi. \tag{2.4}$$

Therefore, to study the properties of Dirichlet-Neumann operator, it is sufficient to study the properties of $\partial_z\varphi$.

**Lemma 2.1.1.** *Let $\psi$ be in the space $\dot{H}^{1/2+p}\cap\mathcal{C}_0$ and $h\in W^{\gamma'}(\mathbb{R})$, then $\varphi(z,x)$ satisfies the following fixed-point type formulation:*

$$\varphi(z)=e^{z|\nabla|}\psi+\frac{1}{2}\int_{-\infty}^{0}e^{(z+z')|\nabla|}[\partial_x|\nabla|^{-1}(\partial_x h\partial_z\varphi)-\partial_x h\partial_x\varphi-(\partial_x h)^2\partial_z\varphi]dz'+$$

$$\frac{1}{2}\int_{-\infty}^{0}e^{-|z-z'||\nabla|}[-\partial_x|\nabla|^{-1}(\partial_x h\partial_z\varphi)+\mathrm{sign}(z-z')(\partial_x h\partial_x\varphi-(\partial_x h)^2\partial_z\varphi)]dz', \tag{2.5}$$

*if $\varphi(\cdot,\cdot)$ satisfies $P\varphi=0$ and $\varphi|_{z=0}=\psi$.*

*Proof.* One can see [2][Lemma 1.1.5] for the detailed proof. We only give a sketch proof here.

We rewrite (1.33) as follows,

$$(\partial_z+|\nabla|)(\partial_z-|\nabla|)\varphi=g(z)=\partial_z g_1(z)+g_2(z), \tag{2.6}$$

where

$$g_1(z)=-(\partial_x h)^2\partial_z\varphi+2\partial_x h\partial_x\varphi,\quad g_2(z)=\partial_x^2 h\partial_z\varphi.$$

By treating the nonlinearity $g(z)$ as given, we can solve $\varphi$ explicitly from (2.6). Note that $g(z)$ has term of type $\partial_z^2\varphi$, which is not in terms of $\nabla_{x,z}\varphi$. Hence we first decompose it as $\partial_z g_1(z)+g_2(z)$, where $g_1(z)$ and $g_2(z)$ are all linearly depending on $\nabla_{x,z}\varphi$. As a result, after doing integration by parts in $z$ once for $\partial_z g_1(z)$, we can derive (2.5). □

From (2.5), we can derive the following fixed-point formulation for $\nabla_{x,z}\varphi$:

$$\nabla_{x,z}\varphi(z)=e^{z|\nabla|}[\partial_x\psi,|D_x|\psi]+\int_{-\infty}^{0}K(z,z')M(\partial_x h)\nabla_{x,z}\varphi(z')dz'+[0,\partial_x h\partial_x\varphi-(\partial_x h)^2\partial_z\varphi], \tag{2.7}$$

where $K(z,z')$ and $M(\partial_x h)$ are the matrices of operators:

$$K(z,z')=\frac{1}{2}e^{(z+z')|\nabla|}\begin{bmatrix}\partial_x & \partial_x\\ |\nabla| & |\nabla|\end{bmatrix}+\frac{1}{2}e^{-|z-z'||D_x|}\begin{bmatrix}-\partial_x & -(\mathrm{sign}(z-z'))\partial_x\\ (\mathrm{sign}(z-z'))|\nabla| & |\nabla|\end{bmatrix}, \tag{2.8}$$

$$M(\partial_x h)=M_1(\partial_x h)+M_2(\partial_x h),$$

$$M_1(\partial_x h):=M(\partial_x h)=\begin{bmatrix}0 & H\circ\partial_x h\\ -\partial_x h & 0\end{bmatrix},\quad M_2(\partial_x h):=M(\partial_x h)=\begin{bmatrix}0 & 0\\ 0 & (\partial_x h)^2\end{bmatrix}. \tag{2.9}$$

Therefore, there exists a fixed point type structure of $\nabla_{x,z}\varphi$ in (2.7). We remark that (2.7) is the starting point of the $L^2-$type and $L^\infty-$type estimates of the Dirichlet-Neumann operator.

Define

$$h_1(z) := e^{z|\nabla|}[\partial_x\psi, |\nabla|\psi], \quad h_2(z) := \int_{-\infty}^{0} K(z,z')M_1(\partial_x h)h_1(z')d\,z' + [0, \partial_x h\partial_x e^{z|\nabla|}\psi], \tag{2.10}$$

$$h_3(z) := \int_{-\infty}^{0} K(z,z')\big[M_2(\partial_x h)h_1(z') + M_1(\partial_x h)h_2(z')\big]\,dz' + [0, \partial_x h h_{2;1}(z) - (\partial_x h)^2 e^{z|\nabla|}(|\nabla|\psi)], \tag{2.11}$$

which can be considered as $\Lambda_1[\nabla_{x,z}\varphi]$, $\Lambda_2[\nabla_{x,z}\varphi]$, and $\Lambda_3[\nabla_{x,z}\varphi]$ respectively. We use $h_{i;1}$ and $h_{i;2}$ to denote the first and second entry of the vector $h_i$ respectively.

From (2.7), we can derive the fixed point type formulation for $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$ and $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$ as follows,

$$\nabla_{x,z}\varphi - h_1(z) = h_2(z) + \int_{-\infty}^{0} K(z,z')M_1(\partial_x h)[\nabla_{x,z}\varphi(z') - h_1(z')] + \int_{-\infty}^{0} K(z,z')M_2(\partial_x h)\nabla_{x,z}\varphi(z')\,dz'$$

$$+ [0, \partial_x h(\partial_x\varphi - h_{1;1}(z)) - (\partial_x h)^2\partial_z\varphi], \tag{2.12}$$

$$\nabla_{x,z}\varphi - h_1(z) - h_2(z) = h_3(z) + \int_{-\infty}^{0} K(z,z')M_1(\partial_x h)[\nabla_{x,z}\varphi(z') - h_1(z') - h_2(z')]\,dz' +$$

$$\int_{-\infty}^{0} K(z,z')M_2(\partial_x h)[\nabla_{x,z}\varphi(z') - h_1(z')]\,dz' + [0, \partial_x h(\partial_x\varphi - h_{1;1}(z) - h_{2;1}(z)) - (\partial_x h)^2(\partial_z\varphi - h_{1;2}(z))], \tag{2.13}$$

$$\nabla_{x,z}\varphi - h_1(z) - h_2(z) - h_3(z) = \int_{-\infty}^{0} K(z,z')M_1(\partial_x h)(\nabla_{x,z}\varphi(z') - [h_1 + h_2 + h_3](z'))\,d\,z' +$$

$$+ [0, \partial_x h(\partial_x\varphi - h_{1;1}(z) - h_{2;1}(z) - h_{3;1}(z)) - (\partial_x h)^2(\partial_z\varphi - h_{1;2}(z) - h_{2;2}(z))]$$

$$+ \int_{-\infty}^{0} K(z,z')M_2(\partial_x h)[\nabla_{x,z}\varphi(z') - [h_1 + h_2](z')]\,d\,z' + \mathfrak{quintic}, \tag{2.14}$$

where

$$\mathfrak{quintic} = \int_{-\infty}^{0} K(z,z')\big[M_1(\partial_x h)h_3(z') + M_2(\partial_x h)h_2(z')\big]\,d\,z' + [0, \partial_x h h_{3;1}(z) - (\partial_x h)^2 h_{2;2}(z)]. \tag{2.15}$$

**Lemma 2.1.2.** *The explicit formula of $h_2(z)$ is given as follows,*

$$h_2(z) = [e^{z|\nabla|}\partial_x(h\partial_x\psi) - \partial_x(he^{z|\nabla|}\partial_x\psi), -e^{z|\nabla|}|\nabla|(h|\nabla|\psi) - \partial_x(he^{z|\nabla|}\partial_x\psi) + \partial_x h\partial_x e^{z|\nabla|}\psi]. \tag{2.16}$$

*Moreover, the following estimate holds*

$$\|h_2(z)\|_{L^\infty_z\widehat{W^\gamma}} \lesssim \|\partial_x h\|_{\widehat{W^\gamma}}\||\nabla|\psi\|_{\widehat{W^\gamma}}, \quad \gamma \geq 0. \tag{2.17}$$

*Proof.* Recall (2.8) and (2.10). The following equality holds,

$$h_2(z)-[0,\partial_x h\partial_x e^{z|\nabla|}\psi]=\int_{-\infty}^{0}\frac{1}{2}e^{(z+z')|\nabla|}\begin{bmatrix}\partial_x & \partial_x\\ |\nabla| & |\nabla|\end{bmatrix}[H(\partial_x h e^{z'|\nabla|}(|\nabla|\psi)),-\partial_x h e^{z'|\nabla|}(\partial_x\psi)]\,d\,z'+$$

$$+\int_{-\infty}^{z}\frac{1}{2}e^{(z'-z)|\nabla|}\begin{bmatrix}-\partial_x & -\partial_x\\ |\nabla| & |\nabla|\end{bmatrix}[H(\partial_x h e^{z'|\nabla|}(|\nabla|\psi)),-\partial_x h e^{z'|\nabla|}(\partial_x\psi)]\,d\,z'+$$

$$\int_{z}^{0}\frac{1}{2}e^{(z-z')|\nabla|}\begin{bmatrix}-\partial_x & \partial_x\\ -|\nabla| & |\nabla|\end{bmatrix}[H(\partial_x h e^{z'|\nabla|}(|\nabla|\psi)),-\partial_x h e^{z'|\nabla|}(\partial_x\psi)]\,d\,z'. \tag{2.18}$$

Therefore

$$\mathcal{F}(h_{2;2}(z)-\partial_x h\partial_x e^{z|\nabla|}\psi)(\xi)=\int_{-\infty}^{0}\int_{\mathbb{R}}\frac{1}{2}e^{(z+z')|\xi|}e^{z'|\sigma|}[-(\xi-\sigma)\xi|\sigma|+(\xi-\sigma)\sigma|\xi|]\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma dz'+$$

$$\int_{-\infty}^{z}\int_{\mathbb{R}}\frac{1}{2}e^{(z'-z)|\xi|}e^{z'|\sigma|}[-(\xi-\sigma)\xi|\sigma|+(\xi-\sigma)\sigma|\xi|]\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma dz'+$$

$$\int_{z}^{0}\frac{1}{2}e^{(z-z')|\xi|+z'|\sigma|}[(\xi-\sigma)\xi|\sigma|+(\xi-\sigma)\sigma|\xi|]\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma dz'=\frac{1}{2}e^{z|\xi|}\int_{\mathbb{R}}(-|\xi||\sigma|+\xi\sigma)\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma$$

$$+\frac{1}{2}\int_{\mathbb{R}}e^{z|\sigma|}(-|\xi||\sigma|+\xi\sigma)\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma+\frac{1}{2}e^{z|\xi|}\int_{\mathbb{R}}(-e^{-z'(|\xi|-|\sigma|)}\Big|_{z}^{0})(|\xi||\sigma|+\xi\sigma)\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma$$

$$=-e^{z|\xi|}\int_{\mathbb{R}}|\xi||\sigma|\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma+\int_{\mathbb{R}}e^{z|\sigma|}\xi\sigma\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\,\sigma. \tag{2.19}$$

In above calculation, we have used the fact that for any $\xi,\sigma\in\mathbb{R}$, the following identity holds,

$$\frac{|\xi|(\xi-\sigma)\sigma-|\sigma|\xi(\xi-\sigma)}{|\xi|+|\sigma|}=\xi\sigma-|\xi||\sigma|. \tag{2.20}$$

Hence from (2.19), we can get the explicit formula of $h_{2;2}(z)$

$$h_{2;2}(z)=-e^{z|\nabla|}|\nabla|(h|\nabla|\psi)-\partial_x(h\partial_x e^{z|\nabla|}\psi)+\partial_x h\partial_x e^{z|\nabla|}\psi. \tag{2.21}$$

Very similarly, we can show that

$$\mathcal{F}(h_{2;1}(z))=-\int_{\mathbb{R}}(e^{z|\xi|}-e^{z|\sigma|})\xi\sigma\widehat{\psi}(\sigma)\widehat{h}(\xi-\sigma)\,d\sigma, \tag{2.22}$$

$$h_{2;1}(z)=e^{z|\nabla|}\partial_x(h\partial_x\psi)-\partial_x(he^{z|\nabla|}\partial_x\psi). \tag{2.23}$$

To prove (2.17), it would be sufficient to prove the High $\times$ Low interaction case, i.e. $|\xi-\sigma| \ll |\sigma| \sim |\xi|$. A important observation in this case is that, for fixed $z \leq 0$, we have

$$|\xi||\sigma| = \xi\sigma, \quad \big|(e^{z|\xi|} - e^{z|\sigma|})|\xi||\sigma|\big| \lesssim |z\sigma|e^{z|\sigma|}|\xi|(|\xi|-|\sigma|) \lesssim |\xi||\xi-\sigma|. \tag{2.24}$$

With above estimate, from Lemma 1.3.8 and $L^\infty - L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, it is easy to verify that the desired estimate (2.17) holds. □

We have the following lemma regarding on the boundedness of the operator $K(z,z')$.

**Lemma 2.1.3.** *For well defined vector function $g : (-\infty, 0] \times \mathbb{R} \longmapsto \mathbb{R}^2$, any $j \in \mathbb{Z}$ and $p_1, q_1$, $p_2$ and $q_2$ s.t. $2 \leq q_1 \leq p_1 \leq +\infty$, $1 \leq q_2 \leq p_2 \leq +\infty$ we have*

$$\| \int_{-\infty}^{0} K(z,z')P_j[g(z')]dz' \|_{L_z^{p_2}L_x^{p_1}} \lesssim 2^{j(1/q_1+1/q_2-1/p_1-1/p_2)} \|P_j g\|_{L_z^{q_2}L_x^{q_1}}. \tag{2.25}$$

*Proof.* From the explicit formula of operator $K(z,z')$ in (2.8), it's easy to find that the kernel of $K(z,z') \circ P_j$ is given as follows,

$$\widetilde{K}_j(z,z',x) := \int_{\mathbb{R}} \frac{1}{2} e^{ix\xi+(z+z')|\xi|} m_1(\xi)\psi_j(\xi) + \frac{1}{2} e^{ix\xi-|z-z'||\xi|} m_2(\xi)\psi_j(\xi) d\xi, \tag{2.26}$$

where matrices $m_1(\xi)$ and $m_2(\xi)$ are defined as follows,

$$m_1(\xi) = \begin{bmatrix} -i\xi & -i\xi \\ |\xi| & |\xi| \end{bmatrix}, \quad m_2(\xi) = \begin{bmatrix} i\xi & i\mathrm{sign}(z-z')\xi \\ \mathrm{sign}(z-z')|\xi| & |\xi| \end{bmatrix}.$$

By doing integration by parts in $\xi$ once, we can gain $(|x|+|z-z'|)^{-1}$ by paying the price of $2^{-j}$. As a result, the following pointwise estimate for the kernel holds,

$$|\widetilde{K}_j(z,z',x)| \lesssim_N 2^{2j}(1+2^j|x|+2^j|z \pm z'|)^{-N}. \tag{2.27}$$

After using Hölder inequality and Bernstein inequality, from (2.27), it's easy to see our desired estimate (2.25) holds. □

### 2.1.1 Main results

Our first result is the following lemma, which will be used constantly and plays an essential role in the estimate of Dirichlet-Neumann operator.

**Lemma 2.1.4.** *Under the assumption* (2.1)*, the following estimates hold for* $\gamma \leq \gamma'$*,*

$$\sup_{z\leq 0}\|\nabla_{x,z}\varphi\|_{\widetilde{W^\gamma}} \lesssim \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}}, \quad \sup_{z\leq 0}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{\widetilde{W^\gamma}} \lesssim \|\partial_x h\|_{\widetilde{W^\gamma}}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}}, \quad (2.28)$$

$$\sup_{z\leq 0}\|\nabla_{x,z}\varphi\|_{H^k} \lesssim \||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{H^k}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}}, \quad (2.29)$$

$$\sup_{z\leq 0}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{H^k} \lesssim \|\partial_x h\|_{\widetilde{W^0}}\||\nabla|\psi\|_{H^k} + \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}}\|\partial_x h\|_{H^k}, \quad (2.30)$$

$$\sup_{z\leq 0}\|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{H^k} \lesssim \|\partial_x h\|^2_{\widetilde{W^0}}\||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{\widetilde{W^0}}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}}\|\partial_x h\|_{H^k}. \quad (2.31)$$

*Proof.* From estimate (2.25) in lemma 2.1.3, we have

$$\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} \lesssim \||\nabla|\psi\|_{H^k} + \|\int_{-\infty}^0 K(z,z')M(\partial_x h)\nabla_{x,z}\varphi dz'\|_{L^\infty_z H^k} \lesssim \||\nabla|\psi\|_{H^k}$$

$$+\|\big|\big[M(\partial_x h)\nabla_{x,z}\varphi\big]\|_{L^\infty_z H^k} \lesssim \||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{\widetilde{W^0}}\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} + \|\nabla_{x,z}\varphi\|_{L^\infty_z \widetilde{W^0}}\|\partial_x h\|_{H^k}. \quad (2.32)$$

From (2.25) we have

$$\sum_{k\geq 0}\|P_k\big[\int_{-\infty}^0 K(z,z')M(\partial_x h)(\nabla_{x,z}\varphi(z'))dz'\big]\|_{L^\infty_z W^\gamma} \lesssim \sum_{k\geq 0}2^{\gamma k_+}\|P_k\big[M(\partial_x h)\nabla_{x,z}\varphi\big]\|_{L^\infty_z L^\infty_x}$$

$$\lesssim \sum_{k\geq 0}\sum_{l=1,2}\|\partial_x h\|^l_{\widetilde{W^0}}2^{\gamma k_+}\|P_k\nabla_{x,z}\varphi\|_{L^\infty_z L^\infty_x} + \big[2^{\gamma k_+}\|P_k((\partial_x h)^l)\|_{L^\infty} \ + 2^{-\gamma k_+}\|\partial_x h\|^l_{\widetilde{W^\gamma}}\big]|\nabla_{x,z}\varphi\|_{L^\infty_z \widetilde{W^0}}$$

$$\lesssim \sum_{l=1,2}\|\partial_x h\|^l_{\widetilde{W^\gamma}}\|\nabla_{x,z}\varphi\|_{L^\infty_z \widetilde{W^\gamma}} \lesssim \|\partial_x h\|_{\widetilde{W^\gamma}}\|\nabla_{x,z}\varphi\|_{L^\infty_z \widetilde{W^\gamma}}. \quad (2.33)$$

Since operator $M(\partial_x h)$ contains Hilbert transform, which is not bounded from $L^\infty$ to $L^\infty$ if the Fourier transform of a function doesn't vanish inside a neighborhood of origin. When $k \leq 0$, we have to gain certain positive power of $k$ in the High$\times$ High interaction case for the purpose of summation with respect to $k$. More precisely, we have

$$\sum_{k\leq 0}\|P_k\big[\int_{-\infty}^0 K(z,z')M(\partial_x h)(\nabla_{x,z}\varphi(z'))dz'\big]\|_{L^\infty_z W^\gamma} \lesssim \sum_{k\leq 0}2^{k/q}\|P_k\big[M(\partial_x h)\nabla_{x,z}\varphi\big]\|_{L^\infty_z L^q_x}$$

$$\lesssim \sum_{l=1,2}\|\partial_x h\|_{L^q}\|\partial_x h\|^{l-1}_{L^\infty}\|\nabla_{x,z}\varphi\|_{L^\infty_z L^\infty} \lesssim \|\partial_x h\|^{2/q}_{L^2}\|\partial_x h\|^{1-2/q}_{L^\infty}\|\nabla_{x,z}\varphi\|_{L^\infty_z \widetilde{W^\gamma}}, \quad (2.34)$$

here $q \in [2, \infty)$ and we can choose it to be sufficiently large. From (2.33) and (2.34), we have

$$\| \int_{-\infty}^{0} K(z, z') M(\partial_x h)(\nabla_{x,z}\varphi(z'))dz' \|_{L_z^\infty W^\gamma} \lesssim \big[\|\partial_x h\|_{L^2}^{2/q}\|\partial_x h\|_{L^\infty}^{1-2/q} + \|\partial_x h\|_{\widetilde{W^\gamma}}\big]\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}}. \tag{2.35}$$

From formula (2.7) and (4.87), the following estimate holds,

$$\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|e^{z|\nabla|}[\partial_x\psi, |\nabla|\psi]\|_{L_z^\infty \widetilde{W^\gamma}} + \| \int_{-\infty}^{0} K(z, z') M(\partial_x h)(\nabla_{x,z}\varphi(z'))d\, z' \|_{L_z^\infty W^\gamma}$$

$$+ \sum_{l=1,2} \|(\partial_x h)^l \nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}} + \big[\|\partial_x h\|_{L^2}^{2/q}\|\partial_x h\|_{L^\infty}^{1-2/q} + \|\partial_x h\|_{\widetilde{W^\gamma}}\big]\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}}, \tag{2.36}$$

which further gives us the following estimate from the smallness assumption (2.1),

$$\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}}. \tag{2.37}$$

Combining estimates (2.37) and (2.32), we have

$$\|\nabla_{x,z}\varphi\|_{L_z^\infty H^k} \lesssim \||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{\widetilde{W^0}}\|\nabla_{x,z}\varphi\|_{L_z^\infty H^k} + \|\partial_x h\|_{H^k}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}}, \tag{2.38}$$

which further gives us the following estimate from the smallness assumption (2.1),

$$\|\nabla_{x,z}\varphi\|_{L_z^\infty H^k} \lesssim \||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{H^k}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}}, \tag{2.39}$$

$$\|\nabla_{x,z}\varphi - e^{z|\nabla|}[\partial_x\psi, |\nabla|\psi]\|_{L_z^\infty H^k} \lesssim \|\partial_x h\|_{\widetilde{W^0}}\||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{H^k}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}}. \tag{2.40}$$

From (2.12), we have

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \big[\|\partial_x h\|_{L^2}^{2/q}\|\partial_x h\|_{L^\infty}^{1-2/q} + \|\partial_x h\|_{\widetilde{W^\gamma}}\big](\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{\widetilde{W^\gamma}} + \|\partial_x h\|_{\widetilde{W^\gamma}}\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}})$$

$$+ \|\partial_x h\|_{\widetilde{W^\gamma}}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}} + \|h_2(z)\|_{L_z^\infty \widetilde{W^\gamma}} + \|\partial_x h\|_{\widetilde{W^\gamma}}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}}[1 + \|\partial_x h\|_{\widetilde{W^\gamma}}]. \tag{2.41}$$

In above estimate, we used (2.37). From the estimate (2.17) in Lemma 2.1.2, the following estimate holds,

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|\partial_x h\|_{\widetilde{W^\gamma}}\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^\gamma}}. \tag{2.42}$$

We proceed to prove (2.31). From the formulas of $h_i$, $i \in \{1, 2, 3\}$ in (2.10) and (2.11) and the

$L^\infty$-type estimate (2.17) of $h_2(z)$ in Lemma 2.1.2, we have

$$\|h_2(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{\widetilde{W^0}} \|h_1(z)\|_{L^\infty_z H^k} + \|\partial_x h\|_{H^k} \|h_1(z)\|_{L^\infty_z \widetilde{W^0}}$$

$$\lesssim \|\partial_x h\|_{\widetilde{W^0}} \||\nabla|\psi\|_{H^k} + \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}} \|\partial_x h\|_{H^k}, \tag{2.43}$$

$$\|h_3(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{\widetilde{W^0}} \|h_2(z)\|_{L^\infty_z H^k} + \|\partial_x h\|_{H^k} \|h_2(z)\|_{L^\infty_z \widetilde{W^0}} + \|\partial_x h\|^2_{\widetilde{W^0}} \|h_1(z)\|_{L^\infty_z H^k}$$

$$+\|h_1(z)\|_{L^\infty \widetilde{W^0}} \|\partial_x h\|_{\widetilde{W^0}} \|\partial_x h\|_{H^k} \lesssim \|\partial_x h\|^2_{\widetilde{W^0}} \||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{\widetilde{W^0}} \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}} \|\partial_x h\|_{H^k}. \tag{2.44}$$

With above estimates, now we are ready to estimate $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$. From (2.13) and (2.44), the following estimate holds,

$$\|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{\widetilde{W^0}} \|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} + \|\partial_x h\|_{H^k} \|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W^0}}$$

$$+\|\partial_x h\|^2_{\widetilde{W^0}} \|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} + \|\partial_x h\|_{H^k} \|\partial_x h\|_{\widetilde{W^0}} \|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W^0}} + \|h_3(z)\|_{L^\infty_z H^k}$$

$$\lesssim \|\partial_x h\|_{\widetilde{W^0}} (1 + \|\partial_x h\|_{\widetilde{W^0}}) [\|\partial_x h\|_{\widetilde{W^0}} \||\nabla|\psi\|_{H^k}$$

$$+\|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}} \|\partial_x h\|_{H^k}] + \|\partial_x h\|_{\widetilde{W^0}} \|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k}, \tag{2.45}$$

which further gives us the following estimate from the smallness assumption (2.1),

$$\|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{H^k} \lesssim \|\partial_x h\|^2_{\widetilde{W^0}} \||\nabla|\psi\|_{H^k} + \|\partial_x h\|_{\widetilde{W^0}} \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}} \|\partial_x h\|_{H^k}. \tag{2.46}$$

Note that, in the estimate of (2.45), we used the following rough estimate directly, which call be derived easily from (2.13) and (2.42).

$$\|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W^0}} \lesssim \big[\|\partial_x h\|^{2/q}_{L^2} \|\partial_x h\|^{1-2/q}_{L^\infty} + \|\partial_x h\|_{\widetilde{W^0}}\big] \|(\partial_x\psi, |\nabla|\psi)\|_{\widetilde{W^0}} \|\partial_x h\|_{\widetilde{W^0}}. \tag{2.47}$$

□

By the same fixed point type argument, we can also derive the following Lemma, which is very helpful when the scaling vector field is applied to the Dirichlet-Neumann operator.

**Lemma 2.1.5.** *Under the smallness assumption* (2.1)*, the following estimates hold for* $i \in \{1, 2, 3\}$ *and* $k \leq \gamma' - 1$*,*

$$\|\Lambda_{\geq i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|^{i-1}_{W^k} \||\nabla|\psi\|_{H^k}, \tag{2.48}$$

$$\|\Lambda_{\geq 4}[\nabla_{x,z}\psi]\|_{L^\infty_z L^2} \lesssim \|\partial_x h\|^3_{W^0}\||\nabla|\psi\|_{L^2}. \tag{2.49}$$

*Proof.* Recall the formulas (2.7), (2.12) and (2.13), we have

$$\|\nabla_{x,z}\varphi(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{W^k}(1+\|\partial_x h\|_{W^k})\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} + \||\nabla|\psi\|_{H^k}, \tag{2.50}$$

$$\|\nabla_{x,z}\varphi(z) - h_1(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{W^k}\|\nabla_{x,z}\varphi(z) - h_1(z)\|_{L^\infty_z H^k} +$$

$$(\|\partial_x h\|_{W^k} + \|\partial_x h\|^2_{W^k})\|\nabla_{x,z}\varphi(z)\|_{L^\infty_z H^k} + \|h_2(z)\|_{L^\infty_z H^k}, \tag{2.51}$$

$$\|\nabla_{x,z}\varphi(z) - h_1(z) - h_2(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{W^k}\|\nabla_{x,z}\varphi(z) - h_1(z) - h_2(z)\|_{L^\infty_z H^k}$$

$$+\|\partial_x h\|^2_{W^k}\|\nabla_{x,z}\varphi(z) - h_1(z)\|_{L^\infty_z H^k} + \|h_3(z)\|_{L^\infty_z H^k}. \tag{2.52}$$

Recall the formulas of $h_2(z)$ and $h_3(z)$: (2.10) and (2.11), we have

$$\|h_2(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{W^k}\||\nabla|\psi\|_{H^k}, \quad \|h_3(z)\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|^2_{W^k}\||\nabla|\psi\|_{H^k}. \tag{2.53}$$

Hence combing the estimates (2.50), (2.51) and (2.52), we can derive

$$\|\Lambda_{\geq i}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|^{i-1}_{W^k}\||\nabla|\psi\|_{H^k}, \quad i \in \{1,2,3\}. \tag{2.54}$$

Let's proceed to estimate $\Lambda_{\geq 4}[\nabla_{x,z}\varphi]$, from (2.14) and (2.15), the following estimate holds,

$$\|\Lambda_{\geq 4}[\nabla_{x,z}\psi]\|_{L^\infty_z L^2} \lesssim \|\partial_x h\|_{W^0}\|\Lambda_{\geq 4}[\nabla_{x,z}\psi]\|_{L^\infty_z L^2} + \|\mathfrak{quintic}\|_{L^\infty_z L^2} + \|\partial_x h\|^2_{W^0}\|\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z L^2},$$

$$\lesssim \|\partial_x h\|_{W^0}\|\Lambda_{\geq 4}[\nabla_{x,z}\psi]\|_{L^\infty_z L^2} + \|\partial_x h\|_{W^0}\|h_3(z,\cdot)\|_{L^\infty_z L^2} + \|\partial_x h\|^2_{W^0}\|h_2(z,\cdot)\|_{L^\infty_z L^2} + \|\partial_x h\|^3_{W^0}\||\nabla|\psi\|_{L^2}$$

$$\lesssim \|\partial_x h\|_{W^0}\|\Lambda_{\geq 4}[\nabla_{x,z}\psi]\|_{L^\infty_z L^2} + \|\partial_x h\|^3_{W^0}\||\nabla|\psi\|_{L^2}(1+\|\partial_x h\|_{W^0}),$$

which further gives us the following estimate from the smallness assumption (2.1),

$$\|\Lambda_{\geq 4}[\nabla_{x,z}\psi]\|_{L^\infty_z L^2} \lesssim \|\partial_x h\|^3_{W^0}\||\nabla|\psi\|_{L^2}. \tag{2.55}$$

□

Note that the main difference between the estimates in Lemma 2.1.5 and the estimates in Lemma 2.1.4 is that only $\psi$ is putted in $L^2$-type space.

The following estimate is very helpful if one wants to estimate the term of type $\psi_2 B(h)\psi_1 - B(h)(\psi_1\psi_2)$. This type of terms will appear when the scaling vector field hits the Dirichlet-Neumann operator.

**Lemma 2.1.6.** *Given any two well defined smooth functions $f(x)$ and $g(x)$, if $\varphi_1(x,z)$ satisfies $P\varphi_1 = 0$ and $\varphi_1\big|_{z=0} = fg$, meanwhile $\varphi_2(x,z)$ satisfies $P\varphi_2 = 0$ and $\varphi_2\big|_{z=0} = g(x)$, then under the smallness assumption that $\|h\|_{\widetilde{W^{\gamma+1}}} \leq \delta < 1$, we have the following estimate for $k \leq \gamma$,*

$$\|\nabla_{x,z}\varphi_1(z) - f\nabla_{x,z}\varphi_2(z)\|_{L^\infty_z H^k} \lesssim \|g\|_{\widetilde{W^k}}\|f\|_{H^{k+1,1-\epsilon}}, \tag{2.56}$$

$$\|\Lambda_{\geq 3}[\nabla_{x,z}\varphi_1 - f\nabla_{x,z}\varphi_2]\|_{L^\infty_z H^k} \lesssim \|\partial_x h\|_{W^k}\|g\|_{\widetilde{W^k}}\|f\|_{H^{k+1,1-\epsilon}}, \tag{2.57}$$

$$\|\Lambda_{\geq 4}[\nabla_{x,z}\varphi_1 - f\nabla_{x,z}\varphi_2]\|_{L^\infty_z L^2} \lesssim \|\partial_x h\|^2_{W^0}\|g\|_{\widetilde{W^0}}\|f\|_{H^{1,1-\epsilon}}. \tag{2.58}$$

*Proof.* Recall (2.7). We have

$$\nabla_{x,z}\varphi_1(z,) = e^{z|\nabla|}[\partial_x(fg), |\nabla|(fg)] + \int_{-\infty}^0 K(z,z')M(\partial_x h)\nabla_{x,z}\varphi_1(z')dz' + [0, \partial_x h\partial_x\varphi_1 - (\partial_x h)^2\partial_z\varphi_1],$$

$$f\nabla_{x,z}\varphi_2(z) = fe^{z|\nabla|}[\partial_x g, |\nabla|g] + \int_{-\infty}^0 fK(z,z')M(\partial_x h)\nabla_{x,z}\varphi_2(z')dz' + [0, \partial_x h f\partial_x\varphi_2 - (\partial_x h)^2 f\partial_z\varphi_2].$$

Hence,

$$\mathcal{F}\big[e^{z|\nabla|}[\partial_x(fg), |\nabla|(fg)] - f(\cdot)e^{z|\nabla|}[\partial_x g, |\nabla|g]\big](z,\xi) =$$

$$\int_{\mathbb{R}} e^{z|\xi|}a(\xi)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta - \int_{\mathbb{R}}\widehat{f}(\xi-\eta)\widehat{g}(\eta)e^{z|\eta|}a(\eta)\,d\eta, \quad \text{where}\, a(\xi) = [i\xi,\ |\xi|].$$

Note that, the following estimate holds for any fixed $z \leq 0$,

$$\big|e^{z|\xi|}a(\xi) - e^{z|\eta|}a(\eta)\big| \lesssim \big||\xi| - |\eta|\big|.$$

Hence, it is easy to derive the following estimate,

$$\|e^{z|\nabla|}[\partial_x(fg), |\nabla|(fg)] - f(\cdot)e^{z|\nabla|}[\partial_x g, |\nabla|g]\|_{L^\infty_z H^k} \lesssim \|f\|_{H^{k+1,1-\epsilon}}\|g\|_{\widetilde{W^k}}. \tag{2.59}$$

We can write the difference of $\nabla_{x,z}\varphi_1$ and $f\nabla_{x,z}\varphi_2$ as follows:

$$\nabla_{x,z}\varphi_1(z) - f\nabla_{x,z}\varphi_2(z) = e^{z|\nabla|}[\partial_x(fg), |\nabla|(fg)] - fe^{z|\nabla|}[\partial_x g, |\nabla|g] +$$

$$[0, \partial_x h(\partial_x\varphi_1 - f\partial_x\varphi_2) - (\partial_x h)^2(\partial_z\varphi_1 - f\partial_z\varphi_2)] + \int_{-\infty}^{0} K(z,z')M(\partial_x h)[\nabla_{x,z}\varphi_1(z') - f\nabla_{x,z}\varphi_2(z')]dz'$$

$$- \int_{-\infty}^{0} fK(z,z')M(\partial_x h)\nabla_{x,z}\varphi_2(z') + K(z,z')M(\partial_x h)[f\nabla_{x,z}\varphi_2(z')]dz'.$$

Very similar to the estimate of (2.59), we have

$$\|\nabla_{x,z}\varphi_1 - f\nabla_{x,z}\varphi_2\|_{L^\infty_z H^k} \lesssim \|f\|_{H^{k+1,1-\epsilon}}\|g\|_{\widetilde{W^k}} +$$

$$\|\partial_x h\|_{W^k}\|\nabla_{x,z}\varphi_1 - f\nabla_{x,z}\varphi_2\|_{L^\infty_z H^k} + \|f\|_{H^{k+1,1-\epsilon}}\|\partial_x h\|_{\widetilde{W^k}}\|\nabla_{x,z}\varphi_2\|_{L^1_z\widetilde{W^k}}. \tag{2.60}$$

From the fixed point type formulation in (2.7) for $\nabla_{x,z}\varphi_2$, the following estimate holds,

$$\|\nabla_{x,z}\varphi_2\|_{L^1_z\widetilde{W^k}} \lesssim \|g\|_{\widetilde{W^k}} + \|M(\partial_x h)\nabla_{x,z}\varphi_2\|_{L^1_z\widetilde{W^k}} \lesssim \|g\|_{\widetilde{W^k}} + \|\partial_x h\|_{\widetilde{W^k}}\|\nabla_{x,z}\varphi_2\|_{L^1_z\widetilde{W^k}}, \tag{2.61}$$

which further gives us the following estimate from the smallness assumption on $\eta$,

$$\|\nabla_{x,z}\varphi_2\|_{L^1_z\widetilde{W^k}} \lesssim \|g\|_{\widetilde{W^k}}. \tag{2.62}$$

From (2.60) and (2.62), the following estimate holds,

$$\|\nabla_{x,z}\varphi_1 - f\nabla_{x,z}\varphi_2\|_{L^\infty_z H^k} \lesssim \|g\|_{\widetilde{W^k}}\|f\|_{H^{k+1,1-\epsilon}}. \tag{2.63}$$

Similar to the proof of (2.49) in Lemma 2.1.5, with minor modifications, we can show our desired estimates (2.57) and (2.58) hold. We omit the details here. □

## 2.2 Scaling vector field acting on the Dirichlet-Neumann operator

In this section, we discuss about what will happen when the scaling vector field $S := t\partial_t + 2x\cdot\partial_x$ hits the Dirichlet-Neumann operator. The scaling vector field $S$ commutes with the linear operator of the gravity water waves system (1.3) in the infinite depth setting. The main results in this section are not quite new. They are based on the results in [2][Chapter 2, Section 3]. However, for our purpose of proving Theorem 1.5.1, we develop these results further to reveal the cancellation structures inside various terms.

Recall that

$$G(h)\psi+\partial_x h\partial_x\psi=(1+(\partial_x h)^2)B(h)\psi,\quad V(h)\psi=\partial_x\psi-\partial_x h\partial_x B(h)\psi. \tag{2.64}$$

**Lemma 2.2.1.** *Under the smallness assumption* (2.1)*, the following identity holds,*

$$S[B(h)\psi]=B(h)(S\psi)-\frac{2\partial_x hV(h)\psi+2G(h)\psi}{1+(\partial_x h)^2}-\frac{2\partial_x h\partial_x hB(h)\psi}{1+(\partial_x h)^2}+$$

$$2\big[B(h)(hB(h)\psi)-hB(h)(B(h)\psi)\big]+\big[ShB(h)(B(h)\psi)-B(h)(ShB(h)\psi)\big]. \tag{2.65}$$

*Moreover, the following estimates hold for* $i\in\{1,2,3\}$*, and any* $0<\epsilon\ll 1$*,*

$$\|\Lambda_{\geq i}[SB(h)\psi]\|_{H^k}\lesssim\|(\partial_x h,\partial_x\psi)\|_{W^k}^{i-1}\big[\|(S\eta,\eta)\|_{H^{k+1,1-\epsilon}}+\|(|\nabla|(S\psi),|\nabla|\psi)\|_{H^k}\big], \tag{2.66}$$

$$\|\Lambda_{\geq 4}[SB(h)\psi]\|_{L^2}\lesssim\big(\|(Sh,h)\|_{H^{1,1-\epsilon}}+\||\nabla|(S\psi,\psi)\|_{L^2}\big)\|(\partial_x h,\partial_x\psi)\|_{W^2}^3. \tag{2.67}$$

*Proof.* From [2][Lemma 2.3.3], the following identity holds,

$$SB(h)\psi=B(h)(S\psi-(B(h)\psi)Sh)+\frac{1}{1+(\partial_x h)^2}\Big[-4(V(h)\psi)\partial_x h+\big[\partial_x h\partial_x(B(h)\psi)-\partial_x(V(h)\psi)\big]Sh\Big]$$

$$+\frac{2}{1+(\partial_x h)^2}\Big[G(h)(hB(h)\psi)-hG(h)(B(h)\psi)+\partial_x hV(h)\psi-G(h)\psi\Big]. \tag{2.68}$$

We remark that above identity is derived from comparing the $S\partial_z\varphi|_{z=0}$ with the $\partial_z\varphi_1\big|_{z=0}$, where $\varphi_1(0)=S\psi$. After studying the two harmonic equations in the $(x,z)$ coordinates formulation, one should derive (2.68) without any problem.

From the facts that $P\varphi=0$ and $B(\eta)\psi=\partial_z\varphi|_{z=0}$, after evaluating $P\varphi(z)$ at the boundary $z=0$, we can derive the following identity,

$$-\partial_x[V(h)\psi]=G(h)(B(h)\psi). \tag{2.69}$$

After combining the identities (2.68), (2.69) , and (2.64), we can see our desired identity (2.65) holds. Since terms inside (2.65) that do not depend on "$S$" can be estimated very easily by using estimates in Lemma 2.1.4, we omit details for those terms. From Lemma 2.1.6, the following estimate holds,

$$\|\Lambda_{\geq i}\big[B(h)(hB(h)\psi)-hB(h)(B(h)\psi)\big]\|_{H^k}+\|\Lambda_{\geq i}\big[ShB(h)(B(h)\psi)-B(h)(ShB(h)\psi)\big]\|_{H^k}$$

$$\lesssim \|(Sh,h)\|_{H^{k+1,1-\epsilon}}\|\Lambda_{\geq i-1}[B(h)\psi]\|_{\widetilde{W}^k} \lesssim \|(Sh,h)\|_{H^{k+1,1-\epsilon}}\|(\partial_x h,\partial_x\psi,|\nabla|\psi)\|^{(i-1)}_{\widetilde{W}^k}.$$

From Lemma 2.1.5, the following estimate holds,

$$\|\Lambda_{\geq i}[B(h)S\psi]\|_{H^k} \lesssim \|\partial_x h\|^{i-1}_{W^k}\||\nabla|(S\psi)\|_{H^k}.$$

To sum up, we can see our desired estimate (2.66) holds. With an extra estimate (2.49) in Lemma 2.1.5, one can derive the desired estimate (2.67) by re-run the argument used in above. We omit details here. □

## 2.3 Paralinearization and Symmetrization of the full system

In this section, we will mainly find good substitution variables $(U_1, U_2)$ to analyze, instead of analyzing $(h,\psi)$ directly. The main processes of finding those substitutions can be summarized as paralinearization and symmmetrization. This method was introduced and studied in a series of works by Alazard-Métivier [1] and Alazard-Burq-Zuily [3, 4]. Interested readers are recommended to read those papers as warming up.

Recall that $P\varphi = 0$. After paralinearizing this equation, we have

$$\mathcal{P}\varphi + 2T_{\partial_z^2\varphi\partial_x h}\partial_x h - 2T_{\partial_x\partial_z\varphi}\partial_x h - T_{\partial_z\varphi}\partial_x^2 h + f_1 = 0, \tag{2.70}$$

where

$$\mathcal{P} := T_{(1+|\partial_x h|^2)}\partial_z^2 + \partial_x^2 - 2T_{\partial_x h}\partial_x\partial_z - T_{\partial_x^2 h}\partial_z,$$

$$f_1 = R_{\mathcal{B}}(|\partial_x h|^2,\partial_z^2\varphi) - 2R_{\mathcal{B}}(\partial_x\partial_z\varphi,\partial_x h) - R_{\mathcal{B}}(\partial_x^2 h,\partial_z\varphi) + T_{\partial_z^2\varphi}R_{\mathcal{B}}(\partial_x h,\partial_x h) + 2\big(T_{\partial_z^2\varphi}T_{\partial_x h} - T_{\partial_z^2\varphi\partial_x h}\big)\partial_x h.$$

Let $W = \varphi - T_{\partial_z\varphi}h$, then we have

$$\mathcal{P}W = f_2, \tag{2.71}$$

where $f_2$ is given as follows,

$$f_2 = -R_{\mathcal{B}}(|\partial_x h|^2,\partial_z^2\varphi) + 2R_{\mathcal{B}}(\partial_x\partial_z\varphi,\partial_x h) + R_{\mathcal{B}}(\partial_x^2 h,\partial_z\varphi) - T_{\partial_z^2\varphi}R_{\mathcal{B}}(\partial_x h,\partial_x h)$$

$$-T_{|\partial_x h|^2}T_{\partial_z^3\varphi}h + T_{\partial_z^3\varphi|\partial_x h|^2}h + 2T_{\partial_x h}T_{\partial_x\partial_z^2\varphi}h - 2T_{\partial_x h\partial_x\partial_z^2\varphi}h + T_{\partial_x^2 h}T_{\partial_z^2\varphi}h - T_{\partial_x^2 h\partial_z^2\varphi}h. \tag{2.72}$$

From the composition Lemma 1.3.5 and Lemma 1.3.4, it is to see that "$f_2$" does not lose derivative.

Follow the same argument in [2], we can decompose $\mathcal{P}W = f_2$ into a forward parabolic equation and a backward parabolic equation. More precisely, we have the following Lemma,

**Lemma 2.3.1.** *The following decomposition holds,*

$$(\partial_z - T_{\tilde{a}})(\partial_z - T_{\tilde{A}})W = f, \tag{2.73}$$

*where*

$$\tilde{a} = a + a', \quad \tilde{A} = A + A', \tag{2.74}$$

$$a = \frac{1}{1+|\partial_x h|^2}(i\partial_x h \cdot \xi - |\xi|), \quad A = \frac{1}{1+|\partial_x h|^2}(i\partial_x h \cdot \xi + |\xi|), \tag{2.75}$$

$$a' = \frac{1}{A-a}\big(i\partial_\xi a \partial_x A - \frac{\partial_x^2 h a}{1+|\partial_x h|^2}\big), \quad A' = \frac{1}{a-A}\big(i\partial_\xi a \partial_x A - \frac{\partial_x^2 h A}{1+|\partial_x h|^2}\big). \tag{2.76}$$

*and $f$ is a good error term. Its precise formula is given in* (2.79)*. Moreover, under the smallness condition* (2.1)*, the following estimate holds for $i \in \{2,3\}$,*

$$\sup_{z\le 0} \|\Lambda_{\ge i}[f(z)]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}} \|(\partial_x h, \partial_x \psi)\|_{W^1}^{(i-1)}. \tag{2.77}$$

*Proof.* Recall (2.71), we have

$$\big(T_{(1+|\partial_x h|^2)}\partial_z^2 + \partial_x^2 - 2T_{\partial_x h}\partial_x\partial_z - T_{\partial_x^2 h}\partial_z\big)W = f_2.$$

Note that the following identities holds from (2.75) and (2.76),

$$\tilde{a} + \tilde{A} = \frac{2i\partial_x h \xi}{1+|\partial_x h|^2} + \frac{\partial_x^2 h}{1+|\partial_x h|^2},$$

$$\tilde{a}\sharp\tilde{A} := \tilde{a}\tilde{A} + \frac{1}{i}\partial_\xi \tilde{a}\partial_x \tilde{A} = -\frac{|\xi|^2}{1+|\partial_x h|} + a'A' - i\partial_\xi \tilde{a}\partial_x A' - i\partial_\xi a'\partial_x A.$$

Therefore, the following decomposition holds,

$$(T_{(1+|\partial_x h|^2)}\partial_z^2 - T_{(\tilde{a}+\tilde{A})(1+|\partial_x h|^2)} + T_{\tilde{a}\sharp\tilde{A}(1+|\partial_x h|^2)})W = f_2 + R_0, \tag{2.78}$$

where

$$R_0 := T_{(a'A' - i\partial_\xi \tilde{a}\partial_x A' - i\partial_\xi a'\partial_x A)(1+|\partial_x h|^2)}W.$$

From (2.75), we can see $(a'A' - i\partial_\xi \tilde{a}\partial_x A' - i\partial_\xi a'\partial_x A)(1+|\partial_x h|^2)$ is a zero order operator, hence $R_0$

does not lose derivative.

From (2.78), we have

$$\big(T_{(1+|\partial_x h|^2)}(\partial_z - T_{\tilde{a}})(\partial_z - T_{\tilde{A}})\big)W = R_1,$$

$$R_1 = f_2 + R_0 + T_{(1+|\partial_x h|^2)}\big(T_{\tilde{a}}T_{\tilde{A}} - T_{\tilde{a}\sharp\tilde{A}}\big) + T_{|\partial_x h|^2}T_{\tilde{a}\sharp\tilde{A}}W - T_{\tilde{a}\sharp\tilde{A}|\partial_x h|^2}W$$

$$-T_{|\partial_x h|^2}T_{\tilde{a}+\tilde{A}}W + T_{(\tilde{a}+\tilde{A})|\partial_x h|^2}W,$$

which further implies the following

$$(\partial_z - T_{\tilde{a}})(\partial_z - T_{\tilde{A}})W = f,$$

$$f = T_{1/(1+|\partial_x h|^2)}R_1 + \big(T_{-|\partial_x h|^4/(1+|\partial_x h|^2)} - T_{-|\partial_x h|^2/(1+|\partial_x h|^2)}T_{|\partial_x h|^2}\big)(\partial_z - T_{\tilde{a}})(\partial_z - T_{\tilde{A}})W. \quad (2.79)$$

Again, from the composition Lemma 1.3.5, we can see that $R_1$ and $f$ do not lose derivative from above explicit formulas. After checking terms in (2.79) very carefully, the desired estimate (2.77) follows very easily. □

**Lemma 2.3.2.** *The following estimate holds for $i \in \{2, 3\}$, $k \geq 0$,*

$$\|\Lambda_{\geq i}\big[(\partial_z - T_A)W\big|_{z=0}\big]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x \psi)\|_{W^1}^{(i-1)}. \quad (2.80)$$

*Proof.* Note that $\Lambda_1[(\partial_z - T_A)W] = 0$, i.e. $(\partial_z - T_A)W$ itself is quadratic and higher. From (2.73) in Lemma 2.3.1, we have

$$(\partial_z - T_{\tilde{a}})(\partial_z - T_{\tilde{A}})W = f, \quad (2.81)$$

where $\tilde{a}$ and $\tilde{A}$ is given in (2.74).

From (2.85) in Lemma 2.3.3 and (2.77) in Lemma 2.3.1 , the following estimate holds,

$$\|(\partial_z - T_A)W\big|_{z=0}\|_{H^k} \lesssim \sup_{z\in[-1/4,0]}\|(\partial_z - T_{\tilde{A}})W(z,x)\|_{H^k} + \|T_{A'}W\big|_{z=0}\|_{H^k}$$

$$\lesssim \sup_{z\in[-1,0]}\|(\partial_z - T_{\tilde{A}})W(z,x)\|_{H^{k-1}} + \|f(z,\cdot)\|_{H^k} + \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x \psi)\|_{W^1}$$

$$\lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x \psi)\|_{W^1}.$$

It remains to consider the case when $i = 3$. We define,

$$\widetilde{W}(z,x) = (\partial_z - T_{\tilde{A}})W(z,x) - h(z,x), \quad z \leq 0, \tag{2.82}$$

where the function $h(z,x)$ is to be determined. Let $h(0,x)$ to be given as follows,

$$h(0,x) := \Lambda_2\big[(\partial_z - T_{\tilde{A}})W(z,x)\big|_{z=0}\big].$$

Hence, from the definition, we know that $\widetilde{W}(0,\cdot)$ is the cubic and higher order terms of $(\partial_z - T_{\tilde{A}})W(z,x)|_{z=0}$.

We let $h(z,x)$ to be the solution of the following parabolic equation:

$$\begin{cases} (\partial_z - T_{\tilde{a}})h = \Lambda_2\big[f\big], \\ \\ h(0,x) = \Lambda_2\big[(\partial_z - T_{\tilde{A}})W(z,x)\big|_{z=0}\big]. \end{cases} \tag{2.83}$$

Therefore, we can derive the following parabolic equation satisfied by $\widetilde{W}(z,x)$ as follows,

$$(\partial_z - T_{\tilde{a}})\widetilde{W} = \widetilde{\mathcal{N}} := f - (\partial_z - T_{\tilde{a}})h = \Lambda_{\geq 3}[f]. \tag{2.84}$$

From (2.85) in Lemma 2.3.3 and (2.77) in Lemma 2.3.1) , the following estimate holds,

$$\|\Lambda_{\geq 3}\big[(\partial_z - T_A)W\big|_{z=0}\big]\|_{H^k} \lesssim \sup_{z\in[-1/4,0]} \|(\partial_z - T_{\tilde{A}})\widetilde{W}(z,x)\|_{H^k} + \|\Lambda_{\geq 3}[T_{A'}W\big|_{z=0}]\|_{H^k}$$

$$\lesssim \sup_{z\in[-1,0]} \|(\partial_z - T_A)\widetilde{W}(z,x)\|_{H^{k-1}} + \|\Lambda_{\geq 3}[f(z,\cdot)]\|_{H^k} + \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x \psi)\|^2_{W^1}$$

$$\lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x \psi)\|^2_{W^1}.$$

□

**Lemma 2.3.3.** *Let $a \in \Gamma^1_2(\mathbb{R}^2)$ and satisfies the following assumption, $Re[a(x,\xi)] \geq c|\xi|$, for some positive constant $c$. If $u$ solves the following equation*

$$(\partial_w + T_a)u(w,\cdot) = g(w,\cdot),$$

*then we have the following estimate holds for any fixed and sufficiently small constant $\tau$, and arbi-*

*trarily small constant* $\epsilon > 0$.

$$\sup_{w\in[\tau,0]} \|u(w)\|_{H^k} \lesssim M_2^1(a)\frac{1+|\tau|}{|\tau|}\big[\sup_{z\in[4\tau,0]} \|u(w)\|_{H^{k-2(1-\epsilon)}} + \sup_{z\in[4\tau,0]} \|g(z)\|_{H^{\mu-(1-\epsilon)}}\big]. \tag{2.85}$$

*Proof.* A detailed proof can be found in [2], above result is the combination of [2][Lemma 2.1.9] and the proof part of [2][Lemma 2.1.10].

□

**Lemma 2.3.4.** *The following paralinearization for the Dirichlet-Neumann operator holds,*

$$G(\eta)\psi = |\nabla|\omega - T_V\partial_x h + F(h)\psi, \quad \omega := \psi - T_{B(h)\psi}\eta, \tag{2.86}$$

*where* $\omega$ *is the good unknown variable and* $F(h)\psi$ *is the good quadratic and higher error term, which does not lose derivative. More precisely, its formula is given as follows,*

$$F(\eta)\psi = \big(T_{1+|\partial_x h|^2}(\partial_z - T_A)W + T_{|\partial_x h|^2}T_{A-|\xi|}W - T_{|\partial_x h|^2(A-|\xi|)}W\big)|_{z=0} + 2T_{\partial_z\varphi}T_{\partial_x h}\partial_x h - 2T_{\partial_z\varphi\partial_x h}\partial_x h$$

$$-T_{\partial_x h}T_{\partial_z\varphi}\partial_x h + T_{\partial_z\varphi\partial_x h}\partial_x h - T_{\partial_x h}T_{\partial_x\partial_z\varphi}h + T_{\partial_x\varphi}R_{\mathcal{B}}(\partial_x h, \partial_x h) \tag{2.87}$$

*where* $W := \varphi - T_{\partial_z\varphi}h$ *and* $A$ *is defined in* (2.75)*. Under the smallness condition* (2.1)*, the following estimate holds for* $i \in \{2,3\}$ *and* $k \geq 0$*,*

$$\|\Lambda_{\geq i}[F(h)\psi]\|_{H^k} \lesssim \|(\partial_x h, \partial_x\psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x\psi)\|_{W^1}^{i-1}. \tag{2.88}$$

*Proof.* Recall that

$$G(h)\psi = \big((1+|\partial_x h|^2)\partial_z\varphi - \partial_x h\partial_x\varphi\big)\big|_{z=0}.$$

Define $W = \varphi - T_{\partial_z\varphi}h$ and $\underline{V} = \partial_x\varphi - \partial_x h\partial_x\partial_z\varphi$. By using (1.7) and composition Lemma 1.3.5, we can do the paralinearization of above equation as follows,

$$\big((1+|\partial_x h|^2)\partial_z\varphi - \partial_x h\partial_x\varphi\big) = T_{1+|\partial_x h|^2}\partial_z\varphi + 2T_{\partial_z\varphi}T_{\partial_x h}\partial_x h + T_{\partial_x\varphi}R_{\mathcal{B}}(\partial_x h, \partial_x h) - T_{\partial_x h}\partial_x\varphi - T_{\partial_x\varphi}\partial_x h$$

$$= T_{1+|\partial_x h|^2}\partial_z(W + T_{\partial_z\varphi}h) + 2T_{\partial_z\varphi}T_{\partial_x h}\partial_x h - T_{\partial_x h}\partial_x(W + T_{\partial_z\varphi}h) - T_{\partial_x\varphi}\partial_x h + T_{\partial_x\varphi}R_{\mathcal{B}}(\partial_x h, \partial_x h)$$

$$= T_{1+|\partial_x h|^2}\partial_z W + T_{(1+|\partial_x h|^2)}T_{\partial_z^2\varphi}h + T_{\partial_z\varphi\partial_x h}\partial_x h - T_{\partial_x h}\partial_x W - T_{\partial_x\varphi}\partial_x h$$

$$+2T_{\partial_z\varphi}T_{\partial_x h}\partial_x h - 2T_{\partial_z\varphi\partial_x h}\partial_x h - T_{\partial_x h}\partial_x\big(T_{\partial_z\varphi}h\big) + T_{\partial_z\varphi\partial_x h}\partial_x h + T_{\partial_x\varphi}R_{\mathcal{B}}(\partial_x h, \partial_x h).$$

$$= |\nabla|W + T_{1+|\partial_x h|^2}\partial_z W - T_{\partial_x h}\partial_x W - |\nabla|W - T_{\underline{V}}\partial_x h + \tilde{\mathcal{R}}_1,$$

$$= |\nabla|W + T_{1+|\partial_x h|^2}(\partial_z - T_A)W + T_{(1+|\partial_x h|)^2}T_A W - T_{(1+|\partial_x h|^2)A}W - T_{\underline{V}}\partial_x h + \tilde{\mathcal{R}}_1,$$

$$= |\nabla|W - T_{\underline{V}}\partial_x h + T_{1+|\partial_x h|^2}(\partial_z - T_A)W + T_{|\partial_x h|^2}T_{A-|\xi|}W - T_{|\partial_x h|^2(A-|\xi|)}W + \tilde{\mathcal{R}}_1, \tag{2.89}$$

where

$$\tilde{\mathcal{R}}_1 := 2T_{\partial_z\varphi}T_{\partial_x h}\partial_x h - 2T_{\partial_z\varphi\partial_x h}\partial_x h - T_{\partial_x h}T_{\partial_z\varphi}\partial_x h + T_{\partial_z\varphi\partial_x h}\partial_x h$$

$$-T_{\partial_x h}T_{\partial_x\partial_z\varphi}h + T_{\partial_x\varphi}R_{\mathcal{B}}(\partial_x h, \partial_x h). \tag{2.90}$$

From (2.89) and (2.90), after evaluation (2.89) at the boundary $z = 0$, we can see our desired decomposition (2.86) holds.

From (2.87), the following estimate holds,

$$\|\Lambda_{\geq i}[F(h)\psi]\|_{H^k} \lesssim \|\Lambda_{\geq i}\big[(\partial_z - T_A)W\big|_{z=0}\big]\|_{H^k} + \|\partial_x h\|_{\widetilde{W}^1}^{i-1}\|\omega\|_{H^{k-1}}$$

$$+\|(\partial_x h, \partial_x\psi, |\nabla|\psi)\|_{\widetilde{W}^1}^{i-1}\|(\partial_x h, \partial_x\psi)\|_{H^{k-1}}, \lesssim \|(\partial_x h, \partial_x\psi)\|_{H^{k-1}}\|(\partial_x h, \partial_x\psi)\|_{W^1}^{(i-1)}.$$

Note that, we used (2.80) in Lemma 2.3.2 in above estimate.

□

**Lemma 2.3.5.** *Under the smallness assumption* (2.1)*, the following estimate holds for* $1 \leq k \leq \gamma'$*,* $p \in (0,1)$*, and* $i \in \{2,3\}$*,*

$$\|\Lambda_{\geq i}[SF(\eta)\psi]\|_{H^{k,p}} \lesssim \big(\|(Sh)\|_{H^{k,p}} + \|h\|_{H^{k+1,1-p}} + \||\nabla|\psi\|_{H^k} + \||\nabla|S\psi\|_{H^{k-1}}\big)\|(\partial_x h, |\nabla|\psi)\|_{W^{k+1}}^{(i-1)}. \tag{2.91}$$

*Proof.* Recall that $F(h)\psi = G(h)\psi - |\nabla|(\psi - T_B h) + \partial_x T_V h$. Hence, from (2.65) in Proposition 2.2.1, we can derive the following equality,

$$S(F(h)\psi) = F(h)(S\psi) + \big(-\partial_x(Sh)V + \partial_x T_V Sh\big) - \big(G(h)(ShB) - Sh(G(h)B) - |\nabla|T_B Sh\big) + \mathcal{G}, \tag{2.92}$$

where the good error term $\mathcal{G}$ is given as follows,

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3,$$

where

$$\mathcal{G}_1 = 2\big(G(h)(hB) - hG(h)(B)\big),$$

$$\mathcal{G}_2 = 2\partial_x hV - 2\Lambda_{\geq 2}[G(h)\psi] - |\nabla|T_{B(\eta)(S\psi)}\eta - \partial_x T_{V(\eta)(S\psi)}\eta + |\nabla|T_{SB}\eta + \partial_x T_{SV}\eta,$$

$$\mathcal{G}_3 = S[|\nabla|(T_B\eta) + \partial_x(T_V\eta)] - |\nabla|T_{SB}\eta - |\nabla|T_B S\eta - \partial_x T_{SV}\eta - \partial_x T_V S\eta.$$

From Lemma 2.1.6, the following estimate holds,

$$\|\Lambda_{\geq i}[\mathcal{G}_1]\|_{H^k} \lesssim \|h\|_{H^{k+1,1-\epsilon}}\|\Lambda_{\geq i-1}[B(h)\psi]\|_{\widetilde{W}^k} \lesssim \|h\|_{H^{k+1,1-\epsilon}}\|(\partial_x h, \partial_x\psi)\|_{W^k}.$$

From (2.66) in Lemma 2.2.1, the following estimate holds,

$$\|\Lambda_{\geq i}[\mathcal{G}_2]\|_{H^k} \lesssim \big(\|\Lambda_{\geq i-1}\big[\big(B(h)(S\psi), V(h)(S\psi)\big)\big]\|_{L^2}$$

$$+\|\Lambda_{\geq i-1}\big[\big(SB(h)\psi, SV(h)\psi\big)\big]\|_{L^2}\big)\|(\partial_x h, \partial_x\psi)\|_{W^k} + \|(\partial_x h, \partial_x\psi)\|_{H^k}\|(\partial_x h, \partial_x\psi)\|^{i-1}_{\widetilde{W}^0}$$

$$\lesssim \big(\|Sh\|_{H^{1,1-\epsilon}} + \|(\partial_x h, \partial_x\psi)\|_{H^k} + \||\nabla|S\psi\|_{L^2}\big)\|(\partial_x h, \partial_x\psi)\|^{i-1}_{W^k}.$$

Note that the commutator term $\mathcal{G}_3$ does not depend on the scaling vector field. It is easy to derive the following estimate,

$$\|\Lambda_{\geq i}[\mathcal{G}_3]\|_{H^k} \lesssim \|\Lambda_{\geq i-1}\big[\big(B(h)\psi, V(h)\psi\big)\big]\|_{H^k}\|(\partial_x h, \partial_x\psi)\|_{W^0}$$

$$+\|(\partial_x h, \partial_x\psi)\|_{H^k}\|\Lambda_{\geq i-1}\big[\big(B(h)\psi, V(h)\psi\big)\big]\|_{\widetilde{W^0}} \lesssim \|(\partial_x h, \partial_x\psi)\|_{H^k}\|(\partial_x h, \partial_x\psi)\|^{i-1}_{W^0}.$$

To sum up, we have

$$\|\Lambda_{\geq i}[\mathcal{G}]\|_{H^k} \lesssim \big(\|Sh\|_{H^{1,1-\epsilon}} + \||\nabla|S\psi\|_{L^2} + \|h\|_{H^{k+1,1-\epsilon}} + \||\nabla|\psi\|_{H^k}\big)\|(\partial_x h, \partial_x\psi)\|^{i-1}_{W^k}. \tag{2.93}$$

To see the structure inside $\big(G(h)(ShB) - Sh(G(h)B) - |\nabla|T_B Sh\big)$, we decompose it as follows,

$$G(h)(ShB) - Sh(G(h)B) - |\nabla|T_B Sh = \sum_{k\in\mathbb{Z}} \mathcal{J}_{1,k} + \mathcal{J}_{2,k}, \tag{2.94}$$

where

$$\mathcal{J}_{1,k} := G(h)(P_k B P_{\leq k-1}(Sh)) - P_{\leq k-1}(Sh)G(h)(P_k B)$$

$$= (1 + (\partial_x h)^2)\big[B(h)(P_k B P_{\leq k-1}(Sh)) - P_{\leq k-1}(Sh)B(h)(P_k B)\big] - \partial_x h P_k B\partial_x(P_{\leq k-1}(Sh)), \tag{2.95}$$

$$\mathcal{J}_{2,k} := G(h)(P_k(Sh)P_{\leq k}B) - P_k(Sh)G(h)(P_{\leq k}B) - |\nabla|T_B Sh := \mathcal{J}^1_{2,k} + \mathcal{J}^2_{2,k}, \tag{2.96}$$

$$\mathcal{J}^1_{2,k} := F(h)(P_k(Sh)P_{\leq k}B) + |\nabla|(P_k(Sh)P_{\leq k-1}B - T_B P_k Sh)$$

$$= F(h)(P_k(Sh)P_{\leq k}B) + |\nabla|(P_k(Sh)P_{k-10\leq\cdot\leq k-1}B). \tag{2.97}$$

$$\mathcal{J}^2_{2,k} := -\big[|\nabla|T_{B(h)(P_k(Sh)P_{\leq k}B)}h + \partial_x T_{V(h)(P_k(Sh)P_{\leq k}B)}h + P_k(Sh)G(h)(P_{\leq k-1}B)\big]. \tag{2.98}$$

Note that in the decomposition (2.96), we used again the fact that $F(h)\psi = G(h)\psi - |\nabla|(\psi - T_B h) + \partial_x T_V h$.

From Lemma 2.1.6, the following estimate holds for some $\epsilon < (1-p)/100$,

$$\|\sum_{j\in\mathbb{Z}} \Lambda_{\geq i}[\mathcal{J}_{1,j}]\|_{H^k} \lesssim \sum_{j\in\mathbb{Z}} \|P_{\leq j-1}(Sh)\|_{H^{k+1,1-\epsilon/2}} \|\Lambda_{\geq i-1}[P_j[B(h)\psi]]\|_{\widetilde{W}^k}$$

$$+\|Sh\|_{H^{k,1-\epsilon}} \|(\partial_x h, \partial_x\psi)\|^{(i-1)}_{W^{k+1}} \lesssim \|Sh\|_{H^{k,1-\epsilon}} \|\Lambda_{\geq i-1}[B(h)\psi]\|_{\widetilde{W}^{k+1}} + \|Sh\|_{H^{k,1-\epsilon}} \|(\partial_x h, \partial_x\psi)\|^{(i-1)}_{W^{k+1}}$$

$$\lesssim \|Sh\|_{H^{k,1-\epsilon}} \|(\partial_x h, \partial_x\psi)\|^{i-1}_{W^{k+1}}. \tag{2.99}$$

From (2.88) in Lemma 2.3.4, the following estimate holds,

$$\|\sum_{j\in\mathbb{Z}} \Lambda_{\geq i}[\mathcal{J}^1_{2,j}]\|_{H^k} \lesssim \sum_{j\in\mathbb{Z}} \|\partial_x\big(P_j(Sh)P_{\leq j}B\big)\|_{H^{k-1}} \|(\partial_x h, \partial_x\psi)\|^{(i-2)}_{W^1}$$

$$\lesssim \|Sh\|_{H^{k,1-\epsilon}} \|(\partial_x h, \partial_x\psi)\|^{i-1}_{W^1}. \tag{2.100}$$

$$\|\sum_{j\in\mathbb{Z}} \Lambda_{\geq i}[\mathcal{J}^2_{2,j}]\|_{H^{k,p}} \lesssim \sum_{j\in\mathbb{Z}} \|\partial_x h\|_{W^k} \|\Lambda_{\geq i-1}\big[\big(B(h)(P_j(Sh)P_{\leq j}B)), V(h)(P_j(Sh)P_{\leq j}B))\big)\big]\|_{L^2}$$

$$+\|Sh\|_{H^{k,p}} \|\Lambda_{\geq i-1}[B(h)\psi]\|_{\widetilde{W}^0} \lesssim \|Sh\|_{H^{k,p}} \|(\partial_x h, \partial_x\psi)\|^{i-1}_{W^k}. \tag{2.101}$$

To sum up, from (2.92), (2.93), (2.99), (2.100), and (2.101), our desired esitmate (2.91) holds. □

**Lemma 2.3.6.** *The following paralinearization for the nonlinearity of equation satisfied by $\psi$ in (1.3) holds,*

$$\frac{1}{2}|\partial_x\psi|^2 - \frac{1}{2}\frac{\big(G(h)\psi + \partial_x h\partial_x\psi\big)^2}{(1+|\partial_x h|^2)} = T_V\partial_x(\psi - T_B h) + T_B G(h)\psi + \tilde{\mathcal{R}}, \tag{2.102}$$

*where $\tilde{\mathcal{R}}$ is the good quadratic and higher error term that does not lose derivative. More precisely,*

*its detailed formula is given as follows,*

$$\tilde{\mathcal{R}} = T_{V\partial_x h}B - T_V T_{\partial_x h}B + T_V T_{\partial_x B}h + \frac{1}{2}R_{\mathcal{B}}(V,V) + R_{\mathcal{B}}(V,\partial_x h) + \frac{1}{2}R_{\mathcal{B}}(B,B). \tag{2.103}$$

*Under the smallness condition* (2.1)*, the following estimate holds for* $i \in \{2,3\}$*,*

$$\|\Lambda_{\geq 3}[\tilde{\mathcal{R}}]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k-1}} \|(\partial_x h, \partial_x \psi)\|_{W^1}^{(i-1)}.$$

*Proof.* Recall that $V = \nabla\psi - \nabla h B$. From (1.7) and the composition Lemma 1.3.5, we have

$$\frac{1}{2}|\partial_x \psi|^2 - \frac{1}{2}\frac{(\partial_x h \partial_x \psi + G(\,h)\psi)^2}{1+|\partial_x\, h|^2} = \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}(1+|\partial_x h|^2)B^2$$

$$= \frac{1}{2}V^2 + \frac{1}{2}(\partial_x h)^2 B^2 + V\partial_x h B - \frac{1}{2}(1+|\partial_x h|^2)B^2$$

$$= T_V V + T_{V\partial_x h}B + T_B V \partial_x h + T_B B + \frac{1}{2}R_{\mathcal{B}}(V,V) + R_{\mathcal{B}}(V,\partial_x h) + \frac{1}{2}R_{\mathcal{B}}(B,B)$$

$$= T_V V + T_{V\partial_x h}B + T_B G(h)\psi + \frac{1}{2}R_{\mathcal{B}}(V,V) + R_{\mathcal{B}}(V,\partial_x h) + \frac{1}{2}R_{\mathcal{B}}(B,B)$$

$$= T_V \partial_x(\psi) - T_V(\partial_x h B) + T_{V\partial_x h}B + T_B G(h)\psi + \frac{1}{2}R_{\mathcal{B}}(V,V) + R_{\mathcal{B}}(V,\partial_x h) + \frac{1}{2}R_{\mathcal{B}}(B,B),$$

$$= T_V \partial_x \psi - T_V T_{\partial_x h}B - T_V T_B \partial_x h + T_{V\partial_x h}B + T_B G(h)\psi + \frac{1}{2}R_{\mathcal{B}}(V,V) + R_{\mathcal{B}}(V,\partial_x h) + \frac{1}{2}R_{\mathcal{B}}(B,B),$$

$$= T_V \partial_x(\psi - T_B h) + T_B G(h)\psi + T_{V\partial_x h}B - T_V T_{\partial_x h}B + T_V T_{\partial_x B}h + \frac{1}{2}R_{\mathcal{B}}(V,V) + R_{\mathcal{B}}(V,\partial_x h) + \frac{1}{2}R_{\mathcal{B}}(B,B).$$

Hence finishing the proof. □

### 2.3.1 Symmetrization of the full system

Recall (1.3). From (2.86) in Lemma 2.3.4 and (2.102) in Lemma 2.3.6, we can derive the following system of equations satisfied by $h$ and the good unknown $\omega$,

$$\begin{cases} \partial_t h = |\nabla|\omega - T_V \partial_x h + F(h)\psi, \\ \\ \partial_t \omega = -T_a h - T_V \partial_x \omega + \mathcal{R}, \end{cases} \tag{2.104}$$

where $a = 1 + \partial_t B + V\partial_x B$ and $\mathcal{R}$ is given as follows,

$$\mathcal{R} = -\tilde{\mathcal{R}} + T_{V\partial_x B}h,$$

and $\tilde{\mathcal{R}}$ is given in (2.103) and "$a$" is the so called Taylor coefficients.

However, one cannot work on the system (2.104) directly as there is a potential problem of losing one derivative. When one tries to do energy estimate by using the system (2.104), one might find that the following term loses one derivative and can't be simply treated.

$$\int_{\mathbb{R}^2} \partial_x^N \omega \partial_x^N [-T_a\, h], \quad \text{here } N \text{ is the prescribed top derivative level.} \tag{2.105}$$

To get around this difficulty, we will symmetrize the system (2.104). Define

$$U_1 = h + T_\alpha h, \quad U_2 = |\nabla|^{1/2}\omega.$$

Then the system of equation satisfied by $U_1$ and $U_2$ is as follows,

$$\begin{cases} \partial_t U^1 = |\nabla|^{1/2} U^2 + T_\alpha |\nabla|^{1/2} U^2 - T_V \partial_x U^1 + \mathcal{R}_1 \\ \\ \partial_t U^2 = -|\nabla|^{1/2} U^1 - T_\alpha |\nabla|^{1/2} U^1 - T_V \partial_x U^2 + \mathcal{R}_2, \end{cases} \tag{2.106}$$

where

$$\mathcal{R}_1 = |\nabla|^{1/2} T_\alpha U^2 - T_\alpha |\nabla|^{1/2} U^2 - T_{\sqrt{a}} T_V \partial_x h + T_V \partial_x T_{\sqrt{a}} h + T_{\sqrt{a}} F(h)\psi, \tag{2.107}$$

$$\mathcal{R}_2 = |\nabla|^{1/2}(T_a h) - T_{\sqrt{a}} |\nabla|^{1/2} T_{\sqrt{a}} h - |\nabla|^{1/2} T_V \partial_x \omega + T_V \partial_x |\nabla|^{1/2} \omega + \mathcal{R}. \tag{2.108}$$

Essentially speaking, the bulk quadratic term in (2.105) becomes the following term,

$$\int \partial_x^N U^1 T_\alpha |\nabla|^{1/2} \partial_x^N U^2 - \partial_x^N U^2 T_\alpha |\nabla|^{1/2} \partial_x^N U^1,$$

which does not lose derivative. One can verify this fact by using symmetries on the Fourier side.

### 2.3.2 Estimates of the Taylor coefficients

The main goal of this subsection is to prove the following Lemma.

**Lemma 2.3.7.** *Under the smallness assumption* (2.1)*, the following estimates hold fork,* $\gamma \geq 0$*,* $\gamma \leq \gamma' - 2$*, and* $i \in \{1, 2, 3\}$*,*

$$\|\Lambda_{\geq i}[\alpha]\|_{\widetilde{W^\gamma}} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{\gamma+1}}^i, \quad \|\Lambda_{\geq i}[\alpha]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^1}^{i-1} \|(\partial_x h, |\nabla|\psi)\|_{H^{k+1}}, \tag{2.109}$$

$$\|\Lambda_{\geq i}[S\alpha]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{k+1}}^{i-1}\big[\|(S\eta,\eta)\|_{H^{k+2,1-\epsilon}} + \|(|\nabla|(S\psi), |\nabla|\psi)\|_{H^{k+1}}\big], \tag{2.110}$$

$$\|\Lambda_{\geq i}[\partial_t \alpha]\|_{\widetilde{W^\gamma}} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{\gamma+2}}^{i}, \quad \|\Lambda_{\geq i}[\partial_t \alpha]\|_{H^k} \lesssim \|\|(\partial_x h, \partial_x \psi)\|\|_{W^2}^{(i-1)}\|(\partial_x h, |\nabla|\psi)\|_{H^{k+2}}, \tag{2.111}$$

$$\|\Lambda_{\geq i}[S\partial_t \alpha]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{k+2}}^{i-1}\big[\|(S\eta,\eta)\|_{H^{k+3,1-\epsilon}} + \|(|\nabla|(S\psi), |\nabla|\psi)\|_{H^{k+2}}\big]. \tag{2.112}$$

*Proof.* Recall that $\alpha = \sqrt{a} - 1$. It is sufficient to estimate the Taylor coefficient "$a$". We cite the following identity from [2][Lemma A.4.2],

$$a = \frac{1}{2(1+(\partial_x h)^2)}\Big[2 + 2V\partial_x B - 2B\partial_x V - G(h)[V^2 + B^2 + 2h]\Big], \tag{2.113}$$

Recall that $a = 1 + \partial_t B + V\partial_x B$ and $\alpha = \sqrt{a} - 1$. Therefore, we have the following identity,

$$\partial_t B = \frac{1}{2(1+(\partial_x h)^2)}\Big[2 + 2V\partial_x B - 2B\partial_x V - G(h)(V^2 + B^2 + 2h)\Big] - 1 - V\partial_x B. \tag{2.114}$$

As we already have $L^2$ type and $L^\infty$ type estimate on $B(h)\psi$, we can derive $L^2$ type and $L^\infty$ type estimate for $\partial_t B$ from (2.114). As a result, from estimates in Lemma 2.1.4 , the following estimate holds for $i \in \{1,2,3\}$,

$$\|\Lambda_{\geq i}[\partial_t B]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k+1}}\|(\partial_x h, \partial_x \psi)\|_{W^1}^{i-1}.$$

$$\|\Lambda_{\geq i}[\partial_t B]\|_{\widetilde{W}^\gamma} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{\gamma+1}}^{i}.$$

From (2.66) in Lemma 2.2.1, the following estimate holds,

$$\|\Lambda_{\geq i}[S\partial_t B]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{k+1}}^{i-1}\big[\|(S\eta,\eta)\|_{H^{k+2,1-\epsilon}} + \|(|\nabla|(S\psi), |\nabla|\psi)\|_{H^{k+1}}\big].$$

After taking another derivative with respect to time "$t$" on both hands side of (2.114), we have an identity for $\partial_t^2 B$. As a result, from the $L^2$ type and $L^\infty$ type estimates of $\partial_t B$ and $B$, the following estimates hold for $i \in \{1,2,3\}$,

$$\|\Lambda_{\geq i}[\partial_t^2 B]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{H^{k+2}}\|(\partial_x h, \partial_x \psi)\|_{W^2}^{i-1}.$$

$$\|\Lambda_{\geq i}[\partial_t^2 B]\|_{\widetilde{W}^\gamma} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{\gamma+2}}^{i}.$$

$$\|\Lambda_{\geq i}[S\partial_t^2 B]\|_{H^k} \lesssim \|(\partial_x h, \partial_x \psi)\|_{W^{k+2}}^{i-1}\big[\|(S\eta,\eta)\|_{H^{k+3,1-\epsilon}} + \|(|\nabla|(S\psi), |\nabla|\psi)\|_{H^{k+2}}\big].$$

Hence, our desired estimates follows from the fact that $a = 1 + \partial_t B + V\partial_x B$, (2.66) in Lemma 2.2.1, and estimates in Lemma 2.1.4. $\square$

# Chapter 3

# The Dirichlet-Neumann operator, Paralinearization, and Symmetrization in the flat bottom setting

## 3.1 The Dirichlet-Neumann operator in the flat bottom setting

In this section, we study the properties of Dirichlet-Neumann operator associated with the $3D$ water waves system in the flat bottom setting. The content of this section is part of the author's paper [37].

From the local wellposedness of water waves system (1.3), we know that the solution remains to be small if it starts with a small initial data. More precisely, if $\|(h, \Lambda\psi)(0)\| \leq \delta$, then there exists a time $T$ such that the following estimate holds,

$$\sup_{t\in[0,T]} \|(h, \Lambda\psi)(t)\|_{\widetilde{W^4}} \leq 2\delta, \quad \Lambda := \sqrt{\tanh|\nabla||\nabla|}. \tag{3.1}$$

Throughout the rest of this paper, estimate (3.1) always holds.

### 3.1.1 Type I formulation of harmonic equation (1.2)

We can do change of variables and map $\Omega(t)$ to strip $\mathcal{S} := \mathbb{R}^2 \times [-1, 0]$ as follows,

$$(x, y) \to (x, z), \quad z := \frac{y - h(t, x)}{h(t, x) + 1}.$$

Very naturally, the inverse of transformation is given as follows,

$$y = h + (h + 1)z.$$

Define $\varphi(x, z) := \phi(x, h + (h + 1)z)$, it's straightforward to see the following identities hold,

$$\phi(x, y) = \varphi(x, \frac{y - h}{1 + h}), \quad \partial_y \phi = \frac{\partial_z \varphi}{1 + h}, \quad \partial_y^2 \phi = \frac{\partial_z^2 \varphi}{(1 + h)^2}, \tag{3.2}$$

$$\partial_{x_i} \phi = \partial_{x_i} \varphi + \partial_z \varphi [\frac{-\partial_{x_i} h}{1 + h} - \frac{(y - h)\partial_{x_i} h}{(1 + h)^2}] = \partial_{x_i} \varphi - \frac{(y + 1)\partial_{x_i} h}{(1 + h)^2} \partial_z \varphi, \tag{3.3}$$

$$\partial_{x_i}^2 \phi = \partial_{x_i}^2 \varphi - 2\frac{(y + 1)\partial_{x_i} h}{(1 + h)^2} \partial_z \partial_{x_i} \varphi + [\frac{-(y + 1)\partial_{x_i}^2 h}{(1 + h)^2} + 2\frac{(y + 1)(\partial_{x_i} h)^2}{(1 + h)^3}]\partial_z \varphi + \frac{(y + 1)^2 (\partial_{x_i} h)^2}{(1 + h)^4} \partial_z^2 \varphi.$$

From above identities and (1.2), it's easy to see the following identity holds,

$$(\Delta_x + \partial_y^2)\phi = 0 \Longrightarrow P\varphi := [\Delta_x + \tilde{a}\partial_z^2 + \tilde{b} \cdot \nabla \partial_z + \tilde{c}\partial_z]\varphi = 0, \tag{3.4}$$

where

$$\tilde{a} = \frac{(y + 1)^2 |\nabla h|^2}{(1 + h)^4} + \frac{1}{(1 + h)^2} = \frac{1 + (z + 1)^2 |\nabla h|^2}{(1 + h)^2}, \tag{3.5}$$

$$\tilde{b} = -2\frac{(y + 1)\nabla h}{(1 + h)^2} = \frac{-2(z + 1)\nabla h}{1 + h}, \quad \tilde{c} = \frac{-(z + 1)\Delta_x h}{(1 + h)} + 2\frac{(z + 1)|\nabla h|^2}{(1 + h)^2}. \tag{3.6}$$

To sum up, the harmonic equation with Dirichlet-Neumann boundary conditions (1.2) can be reformulated in terms of $\varphi$ as follows

$$P\varphi = 0, \quad \varphi\big|_{z=0} = \psi, \quad \partial_z \varphi\big|_{z=-1} = 0, \quad (x, z) \in \mathbb{R}^2 \times [-1, 0]. \tag{3.7}$$

### 3.1.2 Type II formulation of harmonic equation

As the Dirichlet-Neumann operator only locally depends on the interface if we already know the behavior of harmonic potential. In the mean time, the coefficients in equation (3.4) involve dependent variable $h$ in the denominators, it's very difficult to see cancellations during the paralinearization

process for the Dirichlet-Neumann operator. These two facts motivate us to consider another type of formulation.

Under the smallness condition (3.1), $L^\infty$ norm of $h$ is very small, there exists a curve parallel to the interface $\Gamma(t)$ with depth $1/2$ inside $\Omega(t)$. More precisely, we have

$$\Omega_1(t) := \{(x,y) : x \in \mathbb{R}^2, \quad h(t,x) - 1/2 \le y \le h(t,x)\}, \quad \Omega_1(t) \subset \Omega(t).$$

In the later argument, when we use type II formulation, the domain we are considering is $\Omega_1(t)$ instead of $\Omega(t)$. Define

$$\Omega_2(t) := \{(x,y) : x \in \mathbb{R}^2, \quad h(t,x) - 1/4 \le y \le h(t,x)\}, \quad \Omega_2(t) \subset \Omega_1(t) \subset \Omega(t), \tag{3.8}$$

$$\tilde{\phi}(x,y) := \chi(y - h(t,x))\phi(x,y), \quad (x,y) \in \Omega_1(t), \quad \chi(z) = 1 \text{ if } z \ge -1/4, \mathit{supp}(\chi) \subset [-1/2, 0],$$

where $\chi(x)$ in above is a fixed Schwarz function. Recall that $\Delta_{x,y}\phi = 0$, hence the following identities hold,

$$\Delta_{x,y}\tilde{\phi} = \tilde{g} := \Delta_{x,y}[\chi\phi] - \chi\Delta_{x,y}\phi, \quad (x,y) \in \Omega_1(t), \tag{3.9}$$

$$\tilde{\phi}(x,y) = \phi(x,y), \quad \tilde{g}(x,y) = 0, \quad (x,y) \in \Omega_2(t).$$

We can transform $\Omega_1(t)$ to strip $\mathcal{S}' := \mathbb{R}^2 \times [-1/2, 0]$ as follows,

$$(x,y) \to (x,w), \quad w := y - h(t,x).$$

Define $\Phi(x,w) := \tilde{\phi}(x, \omega + h(t,x))$, hence $\tilde{\phi}(x,y) = \Phi(x, y - h(t,x))$. From (3.9), it's easy to verify the following equality holds.

$$\mathcal{P}_{x,w}\Phi := [\Delta_x + a'\partial_w^2 + b' \cdot \nabla\partial_w + c'\partial_w]\Phi = g'(x,w) := \tilde{g}(x, \omega + h(t,x)), \tag{3.10}$$

where

$$a' = 1 + |\nabla h|^2, \quad b' = -2\nabla h, \quad c' = -\Delta h. \tag{3.11}$$

*Remark* 3.1.1. From (3.11), one can see that the coefficients associated with the equation of $\Phi$ (see (3.10)) are much easier and more favorable than the coefficients of the equation of $\varphi$ (see (3.4)) in (3.5) and (3.6). However, the system satisfied by $\Phi$ cannot be used as the starting point because we don't know the properties and the estimates of $\varphi$ at the first place.

From definitions, the following identities hold inside $\Omega_2(t)$,

$$\Phi(x,w)=\varphi(x,\frac{w}{1+h}),\quad \varphi(x,z)=\Phi(x,(1+h)z)\quad (x,w)\in\mathbb{R}^2\times[-1/4,0],$$

$$\partial_{x_i}\Phi=\partial_{x_i}\varphi-\frac{w\partial_z\varphi\partial_{x_i}h}{(1+h)^2}\quad \partial_w\Phi=\frac{\partial_z\varphi}{1+h}.$$

The Dirichlet-Neumann operator in terms of $\varphi$ and $\Phi$ are given as follows,

$$G(h)\psi=[-\nabla h\cdot\nabla\phi+\partial_y\phi]\big|_{y=h}=\frac{1+|\nabla h|^2}{1+h}\partial_z\varphi\big|_{z=0}-\nabla\psi\cdot\nabla h, \tag{3.12}$$

$$G(h)\psi=(1+|\nabla h|^2)\partial_w\Phi\big|_{w=0}-\nabla h\cdot\nabla\psi. \tag{3.13}$$

From (3.12), we can see that the quadratic term of $G(h)\psi$ is given as follows,

$$\Lambda_2[G(h)\psi]=\Lambda_2[\partial_z\varphi]-\Lambda_1[\partial_z\varphi]\,h-\nabla\psi\cdot\nabla h. \tag{3.14}$$

### 3.1.3 A fixed point type formulation for the Dirichlet-Neumann operator

In this subsection, our main goal is to study equation (3.7) and formulate $\nabla_{x,z}\varphi$ into a fixed point type formulation, which enables us to derive $L^2$ and $L^\infty$ estimates for Dirichlet-Neumann operator.

We rewrite the equation $P\varphi=0$ as follows,

$$\partial_z^2\varphi+\Delta_x\varphi=(\partial_z-|\nabla|)(\partial_z+|\nabla|)\varphi=g:=(1-\tilde{a})\partial_z^2\varphi-\tilde{b}\cdot\nabla\partial_z\varphi-\tilde{c}\partial_z\varphi.$$

To see the fixed point structure inside $g$, we need to classify terms inside $g$. Note that

$$g=\partial_z[\frac{2h+h^2-(z+1)^2|\nabla h|^2}{(1+h)^2}\partial_z\varphi+\frac{2(z+1)\nabla h\cdot\nabla\varphi}{1+h}]-\frac{2\nabla h\cdot\nabla\varphi}{1+h}+\frac{(z+1)\Delta h}{1+h}\partial_z\varphi,$$

and

$$\frac{(z+1)\Delta h}{1+h}\partial_z\varphi=\nabla\cdot[\frac{(z+1)\nabla h\partial_z\varphi}{1+h}]+\frac{(z+1)|\nabla h|^2\partial_z\varphi}{(1+h)^2}-\partial_z[\frac{(z+1)\nabla h\cdot\nabla\varphi}{1+h}]+\frac{\nabla h\cdot\nabla\varphi}{1+h}.$$

Hence, we have $g=\partial_z g_1(z)+g_2(z)+\nabla\cdot g_3(z)$, where

$$g_1(z)=\frac{2h+h^2-(z+1)^2|\nabla h|^2}{(1+h)^2}\partial_z\varphi+\frac{(z+1)\nabla h\cdot\nabla\varphi}{1+h},\quad g_1(-1)=0, \tag{3.15}$$

$$g_2(z)=\frac{(z+1)|\nabla h|^2\partial_z\varphi}{(1+h)^2}-\frac{\nabla h\cdot\nabla\varphi}{1+h},\quad g_3(z)=\frac{(z+1)\nabla h\partial_z\varphi}{1+h}. \tag{3.16}$$

Define

$$\tilde{h}_1:=\frac{2h+h^2}{(1+h)^2},\quad \tilde{h}_2:=\frac{h}{1+h},\Longrightarrow \frac{\nabla h}{1+h}=(1+h)\nabla\tilde{h}_2,\quad \frac{|\nabla h|^2}{(1+h)^2}=(1+h)^2|\nabla\tilde{h}_2|^2. \tag{3.17}$$

Therefore, in terms of above notation, we can reformulate $g_1$, $g_2$, and $g_3$ as follows,

$$g_1(z)=\tilde{h}_1\partial_z\varphi-(z+1)^2(1+h)^2|\nabla\tilde{h}_2|^2\partial_z\varphi+(z+1)(1+h)\nabla\tilde{h}_2\cdot\nabla\varphi, \tag{3.18}$$

$$g_2(z)=(z+1)(1+h)^2|\nabla\tilde{h}_2|^2\partial_z\varphi-(1+h)\nabla\tilde{h}_2\cdot\nabla\varphi,\quad g_3(z)=(z+1)(1+h)(\nabla\tilde{h}_2\partial_z\varphi). \tag{3.19}$$

*Remark* 3.1.2. Note that, to find out what the precise formula of quadratic terms in terms of $h$ and $\psi$ is, we will use (3.15) and (3.16). Meanwhile, to estimate $\Lambda_{\geq i}[\nabla_{x,z}\varphi]$, we will use (3.18) and (3.19).

Define $\tilde{h}(x,z):=(\partial_z-|\nabla|)\varphi$, very naturally, we have

$$\begin{cases}(\partial_z+|\nabla|)\tilde{h}=g,\\[2ex] (\partial_z-|\nabla|)\varphi=\tilde{h},\quad \varphi\big|_{z=0}=\psi,\partial_z\varphi\big|_{z=-1}=0.\end{cases} \tag{3.20}$$

We can solve above system of equations with $\tilde{h}(-1)$ to be determined as follows,

$$\tilde{h}(z)=e^{-z|\nabla|}\tilde{h}(-1)+\int_{-1}^{z}e^{-(z-z')|\nabla|}g(z')dz',$$

$$\varphi(z)=e^{z|\nabla|}\varphi(0)+\int_0^z e^{(z-z')|\nabla|}\tilde{h}(z')dz'=e^{z|\nabla|}\psi-\int_z^0 e^{(z-z')|\nabla|}[e^{-z'|\nabla|}\tilde{h}(-1)$$

$$+\int_{-1}^{z'}e^{-(z'-s)|\nabla|}g(s)ds]dz'=e^{z|\nabla|}\psi-\frac{1}{2}|\nabla|^{-1}[e^{-z|\nabla|}-e^{z|\nabla|}]\tilde{h}(-1)$$

$$-\int_{-1}^{z}\int_z^0 e^{(z+s-2z')|\nabla|}g(s)dz'ds-\int_z^0\int_s^0 e^{(z+s-2z')|\nabla|}g(s)dz'ds$$

$$=e^{z|\nabla|}\psi-\frac{1}{2}|\nabla|^{-1}[e^{-z|\nabla|}-e^{z|\nabla|}]\tilde{h}(-1)+\frac{1}{2}\int_{-1}^0|\nabla|^{-1}e^{(z+s)|\nabla|}g(s)ds-\frac{1}{2}\int_{-1}^0|\nabla|^{-1}e^{-|z-s||\nabla|}g(s)ds. \tag{3.21}$$

The unknown $\tilde{h}(-1)$ can be determined from the Neumann condition $\partial_z\varphi|_{z=-1}=0$. We calculate

$\partial_z\varphi\big|_{z=-1}$ from the formula (3.21) as follows,

$$\partial_z\varphi = |\nabla|e^{z|\nabla|}\psi + \frac{1}{2}[e^{z|\nabla|}+e^{-z|\nabla|}]\tilde{h}(-1) + \frac{1}{2}\int_{-1}^{0} e^{(z+s)|\nabla|}g(s)ds - \frac{1}{2}\int_{-1}^{0} e^{-|z-s||\nabla|}\mathrm{sign}(s-z)g(s)ds.$$

Hence

$$\tilde{h}(-1) = -\frac{2|\nabla|e^{-|\nabla|}\psi}{e^{-|\nabla|}+e^{|\nabla|}} - \int_{-1}^{0}\frac{e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}g(s)\,ds, \tag{3.22}$$

which further gives us

$$\partial_z\varphi = \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi - \frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}[e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}]g(s)\,ds \tag{3.23}$$

$$+\frac{1}{2}\int_{-1}^{0} e^{(z+s)|\nabla|}g(s)ds - \frac{1}{2}\int_{-1}^{0} e^{-|z-s||\nabla|}\mathrm{sign}(s-z)g(s)ds. \tag{3.24}$$

Moreover, (3.21) can be reduced as follows,

$$\varphi(z) = \Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\psi + \frac{1}{2}|\nabla|^{-1}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\big[e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}\big]g(s)\,ds$$

$$+\frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{(z+s)|\nabla|}g(s)ds - \frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{-|z-s||\nabla|}g(s)ds. \tag{3.25}$$

After splitting $g$ into $\partial_z g_1$, $g_2$ and $\nabla\cdot g_3$, we can do integration by parts in "$s$" for $g_1$. As a result, we have the following identity,

$$\varphi(z) = \Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\psi + \frac{1}{2}|\nabla|^{-1}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\big[e^{(s-1)|\nabla|}(g_2+\nabla\cdot g_3-|\nabla|g_1)$$

$$-e^{-(s+1)|\nabla|}(g_2+\nabla\cdot g_3+|\nabla|g_1)\big]ds + \frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{(z+s)|\nabla|}[g_2+\nabla\cdot g_3-|\nabla|g_1]ds$$

$$-\frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{-|z-s||\nabla|}[g_2+\nabla\cdot g_3-\mathrm{sign}(z-s)|\nabla|g_1]ds. \tag{3.26}$$

From (3.26) we can derive the fixed point type formulation for $\nabla_{x,z}\varphi$ as follows,

$$\nabla_{x,z}\varphi = \left[\Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\nabla\psi, \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\right] +$$

$$\left[\frac{1}{2}\frac{\nabla}{|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\big[e^{(s-1)|\nabla|}(g_2+\nabla\cdot g_3-|\nabla|g_1) - e^{-(s+1)|\nabla|}(g_2+\nabla\cdot g_3+|\nabla|g_1)\big]ds,\right.$$

$$-\frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\big[e^{(s-1)|\nabla|}(g_2+\nabla\cdot g_3-|\nabla|g_1)-e^{-(s+1)|\nabla|}(g_2+\nabla\cdot g_3+|\nabla|g_1)\big]ds\Big]+$$

$$\Big[\frac{1}{2}\int_{-1}^{0}\frac{\nabla}{|\nabla|}e^{(z+s)|\nabla|}[g_2+\nabla\cdot g_3-|\nabla|g_1]ds-\frac{1}{2}\int_{-1}^{0}\frac{\nabla}{|\nabla|}e^{-|z-s||\nabla|}[g_2+\nabla\cdot g_3-\mathrm{sign}(z-s)|\nabla|g_1]ds,$$

$$\frac{1}{2}\int_{-1}^{0}e^{(z+s)|\nabla|}[g_2+\nabla\cdot g_3-|\nabla|g_1]ds-\frac{1}{2}\int_{-1}^{0}e^{-|z-s||\nabla|}[\mathrm{sign}(s-z)(g_2+\nabla\cdot g_3)+|\nabla|g_1]ds\Big]+[\mathbf{0},g_1(z)]. \tag{3.27}$$

From (3.27), it motives us to define the following operators,

$$K_1(z,s):=\Big[\frac{\nabla}{2|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{(s-1)|\nabla|}+\frac{\nabla}{2|\nabla|}e^{(z+s)|\nabla|},-\frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{(s-1)|\nabla|}+\frac{1}{2}e^{(z+s)|\nabla|}\Big], \tag{3.28}$$

$$K_2(z,s):=\Big[\frac{\nabla}{2|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{-(s+1)|\nabla|}\ ,\ -\frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{-(s+1)|\nabla|}\Big], \tag{3.29}$$

$$K_3(z,s)=\Big[\frac{\nabla}{2|\nabla|}e^{-|z-s||\nabla|}\ ,\ \frac{1}{2}e^{-|z-s||\nabla|}\mathrm{sign}(s-z)\Big]. \tag{3.30}$$

With above definitions, we can rewrite (3.27) as follows,

$$\nabla_{x,z}\varphi=\Big[\Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\nabla\psi,\frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\Big]+[\mathbf{0},g_1(z)]+$$

$$+\int_{-1}^{0}[K_1(z,s)-K_2(z,s)-K_3(z,s)](g_2(s)+\nabla\cdot g_3(s))ds$$

$$+\int_{-1}^{0}K_3(z,s)|\nabla|\mathrm{sign}(z-s)g_1(s)-|\nabla|[K_1(z,s)+K_2(z,s)]g_1(s)\,ds. \tag{3.31}$$

From (3.31), it's easy to see that the linear term of $\nabla_{x,z}\varphi$ is given as follows,

$$\Lambda_1[\nabla_{x,z}\varphi]=\Big[\Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\nabla\psi,\frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\Big]. \tag{3.32}$$

We have the following lemma on the boundednesses of operators $K_i(z,s)$,

**Lemma 3.1.3.** *For $k,\gamma\geq 0$, we have the following estimates,*

$$\sum_{i=1,2,3}\|\int_{-1}^{0}K_i(z,s)\nabla g(s)ds\|_{L_z^\infty H^k}+\|\int_{-1}^{0}K_i(z,s)g(s)ds\|_{L_z^\infty H^k}\lesssim\|g(z)\|_{L_z^\infty H^k}, \tag{3.33}$$

$$\sum_{i=1,2,3}\|\int_{-1}^{0}K_i(z,s)\nabla g(s)ds\|_{L_z^\infty \widetilde{W^\gamma}}$$

$$+\| \int_{-1}^{0} [K_1(z,s) - K_2(z,s) - K_3(z,s)] g(s) ds \|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|g(z)\|_{L_z^\infty \widetilde{W^\gamma}}. \tag{3.34}$$

*Proof.* Recall (3.28), (3.29), and (3.30), the following estimate holds from Lemma 1.3.8,

$$\sup_{z,s\in[-1,0]} \|\mathcal{F}^{-1}[\mathcal{F}\big([K_1(z,s) - K_2(z,s) - K_3(z,s)]\big)(\xi)\psi_{k_1}(\xi)]\|_{L^1} \lesssim 2^{k_1}, \quad k_1 \leq 0, \tag{3.35}$$

$$\sup_{z,s\in[-1,0]} \sum_{i=1,2,3} \|\mathcal{F}^{-1}[\mathcal{F}\big(K_i(z,s)\big)(\xi)\psi_{k_1}(\xi)]\|_{L^1} \lesssim 1. \tag{3.36}$$

Note that (3.36) cannot be used directly to estimate the left hand side of (3.33), otherwise we lose one derivative when $k_1 \geq 0$. An observation is that the integration with respect to "$s$" will compensate the loss. For any fixed $k \geq 0, k \in \mathbb{Z}$, in terms of kernel, we have

$$\int_{-1}^{0} K_i(z,s) \nabla P_k[g(s)] ds = \int_{-1}^{0} \int_{\mathbb{R}^2} K_{i;k}(z,s,y) g(s, x-y) dy ds, \tag{3.37}$$

where

$$K_{i;k}(z,s,y) = \int_{\mathbb{R}^2} e^{iy\cdot\xi} \mathcal{F}(K_1(z,s))(\xi)\psi_k(\xi)\xi d\xi.$$

After integration by parts in $\xi$ many times, the following pointwise estimate holds for $i \in \{1,2,3\}$,

$$|K_{i;k}(z,s,y)| \lesssim 2^{3k}\big(1 + 2^k|y| + 2^k|z-s|\big)^{-10}, \tag{3.38}$$

which further gives us that the kernel belongs to $L^1_{s,y}$ for fixed $z$. Therefore, from (3.36) and (3.38), the following estimate holds,

$$\text{The left hand side of (3.33)}^2 \lesssim \sum_{k_1\leq 0} \|P_{k_1}[g(z)]\|^2_{L_z^\infty L^2} + \sum_{i=1,2,3} \sum_{k_1\geq 0} 2^{2kk_1} \|K_{i;k_1}(z,s)\|^2_{L^1_{s,y}}$$

$$\times \|P_{k_1}[g(z)]\|^2_{L_z^\infty L^2} \lesssim \|g(z)\|^2_{L_z^\infty H^k}.$$

Hence finishing the proof of (3.33). From (3.35), (3.36), and (3.38), our desired estimate (3.34) follows in the same way. □

To formulate $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$ into a fixed point type formulation, it's sufficient to reformulate $\Lambda_{\geq 2}[g_1]$, $\Lambda_{\geq 2}[g_2]$, and $\Lambda_{\geq 2}[g_3]$. Recall (3.18) and (3.19), we can split terms in $g_i(z), i \in \{1,2,3\}$ into two parts: one of them doesn't depend on $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$ and the other part contains $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$. This type of

decomposition enables us to see a fixed point type structure inside. More precisely, we have

$$\Lambda_{\geq 2}[g_1(z)] = \tilde{h}_1\Lambda_{\geq 2}[\partial_z\varphi] - (z+1)^2(1+h)^2|\nabla\tilde{h}_2|^2\Lambda_{\geq 2}[\partial_z\varphi] + (z+1)(1+h)\nabla\tilde{h}_2\cdot\Lambda_{\geq 2}[\nabla\varphi] \quad (3.39)$$

$$+\tilde{h}_1\Lambda_1[\partial_z\varphi] - (z+1)^2(1+h)^2|\nabla\tilde{h}_2|^2\Lambda_1[\partial_z\varphi] + (z+1)(1+h)\nabla\tilde{h}_2\cdot\Lambda_1[\nabla\varphi], \quad (3.40)$$

$$\Lambda_{\geq 2}[g_2(z)] = (z+1)(1+h)^2|\nabla\tilde{h}_2|^2\Lambda_{\geq 2}[\partial_z\varphi] - (1+h)\nabla\tilde{h}_2\cdot\Lambda_{\geq 2}[\nabla\varphi]$$

$$+(z+1)(1+h)^2|\nabla\tilde{h}_2|^2\Lambda_1[\partial_z\varphi] - (1+h)\nabla\tilde{h}_2\cdot\Lambda_1[\nabla\varphi]. \quad (3.41)$$

$$\Lambda_{\geq 2}[g_3(z)] = (z+1)(1+h)\nabla\tilde{h}_2\Lambda_{\geq 2}[\partial_z\varphi] + (z+1)(1+h)\nabla\tilde{h}_2\Lambda_1[\partial_z\varphi]. \quad (3.42)$$

From above fixed point type formulation, we can derive the following lemma.

**Lemma 3.1.4.** *For $\gamma', k' \geq 1$, $0 < \delta \ll 1$, $\alpha \in (0,1]$, if $h \in \widetilde{W^{\gamma'}} \cap H^{k'}$ satisfies the following smallness assumption:*

$$\|h\|_{\widetilde{W^{\gamma'}}} < \delta, \quad (3.43)$$

*then we have the following $L^2$-type and $L^\infty$-type estimates for the solution of* (3.7)*,*

$$\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} \lesssim \|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W^0}}, \quad (3.44)$$

$$\|\nabla_x\varphi\|_{L^\infty_z\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{\widetilde{W^\gamma}}, \quad \|\partial_z\varphi\|_{L^\infty_z\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{W^{\gamma,2\alpha}} + \|h\|_{\widetilde{W^{\gamma+1}}}\|\nabla\psi\|_{\widetilde{W^\gamma}}, \quad (3.45)$$

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{\widetilde{W^\gamma}}\|h\|_{\widetilde{W^{\gamma+1}}}, \quad (3.46)$$

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \lesssim \|h\|_{\widetilde{W^1}}\||\nabla|\psi\|_{H^k} + \|\nabla\psi\|_{\widetilde{W^0}}\|h\|_{H^{k+1}}, \quad (3.47)$$

*where $k \leq k'-1$ and $1 \leq \gamma \leq \gamma'-1$. In above estimates, the range of $z$ for the $L^\infty_z$ norm is restricted in $[-1,0]$.*

*Proof.* Let's first estimate $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$. From (3.31), (3.39), (3.40), (3.41) and (3.42), the following estimate holds after using (3.34) in Lemma 3.1.3,

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z\widetilde{W^\gamma}} \lesssim \|\Lambda_{\geq 2}[(g_1(z), g_2(z), g_3(z))]\|_{L^\infty_z\widetilde{W^\gamma}}$$

$$\lesssim \|h\|_{\widetilde{W^{\gamma+1}}}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z\widetilde{W^\gamma}} + \|h\|_{\widetilde{W^{\gamma+1}}}\|\nabla\psi\|_{\widetilde{W^\gamma}},$$

hence, under the smallness condition (3.43), we have

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \| h\|_{\widetilde{W^{\gamma+1}}} \|\nabla\psi\|_{\widetilde{W^\gamma}}. \tag{3.48}$$

Also the following estimate holds,

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty H^k} \lesssim \|\Lambda_{\geq 2}[(g_1(z), g_2(z), g_3(z))]\|_{L_z^\infty H^k} \lesssim \| h\|_{\widetilde{W^1}} \|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty H^k}$$

$$+\| h\|_{H^{k+1}} \|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty \widetilde{W^0}} + \| h\|_{H^{k+1}} \|\nabla\psi\|_{\widetilde{W^0}} + \|\nabla\psi\|_{H^k} \| h\|_{\widetilde{W^1}}$$

$$\lesssim \| h\|_{\widetilde{W^1}} \|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty H^k} + \| h\|_{H^{k+1}} \|\nabla\psi\|_{\widetilde{W^0}} (1 + \| h\|_{\widetilde{W^1}}) + \|\nabla\psi\|_{H^k} \| h\|_{\widetilde{W^1}},$$

hence, under the smallness assumption (3.43), we have

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L_z^\infty H^k} \lesssim \| h\|_{H^{k+1}} \|\nabla\psi\|_{\widetilde{W^0}} + \|\nabla\psi\|_{H^k} \| h\|_{\widetilde{W^1}}. \tag{3.49}$$

From estimates (3.48) and (3.49) and the explicit formulas of $\Lambda_1[\nabla_{x,z}\varphi]$, we have

$$\|\nabla_{x,z}\varphi\|_{L^\infty H^k} \lesssim \|\nabla\psi\|_{H^k} + \| h\|_{H^{k+1}} \|\nabla\psi\|_{\widetilde{W^0}}, \quad \|\nabla_x\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{\widetilde{W^\gamma}},$$

$$\|\partial_z\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|\Lambda^2\psi\|_{\widetilde{W^\gamma}} + \| h\|_{\widetilde{W^{\gamma+1}}} \|\nabla\psi\|_{\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{W^{\gamma,2\alpha}} + \| h\|_{\widetilde{W^{\gamma+1}}} \|\nabla\psi\|_{\widetilde{W^\gamma}}.$$

Now, it's easy to see our lemma holds. □

### 3.1.4 Expansion of the Dirichlet-Neumann operator: Quadratic terms

In this subsection, we will find out the quadratic terms of $\partial_z\varphi\big|_{z=0}$ explicitly, which helps us to determine the quadratic terms of $G(\,h)\psi$, $B(\,h)\psi$ and $V(\,h)\psi$.

Recall the formula (3.31) , we have

$$\Lambda_2[\partial_z\varphi\big|_{z=0}] = -\frac{1}{e^{|\nabla|}+e^{-|\nabla|}} \int_{-1}^0 [e^{(s-1)|\nabla|} - e^{-(s+1)|\nabla|}][\Lambda_2[g_2 + \nabla\cdot g_3]]ds$$

$$+\frac{1}{e^{|\nabla|}+e^{-|\nabla|}} \int_{-1}^0 [e^{(s-1)|\nabla|} + e^{-(s+1)|\nabla|}]|\nabla|[\Lambda_2[g_1]]ds + \int_{-1}^0 e^{s|\nabla|}\Lambda_2[g_2 + \nabla\cdot g_3 - |\nabla|g_1]ds + \Lambda_2[g_1(0)]$$

$$= \int_{-1}^0 \frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}} \Lambda_2[g_2+\nabla\cdot g_3]ds - \int_{-1}^0 \frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}} |\nabla|\Lambda_2[g_1] + \Lambda_2[g_1(0)]. \tag{3.50}$$

Hence, to determine the quadratic term of $\nabla_{x,z}\varphi$, it is sufficient to find out the quadratic terms of

$g_i(z)$, $i \in \{1,2,3\}$. From (3.15) and (3.16), it is easy to see they are given as follows,

$$\Lambda_2[g_1(z)] = 2\,h\Lambda_1[\partial_z\varphi] + (z+1)\nabla h\cdot\Lambda_1[\nabla\varphi],$$

$$\Lambda_2[g_2(z)] = -\nabla h\cdot\Lambda_1[\nabla\varphi], \quad \Lambda_2[g_3(z)] = (z+1)\nabla h\Lambda_1[\partial_z\varphi].$$

Combining above identities with (3.32), we have

$$\Lambda_2[g_1(s)] = 2\,h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi + (s+1)\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi,$$

$$\Lambda_2[g_1(0)] = 2\,h\Lambda^2\psi + \nabla h\cdot\nabla\psi, \tag{3.51}$$

$$\Lambda_2[g_2(s)] = -\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi, \quad \Lambda_2[g_3(s)] = (s+1)\nabla h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi.$$

We define the first integral of (3.50) as follows,

$$Q_1(h,\psi) := \int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\Lambda_2[g_2+\nabla\cdot g_3]ds = Q_{1,1}(h,\psi) + Q_{1,2}(h,\psi),$$

where

$$Q_{1,1}(h,\psi) = \int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\Big[\nabla\cdot[(s+1)\nabla h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi\Big]ds, \tag{3.52}$$

$$Q_{1,2}(h,\psi) = \int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\Big[-\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi\Big]ds. \tag{3.53}$$

Their corresponding symbols are given as follows,

$$q_{1,1}(\xi-\eta,\eta) = \frac{-\xi\cdot(\xi-\eta)|\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}(s+1)[e^{(s+1)|\xi|}+e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}-e^{-(s+1)|\eta|}]ds$$

$$= \frac{-\xi\cdot(\xi-\eta)|\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{1+(|\xi|+|\eta|-1)e^{|\xi|+|\eta|}}{(|\xi|+|\eta|)^2} - \frac{1+(|\xi|-|\eta|-1)e^{|\xi|-|\eta|}}{(|\xi|-|\eta|)^2}$$

$$+\frac{1+(|\eta|-|\xi|-1)e^{|\eta|-|\xi|}}{(|\xi|-|\eta|)^2} - \frac{1-(-|\xi|-|\eta|-1)e^{-|\xi|-|\eta|}}{(|\xi|+|\eta|)^2}\Big]$$

$$= \frac{-\xi\cdot(\xi-\eta)|\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{(|\xi|+|\eta|-1)e^{|\xi|+|\eta|}-(|\xi|+|\eta|+1)e^{-|\xi|-|\eta|}}{(|\xi|+|\eta|)^2}$$

$$+\frac{(|\eta|-|\xi|-1)e^{|\eta|-|\xi|}-(|\xi|-|\eta|-1)e^{|\xi|-|\eta|}}{(|\xi|-|\eta|)^2}\Big], \tag{3.54}$$

$$q_{1,2}(\xi-\eta,\eta)=\frac{(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}[e^{(s+1)|\xi|}+e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}+e^{-(s+1)|\eta|}]ds$$

$$=\frac{(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}+\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\Big].$$

We define the second integral of (3.50) as follows,

$$Q_2(h,\psi)=-\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\Lambda_2[g_1]ds=Q_{2,1}(h,\psi)+Q_{2,2}(h,\psi),$$

where

$$Q_{2,1}(h,\psi)=-\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\Big[2\,h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi\Big]ds,$$

$$Q_{2,2}(h,\psi)=-\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\Big[(s+1)\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi\Big]ds.$$

Their corresponding symbols are given as follows,

$$q_{2,1}(\xi-\eta,\eta)=\frac{-2|\xi||\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}[e^{(s+1)|\xi|}-e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}-e^{-(s+1)|\eta|}]ds$$

$$=\frac{-2|\xi||\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}-\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\Big],\tag{3.55}$$

$$q_{2,2}(\xi-\eta,\eta)=\frac{|\xi|(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}(s+1)[e^{(s+1)|\xi|}-e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}+e^{-(s+1)|\eta|}]ds$$

$$=\frac{|\xi|(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{(|\xi|+|\eta|-1)e^{|\xi|+|\eta|}-(|\xi|+|\eta|+1)e^{-|\xi|-|\eta|}}{(|\xi|+|\eta|)^2}$$

$$-\frac{(|\eta|-|\xi|-1)e^{|\eta|-|\xi|}-(|\xi|-|\eta|-1)e^{|\xi|-|\eta|}}{(|\xi|-|\eta|)^2}\Big].\tag{3.56}$$

Therefore, after combining (3.14), (3.50) and (3.51), we can see that the quadratic terms of $G(h)\psi$ are given as follows,

$$\Lambda_2[G(h)\psi]:=\widetilde{Q}(h,\psi)=Q_{1,1}(h,\psi)+\widetilde{Q}_{1,2}(h,\psi)+Q_2(h,\psi),\tag{3.57}$$

where

$$\widetilde{Q}_{1,2}(\,h,\psi)=Q_{1,2}(\,h,\psi)+\,h\frac{e^{|\nabla|}-e^{-|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi,$$

and the symbol of $\widetilde{Q}_{1,2}(\,h,\psi)$ is given as follows,

$$\widetilde{q}_{1,2}(\xi-\eta,\eta)=q_{1,2}(\xi-\eta,\eta)+\frac{e^{|\eta|}-e^{-|\eta|}}{e^{|\eta|}+e^{-|\eta|}}|\eta|$$

$$=\frac{(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}+\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\Big]+\frac{e^{|\eta|}-e^{-|\eta|}}{e^{|\eta|}+e^{-|\eta|}}|\eta|$$

$$=\frac{\xi\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}+\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\Big]+$$

$$\frac{|\xi||\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Big[\frac{[e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}]}{|\xi|+|\eta|}-\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\Big]. \tag{3.58}$$

Now, it's not difficult to see that we can gain smallness in the High $\times$ High interaction due to the appearance of "$\xi$" in (1.23), (3.55), (3.56) and (3.58).

**Lemma 3.1.5.** *Let $\widetilde{q}(\cdot,\cdot)$ denotes the symbol of quadratic terms of $G(\,h)\psi$ (see (3.57)), for $k_1,k_2,k\in\mathbb{Z}$, we have the following symbol estimates:*

$$\|\widetilde{q}(\xi-\eta,\eta)\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}}\lesssim 2^{k+k_2}. \tag{3.59}$$

*Proof.* By using Lemma 1.3.8, it is straightforward to derive above estimate after checking the formulas (1.23), (3.55), (3.56) and (3.58) very carefully. □

*Remark* 3.1.6. Note that, when $\xi=\eta$ and $|\xi|\sim 1$, we have

$$\widetilde{q}(0,\xi)=\frac{4|\xi|^2}{(e^{|\xi|}+e^{-|\xi|})^2}\sim 1,$$

which obviously doesn't vanish. This fact is very different to the infinite depth setting, in which case the symbol indeed vanishes. More precisely, the corresponding symbol is given as follows,

$$\widetilde{q}_{\text{infinite depth}}(\xi-\eta,\eta)=-|\xi||\eta|+\xi\cdot\eta,\quad \widetilde{q}_{\text{infinite depth}}(0,\eta)=0.$$

That is to say, only weak "null structure" is available in the finite depth setting and this weak "null structure" is only effective at the high frequency part but not at the low frequency part.

Due to non-existence of "null structure" at low frequency part, one needs to pay special attention

to the $0 \times 1 \to 1$ type interaction in the long run.

## 3.2 Paralinearization and Symmetrization of the full system

We first trying to derive the following proposition, which contains both the paralinearization of the Dirichlet-Neumann operator and the $L^2$ type estimate of the associated remainder term.

**Proposition 3.2.1.** *Let $k \geq 6$, $\alpha \in (0,1]$ and assume that $(h, \Lambda\psi) \in H^k$ and $h$ satisfies the smallness condition* (3.43)*, then we have*

$$G(h)\psi = T_\lambda(\psi - T_B h) - T_V \cdot \nabla h + F(h)\psi = \Lambda^2(\psi - T_B h) + T_{\lambda - |\xi|}(\psi - T_B h) - T_V \cdot \nabla h + \widetilde{F}(h)\psi, \quad (3.60)$$

*where*

$$\lambda := \sqrt{(1 + |\nabla h|^2)|\xi|^2 - (\nabla h \cdot \xi)^2}, \quad (3.61)$$

$$B \overset{abbr}{=} B(h)\psi = \frac{G(h)\psi + \nabla h \cdot \nabla\psi}{1 + |\nabla h|^2}, \quad V \overset{abbr}{=} V(h)\psi = \nabla\psi - B\nabla h. \quad (3.62)$$

*The good remainder terms $F(h)\psi$ and $\widetilde{F}(h)\psi$ don't lose derivatives and they satisfy the following estimate,*

$$\|\Lambda_{\geq 2}[F(h)\psi]\|_{H^k} + \|\widetilde{F}(h)\psi\|_{H^k} \lesssim_k \big[\|(h, \Lambda\psi)\|_{W^{4,2\alpha}} + \|(h, \Lambda\psi)\|_{W^4}^2\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big]. \quad (3.63)$$

*Remark* 3.2.2. We mention that, very different from the infinite depth case, here $F(h)\psi$ actually contains linear term, which is $[\tanh(|\nabla|) - 1]|\nabla|\psi \in H^\infty$ and this term behaves very nicely.

For simplicity, we define the following equivalence relation. For two well defined nonlinearities $A$ and $B$, which are nonlinear with respect to $h$ and $\psi$, we use the following notation,

$$A \approx B, \quad \text{if and only if } A - B \text{ is a good error term in the sense of (3.64),}$$

$$\|\text{good error term}\|_{H^k} \lesssim_k \big[\|(h, \Lambda\psi)\|_{W^{4,2\alpha}} + \|(h, \Lambda\psi)\|_{W^4}^2\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big], \ \alpha = 1/10, k \geq 0. \quad (3.64)$$

**Paralinearization of the harmonic equation**

To find out the cancellation structures inside $G(h)\psi$, we need to go back to the elliptic equation satisfied by $\Phi$ in the region $\Omega_2(t)$ (see (3.8)). Recall (3.10) and the fact that $g'(\cdot, w) = 0$ when

$w \in [-1/4, 0]$, we have

$$[\Delta_x + a'\partial_w^2 + b' \cdot \nabla \partial_w + c'\partial_w]\Phi = 0, \tag{3.65}$$

$$a' = 1 + |\nabla\, h|^2 \approx 1 + 2T_{\nabla\, h} \cdot \nabla\, h, \quad b' = -2\nabla\, h, \quad c' = -\Delta\, h.$$

In the following context, $w$ is restricted inside $[-1/4, 0]$.

Before proceeding to do paralinearization, we need some necessary estimates on "$\Phi$". Essentially speaking, under certain smallness condition, the size of "$\Phi$" is comparable to "$\varphi$", for which we already have necessary estimates. More precisely, from definition of $\Phi$ (see subsection 3.1.2) and estimates of $\varphi$ in Lemma 3.1.4, we have the following lemma,

**Lemma 3.2.3.** *Under the smallness estimate* (3.1)*, we have the following estimates for* $k \geq 1, \gamma \leq 3$,

$$\sup_{w\in[-1/4,0]} \|\nabla_{x,w}\Phi\|_{H^k} \lesssim \|\nabla\psi\|_{H^k} + \|\, h\|_{H^{k+1}} \|\nabla\psi\|_{\widetilde{W^1}}, \tag{3.66}$$

$$\sup_{w\in[-1/4,0]} \|\nabla_x\Phi\|_{\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{\widetilde{W^\gamma}}, \quad \sup_{w\in[-1/4,0]} \|\partial_w\Phi\|_{\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{W^{\gamma,2\alpha}} + \|\, h\|_{\widetilde{W^{\gamma+1}}} \|\nabla\psi\|_{\widetilde{W^\gamma}}, \tag{3.67}$$

$$\sup_{w\in[-1/4,0]} \|\Lambda_{\geq 2}[\nabla_{x,w}\Phi]\|_{L^2} \lesssim \big[\|(\, h, \Lambda\psi)\|_{W^{2,2\alpha}} + \|(\, h, \Lambda\psi)\|_{\widehat{W^2}}^2\big]\big[\|\, h\|_{H^1} + \|\nabla\psi\|_{L^2}\big]. \tag{3.68}$$

*Proof.* Postponed to the end of this subsection for the purpose of improving presentation. □

After paralinearizing (3.65), we have

$$P\Phi + 2T_{\partial_w^2\Phi} T_{\nabla\, h} \cdot \nabla\, h - 2T_{\nabla\partial_w\Phi} \cdot \nabla\, h - T_{\partial_w\Phi}\Delta\, h \approx 0, \tag{3.69}$$

where

$$P := [\Delta + T_{a'}\partial_w^2 + T_{b'}.\nabla\partial_w + T_{c'}\partial_w]. \tag{3.70}$$

To see why the equivalence relation (3.69) holds, we mention that we can always put $\nabla h$ in $L^\infty$ and put $\partial_w\Phi$ and $\partial_w^2\Phi$ in $L^2$.

Define $W := \Phi - T_{\partial_w\Phi}\, h$, same as in [3], we claim that $PW \approx 0$ when $w \in [-1/4, 0]$. After using (1.7) and the composition Lemma 1.3.5, the following equivalence relations hold,

$$PW \approx 0 \Longleftrightarrow P[T_{\partial_w\Phi}\, h] + 2T_{\partial_w^2\Phi} T_{\nabla\, h} \cdot \nabla\, h - 2T_{\nabla\partial_w\Phi} \cdot \nabla\, h - T_{\partial_w\Phi}\Delta\, h \approx 0, \tag{3.71}$$

$$\Longleftrightarrow \big[T_{a'} T_{\partial_w^3\Phi}\, h + T_{b'} \cdot \nabla T_{\partial_w^2\Phi}\, h + T_{c'} T_{\partial_w^2\Phi}\, h + 2T_{\nabla\partial_w\Phi} \cdot \nabla\, h\big] + \big[2T_{\partial_w^2\Phi} T_{\nabla\, h} \cdot \nabla\, h - 2T_{\nabla\partial_w\Phi} \cdot \nabla\, h\big] \approx 0 \tag{3.72}$$

$$\Longleftrightarrow \big[T_{b'}\cdot T_{\partial_w^2\Phi}\nabla h+2T_{\nabla\partial_w\Phi}\cdot\nabla h\big]+\big[2T_{\partial_w^2\Phi}T_{\nabla h}\cdot\nabla h-2T_{\nabla\partial_w\Phi}\cdot\nabla h\big]\approx 0 \tag{3.73}$$

$$\Longleftrightarrow T_{[b'\partial_w^2\Phi+2\partial_w^2\Phi\nabla h]}\cdot\nabla h\approx 0 \tag{3.74}$$

$$\Longleftrightarrow 0\approx 0,\quad \text{as } b'=-2\nabla h. \tag{3.75}$$

Obviously, (3.75) holds, hence we can reverse the directions of all arrows back to derive $PW\approx 0$.

Although tedious, it is not difficult to see all "$\approx$" equivalence relations hold in above equations. As an example, we give a detail proof of (3.72) in a moment, all other equivalence relations can be derived very similarly. To prove (3.72), it's sufficient to estimate $T_{\Delta\partial_w\Phi}h$. From (3.67), we have,

$$\sup_{w\in[-1/4,0]}\|T_{\Delta\partial_w\Phi}\,h\|_{H^k}\lesssim\sup_{w\in[-1/4,0]}\|\,h\|_{H^k}\|\partial_w\Phi\|_{\widetilde{W^2}}\lesssim[\|\Lambda\psi\|_{W^{4,2\alpha}}+\|(\,h,\Lambda\psi)\|_{W^4}^2]\|\,h\|_{H^k}.$$

Hence, the equivalence relation (3.72) holds.

**Forward and backward elliptic evolution equations.**

To see good structures inside the equation $PW\approx 0$. We decompose it into a forward evolution equation and a backward evolution equation. More precisely, we have the following Lemma,

**Lemma 3.2.4.** *There exist two symbols $a=a(x,\xi)$ and $A(x,\xi)$ with*

$$a=a^{(1)}+a^{(0)},\quad A=A^{(1)}+A^{(0)},$$

*where*

$$a^{(1)}(x,\xi)=\frac{1}{1+|\nabla h|^2}(i\nabla h\cdot\xi-\sqrt{(1+|\nabla h|^2)|\xi|^2-(\nabla h\cdot\xi)^2}),$$

$$A^{(1)}(x,\xi)=\frac{1}{1+|\nabla h|^2}(i\nabla h\cdot\xi+\sqrt{(1+|\nabla h|^2)|\xi|^2-(\nabla h\cdot\xi)^2}),$$

$$a^{(0)}(x,\xi)=\frac{1}{A^{(1)}-a^{(1)}}\Big(i\partial_\xi a^{(1)}\cdot\partial_x A^{(1)}-\frac{\Delta\,h\;a^{(1)}}{1+|\nabla h|^2}\Big),$$

$$A^{(0)}(x,\xi)=\frac{1}{a^{(1)}-A^{(1)}}\Big(i\partial_\xi a^{(1)}\cdot\partial_x A^{(1)}-\frac{\Delta\,h\;A^{(1)}}{1+|\nabla h|^2}\Big),$$

*such that*

$$P=T_{a'}(\partial_w-T_a)(\partial_w-T_A)+R_0+R_1\partial_w,\quad a'(a+A)=ib'\cdot\xi+c', \tag{3.76}$$

$$a'[a^{(1)}A^{(1)}+\frac{1}{i}\partial_\xi a^{(1)}\cdot\partial_x A^{(1)}+a^{(1)}A^{(0)}+a^{(0)}A^{(1)}]=a'(a\sharp A)=-|\xi|^2, \tag{3.77}$$

*where*

$$R_0 = T_{a'}T_aT_A - \Delta, \quad R_1 = -T_{a'}T_{a+A} + T_{b'}\cdot\nabla + T_{c'}. \tag{3.78}$$

*Moreover, the following estimate holds for good error operators $R_0$ and $R_1$,*

$$\|R_0 f\|_{H^k} + \|R_1 f\|_{H^{k+1}} \lesssim \|\nabla h\|_{\widetilde{W^{3}}}\|f\|_{H^k}. \tag{3.79}$$

*Proof.* Most parts of above Lemma are cited directly from [3][Lemma 3.18]. Given the a-priori decomposition (3.76), from (3.70), we can calculate explicitly $R_0$ and $R_1$, which are given in (4.135).

Note that $a'$ doesn't depend on $\xi$, from (5.68), (3.77) and (4.135), we have the following identities,

$$R_1 = -T_{a'}T_{a+A} + T_{a'(a+A)} = -T_{a'}T_{a+A} + T_{(a'\sharp(a+A))},$$

$$R_0 = T_{a'}[T_aT_A - T_{a\sharp A}] + T_{a'}T_{a\sharp A} - T_{a'(a\sharp A)} = T_{a'}[T_aT_A - T_{a\sharp A}] + T_{(a'-1)}T_{a\sharp A} - T_{(a'-1)\sharp(a\sharp A)}.$$

From explicit formulations of $a'$, $a$ and $A$, we can see that $a', a'-1 \in \Gamma^0_2(\mathbb{R}^2), a, A, a+A \in \Gamma^1_2(\mathbb{R}^2)$, $a\sharp A \in \Gamma^2_2(\mathbb{R}^2)$. The following estimates on their symbolic bounds hold,

$$M^2_2(a\sharp A) + M^0_2(a') \lesssim 1, \quad M^1_2(a) + M^1_2(A) + M^1_2(a+A) \lesssim \|\nabla h\|_{\widetilde{W^3}}, \quad M^0_2(a'-1) \lesssim \|\nabla h\|^2_{\widetilde{W^3}}.$$

From estimate (1.11) in Lemma 1.3.5, we have

$$\|R_1 f\|_{H^{k+1}} \lesssim M^0_2(a')M^1_2(a+A)\|f\|_{H^k} \lesssim \|\nabla h\|_{\widetilde{W^3}}\|f\|_{H^k},$$

$$\|R_0 f\|_{H^k} \lesssim \big[M^0_2(a')M^1_2(a)M^1_2(A) + M^0_2(a'-1)M^2_2(a\sharp A)\big]\|f\|_{H^k} \lesssim \|\nabla h\|^2_{\widetilde{W^3}}\|f\|_{H^k},$$

hence finishing the proof of (3.79). □

**Paralinearization of Dirichlet-Neumann operator**

Recall the formulation of Dirichlet-Neumann operator in terms of $\Phi$ in (3.13). For readers' convenience, we rewrite it as follows,

$$G(h)\psi = (1+|\nabla h|^2)\partial_w\Phi - \nabla h\cdot\nabla\Phi\big|_{w=0}.$$

Now $w$ is not restricted at boundary but inside the range $[-1/4, 0]$. By using (1.7) and composition Lemma 1.3.5, we can do the paralinearization of above equation as follows,

$$(1+|\nabla h|^2)\partial_w\Phi - \nabla h\cdot\nabla\Phi \approx T_{1+|\nabla h|^2}\partial_w\Phi + 2T_{\partial_w\Phi}T_{\nabla h}\cdot\nabla h - T_{\nabla h}\cdot\nabla\Phi - T_{\nabla\Phi}\cdot\nabla h$$

$$\approx T_{1+|\nabla h|^2}\partial_w\Phi + T_{2\nabla h\partial_w\Phi-\nabla\Phi}\cdot\nabla h - T_{\nabla h}\cdot\nabla\Phi$$

$$= T_{1+|\nabla h|^2}\partial_w(W + T_{\partial_w\Phi}h) + T_{2\nabla h\partial_w\Phi-\nabla\Phi}\cdot\nabla h - T_{\nabla h}\cdot\nabla(W+T_{\partial_w\Phi}h)$$

$$\approx T_{1+|\nabla h|^2}\partial_w W + T_{2\nabla h\partial_w\Phi-\nabla\Phi}\cdot\nabla h - T_{\nabla h}\cdot\nabla W - T_{\nabla h\partial_w\Phi}\cdot\nabla h$$

$$= T_{1+|\nabla h|^2}\partial_w W - T_{\nabla h}\cdot\nabla W - T_{\underline{V}}\cdot\nabla h, \quad \text{where } \underline{V} := \nabla\Phi - \partial_w\Phi\nabla h, \quad \underline{V}\big|_{w=0} = V,$$

$$= T_{1+|\nabla h|^2}[\partial_w W - T_A W] + [T_{1+|\nabla h|^2}T_A - T_{\nabla h}\cdot\nabla]W - T_{\underline{V}}\cdot\nabla h,$$

$$= T_\lambda W - T_{\underline{V}}\cdot\nabla h + T_{1+|\nabla h|^2}[\partial_w W - T_A W] + R_2 W, \quad R_2 := [T_{1+|\nabla h|^2}T_A - T_{(1+|\nabla h|^2)A^{(1)}}], \tag{3.80}$$

where $\lambda$ is given by (3.61), one can see (3.80) holds from the following identity,

$$\lambda = (1+|\nabla h|^2)A^{(1)} - i\xi\cdot\nabla h.$$

Note that

$$R_2 = T_{(1+|\nabla h|^2)}T_{A^{(0)}} + \big[T_{(1+|\nabla h|^2)}T_{A^{(1)}} - T_{(1+|\nabla h|^2)A^{(1)}}\big] = T_{a'}T_{A^0} + \big[T_{(a'-1)}T_{A^{(1)}} - T_{(a'-1)\sharp A^{(1)}}\big].$$

Now, it is easy to see that $R_2$ is an operator of order 0 with an upper bound given by $\|\nabla h\|_{\widetilde{W^3}}$. Hence, the following estimate holds,

$$\|R_2 W\big|_{w=0}\|_{H^k} = \|R_2[P_{\geq 1/2}[W]\big|_{w=0}]\|_{H^k} \lesssim \|\nabla h\|_{\widetilde{W^3}}\|P_{\geq 1/2}[\psi - T_{B(\,h)\psi}\,h]\|_{H^k}$$

$$\lesssim \big[\|(\,h, \Lambda\psi)\|_{W^{4,2\alpha}} + \|(\,h,\Lambda\psi)\|_{W^4}^2\big]\big[\|\,h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big]. \tag{3.81}$$

Combining above discussions with the main result of following lemma, now it's easy to see Proposition 3.2.1 holds.

**Lemma 3.2.5.** *Let $A(x,\xi)$ be defined in Lemma 3.2.4, we have the following estimate for $k \geq 1$,*

$$\|\Lambda_{\geq 2}\big[(\partial_w W - T_A W)\big]\big|_{w=0}\|_{H^k} \lesssim_k \big[\|(\,h,\Lambda\psi)\|_{W^{4,2\alpha}} + \|(\,h,\Lambda\psi)\|_{W^4}^2\big]\big[\|\,h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big].$$

*Proof.* Recall that $PW \approx 0$ and the decomposition of operator $P$ in (5.68), we have

$$T_{a'}(\partial_w - T_a)(\partial_w - T_A)W \approx -R_0 W - R_1 \partial_w W,$$

which further gives us

$$(\partial_w - T_a)(\partial_w - T_A)W \approx \tilde{g},$$

where

$$\tilde{g} = T_{a'^{-1}}[-R_0 W - R_1 \partial_w W] + [I - T_{a'^{-1}} T_{a'}](\partial_w - T_a)(\partial_w - T_A)W,$$

$$= T_{a'^{-1}}[-R_0 W - R_1 \partial_w W] + [I - T_1 + T_{(a'-1)(a'^{-1}-1)} - T_{(a'-1)} T_{(a'^{-1}-1)}](\partial_w - T_a)(\partial_w - T_A)W.$$

From estimate (3.79) in Lemma 3.2.4, and the fact that $T_{(a'-1)(a'^{-1}-1)} - T_{(a'-1)} T_{(a'^{-1}-1)}$ is of order $-2$, we have

$$\sup_{w\in[-1/4,0]} \|\Lambda_{\geq 2}[\tilde{g}(w)]\|_{H^k} \lesssim \|\nabla\, h\|_{\widetilde{W^3}} \big[\|P_{\geq 1/2}[W]\|_{H^k} + \|\partial_w W\|_{H^{k-1}}\big]$$

$$\lesssim \big[\|(\,h, \Lambda\psi)\|_{W^{4,2\alpha}} + \|(\,h, \Lambda\psi)\|_{W^4}^2\big]\big[\|\,h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big].$$

Note that $[\partial_w^2 + \Delta]\Lambda_1[W(w)] = [\partial_w^2 + \Delta]\Lambda_1[\Phi(w)] = 0$ when $w \in [-1/4, 0]$, see (3.65). It's easy to see the following equivalence relation holds,

$$(\partial_w - T_a)\Lambda_{\geq 2}[(\partial_w - T_A)W] + \Lambda_{\geq 2}[(\partial_w - T_a)\Lambda_1[(\partial_w - T_A)W]] \approx \Lambda_{\geq 2}[\tilde{g}]. \tag{3.82}$$

Note that

$$\Lambda_1[\partial_w \Phi] = \Lambda_1[\partial_z \varphi(w/(1+\,h))] = \frac{e^{(w+1)|\nabla|} - e^{-(w+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}} |\nabla| \psi,$$

$$\Lambda_1[\Phi] = \Lambda_1[\varphi(w/(1+\,h))] = \frac{e^{-(w+1)|\nabla|} + e^{(w+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}} \psi,$$

it's easy to see that,

$$\Lambda_1[(\partial_w - T_A)W]] = \Lambda_1[\partial_w \Phi - T_{|\xi|}\Phi] \in H^\infty,$$

$$\|\Lambda_{\geq 2}[(\partial_w - T_a)\Lambda_1[(\partial_w - T_A)W]]\|_{H^k} \lesssim \|\nabla\, h\|_{\widetilde{W^3}} \|\nabla\psi\|_{H^{k-1}}.$$

Therefore, from (3.82), we have

$$(\partial_w - T_a)\Lambda_{\geq 2}[(\partial_w - T_A)W] \approx \Lambda_{\geq 2}[\tilde{g}].$$

We reformulate above equation as follows,

$$(\partial_w + T_{-a})\Lambda_{\geq 2}[(\partial_w - T_A)W] = \Lambda_{\geq 2}[\tilde{g}] + \hat{g},$$

where

$$\hat{g} = \text{error term from "}\approx\text{ equivalence" relation.}$$

Recall the precise formula of "$a$" in Lemma 3.2.4, we know $-a$ satisfies the assumption of Lemma 2.3.3. We can first choose a series of constants $\{\tau_i\}_{i=1}^k$ such that $\tau_{i+1} = 4\tau_i$ and $\tau_k \geq -1/5$ and then keep iterating the estimate (2.85). As a result, the following estimate holds,

$$\|\Lambda_{\geq 2}[(\partial_w - T_A)W\big|_{w=0}]\|_{H^k} \lesssim_k \sup_{w\in[-1/5,0]} \big[\|\Lambda_{\geq 2}[(\partial_w - T_A)W(w,\cdot)]\|_{L^2} + \|\tilde{g}(w)\|_{H^{k-1+\epsilon}}$$

$$+\|\hat{g}(w)\|_{H^{k-1+\epsilon}}\big] \lesssim_k \big[\|(h, \Lambda\psi)\|_{W^{4,2\alpha}} + \|(h, \Lambda\psi)\|_{W^4}^2\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big]. \tag{3.83}$$

□

*Proof of Lemma* 3.2.3. For fixed $w \in [-1/4, 0]$, it's easy to see $\Phi(w) = \varphi(w/(1+h(x)))$ and the following identity holds,

$$\nabla_{x,w}\Phi = \nabla_{x,z}\varphi(w/(1+h)) + [-w\nabla h\partial_z\varphi(w/(1+h))/(1+h)^2, -h\partial_z\varphi(w/(1+h))/(1+h)]. \tag{3.84}$$

Therefore, we can see that the leading term of $\nabla_{x,w}\Phi(w)$ is $\nabla_{x,z}\varphi(w/(1+h))$. Under the smallness estimate (3.1), to estimate $\nabla_{x,w}\Phi$, it's sufficient to estimate $\nabla_{x,z}\varphi(w/(1+h))$.

Recall the fixed point type formulation of $\nabla_{x,z}\varphi$ in (3.31), we study the linear term on the right hand side of (3.31) first. Denotes

$$p_{\pm}(w,x,\xi) := \pm\frac{e^{-(w+1)|\xi|}(e^{h(x)w|\xi|/(1+h(x))} - 1)}{e^{-|\xi|} + e^{|\xi|}} + \frac{e^{(w+1)|\xi|}(e^{-h(x)w|\xi|/(1+h(x))} - 1)}{e^{-|\xi|} + e^{|\xi|}},$$

$$= \sum_{n\geq 1}\frac{1}{n!}\Big[\pm\frac{e^{-(w+1)|\xi|}}{e^{|\xi|} + e^{-|\xi|}}(w|\xi|)^n\Big(\frac{h(x)}{1+h(x)}\Big)^n + \frac{e^{|\xi|}}{e^{|\xi|} + e^{-|\xi|}}e^{w|\xi|}(w|\xi|)^n\Big(\frac{h(x)}{1+h(x)}\Big)^n\Big]$$

$$= \sum_{n\geq 1}\frac{1}{n!}\big[\pm f_n^1(w,\xi)g_n(x) + f_n^2(w,\xi)g_n(x)\big],$$

where

$$f_n^1(w,\xi) := \frac{e^{-(w+1)|\xi|}}{e^{|\xi|} + e^{-|\xi|}}(w|\xi|)^n, \quad f_n^2(w,\xi) := \frac{e^{|\xi|}}{e^{|\xi|} + e^{-|\xi|}}e^{w|\xi|}(w|\xi|)^n, \quad g_n(x) = \Big(\frac{h(x)}{1+h(x)}\Big)^n.$$

It's easy to verify the following identities hold,

$$\Lambda_1[\nabla_x\varphi](x,w/(1+\ h(x))) = \Lambda_1[\nabla_x\varphi](x,w) + P_1\psi(x,w),$$

$$\Lambda_1[\partial_z\varphi](x,w/(1+\ h(x))) = \Lambda_1[\partial_z\varphi](x,w) + P_2\psi(x,w),$$

where

$$P_1\psi(x,w) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\widehat{\psi}(\xi)i\xi p_+(w,x,\xi)d\xi, P_2\psi(x,w) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\widehat{\psi}(\xi)|\xi|p_-(w,x,\xi)d\xi.$$

We will show that, under the smallness estimate (3.1), the size of $\Lambda_1[\nabla_{x,z}\varphi](x,w/(1+\ h(x)))$ is almost same as the size of $\Lambda_1[\nabla_{x,z}\varphi](x,w)$. For $k\in\mathbb{Z}$, we define

$$p_{\pm,k}(w,x,\xi) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\sigma}\mathcal{F}_x(p)(w,\sigma,\xi)\psi_k(\sigma)d\sigma = \frac{1}{4\pi^2}\sum_{n\geq 1}\frac{1}{n!}\big[\pm f_n^1(w,\xi) + f_n^2(w,\xi)\big]P_k[g_n](x), \tag{3.85}$$

$$P_{1,k}\psi(x,w) := \int_{\mathbb{R}^2} e^{ix\cdot\xi}\widehat{\psi}(\xi)i\xi p_{+,k}(w,x,\xi)d\xi, \quad P_{2,k}\psi(x,w) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}\widehat{\psi}(\xi)|\xi|p_{-,k}(w,x,\xi)d\xi.$$

Since $P_2\psi$ can be treated in the same way as $P_1\psi$, we only estimate $P_1\psi$ in details here. We have the following decomposition,

$$P_1\psi = \sum_{k_1,k_2\in\mathbb{Z}} P_{1,k_1}\psi_{k_2} = \mathrm{I}+\mathrm{II}, \quad \mathrm{I} = \sum_{k_2\leq k_1} P_{1,k_2}\psi_{k_1}, \quad \mathrm{II} = \sum_{k_1\leq k_2} P_{1,k_2}\psi_{k_1}.$$

From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, it's easy to see the following estimates hold,

$$\|\mathrm{I}\|_{H^k} \lesssim \Big(\sum_{k_1} 2^{2k_1+2kk_{1,+}}\|P_{k_1}\psi\|_{L^2}^2\Big(\sum_{n\geq 1}\frac{1}{n!}\|P_{\leq k_1}g_n\|_{L^\infty}\Big)^2\Big)^{1/2} \lesssim \|\nabla\psi\|_{H^k}\|\ h\|_{\widetilde{W^1}},$$

$$\|\mathrm{II}\|_{H^s} \lesssim \sum_{n\geq 1}\frac{1}{n!}\sum_{k_2\leq k_1} 2^{k_2+kk_{1,+}}\|P_{k_1}g_n\|_{L^2}\|P_{k_2}\psi\|_{L^\infty} \lesssim \|\ h\|_{H^{s+1}}\|\nabla\psi\|_{\widetilde{W^1}},$$

$$\|\mathrm{I}\|_{\widetilde{W^\gamma}} \lesssim \|\ h\|_{\widetilde{W^\gamma}}\|\nabla\psi\|_{\widetilde{W^\gamma}}, \quad \|\mathrm{II}\|_{\widetilde{W^\gamma}} \lesssim \|\ h\|_{\widetilde{W^\gamma}}\|\nabla\psi\|_{\widetilde{W^\gamma}}.$$

Therefore

$$\|\Lambda_1[\nabla_{x,z}\varphi](x,w/(1+\ h(x)))\|_{H^k} \lesssim \|\nabla\psi\|_{H^k} + \|\ h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W^1}},$$

$$\|\Lambda_1[\nabla_x\varphi](x,w/(1+\ h(x)))\|_{\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{\widetilde{W^\gamma}} + \|\ h\|_{\widetilde{W^\gamma}}\|\nabla\psi\|_{\widetilde{W^\gamma}} \lesssim \|\nabla\psi\|_{\widetilde{W^\gamma}},$$

$$\|\Lambda_1[\partial_z\varphi](x,w/(1+h(x)))\|_{\widetilde{W^\gamma}}\lesssim\|\Lambda^2\psi\|_{\widetilde{W^\gamma}}+\|h\|_{\widetilde{W^\gamma}}\|\nabla\psi\|_{\widetilde{W^\gamma}}.$$

which further gives us the following estimate from (3.84),

$$\|\Lambda_1[\nabla_x\Phi]\|_{\widetilde{W^\gamma}}\lesssim\|\nabla\psi\|_{\widetilde{W^\gamma}},\quad \|\Lambda_1[\partial_w\Phi]\|_{\widetilde{W^\gamma}}\lesssim\|\Lambda^2\psi\|_{\widetilde{W^\gamma}}+\|h\|_{\widetilde{W^\gamma}}\|\nabla\psi\|_{\widetilde{W^\gamma}},\tag{3.86}$$

$$\begin{aligned}\|\Lambda_1[\nabla_{x,w}\Phi]\|_{H^k}&\lesssim\|\Lambda_1[\nabla_{x,z}\varphi](x,w/(1+h))\|_{H^k}+\|\Lambda_1[\nabla_{x,z}\varphi](x,w/(1+h))\|_{\widetilde{W^0}}\|h\|_{H^{k+1}}\\&\lesssim\|\nabla\psi\|_{H^k}+\|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W^1}}.\end{aligned}\tag{3.87}$$

Following a similar procedure, we can handle the integral operator $K_i(z,s)$, $i\in\{1,2,3\}$, in (3.31). Similar to what we did in the proof of Lemma 3.1.3, when $|\xi|\le 1$, we use the size of symbol directly; when $|\xi|\ge 1$, we estimate the associated kernel. As a result, we have the following estimate,

$$\begin{aligned}&\|\text{Everything in the R.H.S. of (3.31) except linear part}(x,w/(1+h(x)))\|_{H^k}\\&\lesssim\sum_{i=1,2,3}\|g_i(z,\cdot)\|_{L^\infty_z H^k}+\|h\|_{H^{k+1}}\|g_i(z,\cdot)\|_{L^\infty_z\widetilde{W^0}}\lesssim\|\nabla\psi\|_{\widetilde{W^0}}\|h\|_{H^{k+1}}+\|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W^0}}\\&\lesssim\|\nabla\psi\|_{\widetilde{W^0}}\|h\|_{H^{k+1}}+\|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W^0}}.\end{aligned}\tag{3.88}$$

$$\begin{aligned}&\|\text{Everything in the R.H.S. of (3.31) except linear part}(x,w/(1+h(x)))\|_{\widetilde{W^\gamma}}\\&\lesssim\sum_{i=1,2,3}\|g_i(z,\cdot)\|_{L^\infty_z\widetilde{W^\gamma}}\lesssim\|h\|_{\widetilde{W^{\gamma+1}}}\|\nabla\psi\|_{\widetilde{W^\gamma}}.\end{aligned}\tag{3.89}$$

From (3.84), (3.86), (3.87), (3.88) and (3.89), now it's easy to see estimates (3.66) and (3.67) hold.

Now, we proceed to prove (3.68). From (3.84) and the same procedure as above, we have the following estimate,

$$\begin{aligned}\|\Lambda_{\ge 2}[\nabla_{x,w}\Phi]\|_{L^2}&\lesssim\|\Lambda_{\ge 2}[\nabla_{x,z}\varphi](x,w/(1+h(x)))\|_{L^2}+\|h\|_{H^1}\|\partial_z\varphi(x,w/(1+h(x)))\|_{\widetilde{W^0}}\\&\lesssim\|h\|_{H^1}\big[\|\Lambda\psi\|_{W^{2,2\alpha}}+\|h\|_{\widetilde{W^1}}\|\nabla\psi\|_{\widetilde{W^1}}\big]+\sum_{i=1,2,3}\|g_i\|_{L^\infty_z L^2}.\end{aligned}\tag{3.90}$$

Recall (3.15) and (3.16), note that $\nabla\varphi$ appears together with $\nabla h$ inside the quadratic terms of $g_i(z)$, $i\in\{1,2,3\}$. Regardless the sizes of frequencies of two inputs, we always put $\nabla\varphi$ in $L^2$ and put $\partial_z\varphi$ in $L^\infty$. As a result, the following estimate holds, i.e., desired estimate (3.68) holds,

$$(3.90)\lesssim\|h\|_{H^1}\|\Lambda\psi\|_{W^{2,2\alpha}}+\big[\|(h,\Lambda\psi)\|_{W^{2,2\alpha}}+\|(h,\Lambda\psi)\|_{W^2}\big](\|h\|_{H^1}+\|\nabla\psi\|_{L^2})$$

$$\lesssim \big[\|(h,\Lambda\psi)\|_{W^{2,2\alpha}}+\|(h,\Lambda\psi)\|_{W^2}\big](\|h\|_{H^1}+\|\nabla\psi\|_{L^2}).$$

□

### 3.2.1 Paralinearization of the full system

In this subsection, we mainly do paralinearization for the nonlinearity of equation satisfied by $\psi$ in (1.3).

**Proposition 3.2.6.** *We have the following paralinearization for the nonlinearity of equation satisfied by* $\psi$,

$$\frac{1}{2}|\nabla\psi|^2-\frac{1}{2}\frac{(\nabla h\cdot\nabla\psi+G(h)\psi)^2}{1+|\nabla h|^2}\approx T_V\cdot\nabla[\psi-T_{B(h)\psi}h]-T_BG(h)\psi. \tag{3.91}$$

*Proof.* Recall that $V=\nabla\psi-\nabla hB$. From (1.7) and the composition Lemma 1.3.5, we have

$$\frac{1}{2}|\nabla\psi|^2-\frac{1}{2}\frac{(\nabla h\cdot\nabla\psi+G(h)\psi)^2}{1+|\nabla h|^2}=\frac{1}{2}|\nabla\psi|^2-\frac{1}{2}(1+|\nabla h|^2)B^2$$

$$=\frac{1}{2}|V|^2+V\cdot\nabla hB+\frac{1}{2}|\nabla h|^2B^2-\frac{1}{2}(1+|\nabla h|^2)B^2=\frac{1}{2}|V|^2+V\cdot\nabla hB-\frac{1}{2}B^2$$

$$\approx T_V\cdot V+T_B(V\cdot\nabla h)+T_{V\cdot\nabla h}B-T_BB=T_V\cdot V+T_{V\cdot\nabla h}B-T_BG(h)\psi$$

$$=T_V\cdot\nabla\psi-T_V\cdot(\nabla hB)+T_{V\cdot\nabla h}B-T_BG(h)\psi$$

$$\approx T_V\cdot\nabla\psi-T_VT_B\cdot\nabla h-[T_V\cdot T_{\nabla h}-T_{V\cdot\nabla h}]B-T_BG(h)\psi$$

$$\approx T_V\cdot[\nabla\psi-T_B\cdot\nabla h]-T_BG(h)\psi\approx T_V\cdot\nabla[\psi-T_{B(h)\psi}h]-T_BG(h)\psi.$$

Hence, finishing the proof of this proposition. □

### 3.2.2 Symmetrization of the full system

Define $\omega=\psi-T_{B(h)\psi}h$, then after combining (3.60) in Proposition 3.2.1 and (3.91) in Proposition 3.2.6, we have the following system of equations,

$$\begin{cases}\partial_t h=\Lambda^2\omega+T_{\lambda-|\xi|}\omega-T_V\cdot\nabla h+\widetilde{F}(h)\psi,\\[2ex]\partial_t\omega=-T_a h-T_V\cdot\nabla\omega+f',\end{cases} \tag{3.92}$$

where

$$a := 1 + \partial_t B + V \cdot \nabla B,$$

and $f'$ is a good error term in the sense of estimate (3.64).

However, one cannot work on the system (3.92) directly as there is a potential problem of losing one derivative. When one tries to do energy estimate by using the system (3.92), one might find that the following term loses one derivative and can't be simply treated.

$$\int_{\mathbb{R}^2} \partial_x^N h \partial_x^N [T_{\lambda - |\xi|}\omega] + \partial_x^N \Lambda\omega \partial_x^N \Lambda[-T_a h], \quad \text{here } N \text{ is the prescribed top derivative level.}$$

To get around this difficulty, we will symmetrize the system (3.92). Define

$$U_1 = h + T_\alpha h, \quad U_2 = \Lambda[\omega + T_\beta \omega] \approx [T_{\sqrt{\lambda} - |\xi|^{1/2}}\omega] + \Lambda\omega, \quad U := U_1 + iU_2, \tag{3.93}$$

where

$$\alpha = \sqrt{a} - 1, \qquad \omega = \psi - T_{B(h)\psi} h, \quad \beta := \sqrt{\lambda/|\xi|} - 1 = \sqrt[4]{(1 + |\nabla h|^2) - (\nabla h \cdot \xi/|\xi|)^2} - 1,$$

$$\Lambda_1[a] = \Lambda_1[\partial_t \Lambda^2 \psi] = -\Lambda^2 h, \quad \Lambda_1[\partial_t a] = -\Lambda^4 \psi, \quad \Lambda_1[\partial_t \alpha] = -\frac{1}{2}\Lambda^4 \psi, \quad \Lambda_1[\alpha] = -\frac{1}{2}\Lambda^2 h.$$

We take the estimates of the Taylor coefficient in Lemma 3.2.7 as granted first. Then it is easy to see the following estimate and equivalence relations hold,

$$\|(U_1, U_2) - (h, \Lambda\psi)\|_{H^k} \lesssim \|(h, \Lambda\psi)\|_{\widetilde{W^3}} \|(h, \Lambda\psi)\|_{H^k}. \tag{3.94}$$

$$\partial_t U_1 \approx \Lambda^2 \omega + [T_{\lambda - |\xi|}\omega] - T_V \cdot \nabla h + T_{\partial_t \alpha} h + T_\alpha [\partial_t h]$$

$$\approx [\Lambda^2 - T_{|\xi|}]\omega + T_\lambda \omega - T_V \cdot \nabla U_1 + T_V \cdot T_\alpha \nabla h + T_\alpha \big[T_\lambda \omega - T_V \cdot \nabla h\big]$$

$$\approx [\Lambda^2 - T_{|\xi|}]\Lambda^{-1} U_2 + T_{\lambda\sqrt{a}}\omega - T_V \cdot \nabla U_1 \approx \Lambda U_2 + [T_{\sqrt{\lambda}\alpha} T_{\sqrt{\lambda}}\omega] - T_V \cdot \nabla U_1$$

$$\approx \Lambda U_2 + [T_{\sqrt{\lambda}\alpha} U_2] - T_V \cdot \nabla U_1. \tag{3.95}$$

$$\partial_t U_2 \approx \Lambda(1 + T_\beta)[-T_a h - T_V \cdot \nabla\omega] + \Lambda T_{\partial_t \beta}\omega \approx -\Lambda U_1 - [T_{\sqrt{\lambda}} T_\alpha U_1] - T_V \cdot \nabla T_{\sqrt{\lambda}}\omega + T_{\partial_t \beta}\Lambda\omega$$

$$\approx -\Lambda U_1 - [T_{\sqrt{\lambda}\alpha} U_1] - T_V \cdot \nabla U_2 + T_{\partial_t \beta} U_2. \tag{3.96}$$

### 3.2.3 Estimates of the Taylor coefficient

The main goal of this subsection is to prove Lemma 3.2.7. Recall (3.62), (1.3), and

$$a = 1 + V \cdot \nabla B + \partial_t B, \quad B = \partial_z \varphi / (1 + h)\big|_{z=0}.$$

To estimate $a$ and $\partial_t a$, it's sufficient to estimate $\partial_z \partial_t \varphi$ and $\partial_z \partial_t^2 \varphi$. From the fixed point type formulation of $\nabla_{x,z}\varphi$ in (3.31), we can derive the following equality,

$$\begin{aligned}\nabla_{x,z}\partial_t\varphi = &\left[\Big[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\Big]\nabla\partial_t\psi, \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\partial_t\psi\right] + [\mathbf{0}, \partial_t g_1(z)] + \\ &+\int_{-1}^{0}[K_1(z,s)-K_2(z,s)-K_3(z,s)](\partial_t g_2(s)+\nabla\cdot\partial_t g_3(s))ds \\ &+\int_{-1}^{0}K_3(z,s)|\nabla|\mathrm{sign}(z-s)\partial_t g_1(s) - |\nabla|[K_1(z,s)+K_2(z,s)]\partial_t g_1(s)\,ds. \end{aligned} \tag{3.97}$$

Following the same fixed point type argument that we used in the proof of Lemma 3.1.4, we can derive the following estimates,

$$\begin{aligned}\|\nabla_{x,z}\partial_t\varphi\|_{L_z^\infty \widetilde{W^\gamma}} &\lesssim \|\nabla h\|_{\widetilde{W^\gamma}} + \|\partial_t h\|_{\widetilde{W^{\gamma+1}}}\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \\ &\lesssim \|\nabla h\|_{\widetilde{W^\gamma}} + \big[\|\nabla\psi\|_{\widetilde{W^{\gamma+1}}} + \|\nabla h\|_{\widetilde{W^\gamma}}\big]^2. \end{aligned} \tag{3.98}$$

$$\begin{aligned}\|\nabla_{x,z}\partial_t\varphi\|_{L_z^\infty H^k} &\lesssim \|\nabla\partial_t\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla_{x,z}\partial_t\varphi\|_{\widetilde{W^1}} + \|\partial_t h\|_{H^{k+1}}\|\nabla_{x,z}\varphi\|_{\widetilde{W^1}} \\ &+\|\partial_t h\|_{\widetilde{W^1}}\|\nabla_{x,z}\varphi\|_{L_z^\infty H^k} \lesssim \|\nabla h\|_{H^k} + \|(h,\nabla\psi)\|_{H^k}\|(h,\nabla\psi)\|_{\widetilde{W^2}}. \end{aligned}$$

We can take another time derivative at the both hands side of equation (3.97) to derive a fixed point type formulation for $\nabla_{x,z}\partial_t^2\varphi$. Following similar argument we can derive the following estimate,

$$\|\nabla_{x,z}\partial_t^2\varphi\|_{L_z^\infty \widetilde{W^\gamma}} \lesssim \|\partial_t^2\psi\|_{\widetilde{W^{\gamma+1}}} + \|\partial_t h\|_{\widetilde{W^{\gamma+1}}}\|\partial_t\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}} + \|\partial_t^2 h\|_{\widetilde{W^{\gamma+1}}}\|\nabla_{x,z}\varphi\|_{L_z^\infty \widetilde{W^\gamma}}.$$

Recall the system of equations satisfied by $h$ and $\psi$ in (1.3), we have

$$\partial_t^2 h = \partial_t G(h)\psi = \partial_t[(1+|\nabla h|^2)B - \nabla h\cdot\nabla\psi],$$

$$\partial_t^2\psi = -\partial_t h + \nabla\psi\cdot\nabla\partial_t\psi + (1+|\nabla h|^2)B\partial_t B + \nabla h\cdot\nabla\partial_t h B^2,$$

hence

$$\|\partial_t^2\psi\|_{\widetilde{W^{\gamma+1}}} \lesssim \|\Lambda^2\psi\|_{\widetilde{W^{\gamma+1}}} + \big[\|\nabla h\|_{\widetilde{W^{\gamma+1}}} + \|\nabla\psi\|_{\widetilde{W^{\gamma+1}}}\big]^2.$$

Combining above estimate, (3.98) and (3.45) in Lemma 3.1.4, we have

$$\|\nabla\partial_t^2\varphi\|_{\widetilde{W^{\gamma}}} \lesssim \|\Lambda^2\psi\|_{\widetilde{W^{\gamma+1}}} + \big[\|\nabla h\|_{\widetilde{W^{\gamma+1}}} + \|\nabla\psi\|_{\widetilde{W^{\gamma+1}}}\big]^2.$$

Follow the same argument, we derive the $L^2$ type estimate of $\partial_t^2\varphi$ as follows,

$$\|\nabla_{x,z}\partial_t^2\varphi\|_{L_z^\infty H^k} \lesssim \|\partial_t^2\psi\|_{H^{k+1}} + \|\partial_t h\|_{H^{k+1}}\|\nabla_{x,z}\partial_t\varphi\|_{L_z^\infty\widetilde{W^1}} + \|\partial_t h\|_{\widetilde{W^1}}\|\nabla_{x,z}\partial_t\varphi\|_{L_z^\infty H^k} + \|\partial_t^2 h\|_{H^k}\times$$

$$\|\nabla_{x,z}\varphi\|_{L_z^\infty\widetilde{W^1}} + \|\partial_t^2 h\|_{\widetilde{W^1}}\|\nabla_{x,z}\varphi\|_{L_z^\infty H^k} + \|h\|_{H^{k+1}}\|\partial_t^2\nabla_{x,z}\varphi\|_{L_z^\infty\widetilde{W^1}} + \|h\|_{\widetilde{W^1}}\|\nabla_{x,z}\partial_t^2\varphi\|_{L_z^\infty H^k},$$

therefore,

$$\|\nabla_{x,z}\partial_t^2\varphi\|_{L_z^\infty H^k} \lesssim \|\Lambda^2\psi\|_{H^{k+1}} + \|\nabla h\|_{\widetilde{W^2}}\|\Lambda^2\psi\|_{H^{k+1}} + \|\nabla\psi\|_{\widetilde{W^2}}\|h\|_{H^{k+1}}$$

$$\lesssim \|\nabla\psi\|_{H^{k+1}} + \|(h,\nabla\psi)\|_{\widetilde{W^3}}\|(h,\nabla\psi)\|_{H^{k+1}}.$$

Now, we can summarize above discussion as the following lemma on the Taylor coefficient.

**Lemma 3.2.7.** *Under the smallness estimate* (3.1)*, we have the following estimates for* $\gamma \le 3, \gamma_1 \le 2$*,*

$$\|a-1\|_{H^k} \lesssim \|\nabla h\|_{H^k} + \|(h,\nabla\psi)\|_{H^k}\|(h,\nabla\psi)\|_{\widetilde{W^2}}, \|a-1\|_{\widetilde{W^\gamma}} \lesssim \|\nabla h\|_{\widetilde{W^\gamma}} + \big[\|\nabla\psi\|_{\widetilde{W^{\gamma+1}}} + \|\nabla h\|_{\widetilde{W^\gamma}}\big]^2,$$

$$\|\partial_t a\|_{H^k} \lesssim \|\nabla\psi\|_{H^{k+1}} + \|(h,\nabla\psi)\|_{\widetilde{W^3}}\|(h,\nabla\psi)\|_{H^{k+1}}, \|\partial_t a\|_{\widetilde{W^{\gamma_1}}} \lesssim \|\Lambda^2\psi\|_{\widetilde{W^{\gamma_1+1}}} + \|(\nabla h,\nabla\psi)\|^2_{\widetilde{W^{\gamma_1+1}}}.$$

# Chapter 4

# The gravity water waves in the infinite depth setting

Recall (2.106). We have the system of equations satisfied by $U^1$ and $U^2$ as follows,

$$\begin{cases} \partial_t U^1 = |\nabla|^{1/2} U^2 + T_\alpha |\nabla|^{1/2} U^2 - T_V \partial_x U^1 + \mathcal{R}_1 \\ \\ \partial_t U^2 = -|\nabla|^{1/2} U^1 - T_\alpha |\nabla|^{1/2} U^1 - T_V \partial_x U^2 + \mathcal{R}_2. \end{cases} \tag{4.1}$$

Till the end of section, we assume that the expansion of the projection operator $\Lambda_{\geq 3}[\cdot]$ is taken with respect to $U^1$ and $U^2$. As we need to cancel out the quadratic terms inside the system (4.1), it is important to know what quadratic terms are first. From (4.1), we have

$$\begin{cases} \partial_t U^1 - |\nabla|^{1/2} U^2 = Q_1(U^1, U^2) + C_1 + \Lambda_{\geq 3}[\mathcal{R}_1] \\ \partial_t U^2 + |\nabla|^{1/2} U^1 = Q_2(U^1, U^1) + Q_3(U^2, U^2) + C_2 + \Lambda_{\geq 3}[\mathcal{R}_2], \end{cases} \tag{4.2}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are given in (2.107) and (2.108), $Q_1(\cdot,\cdot)$, $Q_2(\cdot,\cdot)$ and $Q_3(\cdot,\cdot)$ denote quadratic terms and $C_1$ and $C_2$ denote the cubic and higher order terms that at most lose 1 derivative. More precisely,

$$Q_1(U^1, U^2) = -T_{\partial_x |\nabla|^{-\frac{1}{2}} U^2} \partial_x U^1 - \frac{1}{2} T_{\partial_x^2 |\nabla|^{-\frac{1}{2}} U^2} U^1 - \frac{1}{2} T_{|\nabla| U^1} |\nabla|^{\frac{1}{2}} U^2 +$$

$$-|\nabla|(U^1 |\nabla|^{\frac{1}{2}} U^2) + |\nabla|(T_{|\nabla|^{\frac{1}{2}} U^2} U^1) - \partial_x (U^1 \partial_x |\nabla|^{-\frac{1}{2}} U^2) + \partial_x (T_{\partial_x |\nabla|^{-\frac{1}{2}} U^2} U^1),$$

$$Q_2(U^1, U^1) = \frac{1}{2} |\nabla|^{\frac{1}{2}} T_{|\nabla| U^1} U^1, \quad Q_3(U^2, U^2) = -|\nabla|^{\frac{1}{2}} T_{\partial_x |\nabla|^{-\frac{1}{2}} U^2} \partial_x |\nabla|^{-1/2} U^2,$$

$$C_1 = T_{\Lambda_{\geq 2}[\alpha]}|\nabla|^{1/2}U^2 - T_{\Lambda_{\geq 2}[V]}\partial_x U^1, C_2 = -T_{\Lambda_{\geq 2}[\alpha]}|\nabla|^{1/2}U^1 - T_{\Lambda_{\geq 2}[V]}\partial_x U^2. \tag{4.3}$$

We remark that the good error terms $\mathcal{R}_1$ and $\mathcal{R}_2$ contribute the following quadratic terms,

$$\Lambda_2[\mathcal{R}_1](U^1,U^2) = Q_1(U^1,U^2) + \frac{1}{2}T_{|\nabla|U^1}|\nabla|^{1/2}U^2 + T_{\partial_x|\nabla|^{-1/2}U^2}\partial_x U^1,$$

$$\Lambda_2[\mathcal{R}_2](U^1,U^1) = Q_2(U^1,U^1) - \frac{1}{2}T_{|\nabla|U^1}|\nabla|^{1/2}U^1,$$

$$\Lambda_2[\mathcal{R}_2](U^2,U^2) = Q_3(U^2,U^2) + T_{\partial_x|\nabla|^{-1/2}U^2}\partial_x U^2.$$

The corresponding symbols of quadratic terms $Q_i(\cdot,\cdot)$, $i \in \{1,2,3\}$, are given as follows,

$$q_1(\xi-\eta,\eta) = \sum_{i=1,2,3} q_1^i(\xi-\eta,\eta), q_1^1(\xi-\eta,\eta) = \big((\xi-\eta)\eta|\eta|^{-1/2} + |\eta|^{3/2}/2\big)\theta(\eta,\xi-\eta) \tag{4.4}$$

$$q_1^2(\xi-\eta,\eta) = -|\xi-\eta||\eta|^{1/2}\theta(\xi-\eta,\eta)/2, q_1^3(\xi-\eta,\eta) = (-|\xi||\eta|^{1/2} + \xi\eta|\eta|^{-1/2})\tilde{\theta}(\eta,\xi-\eta), \tag{4.5}$$

$$q_2(\xi-\eta,\eta) = |\xi|^{1/2}|\eta|\theta(\eta,\xi-\eta)/2, \tag{4.6}$$

$$q_3(\xi-\eta,\eta) = |\xi|^{1/2}(\xi-\eta)\eta|\xi-\eta|^{-1/2}|\eta|^{-1/2}\theta(\eta,\ \xi-\eta), \tag{4.7}$$

where $q_i(\cdot,\cdot)$ $i \in \{1,2,3\}$, are the symbols of operators $Q_i(\cdot,\cdot)$.

**Lemma 4.0.8.** *If $|k_1-k_2| \geq 5$, the following estimates hold,*

$$\sum_{i=1,2,3} \|q_i(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{\min\{k_1,k_2\}/2+\max\{k_1,k_2\}}. \tag{4.8}$$

$$\|q_1^1(\xi-\eta,\eta) + q_1^1(-\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} + \|q_2(\eta,\xi-\eta) + q_1^2(\eta-\xi,\xi)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}$$
$$+\|q_3(\xi-\eta,\eta) + q_3(-\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{3\min\{k_1,k_2\}/2}. \tag{4.9}$$

*If $|k_1-k_2| \leq 5$, the following estimate holds,*

$$\sum_{i=1,2,3} \|q_i(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{k+k_1/2}. \tag{4.10}$$

*Proof.* From Lemma 1.3.8 and explicit formulas in (4.4), (4.5), (4.6), and (4.7), it is easy to see that our stated estimates hold. □

### 4.0.4 The bootstrap assumption and proof of the main theorem

As the sizes of $(U^1, U^2)$ and $(h, |\nabla|^{1/2}\psi)$ are comparable, from the smallness assumption of initial data $(\eta_0, \psi_0)$ in (1.21), it's not difficult to see the following estimate holds,

$$\|(U^1, U^2)(0)\|_{H^{N_0,p}} + \|x\partial_x(U^1, U^2)(0)\|_{H^{N_1,p}} \lesssim \epsilon_0. \tag{4.11}$$

We will use the bootstrap argument to show the global existence of solutions of the system (4.2), which further gives us the global existence of $(h, \psi)$. We expect that the energy grows appropriately and the decay rate of $L^\infty$-type norm is sharp. This expectation leads to the following bootstrap assumption,

$$\begin{aligned}\sup_{t\in[0,T]} (1+t)^{-p_0}[\|(U^1, U^2)(t)\|_{H^{N_0,p}} + \|S(U^1, U^2)(t)\|_{H^{N_1,p}}] \\ +(1+t)^{1/2}\|(U^1, U^2)\|_{W^{N_2}} \lesssim \epsilon_1 := \epsilon_0^{5/6} \ll 1.\end{aligned} \tag{4.12}$$

In section 4.1, we will prove the following proposition, which is sufficient to show that the energy appropriately grows.

**Proposition 4.0.9.** *Under the bootstrap assumption* (4.12)*, we can define modified energies* $E_{modi}(t) \approx \|(U^1, U^2)(t)\|^2_{H^{N_0,p}}$ *and* $E^S_{modi}(t) \approx \|(SU^1, SU^2)(t)\|^2_{H^{N_1,p}}$ *and have the following energy estimate,*

$$\sup_{t\in[0,T]} (1+t)^{1-2p_0}\big[|\frac{d}{dt}E_{modi}(t)| + |\frac{d}{dt}E^S_{modi}(t)|\big] \lesssim \epsilon_0. \tag{4.13}$$

*Therefore,*

$$\sup_{t\in[0,T]} (1+t)^{-p_0}\big[\|(U^1, U^2)\|_{H^{N_0,p}} + \|S(U^1, U^2)\|_{H^{N_1,p}}\big] \lesssim \epsilon_0. \tag{4.14}$$

In section 4.2, we will prove the following decay estimate of $L^\infty$-type norm,

**Proposition 4.0.10.** *Under the bootstrap assumption* (4.12) *and the improved energy estimate* (4.14)*, we can derive the following improved decay estimate,*

$$\sup_{t\in[0,T]} (1+t)^{1/2}\|(U^1, U^2)\|_{W^{N_2}} \lesssim \epsilon_0. \tag{4.15}$$

With above two propositions, it's easy to see our main theorem holds.

## 4.1 Energy estimate

### 4.1.1 Normal form transformation

We first find out the normal form transformations that can cancel out the quadratic terms. Let

$$V_1 := U^1 + A_1(U^1, U^1) + A_2(U^2, U^2), \quad V_2 := U^2 + B(U^1, U^2), \tag{4.16}$$

where $A_1(\cdot,\cdot)$ and $A_2(\cdot,\cdot)$ are two symmetric bilinear operators. It's easy to derive the quadratic terms inside the equations satisfied by $V_1$ and $V_2$ as follows,

$$Q_1(U^1, U^2) + 2A_1(|\nabla|^{1/2}U^2, U^1) - 2A_2(|\nabla|^{1/2}U^1, U^2) - |\nabla|^{1/2}B(U^1, U^2),$$

$$Q_2(U^1, U^1) + Q_3(U^2, U^2) + B(|\nabla|^{1/2}U^2, U^2) - B(U^1, |\nabla|^{1/2}U^1)$$

$$+|\nabla|^{1/2}(A_1(U^1, U^1) + A_2(U^2, U^2)).$$

To make above quadratic terms vanish, it would be sufficient if the symbols of bilinear operators $A_1(\cdot,\cdot)$, $A_2(\cdot,\cdot)$ and $B(\cdot,\cdot)$ satisfy the following system of equations:

$$\begin{cases} q_1(\xi-\eta,\eta) + 2|\eta|^{1/2}a_1(\xi-\eta,\eta) - 2|\xi-\eta|^{1/2}a_2(\xi-\eta,\eta) - |\xi|^{1/2}b(\xi-\eta,\eta) = 0, \\ q_2(\xi-\eta,\eta) - b(\xi-\eta,\eta)|\eta|^{1/2} + q_2(\eta,\xi-\eta) - b(\eta,\xi-\eta)|\xi-\eta|^{1/2} + 2|\xi|^{1/2}a_1(\xi-\eta,\eta) = 0, \\ q_3(\xi-\eta,\eta) + q_3(\eta,\xi-\eta) + b(\xi-\eta,\eta)|\xi-\eta|^{1/2} + b(\eta,\xi-\eta)|\eta|^{1/2} + 2|\xi|^{1/2}a_2(\xi-\eta,\eta) = 0. \end{cases} \tag{4.17}$$

The solution of above system is given as follows,

$$a_1(\xi-\eta,\eta) = \frac{b(\xi-\eta,\eta)|\eta|^{1/2} - q_2(\xi-\eta,\eta) + b(\eta,\xi-\eta)|\xi-\eta|^{1/2} - q_2(\eta,\xi-\eta)}{2|\xi|^{1/2}}, \tag{4.18}$$

$$a_2(\xi-\eta,\eta) = -\frac{b(\xi-\eta,\eta)|\xi-\eta|^{1/2} + q_3(\xi-\eta,\eta) + b(\eta,\xi-\eta)|\eta|^{1/2} + q_3(\eta,\xi-\eta)}{2|\xi|^{1/2}}, \tag{4.19}$$

$$b(\xi-\eta,\eta) = \frac{(|\xi-\eta| + |\eta| - |\xi|)A(\xi-\eta,\eta) - 2A(\eta,\xi-\eta)|\xi-\eta|^{1/2}|\eta|^{1/2}}{-(|\xi-\eta| + |\eta| - |\xi|)^2 + 4|\xi-\eta||\eta|}, \tag{4.20}$$

where

$$A(\xi-\eta,\eta) := |\xi|^{1/2}q_1(\xi-\eta,\eta) - (q_2(\xi-\eta,\eta) + q_2(\eta,\xi-\eta))|\eta|^{1/2}$$

$$+(q_3(\xi-\eta,\eta) + q_3(\eta,\xi-\eta))|\xi-\eta|^{1/2}.$$

**Lemma 4.1.1.** *The following estimate holds,*

$$\|a_1(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|a_2(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|b(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{\max\{k_1,k_2\}}. \tag{4.21}$$

*Proof.* If $|k_1-k_2|\geq 10$, then estimate (4.8) in Lemma 5.0.4 holds. From (4.18), (4.19), (4.20) and Lemma 1.3.8, the following estimate holds,

$$\|a_1(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|a_2(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|b(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{\max\{k_1,k_2\}}. \tag{4.22}$$

If $|k_1-k_2|\leq 10$, then estimate (4.10) in Lemma 5.0.4 holds. Note that $q_2(\xi-\eta,\eta)=q_3(\xi-\eta,\eta)=0$ for the case we are considering. Therefore, the following estimate holds,

$$\|A(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|A(\eta,\xi-\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{3k/2+k_1/2}, \tag{4.23}$$

which further gives us the following estimate,

$$\|a_1(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|a_2(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|b(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{k_1}. \tag{4.24}$$

□

### 4.1.2 Energy estimate of $U^1$ and $U^2$

Note that the following cubic terms are problematic when doing energy estimate, as they decay slowly over time,

$$\mathcal{I}_1^{N_0}=\mathfrak{Re}\Big[\int_{\mathbb{R}}\overline{\partial_x^{N_0}U^1}\partial_x^{N_0}\big[Q_1(U^1,U^2)\big]+\overline{\partial_x^{N_0}U^2}\partial_x^{N_0}\big[Q_2(U^1,U^1)+Q_3(U^2,U^2)\big]\Big]. \tag{4.25}$$

Due to the quasilinear nature, to avoid losing derivative again after adding the cubic correction terms, we utilize symmetries inside the system (4.2) first. After switching the roles of same type inputs inside $\mathcal{I}_1^{N_0}$ on the Fourier side, we can rewrite it as follows,

$$\begin{aligned}\mathcal{I}_1^{N_0}=\mathfrak{Re}\Big[\int\frac{1}{2}\overline{\widehat{U^1}(\xi)}\tilde{q}^1_{N_0}(\xi-\eta,\eta)\widehat{U^1}(\xi-\eta)\widehat{U^2}(\eta)+\overline{\widehat{U^2}(\xi)}\tilde{q}^2_{N_0}(\xi-\eta,\eta)\widehat{U^1}(\xi-\eta)\widehat{U^1}(\eta)\\+\int\frac{1}{2}\overline{\widehat{U^2}(\xi)}\tilde{q}^3_{N_0}(\xi-\eta,\eta)\widehat{U^2}(\xi-\eta)\widehat{U^2}(\eta)\Big],\end{aligned} \tag{4.26}$$

where

$$\tilde{q}^1_{N_0}(\xi-\eta,\eta)=q^1_1(\xi-\eta,\eta)|\xi|^{2N_0}+q^1_1(-\xi,\eta)|\xi-\eta|^{2N_0}+2q^3_1(\xi-\eta,\eta)|\xi|^{2N_0},$$

$$\tilde{q}^2_{N_0}(\xi-\eta,\eta):=q_2(\eta,\xi-\eta)|\xi|^{2N_0}+q^2_1(\eta-\xi,\xi)|\eta|^{2N_0},$$

$$\tilde{q}^3_{N_0}(\xi-\eta,\eta):=q_3(\xi-\eta,\eta)|\xi|^{2N_0}+q_3(-\xi,\eta)|\xi-\eta|^{2N_0}. \tag{4.27}$$

Recall (4.5), (4.6) and (4.7). From (4.9) in Lemma 4.0.8, we can see that cancellations happen in $\tilde{q}^i_{N_0}(\cdot,\cdot)$, $i\in\{1,2,3\}$,. From Lemma 1.3.8, the following estimate holds,

$$\sum_{i=1,2,3}\|\tilde{q}^i_{N_0}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{3\min\{k_1,k_2\}/2+2N_0k}. \tag{4.28}$$

We define bilinear operators $\widetilde{Q}_1(U^1,U^2)$, $\widetilde{Q}_2(U^1,U^1),\widetilde{Q}_3(U^2,U^2)$ by the following symbols,

$$\widetilde{q}_i(\xi-\eta,\eta)=\frac{\tilde{q}^i_{N_0}(\xi-\eta,\eta)}{|\xi|^{2N_0}},\quad i\in\{1,2,3\}. \tag{4.29}$$

From (4.28) and Lemma 1.3.8, the following estimate holds:

$$\sum_{i=1,2,3}\|\widetilde{q}_i(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{3\min\{k_1,k_2\}/2}. \tag{4.30}$$

Solving a similar system of equations as in (4.17) with $Q_i(\cdot,\cdot)$ replaced by $\widetilde{Q}_i(\cdot,\cdot)$, we can find bilinear operators $\widetilde{A}_1(U^1,U^1)$, $\widetilde{A}_2(U^2,U^2)$ and $\widetilde{B}(U^1,U^2)$ such that

$$2\widetilde{A}_1(|\nabla|^{\frac12}U^2,U^1)-2\widetilde{A}_2(|\nabla|^{\frac12}U^1,U^2)+\widetilde{Q}_1(U^1,U^2)-|\nabla|^{\frac12}\widetilde{B}(U^1,U^2)=0, \tag{4.31}$$

$$\widetilde{Q}_2(U^1,U^1)+|\nabla|^{\frac12}\widetilde{A}_1(U^1,U^1)-\widetilde{B}(U^1,|\nabla|^{\frac12}U^1)=0, \tag{4.32}$$

$$\widetilde{Q}_3(U^2,U^2)+|\nabla|^{\frac12}\widetilde{A}_2(U^2,U^2)+\widetilde{B}(|\nabla|^{\frac12}U^2,U^2)=0. \tag{4.33}$$

Very similar to the proof of Lemma 4.1.1, from (4.30), we have the following estimate,

$$\|\widetilde{a}_1(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|\widetilde{a}_2(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|\widetilde{b}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{\min\{k_1,k_2\}}. \tag{4.34}$$

So our strategy is to use normal form transformations $\widetilde{A}_1(U^1,U^1)$, $\widetilde{A}_2(U^2,U^2)$ and $\widetilde{B}(U^1,U^2)$ to cancel out the quadratic terms $\widetilde{Q}_i(\cdot,\cdot)$, $i\in\{1,2,3\}$. Although they do not exactly cancel out the quadratic terms $Q_i(\cdot,\cdot)$, $i\in\{1,2,3\}$, they will effectively cancel out the bulk cubic terms

listed in (4.25) when doing energy estimate. An advantage of doing this is that those normal form transformations do not lose derivative, see (4.34).

Define $W^1 = U^1 + \widetilde{A}_1(U^1, U^1) + \widetilde{A}_2(U^2, U^2)$, $W^2 = U^2 + \widetilde{B}(U^1, U^2)$. From the system of equations (4.2) satisfied by $U^1$ and $U^2$, we can first derive the system of equations satisfied by $W^1$ and $W^2$. Then, to see symmetries inside the nonlinearities, we substitute $(U^1, U^2)$ inside the quadratic terms and $C_1$ and $C_2$ by $(W^1, W^2)$. Recall (4.1), (4.2) and (4.3). As a result, we can write the system of equations satisfied by $W^1$ and $W^2$ as follows,

$$\begin{cases} \partial_t W^1 - |\nabla|^{\frac{1}{2}} W^2 = \mathfrak{Q}_1 + GC_1 + \widetilde{C}_1 + \widetilde{\mathcal{R}}_1, \\ \partial_t W^2 + |\nabla|^{\frac{1}{2}} W^1 = \mathfrak{Q}_2 + GC_2 + \widetilde{C}_2 + \widetilde{\mathcal{R}}_2, \end{cases} \tag{4.35}$$

where

$$\mathfrak{Q}_1 = Q_1(W^1, W^2) - \widetilde{Q}_1(W^1, W^2),$$

$$\mathfrak{Q}_2 = Q_2(W^1, W^1) + Q_3(W^2, W^2) - \widetilde{Q}_2(W^1, W^1) - \widetilde{Q}_3(W^2, W^2),$$

$$\widetilde{C}_1 = -T_{\Lambda_{\geq 2}[V]} \partial_x W^1 + T_{\Lambda_{\geq 2}[\alpha]} |\nabla|^{\frac{1}{2}} W^2, \widetilde{C}_2 = -T_{\Lambda_{\geq 2}[V]} \partial_x W^2 - T_{\Lambda_{\geq 2}[\alpha]} |\nabla|^{\frac{1}{2}} W^1,$$

$$GC_1 = T_V \partial_x (\widetilde{A}_1(U^1, U^1) + \widetilde{A}_2(U^2, U^2)) - 2\widetilde{A}_1(T_V \partial_x U^1, U^1) - 2\widetilde{A}_2(T_V \partial_x U^2, U^2)$$

$$-T_\alpha |\nabla|^{\frac{1}{2}} \widetilde{B}(U^1, U^2) + 2\widetilde{A}_1(T_\alpha |\nabla|^{\frac{1}{2}} U^2, U^1) - 2\widetilde{A}_2(T_\alpha |\nabla|^{\frac{1}{2}} U^1, U^2), \tag{4.36}$$

$$GC_2 = T_V \partial_x (\widetilde{B}(U^1, U^2)) - \widetilde{B}(T_V \partial_x U^1, U^2) - \widetilde{B}(U^1, T_V \partial_x U^2) + T_\alpha |\nabla|^{\frac{1}{2}} (\widetilde{A}_1(U^1, U^1)$$

$$+\widetilde{A}_2(U^2, U^2)) + \widetilde{B}(T_\alpha |\nabla|^{\frac{1}{2}} U^2, U^2) - \widetilde{B}(U^1, T_\alpha |\nabla|^{\frac{1}{2}} U^1). \tag{4.37}$$

To improve presentation, we postpone the detailed formulas of $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ to section 4.3, as their formulas are tedious and not very easy to see structures inside without detailed explanation. It is good enough to know that they have sufficient decay rate and do not lose derivatives in high frequency part or the low frequency part.

From the construction of bilinear operators $\widetilde{Q}_i(\cdot, \cdot), i \in \{1, 2, 3\}$, we can see that the quadratic terms $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ vanish when doing energy estimate. We will show that there are cancellations inside the good cubic terms $GC_1$ and $GC_2$ and there are symmetries inside cubic terms $\widetilde{C}_1$ and $\widetilde{C}_2$. Therefore, $W_1$ and $W_2$ are good substitution variable of $U^1$ and $U^2$ at the top regularity. Moreover, as losing derivative is not a issue when there is only $p$ derivatives, we can just cancel the quadratic terms directly by using the normal form transformation. Those intuitions motivate us to define the

following modified energy:

$$E_{modi}(t)=\frac{1}{2}\int\big[|\partial_x^p U^1|^2+|\partial_x^p U^2|^2\big]+\int\partial_x^p U^1\partial_x^p\big[A_1(U^1,U^1)+A_2(U^2,U^2)\big]$$

$$+\partial_x^p U^2\partial_x^p\big[B(U^1,U^2)\big]+\frac{1}{2}\int\big[|\partial_x^{N_0}W^1|^2+|\partial_x^{N_0}W^2|^2\big]. \tag{4.38}$$

**Lemma 4.1.2.** *Under the bootstrap condition* (4.12)*, the following estimate holds,*

$$\sup_{t\in[0,T]}\Big|E_{modi}(t)-\sum_{k=p,N_0}\frac{1}{2}\int\big[|\partial_x^k U^1|^2+|\partial_x^k U^2|^2\big]\Big|\lesssim\epsilon_0^2. \tag{4.39}$$

*Proof.* From (4.21) in Lemma 4.1.1 and $L^2-L^\infty$-type estimate (1.13) in Lemma 1.3.7, the following estimate holds after putting the input with larger frequency in $L^2$ and the smaller one in $L^\infty$,

$$\Big|\int\partial_x^p U^1\partial_x^p\big[A_1(U^1,U^1)+A_2(U^2,U^2)\big]+\partial_x^p U^2\partial_x^p\big[B(U^1,U^2)\big]\Big|$$

$$\lesssim\|(U^1,U^2)(t)\|_{H^{1+p,p}}^2\|(U^1,U^2)(t)\|_{W^0}\lesssim(1+t)^{-1/2+2p_0}\epsilon_1^3.$$

From (4.34) and $L^2-L^\infty$ estimate (1.13) in Lemma 1.3.7, we have

$$\|\partial_x^{N_0}(W^1-U^1,W^2-U^2)(t)\|_{L^2}\lesssim\|(U^1,U^2)(t)\|_{H^{N_0,p}}\|(U^1,U^2)(t)\|_{W^1}\lesssim(1+t)^{-1/2+2p_0}\epsilon_1^2.$$

Therefore, from the definition of $E_{modi}(t)$ in (4.38) and above estimates, it's easy to see our desired estimate (4.39) holds. □

After taking a derivative with respect to time for $E_{modi}(t)$, we have

$$\frac{d}{dt}E_{modi}(t)=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3,$$

where

$$\mathcal{J}_1=\int\partial_x^p U^1\partial_x^p\big[C_1+\Lambda_{\geq3}[\mathcal{R}_1]\big]+\partial_x^p U^2\partial_x^p\big[C_2+\Lambda_{\geq3}[\mathcal{R}_2]\big],$$

$$\mathcal{J}_2=\int\partial_x^p\Lambda_{\geq2}[\partial_t U^1]\partial_x^p\big(A_1(U^1,U^1)+A_2(U^2,U^2)\big)+\partial_x^p\Lambda_{\geq2}[\partial_t U^2]\partial_x^p\big(B(U^1,U^2)\big)$$

$$+2\partial_x^p U^1\partial_x^p\big(A_1(\Lambda_{\geq2}[\partial_t U^1],U^1)+A_2(\Lambda_{\geq2}[\partial_t U^2],U^2)\big)$$

$$+\partial_x^p U^2\partial_x^p\big(B(\Lambda_{\geq2}[\partial_t U^1],U^2)+B(U^1,\Lambda_{\geq2}[\partial_t U^2])\big),$$

$$\mathcal{J}_3=\int \partial_x^{N_0}W^1\partial_x^{N_0}\big[\widetilde{C}_1+GC_1+\widetilde{\mathcal{R}}_1\big]+\partial_x^{N_0}W^2\partial_x^{N_0}\big[\widetilde{C}_2+GC_2+\widetilde{\mathcal{R}}_2\big]. \tag{4.40}$$

**Lemma 4.1.3.** *Under the bootstrap assumption* (4.12)*, we have*

$$\sup_{t\in[0,T]}\sum_{i=1,2,3}(1+t)^{1-2p_0}|\mathcal{J}_i|\lesssim\epsilon_0^2. \tag{4.41}$$

*Proof.* (i) To estimate $\mathcal{J}_1$ and $\mathcal{J}_2$, as $p\in(0,1/4)$ is far away from $N_0$, we can estimate it straightforwardly by putting the input with relatively higher frequency in $L^2$ the input with relatively lower frequency in $L^\infty$. From (4.21) in Lemma 4.1.1, $L^2-L^\infty$-type estimate (1.13) in Lemma 1.3.7 and (4.155) in Lemma 4.3.1, the following estimate holds,

$$|\mathcal{J}_1|+|\mathcal{J}_2|\lesssim\|\Lambda_{\geq 2}(\partial_tU^1,\partial_tU^2)\|_{H^{1+p,p}}\|(U^1,U^2)\|_{H^{1+p,p}}\|(U^1,U^2)\|_{W^0}$$

$$+\|(U^1,U^2)\|_{H^{1+p,p}}^2\|(U^1,U^2)\|_{W^2}^2+\|(U^1,U^2)\|_{H^{N_0,p}}\|\Lambda_{\geq 3}[\mathcal{R}_1,\mathcal{R}_2]\|_{H^{N_0,p}}\lesssim(1+|t|)^{-1+2p_0}\epsilon_0^2.$$

(ii) We can first use symmetries to estimate $\widetilde{C}_1$ and $\widetilde{C}_2$, as a result, the following estimate holds,

$$\Big|\int\partial_x^{N_0}W^1\partial_x^{N_0}\widetilde{C}_1+\partial_x^{N_0}W^2\partial_x^{N_0}\widetilde{C}_2\Big|\lesssim\|(W^1,W^2)\|_{H^{N_0}}^2[\|\Lambda_{\geq 2}[V]\|_{\widetilde{W^1}}+\|\Lambda_{\geq 2}[\alpha]\|_{\widetilde{W^{1/2}}}]$$

$$\lesssim\|(U^1,U^2)\|_{H^{N_0,p}}^2\|(U^1,U^2)\|_{W^3}^2\lesssim(1+|t|)^{-1+2p_0}\epsilon_1^4\lesssim(1+|t|)^{-1+2p_0}\epsilon_0^2. \tag{4.42}$$

From (4.31) and (4.36), we can reduce $GC_1$ further as follows,

$$GC_1=T_V\partial_x(\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2))-2\widetilde{A}_1(T_V\partial_xU^1,U^1)-2\widetilde{A}_2(T_V\partial_xU^2,U^2)$$

$$+T_\alpha[2\widetilde{A}_1(|\nabla|^{\frac{1}{2}}U^2,U^1)-2\widetilde{A}_2(|\nabla|^{\frac{1}{2}}U^1,U^2)]-2\widetilde{A}_1(T_\alpha|\nabla|^{\frac{1}{2}}U^2,U^1)$$

$$+2\widetilde{A}_2(T_\alpha|\nabla|^{\frac{1}{2}}U^1,U^2)+T_\alpha\widetilde{Q}_1(U^1,U^2).$$

Note that $\widetilde{Q}_1(\cdot,\cdot)$ does not lose derivation, see (4.30). Now, it is easy to see there are cancellations inside $GC_1$. After utilizing symmetries on the Fourier side, the following equality holds,

$$\mathcal{F}(GC_1)(\xi)=\int_{\mathbb{R}^2}\widehat{U^1}(\eta-\sigma)\widehat{U^1}(\sigma)\widehat{V}(\xi-\eta)e_1(\xi,\eta,\sigma)+$$

$$\widehat{U^2}(\eta-\sigma)\widehat{U^2}(\sigma)\widehat{V}(\xi-\eta)e_2(\xi,\eta,\sigma)+\widehat{U^1}(\eta-\sigma)\widehat{U^2}(\sigma)\widehat{\alpha}(\xi-\eta)e_3(\xi,\eta,\sigma),$$

where, for $j \in \{1,2\}$,

$$e_j(\xi,\eta,\sigma) = 2i(\eta-\sigma)\big[\theta(\xi-\eta,\eta)\widetilde{a}_j(\eta-\sigma,\sigma) - \widetilde{a}_j(\xi-\sigma,\sigma)\theta(\xi-\eta,\eta-\sigma)\big],$$

$$e_3(\xi,\eta,\sigma) = 2|\sigma|^{1/2}\big[\widetilde{a}_1(\sigma,\eta-\sigma)\theta(\xi-\eta,\eta) - \widetilde{a}_1(\xi-\eta+\sigma,\eta-\sigma)\theta(\xi-\eta,\sigma)\big]$$

$$-2|\eta-\sigma|^{1/2}\big[\widetilde{a}_2(\eta-\sigma,\sigma)\theta(\xi-\eta,\eta) - \widetilde{a}_2(\xi-\sigma,\sigma)\theta(\xi-\eta,\eta-\sigma)\big].$$

From above explicit formulas and Lemma 1.3.8, the following estimates hold,

$$\|e_1(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} + \|e_2(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} \lesssim 2^{2\mathrm{med}\{k_1,k_2,k_3\}},$$

$$\|e_3(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} \lesssim 2^{3\mathrm{med}\{k_1,k_2,k_3\}/2},$$

where $\mathrm{med}\{k_1,k_2,k_3\}$ denotes the medium number of $k_1,k_2,k_3$. Hence, from $L^2-L^\infty-L^\infty$ type estimate in Lemma 1.3.7, the following estimate holds,

$$\|GC_1\|_{\dot{H}^{N_0}} \lesssim \|(U^1,U^2)\|_{H^{N_0,p}}\|\partial_x(U^1,U^2)\|_{\widetilde{W^1}}[\|\alpha\|_{\widetilde{W^0}} + \|V\|_{\widetilde{W^0}}]$$

$$\lesssim \|(U^1,U^2)\|_{H^{N_0,p}}\|(U^1,U^2)\|^2_{W^3} \lesssim (1+|t|)^{-1+p_0}\epsilon_1^3 \lesssim (1+|t|)^{-1+p_0}\epsilon_0^2. \tag{4.43}$$

The estimate of $GC_2$ is very similar, the upper bound in the right hand side of (4.43) still good for $GC_2$. We omit the details here. From (4.155) in Lemma 4.3.1, it is easy to see the following estimate holds,

$$\Big|\int \partial_x^{N_0}W^1\partial_x^{N_0}\widetilde{\mathcal{R}}_1 + \partial_x^{N_0}W^2\partial_x^{N_0}\widetilde{\mathcal{R}}_2\Big| \lesssim \|(W^1,W^2)\|_{H^{N_0,p}}\|(\widetilde{\mathcal{R}}_1,\widetilde{\mathcal{R}}_2)\|_{H^{N_0,p}} \lesssim (1+t)^{-1+2p_0}\epsilon_0^2.$$

Recall (4.40). Now, it is easy to see our desired estimate (4.1.3) holds.

□

### 4.1.3 Energy estimate of $SU^1$ and $SU^2$.

Note that $[\partial_t, S] = \partial_t$ and $[\pm|\nabla|^{\frac{1}{2}}, S] = \pm|\nabla|^{\frac{1}{2}}$. On the one hand, from the system (4.1), we can derive the system of equations satisfied by $SU^1$ and $SU^2$ with highlighted quasilinear structure as follows,

$$\begin{cases} \partial_t SU^1 - |\nabla|^{1/2}SU^2 = T_\alpha|\nabla|^{1/2}SU^2 - T_V\partial_x SU^1 + \mathcal{R}_1^S, \\ \partial_t SU^2 + |\nabla|^{1/2}SU^1 = -T_\alpha|\nabla|^{1/2}SU^1 - T_V\partial_x SU^2 + \mathcal{R}_2^S, \end{cases} \tag{4.44}$$

where $\mathcal{R}_1^S$ and $\mathcal{R}_2^S$ are quadratic and higher good remainder terms that do not lose derivatives in $SU^1$ and $SU^2$. More precisely, we have,

$$\mathcal{R}_1^S := (S+I)\mathcal{R}_1 + S(T_\alpha|\nabla|^{1/2}U^2 - T_V\partial_x U^1) - T_\alpha|\nabla|^{1/2}SU^2 + T_V\partial_x SU^1, \tag{4.45}$$

$$\mathcal{R}_2^S := (S+I)\mathcal{R}_2 + S(-T_\alpha|\nabla|^{1/2}U^1 - T_V\partial_x U^2) + T_\alpha|\nabla|^{1/2}SU^1 + T_V\partial_x SU^2. \tag{4.46}$$

On the other hand, from the system (4.2), we can rewrite the system (4.44) as follows to highlight the structures inside the quadratic terms,

$$\begin{cases} \partial_t SU^1 - |\nabla|^{\frac{1}{2}}SU^2 = \mathfrak{Q}_1^S(SU^1, SU^2) + \mathfrak{Q}_2^S + \mathcal{Q}_1 + \widetilde{C}_1 + \mathfrak{R}_1, \\ \partial_t SU^2 + |\nabla|^{\frac{1}{2}}SU^1 = \mathfrak{Q}_3^S(SU^1, SU^2) + \mathfrak{Q}_4^S + \mathcal{Q}_2 + \widetilde{C}_2 + \mathfrak{R}_2, \end{cases} \tag{4.47}$$

where $\mathfrak{Q}_1^S(\cdot,\cdot)$ and $\mathfrak{Q}_3^S(\cdot,\cdot)$ are quadratic terms that have quasilinear structures inside, $\mathfrak{Q}_2^S$ and $\mathfrak{Q}_3^S$ are quadratic terms that have problematic low frequency part, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are commutator terms that does not depend on the scaling vector field, $\widetilde{C}_1$ and $\widetilde{C}_2$ are cubic and higher order terms that at most lose one derivatives and $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are good cubic and higher order terms that do not lose derivatives. Their detailed formulas are given as follows,

$$\mathfrak{Q}_1^S(SU^1, SU^2) = Q_1(SU^1, U^2) + Q_1(U^1, SU^2) - T_{\partial_x|\nabla|^{-1/2}SU^2}\partial_x U^1,$$

$$\mathfrak{Q}_3^S(SU^1, SU^2) = Q_2(SU^1, U^1) + Q_2(U^1, SU^1)$$

$$+Q_3(SU^2, U^2) + Q_3(U^2, SU^2) + T_{\partial_x|\nabla|^{-\frac{1}{2}}SU^2}\partial_x U^2,$$

$$\mathfrak{Q}_2^S = T_{\partial_x|\nabla|^{-1/2}SU^2}\partial_x U^1,\ \mathfrak{Q}_4^S = -T_{\partial_x|\nabla|^{-\frac{1}{2}}SU^2}\partial_x U^2, \tag{4.48}$$

$$\mathcal{Q}_1 = Q_1(U^1, U^2) + SQ_1(U^1, U^2) - Q_1(SU^1, U^2) - Q_1(U^1, SU^2) \tag{4.49}$$

$$\mathcal{Q}_2 = Q_2(U^1, U^1) + Q_3(U^2, U^2) + SQ_2(U^1, U^1) - Q_2(SU^1, U^1) - Q_2(U^1, SU^2)$$

$$+SQ_2(U^1, U^1) - Q_2(SU^1, U^1) - Q_2(U^1, SU^2), \tag{4.50}$$

$$\widetilde{C}_1 = T_{\Lambda_{\geq 2}[\alpha]}|\nabla|^{\frac{1}{2}}SU^2 - T_{\Lambda_{\geq 2}[V]}\partial_x SU^1, \widetilde{C}_2 = T_{\Lambda_{\geq 2}[\alpha]}|\nabla|^{\frac{1}{2}}SU^2 - T_{\Lambda_{\geq 2}[V]}\partial_x SU^1,$$

$$\mathfrak{R}_1 = \Lambda_{\geq 3}[\mathcal{R}_1^S], \quad \mathfrak{R}_2 = \Lambda_{\geq 3}[\mathcal{R}_2^S].$$

To better see which parts of $\mathfrak{Q}_1^S(\cdot,\cdot)$ and $\mathfrak{Q}_1^S(\cdot,\cdot)$ lose derivatives in $SU^1$ and $SU^2$, especially in later energy estimate part, we define the following auxiliary bilinear forms,

$$\widetilde{\mathfrak{Q}}_1^S(SU^1,SU^2) := \mathfrak{Q}_1^S(SU^1,SU^2) - T_{\Lambda_1[\alpha]}|\nabla|^{1/2}SU^2 + T_{\Lambda_1[V]}\partial_x SU^1, \tag{4.51}$$

$$\widetilde{\mathfrak{Q}}_3^S(SU^1,SU^2) := \mathfrak{Q}_3^S(SU^1,SU^2) + T_{\Lambda_1[\alpha]}|\nabla|^{1/2}SU^1 + T_{\Lambda_1[V]}\partial_x SU^2. \tag{4.52}$$

From (4.44), we can see that $\widetilde{\mathfrak{Q}}_1^S(SU^1,SU^2)$ and $\widetilde{\mathfrak{Q}}_3^S(SU^1,SU^2)$ do not lose derivatives in $SU^1$ and $SU^2$.

**Handling the bulk quadratic terms $\mathfrak{Q}_1^S$ and $\mathfrak{Q}_3^S$**

Similar to what we did in the subsection 4.1.2, we will also utilize symmetries for the high frequency part. For $\tau = p, N_1$, we can utilize symmetries of the same type of inputs and do changing of variables on the Fourier side. As a result, the following equality holds,

$$\begin{aligned}\mathfrak{Re}\big[\int \overline{\partial_x^\tau SU^1}\partial_x^\tau \mathfrak{Q}_1^S + \overline{\partial_x^\tau SU^2}\partial_x^\tau \mathfrak{Q}_3^S\big] = \mathfrak{Re}\big[\int \overline{\widehat{SU^1}(\xi)}\hat{q}_\tau^1(\xi-\eta,\eta)\widehat{SU^1}(\xi-\eta)\widehat{U^2}(\eta)\\ +\overline{\widehat{SU^1}(\xi)}\hat{q}_\tau^2(\xi-\eta,\eta)\widehat{SU^2}(\xi-\eta)\widehat{U^1}(\eta) + \overline{\widehat{SU^2}(\xi)}\hat{q}_\tau^3(\xi-\eta,\eta)\widehat{SU^2}(\xi-\eta)\widehat{U^2}(\eta)\big],\end{aligned} \tag{4.53}$$

where

$$\hat{q}_\tau^1(\xi-\eta,\eta) = |\xi|^{2\tau}\big(q_1^1(\xi-\eta,\eta)/2 + q_1^2(\xi-\eta,\eta) + q_1^3(\xi-\eta,\eta)\big) + q_1^1(-\xi,\eta)|\xi-\eta|^{2\tau}/2,$$

$$\hat{q}_\tau^2(\xi-\eta,\eta) = |\xi|^{2\tau}\big(q_1(\eta,\xi-\eta) + \eta(\xi-\eta)|\eta|^{-1/2}\theta(\xi-\eta,\eta)\big) + |\xi-\eta|^{2\tau}\big(q_2(\eta,-\xi) + q_2(-\xi,\eta)\big),$$

$$\hat{q}_\tau^3(\xi-\eta,\eta) = |\xi|^{2\tau}\big(q_3(\xi-\eta,\eta) + q_3(\eta,\xi-\eta)\big)/2 + |\xi-\eta|^{2\tau}\big(q_3(\eta,-\xi) + q_3(-\xi,\eta)\big)/2$$

$$-(\xi-\eta)\eta|\xi-\eta|^{-1/2}|\eta|^{-1/2}|\xi|^{1/2+2\tau}\theta(\xi-\eta,\eta).$$

Note that $q_1(\xi-\eta,\eta)\theta(\eta,\xi-\eta) = q_1^1(\xi-\eta,\eta)$ and $q_1(\xi-\eta,\eta)\theta(\xi-\eta,\eta) = q_1^2(\xi-\eta,\eta)$ . From (4.9) in Lemma 4.0.8, we can see cancellations also happen when $|\eta| \ll |\xi-\eta|$. Moreover, when $|\xi-\eta| \ll |\eta|$, we can gain at least one degree of smallness from the symbols as we put those exception terms into $\mathfrak{Q}_2^S$ and $\mathfrak{Q}_4^S$. For the Hight $\times$ High type interaction, all terms except $q_1^3(\cdot,\cdot)$ vanish, see

(4.4), (4.6), and (4.7). As a result, the following estimate holds,

$$\sum_{i=1,2,3}\|\hat{q}_\tau^3(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim\begin{cases}2^{3k_2/2+2\tau k} & \text{if } k_2\le k_1-5\\ 2^{(2\tau+1)k+k_1/2} & \text{if } |k_1-k_2|\le 5\\ 2^{k_1+(2\tau+1/2)k} & \text{if } k_1\le k_2-5.\end{cases}\tag{4.54}$$

For $\tau=p,N_1$, we define $\widetilde{Q}_{1,\tau}(SU^1,U^2)$, $\widetilde{Q}_{2,\tau}(SU^2,U^1)$, $\widetilde{Q}_{3,\tau}(SU^1,U^1)$ and $\widetilde{Q}_{4,\tau}(SU^2,U^2)$ by the following symbols,

$$\widetilde{q}_{1,\tau}(\xi-\eta,\eta)=\frac{\hat{q}_\tau^1(\xi-\eta,\eta)}{|\xi|^{2\tau}},\quad \widetilde{q}_{2,\tau}(\xi-\eta,\eta)=\frac{\hat{q}_\tau^2(\xi-\eta,\eta)}{2|\xi|^{2\tau}},$$

$$\widetilde{q}_{3,\tau}(\xi-\eta,\eta)=\frac{\hat{q}_\tau^2(\xi,-\eta)}{2|\xi|^{2\tau}},\quad \widetilde{q}_{4,\tau}(\xi-\eta,\eta)=\frac{\hat{q}_\tau^3(\xi-\eta,\eta)}{|\xi|^{2\tau}}.$$

We also define the following substitution variables of $SU^1$ and $SU^2$,

$$W_{1,\tau}:=SU^1+\widetilde{C}_{1,\tau}(SU^1,U^1)+\widetilde{C}_{2,\tau}(SU^2,U^2),\tag{4.55}$$

$$W_{2,\tau}:=SU^2+\widetilde{D}_{1,\tau}(SU^1,U^2)+\widetilde{D}_{2,\tau}(SU^2,U^1).\tag{4.56}$$

The goal is to use bilinear operators $\widetilde{C}_{i,\tau}(\cdot,\cdot)$ and $\widetilde{D}_{i,\tau}(\cdot,\cdot)$, $i\in\{1,2\}$ , to cancel out $\widetilde{Q}_{j,\tau}(\cdot,\cdot)$, $j\in\{1,2,3,4\}$. Effectively speaking, it cancel out (4.53) in energy estimate. To this end, it would be sufficient if symbols $c_\tau^i(\cdot,\cdot)$ and $d_\tau^i(\cdot,\cdot)$ of bilinear operators $\widetilde{C}_{i,\tau}(\cdot,\cdot)$ and $\widetilde{D}_{i,\tau}(\cdot,\cdot)$ solve the following system of equations,

$$\begin{cases}|\eta|^{\frac12}c_\tau^1(\xi-\eta,\eta)-|\xi-\eta|^{\frac12}c_\tau^2(\xi-\eta,\eta)-|\xi|^{\frac12}d_\tau^1(\xi-\eta,\eta)+\widetilde{q}_{1,\tau}(\xi-\eta,\eta)=0\\ |\xi-\eta|^{\frac12}c_\tau^1(\xi-\eta,\eta)-|\eta|^{\frac12}c_\tau^2(\xi-\eta,\eta)-|\xi|^{\frac12}d_\tau^2(\xi-\eta,\eta)+\widetilde{q}_{2,\tau}(\xi-\eta,\eta)=0\\ |\xi|^{\frac12}c_\tau^1(\xi-\eta,\eta)-|\eta|^{\frac12}d_\tau^1(\xi-\eta,\eta)-|\xi-\eta|^{\frac12}d_\tau^2(\xi-\eta,\eta)+\widetilde{q}_{3,\tau}(\xi-\eta,\eta)=0\\ |\xi|^{\frac12}c_\tau^2(\xi-\eta,\eta)+|\xi-\eta|^{\frac12}d_\tau^1(\xi-\eta,\eta)+|\eta|^{\frac12}d_\tau^2(\xi-\eta,\eta)+\widetilde{q}_{4,\tau}(\xi-\eta,\eta)=0.\end{cases}\tag{4.57}$$

It's not difficult to solve above system of equations and derive the following,

$$d_\tau^1(\xi-\eta,\eta)=\frac{(|\xi-\eta|+|\eta|-|\xi|)F_1(\xi-\eta,\eta)-2F_2(\xi-\eta,\eta)|\xi-\eta|^{\frac12}|\eta|^{\frac12}}{-(|\xi-\eta|+|\eta|-|\xi|)^2+4|\xi-\eta||\eta|},\tag{4.58}$$

$$d_\tau^2(\xi-\eta,\eta)=\frac{(|\xi-\eta|+|\eta|-|\xi|)F_2(\xi-\eta,\eta)-2F_1(\xi-\eta,\eta)|\xi-\eta|^{\frac12}|\eta|^{\frac12}}{-(|\xi-\eta|+|\eta|-|\xi|)^2+4|\xi-\eta||\eta|},\tag{4.59}$$

$$c_\tau^1(\xi-\eta,\eta)=\frac{|\eta|^{\frac12}d_\tau^1(\xi-\eta,\eta)+|\xi-\eta|^{\frac12}d_\tau^2(\xi-\eta,\eta)-\widetilde{q}_{3,\tau}(\xi-\eta,\eta)}{|\xi|^{\frac12}},\tag{4.60}$$

$$c_\tau^2(\xi-\eta,\eta)=\frac{-|\xi-\eta|^{\frac12}d_\tau^1(\xi-\eta,\eta)-|\eta|^{\frac12}d_\tau^2(\xi-\eta,\eta)-\widetilde{q}_{4,\tau}(\xi-\eta,\eta)}{|\xi|^{\frac12}},\tag{4.61}$$

where $F_1(\xi-\eta,\eta)$ and $F_2(\xi-\eta,\eta)$ are defined as follows,

$$F_1(\xi-\eta,\eta)=|\xi|^{\frac12}\widetilde{q}_{1,\tau}(\xi-\eta,\eta)-|\eta|^{\frac12}\widetilde{q}_{3,\tau}(\xi-\eta,\eta)+|\xi-\eta|^{\frac12}\widetilde{q}_{4,\tau}(\xi-\eta,\eta),$$

$$F_2(\xi-\eta,\eta)=|\xi|^{\frac12}\widetilde{q}_{2,\tau}(\xi-\eta,\eta)-|\xi-\eta|^{\frac12}\widetilde{q}_{3,\tau}(\xi-\eta,\eta)+|\eta|^{\frac12}\widetilde{q}_{4,\tau}(\xi-\eta,\eta).$$

**Lemma 4.1.4.** *The following estimate holds for $i\in\{1,2\}$,*

$$\|c_\tau^i(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|d_\tau^i(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim\begin{cases}2^{k_2} & \text{if } k_2\le k_1+5\\ 2^{k_1/2+k_2/2} & \text{if } k_1\le k_2-5.\end{cases}\tag{4.62}$$

*Proof.* Recall (4.58), (4.59), (4.60) and (4.58). From (4.54), Lemma 1.3.8, and (1.12) in Lemma 1.3.7, our desired estimate (4.62) follows straightforwardly. □

Recall (4.44) and (4.47). From (4.55) and (4.56), after substituting $SU^i$ by $W_{i,\tau}, i\in\{1,2\}$, we can derive the equations satisfied by $W_{1,\tau}$ and$W_{2,\tau}$ with good structures as follows,

$$\partial_tW_{1,\tau}-|\nabla|^{\frac12}W_{2,\tau}=\mathfrak{Q}_1^S(W_{1,\tau},U^2)+\mathfrak{Q}_1^S(W_{2,\tau},U^1)-\widetilde{Q}_{1,\tau}(W_{1,\tau},U^2)-\widetilde{Q}_{2,\tau}(W_{2,\tau},U^1)$$

$$-T_{\Lambda_{\ge2}[V]}\partial_xW_{1,\tau}+T_{\Lambda_{\ge2}[\alpha]}|\nabla|^{\frac12}W_{2,\tau}+\mathfrak{Q}_2^S+\mathcal{Q}_1+\widetilde{GC}_1+\widetilde{\mathfrak{R}}_1,\tag{4.63}$$

$$\partial_tW_{2,\tau}+|\nabla|^{\frac12}W_{1,\tau}=\mathfrak{Q}_3^S(W_{1,\tau},U^1)+\mathfrak{Q}_3^S(W_{2,\tau},U^2)-\widetilde{Q}_{3,\tau}(W_{1,\tau},U^1)-\widetilde{Q}_{4,\tau}(W_{2,\tau},U^2)$$

$$-T_{\Lambda_{\ge2}[V]}\partial_xW_{2,\tau}-T_{\Lambda_{\ge2}[\alpha]}|\nabla|^{\frac12}W_{1,\tau}+\mathfrak{Q}_4^S+\mathcal{Q}_2+\widetilde{GC}_2+\widetilde{\mathfrak{R}}_2,\tag{4.64}$$

where

$$\widetilde{GC}_1=T_V\partial_x(\widetilde{C}_{1,\tau}(SU^1,U^1)+\widetilde{C}_{2,\tau}(SU^2,U^2))-T_\alpha|\nabla|^{\frac12}[\widetilde{D}_{1,\tau}(SU^1,U^2)$$

$$+\widetilde{D}_{2,\tau}(SU^2,U^1)]-\widetilde{C}_{1,\tau}(T_V\partial_xSU^1,U^1)+\widetilde{C}_{1,\tau}(T_\alpha|\nabla|^{\frac12}SU^2,U^1)$$

$$-\widetilde{C}_{2,\tau}(T_V\partial_xSU^2,U^2)-\widetilde{C}_{2,\tau}(T_\alpha|\nabla|^{\frac12}SU^1,U^2),$$

$$\widetilde{GC}_2=T_V\partial_x[\widetilde{D}_{1,\tau}(SU^1,U^2)+\widetilde{D}_{2,\tau}(SU^2,U^1)]+T_\alpha|\nabla|^{\frac12}[\widetilde{C}_{1,\tau}(SU^1,U^1)$$

$$+\widetilde{C}_{2,\tau}(SU^2,U^2)]-\widetilde{D}_{1,\tau}(T_V\partial_xSU^1,U^2)+\widetilde{D}_{1,\tau}(T_\alpha|\nabla|^{\frac12}SU^2,U^2)$$

$$-\widetilde{D}_{2,\tau}(T_V\partial_x SU^2,U^1)-\widetilde{D}_{2,\tau}(T_\alpha|\nabla|^{\frac{1}{2}}SU^1,U^1).$$

To improve presentation, we postpone the formulas of $\widetilde{\mathfrak{R}}_1$ and $\widetilde{\mathfrak{R}}_2$ to the section 4.3. It is enough to know that they are good cubic and higher remainder terms in following sense: (i) They don't lose derivatives in $SU^1$ and $SU^2$. (ii) We can gain at least half derivative of smallness in $SU^1$ and $SU^2$. Hence those remainder terms can be estimated straightforwardly in the energy estimate.

Similar to what we did in the energy estimate of $U^1$ and $U^2$, we define the high frequency part of the modified energy for $SU^1$ and $SU^2$ as follows:

$$E^{S,high}_{modi}(t):=\sum_{\tau=p,N_1}\int\frac{1}{2}\Big[|\partial_x^\tau W_{1,\tau}|^2+\int\partial_x^\tau W_{1,\tau}\partial_x^\tau[E_1(U^1,U^1)+E_2(U^2,U^2)]$$

$$+\partial_x^\tau W_{2,k}\partial_x^\tau F(U^1,U^2),\tag{4.65}$$

where bilinear forms $E_1(U^1,U^1)$, $E_2(U^2,U^2)$ and $F(U^1,U^2)$ are the normal form transformations, which cancel out the commutator terms $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

By solving a similar system of equations as in (4.17) with $Q_1(U^1,U^2)$ replaced by $\mathcal{Q}_1$ and $Q_2(U^1,U^1)+Q_3(U^2,U^2)$ replaced by $\mathcal{Q}_2$, we can explicitly solve symbols $e_1(\xi-\eta,\eta)$, $e_2(\xi-\eta,\eta)$, and $f(\xi-\eta,\eta)$ of bilinear operators $E_1(\cdot,\cdot)$, $E_2(\cdot,\cdot)$ and $F(\cdot,\cdot)$. Their precise formulas are not so important here, hence we omit them here. Very similar to the proof of estimate (4.21) in Lemma 4.1.1, the following estimate holds,

$$\|e_1(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|e_2(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}+\|f(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{\max\{k_1,k_2\}}.\tag{4.66}$$

**Lemma 4.1.5.** *Under the bootstrap smallness assumption* (4.12)*, we have*

$$\sup_{t\in[0,T]}\big|E^{S,high}_{modi}(t)-\sum_{k=p,N_0}\frac{1}{2}\int\big[|\partial_x^k U^1|^2+|\partial_x^k U^2|^2\big]\big|\lesssim\epsilon_0^2.\tag{4.67}$$

*Proof.* From (4.62) in Lemma 4.1.4 and the $L^2-L^\infty$-type bilinear estimate (1.13) in Lemma 1.3.7, the following estimate holds,

$$\sum_{\tau=p,N_1}\|W_{1,\tau}(t)-SU^1(t)\|_{\dot{H}^\tau}+\|W_{2,\tau}(t)-SU^2(t)\|_{\dot{H}^\tau}$$

$$\lesssim\|(SU^1,SU^2)(t)\|_{H^{N_1,p}}\|(U^1,U^2)(t)\|_{W^1}\lesssim(1+|t|)^{-1/2+p_0}\epsilon_1^2.\tag{4.68}$$

From (4.66), (4.68), and the $L^2-L^\infty$-type bilinear estimate (1.13) in Lemma 1.3.7, our desired

estimate holds as follows,

$$\Big|E^{S,high}_{modi}(t)-\sum_{k=p,N_1}\int|\partial_x^k SU^1|^2+|\partial_x^k SU^2|^2\Big|\lesssim(1+|t|)^{-1/2+p_0}\epsilon_1^3$$

$$+\|(SU^1,SU^2)(t)\|_{H^{N_1,p}}\|(U^1,U^2)\|_{H^{N_1+1,p}}(U^1,U^2)(t)\|_{W^1}\lesssim\epsilon_0^2.$$

□

From direct computations, we have the following estimate,

$$\Big|\frac{d}{dt}E^{S,high}_{modi}(t)+\sum_{\tau=p,N_1}\int\partial_x^\tau SU^1\partial_x^\tau\mathfrak{Q}_2^S+\partial_x^\tau SU^2\partial_x^\tau\mathfrak{Q}_4^S\Big|\lesssim\sum_{i=1,2,3}\big|\mathcal{J}_i^S\big|,\tag{4.69}$$

where

$$\mathcal{J}_1^S=\sum_{\tau=p,N_1}\int\partial_x^k W_{1,\tau}\partial_x^\tau[\widetilde{GC}_1+\widetilde{\mathfrak{R}}_1]+\partial_x^\tau W_{2,\tau}\partial_x^\tau[\widetilde{GC}_2+\widetilde{\mathfrak{R}}_2]-\partial_x^\tau\big(\widetilde{C}_{1,\tau}(SU^1,U^1),$$

$$+\widetilde{C}_{2,\tau}(SU^2,U^2)\big)\partial_x^\tau\mathfrak{Q}_2^S+\partial_x^\tau\big(\widetilde{D}_{1,\tau}(SU^1,U^2)+\widetilde{D}_{2,\tau}(SU^2,U^1)\big)\partial_x^\tau\mathfrak{Q}_4^S,$$

$$\mathcal{J}_2^S=\sum_{\tau=p,N_1}\int\partial_x^\tau\Lambda_{\geq2}[\partial_t W_{1,\tau}]\partial_x^\tau[E_1(U^1,U^1)+E_2(U^2,U^2)]+\partial_x^\tau\Lambda_{\geq2}[\partial_t W_{2,\tau}]\partial_x^\tau F(U^1,U^2)$$

$$+2\partial_x^\tau W_{1,k}\partial_x^\tau[E_1(\Lambda_{\geq2}[\partial_t U^1],U^1)+E_2(\Lambda_{\geq2}[\partial_t U^2],U^2)]+\partial_x^\tau W_{2,k}\partial_x^\tau[F(\Lambda_{\geq2}[\partial_t U^1],U^2)+F(U^1,\Lambda_{\geq2}[\partial_t U^2])],$$

$$\mathcal{J}_3^S=\sum_{\tau=p,N_1}\int\partial_x^\tau W_{1,\tau}\partial_x^\tau[-T_{\Lambda_{\geq2}[V]}\partial_x W_{1,\tau}+T_{\Lambda_{\geq2}[\alpha]}|\nabla|^{\frac{1}{2}}W_{2,\tau}]$$

$$-\partial_x^\tau W_{2,\tau}\partial_x^\tau[T_{\Lambda_{\geq2}[V]}\partial_x W_{2,\tau}+T_{\Lambda_{\geq2}[\alpha]}|\nabla|^{\frac{1}{2}}W_{1,\tau}].$$

We remark that the quartic terms that depend on $\mathfrak{D}_2^S$ and $\mathfrak{D}_4^S$ inside $\mathcal{J}_1^S$ are resulted from replacing $SU^1$ and $SU^2$ by $W_{1,\tau}$ and $W_{2,\tau}$ respectively.

**Lemma 4.1.6.** *Under the bootstrap condition* (4.12)*, we have the following estimate*

$$\sum_{i=1}|\mathcal{J}_i^S|\lesssim(1+t)^{-1+2p_0}\epsilon_0^2.\tag{4.70}$$

*Proof.* We first estimate $\mathcal{J}_1^S$ and $\mathcal{J}_3^S$. From (4.158) in Lemma 4.3.2, we have the estimate of $H^{N_1,p}$ norm of $\widetilde{\mathfrak{R}}_1$ and $\widetilde{\mathfrak{R}}_2$. Similar to the estimate of $GC_i$ we did in the proof of Lemma 4.1.3, one can see cancellation happens once write $\widetilde{GC}_i$ on the Fourier side after utilizing the equalities satisfied by normal form transformations in (4.57). Also after utilizing symmetries, we can see cancellations

inside $\mathcal{J}_3^S$. As a result, the following estimate holds,

$$|\mathcal{J}_1^S| + |\mathcal{J}_3^S| \lesssim \left[\|(SU^1, SU^2)\|_{H^{N_1,p}} + \|(U^1, U^2)\|_{H^{N_0,p}}\right]^2 \|(U^1, U^2)\|_{W^3}^2 + (1+t)^{-1+2p_0}\epsilon_0^2$$

$$\lesssim (1+t)^{-1+2p_0}\epsilon_0^2.$$

We proceed to estimate $\mathcal{J}_2$. Note that the inputs inside bilinear operators $E_1(\cdot,\cdot)$, $E_2(\cdot,\cdot)$ and $F(\cdot,\cdot)$ only depend on $(U^1, U^2)$. Hence we can always put the input with higher frequency in $L^2$ to avoid losing $p$ derivatives at the low frequency part. As a result, we have

$$|\mathcal{J}_2^S| \lesssim \left[\|(SU^1, SU^2)\|_{H^{N_1,p}} + \|(U^1, U^2)\|_{H^{N_0,p}}\right]^2 \|(U^1, U^2)\|_{W^3}^2 \lesssim (1+t)^{-1+2p_0}\epsilon_0^2.$$

To sum up, we can see the desired estimate (4.70) holds. □

**Handling the bulk quadratic terms $\mathfrak{Q}_2^S$ and $\mathfrak{Q}_4^S$**

It remains to deal with the quadratic terms $\mathfrak{Q}_2^S$ and $\mathfrak{Q}_4^S$. As losing a derivative is not a issue, it is unnecessary to utilize symmetries to see cancellations any more. We choose to work in the complex variables setting, which is more convenient in the Fourier transform based method. We define

$$U = U^1 + iU^2, \quad SU = SU^1 + iSU^2, \quad c_\mu = \mu/(2i),\ \mu \in \{+,-\}.$$

From (4.2) and (4.47), we can write the equations satisfied by $U$ and $SU$ as follows

$$\partial_t U + i|\nabla|^{1/2}U = \sum_{\mu,\nu\in\{+,-\}} Q_{\mu,\nu}(U_\mu, U_\nu) + \Lambda_{\geq 3}[\partial_t U],$$

$$\partial_t SU + i|\nabla|^{1/2}SU = \sum_{\mu,\nu\in\{+,-\}} Q^1_{\mu,\nu}((SU)_\mu, U_\nu) + Q^2_{\mu,\nu}(U_\mu, U_\nu) + \Lambda_{\geq 3}[\partial_t SU],$$

where $f_+ := f$ and $f_- := \overline{f}$. From (4.8) in Lemma 4.0.8, it's easy to see the following estimate holds,

$$\sum_{\mu,\nu\in\{+,-\}} \|q_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} + \|q^1_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} + \|q^2_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}$$

$$\lesssim 2^{\min\{k_1,k_2\}/2+\max\{k_1,k_2\}}. \tag{4.71}$$

Recall (4.48). We can write those problematic cubic terms on the Fourier side and in terms of $SU$ and $U$ as follows,

$$\sum_{\tau=p,N_1}\mathfrak{Re}\big[\int\overline{\partial_x^\tau SU^1}\partial_x^\tau\mathfrak{Q}_2^S+\overline{\partial_x^\tau SU^2}\partial_x^\tau\mathfrak{Q}_4^S\big]$$

$$=\sum_{\tau=p,N_1}\sum_{\mu,\nu,\kappa\in\{+,-\}}\mathfrak{Re}\big[\int\overline{\widehat{(SU)_\mu}}(\xi)\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_\kappa}(\eta)p_{\mu,\nu}^{\kappa,\tau}(\xi-\eta,\eta)d\eta d\xi, \tag{4.72}$$

where

$$p_{\mu,\nu}^{\kappa,\tau}(\xi-\eta,\eta)=-c_\nu(\frac{1}{4}+c_{-\mu}c_\kappa)(\xi-\eta)\eta|\xi-\eta|^{-1/2}|\xi|^{2\tau}\theta(\xi-\eta,\eta). \tag{4.73}$$

Note that

$$\begin{aligned}&\Big|\int\overline{\widehat{(SU)_\mu}}(\xi)\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_\kappa}(\eta)p_{\mu,\nu}^{\kappa,\tau}(\xi-\eta,\eta)\psi_{\le0}((1+t)^{5/4+3p_0}|\xi-\eta|)d\eta d\xi\Big|\\&\lesssim\sum_{k_1\le k-10}\sum_{2^{k_1}\le(1+t)^{-5/4-3p_0}}2^{k_1/2+k}\|P_k[SU]\|_{L^2}\|P_{k_1}[SU]\|_{L^\infty}\|P_kU\|_{L^2}\\&\lesssim\sum_{2^{k_1}\le(1+t)^{-5/4-3p_0}}2^{(1-p)k_1}\|SU\|_{H^{N_1,p}}^2\|U\|_{H^{N_1,p}}\lesssim(1+t)^{-1+2p_0}\epsilon_0^2.\end{aligned} \tag{4.74}$$

From above estimate, we can see that the low frequency part already has sufficient decay rate. It is not needed to cancel it out. This observation motives us to define the following cubic correction terms with time dependent cutoff functions, which only cancel out the case when $|\xi-\eta|\gtrsim(1+t)^{-5/4-3p_0}$,

$$E_{modi}^{S,low1}(t):=\sum_{\tau=p,N_1}\sum_{\mu,\nu,\kappa\in\{+,-\}}\mathfrak{Re}\big[\int\overline{\widehat{(SU)_\mu}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_\kappa}(\eta)q_{\mu,\nu}^{\kappa,\tau}(t,\xi-\eta,\eta)d\eta d\xi\big], \tag{4.75}$$

where

$$q_{\mu,\nu}^{\kappa,\tau}(t,\xi-\eta,\eta)=\frac{p_{\mu,\nu}^{\kappa,\tau}(\xi-\eta,\eta)\psi_{\ge0}((1+t)^{5/4+3p_0}|\xi-\eta|)}{\mu|\xi|^{1/2}-\nu|\xi-\eta|^{1/2}-\kappa|\eta|^{1/2}}. \tag{4.76}$$

Note that the following estimate holds for any $\mu,\nu,\kappa\in\{+,-\}$,

$$\begin{aligned}\big|\mu|\xi|^{1/2}-\nu|\xi-\eta|^{1/2}-\kappa|\eta|^{1/2}\big|\theta(\xi-\eta,\eta)&\gtrsim\big(\big|\xi-\eta|^{1/2}-\big|\frac{|\xi|-|\eta|}{|\xi|^{1/2}+|\eta|^{1/2}}\big|\big)\theta(\xi-\eta,\eta)\\&\gtrsim|\xi-\eta|^{1/2}\theta(\xi-\eta,\eta).\end{aligned}$$

With above estimate, from Lemma 1.3.8 and (4.73), the following estimates hold for $\tau\in\{p,N_1\}$,

$$\sum_{\mu,\nu,\kappa\in\{+,-\}}\|q_{\mu,\nu}^{\kappa,\tau}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim2^{(2\tau+1)k}, \tag{4.77}$$

$$\sum_{\mu,\nu,\kappa\in\{+,-\}}\|\partial_t q^{\kappa,\tau}_{\mu,\nu}(t,\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim(1+t)^{-1}2^{(2\tau+1)k}. \tag{4.78}$$

Note that, from (4.77) and (4.78), the symbols $q^{\kappa,\tau}_{\mu,\nu}(\xi-\eta,\eta)$ and $\partial_t q^{\kappa,\tau}_{\mu,\nu}(t,\xi-\eta,\eta)$ now do not contribute any smallness when $|\xi-\eta|\ll|\eta|$. As $SU$ is forced to be putted in $L^2$ and the symbol can not cover the loss of $p$-derivatives of smallness, there is a potential problem when $|\xi-\eta|$ is the smallest inside the time derivative of $E^{S,low1}_{modi}(t)$.

To get around this issue, we will add quartic correction terms to cancel problematic quartic terms inside $dE^{S,low1}_{modi}(t)/dt$. We first identify those problematic quartic terms by calculating $dE^{S,low1}_{modi}/dt$ as follows,

$$\Lambda_{\geq4}[\frac{d}{dt}E^{S,low1}_{modi}]=\sum_{\tau=p,N_1}\sum_{\mu,\nu,\kappa\in\{+,-\}}\mathfrak{G}^{\kappa,\tau}_{\mu,\nu}+\sum_{\mu,\nu,\mu',\nu'\in\{+,-\}}\mathfrak{G}^{\tau}_{\mu,\nu;\mu',\nu'},$$

where

$$\mathfrak{G}^{\kappa,\tau}_{\mu,\nu}=\mathfrak{Re}\Big[\int\Big(\overline{\Lambda_{\geq3}\widehat{[\partial_t(SU)_\mu]}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_\kappa}(\eta)$$

$$+\overline{\widehat{(SU)_\mu}(\xi)}\Lambda_{\geq2}\widehat{[(\partial_tSU)_\nu]}(\xi-\eta)\widehat{U_\kappa}(\eta)+\overline{\widehat{(SU)_\mu}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\Lambda_{\geq3}\widehat{[\partial_tU_\kappa]}(\eta)\Big)$$

$$\times q^{\kappa,\tau}_{\mu,\nu}(t,\xi-\eta,\eta)+\overline{\widehat{(SU)_\mu}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_\kappa}(\eta)\partial_t q^{\kappa,k}_{\mu,\nu}(t,\xi-\eta,\eta)d\eta d\xi\Big],$$

$$\mathfrak{G}^{\tau}_{\mu,\nu;\mu',\nu'}=\mathfrak{Re}\Big[\int\overline{\widehat{(SU)_\mu}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_{\mu'}}(\eta-\sigma)\widehat{U_{\nu'}}(\sigma)e^{\tau,1}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)$$

$$+\overline{\widehat{U_\mu}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_{\mu'}}(\eta-\sigma)\widehat{U_{\nu'}}(\sigma)e^{\tau,2}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)d\sigma d\eta d\xi\Big], \tag{4.79}$$

where

$$e^{\tau,1}_{\mu,\nu}(t,\xi,\eta,\sigma)=\sum_{\kappa\in\{+,-\}}q^{\kappa,\tau}_{\mu,\nu}(t,\xi-\eta,\eta)P_\kappa[q_{\kappa\mu',\kappa\nu'}(\eta-\sigma,\sigma)]$$

$$+q^{\mu',\tau}_{\kappa,\nu}(t,\xi-\eta,\eta-\sigma)P_{-\kappa}[q^1_{\mu,-\kappa\nu'}(\xi,-\sigma)], \tag{4.80}$$

$$e^{\tau,2}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)=\sum_{\kappa\in\{+,-\}}q^{\mu',\tau}_{\kappa,\nu}(t,\xi-\eta,\eta-\sigma)P_{-\kappa}[q^2_{\mu,-\kappa\nu'}(\xi,-\sigma)], \tag{4.81}$$

where $P_\mu[f]:=f_\mu$. From (4.77), (4.78), (4.71), and (1.12) in Lemma 1.3.7, the following estimate holds,

$$\|e^{\tau,1}_{\mu,\nu}(t,\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}+\|e^{\tau,2}_{\mu,\nu}(t,\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}\lesssim2^{k/2+(2\tau+2)\max\{k_1,k_2,k_3\}}. \tag{4.82}$$

There are two types of terms inside $\mathfrak{G}^{\kappa,\tau}_{\mu,\nu}$: (i) term like $\overline{\widehat{(SU)_\mu}(\xi)}\Lambda_{\geq2}\widehat{[(\partial_tSU)_\nu]}(\xi-\eta)\widehat{U_\kappa}(\eta)$, which is quartic and higher. As we can gain half degree of smallness from the symbol of $\Lambda_{\geq2}\widehat{[(\partial_tSU)_\nu]}(\xi-\eta)$, losing $p$ derivatives in $SU$ is not a issue; (ii) all other terms inside $\mathfrak{G}^{\kappa,\tau}_{\mu,\nu}$. Note that the decay rate of

those terms is at least $(1+t)^{-3/2+2p_0}$. The extra gain of $(1+t)^{-1/2}$ can cover the loss of $p$ derivatives in $SU$. More precisely, we at most lose $|\xi-\eta|^{-p}$. Recall that $|\xi-\eta| \gtrsim (1+t)^{-5/4-3p_0}$ inside the support of cut-off function, hence we at most lose $(1+t)^{1/4+p_0}$, which can be covered from the extra gain of $(1+t)^{-1/2}$.

Recall (4.80), (4.81), (4.71), (4.77) and (4.78). We know that the new introduced symbols $q_{\mu,\nu}(\cdot,\cdot)$ and $q^i_{\mu,\nu}(\cdot,\cdot)$ inside $e^{\tau,i}_{\mu,\nu}(t,\xi,\eta,\sigma)$, $i\in\{1,2\}$, contribute half degree of smallness, which is less than or equal to the second smallest number among $\xi$, $\xi-\eta$, $\eta-\sigma$, and $\sigma$. Intuitively speaking, as $\xi-(\xi-\eta+\eta-\sigma+\sigma)=0$, we know the largest two number are comparable and the half degree of smallness is less than the biggest number, hence the second largest. Therefore, it less than or equal to the second smallest number.

As a result, combining the estimate (4.82), the following estimate holds,

$$\|e^{\tau,1}_{\mu,\nu}(t,\xi,\eta,\sigma)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}+\|\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}$$

$$\lesssim \min\{2^{k/2+(2\tau+2)\max\{k_1,k_2,k_3\}},2^{\mathrm{Smin}\{k,k_1,k_2,k_3\}/2+(2\tau+2)\max\{k_1,k_2,k_3\}}\},\tag{4.83}$$

where "Smin$\{k,k_1,k_2,k_3\}$"denotes the second smallest number among $k,k_1,k_2$, and $k_3$.

Recall the detail formula of $\mathfrak{G}^\tau_{\mu,\nu;\mu',\nu'}$ in (4.79). By multilinear estimate, we put all $SU$ in $L^2$ and all $U$ in $L^\infty$. As a result, the lose is of size $|\xi|^{-p}|\xi-\eta|^{-p}=2^{-pk-pk_1}$. From (4.83), we can see that the loss of $2^{-pk-pk_1}$ can be covered by the symbol if $2^{k_1}\gtrsim 2^{\mathrm{Smin}\{k,k_1,k_2,k_3\}}$.

Recall that, due to the time dependent cutoff function of $|\xi-\eta|$ in (4.76), we have $|\xi-\eta|\geq (1+t)^{-5/4-3p_0}$. Again, from (4.83), we know that the loss can be covered by the symbol if $2^{(1/2-p)k}$ is less than $2^{-pk_1}\sim|\xi-\eta|^{-p}\leq(1+t)^{1/4+p_0}$. Hence, we can further rule out the case when $|\xi|\sim 2^k\leq(1+t)^{-5/6-5p_0}$.

To sum up, the quartic terms inside $\mathfrak{G}^\tau_{\mu,\nu;\mu',\nu'}$ are only problematic if $|\xi-\eta|$ is the smallest number and not comparable with the second smallest number among $\xi$, $\xi-\eta$, $\eta-\sigma$, and $\sigma$ and $|\xi|\sim 2^k\gtrsim(1+t)^{-5/6-5p_0}$.

For this problematic scenario, a key observation is that the size of phases in this case is greater than the second smallest number instead of the smallest number among $\xi$, $\xi-\eta$, $\eta-\sigma$, and $\sigma$. More precisely, the following estimate holds,

$$\big|\mu|\xi|^{1/2}-\nu|\xi-\eta|^{1/2}-\mu'|\eta-\sigma|^{1/2}-\nu'|\sigma|^{1/2}\big|\theta(\xi-\eta,\eta-\sigma)\theta(\xi-\eta,\sigma)\theta(\xi-\eta,\xi)$$

$$\gtrsim \min\{|\xi|,|\eta-\sigma|,|\sigma|\}^{1/2},\quad \mu,\nu,\mu',\nu'\in\{+,-\}.\tag{4.84}$$

Above estimate and a similar type estimate will be used in the $L^\infty$ decay estimate part, hence we postpone the prove of (4.84) to the end of subsubsection 4.2.2.

As the size of phase is not small, we can dividing the phase again to gain another $t^{-1/2}$ decay with the price of $2^{-\mathrm{Smin}\{k,k_1,k_2,k_3\}/2}$, which can be covered from the symbol, see (4.83). This observation motives us to define the following quartic correction terms, which cancel out the problematic quartic terms inside $\mathfrak{G}^{\tau}_{\mu,\nu;\mu',\nu'}$,

$$E^{S,low2}_{modi}(t)=\sum_{\tau=p,N_1}\sum_{\mu,\nu,\mu',\nu'\in\{+,-\}}\mathfrak{Re}\Big[\int\overline{\widehat{(SU)_\mu}(\xi)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_{\mu'}}(\eta-\sigma)\widehat{U_{\nu'}}(\sigma) \tag{4.85}$$

$$\times\tilde{e}^{\tau,1}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)+\overline{\widehat{U_\mu}(\xi-\sigma)}\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_{\mu'}}(\eta)\widehat{U_{\nu'}}(\sigma)\tilde{e}^{\tau,2}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)d\sigma d\eta d\xi\Big],$$

where for $i\in\{1,2\}$,

$$\tilde{e}^{\tau,i}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)=\frac{e^{\tau,i}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)\theta(\xi-\eta,\min\{|\sigma|,|\eta-\sigma|,|\xi|\})\psi_{\geq 0}(\xi(1+t)^{5/6+5p_0})}{\mu|\xi|^{1/2}-\nu|\xi-\eta|^{1/2}-\mu'|\eta-\sigma|^{1/2}-\nu'|\sigma|^{1/2}}.$$

From (4.84), we can see that the loss from the denominator of $\tilde{e}^{\tau,i}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)$ can be covered by the size of $e^{\tau,i}_{\mu,\nu}(t,\xi,\eta,\sigma)$, $i\in\{1,2\}$. As a result, from (4.71), (4.80), (4.81), (4.77), (4.78), Lemma 1.3.8, the following estimate holds,

$$\|\tilde{e}^{\tau,1}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}+\|\tilde{e}^{\tau,2}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}\lesssim 2^{(2\tau+2)\max\{k_1,k_2,k_3\}}, \tag{4.86}$$

$$\begin{aligned}&\|\partial_t\tilde{e}^{\tau,1}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}+\|\partial_t\tilde{e}^{\tau,2}_{\mu,\nu;\mu',\nu'}(t,\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}\\&\lesssim(1+t)^{-1}2^{(2\tau+2)\max\{k_1,k_2,k_3\}}.\end{aligned} \tag{4.87}$$

Note that $|\xi|^{-p}|\xi-\eta|^{-p}\lesssim(1+t)^{5/12+2p_0}\lesssim(1+t)^{1/2}$ inside the size of support, hence the extra gain of $t^{-1/2}$ is sufficient to close the argument.

From above discussions, it motivates us to define the total correction terms for the low frequency part as follows,

$$E^{S,low}_{modi}(t):=E^{S,low1}_{modi}(t)+E^{S,low2}_{modi}(t). \tag{4.88}$$

**Lemma 4.1.7.** *Under the bootstrap assumption* (4.12)*, we have*

$$\sup_{t\in[0,T]}|E^{S,low}_{modi}(t)|\lesssim\epsilon_0^2, \tag{4.89}$$

$$\Big|\frac{dE^{S,low}_{modi}(t)}{dt}-\sum_{\tau=p,N_1}\mathfrak{Re}\big[\int\overline{\partial_x^\tau SU^1}\partial_x^\tau\mathfrak{Q}_2^S+\overline{\partial_x^\tau SU^2}\partial_x^\tau\mathfrak{Q}_4^S\big]\Big|\lesssim(1+t)^{-1+2p_0}\epsilon_1^3. \tag{4.90}$$

*Proof.* Recall (4.75). From (4.77) and the $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, the following estimate holds,

$$|E^{S,low1}_{modi}(t)|\lesssim\|SU(t)\|^2_{H^{N_1,p}}\|U(t)\|_{W^3}\sum_{k\in\mathbb{Z},2^k\geq(1+t)^{-5/4-3p_0}}2^{-pk}\lesssim(1+t)^{-1/4+10p_0}\epsilon_1^3\lesssim\epsilon_0^2. \tag{4.91}$$

Recall (4.85). From (4.86) and the $L^2-L^\infty-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, we have

$$\begin{aligned}|E^{S,low2}_{modi}(t)|&\lesssim\sum_{2^{k_1}\geq(1+t)^{-5/4-3p_0}}\sum_{2^k\geq(1+t)^{-5/6-5p_0}}2^{-pk_1-pk}(\|SU\|_{H^{N_1,p}}+\|U\|_{H^{N_0,p}})^2\|U\|^2_{W^3}\\&\lesssim(1+t)^{-1/2}\epsilon_1^4\lesssim\epsilon_0^2.\end{aligned} \tag{4.92}$$

From (4.91) and (4.92), we can see the desired estimate (4.89) holds. From (4.74), (4.86) and (4.87), the following estimate holds,

$$\begin{aligned}\text{L.H.S. of (4.90)}\ &\lesssim\sum_{\tau=p,N_1}\sum_{\mu,\nu,\kappa\in\{+,-\}}|\mathfrak{G}^{\kappa,\tau}_{\mu,\nu}|+\Big|\sum_{\tau=p,N_1}\sum_{\mu,\nu,\mu',\nu'\in\{+,-\}}\mathfrak{G}^{\tau}_{\mu,\nu;\mu',\nu'}+\frac{d}{dt}E^{S,low2}_{modi}(t)\Big|\\&+\sum_{\mu,\nu,\kappa\in\{+,-\}}\Big|\int\widehat{(SU)_\mu}(\xi)\widehat{(SU)_\nu}(\xi-\eta)\widehat{U_\kappa}(\eta)p^{\kappa,\tau}_{\mu,\nu}(\xi-\eta,\eta)\psi_{\leq0}((1+t)^{5/4+3p_0}|\xi-\eta|)d\eta d\xi\Big|\\&\lesssim\|SU\|^2_{H^{N_1,p}}[(1+t)^{-1/2}\|U\|_{W^3}+\|U\|_{H^{N_0,p}}(1+t)^{-(1+2p_0)}]+(\|SU\|_{H^{N_1,p}}+\|U\|_{H^{N_0,p}})^2\times\\&\|U\|^2_{W^3}+\sum_{2^{k_1}\geq(1+t)^{-5/4-3p_0}}\sum_{2^k\geq(1+t)^{-5/6-5p_0}}2^{-pk_1-pk}\big(\|\Lambda_{\geq2}[\partial_tSU]\|_{H^{N_1,p}}\|SU\|_{H^{N_1,p}}\|U\|^2_{W^3}\\&+\|SU\|^2_{H^{N_1,p}}\|U\|_{W^2}\|\Lambda_{\geq2}[\partial_tU]\|_{W^2}\big)+\sum_{2^{k_1}\geq(1+t)^{-5/4-3p_0}}2^{-pk_1}\|SU\|^2_{H^{N_1,p}}\|U\|_{W^3}\|\Lambda_{\geq2}[\partial_tU]\|_{W^3}\\&\lesssim(1+t)^{-1+2p_0}\epsilon_1^3.\end{aligned} \tag{4.93}$$

In above estimate, we used the following fact, which is derived from Sobolev embedding,

$$\|\Lambda_{\geq2}[\partial_tU]\|_{W^3}\lesssim\|U\|_{W^3}\|U\|_{W^4}\lesssim\|U\|^{3/2+10p_0}_{W^{N_2}}\|U\|^{1/2-2p_0}_{H^{N_0,p}}.$$

□

## 4.2 Improved dispersion estimate

The goal of this section is to derive the sharp $L^\infty$-type estimate under the bootstrap assumptions (4.12), hence finish the bootstrap argument. From the main result in the energy estimate part, the following estimate holds,

$$\sup_{t\in[0,T]}(1+t)^{-p_0}\left[\|(U^1,U^2)\|_{H^{N_0,p}}+\|S(U^1,U^2)\|_{H^{N_1,p}}\right]\lesssim\epsilon_0. \tag{4.94}$$

From the improved energy estimate (4.94), we can first rule out the very low frequency case and very high frequency case as follows,

$$\sum_{2^k\leq(1+t)^{-5/3-2p_0}\text{ or }2^k\geq(1+t)^{10/89+p_0}}\|P_k(U^1,U^2)\|_{W^{N_2}}$$

$$\lesssim\sum_{2^k\leq(1+t)^{-5/3-2p_0}\text{ or }2^k\geq(1+t)^{10/89+p_0}}2^{k/2+N_2k_+}\|P_k(U^1,U^2)\|_{L^2}\lesssim(1+t)^{-1/2}\epsilon_0. \tag{4.95}$$

To derive the sharp decay estimate for the case when $(1+t)^{-5/3-2p_0}\leq|\xi|\leq(1+t)^{10/89+p_0}$, we use the following auxiliary $Z$-normed space

$$\|h\|_Z:=\||\xi|^\beta(1+|\xi|^\gamma)h(\xi)\|_{L^\infty_\xi},\quad \beta=\frac{3}{4}-p_0,\gamma=N_2+2p_0, \tag{4.96}$$

and the linear decay estimate (4.97) in Lemma 5.0.5. To simplify notations, we define $\Lambda:=|\nabla|^{1/2}$ and $\Lambda(\xi):=|\xi|^{1/2}$ throughout this section.

**Lemma 4.2.1.** *For any $t,|t|\geq1$, $k\in\mathbb{Z}$, $\epsilon\in(0,1/2]$ and $f\in L^2(\mathbb{R})$, the following estimate holds,*

$$\|e^{it\Lambda}P_kf\|_{L^\infty}\lesssim|t|^{-\frac{1}{2}}2^{3k/4}\|\widehat{f}\|_{L^\infty}+|t|^{-(1/2+\epsilon/2)}2^{(1-\epsilon)k/4}[2^k\|\partial_\xi\widehat{f}\|_{L^2}+\|f\|_{L^2}]. \tag{4.97}$$

*Proof.* Note that estimate (4.97) is scaling invariant, it is sufficient to prove it for the case when $k=0$. The proof is very similar to the proof of linear decay estimates in [24, 27, 28]. With minor modifications, one can rerun the argument used in [24] to derive (4.97) with any problems. □

The strategy that we will use to get the sharp decay estimate for $U^1$ and $U^2$ is as follows: (i) Recall (4.16). We first derive the equation satisfied by the normal form transformation($V_1$ and $V_2$) of $U^1$ and $U^2$, which is cubic and higher. Then we show that the decay rates of $(U^1,U^2)$ and $(V_1,V_2)$ are comparable. (ii) We prove that the $Z$-norm of the profile of $(V_1,V_2)$ does not grow with respect to time and of size $\epsilon_0$. Hence $V_1$ and $V_2$ decay sharply from Lemma 4.2.1. As the decay rate of

$(U^1, U^2)$ is same as $(V_1, V_2)$, we know that $U^1$ and $U^2$ also decay sharply.

Recall (4.16). Define $V = V_1 + i\,V_2$ and $\mathcal{S} := \{(+,+,+),(+,+,-),(+,-,-),(-,-,-)\}$. As the equation satisfied by $V$ is cubic and higher, we can write it as follows,

$$(\partial_t + i\Lambda)V = \sum_{(\iota_1,\iota_2,\iota_3)\in\mathcal{S}} C_{\iota_1,\iota_2,\iota_3}(V^{\iota_1}, V^{\iota_2}, V^{\iota_3}) + R(t), \tag{4.98}$$

where $R(t)$ represents the quintic and higher order terms. Define $f(t) := e^{it\Lambda}V$.

**Lemma 4.2.2.** *Under the smallness conditions* (4.12) *and* (4.94)*, the following estimates hold for* $t \in [0,T]$*,*

$$\|(V_1 - U_1, V_2 - U_2)(t)\|_{W^{N_2}} \lesssim (1+t)^{-5/8}\epsilon_1^2, \quad \|V(t)\|_{W^{N_2}} \lesssim (1+t)^{-1/2}\epsilon_1, \tag{4.99}$$

$$\sup_{k\in\mathbb{Z}} \left[2^{k_-/5+7k_+}\|P_k f(t)\|_{L^2} + 2^{k_-/5}\|\xi\partial_\xi \widehat{f}(\xi,t)\psi_k(\xi)\|_{L^2}\right] \lesssim (1+t)^{p_0}\epsilon_0. \tag{4.100}$$

*Proof.* From estimate (4.21), bilinear estimate (1.13) in Lemma 1.3.7, and Sobolev embedding, our desired estimate (4.99) holds as follows,

$$\begin{aligned}
&\|A_1(U^1,U^1)\|_{W^{N_2}} + \|A_2(U^2,U^2)\|_{W^{N_2}} + \|B(U^1,U^2)\|_{W^{N_2}} \\
\lesssim &\sum_{k\in\mathbb{Z}} 2^{5k/4+N_2k_+}\|(P_{\le k}U^1, P_{\le k}U^2)\|_{W^0}\|(P_kU^1,P_kU^2)\|_{L^\infty}^{1/2}\|(P_kU^1,P_kU^2)\|_{L^2}^{1/2} \\
&+ \sum_{k\in\mathbb{Z}}\sum_{k_1\le k} 2^{k_1/4+k+N_2k_{1,+}}\|P_kU^1\|_{L^4}\|P_kU^2\|_{L^\infty} \\
\lesssim &\|(U^1,U^2)\|_{H^{N_0,p}}^{1/2}\|(U^1,U^2)\|_{W^{N_2}}^{3/2} \lesssim (1+t)^{-5/8}\epsilon_1^2,
\end{aligned}$$

We also have the following $L^2$- type estimate,

$$\begin{aligned}
&\|A_1(U^1,U^1)(t)\|_{H^{N_0-1,p}} + \|A_2(U^2,U^2)(t)\|_{H^{N_0-1,p}} + \|B(U^1,U^2)(t)\|_{H^{N_0-1,p}} \\
\lesssim &\|(U^1,U^2)(t)\|_{H^{N_0,p}}\|(U^1,U^2)(t)\|_{W^0} \lesssim (1+t)^{-(1/2-p_0)}\epsilon_1^2,
\end{aligned}$$

which further gives us the following estimate,

$$\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}}(1+t)^{-p_0}2^{k_-/5+7k_+}\|P_kf(t)\|_{L^2} \lesssim \epsilon_0. \tag{4.101}$$

Recall that $A(\cdot,\cdot)$ is a symmetric bilinear operator, the following estimate holds for fixed $k \in \mathbb{Z}$,

$$\|P_k[SA_1(U^1,U^1)]\|_{L^2} \lesssim \|P_k[A_1(SU^1,U^1)]\|_{L^2} + \|P_k[A_1(U^1,U^1)]\|_{L^2}$$

$$\lesssim \sum_{k_1\le k-4} 2^k \big[\|P_kU^2\|_{L^2}\|P_{k_1}(SU^1)\|_{L^\infty}\|P_kU^1\|_{L^\infty}\|P_{k_1}(SU^1)\|_{L^2}\big]^{1/2}$$

$$+2^k\|P_k(SU^1,U^1)\|_{L^2}\|U^1\|_{W^0} \lesssim \sum_{k_1\le k-4} 2^{k_1/4}2^k\|P_{k_1}(SU^1)\|_{L^2}\|P_kU^1\|_{L^2}^{1/2}\|P_kU^1\|_{L^\infty}^{1/2}$$

$$+(1+|t|)^{-1/2+p_0}\epsilon_1^2 \lesssim (1+t)^{-1/4+2p_0}\epsilon_1^2 \lesssim \epsilon_0.$$

Following the same argument, we can show that

$$\sup_{k\in\mathbb{Z}} \|P_k[SA_2(U^2,U^2)]\|_{L^2} + \|P_k[SB(U^1,U^2)]\|_{L^2} \lesssim \epsilon_0. \tag{4.102}$$

Note that $\widehat{SV}(\xi) = e^{-it|\xi|^{1/2}}(t\partial_t - 2\xi\partial_\xi - 2)\widehat{f}(t,\xi)$. Hence

$$\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}}(1+t)^{-p_0}2^{k_-/5}\|(t\partial_t - 2\xi\partial_\xi)\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^2} \lesssim \epsilon_0. \tag{4.103}$$

From the uniform upper bound of cubic symbols in (4.106), we can see that

$$\|t\partial_t f\|_{L^2} \lesssim t\sum_{k_1,k_2,k_3\in\mathbb{Z}} 2^{\mathrm{med}\{k_i\}/2}2^{2\max\{k_i\}}\|P_{\min\{k_i\}}(e^{-it\Lambda}f)\|_{L^\infty}\|P_{\mathrm{med}\{k_i\}}(e^{-it\Lambda}f)\|_{L^\infty}\times$$

$$\|P_{\max\{k_i\}}f\|_{L^2} \lesssim t\|V(t)\|_{W^{1/2}}^2\|V\|_{H^{2,p}} \lesssim (1+|t|)^{p_0}\epsilon_1^3 \lesssim (1+|t|)^{p_0}\epsilon_0. \tag{4.104}$$

Combining (4.101), (4.103) and (4.104), it's easy to see the desired estimate (4.100) holds. With minor modifications, we can also derive the following estimate,

$$\sup_{(1+t)^{-5/3-2p_0}\le 2^k\le(1+t)^{10/89+p_0}} 2^{k_-/5+N_1k}\|\xi\partial_\xi\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^2} \lesssim (1+t)^{p_0}\epsilon_0. \tag{4.105}$$

□

From (4.100) and (4.105), the following estimate holds,

$$\sum_{(1+t)^{-5/3-2p_0}\le 2^k\le(1+t)^{10/89+p_0}} 2^{k/8+N_2k_+}[2^k\|\partial_\xi\widehat{f}\psi_k(\xi)\|_{L^2} + \|P_kf\|_{L^2}] \lesssim (1+|t|)^{1/4-2p_0}\epsilon_0.$$

Hence, from above estimate and (4.97) in Lemma 5.0.5, it is sufficient to derive the improved $Z$-norm

estimate of $\widehat{f}(t,\xi)$ to derive the sharp decay estimate for $V$.

### 4.2.1 Set-up of the $Z$-norm estimate

After writing the cubic terms on the Fourier side and do Littlewood-Paley decomposition for all inputs, we have

$$\mathcal{F}(e^{it\Lambda}C_{\iota_1,\iota_2,\iota_3}(V^{\iota_1},V^{\iota_2},V^{\iota_3}))(\xi):=iI^{\iota_1,\iota_2,\iota_3}(t,\xi)=i\sum_{k_1,k_2,k_3\in\mathbb{Z}}I^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}(t,\xi),$$

$$I^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}(t,\xi)=\int_{\mathbb{R}\times\mathbb{R}}e^{it\,\Phi_{\iota_1,\iota_2,\iota_3}(\xi,\eta,\sigma)}\,c^{\iota_1,\iota_2,\iota_3}(\xi-\eta,\eta-\sigma,\sigma)\widehat{f^{\iota_1}_{k_1}}(t,\xi-\eta)\times$$

$$\widehat{f^{\iota_2}_{k_2}}(t,\eta-\sigma)\widehat{f^{\iota_3}_{k_3}}(t,\sigma)d\eta d\sigma,$$

where the phase $\Phi_{\iota_1,\iota_2,\iota_3}(\cdot,\cdot,\cdot)$ is defined as follows,

$$\Phi_{\iota_1,\iota_2,\iota_3}(\xi,\eta,\sigma)=\Lambda(\xi)-\iota_1\Lambda(\xi-\eta)-\iota_2\Lambda(\eta-\sigma)-\iota_3\Lambda(\sigma),\quad(\iota_1,\iota_2,\iota_3)\in\mathcal{S}.$$

The following Lemma on the $\mathcal{S}^\infty$ norm of symbols $c^{\iota_1,\iota_2,\iota_3}(\cdot,\cdot,\cdot)$ holds,

**Lemma 4.2.3.** *For any $(\iota_1,\iota_2,\iota_3)\in\mathcal{S}$, $k,k_1,k_2,k_3\in\mathbb{Z}$, the following estimate holds,*

$$\|c^{\iota_1,\iota_2,\iota_3}(\xi-\eta,\eta-\sigma,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}}\lesssim 2^{\mathrm{med}\{k_1,k_2,k_3\}/2}2^{2\max\{k_1,k_2,k_3\}},\tag{4.106}$$

*where* $\mathrm{med}\{k_1,k_2,k_3\}$ *denotes the medium number among $k_1,k_2$, and $k_3$.*

*Proof.* Postponed to Appendix 4.3.3. □

Same as in [27], to successfully close the argument and see the modified scattering property, we need to modified the phase of the profile first. We define

$$c^*(\xi,x,y):=c^{+,+,-}(\xi+x,\xi+y,-\xi-x-y),\quad\widetilde{c}(\xi):=-8\pi|\xi|^{3/2}c^*(\xi,0,0),\tag{4.107}$$

and the modified phase is defined as

$$L(t,\xi):=\widetilde{c}(\xi)\int_0^t|\widehat{f}(s,\xi)|^2\frac{d\,s}{1+s},\quad g(t,\xi):=e^{iL(t,\xi)}\widehat{f}(t,\xi),\tag{4.108}$$

hence

$$\partial_t g(\xi,t)=ie^{iL(t,\xi)}\big[I^{+,+,-}(t,\xi)+\widetilde{c}(\xi)\frac{|\widehat{f}(t,\xi)|^2}{1+t}\widehat{f}(t,\xi)\big]+$$

$$ie^{iL(t,\xi)}\Big[I^{+,+,+}(t,\xi)+I^{+,-,-}(t,\xi)+I^{-,-,-}(t,\xi)\Big]+e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi). \tag{4.109}$$

From (4.109), we can see that the modified phase is only effective for the cubic term $I^{+,+,-}(t,\xi)$, which is the only cubic term that has non empty space-time resonance set.

Now, our main goal is to prove the following proposition.

**Proposition 4.2.4.** *Under the bootstrap assumption* (4.12) *and the following assumption*

$$\sup_{t\in[0,T']}\|\widehat{f}(t)\|_Z\lesssim\epsilon_1,\quad T'\in[0,T], \tag{4.110}$$

*there exist $p_1>0$ such that the following estimate holds for any $m\in\mathbb{N}$ and any $t_1,t_2\in[2^{m-1},2^{m+1}]\subset[0,T']$.*

$$\||\xi|^{3/4-p_0}(1+|\xi|^{N_2+2p_0})\big(g(t_2,\xi)-g(t_1,\xi)\big)\|_{L^\infty_\xi}\lesssim 2^{-p_1m}\epsilon_0, \tag{4.111}$$

*Hence, we have $T'=T$ and*

$$\sup_{t\in[0,T]}\|\widehat{f}(t)\|_Z=\sup_{t\in[0,T]}\|g(t)\|_Z\lesssim\epsilon_0,\quad \sup_{t\in[0,T]}(1+t)^{1/2}\|V(t)\|_{W^{N_2}}\lesssim\epsilon_0. \tag{4.112}$$

Now we restrict ourself inside the time interval $[2^{m-1},2^{m+1}]\subset[0,T']$ and reformulate estimates (4.99), (4.100), and (4.110) as follows,

$$\sup_{k\in\mathbb{Z}}\|P_kf\|_{L^2}\lesssim\epsilon_0 2^{p_0m}2^{-k_-/5-7k_+},\quad \sup_{k\in\mathbb{Z}}\|\mathcal{F}(P_kf)(\xi)\|_{L^\infty_\xi}\lesssim\epsilon_1 2^{-\beta k}2^{-\gamma k_+}, \tag{4.113}$$

$$\sup_{k\in\mathbb{Z}}\|\xi\partial_\xi\widehat{f}(\xi)\psi_k(\xi)\|_{L^2}\lesssim\epsilon_0 2^{p_0m}2^{-k_-/5},\quad \sup_{k\in\mathbb{Z}}\|e^{-it\Lambda}P_kf\|_{L^\infty_x}\lesssim\epsilon_1 2^{-m/2}2^{-N_2k_+}. \tag{4.114}$$

From estimate (1.16) in Lemma 1.3.9, the following estimates hold when $|\xi|\le 2^{-21p_0m}$ or $|\xi|\ge 2^{20p_0m}$,

$$\begin{aligned}&\sup_{|\xi|\le 2^{-21p_0m}\text{or}|\xi|\ge 2^{20p_0m}}||\xi|^{3/4-p_0}(1+|\xi|^{N_2+2p_0})g(t,\xi)|\\ =&\sup_{|\xi|\le 2^{-21p_0m}\text{or}|\xi|\ge 2^{20p_0m}}||\xi|^{3/4-p_0}(1+|\xi|^{N_2+2p_0})f(t,\xi)|\lesssim\sup_{k\le -21p_0m}\epsilon_0 2^{(1/4-p_0)k-k/5}2^{p_0m}\\ &+\sup_{k\ge 20p_0m}\epsilon_0 2^{(1/4+N_2+p_0)k-(N_0-1)k/2}2^{p_0m}\lesssim\epsilon_0 2^{-p_0m}.\end{aligned}$$

Hence, to prove (4.111), it remains to consider the case when $2^{-21p_0m}\le|\xi|\le 2^{20p_0m}$. For this case, we need to use the equation satisfied by the modified profile "$g(t,\xi)$". Recall (4.109), we have

the following identity,

$$g(t_2,\xi)-g(t_1,\xi)=\sum_{k_1,k_2,k_3\in\mathbb{Z}}\sum_{(\iota_1,\iota_2,\iota_3)\in\mathcal{S}}J^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}+\int_{t_1}^{t_2}e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)dt,$$

$$J^{+,+,-}_{k_1,k_2,k_3}:=\int_{t_1}^{t_2}ie^{iL(t,\xi)}\big[I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)+\widetilde{c}(\xi)\frac{\widehat{f_{k_1}}(t,\xi)\widehat{\overline{f}}_{k_3}(t,-\xi)}{1+t}\widehat{f_{k_2}}(t,\xi)\big]ds,$$

$$J^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}:=\int_{t_1}^{t_2}ie^{iL(t,\xi)}I^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}(t,\xi)dt,(\iota_1,\iota_2,\iota_3)\in\{(+,+,+),(+,-,-),(-,-,-)\}.$$

The argument naturally splits into two parts : $Z$-norm estimate for the cubic terms and $Z$-norm estimate for the remainder term.

### 4.2.2 $Z$-norm estimate for the cubic terms

The goal of this subsection is to prove the following proposition,

**Proposition 4.2.5.** *For $t_1,t_2\in[2^{m-1},2^{m+1}]\subset[0,T']$, $|\xi|:=2^k\in[2^{-21p_0m},2^{20p_0m}]$, the following estimates hold under the bootstrap assumptions* (4.12) *and* (4.110).

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}}\sum_{(\iota_1,\iota_2,\iota_3)\in\mathcal{S}}\|J^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}\psi_k(\xi)\|_Z\lesssim 2^{-p_0m}\epsilon_1^3. \tag{4.115}$$

**Lemma 4.2.6.** *Under the assumptions in Proposition* 4.2.5*, the following estimate holds if $k_1,k_2,k_3\in[k-10,k+10]$,*

$$\|J^{+,+,-}_{k_1,k_2,k_3}\psi_k(\xi)\|_Z\lesssim 2^{-2p_0m}. \tag{4.116}$$

*Proof.* The idea of proof is very similar to the proof of [24][Lemma 6.4]. Because the setting of function spaces is changed, and also for self-completeness, we still give a detailed proof here.

We first do change of variables to transform $(\xi-\eta,\eta-\sigma,\sigma)$ into $(\xi+\eta,\xi+\sigma,-\xi-\eta-\sigma)$, hence near the critical point $(\xi,\xi,-\xi)$ ( corresponds to the space-time resonance set), $(\eta,\sigma)=(0,0)$. After changing of variables and decomposing $I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)$ with the threshold $\bar{l}=-(1-100p_0)m/2+3k/4$, we have the following decomposition,

$$I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)=\sum_{l_1,l_2=\bar{l}}^{k+10}J_{l_1,l_2}(t,\xi), \tag{4.117}$$

$$J_{l_1,l_2}(t,\xi)=\int_{\mathbb{R}^2}e^{it\Phi(\xi,\eta,\sigma)}\widehat{f}_{k_1}(t,\xi+\eta)\widehat{f}_{k_2}(t,\xi+\sigma)\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)$$

$$c^*(\xi,\eta,\sigma)\psi^{\bar{l}}_{l_1}(\eta)\psi^{\bar{l}}_{l_2}(\sigma)d\eta d\sigma,$$

where $c^*(\cdot,\cdot,\cdot)$ is defined in (4.107), the phase $\Phi(\xi,\eta,\sigma)$ is defined as follows,

$$\Phi(\xi,\eta,\sigma) := \Lambda(\xi) - \Lambda(\xi+\eta) - \Lambda(\xi+\sigma) + \Lambda(-\xi-\eta-\sigma),$$

and the cutoff function $\psi^{l_1}_{l_2}(\cdot)$ is defined as follows,

$$\psi^{l_1}_{l_2}(\xi) = \begin{cases} \psi_{l_2}(\xi) & \text{if} l_2 > l_1 \\ \psi_{\leq l_1}(\xi) & \text{if} l_2 = l_1. \end{cases}$$

**Case 1:** We first consider the case when $l_2 \geq \max\{l_1, \bar{l}+1\}$. For this case, we have

$$|\partial_\eta \Phi(\xi,\eta,\sigma)|\psi^{\bar{l}}_{l_2}(\sigma) = |\Lambda'(\xi+\eta+\sigma) - \Lambda'(\xi+\eta)|\psi^{\bar{l}}_{l_2}(\sigma) \gtrsim 2^{l_2}2^{-3k/2}. \tag{4.118}$$

After integrating by parts in $\eta$, we can derive the following:

$$|J_{l_1,l_2}(t,\xi)| \lesssim 2^{-m}\big[|J^1_{l_1,l_2}(t,\xi)| + |J^2_{l_1,l_2}(t,\xi)|\big], \tag{4.119}$$

where

$$J^1_{l_1,l_2}(t,\xi) = \int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta,\sigma)}\widehat{f}_{k_1}(t,\xi+\eta)\widehat{f}_{k_2}(t,\xi+\sigma)\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)\partial_\eta r_1(\xi,\eta,\sigma)\,d\eta d\sigma,$$

$$J^2_{l_1,l_2}(\xi,s) = \int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta,\sigma)}\partial_\eta\big(\widehat{f}_{k_1}(t,\xi+\eta)\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)\big)\widehat{f}_{k_2}(t,\xi+\sigma)r_1(\xi,\eta,\sigma)d\eta d\sigma,$$

where

$$r_1(\xi,\eta,\sigma) := \frac{c^*(\xi,\eta,\sigma)\psi^{\bar{l}}_{l_1}(\eta)\psi^{\bar{l}}_{l_2}(\sigma).}{\partial_\eta\Phi(\xi,\eta,\sigma)}.$$

From Lemma 1.3.8, (4.106) in Lemma 4.2.3, and (4.118), the following estimate holds,

$$\|r_1(\xi,\eta,\sigma)\psi_k(\xi)\|_{\mathcal{S}^\infty} \lesssim 2^{-l_2+4k}, \quad \|r_1(\xi,\eta,\sigma)\psi_k(\xi)\|_{\mathcal{S}^\infty} \lesssim 2^{-l_2-l_1+4k} + 2^{-l_2+3k}. \tag{4.120}$$

Therefore, from (4.120) and the $L^2-L^2-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, the following estimates hold,

$$|J^1_{l_1,l_2}(t,\xi)| \lesssim \big(2^{-l_2-l_1}2^{4k} + 2^{-l_2+3k}\big)\|\widehat{f_{k_1}}(t,\xi+\eta)\psi^{\bar{l}}_{l_1}(\eta)\|_{L^2}\|\widehat{f}_{k_2}(t,\xi+\sigma)\psi^{\bar{l}}_{l_2}(\sigma)\|_{L^2}$$

$$\times\|e^{-it\Lambda}P_{k_3}f\|_{L^\infty}\lesssim\big(2^{-l_2-l_1}2^{4k}+2^{-l_2+3k}\big)2^{(l_1+l_2)/2}\|\widehat{f}_k(t,\xi)\|_{L^\infty}^2 2^{-m/2-N_2k_+}\epsilon_1$$

$$\lesssim 2^{-50p_0m}\epsilon_1^3 2^{-6k_+},$$

$$|J^2_{l_1,l_2}(t,\xi)|\lesssim\sum_{\{i,j\}=\{1,3\}}2^{-l_2+4k}\|\partial_\xi\widehat{f}(\xi,s)\psi_{k_i}(\xi)\|_{L^2}\|P_{k_j}f\|_{L^2}\|e^{-it\Lambda}P_{k_2}f\|_{L^\infty}$$

$$\lesssim 2^{-l_2-m/2+2p_0m}2^{(3-2p)k-(N_2+N_0-1)k_+}\epsilon_1^3\lesssim 2^{-50p_0m}2^{-6k_+}\epsilon_1^3.$$

To sum up, we have

$$|J_{l_1,l_2}(t,\xi)|\lesssim\epsilon_1^3 2^{-(1+50p_0)m}2^{-6k_+}. \tag{4.121}$$

The symmetric case when $l_1\geq\max\{l_2,\bar{l}\}$ can be handled very similarly.

**Case 2:** It's remained to consider the case when $l_1=l_2=\bar{l}$. It is easy to see the following estimate holds,

$$|J_{\bar{l},\bar{l}}(\xi,s)+\frac{\tilde{c}(\xi)\widehat{f}_{k_1}(t,\xi)\widehat{f}_{k_2}(t,\xi)\widehat{\overline{f}}_{k_3}(t,-\xi)}{t+1}|\lesssim|\mathcal{I}_1|+|\mathcal{I}_2|+|\mathcal{I}_3|, \tag{4.122}$$

where

$$\mathcal{I}_1=\int_{\mathbb{R}^2}\big(e^{it\Phi(\xi,\eta,\sigma)}-e^{-it\eta\sigma/(4|\xi|^{3/2})}\big)\widehat{f}_{k_1}(t,\xi+\eta)\widehat{f}_{k_2}(t,\xi+\sigma)$$

$$\times\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)c^*(\xi,\eta,\sigma)\psi_{\bar{l}}^{\bar{l}}(\eta)\psi_{\bar{l}}^{\bar{l}}(\sigma)\,d\eta d\sigma$$

$$\mathcal{I}_2=\int_{\mathbb{R}^2}e^{-it\eta\sigma/(4|\xi|^{3/2})}\big[\widehat{f}_{k_1}(t,\xi+\eta)\widehat{f}_{k_2}(t,\xi+\sigma)\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)c^*(\xi,\eta,\sigma)-$$

$$\widehat{f}_{k_1}(t,\xi)\widehat{f}_{k_2}(t,\xi)\widehat{\overline{f}}_{k_3}(t,-\xi)c^*(\xi,0,0)\big]\psi_{\bar{l}}^{\bar{l}}(\eta)\psi_{\bar{l}}^{\bar{l}}(\sigma)\,d\eta d\sigma,$$

$$\mathcal{I}_3=\int_{\mathbb{R}^2}e^{-is\eta\sigma/(4|\xi|^{3/2})}\widehat{f}_{k_1}(t,\xi)\widehat{f}_{k_2}(t,\xi)\widehat{\overline{f}}_{k_3}(t,-\xi)c^*(\xi,0,0)\psi_{\bar{l}}^{\bar{l}}(\eta)\psi_{\bar{l}}^{\bar{l}}(\sigma)\,d\eta d\sigma+$$

$$\frac{\tilde{c}(\xi)\widehat{f}_{k_1}(t,\xi)\widehat{f}_{k_2}(t,\xi)\widehat{\overline{f}}_{k_3}(t,-\xi)}{t+1}.$$

Note that,

$$\Big|\Phi(\xi,\eta,\sigma)+\frac{\eta\sigma}{4|\xi|^{3/2}}\Big|\lesssim 2^{-5k/2}(|\eta|+|\sigma|)^3.$$

Hence, after using estimate (4.106) in Lemma 4.2.3 and the size of support of $\eta$ and $\sigma$, the following estimate holds,

$$|\mathcal{I}_1|\lesssim\epsilon_1^3 2^m 2^{5\bar{l}}2^{-9k/4-3N_2k_+}\lesssim\epsilon_1^3 2^{-6k_+-(3-1000p_0)m/2}\lesssim\epsilon_0 2^{-6k_+}2^{-(1+50p_0)m}. \tag{4.123}$$

Note that

$$\big|\widehat{f_l}(\xi+\rho,s)-\widehat{f_l}(\xi,s)\big|\lesssim|\rho|^{\frac{1}{2}}\|\partial_\xi\widehat{f_l}(s)\|_{L^2},|c^*(\xi,\eta,\sigma)-c^*(\xi,0,0)|\psi_{\bar{l}}^{\bar{l}}(\eta)\psi_{\bar{l}}^{\bar{l}}(\sigma)\lesssim 2^{3k/2}2^{\bar{l}}.$$

From above estimates, the following estimate holds after using estimate (4.106) in Lemma 4.2.3 and the size of support of $\eta$ and $\sigma$,

$$|\mathcal{I}_2|\lesssim 2^{-k_-/5}2^{5\bar{l}/2-2N_2k_++k}\epsilon_1^3+2^{3k/2-9k/4+3\bar{l}-3N_2k_+}\epsilon_1^3\lesssim\epsilon_1^3 2^{-6k_+-(1+50p_0)m}. \tag{4.124}$$

We proceed to estimate $\mathcal{I}_3$. Note the fact (see also [25]) that

$$\int_{\mathbb{R}\times\mathbb{R}}e^{-ixy}\tilde{\psi}(x/N)\tilde{\psi}(y/N)dxdy=2\pi+\mathcal{O}(N^{-1/2}). \tag{4.125}$$

Through scaling, the following estimate holds from (4.125),

$$\Big|\int_{\mathbb{R}^2}e^{-is\eta\sigma/(4|\xi|^{3/2})}\psi_{\bar{l}}^{\bar{l}}(\eta)\psi_{\bar{l}}^{\bar{l}}(\sigma)d\eta d\sigma-\frac{4|\xi|^{3/2}}{s}(2\pi)\Big|\lesssim 2^{-(1+25p_0)m}2^{3k/2}, \tag{4.126}$$

which further gives us

$$|\mathcal{I}_3|\lesssim\epsilon_1^3 2^{7k/4-3N_2k_+}2^{-(1+25p_0)m}\lesssim\epsilon_1^3 2^{-6k_+}2^{-(1+25p_0)m}. \tag{4.127}$$

To sum up, from (4.123), (4.124) and (4.127), the following estimate holds,

$$(4.122)\lesssim\epsilon_1^3 2^{-(1+25p_0)m}2^{-6k_+}.$$

Therefore finishing the proof. □

Now, it remains to consider the case when $k_1\notin[k-10,k+10]$ for some $i\in\{1,2,3\}$. Therefore the term $\widehat{f_{k_1}}(t,\xi)\widehat{\overline{f}}_{k_3}(t,-\xi)\widehat{f_{k_2}}(t,\xi)\psi_k(\xi)$ vanishes. From now on, we drop this term inside $J_{k_1,k_2,k_3}^{+,+,-}$.

**Lemma 4.2.7.** *For any $(\iota_1,\iota_2,\iota_3)\in\mathcal{S}$, we have the following rough estimates:*

$$|I_{k_1,k_2,k_3}^{\iota_1,\iota_2,\iota_3}(t,\xi)|\lesssim 2^{\min\{k_i\}/4}2^{3\mathrm{med}\{k_i\}/4-N_2\mathrm{med}\{k_i\}_+}2^{5\max\{k_i\}/4-N_2\max\{k_i\}_+}\epsilon_1^3, \tag{4.128}$$

$$|I_{k_1,k_2,k_3}^{\iota_1,\iota_2,\iota_3}(t,\xi)|\lesssim 2^{\mathrm{med}\{k_i\}/2}2^{2\max\{k_i\}}\|e^{-it\Lambda}P_{\min\{k_i\}}f\|_{L^\infty}\|P_{\mathrm{med}\{k_i\}}f\|_{L^2}\|P_{\max\{k_i\}}f\|_{L^2}. \tag{4.129}$$

*Proof.* From (4.106) in Lemma 4.2.3, the desired estimate (4.128) follows straightforwardly after

putting all inputs into the $Z$-normed space. From (4.106) in Lemma 4.2.3 and $L^2-L^\infty-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, (4.129) holds straightforwardly. □

Recall that $2^{-21p_0m}\le|\xi|\le 2^{20p_0m}$. From (4.128) and (4.129), the following estimate holds when $\mathrm{med}\{k_i\}\le -(1+100p_0)m$ or $\min\{k_i\}\le -4(1+100p_0)m$ or $\max\{k_i\}\ge(1+100p_0)m/5$,

$$\sup_{t\in[2^{m-1},2^{m+1}]}\big||\xi|^{3/4-p_0}(1+|\xi|^{N_2+2p_0})I^{\iota_1,\iota_2,\iota_3}_{k_1,k_2,k_3}(\xi,t)\big|\lesssim\epsilon_1^3 2^{-(1+p_0)m}, \tag{4.130}$$

which is sufficient for most of cases in our desired estimate (4.115). Therefore, it is sufficient to consider fixed $k_1,k_2$, and $k_3$ in the following range:

$$-(1+100p_0)m\le\mathrm{med}\{k_i\}\le\max\{k_i\}\le(1+100p_0)m/5,\min\{k_i\}\ge-4(1+100p_0)m. \tag{4.131}$$

**Lemma 4.2.8.** *If $k_1,k_2,k_3$ satisfies the estimate* (4.131) *and one of the following two conditions is satisfied:*

$$\max\{|k_1-k|,|k_2-k|,|k_3-k|\}\ge 20,\quad \mathrm{med}\{k_i\}-\min\{k_i\}\le 10,$$

$$\max\{|k_1-k|,|k_2-k|,|k_3-k|\}\ge 20,\quad \min\{k_i\}\ge-(5/7-1000p_0)m,$$

*then the following estimate holds,*

$$\sup_{t\in[2^{m-1},2^{m+1}]}\big||\xi|^{3/4-p_0}(1+|\xi|)^{N_2+2p_0}I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)\big|\psi_k(\xi)\lesssim\epsilon_1^3 2^{-(1+p_0)m}. \tag{4.132}$$

*Proof.* **Case 1:** We first consider the case when $\max\{|k_1-k_3|,|k_2-k_3|\}\ge 5$ and $\min\{k_i\}\ge-(5/7-1000p_0)m$. Note that

$$\partial_\eta\Phi=\Lambda'(\xi+\eta+\sigma)-\Lambda'(\xi+\eta),\quad \partial_\sigma\Phi(\xi,\eta,\sigma)=\Lambda'(\xi+\eta+\sigma)-\Lambda'(\xi+\sigma).$$

From symmetry and without loss of generality, we assume that $|k_1-k_3|\ge 5$. Hence

$$|\partial_\eta\Phi(\xi,\eta,\sigma)|=\big|\Lambda'(\xi+\eta)-\Lambda'(\xi+\eta+\sigma)\big|\gtrsim 2^{-\min\{k_1,k_3\}/2}. \tag{4.133}$$

After integration by parts in $\eta$, we can derive the following estimate:

$$|I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)|\lesssim\big(|\mathcal{J}^1_{k_1,k_2,k_3}|+|\mathcal{J}^2_{k_1,k_2,k_3}|\big), \tag{4.134}$$

$$\mathcal{J}^1_{k_1,k_2,k_3} := \frac{1}{t}\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta,\sigma)}\widehat{f^+_{k_1}}(t,\xi+\eta)\widehat{f^+_{k_2}}(t,\xi+\sigma)\widehat{f^-_{k_3}}(t,-\xi-\eta-\sigma)\partial_\eta r_2(\xi,\eta,\sigma)d\eta d\sigma,$$

$$\mathcal{J}^2_{k_1,k_2,k_3} := \frac{1}{t}\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta,\sigma)}\partial_\eta\big(\widehat{f^+_{k_1}}(t,\xi+\eta)\widehat{f^-_{k_3}}(t,-\xi-\eta-\sigma)\big)\widehat{f^+_{k_2}}(t,\xi+\sigma)r_2(\xi,\eta,\sigma)d\eta d\sigma,$$

where

$$r_2(\xi,\eta,\sigma) = \frac{c^*(\xi,\eta,\sigma)\psi_{k_1}(\xi+\eta)\psi_{k_2}(\xi+\sigma)\psi_{k_3}(\xi+\eta+\sigma)}{\partial_\eta\Phi(\xi,\eta,\sigma)}.$$

From (4.133), (4.106) in Lemma 4.2.3 and Lemma 1.3.8, the following estimates hold,

$$\|r_2(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty} \lesssim 2^{\min\{k_1,k_3\}/2}2^{\mathrm{med}\{k_1,k_2,k_3\}/2}2^{2\max\{k_1,k_2,k_3\}}, \tag{4.135}$$

$$\|\partial_\eta r_2(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty} \lesssim 2^{-\min\{k_1,k_3\}/2}2^{\mathrm{med}\{k_1,k_2,k_3\}/2}2^{2\max\{k_1,k_2,k_3\}}. \tag{4.136}$$

Therefore, from (4.135), (4.136) and $L^2-L^2-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, the following estimates hold,

$$|\mathcal{J}^1_{k_1,k_2,k_3}| \lesssim 2^{-\min\{k_i\}/2+\mathrm{med}\{k_i\}/2+2\max\{k_i\}}2^{-m}\|e^{-is\Lambda}P_{\min}f\|_{L^\infty}\|P_{\mathrm{med}}f\|_{L^2}\|P_{\max}f\|_{L^2}$$

$$\lesssim \epsilon_1^3 2^{2\max\{k_i\}}2^{-(1+100p_0)m}2^{-4\max\{k_1,k_2,k_3\}_+} \lesssim \epsilon_1^3 2^{-(1+100p_0)m}, \tag{4.137}$$

$$|\mathcal{J}^2_{k_1,k_2,k_3}| \lesssim 2^{-m}2^{\min\{k_1,k_3\}/2}2^{\mathrm{med}\{k_1,k_2,k_3\}/2}2^{2\max\{k_1,k_2,k_3\}}\big(\|\partial_\xi\widehat{f}(\xi,s)\psi_{k_1}(\xi)\|_{L^2}$$

$$\times\|P_{\max\{k_2,k_3\}}f\|_{L^2}\|e^{-is\Lambda}P_{\min\{k_2,k_3\}}f\|_{L^\infty} + \|\partial_\xi\widehat{f}(\xi,s)\psi_{k_3}(\xi)\|_{L^2}\|e^{-is\Lambda}P_{\min\{k_1,k_2\}}f\|_{L^\infty}$$

$$\times\|P_{\max\{k_1,k_2\}}f\|_{L^2}\big) \lesssim \epsilon_1^3 2^{-7\min\{k_i\}/10}2^{-3m/2} + \epsilon_1^3 2^{-3m/2}2^{3\max\{k_i\}_+/2} \lesssim \epsilon_1^3 2^{-(1+100p_0)m}.$$

From above estimates, (4.134) and the fact that $|\xi| \le 2^{20p_0m}$, the following estimate holds,

$$\big||\xi|^{3/4-p_0}(1+|\xi|)^{N_2+2p_0}I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)\big| \lesssim 2^{-(1+100)p_0m+80p_0m}\epsilon_1^3 \lesssim 2^{-(1+p_0)m}\epsilon_1^3. \tag{4.138}$$

Therefore, finishing the proof. We remark that when $|k_2-k_3|\ge 5$, we can do integration by parts with respect to $\sigma$ and the argument is very similar.

**Case 2:** Now we proceed to consider the case when $\max\{|k_1-k_3|,|k_2-k_3|\}\le 5$ and $\min \ge -(5/7-1000p_0)m$. Recall that $\max\{|k_1-k|,|k_2-k|,|k_3-k|\}\ge 20$, hence $\min\{k_1,k_2,k_3\}\ge k+10$. As a result, the following estimate holds,

$$|\xi+\eta|\sim|\xi+\sigma|\sim|\xi+\sigma+\eta|\sim 2^{k_1}, |\xi|\sim 2^k \Longrightarrow |\eta|\sim|\sigma|\sim 2^{k_1}, \tag{4.139}$$

$$|\partial_\eta \Phi(\xi,\eta,\sigma)| = |\Lambda'(\xi+\eta) - \Lambda'(\xi+\eta+\sigma)| \gtrsim 2^{-k_1/2} \sim 2^{-\min\{k_1,k_3\}/2}. \tag{4.140}$$

Note that estimate (4.140) is of same type as (4.133). Hence the argument used in **Case 1** can be applied on this case. We can use integration by parts in "$\eta$" to derive our desired estimate (4.138).

**Case 3:** For the case when $\text{med}\{k_i\} - \min\{k_i\} \leq 10$, we can use the same method used in **Case 1** and **Case 2**. After integration by parts in $\eta$ or $\sigma$, the loss of $2^{-\min\{k_i\}/2}$ can be covered by $2^{\text{med}\{k_i\}/2}$ from the symbol. As a result, the following estimate holds,

$$\big|I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)\big| \lesssim \epsilon_1^3 2^{-\text{med}\{k_i\}/5} 2^{-3m/2} + \epsilon_1^3 2^{-3m/2} 2^{3\max\{k_i\}_+/2} \lesssim \epsilon_1^3 2^{-(21-1000p_0)m/20},$$

where we used the fact that $\text{med}\{k_i\} \geq -(1+100p_0)m$ and $\max\{k_i\} \leq (1+100p_0)m/5$, see (4.131).

□

**Lemma 4.2.9.** *If $k_1, k_2, k_3$ satisfies the estimate* (4.131) *and the following condition holds,*

$$\max\{|k_1-k|,|k_2-k|,|k_3-k|\} \geq 20,\ \min\{k_i\} < -(5/7-1000p_0)m, \text{med}\{k_i\} - \min\{k_i\} \geq 10,$$

*then the following estimate holds,*

$$\Big||\xi|^{3/4-2p_0}(1+|\xi|^{N_2+2p_0}) \int_{t_1}^{t_2} e^{iL(t,\xi)}[I^{+,+,-}_{k_1,k_2,k_3}(t,\xi)]\,dt\Big|\psi_k(\xi) \lesssim 2^{-p_0 m}\epsilon_1^3. \tag{4.141}$$

*Proof.* Define $\tilde{k} = \min\{k, \text{med}\{k_i\}\}$. Recall that $-21p_0m \leq k \leq 20p_0m$ and $\min\{k_i\} < -(5/7 - 1000p_0)m$. Hence $\min\{k_1,k_2,k_3\}$ is much smaller than $k$, and as $\text{med}\{k_i\} - \min\{k_i\} \geq 10$, we could see that the smallest number among $k, k_1, k_2, k_3$, which is $\min\{k_i\}$, is much smaller than the second smallest number, which is $\tilde{k}$. A very important observation is that the size of phase $\Phi(\xi,\eta,\sigma)$ is determined by the second smallest number instead of the smallest number. More precisely, the following estimate holds,

$$|\Phi(\xi,\eta,\sigma)| \gtrsim 2^{\tilde{k}/2}. \tag{4.142}$$

To improve presentation, we postpone the proof of (4.142) to the end of this subsection. We can take it as granted first.

To prove (4.141), we will integrate by parts in time. Hence, after integration by parts in time, we have

$$\Big|\int_{t_1}^{t_2} e^{iL(\xi,s)}[I^{+,+,-}_{k_1,k_2,k_3;1}(\xi,s)]\,ds\Big| \lesssim |B_1(t_1,\xi)| + |B_2(t_2,\xi)| + |T_1(\xi)| + |T_2(\xi)|,$$

where for $i \in \{1,2\}$,

$$B_i(t_i,\xi) = \int_{\mathbb{R}^2} e^{it_i\Phi(\xi,\eta,\sigma)+iL(\xi,t_i)} \widehat{f_{k_1}}(\xi+\eta,t_i)\widehat{f_{k_2}}(\xi+\sigma,t_i)\widehat{\overline{f}}_{k_3}(-\xi-\eta-\sigma,t_i) r_3(\xi,\eta,\sigma)d\eta d\sigma,$$

$$T_1(\xi) = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta,\sigma)+iL(t,\xi)}\partial_t L(t,\xi)\widehat{f_{k_1}}(t,\xi+\eta)\widehat{f_{k_2}}(t,\xi+\sigma)\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)$$

$$\times r_3(\xi,\eta,\sigma)d\eta d\sigma dt,$$

$$T_2(\xi) = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta,\sigma)+iL(t,\xi)}\partial_t\Big(\widehat{f_{k_1}}(t,\xi+\eta)\widehat{f_{k_2}}(t,\xi+\sigma)\widehat{\overline{f}}_{k_3}(t,-\xi-\eta-\sigma)\Big)$$

$$\times r_3(\xi,\eta,\sigma)d\eta d\sigma dt,\ r_3(\xi,\eta,\sigma) = \frac{c^*(\xi,\eta,\sigma)\psi_{k_1}(\xi+\eta)\psi_{k_2}(\xi+\sigma)\psi_{k_3}(\xi+\eta+\sigma)}{\Phi(\xi,\eta,\sigma)}.$$

From (4.142), Lemma 1.3.8, and (4.106) in Lemma 4.2.3, the following estimates hold,

$$\|r_3(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty} \lesssim 2^{\mathrm{med}\{k_i\}/2+2\max\{k_i\}-\tilde{k}/2} \lesssim 2^{5\max\{k_i\}/2+11p_0m}. \tag{4.143}$$

From (4.143), the $L^2-L^2-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7 and the fact that $\mathrm{med}\{k_i\} \geq -(1+100p_0)m$, the following estimate holds,

$$|B_i(t_i,\xi)| \lesssim 2^{5\max\{k_i\}/2+11p_0m}\|e^{-it_i\Lambda}P_{\min\{k_i\}}f\|_{L^\infty}\|P_{\mathrm{med}\{k_i\}}f\|_{L^2}\|P_{\max\{k_i\}}f\|_{L^2}$$

$$\lesssim 2^{-\mathrm{med}\{k_i\}/5+23\max\{k_i\}/10-4\max\{k_i\}_+-m/2+11p_0m}\epsilon_1^3 \lesssim \epsilon_1^3 2^{-(3-1000p_0)m/10}. \tag{4.144}$$

Recall (4.108). From (4.106) in Lemma 4.2.3, the following estimate holds,

$$|\partial_t L(t,\xi)\psi_k(\xi)| \lesssim 2^{-m}|\xi|^4\|\widehat{f}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}^2 \lesssim 2^{-m+4k_+}\epsilon_1^2 \lesssim 2^{-(1-100p_0)m}\epsilon_0. \tag{4.145}$$

From (4.145), (4.143), and the $L^2-L^2-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, the following estimate holds,

$$|T_1(\xi)| \lesssim \epsilon_0 2^{m-(1-111p_0)m+5\max\{k_i\}/2}\|e^{-it\Lambda}P_{\min}f\|_{L^\infty}\|P_{\mathrm{med}}f\|_{L^2}\|P_{\max}f\|_{L^2}$$

$$\lesssim \epsilon_1^3 2^{111p_0m-m/2-\mathrm{med}\{k_i\}/5} \lesssim 2^{-(3-1000p_0)m/10}\epsilon_1^3. \tag{4.146}$$

To estimate $T_2(\xi)$, we put the input $\partial_t\widehat{f_{k_i}^{\pm}}$ and the input with the smallest frequency in $L^2$ and the other one in $L^\infty$. More precisely, from (4.143) and the $L^2-L^2-L^\infty$ type trilinear estimate

(1.14) in Lemma 1.3.7, the following estimate holds,

$$|T_2(\xi)| \lesssim \sum_{\{l,m,n\}=\{1,2,3\}} 2^{m+5\max\{k_i\}/2+11p_0m}\|\partial_t\widehat{f}(t,\xi)\psi_{k_l}(\xi)\|_{L^2}\|P_{\max\{k_m,k_n\}}f\|_{L^2}$$

$$\times\|e^{-it\Lambda}P_{\min\{k_m,k_n\}}f\|_{L^\infty} \lesssim \epsilon_1^5 2^{-(3-1000p_0)m/10}. \tag{4.147}$$

To sum up, our desire estimate (4.141) holds. □

It remains to prove (4.115) for the case when $(\iota_1,\iota_2,\iota_3) \in \{(+,+,+),(+,-,-),\ (-,-,-)\}$ and $k_1,k_2$ and $k_3$ satisfy (4.131). An important observation is that we have the following weak ellipticity estimate for phases $\Phi^{-,-,-}, \Phi^{+,-,-}, \Phi^{+,+,+}$:

$$\sum_{(\iota_1,\iota_2,\iota_3)\in\mathcal{S},(\iota_1,\iota_2,\iota_3)\neq(+,+,-)} |\Phi^{\iota_1,\iota_2,\iota_3}(\xi_1,\xi_2,\xi_3)\psi_{k_1}(\xi_1)\psi_{k_2}(\xi_2)\psi_{k_3}(\xi_3)| \gtrsim 2^{\mathrm{med}\{k_i\}/2}. \tag{4.148}$$

Hence, after doing integration by parts in time, the loss of $2^{-\mathrm{med}\{k_i\}/2}$ can be covered by the size of the symbol, see (4.131) in Lemma 4.2.3. Similar to the argument we used in the proof of Lemma 4.2.9, with minor modifications, one can show the desired estimate (4.115) holds without any problems.

*Proof of (4.142) and (4.148).* Without loss of generality, we assume that $|\xi| \geq |\xi-\eta| \geq |\eta-\sigma| \geq |\sigma|$. As a result, we have $|\eta| \leq 2|\xi-\eta|$ and $\xi\eta > 0$, which further gives us $|\eta| \leq 2|\xi|/3$ or $\eta = 2\xi$. Recall that $\Lambda(\xi) := |\xi|^{1/2}$. Note that, the following estimate holds,

$$\big|\Lambda(\xi)-\Lambda(\xi-\eta)\big| = \Big|\frac{|\xi|-|\xi-\eta|}{|\xi|^{1/2}+|\xi-\eta|^{1/2}}\Big| \leq \frac{1}{2}|\eta|^{1/2}. \tag{4.149}$$

When $(\iota_1,\iota_2,\iota_3) \in \{(+,+,+),(+,-,-),\ (-,-,-)\}$, from (4.149) , the following estimate holds,

$$|\Phi^{\iota_1,\iota_2,\iota_3}(\xi,\eta,\sigma)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)|$$

$$\geq |\Lambda(\eta-\sigma)| + |\Lambda(\sigma)| - |\Lambda(\xi)-\Lambda(\xi-\eta)| \gtrsim \max\{|\eta-\sigma|^{1/2},|\sigma|^{1/2}\}.$$

Therefore desired estimate (4.148) holds. To prove (4.142), we have another assumption, which is $|\sigma| \leq 2^{-10}|\eta-\sigma|$. As a result, we have $\Lambda(|\eta-\sigma|)-\Lambda(|\sigma|) \geq (1-2^{-5})|\eta-\sigma|^{1/2} \geq (1-2^{-5})^2|\eta|^{1/2}$. Hence, from (4.149), the following estimate holds,

$$|\Phi^{+,+,-}(\xi,\eta,\sigma)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)| \geq |\Lambda(\eta-\sigma)| - |\Lambda(\sigma)|$$

$$-|\Lambda(\xi)-\Lambda(\xi-\eta)|\gtrsim|\eta-\sigma|^{1/2}\sim\max\{|\eta-\sigma|^{1/2},|\sigma|^{1/2}\}.$$

Therefore, finishing the proof of (4.142). □

## 4.3 Remainder Estimates

### 4.3.1 Explicit formulas and estimates of good remainder terms

In this subsection, we give all detail formulas for all good remainder terms that are postponed in section 4.1 and section 4.2.

The detailed formulas of $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ inside the system (4.35) are given as follows,

$$\begin{aligned}\widetilde{\mathcal{R}}_1=&\widetilde{Q}_1(\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2),U^2)+\widetilde{Q}_1(U^1,\widetilde{B}(U^1,U^2))\\&+2\widetilde{A}_1(\mathcal{R}_1,U^1)+2\widetilde{A}_2(\mathcal{R}_2,U^2)-\Lambda_2[\mathcal{R}_1](U^1,\widetilde{B}(U^1,U^2))\\&-\Lambda_2[\mathcal{R}_1](\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2),U^2)+\Lambda_{\geq3}[\mathcal{R}_1],\end{aligned}\tag{4.150}$$

$$\begin{aligned}\widetilde{\mathcal{R}}_2=&\widetilde{Q}_2(\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2),U^1)+\widetilde{Q}_2(U^1,\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2))\\&+\widetilde{Q}_3(U^2,\widetilde{B}(U^1,U^2))+\widetilde{Q}_3(\widetilde{B}(U^1,U^2),U^2)+\widetilde{B}(\mathcal{R}_1,U^2)+\widetilde{B}(U^1,\mathcal{R}_2)+\Lambda_{\geq3}[\mathcal{R}_2]\\&-\Lambda_2[\mathcal{R}_2](\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2),U^1)-\Lambda_2[\mathcal{R}_2](U^1,\widetilde{A}_1(U^1,U^1)+\widetilde{A}_2(U^2,U^2))\\&-\Lambda_2[\mathcal{R}_2](\widetilde{B}(U^1,U^2),U^2)-\Lambda_2[\mathcal{R}_2](U^2,\widetilde{B}(U^1,U^2))\end{aligned}\tag{4.151}$$

where the detail formulas of $\mathcal{R}_1$ and $\mathcal{R}_2$ are given in (2.107) and (2.108).

The detailed formula of $\widetilde{\mathfrak{R}}_1$ and $\widetilde{\mathfrak{R}}_2$ in (4.63) and (4.64) are given as follows,

$$\begin{aligned}\widetilde{\mathfrak{R}}_1=&\widetilde{C}_{1,\tau}(\mathcal{R}_1^S,U_2)+\widetilde{C}_{2,\tau}(\mathcal{R}_2^S,U^1)+\widetilde{C}_{1,\tau}(SU^1,\Lambda_{\geq2}[\partial_tU^2])+\widetilde{C}_{2,\tau}(SU^2,\Lambda_{\geq2}[\partial_tU^1])\\&+\Lambda_{\geq3}[\mathcal{R}_1^S]-\widetilde{\mathfrak{Q}}_1^S(W_{1,\tau}-SU^1,SU^2)-\widetilde{\mathfrak{Q}}_1^S(SU^1,W_{2,\tau}-SU^2),\end{aligned}\tag{4.152}$$

$$\begin{aligned}\widetilde{\mathfrak{R}}_2=&\widetilde{C}_{3,\tau}(\mathcal{R}_1^S,U_1)+\widetilde{C}_{4,\tau}(\mathcal{R}_2^S,U^2)+\widetilde{C}_{3,\tau}(SU^1,\Lambda_{\geq2}[\partial_tU^1])+\widetilde{C}_{4,\tau}(SU^2,\Lambda_{\geq2}[\partial_tU^2])\\&+\Lambda_{\geq3}[\mathcal{R}_2^S]-\widetilde{\mathfrak{Q}}_3^S(W_{1,\tau}-SU^1,SU^2)-\widetilde{\mathfrak{Q}}_3^S(SU^1,W_{2,\tau}-SU^2),\end{aligned}\tag{4.153}$$

where $\mathcal{R}_1^S$ and $\mathcal{R}_2^S$ are defined in (4.45) and (4.46).

At last, we give the detail formula for the remainder term $R(t)$ in (4.98). Recall (4.2) and the

fact that $R(t)$ is quintic and higher. The following identity holds,

$$R(t) = \Lambda_{\geq 4}[C_1 + \mathcal{R}_1] + i\Lambda_{\geq 4}\big[C_2 + \mathcal{R}_2\big] + 2A_1(C_1 + \Lambda_{\geq 3}[\mathcal{R}_1], U^1)$$

$$+2A_2(C_2 + \Lambda_{\geq 3}[\mathcal{R}_2], U^2) + iB(C_1 + \Lambda_{\geq 3}[\mathcal{R}_1], U^2) + iB(U^1, C_2 + \Lambda_{\geq 3}[\mathcal{R}_2]). \tag{4.154}$$

### 4.3.2 $L^2$ type estimate of remainder terms

**Lemma 4.3.1.** *Under the bootstrap assumption* ( 4.12)*, we have following estimates for the remainder terms,*

$$\sup_{t\in[0,T]} (1+t)^{1-p_0}\big(\|\Lambda_{\geq 3}[\mathcal{R}_1]\|_{H^{N_0,p}} + \|\Lambda_{\geq 3}[\mathcal{R}_2]\|_{H^{N_0,p}} + \|\widetilde{\mathcal{R}}_1\|_{H^{N_0,p}} + \|\widetilde{\mathcal{R}}_2\|_{H^{N_0,p}}\big) \lesssim \epsilon_0^2 \tag{4.155}$$

$$\sup_{t\in[2^{m-1},2^m]} \sup_{k\in[2^{-100p_0m},2^{100p_0m}]} 2^{3m/2-200p_0m}2^{pk}\|P_k(R(t))\|_{L^2} \lesssim \epsilon_0^2. \tag{4.156}$$

*Proof.* Recall the detailed formulas of $\mathcal{R}_1$ and $\mathcal{R}_2$ in(2.107) and (2.108). From (2.88) in Lemma 2.3.4, estimates in Lemma 2.3.7, (1.8) in Lemma 1.3.4 and estimates in Lemma 2.1.4, the following estimate holds for $i \in \{2, 3\}$,

$$\|\Lambda_{\geq i}[\mathcal{R}_1]\|_{H^{N_0,p}} + \|\Lambda_{\geq i}[\mathcal{R}_2]\|_{H^{N_0,p}} \lesssim \|\Lambda_{\geq i}[F(h)\psi]\|_{H^{N_0}} + \|(B, \partial_x h, V)\|_{H^{N_0-1}}\|(V, \partial_x h, B)\|_{\widetilde{W}^1}^{i-1}$$

$$+\|h\|_{H^{N_0,p}}\big(\|\Lambda_{\geq i-1}\partial_t\alpha\|_{\widetilde{W}^0} + \|\alpha\|_{\widetilde{W}^1}\|V\|_{\widetilde{W}^0}\big) + \|\Lambda_{\geq 2}[(B, V)]\|_{H^{N_0-1}}\|(B, V)\|_{\widetilde{W}^1} + \|h\|_{H^{N_0,p}}\|\alpha\|_{\widetilde{W}^0}^2$$

$$\lesssim \big(\|h\|_{H^{N_0,p}} + \|\partial_x\psi\|_{H^{N_0-1}}\big)\|(\partial_x h, \partial_x\psi)\|_{W^1}^{i-1} \lesssim (1+t)^{-1+2p_0}\epsilon_0^2. \tag{4.157}$$

From (4.30), (4.34), Lemma 1.3.7, (4.157), and estimates in Lemma 2.1.4, the following estimate holds,

$$\|\Lambda_{\geq 3}[\widetilde{\mathcal{R}}_1]\|_{H^{N_0,p}} + \|\Lambda_{\geq 3}[\widetilde{\mathcal{R}}_2]\|_{H^{N_0,p}} \lesssim \|(U^1, U^2)\|_{H^{N_0,p}}\|(U^1, U^2)\|_{W^3}^2$$

$$+\|(U^1, U^2)\|_{H^{N_0,p}}\|(\mathcal{R}_1, \mathcal{R}_2)\|_{\widetilde{W}^1} + \|(U^1, U^2)\|_{W^1}\|(\mathcal{R}_1, \mathcal{R}_2)\|_{H^{N_0,p}}$$

$$\lesssim \big(\|(U^1, U^2, h)\|_{H^{N_0,p}} + \||\nabla|\psi\|_{H^{N_0-1}}\big(\|(\partial_x h, \partial_x\psi)\|_{W^2} + \|(U^1, U^2)\|_{W^3}\big)$$

$$\lesssim (1+t)^{-1+2p_0}\epsilon_0^2.$$

In above estimate, we used the following rough estimate on the good error terms

$$\|\mathcal{R}_1\|_{\widetilde{W}^1} + \|\mathcal{R}_2\|_{\widetilde{W}^1} \lesssim \|(\partial_x h, \partial_x\psi)\|_{W^2}^2,$$

which is derived from estimating each term inside (2.107) and (2.108).

To prove (4.156), we only need to estimate the $\dot{H}^p$ norm of each term inside (4.156), as we do not worry about losing derivative for the desired estimate (4.156). Recall (2.107), (2.108), and (4.154). From (2.49) in Lemma 2.1.5, (2.109) and (2.112) in Lemma 2.3.7, our desired estimate (4.156) follows straightforwardly as it is quintic and higher.

□

**Lemma 4.3.2.** *Under the bootstrap assumption (4.12), the following estimates hold on the remainder terms,*

$$\sup_{t\in[0,T]}(1+t)^{1-p_0}\big(\|\Lambda_{\geq 3}[\mathcal{R}_1^S]\|_{H^{N_1,p}}+\|\Lambda_{\geq 3}[\mathcal{R}_2^S]\|_{H^{N_1,p}}+\|\widetilde{\mathfrak{R}}_1\|_{H^{N_1,p}}+\|\widetilde{\mathfrak{R}}_2\|_{H^{N_1,p}}\big)\lesssim\epsilon_0^2. \tag{4.158}$$

$$\sup_{t\in[2^{m-1},2^m]}\sup_{k\in[2^{-100p_0m},2^{100p_0m}]}2^{3m/2-200p_0m}2^{pk}\|P_k(SR(t))\|_{L^2}\lesssim\epsilon_0^2. \tag{4.159}$$

*Proof.* The proof is very straightforward, as we already have all necessary ingredients. Also those error terms do not lose derivatives, one can close the argument by using the multilinear estimate carefully.

Recall (4.45) and (4.46). From we can derive the following estimate for $i\in\{2,3\}$,

$$\begin{aligned}\|\Lambda_{\geq i}[\mathcal{R}_1^S]\|_{H^{N_1,p}}+\|\Lambda_{\geq i}[\mathcal{R}_2^S]\|_{H^{N_1,p}}&\lesssim\|\Lambda_{\geq i}[SF(\eta)\psi]\|_{H^{N_1,p}}+\|S\alpha\|_{H^{-1}}\|(\partial_x h,\partial_x\psi)\|_{W^1}^2\\&+\|Sh\|_{H^{N_1,p}}\big(\|\Lambda_{\geq i-1}[\partial_t\alpha]\|_{\widetilde{W}^0}+\|\partial_x\alpha\|_{\widetilde{W}^0}\|V\|_{\widetilde{W}^1}\big)+\|\Lambda_{\geq i-1}S\partial_t\alpha\|_{H^{-2}}\|h\|_{W^{2+N_1}}\\&+\sum_{i_1+i_2=i,i_1,i_2\in\mathbb{N}_+}\|\Lambda_{\geq i_1}[(SB,SV)]\|_{L^2}\|\Lambda_{\geq i_2}[(B,V)]\|_{\widetilde{W}^2}\\&+\|(SB,SV,S\partial_x h)\|_{L^2}\|\partial_x h\|_{\widetilde{W}^1}\|(B,V)\|_{\widetilde{W}^1}+\|(\partial_x h,\partial_x\psi)\|_{H^{N_1+3}}\|(\partial_x h,\partial_x\psi)\|_{W^2}^2.\end{aligned} \tag{4.160}$$

From (2.109), (2.110), (2.111) and (2.112) in Lemma 2.3.7, (2.91) in Lemma 2.3.5, (2.66) in Lemma 2.65, we have the following estimate,

$$(4.160)\lesssim\big(\|Sh\|_{H^{N_1,p}}+\||\nabla|S\psi\|_{L^2}\big)\|(\partial_x h,\partial_x\psi)\|_{W^2}^{i-1}\lesssim(1+t)^{-i/2+p_0}\epsilon_0^2. \tag{4.161}$$

Recall (4.152) and (4.153). From (4.62) in Lemma 4.1.4, (4.51), (4.52), and (4.161), the following estimate holds,

$$\|\widetilde{\mathfrak{R}}_1\|_{H^{N_1,p}}+\|\widetilde{\mathfrak{R}}_2\|_{H^{N_1,p}}\lesssim(1+t)^{-1+p_0}+(1+t)^{-1/2+p_0}\|(U^1,U^2)\|_{W^3}$$

$$+\|(SU^1,SU^2)\|_{H^{N_1,p}}\big(\|\Lambda_{\geq 2}[(\partial_t U^1,\partial_t U^2)]\|_{\widetilde{2}}+\|(U^1,U^2)\|_{W^3}^2\big)+\|(U^1,U^2)\|_{H^{N_0,p}}\|(U^1,U^2)\|_{W^3}^2$$

$$\lesssim (1+t)^{-1+p_0}\epsilon_0.$$

Note that we do not worry about losing derivative for the desired estimate (4.159). From (2.49) in Lemma 2.1.5, (2.109) and (2.112) in Lemma 2.3.7, (2.67) and (2.66) in Lemma 2.2.1 our desired estimate (4.159) follows straightforwardly as it is quintic and higher. □

### 4.3.3 $Z$-norm estimate of the remainder term

**Lemma 4.3.3.** *Under the bootstrap assumptions* (4.12) *and* (4.110)*, the following estimate holds for* $k\in[-21p_0m,20p_0m]$*,*

$$\sup_{t\in[2^{m-1},2^m]}\|\int_{t_1}^{t_2}e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)\psi_k(\xi)dt\|_Z\lesssim 2^{-(1+p_0)m}\epsilon_1^3. \tag{4.162}$$

*Proof.* From (1.16) in Lemma 1.3.9, (4.156) in Lemma 4.3.1, and (4.159) in Lemma 4.3.2, we have the following estimate,

$$\sup_{t_1,t_2\in[2^{m-1},2^{m+1}]}\|\int_{t_1}^{t_2}e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)\psi_k(\xi)dt\|_{L^\infty_\xi}$$

$$\lesssim\sup_{t\in[2^{m-1},2^{m+1}]}2^{-k/2}\|\int_{t_1}^{t_2}e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)\psi_k(\xi)dt\|_{L^2}^{\frac12}\Big[\|\int_{t_1}^{t_2}t\partial_t\big(e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)\big)\psi_k(\xi)dt\|_{L^2}$$

$$+2^m\big(\|SP_k[R(t)]\|_{L^2}+\|P_kR(t)\|_{L^2}\big)\Big]^{\frac12}$$

$$\lesssim 2^{-m/2+220p_0m}\epsilon_0^2+2^{-m/4+20p_0m}\|\int_{t_1}^{t_2}t\partial_t\big(e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)\big)\psi_k(\xi)dt\|_{L^2}^{1/2}. \tag{4.163}$$

Note that, in above estimate, we used the following fact,

$$\sup_{t\in[2^{m-1},2^m]}\|(t\partial_t-2\xi\cdot\partial_\xi)e^{iL(t,\xi)}\|_{L^\infty_\xi}\lesssim 2^{p_0m}\epsilon_1^2,$$

which follows from the definition of $L(t,\xi)$ in (4.108).

After integration by parts in time, from (4.156) in Lemma 4.3.1, it is easy to see the following estimate holds,

$$\|\int_{t_1}^{t_2}t\partial_t\big(e^{iL(t,\xi)+it\Lambda(\xi)}\widehat{R}(t,\xi)\big)\psi_k(\xi)dt\|_{L^2}\lesssim\sup_{t\in[2^{m-1},2^m]}2^m\|P_k(R)(t)\|_{L^2}$$

$$\lesssim 2^{-m/2+200p_0m}\epsilon_0^2. \tag{4.164}$$

Combining (4.163) and (4.164), it is easy to see our desired estimate (4.162) holds. Hence finishing the proof. □

# Appendix: Cubic terms inside the equation (4.98)

The goal of this section is to find out explicitly what the symbols of the cubic terms inside (4.98) are. Recall that $V = V_1 + iV_2$, where $V_1 = U^1 + A_1(U^1, U^1) + A_2(U^2, U^2)$ and $V_2 = U^2 + B(U^1, U^2)$.

From (4.2), we know that the cubic terms inside the system of equation satisfied by $(U^1, U^2)$ are given by $\Lambda_3[C_1 + \mathcal{R}_1]$ and $\Lambda_3[C_2 + \mathcal{R}_2]$. We define,

$$\mathfrak{cubic}_1(U^1, U^2) := \text{cubic terms of } \partial_t U^1 = \widetilde{C}_1(U^1, U^1, U^2) + \widetilde{C}_2(U^2, U^2, U^2),$$

$$\mathfrak{cubic}_2(U^1, U^2) := \text{cubic terms of } \partial_t U^2 = \widetilde{C}_3(U^1, U^1, U^1) + \widetilde{C}_4(U^1, U^2, U^2).$$

From the definitions of $V_1$ and $V_2$ and (4.2), naturally, we have

$$\text{cubic terms of } \partial_t V_1 = \mathfrak{cubic}_1(V_1, V_2) + 2A_1(Q_1(V_1, V_2), V_1) + 2A_2(Q_2(V_1, V_1) + Q_3(V_2, V_2), V_2), \tag{4.165}$$

$$\text{cubic terms of } \partial_t V_2 = \mathfrak{cubic}_2(V_1, V_2) + B(Q_1(V_1, V_2), V_2) + B(V_1, Q_2(V_1, V_1) + Q_3(V_2, V_2)). \tag{4.166}$$

Recall the notation that $V^+ := V$, $V^- := \overline{V}$ and $c_\mu := \mu/(2i)$, $\mu \in \{+, -\}$. For $\mu, \nu, \kappa \in \{+, -\}$, we define

$$\begin{aligned} C_{\mu,\nu,\kappa} :=& \frac{c_\kappa}{4}\widetilde{C}_1(V^\mu, V^\nu, V^\kappa) + c_\mu c_\nu c_\kappa \widetilde{C}_2(V^\mu, V^\nu, V^\kappa) + \frac{i}{8}\widetilde{C}_3(V^\mu, V^\nu, V^\kappa) + \frac{ic_\nu c_\kappa}{2}\widetilde{C}_4(V^\mu, V^\nu, V^\kappa) \\ &+ \frac{c_\nu}{2}[A_1(Q_1(V^\mu, V^\nu), V^\kappa) + A_2(Q_2(V^\mu, V^\kappa), V^\mu)] + 2c_\mu c_\nu c_\kappa A_2(Q_3(V^\mu, V^\nu), V^\kappa) \\ &+ \frac{ic_\nu c_\kappa}{2}[B(Q_1(V^\mu, V^\nu), V^\kappa) + B(V^\mu, Q_3(V^\nu, V^\kappa)] + \frac{i}{8}B(V^\mu, Q_2(V^\nu, V^\kappa)). \end{aligned} \tag{4.167}$$

Recall (4.98), we have

$$C_{+,+,+}(V^+, V^+, V^-) = C_{+,+,+}, \quad C_{+,+,-}(V^+, V^+, V^-) = \sum_{\{\mu,\nu,\kappa\}=\{+,+,-\}} C_{\mu,\nu,\kappa},$$

$$C_{+,-,-}(V^+, V^-, V^-) = \sum_{\{\mu,\nu,\kappa\}=\{+,-,-\}} C_{\mu,\nu,\kappa}, \quad C_{-,-,-}(V^-, V^-, V^-) = C_{-,-,-}.$$

Hence, to find out the symbols $c^{\iota_1,\iota_2,\iota_3}(\cdot,\cdot,\cdot)$, $(\iota_1,\iota_2,\iota_3)\in\mathcal{S}$, from (4.167), it's sufficient to find out the symbol $\widetilde{c}_i(\cdot,\cdot,\cdot)$ of cubic term $\widetilde{C}_i(\cdot,\cdot,\cdot)$, $i\in\{1,2,3,4\}$.

The precise formulations of those postponed cubic terms $\widetilde{C}_i(\cdot,\cdot,\cdot), i\in\{1,2,3,4\}$, are given as follows,

$$\begin{aligned}
\widetilde{C}_1(U^1,U^1,U^2)&=T_{\partial_xU^1|\nabla|^{\frac12}U^2}\partial_xU^1+\frac12T_{[|\nabla|(U^1|\nabla|U^1)+U^1\partial_x^2U^1)]}|\nabla|^{\frac12}U^2+|\nabla|(U^1|\nabla|(U^1|\nabla|^{\frac12}U^2))\\
&+\frac12|\nabla|((U^1)^2\partial_x^2|\nabla|^{-\frac12}U^2)+\frac12\partial_x^2((U^1)^2|\nabla|^{\frac12}U^2)-|\nabla|T_{[|\nabla|(U^1|\nabla|^{\frac12}U^2)+U^1\partial_x^2|\nabla|^{-\frac12}U^2]}U^1-\\
&\partial_xT_{\partial_xU^1|\nabla|^{\frac12}U^2}U^1+T_{\partial_x(\partial_xU^1|\nabla|^{\frac12}U^2)}U^1-\frac12T_{\partial_x|\nabla|^{-\frac12}U^2}T_{\partial_x|\nabla|U^1}U^1-\frac12[T_{\partial_x|\nabla|^{-\frac12}U^2},T_{|\nabla|U^1}]\partial_xU^1\\
&-\frac12T_{|\nabla|U^1}\big[-|\nabla|(U^1|\nabla|^{\frac12}U^2)-\partial_x(U^1\partial_x|\nabla|^{-\frac12}U^2)+|\nabla|T_{|\nabla|^{\frac12}U^2}U^1+\partial_xT_{\partial_x|\nabla|^{-\frac12}U^2}U^1\\
&-T_{\partial_x^2|\nabla|^{-\frac12}U^2}U^1\big]-T_{\partial_xT_{|\nabla|^{\frac12}U^2}U^1}\partial_xU^1-\frac14T_{|\nabla|T_{|\nabla|U^1}U^1}|\nabla|^{\frac12}U^2-\frac12|\nabla|(T_{|\nabla|U^1}U^1|\nabla|^{\frac12}U^2)\\
&-|\nabla|(U^1|\nabla|T_{|\nabla|^{\frac12}U^2}U^1)-\frac12\partial_x(T_{|\nabla|U^1}U^1\partial_x|\nabla|^{-\frac12}U^2)-\partial_x(U^1\partial_xT_{|\nabla|^{\frac12}U^2}U^1)+\\
&|\nabla|T_{|\nabla|T_{|\nabla|^{\frac12}U^2}U^1}U^1+\frac12|\nabla|T_{|\nabla|^{\frac12}U^2}T_{|\nabla|U^1}U^1+\frac12T_{|\nabla|^2T_{|\nabla|^{\frac12}U^2}U^1}U^1+\partial_xT_{\partial_xT_{|\nabla|^{\frac12}U^2}U^1}U^1\\
&+\frac12\partial_xT_{\partial_x|\nabla|^{-\frac12}U^2}T_{|\nabla|U^1}U^1+\frac14T_{|\nabla|^{\frac32}U^2}T_{|\nabla|U^1}U^1+T_{\Lambda_2[\partial_t\alpha](U^1,U^2)}U^1,
\end{aligned}\tag{4.168}$$

where

$$\begin{aligned}
\Lambda_2[\partial_t\alpha](U^1,U^2)&=\frac12|\nabla|\Big[|\nabla|(U^1|\nabla|^{\frac12}U^2)+\partial_x(U^1\partial_x|\nabla|^{-\frac12}U^2)\Big]+\frac12\Big[-\partial_xU^1\partial_x|\nabla|^{\frac12}U^2\\
&-\partial_x|\nabla|^{-\frac12}U^2\partial_x|\nabla|U^1+|\nabla|U^1\partial_x^2|\nabla|^{-\frac12}U^2+|\nabla|^{\frac12}U^2\partial_x^2U^1\Big]+\frac12|\nabla|\Big[\partial_x|\nabla|^{-\frac12}U^2\partial_xU^1\\
&+|\nabla|^{\frac12}U^2|\nabla|U^1\Big]+\frac12\Big[|\nabla|(|\nabla|^{\frac12}U^2|\nabla|U^1)+|\nabla|(U^1|\nabla|^{\frac32}U^2)+|\nabla|^{\frac12}U^2\partial_x^2U^1+U^1\partial_x^2|\nabla|^{\frac12}U^2\Big],
\end{aligned}$$

$$\begin{aligned}
\widetilde{C}_2(U^2,U^2,U^2)&=\frac14\Big[T_{2\partial_x|\nabla|^{-\frac12}U^2\partial_x|\nabla|^{\frac12}U^2-2|\nabla|^{\frac12}U^2\partial_x^2|\nabla|^{-\frac12}U^2}\\
&+T_{-|\nabla|[(\partial_x|\nabla|^{-\frac12}U^2)^2+(|\nabla|^{\frac12}U^2)^2]}\Big]|\nabla|^{\frac12}U^2,
\end{aligned}\tag{4.169}$$

$$\begin{aligned}
\tilde{C}_3(U^1,U^1,U^1)&=-\frac14|\nabla|^{\frac12}\Big[2T_{[|\nabla|(U^1|\nabla|U^1)+U^1\partial_x^2U^1]}U^1-\\
&[T_{|\nabla|U^1}T_{|\nabla|U^1}-T_{(|\nabla|U^1)^2}]-T_{|\nabla|T_{|\nabla|U^1}U^1}\Big]U^1,
\end{aligned}\tag{4.170}$$

$$\widetilde{C}_4(U^1,U^2,U^2)=|\nabla|^{\frac{1}{2}}\Big[T_{\partial_x U^1|\nabla|^{\frac{1}{2}}U^2|\xi|^{-\frac{1}{2}}}\partial_x U^2-T_{\partial_x T_{|\nabla|^{\frac{1}{2}}U^2}U^1|\xi|^{-\frac{1}{2}}}\partial_x U^2+\big[R_{\mathcal{B}}(|\nabla|T_{|\nabla|^{\frac{1}{2}}U^2}U^1,$$

$$|\nabla|^{\frac{1}{2}}U^2)\big]-\frac{1}{4}T_{2\partial_x|\nabla|^{-\frac{1}{2}}U^2\partial_x|\nabla|^{\frac{1}{2}}U^2-2|\nabla|^{\frac{1}{2}}U^2\partial_x^2|\nabla|^{-\frac{1}{2}}U^2-|\nabla|[(\partial_x|\nabla|^{-\frac{1}{2}}U^2)^2+(|\nabla|^{\frac{1}{2}}U^2)^2]}U^1+$$

$$[T_{\partial_x|\nabla|^{-\frac{1}{2}}U^2}T_{\partial_x U^1}-T_{\partial_x|\nabla|^{-\frac{1}{2}}U^2\partial_x U^1}]|\nabla|^{\frac{1}{2}}U^2+[T_{\partial_x|\nabla|^{-\frac{1}{2}}U^2\partial_x|\nabla|^{\frac{1}{2}}U^2}-T_{\partial_x|\nabla|^{-\frac{1}{2}}U^2}T_{\partial_x|\nabla|^{\frac{1}{2}}U^2}]U^1$$

$$-R_{\mathcal{B}}(|\nabla|^{\frac{1}{2}}U^2,|\nabla|(U^1|\nabla|^{\frac{1}{2}}U^2)+U^1\partial_x^2|\nabla|^{-\frac{1}{2}}U^2)+R_{\mathcal{B}}(\partial_x|\nabla|^{-\frac{1}{2}}U^2,\partial_x U^1|\nabla|^{\frac{1}{2}}U^2)+$$

$$T_{\partial_x|\nabla|^{-\frac{1}{2}}U^2}R_{\mathcal{B}}(|\nabla|^{\frac{1}{2}}U^2,\partial_x U^1)-R_{\mathcal{B}}(|\nabla|^{\frac{1}{2}}U^2,\partial_x|\nabla|^{-\frac{1}{2}}U^2\partial_x U^1)-R_{\mathcal{B}}(\partial_x T_{|\nabla|^{\frac{1}{2}}U^2}U^1,\partial_x|\nabla|^{-\frac{1}{2}}U^2)\Big]. \tag{4.171}$$

Their symbols are given as follows,

$$\widetilde{c}_1(\xi_1,\xi_2,\xi_3)=-\xi_1\xi_2|\xi_3|^{\frac{1}{2}}\theta(\xi_2+\xi_3,\xi_1)+\frac{1}{2}|\xi_3|^{\frac{1}{2}}|\xi_2|(|\xi_1+\xi_2|-|\xi_2|)\theta(\xi_1+\xi_2,\xi_3)$$

$$+\frac{1}{2}|\xi_1+\xi_2+\xi_3||\xi_3|^{\frac{1}{2}}[2|\xi_2+\xi_3|-|\xi_3|-|\xi_1+\xi_2+\xi_3|]-|\xi_3|^{\frac{1}{2}}|\xi_1+\xi_2+\xi_3|\theta(\xi_2+\xi_3,\xi_1)[|\xi_2+\xi_3|-|\xi_3|]$$

$$-\xi_1\xi_2|\xi_3|^{\frac{1}{2}}|\theta(\xi_2+\xi_3,\xi_1)+\frac{1}{2}\xi_2\xi_3|\xi_2||\xi_3|^{-\frac{1}{2}}\theta(\xi_3,\xi_1+\xi_2)\theta(\xi_2,\xi_1)+\frac{1}{2}\xi_1\xi_3|\xi_2||\xi_3|^{-\frac{1}{2}}\big[\theta(\xi_3,\xi_1+\xi_2)\theta(\xi_2,\xi_1)$$

$$-\theta(\xi_2,\xi_1+\xi_3)\theta(\xi_3,\xi_1)\big]+\frac{1}{2}|\xi_2|\theta(\xi_2,\xi_1+\xi_3)\Big[[|\xi_1+\xi_3||\xi_3|^{\frac{1}{2}}-(\xi_1+\xi_3)\xi_3|\xi_3|^{-\frac{1}{2}}]\tilde{\theta}(\xi_1,\xi_3)+|\xi_3|^{\frac{3}{2}}\theta(\xi_3,\xi_1)\Big]$$

$$+\theta(\xi_2+\xi_3,\xi_1)\frac{1}{2}|\xi_3|^{\frac{1}{2}}\Big[|\xi_2+\xi_3|^2(1-\mathrm{sign}(\xi_3(\xi_2+\xi_3)))+|\xi_2|(|\xi_2+\xi_3|-|\xi_2|-|\xi_3|)(1-\mathrm{sign}(\xi_2\xi_3))$$

$$+|\xi_2+\xi_3|(|\xi_2|+|\xi_3|)-|\xi_2|^2-|\xi_3|^2\Big]+(\xi_2+\xi_3)\xi_1|\xi_3|^{\frac{1}{2}}\theta(\xi_2+\xi_3,\xi_1)\theta(\xi_3,\xi_2)-\frac{1}{4}|\xi_1+\xi_2||\xi_2||\xi_3|^{\frac{1}{2}}\theta(\xi_2,\xi_1)\times$$

$$\theta(\xi_1+\xi_2,\xi_3)-\frac{1}{2}|\xi_1+\xi_2+\xi_3||\xi_3|^{\frac{1}{2}}\Big[|\xi_2|\theta(\xi_2,\xi_1)(1-\mathrm{sgn}[\xi_3(\xi_1+\xi_3)])\tilde{\theta}(\xi_1+\xi_2,\xi_3)+2|\xi_2+\xi_3|\theta(\xi_3,\xi_2)\times(1-$$

$$\mathrm{sign}[\xi_2(\xi_1+\xi_2)])\tilde{\theta}(\xi_1,\xi_2+\xi_3)\Big]+\frac{1}{4}|\xi_2||\xi_3|^{\frac{3}{2}}\theta(\xi_2,\xi_1)\theta(\xi_3,\xi_2+\xi_1)+\frac{1}{2}|\xi_2+\xi_3|^2|\xi_3|^{\frac{1}{2}}\theta(\xi_3,\xi_2)\theta(\xi_3+\xi_2,\xi_1), \tag{4.172}$$

$$\widetilde{c}_2(\xi_1,\xi_2,\xi_3)=\frac{1}{4}|\xi_2|^{\frac{1}{2}}|\xi_1|^{\frac{1}{2}}|\xi_3|^{\frac{1}{2}}(2|\xi_3|-|\xi_2+\xi_3|)[1-\mathrm{sign}(\xi_2\xi_3)]\theta(\xi_2+\xi_3,\xi_1), \tag{4.173}$$

$$\widetilde{c}_3(\xi_1,\xi_2,\xi_3)=|\xi_2|^{\frac{1}{2}}|\xi_1+\xi_2+\xi_3|^{\frac{1}{2}}\Big[-\xi_1\xi_3|\xi_3|^{-\frac{1}{2}}\theta(\xi_1+\xi_2,\xi_3)-\frac{1}{4}|\xi_3|^{\frac{1}{2}}(2|\xi_3|-|\xi_2+\xi_3|)\times$$

$$[1-\mathrm{sign}(\xi_2\xi_3)]\theta(\xi_2+\xi_3,\xi_1)-\xi_1\xi_2|\xi_2|^{-\frac{1}{2}}[\theta(\xi_2,\xi_1+\xi_3)\theta(\xi_1,\xi_3)-\theta(\xi_1+\xi_2,\xi_3)]-\xi_2\xi_3|\xi_2|^{-\frac{1}{2}}\times$$

$$[\theta(\xi_2+\xi_3,\xi_1)-\theta(\xi_2,\xi_3+\xi_1)\theta(\xi_3,\xi_1)]-(|\xi_1+\xi_2|-|\xi_2|)|\xi_2|^{\frac{1}{2}}\tilde{\theta}(\xi_3,\xi_1+\xi_2)-\xi_1\xi_2|\xi_2|^{-\frac{1}{2}}\times$$

$$\big[\tilde{\theta}(\xi_2,\xi_1+\xi_3)-\theta(\xi_2,\xi_1+\xi_3)\tilde{\theta}(\xi_1,\xi_3)+\tilde{\theta}(\xi_3,\xi_1+\xi_2)\big]+(\xi_1+\xi_3)\xi_2|\xi_2|^{-\frac{1}{2}}\theta(\xi_3,\xi_1)\theta(\xi_1+\xi_3,\xi_2)+$$

$$|\xi_1+\xi_3||\xi_2|^{\frac{1}{2}}\tilde{\theta}(\xi_1+\xi_3,\xi_2)+(\xi_1+\xi_3)\xi_2|\xi_2|^{-\frac{1}{2}}\tilde{\theta}(\xi_1+\xi_3,\xi_2)\Big], \tag{4.174}$$

$$\widetilde{c}_4(\xi_1,\xi_2,\xi_3)=-\frac{1}{4}|\xi_1+\xi_2+\xi_2|^{\frac{1}{2}}\Big[\theta(\xi_3+\xi_2,\xi_1)\big[2(|\xi_2+\xi_3||\xi_3|-|\xi_3|^2)+|\xi_2||\xi_3|-|\xi_2+\xi_3||\xi_3|\theta(\xi_3,\xi_2)\big]$$

$$-|\xi_2||\xi_3|\theta(\xi_3,\xi_2+\xi_1)\theta(\xi_2,\xi_1)\Big]. \tag{4.175}$$

*Proof of Lemma* 4.2.3.

We give two proofs here. First one is straightforward but it requires certain amount of time. From Lemma 1.3.8, one can estimate all symbols in above to see our desired estimate (4.106) holds.

In the following, we provide a more intuitive "proof", which explains why our desired estimate (4.106) should hold. Recall (4.165) and (4.166). There are two sources of cubic terms that contribute to the cubic terms inside the equation satisfied by $V$ (4.98). One of them comes from $\Lambda_3[\partial_t U_1]$ and $\Lambda_3[\partial_t U_2]$. The other one comes from $A_1(\Lambda_2[\partial_t U^1],U^1)$, $A_2(\Lambda_2[\partial_t U^2],U^2)$, $B(\Lambda_2[\partial_t U^1],U^2)$, and $B(U^1,\Lambda_2[\partial_t U^1])$.

Our desired estimate (4.106) follows from the following facts: (i) Note that the symbols of all cubic terms are homogeneous of degree $5/2$ in terms of $U^1$ and $U^2$. Hence to prove (4.106), we only have to consider the case when $\text{med}\{k_i\}\leq \max\{k_i\}-10$. (ii) Cubic terms at most lose 2 derivatives. From (4.1), we know that $\Lambda_3[\partial_t U_1]$ and $\Lambda_3[\partial_t U_2]$ at most lose one derivative. From (4.21) in Lemma 4.1.1 and (4.8) in Lemma 4.0.8, we know that cubic terms, $A_1(\Lambda_2[\partial_t U^1],U^1)$, $A_2(\Lambda_2[\partial_t U^2],U^2)$, $B(\Lambda_2[\partial_t U^1],U^2)$, and $B(U^1,\Lambda_2[\partial_t U^1])$, at most lose two derivatives. To sum up, we know that the size of symbol contributes at least half degree smallness of the medium frequency.

To see why fact (i) holds. Note that the symbols of normal form transformations $A_1(\cdot,\cdot)$, $A_2(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are all homogeneous of degree "1" and the symbols of quadratic terms $\Lambda_2[\partial_t U^1]$ and $\Lambda_2[\partial_t U^2]$ are all homogeneous of degree "3/2". Therefore the symbols of all cubic terms that come from the normal form transformation are homogeneous of degree $5/2$. Intuitively speaking, the cubic term of $\partial_t U^1$ is either of type $\partial_x h\partial_x h\partial_x\psi$ or of type $\partial_x|\nabla|\psi|\nabla|\psi|\nabla|\psi$; the cubic term of $\partial_t U^2$ is of type $|\nabla|^{1/2}(\partial_x\psi\partial_x\psi\partial_x h)$ or of type $|\nabla|^{1/2}(h\partial_x h\partial_x h)$. Recall that $\Lambda_1[\psi]=|\nabla|^{-1/2}U^2$. Now it is easy to see that they are all homogeneous of degree $5/2$.

□

# Chapter 5

# The gravity water waves in the flat bottom setting

Recall that,

$$B(h)\psi = \big[G(h)\psi + \nabla h \cdot \nabla\psi\big]/(1+|\nabla h|^2), \tag{5.1}$$

and recall the system of equations (1.3) satisfied by $h$ and $\psi$, we can reformulate it as follows,

$$\begin{cases} \partial_t h = \Lambda^2\psi + \Lambda_2[G(h)\psi] + \Lambda_{\geq 3}[G(h)\psi], \\ \\ \partial_t\psi = -h - \dfrac{1}{2}|\nabla\psi|^2 + \dfrac{1}{2}|\Lambda^2\psi|^2 + \dfrac{1}{2}\Lambda_{\geq 3}\Big[(1+|\nabla h|^2)(B(h)\psi)^2\Big]. \end{cases} \tag{5.2}$$

Define $u = h + i\Lambda\psi$ and $Q(h,\psi) = \Lambda_2[G(h)\psi]$. Very naturally, we have

$$h = \frac{u+\bar{u}}{2}, \quad \psi = \Lambda^{-1}\big(\frac{u-\bar{u}}{2i}\big). \tag{5.3}$$

With above notation, we can derive the equation satisfied by $u$ from (5.2) as follows,

$$(\partial_t + i\Lambda)u = \sum_{\mu,\nu\in\{+,-\}} Q_{\mu,\nu}(u^\mu, u^\nu) + \mathcal{R}, \tag{5.4}$$

where

$$Q_{\mu,\nu}(u^\mu,u^\nu) = \frac{c_\nu}{2}Q(u^\mu,\Lambda^{-1}u^\nu) + \frac{ic_\mu c_\nu}{2}\Lambda\big[-\frac{\nabla}{\Lambda}u^\mu\cdot\frac{\nabla}{\Lambda}u^\nu + \Lambda u^\mu\Lambda u^\nu\big], \quad \mu,\nu\in\{+,-\}, \tag{5.5}$$

$$\mathcal{R} = \Lambda_{\geq 3}[\partial_t h] + i\Lambda\Lambda_{\geq 3}[\partial_t \psi] = \Lambda_{\geq 3}[G(h)\psi] + i\Lambda\Lambda_{\geq 3}[(1+|\nabla h|^2)(B(h)\psi)^2], \tag{5.6}$$

and $c_+ := -i/2$ and $c_- := i/2$.

**Lemma 5.0.4.** *Suppose that the symbol of quadratic term $Q_{\mu,\nu}(u_\mu, u_\nu)$ is given by $q_{\mu,\nu}(\xi-\eta,\eta)$, for $k, k_1, k_2 \in \mathbb{Z}$ and any $\mu, \nu \in \{+,-\}$ we have the following estimates*

$$\|q_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{k/2+k_-/2+k_{1,+}}, \quad \text{if } |k_1-k_2| \leq 10, \tag{5.7}$$

$$\|q_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{\max\{k_1,k_2\}+\max\{k_1,k_2\}_+/2}, \quad \text{if } |k_1-k_2| \geq 10. \tag{5.8}$$

*Proof.* From (5.5) and Lemma 1.3.8, it's easy to see that above estimates hold for the symbols determined other than quadratic term $Q(\cdot,\cdot)$. From (3.59) in Lemma 3.1.5, we can see that above estimates also hold for the symbol of quadratic term $Q(\cdot,\cdot)$. □

We have the following lemma on the $L^\infty$-decay estimate for the linear solution,

**Lemma 5.0.5.** *For $f \in L^1(\mathbb{R}^2)$, we have the following $L^\infty$ type estimates:*

$$\|e^{it\Lambda}P_k f\|_{L^\infty} \lesssim (1+|t|)^{-1}2^{3k/2}\|f\|_{L^1}, \quad \text{if } k \geq 0. \tag{5.9}$$

$$\|e^{it\Lambda}P_k f\|_{L^\infty} \lesssim (1+|t|)^{-\frac{1+\theta}{2}}2^{\frac{(3-3\theta)k}{2}}\|f\|_{L^1}, \quad 0 \leq \theta \leq 1, \quad \text{if } k \leq 0. \tag{5.10}$$

*Proof.* After checking the expansion of the phase, see (5.58), we can apply the main result in [19][Theorem 1:(a)&(b)] directly to derive above results. □

We will prove our main theorem 1.5.1 via the standard bootstrap argument. The bootstrap assumption is as follows,

$$\sup_{t\in[0,T]}(1+t)^{-\delta}\|(h,\Lambda\psi)(t)\|_{H^{N_0}} + \|e^{it\Lambda}(h+i\Lambda\psi)\|_Z + (1+t)\|(h,\Lambda\psi)(t)\|_{W^{4,2\alpha}} \lesssim \epsilon_1 := \epsilon_0^{5/6}. \tag{5.11}$$

We split the argument into two parts, which corresponding to the following two propositions, which are sufficient to close the argument,

**Proposition 5.0.6.** *Under the bootstrap assumption* (5.11)*, we have the following estimate,*

$$\sup_{t\in[0,T]}(1+t)^{-\delta}\|(h,\Lambda\psi)(t)\|_{H^{N_0}} \lesssim \epsilon_0. \tag{5.12}$$

**Proposition 5.0.7.** *Under the bootstrap assumption* (5.11) *and the energy estimate* (5.12)*, we have the following estimate,*

$$\sup_{t\in[0,T]} \|e^{it\Lambda}(h+i\Lambda\psi)\|_Z + (1+t)\|(h,\Lambda\psi)(t)\|_{W^{4,2\alpha}} \lesssim \epsilon_0. \tag{5.13}$$

## 5.1 Energy estimate

The main goal in this section is to prove our main Theorem 1.5.1. Since the energy of $(U_1, U_2)$ is comparable with the energy of $(h, \Lambda\psi)$ (see(3.94)), it is sufficient to do energy estimate for the system of equations satisfied by $U_1$ and $U_2$. Let $N_0$ be the prescribed top regularity level. From (3.95) and (3.96), we know that the system of equations satisfied by $(U_1, U_2)$ reads as follows,

$$\begin{cases} \partial_t U_1 - \Lambda U_2 = T_{\sqrt{\lambda}\alpha} U_2 - T_V.\nabla U_1 + \mathcal{R}_1, \\ \\ \partial_t U_2 + \Lambda U_1 = -T_{\sqrt{\lambda}\alpha} U_1 - T_V \cdot \nabla U_2 + \mathcal{R}_2. \end{cases} \tag{5.14}$$

The precise formulations of good remainder terms $\mathcal{R}_1$ and $\mathcal{R}_2$ are not so important in the energy estimate part. From (3.95) and (3.96), we know that they are good error terms, i.e.,

$$\|\mathcal{R}_1\|_{H^{N_0}} + \|\mathcal{R}_2\|_{H^{N_0}} \lesssim_{N_0} \big[\|(h,\Lambda\psi)\|_{W^{4,2\alpha}} + \|(h,\Lambda\psi)\|_{W^4}^2\big]\|(\eta,\Lambda\psi)\|_{H^{N_0}}. \tag{5.15}$$

Define the energy of $U_1$ and $U_2$ as follows,

$$E_{N_0}(t) := \frac{1}{2}\Big[\|U_1\|_{L^2} + \|U_2\|_{L^2} + \sum_{k+j=N_0, 0\le k,j\in\mathbb{Z}} \|\partial_1^k\partial_2^j U_1\|_{L^2}^2 + \|\partial_1^k\partial_2^j U_2\|_{L^2}^2\Big]. \tag{5.16}$$

From (5.14), we have

$$\begin{aligned} \Big|\frac{d}{dt}E_{N_0}(t)\Big| &\lesssim \|(U_1,U_2)\|_{H^{N_0}}\|(\mathcal{R}_1,\mathcal{R}_2)\|_{H^{N_0}} + \sum_{k+j=N_0, 0\le k,j\in\mathbb{Z}} \Big|\int_{\mathbb{R}^2} \big[\partial_1^k\partial_2^j U_1 \partial_1^k\partial_2^j[-T_V\cdot\nabla U_1] \\ &+\partial_1^k\partial_2^j U_2 \partial_1^k\partial_2^j[-T_V\cdot\nabla U_2]\big]dx\Big| + \Big|\int_{\mathbb{R}^2} \partial_1^k\partial_2^j U_1 \partial_1^k\partial_2^j([T_{\sqrt{\lambda}\alpha}U_2]) - \partial_1^k\partial_2^j U_2\partial_1^k\partial_2^j([T_{\sqrt{\lambda}\alpha}U_1])\Big| \\ &\lesssim_{N_0} \big[\|\nabla V\|_{\widetilde{W}^1} + \|(h,\Lambda\psi)\|_{\widehat{W^{4,\alpha}}} + \|(h,\Lambda\psi)\|_{\widehat{W^4}}^2\big]\|(U_1,U_2)\|_{H^{N_0}}^2 + \mathcal{E}_{N_0}, \\ &\lesssim_{N_0} \big[\|(h,\Lambda\psi)\|_{\widehat{W^{4,\alpha}}} + \|(h,\Lambda\psi)\|_{\widehat{W^4}}^2\big]\|(U_1,U_2)\|_{H^{N_0}}^2 + \mathcal{E}_{N_0}, \end{aligned} \tag{5.17}$$

where

$$\mathcal{E}_{N_0} = \sum_{k+j=N_0, 0\le k,j\in\mathbb{Z}} \Big| \int_{\mathbb{R}^2} \Big[\partial_1^k\partial_2^j U_1 T_{\sqrt{\lambda}\alpha}\partial_1^k\partial_2^j U_2 - \partial_1^k\partial_2^j U_2 T_{\sqrt{\lambda}\alpha}\partial_1^k\partial_2^j U_1\Big]\Big|$$

$$= \sum_{k+j=N_0, 0\le k,j\in\mathbb{Z}} \Big| \int_{\mathbb{R}^2} \Big[\partial_1^k\partial_2^j U_1 \big(T_{\sqrt{\lambda}\alpha} - (T_{\sqrt{\lambda}\alpha})^*\big)\partial_1^k\partial_2^j U_2\Big]\Big|.$$

Recall that $\sqrt{\lambda}\alpha \in M_1^{1/2}$ is a symbol of order $1/2$ and it's real, hence from Lemma 1.3.6, we know that the symbol of adjoint operator is itself, and operator $(T_{\sqrt{\lambda}\alpha})^* - T_{\sqrt{\lambda}\alpha}$ is of order $-1/2$, hence the following estimate holds,

$$\mathcal{E}_{N_0} \lesssim M_1^{1/2}(\sqrt{\lambda}\alpha)\|(U_1,U_2)\|_{H^{N_0}}^2 \lesssim \|\nabla h\|_{\widehat{W^2}}\|(U_1,U_2)\|_{H^{N_0}}^2. \tag{5.18}$$

Combining above estimate with (5.17) and (3.94), we have

$$\Big|\frac{d}{dt}E_{N_0}(t)\Big| \lesssim_{N_0} \big[\|(h,\Lambda\psi)\|_{\widehat{W^{4,\alpha}}} + \|(h,\Lambda\psi)\|_{\widehat{W^4}}^2\big]\|(h,\Lambda\psi)\|_{H^{N_0}}^2. \tag{5.19}$$

*Proof of Proposition* 5.0.6. Note that the following estimate holds under the bootstrap assumption (5.11),

$$\|(h,\Lambda\psi)(t)\|_{W^4} \lesssim \|(h,\Lambda\psi)(t)\|_{H^{N_0}}^{1/4}\|(h,\Lambda\psi)(t)\|_{W^{4,2\alpha}}^{3/4} \lesssim (1+t)^{-3/4+2\delta}\epsilon_0.$$

From (5.19), the following estimate holds for any $t\in[0,T]$,

$$\|(h,\Lambda\psi)(t)\|_{H^{N_0}}^2 \lesssim \epsilon_0^2 + \int_0^t \frac{\epsilon_1^3}{(1+s)^{1-2\delta}}ds \lesssim (1+t)^{2\delta}\epsilon_0^2.$$

Hence (5.12) holds. □

### 5.1.1 Quadratic terms of the good remainders

In this subsection, we will find out explicitly what the quadratics terms of $\mathcal{R}_1$ and $\mathcal{R}_2$ are to help readers to understand the fact that we can gain "1" derivative for the inputs putted in $L^\infty$–type space in energy estimate (5.19). Recall that

$$U_1 = h + T_\alpha h, \quad U_2 = \Lambda[\omega + T_\beta\omega], \quad \omega = \psi - T_{B(\eta)\psi}h, \quad \alpha = \sqrt{a} - 1, \quad \beta = \sqrt{\lambda/|\xi|} - 1$$

$$\Lambda_1[B] = \Lambda^2\psi, \quad \Lambda_1[a] = \Lambda_1[\partial_t B] = -\Lambda^2 h, \quad \Lambda_1[\alpha] = -\frac{1}{2}\Lambda^2 h, \quad \Lambda_1[\beta] = 0.$$

From above definition and (5.14), we can reformulate it in terms of $h$ and $\psi$ as the following system of equations,

$$\begin{cases} \partial_t\, h - \Lambda^2\psi = \widetilde{Q}_1(\, h, \psi) + \Lambda_2[\mathcal{R}_1](\, h, \psi) + \text{cubic and higher}_1, \\ \\ \partial_t \Lambda\psi + \Lambda\, h = \widetilde{Q}_2(\, h, \psi) + \Lambda_2[\mathcal{R}_2](\, h, \psi) + \text{cubic and higher}_2, \end{cases} \tag{5.20}$$

where

$$\widetilde{Q}_1(\, h, \psi) = -\Lambda^2(T_{\Lambda^2\psi}\, h) + \frac{1}{2}T_{\Lambda^2\, h}\Lambda^2\psi + \frac{1}{2}T_{\Lambda^4\psi}\, h - \frac{1}{2}(T_{\Lambda^2\, h}|\nabla|^{1/2}\Lambda\psi) - T_{\nabla\psi}\cdot\nabla\, h,$$

$$\widetilde{Q}_2(\, h, \psi) = \Lambda(T_{\Lambda^2\psi}\Lambda^2\psi - T_{\Lambda^2\, h}\, h) + \Lambda(\frac{1}{2}T_{\Lambda^2\, h}\, h) + \frac{1}{2}(T_{\Lambda^2\, h}|\nabla|^{1/2}\, h) - T_{\nabla\psi}\cdot\nabla\Lambda\psi.$$

Recall that $h$ and $\psi$ satisfy the system of equations (1.3), hence we have

$$\Lambda_2[\mathcal{R}_1](h,\psi) = \widetilde{Q}(h,\psi) - \widetilde{Q}_1(h,\psi), \quad \text{recall (3.57) for details of } \widetilde{Q}(h,\psi),$$

$$\Lambda_2[\mathcal{R}_2](h,\psi) = \Lambda\big[-\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}|\Lambda^2\psi|^2\big] - \widetilde{Q}_2(h,\psi).$$

For the High $\times$ High interaction case and the case when the output frequency is less than 1, we can safely put the input with higher frequency in $L^\infty$ and the other input in $L^2$. Hence, we only need to consider the Low $\times$ High interaction and High $\times$ Low interaction with the size of frequency of output greater than 1.

In the following, we use notation $f(\xi-\sigma,\sigma) \doteqdot g(\xi-\sigma,\sigma)$ to represent that the difference between symbol $f(\xi-\sigma,\sigma)$ and symbol $g(\xi-\sigma,\sigma)$ contains at least two degrees of smallness and does not lose derivative at the same time. More precisely, $f(\xi-\sigma,\sigma) \doteqdot g(\xi-\sigma,\sigma)$ if and only if

$$\|f(\xi-\sigma,\sigma) - g(\xi-\sigma,\sigma)\psi_k(\xi)\psi_{k_1}(\xi-\sigma)\psi_{k_2}(\sigma)\|_{\mathcal{S}^\infty} \lesssim 2^{2\min\{k_1,k_2\}}, \quad k,k_1,k_2\in\mathbb{Z}.$$

Let us check $\Lambda_2[\mathcal{R}_2](h,\psi)$ first, it's easy to see that we only have to check the following term

$$T_{\nabla\psi}\cdot\nabla\Lambda\psi - \Lambda(T_{\nabla\psi}\cdot\nabla\psi).$$

Its corresponding symbol is given as $(\xi-\sigma)\cdot\sigma(\tanh(|\xi|)-\tanh(|\sigma|))\theta(\xi-\sigma,\sigma)$ and

$$|(\xi-\sigma)\cdot\sigma(\tanh(|\xi|)-\tanh(|\sigma|))\theta(\xi-\sigma,\sigma)| \lesssim |\xi-\sigma|^2\theta(\xi-\sigma,\sigma). \tag{5.21}$$

That is to say, there are two degree of smallness inside the symbol and it does not lose derivative at the same time.

To see the good structure inside $\Lambda_2[\mathcal{R}_1](h,\psi)$, we only have to check the following term,

$$\widetilde{Q}(h,\psi)+\Lambda^2(T_{\Lambda^2\psi}h)+\nabla\cdot(T_{\nabla\psi}h),$$

and its corresponding symbol is given as follows,

$$\widetilde{q}(\xi-\sigma,\sigma)+\Big[\frac{e^{|\xi|}-e^{-|\xi|}}{e^{|\xi|}+e^{-|\xi|}}\frac{e^{|\sigma|}-e^{-|\sigma|}}{e^{|\sigma|}+e^{-|\sigma|}}|\xi||\sigma|-\xi\cdot\sigma\Big]\theta(\sigma,\xi)=r_1(\xi-\sigma,\sigma)+r_2(\xi-\sigma,\sigma),$$

where

$$r_1(\xi-\sigma,\sigma)=\widetilde{q}(\xi-\sigma,\sigma)\theta(\xi-\sigma,\xi),$$

$$r_2(\xi-\sigma,\sigma)=\Big[\widetilde{q}(\xi-\sigma,\sigma)+\frac{e^{|\xi|}-e^{-|\xi|}}{e^{|\xi|}+e^{-|\xi|}}\frac{e^{|\sigma|}-e^{-|\sigma|}}{e^{|\sigma|}+e^{-|\sigma|}}|\xi||\sigma|-\xi\cdot\sigma\Big]\theta(\sigma,\xi),$$

and $r_1(\xi-\sigma,\sigma)$ corresponding to the aforementioned "weak null structure" inside Dirichlet-Neumann operator. Recall (3.57), as a result of calculation, we have the following two estimates,

$$\begin{aligned}|r_1(\xi-\sigma,\sigma)|\doteqdot&\Big|\frac{\xi\cdot\sigma+|\xi||\sigma|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\sigma|}+e^{-|\sigma|})}\frac{e^{|\xi|-|\sigma|}-e^{|\sigma|-|\xi|}}{|\xi|-|\sigma|}\\&+\frac{-2\xi\cdot(\xi-\sigma)|\sigma|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\sigma|}+e^{-|\sigma|})}\frac{(|\sigma|-|\xi|-1)e^{|\sigma|-|\xi|}-(|\xi|-|\sigma|-1)e^{|\xi|-|\sigma|}}{(|\xi|-|\sigma|)^2}\Big|\theta(\xi-\sigma,\xi)\\\lesssim&\ e^{-|\xi|}\theta(\xi-\sigma,\xi),\end{aligned}\tag{5.22}$$

$$\begin{aligned}|r_2(\xi-\sigma,\sigma)|\doteqdot&\Big|\frac{2e^{-|\xi|-|\sigma|}}{(e^{|\xi|}+e^{-|\xi|})(e^{|\sigma|}+e^{-|\sigma|})}[|\sigma||\xi|+|\sigma|-\xi\cdot\sigma-\xi\cdot\sigma/|\xi|]\Big|\theta(\sigma,\xi)\\\lesssim&\ e^{-|\xi|}\theta(\sigma,\xi).\end{aligned}\tag{5.23}$$

From (5.21), (5.22) and (5.23), it's easy to see that the symbols either have two degrees of smallness inside or are bounded by $e^{-|\xi|}$, which corresponding to the N-steps iterations we did before in (3.83) to lower the regularity level to $L^2$.

We end this subsection with a quick summary about the structures of quadratic terms inside the remainder terms $\mathcal{R}_1$ and $\mathcal{R}_2$. Only two types of quadratic terms appear: (i) one part doesn't lose derivative and has two degrees of smallness inside the symbol. (ii) the other part always allows us to put the input with relatively larger frequency in $L^\infty$ type space. It consists of the following three categories: (1) the output frequency is less than 1, (2) High $\times$ High type interaction, (3) $H^\infty$-part,

its size of symbol is bounded from above by $e^{-|\xi|}$.

## 5.2 The Set-up of The Improved $Z$-norm Estimate

In this section, we introduce the set-up of doing the improved $Z$-norm estimate for the solutions of (5.4), which further enables us to derive the improved $L^\infty$ estimate from Lemma 5.0.5. The argument is reduced naturally to control the following three parts: good type quadratic terms, bad type quadratic terms and remainder terms, which will be specified in this section. We will deal with each of them separately in section 5.4, 5.5 and 5.6.

### 5.2.1 The set-up

For any fixed $k \in \mathbb{Z}$, we define associated sets as follows,

$$\chi_k^1 := \{(k_1, k_2) : k_1, k_2 \in \mathbb{Z}, k \leq \max\{k_1, k_2\} - 5 \leq \min\{k_1, k_2\}\},$$

$$\chi_k^2 := \{(k_1, k_2) : k_1, k_2 \in \mathbb{Z}, \min\{k_1, k_2\} \leq k - 5, |\max\{k_1, k_2\} - k| \leq 4\},$$

$$\chi_k^3 := \{(k_1, k_2) : k_1, k_2 \in \mathbb{Z}, k - 5 < \min\{k_1, k_2\} \leq \max\{k_1, k_2\} < k + 5\}.$$

The set $\chi_k^1$ corresponds to the High $\times$ High interaction with the very low frequency output; the set $\chi_k^2$ corresponds to the Low $\times$ High and the High $\times$ Low type interaction with a very low frequency input; the set $\chi_k^3$ corresponds to the type of interaction where the frequencies of two inputs and the output are all comparable. From the symmetry between two inputs, we assume that $k_2 = \min\{k_1, k_2\}$ in the rest of this paper, otherwise we can do change of variables to switch the roles of $k_1$ and $k_2$.

Recall the equation satisfied by $u$ in (5.4), we define the associated profile of $u$ as $f(t) := e^{it\Lambda}u(t)$ and then rewrite the equation (5.4) in terms of profile $f(t)$ as follows,

$$\partial_t f = \sum_{\mu,\nu\in\{+,-\}} T^{\mu,\nu}(f^\mu, f^\nu) + \mathcal{R}', \tag{5.24}$$

where $\mathcal{R}' = e^{it\Lambda}\mathcal{R}$. The bilinear operators $T^{\mu,\nu}(\cdot,\cdot)$ in (5.24) is defined as follows,

$$T^{\mu,\nu}(g,h) := \mathcal{F}^{-1}\Big[\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{g}(\xi-\eta)\widehat{h}(\eta)q_{\mu,\nu}(\xi-\eta,\eta)d\eta\Big].$$

In later argument, we will find that the decomposition of time interval contributes a "$\log t$" growth for the endpoint of the case $\xi = 2\eta$. This growth caused by technicality can be avoid by doing the following normal form transformation at the very beginning as follows,

$$v(t) := u(t) + \sum_{\mu,\nu\in\{+,-\}} \sum_{k\in\mathbb{Z},(k_1,k_2)\in\chi_k^3} A_{\mu,\nu}(u_{k_1}^{\mu}(t), u_{k_2}^{\nu}(t)), \tag{5.25}$$

where the symbol $m_{\mu,\nu}(\cdot,\cdot)$ of $A_{\mu,\nu}(\cdot,\cdot)$ is defined as follows,

$$m_{\mu,\nu}(\xi-\eta,\eta) := -\frac{q_{\mu,\nu}(\xi-\eta,\eta)}{i\Phi^{\mu,\nu}(\xi,\eta)}. \tag{5.26}$$

Define $g(t) := e^{it\Lambda}v(t)$. From (5.24) and (5.25), we have

$$\widehat{g}(t,\xi) = \widehat{f}(t,\xi) + \sum_{\mu,\nu\in\{+,-\}} \sum_{k\in\mathbb{Z},(k_1,k_2)\in\chi_k^3} \int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)} m_{\mu,\nu}(\xi-\eta,\eta)\widehat{f_{k_1}^{\mu}}(t,\xi-\eta)\widehat{f_{k_2}^{\nu}}(t,\eta)d\eta. \tag{5.27}$$

From estimate (5.29) in Lemma 5.2.1, we can see that the $Z$-norm of $f(t)$ and $g(t)$ are comparable, Hence, it would be sufficient to prove the following estimate to close the argument,

$$\sup_{t_1,t_2\in[2^{m-1},2^{m+1}]} \|g(t_2)-g(t_1)\|_Z \lesssim 2^{-\delta m}\epsilon_0, \quad [2^{m-1},2^{m+1}] \subset [0,T]. \tag{5.28}$$

In the rest of this paper, time "$t$" will be naturally restricted inside the time interval $[2^{m-1},2^{m+1}]$ and "$m$" is a fixed and sufficiently large number.

**Lemma 5.2.1.** *Under the bootstrap assumption* (3.43)*, the following estimate holds,*

$$\sup_{t\in[0,T]} \Big\| \sum_{\mu,\nu\in\{+,-\}} \sum_{k\in\mathbb{Z},(k_1,k_2)\in\chi_k^3} e^{it\Lambda} A_{\mu,\nu}(u_{k_1}^{\mu}(t), u_{k_2}^{\nu}(t)) \Big\|_Z \lesssim \epsilon_0. \tag{5.29}$$

*Proof.* The proof of this Lemma requires the techniques established in section 5.5. Hence we postpone the proof to subsubsection 5.5.3. □

From the definition of $Z$-norm, the bootstrap assumption (3.43), and the linear decay estimates in Lemma 5.0.5, we have the following estimates,

$$\|P_k f(t)\|_{L^2} \lesssim \begin{cases} 2^{-N_0 k_+} 2^{\delta m}\epsilon_0 & \text{if } k \geq 0 \\ \\ \sum_{j\geq -k} \|Q_{k,j} f\|_{L^2} \lesssim 2^{(1-\alpha)k}\epsilon_1 & \text{if } k \leq 0, \end{cases} \tag{5.30}$$

$$\|\widehat{f_k}(\xi)\|_{L^\infty} \lesssim 2^{-\alpha k-6k_+}2^{\delta m}\epsilon_1, \quad \|e^{-it\Lambda}P_k f\|_{L^\infty} \lesssim 2^{-\alpha k-m-4.5k_+}\epsilon_1. \tag{5.31}$$

Note that the $L^\infty$- estimate in (5.31) is not sharp when $k$ is sufficiently small. There is another way to estimate it by choosing $\theta = 1-2\alpha/3$ in (5.10) in Lemma 5.0.5. As a result, the following estimate holds,

$$\|e^{it\Lambda}P_k f\|_{L^\infty} \lesssim 2^{-m+\alpha m/3}2^{\alpha k}\|f_k\|_{L^1} \lesssim 2^{-m+\alpha m/3}\epsilon_1, \quad \text{if } k \le 0.$$

To sum up, we have the following linear decay estimate at low frequency part,

$$\|e^{-it\Lambda}P_k f\|_{L^\infty} \lesssim \min\{2^{-m-\alpha k}, 2^{-(1-\alpha/3)m}\}\epsilon_1, \quad \text{if } k \le 0. \tag{5.32}$$

### 5.2.2 Splitting the quadratic terms

In this subsection, we split the quadratic terms into two types, good type and bad type, based on the properties of their associated phases. For good type phase, the size of phase is not highly degenerated. For bad type phase, the size of phase is highly degenerate but the two types of angle inside $\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)$ and $\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)$ agree. One of them proportionally decreases to zero as the other one decreases to zero. With this key observation, we can utilize the fact that the angle is very small when a bad phase is highly degenerated.

From the normal form transformation that we did in (5.25), we know that quadratic terms with $(k_1,k_2) \in \chi_k^3$ are canceled out. We only have to consider the case when $(k_1,k_2) \in \chi_k^1 \cup \chi_k^2$.

#### Good type phases

Recall that phase $\Phi^{\mu,\nu}(\xi,\eta)$ is defined as follows,

$$\Phi^{\mu,\nu}(\xi,\eta) = \big(\Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|)\big).$$

It is easy to verify the following estimate holds when $k_2 \le k_1$,

$$|\Phi^{\mu,\nu}(\xi,\eta)|\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta) \sim 2^{k_1/2+k_{1,-}/2}, \ (\mu,\nu) \in \{(-,-),(+,+)\}, (k_1,k_2) \in \chi_k^1, \tag{5.33}$$

$$|\Phi^{\mu,\nu}(\xi,\eta)|\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta) \sim 2^{k_1/2+k_{1,-}/2}, \ (\mu,\nu) \in \{(-,-),(-,+)\}, (k_1,k_2) \in \chi_k^2. \tag{5.34}$$

From (5.33) and (5.34), we can see that the sizes of phases in various scenarios are not highly degenerated. We refer above mentioned scenarios as good type phases in later context.

**Bad type phases**

For the other possible cases that were not mentioned in previous subsubsection, we call them bad type phases. For simplicity, we drop frequencies localization functions in the following context but one should remember that $|\xi| \sim 2^k$, $|\xi-\eta| \sim 2^{k_1}$ and $|\eta| \sim 2^{k_2}$ with $(k_1,k_2) \in \chi_k^1 \cup \chi_k^2$ and $k_2 \le k_1$. Note that

$$\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta) = \Lambda'(|\xi|)\frac{\xi}{|\xi|} - \mu\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|},$$

$$\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta) = \mu\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} - \nu\Lambda'(|\eta|)\frac{\eta}{|\eta|}.$$

It turns out that the relative size between $\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)$ and $\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)$ plays an essential role to close the argument. Hence it is necessary to consider the relation between $\angle(\xi,\mu(\xi-\eta))$ and $\angle(\mu(\xi-\eta),\nu\eta)$ with the angle $\angle(\xi,\nu\eta)$ dyadically localized and fixed. We will see that, in the bad type phases scenario, these two angles agree and their sizes are proportional.

For $(k_1,k_2) \in \chi_k^1$ and $(\mu,\nu) \in \{(+,-),(-,+)\}$, we have $\nu = -\mu$ and

$$\angle(\xi,\mu(\xi-\eta)) \sim \angle(\xi,\nu\eta), \quad \frac{\angle(\xi-\eta,-\eta)}{\angle(\xi,\nu\eta)} = \frac{|\xi|}{|\xi-\eta|}, \Longrightarrow \angle(\xi-\eta,-\eta) = 2^{k-k_1}\angle(\xi,\nu\eta).$$

For $(k_1,k_2) \in \chi_k^2$, and $(\mu,\nu) \in \{(+,-),(+,+)\}$, we have $\mu = +$ and

$$\angle(\xi-\eta,\nu\eta) \sim \angle(\xi,\nu\eta), \quad \angle(\xi,\xi-\eta) \sim \frac{|\eta|}{|\xi-\eta|}\angle(\xi,\nu\eta) = 2^{k_2-k_1}\angle(\xi,\nu\eta).$$

To sum up, from above discussion, (5.24) and (5.27), we can rewrite the equation satisfied by the frequency localized profile $g(t)$ as follows,

$$\begin{aligned}\partial_t g_k = &\sum_{\mu,\nu\in\{+,-\}}\sum_{i=1,2,3}\sum_{(k_1,k_2)\in\chi_k^i} T^{\mu,\nu}(f_{k_1},f_{k_2}) + P_k[\mathcal{R}'] - \sum_{\mu,\nu\in\{+,-\}}\sum_{k_1,k_2\in\chi_k^3}\mathcal{F}^{-1}\Big[\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\\ &\times\big[q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f_{k_1}^\mu}(t,\xi-\eta)\widehat{f_{k_2}^\nu}(t,\eta) - m_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\widehat{f_{k_1}^\mu}(t,\xi-\eta)\widehat{f_{k_2}^\nu}(t,\eta)\big)\big]d\eta\Big]\\ &= \mathrm{good}_k(t) + \mathrm{bad}_k(t) + P_k[\mathcal{R}'], \end{aligned} \tag{5.35}$$

where

$$\mathrm{good}_k(t) = \sum_{(k_1,k_2,\mu,\nu)\in\mathcal{P}^k_{\mathrm{good}}} T^{\mu,\nu}(f_{k_1}^\mu, f_{k_2}^\nu), \tag{5.36}$$

$$\mathrm{bad}_k(t) = \sum_{(k_1,k_2,\mu,\nu)\in\mathcal{P}^k_{\mathrm{bad}}} T^{\mu,\nu}(f_{k_1}^\mu, f_{k_2}^\nu) + \sum_{(k_1,k_2)\in\chi_k^3}\sum_{\mu,\nu\in\{+,-\}}\mathcal{F}^{-1}\big[K^{\mu,\nu}(f_{k_1}^\mu, f_{k_2}^\nu)\big], \tag{5.37}$$

where $\mathcal{P}^k_{\text{good}}$ and $\mathcal{P}^k_{\text{bad}}$ denote the set of possible combination of $(k_1, k_2)$ and $(\mu, \nu)$ as follows,

$$\mathcal{P}^k_{\text{good}} = \chi^1_k \times \{(-,-),(+,+)\} \cup \chi^2_k \times \{(-,-),(-,+)\},$$

$$\mathcal{P}^k_{\text{bad}} = \chi^1_k \times \{(+,-),(-,+)\} \cup \chi^2_k \times \{(+,-),(+,+)\},$$

and bilinear terms $K^{\mu,\nu}(\cdot,\cdot)$ is defined as follows,

$$K^{\mu,\nu}(f^\mu_{k_1}, f^\nu_{k_2}) = \int e^{it\Phi^{\mu,\nu}(\xi,\eta)} m_{\mu,\nu}(\xi-\eta,\eta)\partial_t\big(\widehat{f^\mu_{k_1}}(t,\xi-\eta)\widehat{f^\nu_{k_2}}(t,\eta)\big)d\eta. \tag{5.38}$$

From (5.35), the following identity holds,

$$P_k g(t_2) - P_k g(t_1) = \int_{t_1}^{t_2} \text{good}_k(t) + \text{bad}_k(t) + P_k[\mathcal{R}'(t)]dt.$$

Hence to prove (5.28), it will be sufficient if we can derive the following two propositions,

**Proposition 5.2.2.** *Under the bootstrap assumption* (3.43) *and the energy estimate* (4.1)*, we have the following estimates for the remainder term* $\mathcal{R}'$*:*

$$\sup_{t_1,t_2\in[2^{m-1},m+1]} \sup_{k\in\mathbb{Z},j\geq\max\{-k,0\}} 2^{\delta j}\| \int_{t_1}^{t_2} P_k[\mathcal{R}'(t)]dt\|_{B_{k,j}} \lesssim 2^{-\delta m}\epsilon_0. \tag{5.39}$$

$$\sup_{t\in[2^{m-1},m+1]} \|\mathcal{R}'(t)\|_Z + 2^{-(1-\theta)k+\theta m}\|P_k\big(\mathcal{R}'(t)\big)\|_{L^2} \lesssim 2^{-m}\epsilon_0, \quad \theta\in[0,1]. \tag{5.40}$$

*Proof.* Postponed to section 5.6. □

**Proposition 5.2.3.** *Under the bootstrap assumption* (3.43) *and the energy estimate* (4.1)*, we have following Z-norm estimates for any* $t_1, t_2 \in [2^{m-1}, 2^{m+1}]$*,*

$$\sup_{k\in\mathbb{Z},j\geq\max\{-k,0\}} 2^{\delta j}\| \int_{t_1}^{t_2} \text{good}_k(t)dt\|_{B_{k,j}} \lesssim 2^{-\delta m}\epsilon_0, \tag{5.41}$$

$$\sup_{k\in\mathbb{Z},j\geq\max\{-k,0\}} 2^{\delta j}\| \int_{t_1}^{t_2} \text{bad}_k(t)dt\|_{B_{k,j}} \lesssim 2^{-\delta m}\epsilon_0. \tag{5.42}$$

*Proof.* In section 5.4 and 5.5, we will mainly prove (5.41) and (5.42) respectively under the assumption that Proposition 5.2.2 holds. □

**Lemma 5.2.4.** *Under the bootstrap assumption* (3.43)*, the energy estimate* (4.1) *and Proposition*

*5.2.2, the following estimates hold,*

$$\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}}2^{-k}t^{-\delta}\|f_k(t,x)\|_{L^2}\lesssim\epsilon_0, \tag{5.43}$$

$$\sup_{t\in[2^{m-1},2^{m+1}]}\sup_{k\in\mathbb{Z}}2^{-k}\|\partial_t f_k(t,x)\|_{L^2}\lesssim 2^{-m}\epsilon_0, \tag{5.44}$$

$$\sup_{t\in[2^{m-1},2^{m+1}]}\sup_{k\leq 0}\|\widehat{f_k}(t,\xi)\|_{L^\infty_\xi}\lesssim 2^{3\delta m}\epsilon_0. \tag{5.45}$$

*Proof.* Recall (5.24). From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, (5.7) and (5.8) in Lemma 5.0.4, and (5.40) in Proposition 5.2.2, the following estimate holds,

$$\sup_{t\in[2^{m-1},2^{m+1}]}\sup_{k\in\mathbb{Z}}2^{-k}\|\partial_t f_k(t,x)\|_{L^2}\lesssim\sup_{t\in[2^{m-1},2^{m+1}]}2^{-k}\|P_k[\mathcal{R}'](t)\|_{L^2}$$

$$+\sum_{k_2\leq k_1}2^{-k+k+k_{1,+}}\|f_{k_2}(t)\|_{L^2}\|e^{-it\Lambda}f_{k_1}(t)\|_{L^\infty}\lesssim 2^{-m}\epsilon_0+\sum_{k_2\leq k_1}2^{-m+(1-2\alpha)k_2-3k_{1,+}}\epsilon_1^2\lesssim 2^{-m}\epsilon_0. \tag{5.46}$$

Recall that the initial data belongs to $L^1$, hence initially it is of size $\epsilon_0$. More precisely,

$$\sup_{k\in\mathbb{Z}}2^{-k}\|f_k(0,x)\|_{L^2}\lesssim\|f(0,x)\|_{L^1}\lesssim\epsilon_0. \tag{5.47}$$

Combining (5.47) and (5.46), it is easy to see our desired estimate (5.43) holds. From (5.24), the following estimate holds straightforwardly for any $k\in\mathbb{Z},k\leq 0$,

$$\|\widehat{f_k}(t,\xi)\|_{L^\infty_\xi}\lesssim\epsilon_0+\sum_{\mu,\nu\in\{+,-\},k_1,k_2\in\mathbb{Z}}\|\int_0^t e^{it\Phi^{\mu,\nu}(\xi,\eta)}q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^\mu_{k_1}}(\xi-\eta)\times$$

$$\widehat{f^\nu_{k_2}}(\eta)\psi_k(\xi)d\eta\|_{L^\infty_\xi}+\|\int_0^t\widehat{\mathcal{R}}(s,\xi)\psi_k(\xi)dt\|_{L^\infty_\xi}.$$

Note that the remainder term $\mathcal{R}$ is cubic and higher. By $L^2-L^2-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, it is easy to derive the following estimate,

$$\|\int_0^t\widehat{\mathcal{R}}(s,\xi)\psi_k(\xi)dt\|_{L^\infty_\xi}\lesssim 2^{2\delta m}\epsilon_0. \tag{5.48}$$

For the quadratic terms, we do integration by parts in time when $-2m\leq k_1,k_2\leq 2\beta m$. Note that

there are only "$m^2$" cases. From estimates in Lemma 5.0.4 and Lemma 5.3.1, we have

$$\Big\| \frac{q_{\mu,\nu}(\xi-\eta,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)} \Big\|_{L^\infty_{\xi,\eta}} \lesssim 2^{k-\min\{k,k_2\}-2k_{1,-}+3k_{1,+}/2}.$$

Therefore, from above estimate, $L^2-L^2$ type estimate, (5.46) and (5.43), we have

$$\sum_{\mu,\nu\in\{+,-\},k_1,k_2\in\mathbb{Z}} \| \int_0^t e^{it\Phi^{\mu,\nu}(\xi,\eta)} q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^\mu_{k_1}}(\xi-\eta)\widehat{f^\nu_{k_2}}(\eta)d\eta\|_{L^\infty_\xi} \lesssim \sum_{k_2\le -2m, \text{or } k_1\ge 2\beta m} 2^{m+k+k_{1,+}}$$

$$\times \sup_{t\le 2^m}\|f_{k_1}(t)\|_{L^2}\|f_{k_2}\|_{L^2} + \sum_{-2m\le k_2\le k_1\le 2\beta m} 2^{k-\min\{k,k_2\}-2k_{1,-}+3k_{1,+}/2} \sup_{t\le 2^m}\|f_{k_1}(t)\|_{L^2}\|f_{k_2}(t)\|_{L^2}$$

$$+ \sum_{-2m\le k_2\le k_1\le 2\beta m} 2^{k-\min\{k,k_2\}-2k_{1,-}+3k_{1,+}/2} \int_0^t \big(\|\partial_t f_{k_1}(s)\|_{L^2}\|f_{k_2}(s)\|_{L^2}+\|f_{k_1}(s)\|_{L^2}\|\partial_t f_{k_2}(s)\|_{L^2}\big)ds$$

$$\lesssim m^2 2^{2\delta m}\epsilon_0 \lesssim 2^{3\delta m}\epsilon_0. \tag{5.49}$$

From (5.48) and (5.49), it is easy to see our desired estimate (5.45) holds. We can see that it is better than estimate (5.31) when $k$ is sufficiently small. □

### 5.2.3 Some Lemmas

We will use the following lemma for operators with angular cutoff function very often in the bad type phases scenarios. We state it and prove it here.

**Lemma 5.2.5.** *For $l,k,k_1,k_2\in\mathbb{Z}$, $l\le 0$, $k_2\le k_1$, and $f,g\in L^2\cap L^1$, we define a bilinear form as follows,*

$$T(f,g)=\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\psi_l(\angle(\xi,\nu\eta))m(\xi,\eta)f^\mu(\xi-\eta)g^\nu(\eta)d\eta,$$

*where $\mu,\nu\in\{+,-\}$, $m(\xi,\eta)\in S^\infty$, then following estimates hold,*

$$\|T(f,g)\|_{L^2}\lesssim \|m\|_{L^\infty_{\xi,\eta}}\min\big\{2^{2k_2+l}\|f\|_{L^2}\|g\|_{L^\infty_\xi}, 2^{k_2+l/2}\|f\|_{L^2}\|g\|_{L^2}, 2^{k_2+k_1+l}\|f\|_{L^\infty_\xi}\|g\|_{L^2}\big\}, \tag{5.50}$$

$$\|T(f,g)\|_{L^2}\lesssim \|m\|_{L^\infty_{\xi,\eta}}\min\{2^{k+k_1+l}\|f\|_{L^\infty_\xi}\|g\|_{L^2}, 2^{k+l/2}\|f\|_{L^2}\|g\|_{L^2}, 2^{k_2+k+l}\|g\|_{L^\infty_\xi}\|f\|_{L^2}\}, \tag{5.51}$$

$$\|T(f,g)\|_{L^2}\lesssim \|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\min\{\|f_{k_1}\|_{L^2}\|\mathcal{F}^{-1}[e^{-it\Lambda(\xi)}g(\xi)\psi_{k_2}(\xi)]\|_{L^\infty_x},$$

$$2^{(k_1-k_2)/2}\|g(\eta)\psi_{k_2}(\eta)\|_{L^2}\|\mathcal{F}^{-1}[e^{-it\Lambda(\xi)}f(\xi)\psi_{k_1}(\xi)]\|_{L^\infty_x}\}. \tag{5.52}$$

*Proof.* • We first estimate (5.50) and (5.51).To see the role of the angle $\angle(\xi,\nu\eta)$, we first do angular localizations for two inputs. For any given small number $0<2^n\lesssim 1$, we can decompose the unite circle $\mathbb{S}^1$ into the union of angular sections with bounded (with upper bound given by an absolute constant) overlaps, where each sector has angular size $2^n$. These cutoff functions form a partition of unity. We label those sectors by their angles $\omega=\xi/|\xi|$, use $|\omega|$ to denotes the size of angle and use $b_n^{\omega}(\xi)$ to denote a fixed standard bump function that supported in this sector.

With above definitions and notation, we are ready to prove (5.50). From orthogonality in $L^2$, we have

$$\|T(f,g)\|_{L^2}^2\lesssim \sum_{\substack{|\omega_1|,|\omega_2|\sim 2^{l+k_2-k_1}\\ |\omega_1\pm\omega_2|\sim 2^{l+k_2-k_1}\\ |\omega_3|\sim 2^l,|\omega_1-\nu\omega_3|\sim 2^l}} \Big\| \int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)m(\xi,\eta)$$

$$\times b_{l+k_2-k_1}^{\omega_1}(\xi)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)f^{\mu}(\xi-\eta)g^{\nu}(\eta)b_l^{\omega_3}(\eta)d\eta\Big\|_{L^2_\xi}^2 \tag{5.53}$$

$$\lesssim \|m\|_{L^\infty_{\xi,\eta}}^2\min\{\sum_{\omega_2}\sup_{\omega_1,\omega_3}2^{4k_2+2l}\|f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L^2}^2\|g(\eta)\|_{L^\infty_\eta}^2,$$

$$\sum_{\omega_3}\sup_{\omega_1,\omega_2}2^{2k_2+l}\|f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L^2}^2\|b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)g(\eta)\|_{L^2_\eta}^2\}. \tag{5.54}$$

In the above summations, $\omega_1$, $\omega_2$, and $\omega_3$ are all same as listed in the first summation of (5.54).We will use the following two facts: (i) when sum with respect to $\omega_2$, the multiplicity (i.e., how many times it has been counted) of the summation is a finite number; (ii) when sum with respect to $\omega_3$, the multiplicity of the summation is $2^{k_1-k_2}$ because there are $2^{k_1-k_2}$ number of sectors $\omega_2$ corresponds to the same sector $\omega_3$. Hence

$$\sum_{\omega_3}\sup_{\omega_1,\omega_2}2^{2k_2+l}\|f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L^2}^2\|b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)g(\eta)\|_{L^2_\eta}^2$$

$$\lesssim 2^{2k_2+l+k_1+k_2+l}\|f\|_{L^\infty_\xi}^2\sum_{\omega_3}\sup_{\omega_1,\omega_2}\|b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)g(\eta)\|_{L^2_\eta}^2\lesssim 2^{2k_1+2k_2+2l}\|f\|_{L^\infty_\xi}^2\|g\|_{L^2}^2.$$

Therefore, it's easy to derive our desired estimate (5.50).

On the other hand, if we first use the size of support of $\xi$, then the following estimate holds

$$\text{L.H.S. of (5.54)}\ \lesssim \|m\|_{L^\infty_{\xi,\eta}}^2\sum_{\substack{\omega_1,\omega_2,\omega_3\\ \text{same as above}}}2^{2k+l}\|f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L^2}^2$$

$$\times\|b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)g(\eta)\|_{L^2_\eta}^2\lesssim\|m\|_{L^\infty_{\xi,\eta}}^2\min\big\{\sum_{\omega_3}\sup_{\omega_1,\omega_2}2^{2k+2l+k_1+k_2}\|f\|_{L^\infty_\xi}^2\|b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)g\|_{L^2_\eta}^2,$$

$$2^{2k+l}\|f\|_{L^2}^2\|g\|_{L^2}^2, \sum_{\omega_2}\sup_{\omega_1,\omega_3} 2^{2k+2k_2+2l}\|f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L^2}^2\|g\|_{L_\xi^\infty}^2\}$$

$$\lesssim \|m\|_{L_{\xi,\eta}^\infty}^2 \min\{2^{2k+2k_1+2l}\|f\|_{L_\xi^\infty}^2\|g\|_{L^2}^2, 2^{2k+l}\|f\|_{L^2}^2\|g\|_{L^2}^2, 2^{2k_2+2k+2l}\|g\|_{L_\xi^\infty}^2\|f\|_{L^2}^2\}.$$

Hence finish the proof of (5.51).

- Now we proceed to prove (5.52). From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, the following estimate holds,

$$(5.53)\lesssim \|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}}^2 \min\big\{\sum_{\omega_2}\sup_{\omega_1,\omega_3}\|f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L^2}^2$$

$$\times\|\mathcal{F}^{-1}[e^{-it\Lambda(\eta)}g(\eta)b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)]\|_{L_x^\infty}^2, \sum_{\omega_3}\sup_{\omega_1,\omega_2}\|g(\eta)b_l^{\omega_3}(\eta)\psi_{k_2}(\eta)\|_{L^2}^2$$

$$\times\|\mathcal{F}^{-1}[e^{-it\Lambda(\xi-\eta)}f(\xi-\eta)b_{l+k_2-k_1}^{\omega_2}(\xi-\eta)\psi_{k_1}(\xi-\eta)\|_{L_x^\infty}^2\big\}$$

$$\lesssim \|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}}^2 \min\big\{\|f_{k_1}\|_{L^2}^2\|\mathcal{F}^{-1}[e^{-it\Lambda(\eta)}g_{k_2}(\eta)]\|_{L_x^\infty}^2, 2^{(k_1-k_2)}\|g_{k_2}\|_{L^2}^2\|\mathcal{F}^{-1}[e^{-it\Lambda(\eta)}f_{k_1}(\eta)]\|_{L_x^\infty}^2\big\}.$$

Hence finishing the proof of (5.52). Note that in above estimate, we used the facts that the multiplicity of summation with respect to $\omega_3$ is $2^{k_1-k_2}$ and the kernel of symbol $b_l^\omega(\xi)\psi_k(\xi)$ belongs $L^1$, where $l,k\in\mathbb{Z}$. □

**Lemma 5.2.6.** *For $m,k,k_1,k_2,\kappa,l\in\mathbb{Z}$, $t\in[2^{m-1},2^m]$, $2^{-\kappa}\le 2^{m-2\delta m}$, and $f,g\in L^2$, we define bilinear forms as follow,*

$$T_1(f,g)=\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\frac{\psi_0(2^{-\kappa}\Phi^{\mu,\nu}(\xi,\eta))}{\Phi^{\mu,\nu}(\xi,\eta)}\psi_k(\xi)m(\xi,\eta)\widehat{f}_{k_1}^{\mu}(\xi-\eta)\widehat{g}_{k_2}^{\nu}(\eta)d\eta,$$

$$T_2(f,g)=\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\frac{\psi_0(2^{-\kappa}\Phi^{\mu,\nu}(\xi,\eta))}{\Phi^{\mu,\nu}(\xi,\eta)}\psi_k(\xi)m(\xi,\eta)\widehat{f}_{k_1}^{\mu}(\xi-\eta)\widehat{g}_{k_2}^{\nu}(\eta)\varphi_l(\angle(\xi,\nu\eta))d\eta,$$

*where $\mu,\nu\in\{+,-\}$, $m(\xi,\eta)\in S^\infty$, then the following estimates hold,*

$$\|T_1(f,g)\|_{L^2}\lesssim 2^{-\kappa}\|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\sup_{|\lambda|\le 2^{\delta m}}\min\{\|e^{-i(t+\lambda 2^{-\kappa})\Lambda}f_{k_1}\|_{L^\infty}\|g_{k_2}\|_{L^2},$$

$$\|e^{-i(t+\lambda 2^{-\kappa})\Lambda}g_{k_2}\|_{L^\infty}\|f_{k_1}\|_{L^2}\}+2^{-10m-\kappa+k}\|m\|_{L_{\xi,\eta}^\infty}\|f_{k_1}\|_{L^2}\|g_{k_2}\|_{L^2}, \tag{5.55}$$

$$\|T_2(f,g)\|_{L^2}\lesssim 2^{-\kappa}\|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\sup_{|\lambda|\le 2^{\delta m}}\min\{2^{(k_1-k_2)/2}\|e^{-i(t+\lambda 2^{-\kappa})\Lambda}f_{k_1}\|_{L^\infty}\|g_{k_2}\|_{L^2},$$

$$\|e^{-i(t+\lambda 2^{-\kappa})\Lambda}g_{k_2}\|_{L^\infty}\|f_{k_1}\|_{L^2}\}+2^{-10m-\kappa+k}\|m\|_{L_{\xi,\eta}^\infty}\|f_{k_1}\|_{L^2}\|g_{k_2}\|_{L^2}. \tag{5.56}$$

*Proof.* To prove (5.55), we use inverse Fourier transform to rewrite $T_1(f,g)$ as follows,

$$T_1(f,g)=\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2}2^{-\kappa}e^{i(t+\lambda 2^{-\kappa})\Phi^{\mu,\nu}(\xi,\eta)}\widehat{\chi}(\lambda)\psi_k(\xi)m(\xi,\eta)\widehat{f}^{\mu}_{k_1}(\xi-\eta)\widehat{g}^{\nu}_{k_2}(\eta)d\eta d\lambda,$$

where

$$\widehat{\chi}(\lambda)=\int_{\mathbb{R}}e^{-i\lambda x}\frac{\psi_0(x)}{x}dx,\Longrightarrow|\widehat{\chi}(\lambda)|\lesssim(1+|\lambda|)^{-N_0^3}. \tag{5.57}$$

Hence, when $|\lambda|\leq 2^{\delta m}$, we use $L^2-L^\infty$ bilinear estimate (1.13) in Lemma 1.3.7, which gives the first part of estimate (5.55). When $|\lambda|\geq 2^{\delta m}$, from (5.57), $\widehat{\chi}(\lambda)$ provides fast decay. After using the size of support of $\xi$ first and then use $L^2-L^2$ type estimate, we derive the second part of estimate (5.55). The estimate (5.56) can be proved very similarly after combining above argument with the proof of (5.52). We omit details here. □

## 5.3 Analysis of Phases

In this section, we will mainly study the properties of phases associated with different types of quadratic terms. Recall that $\Lambda(|\xi|)=\sqrt{\tanh|\xi||\xi|}$. After direct computations, we have

$$\Lambda'(r)=\frac{4re^{2r}+e^{4r}-1}{2(1+e^{2r})^{3/2}(e^{2r}-1)^{1/2}r^{1/2}}\geq 0,\quad r\geq 0,$$

$$\Lambda''(r)=\frac{2e^{4r}(1+8r^2)-1-e^{8r}-8e^{6r}r(2r-1)-8e^{2r}r(1+2r)}{4(1+e^{2r})^{5/2}(e^{2r}-1)^{3/2}r^{3/2}},\quad r\geq 0.$$

An important observation is that $\Lambda''(r)\leq 0$ and $\Lambda''(r)=0$ if and only if $r=0$. To show this fact, we only have to prove that the numerator is non-positive. We define $f(r)$ to be the numerator of $\Lambda''(r)$ and then have the following decomposition,

$$f(r):=g(r)+h(r),\quad g(r):=16e^{4r}r^2-8e^{6r}r^2-8e^{2r}r^2=-8e^{2r}r^2(e^{2r}-1)^2,$$

$$h(r):=2e^{4r}-1-e^{8r}-8e^{6r}r(r-1)-8e^{2r}r(1+r).$$

Obviously, $g(r)\leq 0$ and $g(r)=0$ if and only $r=0$. It remains to check $h(r)$. After taking up to four derivatives for $h(r)$, we have $h^{(n)}(0)=0$ for $n\in\{0,1,2,3\}$, and the following estimate holds,

$$h^{(4)}(r)=-128e^{2r}(5+5r+r^2+27e^{4r}(r+3r^2)+32e^{6r}-27e^{4r}-4e^{2r})<0,$$

hence $h(r) \leq 0$ and $h(r) = 0$ if and only $r = 0$.

The asymptotic behavior of $\Lambda(r)$ at 0 and $\infty$ can be described as follows,

$$\lim_{r\to 0} \Lambda'(r) = 1, \quad \lim_{r\to\infty} \Lambda'(r) r^{1/2} = \frac{1}{2}, \quad \Lambda(r) = r - \frac{1}{6} r^3 + o(r^4), \text{if } r \ll 1. \tag{5.58}$$

Now, we proceed to consider the phases associated with the quadratic terms. Recall that they are given as follows,

$$\Phi^{\mu,\nu}(\xi,\eta) := \Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|), \quad \mu,\nu \in \{+,-\}, \xi,\eta \in \mathbb{R}^2.$$

As an example (other types can be analyzed similarly), we analyze the phase given as follows,

$$\Lambda(|\xi|) - \Lambda(|\xi+\eta|) + \Lambda(|\eta|) = \mathrm{I} + \mathrm{II}, \quad \mathrm{I} := \Lambda(|\xi| + |\eta|) - \Lambda(|\xi+\eta|),$$

$$\mathrm{II} := \Lambda(|\xi|) + \Lambda(|\eta|) - \Lambda(|\xi| + |\eta|).$$

As mentioned in the Introduction section, an very important fact in above decomposition is that both I and II are positive, which can be derived from the fact that $\Lambda'(r) \geq 0$ and $\Lambda''(r) \leq 0$.

Note that I is zero if and only if $\xi$ and $\eta$ are parallel and in the same direction. Now, we proceed to estimate the size of II. Define

$$f(r,s) = \Lambda(r) + \Lambda(s) - \Lambda(r+s), 0 \leq s \leq r,$$

then, we have

$$f(r,s) = \int_0^r [\Lambda'(\tau_1) - \Lambda'(s+\tau_1)] d\tau_1 = \int_0^r \int_0^s -\Lambda''(\tau_1+\tau_2) d\tau_2 d\tau_1 \geq 0. \tag{5.59}$$

Note that, if $r \leq 1$, then $-\Lambda''(r) \sim r$, and if $r \geq 1$, then $-\Lambda''(r) \sim r^{-3/2}$. From (5.59), we can see that the following estimate holds if $r, s \leq 1$,

$$r^2 s \lesssim f(r,s) \lesssim r^2 s.$$

If $s \leq 1 \leq r$, then we have

$$s \lesssim \int_{1/2}^1 \int_0^s -\Lambda''(\tau_1+\tau_2) d\tau_1 d\tau_2 \leq f(r,s) \leq \Lambda(s) \leq s.$$

At last, if $1 \le s \le r$, then we have

$$s^{\frac{1}{2}} \lesssim \int_{s/2}^{s}\int_{s/2}^{s} -\Lambda''(\tau_1+\tau_2)d\tau_2 d\tau_1 \le f(r,s) \le \Lambda(s) \le |s|^{\frac{1}{2}}.$$

To sum up, we have

$$f(r,s) \sim s^{\frac{1}{2}} \min\{s,1\}^{\frac{1}{2}} \min\{r,1\}^2, \quad 0 \le s \le r. \tag{5.60}$$

With warming up from above discussions, we are ready to proceed into more comprehensive analysis.

**Lemma 5.3.1.** *For $k_1, k_2, k, l \in \mathbb{Z}$, $l \le 2$, $k_2 \le k_1$, $\mu, \nu \in \{+,-\}$, assume that $|\xi| \sim 2^k, |\xi-\eta| \sim 2^{k_1}, |\eta| \sim 2^{k_2}$. We have the following estimates on the size of phase. If $k_2 \le k_1 - 5$, we have*

$$|\Lambda(|\xi|) - \Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|)|\psi_l(\angle(\xi,\nu\eta)) \sim 2^{k_2}\max\{2^{-k_{1,+}/2+2l}, 2^{-k_{2,+}/2+2k_{1,-}}\}, \tag{5.61}$$

$$|\Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|)|\psi_l(\angle(\xi,\nu\eta)) \gtrsim 2^{k_2}\max\{2^{-k_{1,+}/2+2l}, 2^{-k_{2,+}/2+2k_{1,-}}\}. \tag{5.62}$$

*If $k_1 - 5 \le k_2 \le k_1$, we have*

$$|\Lambda(|\xi|) - \Lambda(|\xi-\eta|) + \Lambda(|\eta|)|\psi_l(\angle(\xi,-\eta)) \sim 2^{k}\max\{2^{-k_{+}/2+2k_{1,-}}, 2^{-k_{1,+}/2+2l}\}, \tag{5.63}$$

$$|\Lambda(|\xi|) + \Lambda(|\xi-\eta|) - \Lambda(|\eta|)| \sim 2^{(k+k_-)/2}, \tag{5.64}$$

$$|\Lambda(|\xi|) - \mu\Lambda(|\xi-\eta|) - \nu\Lambda(|\eta|)|\psi_l(\angle(\xi,\nu\eta)) \gtrsim 2^{k}\max\{2^{-k_{+}/2+2k_{1,-}}, 2^{-k_{1,+}/2+2l}\}. \tag{5.65}$$

*We also have several estimates on the gradient of phase. If $k_2 \le k_1 - 5$, we have*

$$2^{-k_{1,+}/2+\max\{l,2k_{1,-}\}} \lesssim \big|\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} - \nu\Lambda'(|\eta|)\frac{\eta}{|\eta|}\big|\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{\max\{l,2k_{1,-}\}}. \tag{5.66}$$

*If $k_1 - 5 \le k_2 \le k_1$, $k \le k_1 - 5$, we have*

$$\big|\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} + \Lambda'(|\eta|)\frac{\eta}{|\eta|}\big|\psi_l(\angle(\xi,\nu\eta)) \sim 2^{k-k_1+\max\{l,2k_{1,-}\}-k_{1,+}/2}, \tag{5.67}$$

$$\big|\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} - \Lambda'(|\eta|)\frac{\eta}{|\eta|}\big|\psi_l(\angle(\xi,\nu\eta)) \sim 2^{-k_{1,+}/2}. \tag{5.68}$$

*Proof.* As in the assumption, $|\xi| \sim 2^k, |\xi-\eta| \sim 2^{k_1}$, and $|\eta| \sim 2^{k_2}$, we omit the frequency localization functions in the following for convenience, but readers should keep the sizes of frequencies in mind.

- We first prove (5.61) and (5.62). Recall that $k_2 \leq k_1 - 5$, if $\nu = -$, then the following estimate holds from (5.60),

$$|\Lambda(|\xi|) - \Lambda(|\xi-\eta|) + \Lambda(\eta)|\psi_l(\angle(\xi,-\eta)) = \big[\Lambda(|\xi|) - \Lambda(|\xi|+|\eta|) + \Lambda(|\eta|)$$

$$+\Lambda(|\xi|+|\eta|) - \Lambda(|\xi-\eta|)\big]\psi_l(\angle(\xi,-\eta)) \sim 2^{(k_2+k_{2,-})/2+2k_{1,-}} + 2^{-k_{1,+}/2}[|\xi|+|\eta|-|\xi-\eta|]\psi_l(\angle(\xi,-\eta))$$

$$\sim 2^{(k_2+k_{2,-})/2+2k_{1,-}} + 2^{-k_{1,+}/2+k_2+2l} \sim 2^{k_2}\max\{2^{-k_{1,+}/2+2l}, 2^{-k_{2,+}/2+2k_{1,-}}\}.$$

If $\nu = +$, then the following estimate holds from (5.60),

$$|\Lambda(|\xi|) - \Lambda(|\xi-\eta|) - \Lambda(|\eta|)|\psi_l(\angle(\xi,\eta)) \sim \big[\Lambda(|\xi-\eta|) - \Lambda(|\xi-\eta|+|\eta|) + \Lambda(|\eta|)$$

$$+\Lambda(|\xi-\eta|+|\eta|) - \Lambda(|\xi|)\big]\psi_l(\angle(\xi,\eta)) \sim 2^{(k_2+k_{2,-})/2+2k_{1,-}} + 2^{-k_{1,+}/2}[|\xi-\eta|+|\eta|-|\xi|]\psi_l(\angle(\xi,\eta)) \quad (5.69)$$

$$\sim 2^{(k_2+k_{2,-})/2+2k_{1,-}} + 2^{-k_{1,+}/2+k_2+2l} \sim 2^{k_2}\max\{2^{-k_{1,+}/2+2l}, 2^{-k_{2,+}/2+2k_{1,-}}\}.$$

When $\mu = -$, the size of phase is bigger than the case when $\mu = +$, hence (5.62) holds.

- We move on to prove (5.63), (5.64), and (5.65). Note that $k_1 - 5 \leq k_2 \leq k_1$. If $\nu = -$, then the following estimate holds from (5.60)

$$|\Lambda(|\xi|) - \Lambda(|\xi-\eta|) + \Lambda(|\eta|)|\psi_l(\angle(\xi,-\eta)) \sim \big[\Lambda(|\xi|) - \Lambda(|\eta|+|\xi|) + \Lambda(|\eta|)$$

$$+\Lambda(|\eta|+|\xi|) - \Lambda(|\xi-\eta|)\big]\psi_l(\angle(\xi,-\eta)) \sim 2^{(k+k_-)/2+2k_{1,-}} + 2^{-k_{1,+}/2}[|\xi|+|\eta|-|\xi-\eta|]\psi_l(\angle(\xi,-\eta))$$

$$\sim 2^{(k+k_-)/2+2k_{1,-}} + 2^{-k_{1,+}/2+k+2l} \sim 2^{k}\max\{2^{-k_+/2+2k_{1,-}}, 2^{-k_{1,+}/2+2l}\}.$$

As $k_2 \leq k_1$, the following estimate holds,

$$2^{(k+k_-)/2} \leq \Lambda(|\xi|) \leq \Lambda(|\xi|) + \Lambda(|\xi-\eta|) - \Lambda(|\eta|) \leq \Lambda(|\xi|) + 2^{-k_{1,+}/2}\big||\xi-\eta|-|\eta|\big| \lesssim 2^{(k+k_-)/2}.$$

In other words, desired estimate (5.64) holds. Since the case when $(\mu,\nu) = (+,-)$ is the smallest, it is easy to see (5.65) holds.

- We proceed to prove (5.66). Note that

$$\big|\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|} - \nu\Lambda'(|\eta|)\frac{\eta}{|\eta|}\big|\psi_l(\angle(\xi,\nu\eta)) \sim \max\{|\Lambda'(|\xi-\eta|) - \Lambda'(|\eta|)|, 2^{-k_{1,+}/2}\angle(\xi-\eta,\nu\eta)\}, \quad (5.70)$$

and the following estimates hold when $k_2 \leq k_1 - 5$,

$$\angle(\xi, \nu\eta) \sim \angle(\xi - \eta, \nu\eta),$$

$$2^{-k_{1,+}/2+2k_{1,-}} \leq \int_{|\xi-\eta|/2}^{|\xi-\eta|} |\Lambda''(r)|dr \leq |\Lambda'(|\xi-\eta|) - \Lambda'(|\eta|)| \lesssim \int_0^{|\xi-\eta|} |\Lambda''(r)|dr \leq 2^{2k_{1,-}}.$$

Hence, it's easy to see (5.66) holds.

• We proceed to prove (5.67). The estimate (5.70) also holds. Note that there exists a certain incompatibility. Recall that $0 < |\xi| \ll |\xi - \eta| \approx |\eta|$ . Hence, if $|\xi - \eta| = |\eta|$ then the angle between $\xi$ and $\eta$ can't be zero; meanwhile, if $\xi - \eta$ and $\eta$ are parallel, then $|\xi - \eta| \neq |\eta|$. Note that

$$\angle(\xi, \nu\eta) = 2^l, k \leq k_1 - 5, \Longrightarrow \angle(\xi - \eta, -\eta) \sim 2^{k-k_1+l} \sim 2^{k-k_{1,-}+l-k_{1,+}}, \tag{5.71}$$

$$\big|\Lambda'(|\xi-\eta|) - \Lambda'(|\eta|)\big| \sim 2^{-3k_{1,+}/2+k_{1,-}} \big||\xi-\eta| - |\eta|\big|. \tag{5.72}$$

From (5.71), we know that (5.67) holds if $l \geq 2k_{1,-} - 10$. It remains to consider the case when $l \leq 2k_{1,-} - 10$. As the angle between $\xi$ and $\nu\eta$ is very small in this case, we have $\big||\xi-\eta| - |\eta|\big| \sim 2^k$, which further implies our desired estimate (5.67) by combining (5.70) and (5.72).

• At last, we prove (5.68). After observing that $(\eta - \xi)/|\eta - \xi|$ and $\eta/|\eta|$ are almost in the same direction (their angle is less than $\pi/2$) as $|\xi| \ll |\xi - \eta|$, it's very straightforward to see the desired estimate (5.68) holds. □

**Lemma 5.3.2.** *For $k_1, k_2, k, l \in \mathbb{Z}$, $l \leq 2$, $k_2 \leq k_1$, $\mu, \nu \in \{+, -\}$, we localize frequencies as follows $|\xi| \sim 2^k, |\xi - \eta| \sim 2^{k_1}, |\eta| \sim 2^{k_2}$. If either $k_2 \leq k_1 - 5$, $\mu = -$ or $k \leq k_1 - 5 \leq k_2$, $\mu\nu = +$, we have*

$$\Big(\Big|\frac{\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big| + \Big|\frac{\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big|\Big) \lesssim 2^{-k_1+k_{1,+}/2}, \tag{5.73}$$

$$\Big[2^{k_1-k_2}\Big|\frac{\partial_\xi\partial_\eta \Phi^{\mu,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)|^2}\Big| + \Big|\frac{\partial_\eta\partial_\eta \Phi^{\mu,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)|^2}\Big|\Big]\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{-k_2-2\max\{l,2k_{1,-}\}+k_{1,+}}. \tag{5.74}$$

*If either $k_2 \leq k_1 - 5$, $\mu = +$, or $k \leq k_1 - 5 \leq k_2$, $\mu\nu = -$, we have,*

$$\Big(2^k\Big|\frac{\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big| + 2^{k_2}\Big|\frac{\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big|\Big) \lesssim 2^{-k_{1,-}+k_{1,+}/2}, \tag{5.75}$$

$$\Big[2^{k-k_2}\Big|\frac{\partial_\xi\partial_\eta \Phi^{\mu,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)|^2}\Big| + \Big|\frac{\partial_\eta\partial_\eta \Phi^{\mu,\nu}(\xi,\eta)}{|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)|^2}\Big|\Big]\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{-\min\{k,k_2\}-2\max\{l,2k_{1,-}\}+k_{1,+}}. \tag{5.76}$$

*Proof.* • We first prove prove (5.73). Under the assumption that $k_2 \leq k_1 - 5$, $\mu = -$ or $k \leq$

$k_1 - 5 \le k_2$, $\mu\nu = +$, we have

$$|\Phi^{\mu,\nu}(\xi,\eta)| \gtrsim 2^{k_1 - k_{1,+}/2}, \quad |\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)| + |\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)| \lesssim 2^{-k_{2,+}/2} \Longrightarrow (5.73) \text{ holds.}$$

- We proceed to prove (5.74). Note that the worst scenario is when $\nabla_\xi$ or $\nabla_\eta$ hits the numerator of $(\xi-\eta)/|\xi-\eta|$ or $\eta/|\eta|$ of $\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)$, as a result, the following estimate holds,

$$|\partial_\xi \partial_\eta \Phi^{\mu,\nu}(\xi,\eta)| \lesssim 2^{-k_1 - k_{1,+}/2}, \quad |\partial_\eta \partial_\eta \Phi^{\mu,\nu}(\xi,\eta)| \lesssim 2^{-k_2 - k_{2,+}/2}. \tag{5.77}$$

For the case we are considering, either estimate (5.66) or estimate (5.68) holds for $\nabla_\eta \Phi^{\mu,\nu}$. Combining (5.77), (5.66) and (5.68), we can see desired estimate (5.74) holds.

- We proceed to prove (5.75). If $k_2 \le k_1 - 5$ and $\mu = +$, then from (5.62), (5.66), and (5.67), we have

$$|\Phi^{\mu,\nu}(\xi,\eta)|\psi_l(\angle(\xi,\nu\eta)) \gtrsim 2^{k_2 + 2\max\{l,k_{1,-}\} - k_{1,+}/2}, \quad |\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)|\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{\max\{l,2k_{1,-}\}},$$

$$|\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)|\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{k_2 - k_1 + \max\{l,2k_{1,-}\} - k_{1,+}/2},$$

hence,

$$\Big(2^k \Big|\frac{\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big| + 2^{k_2}\Big|\frac{\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big|\Big) \lesssim \sup_{l\le 2}\Big(2^{k_1}\Big|\frac{\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big| + 2^{k_2}\Big|\frac{\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)}{\Phi^{\mu,\nu}(\xi,\eta)}\Big|\Big)\psi_l(\angle(\xi,\nu\eta))$$

$$\lesssim \sup_{l\le 2} 2^{\max\{l,2k_{1,-}\} - 2\max\{l,k_{1,-}\} + k_{1,+}/2} \lesssim 2^{-k_{1,-} + k_{1,+}/2}. \tag{5.78}$$

Therefore (5.75) holds for this case. If $k \le k_1 - 5 \le k_2$ and $\mu\nu = -$, then from (5.65), (5.66), and (5.67), we have

$$|\Phi^{\mu,\nu}(\xi,\eta)|\psi_l(\angle(\xi,\nu\eta)) \gtrsim 2^{k + 2\max\{l,k_{1,-}\} - k_{1,+}/2}, \quad |\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)|\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{\max\{l,2k_{1,-}\}},$$

$$|\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta)|\psi_l(\angle(\xi,\nu\eta)) \lesssim 2^{k - k_1 + \max\{l,2k_{1,-}\} - k_{1,+}/2}.$$

Same as we did in the estimate (5.78), it's easy to see (5.75) also holds for this case.

- We proceed to prove (5.76). The proof and the estimate are exactly same as (5.74) except when $k \le k_1 - 5 \le k_2$ and $\mu\nu = -$. For this case, we have

$$\nabla_\eta \Phi^{\mu,\nu}(\xi,\eta) = -\mu(\Lambda'(|\eta-\xi|)\frac{\eta-\xi}{|\eta-\xi|} - \Lambda'(|\eta|)\frac{\eta}{|\eta|}), \tag{5.79}$$

hence

$$|\partial_\xi \partial_\eta \Phi^{\mu,\nu}(\xi,\eta)| \lesssim 2^{-k_1-k_{1,+}/2}, \tag{5.80}$$

$$|\partial_\eta \partial_\eta \Phi^{\mu,\nu}(\xi,\eta)| \lesssim \sup_{|\sigma|\sim 2^{k_2},|\beta|=3} |\partial_\sigma^\beta(\Lambda(|\sigma|))|\big||\eta|-|\eta-\xi|\big| \lesssim 2^{k-2k_1-k_{1,+}/2}. \tag{5.81}$$

Combining (5.80) and (5.81) with estimate (5.67) in Lemma 5.3.1, we can see the desired estimate (5.76) holds. Therefore, finishing the proof of this lemma. □

We proceed to study the phase associated with the cubic terms, which generally has the following form,

$$\Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma) := \Lambda(\xi) - \mu\Lambda(\xi-\eta) - \nu\Lambda(\eta-\sigma) - \tau\Lambda(\sigma), \quad \mu,\nu,\tau \in \{+,-\}. \tag{5.82}$$

The structure inside the phase $\Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma)$ is very delicate when all frequencies are comparable, as both $\nabla_\sigma \Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma)$ and $\nabla_\eta \Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma)$ could be "0" (see also subsection 5.5.3). However, when two inputs are not comparable, the structures inside the phase can be summarized as the following lemma.

**Lemma 5.3.3.** *Assume that* $|\xi| \sim 2^k, |\xi-\eta| \sim 2^{k_1}, |\eta| \sim 2^{k_2}, |\eta-\sigma| \sim 2^{k_1'}$, *and* $|\sigma| \sim 2^{k_2'}$, $k, k_1, k_2, k_1', k_2' \in \mathbb{Z}$, $k_2' \le k_1'$. *If either* $k \le k_2 - 5 \le k_1 - 4$ *or* $k_2 \le k_1 - 5 \le k - 4$, *then the following estimate holds,*

$$|\nabla_\sigma \Phi^{\mu,\nu,-\nu}(\xi,\eta,\sigma)| \gtrsim 2^{-3k_{1,+}'/2+k_{1,-}'+k_{2,-}}, \tag{5.83}$$

$$|\nabla_\sigma \Phi^{\mu,\nu,\nu}(\xi,\eta,\sigma)|\psi_{\ge\tilde{l}}(\eta-2\sigma) \gtrsim 2^{-3k_{1,+}'/2+k_{1,-}'+\tilde{l}}, \tag{5.84}$$

$$|\nabla_\eta \Phi^{\mu,\nu,\nu}(\xi,\eta,\sigma)|\psi_{\le\tilde{l}}(\eta-2\sigma) \gtrsim 2^{-3\max\{k_{1,+},k_{1,+}'\}/2+\max\{k_{1,-}',k_{1,-}\}+k_1}, \tag{5.85}$$

*for any* $\tilde{l} \le k_{2,-} - 10$.

*Proof.* From (5.82), we have

$$\nabla_\sigma \Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma) = \nu\Lambda'(|\eta-\sigma|)\frac{\eta-\sigma}{|\eta-\sigma|} - \tau\Lambda'(|\sigma|)\frac{\sigma}{|\sigma|}, \tag{5.86}$$

$$\nabla_\eta \Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma) = \mu\Lambda'(\xi-\eta)\frac{\xi-\eta}{|\xi-\eta|} - \nu\Lambda'(\eta-\sigma)\frac{\eta-\sigma}{|\eta-\sigma|}. \tag{5.87}$$

Therefore

$$|\nabla_\sigma \Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma)| \gtrsim \max\{2^{-3k_{1,+}'/2}2^{k_{1,-}'}\big||\eta-\sigma|-|\sigma|\big|, 2^{-k_{1,+}'/2}\angle(\eta-\sigma,\nu\tau\sigma)\}, \tag{5.88}$$

$$|\nabla_\eta \Phi^{\mu,\nu,\tau}(\xi,\eta,\sigma)| \gtrsim 2^{-3\max\{k_{1,+},k'_{1,+}\}/2}\max\{2^{\max\{k'_{1,-},k_{1,-}\}}\big||\xi-\eta|-|\eta-\sigma|\big|, \angle(\xi-\eta,\mu\nu(\eta-\sigma))\}. \tag{5.89}$$

Note that when $\nu\tau = -$ and $\angle(\eta-\sigma,\nu\tau\sigma) \le 2^{-5}$, we have $|\eta-\sigma|-|\sigma| \sim |\eta|$. From (5.88), now it is easy to see our desired estimate (5.83) holds.

When $\nu\tau = +$, we know that $|\eta-\sigma|-|\sigma|$ and $\angle(\eta-\sigma,\sigma)$ cannot be small at the same time when $\sigma$ is away from $\eta/2$. More precisely, the following estimate holds,

$$2^{k'_{1,-}}\big||\eta-\sigma|-|\sigma|\big| + \angle(\eta-\sigma,\sigma) \gtrsim 2^{\tilde{l}+k'_{1,-}-k'_{1,+}}, \quad \text{if } |\eta-2\sigma| \ge 2^{\tilde{l}}.$$

Combining above estimate with (5.88), it is easy to see our desired estimate (5.84) holds. When $\sigma$ is very close to $\eta/2$, we have $\eta-\sigma \approx \eta/2$. As either $|\eta| \ll |\xi-\eta|$ or $|\eta| \approx |\xi-\eta|$, we know that $|\xi-\eta|-|\eta-\sigma| \sim |\xi-\eta|/2$. As a result, from (5.89), our desired estimate (5.85) holds. □

## 5.4 The Improved $Z$-norm Estimate: Good Type Phases

In this section, we estimate all bilinear terms listed in (5.36). We let $j$ to be fixed and consider all possible scenarios. We expect the argument used in this relatively easier case can help readers to get familiar with how this argument works.

### 5.4.1 When $(k_1,k_2) \in \chi_k^1, (\mu,\nu) \in \{(-,-),(+,+)\}$

Recall that $k_2 \le k_1$. We first use $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.7) in Lemma 5.0.4 to rule out the very high frequency of inputs as follows,

$$\sum_{(k_1,k_2)\in\chi_k^1, k_2\ge((1+\delta)j+\delta m)/(N_0-8)} \|\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1},f_{k_2})dt\|_{B_{k,j}}$$

$$\lesssim \sum_{(k_1,k_2)\in\chi_k^1, k_2\ge((1+\delta)j+\delta m)/(N_0-8)} 2^{\alpha k+6k_++m+j}\|q_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\|e^{-it\Lambda}P_{k_1}f\|_{L^\infty}\|P_{k_2}f\|_{L^2}$$

$$\lesssim \sum_{(k_1,k_2)\in\chi_k^1, k_2\ge((1+\delta)j+\delta m)/(N_0-8)} 2^{\alpha k+6k_++m+j+2k_2-N_0k_1-m-4k_1}\epsilon_1^2 \lesssim 2^{-\delta m-\delta j}\epsilon_0.$$

Then we rule out the very low output frequency. We first use the size of support of $\xi$ and then use $L^2-L^2$ type bilinear estimate (1.13) in Lemma 1.3.7 to derive the following estimate,

$$\sup_{-j\le k\le -(1+\delta)(m+j)/(2+\alpha)} \sum_{(k_1,k_2)\in\chi_k^1} \|\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1},f_{k_2})dt\|_{B_{k,j}}$$

$$\lesssim \sup_{-j\le k\le -(1+\delta)(m+j)/(2+\alpha)} \sum_{|k_1-k_2|\le 10} 2^{\alpha k+6k_++m+j+k}\|q_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\|P_{k_1}f\|_{L^2}\|P_{k_2}f\|_{L^2}$$

$$\lesssim \sup_{-j\le k\le -(1+\delta)(m+j)/(2+\alpha)} 2^{(2+\alpha)k+m+j}\epsilon_1^2 \lesssim 2^{-\delta m-\delta j}\epsilon_0.$$

Therefore, it is sufficient to consider fixed $k$ and $k_1$ in the following range:

$$-(1+\delta)(m+j)/(2+\alpha) \le k+5 \le k_1 \le ((1+\delta)j+\delta m)/(N_0-8), \tag{5.90}$$

since we have at most $(m+j)^2$ cases in total, which is only a logarithmic loss.

**If** $j \ge (1+20\delta)m$

We first do spatial localizations for two inputs and then have the following decomposition,

$$\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1}(t),f_{k_2}(t))dt = \sum_{j_1\ge -k_{1,-},j_2\ge -k_{2,-}} \int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1,j_1}(t),f_{k_2,j_2}(t))dt.$$

In the rest of this paper, we use the convention that $(k_1,j_1)$ and $(k_2,j_2)$ are naturally restricted in $\mathcal{J}$ even without explicitly written. If $\min\{j_1,j_2\}\ge j-\delta j-\delta m$, then we have

$$\sum_{\min\{j_1,j_2\}\ge j-\delta j-\delta m} \|\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1,j_1}(t),f_{k_2,j_2}(t))dt\|_{B_{k,j}} \lesssim \sum_{\min\{j_1,j_2\}\ge j-\delta j-\delta m} 2^{\alpha k+m+j}$$

$$\times 2^{k+(1-2\alpha)k_2-j_1-j_2-3k_{1,+}}\|f_{k_1,j_1}\|_Z\|f_{k_2,j_2}\|_Z \lesssim 2^{m+2\delta m-(1-2\delta)j}\epsilon_0 \lesssim 2^{-2\delta j-2\delta m}\epsilon_0.$$

It remains to consider the case $\min\{j_1,j_2\}\le j-\delta j-\delta m$. Since $|k_1-k_2|\le 5$, we can assume that $j_1=\min\{j_1,j_2\}$, otherwise, we can simply do change of variables to switch the roles of $\xi-\eta$ and $\eta$. For this case, we can do integration by parts in "$\xi$" to see rapidly decay. More precisely, after integration by parts in time once, we have

$$\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1,j_1}(t),f_{k_2,j_2}(t))dt$$

$$= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}\nabla_\xi\cdot\big(\widehat{f^\mu_{k_1,j_1}}(t,\xi-\eta)\widehat{f^\nu_{k_2,j_2}}(t,\eta)a_{\mu,\nu}(t,x,\xi,\eta)\big)d\eta d\xi dt,$$

where

$$a_{\mu,\nu}(t,x,\xi,\eta) = q_{\mu,\nu}(\xi-\eta,\eta)\frac{x+t\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)}{|x+t\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)|^2}. \tag{5.91}$$

Note that

$$|\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)| \lesssim 1 \Longrightarrow \frac{x+t\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)}{|x+t\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)|^2}\varphi_j^k(x) \sim 2^{-j}. \tag{5.92}$$

Hence, we can gain $2^{-j}$ if we do integration by parts in "$\xi$" once. We also need to find out what the maximal loss is. If $\nabla_\xi$ hits input $\widehat{f^\mu_{k_1,j_1}}(\cdot)$, then we lose $2^{j_1}$. If $\nabla_\xi$ hits the cutoff functions or the symbol $a_{\mu,\nu}(t,x,\xi,\eta)$, the total loss is at most $\max\{2^{-k_1},2^{-k},1\}$ and this can be derived from (5.77), (5.81), and (5.7). Since $j_1 \geq \max\{-k_1,0\}$, we at most lose $\max\{2^{j_1},2^{-k}\}$. Therefore, the net gain of doing integration by parts in "$\xi$" once is $\max\{2^{-j+j_1},2^{-j+k}\} \lesssim 2^{-\delta j-\delta m}$. Note that we used the fact that $j+k \geq \delta j+\delta m$, which can be derived from (5.90). We can do this process as many times as we want to see rapidly decay. As a result, we have the following estimate after using $L^\infty_{\xi,\eta}$-norm of symbol and $L^2-L^2$ type estimate,

$$\Big|\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1,j_1}(t),f_{k_2,j_2}(t))dt\Big|\varphi_j^k(x) \lesssim 2^{-10j}\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^2} \tag{5.93}$$

which further gives us

$$\sum_{\min\{j_1,j_2\}\leq j-\delta j-\delta m} \|\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1,j_1}(t),f_{k_2,j_2}(t))dt\|_{B_{k,j}} \lesssim 2^{m+2j-10j}\epsilon_1^2 \lesssim 2^{-2\delta m-2\delta j}\epsilon_0.$$

Therefore finishing the proof for this case.

**If** $j \leq (1+20\delta)m$

For this case, we have

$$k+5 \leq k_1 \leq ((1+\delta)j+\delta m)/(N_0-8) \leq \beta m, \quad \beta := 1/980.$$

Note that

$$\mathcal{F}[\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1}(t),f_{k_2}(t))dt](\xi) = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^\mu_{k_1}}(t,\xi-\eta)\widehat{f^\nu_{k_2}}(t,\eta)q_{\mu,\nu}(\xi-\eta,\eta)d\eta dt,$$

we do integration by parts in time once for above integral and have the following identity,

$$\mathcal{F}[\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1}(t),f_{k_2}(t))dt](\xi) = \sum_{i=1,2} \mathrm{End}^{\mu,\nu,i}_{k_1,k_2} + J^{\mu,\nu,i}_{k_1,k_2}, \tag{5.94}$$

where

$$\mathrm{End}_{k_1,k_2}^{\mu,\nu,i} = (-1)^{i-1}\int_{\mathbb{R}^2} e^{it_i\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}}(t_i,\xi-\eta)\widehat{f^{\nu}}(t_i,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta, \quad (5.95)$$

$$J_{k_1,k_2}^{\mu,\nu,1} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}}(t,\xi-\eta)\widehat{\partial_t f^{\nu}}(t,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta dt, \quad (5.96)$$

$$J_{k_1,k_2}^{\mu,\nu,2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{\partial_t f^{\mu}}(t,\xi-\eta)\widehat{f^{\nu}}(t,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta dt, \quad (5.97)$$

and the symbol $m_{\mu,\nu}(\cdot,\cdot)$ is same as in (5.26).

From Lemma 1.3.8, (5.7) and (5.8) in Lemma 5.0.4, (5.33), and (5.73) in Lemma 5.3.2, we know that the following estimate holds uniformly,

$$\|m_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}} \lesssim 2^{k/2+k_-/2+3k_{1,+}-k_1} \lesssim 2^{3\beta m}. \quad (5.98)$$

From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, (5.44) in Lemma 5.2.4 and (5.98), we have the following estimate if $k_1 \le -4\beta m$ or $k \le -40\beta m$,

$$\sum_{i=1,2}\|\mathrm{End}_{k_1,k_2}^{\mu,\nu,i}\|_{B_{k,j}} + \|J_{k_1,k_2}^{\mu,\nu,i}\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++j+3\beta m}\|f_{k_1}\|_{L^2}\|e^{-it\Lambda}f_{k_2}\|_{L^\infty} + 2^{\alpha k+6k_++j+m+3\beta m}\times$$

$$\Big(\|\partial_t f_{k_1}\|_{L^2}\|e^{-it\Lambda}f_{k_2}\|_{L^\infty} + \|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|\partial_t f_{k_2}\|_{L^2}\Big) \lesssim 2^{\alpha k+(1-2\alpha)k_1+3\beta m}\epsilon_1^2 \lesssim 2^{-\delta m-\delta j}\epsilon_0.$$

It remains to consider the case when $k_1 \ge -4\beta m$ and $k \ge -40\beta m$.

- *Estimate of endpoint case.* For this case, we first do spatial localizations for two inputs and have the following decomposition,

$$\mathrm{End}_{k_1,k_2}^{\mu,\nu,i} = \sum_{j_1\ge -k_{1,-},j_2\ge -k_{2,-}} \mathrm{End}_{k_1,j_1,k_2,j_2}^{\mu,\nu,i}, \quad (5.99)$$

$$\mathrm{End}_{k_1,j_1,k_2,j_2}^{\mu,\nu,i} = (-1)^{i-1}\int_{\mathbb{R}^2} e^{it_i\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t_i,\eta)m_{\mu,\nu}(\xi-\eta,\eta)d\eta.$$

We first consider the case when $\max\{j_1,j_2\} \ge m+k+k_1-4\beta m$. From the $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.98), we can put the input with larger spatial concentration in $L^2$ and the other one in $L^\infty$. As a result, the following estimate holds,

$$\sum_{\max\{j_1,j_2\}\ge m+k+k_1-4\beta m}\|\mathrm{End}_{k_1,j_1,k_2,j_2}^{\mu,\nu,i}\|_{B_{k,j}} \lesssim \sum_{\max\{j_1,j_2\}\ge m+k+k_1-4\beta m} 2^{\alpha k+6k_++j+3\beta m}$$

$$\times 2^{-\max\{j_1,j_2\}-m-\alpha(k_1+k_2)}\|f_{k_1,j_1}\|_Z\|f_{k_2,j_2}\|_Z \lesssim 2^{-m-(1+\alpha)(k+k_1)+8\beta m}\epsilon_1^2 \lesssim 2^{-2\delta m-2\delta j}\epsilon_0.$$

Then we consider the case when $\max\{j_1,j_2\} \leq m+k+k_1-4\beta m$. Note that, we only have at most $m^3$ cases to consider. Hence, it's sufficient to consider fixed $j_1$ and $j_2$. For this case, we do integration by parts in "$\eta$" many times to see rapidly decay. More precisely, after integration by parts in $\eta$, we have the following identity,

$$\mathrm{End}^{\mu,\nu,i}_{k_1,j_1,k_2,j_2} = \frac{(-1)^{i-1}}{t_i}\int_{\mathbb{R}^2} e^{it_i\Phi^{\mu,\nu}(\xi,\eta)}\nabla_\eta\cdot\big(\widehat{f^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t_i,\eta)\tilde{m}_{\mu,\nu}(\xi-\eta,\eta)\big)d\eta, \quad (5.100)$$

$$\tilde{m}_{\mu,\nu}(\xi-\eta,\eta) = -\frac{m_{\mu,\nu}(\xi-\eta,\eta)\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)}{i|\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)|^2}. \quad (5.101)$$

If $\nabla_\eta$ hits $\widehat{f}_{k_1,j_1}$ and $\widehat{f}_{k_2,j_2}$, we at most lose $2^{\max\{j_1,j_2\}-k-k_1+5k_{1,+}/2}$. If $\nabla_\eta$ hits the symbol $\tilde{m}_{\mu,\nu}(\cdot,\cdot)$, then from estimates (5.74) and (5.73) in Lemma 5.3.2, we at most lose $2^{-5k_{1,-}+k_{1,+}}$. Therefore, through integration by parts in "$\eta$" once, we can gain at least $2^{-\beta m}$ as the total gain is $2^{-m}$ but the total loss is at most $\max\{2^{\max\{j_1,j_2\}-k-k_1+5k_{1,+}/2}, 2^{-5k_{1,-}+k_{1,+}}\} \leq 2^{m-\beta m}$. We can do this process many times to see rapidly decay, more precisely, the following estimate holds,

$$\sum_{\max\{j_1,j_2\}\leq m+k+k_1-4\beta m}\|\mathrm{End}^{\mu,\nu,i}_{k_1,j_1,k_2,j_2}\|_{B_{k,j}} \lesssim \sum_{\max\{j_1,j_2\}\leq m+k+k_1-4\beta m} 2^{-10m}\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^2}$$

$$\lesssim 2^{-2\delta m-2\delta j}\epsilon_0.$$

- *Estimate of* $J^{\mu,\nu,i}_{k_1,k_2}$, $i\in\{1,2\}$. After plugging in the equation satisfied by $\partial_t f$ and doing dyadic decompositions for the quadratic terms of $\partial_t f$, we have

$$J^{\mu,\nu,i}_{k_1,k_2} = \sum_{k_1',k_2'\in\mathbb{Z}}\sum_{\mu',\kappa'\in\{+,-\}} J^{\mu,\nu,\tau,\kappa,i}_{k_1',k_2'} + \mathrm{JR}^i_{k_1,k_2}, \quad J^{\mu,\nu,\tau,\kappa,i}_{k_1',k_2'} := \sum_{j_1'\geq -k'_{1,-},j_2'\geq -k'_{2,-},j_i\geq -k_{i,-}} H^{\mu,\nu,\tau,\kappa,i}_{j_i;j_1',j_2'},$$

$$H^{\mu,\nu,\tau,\kappa,i}_{j_1',j_2'} = \sum_{j_i\geq -k_{i,-}} H^i_{j_i;j_1',j_2'}, \quad H^{\mu,\nu,\tau,\kappa,i}_{j_i;j_1'} = \sum_{j_2'\geq -k'_{2,-}} H^i_{j_i;j_1',j_2'}, \quad i\in\{1,2\}, \quad (5.102)$$

where

$$H^{\mu,\nu,\tau,\kappa,1}_{j_i;j_1',j_2'} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)}m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma)d\eta d\sigma dt,$$

$$H^{\mu,\nu,\tau,\kappa,2}_{j_i;j_1',j_2'} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\Phi_2^{\tau,\kappa,\nu}(\xi,\eta,\sigma)}m^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma)\widehat{f^{\tau}_{k_1',j_1'}}(t,\xi-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t,\eta)d\eta d\sigma dt,$$

$$\mathrm{JR}^1_{k_1,k_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}}(t,\xi-\eta)\widehat{\mathcal{R}'^{\nu}}(t,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta dt,$$

$$\mathrm{JR}^2_{k_1,k_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{\mathcal{R}'^{\mu}}(t,\xi-\eta)\widehat{f^{\nu}}(t,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)d\eta dt,$$

$$\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma) = \Lambda(\xi) - \mu\Lambda(\xi-\eta) - \tau\Lambda(\eta-\sigma) - \kappa\Lambda(\sigma), \tag{5.103}$$

$$\Phi_2^{\tau,\kappa,\nu}(\xi,\eta,\sigma) = \Lambda(\xi) - \tau\Lambda(\xi-\sigma) - \kappa\Lambda(\sigma-\eta) - \nu\Lambda(\eta), \tag{5.104}$$

$$m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma) = m_{\mu,\nu}(\xi-\eta,\eta)\big(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma)\big)^{\nu}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta), \tag{5.105}$$

$$m^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma) = m_{\mu,\nu}(\xi-\eta,\eta)\big(q_{\tau\mu,\kappa\mu}(\xi-\sigma,\sigma-\eta)\big)^{\mu}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta). \tag{5.106}$$

Here, we remind readers that "$\tau\nu$ " is understood as the product of signs, e.g., $+- = -$. Also, as before, $f^+ := f$ and $f^- := \bar{f}$.

From (1.12) in Lemma 1.3.7, (5.98), and estimate (5.7) in Lemma 5.0.4, we have the following estimate,

$$\|m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\psi_{k_1'}(\eta-\sigma)\psi_{k_2'}(\sigma)\|_{\mathcal{S}^\infty} + \|m^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma)\psi_{k_1'}(\xi-\sigma)\psi_{k_2'}(\sigma-\eta)\|_{\mathcal{S}^\infty} \lesssim 2^{k_1+4\beta m}. \tag{5.107}$$

From (5.40) in Proposition 5.2.2, we know that the $Z$-norm of $\mathcal{R}'(t)$ decays at rate $2^{-m}$, which compensates the loss from integration with respect to time. The method used in the estimate of $\mathrm{End}^{\mu,\nu,i}_{k_1,k_2}$ can be applied directly to the estimate of $\mathrm{JR}^1_{k_1,k_2}$ and $\mathrm{JR}^2_{k_1,k_2}$. We omit details here.

Now let us proceed to estimate $J^{\mu,\nu,\tau,\kappa,1}_{k_1',j_1',k_2',j_2'}$ and $J^{\mu,\nu,\tau,\kappa,2}_{k_1',j_1',k_2',j_2'}$. Without loss of generality, we assume that $k_2' \le k_1'$. From $L^\infty - L^\infty - L^2$ type trilinear estimate (1.14) in Lemma 1.3.7 and (5.107), the following estimate holds for fixed $k_1'$ and $k_2'$

$$\sum_{i=1,2}\|J^{\mu,\nu,\tau,\kappa,i}_{k_1',k_2'}\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_+ +m+j+k_1+4\beta m}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}$$

$$\lesssim \min\{2^{\alpha k+(1-\alpha)k_1+(1-2\alpha)k_2'-2k_{1,+}'+4\beta m+20\delta m}, 2^{m+\alpha k+(1-\alpha)k_1+(1-2\alpha)k_2'-(N_0-6)k_{1,+}'+4\beta m+20\delta m}\}\epsilon_0. \tag{5.108}$$

For the second bound, we used the fact that $\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty} \lesssim 2^{k_1'}\|f_{k_1'}\|_{L^2} \lesssim 2^{k_1'-N_0k_{1,+}'}\epsilon_1$.

From (5.108), we can rule out the case when $k_1' \ge \beta m$ or $k_2' \le -6\beta m$. Hence, it is sufficient to consider fixed $k_1'$, and $k_2'$ such that $-6\beta m \le k_2' \le k_1' \le \beta m$. Recall that $|\xi-\eta| \approx |\eta|$. Hence, with minor changes, we can estimate $J^{\mu,\nu,\tau,\kappa,2}_{k_1',k_2'}$ and $J^{\mu,\nu,\tau,\kappa,1}_{k_1',k_2'}$ in the same way. In the following, we only show the estimate of $J^{\mu,\nu,\tau,\kappa,1}_{k_1',k_2'}$ in details.

For $J^{\mu,\nu,\tau,-\tau,1}_{k_1',k_2'}$, we can do integration by parts in $\sigma$ many times to rule out the case when $\max\{j_1', j_2'\} \leq m + k_2 + k_1' - 10\beta m$. More precisely, after doing integration by parts in "$\sigma$", we have

$$H^{\mu,\nu,\tau,-\tau,1}_{j_1',j_2'} = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{t} e^{it\Phi_1^{\mu,\tau,-\tau}(\xi,\eta,\sigma)} \nabla_\sigma \cdot \big( \tilde{m}^{\tau,-\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)$$

$$\times \widehat{f^{\mu}_{k_1}}(t,\xi-\eta) \widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma) \widehat{f^{-\tau}_{k_2',j_2'}}(t,\sigma) \big) d\eta d\sigma dt, \tag{5.109}$$

where

$$\tilde{m}^{\tau,-\tau}_{\mu,\nu,1}(\xi,\eta,\sigma) := -\frac{m^{\tau,-\tau}_{\mu,\nu,1}(\xi,\eta,\sigma) \nabla_\sigma \Phi_1^{\mu,\tau,-\tau}(\xi,\eta,\sigma)}{i|\nabla_\sigma \Phi_1^{\mu,\tau,-\tau}(\xi,\eta,\sigma)|^2}. \tag{5.110}$$

From (5.83) in Lemma 5.3.3 and (5.107), we have

$$|\nabla_\sigma \tilde{m}^{\tau,-\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)| \lesssim 2^{-2(k_{2,-}+k'_{1,-})-k_2'+9\beta m} \lesssim 2^{m/2}.$$

With above estimate, we can see that the net gain of doing integration by parts in "$\sigma$" once is at least $\max\{2^{-m/2}, 2^{-m+\max\{j_1',j_2'\}-k'_{2,-}-k'_{1,-}+9\beta m}\}$, which is less than $2^{-\beta m}$. Therefore, we can do this process many times to see rapidly decay.

From (5.107) and the $L^2 - L^\infty - L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, the following estimate holds after putting the input with the higher spatial concentration in $L^2$ and other inputs in $L^\infty$,

$$\sum_{\max\{j_1',j_2'\} \geq m+k_2+k_1'-4\beta m} \|H^{\mu,\nu,\tau,-\tau,1}_{j_1',j_2'}\|_{B_{k,j}} \lesssim \sum_{\max\{j_1',j_2'\} \geq m+k_2+k_1'-4\beta m} 2^{\alpha k + 6k_+ + m + j}$$

$$\times 2^{k_1+4\beta m} 2^{-\alpha k_1 - \alpha k_1' - \alpha k_2'} 2^{-2m-\max\{j_1',j_2'\}} \|f_{k_1',j_1'}\|_Z \|f_{k_2',j_2'}\|_Z \|f_{k_1}\|_Z \lesssim 2^{-m/2}\epsilon_0. \tag{5.111}$$

Now, we proceed to estimate $J^{\mu,\nu,\tau,\tau,1}_{k_1',k_2'}$. By doing integration by parts in $\eta$ (when $\sigma$ is close to $\eta/2$) and $\sigma$ (when $\sigma$ is away from $\eta/2$), we can first rule out the case when $\max\{j_1, j_1', j_2'\} \leq m + k_2 + k_2' - 10\beta m$. More precisely, we first split $H^{\mu,\nu,\tau,\tau,1}_{j_i;j_1',j_2'}$ into two parts as follows,

$$H^{\mu,\nu,\tau,\tau,1}_{j_i;j_1',j_2'} = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)} m^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma) \widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta) \widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)$$

$$\times \widehat{f^{\tau}_{k_2',j_2'}}(t,\sigma) \psi_{\leq k_2-10}(\eta-2\sigma) d\eta d\sigma dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)} m^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)$$

$$\times \widehat{f^{\mu}_{k_1}}(t,\xi-\eta) \widehat{f^{\tau}_{k_1'}}(t,\eta-\sigma) \widehat{f^{\tau}_{k_2'}}(t,\sigma) \psi_{>k_2-10}(\eta-2\sigma) \big) d\eta d\sigma dt. \tag{5.112}$$

For the first integral in (5.112), we do integration by parts in $\eta$. For the second integral in (5.112), we do integration by parts in $\sigma$. As a result, we have

$$H^{\mu,\nu,\tau,\tau,1}_{j_i;j_1',j_2'} = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{t} e^{it\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)} \nabla_\sigma \cdot \big(\tilde{m}^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)\widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{f^{\tau}_{k_2',j_2'}}(t,\sigma)\big)$$

$$+\frac{1}{t} e^{it\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)} \nabla_\eta \cdot \big(\tilde{m}^{\tau,\tau}_{\mu,\nu,2}(\xi,\eta,\sigma)\widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{f^{\tau}_{k_2',j_2'}}(t,\sigma)\big) d\eta d\sigma dt, \tag{5.113}$$

where

$$\tilde{m}^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma) := -\frac{m^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)\nabla_\sigma \Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)\psi_{>k_2-10}(\eta-2\sigma)}{i|\nabla_\sigma \Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)|^2}, \tag{5.114}$$

$$\tilde{m}^{\tau,\tau}_{\mu,\nu,2}(\xi,\eta,\sigma) := -\frac{m^{\mu,\tau,\tau}_{1}(\xi,\eta,\sigma)\nabla_\eta \Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)\psi_{\leq k_2-10}(\eta-2\sigma)}{i|\nabla_\eta \Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)|^2}. \tag{5.115}$$

From (5.84) and (5.85) in Lemma 5.3.3 and (5.107), we have

$$|\nabla_\sigma \tilde{m}^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)| + |\nabla_\eta \tilde{m}^{\tau,\tau}_{\mu,\nu,2}(\xi,\eta,\sigma)| \lesssim 2^{-2(k_{2,-}+k'_{1,-})-k'_2+9\beta m} \lesssim 2^{m/2}. \tag{5.116}$$

With above estimate, we can see that the net gain of doing integration by parts in "$\sigma$" and "$\eta$" once is at least $\max\{2^{-m/2},\, 2^{-m+\max\{j_1,j_1',j_2'\}-k'_{2,-}-k'_{1,-}+2\beta m}\}$, which is less than $2^{-\beta m}$. Therefore, we can do this process many times to see rapidly decay.

It remains to consider the case when $\max\{j_1, j_1', j_2'\} \geq m + k_2 + k_2' - 10\beta m$. From (5.107) and the $L^2 - L^\infty - L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, the following estimate holds after putting the input with the higher spatial concentration in $L^2$ and other inputs in $L^\infty$,

$$\sum_{\max\{j_1,j_1',j_2'\}\geq m+k_2+k_2'-10\beta m} \|H^{\mu,\nu,\tau,\tau,1}_{j_i;j_1',j_2'}\|_{B_{k,j}} \lesssim \sum_{\max\{j_1,j_1',j_2'\}\geq m+k_2+k_2'-10\beta m} 2^{\alpha k+6k_++m+j}2^{k_1+4\beta m}$$

$$\times 2^{-\alpha k_1'-\alpha k_2'-\alpha k_2}2^{-2m-\max\{j_1,j_1',j_2'\}}\|f_{k_1,j_1}\|_Z\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z \lesssim 2^{-m/2}\epsilon_0. \tag{5.117}$$

### 5.4.2 When $(k_1, k_2) \in \chi_k^2, (\mu,\nu) \in \{(-,-),(-,+)\}$

Recall that we previously assumed $k_2 = \min\{k_1,\ k_2\}$ for this case, we have $|k_1 - k| \leq 5$ and $k_2 \leq k_1 - 5$. As what we did before, we can first rule out the very low frequency case and the very high frequency case.

From estimate (1.13) in Lemma 1.3.7 and estimate (5.8) in Lemma 5.0.4, we have

$$\|\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1},f_{k_2})dt\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++m+j}\|q_{\mu,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\|P_{k_1}f\|_{L^2}\|e^{-it\Lambda}P_{k_2}f\|_{L^\infty} \tag{5.118}$$

$$\lesssim 2^{\alpha k_1+m+j+k_1+(1-\alpha)k_1+(2-\alpha)k_2-(N_0-7)k_{1,+}-4k_{2,+}}\epsilon_1^2. \tag{5.119}$$

From (5.119), it's easy to see the following estimates hold,

$$\sup_{k\le -(1+\delta)(m+j+10\delta m)/(4-\alpha)}\sum_{k_2\le k-10}\|\int_{t_1}^{t_2}T^{\mu,\nu}(f_{k_1},f_{k_2})dt\|_{B_{k,j}}\lesssim 2^{-10\delta m-\delta j}\epsilon_0,$$

$$\sup_{k\ge (1+\delta)(m+j+10\delta m)/(N_0-10)}\sum_{k_2\le k-10}\|\int_{t_1}^{t_2}T^{\mu,\nu}(f_{k_1},f_{k_2})dt\|_{B_{k,j}}\lesssim 2^{-10\delta m-\delta j}\epsilon_0.$$

For a fixed $k$ in the range $[-(1+\delta)(m+j+10\delta m)/(4-\alpha),(1+\delta)(m+j+10\delta m)/(N_0-10)]$, we have

$$\sum_{k_2\le -(1+\delta)(m+j+10\delta m)/(2-\alpha)}\|\int_{t_1}^{t_2}T^{\mu,\nu}(f_{k_1},f_{k_2})dt\|_{B_{k,j}}\lesssim \sum_{k_2\le -(1+\delta)(m+j+10\delta m)/(2-\alpha)}\text{R.H.S. of (5.118)}$$

$$\lesssim \sum_{k_2\le -(1+\delta)(m+j+10\delta m)/(2-\alpha)}2^{\alpha k+6k_++m+j+2k_{1,+}+(2-\alpha)k_2-(N_0-6)k_{1,+}}\epsilon_0\lesssim 2^{-10\delta m-\delta j}.$$

Now, it is sufficient to consider fixed $k$ and $k_2$ in the following range,

$$-(1+\delta)(m+j+10\delta m)/(2-\alpha)\le k_2\le k-10, \tag{5.120}$$

$$-(1+\delta)(m+j+10\delta m)/(4-\alpha)\le k\le (1+\delta)(m+j+10\delta m)/(N_0-10). \tag{5.121}$$

**If $j\ge m+20\delta m$.**

Note that the rough estimate (5.92) still holds in this case. The relative sizes of frequencies of inputs do not play a role there. With minor modifications, we can use the method we used in subsubsection 5.4.1 to show that the desired estimate also holds for this case. We omit details for this case here.

**If $j\le m+20\delta m$.**

Under the this assumption, we have at most $m^3$ possible cases in the range listed in (5.120) and (5.121) to consider. We will same notations used in subsection 5.4.1 but the only difference is that now $(k_1,k_2)\in\chi_k^2$ instead of inside $\chi_k^1$. Same as we did before, we do integration by parts in time first and have the same identity as (5.94).

From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.98), the following estimate

holds when $k_2 \leq -\alpha m$,

$$\sum_{i=1,2} \|\mathrm{End}^{\mu,\nu,i}_{k_1,k_2}\|_{B_{k,j}} + \|J^{\mu,\nu,i}_{k_1,k_2}\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++j+3\beta m}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|f_{k_2}\|_{L^2} + 2^{\alpha k+6k_++j+m+3\beta m}\times$$

$$\Big(\|e^{-it\Lambda}\partial_t f_{k_1}\|_{L^\infty}\|f_{k_2}\|_{L^2} + \|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|\partial_t f_{k_2}\|_{L^2}\Big) \lesssim 2^{(1-\alpha)k_2+\alpha m/2+10\beta m}\epsilon_1^2 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

We used the following estimate in above estimate, which can be derived from $L^\infty - L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.40) in Proposition 5.2.2,

$$\|e^{-it\Lambda}\partial_t f_{k_1}\|_{L^\infty} \lesssim \sum_{\mu,\nu\in\{+,-\},k_2'\leq k_1'} \|e^{-it\Lambda}T^{\mu,\nu}(f^\mu_{k_1'}, f^\nu_{k_2'})\|_{L^\infty} + 2^{-m}\|P_{k_1}(\mathcal{R}')\|_Z$$

$$\lesssim \sum_{k_2'\leq k_1'} 2^{k_1'+k_{1,+}'}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|e^{-it\Lambda}f_{k_2'}\|_{L^\infty} + 2^{-2m}\epsilon_0 \lesssim 2^{-2m+\alpha m/2}\epsilon_0.$$

In above estimate, we scarified a little bit decay rate to guarantee the summability with respect to $k_2'$.

Now it is sufficient to consider the case when $k_2 \geq -\alpha m$.

- *Estimate of the endpoint case.* Same as before, we can use $L^2 - L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.98) to rule out the case when $\max\{j_1, j_2\} \geq m + 2k_2 - 4\beta m$ as follows,

$$\sum_{\max\{j_1,j_2\}\geq m+2k_2-4\beta m} \|\mathrm{End}^{\mu,\nu,i}_{k_1,j_1,k_2,j_2}\|_{B_{k,j}} \lesssim \sum_{\max\{j_1,j_2\}\geq m+2k_2-4\beta m} 2^{\alpha k+6k_++j+3\beta m}$$

$$\times 2^{-\max\{j_1,j_2\}-m-\alpha(k_1+k_2)}\|f_{k_1,j_1}\|_Z\|f_{k_2,j_2}\|_Z \lesssim 2^{-m-(2+2\alpha)k_2+15\beta m}\epsilon_1^2 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

For the case when $\max\{j_1, j_2\} \leq m + 2k_2 - 4\beta m$, we will do integration by parts in" $\eta$" to see rapidly decay. If $\nabla_\eta$ hits $\widehat{f}_{k_1,j_1}$ and $\widehat{f}_{k_2,j_2}$, we at most lose $2^{\max\{j_1,j_2\}-2k_2+5k_{1,+}/2}$, which is less than $2^{m-\beta m}$. If $\nabla_\eta$ hits the symbol $\tilde{m}_{\mu,\nu}(\cdot,\cdot)$, then from estimates (5.74) and (5.73) in Lemma 5.3.2, we at most lose $2^{-k_2-4k_{1,-}+k_{1,+}}$, which is less than $2^{m-\beta m}$. Hence the net gain is at least $2^{-\beta m}$ from integration by parts in "$\eta$" once. We can keep doing this process to see rapidly decay.

- *Estimate of $J^{\mu,\nu,i}_{k_1,k_2}$, $i \in \{1,2\}$.* Same as before, the method used in the estimate of $\mathrm{End}^{\mu,\nu,i}_{k_1,k_2}$ can be applied directly to the estimate of $\mathrm{JR}^1_{k_1,k_2}$ and $\mathrm{JR}^2_{k_1,k_2}$. Now we proceed to estimate $J^{\mu,\nu,\tau,\kappa,1}_{k_1',k_2'}$ and $J^{\mu,\nu,\tau,\kappa,2}_{k_1',k_2'}$. Without loss of generality, we assume that $k_2' = \min\{k_1', k_2'\}$.

Recall (5.105) and (5.106). From (1.12) in Lemma 1.3.7, (5.8) in Lemma 5.0.4, and (5.98), we

have the following estimate,

$$\|m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\psi_{k'_1}(\eta-\sigma)\psi_{k'_2}(\sigma)\|_{\mathcal{S}^\infty}+\|m^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma)\psi_{k'_1}(\xi-\sigma)\psi_{k'_2}(\sigma-\eta)\|_{\mathcal{S}^\infty}\lesssim 2^{k'_1+4\beta m}. \quad (5.122)$$

From (5.122) and $L^2-L^\infty-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7, the following estimate holds for fixed $k'_1$ and $k'_2$,

$$\sum_{i=1,2}\|J^{\mu,\nu,\tau,\kappa,i}_{k'_1,k'_2}\|_{B_{k,j}}\lesssim\sum_{i=1,2}2^{\alpha k+6k_++m+j+k'_1+4\beta m}\|e^{-it\Lambda}f_{k_i}\|_{L^\infty}\|e^{-it\Lambda}f_{k'_1}\|_{L^\infty}\|f_{k'_2}\|_{L^2}$$

$$\lesssim\min\{2^{(1-\alpha)(k'_1+k'_2)+11\beta m},2^{(1-\alpha)k'_2+m+4\beta m-(N_0-8)k'_{1,+}}\}\epsilon_0.$$

From above estimate, we can rule out the case when $k'_2\le -14\beta m$ or $k'_1\ge\beta m$. Now It is sufficient to consider the case when $k'_1$ and $k'_2$ are fixed and $-14\beta m\le k'_2\le k'_1\le\beta m$.

Although the $|\eta|$ we consider here could be slightly worse, it is possible to be $2^{-\alpha m}$. The methods used in the $\chi^1_k$ case can be applied directly. Note that $\xi-\eta$ and $\eta/2$ are still not close in the case we are considering, therefore either $\nabla_\sigma\Phi^{\mu,\tau,\kappa}_i$ or $\nabla_\eta\Phi^{\mu,\tau,\kappa}_i$ is big. This fact allows us to rule out the case when all inputs have small spatial concentration.

From (5.83) in Lemma 5.3.3, we know that $\nabla_\sigma\Phi^{\mu,\tau,-\tau}_1$ still has a good lower bound of size $2^{-2\alpha m-2\beta m}$. Hence, as what we did in (5.109), we do integration by parts in "$\sigma$" to rule out the case when $\max\{j'_1,j'_2\}\le m+k'_{2,-}+k'_{1,-}-10\beta m$ for $J^{\mu,\nu,\tau,-\tau,i}_{k'_1,k'_2}$, $i\in\{1,2\}$ . For the case when $\max\{j'_1,j'_2\}\ge m+k'_{2,-}+k'_{1,-}-10\beta m$, estimate (5.111) still holds for the case we are considering.

From (5.84) and (5.85) in Lemma 5.3.3, we know that $\nabla_\sigma\Phi^{\mu,\tau,\tau}_1$ ($\nabla_\sigma\Phi^{\tau,\tau,\nu}_2$) and $\nabla_\eta\Phi^{\mu,\tau,\tau}_1$ ($\nabla_\eta\Phi^{\tau,\tau,\nu}_2$ ) still have a good lower bound of size $2^{-2\alpha m-2\beta m}$. Hence, as what we did in (5.112), we can rule out the case when $\max\{j_1(j_2),j'_1,j'_2\}\le m+k_2+k'_2-10\beta m$ by doing integration by parts in "$\sigma$" when $\sigma$ is away from $\eta/2$ $\big((\xi+\eta)/2\big)$ and do integration by parts in "$\eta$" when $\sigma$ is close to $\eta/2$ $\big((\xi+\eta)/2\big)$ for $J^{\mu,\nu,\tau,\tau,1}_{k'_1,k'_2}$ $\big(J^{\mu,\nu,\tau,\tau,2}_{k'_1,k'_2}\big)$. For the case when $\max\{j_1(j_2),j'_1,j'_2\}\le m+k_2+k'_2-10\beta m$, it is easy to verify that the estimate we did in (5.117) still holds. As we gain extra $2^{-m}$ but only pay the price of at most $2^{4\alpha m}=2^{0.4m}$.

## 5.5 The Improved $Z$-Norm Estimate: Bad Type Phases

In previous section, we study the quadratic terms with good type phases, in which the largest two inputs have the same sign inside good phases. As a result, we do not actually see the high degeneracy of phases. Now, things become different in the bad type phases. The opposite sign of the largest two

inputs will cause cancellations and high degeneracy of bad phases. For this case, delicate analysis of roles of the angles between the output and inputs is required.

### 5.5.1 When $(k_1, k_2) \in \chi_k^1, (\mu, \nu) \in \{(-,+),(+,-)\}$

Similar to what we did in subsection 5.4.1, we can first rule out the very high input frequency case and the very low output frequency case. For the rough multilinear estimates derived at the beginning of subsection 5.4.1, the type of phases does not play a role, hence we will derive the same result. To sum up, $k$, $k_1$, and $j$ are fixed in the range as follows,

$$-(1+\delta)(m+j)/(2+\alpha) \le k+10 \le k_1 \le (1+\delta)(m+j+10\delta m)/(N_0-10) \le \beta m, j \le m+20\delta m, \quad (5.123)$$

which is same as (5.90).

For fixed $k$ and $k_1$ in above range, we dyadically localize the angle between $\xi$ and $\nu\eta$ with threshold chosen to be $\bar{l} := 2k_{1,-}$ and have the following decomposition,

$$\mathcal{F}[\int_{t_1}^{t_2} T^{\mu,\nu}(f_{k_1}(t), f_{k_2}(t))dt](\xi) = \sum_{\bar{l} \le l \le 2} I_l^{\mu,\nu}, \quad I_l^{\mu,\nu} = \sum_{j_1 \ge -k_{1,-}, j_2 \ge -k_{2,-}} I_{l;j_1,j_2}^{\mu,\nu}, \quad (5.124)$$

where

$$I_{l;j_1,j_2}^{\mu,\nu} = \int_{t_1}^{t_2} \int e^{it\Phi^{\mu,\nu}(\xi,\eta)} \varphi_{\bar{l};l}(\angle(\xi,\nu\eta)) q_{\mu,\nu}(\xi-\eta,\eta) \widehat{f_{k_1,j_1}^{\mu}}(t,\xi-\eta) \widehat{f_{k_2,j_2}^{\nu}}(t,\eta) d\eta dt, \quad (5.125)$$

and $\varphi_{\bar{l};l}(\cdot)$ is defined as follows,

$$\varphi_{\bar{l};l}(x) = \begin{cases} \psi_l(x) & \text{if } \bar{l} < l \le 2 \\ \psi_{\le \bar{l}}(x) & \text{if } l = \bar{l}. \end{cases} \quad (5.126)$$

In the following, we mainly estimate $I_l^{\mu,\nu}$, for fixed $l, k$ and $k_1$, as there are at most $m^3$ cases in total, which is only a logarithmic loss.

**If** $\max\{m+l, -k-l\} + 100\delta m \le j \le m + 20\delta m$**.**

For this case, we have $k \le -30\delta m$, otherwise $\max\{m+l,-k-l\} + 100\delta m > m + 20\delta m$ . To deal with this case, an observation is that the estimate (5.92) can be improved if the angle between $\xi$ and $\nu\eta$ is well localized. As a result, through integration by parts in " $\xi$", we can further rule out the case when $j \ge \max\{m+l,-k-l\} + 100\delta m$.

- We first consider the case when $\min\{j_1, j_2\} \geq j - \delta m$. From (5.51) in Lemma 5.2.5 and (5.7) in Lemma 5.0.4, we can derive the following esitmate,

$$\sum_{\min\{j_1,j_2\}\geq j-\delta m} \|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_{B_{k,j}} \lesssim \sum_{\min\{j_1,j_2\}\geq j-\delta m} 2^{\alpha k+6k_++m+j+k+k_{1,+}+k+l/2}$$

$$\times \|Q_{k_1,j_1} f\|_{L^2} \|Q_{k_2,j_2} f\|_{L^2} \lesssim 2^{(2+\alpha)k+m+2\delta m-j-2\alpha k_1+l/2}\epsilon_1^2$$

$$\lesssim 2^{(2+\alpha)k-l/2-2\alpha k_1+2\delta m}\epsilon_0 \lesssim 2^{(1-\alpha)k+2\delta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0. \tag{5.127}$$

- It remains to consider the case when $\min\{j_1, j_2\} \leq j - \delta m$. For this case, we will mainly do integration by parts in "$\xi$" to see rapidly decay. Recall that $\mu\nu = -$. From (5.66), the following estimate holds,

$$|\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)|\varphi_l(\angle(\xi,\nu\eta)) = \Big|\Lambda'(|\xi|)\frac{\xi}{|\xi|} - \mu\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}\Big|\varphi_l(\angle(\xi,\nu\eta)) \lesssim 2^l,$$

hence the following estimate holds as $j \geq m + l + 100\delta m$,

$$|\nabla_\xi[x\cdot\xi + s\Phi^{\mu,\nu}(\xi,\eta)]|\varphi_l(\angle(\xi,\nu\eta))\varphi_j^k(x) \sim 2^j. \tag{5.128}$$

For every time of integration by part in $\xi$, we can gain $2^{-j}$ by paying the price of at most $\max\{2^{\min\{j_1,j_2\}},\ 2^{-k-l}\}$, here $2^{-k-l}$ comes from the fact that $\nabla_\xi$ might hit the angular cutoff function $\varphi_l(\angle(\xi,\nu\eta))$ or the symbol $a_{\mu,\nu}(t,x,\xi,\eta)$ (see (5.91)). As $j \geq -k-l+100\delta m$ and $\min\{j_1,j_2\} \leq j-\delta m$, we can see that the total gain of each step of integration by parts in "$\xi$" is at least $2^{-\delta m}$ and we can keep doing this process to see rapidly decay. More precisely, the following estimate holds,

$$\big|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}](x)\big|\varphi_j^k(x) \lesssim 2^{-10m}\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^2},$$

therefore, it's easy to see the following estimate holds

$$\sum_{\min\{j_1,j_2\}\leq j-\delta m} \|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_{B_{k,j}} \lesssim 2^{-2\delta m-2\delta j}\epsilon_0. \tag{5.129}$$

**If $j \le \max\{m+l, -k-l\} + 100\delta m$, $k+2k_1 \le -m+\beta m$, and $k+2l \le -m+2\beta m$.**

For this case, we have $j \le -k-l+2\beta m+100\delta m$ and $k \le -m/3+\beta m$. From (5.51) in Lemma 5.2.5, (5.43) and (5.45) in Lemma 5.2.4, the following estimate holds,

$$\|\mathcal{F}^{-1}[I_l^{\mu,\nu}]\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++m+j+k+k+k_1+l}\|f_{k_1}\|_{L^2}\|\widehat{f_{k_2}}(t,\xi)\|_{L^\infty_\xi}$$

$$\lesssim 2^{(1+\alpha)k+m+2k_1+3\beta m}\epsilon_0 \lesssim 2^{(1+\alpha)k+2k_1+3\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

*Remark* 5.5.1. Note that, if without the improved $L^2$ estimates (5.43) and (5.45) in Lemma 5.2.4, it is impossible to easily close the argument for the case we are considering. If one uses estimates (5.30) and (5.31) instead, then we only have the following estimate,

$$\|\mathcal{F}^{-1}[I_l^{\mu,\nu}]\|_{B_{k,j}} \lesssim 2^{(1+\alpha)k+2k_1-2\alpha k_1+m+3\beta m}\epsilon_0,$$

which is not sufficient when $k \ge 2k_1$ and $k+2k_1 \approx -m+\beta m$.

**If $j \le \max\{m+l, -k-l\} + 100\delta m$, $k+2k_1 \le -m+\beta m$, and $k+2l \ge -m+2\beta m$.**

For this case, we have $j \le m+l+100\delta m$ and $k \le -m/3+\beta m$. When do integration by parts in "$\eta$", it is not a problem anymore when $\nabla_\eta$ hits the symbol $\tilde{m}_{\mu,\nu}(\xi-\eta,\eta)$(see (5.101)), as $k+2l \ge -m+2\beta m$. See (5.76) in Lemma 5.3.2.

- We first consider the case when $\max\{j_1,j_2\} \ge m+k-k_1+l-4\beta m$. For this case, by using estimate (5.51) in Lemma 5.2.5 and putting the input with higher spatial concentration in $L^2$ and the other one in $L^\infty_\xi$, we have the following estimate if $l \le 2\alpha k_1$,

$$\sum_{\max\{j_1,j_2\}\ge m+k-k_1+l-4\beta m}\|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_{B_{k,j}} \lesssim \sum_{\max\{j_1,j_2\}\ge m+k-k_1+l-4\beta m} 2^{\alpha k+6k_++m+j+k+k+k_1+l}$$

$$\times 2^{-\max\{j_1,j_2\}-2\alpha k_1}\|f_{k_1,j_1}\|_Z\|f_{k_2,j_2}\|_Z \lesssim 2^{(1+\alpha)k+(2-2\alpha)k_1+l+m+4\beta m}\epsilon_1^2$$

$$\lesssim 2^{\alpha k+(k+2k_1)+m+4\beta m}\epsilon_1^2 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

It remains to consider the case when $l \ge 2\alpha k_1$. From Lemma 1.3.8, the following estimate holds,

$$\|q_{\mu,\nu}(\xi-\eta,\eta)\varphi_l(\angle(\xi,\nu\eta))\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{k-3l+\beta m},$$

as when $\nabla_{\xi,\eta}$ hits the angular cutoff function once, we lose extra $2^{-l}$. As we need to take derivative

three times, which explains $2^{-3l}$ in above estimate. From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, we have the following estimate after putting the input with higher spatial localization in $L^2$ and the other one in $L^\infty$,

$$\sum_{\max\{j_1,j_2\}\geq m+k-k_1+l-4\beta m}\|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_{B_{k,j}}\lesssim\sum_{\max\{j_1,j_2\}\geq m+k-k_1+l-4\beta m}2^{\alpha k+6k_++m+j}$$

$$\times 2^{k-3l+\beta m-m-2\alpha k_1-\max\{j_1,j_2\}-6k_{1,+}}\|f_{k_1,j_1}\|_Z\|f_{k_2,j_2}\|_Z\lesssim 2^{\alpha k+(1-2\alpha)k_1-3l+6\beta m}\epsilon_1^2$$

$$\lesssim 2^{\alpha(k+2k_1)+7\beta m}\epsilon_0\lesssim 2^{-2\delta m-2\delta j}\epsilon_0.$$

- We proceed to consider the case when $\max\{j_1,j_2\}\leq m+k-k_1+l-4\beta m$. For this case, we do integration by parts in " $\eta$" to see rapidly decay. From (5.76) in Lemma 5.3.2 and (5.67) in Lemma 5.3.1, we can see that the net gain is at least $2^{-m}\max\{2^{-k-2l+\beta m},\,2^{\max\{j_1,j_2\}-k+k_1-l+3\beta m}\}$, which is less than $2^{-\beta m}$, for each time of integration by parts in "$\eta$". Therefore, we can keep doing this process to see rapidly decay.

**If $j\leq\max\{m+l,-k-l\}+100\delta m$, $k+2k_1\geq -m+\beta m$, and $k+2l\leq -m+2\beta m$.**

For this case, we have $j\leq -k-l+\beta m+100\delta m$, $k\leq -m/5+\beta m$ and $l\leq k_1+\beta m/2$. For this case, we do integration by parts in time first. The formulas are very similar as in (5.94) and (5.102). We only need to add angular cutoff function $\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))$ into the symbol. For readers' convenience, we state them in details as follows,

$$I^{\mu,\nu}_l=\sum_{i=1,2}\mathrm{End}^{\mu,\nu,i}_{l;k_1,k_2}+J^{\mu,\nu,i}_{l;k_1,k_2},\quad J^{\mu,\nu,i}_{l;k_1,k_2}=\sum_{k_1',k_2'\in\mathbb{Z}}\sum_{\mu',\kappa'\in\{+,-\}}J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',k_2'}+\mathrm{JR}^i_{l;k_1,k_2},\quad i\in\{1,2\},\tag{5.130}$$

$$J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',k_2'}:=\sum_{j_1'\geq -k'_{1,-},j_2'\geq -k'_{2,-},j_i\geq -k_{i,-}}H^{\mu,\nu,\tau,\kappa,i}_{l,j_i;j_1',j_2'},,\quad \mathrm{End}^{\mu,\nu,i}_{l;k_1,k_2}=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2},\tag{5.131}$$

$$H^{\mu,\nu,\tau,\kappa,i}_{l;j_1',j_2'}=\sum_{j_i\geq -k_{i,-}}H^{\mu,\nu,\tau,\kappa,i}_{l,j_i;j_1',j_2'},\quad H^{\mu,\nu,\tau,\kappa,i}_{l,j_i;j_1'}=\sum_{j_2'\geq -k'_{2,-}}H^{\mu,\nu,\tau,\kappa,i}_{l,j_i;j_1',j_2'},\quad i\in\{1,2\},\tag{5.132}$$

where

$$\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}=(-1)^{i-1}\int_{\mathbb{R}^2}e^{it_i\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}_{k_1,j_1}}(t_i,\xi-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t_i,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))d\eta,$$

$$H^{\mu,\nu,\tau,\kappa,1}_{l,j_i;j_1',j_2'}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)}m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))$$

$$\times\widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma)d\eta d\sigma dt,$$

$$H^{\mu,\nu,\tau,\kappa,2}_{l,j_i;j_1',j_2'}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it_i\Phi_2^{\tau,\kappa,\nu}(\xi,\eta,\sigma)}m^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))$$

$$\times\widehat{f^{\tau}_{k_1',j_1'}}(t,\xi-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t,\eta)d\eta d\sigma dt,$$

$$\mathrm{JR}^1_{l;k_1,k_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}}(t,\xi-\eta)\widehat{\mathcal{R}'^{\nu}}(t,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))d\eta dt,$$

$$\mathrm{JR}^2_{l;k_1,k_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{\mathcal{R}'^{\mu}}(t,\xi-\eta)\widehat{f^{\nu}}(t,\eta)m_{\mu,\nu}(\xi-\eta,\eta)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))d\eta dt,$$

where $m_{\mu,\nu}(\xi-\eta,\eta)$, $m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)$, and $m^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma)$ are same as in (5.26), (5.105), and (5.106).

From Lemma 1.3.8, (5.7) in Lemma 5.0.4, (5.65) in Lemma 5.3.1, and (5.75) in Lemma 5.3.2, the following estimates hold,

$$\|m_{\mu,\nu}(\xi-\eta,\eta)\varphi_l(\angle(\xi,\nu\eta))\|_{L^\infty_{\xi,\eta}}\lesssim 2^{k-k-2\max\{k_{1,-},l\}+2k_{1,+}}\lesssim 2^{-2\max\{k_{1,-},l\}+2\beta m},\tag{5.133}$$

$$\|m_{\mu,\nu}(\xi-\eta,\eta)\varphi_l(\angle(\xi,\nu\eta))\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim\max\{2^{k-k-2\max\{k_{1,-},l\}+2k_{1,+}-3k_{1,-}},$$

$$2^{k-k-2\max\{k_{1,-},l\}+2k_{1,+}-3l}\}\lesssim 2^{-2\max\{k_{1,-},l\}-6k_{1,-}+2\beta m}.\tag{5.134}$$

- *Estimate of the endpoint case.* From (5.51) in Lemma 5.2.5, (5.43) and (5.45) in Lemma 5.2.4, and (5.133), the following estimate holds,

$$\sum_{i=1,2}\|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,k_2}]\|_Z\lesssim 2^{\alpha k+j-2\max\{k_{1,-},l\}+2\beta m+k+k_1+l}\|\widehat{f_{k_1}}(\xi)\|_{L^\infty_\xi}\|f_{k_2}\|_{L^2}$$

$$\lesssim 2^{\alpha k+3\beta m}\epsilon_1^2\lesssim 2^{\alpha k+3\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

- *Estimate of $J^{\mu,\nu,i}_{l;k_1,k_2}$, $i\in\{1,2\}$.* From (5.51) in Lemma 5.2.5, (5.44) in Lemma 5.2.4, (5.45), and (5.133) the following estimate holds,

$$\sum_{i=1,2}\|\mathcal{F}^{-1}[J^{\mu,\nu,i}_{l;k_1,k_2}]\|_Z\lesssim\sum_{i=1,2}2^{\alpha k+m+j-2\max\{k_{1,-},l\}+2\beta m+k+k_1+l}\|\widehat{f_{k_i}}(t,\xi)\|_{L^\infty}\|\partial_t f_{k_{3-i}}\|_{L^2}$$

$$\lesssim 2^{\alpha k+3\beta m}\epsilon_1^2\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

**If** $j \le \max\{m+l, -k-l\} + 100\delta m$**,** $k + 2k_1 \ge -m + \beta m$**, and** $k + 2l \ge -m + 2\beta m$**.**

For this case, we have $j \le m + l + 100\delta m$ and $k_1 \ge -m/3$. Same as the previous case, we do integration by parts in time first.

- *Estimate of the endpoint case.* Again, as $k + 2l \ge -m + 2\beta m$, it is not problematic when doing integration by parts in "$\eta$". Same as before, we can first rule out the case when $\max\{j_1, j_2\} \le m + k - k_1 + l - 4\beta m$ by doing integration by parts in " $\eta$" many times.

It remains to consider the case when $\max\{j_1, j_2\} \ge m + k - k_1 + l - 4\beta m$. From (5.51) in Lemma 5.2.5 and (5.133), the following estimate holds after putting the input with higher spatial concentration in $L^2$ and the other one in $L^\infty_\xi$.

$$\sum_{\max\{j_1,j_2\}\ge m+k-k_1+l-4\beta m} \sum_{i=1,2} \|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}]\|_Z$$

$$\lesssim \sum_{\max\{j_1,j_2\}\ge m+k-k_1+l-4\beta m} 2^{\alpha k+6k_+ +j-2\max\{k_{1,-},l\}+2\beta m+k+k_1+l-\max\{j_1,j_2\}-6k_{1,+}-2\alpha k_1}$$

$$\times \|f_{k_1,j_1}\|_Z \|f_{k_2,j_2}\|_Z \lesssim 2^{\alpha k+2k_1+l-2\max\{k_{1,-},l\}-2\alpha k_1+6\beta m}\epsilon_1^2 \lesssim 2^{\alpha k+(1-2\alpha)k_1+6\beta m}\epsilon_0.$$

From above estimate, we can rule out the case when $k_1 \le -7\beta m$ or $k \le -70\beta m$. It is sufficient to consider the case when $k_1 \ge -7\beta m$ and $k \ge -70\beta m$. From (5.134), the following estimate holds,

$$\|m_{\mu,\nu}(\xi-\eta,\eta)\varphi_l(\angle(\xi,\nu\eta))\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{-8k_{1,-}+2\beta m} \lesssim 2^{58\beta m}. \tag{5.135}$$

From $L^2 - L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.135), the following estimate holds after putting the input with higher spatial localization in $L^2$ and the other one in $L^\infty$.

$$\sum_{\max\{j_1,j_2\}\ge m+k-k_1+l-4\beta m} \sum_{i=1,2} \|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}]\|_Z$$

$$\lesssim \sum_{\max\{j_1,j_2\}\ge m+k-k_1+l-4\beta m} 2^{\alpha k+6k_+ +j+58\beta m-\max\{j_1,j_2\}-m-2\alpha k_1}\|f_{k_1,j_1}\|_Z\|f_{k_2,j_2}\|_Z$$

$$\lesssim 2^{-m-(1-\alpha)k+(1-2\alpha)k_1+70\beta m}\epsilon_1^2 \lesssim 2^{-m+140\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

- *Estimate of* $J^{\mu,\nu,i}_{l;k_1,k_2}$, $i \in \{1,2\}$. As before, since the decay rate of $Z$-norm of $\mathcal{R}'$ is $2^{-m}$. With minor modification, we can estimate of $\mathrm{JR}^1_{l;k_1,k_2}$ and $\mathrm{JR}^2_{l;k_1,k_2}$ in the same way as we did for $\mathrm{End}^{\mu,\nu,i}_{l;k_1,k_2}$. We omit details here.

Without loss of generality, we may assume that $k_2' = \min\{k_1', k_2'\}$. From (5.63) and (5.64) in

Lemma 5.3.1, the size of $\Phi^{\mu,\nu}(\xi-\eta,\eta)$ is either $2^k\max\{2^{-k_+/2+2k_{1,-}},2^{-k_{1,+}/2+2l}\}$ or $2^{k-k_+/2}$, hence we can put a cutoff function $\psi_0(2^{-\kappa}\Phi^{\mu,\nu}(\xi-\eta,\eta))$ inside the symbols $m_{\mu,\nu}(\xi-\eta,\eta)$, $m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)$, and $m_2^{\tau,\kappa,\nu}(\xi,\eta,\sigma)$, with $2^\kappa$ choose to be either $2^k\max\{2^{-k_+/2+2k_{1,-}},2^{-k_{1,+}/2+2l}\}$ or $2^{k-k_+/2}$. This fact enables us to use the Lemma 5.2.6.

From (5.56) in Lemma 5.2.6, after putting $f_{k_i}(t)$ in $L^\infty$ and $T^{\tau,\kappa}(f_{k_1'},f_{k_2'})$ in $L^2$, the following estimate holds

$$\sum_{i=1,2}\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',k_2'}]\|_Z\lesssim\sup_{|\lambda|\le 2^{\beta m}}2^{\alpha k+6k_++m+j-2\max\{l,k_{1,-}\}+2\beta m}\|e^{-i(t+2^{-k-2l}\lambda)}f_{k_1}(t)\|_{L^\infty}$$

$$\times 2^{k_1}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}+2^{-10m+k+\alpha k+m+j-2\max\{l,k_{1,-}\}}\|f_{k_1}\|_{L^2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}$$

$$\lesssim\min\{2^{(1-2\alpha)k_2'+3\beta m},2^{(1-2\alpha)k_2'+m+3\beta m-(N_0-8)k_{1,+}'}\}\epsilon_0. \tag{5.136}$$

For the second bound of above estimate, we used again the fact that $\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\lesssim 2^{k_1'}\|f_{k_1'}\|_{L^2}$ $\lesssim 2^{k_1'-N_0k_{1,+}'}\epsilon_0$. From (5.136), we can rule out the case when $k_1'\ge\beta m$ or $k_2'\le -4\beta m$. Now, it is sufficient to consider fixed $k_1'$ and $k_2'$ such that $-4\beta m\le k_2'\le k_1'\le\beta m$.

We first consider the case when $k_1\le -10\beta m$. As $-4\beta m\le k_2'\le k_1'\le\beta m$, we know that $|\eta|\ll|\eta-\sigma|$( or $|\xi-\eta|\ll|\xi-\sigma|$). Hence, from (5.67) and (5.68) in Lemma 5.3.1, $\nabla_\sigma\Phi^{\mu,\tau,\kappa}$ always has a good lower bound of size $2^{k_1+k_1'-2\beta m}\gtrsim 2^{-m/3-2\beta m}$. From (5.74) and (5.76) in Lemma 5.3.2, we know that the net gain of doing integration by parts in "$\sigma$" is at least $\max\{2^{-m+\max\{j_1',j_2'\}-k_1-k_1'+3\beta m}$, $2^{-m-\min\{k_1,k_2'\}-4k_1'+5\beta m}\}$, which is less than $2^{-\beta m}$. Therefore, we can do integration by parts in "$\sigma$" many times to see rapidly decay to rule out the case when $\max\{j_1',j_2'\}\le m+k_1+k_1'-4\beta m$.

Next, we consider the case when $k_1\ge -10\beta m$. As $k_1$ is not small, same as what we did in the good type phase setting, either $\nabla_\sigma\Phi^{\mu,\tau,\kappa}$ or $\nabla_\eta\Phi^{\mu,\tau,\kappa}$ has a good lower bound. Through doing integration by parts in "$\sigma$" or doing integration by parts in "$\sigma$" and "$\eta$", we can rule out the case when $\max\{j_i,j_1',j_2'\}\le m+k_1+k_1'-10\beta m$.

Therefore, in whichever case, we can always rule out the case when $\max\{j_i,j_1',j_2'\}\le m+k_1+k_1'-10\beta m$. It remains to consider the case when $\max\{j_i,j_1',j_2'\}\ge m+k_1+k_1'-10\beta m$. For this case, we use estimate (5.56) in Lemma 5.2.6. After putting the input with the maximum spatial concentration in $L^2$ and the other two inputs in $L^\infty$, the following estimate holds,

$$\sum_{i=1,2}\sum_{\max\{j_i,j_1',j_2'\}\ge m+k_1+k_1'-10\beta m}\|\mathcal{F}^{-1}[H^{\mu,\nu,\tau,\kappa,i}_{l,j_i;j_1',j_2'}]\|_Z$$

$$\lesssim \sum_{\max\{j_i,j_1',j_2'\}\geq m+k_1+k_1'-10\beta m} \big(2^{\alpha k+6k_++m+j-2\max\{l,k_{1,-}\}+2\beta m}2^{-m-\alpha k_1}$$

$$+2^{-10m+k+\alpha k+m+j-2\max\{l,k_{1,-}\}+2\beta m}\big)\times 2^{k_1}2^{-\max\{j_i,j_1',j_2'\}-m-\alpha(k_1'+k_2')}\|f_{k_1',j_1'}\|_Z\|f_{k_i,j_i}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$\lesssim 2^{-m-k_1-(1+\alpha)(k_1'+k_2')+24\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0. \tag{5.137}$$

In above estimate, we used the fact that $k_1\geq -m/3$.

### 5.5.2 When $(k_1,k_2)\in\chi_k^2, (\mu,\nu)\in\{(+,+),(+,-)\}$

Recall that $k_2=\min\{k_1,k_2\}$. Same as what we did in subsection 5.4.2, we can use the rough estimates to rule out the very high frequency case, the very low frequency case and the very high spatial localization $j$. To sum up, from now on $k$, $k_1$, $k_2$, and $j$ are restricted in the ranges as follows

$$-(1+\delta)(m+j+10\delta m)/(2-\alpha)\leq k_2\leq k-10,\quad j\leq m+20\delta m, \tag{5.138}$$

$$-(1+\delta)(m+j+10\delta m)/(4-\alpha)\leq k\leq (1+\delta)(m+j+10\delta m)/(N_0-10)\leq \beta m. \tag{5.139}$$

For fixed $k_1$ and $k_2$ in above range, we dyadically localize the angle between $\xi$ and $\nu\eta$ with the threshold $\bar{l}$ chosen to be $2k_{1,-}$ and then spatially localize two inputs as in (5.125). For simplicity, we use same notations listed in (5.124), (5.130) and (5.131), but one should keep in mind that now $(k_1,k_2)\in\chi_k^2$ instead of $\chi_k^1$.

**If** $\max\{m+l,\min\{-k_2-l,m\}\}+100\delta m\leq j\leq m+20\delta m$

The case when $j\geq m+100\delta m$ is already considered. We only have to consider the case when $k_2+l\geq -m$, $j\geq\max\{m+l,-k_2-l\}+100\delta m$ and $k_1\leq -20\delta m$. Note that $k_1$ and $k_2$ are not comparable in the case we are considering. Since $\nabla_\xi$ might hit $\widehat{f_{k_1,j_1}}(\xi-\eta)$ when doing integration by parts in $\xi$, it is problematic when $j_1\geq j_2$. So, if $j_1\geq j_2$, we do change of coordinates to switch the role of $\xi-\eta$ and $\eta$. However, in any case, the following estimate holds,

$$|\nabla_\xi\Phi^{\mu,\nu}(\xi,\eta)|\varphi_l(\angle(\xi,\nu\eta))=\Big|\Lambda'(|\xi|)\frac{\xi}{|\xi|}-\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}\Big|\varphi_l(\angle(\xi,\nu\eta))\lesssim 2^l.$$

Recall that $j\geq\max\{m+l,-k_2-l\}+100\delta m$, by doing integration by parts in $\xi$ once, we gain $2^{-j}$ by paying the price of $\max\{2^{\min\{j_1,j_2\}},2^{-k_2-l}\}$, where $2^{-k_2-l}$ comes from the fact that $\nabla_\xi$ might hit the angular cutoff function or $a_{\mu,\nu}(t,x,\xi-\eta)$ (see (5.91)). We can rule out the case when $\min\{j_1,j_2\}\leq j-\delta m$ by doing integration by parts in $\xi$ many times. For the case when

$\min\{j_1, j_2\} \geq j - \delta m$, from (5.50) in Lemma 5.2.5, the following estimate holds,

$$\sum_{\min\{j_1,j_2\}\geq j-\delta m} \|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_{B_{k,j}} \lesssim \sum_{\min\{j_1,j_2\}\geq j-\delta m} 2^{\alpha k+6k_++m+j+k+k_{1,+}+k_2+l/2}$$

$$\times \|Q_{k_1,j_1} f\|_{L^2} \|Q_{k_2,j_2} f\|_{L^2} \lesssim 2^{k+(1-\alpha)k_2+m+2\delta m-j+l/2}\epsilon_1^2$$

$$\lesssim 2^{k+(1-\alpha)k_2-l/2+2\delta m}\epsilon_0 \lesssim 2^{(1-\alpha)k+2\delta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0. \tag{5.140}$$

We can summarize the result in this subsubsection and the result in subsubsection 5.5.1 as the following lemma,

**Lemma 5.5.2.** *Under the bootstrap assumption* (3.43) *and the improved energy estimate* (4.1)*, the following estimate holds if* $(k_1, k_2, \mu, \nu) \in \mathcal{P}^k_{bad}$*,*

$$\|Q_{k,j}\Big[\mathcal{F}^{-1}\big[\int_{\mathbb{R}^2} e^{it\Phi^{\mu,\nu}(\xi,\eta)}\widehat{f^{\mu}_{k_1}}(t,\xi-\eta)\widehat{f^{\nu}_{k_2}}(t,\eta)q_{\mu,\nu}(\xi-\eta,\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))d\eta\big]\Big]\|_{L^2}$$

$$\lesssim 2^{(1-2\alpha)k-m-j+2\delta m}\epsilon_0, \quad \text{if } j \geq \max\{m+l, -k-l, -k_2-l\} + 100\delta m \text{ and } \bar{l} \geq 2k_{1,-}. \tag{5.141}$$

*Proof.* One can redo above argument and the argument in subsubsection 5.5.1 to see the lemma holds. We keep the second estimate in (5.127) and (5.140) instead of the last estimate for (5.141). □

**If** $j \leq \max\{m+l, \min\{-k_2-l, m\}\}+100\delta m$**,** $(2-2\alpha)k_2 \leq -m-20\beta m$**, and** $k_2+2l \leq -m+4\beta m$**.**

For this case, we have $j \leq \min\{-k_2 - l + 4\beta m, m\} + 100\delta m$. As $k_1$ and $k_2$ are not comparable in the case we are considering, it turns out that whether $j_2$ is the smaller one between $j_1$ and $j_2$ makes a difference.

- We first consider the case when $j_2 \leq j_1$. Recall that $j_2 \geq -k_2$. Therefore, from (5.50) in Lemma 5.2.5, (5.43) in Lemma 5.2.4, and (5.45), the following estimate holds,

$$\sum_{j_2\leq j_1} \|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_Z \lesssim \sum_{-k_2\leq j_2\leq j_1} 2^{\alpha k+6k_++m+j+k_1+k_{1,+}+k_2+l/2}\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^2}$$

$$\lesssim 2^{12\beta m+m+k_1-l/2+(2-\alpha)k_2}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

- We proceed to consider the case when $j_1 \leq j_2$. For this case, we can do better for $j$. More precisely, as $j_1 \leq j_2$, it is not necessary to do change of coordinates to switch the role of $\xi - \eta$ and

$\eta$. As a result, the following estimate holds,

$$|\nabla_\xi \Phi^{\mu,\nu}(\xi,\eta)|\varphi_l(\angle(\xi,\nu\eta) = |\Lambda'(|\xi|)\frac{\xi}{|\xi|} - \Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}|\varphi_{\bar{l};l}(\angle(\xi,\nu\eta)) \lesssim 2^{k_2-k_1+l}.$$

Hence, we can further rule out the case when $\max\{m+(1-\alpha)(k_2-k_1)+l, -k_1-l\}+3\beta m \leq j \leq \max\{m+l,-k_2-l\}+100\delta m$ by run the argument used in subsubsection 5.5.2 again. Use similar estimate used in (5.140), we can rule out the case when $j_1 \geq j-\delta m$. Use integration by parts in "$\xi$" many times, we can rule out the case when $j_1 \leq j-\delta m$. Note that the price to pay is $2^{-k_1-l}$ instead of $2^{-k_2-l}$ when $\nabla_\xi$ hits $a_{\mu,\nu}(t,x,\xi,\eta)$. This fact is crucial here. Using (5.81), we know the size of numerator changes from $2^{k_2-k_1+l}$ to $2^{k_2-2k_1}$ when $\nabla_\xi$ hits $a_{\mu,\nu}(t,x,\xi,\eta)$. Hence we at most lose $2^{-k_1-l}$ when $\nabla_\xi$ hits $a_{\mu,\nu}(t,x,\xi,\eta)$.

Hence, it remains to consider the case when $j \leq \max\{m+(1-\alpha)(k_2-k_1)+l,-k_1-l\}+3\beta m$. From (5.50) in Lemma 5.2.5, (5.43) in Lemma 5.2.4, and (5.45), the following estimate holds,

$$\sum_{j_1\leq j_2}\|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_Z \lesssim \sum_{j_1\leq j_2} 2^{\alpha k+6k_++m+j+k_1+k_{1,+}+k_1+k_2+l}\|f_{k_1,j_1}\|_{L^1}\|f_{k_2,j_2}\|_{L^2}$$

$$\lesssim \max\{2^{7\beta m+2m+(3-2\alpha)k_2+k_1+2l}\epsilon_0, 2^{7\beta m+m+(2-\alpha)k_2+k_1}\epsilon_0\} \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

**If** $j \leq \max\{m+l,\min\{-k_2-l,m\}\}+100\delta m$**,** $(2-2\alpha)k_2 \leq -m-20\beta m$**, and** $k_2+2l \geq -m+4\beta m$

For this case, we have $j \leq m+l+100\delta m$ and $l \geq -m/4$. Same as the previous case, whether $j_2$ is the smaller one between $j_1$ and $j_2$ makes a difference. We separate into two cases as follows.

- We first consider the case when $j_2 \leq j_1$. If moreover $j_1 \leq m+l-4\beta m$, then we can do integration by parts in "$\eta$" many times to see rapidly decay. Because the net gain of doing integration by parts in $\eta$ once is at least $2^{-m}\max\{2^{\max\{j_1,j_2\}-l+3\beta m}, 2^{-k_2-2l+\beta m}\}$, which is less than $2^{-\beta m}$. It remains to consider the case when $j_1 \geq m+l-4\beta m$. From (5.50) in Lemma 5.2.4, the following estimate holds,

$$\sum_{j_2\leq j_1, m+l-4\beta m\leq j_1}\|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_Z \lesssim \sum_{j_2\leq j_1, m+l-4\beta m\leq j_1} 2^{\alpha k+6k_++m+j+k_1+k_{1,+}+2k_2+l}$$

$$\times\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^1} \lesssim 2^{m+(2-\alpha)k_2+k_1+l+6\beta m} \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

- We proceed to consider the case when $j_1 \leq j_2$. As before, we can do better for $j$ for this case. As a result, we can rule out the case when $\max\{m+(1-\alpha)(k_2-k_1)+l,-k_1-l\}+3\beta m \leq j \leq \max\{m+l,-k_2-l\}+100\delta m$ by running the argument used in subsubsection 5.5.2 again. It is sufficient

to consider the case when $j \le \max\{m+(1-\alpha)(k_2-k_1)+l, -k_1-l\}+3\beta m$. Moreover, we can further rule out the case when $j_2 \le m+l-4\beta m$ by doing integration by parts in "$\eta$" many times to see rapidly decay. Hence, it remains to consider the case when $j \le \max\{m+(1-\alpha)(k_2-k_1)+l, -k_1-l\}+3\beta m$ and $j_2 \ge m+l-4\beta m$. For this case, we use (5.50) in Lemma 1.13 and (5.8) in Lemma 5.0.4 to derive the following estimate,

$$\sum_{j_1\le j_2, m+l-4\beta m\le j_2} \|\mathcal{F}^{-1}[I^{\mu,\nu}_{l;j_1,j_2}]\|_Z \lesssim \sum_{j_1\le j_2, m+l-4\beta m\le j_2} 2^{\alpha k+6k_++m+j+k_1+k_{1,+}+k_1+k_2+l}$$

$$\times \|f_{k_1,j_1}\|_{L^1}\|f_{k_2,j_2}\|_{L^2} \lesssim \max\{2^{m+(2-2\alpha)k_2+k_1+l+12\beta m}, 2^{12\beta m+(1-\alpha)k_2+k_1-l}\}\epsilon_1^2 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

In above estimate, we used the fact that $k_2 \le -m/(2-2\alpha)-12\beta m$ and $l \ge -m/4$.

**If** $j \le \max\{m+l, \min\{-k_2-l, m\}\}+100\delta m$**,** $(2-2\alpha)k_2 \ge -m-20\beta m$**, and** $k_2+2l \le -m+4\beta m$

For this case, we have $j \le -k_2-l+100\delta m+4\beta m$, $l \le -m/5$ and $k_1 \le -m/10$. We first do integration by parts in time and have the same equality as in (5.130).

From Lemma 1.3.8, (5.7) in Lemma 5.0.4, (5.65) in Lemma 5.3.1, and (5.75) in Lemma 5.3.2, the following estimates hold,

$$\|m_{\mu,\nu}(\xi-\eta,\eta)\varphi_l(\angle(\xi,\nu\eta))\|_{L^\infty_{\xi,\eta}} \lesssim 2^{k_1-k_2-2\max\{k_{1,-},l\}+2k_{1,+}}, \tag{5.142}$$

$$\|m_{\mu,\nu}(\xi-\eta,\eta)\varphi_l(\angle(\xi,\nu\eta))\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim \max\{2^{k_1-k_2-2\max\{k_{1,-},l\}+2k_{1,+}-3k_{1,-}},$$

$$2^{k_1-k_2-2\max\{k_{1,-},l\}+2k_{1,+}-3l}\} \lesssim 2^{k_1-k_2-2\max\{k_{1,-},l\}-6k_{1,-}+2k_{1,+}}. \tag{5.143}$$

- *Estimate of the endpoint case.* From (5.50) in Lemma 5.2.5, (5.43) and (5.45) in Lemma 5.2.4, and (5.142), the following estimate holds,

$$\sum_{i=1,2}\|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,k_2}]\|_Z \lesssim \sum_{i=1,2} 2^{\alpha k+6k_++j-k_1-k_2+k_1+k_2+l}\|\widehat{f_{k_1}}(t_i,\xi)\|_{L^\infty_\xi}\|f_{k_2}(t_i)\|_{L^2}$$

$$\lesssim 2^{\alpha k+4\beta m+200\delta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}.$$

- *Estimate of $J^{\mu,\nu,i}_{l;k_1,k_2}$, $i\in\{1,2\}$.* As usual, with minor modification, we can estimate of $\mathrm{JR}^1_{l;k_1,k_2}$ and $\mathrm{JR}^2_{l;k_1,k_2}$ in the same way as we did for $\mathrm{End}^{\mu,\nu,i}_{l;k_1,k_2}$. We omit details here and proceed to the estimate of $J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',k_2'}$. Without loss of generality, we may assume that $k_2' = \min\{k_1',k_2'\}$.

From (5.50) in Lemma 5.2.5, (5.43) and (5.45) in Lemma 5.2.4, and (5.142), the following estimate

holds after putting $T^{\tau,\kappa}(f_{k_1'}, f_{k_2'})$ in $L^2$ and the other one in $L^\infty_\xi$

$$\sum_{i=1,2}\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',k_2'}]\|_Z \lesssim 2^{\alpha k+m+j-k_1-k_2+k_1+k_2+l}\|\widehat{f_{k_1}}(t,\xi)\|_{L^\infty_\xi}2^{k_2+k'_{1,+}}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}$$

$$+2^{\alpha k+m+j-k_1-k_2+2k_2+l}\|\widehat{f_{k_2}}(t,\xi)\|_{L^\infty_\xi}2^{k_1+k'_{1,+}}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}$$

$$\lesssim \min\{2^{\alpha k-\alpha k_2/2+(1-\alpha)k_2'-4k'_{1,+}+3\beta m}, 2^{m+\alpha k-\alpha k_2/2+(1-\alpha)k_2'+k_1'-N_0k'_{1,+}+3\beta m}\}\epsilon_0 \tag{5.144}$$

$$\lesssim \min\{2^{\alpha m/5+4\beta m+(1-\alpha)k_2'-4k'_{1,+}}, 2^{\alpha m/5+m+4\beta m+(1-\alpha)k_2'-(N_0-1)k'_{1,+}}\}\epsilon_0.$$

With above estimate, we can rule out the case when $k_2' \le -\alpha m/4$ or $k_1' \ge \beta m$. It remains to consider fixed $k_1'$ and $k_2'$ such that $-\alpha m/4 \le k_2' \le k_1' \le \beta m$. Recall that $k_1 \le -m/10$. Hence, either $|\eta| \ll |\eta-\sigma|$ or $|\xi-\eta| \ll |\xi-\sigma|$, which implies that $\nabla_\sigma \Phi_i^{\mu,\tau,\kappa}(\xi,\eta,\sigma)$ always has a good lower bound. Hence, we can rule out the case when $\max\{j_1', j_2'\} \le m+k_2+k_1'-4\beta m$ by doing integration by parts in "$\sigma$" many times.

It remains to consider the case when $\max\{j_1', j_2'\} \ge m+k_2+k_1'-4\beta m$. From (5.50) in Lemma 5.2.5, the following estimate holds after first putting $T^{\tau,\kappa}(f_{k_1',j_1'}, f_{k_2',j_2'})$ in $L^2$ and then putting the input with higher spatial localization in $L^2$ and the other one in $L^\infty_x$,

$$\sum_{i=1,2}\sum_{\max\{j_1,j_2\}\ge m+k_2+k_1'-4\beta m}\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',j_1',k_2',j_2'}]\|_Z \lesssim \sum_{\max\{j_1,j_2\}\ge m+k_2+k_1'-4\beta m}2^{\alpha k+m+j}$$

$$\times 2^{-k_1-k_2+k_1+k_2+l}\|\widehat{f_{k_1}}(t,\xi)\|_{L^\infty_\xi}2^{k_2+k'_{1,+}}2^{-m-\alpha k_1'-\alpha k_2'-\max\{j_1',j_2'\}}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$+2^{\alpha k+m+j-k_1-k_2+2k_2+l}\|\widehat{f_{k_2}}(t,\xi)\|_{L^\infty_\xi}2^{k_1+k'_{1,+}}2^{-m-\alpha k_1'-\alpha k_2'-\max\{j_1',j_2'\}}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$\lesssim 2^{-m-(1+\alpha)k_2-(1+2\alpha)k_2'+4\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

**If** $j \le \max\{m+l, \min\{-k_2-l, m\}\}+100\delta m$**,** $(2-2\alpha)k_2 \ge -m-20\beta m$**, and** $k_2+2l \ge -m+4\beta m$

For this case, we have $j \le m+l+100\delta m$. We do integration by parts in time and have the same identity as in (5.130).

- *Estimate of the endpoint case.* $(i)$ We first consider the case when $j_2 \le j_1$. By doing integration by parts in "$\eta$" many times, we can rule out the case when $j_1 \le m+l-4\beta m$. It remains to consider the case when $j_1 \ge m+l-4\beta m$. From (5.50) in Lemma 5.2.5, the following estimate

holds if $k_2 \leq -30\beta m$

$$\sum_{j_2\leq j_1, m+l-4\beta m\leq j_1}\sum_{i=1,2}\|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}]\|_Z \lesssim \sum_{j_2\leq j_1, m+l-4\beta m\leq j_1} 2^{\alpha k+6k_++j+k_1-k_2-2\max\{l,k_{1,-}\}}$$

$$\times 2^{2\beta m+2k_2+l}\|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^1} \lesssim 2^{(1-\alpha)k_2+15\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

If $k_2 \geq -30\beta m$, then we can use Lemma 1.3.8, $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.143) to derive the following estimate,

$$\sum_{j_2\leq j_1, m+l-4\beta m\leq j_1}\sum_{i=1,2}\|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}]\|_Z \lesssim \sum_{j_2\leq j_1, m+l-4\beta m\leq j_1} 2^{\alpha k+6k_++j-k_2-2k_1-3l}\|f_{k_1,j_1}\|_{L^2}$$

$$\times\|e^{-it\Lambda}f_{k_2,j_2}\|_{L^\infty} \lesssim 2^{-m-(1+\alpha)k_2-2k_1-3l}\epsilon_1^2 \lesssim 2^{-m-10k_2}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

$(ii)$ We proceed to consider the case when $j_1 \leq j_2$. By running the argument used in subsubsection 5.5.2 again, we can rule out the case when $\max\{m+(1-\alpha)(k_2-k_1)+l,-k_1-l\}+3\beta m \leq j \leq \max\{m+l,-k_2-l\}+100\delta m$. Moreover, by using integration by parts in "$\eta$" many times, we can further rule out the case when $j_2 \leq m+l-4\beta m$. Therefore, it is sufficient to consider the case when $j \leq \max\{m+(1-\alpha)(k_2-k_1)+l,-k_1-l\}+3\beta m = m+(1-\alpha)(k_2-k_1)+l+3\beta m$ and $j_2 \geq m+l-4\beta m$. From (5.50) in Lemma 5.2.5, the following estimate holds if $k_2 \leq -30\beta m$

$$\sum_{j_1\leq j_2, m+l-4\beta m\leq j_2}\sum_{i=1,2}\|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}]\|_Z \lesssim \sum_{j_1\leq j_2, m+l-4\beta m\leq j_2} 2^{\alpha k+j+6k_++k_1-k_2-2\max\{l,k_{1,-}\}}$$

$$\times 2^{2\beta m+k_1+k_2+l}\|f_{k_1,j_1}\|_{L^1}\|f_{k_2,j_2}\|_{L^2} \lesssim 2^{(1-2\alpha)k_2+15\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

If $k_2 \geq -30\beta m$, then we can use $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.143) to derive the following estimate,

$$\sum_{j_1\leq j_2, m+l-4\beta m\leq j_2}\sum_{i=1,2}\|\mathcal{F}^{-1}[\mathrm{End}^{\mu,\nu,i}_{l;k_1,j_1,k_2,j_2}]\|_Z \lesssim \sum_{j_1\leq j_2, m+l-4\beta m\leq j_2} 2^{\alpha k+j+6k_+-k_2-2k_1-3l+2\beta m}$$

$$\times\|f_{k_2,j_2}\|_{L^2}\|e^{-it\Lambda}f_{k_1,j_1}\|_{L^\infty} \lesssim 2^{-m-(1+\alpha)k_2-2k_1-3l+15\beta m}\epsilon_1^2 \lesssim 2^{-m-10k_2+15\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

- *Estimate of* $J^{\mu,\nu,i}_{l;k_1,k_2}$, $i\in\{1,2\}$. Same as before, we omit the detailed proof of $\mathrm{JR}^1_{l;k_1,k_2}$ and $\mathrm{JR}^2_{l;k_1,k_2}$ here and proceed to the estimate of $J^{\mu,\nu,\tau,\kappa,i}_{l;k_1',k_2'}$. Without loss of generality, we may assume that $k_2' = \min\{k_1',k_2'\}$.

From (5.50) in Lemma 5.2.5 and (5.45) in Lemma 5.2.4, the following estimates hold,

$$\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,1}_{l;k_1',k_2'}]\|_Z \lesssim 2^{\alpha k+6k_++m+j+k_1-k_2-2\max\{l,k_{1,-}\}+k_1+k_2+l}$$

$$\times\|\widehat{f_{k_1}}(t,\xi)\|_{L^\infty_\xi}2^{k_2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}. \tag{5.145}$$

$$\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,2}_{l;k_1',k_2'}]\|_Z \lesssim 2^{\alpha k+6k_++m+j+k_1-k_2-2\max\{l,k_{1,-}\}+2k_2+l}$$

$$\times\|\widehat{f_{k_2}}(t,\xi)\|_{L^\infty_\xi}2^{k_1}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}. \tag{5.146}$$

From (5.145) and (5.146), the following estimate holds if $k_2' \le -2m$ or $k_1' \ge 2\beta m$,

$$\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,1}_{l;k_1',k_2'}]\|_Z + \|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,2}_{l;k_1',k_2'}]\|_Z \lesssim \min\{2^{m+(1-\alpha)k_2'+10\beta m},$$

$$2^{2m+(1-\alpha)k_2'+10\beta m-(N_0-8)k'_{1,+}}\}\epsilon_1^2 \lesssim 2^{-2\delta m-2\delta j}\epsilon_0.$$

Therefore, now it's sufficient to consider the case when $k_1'$ and $k_2'$ are fixed and $-2m \le k_2' \le k_1' \le 2\beta m$.

We first estimate $J^{\mu,\nu,\tau,\kappa,1}_{l;k_1',k_2'}$. For this case, we have $|k_2-k_1'|\le 5$ if $k_1' \ge k_2'+5$ and $k_2 \le k_1'+5$ if $k_1' \le k_2'+5$.

**Case 1:** If $k_2' - 3\beta m \le k_2$. From (5.56) in Lemma 5.2.6, after putting $T^{\nu\tau,\nu\kappa}(f_{k_1'},f_{k_2'})$ in $L^2$ and the other one in $L^\infty$, the following estimate holds,

$$\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,1}_{l;k_1',k_2'}]\|_Z \lesssim \sup_{|\lambda|\le 2^{\beta m}} 2^{\alpha k+6k_++m+j+k_1-k_2-2\max\{l,k_{1,-}\}}2^{(k_1-k_2)/2}2^{k_2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}$$

$$\times\|e^{-i(t+2^{-k_2-2\max\{l,k_{1,-}\}}\lambda)}f_{k_1}(t)\|_{L^\infty} + 2^{-10m+k+\alpha k+m+j+k_1-k_2-2\max\{l,k_{1,-}\}+k_2}$$

$$\times\|f_{k_1}\|_{L^2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2} \lesssim 2^{k_2'-\alpha k_1'+(k_1-k_2)/2+4\beta m}\epsilon_0 \lesssim 2^{(1-2\alpha)k_2'/2+k_1/2+4\beta m}\epsilon_0.$$

From above estimate, we can rule out the case when $k_2' \le -12\beta m$. It remains to consider the case when $k_2' \ge -12\beta m$. For this case, we have $k_2 \ge -15\beta m$. All frequencies are relatively big. From (5.83), (5.84), and (5.85) in Lemma 5.3.3, we know that either $\nabla_\sigma\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)$ or $\nabla_\eta\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)$ is big. Through doing integration by parts in "$\sigma$" or doing integration by parts in "$\sigma$" and "$\eta$", we can rule out the case when $\max\{j_1,j_1',j_2'\} \le m+k_2+k_2'-10\beta m$.

Now, it's sufficient to consider the case when $\max\{j_1,j_1',j_2'\} \ge m+k_2+k_2'-10\beta m$. From (5.56) in Lemma 5.2.6, we the input with the maximum spatial concentration in $L^2$ and the other two

inputs in $L^\infty$. As a result, the following estimate holds,

$$\sum_{\max\{j_1,j_1',j_2'\}\geq m+k_2+k_2'-4\beta m}\|\mathcal{F}^{-1}[\|\mathcal{F}^{-1}[H^{\mu,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}]\|_Z]\|_Z\lesssim \sum_{\max\{j_1,j_1',j_2'\}\geq m+k_2+k_2'-4\beta m}2^{\alpha k+6k_++m+j}$$

$$\times 2^{k_1-k_2-2\max\{l,k_{1,-}\}}2^{(k_1-k_2)/2}2^{-m-\alpha k_1}2^{k_2-m-\max\{j_1,j_1',j_2'\}-2\alpha k_2'}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z\|f_{k_1,j_1}\|_Z$$

$$+2^{-10m+k+m+j+k_1-k_2-2\max\{l,k_{1,-}\}+k_2}\|f_{k_1,j_1}\|_Z2^{-m-\max\{j_1,j_1',j_2'\}-2\alpha k_2'}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$\lesssim 2^{-m-3k_2'+14\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

**Case 2:** If $k_2'-3\beta m\geq k_2$ and $(k_1',k_2',\nu\tau,\nu\kappa)\in\mathcal{P}^{k_2}_{good}$. A key observation is that the phase is relatively big. More precisely,

$$2^{k_2'-k_{2,+}'/2}\lesssim 2^{k_2'-k_{2,+}'/2}-2^{k_2+2\max\{l,k_{1,-}\}}\leq|\Phi_1^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)|$$

$$\leq 2^{k_2'-k_{2,+}'/2}+2^{k_2+2\max\{l,k_{1,-}\}}\lesssim 2^{k_2'-k_{2,+}'/2}.$$

Hence, we can take advantage of this fact by doing integration by parts in time again. As a result, we have

$$J^{\mu,\nu,\tau,\kappa,1}_{l;k_1',k_2'}=\sum_{i=1,2}(-1)^i\mathrm{E}_i+\mathrm{H}_1,\quad \mathrm{E}_i=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it_i\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)}\widetilde{m}^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))\widehat{f^{\mu}_{k_1}}(t_i,\xi-\eta)$$

$$\times\widehat{f^{\tau}_{k_1'}}(t_i,\eta-\sigma)\widehat{f^{\kappa}_{k_2'}}(t_i,\sigma)d\eta d\sigma,\quad \mathrm{H}_1=-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)}$$

$$\times\widetilde{m}^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))\partial_t\big(\widehat{f^{\mu}_{k_2}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1'}}(t,\eta-\sigma)\widehat{f^{\kappa}_{k_2'}}(t,\sigma)\big)d\eta d\sigma dt,\tag{5.147}$$

where

$$\widetilde{m}^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)=\frac{m^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)}{i\Phi_1^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)}=\frac{q_{\mu,\nu}(\xi-\eta,\eta)}{-\Phi^{\mu,\nu}(\xi,\eta)}\frac{(q_{\nu\tau,\nu\kappa}(\eta-\sigma,\sigma))^\nu}{\Phi_1^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)}$$

$$=\frac{q_{\mu,\nu}(\xi-\eta,\eta)}{-\Phi^{\mu,\nu}(\xi,\eta)}\frac{(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma)^\nu}{\Phi_1^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)}\psi_0(2^{-\kappa_1}\Phi^{\mu,\nu}(\xi,\eta))\psi_0(2^{-\kappa_2}\Phi_1^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)),$$

$$\kappa_1=k_2+2\max\{l,k_{1,-}\},\kappa_2=k_2'-k_{2,+}'/2.$$

We only do the endpoint case in details here, as $\mathrm{H}_1$ term can be dealt with very similarly, Use the inverse Fourier transform twice, we have

$$\mathrm{E}_i=\frac{1}{16\pi^4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}2^{-\kappa_2-\kappa_1}\widehat{\chi}(\lambda_2)\widehat{\chi}(\lambda_1)e^{i(t_i+2^{-\kappa_2}\lambda_2)\Phi_1^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)+i2^{-\kappa_1}\lambda_1\Phi^{\mu,\nu}(\xi,\eta)}\varphi_{\bar{l};l}(\angle(\xi,\nu\eta))$$

$$\times q_{\mu,\nu}(\xi-\eta,\eta)\big(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma)\big)^{\nu}\widehat{f^{\mu}_{k_1}}(t_i,\xi-\eta)\widehat{f^{\tau}_{k'_1}}(t_i,\eta-\sigma)\widehat{f^{\kappa}_{k'_2}}(t_i,\sigma)d\eta d\sigma d\lambda_2 d\lambda_1$$

$$=\frac{1}{16\pi^4}2^{-\kappa_1-\kappa_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\widehat{\chi}(\lambda_1)\widehat{\chi}(\chi_2)e^{i(t_i+2^{-\kappa_2}\lambda_2+2^{-\kappa_1}\lambda_1)\Phi^{\mu,\nu}(\xi,\eta)+i\nu(t_i+2^{-\kappa_2}\lambda_2)\Lambda(\eta)}\varphi_{\bar{l},l}(\angle(\xi,\nu\eta))$$

$$q_{\mu,\nu}(\xi-\eta,\eta)\widehat{f^{\mu}_{k_1}}(\xi-\eta)T^{\tau,\kappa}_{\lambda_2}(f_{k'_1},f_{k'_2})(\eta)d\eta d\lambda_1 d\lambda_2,$$

where

$$\widehat{\chi}(\lambda)=\int e^{-i\lambda x}\frac{\psi_0(x)}{x}dx,$$

$$T^{\tau,\kappa}_{\lambda_2}(f_{k'_1},f_{k'_2})(\eta)=\int_{\mathbb{R}^2}e^{-i(t_i+2^{-\kappa_2}\lambda_2)(\tau\Lambda(|\eta-\sigma|)+\kappa\Lambda(|\sigma|))}(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma))^{\nu}\widehat{f^{\tau}_{k'_1}}(t_i,\eta-\sigma)\widehat{f^{\kappa}_{k'_2}}(t_i,\sigma)d\sigma.$$

Using the rapidly decay property of $\widehat{\chi}(\lambda)$, very similar to the proof of (5.55) in Lemma 5.2.6, we can derive the following estimate,

$$\|\mathcal{F}^{-1}[\mathrm{E}_i]\|_{B_{k,j}}\lesssim\sup_{|\lambda_1|,|\lambda_2|\leq 2^{\beta m/10}}2^{\alpha k+6k_++j+k_1-\kappa_1-\kappa_2}2^{(k_1-k_2)/2}\|e^{i(t_i+2^{-\kappa_1}\lambda_1+2^{-\kappa_2}\lambda_2)\Lambda}f_{k_1}\|_{L^\infty}$$

$$\times\|T^{\tau,\kappa}_{\lambda_2}(f_{k'_1},f_{k'_2})(\eta)\|_{L^2}+2^{-10m+k_1+k_2-\kappa_1-\kappa_2+2k_2+2k'_2}\|\widehat{f_{k_1}}\|_{L^\infty_\xi}\|\widehat{f_{k'_1}}\|_{L^\infty_\xi}\|\widehat{f_{k'_2}}\|_{L^\infty_\xi}$$

$$\lesssim\sup_{|\lambda_1|,|\lambda_2|\leq 2^{\beta m/10}}2^{-3k_2/2+k_1/2-k'_2+k_2+12\beta m}\|e^{i(t_i+2^{-\kappa_2}\lambda_2)\Lambda}f_{k'_1}\|_{L^\infty}\|f_{k'_2}\|_{L^2}+2^{-2\delta m-\delta j}\epsilon_0$$

$$\lesssim 2^{-m-(1+2\alpha)k_2/2+k_1/2+13\beta m}\epsilon_0+2^{-2\delta m-\delta j}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0. \tag{5.148}$$

In above estimate, we used the fact that $k_2\geq -m/(2-2\alpha)-12\beta m$ and also used (5.43) in Lemma 5.2.4 and (5.7) in Lemma 5.0.4.

The estimate of $\mathrm{H}_1$ can be done very similarly. Following the same argument, (5.50) in Lemma 5.2.5, and (5.44) in Lemma 5.2.4, we can prove the following estimate if $k_2\leq -10\beta m$

$$\|\mathcal{F}^{-1}[\mathrm{H}_1]\|_Z\lesssim\sup_{|\lambda_1|,|\lambda_2|\leq 2^{\beta m/10}}\big[2^{\alpha k+6k_++m+j+k_1-\kappa_1-\kappa_2}2^{(k_1-k_2)/2}\|e^{i(t_i+2^{-\kappa_1}\lambda_1+2^{-\kappa_2}\lambda_2)\Lambda}f_{k_1}\|_{L^\infty}$$

$$\times\big(\|T^{\tau,\kappa}_{\lambda_2}(\partial_t f_{k'_1},f_{k'_2})(\eta)\|_{L^2}+\|T^{\tau,\kappa}_{\lambda_2}(f_{k'_1},\partial_t f_{k'_2})(\eta)\|_{L^2}\big)+2^{\alpha k+6k_++m+j+k_1-\kappa_1-\kappa_2+k_2+l/2}\|\partial_t f_{k_1}\|_{L^2}\times$$

$$\|T^{\tau,\kappa}_{\lambda_2}(f_{k'_1},f_{k'_2})(\eta)\|_{L^2}\big]+2^{-10m+k_1+k_2-\kappa_1-\kappa_2+k_2+k'_2}\big(\|\partial_t f_{k_1}\|_{L^2}\|f_{k'_1}\|_{L^2}\|f_{k'_2}\|_{L^2}$$

$$+\|f_{k_1}\|_{L^2}\|\partial_t f_{k'_1}\|_{L^2}\|f_{k'_2}\|_{L^2}+\|f_{k_1}\|_{L^2}\|f_{k'_1}\|_{L^2}\|\partial_t f_{k'_2}\|_{L^2}\big)$$

$$\lesssim 2^{-(1+2\alpha)k_2/2+k_1/2-m+13\beta m}\epsilon_0+2^{-2\delta m-\delta j}\epsilon_0+2^{(1-\alpha)k_2+k_1/2+8\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0. \tag{5.149}$$

If $k_2\geq -10\beta m$, then instead of using the inverse Fourier transform twice, we use the $L^2-L^\infty-L^\infty$ type trilinear estimate directly. From Lemma 1.3.8, (1.14) in Lemma 1.3.7, (5.73) and (5.75) in

Lemma 5.3.2, and (5.44) in Lemma 5.2.4, the following estimate holds,

$$\|\mathcal{F}^{-1}[\mathrm{H}_1]\|_Z \lesssim 2^{\alpha k+6k_++m+j-20k_2}\big(\|\partial_t f_{k_1}\|_{L^2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|e^{-it\Lambda}f_{k_2'}\|_{L^\infty}$$

$$+\|\partial_t f_{k_1'}\|_{L^2}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|e^{-it\Lambda}f_{k_2'}\|_{L^\infty}+\|\partial_t f_{k_2'}\|_{L^2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\big)$$

$$\lesssim 2^{-m-22k_2+7\beta m}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0. \tag{5.150}$$

**Case 3:** If $k_2'-3\beta m\geq k_2$ and $(k_1',k_2',\nu\tau,\nu\kappa)\in\mathcal{P}_{bad}^{k_2}$. For this case, we first localize the angle between $\eta$ and $\nu\kappa\sigma$ and then decompose $J_{l;k_1',k_2'}^{\mu,\nu,\tau,\kappa,1}$ as follows,

$$J_{l;k_1',k_2'}^{\mu,\nu,\tau,\kappa,1}=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-}}\sum_{j_1'\geq -k_{1,-}',j_2'\geq -k_{2,-}'}\sum_{\bar{\tilde{l}}\leq\tilde{l}\leq 2}H_{l,\tilde{l};j_1',j_2'}^{j_1,j_2} \tag{5.151}$$

$$H_{l,\tilde{l};j_1',j_2'}^{j_1,j_2}:=\int_{t_1}^{t_2}e^{it\Phi^{\mu,\nu}(\xi,\eta)}\varphi_{\tilde{l};l}(\angle(\xi,\nu\eta))m_{\mu,\nu}(\xi,\eta)\widehat{f_{k_1,j_1}^{\mu}}(t,\xi-\eta)Q_{k_2,j_2}\widehat{[T_{\tilde{l};j_1',j_2'}^{\nu\tau,\nu\kappa}(t)]^\nu}(\eta)d\eta dt, \tag{5.152}$$

where $\bar{\tilde{l}}=\max\{l-6\beta m/5,2k_{1,-}'\}$ and

$$T_{\tilde{l};j_1',j_2'}^{\nu\tau,\nu\kappa}(t)=\mathcal{F}^{-1}\big[\int_{\mathbb{R}}e^{it\Phi^{\tau,\kappa}(\eta,\sigma)}\widehat{f_{k_1',j_1'}^{\nu\tau}}(t,\eta-\sigma)\widehat{f_{k_2',j_2'}^{\nu\kappa}}(t,\sigma)q_{\nu\tau,\nu\kappa}(\eta-\sigma,\sigma)\varphi_{\bar{\tilde{l}};\tilde{l}}(\angle(\eta,\nu\kappa\sigma))d\sigma\big].$$

For simplicity, we also use the following notation,

$$H_{l,\tilde{l};j_1',j_2'}:=\sum_{j_1\geq -k_{1,-},j_2\geq -k_{2,-},}H_{l,\tilde{l};j_1',j_2'}^{j_1,j_2},$$

$$H_{l,\tilde{l}}^{j_1,j_2}=\sum_{j_1'\geq -k_{1,-}',j_2'\geq -k_{2,-}'}H_{l,\tilde{l};j_1',j_2'}^{j_1,j_2},\quad T_{\tilde{l}}^{\nu\tau,\nu\kappa}(t)=\sum_{j_1'\geq -k_{1,-}',j_2'\geq -k_{2,-}'}T_{\tilde{l};j_1',j_2'}^{\nu\tau,\nu\kappa}(t).$$

- If either $2k_{1,-}'\geq l-6\beta m/5$ or $2k_{1,-}'<l-6\beta m/5$, $\tilde{l}>\bar{\tilde{l}}=l-6\beta m/5$, we have $k_2+2\tilde{l}\geq k_2+2l-12\beta m/5\geq -m+8\beta m/5$ and $l-\tilde{l}\leq 6\beta m/5$. Recall that $k_2\leq k_2'-3\beta m$, i.e., $|\eta|\ll|\sigma|$. Hence, we can do integration by parts in "$\sigma$" many times to rule out the case when $\max\{j_1',j_2'\}\leq m+k_2-k_2'+\tilde{l}-2\beta m$. From (5.55) in Lemma 5.2.6 and (5.52) in Lemma 5.2.5, the following estimate holds,

$$\sum_{\max\{j_1',j_2'\}\geq m+k_2-k_2'+\tilde{l}-2\beta m}\|\mathcal{F}^{-1}[H_{l,\tilde{l};j_1',j_2'}]\|_Z\lesssim\sum_{\max\{j_1',j_2'\}\geq m+k_2-k_2'+\tilde{l}-2\beta m}\sup_{|\lambda|\leq 2^{\beta m}}2^{\alpha k+6k_++m+j}$$

$$\times 2^{k_1-k_2-2\max\{l,k_{1,-}\}}2^{(k_1-k_2)/2}\|e^{-i(t+2^{-k_2-2\max\{l,k_{1,-}\}}\lambda)}f_{k_1}(t)\|_{L^\infty}2^{k_2-m-\max\{j_1',j_2'\}-2\alpha k_2'}\|f_{k_1',j_1'}\|_Z$$

$$\times\|f_{k_2',j_2'}\|_Z+2^{-10m+k+\alpha k+m+j+k_1-k_2-2\max\{l,k_{1,-}\}+k_2}\|f_{k_1}\|_{L^2}2^{-m-\max\{j_1',j_2'\}-2\alpha k_2'}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$\lesssim 2^{-3k_2/2-k_1/2+k_2'+10\beta m-m}\epsilon_0\lesssim 2^{-m/10+11\beta m-k_1/2}\epsilon_0\lesssim 2^{-\alpha m/2-k_1/2}\epsilon_0. \tag{5.153}$$

From (5.50) and (5.52) in Lemma 5.2.5, the following estimate also holds,

$$\sum_{\max\{j_1',j_2'\}\geq k_2-k_2'+\tilde{l}-2\beta m}\|\mathcal{F}^{-1}[H_{l,\tilde{l};j_1',j_2'}]\|_Z\lesssim \sum_{\max\{j_1',j_2'\}\geq k_2-k_2'+\tilde{l}-2\beta m}2^{\alpha k+6k_++m+j+k_1-k_2-2\max\{l,k_{1,-}\}}$$

$$\times 2^{k_1+k_2+l}\|\widehat{f_{k_1}}(t,\xi)\|_{L^\infty_\xi}2^{k_2-m-\max\{j_1',j_2'\}-2\alpha k_2'}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z\lesssim 2^{k_1+10\beta m}\epsilon_0. \tag{5.154}$$

Therefore, combining estimates (5.153) and (5.154), we can derive the following estimates,

$$\sum_{\max\{j_1',j_2'\}\geq k_2-k_2'+\tilde{l}-2\beta m}\|\mathcal{F}^{-1}[H_{l,\tilde{l};j_1',j_2'}]\|_Z\lesssim \left(2^{-\alpha m/2-k_1/2}\epsilon_0\right)^{1/2}(2^{k_1+10\beta m}\epsilon_0)^{1/2}\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

- If $2k_{1,-}'<l-6\beta m/5$ and $\tilde{l}=\bar{\bar{l}}=l-6\beta m/5$. The estimate of $J^{\mu,\nu,\tau,\kappa,1}_{l;k_1',k_2'}$ is exactly same as the estimate of the previous endpoint case. We view the bilinear term $T^{\nu\tau,\nu\kappa}_{\tilde{l}}(t)$ as a input of a bilinear operator. As in the endpoint case, whether $j_1\leq j_2$ makes a difference.

(i) If $j_2\leq j_1$. Then we can first rule out the case when $j_1\leq m+l-\beta m$ by doing integration by parts in "$\eta$" many times for (5.152). It remains to consider the case when $j_1\geq m+l-\beta m$. From (5.50) and (5.52) in Lemma 5.2.5, the following estimate holds if $k_2\leq -30\beta m$.

$$\|\sum_{j_2\leq j_1,m+l-\beta m\leq j_1}\mathcal{F}^{-1}[H^{j_1,j_2}_{l,\tilde{l}}]\|_Z\lesssim \sum_{m+l-\beta m\leq j_1}2^{\alpha k+6k_++m+j+k_1-k_2-2\max\{l,k_{1,-}\}+k_2+l/2}$$

$$\times\|f_{k_1,j_1}\|_{L^2}\|T^{\nu\tau,\nu\kappa}_{\tilde{l}}(t)\|_{L^2}\lesssim \sum_{j_1\geq m+l-\beta m}2^{-k_1/2+2m+l+8\beta m-j_1}\|f_{k_1,j_1}\|_Z\times 2^{k_2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}$$

$$\lesssim 2^{k_2/2+12\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

If $k_2\geq -30\beta m$, then we can use $L^2-L^\infty-L^\infty$ type trilinear estimate (1.14) in Lemma 1.3.7 and Lemma 1.3.8 to derive the following estimate

$$\|\sum_{j_2\leq j_1,m+l-\beta m\leq j_1}\mathcal{F}^{-1}[H^{j_1,j_2}_{l,\tilde{l}}]\|_Z\lesssim \sum_{m+l-\beta m\leq j_1}2^{\alpha k+6k_++m+j-12k_2}\|f_{k_1,j_1}\|_{L^2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}$$

$$\times\|e^{-it\Lambda}f_{k_2'}\|_{L^\infty}\lesssim 2^{-m-13k_2+10\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

(ii) If $j_2\geq j_1$. For this case, we can first rule out the case when $\max\{m+(1-\alpha)(k_2-k_1)+$

$l, -k_1 - l\} + 3\beta m \le j \le \max\{m + l, -k_2 - l\} + 20\delta m$ by running the argument used in subsubsection 5.5.2 again for (5.152). Moreover, we can further rule out the case when $j_2 \le m + l - \beta m$ by doing integration by parts in "$\eta$" many times for (5.152).

Therefore, it remains to consider the case when $j \le \max\{m+(1-\alpha)(k_2-k_1)+l, -k_1-l\}+3\beta m = m + (1-\alpha)(k_2 - k_1) + l + 3\beta m$ and $j_2 \ge m + l - \beta m$. Note that $j_2 \ge m + l - \beta m \ge m + \tilde{l} + \beta m/6$, $k_2 + 2\tilde{l} \ge k_2 + 2l - 12\beta m/5 \ge -m + 8\beta m/5$, and $k_2' \ge k_2 + 3\beta m$, all conditions of Lemma 5.5.2 are satisfied. Therefore, from (5.141) in Lemma 5.5.2 and (5.43) in Lemma 5.2.4, the following estimate holds if $k_2 \le -15\beta m$,

$$\|\sum_{j_1 \le j_2, m+l-\beta m \le j_2} \mathcal{F}^{-1}[H_{l,\tilde{l}}^{j_1,j_2}]\|_Z \lesssim \sum_{m+l-\beta m \le j_2} 2^{\alpha k + 6k_+ + m + j + k_1 - k_2 - 2\max\{l, k_{1,-}\} + k_2 + l/2}$$

$$\times \|f_{k_1}\|_{L^2} \|Q_{k_2,j_2}[\mathcal{F}^{-1}[T_{\tilde{l}}^{\nu\tau,\nu\kappa}(t)]]\|_{L^2} \lesssim \sum_{j_2 \ge m+l-\beta m} 2^{m+(1-2\alpha)k_2 + 3l/2 - j_2 + 9\beta m}\epsilon_0 \lesssim 2^{-2\delta m - \delta j}\epsilon_0.$$

If $k_2 \ge -15\beta m$, then by $L^2 - L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, (5.141) in Lemma 5.5.2 and Lemma 1.3.8, the following estimate holds,

$$\|\sum_{j_1 \le j_2, m+l-\beta m \le j_2} \mathcal{F}^{-1}[H_{l,\tilde{l}}^{j_1,j_2}]\|_Z \lesssim \sum_{j_1 \le j_2, m+l-\beta m \le j_2} 2^{8\beta m + m + j - 12k_2} \|e^{-it\Lambda} f_{k_1,j_1}\|_{L^\infty}$$

$$\times \|Q_{k_2,j_2}[\mathcal{F}^{-1}[T_{\tilde{l}}^{\nu\tau,\nu\kappa}(t)]]\|_{L^2} \lesssim 2^{-m - 15k_2 + 8\beta m}\epsilon_0 \lesssim 2^{-2\delta m - \delta j}\epsilon_0.$$

Now we proceed to estimate $J_{l;k_1',k_2'}^{\mu,\nu,\tau,\kappa,2}$. Note that, for $J_{l;k_1',k_2'}^{\mu,\nu,\tau,\kappa,2}$, $|k_1 - k_1'| \le 5$ if $k_1' \ge k_2' + 5$ and $k_1 \le k_1' + 5$ if $k_1' \le k_2' + 5$. Same as what we did in the estimate of $J_{l;k_1',k_2'}^{\mu,\nu,\tau,\kappa,2}$, when $\tau = \kappa = -$, we can do integration by parts in time to take the advantage of the fact that the size of phase is big. With minor modifications in (5.148), (5.149) and (5.150), we can see our desired estimate still holds. For the case when $(\tau, \kappa) \ne (-,-)$, we divide it into four cases as follows.

**Case 1:** If $l \ge -2\alpha m/3$ and $k_2' + 2k_1 \le -m/2 - \alpha m + 3\beta m$. From (5.61) in Lemma 5.3.1, we can see that the size of $\Phi^{\mu,\nu}(\xi - \eta, \eta)$ is $2^{k_2 + 2\max\{k_{1,-}, l\} - k_{1,+}/2}$, hence we can put a cutoff function $\psi_0(2^{-\kappa}\Phi^{\mu,\nu}(\xi - \eta, \eta))$ inside the symbols $m_{\mu,\nu}(\xi - \eta, \eta)$, $m_{\mu,\nu,1}^{\tau,\kappa}(\xi, \eta, \sigma)$, and $m_{\mu,\nu,2}^{\tau,\kappa}(\xi, \eta, \sigma)$, with $\kappa$ choose to be $2^{k_2 + 2\max\{k_{1,-}, l\} - k_{1,+}/2}$. This fact enables us to use the Lemma 5.2.6. From (5.56) in Lemma 5.2.6 and (5.43), after putting the input $f_{k_2}$ in $L^\infty$ and $T^{\tau,\kappa}(f_{k_1'}, f_{k_2'})$ in $L^2$, the following estimate holds,

$$\|\mathcal{F}^{-1}[J_{l;k_1',k_2'}^{\mu,\nu,\tau,\kappa,2}]\|_Z \lesssim \sup_{|\lambda| \le 2^{\beta m}} 2^{\alpha k + 6k_+ + m + j + k_1 - k_2 - 2\max\{l, k_{1,-}\} + k_1} \|e^{i(t + 2^{-k_2 - 2\max\{l, k_{1,-}\}}\lambda)} f_{k_2}(t)\|_{L^\infty}$$

$$\times\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}+2^{-10m+k+\alpha k+m+j+k_1-k_2-2\max\{l,k_{1,-}\}+k_1}\|f_{k_2}\|_{L^2}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2} \tag{5.155}$$

$$\lesssim 2^{m+2k_1+2\beta m+k_2'-k_2-l}(2^{2k_2+\delta m})^{1/2}(2^{-m+\alpha m/3})^{1/2}\epsilon_0\lesssim 2^{-\alpha m/6+6\beta m}\epsilon_0\lesssim 2^{-2\delta m-\delta j}\epsilon_0$$

In above estimate, we used the following fact,

$$\|e^{-it\Lambda}f_{k_2}(t)\|_{L^\infty}\lesssim\min\{2^{-m+\alpha m/3}\epsilon_1,2^{k_2}\|P_{k_2}f(t)\|_{L^2}\}\lesssim\min\{2^{-m+\alpha m/3},2^{2k_2+\delta m}\}\epsilon_1.$$

**Case 2:** If $l\geq -2\alpha m/3$ and $k_2'+2k_1\geq -m/2-\alpha m+3\beta m$. For this case we have $\max\{k_1,k_1'\}\geq -m/6-\alpha m/3+\beta m$ and $\min\{k_1,k_2'\}+4k_1'\geq -5m/6-5\alpha m/3+5\beta m\geq -m+5\beta m$.

When $\tau\kappa=-$, from (5.83) in Lemma 5.3.3, we can do integration by parts in $\sigma$ many times to rule out the case $\max\{j_1',j_2'\}\leq m+k_1'+k_1-4\beta m$.

For the case when $\tau\kappa=+$, we do integration by parts in $\sigma$ when $\angle(\xi-\sigma,\sigma-\eta)\geq 2^{-\alpha m}$. The net gain of doing integration by parts in "$\sigma$ " once is at least $\max\{2^{-m+\alpha m-\max\{j_1',j_2'\}+2\beta m},2^{-m-k_1+2\alpha m+2\beta m}\}$, which is less than $2^{-\beta m}$ when $\max\{j_1',j_2'\}\leq m-2\alpha m$. When $\angle(\xi-\sigma,\sigma-\eta)\leq 2^{-\alpha m}$, then we have $\angle(\sigma-\eta,\nu\eta)\sim\angle(\xi-\eta,\nu\eta)\sim 2^l\gtrsim 2^{-2\alpha m/3}$. For this case, we do integration by parts in "$\eta$". The net gain of doing integration by parts in "$\eta$" once is at least $\max\{2^{-m-l-\max\{j_2,j_2'\}+4\beta m},2^{-m-k_2-2l+2\beta m}\}$, which is less than $2^{-\beta m}$ when $\max\{j_2,j_2'\}\leq m-2\alpha m$.

Therefore, in whichever case, we can rule out the case when $\max\{j_2,j_1',j_2'\}\leq\min\{m+k_1+k_1'-4\beta m,m-2\alpha m\}$. It is sufficient to consider the case when $\max\{j_2,j_1',j_2'\}\geq\min\{m+k_1+k_1'-4\beta m,m-2\alpha m\}\geq m+k_1+k_1'-4\beta m-2\alpha m$. From (5.56) in Lemma 5.2.6 and (5.43), the following estimate holds,

$$\sum_{\max\{j_2,j_1',j_2'\}\geq m+k_1+k_1'-4\beta m-2\alpha m}\|\mathcal{F}^{-1}[H^{\mu,\nu,\tau,\kappa,2}_{l,j_2;j_1',j_2'}]\|_Z\lesssim\sum_{\max\{j_2,j_1',j_2'\}\geq m+k_1+k_1'-4\beta m-2\alpha m}2^{\alpha k+6k_++m+j}$$

$$\times 2^{k_1-k_2-2\max\{l,k_{1,-}\}}2^{-m-\alpha k_2}\|f_{k_2,j_2}\|_Z 2^{k_1-m-\max\{j_2,j_1',j_2'\}-\alpha k_1'-\alpha k_2'}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$+2^{-10m+k+\alpha k+m+j+k_1-k_2-2\max\{l,k_{1,-}\}}\|f_{k_2,j_2}\|_Z 2^{k_1-m-\max\{j_2,j_1',j_2'\}-\alpha k_1'-\alpha k_2'}\|f_{k_1',j_1'}\|_Z\|f_{k_2',j_2'}\|_Z$$

$$\lesssim 2^{-(1+\alpha)k_2-l+2\alpha m-\alpha k_2'+4\beta m-m}\lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

**Case 3:** If $l\leq -2\alpha m/3$ and $k_2'+2k_1\leq k_2+2\max\{l,k_{1,-}\}+\beta m$. From (5.56) in Lemma 5.2.6 and (5.43), estimate (5.155) also holds and

$$(5.155)\lesssim 2^{l+2k_1+k_2'-k_2-2\max\{l,k_{1,-}\}+\alpha m/3+8\beta m}\epsilon_0+2^{-2\delta m-\delta j}\epsilon_0$$

$$\lesssim 2^{-2\alpha m/3+\alpha m/3+9\beta m}\epsilon_0 + 2^{-2\delta m-\delta j}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

**Case** 4: If $l \le -2\alpha m/3$ and $k_2' + 2k_1 \ge k_2 + 2\max\{l, k_{1,-}\} + \beta m$. For this case, we have $k_1 \le l/2 \le -\alpha m/3$ and $k_2' \ge k_2$. Note that,

$$|\Phi_2^{\mu,\nu,\tau,\kappa}(\xi,\eta,\sigma)| \gtrsim 2^{k_2'+2k_1} - 2^{k_2+2\max\{l,k_{1,-}\}} \gtrsim 2^{k_2'+2k_1}. \tag{5.156}$$

With this observation, it motives us to do integration by parts in time again and have a similar identity as in (5.147) for $J^{\mu,\nu,\tau,\kappa,2}_{l;k_1',k_2'}$. The main difference is that now $\kappa_1 = k_2 + 2\max\{l, k_{1,-}\}$ and $\kappa_2 \ge k_2' + 2k_1$ in the case we are considering.

After using the inverse Fourier transform twice, we can derive the following estimate by using (5.50) in Lemma 5.2.5, (5.43), (5.44) and (5.45) in Lemma 5.2.4,

$$\|\mathcal{F}^{-1}[J^{\mu,\nu,\tau,\kappa,2}_{l;k_1',k_2'}]\|_Z \lesssim \sum_{\kappa_2\ge k_2'+2k_1}\sup_{|\lambda_1|,|\lambda_2|\le 2^{\beta m/10}}\sum_{i=1,2} 2^{\alpha k+j+2k_1-\kappa_1-\kappa_2+k_2+l/2}\|f_{k_2}(t_i)\|_{L^2}$$

$$\times\|e^{-i(t_i+2^{-\kappa_2}\lambda_2)\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2} + 2^{\alpha k+m+j+2k_1-\kappa_1-\kappa_2+k_2+l/2}$$

$$\times\big(\|\partial_t f_{k_2}\|_{L^2}\|e^{-i(t_i+2^{-\kappa_2}\lambda_2)\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2} + \|f_{k_2}\|_{L^2}\|e^{-i(t_i+2^{-\kappa_2}\lambda_2)\Lambda}f_{k_1'}\|_{L^\infty}\|\partial_t f_{k_2'}\|_{L^2}\big)$$

$$+2^{\alpha k+m+j+2k_1-\kappa_1-\kappa_2+2k_2+l}\|\widehat{f_{k_2}}(t,\xi)\|_{L^\infty_\xi}\|e^{-i(t_i+2^{-\kappa_2}\lambda_2)\Lambda}f_{k_2'}\|_{L^\infty}\|\partial_t f_{k_1'}\|_{L^2}$$

$$+2^{-10m-\kappa_1-\kappa_2}\big(\|f_{k_2}\|_{L^2}\|f_{k_1'}\|_{L^2}\|f_{k_2'}\|_{L^2} + \|f_{k_2}\|_{L^2}\|\partial_t f_{k_1'}\|_{L^2}\|f_{k_2'}\|_{L^2}$$

$$+\|\partial_t f_{k_2}\|_{L^2}\|f_{k_1'}\|_{L^2}\|f_{k_2'}\|_{L^2} + \|f_{k_2}\|_{L^2}\|f_{k_1'}\|_{L^2}\|\partial_t f_{k_2'}\|_{L^2}\big)$$

$$\lesssim 2^{k_2-\alpha k_2'-\max\{l,k_{1,-}\}/2+\beta m}\epsilon_0 + 2^{(1-\alpha/2)k_2+k_1'+2l-k_2'-2\max\{l,k_{1,-}\}+\beta m+\alpha m/2+\alpha k_2'/2}\epsilon_0 + 2^{-2\delta m-\delta j}\epsilon_0$$

$$\lesssim 2^{(1-2\alpha)k_2/2+2\beta m}\epsilon_0 + 2^{(1-\alpha/2)l+\alpha m/2+2\beta m}\epsilon_0 + 2^{-2\delta m-\delta j}\epsilon_0 \lesssim 2^{-2\delta m-\delta j}\epsilon_0.$$

Note that we used the fact that $k_2 \le k_2'$, $l \le -2\alpha m/3$ and also used the following estimate in above estimate,

$$\|e^{-it\Lambda}f_{k_2'}\|_{L^\infty} \lesssim 2^{-m+\alpha m/2+3\alpha k_2'/2}\|f_{k_2'}\|_{L^1} \lesssim 2^{-m+\alpha m/2+\alpha k_2'/2}\epsilon_1,$$

which can be derived by letting $\theta = 1-\alpha$ in estimate (5.10) in Lemma 5.0.5.

### 5.5.3 Estimate of $K^{\tau,\kappa}(\cdot,\cdot)$ in $\mathbf{bad}_k$.

In this subsection, we mainly study the case when $(k_1,k_2)\in\chi_k^3$. The main difference between this case and what we studied before (where $(k_1,k_2)\in\chi_k^1\cup\chi_k^2$) is that both $\nabla_\eta\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)$ $(\nabla_\eta\Phi_2^{\tau,\tau,\nu}(\xi,\eta,\sigma))$ and $\nabla_\sigma\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)(\nabla_\sigma\Phi_2^{\tau,\tau,\nu}(\xi,\eta,\sigma))$ can be zero at the same time. To get around difficulty, we need to utilize different properties of phase near the space resonance set in " $\eta$" and "$\sigma$". Following the same argument as we did in subsection 5.4.1, we restrict ourself to the case where $-m+\alpha m/3\le k\le\beta m$ and $j\le m+20\delta m$.

As we need to gain some smallness from the angle when the phase is highly degenerated, it is crucial to classify phases first. Note that, phases of the following types are highly degenerated.

$$\Phi^{-,-}(\xi,\eta)\sim 2^k,\quad \Phi^{-,+}(\xi,\eta)\sim 2^k. \tag{5.157}$$

Note that the linear part of $\Phi^{+,-}(\xi,\eta)$ vanishes when $\angle(\xi,-\eta)=0$ and the linear part of $\Phi^{+,+}(\xi,\eta)$ vanishes when $\angle(\xi,\eta)=\angle(\xi,\xi-\eta)=0$. Hence, the following estimate holds,

$$\Phi^{+,-}(\xi,\eta)\psi_{\ge -10}(\angle(\xi,-\eta))+\Phi^{+,+}(\xi,\eta)\big(1-\psi_{\le -10}(\angle(\xi,\xi-\eta))\psi_{\le -5}(\angle(\xi,-\eta))\big)\sim 2^k. \tag{5.158}$$

Therefore, we can decompose $A_{+,\nu}(\cdot,\cdot)$ and $K^{+,\nu}(\cdot,\cdot)$ into two parts as follows,

$$A_{+,\nu}(u(t),u^\nu(t)):=\sum_{i=1,2}A^i_{+,\nu}(u(t),u^\nu(t)),\quad K^{+,\nu}(f_{k_1},f^\nu_{k_2})=\sum_{i=1,2}K_i^{+,\nu}(f_{k_1},f^\nu_{k_2}), \tag{5.159}$$

where symbols $m^i_{+,\nu}(\cdot,\cdot)$ of $A^i_{+,\nu}(\cdot,\cdot)$, $\nu\in\{+,-\},i\in\{1,2\}$, are defined as follows,

$$m^1_{+,-}(\xi-\eta,\eta)=m_{+,-}(\xi-\eta,\eta)\psi_{\ge -10}(\angle(\xi,-\eta),\quad m^2_{+,-}(\xi-\eta,\eta)=a_{+,-}(\xi-\eta,\eta)\psi_{<-10}(\angle(\xi,-\eta),$$

$$m^1_{+,+}(\xi-\eta,\eta)=m_{+,+}(\xi-\eta,\eta)\big(1-\psi_{\le -10}(\angle(\xi,\xi-\eta))\psi_{\le -5}(\angle(\xi,-\eta))\big),$$

$$m^2_{+,+}(\xi-\eta,\eta)=m_{+,+}(\xi-\eta,\eta)\psi_{\le -10}(\angle(\xi,\xi-\eta))\psi_{\le -5}(\angle(\xi,-\eta)),$$

and

$$K_i^{+,\nu}(f^\mu_{k_1},f^\nu_{k_2})=-\int_{t_1}^{t_2}\int e^{it\Phi^{+,\nu}(\xi,\eta)}m^i_{+,\nu}(\xi-\eta,\eta)\partial_t\big(\widehat{f_{k_1}}(t,\xi-\eta)\widehat{f^\nu_{k_2}}(t,\eta)\big)d\eta dt,\quad i\in\{1,2\}.$$

The angle $\angle(\xi,\nu\eta)$ inside $K_2^{+,\nu}(f_{k_1},f^\nu_{k_2})$ is very crucial. We localize the angle and do decomposition

as follows,

$$K_2^{+,\nu}(f_{k_1},f_{k_2}^{\nu})=\sum_{\bar{l}\le l\le 2}K_{2,l}^{+,\nu}(f_{k_1},f_{k_2}^{\nu}),\quad \nu\in\{+,-\},\quad \bar{l}:=2k_-,$$

$$K_{2,l}^{+,\nu}(f_{k_1},f_{k_2}^{\nu})=-\int_{t_1}^{t_2}\int e^{it\Phi^{+,\nu}(\xi,\eta)}m_{+,\nu}^2(\xi-\eta,\eta)\partial_t\big(\widehat{f_{k_1}^{\mu}}(t,\xi-\eta)\widehat{f_{k_2}^{\nu}}(t,\eta)\big)\varphi_{\bar{l},l}(\angle(\xi,\nu\eta))d\eta dt.$$

From above estimates, (5.26) and Lemma 1.3.8, we have

$$\sum_{\nu\in\{+,-\}}\|m_{-,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}}+\|m^1_{+,\nu}(\xi-\eta,\eta)\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}}\lesssim 1. \tag{5.160}$$

As the angle is well localized for $K_2^{+,\nu}(\cdot,\cdot)$, we can do better for $j$. After doing integration by parts in "$\xi$" many times, we can rule out the case when $j\ge\max\{m+\max\{2k,l\},-k-l\}+100\delta m$. It is sufficient to consider the case when $j\le\max\{m+\max\{2k,l\},-k-l\}+100\delta m$.

Recall (5.38), after plug-in the equation satisfied by $\partial_t f$ in (5.24), we have essentially same formulas as in (5.102) and (5.132) for $K^{-,\nu}(\cdot,\cdot)$, $K_1^{+,\nu}(\cdot,\cdot)$, and $K_2^{+,\nu}(\cdot,\cdot)$. More precisely, we have,

$$K^{-,\nu}(\bar{f}_{k_1},f_{k_2}^{\nu})=\sum_{(k_1',j_1'),(k_2',j_2')\in\mathcal{J}}\sum_{\tau,\kappa\in\{+,-\}}\Big(\sum_{j_1\ge -k_{1,-}}K^{-,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}+\sum_{j_2\ge -k_{2,-}}K^{-,\nu,\tau,\kappa,2}_{j_2;j_1',j_2'}\Big)+\mathrm{JR}^{-,\nu}_{k_1,k_2},$$

$$K_1^{+,\nu}(f_{k_1},f_{k_2}^{\nu})=\sum_{(k_1',j_1'),(k_2',j_2')\in\mathcal{J}}\sum_{\tau,\kappa\in\{+,-\}}\Big(\sum_{j_1\ge -k_{1,-}}K^{+,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}+\sum_{j_2\ge -k_{2,-}}K^{+,\nu,\tau,\kappa,2}_{j_2;j_1',j_2'}\Big)+\mathrm{JR}^{+,\nu}_{k_1,k_2;1},$$

$$K_{2,l}^{+,\nu}(f_{k_1},f_{k_2}^{\nu})=\sum_{(k_1',j_1'),(k_2',j_2')\in\mathcal{J}}\sum_{\tau,\kappa\in\{+,-\}}\Big(\sum_{j_1\ge -k_{1,-}}K^{+,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}+\sum_{j_2\ge -k_{2,-}}K^{+,\nu,\tau,\kappa,2}_{l,j_2;j_1',j_2'}\Big)+\mathrm{JR}^{+,\nu}_{k_1,k_2;2},$$

where

$$K^{\mu,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}=-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi_1^{\mu,\tau,\kappa}(\xi,\eta,\sigma)}c^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)\widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma)d\eta d\sigma dt,$$

$$K^{\mu,\nu,\tau,\kappa,2}_{j_2;j_1',j_2'}=-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi_2^{\tau,\kappa,\nu}(\xi,\eta,\sigma)}c^{\tau,\kappa}_{\mu,\nu,2}(\xi,\eta,\sigma)\widehat{f^{\tau}_{k_1',j_1'}}(t,\xi-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t,\eta)d\eta d\sigma dt,$$

$$K^{+,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}=-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi_1^{+,\tau,\kappa}(\xi,\eta,\sigma)}c^{\tau,\kappa}_{+,\nu,l,1}(\xi,\eta,\sigma)\widehat{f^{\mu}_{k_1,j_1}}(t,\xi-\eta)\widehat{f^{\tau}_{k_1',j_1'}}(t,\eta-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma)d\eta d\sigma dt,$$

$$K^{+,\nu,\tau,\kappa,2}_{l,j_2;j_1',j_2'}=-\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{it\Phi_2^{\tau,\kappa,\nu}(\xi,\eta,\sigma)}c^{\tau,\kappa}_{+,\nu,l,2}(\xi,\eta,\sigma)\widehat{f^{\tau}_{k_1',j_1'}}(t,\xi-\sigma)\widehat{f^{\kappa}_{k_2',j_2'}}(t,\sigma-\eta)\widehat{f^{\nu}_{k_2,j_2}}(t,\eta)d\eta d\sigma dt,$$

$$\mathrm{JR}^{-,\nu}_{k_1,k_2}=-\int_{t_1}^{t_2}\int e^{it\Phi^{-,\nu}(\xi,\eta)}a_{-,\nu}(\xi-\eta,\eta)\big(\widehat{\mathcal{R}'}_{k_1}(t,\xi-\eta)\widehat{f^{\nu}_{k_2}}(t,\eta)+\widehat{\bar{f}_{k_1}}(t,\xi-\eta)\widehat{\mathcal{R}'^{\nu}_{k_2}}(t,\eta)\big)d\eta dt,$$

$$\mathrm{JR}^{+,\nu}_{k_1,k_2;i}=-\int_{t_1}^{t_2}\int e^{it\Phi^{+,\nu}(\xi,\eta)}d^i_{+,\nu}(\xi-\eta,\eta)\big(\widehat{\mathcal{R}'}_{k_1}(t,\xi-\eta)\widehat{f^{\nu}_{k_2}}(t,\eta)+\widehat{f_{k_1}}(t,\xi-\eta)\widehat{\mathcal{R}'^{\nu}_{k_2}}(t,\eta)\big)d\eta dt,$$

where the explicit formulas of all symbols appeared above are given as follows,

$$d^1_{+,\nu}(\xi-\eta,\eta)=m^1_{+,\nu}(\xi-\eta,\eta),\quad d^2_{+,\nu}(\xi-\eta,\eta)=m^2_{+,\nu}(\xi-\eta,\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta)),$$

$$c^{\tau,\kappa}_{-,\nu,1}(\xi,\eta,\sigma):=m_{-,\nu}(\xi-\eta,\eta)\big(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma)\big)^{\nu}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta),$$

$$c^{\tau,\kappa}_{+,\nu,1}(\xi,\eta,\sigma):=m^1_{+,\nu}(\xi-\eta,\eta)\big(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma)\big)^{\nu}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta),$$

$$c^{\tau,\kappa}_{-,\nu,2}(\xi,\eta,\sigma):=m_{-,\nu}(\xi-\eta,\eta)\overline{q_{-\tau,-\kappa}(\xi-\sigma,\sigma-\eta)}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta),$$

$$c^{\tau,\kappa}_{+,\nu,2}(\xi,\eta,\sigma):=m^1_{+,\nu}(\xi-\eta,\eta)q_{\tau,\kappa}(\xi-\sigma,\sigma-\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta),$$

$$c^{\tau,\kappa}_{+,\nu,l,1}(\xi,\eta,\sigma):=m^2_{+,\nu}(\xi-\eta,\eta)\big(q_{\tau\nu,\kappa\nu}(\eta-\sigma,\sigma)\big)^{\nu}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta)),$$

$$c^{\tau,\kappa}_{+,\nu,l,2}(\xi,\eta,\sigma):=m^2_{+,\nu}(\xi-\eta,\eta)q_{\tau,\kappa}(\xi-\sigma,\sigma-\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\varphi_{\bar{l};l}(\angle(\xi,\nu\eta)).$$

**The $Z$-norm estimate of $K^{\mu,\nu,\tau,\kappa,i}_{j_1;j_1',j_2'}$ and $K^{\mu,\nu,\tau,\kappa,i}_{l,j_1;j_1',j_2'}$, $i\in\{1,2\}$.**

As $\eta$ and $\xi-\eta$ now are comparable, there is little difference to estimate $K^{\mu,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}$ ($K^{\mu,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}$) and $K^{\mu,\nu,\tau,\kappa,2}_{j_1;j_1',j_2'}$ ($K^{\mu,\nu,\tau,\kappa,2}_{l,j_1;j_1',j_2'}$). For simplicity, we only estimate $K^{\mu,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}$ and $K^{\mu,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}$ in details here.

If either $(k_1',k_2')\in\chi^1_{k_2}\cup\chi^2_{k_2}$ or $(k_1',k_2')\in\chi^3_{k_2},\tau\kappa=-$ , from estimates in Lemma 5.3.1 and (5.83) in Lemma 5.3.3, we still have a good lower bound for $\nabla_\sigma\Phi^{\mu,\tau,\kappa}_1(\xi,\eta,\sigma)$. Therefore, the methods we used in the good type phases and the bad type phases can still be used to estimate those scenarios. Therefore, we restrict ourself to the case when $(k_1',k_2')\in\chi^3_{k_2}$ and $\tau\kappa=+$.

From $L^2-L^\infty$ type bilinear estimate (1.13) in Lemma 1.3.7, (5.160), and (5.7) in Lemma 5.0.4, the following estimate holds when $k\le-\beta m/100$,

$$\sum_{\mu,\nu,\tau,\kappa\in\{+,-\}}\|\sum_{j_1\ge-k_{1,-},j_1'\ge-k_{1,-},j_2'\ge-k'_{2,-}}K^{\mu,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}\|_{B_{k,j}}$$

$$\lesssim\sup_{t\in[t_1,t_2]}2^{\alpha k+m+j+k}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2}\|_{L^2}\lesssim2^{(2-\alpha)k+100\delta m}\epsilon_0\lesssim2^{-4\delta m}\epsilon_0.\quad(5.161)$$

From (5.50) in Lemma 5.2.5 and (5.44) in Lemma 5.2.4 , the following estimate holds if $3k\le-m+\beta m$,

$$\sum_{\nu\in\{+,-\}}\|\sum_{j_1\ge-k_{1,-},j_1'\ge-k_{1,-},j_2'\ge-k'_{2,-}}K^{+,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}\|_{B_{k,j}}\lesssim\sup_{t\in[t_1,t_2]}2^{\alpha k+m+j}2^{-2\max\{k,l\}}2^{2k+l}2^k\|\widehat{f}_{k_2}(t,\xi)\|_{L^\infty_\xi}$$

$$\times\|e^{-it\Lambda}f_{k_1'}\|_{L^\infty}\|f_{k_2'}\|_{L^2}\lesssim2^{200\delta m}\max\{2^{m+l+4k},2^{m+4k},2^k\}\epsilon_0\lesssim2^{-10\delta m}\epsilon_0.\quad(5.162)$$

If $-m+\beta m \le 3k \le -3\beta m/100$, then $|\Phi^{+,\nu}(\cdot,\cdot)|$ is greater than $2^{-m+\beta m}$. This fact enables us to use the estimate (5.56) in Lemma 5.2.6. Together with (5.50) in Lemma 5.2.5, the following estimate holds,

$$\sum_{\nu\in\{+,-\}} \|K^{+,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}\|_{B_{k,j}} \lesssim 2^{\alpha k+m+j-2\max\{k,l\}} \min\{2^{-m-\alpha k}2^{-m+(2-\alpha)k}, 2^{2k+l+10\delta m}2^{-m+(2-\alpha)k}\}\epsilon_0$$

$$\lesssim \max\{2^{(1-\alpha)k+200\delta m}, 2^{k+200\delta m}\}\epsilon_0 \lesssim 2^{(1-\alpha)k+200\delta m}\epsilon_0 \lesssim 2^{-2\delta m}. \tag{5.163}$$

To sum up, we only need to consider the case when $-\beta m/100 \le k \le \beta m$, $(k_1', k_2') \in \chi_k^3$. As now $k$ is almost like "1", the angle "$l$", which is greater than $2k_-$, plays little role. there is little difference of estimating $K^{+,\nu,\tau,\kappa,1}_{l,j_1;j_1',j_2'}$ and $K^{+,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}$. We only consider $K^{+,\nu,\tau,\kappa,1}_{j_1;j_1',j_2'}$ in details here.

From $L^2-L^\infty-L^\infty$ type estimate, after putting the input with the maximum spatial concentration in $L^2$ and the other two inputs in $L^\infty$, we can rule out the case when $\max\{j_1, j_1', j_2'\} \ge \beta m$. It remains to consider the case when $\max\{j_1, j_1', j_2'\} \le \beta m$.

We decompose $K^{\mu,\nu,\tau,\tau,1}_{j_1;j_1',j_2'}$ into three parts: (i) the region where integration by parts in $\sigma$ works; (ii) the region where integration by parts in $\eta$ works; (iii) the region where neither integration by parts in "$\sigma$" nor in "$\eta$" works. More precisely, we decompose the symbol $c^{\tau,\kappa}_{\mu,\nu,1}(\xi,\eta,\sigma)$ into three pieces as follows,

$$c^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma) = \sum_{i=1,2,3} e^{\tau,\tau}_{\mu,\nu,i}(\xi,\eta,\sigma), e^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma) = c^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)\psi_{\ge \tilde{l}_{\mu,\tau}}(\sigma-\eta/2),$$

$$e^{\tau,\tau}_{\mu,\nu,2}(\xi,\eta,\sigma) = c^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)\psi_{\ge \tilde{l}_{\mu,\tau}}\big((\xi-\eta)-\mu\tau(\eta-\sigma)\big)\psi_{<\tilde{l}_{\mu,\tau}}(\sigma-\eta/2),$$

$$e^{\tau,\tau}_{\mu,\nu,3}(\xi,\eta,\sigma) = c^{\tau,\tau}_{\mu,\nu,1}(\xi,\eta,\sigma)\psi_{<\tilde{l}_{\mu,\tau}}\big((\xi-\eta)-\mu\tau(\eta-\sigma)\big)\psi_{<\tilde{l}_{\mu,\tau}}(\sigma-\eta/2),$$

$$\tilde{l}_{+,-} = \tilde{l}_{+,+} = \tilde{l}_{-,-} = -2\beta m, \quad \tilde{l}_{-,+} = -m/2+10\beta m.$$

As long as the threshold $\tilde{l}_{\mu,\nu}$ we choose is greater than $-m/2+5\beta m$, then we can safely do integration by parts in "$\sigma$" in region (i) and do integration by parts in "$\eta$" in region (ii) to rule out the case when $\max\{j_1, j_1', j_2'\} \le \beta m$. We mention that the threshold we chose for $\tilde{l}_{-,+}$ is different because the strategy we use to control the region (iii) is different.

A very important observation for $\Phi_1^{+,+,+}(\xi,\eta,\sigma)$, $\Phi_1^{-,-,-}(\xi,\eta,\sigma)$, and $\Phi_1^{+,-,-}(\xi,\eta,\sigma)$ is that the sizes of phases are very big, which are greater than $2^{-\beta m/10}$, inside the region (iii). We can take advantage of this fact by doing integration by parts in time again to gain extra $2^{-m}$, which is

sufficient to close the argument. More precisely, the following estimate holds,

$$|\Phi_1^{\mu,\tau,\tau}(\xi,\eta,\sigma)|\psi_{<\tilde{l}_{\mu,\tau}}\big((\xi-\eta)-\mu\tau(\eta-\sigma)\big)\psi_{<\tilde{l}_{\mu,\tau}}(\sigma-\eta/2)\gtrsim 2^{-\beta m/10},\quad (\mu,\tau,\tau)\neq(-,+,+). \tag{5.164}$$

For example, when $(\mu,\tau,\tau)=(+,+,+)$, we know that $\sigma$ is very close (in a neighborhood of radius $2^{-\beta m}$) to $\eta/2$ and $\xi$ is very close to $3\eta/2$ (in a neighborhood of radius $2^{-\beta m}$). Hence $\Phi_1^{+,+,+}(\xi,\eta,\sigma)$ is very close (in a neighborhood of radius $2^{-\beta m}$) to $\Phi_1^{+,+,+}(3\eta/2,\eta,\eta/2)$, which is greater than $2^{-\beta m/10}$. Therefore (5.164) holds.

Unfortunately, the good lower bound doesn't hold for $\Phi_1^{-,+,+}(\xi,\eta,\sigma)$. As $\Phi_1^{-,+,+}(\xi,\eta,\sigma)$ is also very close (in a neighborhood of radius $2^{-\beta m}$) to $\Phi_1^{-,+,+}(\eta/2,\eta,\eta/2)$, which equals to "0". Another very important observation is that

$$\nabla_\xi\Phi_1^{-,+,+}(\xi,\eta,\sigma)\big|_{(\eta/2,\eta,\eta/2)}=\Lambda'(|\xi|)\frac{\xi}{|\xi|}+\Lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}\big|_{(\eta/2,\eta,\eta/2)}=0.$$

Therefore, the following estimate holds,

$$|\nabla_\xi\Phi_1^{-,+,+}(\xi,\eta,\sigma)|\psi_{<\tilde{l}_{-,+}}\big((\xi-\sigma)\big)\psi_{<\tilde{l}_{-,+}}(\sigma-\eta/2)\lesssim 2^{-k_-+\beta m}\tilde{l}_{-,+}\lesssim 2^{-m/2+12\beta m}.$$

Therefore, by doing integration by parts in "$\xi$" many times, we can rule out the case when $j\geq m/2+13\beta m$. For the case when $j\leq m/2+13\beta m$, we use the size of support of $\eta$ and $\sigma$ for fixed $\xi$, which is $2^{4\tilde{l}_{-,+}}$ in total, which is less than $2^{-2m+40\beta m}$. This is sufficient to cover the loss of $2^{m+j}\leq 2^{3m/2+13\beta m}$. Therefore, close the argument.

**The $Z$-norm estimate of the reminder terms.**

The estimate of $\mathrm{JR}_{k_1,k_2}^{-,\nu}$ and $\mathrm{JR}_{k_1,k_2;i}^{+,\nu}$, $i\in\{1,2\}$, is straightforward. From $L^\infty-L^2$ type bilinear estimate (1.13) in Lemma 1.3.7, (5.160), and (5.40) in Proposition 5.2.2, the following estimate holds,

$$\sum_{\nu\in\{+,-\}}\|\mathrm{JR}_{k_1,k_2}^{-,\nu}\|_{B_{k,j}}+\|\mathrm{JR}_{k_1,k_2;1}^{+,\nu}\|_{B_{k,j}}\lesssim\sum_{i=1,2}\sup_{t\in[t_1,t_2]}2^{\alpha k+6k_++m+j}\|P_{k_i}\mathcal{R}(t)\|_{L^2}$$

$$\times\|e^{-it\Lambda}f_{k_{3-i}}\|_{L^\infty}\lesssim 2^{-m+10\beta m}\epsilon_0.$$

From (5.50) in Lemma 5.2.5 and (5.40) in Proposition 5.2.2, the following estimate holds if $k \le -\beta m$,

$$\sum_{\nu\in\{+,-\}} \|\mathrm{JR}^{+,\nu}_{k_1,k_2;2}\|_{B_{k,j}} \lesssim \sum_{i=1,2} \sup_{t\in[t_1,t_2]} 2^{\alpha k+m+j-2\max\{l,k\}} 2^{2k+l} \|P_{k_i}\mathcal{R}(t)\|_{L^2}$$

$$\times \|\widehat{f}_{k_{3-i}}(t,\xi)\|_{L^\infty_\xi} \lesssim 2^{k+l+200\delta m}\epsilon_0 \lesssim 2^{-4\delta m}\epsilon_0.$$

From $L^\infty - L^2$ type bilinear estimate (1.13) in Lemma 1.3.7 and (5.40) in Proposition 5.2.2, the following estimate holds if $k \ge -\beta m$,

$$\sum_{\nu\in\{+,-\}} \|\mathrm{JR}^{+,\nu}_{k_1,k_2;2}\|_{B_{k,j}} \lesssim \sum_{i=1,2} \sup_{t\in[t_1,t_2]} 2^{m+j+50\beta m} \|P_{k_i}\mathcal{R}(t)\|_{L^2} \|e^{-it\Lambda} f_{k_{3-i}}\|_{L^\infty} \lesssim 2^{-\alpha m/4}\epsilon_0.$$

**Proof of Lemma 5.2.1.**

Assume that $t \in [2^{m-1}, 2^m]$ for some $m \in \mathbb{N}$. Note that $|k_1 - k|, |k_2 - k| \le 5$.

- We first estimate $A_{-,\nu}(\cdot,\cdot)$ and $A^1_{+,\nu}(\cdot,\cdot)$. By do integration by parts in "$\xi$" many times, we can first rule out the case when $j \ge m+5$. It remains to consider the case when $j \le m+5$. For this case, we have

$$\sum_{\nu\in\{+,-\}} \|e^{it\Lambda} A_{-,\nu}(u^\mu(t), u^\nu(t))\|_{B_{k,j}} + \|e^{it\Lambda} A^1_{+,\nu}(u^\mu(t), u^\nu(t))\|_{B_{k,j}} \lesssim \sum_{(k_1,k_2)\in\chi^3_k} \sum_{j\le m+5} 2^{\alpha k+6k_+ +j}$$

$$\times \|e^{-it\Lambda} f_{k_1}\|_{L^\infty} \|f_{k_2}\|_{L^2} \lesssim 2^{(1-\alpha)k-4k_+} \|f_{k_1}\|_Z \|f_{k_2}\|_Z \lesssim \epsilon_1^2 \lesssim \epsilon_0.$$

- Now we estimate $A^2_{+,\nu}(\cdot,\cdot)$. Although associated phases are highly degenerated, we know that the angle of $\nabla_\xi \Phi^{+,\nu}(\cdot,\cdot)$ is comparable to the angle of $\nabla_\eta \Phi^{+,\nu}(\cdot,\cdot)$. For this case, we do angular decomposition as follows,

$$\big[e^{it\Lambda} A^2_{+,\nu}(u(t), u(t))\big](x) = \sum_{k\in\mathbb{Z}} \sum_{(k_1,k_2)\in\chi^3_k} \sum_{l\ge \bar{l}} H^l_{k_1,k_2}, \quad H^l_{k_1,k_2} = \sum_{j_1\ge -k_{1,-}, j_2\ge -k_{2,-}} H^l_{k_1,k_2;j_1,j_2},$$

where the threshold $\bar{l} := 2k_-$ and

$$H^l_{k_1,k_2;j_1,j_2}, = \mathcal{F}^{-1}\Big[ \int_{\mathbb{R}^2} e^{it\Phi^{+,\nu}(\xi,\eta)} m^2_{+,\nu}(t,\xi-\eta,\eta) \widehat{f_{k_1,j_1}}(t,\xi-\eta) \widehat{f^\nu_{k_2,j_2}}(t,\eta) \varphi_{\bar{l},l}(\angle(\xi,\nu\eta)) \psi_k(\xi) d\eta \Big].$$

By doing integration by parts many times in $\xi$, we can rule out the case when $j \ge \max\{m+l, -k-l\}+5$ and $\min\{j_1,j_2\} \le j - \delta j$. For the case when $j \ge \max\{m+l,-k-l\}+5$ and

$\min\{j_1, j_2\} \geq j - \delta j$, we can derive the following estimate from (5.50) in Lemma 5.2.5,

$$\sum_{j\geq \max\{(m+l),-k-l\}+5} \sum_{\min\{j_1,j_2\}\geq j-\delta j} \|H^l_{k_1,k_2;j_1,j_2}\|_{B_{k,j}} \lesssim \sum_{j\geq \max\{m+l,-k-l\}+5} \sum_{\min\{j_1,j_2\}\geq j-\delta j} 2^{\alpha k+6k_++j}$$

$$\times 2^{-2\max\{k,l\}} 2^{k+l/2} 2^{-2\alpha k_1-12k_+-j_1-j_2}\epsilon_1^2 \lesssim 2^{(1-2\alpha-2\delta)k/2-6k_+}\epsilon_1^2 \lesssim \epsilon_0.$$

We split the case when $j \leq \max\{m+l, -k-l\}+5$ into several cases as follows.

**Case** 1: If $l = \bar{l}$ and $k \geq -m/5$, then $j \leq m + \bar{l} = m + 2k_- + 5$ and $|\Phi^{+,\nu}(\xi,\eta)| \gtrsim 2^{3k} \gtrsim 2^{-3m/5}$. From (5.56) in Lemma 5.2.6, we have

$$\sum_{j\leq m+\bar{l}+5} \|H^{\bar{l}}_{k_1,k_2}\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++j} 2^{-2k} 2^{-m-\alpha k-4k_+} 2^{(1-\alpha)k-6k_+} \|f_{k_1}\|_Z \|f_{k_1}\|_Z \lesssim 2^{(1-\alpha)k-4k_+}\epsilon_1^2 \lesssim \epsilon_0.$$

**Case** 2: If $l = \bar{l}$, $k \leq -m/5$, then $j \leq -k - \bar{l} = -3k$. From (5.50 ) in Lemma 5.2.5 and (5.43) and (5.45) in Lemma 5.2.4, the following estimate holds,

$$\sum_{j\leq -3k} \|H^{\bar{l}}_{k_1,k_2}\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++j} 2^{-2\max\{k_-,\bar{l}\}} 2^{2k+\bar{l}} \|\widehat{f_{k_1}}(t,\xi)\|_{L^\infty_\xi} \|f_{k_2}\|_{L^2} \lesssim 2^{\alpha k+10\delta m}\epsilon_1^2 \lesssim 2^{-10\delta m}\epsilon_0.$$

**Case** 3: If $l > \bar{l}$, $k + 2l \leq -m + 4\beta m$, then $j \leq -k - l + 4\beta m + 5$. From (5.50 ) in Lemma 5.2.5 and (5.43) and (5.45) in Lemma 5.2.4, the following estimate holds,

$$\|H^l_{k_1,k_2}\|_{B_{k,j}} \lesssim 2^{\alpha k+6k_++j} 2^{-2\max\{k_-,l\}} 2^{2k+l} \|\widehat{f_{k_1}}(t,\xi)\|_{L^\infty_\xi} \|f_{k_2}\|_{L^2} \lesssim 2^{\alpha k+10\delta m}\epsilon_1^2 \lesssim 2^{-10\delta m}\epsilon_0.$$

**Case** 4: If $l > \bar{l}$, $k + 2l \geq -m + 4\beta m$, then $j \leq m + l + 5$ and $\max\{k, l\} \geq -m/3 + 2\beta m$. Note that,

$$|\nabla_\eta \Phi^{+,+}(\xi,\eta)| \geq 2^{-k_+/2} \angle(\xi-\eta,\eta) \geq 2^{-k_+/2+l}, \quad |\Phi^{+,+}(\xi,\eta)| \gtrsim 2^{k+2\max\{k_-,l\}} \gtrsim 2^{m-4\beta m}.$$

As $k + 2l \geq -m + 4\beta m$, we can do integration by parts in "$\eta$" many times to rule out the case when $\max\{j_1, j_2\} \leq m + l - k_+/2 - 10\delta m$. For the case when $\max\{j_1, j_2\} \geq m + l - k_+/2 - 10\delta m$. From (5.56) in Lemma 5.2.6, we put the input with higher spatial concentration in $L^2$ and the other one in $L^\infty$. As a result, the following estimate holds,

$$\sum_{\max\{j_1,j_2\}\geq m+l-k_+/2-10\delta m} \|H^l_{k_1,k_2;j_1,j_2}\|_{B_{k,j}} \lesssim \sum_{\max\{j_1,j_2\}\geq m+l-k_+/2-10\delta m} 2^{\alpha k+j+6k_+}$$

$$\times 2^{-2\max\{k_-,l\}}2^{-m-\alpha k-3k_+}2^{-\max\{j_1,j_2\}-\alpha k-6k_+}\|f_{k_1,j_1}\|_Z\|f_{k_1,j_2}\|_Z \lesssim 2^{-m/3+\alpha m-2k_+}\epsilon_0.$$

As there are at most $m^3$ number of cases if $l > \bar{l}$, from above estimates, it is easy to see our desired estimate (5.29) in Lemma 5.2.1 holds. □

## 5.6 Remainder estimate

This section is devoted to prove Proposition 5.2.2. Intuitively speaking, we can prove Proposition 5.2.2 by the following three steps:

(i) we can first decompose the remainder term $\mathcal{R}$ into two parts: cubic type terms and terms depend on $\Lambda_{\geq 3}[B(h)\psi]$. Cubic type terms can be easily estimated.

(ii) To estimate the $Z$– norm of the profile of $\Lambda_{\geq 3}[B(h)\psi]$, it's sufficient to estimate the profile of $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$ in the $L^\infty_z Z$–normed space, which can be done by a fixed point type argument.

(iii) We can group all information together to prove Proposition 5.2.2.

Step (i) is straightforward, recall (5.6), we have

$$\mathcal{R} = \Lambda_{\geq 3}[(1+|\nabla h|^2)B(h)\psi] + i\Lambda\Lambda_{\geq 3}[(1+|\nabla h|^2)(B(h)\psi)^2] = \Lambda_{\geq 3}[(1+|\nabla h|^2)(\Lambda_{\leq 2}[B(h)\psi]$$

$$+\Lambda_{\geq 3}[B(h)\psi])] + i\Lambda\Lambda_{\geq 3}[(1+|\nabla h|^2)(\Lambda_{\leq 2}[B(h)\psi]+\Lambda_{\geq 3}[B(h)\psi])^2] = I_{\text{cubic}} + I_{\text{fps}},$$

where

$$I_{\text{cubic}} = |\nabla h|^2\Lambda_{\leq 2}[B(h)\psi] + i\Lambda\Big(|\nabla h|^2(\Lambda_{\leq 2}[B(h)\psi])^2 + (\Lambda_2[B(h)\psi])^2 + 2\Lambda_2[B(h)\psi]\Lambda_1[B(h)\psi]\Big), \tag{5.165}$$

$$I_{\text{fps}} = (1+|\nabla h|^2)\Lambda_{\geq 3}[B(h)\psi] + i\Lambda\Big((1+|\nabla h|^2)(\Lambda_{\geq 3}[B(h)\psi])^2$$

$$+2(1+|\nabla h|^2)(\Lambda_{\leq 2}[B(h)\psi])(\Lambda_{\geq 3}[B(h)\psi])\Big). \tag{5.166}$$

Since we already know the explicit formula of $\Lambda_{\leq 2}[B(h)\psi]$. That is to say (5.165) can be explicitly represented in terms of $h$ and $\psi$, hence also can be represented in terms of $u$ and $\bar{u}$.

Intuitively speaking, we can rewrite (5.165) as follows,

$$I_{\text{cubic}} = \sum_{\mu,\nu,\tau\in\{+,-\}} C_{\mu,\nu,\tau}(u^\mu,u^\nu,u^\tau) + C'_{\mu,\nu}(u^\mu,u^\nu,h_1) + C_\mu(u^\mu,h_2,h_3) + C(h_4,h_5,h_6). \tag{5.167}$$

where $h_i$, $1 \le i \le 6$, denotes some quadratic term depends on $u$ and $\bar{u}$. They can be represented as follows,

$$h_i = \sum_{\mu,\nu\in\{+,-\}} T^i_{\mu,\nu}(u^\mu, u^\nu), \quad 1 \le i \le 6.$$

*Proof of Proposition 5.2.2 .* Recall (3.43) and (4.1). From (5.176) in Lemma 5.6.2, we have

$$\sup_{1\le i\le 6}\sup_{t\in[2^{m-1},2^{m+1}]} \|e^{it\Lambda}h_i\|_Z \lesssim \epsilon_0.$$

From above estimate and estimates (5.178), (5.179), and (5.180) in Lemma 5.6.2, the following estimate holds for $k \in \mathbb{Z}$, $\theta \in [0,1]$, and $t, t_1, t_2 \in [2^{m-1}, 2^{m+1}]$,

$$\|e^{it\Lambda}\big[I_{\text{cubic}}\big]\|_Z + 2^{-(1-\theta)k+\theta m}\|P_k\big(e^{it\Lambda}\big[I_{\text{cubic}}\big]\big)\|_{L^2} \lesssim 2^{-m}\epsilon_0,$$

$$\sup_{k\in\mathbb{Z},j\ge\max\{-k,0\}} 2^{\delta j}\|\int_{t_1}^{t_2} e^{it\Lambda}\big[I_{\text{cubic}}\big]dt\|_{B_{k,j}} \lesssim 2^{-\delta m}\epsilon_0.$$

Combing above estimates with estimates (5.168) and (5.169), it's easy to see Proposition 5.2.2 holds. □

### 5.6.1 $Z-$ norm estimate of terms depend on $\Lambda_{\ge 3}[B(h)\psi]$

In this subsection, we mainly do step (ii). More precisely, we have the following lemma,

**Lemma 5.6.1.** *Under the bootstrap assumption*(3.43) *and the improved energy estimate* (4.1)*, the following estimates hold for any $k \in \mathbb{Z}$, $\theta \in [0,1]$, and $t, t_1, t_2 \in [2^{m-1}, 2^{m+1}]$,*

$$\|e^{it\Lambda}\big[I_{\text{fps}}\big]\|_Z + 2^{-(1-\theta)k+\theta m}\|P_k\big(e^{it\Lambda}\big[I_{\text{fps}}\big]\big)\|_{L^2} \lesssim 2^{-m}\epsilon_0, \tag{5.168}$$

$$\sup_{k\in\mathbb{Z},j\ge\max\{-k,0\}} 2^{\delta j}\|\int_{t_1}^{t_2} e^{it\Lambda}\big[I_{\text{fps}}\big]dt\|_{B_{k,j}} \lesssim 2^{-\delta m}(A+B)^3. \tag{5.169}$$

*Proof.* To estimate the $Z$-norm of $\Lambda_{\ge 3}[B(h)\psi]$, it is sufficient to estimate the $L^\infty_z Z$-norm of $\Lambda_{\ge 3}[\nabla_{x,z}\varphi]$. Recall (3.31), (3.18) and (3.19). We can rewrite $\Lambda_{\ge 3}[\nabla_{x,z}\varphi]$ as follows,

$$\Lambda_{\ge 3}[\nabla_{x,z}\varphi] = \sum_{i=1,2} C^i_z(h,\psi,\tilde{h}_i) + h\tilde{C}^i_z(h,\psi,\tilde{h}_i) + (1+h)^2\tilde{C}_z(\tilde{h}_2,\tilde{h}_2,\Lambda_{\le 2}[\nabla_{x,z}\varphi]) + h^2\hat{C}_z(h,\tilde{h}_2,\psi)$$

$$+T^i_z(\tilde{h}_i,\Lambda_{\ge 3}[\nabla_{x,z}\varphi]) + \widetilde{C}^1_z(h,\tilde{h}_2,\Lambda_{\ge 3}[\nabla_{x,z}\varphi]) + (1+h)^2\widetilde{C}^2_z(\tilde{h}_2,\tilde{h}_2,\Lambda_{\ge 3}[\nabla_{x,z}\varphi]), \tag{5.170}$$

where $\tilde{h}_1$ and $\tilde{h}_2$ are defined in (3.16), $C^i_z$, $\tilde{C}^i_z$, $\tilde{C}_z$, $\hat{C}_z$, $\widetilde{C}^i_z$, $i \in \{1,2\}$ are some trilinear operators, $T^i_z$

is some bilinear operator. The symbols of bilinear operators $T_z^i$ satisfy the following type of estimate

$$\|q(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{4\max\{k_1,k_2\}_+}, \tag{5.171}$$

and symbols of all above trilinear operators satisfy the following estimate

$$\|c(\xi-\eta,\eta-\sigma,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} \lesssim 2^{4\max\{k_1,k_2,k_3\}_+}. \tag{5.172}$$

To see structure of the profile of $\tilde{h}_i$ for $i\in\{1,2\}$, we define

$$\tilde{u}_1=\tilde{h}_1+i2\Lambda\psi,\quad \tilde{u}_2=\tilde{h}_2+i\Lambda\psi,$$

hence

$$\begin{aligned}\tilde{h}_1=\frac{2h+h^2}{(1+h)^2}\Longrightarrow \tilde{u}_1&=2u-\frac{3h}{2}\frac{2h+h^2}{(1+h)^2}-\frac{h^3}{2(1+h)^2}\\&=2u-\frac{3(u+\bar{u})}{4}\frac{\tilde{u}_1+\overline{\tilde{u}_1}}{2}-\frac{(u+\bar{u})}{4}\Big(\frac{\tilde{u}_2+\overline{\tilde{u}_2}}{2}\Big)^2,\end{aligned} \tag{5.173}$$

$$\tilde{h}_2=\frac{h}{1+h}\Longrightarrow \tilde{u}_2=u-h\frac{h}{1+h}=u-\frac{(u+\bar{u})(\tilde{u}_2+\overline{\tilde{u}_2})}{4}. \tag{5.174}$$

With above discussions, we can easily transfer the fixed point type formulation (5.170) into a fixed point type formulation in terms of $u$, $\bar{u}$, $\tilde{u}_i$ and $\overline{\tilde{u}_i}$, $i\in\{1,2\}$.

Before estimating $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$, we need to know the $Z$–norm of the profile of $\tilde{u}_i$. From (5.173) and (5.174), we have following estimate by using the main result in Lemma 5.6.2,

$$\sum_{i=1,2}\|e^{it\Lambda}\tilde{u}_i\|_Z\lesssim\epsilon_1+\epsilon_1\sum_{i=1,2}\|e^{it\Lambda}\tilde{u}_i\|_Z,\Longrightarrow\sum_{i=1,2}\|e^{it\Lambda}\tilde{u}_i\|_Z\lesssim\epsilon_1.$$

Now we are ready to prove Lemma 5.6.1. From estimates (5.178), (5.179), and (5.180) in Lemma 5.6.2, and Hölder type estimates, we can derive the following estimate from (5.170),

$$\|e^{it\Lambda}\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z Z}\lesssim 2^{-m}\epsilon_0+\epsilon_1\|e^{it\Lambda}\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z Z},$$

$$2^{-(1-\theta)k+\theta m}\|P_k[\Lambda_{\geq 3}[\nabla_{x,z}\varphi]]\|_{L^\infty_z L^2}\lesssim 2^{-m}\epsilon_0+\epsilon_0\|e^{it\Lambda}[\Lambda_{\geq 3}[\nabla_{x,z}\varphi]]\|_{L^\infty_z Z},$$

$$\sup_{k\in\mathbb{Z},j\geq -k_-}2^{\delta j}\|\int_{t_1}^{t_2}e^{it\Lambda}\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z B_{k,j}}\lesssim 2^{-\delta m}\epsilon_0+2^{3m/2}\sup_{k\in\mathbb{Z}}\|P_k[\Lambda_{\geq 3}[\nabla_{x,z}\varphi]]\|_{L^\infty_z L^2}\lesssim 2^{-\delta m}\epsilon_0.$$

In above estimate, we used the fact that $j\leq m+100\delta m$ and used $L^2-L^\infty$ type estimate for all

quartic and higher order terms in (5.170). Above estimates further imply the following estimates hold,

$$\|e^{it\Lambda}\Lambda_{\geq 3}[B(h)\psi]\|_Z + 2^{-(1-\theta)k+\theta m}\|P_k[B(h)\psi]\|_{L^\infty_z Z} \lesssim \|e^{it\Lambda}\Lambda_{\geq 3}[\nabla_{x,z}\varphi]\|_{L^\infty_z Z}$$

$$+2^{-(1-\theta)k+\theta m}\|P_k[\Lambda_{\geq 3}[\nabla_{x,z}\varphi]]\|_{L^\infty_z Z} \lesssim 2^{-m}\epsilon_0. \tag{5.175}$$

$$\sup_{k\in\mathbb{Z}, j\geq -k_-} 2^{\delta j}\| \int_{t_1}^{t_2} e^{it\Lambda}\Lambda_{\geq 3}[B(h)\psi]\|_{L^\infty_z B_{k,j}} \lesssim 2^{-\delta m}\epsilon_0.$$

Following the same procedure, recall (5.166), it's easy to see our desired estimates (5.168) and (5.169) hold. □

**Lemma 5.6.2.** *For any $\mu,\nu,\kappa \in \{+,-\}$ and $f$, $g$,$h \in H^{N_0}\cap Z$, which satisfy the following estimates,*

$$\|f\|_{H^{N_0}} + \|g\|_{H^{N_0}} + \|h\|_{H^{N_0}} \leq A, \quad \|f\|_Z + \|g\|_Z + \|h\|_Z \leq B,$$

*we have following estimates for any $t, t_1, t_2 \in [2^{m-1}, 2^{m+1}]$ and $\theta \in [0,1]$,*

$$\|e^{it\Lambda}Q((e^{-it\Lambda}f)^\mu, (e^{-it\Lambda}g)^\nu)\|_Z \lesssim B^2 + 2^{-10\delta m}(A+B)^2, \tag{5.176}$$

$$\sup_{k\in\mathbb{Z}} 2^{-(1-\theta)k}\|P_k\big[e^{it\Lambda}Q((e^{-it\Lambda}f)^\mu, (e^{-it\Lambda}g)^\nu)\big]\|_{L^2} \lesssim 2^{-\theta m}B^2. \tag{5.177}$$

$$\|e^{it\Lambda}C\big((e^{-it\Lambda}f)^\mu, (e^{-it\Lambda}g)^\nu, (e^{-it\Lambda}h)^\kappa\big)\|_Z \lesssim 2^{-m}B^2 + 2^{-m-10\delta m}(A+B)^2, \tag{5.178}$$

$$\sup_{k\in\mathbb{Z}} 2^{-(1-\theta)k}\|P_k\big[e^{it\Lambda}C\big((e^{-it\Lambda}f)^\mu, (e^{-it\Lambda}g)^\nu, (e^{-it\Lambda}h)^\kappa\big)\big]\|_{L^2} \lesssim 2^{-(1+\theta)m}B^3, \tag{5.179}$$

$$\sup_{k\in\mathbb{Z}, j\geq \max\{-k,0\}} 2^{\delta j}\| \int_{t_1}^{t_2} e^{it\Lambda}C\big((e^{-it\Lambda}f)^\mu, (e^{-it\Lambda}g)^\nu, (e^{-it\Lambda}h)^\kappa\big)dt\|_{B_{k,j}} \lesssim 2^{-\delta m}(A+B)^3, \tag{5.180}$$

*where the symbol $q(\xi-\eta,\eta)$ of bilinear operator $Q(\cdot,\cdot)$ and the symbol $c(\xi-\eta,\eta-\sigma,\sigma)$ of trilinear operator $C(\cdot,\cdot,\cdot)$ satisfy the following estimates respectively,*

$$\|q(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{4\max\{k_1,k_2\}_+}, \tag{5.181}$$

$$\|c(\xi-\eta,\eta-\sigma,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} \lesssim 2^{4\max\{k_1,k_2,k_3\}_+}. \tag{5.182}$$

*Proof.* • We first prove estimate (5.177) and (5.179). We only prove (5.177) in details here. (5.179) can be proved very similarly. From bilinear estimate (1.13) in Lemma 1.3.7 and (5.43) in Lemma

5.2.4, the following estimate holds for any $k \in \mathbb{Z}$ and any $\theta \in [0,1]$,

$$\sup_{k\in\mathbb{Z}} 2^{-(1-\theta)k}\|P_k\big[e^{it\Lambda}Q((e^{-it\Lambda}f)^{\mu},(e^{-it\Lambda}g)^{\nu})\big]\|_{L^2} \lesssim \sum_{k_2\leq k_1-10} 2^{-(1-\theta)k_1+4k_{1,+}}\big[\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}\|g_{k_2}\|_{L^2}$$

$$+\|e^{-it\Lambda}g_{k_1}\|_{L^\infty}\|f_{k_2}\|_{L^2}\big]+\sum_{|k_2-k_1|\leq 10} 2^{-(1-\theta)k+(1-\theta)k+4k_{1,+}}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}^{\theta}\|f_{k_1}\|_{L^2}^{(1-\theta)}\|g_{k_2}\|_{L^2} \lesssim 2^{-\theta m}B^2.$$

- Now we proceed to prove estimate (5.176) and (5.178). The proof of (5.178) is very similar, we only estimate (5.176) in details here. Through doing integration by parts in "$\xi$" many times, we can rule out the case when $j \geq m+5$. It remains to consider the case when $j \leq m+5$. Intuitively speaking, our desired estimates follow straightforwardly from $L^2-L^\infty$ type estimate. However, as the symbols have certain derivatives, we only need to get around this issue.

We first do dyadic decomposition and spatial localization for two inputs. Without loss of generality, we assume that $k_2 \leq k_1$. When $k_1-k_2\geq 5$, $k_2 \geq -2m/3$, $0\leq k_1 \leq 2\beta m$, we do integration by parts in "$\eta$" many times to rule out the case when $\max\{j_1,j_2\}\leq m-10\beta m$. As a result, we have

$$\sum_{j\leq m+5}\|e^{it\Lambda}Q((e^{-it\Lambda}f)^{\mu},(e^{-it\Lambda}g)^{\nu})\|_{B_{k,j}} \lesssim \sum_{j\leq m+5} 2^{\alpha k+j+6k_+}\Big(\sum_{|k_1-k_2|\leq 5,\text{ or } k_1\leq 0} 2^{4k_{1,+}}\|e^{-it\Lambda}f_{k_1}\|_{L^\infty}$$

$$\times\|g_{k_2}\|_{L^2}+\Big(\sum_{k_2\leq -2m/3,k_2\leq k_1}+\sum_{k_1\geq 2\beta m,k_2\leq k_1}\Big)2^{4k_{1,+}}\|f_{k_1}\|_{L^2}2^{k_2}\|g_{k_2}\|_{L^2}+2^{-10m}(A+B)^2$$

$$+\sum_{k_2\geq -2m/3,0\leq k_1\leq 2\beta m}\sum_{\max\{j_1,j_2\}\geq m-10\beta m} 2^{4k_{1,+}}2^{-m-\max\{j_1,j_2\}-\alpha k_1-\alpha k_2}\|f_{k_1,j_1}\|_Z\|g_{k_2,j_2}\|_Z\Big)$$

$$\lesssim B^2+2^{-10\delta m}(A+B)^2.$$

- Now, we proceed to prove (5.180). The major difference between (5.179) and (5.180) is that now we can take the advantage of the oscillation in time. From the straightforward $L^2-L^\infty-L^\infty$ type estimate, we only need to consider the case where all frequencies are like "1" and all inputs are spatially concentrated around "1". The strategies for those cases are already discussed in subsubsection 5.5.3. We omit details here. □

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