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IN SEARCH OF MINIMAL HYPERSURFACES

ANTOINE SONG

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
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Abstract

We study minimal hypersurfaces from the point of view of min-max theory. We present a proof of Yau's conjecture for the abundance of minimal surfaces, which builds on previous works by F. C. Marques and A. Neves, and extend it to some non-compact ambient manifolds. We show a generic equidistribution result for minimal hypersurfaces (joint with F. C. Marques and A. Neves). Then we give a proof of a conjecture by H. J. Rubinstein on realizing strongly irreducible Heegaard splittings of 3-manifolds by minimal surfaces (joint with D. Ketover and Y. Liokumovich). Other results related to minimal surfaces are explained.

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Chapter 1

Introduction

Given a compact Riemannian manifold (M^{n+1}, g) , the space of closed embedded hypersurfaces comes with a natural functional, called area functional, which associates to each hypersurface its n -volume computed with g . The critical points of the area functional are minimal hypersurfaces: their special geometric properties give them intrinsic richness and explain their usefulness in studying other geometric problems. Stable minimal hypersurfaces have a long history of applications in geometry and topology, while higher Morse index minimal hypersurfaces became of significant interest more recently, with the emergence of min-max theories, which are designed to construct non-stable minimal hypersurfaces with a mountain-pass type argument. An influential example where min-max theory and higher index minimal hypersurfaces played a key role is the resolution of the Willmore conjecture by F. C. Marques and A. Neves [101].

Min-max theories which produce minimal hypersurfaces in dimensions at least 3 are essentially higher dimensional versions of Morse theory for geodesics. The main min-max theory we will be dealing with is due to Almgren and Pitts, whose successive works showed the existence of an embedded closed minimal hypersurface in any closed Riemannian manifold of dimension at least 3 [125]. Their theory has been considerably extended by Marques-Neves, who could construct an entire sequence of minimal hypersurfaces with Morse index bounds and with area corresponding to naturally defined min-max numbers called widths [104, 102]. As M. Gromov suggested, these widths can be thought of as a nonlinear analogue of the Laplacian spectrum. In that analogy, minimal hypersurfaces

then correspond to eigenfunctions. Interestingly, the min-max widths also satisfy a Weyl law [94].

A central problem that motivated the more recent developments in min-max theory is a conjecture of S.-T. Yau, asking for the existence of infinitely many geometrically distinct minimal surfaces in closed 3-manifolds. The difficulty when trying to apply min-max theory to this problem is that, a priori, the sequence of minimal hypersurfaces mentioned in the previous paragraph could be different integer linear combinations of only a finite number of minimal hypersurfaces. In other words, multiplicity is an important obstacle to the construction of infinitely many minimal hypersurfaces.

In this thesis, most results are about finding, with the help of min-max theory, minimal hypersurfaces in various geometric situations. This search for minimal hypersurfaces lead us to quite different parts of differential geometry and even topology. Let us briefly describe the results we obtained.

1.1 On the abundance of minimal surfaces

In 1982, Yau formulated the following conjecture.

Conjecture: In any closed Riemannian 3-manifold, there is infinitely many immersed smooth minimal surfaces.

After Almgren-Pitts' result about the existence of one minimal surface, some related results appeared for special metrics, but not much more was known for general metrics until Marques and Neves proved the following [104]: let (M^{n+1}, g) be a closed manifold with $2 \leq n \leq 6$; if the Frankel property is satisfied, then the conjecture is true.

This result was the start of a series of results revolving around Yau's conjecture. In particular it was proved to be true assuming that the metric g is generic in the sense of Baire; this is due to K. Irie-Marques-Neves [75], then later O. Chodosh-C. Mantoulidis [26], and more recently X. Zhou [158].

Building on methods developed by Marques-Neves, we were able to prove Yau's conjecture for general (not necessarily generic) metrics with min-max theory [146]. To achieve that, we localized the multi-parameter min-max theory to compact manifolds with strictly

stable minimal boundaries. In fact, we could prove that in any closed manifold with dimension between 3 and 7, there are infinitely many smoothly embedded closed minimal hypersurfaces which are saddle points of the area functional.

Naturally one can wonder to what extent the conjecture is true when the ambient manifold M is complete non-compact. We proposed a generalization of the resolution of Yau's conjecture to complete manifolds that are "thick at infinity", under the form of a dichotomy: either there are infinitely many saddle point minimal hypersurfaces, or there is none [147]. Being thick at infinity is for instance implied by having bounded geometry.

For generic metrics, together with Marques and Neves, we also quantified the result of Irie-Marques-Neves which says that generically the union of closed minimal hypersurfaces is dense in closed manifolds of dimension between 3 and 7. We proved that actually, in such manifolds, generically a sequence of minimal hypersurfaces becomes equidistributed [106]. Is that result true for non-generic metrics? The situation remains unclear at the moment...

1.2 Minimal surfaces and topology of 3-manifolds

When studying the topology of a 3-manifold M , minimal surfaces are useful, especially those isotopic to Heegaard splittings. These surfaces separate M into two handlebodies, and an important question is to classify them, or at least obtain some finiteness theorems up to isotopy for certain Heegaard splittings, called strongly irreducible, in non-Haken manifolds: this is the goal of a conjecture of Waldhausen. A topological solution was given by T. Li [91, 90], but H. J. Rubinstein [128] previously suggested that if one could isotop strongly irreducible Heegaard splittings to minimal surfaces of index 1 by min-max theory, then since these minimal surfaces enjoy good compactness properties, we would automatically get the desired finiteness theorems. Rubinstein sketched an explanation of how to move strongly irreducible Heegaard splittings to minimal surfaces by min-max theory. Together with D. Ketover and Y. Liokumovich [145, 82, 83], we gave a rigorous proof of that picture, which was a tool in the new effective proof of T. Li's result by D. Gabai-T. Colding, and Colding-Gabai-Ketover [30, 31] in the hyperbolic case.

An important aspect of our result is a localization of a version of min-max theory due to Simon and Smith to compact manifolds with strictly stable minimal boundaries.

Here it is important to keep track of the topology of the resulting minimal surface. The techniques involved are more topological, while those used in results of the previous section come more often from geometric measure theory.

1.3 Other results

Some time ago we were interested in the following question. Since the Ricci flow tends to make 3-manifolds rounder, and since stable minimal spheres are an obstruction to “roundness”, can stable minimal spheres appear along the Ricci flow if the scalar curvature is positive? It turns out that the answer is yes [144], and we will present explicit examples where that happens. The meaning of these constructions is that one cannot rule out the appearance of true singularities along the Ricci flow simply by looking at the nonexistence of stable minimal spheres in the metric at time 0.

Under the guidance of Fernando Codá Marques, we also investigated least area minimal hypersurfaces, i.e. immersed closed minimal hypersurfaces that have least area among all closed immersed minimal hypersurfaces. We proved that they are always embedded [142] and the proof uses min-max theory in the same fashion as a similar result of Calabi-Cao for least length geodesics in convex spheres [17] .

Finally, we will mention an earlier result about the asymptotic geometry of self-shrinkers of the mean curvature flow. Note that these are also minimal surfaces with respect to a conformally changed metric in the Euclidean space. We found a graphical maximum principle which implied for instance that the second fundamental form of certain self-shrinkers in $\mathbb{R}^3$ grows at most linearly [143]. That theorem was later used by L. Wang [148] in her work on a conjecture of Ilmanen, where she showed the stronger statement that self-shrinkers of finite topology have ends asymptotic to either regular cones or straight cylinders.

1.4 Organization of the thesis

In Chapter 2, we recall some fundamental aspects of min-max theory, in various settings. In Chapter 3 and 4, we give a proof of Yau’s conjecture and we extend it to manifolds thick at infinity. In Chapter 5, we show the generic equidistribution theorem (joint work

with Marques-Neves). In Chapter 6, we explain the proof of Rubinstein's conjecture (joint work with Ketover-Liokumovich). In Chapter 7, 8, and 9, we present some other results.

The impatient reader can focus on Chapters 3, 5 and 6. We decided to remove most proofs of intermediate results from this dissertation, in an effort to make it more readable. All proofs can be found in the original papers.

Chapter 2

Preliminaries on min-max theory

In this chapter, we give a brief overview of some important notions in Almgren and Pitts' theory, and the continuous version of min-max theory by De Lellis-Tasnady, Simon-Smith. We will not have time to touch upon the Allen-Cahn min-max theory, which recently received a lot of attention (see [59, 51, 26] for instance).

2.1 Min-max theory in the setting of Almgren and Pitts

2.1.1 1-parameter min-max in closed manifolds

Assuming a reasonable amount of Geometric Measure Theory, we review some notions of the Almgren and Pitts' theory used thereafter. For a complete presentation, one can refer to the book of Pitts [125] or to Section 2 in [104]. Note that what follows could be presented in a simpler form using a “continuous” formulation due to Marques-Neves [102], and it would avoid the rather intricate formulation using “discrete” sweepouts that you will see below. However the regularity proofs of Pitts seem to need this “discrete” formulation, so we decided to present it nevertheless. The “continuous” formulation will be mentioned in the next two subsections anyway.

Let M be a closed Riemannian $(n+1)$ -manifold, assumed to be isometrically embedded in $\mathbb{R}^P$. We will always suppose M connected. Because we will have to consider non-orientable submanifolds, we will use flat chains modulo 2 (see [48, 4.2.26]). We will work with the space $\mathbf{I}_k(M, \mathbb{Z}_2)$ of flat chains modulo 2 in the congruence class of some k -dimensional integral currents with support contained in M , the subspace $\mathcal{Z}_k(M, \mathbb{Z}_2) \subset \mathbf{I}_k(M, \mathbb{Z}_2)$ whose elements have no boundary, and with the space $\mathcal{V}_k(M)$ of the closure, in the weak topology, of the set of k -dimensional rectifiable varifolds in $\mathbb{R}^P$ with support in M .

As in [125], we will suppress the superscript 2 in the notation of [48]. An integral current $T \in \mathbf{I}_k(M, \mathbb{Z}_2)$ determines an integral varifold $|T|$ and a Radon measure $\|T\|$ ([125, Chapter 2, 2.1, (18) (e)]). If $V \in \mathcal{V}_k(M)$, denote by $\|V\|$ the associated Radon measure on M . Given an open set $U \subset M$, if the associated rectifiable current is an integral current in $\mathbf{I}_{n+1}(M, \mathbb{Z}_2)$, it will be written as $[|U|]$. To a rectifiable subset R of M corresponds an integral varifold called $|R|$. The support of a current or a measure is denoted by spt . The notation $\mathbf{M}$ stands for the mass of an element in $\mathbf{I}_k(M, \mathbb{Z}_2)$. On $\mathbf{I}_k(M, \mathbb{Z}_2)$ there is also the flat metric $\mathcal{F}(\cdot, \cdot)$ which induces the so-called flat topology. The space $\mathcal{V}_k(M)$ is endowed with the topology of the weak convergence of varifolds. The notations $\mathcal{Z}_k(M, \underline{\nu}, \mathbb{Z}_2)$ and $\mathbf{I}_k(M, \underline{\nu}, \mathbb{Z}_2)$ mean that the respective spaces of currents are considered with the topology induced by $\underline{\nu}$, where $\underline{\nu}$ is either $\mathbf{M}$ or $\mathcal{F}$.

We denote $[0, 1]$ by I . For each $j \in \mathbb{N}$, $I(1, j)$ stands for the cell complex on I whose 1-cells and 0-cells are, respectively,

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \dots, [1 - 3^{-j}, 1] \text{ and } [0], [3^{-j}], \dots, [1 - 3^{-j}], [1].$$

$I(1, j)_p$ denotes the set of all p -cells in $I(1, j)$.

In the theory of Almgren and Pitts, instead of considering continuous maps from I to $\mathcal{Z}_n(M, \mathbb{Z}_2)$, one consider a sequence of mappings from $I(1, n_i)$ to $\mathcal{Z}_n(M, \mathbb{Z}_2)$, where $n_i \rightarrow \infty$ and the discrete slices corresponding to adjacent vertices in $I(1, n_j)$ are closer and closer. This leads to the following notions:

Definition 2.1.1. *Whenever $\phi : I(1, j)_0 \rightarrow \mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2)$, we define the fineness of ϕ to*

be

$$\mathbf{f}_{\underline{\nu}}(\phi) = \sup \left\{ \frac{\underline{\nu}(\phi(x) - \phi(y))}{\mathbf{d}(x, y)}; x, y \in I(1, j)_0, x \neq y \right\}$$

where $\mathbf{d}(x, y) = 3^j |x - y|$.

For each $x \in I(1, j)_0$, define $\mathbf{n}(i, j)(x)$ to be the unique element of $I(1, j)_0$ such that

$$\mathbf{d}(x, \mathbf{n}(i, j)(x)) = \inf \{ \mathbf{d}(x, y); y \in I(1, j)_0 \}.$$

Definition 2.1.2. 1. Let $\delta > 0$. We say that ϕ_1 and ϕ_2 are homotopic in $(\mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2), \{0\})$ with fineness δ if and only if there exist positive integers k_1, k_2, k_3 and a map

$$\psi : I(1, k_3)_0 \times I(1, k_3)_0 \rightarrow \mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2)$$

such that $\mathbf{f}_{\underline{\nu}}(\psi) < \delta$ and whenever $j = 1, 2$ and $x \in I(1, k_3)_0$,

$$\phi_j : I(1, k_j) \rightarrow \mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2), \quad \phi_j([0]) = \phi_j([1]) = 0,$$

$$\psi([j-1], x) = \phi_j(\mathbf{n}(k_3, k_j)(x)), \quad \psi(x, [0]) = \psi(x, [1]) = 0.$$

2. A $(1, \underline{\nu})$ -homotopy sequence of mappings into $(\mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2), \{0\})$ is a sequence $S = \{\phi_1, \phi_2, \dots\}$ for which there exist positive numbers $\delta_1, \delta_2, \dots$ such that ϕ_i is homotopic to ϕ_{i+1} in $(\mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2), \{0\})$ with fineness δ_i for each positive integer i , $\lim_i \delta_i = 0$ and

$$\sup \{ \mathbf{M}(\phi_i(x)); x \in \text{domain}(\phi_i), i > 0 \} < \infty.$$

3. If $S_1 = \{\phi_i^1\}$ and $S_2 = \{\phi_i^2\}$ are $(1, \underline{\nu})$ -homotopy sequences of mappings into $(\mathcal{Z}_n(M, \underline{\nu}, \mathbb{Z}_2), \{0\})$, then S_1 is homotopic with S_2 if and only if there is a sequence of positive real numbers $\delta_1, \delta_2, \dots$ such that ϕ_i^1 is homotopic to ϕ_i^2 in $(\mathcal{Z}_n(M, \mathbf{M}), \{0\})$ with fineness δ_i for $i > 0$ and $\lim_i \delta_i = 0$.

"To be homotopic with" is an equivalence relation on the set of $(1, \mathbf{M})$ -homotopy sequences of mappings into $(\mathcal{Z}_n(M, \mathbb{Z}_2), \{0\})$. An equivalence class of such sequences is a $(1, \underline{\nu})$ -homotopy class of mappings into $(\mathcal{Z}_n(M, \underline{\nu}), \{0\})$. The space of these

equivalence classes is denoted by

$$\pi_1^\sharp(\mathcal{Z}_n(M, \mathbb{Z}_2), \{0\}).$$

Remark 2.1.1. By [125, Theorem 4.6],

$$\pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\}), \pi_1^\sharp(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}) \text{ and } \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$$

are all naturally isomorphic.

Given $\Pi \in \pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$, consider the function $\mathbf{L} : \Pi \rightarrow [0, \infty]$ defined such that if $S = \{\phi_i\}_{i \in \mathbb{M}} \in \Pi$ and $\phi_i : I(1, n_i) \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$, then

$$\mathbf{L}(S) = \limsup_{i \rightarrow \infty} \max\{\mathbf{M}(\phi_i(x)) : x \in I(1, n_i)\}.$$

The width of Π is then the following quantity:

$$\mathbf{L}(\Pi) = \inf\{\mathbf{L}(S); S \in \Pi\}.$$

Using the isomorphism 2.1, Almgren showed that for any closed manifold M , there is $\Pi \in \pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ so that $\mathbf{L}(\Pi) > 0$. Such a Π is said to be non-trivial.

A sequence $S = \{\phi_i : I(1, n_i) \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)\}_i \in \Pi$ is said to be a min-max sequence for Π if $\mathbf{L}(S) = \mathbf{L}(\Pi)$. Define the critical set $\mathbf{C}(S) \subset \mathcal{V}_n(M)$ of $S \in \Pi$ as

$$\begin{aligned} \mathbf{C}(S) &= \{V; \exists \{i_j\}_j, \exists \{x_j\}, i_j \rightarrow \infty, x_j \in I(1, n_{i_j}), \\ &V = \lim_{j \rightarrow \infty} |\phi_{i_j}(x_j)| \text{ and } ||V||(M) = \mathbf{L}(S)\}. \end{aligned}$$

We recall that one can define the $\mathbf{F}$ -metric on $\mathcal{V}_n(M)$ ([125, Chapter 2, 2.1, (19)]). The $\mathbf{F}$ -metric induces the weak topology on any set of varifolds whose mass is bounded by a certain constant. We now give Pitts' definition of almost minimizing varifolds (see [125, Chapter 3]). Let U be an open set of $\mathbb{R}^L$. Denote by $\mathcal{Z}_n(M, M \setminus U, \mathbb{Z}_2)$ the set of currents $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ such that $\text{spt}(\partial T) \subset M \setminus U$. For each pair of positive numbers ϵ and δ , we define

$$\mathbf{a}_n(U, \epsilon, \delta, \mathbb{Z}_2)$$

to be the set of all currents $T \in \mathcal{Z}_n(M, M \setminus U, \mathbb{Z}_2)$ with the following property. If $T = T_0, T_1, \dots, T_m \in \mathcal{Z}_n(M, M \setminus U, \mathbb{Z}_2)$,

$$\text{spt}(T - T_i) \subset U \quad \text{for } i = 1, \dots, m$$

$$\mathcal{F}(T_i, T_{i-1}) \leq \delta \quad \text{for } i = 1, \dots, m$$

$$\mathbf{M}(T_i) \leq \mathbf{M}(T) + \delta \quad \text{for } i = 1, \dots, m$$

then $\mathbf{M}(T) - \mathbf{M}(T_m) \leq \epsilon$.

Definition 2.1.3. We say that $V \in \mathcal{V}_n(M)$ is $\mathbb{Z}_2$ almost minimizing in U if and only if for each $\epsilon > 0$, there exist a $\delta > 0$ and a current $T \in \mathbf{a}_n(U, \epsilon, \delta, \mathbb{Z}_2)$ such that $\mathbf{F}(V, |T|) \leq \epsilon$.

Remark 2.1.2. The 1-parameter min-max theory of Almgren-Pitts works equally well for $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients.

The min-max theorem of Almgren-Pitts says the following (see [125]).

Theorem 1 (Min-max theorem). 1. Let (M^{n+1}, g) be a closed manifold. Suppose that

$\mathbf{L}(\Pi) > 0$ and let $\{\Phi_i\}_i$ be a min-max sequence for Π . Then there exists a stationary varifold $V \in \mathbf{C}(\{\Phi_i\}_i)$ (hence $\|V\|(M) = \mathbf{L}(\Pi)$), which is almost minimizing in small annuli.

2. Any stationary varifold almost minimizing in small annuli is an integral varifold whose support is a closed minimal hypersurface embedded and smooth outside a set of dimension $n - 7$.

Corollary 2. In any closed manifold there exists at least one embedded minimal hypersurface whose singular set has dimension at most $n - 7$.

Finally, we describe Almgren's isomorphism

$$H_{n+1}(M, \mathbb{Z}_2) \simeq \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}) \quad (2.1)$$

constructed in [2, Section 3]. Actually, Almgren treated the case where the coefficient group is $G = \mathbb{Z}$, but as Pitts noted in [125], all the methods extend to the case $G = \mathbb{Z}_2$.

There is a number $\mu > 0$ such that if $T \in \mathbf{I}_k(M, \mathbb{Z}_2)$ has no boundary and $\mathcal{F}(T) \leq \mu$, then there is an $S \in \mathbf{I}_{k+1}(M, \mathbb{Z}_2)$ such that $\partial S = T$ and

$$\mathbf{M}(S) = \mathcal{F}(T) = \inf\{\mathbf{M}(S'); S' \in \mathbf{I}_{k+1}(M, \mathbb{Z}_2) \text{ and } \partial S' = T\}.$$

Such an S is called an $\mathcal{F}$ -isoperimetric choice for T . A chain map

$$\Phi : I(1, j) \rightarrow \mathbf{I}_*(M)$$

of degree n is a graded homomorphism Φ of degree n , such that $\partial \circ \Phi = \Phi \circ \partial$ and $\Phi(\alpha)$ is $\mathcal{F}$ -isoperimetric for $\Phi(\partial\alpha)$ where $\alpha \in I(1, j)_1$. Now let

$$[f] \in \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$$

be a class whose one of the representative maps is

$$f : (I, \{0, 1\}) \rightarrow (\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}).$$

The isomorphism $F : \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}) \rightarrow H_{n+1}(M, \mathbb{Z}_2)$ is defined as follows: take any integer m sufficiently large, there is a chain map

$$\Phi : I(1, m) \rightarrow \mathbf{I}_*(M, \mathbb{Z}_2)$$

of degree n , such that

$$\Phi([x]) = f(x) \quad \forall [x] \in I(1, m)_0.$$

Then

$$\sum_{\alpha \in I(1, m)_1} \Phi(\alpha)$$

is a cycle in $\mathbf{I}_{n+1}(M, \mathbb{Z}_2)$ which depends neither on m if the latter is chosen large enough, nor on the representative f . Because the homology groups of the chain complex $\mathbf{I}_*(M, \mathbb{Z}_2)$ are isomorphic with the singular homology groups of M with coefficient group $\mathbb{Z}_2$, it makes

sense to set

$$F([f]) = \left[\sum_{\alpha \in I(1,m)_1} \Phi(\alpha) \right] \in H_{n+1}(M, \mathbb{Z}_2).$$

2.1.2 Localization of 1-parameter min-max to compact manifolds with strictly stable boundary

We explain the local 1-parameter min-max theory in the Almgren-Pitts setting: it is essentially a mixture of [125] and [102, Theorem 1.7]. By “local”, we mean that the result applies to compact manifolds with boundary satisfying some assumptions. We use the “continuous” formulation of Marques-Neves, which is formally simpler and equivalent to the original one by interpolation results [101, 104, 102].

Let N be a compact connected Riemannian $(n+1)$ -manifold. Suppose that $\partial_0 N$ and $\partial_1 N$ are disjoint closed sets (which can be empty), $\partial_0 N \cup \partial_1 N = \partial N$, and C_0 (resp. C_1) is the cycle in $\mathcal{Z}_n(N; \mathbb{Z}_2)$ which is determined by $\partial_0 N$ (resp. $\partial_1 N$). Note that $C_0 = C_1 = 0$ if $\partial N = \emptyset$. Let $\Phi : [0, 1] \rightarrow \mathcal{Z}_n(N; \mathbf{F}; \mathbb{Z}_2)$ be a continuous map so that $\Phi(0) = C_0$, $\Phi(1) = C_1$. Let Π be the class of all continuous maps $\Phi' : [0, 1] \rightarrow \mathcal{Z}_n(N; \mathbf{F}; \mathbb{Z}_2)$ homotopic to Φ in the flat topology, with endpoints fixed, i.e. there is a continuous map $H : [0, 1] \times [0, 1] \rightarrow \mathcal{Z}_n(N; \mathbf{F}; \mathbb{Z}_2)$ so that $H(., 0) = \Phi(.)$, $H(., 1) = \Phi'(.)$, $H(j, s) = C_j$ for $j = 0, 1$ and all $s \in [0, 1]$. Let $\pi^\#(C_0, C_1)$ be the family of such homotopy classes Π of maps starting at C_0 and ending at C_1 .

Let us repeat a bit differently what we presented in the previous subsection. In [2], Almgren describes how to associate to a continuous map $\Phi : [0, 1] \rightarrow \mathcal{Z}_n(N; \mathbf{F}; \mathbb{Z}_2)$ an element of $\mathbf{I}_{n+1}(N, \mathbb{Z}_2)$. Let us explain this construction. There is a number $\mu > 0$ such that if $T \in \mathbf{I}_n(N, \mathbb{Z}_2)$ has no boundary and $\mathcal{F}(T) \leq \mu$, then there is an $S_T \in \mathbf{I}_{n+1}(N, \mathbb{Z}_2)$ such that $\partial S_T = T$ and

$$\mathbf{M}(S_T) = \mathcal{F}(T) = \inf \{ \mathbf{M}(S'); S' \in \mathbf{I}_{n+1}(N, \mathbb{Z}_2) \text{ and } \partial S' = T \}.$$

Such an S_T is called an $\mathcal{F}$ -isoperimetric choice for T . Now let k_0 be large enough so that for all $i = 1, \dots, k_0$, $\mathcal{F}(\Phi(\frac{i}{k_0}) - \Phi(\frac{i-1}{k_0})) \leq \mu$. With the previous notation for $\mathcal{F}$ -

isoperimetric choices, consider the $(n+1)$ -dimensional integral current

$$\sum_{i=1}^{k_0} S_{\Phi(\frac{i}{k_0}) - \Phi(\frac{i-1}{k_0})}. \quad (2.2)$$

This current does not depend of k_0 provided it is sufficiently large. By the interpolation formula of [2, Section 6], this sum is also invariant by homotopies. Hence when $\Phi \in \Pi \in \pi^\sharp(C_0, C_1)$, the map which associates to Π the $(n+1)$ -dimensional current (2.2), defined with Φ , is well defined. We call this map the Almgren map and we denote it by

$$\mathcal{A} : \pi^\sharp(C_0, C_1) \rightarrow \mathbf{I}_{n+1}(N, \mathbb{Z}_2).$$

Given $\pi^\sharp(C_0, C_1)$, consider the function $\mathbf{L} : \pi^\sharp(C_0, C_1) \rightarrow [0, \infty]$ defined by

$$\mathbf{L}(\Pi) = \inf \left\{ \sup_{x \in [0,1]} \mathbf{M}(\Phi(x)); \Phi \in \Pi \right\}.$$

A sequence $\{\Phi_i\}_i \subset \Pi$ is a min-max sequence if

$$\limsup_{i \rightarrow \infty} \left(\sup_{x \in [0,1]} \mathbf{M}(\Phi_i(x)) \right) = \mathbf{L}(\Pi).$$

The image set of $\{\Phi_i\}_i$ is

$$\begin{aligned} \mathbf{\Lambda}(\{\Phi_i\}_i) = & \{V \in \mathcal{V}_n(N); \exists \{i_j\} \rightarrow \infty, x_{i_j} \in [0, 1], \\ & \text{such that } \lim_{j \rightarrow \infty} \mathbf{F}(|\Phi_{i_j}(x_{i_j})|, V) = 0\}. \end{aligned}$$

If $\{\Phi_i\}_i \subset \Pi$ is a min-max sequence, define the critical set $\mathbf{C}(S) \subset \mathcal{V}_n(N)$ of $\{\Phi_i\}_i$ as

$$\mathbf{C}(\{\Phi_i\}_i) = \{V \in \mathbf{\Lambda}(\{\Phi_i\}_i); \|V\|(N) = \mathbf{L}(\Pi)\}.$$

A min-max sequence $\{\Phi_i\}_i \subset \Pi$ such that every element of $\mathbf{C}(\{\Phi_i\}_i)$ is stationary is called pulled-tight.

We finally define the width W of (N, g) to be

$$W := \inf_{\mathcal{A}(\Pi)=[N]} \mathbf{L}(\Pi), \quad (2.3)$$

the infimum being taken over all the possible following choices: we start with a partition $\partial N = X_1 \cup X_2$ (X_1, X_2 are closed), C_i is the cycle in $\mathcal{Z}_n(N, \mathbb{Z}_2)$ determined by X_i ($i = 1, 2$), and $\Pi \in \pi_1^\#(C_0, C_1)$ satisfies

$$\mathcal{A}(\Pi) = [|N|].$$

This width should not be confused with the first width $\omega_1(N, g)$ which is defined with relative cycles (see next subsection or [54, 61, 104, 94]).

The following theorem is essentially a consequence of the index bound of Marques-Neves [102].

Theorem 3. *Let (N^{n+1}, g) be a compact manifold with boundary, with $2 \leq n \leq 6$. Suppose that the boundary ∂N is a strictly stable minimal hypersurface. Then there exists a stationary integral varifold V whose support is a smooth embedded minimal hypersurface $\Sigma \subset N$ of index bounded by one, such that*

$$||V||(N) = W.$$

Moreover one of the components of Σ is contained in the interior $\text{Int}(N)$.

2.1.3 Multi-parameter min-max in closed manifolds

In their works tackling Yau's conjecture for minimal surfaces, Marques and Neves developed some tools in order to systematically extend the theory of Almgren-Pitts. Let us describe some of their contribution to this framework.

Let (M, g) be an $(n + 1)$ -dimensional closed Riemannian manifold. Previously in the first subsection, we saw how to define a non-trivial homotopy class of mappings. In the “continuous” formulation, this notion can naturally be replaced by that of a sweepout in the following sense: $\Phi : [0, 1] \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2, \{0\})$ is a sweepout if the the Almgren map does not send Φ to $0 \in H_{n+1}(M; \mathbb{Z}_2)$ (see previous subsection). More generally, one considers maps Φ from a cubical subcomplex of $[0, 1]^m$ for some m , with value in $\mathcal{Z}_n(M^{n+1}, g)$ and we can define multi-parameter sweepouts as follows. By Almgren's isomorphism 2.1, $\mathcal{Z}_n(M^{n+1}, g)$ is weakly homotopy equivalent to $\mathbb{R}P^\infty$. Let $\bar{\lambda} \in H^1(\mathcal{Z}_n(M^{n+1}, g))$ be a generator of the cohomology algebra of $\mathcal{Z}_n(M^{n+1}, g)$.

Definition 2.1.4. A continuous map $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a p -sweepout if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2).$$

As Guth [61], one defines the widths:

Definition 2.1.5. The p -width of M is

$$\omega_p(M) = \inf_{\Phi} \sup \{ \mathbf{M}(\Phi(x)); x \in \text{dmn}(\Phi) \},$$

where the infimum is taken over all p -sweepouts Φ (which are “admissible”), and $\text{dmn}(\Phi)$ is the domain of Φ .

The set of these numbers can be considered as a geometric spectrum, and satisfy a Weyl law proved by Liokumovich-Marques-Neves [94].

By the use of general upper index bounds proved by Marques-Neves [102] for minimal hypersurfaces arising in min-max constructions, they showed the following.

Theorem 4 (Multi-parameter min-max theorem). *Let (M^{n+1}, g) be a closed manifold of dimension $n + 1$, where $2 \leq n \leq 6$. For all p , there is a stationary integral varifold V with mass $\|V\|(M) = \omega_p(M)$, whose support is a closed embedded smooth minimal hypersurface of index at most p .*

The dimension assumption is mostly technical, in higher dimension one expects the minimal hypersurface to be smooth outside a set of dimension $n - 7$.

2.2 Min-max theory in the continuous setting

Here we will quickly talk about the works of Simon-Smith and De Lellis-Tasnady, which describe a 1-parameter min-max theory using less Geometric Measure Theory compared to Almgren-Pitts. Note that a “continuous min-max theory” can also be written in the setting of Almgren-Pitts: that was realized by Marques-Neves [102, 105] (see previous two subsections).

2.2.1 De Lellis-Tasnady min-max theory

The theory of Almgren and Pitts deals with discrete sweepouts, and this can bring some technical complications when constructing explicit sweepouts. For this reason, De Lellis and Tasnady [42] wrote a version of this theory in the continuous setting, based on ideas in [140] and [29]. In this subsection, we recall the basic notions in this setting.

Let (M^{n+1}, g) be a closed Riemannian manifold. In what follows, the topological boundary of a subset of M will be denoted by ∂ . Consider an open subset $N \subset M$ whose boundary ∂N , when non-trivial, is a rectifiable hypersurface of finite n -dimensional Hausdorff measure. Suppose also that if ∂N is non empty, each of its connected components C separates M (i.e. $M \setminus C$ has two connected components). The notation for the m -dimensional Hausdorff measure will be $\mathcal{H}^m$. Take $a < b \in \mathbb{R}$, $k \in \mathbb{N}$.

Definition 2.2.1. *A family of $\mathcal{H}^n$ -measurable closed subsets $\{\Gamma_t\}_{t \in [a, b]^k}$ in $N \cup \partial N$ with finite $\mathcal{H}^n$ -measure is called a generalized smooth family if*

- *for each t there is a finite subset $P_t \subset N$ such that $\Gamma_t \cap N$ is a smooth hypersurface in $N \setminus P_t$,*
- *$t \mapsto \mathcal{H}^n(\Gamma_t)$ is continuous and $t \mapsto \Gamma_t$ is continuous in the Hausdorff topology,*
- *$\Gamma_t \rightarrow \Gamma_{t_0}$ smoothly in any compact $U \subset \subset N \setminus P_{t_0}$ as $t \rightarrow t_0$.*

If $\partial N = \emptyset$, a generalized smooth family $\{\Sigma_t\}_{t \in [a, b]}$ is called a continuous sweepout of N if there exists a family of open subsets $\{\Omega_t\}_{t \in [a, b]}$ of N such that

- (i) $(\Sigma_t \setminus \partial \Omega_t) \subset P_t$ for any $t \in (a, b]$,
- (ii) $\mathcal{H}^{n+1}(\Omega_t \setminus \Omega_s) + \mathcal{H}^{n+1}(\Omega_s \setminus \Omega_t) \rightarrow 0$, as $s \rightarrow t$,
- (iii) $\Omega_a = \emptyset$, and $\Omega_b = N$.

When $\partial N \neq \emptyset$, a continuous sweepout is required to satisfy the above conditions, except that $\partial \Omega_t$ denotes the boundary of Ω_t in N and (iii) is replaced by

- (iii) $\Omega_b = N$, $\Sigma_a = \partial N$ and $\Sigma_t \subset N$ for $t \in (a, b]$.

Definition 2.2.2. *When $\partial N = \emptyset$, two continuous sweepouts $\{\Sigma_t^1\}_{t \in [a, b]}$ and $\{\Sigma_t^2\}_{t \in [a, b]}$ are homotopic if there is a generalized smooth family $\{\Gamma_{(s, t)}\}_{(s, t) \in [a, b]^2}$, such that $\Gamma_{(a, t)} =$*

Σ_t^1 and $\Gamma_{(b,t)} = \Sigma_t^2$. When $\partial N \neq \emptyset$, we also require the following condition: $\Gamma_{(s,t)} \subset N$ for $t \in (a, b]$ and there exists a small $\alpha > 0$ such that $\Gamma_{(s,t)} = \Gamma_{(a,t)}$ for $(s, t) \in [a, b] \times [a, a + \alpha]$.

A family Λ of continuous sweepouts is called homotopically closed if it contains the homotopy class of each of its element.

If Λ is a homotopically closed family of continuous sweepouts, the width of Λ in N is defined as the min-max quantity

$$W(N, \partial N, \Lambda) = \inf_{\{\Sigma_t\} \in \Lambda} \max_t \mathcal{H}^n(\Sigma_t).$$

A sequence $\{\{\Sigma_t^k\}_{t \in [a,b]}\}_{k \in \mathbb{N}} \subset \Lambda$ is called a minimizing sequence if

$$\max_t \mathcal{H}^n(\Sigma_t^k) \rightarrow W(N, \partial N, \Lambda) \text{ as } k \rightarrow \infty.$$

A sequence of slices $\{\Sigma_{t_k}^k\}_{k \in \mathbb{N}}$ is called a min-max sequence if

$$\mathcal{H}^n(\Sigma_{t_k}^k) \rightarrow W(N, \partial N, \Lambda) \text{ as } k \rightarrow \infty.$$

The min-max theorem that results from this formulation is completely similar to Theorem 1 (when the boundary ∂N is not empty, we need to assume that it is mean convex), see [42] [156].

2.2.2 Simon-Smith min-max theory in 3-dimensional manifolds

The work of Smith [140] appeared right after Pitt's completion of the existence theory started by Almgren. It partly inspired the formulation of the min-max theory proposed by De Lellis-Tasnady, see previous subsection. [29] is an excellent place to learn about Simon-Smith's min-max theory. We will just mention some key properties of this theory: it applies to closed 3-dimensional manifolds, the relevant sweepouts are continuous families of smooth surfaces and in the tightening process, one can keep track of the topology so that eventually the genus of the minimal surface constructed is bounded (with multiplicities) by the maximal genus of slices that appear in the sweepouts [41, 81].

Chapter 3

Yau's conjecture on the abundance of minimal hypersurfaces

After introducing Yau's conjecture, we review some previous contributions to the question and present a proof of the conjecture which builds on methods developed by Marques-Neves.

3.1 Introduction

An important theme in Riemannian geometry is the study of the space of k -dimensional submanifolds in a Riemannian manifold with the k -volume functional by means of variational methods. For instance in the case of codimension one, a closed hypersurface Σ in an $(n + 1)$ -manifold (M, g) is a minimal hypersurface when it is a critical point of the n -volume functional, and a natural question which arises is how to construct these special objects. Birkhoff [14] was the first to use a mountain-pass type argument to construct a closed geodesic (which is a minimal hypersurface of dimension one) in any 2-sphere. In higher dimension, Almgren and Pitts developed a similar but much more complex theory in order to construct minimal hypersurfaces: they showed that in any closed Riemannian

nian manifold of dimension at least 3 and less than 7, there exists a smoothly embedded closed minimal hypersurface [125]. This result was shortly after extended to the case of dimension 7 by Schoen and Simon [133]. For dimensions higher than 7 there still exists a minimal hypersurface, but it may be singular along a set of Hausdorff codimension at least 7. Motivated by these results, Yau formulated in the early 80's the following conjecture.

Conjecture (S.-T. Yau [154]): *Any closed three-dimensional manifold must contain an infinite number of immersed minimal surfaces.*

A recurring difficulty when dealing with minimal hypersurfaces produced by min-max theory is the issue of multiplicity. A given n -dimensional minimal hypersurface endowed with different integer multiplicities are different critical points of the n -volume functional from the variational point of view, which is problematic if one wants to find geometrically distinct minimal hypersurfaces.

Even in the case of geodesics ($n = 1$), the analogous conjecture was settled only later, by the combined work of Franks [49] and Bangert [13]. The first mentioned paper relies on dynamical systems methods while the second one uses Morse theoretic arguments specific to closed curves in surfaces.

Up until a few years ago, general cases of the conjecture in higher dimensions remained elusive. In a paper which gave the impulsion to most of the recent activity on this question, Marques and Neves [104] proved the conjecture of Yau in the case where (M^{n+1}, g) is closed, $2 \leq n \leq 6$ and M satisfies the so-called Frankel property: any two closed embedded minimal hypersurfaces intersect each other. For instance this property is implied by $\text{Ric}_g > 0$. In their proof, they recast the sublinear bounds for a certain sequence of min-max widths, proved by Gromov [55] and Guth [61], in the context of an extended Almgren-Pitts' min-max theory and argue as follows. If the Frankel property is satisfied, then each width is realized as an integer multiple of the area of a connected minimal hypersurface. But since the widths grow sublinearly, when they are all distinct a counting argument leads to a contradiction if there was only finitely many minimal hypersurfaces

to begin with. When two widths coincide then a Lusternick-Schnirelmann type argument gives the result. Following this work, Marques and Neves went on with the development of a higher dimensional Morse theory for the n -volume functional [102] [105].

Afterwards, Irie, Marques and Neves [75] settled the conjecture for generic metrics (in the Baire sense) by showing that for generic metrics on a closed manifold with the usual restriction on the dimension, the union of minimal hypersurfaces is dense. This was later quantified in [106] by Marques, Neves and the author. The central ingredient of their proof is the Weyl law for the volume spectrum proved by Liokumovich, Marques and Neves [94].

Previously, Marques and Neves had devised another strategy to tackle the question: if for generic metrics the multiplicity of the minimal hypersurfaces produced by min-max was one, then since the widths are a sequence going to infinity, it would automatically lead to another proof of Yau's conjecture in the generic case. This is the Multiplicity One conjecture [102]. In dimension 3, it was proved by Chodosh and Mantoulidis [26] for bumpy metrics in the context of the Allen-Cahn version of min-max theory. The latter was an alternative to the Almgren-Pitts theory, proposed by Guaraco [59] and later extended by Gaspar and Guaraco [51]. Several consequences on the area, genus and index of minimal surfaces in bumpy metrics are derived in [26].

The goal of this chapter is to prove Yau's conjecture in the general case where the metric may not be generic. Our proof builds on [104].

Theorem 5. *In any closed Riemannian manifold of dimension at least 3 and at most 7, there exist infinitely many smoothly embedded closed minimal hypersurfaces.*

Actually as a byproduct of our proof, we produce many closed embedded minimal hypersurfaces *locally*: for instance if the metric is bumpy, then any compact manifold (of dimension $3 \leq n + 1 \leq 7$) with stable boundary contains infinitely many closed minimal hypersurfaces. A more precise formulation is stated in Remark 3.3.1.

We note that in the special case of closed hyperbolic 3-manifolds, the incompressible surfaces found by Kahn and Marković [78] produce after minimization an infinite number of closed immersed minimal surfaces. Let us finally mention other approaches which have been less successful in handling general cases of the conjecture. In [128], Rubinstein sketched an argument to produce infinitely many minimally immersed surfaces in any

hyperbolic 3-manifold with finite volume. He relies among other things on a construction of certain minimal surfaces of index at most one using Heegaard splittings via the smooth one-parameter min-max theory of Simon and Smith [140], which we justified rigorously in [83] jointly with Ketover and Liokumovich. Kapouleas [79] [80] proposed a strategy to construct infinitely many different embedded minimal surfaces in 3-manifolds with a generic metric by desingularizing two intersecting minimal surfaces or by gluing small necks between two surfaces close to an unstable minimal surface.

Outline of proof:

To simplify the discussion, let us consider a closed manifold M^{n+1} ($2 \leq n \leq 6$) endowed with a bumpy metric g and not containing one-sided minimal hypersurfaces. Suppose by contradiction that it has only finitely many minimal hypersurfaces. Then by the previous work of Marques and Neves [104], there exists at least one stable minimal hypersurface. We start by cutting M along stable minimal hypersurfaces a maximal number of times, keeping one component at each step, to get a connected compact “core” U with stable minimal boundary. Then one can show that any two minimal hypersurfaces in the interior of U intersect, i.e. the core satisfies the Frankel property. The main thrust of the proof is then to produce by min-max theory closed minimal hypersurfaces confined inside the interior of U . The difficulty of this is that while there is a version of min-max theory for manifolds with boundary due to M. Li and X. Zhou [89], the minimal hypersurfaces obtained have themselves non-empty boundaries in general. We are able to produce closed minimal hypersurfaces by considering the non-compact manifold with cylindrical ends $\mathcal{C}(U)$ obtained by gluing to U the straight cylindrical manifold $\partial U \times [0, \infty)$ along the boundary ∂U . Note that the metric may not be smooth. We will prove that min-max theory applied to the non-compact manifold $\mathcal{C}(U)$ associates to each min-max widths $\omega_p = \omega_p(\mathcal{C}(U))$ an integer multiple of one closed connected minimal hypersurface in the interior of the core U , whose total n -volume is ω_p . The next step is to prove that, in contrast to the sublinear bounds of Gromov-Guth in the compact case [55][61], the widths ω_p here behave asymptotically linearly and the leading factor is the n -volume of the largest component Σ_1 of ∂U . Since all the closed minimal hypersurfaces inside $U \setminus \partial U$ (which we assumed are in finite number) have their n -volume larger than that of Σ_1 , an

arithmetic lemma shows that the widths ω_p eventually become much larger than p times the n -volume of Σ_1 , a contradiction. These arguments do not rely on the bumpiness of the metric, in general to obtain the core we will only cut along the stable minimal hypersurfaces which are area minimizing at least on one side.

3.2 Min-max theory on a non-compact manifold with cylindrical ends

All manifolds considered in this chapter have dimension $n + 1$ with $3 \leq n + 1 \leq 7$. The minimal hypersurfaces are smooth embedded. In this section, we construct a certain non-compact manifold with cylindrical ends, and explain how to produce closed minimal hypersurfaces by min-max theory. Their areas satisfy a linear asymptotic behavior depending on the ends. This section can be thought of as a localization of multi-parameter min-max theory to compact manifolds with stable minimal boundaries, and should be compared to Subsections 2.1.1, 2.1.2, 2.1.3 of Chapter 2.

Before starting, we point out that previously there were already some results on min-max minimal hypersurfaces in non-compact manifolds. In [56], Gromov constructed minimal hypersurfaces in some special classes of non-compact manifolds. The existence of a closed embedded minimal surface in any finite volume hyperbolic 3-manifold was proved by Collin-Hauswirth-Mazet-Rosenberg in [38]. Montezuma showed in [113] the existence of one closed minimal hypersurface in manifolds which contain a mean-concave domain and whose ends have controlled geometry. Inspired by the proof of the Willmore conjecture by Marques and Neves [101], Ketover and Zhou [85] proved an entropy lower bound for closed surfaces of genus not equal to 1, embedded in $\mathbb{R}^3$, by min-max arguments applied to the Gaussian metric on $\mathbb{R}^3$. In [21], Chambers and Liokumovich produced a complete minimal hypersurface in any finite volume manifolds. Our situation is very different from the previous ones, in that our goal is to localize precisely the min-max minimal hypersurfaces, which should lie inside a given compact domain. In particular, we have to choose a potentially non-smooth metric in order to force them to stay in that domain.

3.2.1 Preliminaries in geometric measure theory

We recall some definitions about varifolds (see [138]) and observe that a version of the usual monotonicity formula holds uniformly for stationary varifolds in metrics close to the Euclidean metric in the C^1 norm. As a consequence, we prove an integrality result for certain varifolds which are limits of integral varifolds stationary for different metrics close in the C^0 norm and bounded in the C^1 topology.

Let Ω be an open subset of the Euclidean space $\mathbb{R}^{n+1}$. Let $\mathbf{V}_n(\Omega)$ be the set of n -varifold in Ω , i.e. the set of Radon measures on the Grassmannian $\Omega \times \mathbf{G}(n+1, n)$ where $\mathbf{G}(n+1, n)$ denotes the family of unoriented n -dimensional subspaces in $\mathbb{R}^{n+1}$. More generally by abuse of notations, if Ω is an open $(n+1)$ -dimensional manifold, we will still write $\Omega \times \mathbf{G}(n+1, n)$ to denote the Grassmannian bundle of n -planes over Ω . In this setting, one defines a varifold similarly. The weight of a varifold $V \in \mathbf{V}_n(\Omega)$ is denoted by $\|V\|$ and its mass is $\mathbf{M}(V) = \|V\|(\Omega)$. If A is a Borel subset of Ω , $V \llcorner A$ is the restriction of V to $A \times \mathbf{G}(n+1, n)$. The support of V , $\text{spt}(V)$, is the smallest relatively closed subset $B \subset \Omega$ such that $V \llcorner (\Omega \setminus B) = 0$. We will be mostly interested in rectifiable n -varifolds V . They are determined by an n -dimensional rectifiable set W and a density function $\Theta(\cdot) = \Theta^n(dV, \cdot)$, i.e. a nonnegative function measurable and locally integrable with respect to the n -dimensional Hausdorff measure $\mathcal{H}^n$. For Borel set $A \subset \Omega \times \mathbf{G}(n+1, n)$,

$$V(A) := \int_{\{x \in W; (x, T_x W) \in A\}} \Theta d\mathcal{H}^n(x).$$

If the density function of V takes integer values, then V is said to be integral. Again these definitions extend more generally to an open manifold Ω endowed with a general metric g . For instance let g_1, g_2 be two smooth metrics on an open domain Ω of the Euclidean n -space, and let $\mathcal{H}_1^n, \mathcal{H}_2^n$ be the respective Hausdorff measures. If V is a rectifiable varifold in Ω , and Θ_1, Θ_2 are the density functions of V respectively for g_1, g_2 , then

$$\Theta_1 \mathcal{H}_1^n = \Theta_2 \mathcal{H}_2^n$$

and if S is an n -plane and $\det_S g_i$ is the determinant of the restriction of g_i to S in an

orthonormal basis of S for the Euclidean metric, we have at dV -a.e. $x \in W = \text{spt } V$:

$$\Theta_2(x) = \sqrt{\frac{\det_{T_x W} g_1}{\det_{T_x W} g_2}} \Theta_1(x). \quad (3.1)$$

A varifold V is called stationary for a metric $\tilde{g}$ (or $\tilde{g}$ -stationary) in Ω when for all smooth vector field X with compact support in Ω ,

$$\delta V(X) := \int_{\Omega \times \mathbf{G}(n+1, n)} \tilde{\text{div}}_S X \, dV(x, S) = 0.$$

Here $\tilde{\text{div}}_S = \sum_{i=1}^n \langle \tilde{\nabla}_{e_i} X, e_i \rangle$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$ and $\{e_1, \dots, e_n\}$ is any $\tilde{g}$ -orthonormal basis of the n -subspace S . Note that the notion of rectifiability does not depend on the metric while being integral or stationary depends strongly on the metric.

Let B_r be the Euclidean ball of radius r in $\mathbb{R}^{n+1}$ and let g_{eucl} be the standard Euclidean metric. We start by stating the known fact that the monotonicity formula holds uniformly for stationary varifolds in metrics which are bounded in the C^1 topology.

Lemma 6. *Let $\eta \in (0, 1)$. Consider a metric $\tilde{g}$ on B_3 such that*

$$\|\tilde{g} - g_{eucl}\|_{C^1(\Omega)} \leq \eta.$$

Then there exists nonnegative real numbers $\mathfrak{c} = \mathfrak{c}(\eta)$ and $\mathfrak{a} = \mathfrak{a}(\eta)$ such that the following holds. Let V be a $\tilde{g}$ -stationary varifold in B_3 . Then for any $\xi \in B_1$, for any $0 < \sigma \leq \rho < 1$:

$$\exp(\mathfrak{c}\sigma) \frac{\|V\|(B(\xi, \sigma))}{\sigma^n} \leq (1 + \mathfrak{a}) \exp(\mathfrak{c}(1 + \mathfrak{a})\rho) \frac{\|V\|(B(\xi, (1 + \mathfrak{a})\rho))}{\rho^n},$$

where $B(\xi, r)$ is the Euclidean ball of radius r centered at ξ . More generally, for any smooth function $h : B_3 \rightarrow [0, 1]$ and ξ, σ, ρ as above:

$$\begin{aligned} & \exp(\mathfrak{c}\sigma) \frac{1}{\sigma^n} \int_{B(\xi, \sigma)} h \, d\|V\| \\ & \leq (1 + \mathfrak{a}) \exp(\mathfrak{c}(1 + \mathfrak{a})\rho) \left[\frac{1}{\rho^n} \int_{B(\xi, (1 + \mathfrak{a})\rho)} h \, d\|V\| \right. \\ & \quad \left. + \int_{\sigma}^{(1 + \mathfrak{a})\rho} \tau^{-n} \int_{B(\xi, \tau) \times \mathbf{G}(n+1, n)} |\nabla_S h| \, dV(x, S) d\tau \right] \end{aligned}$$

where $|\nabla_S h|$ is the norm of the gradient of h along S computed with g_{eucl} . Moreover, the constants $\mathfrak{c}(\eta)$ and $\mathfrak{a}(\eta)$ converge to 0 with η :

$$\lim_{\eta \rightarrow 0} \mathfrak{c} = \lim_{\eta \rightarrow 0} \mathfrak{a} = 0.$$

The following proposition, which will be useful in Subsection 3.2.4, is a consequence of the proof of Allard's compactness theorem for stationary integral varifolds.

Proposition 7. *Fix $\eta > 0$. Let g and $\{g_k\}_k \geq 0$ be metrics on an open domain $\Omega \subset \mathbb{R}^n$ such that*

$$\|g_k\|_{C^1(\Omega)} \leq \eta.$$

Let $\{V_k\}$ be a sequence of varifolds which are integral and stationary with respect to g_k . Suppose that V_k converges to a varifold V in the varifold sense, and that V is g -stationary. Suppose moreover that

$$\forall x \in \text{spt}(V), \quad \lim_{k \rightarrow \infty} |g_k(x) - g(x)| = 0. \quad (3.2)$$

Then V is integral.

3.2.2 Construction of a non-compact manifold with cylindrical ends

Let (U, g) be a connected compact Riemannian manifold with boundary endowed with a smooth metric g . Suppose that ∂U is a minimal surface such that a neighborhood of ∂U in U is smoothly foliated by closed leaves whose mean curvature vectors are pointing towards ∂U . In other words, we assume that there is a diffeomorphism

$$\Phi : \partial U \times [0, \hat{t}] \rightarrow U$$

where $\Phi(\partial U \times \{0\}) = \partial U$ is a minimal hypersurface, and for all $t \in (0, \hat{t}]$, the leaf $\Phi(\partial U \times \{t\})$ has non-zero mean curvature vector pointing towards ∂U . By convention, $\Phi(\partial U \times \{t\})$ has positive mean curvature with respect to the normal vector in the direction given by $\frac{\partial}{\partial t}$.

Let $\varphi : \partial U \times \{0\} \rightarrow \partial U$ be the canonical identifying map. Define the following non-compact manifold with cylindrical ends:

$$\mathcal{C}(U) := U \cup_{\varphi} (\partial U \times [0, \infty)).$$

We endow it with the metric h such that $h = g$ on U and $h = g_{\perp} \partial U \oplus ds^2$. Here $g_{\perp} \partial U$ is the restriction of g to the tangent bundle of the boundary ∂U and $g_{\perp} \partial U \oplus ds^2$ is the product metric on $\partial U \times [0, \infty)$. Note that the metric h is only Lipschitz continuous in general.

Next, we define for any small $\epsilon > 0$ a compact Riemannian manifold with boundary $(U_{\epsilon}, h_{\epsilon})$ diffeomorphic to U and converging to $(\mathcal{C}(U), h)$ as $\epsilon \rightarrow 0$ in a sense to be defined later. These U_{ϵ} are bigger and bigger chopped pieces of $\mathcal{C}(U)$ with a slightly modified smoothed metric h_{ϵ} . For $0 < \epsilon < \hat{t}$, let us first define

$$\tilde{U}_{\epsilon} = U \setminus \Phi(\partial U \times [0, \epsilon)).$$

Its boundary is $\partial \tilde{U}_{\epsilon} = \Phi(\partial U \times \{\epsilon\})$. For a small positive number $\delta_{\epsilon} > 0$, the following map

$$\tilde{\gamma}_{\epsilon} : \partial \tilde{U}_{\epsilon} \times [-\delta_{\epsilon}, 0] \rightarrow M$$

$$\tilde{\gamma}_{\epsilon}(x, t) = \exp(x, t\nu)$$

is well-defined and gives Fermi coordinates on one side of $\partial \tilde{U}_{\epsilon}$. By “coordinates”, we mean that $\tilde{\gamma}_{\epsilon}$ is a diffeomorphism onto its image. Here $\exp$ is the exponential map with respect to g , ν is the inward unit normal of $\partial \tilde{U}_{\epsilon}$. Note that $\tilde{\gamma}_{\epsilon}(\partial \tilde{U}_{\epsilon} \times \{0\}) = \partial \tilde{U}_{\epsilon}$ and $\tilde{\gamma}_{\epsilon}(\partial \tilde{U}_{\epsilon} \times [-\delta_{\epsilon}, 0]) \subset M \setminus \tilde{U}_{\epsilon}$. By reducing $\delta_{\epsilon} > 0$ if necessary, we can moreover suppose $\delta_{\epsilon} > 0$ small enough so that

- $\lim_{\epsilon \rightarrow 0} \delta_{\epsilon} = 0$,
- $\tilde{\gamma}_{\epsilon}(\partial \tilde{U}_{\epsilon} \times [-\delta_{\epsilon}, 0]) \subset \Phi(\partial U \times [0, \epsilon])$,
- and for all $t \in [-\delta_{\epsilon}, 0]$, the hypersurface $\tilde{\gamma}_{\epsilon}(\partial \tilde{U}_{\epsilon} \times \{t\})$ has positive mean curvature (the normal vector is in the direction of ν).

For $0 < \epsilon < \hat{t}$, we define

$$U_\epsilon = \tilde{U}_\epsilon \cup \tilde{\gamma}_\epsilon(\partial\tilde{U}_\epsilon \times [-\delta_\epsilon, 0])$$

and by convention $U_0 = U$.

For a positive number $\hat{d} > 0$ small enough and fixed independently of $\epsilon \in [0, \hat{t})$, consider the following Fermi coordinates on a $\hat{d}$ -neighborhood of ∂U_ϵ :

$$\gamma_\epsilon : \partial U_\epsilon \times [-\hat{d}, \hat{d}] \rightarrow M$$

$$\gamma_\epsilon(x, t) = \exp(x, t\nu)$$

where $\exp$ is the exponential map with respect to g , ν is the inward unit normal of ∂U_ϵ . We remark for clarity that for all $s \in [0, \delta_\epsilon]$,

$$\gamma_\epsilon(\partial U_\epsilon \times \{s\}) = \tilde{\gamma}_\epsilon(\partial\tilde{U}_\epsilon \times \{-\delta_\epsilon + s\}).$$

For each small positive ϵ , choose a smooth function $\vartheta_\epsilon : [0, \delta_\epsilon] \rightarrow \mathbb{R}$ and $z_\epsilon \in (0, \delta_\epsilon)$ with the following properties:

- $1 \leq \vartheta_\epsilon$ and $\frac{\partial}{\partial t} \vartheta_\epsilon \leq 0$,
- $\vartheta_\epsilon \equiv 1$ in a neighborhood of δ_ϵ ,
- ϑ_ϵ is constant on $[0, z_\epsilon]$,
- $\lim_{\epsilon \rightarrow 0} \int_{[0, \delta_\epsilon]} \vartheta_\epsilon = \infty$,
- $\lim_{\epsilon \rightarrow 0} \int_{[z_\epsilon, \delta_\epsilon]} \vartheta_\epsilon = 0$.

This function naturally induces a function on U_ϵ still called ϑ_ϵ , defined by

$$\vartheta_\epsilon(\gamma_\epsilon(x, t)) = \vartheta_\epsilon(t) \text{ for all } (x, t) \in \partial U_\epsilon \times [0, \delta_\epsilon]$$

and extended continuously by 1. The original metric g can be written in the Fermi coordinates γ_ϵ as $g_t \oplus dt^2$. Now define on U_ϵ the following smooth metric h_ϵ :

$$h_\epsilon(q) = \begin{cases} g_t(q) \oplus (\vartheta_\epsilon(q)dt)^2 & \text{for } q \in \gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon]) \\ g(q) & \text{for } q \in U_\epsilon \setminus \gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon]) \end{cases}.$$

We just defined a compact manifold with boundary (U_ϵ, h_ϵ) . Let us mention some useful properties of h_ϵ . A first lemma controls the extrinsic curvature of the slices $\gamma_\epsilon(\partial U_\epsilon \times \{t\})$ for the new metric h_ϵ .

Lemma 8. *Let (U_ϵ, h_ϵ) defined as above. By abuse of notations, consider $\partial U_\epsilon \times [0, \delta_\epsilon]$ as a subset of U_ϵ via γ_ϵ . Then for $t \in [0, \delta_\epsilon]$, the slices $\partial U_\epsilon \times \{t\}$ satisfy the following with respect to the new metric h_ϵ :*

1. *they have non-zero mean curvature vector pointing in the direction of $-\frac{\partial}{\partial t}$,*
2. *their mean curvature goes uniformly to 0 as ϵ converges to 0,*
3. *their second fundamental form is bounded by a constant C independent of ϵ .*

Remark 3.2.1. *For later use, we point out that, as a consequence of the previous lemma, the union of $\Phi(\partial U \times \{t\})$ for $t \in [\epsilon, \hat{t}]$ and $\gamma_\epsilon(\partial U_\epsilon \times \{t\})$ for $t \in [0, \delta_\epsilon]$ form a continuous foliation of a neighborhood of ∂U_ϵ starting from $\Phi(\partial U \times \{\hat{t}\})$; each leaf is smooth and has a mean curvature vector which is non-zero and pointing towards ∂U_ϵ with respect to the metric h_ϵ .*

The next two lemmas give information on the asymptotic behavior of the metrics h_ϵ . The following Lemma 9 shows that (U_ϵ, h_ϵ) converges to the non-compact manifold with cylindrical ends $\mathcal{C}(U)$ and the convergence is smooth outside of the folding region where the curvature can be unbounded. In Lemma 10 we will describe how the folding region of (U_ϵ, h_ϵ) is controlled in the C^1 topology by g .

Recall that a metric g on a manifold Ω naturally determines a C^k norm for smooth tensors on Ω and any nonnegative integer k . The notion of geometric convergence is defined for instance in [114, Chapter 5, Section 1]. We will say that a sequence (U_k, g_k, x_k) converges geometrically to $(U_\infty, g_\infty, x_\infty)$ in the C^k topology if (U_k, g_k, x_k) , $(U_\infty, g_\infty, x_\infty)$ satisfy the conditions in Definition 5.3 of [114], with the C^∞ -convergence on compact subsets replaced by C^k -convergence on compact subsets.

Lemma 9. *Let q be a point of $U \setminus \partial U$; for small ϵ , $q \in U_\epsilon$. Then $(U_\epsilon, h_\epsilon, q)$ converges geometrically to $(\mathcal{C}(U), h, q)$ in the C^0 topology. Moreover the geometric convergence is smooth outside of $\partial U \subset \mathcal{C}(U)$ in the following sense.*

1. Let $q \in U \setminus \partial U$; for small ϵ , $q \in U_\epsilon \setminus \gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon])$. Then as $\epsilon \rightarrow 0$,

$$(U_\epsilon \setminus \gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon]), h_\epsilon, q)$$

converges geometrically to $(U \setminus \partial U, g, q)$ in the C^∞ topology.

2. Fix any connected component C of ∂U ; for ϵ small we can choose a component C_ϵ of ∂U_ϵ so that C_ϵ converges to C as $\epsilon \rightarrow 0$ and the following holds. Let $\epsilon_k > 0$ be a sequence converging to 0. For all k let $q_k \in \gamma_\epsilon(C_{\epsilon_k} \times [0, \delta_{\epsilon_k}])$ be a point at fixed distance $\hat{d} > 0$ from $\gamma_\epsilon(C_{\epsilon_k} \times \{\delta_{\epsilon_k}\})$ for the metric h_{ϵ_k} , $\hat{d}$ being independent of k . Then

$$(\gamma_{\epsilon_k}(C_{\epsilon_k} \times [0, \delta_{\epsilon_k}]), h_{\epsilon_k}, q_k)$$

subsequently converges geometrically to $(C \times (0, \infty), g_{\text{prod}}, q_\infty)$ in the C^∞ topology, where g_{prod} is the product of the restriction of g to C and the standard metric on $(0, \infty)$, and q_∞ is a point of $C \times (0, \infty)$ at distance $\hat{d}$ from $C \times \{0\}$.

We describe more precisely the folding region in U_ϵ by finding explicit local charts where the metrics h_ϵ converge in the C^0 topology while remaining bounded in the C^1 topology. The sign $\lrcorner$ stands for the restriction of a metric g to a submanifold.

Lemma 10. For any $\mathfrak{d}_1 \in (0, \hat{d})$, there exists $\eta > 0$ such that for all $\epsilon > 0$ small, there is an embedding $\theta_\epsilon : \partial U \times [-\mathfrak{d}_1, \mathfrak{d}_1] \rightarrow U_\epsilon$ satisfying the following properties:

1. $\theta_\epsilon(\partial U \times \{0\}) = \gamma_\epsilon(\partial U_\epsilon \times \{\delta_\epsilon\})$ and

$$\theta_\epsilon(\partial U \times [-\mathfrak{d}_1, \mathfrak{d}_1]) = \{q \in U_\epsilon; d_{h_\epsilon}(q, \gamma_\epsilon(\partial U_\epsilon \times \{\delta_\epsilon\})) \leq \mathfrak{d}_1\},$$

2. $\|\theta_\epsilon^* h_\epsilon\|_{C^1(\partial U \times [-\mathfrak{d}_1, \mathfrak{d}_1])} \leq \eta$ where $\|\cdot\|_{C^1(\partial U \times [-\mathfrak{d}_1, \mathfrak{d}_1])}$ is computed with the product metric $h' := g \lrcorner \partial U \oplus ds^2$,
3. the metrics $\theta_\epsilon^* h_\epsilon$ converge in the C^0 topology to a Lipschitz continuous metric which is smooth outside of $\partial U \times \{0\} \subset \partial U \times [-\mathfrak{d}_1, \mathfrak{d}_1]$ and

$$\lim_{\epsilon \rightarrow 0} \|\theta_\epsilon^* h_{\epsilon \lrcorner}(\partial U \times [0, \mathfrak{d}_1]) - \gamma_0^* g \lrcorner(\partial U \times [0, \mathfrak{d}_1])\|_{C^0(\partial U \times [0, \mathfrak{d}_1])} = 0$$

where $||\cdot||_{C^0(\partial U \times [0, \mathfrak{d}_1])}$ is computed with h' .

3.2.3 Definitions and asymptotics for the min-max widths

Let (M, g) be a connected compact Riemannian manifold with boundary. The space of k -dimensional rectifiable mod 2 flat chains with rectifiable boundary in a manifold (M, g) is denoted by $\mathbf{I}_k(M; \mathbb{Z}_2)$, and $\mathbf{M}$ stands for the mass of a mod 2 flat chain. We denote by $\mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ the space of $T \in \mathbf{I}_n(M; \mathbb{Z}_2)$ in (M, g) with $T = \partial U + T_1$ for some $(n+1)$ -dimensional mod 2 flat chain U in M and some n -dimensional mod 2 flat chain T_1 with support in ∂M . As in [2, Definition 1.20] (see also [94, 2.2]), we define from $\mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ the space of relative cycles $\mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$. The latter coincides with $\mathcal{Z}_n(M; \mathbb{Z}_2)$ when the boundary ∂M is empty. The space of relative cycles is endowed with the flat topology and is weakly homotopically equivalent to $\mathbb{RP}^\infty$ (see [2] and Section 5 of [105] for the case $\partial M = \emptyset$). We denote by $\bar{\lambda}$ the generator of $H^1(\mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$.

Let X be a finite dimensional simplicial complex. A map $\Phi : X \rightarrow \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$ continuous in the flat topology is called a *p-sweepout* if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2).$$

By [94, Theorem 2.10, Theorem 2.11] (based on [101]), it will make no difference if we restrict ourselves to maps $\Phi : X \rightarrow \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$ which are continuous in the much stronger mass topology. We say that a *p-sweepout* $\Phi : X \rightarrow \mathcal{Z}_{n,rel}(M, \partial M; \mathbb{Z}_2)$ has *no concentration* of mass when

$$\limsup_{r \rightarrow 0} \{\mathbf{M}(\Phi(x) \cap B_r(p)) : x \in X, p \in M\} = 0.$$

The set of all *p-sweepouts* Φ that have no concentration of mass is denoted by $\mathcal{P}_p = \mathcal{P}_p(M, g)$. Note that two maps in $\mathcal{P}_p$ can have different domains.

In [104] and then [94], the widths of M were defined as

$$\omega_p(M, g) = \inf_{\Phi \in \mathcal{P}_p} \sup\{M(\Phi(x)) : x \in \text{dmn}(\Phi)\},$$

where $\text{dmn}(\Phi)$ is the domain of Φ (see [54] and [61] for previous works on the subject). As explained in [104, Lemma 4.7] for the case $\partial M = \emptyset$, the p -width can be expressed as the infimum of the widths of homotopy classes of discrete p -sweepouts (see [104] for definitions and notations):

$$\omega_p(M, g) = \inf_{\Pi \in \mathcal{D}_p} \mathbf{L}(\Pi). \quad (3.3)$$

A rather technical point in the theory is that usually one proves the existence of smooth minimal hypersurfaces and Morse index bounds with Almgren-Pitts's theory by working on the classes of discrete sweepouts Π . However it is not clear that the infimum in (3.3) is realized by one particular Π , so for instance in the the case $\partial M = \emptyset$, one can prove that $\omega_p(M, g)$ is achieved by an integral varifold whose support is a smooth minimal hypersurface of index bounded by p , by reducing to discrete sweepouts with k -dimensional domains X , then using the index bound of Marques-Neves [102] and the compactness result of Sharp [136] (see [75, Proposition 2.2] for a detailed explanation). When M has a non-trivial boundary, M. Li and X. Zhou [89] extended the arguments of Almgren-Pitts to show that for each Π as above, one can produce an integral varifold of mass $\mathbf{L}(\Pi)$ whose support is a smooth “almost properly embedded” free boundary minimal hypersurface (see [89, Section 2] for definitions). Note that [89, Theorem 4.21] holds for discrete sweepouts with more general domains X which are finite dimensional simplicial complex (see [104, Section 2]). The analogue of the index bounds in [102] is not available in the literature yet, so for the moment one cannot take a limit but can only produce such varifolds whose mass is arbitrarily close to $\omega_p(M, g)$. However when M has a non-empty boundary, if we suppose that when applying the min-max theorem of [89] to any (M, g') where g' is in a neighborhood of g , all the varifolds V as in [89, Theorem 4.21] have support inside the interior of M , i.e. all the smooth minimal hypersurfaces produced are contained in $\text{Int}(M)$, then the index bound [102, Theorem 1.2] still holds. This will be useful in next subsection. Another technical detail is the definitions of equivalence classes of relative cycles in [94, Section 2] and [89, Section 3]: in [94] the authors use integral currents before defining the quotient space whereas in [89], the authors use currents which are only integer rectifiable. These formulations are equivalent by [89, Lemma 3.8], they lead to the same notion of space of relative cycles $\mathcal{Z}_{n, \text{rel}}(M, \partial M; \mathbb{Z}_2)$ and the same functionals mass $\mathbf{M}$ and flat norm $\mathcal{F}$.

In our paper, we will need to consider the non-compact setting, so naturally we define the following.

Definition 3.2.1. *Let (N, g) be a complete non-compact manifold. Let $K_1 \subset K_2 \subset \dots \subset K_i \subset \dots$ be an exhaustion of N by compact $(n+1)$ -submanifolds with smooth boundary. The p -width of (N, g) is the number*

$$\omega_p(N, g) = \lim_{i \rightarrow \infty} \omega_p(K_i, g) \in [0, \infty].$$

For any two compact $(n+1)$ -submanifolds with smooth boundary $A \subset B \subset N$, we have $\omega_p(A, g) \leq \omega_p(B, g)$: this follows from adapting the proof of [94, Lemma 2.15, (1)] to the case of general metrics. Therefore $\omega_p(N, g)$ is well-defined since $\omega_p(K_i, g)$ is a nondecreasing sequence of nonnegative numbers. Moreover it does not depend on the choice of the exhaustion $\{K_i\}$.

Let $(\mathcal{C}, h)$ be a complete manifold with cylindrical ends, i.e. outside of a compact subset, the manifold is isometric to $\Sigma \times [0, \infty)$ endowed with a product metric (here Σ is a smooth n -dimensional manifold). The metric h is allowed to be only Lipschitz continuous. In the following proposition, we show that the asymptotic behavior of the widths $\omega_p(\mathcal{C}) = \omega_p(\mathcal{C}, h)$ is linear, and the leading term depends on the largest cylindrical end of $\mathcal{C}$. We denote by $\mathcal{H}^n$ the Hausdorff n -dimensional volume.

Theorem 11. *Let $(\mathcal{C}, h)$ be an $(n+1)$ -dimensional connected non-compact manifold with cylindrical ends, which is isometric to a product metric $(\Sigma \times [0, \infty), h_1 \oplus dt^2)$ outside of a compact subset. Let $\Sigma_1, \dots, \Sigma_m$ be the connected components of Σ and suppose that Σ_1 has the largest n -volume among these components:*

$$\mathcal{H}^n(\Sigma_1) \geq \mathcal{H}^n(\Sigma_j) \text{ for all } j \in \{1, \dots, m\},$$

$\mathcal{H}^n$ being computed with h_1 . Then $\omega_p(\mathcal{C})$ is finite for all p and the following holds:

1. *for all $p \in \{1, 2, 3, \dots\}$,*

$$\omega_{p+1}(\mathcal{C}) - \omega_p(\mathcal{C}) \geq \mathcal{H}^n(\Sigma_1),$$

2. moreover, there exists a constant $\hat{C}$ depending on h such that for all $p \in \{1, 2, 3, \dots\}$:

$$p\mathcal{H}^n(\Sigma_1) \leq \omega_p(\mathcal{C}) \leq p\mathcal{H}^n(\Sigma_1) + \hat{C}p^{\frac{1}{n+1}}.$$

Proof. Suppose first that each $\omega_p(\mathcal{C})$ is finite. We will use a few times [94, Lemma 2.15] which holds true more generally for Riemannian manifolds with smooth boundaries. By hypothesis, there is a compact subset $A \subset \mathcal{C}$ such that $(\mathcal{C} \setminus A, h)$ is isometric to $(\Sigma \times [0, \infty), h_1 \times dt^2)$. We need the following basic fact on the first width of cylinders:

$$\exists R_0 > 0, \forall R \geq R_0, \quad \omega_1(\Sigma_1 \times [0, R], h_1 \times dt^2) = \mathcal{H}^n(\Sigma_1). \quad (3.4)$$

This follows by noticing two things. On one hand, the hypersurfaces $\{\Sigma_1 \times \{r\}\}_{r \in [0, R]}$ give an explicit sweepout in $\mathcal{P}_1$ for which each non-trivial slice has n -volume equal to $\mathcal{H}^n(\Sigma_1)$, so

$$\omega_1(\Sigma_1 \times [0, R], h_1 \times dt^2) \leq \mathcal{H}^n(\Sigma_1).$$

On the other hand by applying the min-max theory in the setting with boundary of M. Li and X. Zhou [89], we get a varifold V such that

- the mass of V is arbitrarily close to $\omega_1(\Sigma_1 \times [0, R], h_1 \times dt^2)$ (see (3.3 and the following comments),
- the support of V is a smooth almost properly embedded minimal hypersurface (see [89, Subsection 2.3] for the definition of “almost properly embedded”).

By the maximum principle each connected component of $\text{spt}(V)$ is either of the form $\Sigma_1 \times \{r\}$ or intersect all such slices. Hence by the monotonicity formula, if R is large enough,

$$\omega_1(\Sigma_1 \times [0, R], h_1 \times dt^2) \geq \mathcal{H}^n(\Sigma_1) \text{ for } R \text{ large enough}$$

and so (3.4) is proved.

Now we show that

$$\omega_1(\mathcal{C}) \geq \mathcal{H}^n(\Sigma_1) \text{ and } \omega_{p+1}(\mathcal{C}) - \omega_p(\mathcal{C}) \geq \mathcal{H}^n(\Sigma_1) \text{ for } p \in \{1, 2, \dots\}. \quad (3.5)$$

This will immediately yield item (1) and the first inequality in item (2). Let E_0 be a subset of $\mathcal{C}$ isometric to $(\Sigma_1 \times [0, R_0], h_1 \times dt^2)$. By [94, Lemma 2.15 (1)] and by (3.4), for all R large enough and point $q \in \mathcal{C}$, we have $E_0 \subset B(q, R)$:

$$\omega_1(\mathcal{C}) \geq \omega_1(B(q, R), h) \geq \omega_1(E_0, h) = \mathcal{H}^1(\Sigma_1).$$

The second formula in (3.5) follows from a Lusternick-Schnirelmann type argument in the setting of Almgren-Pitts theory already used by Marques-Neves [104] and then by Liokumovich-Marques-Neves [94]. Let $\mu > 0$, fix a point $q \in \mathcal{C}$ and suppose that R_1 is big enough so that

$$\forall R \geq R_1, \quad \omega_p(B(q, R), h) > \omega_p(\mathcal{C}) - \mu. \quad (3.6)$$

Let $R > R_1$ be large enough so that $B(q, R)$ contains the disjoint union of $B(q, R_1)$ and a subset isometric to E_0 . Given $\Phi \in \mathcal{P}_{p+1}(B(q, R))$ continuous in the mass topology, with $X = \text{dmn}(\Phi)$, consider U_1 and U_2 the open subsets of X given by [94, Lemma 2.15 (2)] containing respectively the open subsets:

$$\{x \in X; \mathbf{M}(\Phi(x) \llcorner B(q, R_1)) < \omega_p(B(q, R_1), h) - \mu\}$$

$$\text{and } \{x \in X; \mathbf{M}(\Phi(x) \llcorner E_0) < \omega_1(E_0, h) - \mu\}.$$

Arguing as in [94, Theorem 3.1], we obtain $X \neq U_1 \cup U_2$. Let $x \in X \setminus (U_1 \cup U_2)$, then by (3.4) and (3.6):

$$\begin{aligned} \omega_{p+1}(\mathcal{C}) &\geq \omega_{p+1}(B(q, R), h) \\ &\geq \omega_p(B(q, R_1), h) + \omega_1(E_0, h) - 2\mu \\ &\geq \omega_p(\mathcal{C}) + \mathcal{H}^n(\Sigma_1) - 3\mu. \end{aligned}$$

Using [94, Corollary 2.13], and making μ go to 0 we obtain

$$\omega_{p+1}(\mathcal{C}) \geq \omega_p(\mathcal{C}) + \mathcal{H}^n(\Sigma_1).$$

We need to show that the widths $\omega_p(\mathcal{C})$ are finite and satisfy the second inequality in item (2). One way would be to construct an explicit p -sweepout in $\mathcal{P}_p$ as in [104, Theorem 5.1] using the bend-and-cancel argument of Guth [61]. Instead we use the gluing technique of Liokumovich-Marques-Neves [94] which enables to combine the p -sweepouts over the same domain X of several compact submanifolds $A_1, \dots, A_k$ with disjoint interiors into one p -sweepout over X of the union $A_1 \cup \dots \cup A_k$. By assumption there is a connected compact submanifold with boundary A such that $(\mathcal{C} \setminus A, g)$ is isometric to $(\Sigma \times [0, \infty), h_1 \times dt^2)$. We view $\Sigma \times [0, \infty)$ as an $(n+1)$ -submanifold of $\mathcal{C}$. Let $L > 0$ and define $B_L := \Sigma \times [0, L]$. The boundaries of the two submanifolds A and B_L intersect along the closed hypersurface $\Sigma \times \{0\}$. Fix p , by the sublinear bound on the p -widths [104, Theorem 5.1] (see also [55, Section 8] [61, Theorem 1]) which holds for compact manifolds with boundary with a Lipschitz metric, there is a p -sweepout $\Phi : \mathbb{RP}^p \rightarrow \mathcal{Z}_{n,rel}(A, \partial A; \mathbb{Z}_2)$ in $\mathcal{P}_p$ and there is a constant $C > 0$ depending on (A, h) but independent of p such that

$$\sup_{x \in \mathbb{RP}^p} \mathbf{M}(\Phi(x)) \leq Cp^{\frac{1}{n+1}}. \quad (3.7)$$

Recall that $\Sigma_1, \dots, \Sigma_m$ are the connected components of Σ . Let $f : B_L \rightarrow \mathbb{R}$ be the Morse function defined by $f(x, t) := (j-1)L + t$ if $(x, t) \in \Sigma_j \times [0, L]$. As in [104, Theorem 5.1], f determines a p -sweepout in $\mathcal{P}_p$ as follows. Define

$$\begin{aligned} \hat{\Psi} : \{a \in \mathbb{R}^{p+1}; |a| = 1\} &\rightarrow \mathcal{Z}_{n,rel}(B_L, \partial B_L; \mathbb{Z}_2) \\ \hat{\Psi}(a_0, \dots, a_p) &= \partial\{x \in B_L; \sum_{i=0}^p a_i f(x)^i < 0\}. \end{aligned}$$

Since $\hat{\Psi}(a) = \hat{\Psi}(-a)$, $\hat{\Psi}$ induces a map $\Psi : \mathbb{RP}^p \rightarrow \mathcal{Z}_{n,rel}(B_L, \partial B_L; \mathbb{Z}_2)$ which is a p -sweepout in $\mathcal{P}_p$. Note that since we are assuming $\mathcal{H}^n(\Sigma_1) \geq \mathcal{H}^n(\Sigma_j)$ for all $j \in \{1, \dots, m\}$ where the n -volume is computed with h_1 , we have

$$\sup_{x \in \mathbb{RP}^p} \mathbf{M}(\Psi(x)) \leq p\mathcal{H}^n(\Sigma_1). \quad (3.8)$$

Note that $B_L = \bigcup_{j=1}^m (\Sigma_j \times [0, L])$ and that each $\Sigma_j \times [0, L]$ is connected. In [94, Section

4], it is shown that for all $i \in \{1, \dots, m\}$,

$$\begin{aligned} SX_0 &:= \{(x, Z); x \in \mathbb{RP}^p, \Phi(x) - \partial Z \in \mathbf{I}_n(\partial A; \mathbb{Z}_2)\} \\ &\subset \mathbb{RP}^p \times \mathbf{I}_{n+1}(A; \mathbb{Z}_2), \end{aligned}$$

$$\begin{aligned} SX_i &:= \{(x, Z); x \in \mathbb{RP}^p, \Psi(x) \llcorner (\Sigma_i \times [0, L]) - \partial Z \in \mathbf{I}_n(\Sigma_i \times \{0, L\}; \mathbb{Z}_2)\} \\ &\subset \mathbb{RP}^p \times \mathbf{I}_{n+1}(\Sigma_i \times [0, L]; \mathbb{Z}_2) \end{aligned}$$

are isomorphic 2-covers of $\mathbb{RP}^p$ (they are actually isomorphic to the sphere S^p). Let $F_i : S^p \rightarrow SX_i$ ($i = 0, \dots, m$) be the corresponding isomorphism. Let $\Xi_i : SX_i \rightarrow \mathbf{I}_{n+1}(A \cup B_L; \mathbb{Z}_2)$ ($i = 0, \dots, m$) be the natural projection. Set

$$\hat{\varkappa}_L : S^p \rightarrow \mathbf{I}_n(A \cup B_L; \mathbb{Z}_2)$$

$$\hat{\varkappa}_L(y) = \sum_{i=0}^m \partial(\Xi_i \circ F_i(y)).$$

Since for any $y \in S^p$, $\hat{\varkappa}_L(y) = \hat{\varkappa}_L(-y) + \partial(A \cup B_L)$ in $\mathbf{I}_n(A \cup B_L; \mathbb{Z}_2)$, $\hat{\varkappa}_L$ induces a map

$$\varkappa_L : \mathbb{RP}^p \rightarrow \mathcal{Z}_{n,rel}(A \cup B_L, \partial(A \cup B_L); \mathbb{Z}_2).$$

By (3.7) and (3.8),

$$\sup_{x \in \mathbb{RP}^p} \mathbf{M}(\varkappa_L(x)) \leq p\mathcal{H}^n(\Sigma_1) + Cp^{\frac{1}{n+1}} + \mathcal{H}^n(\Sigma).$$

The map $\varkappa_L$ is a p -sweepout without concentration of mass (see [94]) thus for $\hat{C} = C + \mathcal{H}^n(\Sigma)$,

$$\omega_p(C) = \lim_{L \rightarrow \infty} \omega_p(A \cup B_L) \leq \sup_{x \in \mathbb{RP}^p} \mathbf{M}(\varkappa_L(x)) \leq p\mathcal{H}^n(\Sigma_1) + \hat{C}p^{\frac{1}{n+1}}$$

and so the widths $\omega_p(C)$ are finite and satisfy the second inequality of item (2). The theorem is proved. □

3.2.4 Confined min-max closed minimal hypersurfaces

Let (U, g) and $(\mathcal{C}(U), h)$ be as in Subsection 3.2.2. In the following, we will prove that the widths $\omega_p(\mathcal{C}(U))$ are realized as the n -volume (with multiplicities) of smooth closed minimal hypersurfaces confined in the non-cylindrical part, i.e. the interior of U . The proof will use the approximations (U_ϵ, h_ϵ) previously constructed.

Theorem 12. *Let $(\mathcal{C}(U), h)$ be as in Subsection 3.2.2. For all $p \in \{1, 2, 3, \dots\}$, there exist disjoint smooth closed connected embedded minimal hypersurfaces $\Gamma_1, \dots, \Gamma_N$ contained in $U \setminus \partial U$ and positive integers $m_1, \dots, m_N$ such that*

$$\omega_p(\mathcal{C}(U)) = \sum_{j=1}^N m_j \mathcal{H}^n(\Gamma_j).$$

Besides, if Γ_j is one-sided then the corresponding multiplicity m_j is even.

Proof. Consider the compact smooth approximations (U_ϵ, h_ϵ) constructed in Subsection 3.2.2. We fix p and apply the Almgren-Pitts min-max theorem for p -sweepouts with relative cycles in the setting with boundary, developped by M. Li and X. Zhou [89] (their results hold for more general sweepouts as defined in [104, 94]). We obtain a varifold V_ϵ of total mass ϵ -close to $\omega_p(U_\epsilon, h_\epsilon)$ by (3.3) and the following remarks. The support of V_ϵ is a smooth compact almost properly embedded free boundary minimal hypersurface $S_\epsilon \subset (U_\epsilon, h_\epsilon)$ (see [89, Subsection 2.3] for the definition of “almost properly embedded”). Since the boundary ∂U_ϵ is strictly mean concave with respect to the unit normal pointing outside of U_ϵ , no component of the hypersurface S_ϵ is contained in the boundary ∂U_ϵ . Moreover by the maximum principle and Remark 3.2.1, if a connected component of S_ϵ intersects $\gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon])$ then it also has to intersect $\Phi(\partial U \times \hat{t})$ (see Subsection 3.2.2 for notations). Since by Lemma 9 (2), the regions $\gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon])$ look more and more cylindrical, the monotonicity formula indicates that there is a radius $\tilde{R} > 0$ and a point $q \in U \setminus \partial U$ such that for all $\epsilon > 0$ small enough, the varifold V_ϵ is supported in the ball $B_{h_\epsilon}(q, \tilde{R})$. In particular for ϵ small, all the smooth minimal hypersurfaces produced by the min-max theory are closed minimal hypersurface embedded in the interior of U_ϵ . Besides the one-sided components have even multiplicities. Note that for any ϵ fixed and small, these facts are true for small perturbations of the metric h_ϵ and so [102, Theorem 1.2] is true for (U_ϵ, h_ϵ) (see the comment after (3.3)).

The next step is to take a limit as a sequence $\epsilon_k \rightarrow 0$ and argue that S_{ϵ_k} converges to a smooth closed minimal hypersurface contained in the interior of U . Note that $\omega_p(U_\epsilon, h_\epsilon)$ converges to $\omega_p(\mathcal{C}(U), h)$. Thus for a sequence $\epsilon_k \rightarrow 0$, the varifolds V_{ϵ_k} produced by min-max subsequently converge in the varifold sense to a varifold V_∞ in $\mathcal{C}(U)$ of total mass $\omega_p(\mathcal{C}(U), h)$, whose support is denoted by S_∞ . Moreover S_∞ is contained in the ball $B_h(q, 2\tilde{R})$. Even though the metrics h_ϵ converge to h only in the C^0 topology and h is not smooth, the varifold convergence makes sense as usual by using Lemma 9 and Lemma 10.

By the index bound of Marques-Neves [102, Theorem 1.2] which still holds in our situation with boundary, the minimal hypersurfaces S_ϵ can be chosen to have index bounded by p when ϵ is small. Thus by Sharp's compactness result [136], for a well chosen sequence $\epsilon_k \rightarrow 0$, the restriction of the support S_∞ of V_∞ to $\mathcal{C}(U) \setminus \partial U$ is a smooth minimal hypersurface, since the metrics h_ϵ converge smoothly in this region by Lemma 9. We first observe that by the maximum principle any component of S_∞ in the cylindrical part $\mathcal{C}(U) \setminus U$ would be a slice isometric to $(\partial U, g)$. But as we explained two paragraphs before, any component of S_{ϵ_k} intersects $\Phi(\partial U \times \hat{t})$ so it shows that the support S_∞ is contained in the compact set (U, g) . Secondly, we wish to prove that V_∞ is g -stationary. Indeed supposing this is verified, then applying Lemma 10 and Proposition 7, we obtain that V_∞ is actually a g -stationary integral varifold. By the monotonicity formula, no component of S_∞ is contained in ∂U . Hence the maximum principle proved by White [152] (based on [141]) implies that $S_\infty \cap \partial U = \emptyset$. Therefore the support S_∞ of V_∞ is a closed minimal hypersurface smoothly embedded inside $\text{Int}(U)$. By the description of convergence of finite index minimal hypersurfaces [136], the one-sided components of S_∞ have even multiplicities. This would complete the proof.

In the remaining of the proof, we show that V_∞ is g -stationary in U . It is non-trivial only because the limit metric h is not smooth in general. For $\epsilon \geq 0$, we will denote by div^ϵ the divergence computed in the metric h_ϵ (by convention $h_0 = g$). Let X be any vector field X smooth until the boundary U . Our goal is to prove that the first variation along X vanishes:

$$\delta V_\infty(X) = \int \text{div}_S^0 X(x) dV_\infty(x, S) = 0. \quad (3.9)$$

Recall that for all $\epsilon \geq 0$ small, the map

$$\gamma_\epsilon : \partial U_\epsilon \times [0, \hat{d}] \rightarrow U$$

$$\gamma_\epsilon(x, t) = \exp(x, t\nu)$$

is a diffeomorphism onto its image, where $\hat{d} > 0$ is independent of ϵ . Here $\exp$ is the exponential map for g , ν is the inward unit normal of ∂U_ϵ for g . By abuse of notations, we define the following vector fields:

$$\frac{\partial}{\partial t} := (\gamma_\epsilon)_* \left(\frac{\partial}{\partial t} \right),$$

$$\frac{\partial}{\partial s} := \frac{(\gamma_\epsilon)_* \left(\frac{\partial}{\partial t} \right)}{\|(\gamma_\epsilon)_* \left(\frac{\partial}{\partial t} \right)\|_{h_\epsilon}},$$

so that $\frac{\partial}{\partial t}$ (resp. $\frac{\partial}{\partial s}$) is a unit vector field for g (resp. h_ϵ). Let s be the coordinate obtained by integrating $\frac{\partial}{\partial s}$ from $\gamma_\epsilon(\partial U_\epsilon \times \{\delta_\epsilon\})$, so that $\{s = 0\} = \gamma_\epsilon(\partial U_\epsilon \times \{\delta_\epsilon\})$ and $s < 0$ on $\gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon])$. Because the support of V_∞ restricted to $\text{Int}(U)$ is a smooth minimal hypersurface, we can assume that the vector field X is supported inside $\gamma_0(\partial U \times [0, \hat{d}/2])$. Thus for all ϵ small enough, the vector field X restricted to U_ϵ can be decomposed into two components

$$X = X_\perp^\epsilon + X_\parallel^\epsilon$$

where $X_\perp^\epsilon$ is orthogonal to $\frac{\partial}{\partial t}$ (for either g or h_ϵ) and $X_\parallel^\epsilon$ is a multiple of $\frac{\partial}{\partial t}$.

Let $a_1 < b_1 < b_2 < a_2$ be four numbers in $(-\infty, \hat{d})$. Consider a smooth nondecreasing function $\beta : \mathbb{R} \rightarrow [0, 1]$ such that

- $\beta(s) \equiv 0$ (resp. 1) when $s \leq a_1$ (resp. $s \geq a_2$),
- on $[b_1, b_2]$, $\frac{\partial \beta}{\partial s} \geq \frac{1}{2(a_2 - a_1)}$.

Define the following vector field on U_ϵ :

$$Y^\epsilon := (1 - \beta(s)) \exp(-Cs) \frac{\partial}{\partial s},$$

Let (y, S) be a point of the Grassmannian bundle of U and let $(e_1, \dots, e_n)$ be an h_ϵ -orthonormal basis of S so that $e_1, \dots, e_{n-1}$ are h_ϵ -orthogonal to $\frac{\partial}{\partial s}$. Let e_n^* be a unit

vector such that $(e_1, \dots, e_n^*)$ is an h_ϵ -orthonormal basis of the n -plane h_ϵ -orthogonal to $\frac{\partial}{\partial s}$ at y . If $y = \gamma_\epsilon(x, t) \in \gamma_\epsilon(\partial U_\epsilon \times [0, \hat{d}])$, let H^ϵ (resp. $\mathbf{A}^\epsilon$) denote the mean curvature (resp. second fundamental form) at y of $\gamma_\epsilon(\partial U_\epsilon \times \{t\})$, with respect to h_ϵ and the unit normal vector $\frac{\partial}{\partial s}$. Let $C > 0$ be a constant larger than the norm of $\mathbf{A}^\epsilon$: in view of Lemma 8 (3), C can be chosen independently of y and ϵ . For all $\epsilon > 0$, we compute similarly to the proof of [100, Lemma 2.2] the divergence of Y^ϵ (with respect to the metric h_ϵ) at (y, S) :

$$\begin{aligned}
\operatorname{div}_S^\epsilon Y^\epsilon &= \frac{\partial(1 - \beta(s))}{\partial s} \exp(-Cs) |h_\epsilon(e_n, \frac{\partial}{\partial s})|^2 \\
&\quad - C(1 - \beta(s)) \exp(-Cs) |h_\epsilon(e_n, \frac{\partial}{\partial s})|^2 \\
&\quad + (1 - \beta(s)) \exp(-Cs) H^\epsilon \\
&\quad - (1 - \beta(s)) \exp(-Cs) \mathbf{A}^\epsilon(e_n^*, e_n^*) |h_\epsilon(e_n, \frac{\partial}{\partial s})|^2 \\
&= \exp(-Cs) |h_\epsilon(e_n, \frac{\partial}{\partial s})|^2 \left(\frac{\partial(1 - \beta(s))}{\partial s} - C(1 - \beta(s)) - (1 - \beta(s)) \mathbf{A}^\epsilon(e_n^*, e_n^*) \right) \\
&\quad + (1 - \beta(s)) \exp(-Cs) H^\epsilon \\
&\leq - \exp(-Cs) \frac{\partial \beta(s)}{\partial s} |h_\epsilon(e_n, \frac{\partial}{\partial s})|^2 + (1 - \beta(s)) \exp(-Cs) H^\epsilon.
\end{aligned} \tag{3.10}$$

For the inequality, we used that C is larger than the norm of the second fundamental forms. Note that since the limit varifold V_∞ has support in U , the coordinate s stays bounded independently of ϵ on $\operatorname{spt}(V_\epsilon)$ when defined. Recall that by Lemma 8 (2), the term $(1 - \beta(s)) H^\epsilon$ goes to 0 as $a_2 \rightarrow 0$, uniformly in ϵ . Since the varifold V_ϵ is h_ϵ -stationary, for all $\epsilon > 0$ small:

$$\delta V_\epsilon(Y^\epsilon) = \int \operatorname{div}^\epsilon Y^\epsilon dV_\epsilon = 0$$

computed with h_ϵ , for any ϵ small and any choice of $a_1 < b_1 < a_2 < b_2$ in the definition of β . Recall also that the mass of V_ϵ is bounded uniformly in ϵ and that its support stays uniformly far from the boundary of (U_ϵ, h_ϵ) for ϵ small.

Collecting these facts together, the previous computation implies the following. If we fix $b_1 < 0$ and let a_2 and $b_2 > 0$ tend to 0, inequality (3.10) and the fact that $\delta V_\epsilon(Y^\epsilon) = 0$

show that with the previous notation for e_n :

$$\begin{aligned} & \forall \hat{a} > 0, \forall \hat{b} > 0, \exists \hat{r} > 0, \exists \epsilon_0, \forall \epsilon \in [0, \epsilon_0], \\ & \int_{\gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon + \hat{r}]) \times \mathbf{G}(n+1, n)} \chi_{\{|h_\epsilon(e_n, \frac{\partial}{\partial s})| > \hat{b}\}} dV_\epsilon(x, S) < \hat{a}, \end{aligned} \quad (3.11)$$

where $\chi_{\{|h_\epsilon(e_n, \frac{\partial}{\partial s})| > \hat{b}\}}$ is the characteristic function of the set of (x, S) such that $|h_\epsilon(e_n, \frac{\partial}{\partial s})| > \hat{b}$. In particular by taking $\epsilon = 0$ in the previous expression,

$$V_\infty \llcorner \{(x, S); x \in \partial U, S \neq T_x \partial U\} = 0. \quad (3.12)$$

We now explain how to deduce (3.9) from the previous estimates. Let $\{\hat{r}_k\}$ be a sequence of positive numbers converging to 0. Consider

$$A_k := \gamma_\epsilon([0, \delta_\epsilon + \hat{r}_k]),$$

$$B_k := U_\epsilon \setminus A_k.$$

Then, by taking a subsequence of $\{\epsilon_k\}$ and renumbering if necessary, we can assume that there are two varifolds V'_∞, V''_∞ in U so that as $k \rightarrow \infty$, the following convergences in the varifold sense take place:

$$V_k := V_{\epsilon_k} \rightharpoonup V_\infty$$

$$V'_k := V_{\epsilon_k} \llcorner (A_k \times \mathbf{G}(n+1, n)) \rightharpoonup V'_\infty,$$

$$V''_k := V_{\epsilon_k} \llcorner (B_k \times \mathbf{G}(n+1, n)) \rightharpoonup V''_\infty.$$

Recall that we decomposed $X = X_\perp^\epsilon + X_\parallel^\epsilon$. We will show first that

$$\lim_{k \rightarrow \infty} \int \operatorname{div}^{\epsilon_k} X_\perp^{\epsilon_k} dV_k = \int \operatorname{div}^0 X_\perp^0 dV_\infty \quad (3.13)$$

and this will imply

$$\int \operatorname{div}^0 X_\perp^0 dV_\infty = 0$$

since the left-hand side of (3.13) is zero by h_ϵ -stationarity of V_k . Then we will show less

directly that

$$\int \operatorname{div}^0 X_{\parallel}^0 dV_{\infty} = 0. \quad (3.14)$$

and that would imply (3.9). In both cases, the decomposition $V_k = V'_k + V''_k$ is useful.

We record the fact that, since on B_k the metric h_{ϵ} coincides with the original metric g , for any smooth vector field $\tilde{X}$ on M :

$$\lim_{k \rightarrow \infty} \int \operatorname{div}^{\epsilon_k} \tilde{X} dV''_k = \lim_{k \rightarrow \infty} \int \operatorname{div}^0 \tilde{X} dV''_k = \int \operatorname{div}^0 \tilde{X} dV''_{\infty}. \quad (3.15)$$

In particular we already obtain

$$\lim_{k \rightarrow \infty} \int \operatorname{div}^{\epsilon_k} X_{\perp}^{\epsilon_k} dV''_k = \lim_{k \rightarrow \infty} \int \operatorname{div}^{\epsilon_k} X_{\perp}^0 dV''_k = \int \operatorname{div}^0 X_{\perp}^0 dV''_{\infty}$$

since $X_{\perp}^{\epsilon}$ converges smoothly to $X_{\perp}^0$ on U . To prove the convergence for V'_k , we evaluate the divergence of $X_{\perp}^{\epsilon}$ in h_{ϵ} for $\epsilon > 0$ as follows. Let ∇^{ϵ} be the Levi-Civita connection of h_{ϵ} . By definition of h_{ϵ} , for any $\epsilon > 0$ the restriction of h_{ϵ} and g to a slice $\gamma_{\epsilon}(\partial U_{\epsilon} \times \{t\})$ coincide. Thus for any vector e' in the tangent bundle of a slice $\gamma_{\epsilon}(\partial U_{\epsilon} \times \{t\})$, we have

$$h_{\epsilon}(\nabla_{e'}^{\epsilon} X_{\perp}^{\epsilon}, e') = g(\nabla_{e'}^0 X_{\perp}^{\epsilon}, e'). \quad (3.16)$$

If (x, S) , let $e_1, \dots, e_n, e_n^*$ be defined as before and let $S_{\perp}$ denote the n -plane at x orthogonal to $\frac{\partial}{\partial s}$. For all (x, S) such that $x \in A_k$, we have:

$$\begin{aligned} \operatorname{div}_S^{\epsilon} X_{\perp}^{\epsilon} &= \sum_{i=1}^n h_{\epsilon}(\nabla_{e_i}^{\epsilon} X_{\perp}^{\epsilon}, e_i) \\ &= \sum_{i=1}^{n-1} h_{\epsilon}(\nabla_{e_i}^{\epsilon} X_{\perp}^{\epsilon}, e_i) + h_{\epsilon}(\nabla_{e_n^*}^{\epsilon} X_{\perp}^{\epsilon}, e_n^*) + (h_{\epsilon}(\nabla_{e_n}^{\epsilon} X_{\perp}^{\epsilon}, e_n) - h_{\epsilon}(\nabla_{e_n^*}^{\epsilon} X_{\perp}^{\epsilon}, e_n^*)) \\ &= \sum_{i=1}^{n-1} g(\nabla_{e_i}^0 X_{\perp}^{\epsilon}, e_i) + g(\nabla_{e_n^*}^0 X_{\perp}^{\epsilon}, e_n^*) + (h_{\epsilon}(\nabla_{e_n}^{\epsilon} X_{\perp}^{\epsilon}, e_n) - h_{\epsilon}(\nabla_{e_n^*}^{\epsilon} X_{\perp}^{\epsilon}, e_n^*)) \\ &= \operatorname{div}_{S_{\perp}}^0 X_{\perp}^{\epsilon} + \Upsilon(\epsilon, x, S, X), \end{aligned} \quad (3.17)$$

where $\Upsilon(\cdot)$ is a real function converging uniformly to 0 as $|h_{\epsilon}(e_n, \frac{\partial}{\partial s})| \rightarrow 0$. The third equality comes from (3.16). For the last equality we used the fact that $\|\nabla^{\epsilon} X_{\perp}^{\epsilon}\|_{h_{\epsilon}}$ is

bounded uniformly in $\epsilon > 0$: to see this it is helpful to use the chart θ_ϵ given in Lemma 10.

Now by the fact that $S_\infty = \text{spt}(V_\infty) \subset U$ and (3.12), we see that V'_∞ is entirely supported on $\{(x, S); x \in \partial U, S = T_x \partial U\}$ as a Radon measure. For any $\hat{a} > 0$, let $\epsilon_k > 0$ and $\hat{b} > 0$ be small enough so that (3.11) is true with $\hat{r} = \hat{r}_k$ and $|\Upsilon(b)| \leq \hat{a}$. Then from (3.17) we can write that for any k large enough:

$$\begin{aligned}
& \left| \int \text{div}_S^{\epsilon_k} X_\perp^{\epsilon_k} dV'_k(x, S) - \int \text{div}^0 X_\perp^0 dV'_\infty \right| \\
&= \left| \int \chi_{\{|h_{\epsilon_k}(e_n, \frac{\partial}{\partial s})| \leq \hat{b}\}} \text{div}_S^{\epsilon_k} X_\perp^{\epsilon_k} dV'_k(x, S) + \int \chi_{\{|h_{\epsilon_k}(e_n, \frac{\partial}{\partial s})| > \hat{b}\}} \text{div}^{\epsilon_k} X_\perp^{\epsilon_k} dV'_k - \int \text{div}^0 X_\perp^0 dV'_\infty \right| \\
&\leq \left| \int \chi_{\{|h_{\epsilon_k}(e_n, \frac{\partial}{\partial s})| \leq \hat{b}\}} \text{div}_S^{\epsilon_k} X_\perp^{\epsilon_k} dV'_k(x, S) - \int \text{div}^0 X_\perp^0 dV'_\infty \right| + C\hat{a} \\
&\leq \left| \int \chi_{\{|h_{\epsilon_k}(e_n, \frac{\partial}{\partial s})| \leq \hat{b}\}} \text{div}_{S_\perp}^0 X_\perp^{\epsilon_k} dV'_k(x, S) - \int \text{div}^0 X_\perp^0 dV'_\infty \right| + 2C\hat{a} \\
&\leq \left| \int \text{div}_{S_\perp}^0 X_\perp^{\epsilon_k} dV'_k(x, S) - \int \text{div}_{S_\perp}^0 X_\perp^0 dV'_\infty(x, S) \right| + 3C\hat{a} \\
&\leq 4C\hat{a}.
\end{aligned}$$

Here C is a constant depending on X but neither on ϵ_k nor on $\hat{a}, \hat{b}$. The last inequality can be seen in the chart θ_ϵ of Lemma 10. Hence we just proved (3.13).

Next we study $X_\parallel^0$. Write $X_\parallel^\epsilon = \varphi_\epsilon \frac{\partial}{\partial t}$ where φ_ϵ is a smooth function on U_ϵ bounded independently of ϵ . Then we define the vector field

$$Z^\epsilon := \varphi_\epsilon \frac{\partial}{\partial s},$$

that is, we rescale $X_\parallel^\epsilon$ by ϑ_ϵ^{-1} so that Z^ϵ is bounded for h_ϵ . This particular definition of Z^ϵ is not essential, what counts is that

- Z^ϵ is a multiple of $\frac{\partial}{\partial s}$,
- $Z^\epsilon = X_\parallel^\epsilon$ on $U_\epsilon \setminus (\gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon]))$ and
- $\|\nabla^\epsilon Z^\epsilon\|_{h_\epsilon}$ is bounded uniformly in $\epsilon > 0$.

Again to see the second bullet, it is helpful to use the chart θ_ϵ given in Lemma 10. Let

H^ϵ be defined as before. For all (x, S) such that $x \in A_k$,

$$\begin{aligned}
\operatorname{div}_S^\epsilon Z^\epsilon &= \sum_{i=1}^n h_\epsilon(\nabla_{e_i}^\epsilon Z^\epsilon, e_i) \\
&= \sum_{i=1}^{n-1} h_\epsilon(\nabla_{e_i}^\epsilon Z^\epsilon, e_i) + h_\epsilon(\nabla_{e_n^*}^\epsilon Z^\epsilon, e_n^*) + (h_\epsilon(\nabla_{e_n}^\epsilon Z^\epsilon, e_n) - h_\epsilon(\nabla_{e_n^*}^\epsilon Z^\epsilon, e_n^*)) \\
&= \sum_{i=1}^{n-1} h_\epsilon(\nabla_{e_i}^\epsilon \varphi_\epsilon \frac{\partial}{\partial S}, e_i) + h_\epsilon(\nabla_{e_n^*}^\epsilon \varphi_\epsilon \frac{\partial}{\partial S}, e_n^*) + (h_\epsilon(\nabla_{e_n}^\epsilon Z^\epsilon, e_n) - h_\epsilon(\nabla_{e_n^*}^\epsilon Z^\epsilon, e_n^*)) \\
&= \varphi_\epsilon H^\epsilon + (h_\epsilon(\nabla_{e_n}^\epsilon Z^\epsilon, e_n) - h_\epsilon(\nabla_{e_n^*}^\epsilon Z^\epsilon, e_n^*)) \\
&= \Upsilon'(\epsilon, x, S, X),
\end{aligned} \tag{3.18}$$

where $\Upsilon'(\cdot)$ is a real function converging to 0 uniformly as $|h_\epsilon(e_n, \frac{\partial}{\partial S})| \rightarrow 0$ and $x \in A_k$. The last equality used that the mean curvature H^ϵ goes uniformly to 0 by Lemma 8 and that $\|\nabla^\epsilon Z^\epsilon\|_{h_\epsilon}$ is bounded uniformly in $\epsilon > 0$. For any $\hat{a} > 0$, let $\epsilon_k > 0$ and $\hat{b} > 0$ be small enough so that (3.11) is true with $\hat{r} = \hat{r}_k$ and $|\Upsilon'(b)| \leq \hat{a}$. The computation (3.18) and the fact that $Z^\epsilon = X_\parallel^\epsilon$ on $U_\epsilon \setminus (\gamma_\epsilon(\partial U_\epsilon \times [0, \delta_\epsilon]))$ imply that for k large:

$$\begin{aligned}
\left| \int \operatorname{div}^{\epsilon_k} X_\parallel^{\epsilon_k} dV_k'' \right| &= \left| \int \operatorname{div}^{\epsilon_k} Z^{\epsilon_k} dV_k'' \right| \\
&= \left| \int \operatorname{div}^{\epsilon_k} Z^{\epsilon_k} dV_k' \right| \\
&\leq \left| \int \chi_{\{|h_{\epsilon_k}(e_n, \frac{\partial}{\partial S})| \leq \hat{b}\}} \operatorname{div}_S^{\epsilon_k} Z^{\epsilon_k} dV_k'(x, S) \right| + C\hat{a} \\
&\leq 2C\hat{a}.
\end{aligned}$$

As previously, C is a constant depending on X but neither on ϵ_k nor on $\hat{a}, \hat{b}$. The second equality above comes from the stationarity of V_k for h_{ϵ_k} . Thus combining with (3.15) we obtain

$$\int \operatorname{div}^0 X_\parallel^0 dV_\infty'' = \lim_{k \rightarrow \infty} \int \operatorname{div}^{\epsilon_k} X_\parallel^0 dV_k'' = \lim_{k \rightarrow \infty} \int \operatorname{div}^{\epsilon_k} X_\parallel^{\epsilon_k} dV_k'' = 0$$

since $X_\parallel^\epsilon$ converges smoothly to $X_\parallel^0$. On the other hand, (3.12), (3.18) for $\epsilon = 0$ and

$x \in \partial U$, $S = T_x \partial U$, and the minimality of ∂U implies that

$$\int \operatorname{div}^0 X_{\parallel}^0 dV'_{\infty} = 0.$$

Finally we conclude that

$$\int \operatorname{div}^0 X_{\parallel}^0 dV_{\infty} = \int \operatorname{div}^0 X_{\parallel}^0 dV'_{\infty} + \int \operatorname{div}^0 X_{\parallel}^0 dV''_{\infty} = 0$$

and (3.14) follows. This ends the proof. □

3.3 Proof of Theorem 5

Let (M, g) be a connected closed Riemannian manifold. In this section, all the minimal hypersurfaces considered are closed and smoothly embedded. We say that a two-sided minimal hypersurface is *degenerate* if its Jacobi operator has a non-trivial kernel. If such a hypersurface is degenerate and stable, then the kernel of its Jacobi operator is spanned by a positive eigenfunction. Note that for a two-sided minimal hypersurface which is either unstable or non-degenerate stable, it is well-known that the hypersurface has a neighborhood foliated by closed leaves which, when not equal to the minimal hypersurface itself, have non-zero mean curvature vector. Actually a similar result is true for degenerate stable minimal hypersurfaces, as the following lemma shows.

Lemma 13. *Let Γ be a two-sided degenerate stable minimal hypersurface in (M, g) and ν a choice of unit normal vector on Γ . Then there exist a positive number δ_1 and a smooth map $w : \Gamma \times (-\delta_1, \delta_1) \rightarrow \mathbb{R}$ with the following properties:*

1. *for each $x \in \Gamma$, we have $w(x, 0) = 0$ and $\phi_0 := \frac{\partial}{\partial t} w(x, t)|_{t=0}$ is a positive function in the kernel of the Jacobi operator of Γ ,*
2. *for each $t \in (-\delta_1, \delta_1)$, we have $\int_{\Gamma} (w(\cdot, t) - t\phi_0)\phi_0 = 0$,*
3. *for each $t \in (-\delta_1, \delta_1)$, the mean curvature of the hypersurface*

$$\{\exp(x, w(x, t)\nu(x)); x \in \Gamma\}$$

is either positive or negative or identically zero.

If the minimal hypersurface Γ is one-sided (M, g) , one can still apply the previous discussion (in particular Lemma 13) in a double-cover of M where Γ lifts to a two-sided hypersurface.

Let S be a minimal hypersurface embedded in a compact $(n+1)$ -dimensional compact manifold $\hat{M}$ (possibly with boundary). Let $\mu > 0$, consider a neighborhood $\mathcal{N}$ of S and a diffeomorphism

$$F : \{x \in \hat{M}; d(x, S) \leq \mu\} \rightarrow \mathcal{N}$$

such that $F(x) = x$ for $x \in S$. We define the following:

- S has a *contracting neighborhood* if there are such μ , $\mathcal{N}$ and F such that for all $t \in (0, \mu]$, $F(\{x \in \hat{M}; d(x, S) = t\})$ has mean curvature vector pointing towards S ,
- S has a *expanding neighborhood* if there are such μ , $\mathcal{N}$ and F such that for all $t \in (0, \mu]$, $F(\{x \in \hat{M}; d(x, S) = t\})$ has mean curvature vector pointing away from S ,
- S has a *mixed neighborhood* if S is two-sided, contained in the interior of $\hat{M}$ and there are such μ , $\mathcal{N}$ and F such that for all $t \in [-\mu, 0)$ (resp. $t \in (0, \mu]$), $F(\{x \in \hat{M}; d_{signed}(x, S) = t\})$ has mean curvature vector pointing towards (resp. pointing away from) S . Here d_{signed} is a choice of signed distance locally around S .

By the first variation formula, it is clear that if S has a contracting (resp. expanding) neighborhood, the slices $F(\{x \in \hat{M}; d(x, S) = t\})$ have larger (resp. smaller) n -volume than S .

Given (M, g) we will need to cut M along some of its minimal hypersurfaces and get a compact manifold with boundary. Let Γ be a connected minimal hypersurface in M . There are three possibilities: Γ is either two-sided separating, or two-sided non-separating, or one-sided. In the first case, we cut M along Γ , then choose any one of the two components $M \setminus \Gamma$: its metric completion is a compact manifold whose boundary has one connected component. In the second case, we cut M along Γ and obtain by completion a compact manifold whose boundary has two connected components. In the third case, we consider the metric completion of the complement of Γ in M : it is a compact manifold

whose boundary has one component. In all cases, let M_1 be the compact manifold we get. It is naturally endowed with a metric still denoted by g . We can repeat this procedure for any minimal hypersurface $\Gamma_1 \subset (\text{Int } M_1, g)$, obtain M_2 and go on. Thus we construct a finite sequence $(M, g), (M_1, g), \dots (M_J, g)$ by successive cuts. Note that no two of them are isometric: each cut either adds new boundary components or strictly reduces the $(n+1)$ -volume. We will say that a compact manifold U is *obtained by cutting M along minimal hypersurfaces* if U is the manifold obtained after a positive finite number of steps we just described.

Lemma 14. *Let (N, g') be a connected compact manifold, possibly with boundary. Suppose that the boundary, if non-empty, has a contracting neighborhood and that any minimal hypersurface in N has a neighborhood which is either contracting or expanding or mixed. Then the following is true.*

1. *If the interior of (N, g') contains two disjoint connected minimal hypersurfaces, then the interior of N contains a minimal hypersurface with a contracting or mixed neighborhood.*
2. *If the interior of (N, g') contains a minimal hypersurface with a contracting or mixed neighborhood, then one can cut N along some minimal hypersurfaces and get a different manifold (N', g') such that $\partial N'$ has a contracting neighborhood in N' .*

Next, suppose by contradiction that (M, g) only contains finitely many minimal hypersurfaces. Then by Lemma 13 each one of them has either a contracting neighborhood or an expanding neighborhood or a mixed neighborhood. Let us cut M along minimal hypersurfaces a certain number of times in such a way that we get a new manifold U whose boundary, if not empty, has a contracting neighborhood. We choose U so that the number of cuts is maximal: it is possible since the cuts are realized along minimal hypersurfaces, which are in finite number in M . By Lemma 14, we know that all minimal hypersurfaces embedded in the interior of U have an expanding neighborhood and any two such hypersurfaces intersect. By the work of Marques and Neves [104], the manifold (M, g) also necessarily contains at least two disjoint minimal hypersurfaces, otherwise it would contain infinitely many minimal hypersurfaces. Thus we deduce from Lemma 14 that U is not equal to M or in other words the boundary ∂U is not empty.

The following lemma is true in general.

Lemma 15. *Let $(\tilde{U}, \tilde{g})$ be a connected compact manifold such that $\partial\tilde{U}$ is a minimal hypersurface with a contracting neighborhood. Let $\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_q$ be the connected components of $\partial\tilde{U}$. Assume that every minimal hypersurface in the interior of $\tilde{U}$ has an expanding neighborhood and that any two of them intersect. Then for any minimal hypersurface Γ in the interior of $\tilde{U}$:*

1. *if Γ is two-sided,*

$$\mathcal{H}^n(\Gamma) > \max\{\mathcal{H}^n(\tilde{\Sigma}_1), \dots, \mathcal{H}^n(\tilde{\Sigma}_q)\},$$

2. *if Γ is one-sided,*

$$2\mathcal{H}^n(\Gamma) > \max\{\mathcal{H}^n(\tilde{\Sigma}_1), \dots, \mathcal{H}^n(\tilde{\Sigma}_q)\}.$$

Since we are supposing by contradiction that M contains finitely many minimal hypersurfaces, let $\Gamma_1, \dots, \Gamma_k$ be the minimal hypersurfaces contained in the interior of U . Let $\Sigma_1, \dots, \Sigma_q$ be the connected components of ∂U , which is non-empty as previously explained. Suppose that

$$\mathcal{H}^n(\Sigma_1) \geq \mathcal{H}^n(\Sigma_j) \text{ for all } j \in \{1, \dots, q\}.$$

Then by Lemma 15 and by the construction of U , for all $i \in \{1, \dots, k\}$ we have:

$$\begin{aligned} \mathcal{H}^n(\Gamma_i) &> \mathcal{H}^n(\Sigma_1) \text{ if } \Gamma_i \text{ is two-sided,} \\ 2\mathcal{H}^n(\Gamma_i) &> \mathcal{H}^n(\Sigma_1) \text{ if } \Gamma_i \text{ is one-sided.} \end{aligned} \tag{3.19}$$

Let $\mathcal{C}(U)$ as constructed in Subsection 3.2.2. By Theorem 12 and since any two of the Γ_i intersect, the widths $\omega_p = \omega_p(\mathcal{C}(U))$ are realized as an integer multiple of the n -volume of one of the hypersurfaces $\Gamma_1, \dots, \Gamma_k$:

$$\forall p, \exists i \in \{1, \dots, k\}, \exists m_{p,i} \in \{1, 2, 3, \dots\}, \quad \omega_p = m_{p,i} \mathcal{H}^n(\Gamma_i), \tag{3.20}$$

and $m_{p,i}$ is even if Γ_i is one-sided. We would like to derive a contradiction from the asymptotic behavior of the widths ω_p described in Theorem 11 and inequality (3.19). The next lemma is an elementary arithmetic result.

Lemma 16. *Let $\alpha_1, \dots, \alpha_R$ be a collection of reals numbers strictly larger than 1. Consider a sequence of increasing positive real numbers $\{u_p\}_{p \geq 1}$ such that for all $p \geq 1$:*

- $u_1 \geq 1$, $u_{p+1} \geq u_p + 1$ and
- $u_p \in \{m\alpha_r; m \in \{1, 2, 3, \dots\}, r \in \{1, \dots, R\}\}$.

Then there exists an $\bar{\epsilon}_0 = \bar{\epsilon}_0(\alpha_1, \dots, \alpha_R) > 0$ such that for p large enough,

$$u_p > (1 + \bar{\epsilon}_0)p.$$

Proof. We can assume that $(\alpha_1^{-1}, \dots, \alpha_S^{-1})$ (where $S \leq R$) form a $\mathbb{Q}$ -basis of the $\mathbb{Q}$ -vector space generated by $\alpha_1^{-1}, \dots, \alpha_R^{-1}$. Suppose that

$$\forall r \in \{S+1, \dots, R\}, \exists c_{r,1}, \dots, c_{r,S} \in \mathbb{Z}, \quad c_r \alpha_r^{-1} = \sum_{i=1}^S c_{r,i} \alpha_i^{-1}. \quad (3.21)$$

Since $1, \frac{\alpha_1}{\alpha_2}, \dots, \frac{\alpha_1}{\alpha_S}$ are independent real numbers over $\mathbb{Q}$, by Weyl's equidistribution theorem, the sequence

$$\{(l \frac{\alpha_1}{\alpha_2}, \dots, l \frac{\alpha_1}{\alpha_S}); l \in \mathbb{N}\}$$

is equidistributed modulo $\mathbb{Z}^{S-1}$. In particular, if $\epsilon \in (0, 1)$, then the density of the set

$$\mathcal{L} := \{l \in \mathbb{N}; \forall i \in \{2, \dots, S\}, \exists q \in \mathbb{N}, |l \frac{\alpha_1}{\alpha_i} - q| < \frac{\epsilon}{2}\}$$

exists and is equal to $\mathbf{d}(\mathcal{L}) = \epsilon^{S-1}$, where the density $\mathbf{d}(\mathcal{S})$ of a discrete subset $\mathcal{S}$ of $\mathbb{R}$ is defined (when it exists) by

$$\mathbf{d}(\mathcal{L}) := \lim_{x \rightarrow \infty} \frac{\#\{l \in \mathcal{L}; l \leq x\}}{x}.$$

Now note that by (3.21), for all $l \in \mathcal{L}$ and for all $r \in \{S+1, \dots, R\}$, there is an integer $q(r, l)$ such that

$$|c_r l \frac{\alpha_1}{\alpha_r} - q(r, l)| < (|c_{r,2}| + \dots + |c_{r,S}|)\epsilon.$$

Thus, if we define $M := \prod_{r=S+1}^R c_r$, there is a constant C depending only on $\alpha_1, \dots, \alpha_R$

such that for all $l \in \mathcal{L}$ and $r \in \{2, \dots, R\}$, there is an integer $q'(l, r)$ satisfying

$$|Ml\alpha_1 - q'(l, r)\alpha_r| < C\epsilon. \quad (3.22)$$

Define $M\alpha_1\mathcal{L} := \{Ml\alpha_1; l \in \mathcal{L}\}$. We can compute the density of this discrete subset of $\mathbb{R}$:

$$\mathbf{d}(M\alpha_1\mathcal{L}) = \frac{\epsilon^{S-1}}{M\alpha_1}.$$

Let us define $\mathcal{A} := \{m\alpha_r; m \in \{1, 2, 3, \dots\}, r \in \{1, \dots, R\}\}$. Note that by assumption $\{u_p\}_{p \geq 1} \subset \mathcal{A}$. Fix $\epsilon_1 \in (0, 1)$ such that $\min_{r \in \{1, \dots, R\}} \alpha_r > 1 + 3\epsilon_1$. We can choose $\epsilon < \epsilon_1/C$ and then (3.22) gives for all $l' \in M\alpha_1\mathcal{L}$:

$$[l' + \epsilon_1, l' + 2\epsilon_1 + 1] \cap \mathcal{A} = \emptyset. \quad (3.23)$$

For a positive real number P , consider the integer $\bar{n}(P) := \#\{l' \in M\alpha_1\mathcal{L}; l' + 2\epsilon_1 \leq P\}$. Note that

$$\lim_{P \rightarrow \infty} \frac{\bar{n}(P)}{P} = \mathbf{d}(M\alpha_1\mathcal{L}). \quad (3.24)$$

From (3.23), (3.24) and the fact that the gap between any two consecutive u_p, u_{p+1} is at least 1, we deduce that for all $P \in \mathbb{R}$ large enough:

$$\begin{aligned} \#\{p; u_p \leq P\} &\leq P - \bar{n}(P)\epsilon_1 \\ &\leq P(1 - \mathbf{d}(M\alpha_1\mathcal{L})\frac{\epsilon_1}{2}). \end{aligned}$$

Consequently for $p \in \mathbb{N}$ large enough,

$$u_p \geq (p-1) \frac{1}{1 - \mathbf{d}(M\alpha_1\mathcal{L})\frac{\epsilon_1}{2}}$$

so the lemma is proved by taking for instance

$$\bar{\epsilon}_0 := \mathbf{d}(M\alpha_1\mathcal{L})\frac{\epsilon_1}{4} > 0.$$

□

Of course by rescaling, we can assume that $\mathcal{H}^n(\Sigma_1) = 1$. Let us apply the previous lemma to the volume spectrum $\{\omega_p\}_{p \geq 1}$ (see (3.20) and Theorem 11 (1)), and the n -volumes (resp. twice the n -volumes) of the two-sided (resp. one-sided) hypersurfaces Γ_i (see (3.19)). We obtain for a certain $\bar{\epsilon}_0 > 0$ and all p large enough:

$$\omega_p > (1 + \bar{\epsilon}_0)p.$$

But Theorem 11 (2) implies that $\lim_{p \rightarrow \infty} p^{-1}\omega_p = 1$ and in particular for p large enough,

$$\omega_p < (1 + \bar{\epsilon}_0)p.$$

These two inequalities give the desired contradiction.

Remark 3.3.1. *The arguments in this section actually yield a more precise version of Theorem 5: any compact manifold M^{n+1} ($2 \leq n \leq 6$) whose boundary has a contracting neighborhood in M contains infinitely many closed embedded minimal hypersurfaces.*

Chapter 4

A generalization of Yau’s conjecture for some non-compact manifolds

To what extent can Yau’s conjecture be generalized to non-compact ambient manifolds? By defining a notion of saddle point minimal hypersurfaces, we find a satisfying answer for a class of manifolds called “thick at infinity”. The proof builds on a result of Gromov and in fact also implies that Yau’s conjecture is true for finite volume hyperbolic 3-manifolds. Additionally, we extend a result of Irie-Marques-Neves.

4.1 Introduction

The search for minimal hypersurfaces in compact manifolds has enjoyed significant progress recently, thanks to the development of various min-max methods, such as the systematic extension of Almgren-Pitts’ min-max theory [125] led by Marques and Neves [101, 104, 102, 105], the Allen-Cahn approach [59, 51, 26], or others [129, 77, 34, 157, 127, 123]. One central motivation was the following conjecture of S.-T. Yau:

Yau’s conjecture [154]: In any closed three-dimensional manifold, there are in-

finitely many minimal surfaces.

Strong results implying the conjecture were obtained for generic metrics by Irie-Marques-Neves [75] (see [106] for a quantified version), Chodosh-Mantoulidis [26], X. Zhou [158], Y. Li [93]. Concurrently to these results, the conjecture for non-generic metrics was treated with a different line of arguments. When the manifold satisfies the “Frankel property”, it was solved by Marques-Neves in [104]. We recently settled the general case in [146], where we localized min-max constructions appearing in [104] to some compact manifold with stable minimal boundary by introducing a non-compact manifold with cylindrical ends.

On the other hand, results about minimal hypersurfaces in complete non-compact manifolds are comparatively few and far between, and most of them are existence results. We give here a non exhaustive list. In [38, 39], Collin-Hauswirth-Mazet-Rosenberg constructed one closed embedded minimal surfaces in any finite volume hyperbolic 3-manifold; there is also the work of Z. Huang and B. Wang [68], and of Coskunuzer [40]. In [113], Montezuma showed that a strictly mean concave compact domain in a complete manifold intersects a finite volume embedded minimal hypersurface. In [56], Gromov proved the following existence theorem that we interpret as the analogue of Almgren-Pitts existence result [125] for non-compact manifolds:

Gromov’s result [56]: In a complete non-compact manifold M , either there is an embedded finite volume complete minimal hypersurface, or there is a possibly singular strictly mean convex foliation of any compact domain of M .

In [21], Chambers and Liokumovich showed the existence of a finite volume embedded minimal hypersurface in finite volume complete manifolds; in fact they proved the existence of such a minimal hypersurface if there is a region whose boundary is, say, ten times smaller than its width. In asymptotically flat 3-manifolds, Chodosh and Ketover constructed minimal planes in [24], using a degree argument (see Mazet-Rosenberg [108] for generalizations).

The goal of this chapter is to propose a relevant generalization of the solution of

Yau’s conjecture to non-compact manifolds, by building on Gromov’s result. Motivations came from our solution of the conjecture when the Frankel property is not satisfied [146], where we perform min-max in a non-compact manifold with cylindrical ends. Besides some classes of manifolds naturally contain non-compact manifolds, for instance finite volume hyperbolic 3-manifolds. The non-compact situation substantially differs from the compact case: there are many non-compact manifolds without any closed (or finite volume) minimal hypersurfaces. For any integer $m > 0$, it is easy to construct a metric on say $\mathbb{S}^2 \times \mathbb{R}$ with exactly m closed minimal surfaces. That metric can look like a long tube which gets thinner around $\mathbb{S}^2 \times \{0\}$, and the minimal surfaces are $\mathbb{S}^2 \times \{0\}$ and some other slices $\mathbb{S}^2 \times \{t\}$ which are degenerate stable. At first sight, it seems hard to come up with essentially different examples of manifolds that would contain only finitely many closed minimal hypersurfaces. In our main result, we confirm this intuition for manifolds called “thick at infinity”, which we define next.

The class $\mathcal{T}_\infty$ of manifolds thick at infinity

Before stating our main theorem, we recall the notion of “thickness at infinity” introduced by Gromov [56]. Unless mentioned, we consider hypersurfaces without boundary.

Definition 4.1.1. *Let (X^{n+1}, g) be a complete $(n+1)$ -dimensional Riemannian manifold. (X, g) is said to be thick at infinity (in the weak sense) if any connected finite volume complete minimal hypersurface in (X, g) is closed. We denote by $\mathcal{T}_\infty$ the class of manifolds thick at infinity.*

In [56], Gromov actually uses a slightly stronger notion of thickness at infinity, since he asks that any connected finite volume minimal hypersurface with maybe non-empty compact boundary is compact.

The property of “thickness at infinity” is checkable. Instances are given in [56, Section 1.3], another more general example is the condition $\star_k$ in [113]. Special cases of the previous conditions include coverings of closed manifolds and asymptotically flat manifolds.

Note that M can be thick at infinity and at the same time “thin” in a certain sense. Indeed, using the monotonicity formula, it is easy to construct a warped product metric $g_t \oplus dt^2$ on a cylinder $N^n \times \mathbb{R}$ (where N is any closed n -dimensional manifold) such that

$(N^n \times \mathbb{R}, g_t \oplus dt^2)$ is thick at infinity, has finite volume, and the n -volume of the cross section $N \times \{t\}$ decreases to zero as $t \rightarrow \pm\infty$.

A zero-infinity dichotomy for manifolds thick at infinity

Almost by definition, closed minimal hypersurfaces are critical points of the n -volume functional. By the properties of the Jacobi operator, which encodes the second variation of the n -volume at a minimal hypersurface, the space of deformations that do not increase the area at second order is finite dimensional. It is natural to define *saddle points of the n -volume functional* (or simply *saddle point minimal hypersurfaces*) as follows. Consider a connected closed embedded minimal hypersurface Γ . If it is 2-sided then we call it a saddle point minimal hypersurface if there is a smooth family of hypersurfaces $\{\Gamma_t\}_{t \in (-\epsilon, \epsilon)}$ ($\epsilon > 0$) which are small graphical perturbations of $\Gamma = \Gamma_0$ so that $\{\Gamma_t\}_{t \in (-\epsilon, 0)}$ and $\{\Gamma_t\}_{t \in (0, \epsilon)}$ are on different sides of Γ and distinct from Γ , and

$$\text{Vol}_n(\Gamma) = \max_{t \in (-\epsilon, \epsilon)} \text{Vol}_n(\Gamma_t).$$

If Γ is 1-sided, we call it a saddle point minimal hypersurface if its connected double cover is a saddle point minimal hypersurface in a double cover of the ambient manifold. Note that if the metric is bumpy (i.e. no closed minimal hypersurface has a non-trivial Jacobi field), then saddle point minimal hypersurfaces are exactly unstable 2-sided closed minimal hypersurfaces and 1-sided closed minimal hypersurfaces with unstable double cover. By “compact domain”, we mean a compact $(n+1)$ -dimensional submanifold of M with smooth boundary.

Our main theorem is a dichotomy for the space of closed hypersurfaces embedded in a manifold thick at infinity. It says that either this space has infinite complexity from a Morse theoretic point of view, or its structure is locally simple.

Theorem 17. *Let (M, g) be an $(n+1)$ -dimensional complete manifold with $2 \leq n \leq 6$, thick at infinity. Then the following dichotomy holds true:*

1. *either (M, g) contains infinitely many saddle point minimal hypersurfaces,*
2. *or there is none; in that case for any compact domain B , there is an embedded closed area minimizing hypersurface Σ_B such that $B \setminus \Sigma_B$ has a singular weakly mean*

convex foliation.

We make the following comments, which will be developed in Sections 4.3, 4.4 and 4.5:

- In the second case, the minimal hypersurface Σ_B may be empty or disconnected. The foliation comes from the mean curvature flow so it has the corresponding regularity [151], and it is shrinking towards Σ_B .
- This theorem still holds if M has minimal boundary and if each component of ∂M is compact. Closed manifolds are trivially in $\mathcal{T}_\infty$. We will see in Section 4.3 that 1-parameter min-max produces a saddle point minimal hypersurface, thus Theorem 17 implies the existence of infinitely many saddle point minimal hypersurfaces. This fact does not follow directly from the solution of Yau's conjecture [104][146].
- Finite volume hyperbolic 3-manifolds do not belong to $\mathcal{T}_\infty$ in general. Nevertheless we will show in Section 4.5 that they satisfy Yau's conjecture since they contain infinitely many saddle point minimal hypersurfaces, extending the existence result of Collin-Hauswirth-Mazet-Rosenberg [38, 39].
- The situation for geodesics in surfaces is different: some 2-spheres contain only three simple closed geodesics, and immersed closed geodesics in hyperbolic surfaces are all strictly stable.

In the process of proving the Theorem 17, we will explain the following local version of Gromov's result for complete manifolds that are not necessarily in $\mathcal{T}_\infty$.

Theorem 18. *Let (M, g) be an $(n + 1)$ -dimensional complete manifold with $2 \leq n \leq 6$, and let B be a compact domain. Then*

1. *either M contains a complete embedded minimal hypersurface intersecting B and with finite n -volume,*
2. *or B has a singular strictly mean-convex foliation.*

This theorem is essentially already contained in [56], but we find it useful for the reader to present a detailed proof with some new arguments, for instance the use of Marques-Neves lower index bound [102] and the mean curvature flow. We think that

after some technical improvements, our proof should also cover higher dimensions (the minimal hypersurface then may have a codimension at least 7 singularity set).

Let us list a few corollaries (see Corollary 21 for more details). If M has finite volume, then a complete finite volume embedded minimal hypersurface exists (that was known since [21]). Moreover, if there is a compact subset $X \subset M$ whose boundary is mean concave in the sense that the mean curvature vector is pointing outside of X , then such a hypersurface also exists and intersects X . In particular this recovers a result of Montezuma [113].

A density result à la Irie-Marques-Neves

A complete $(n + 1)$ -dimensional manifold (M, g) is said to have a *thin foliation at infinity* if there is a proper Morse function $f : M \rightarrow [0, \infty)$ so that the n -volume of the level sets $f^{-1}(t)$ converges to zero as t goes to infinity. The relevant topology on the space of complete metrics on M is the strong (Whitney) C^∞ -topology. Let $\mathcal{F}_{thin}$ be the family of complete metrics on M with a thin foliation at infinity; it is an open subset for the strong topology. Similarly, the intersection $\mathcal{F}_{thin} \cap \text{Int}(\mathcal{T}_\infty)$ is a non-empty open subset for that topology (here $\text{Int}(\mathcal{T}_\infty)$ denotes the interior of $\mathcal{T}_\infty$ in the space of complete metrics). The following theorem generalizes the density theorem of Irie, Marques and Neves to these metrics:

Theorem 19. *Let M be an $(n + 1)$ -dimensional manifold with $2 \leq n \leq 6$.*

1. *For any metric g in a C^∞ -dense subset of $\mathcal{F}_{thin}$, the union of complete finite volume embedded minimal hypersurfaces in (M, g) is dense.*
2. *For any metric g' in a C^∞ -generic subset of $\mathcal{F}_{thin} \cap \text{Int}(\mathcal{T}_\infty)$, the union of closed embedded minimal hypersurfaces in (M, g') is dense.*

The proof borrows an idea of Irie, Marques and Neves in [75], where they use an elegant argument based on the Weyl law for the volume spectrum proved by Liokumovich, Marques and Neves [94]. Many non-compact manifolds of finite volume do not obey the Weyl law, even if all the min-max widths are finite (see Remark 4.6.1 for an informal justification). Thus, we have to find a more robust property of the min-max widths which in fact gives an alternative argument even in the compact case, not based on the

Weyl law. On the other hand, the Weyl law seems essential in the quantified result we obtained with Marques and Neves [106] about the generic equidistribution of a sequence of minimal hypersurfaces. Another remark is that we cannot prove the result for a C^∞ -generic subset of $\mathcal{F}_{thin}$, but only for a C^∞ -dense subset, because non-compact minimal hypersurfaces may appear and structural results like White’s bumpy metric theorems [149, 153] become false.

Organisation

In Section 4.2, we reprove a local version of Gromov’s result [56] and derive a few corollaries. After explaining how to construct saddle point minimal hypersurfaces with 1-parameter min-max in Section 4.3, we show the zero-infinity dichotomy for manifolds thick at infinity in Section 4.4. We also check that finite volume hyperbolic 3-manifolds satisfy Yau’s conjecture in Section 4.5, and extend in the last section the density result of Irie-Marques-Neves.

4.2 Local existence of finite volume minimal hypersurfaces

Let (M^{n+1}, g) be a complete possibly non-compact $(n+1)$ -dimensional Riemannian manifold. We will use in this section a local version of Almgren-Pitts theory; definitions and some relevant results were stated in Subsection 2.1.2 of Chapter 2. We will also need the mean curvature flow, in its level set flow formulation, whose basic properties are recalled in Appendix B. Strict and weak mean convexity in the sense of level set flow are also defined there. By “compact domain”, we mean a compact $(n+1)$ -dimensional submanifold of M with smooth boundary.

We say that B has a singular (strictly) mean convex foliation if there is a bigger compact domain B_0 containing B and endowed with a metric g' coinciding with g on B , so that there is a family of closed subsets of B_0 , $\{K_t\}_{t \in [0,1]}$, satisfying:

- $K_0 = B_0$, $K_1 \cap B = \emptyset$,
- ∂K_t has the regularity of the level set flow,

- ∂K_t is (strictly) mean convex for g' in the sense of mean curvature flow.

The main theorem of this section is essentially proved in [56]. We give here a detailed proof with new arguments, especially for the min-max part. Another more formal difference is the alternative use of level set flow instead of constructing foliations “by hand” (already suggested by B. Kleiner, see [56]). Here M can be compact or non-compact.

Theorem 20 (Local version of Gromov’s theorem). *Let (M, g) be an $(n+1)$ -dimensional complete manifold with $2 \leq n \leq 6$, and let B be a compact domain. Then*

1. *either M contains a complete embedded minimal hypersurface intersecting B and with finite n -volume,*
2. *or B has a singular strictly mean-convex foliation.*

That theorem immediately implies the following corollaries which were proved in [21] and [113] respectively (it seems that the relation with [56] was not in the literature).

Corollary 21. *Let M be as in the previous theorem.*

1. *If M has finite volume or more generally if there is an exhaustion $X_1 \subset \dots \subset X_i \subset \dots$ of M by compact subsets with smooth boundaries such that*

$$\lim_{i \rightarrow \infty} \text{Vol}_n(\partial X_i) = 0,$$

then there is a complete finite volume embedded minimal hypersurface.

2. *If M contains a compact subset X with mean concave smooth boundary (the mean curvature vector is non-zero pointing outwards), then there is a complete finite volume embedded minimal hypersurface intersecting X .*

We will say that a hypersurface S embedded in (N, g) is *locally* (resp. *globally*) *area minimizing* if S is a minimal hypersurface and if any hypersurface isotopic to S in a neighborhood of S in N (resp. any hypersurface in the same $\mathbb{Z}_2$ -homology class as $S \subset N$) has n -volume at least $\text{Vol}_n(S)$. Before giving the proof of Theorem 20 and Corollary 21, we need the following local min-max theorem for non-bumpy metrics. In our setting, the width W of a compact manifold (N, g) is defined in Subsection 2.1.2 of Chapter 2,

(2.3). Condition [M] and Type I, II, III stable minimal hypersurfaces are introduced in the discussion of Appendix A

Proposition 22. *Let (N^{n+1}, g) be a compact manifold with minimal boundary, with $2 \leq n \leq 6$, such that ∂N is locally area minimizing inside N . Then there is a closed embedded minimal hypersurface Γ inside the interior $\text{Int}(N)$ whose index is at most one and whose n -volume is bounded by the width W of (N, g) .*

Proof of Theorem 20. Fix $p \in M$. Let B be a compact domain which we can assume to be connected, and let D be a compact domain containing a geodesic ball $B_r(p)$ which itself contains the closure of B in M . We fix r temporarily but at the end of the argument we will make r go to ∞ . We modify the metric in a thin neighborhood of ∂D to make it mean convex (the mean curvature vector points inwards). Let $D' \subset D$ be a domain containing B with $C^{1,1}$ weakly mean convex boundary (the mean curvature is nonnegative), consider a closed minimal hypersurface S embedded inside $\text{Int}(D') \setminus B$ and let $\mathbf{D}$ be the metric completion of $D' \setminus S$. B is isometrically embedded inside $\mathbf{D}$.

We consider the following constrained minimization problem: minimize the n -volume of $\partial B'$ over open sets with rectifiable boundary B' containing B , such that there is a family of integral currents $\{b_t\}_{t \in [0,1]} \subset \mathbf{I}_{n+1}(\mathbf{D}; \mathbb{Z}_2)$ verifying:

- $\{\partial b_t\}_{t \in [0,1]} \subset \mathcal{Z}_n(\mathbf{D}; \mathbb{Z}_2)$ is continuous in the $\mathbf{F}$ -topology,
- $\text{spt}(\partial b_0) = \partial B$, $\text{spt}(\partial b_1) = \partial B'$,
- for all $t \in [0, 1]$, $B \subset \text{spt}(b_t) \subset \mathbf{D}$
- for all $t \in [0, 1]$, $\mathbf{M}(\partial b_t) \leq \text{Vol}_n(\partial B) + 1$.

This is a constrained Plateau problem since there is an n -volume constraint (which does not affect regularity) and a geometric constraint given by B , $\mathbf{D}$ (which does). A solution of this minimization problem exists by weak mean convexity of $\partial \mathbf{D}$, by compactness of cycles for the flat topology and by interpolation results [102, Proposition A.2]: it gives a $C^{1,1}$ -hypersurface Γ smooth except maybe at points touching ∂B and with n -volume at most $\text{Vol}_n(\partial B)$. Let B'' be the metric completion of the component of $\mathbf{D} \setminus \Gamma$ containing B .

Given $B \subset B''$ as above, two possibilities can occur:

- (i) either one component A of $\partial B''$ is strictly mean convex in the sense of mean curvature flow,
- (ii) or the boundary $\partial B''$ is a smooth minimal hypersurface.

In the first case, the strictly mean convex component A has to touch ∂B . We can run the mean curvature flow starting from A (see Appendix B) and get $\{A_t\}_{t \geq 0}$: either the level sets A_t are all strictly mean convex and sweep out the whole domain B (the conclusion of the theorem is then true) or it converges to a non empty stable closed embedded minimal hypersurface S intersecting B and having n -volume less than that of ∂B . In the latter situation occurs for every r and some $\mathbf{D}$, B'' , then by sending r to infinity (i.e. D covers larger and larger regions of M), we get subsequently a limiting complete embedded finite volume minimal hypersurface ([136]) which intersects B . This proves the theorem in the first case.

In the second case we can assume that $\partial B''$ does not touch B (otherwise the theorem is proved). It means we can suppose that $\partial B''$ is locally area minimizing inside B'' . If for any choice of $\mathbf{D}$ and B'' , this situation occurs then we can consider such a B'' of minimal volume, where the minimum is taken over all B'' obtained from the constrained minimization problem in a choice of $\mathbf{D}$ constructed as previously. Such a minimizer exists by compactness because of the n -volume bound on Γ and the stability of $\partial B''$ (see [136]). This manifold B'' is compact. Given such a minimizer B'' , we remove a maximal number of disjoint 1-sided minimal hypersurfaces $S_1, \dots, S_q$ and 2-sided non-separating minimal hypersurfaces $T_1, \dots, T_r$ contained in the interior of $B'' \setminus B$ so that

$$\text{Vol}_n(\partial B'') + \sum_{i=1}^q 2 \text{Vol}_n(S_i) + \sum_{i=1}^r \text{Vol}_n(T_i) \leq \text{Vol}_n(\partial B) + 1$$

and consider the metric completion $\mathbf{B}$ of $B'' \setminus (\bigcup_{i=1}^q S_i \cup \bigcup_{i=1}^r T_i)$. This manifold $\mathbf{B}$ is compact and has a non empty boundary. Besides, the original domain B is isometrically embedded in $\mathbf{B}$. We used similar ideas of considering a “core” in [146].

We can apply local min-max to $\mathbf{B}$: by Proposition 22, we get a closed connected embedded minimal hypersurface S inside the interior $\text{Int}(\mathbf{B})$, which has Morse index at most one. Moreover the width of $\mathbf{B}$ and thus the n -volume of S are bounded in terms of B only. To see this, it suffices to understand that one can deform ∂B inside $\mathbf{B}$ continuously in

the $\mathbf{F}$ -topology to $\partial\mathbf{B}$, such that along the deformation the n -volume of the hypersurfaces is say less than $\text{Vol}_n(\partial B) + 1$; this in turn follows from the volume minimality property of $\mathbf{B}$ and [102, Proposition A.2]. We wish to show that S intersects B (for any choice of radius r) because then the theorem is proved by taking a limit of such hypersurfaces S using [136], as $r \rightarrow \infty$. To argue towards a contradiction, assume that $S \subset \text{Int}(\mathbf{B})$ does not intersect B . Suppose first that S is 2-sided: then consider the metric completion of $\mathbf{B} \setminus S$. If S is 2-sided separating $\mathbf{B}$ into two components then it is clear that one can find a competitor $\hat{\mathbf{B}}$ to $\mathbf{B}$ contradicting the minimality of its volume. If S is 1-sided or 2-sided non-separating, then by removing S from $\mathbf{B}$ and taking the metric completion $\mathbf{B}'$, we could run again the minimization process described earlier to ∂B inside $\mathbf{B}'$. Either we find a minimizer not entirely contained in the boundary $\partial\mathbf{B}'$ or a minimizer is contained in $\partial\mathbf{B}'$ (then we have $\text{Vol}_n(\partial\mathbf{B}') \leq \text{Vol}_n(\partial B) + 1$). In any case, this would contradict either the minimality of the volume of $\mathbf{B}$ or the maximality of the number of boundary components of $\partial\mathbf{B}$, since $\partial\mathbf{B}'$ has more boundary components than $\partial\mathbf{B}$.

□

4.3 Local min-max and saddle point minimal hypersurfaces

Consider a complete manifold (M^{n+1}, g) . We define *saddle points of the n -volume functional* (or simply *saddle point minimal hypersurfaces*) as follows. Let Γ be a connected closed embedded minimal hypersurface. If it is 2-sided then we call it a saddle point if there is a smooth family of hypersurfaces $\{\Gamma_t\}_{t \in (-\epsilon, \epsilon)}$ ($\epsilon > 0$) which are small graphical perturbations of $\Gamma = \Gamma_0$ so that $\{\Gamma_t\}_{t \in (-\epsilon, 0)}$ and $\{\Gamma_t\}_{t \in (0, \epsilon)}$ are on different sides of Γ and distinct from Γ , and

$$\text{Vol}_n(\Gamma) = \max_{t \in (-\epsilon, \epsilon)} \text{Vol}_n(\Gamma_t).$$

If Γ is 1-sided, we call it a saddle point if its connected double cover is a saddle point in a double cover of the ambient manifold. Note that if the metric is bumpy (i.e. no closed minimal hypersurface has a non-trivial Jacobi field), then saddle point minimal hypersurfaces are exactly 2-sided unstable closed embedded minimal hypersurfaces and 1-sided closed embedded minimal hypersurfaces with unstable double cover.

In this section, we explain how to construct saddle points for general metrics from a localized 1-parameter min-max procedure. When the metric is bumpy, this was achieved by the index bounds of Marques-Neves in [102]. Most notations appeared in the first chapter.

4.3.1 Deformation Theorems

In this subsection, we consider a compact manifold (N, g) . Let $\{\Phi_i\}_{i \in \mathbb{N}}$ be a sequence of continuous maps from $[0, 1]$ into $\mathcal{Z}_n(N; \mathbf{F}; \mathbb{Z}_2)$. Set

$$L = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) := \limsup_{i \rightarrow \infty} \sup_{x \in [0, 1]} \mathbf{M}(\Phi_i(x))$$

and suppose that $\liminf_{i \rightarrow \infty} (L - \max_{j=0,1} \mathbf{M}(\Phi_i(j))) > 0$. To simplify the presentation, let us assume that each component of ∂N is minimal and either strictly stable or degenerate stable of Type II.

We first need a 1-parameter version of Deformation Theorem A of [102] for non-bumpy metrics. Let $\mathcal{S}(L)$ be the family of stationary integral varifolds in $\mathcal{V}_n(N)$ of total mass L with support a smooth closed embedded minimal hypersurface in N , which are 2-unstable (see [102, Definitions 4.1, 4.2]). The unstable components of $\text{spt}(V)$, where $V \in \mathcal{S}(L)$, are inside the interior of N by assumption on the boundary ∂N .

Theorem 23 (Deformation Theorem A, [102]). *Given $\{\Phi_i(x)\}_{i \in \mathbb{N}}$ and a compact set $K \subset \mathcal{V}_n(N)$ which is at positive $\mathbf{F}$ -distance of $\mathcal{S}(L) \cup |\Phi_i|([0, 1])$ for all i large, there exist another sequence $\{\Psi_i\}_{i \in \mathbb{N}}$ such that*

- (i) Ψ_i is homotopic to Φ_i with fixed endpoints in the $\mathbf{F}$ -topology for all $i \in \mathbb{N}$,
- (ii) $\mathbf{L}(\{\Psi_i\}_{i \in \mathbb{N}}) \leq L$,
- (iii) for any $\Sigma \in \mathcal{S}(L)$, there exists $\bar{\epsilon} > 0$, $j_0 \in \mathbb{N}$, so that for all $i \geq j_0$, $|\Psi_i|([0, 1]) \cap (\bar{\mathbf{B}}_{\bar{\epsilon}}^{\mathbf{F}}(\Sigma) \cup K) = \emptyset$.

We will not need Deformation Theorem B of [102]. However, Deformation Theorem C of [102] will be useful. Before stating it, consider a minimal hypersurface S which is degenerate stable of Type II. Then we associate to S some squeezing maps like the ones in [102, Proposition 5.7]. Suppose that S is 2-sided embedded in the interior of N ,

the other cases (1-sided or boundary component) are similar. A neighborhood of S is foliated by hypersurfaces with mean curvature vector pointing towards S when non-zero (Appendix A, Lemma 38). Let f be a real function defined on such a neighborhood with $\nabla f \neq 0$ such that $S(s) = f^{-1}(s)$ ($s \in [-1, 1]$) are the smooth embedded hypersurfaces of the foliation and $S(0) = S$ (in particular, $s\langle \nabla f, \vec{H}(S(s)) \rangle \leq 0$). Write $X = \nabla f / |\nabla f|^2$, and let $\phi : S \times [-1, 1] \rightarrow N$ such that $\frac{\partial \phi}{\partial s}(x, s) = X(\phi(x, s))$ and $\phi(S, s) = S(s)$ for all $s \in [-1, 1]$. Set $\Omega_r = \phi(S \times (-r, r))$ and define the maps

$$P_t : \Omega_1 \rightarrow \Omega_1 \text{ such that } P_t(\phi(x, s)) = \phi(x, s(1 - t)) \text{ for } t \in [0, 1].$$

Lemma 24. *Let S be as above. There exists $r_0 > 0$ such that*

1. P_t satisfies items (i), (ii), (iii) of [102, Proposition 5.7],
2. for all $V \in \mathcal{V}_n(\Omega_{r_0})$ and every connected component Ω of Ω_{r_0} , the function $t \mapsto \|(P_t)_\# V\|(\Omega)$ is a strictly decreasing function of t , unless $\text{spt}(V) \cap \Omega \subset S \cap \Omega$, in which case it is constant.

We observe that if S is 1-sided, or a boundary component of N , the previous discussion still applies to a neighborhood of S , on which one can define squeezing maps P_t .

Theorem 25 (Deformation Theorem C, [102]). *Suppose the sequence $\{\Phi_i\}$ is pulled-tight (every varifold in $\mathbf{\Lambda}(\{\Phi_i\})$ with $\|V\|(N) = L$ is stationary). Let $\{\Sigma^{(1)}, \dots, \Sigma^{(Q)}\} \subset \mathcal{V}_n(N)$ be a collection of stationary integral varifolds such that for every $1 \leq q \leq Q$:*

- the support of $(\Sigma^{(q)})$ for $1 \leq q \leq Q$ is a closed embedded minimal hypersurface $S^{(q)}$ whose components are either in the interior of N or a component of ∂N ,
- each component of $S^{(q)}$ (its double cover if not 2-sided) is strictly stable or degenerate stable of Type II,
- $L = \|\Sigma^{(q)}\|(N)$.

Then there exist $\xi > 0$, $j_0 \in \mathbb{N}$ so that for all $i \geq j_0$ one can find $\Psi_i : [0, 1] \rightarrow \mathcal{Z}_n(N; \mathbf{F}; \mathbb{Z}_2)$ such that

- (i) Ψ_i is homotopic to Φ_i with fixed endpoints in the flat topology,

$$(ii) \quad \mathbf{L}(\{\Psi_i\}_{i \in \mathbb{N}}) \leq L,$$

(iii)

$$\mathbf{A}(\{\Psi_i\}) \subset (\mathbf{A}(\{\Phi_i\}) \setminus \bigcup_{q=1}^Q \mathbf{B}_\xi^{\mathbf{F}}(\Sigma^{(q)})) \cup (\mathcal{V}_n(N) \setminus \mathbf{B}_\xi^{\mathbf{F}}(\Gamma)),$$

where Γ is the collection of all stationary integral varifolds $V \in \mathcal{V}_n(N)$ with $L = ||V||(N)$.

Finally we need a last deformation theorem in order to deal with degenerate stable minimal hypersurfaces of Type I. Let S be such a connected hypersurface (it is necessarily 2-sided and contained in the interior of N). Note that one can again define squeezing maps here. A neighborhood inside $\text{Int}(N)$ of S is foliated by hypersurfaces with mean curvature vector pointing towards a fixed direction when non-zero. Let f be a real function defined on such a neighborhood with $\nabla f \neq \emptyset$ such that $S(s) = f^{-1}(s)$ ($s \in [-1, 1]$) are the smooth embedded hypersurfaces of the foliation, verifying $S(0) = S$ and $\langle \nabla f, \vec{H}(S(s)) \rangle \leq 0$. Write $X = \nabla f / |\nabla f|^2$, and let $\phi : S \times [-1, 1] \rightarrow N$ such that $\frac{\partial \phi}{\partial s}(x, s) = X(\phi(x, s))$ and $\phi(S, s) = S(s)$ for all $s \in [-1, 1]$. Set $\Omega'_r = \phi(S \times [-r, r])$ and define the maps

$$P'_{r,t} : \Omega_r \rightarrow \Omega_r \text{ such that } P'_{r,t}(\phi(x, s)) = \phi(x, (s+r)(1-t) - r) \text{ for } t \in [0, 1].$$

The following lemma is proved as Lemma 24 and [102, Proposition 5.7].

Lemma 26. *Let S be as above. There exists $r_1 > 0$ such that $P'_{r_1,t} : \Omega'_{r_1} \rightarrow \Omega'_{r_1}$ satisfies:*

1. $P'_{r_1,0}(x) = x$ for all $x \in \Omega'_{r_1}$, $P'_{r_1,t}(y) = y$ for all $y \in S(-r_1)$, $t \in [0, 1]$,
2. $P'_{r_1,t}(\phi(S \times [-r_1, r])) \subset \phi(S \times [-r_1, r])$ for all $t \in [0, 1]$, $r \leq r_1$, and $P'_{r_1,1}(\Omega'_{r_1}) = S(-r_1)$,
3. the map $P'_{r_1,t} : \Omega'_{r_1} \rightarrow \Omega'_{r_1}$ is a diffeomorphism onto its image for $0 \leq t < 1$,
4. for all $V \in \mathcal{V}_n(\Omega'_{r_1})$, the function $t \mapsto |(P'_{r_1,t})_\# V|(\Omega'_{r_1})$ is a strictly decreasing function of t , unless $\text{spt}(V) \subset S(-r_1)$, in which case it is constant,
5. if V_0 is a stationary integral varifold with support S , then for all $\epsilon > 0$, there are $\kappa = \kappa(V_0, \epsilon) > 0$, $\epsilon' = \epsilon'(V_0, \epsilon) > 0$ and $t' = t'(V_0, \epsilon) \in (0, 1)$ so that for all

$V \in \mathcal{V}_n(\Omega'_{r_1}) \cap \mathbf{B}_\epsilon^\mathbf{F}(V_0)$ and for all $s \in [0, t']$,

$$(P'_{r_1, s})_\# V \in \mathbf{B}_\epsilon^\mathbf{F}(V_0) \text{ and}$$

$$|(P'_{r_1, t'})_\# V|(\Omega'_{r_1}) \leq |V_0|(\Omega'_{r_1}) - \kappa.$$

For the last deformation theorem, we will assume that

$[\star]_L$: any degenerate stable minimal hypersurface in N
of n -volume at most L is of Type I or II.

Let $\mathcal{T}(L)$ be the family of stationary integral varifolds in $\mathcal{V}_n(N)$ of total mass L whose support is a stable smooth closed embedded minimal hypersurface and for which at least one of the components is degenerate stable of Type I.

Remark 4.3.1. By [133] for any sequence $V_i \in \mathcal{T}(L)$, the supports $\text{spt}(V_i)$ converge sub-sequently smoothly to a stable minimal hypersurface S_0 . Assuming $[\star]_L$, the components of S_0 are either strictly stable, or degenerate stable of Type I, II. At least one of the components of S_0 is degenerate stable of Type I. Indeed, we can write $V_i = a_{1,i}|S_i^1| + \dots + a_{k_i,i}|S_i^{k_i}|$ where $S_i^1, \dots, S_i^{k_i}$ are disjoint minimal hypersurfaces, $k_i, a_{1,i}, \dots, a_{k_i,i}$ are bounded independently of i and we can suppose that these sequence of integers all stabilize to respectively $k, a_1, \dots, a_k$. Then $V_i = a_{1,i}|S_i^1| + \dots + a_{k_i,i}|S_i^{k_i}|$ converges to $V = a_1|S^1| + \dots + a_k|S^k|$ where S^l are the components of S_0 and are the smooth limit of S_i^l . But if S^l are all strictly stable or degenerate stable of Type II, [102, Proposition 5.7] and Lemma 24 would imply that $S_i^l = S^l$ for l large since the mass $|V_i|(N)$ is constant equal to L . That contradicts the fact that $V_i \in \mathcal{T}(L)$. As a consequence of this discussion, assuming $[\star]_L$, $\mathcal{T}(L)$ is a compact subset of $\mathcal{V}_n(N)$ and the number r_1 in Lemma 26 can be chosen independently of $V \in \mathcal{T}(L)$.

Here is the last deformation theorem.

Theorem 27 (Deformation Theorem D). *Assuming $[\star]_L$, given $\{\Phi_i\}_{i \in \mathbb{N}}$, there is another sequence $\{\Psi_i\}_{i \in \mathbb{N}}$ such that*

(i) $\{\Psi_i\}_{i \in \mathbb{N}}$ is homotopic to $\{\Phi_i\}_{i \in \mathbb{N}}$ with fixed endpoints in the flat topology for all

$$i \in \mathbb{N},$$

$$(ii) \quad \mathbf{L}(\{\Psi_i\}_{i \in \mathbb{N}}) \leq L,$$

(iii) *there exists $\hat{\epsilon} > 0$, $j_1 \in \mathbb{N}$ so that for all $i \geq j_1$,*

$$\inf\{\mathbf{F}(\Sigma, |\Psi_i|([0, 1])); \Sigma \in \mathcal{T}(L)\} \geq \hat{\epsilon}.$$

4.3.2 Existence of saddle point minimal hypersurfaces

Equipped with the previous deformation theorems, we prove the existence of saddle points in non-bumpy metrics. Remember that if a degenerate stable minimal hypersurface satisfies Condition [M] then it is of type I, II, or III (Appendix A). The following theorem is an extension of Proposition 22.

Theorem 28. *Let (N^{n+1}, g) be a compact manifold with boundary, with $2 \leq n \leq 6$, such that ∂N is locally area minimizing inside N . Then there is a saddle point minimal hypersurface Γ inside the interior $\text{Int}(N)$, whose index is at most one and whose n -volume is bounded by $W + 1$ where W is the width of (N, g) .*

4.4 Zero-infinity dichotomy for manifolds thick at infinity

4.4.1 Local version of Gromov's result for manifolds thick at infinity

Consider a complete manifold (M^{n+1}, g) . Recall that saddle point minimal hypersurfaces are closed embedded minimal hypersurfaces satisfying a natural saddle point condition. By “compact domain”, we mean a compact $(n + 1)$ -dimensional submanifold of M with smooth boundary.

Let B be a compact domain and $\Sigma \subset (M, g)$ be a closed embedded minimal hypersurface which may be empty. Suppose that Σ is locally area-minimizing. We say that $B \setminus \Sigma$ has a singular weakly mean convex foliation if there is a bigger compact domain B_0

containing $B \cup \Sigma$ and endowed with a metric g' coinciding with g on a neighborhood of $B \cup \Sigma$, so that there is a family of closed subsets of B_0 , $\{K_t\}_{t \in [0,1]}$, satisfying:

- $K_0 = B_0$, for all neighborhood N of Σ , $K_t \cap (B \setminus N) = \emptyset$ for t close enough to 1 and $\Sigma \subset K_t$ for all $t \in [0, 1)$,
- ∂K_t has the regularity of the level set flow,
- ∂K_t is mean convex in the sense of mean curvature flow, and the non strictly mean convex level sets ∂K_t are smoothly embedded closed minimal hypersurfaces in (B_0, g') ,

The following theorem is a more precise version of Theorem 20 for manifolds thick at infinity.

Theorem 29. *Let (M, g) be an $(n + 1)$ -dimensional complete manifold with $2 \leq n \leq 6$, thick at infinity, and let $B \subset M$ be a compact domain. Then*

1. *either M contains a saddle point minimal hypersurface intersecting B ,*
2. *or there is an embedded closed locally area minimizing hypersurface $\Sigma_B \subset (M, g)$ (maybe empty) such that $B \setminus \Sigma_B$ has a singular weakly mean convex foliation.*

Remark 4.4.1. *By inspecting the proof of Theorem 29, we can check that it also holds if (M, g) , thick at infinity, has a non-empty boundary and if the components of ∂M are closed minimal hypersurfaces: for any compact domain with smooth boundary $B \subset M$, the dichotomy of Theorem 29 holds true. Here some components of ∂M might be included in B .*

4.4.2 Min-max in a manifold generated by a saddle point minimal hypersurface with the level set flow

Let (M^{n+1}, g) be a complete manifold thick at infinity. We assume in this subsection that the metric g satisfies Condition [M].

We will often use $\{\Sigma_t\}_{t \in [0,1]}$ or similar notations for 1-sweepouts of a region R of M . Sometimes we will define Σ_t as a hypersurface, even though rigorously speaking, each Σ_t should be a current in $\mathcal{Z}_{n,rel}(R, \mathbb{Z}_2)$ (see [2, Definition 1.20], [94, 2.2]). For simplicity we will also denote by $\text{Vol}_n(\Sigma_t)$ its mass instead of using $\mathbf{M}(\Sigma_t)$.

Let Γ be a saddle point minimal hypersurface in (M, g) . We will treat the case where Γ is 2-sided for simplicity, but the case where it is 1-sided is completely analogous. Γ has to be either unstable, or degenerate stable of Type III, and so we can find a neighborhood N_Γ of Γ and a diffeomorphism $\phi : \Gamma \times (-\delta_1, \delta'_1) \rightarrow N_\Gamma$ such that $\phi(\Gamma \times \{0\}) = \Gamma$, the mean curvature of $\phi(\Gamma \times \{s\})$ is either vanishing or non-zero pointing away from Γ , and $\phi(\Gamma \times \{-\delta_1\})$, $\phi(\Gamma \times \{\delta'_1\})$ have non-zero mean curvature.

Let $K_0 := M \setminus N_\Gamma$. Let B_k be an exhausting sequence of compact domains with smooth boundary, containing N_Γ . We choose a metric g_k on B_k verifying:

1. ∂B_k is minimal strictly stable with respect to g_k ,
2. $\|g_k - g\|_{C^0} \leq \mu_k$, where the right-hand side is the C^0 distance between g and g_k on B_k , computed with g , and μ_k converges to zero as $k \rightarrow \infty$,
3. $g = g_k$ except in a $1/k$ -neighborhood of ∂B_k .

For each integer k , let $K_0^{(k)} := (B_k \cap K_0, g_k)$. It is a strictly mean convex subset of (B_k, g_k) . We run the level set flow to $K_0^{(k)}$ and get a family $\{K_t^{(k)}\}_{t \geq 0}$. Denote by X_k the metric completion of $B_k \setminus \bigcap_{t \geq 0} K_t^{(k)}$. X_k is a compact manifold with closed minimal stable boundary components. Each of them has an n -volume bound coming from monotonicity properties of the level set flow and is locally area minimizing inside X_k . Besides the level set flow $\partial K_t^{(k)}$ converges to ∂X_k smoothly (see Appendix B). Since interior points of X_k are identified with points of M , there is a natural map $X_k \rightarrow M$.

As $k \rightarrow \infty$, X_k subsequently converges in the Gromov-Hausdorff topology to a manifold X that is naturally endowed with the metric g . X is a (maybe non-compact) manifold with minimal stable boundary components. Their total n -volume is finite, each of them is compact because (M, g) is thick at infinity, and locally area minimizing inside X . By Condition [M], it means that each component of ∂X is either strictly stable or degenerate stable of Type II. We will refer to this construction by saying that X is *generated* by Γ , K_0 in (M, g) .

Let $\mathcal{C}(X_k)$ denote the result of gluing the compact manifold X_k to straight half-cylinders $(\partial X_k \times [0, \infty), g|_{\partial X} \oplus dt^2)$ along ∂X_k . The final metric (still denoted by g_k) is

Lipschitz continuous around ∂X_k in general. Let us define

$$\tilde{\omega}_p(X_k, g_k) := \omega_p(\mathcal{C}(X_k), g_k)$$

where the p -widths ω_p of a possibly non-compact manifold are defined in [146, Definition 8]. Note that if X_k has no boundary then $\tilde{\omega}_p(X_k, g) = \omega_p(X_k, g)$. It is clear that the first width of $(\mathcal{C}(X_k), g_k)$ is finite. Indeed let $\{\Sigma_t\}_{t \in [0,1]}$ be an arbitrary 1-sweepout of $X_k \setminus K_0^{(k)} = X \setminus K_0$ with $\Sigma_0 = 0$ and $\Sigma_1 = \partial K_0$, let $\{S_t^{(k)}\}_{t \in [0,\infty)}$ with $S_t^{(k)} := \partial X_k \times \{t\} \subset \mathcal{C}(X_k)$. Then the three families $\{\Sigma_t\}_{t \in [0,1]}$, $\{\partial K_t^{(k)}\}_{t \geq 0}$, $\{S_t^{(k)}\}_{t \geq 0}$ concatenated gives an explicit 1-sweepout of any bounded subset of $(\mathcal{C}(X_k), g_k)$ and

$$\begin{aligned} \omega_1(\mathcal{C}(X_k), g_k) &\leq \max\left\{ \sup_{t \in [0,1]} \text{Vol}_n(\Sigma_t), \sup_t \text{Vol}_n(\partial K_t^{(k)}), \sup_t \text{Vol}_n(S_t^{(k)}) \right\} \\ &\leq \max\left\{ \sup_{t \in [0,1]} \text{Vol}_n(\Sigma_t), \text{Vol}_n(\partial K_0) \right\} < \infty. \end{aligned}$$

In [104, proof of Theorem 5.1, Claim 5.6], it is explained that given a compact manifold Y , given a 1-sweepout $\Phi_1 : [0, 1] \rightarrow \mathcal{Z}_{n,rel}(Y, \mathbb{Z}_2)$ with $\sup_{t \in [0,1]} \mathbf{M}(\Phi_1(t)) \leq C_1$, one can construct p -sweepouts $\Phi_p : \mathbb{R}P^p \rightarrow \mathcal{Z}_{n,rel}(Y, \mathbb{Z}_2)$ for any p , satisfying $\sup_{t \in \mathbb{R}P^p} \mathbf{M}(\Phi_p(t)) \leq p.C_1$. In particular, we have the following general

Fact: if the first width is finite, then all the widths are finite and the p -width is bounded by p times the first width.

Using the Fact, it is simple to check that for each p , $\tilde{\omega}_p(X_k, g_k)$ is bounded between to positive constants independent of k . We choose a subsequence of B_k (that we do not rename) in a way that $\tilde{\omega}_p(X_k, g_k)$ converges for each p and we define

$$\tilde{\omega}_p(X, g) := \lim_{k \rightarrow \infty} \tilde{\omega}_p(X_k, g_k).$$

For each k we also set

$$\mathcal{A}(\Gamma, g_k) := \max\{\text{Vol}_n(\hat{C}); \hat{C} \text{ component of } \partial X_k\},$$

where the n -volume of $\hat{C}$ is computed with g_k . By the monotonicity property of the level set flow, the sequence $\mathcal{A}(\Gamma, g_k)$ is bounded. We can assume that this sequence converges (by taking a subsequence if necessary) and we define

$$\mathcal{A}(\Gamma, g) := \lim_{k \rightarrow \infty} \mathcal{A}(\Gamma, g_k).$$

In what follows, ∂X (resp. X , $\mathcal{A}(\Gamma, g)$) will roughly play the role of ∂U (resp. U , the n -volume of the largest boundary component of U) in [146, Section 2]. For simplicity, we will write $\mathcal{A}$, $\mathcal{A}_k$, $\tilde{\omega}_p(g)$, $\tilde{\omega}_p(g_k)$, for $\mathcal{A}(\Gamma, g)$, $\mathcal{A}(\Gamma, g_k)$, $\tilde{\omega}_p(X, g)$, $\tilde{\omega}_p(X_k, g_k)$ respectively.

By [146, Theorem 10], for any fixed p , there is a stationary integral varifold V_k with support a closed minimal hypersurfaces of Morse index at most p embedded inside the interior of (X_k, g_k) , each component intersecting $M \setminus K_0$ by the maximum principle, such that the total mass of V_k is $\tilde{\omega}_p(g_k)$. Multiplicities of the 1-sided components of $\text{spt}(V_k)$ are even. Making $k \rightarrow \infty$ and taking a subsequence, V_k converges in the varifold sense to a stationary integral varifold V_k with support a minimal hypersurface embedded inside the interior of X ([136]), and closed because (M, g) is thick at infinity. Moreover, since $g = g_k$ except very close to ∂B_k , V_k is stationary for the original metric g if k is large. In other words, we just proved the following

Proposition 30. *For all p there are disjoint connected closed embedded minimal hypersurface $\Gamma_i^{(p)} \subset (\text{Int}(X), g)$ ($1 \leq i \leq J$) and integers $m_{p,i}$ ($m_{p,i}$ is even when $\Gamma_i^{(p)}$ is 1-sided) satisfying*

$$\tilde{\omega}_p(X, g) = \sum_{i=1}^J m_{p,i} \text{Vol}_n(\Gamma_i^{(p)}).$$

We denote by $\text{Vol}_n(\cdot, g_k)$ the n -volume computed with g_k . If $\bar{D}$ is a compact domain of M , for all k it induces a compact subset $\bar{D}'$ in X_k by pulling back with the natural map $X_k \rightarrow M$, and since $X_k \subset \mathcal{C}(X_k)$, $\bar{D}'$ in turn induces a compact subset of $\mathcal{C}(X_k)$ that we call $\mathbf{i}_k(\bar{D})$. For a region $R \subset \mathcal{C}(X_k)$ with rectifiable boundary, we will denote by $\mathcal{Z}_{n, \text{rel}}(R; \mathbb{Z}_2)$ the space of relative cycles mod 2 in the closure of R (see [2, Definition 1.20], [94, 2.2]).

Before continuing, let us remark the following.

Lemma 31. *For all $\bar{\epsilon} > 0$, there is a compact domain $\bar{D} \subset M$ so that for any k*

large enough, the region $X_k \setminus i_k(\bar{D}) \subset \mathcal{C}(X_k)$ has a 1-sweepout $\{S_t\}_{t \in [0,1]}$ where $S_t \in \mathcal{Z}_{n,rel}(X_k \setminus i_k(\bar{D}); \mathbb{Z}_2)$ for each $t \in [0,1]$, satisfying

$$\sup_{t \in [0,1]} \{\text{Vol}_n(S_t, g_k)\} \leq \mathcal{A} + \bar{\epsilon}.$$

We want the analogue of [146, Theorem 9] for $\tilde{\omega}_p(g)$:

Proposition 32.

$$\forall p \geq 1 \quad \tilde{\omega}_{p+1}(g) - \tilde{\omega}_p(g) \geq \mathcal{A}, \quad \lim_{p \rightarrow \infty} \frac{\tilde{\omega}_p(g)}{p} = \mathcal{A}.$$

Finally we have:

Lemma 33. *Suppose that any closed embedded minimal hypersurface in X intersecting Γ is a saddle point minimal hypersurface. Then for all closed embedded minimal hypersurface $\Sigma \subset \text{Int}(X)$ intersecting Γ ,*

$$\text{Vol}_n(\Sigma) > \mathcal{A}(\Gamma, g) \text{ if } \Sigma \text{ is 2-sided,}$$

$$2 \text{Vol}_n(\Sigma) > \mathcal{A}(\Gamma, g) \text{ if } \Sigma \text{ is 1-sided.}$$

4.4.3 Zero-infinity dichotomy for the space of cycles in manifolds thick at infinity

We now state our main theorem, which can be thought of as an extension of Yau's conjecture to non-compact manifolds thick at infinity:

Theorem 34 (Zero-infinity dichotomy). *Let (M, g) be an $(n+1)$ -dimensional complete manifold with $2 \leq n \leq 6$, thick at infinity. Then the following dichotomy holds true:*

1. *either (M, g) contains infinitely many saddle point minimal hypersurfaces,*
2. *or there is none; in that case for any compact domain B , there is an embedded closed area minimizing hypersurface $\Sigma_B \subset (M, g)$ (maybe empty) such that $B \setminus \Sigma_B$ has a singular weakly mean convex foliation.*

By inspecting the proof of the above theorem, we can check that it also holds more generally for manifolds M^{n+1} ($2 \leq n \leq 6$) thick at infinity with minimal boundary, such that each component of ∂M is closed. In particular, if M is compact with minimal boundary, then the following dichotomy holds: either there are infinitely many saddle point minimal hypersurfaces in the interior of M , or there is a closed embedded area minimizing minimal hypersurface $\Sigma \subset M$ such that $M \setminus \Sigma$ has a singular weakly mean convex foliation.

As briefly mentioned in the introduction, there is an interpretation of the zero-infinity dichotomy in Morse theoretic terms. Let $\mathcal{Z}_n(M, \mathbb{Z}_2)$ be the space of integral cycles with bounded support in M , endowed with the flat topology. It is a space of generalized closed hypersurfaces and the mass functional $\mathbf{M}$ extends the notion of n -volume for smooth hypersurfaces. The dichotomy then says that either $\mathcal{Z}_n(M, \mathbb{Z}_2)$ contains infinitely many “non-trivial critical points” of $\mathbf{M}$ whose supports are smooth minimal hypersurfaces, or $\mathcal{Z}_n(M, \mathbb{Z}_2)$ is locally simple: the only “critical points” are supported on smooth stable minimal hypersurfaces and for each bounded region B of M , the set of elements of $\mathcal{Z}_n(M, \mathbb{Z}_2)$ with support inside B can be contracted to $\mathbf{M}$ -minimizing elements of $\mathcal{Z}_n(M, \mathbb{Z}_2)$ by a retraction flow which is $\mathbf{M}$ -nonincreasing and continuous in the flat topology.

The proof of Theorem 34 will follow from Theorem 29 and the following:

Theorem 35. *Let (M, g) be an $(n + 1)$ -dimensional complete manifold with $2 \leq n \leq 6$, thick at infinity. If there exists a saddle point minimal hypersurface, then there exists infinitely many.*

Proof. We suppose that M is thick at infinity. Unless specified, minimal hypersurfaces are closed embedded. We assume that the metric g satisfies Condition [M], otherwise there are already infinitely many saddle points (Appendix A).

Let Γ be a saddle point minimal hypersurface in (M, g) . We are now in the situation treated by Subsection 4.4.2. We will explain the case where Γ is 2-sided for simplicity, but the case where it is 1-sided is completely analogous. Γ has to be either unstable, or degenerate stable of Type III, and so we can find a neighborhood N_Γ of Γ and a diffeomorphism $\phi : \Gamma \times (-\delta_1, \delta'_1) \rightarrow N_\Gamma$ such that $\phi(\Gamma \times \{0\}) = \Gamma$, the mean curvature of $\phi(\Gamma \times \{s\})$ is either vanishing or non-zero pointing away from Γ , and $\phi(\Gamma \times \{-\delta_1\})$,

$\phi(\Gamma \times \{\delta'_1\})$ have non-zero mean curvature.

Let $K_0 := M \setminus N_\Gamma$. Let X be generated by Γ , K_0 in (M, g) as explained in Subsection 4.4.2. Recall that X is the limit of compact manifolds X_k .

Suppose first that there is a closed minimal hypersurface S embedded inside the interior of X , intersecting Γ , which is either strictly stable or degenerate stable of Type I or II. If S is not non-separating degenerate stable of Type I, let Y denote the metric completion of $X \setminus S$. Otherwise S is 2-sided, and if X' is the metric completion of $X \setminus S$, two boundary components T_1, T_2 of same n -volume come from S . One (say T_1) is locally area minimizing inside X' , the other component T_2 is not. We minimize the n -volume of a hypersurface in X' close to T_2 in its homology class and get a 2-sided locally area minimizing closed minimal hypersurface W that separates T_1 from T_2 : let Y be defined as the metric completion of the component of $X' \setminus W$ containing T_1 . In any case, Y has locally area minimizing boundary, whose components are closed. Let $\Gamma^* \subset Y$ denote the preimage of Γ under the natural map $Y \rightarrow M$. We apply Theorem 29 to a domain $B \subset Y$ containing Γ^* (see Remark 4.4.1), and we claim that only case (1) can happen: if case (2) was true, there would be a locally area minimizing minimal hypersurface Σ_B and a weakly mean convex foliation $\{A_t\}_{t \geq 0}$ of $B \setminus \Sigma_B$. Since $\partial B \cap \partial Y$ is locally area minimizing, there is a first time T when A_T touches Γ^* , which of course contradicts the maximum principle. Thus we deduce that there is a saddle point $\Gamma_1 \subset Y$, different from Γ . We can reapply this discussion to the new saddle point Γ_1 , get a manifold $X^{(1)}$ with compact locally area minimizing boundary, and if $X^{(1)}$ contains in its interior a closed minimal hypersurface S_1 intersecting Γ_1 , which is either strictly stable or degenerate stable of Type I or II, we get a new saddle point Γ_2 different from Γ and Γ_1 . If at each step, $X^{(j)}$ contains in its interior a minimal hypersurface either strictly stable or degenerate stable of Type I or II, then we produce an infinite sequence of distinct saddle points Γ_j , and the conclusion of the theorem is verified.

For these reasons, we now only need to assume that if X, Γ are as above, minimal hypersurfaces embedded in the interior of X intersecting Γ are either unstable or degenerate

stable of Type III, that is, they are all saddle points. Note that this condition implies the

Frankel property inside X for minimal hypersurfaces *intersecting* Γ :

$$\text{any two closed embedded minimal hypersurfaces in } \text{Int}(X) \quad (4.1)$$

intersecting Γ have to intersect each other.

To check this, first notice by the maximum principle applied to (X_k, g_k) for k large, that any minimal hypersurface in $\text{Int}(X)$ disjoint from Γ is of the form $\phi(\Gamma \times \{s\})$ for some $s \in (-\delta_1, \delta'_1) \setminus \{0\}$, so it is degenerate stable of Type I. Secondly, if there were two disjoint saddle point minimal hypersurfaces S_1, S_2 in $\text{Int}(X)$ (intersecting Γ), by a minimization argument we would get a minimal hypersurface S in $\text{Int}(X)$ which is

- either strictly stable or degenerate stable of Type II, and in these cases S intersects Γ , contradicting our assumption on X ,
- or degenerate stable of Type I, in that case S has to be S_1 or S_2 , a contradiction.

Before continuing, we explain why all connected components of the minimal hypersurfaces we will construct from now on will always intersect Γ . If Γ is unstable then we could have chosen ϕ so that all the hypersurfaces $\phi(\Gamma \times \{s\})$ with $s \neq 0$ are non-minimal with mean curvature vector pointing away from Γ . In that case by the maximum principle any minimal hypersurface in $(\text{Int}(X), g)$ intersects Γ so we are done. However if Γ is degenerate stable of Type III, it might happen that some $\phi(\Gamma \times \{s\})$ distinct from Γ are also minimal. By Lemma 39 there is a sequence of metrics $h^{(q)}$ converging to g so that Γ is still a saddle point and any $\phi(\Gamma \times \{s\})$ with $s \neq 0$ is non-minimal. Hence any minimal hypersurface in $(\text{Int}(X), h^{(q)})$ intersects Γ by the maximum principle. Applying min-max theory to $(X, h^{(q)})$ will produce for each integer p a minimal hypersurface with integer multiplicities in $\text{Int}(X)$, intersecting Γ , of Morse index and n -volume bounded independently of q . By [136], we take a subsequence limit as $q \rightarrow \infty$ and get a minimal hypersurface with similar properties in $(\text{Int}(X), g)$: in particular each component intersects Γ . Thus by (4.1), the usual Frankel property, even though it may not be satisfied for all minimal hypersurfaces in $\text{Int}(X)$, will be satisfied for all the minimal hypersurfaces we will consider until the end of this proof (since they are constructed by min-max). In what follows, we will implicitly assume the use of such a limiting procedure involving $h^{(q)}$.

For the end of the proof, the strategy to produce infinitely many saddle points is to use arguments of the solution of Yau's conjecture [104] [146]. In the simplest case where X is actually closed compact then by the Frankel property in X satisfied by minimal hypersurfaces intersecting Γ , [104] implies the existence of infinitely many minimal hypersurfaces in X which are saddle point minimal hypersurfaces by assumption on X .

In Subsection 4.4.2, we defined some numbers $\tilde{\omega}_p(X, g)$ ($p \geq 1$) and $\mathcal{A}(\Gamma, g)$. Suppose that X is non-compact without boundary and $\mathcal{A}(\Gamma, g) = 0$. By Proposition 30 and by (4.1), the widths $\tilde{\omega}_p(X, g)$ are all finite, for each p there is a connected closed embedded minimal hypersurface $\Gamma^{(p)} \subset (\text{Int}(X), g)$ and a positive integer m_p (which is even when $\Gamma^{(p)}$ is 1-sided) satisfying

$$\tilde{\omega}_p(X, g) = m_p \text{Vol}_n(\Gamma^{(p)}). \quad (4.2)$$

Moreover we have the following asymptotics (Proposition 32):

$$\lim_{p \rightarrow \infty} \tilde{\omega}_p(X, g)/p = 0.$$

We claim that this implies the existence of infinitely many saddle point minimal hypersurfaces in $\text{Int}(X)$. Suppose by contradiction that there are only finitely many minimal hypersurfaces $\Sigma_1, \dots, \Sigma_L \subset \text{Int}(X)$ intersecting Γ . Then $\tilde{\omega}_p(X, g)$ is a strictly increasing sequence. To explain this, consider the compact manifolds (X_k, g_k) defined in Subsection 4.4.2; by thickness at infinity, for any p and for k large the min-max minimal hypersurface in (X_k, g_k) whose n -volume with multiplicity is $\tilde{\omega}_p(X_k, g_k)$ is actually a minimal hypersurface for the original metric g (see paragraph right before Proposition 30). Consequently we can find a subsequence $\{X_{k_i}\}$ of $\{X_k\}$, so that both sequences $\{\tilde{\omega}_p(X_{k_i}, g_{k_i})\}$ and $\{\tilde{\omega}_{p+1}(X_{k_i}, g_{k_i})\}$ stabilize to a constant, i.e.

$$\forall i \geq 0 \quad \tilde{\omega}_p(X_{k_i}, g_{k_i}) = \tilde{\omega}_p(X, g), \quad \tilde{\omega}_{p+1}(X_{k_i}, g_{k_i}) = \tilde{\omega}_{p+1}(X, g).$$

So if $\tilde{\omega}_p(X, g) = \tilde{\omega}_{p+1}(X, g)$, then in particular $\tilde{\omega}_p(X_{k_1}, g_{k_1}) = \tilde{\omega}_{p+1}(X_{k_1}, g_{k_1})$ but this is not possible by [146, Theorem 9 (1)] and since $\partial X_k \neq \emptyset$ (X is non-compact). The counting argument of [104, Section 7] is then enough to get the existence of infinitely many minimal hypersurfaces (contradicting our assumption that there were only finitely

many). The theorem is then proved in that case.

Suppose finally that X is non-compact (with empty or non-empty boundary) and $\mathcal{A}(\Gamma, g) > 0$. We appeal to the method of [146] as follows. In Subsection 4.4.2, we saw that $\tilde{\omega}_p(X, g)$ are finite numbers, and by Proposition 30 combined with (4.1) for all p there is a connected closed embedded minimal hypersurface $\Gamma^{(p)} \subset \text{Int}(X)$ and a positive integer m_p (which is even when $\Gamma^{(p)}$ is 1-sided) satisfying

$$\tilde{\omega}_p(X, g) = m_p \text{Vol}_n(\Gamma^{(p)}).$$

Furthermore, by Proposition 32, for all p

$$\tilde{\omega}_{p+1}(X, g) - \tilde{\omega}_p(X, g) \geq \mathcal{A}(\Gamma, g),$$

$$\lim_{p \rightarrow \infty} \tilde{\omega}_p(X, g)/p = \mathcal{A}(\Gamma, g).$$

Additionally, by Lemma 33 for any closed embedded minimal hypersurface $\Sigma \subset \text{Int}(X)$,

$$\text{Vol}_n(\Sigma) > \mathcal{A}(\Gamma, g) \text{ if } \Sigma \text{ is 2-sided,}$$

$$2 \text{Vol}_n(\Sigma) > \mathcal{A}(\Gamma, g) \text{ if } \Sigma \text{ is 1-sided.}$$

Consequently the previous identities and estimates combined with the arithmetic lemma [146, Lemma 14] imply that there are infinitely many closed embedded minimal hypersurfaces inside $\text{Int}(X)$ intersecting Γ , which we recall have to be saddle points by assumption on X . This finishes the proof of the theorem.

□

Proof of Theorem 34. Theorem 34 readily ensues from Theorem 29 and Theorem 35. The only non trivial point is that in item (2), for any compact domain B , the minimal hypersurface Σ_B is globally area minimizing in its $\mathbb{Z}_2$ -homology class in (M, g) (instead of just locally area minimizing as in item (2) of Theorem 29). Assume that there are no saddle points, let B be a bounded domain and Σ_B the associated locally area minimizing minimal hypersurface given by Theorem 29 (2). Suppose that there is another closed embedded hypersurface Γ with $\text{Vol}_n(\Gamma) \leq \text{Vol}_n(\Sigma_B)$ and such that $\partial D = \Gamma \cup \Sigma_B$ for

some compact domain D . Let X be a compact domain containing D , by Theorem 29 (2), there are a locally area minimizing hypersurface $\Sigma_X \subset (M, g)$, a bigger region $Y \supset X$ with a metric g_Y coinciding with g on X , and a collection of weakly mean convex closed sets $\{K_t\}_{t \geq 0}$ such that K_t foliates $X \setminus \Sigma_X$. We can minimize the n -volume of Γ inside K_0 (which is mean convex) and obtain a locally area minimizing minimal hypersurface $\Gamma' \subset (K_0, g_Y)$, of n -volume at most $\text{Vol}_n(\Sigma_B)$. The non strictly mean convex K_t have smooth minimal boundary ∂K_t which have to be degenerate stable of Type I. Hence $\Sigma_B \subset \Sigma_X$ and $\Gamma' \subset \Sigma_X$. But since these two hypersurfaces are $\mathbb{Z}_2$ -homologous inside K_0 , $\Sigma_B = \Gamma'$. Moreover a posteriori, we now know that Γ was actually already locally area minimizing, hence $\Gamma \subset \Sigma_X$ and so again $\Gamma = \Sigma_B$. This proves that $\Sigma_B \subset (M, g)$ is the unique area minimizer in its $\mathbb{Z}_2$ -homology class inside (M, g) .

□

4.5 Yau's conjecture for finite volume hyperbolic 3-manifolds

Combining methods from previous sections and [38, 39], we prove that Yau's conjecture holds true for finite volume hyperbolic 3-manifolds.

Theorem 36. *In any finite volume hyperbolic 3-manifold M , there are infinitely many saddle point minimal surfaces. In particular, there are infinitely many closed embedded minimal surfaces.*

4.6 Density of the union of finite volume minimal hypersurfaces

In our setting the natural topology on the space of complete metrics is the usual strong C^∞ -topology, or “Whitney C^∞ -topology” (see [52, Chapter II, §3]). It can be described as follows. Let $b_1, b_2, \dots$ be a sequence of open balls forming a locally finite covering M , and for each complete metric g , and i, k, ϵ , set

$$O(g, i, k, \epsilon) := \{g'; \|(g' - g)|_{b_i}\|_{C^k} \leq \epsilon\}.$$

Now consider g be a complete metric, $\mathbf{e} = (\epsilon_1, \epsilon_2, \dots)$ a sequence of positive numbers, $\mathbf{k} = (k_1, k_2, \dots)$ a sequence of integers and set

$$O(g, \mathbf{e}, \mathbf{k}) := \bigcap_{i=1}^{\infty} O(g, i, k_i, \epsilon_i).$$

Then by definition the strong C^∞ topology on the space of complete metrics of M has a basis of open sets given by $\{O(g, \mathbf{e}, \mathbf{k})\}_{g, \mathbf{e}, \mathbf{k}}$. The weak C^∞ -topology where a basis of open sets is given by $\{O(g, i, k, \epsilon)\}_{g, i, k, \epsilon}$ is less relevant for us, because it is “easier” to change the behavior at infinity and to be generic in this topology. The space of complete metrics endowed with the strong topology is a Baire space.

As usual, a closed minimal hypersurface is said to be non-degenerate when it has no non-trivial Jacobi fields.

Recall from the introduction that $\mathcal{F}_{thin}$ (resp. $\mathcal{T}_\infty$) is the family of complete metrics on M with a thin foliation at infinity (resp. thick at infinity). $\mathcal{F}_{thin}$ and $\mathcal{F}_{thin} \cap \text{Int}(\mathcal{T}_\infty)$ are non-empty open subsets for the strong topology. For instance, consider the following example already mentioned in the introduction: if N is a closed manifold, there is a metric h on $N \times \mathbb{R}$ with a thin foliation at infinity while also satisfying condition $\star_k$ of [113], hence being thick at infinity. Since $\star_k$ is an open condition, a neighborhood of h is in $\mathcal{F}_{thin} \cap \text{Int}(\mathcal{T}_\infty)$. Note that if M is compact, any metric on M is in $\mathcal{F}_{thin} \cap \text{Int}(\mathcal{T}_\infty)$. The following theorem generalizes the density theorem of Irie, Marques and Neves [75] to these manifolds:

Theorem 37. *Let M be a complete $(n+1)$ -dimensional manifold with $2 \leq n \leq 6$.*

1. *For any metric g in a C^∞ -dense subset of $\mathcal{F}_{thin}$, the union of complete finite volume embedded minimal hypersurfaces in (M, g) is dense.*
2. *For any metric g' in a C^∞ -generic subset of $\mathcal{F}_{thin} \cap \text{Int}(\mathcal{T}_\infty)$, the union of closed embedded minimal hypersurfaces in (M, g') is dense.*

We end this section with a slightly non rigorous construction of non-compact manifolds of finite volume which do not obey the Weyl law [94], which justify our use of a new argument in the proof of Theorem 37 compared to [75].

Remark 4.6.1. By [119], for any $A, \epsilon > 0$ there is a metric on the 3-sphere S^3 of volume 1 such that any surface separating S^3 into two regions of volume more than ϵ has area greater than A . From this, one should be able to construct a sequence of spheres $S_k := (S^3, g_k)$ of volume less than $\frac{1}{k^2}$ with first width in the sense of Almgren-Pitts equal to 1. Then by forming the infinite connected sum of S_i with thin necks, one gets a non-compact manifold of finite volume but with widths ω_p growing linearly, by a Lusternik-Schnirelmann type argument. In this construction, there is a lot a freedom in the rate of growth of ω_p by playing with the sizes of S_k . In particular, such a finite volume manifold generally does not satisfy a Weyl law.

4.7 Appendix A: Degenerate stable minimal hypersurfaces

We collect some simple facts about the structure of neighborhoods of minimal hypersurfaces.

Let (N, g) be a compact Riemannian manifold. In this Appendix, minimal hypersurfaces are smooth closed embedded. We say that a 2-sided minimal hypersurface is *degenerate* if its Jacobi operator has a non-trivial kernel. If such a hypersurface is degenerate and stable, then the kernel of its Jacobi operator is spanned by a positive eigenfunction. Note that for a 2-sided minimal hypersurface which is either unstable or non-degenerate stable, it is well-known that the hypersurface has a neighborhood foliated by closed leaves which, when not equal to the minimal hypersurface itself, have non-zero mean curvature vector. A similar result is true for degenerate stable minimal hypersurfaces, as we noted in [146, Lemma 11].

Lemma 38. *Let Γ be a 2-sided degenerate stable minimal hypersurface in the interior of (N, g) and ν a choice of unit normal vector on Γ . Then there exist a positive number δ_1 and a smooth map $w : \Gamma \times (-\delta_1, \delta_1) \rightarrow \mathbb{R}$ with the following properties:*

1. *for each $x \in \Gamma$, we have $w(x, 0) = 0$ and $\phi_0 := \frac{\partial}{\partial t} w(x, t)|_{t=0}$ is a positive function in the kernel of the Jacobi operator of Γ ,*
2. *for each $t \in (-\delta_1, \delta_1)$, we have $\int_{\Gamma} (w(\cdot, t) - t\phi_0)\phi_0 = 0$,*

3. for each $t \in (-\delta_1, \delta_1)$, the mean curvature of the hypersurface

$$\Gamma_t := \{\exp(x, w(x, t)\nu(x)); x \in \Gamma\}$$

is either positive or negative or identically zero,

4. if Γ_t is minimal for a $t \in (-\delta_1, \delta_1)$, its Morse index is at most one.

If the minimal hypersurface Γ is 1-sided, one can still apply the previous lemma in a double-cover of N where Γ lifts to a 2-sided hypersurface. If Γ is a boundary component of N then the lemma is still valid on the interior side of Γ .

Discussion: Note that Lemma 38 implies the following for a 2-sided degenerate stable minimal hypersurface Γ (the situation is completely similar for 1-sided minimal hypersurfaces): by the first variation formula if Γ is degenerate stable, the n -volume of the hypersurfaces Γ_t is a smooth function

$$A : (-\delta_1, \delta_1) \rightarrow \mathbb{R}$$

so that the sign of $\partial_t A$ is the sign of the mean curvature of Γ_t (after the correct continuous choice of unit normal). This function A associated to Γ is defined for 2-sided Γ embedded inside $\text{Int}(N)$; if Γ is 1-sided, then we call A the analogue function associated to the double 2-sided cover and if Γ is a boundary component of N , then A is only defined on $[0, \delta_1)$. If the function A is not strictly monotonous on any intervals of the form $(-\delta_2, 0)$ or $(0, \delta_2)$ where $\delta_2 \leq \delta_1$, then there are clearly an infinite sequence of $t_k \in (0, \delta_1)$ converging to 0 so that for each k , A restricted to an open interval containing t_k achieves a maximum at t_k . In other words, the minimal hypersurfaces Γ_{t_k} are saddle point minimal hypersurfaces (as defined in the introduction). If on the contrary the function A is strictly monotonous on both $(-\delta_2, 0)$ and $(0, \delta_2)$, then we have three possibilities:

- Type I: A is monotonous on $(-\delta_2, \delta_2)$,
- Type II: A achieves a strict minimum at 0,
- Type III: A achieves a strict maximum at 0 (in this last case Γ is a saddle point).

Because we are interested in constructing saddle point minimal hypersurfaces, we introduce the following condition: we say that

a degenerate stable minimal hypersurface Γ
(resp. the metric g) satisfies Condition [M]

if, with the previous notations, the function A associated to Γ (resp. to any degenerate stable minimal hypersurface) is strictly monotonous on both $(-\delta_2, 0)$ and $(0, \delta_2)$ for $\delta_2 > 0$ small enough. By what we just said, if a degenerate stable minimal hypersurface does not satisfy Condition [M] then there are infinitely many saddle point minimal hypersurfaces. On the other hand if a degenerate stable minimal hypersurface satisfies Condition [M], then it is either of Type I, II or III. Again this classification clearly extend to 1-sided hypersurfaces or boundary components, by convention a degenerate stable minimal hypersurface of Type I is 2-sided embedded in the interior of (N, g) .

Let Γ be stable degenerate of Type II or III. The next lemma enables to construct approximations of g for which the hypersurfaces Γ_t are non-minimal if $t \neq 0$.

Lemma 39. *Let (N, g) be a compact manifold with boundary and let Γ be a disjoint union of degenerate stable minimal hypersurfaces of Type II or III, embedded either in $\text{Int}(N)$ or in ∂N . Then there is a sequence of metrics $h^{(q)}$ converging to g in the C^∞ -topology so that Γ is still minimal for $h^{(q)}$ and with the previous notations, for each component Γ' of Γ :*

1. *if Γ' is of Type II, with respect to $h^{(q)}$, Γ' is strictly stable, moreover for all $t \in (-\delta_2, 0) \cup (0, \delta_2)$ the mean curvature vector of Γ'_t is never zero and points towards Γ' ,*
2. *if Γ' is of Type III, with respect to $h^{(q)}$, Γ' is unstable, moreover for all $t \in (-\delta_2, 0) \cup (0, \delta_2)$, the mean curvature vector of Γ'_t is never zero and points away from Γ' .*

4.8 Appendix B: Facts about mean curvature flow

We find it convenient to use the level set flow formulation of the mean curvature flow. Under the level set flow introduced by Chen-Giga-Goto [23] and Evans-Spruck [47], any

closed set K in $\mathbb{R}^{n+1}$ generates a one-parameter family of closed sets $\{K_t\}_{t \geq 0}$ with $K_0 = K$. The closed set K is said to be weakly (resp. strictly) mean convex if $K_t \subset K$ (resp. $K_t \subsetneq K$) for all $t > 0$. When $X := \partial K$ happens to be smooth, then K is mean convex if and only if the mean curvature of X is nonnegative.

When K is mean convex, the level set flow is given by the level sets of a Lipschitz function u satisfying in the viscosity sense

$$-1 = |\nabla u| \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right),$$

and $\partial K_t = \{x \in \mathbb{R}^{n+1}; u(x) = t\}$. In that case, there is a unique Lipschitz function u giving a viscosity solution of the level set flow [47, Theorem 7.4], [23] (see also [32]). The flow is non-fattening (the level sets have no interior) [151, Corollary 3.3]. The singular set at each time $t > 0$ has Hausdorff dimension at most $n - 1$, in particular at all positive time t , ∂K_t is a closed hypersurface smooth outside a subset of dimension $n - 1$ [151] (see also [33]). Moreover for all $T > 0$, $\{\partial K_t\}_{t \in [0, T]}$ form a possibly singular mean convex foliation of $K \setminus K_T$ [151]. By monotonicity, the n -dimensional Hausdorff measure (that we usually denote by Vol_n) of the set ∂K_t is nonincreasing in t [151, Corollary 3.6].

The previous facts extend naturally to closed ambient manifolds (N^{n+1}, g) [71, 151]. The long term behavior can be different: in general a closed mean convex non-minimal n -dimensional hypersurface $X \subset N$ evolving under the level set flow will sweep out an $(n + 1)$ -dimensional manifold with boundary $Y \subset N$, and $\partial Y = X \cup X_\infty$, where X_∞ is a closed embedded stable minimal hypersurface. If $2 \leq n \leq 6$, X_∞ is smooth, the convergence of X_t to X_∞ is smooth and one-sheeted (resp. two-sheeted) for 2-sided (resp. 1-sided) components of X_∞ [151, Section 11]. By the avoidance principle (or maximum principle, see [71, Lemma 6.3]), if K is a mean convex closed set generating $\{K_t\}_{t \geq 0}$ then any strictly mean concave closed set $W \subset K$ satisfies

$$W \subset \operatorname{Int}(\cap_{t \geq 0} K_t).$$

Chapter 5

Generic equidistribution of minimal hypersurfaces

A few years ago, Irie-Marques-Neves proved that generically, the union of minimal hypersurfaces is dense in a closed manifold. In this chapter, which is joint work with F. C. Marques and A. Neves, we wish to quantify that result and prove a generic equidistribution property for minimal hypersurfaces.

5.1 Introduction

In 1982, S. T. Yau ([154]) conjectured that every closed Riemannian three-manifold contains infinitely many smooth, closed, immersed minimal surfaces. In [75], Irie-Marques-Neves settled Yau's conjecture in the generic case by proving a much stronger property holds true:

Theorem (Irie, Marques, Neves, 2017): *Let M^{n+1} be a closed manifold of dimension $(n+1)$, with $3 \leq (n+1) \leq 7$. Then for a C^∞ -generic Riemannian metric g on M , the union of all closed, smooth, embedded minimal hypersurfaces is dense.*

In this chapter, we use the methods of [75] in a more quantitative way and prove an even stronger property: there is a sequence of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M .

Main Theorem: *Let M^{n+1} be a closed manifold of dimension $n+1$, with $3 \leq (n+1) \leq 7$. Then for a C^∞ -generic Riemannian metric g on M , there exists a sequence $\{\Sigma_j\}_{j \in \mathbb{N}}$ of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M : for any $f \in C^\infty(M)$ we have*

$$\lim_{q \rightarrow \infty} \frac{1}{\sum_{j=1}^q \text{vol}_g(\Sigma_j)} \sum_{j=1}^q \int_{\Sigma_j} f d\Sigma_j = \frac{1}{\text{vol}_g M} \int_M f dM. \quad (5.1)$$

Even more, for any symmetric $(0, 2)$ -tensor h on M we have:

$$\lim_{q \rightarrow \infty} \frac{1}{\sum_{j=1}^q \text{vol}_g(\Sigma_j)} \sum_{j=1}^q \int_{\Sigma_j} \text{Tr}_{\Sigma_j}(h) d\Sigma_j = \frac{1}{\text{vol}_g M} \int_M \frac{n \text{Tr}_M h}{n+1} dM. \quad (5.2)$$

Equidistribution theorems have an old history in fields like number theory, ergodic theory and harmonic analysis. Equidistribution of closed geodesics is known in some cases, like for compact hyperbolic manifolds (Bowen' 72 [15] or Margulis [98], see also [155]). Equidistribution of totally geodesic surfaces is a well-studied problem for hyperbolic 3-manifolds ([46], [109], [115], [126], [135]). Our theorem is the first of its kind for the higher-dimensional setting of minimal surfaces in general manifolds.

Remark: Yau's Conjecture has been fully resolved in [146], by localizing the methods initially developed by the first two authors in [104], and proved that any compact (M^{n+1}, g) , $3 \leq (n+1) \leq 7$, contains infinitely many smooth, embedded, closed minimal hypersurfaces. It would be interesting to know whether density and equidistribution of minimal hypersurfaces hold for all Riemannian metrics.

As in the Irie-Marques-Neves paper ([75]), the crucial tool in our proof is the Weyl law for the Volume Spectrum conjectured by Gromov ([55],) and recently proven by the first two authors jointly with Liokumovich in [94]:

Weyl Law for the Volume Spectrum (Liokumovich, Marques, Neves, 2016): *There exists a universal constant $a(n) > 0$ such that for any compact Riemannian manifold (M^{n+1}, g) we have:*

$$\lim_{p \rightarrow \infty} \omega_p(M, g) p^{-\frac{1}{n+1}} = a(n) \text{vol}(M, g)^{\frac{n}{n+1}}.$$

The volume spectrum of a compact Riemannian manifold (M^{n+1}, g) is a nondecreasing sequence of numbers $\{\omega_p(M, g) : p \in \mathbb{N}\}$ defined variationally by performing a min-max procedure for the area functional over multiparameter sweepouts. The first estimates for these numbers were proven in fundamental papers by Gromov in the late 1980s [54] and by Guth [61] more recently.

Our proof also uses a transversality argument, based on the Structure Theorem of White ([149], Theorem 2.1), that allows one to compute the derivative of the p -width as the derivative of the area of some minimal hypersurface. We combine this information with appropriately chosen N -parameter deformations of the metric, for N large, that generalize the one-parameter deformations of [74] and [75]. A key idea in the paper (that can be deduced from Lemma (41)) is that metrics which are critical points of the functional $g \mapsto \omega_p(M, g)p^{-\frac{1}{n+1}} - a(n)\text{vol}(M, g)^{\frac{n}{n+1}}$ when restricted to the N -parameter family of deformations have minimal hypersurfaces that obey some form of equidistribution. The fact that this functional is only Lipschitz continuous and thus not differentiable everywhere is a serious technical issue that the authors had to overcome.

We note that Property (5.1) is equivalent to saying that

$$\frac{\sum_{j=1}^q \mu_{\Sigma_j}}{\sum_{j=1}^q \mu_{\Sigma_j}(M)} \rightarrow \frac{\mu}{\mu(M)}$$

as measures, where $\mu_{\Sigma_j} = ||\Sigma_j||$ is the Radon measure $\mu_{\Sigma_j}(U) = \text{vol}_g(\Sigma_j \cap U)$, $U \subset M$, and $\mu = dv_g$ is the Riemannian volume measure of (M, g) . Property (5.1) follows from Property (5.2) by choosing $h = f \cdot g$.

The dimensional restriction in the Main Theorem is due to the fact that in higher dimensions min-max (even area-minimizing) minimal hypersurfaces can have singular sets. We use Almgren-Pitts theory ([3], [125]), which together with SchoenSimon regularity ([133]) produces smooth minimal hypersurfaces when $3 \leq (n+1) \leq 7$. We expect that the methods of this chapter can be generalized to handle the higher-dimensional singular case.

The Main Theorem raises many interesting and exciting new questions: namely whether equidistribution holds in the Grassmanian bundle, whether the minimal hypersurfaces realizing the width are the ones that become equidistributed, or whether there

are conditions which ensure that the sequence of unit measures $\mu_{\Sigma_j}/\mu_{\Sigma_j}(M)$ converges to the normalized volume measure. Any progress related with these directions would be highly desirable.

5.2 Preliminaries

We suppose that M is a closed manifold of dimension $3 \leq (n+1) \leq 7$. For each $2 \leq q \leq \infty$, we denote by Γ_q the space of all C^q Riemannian metrics on M , endowed with the C^q topology. Given $g \in \Gamma_q$, we let $\mathcal{V}(g)$ be the set of stationary integral varifolds in (M, g) whose support is a closed, C^2 , embedded, minimal hypersurface. Hence $V \in \mathcal{V}(g)$ if and only if there exist a disjoint collection $\{\Sigma_1, \dots, \Sigma_s\}$ of closed, C^2 , embedded, connected minimal hypersurfaces in (M, g) and integers $\{m_1, \dots, m_s\} \subset \mathbb{N}$ such that $V = m_1 \Sigma_1 + \dots + m_s \Sigma_s$. By elliptic regularity, each Σ_i is in fact of class C^q . The support of V is denoted by $\text{spt}(V)$ and is equal to $\cup_{i=1}^s \Sigma_i$, while $\|V\|$ denotes the Radon measure induced by V on M .

We denote by $\mathcal{Z}_n(M; \mathbb{Z}_2)$ the space of modulo two n -dimensional flat chains T in M with $T = \partial U$ for some $(n+1)$ -dimensional modulo two flat chain U in M , endowed with the flat topology. This space is weakly homotopically equivalent to $\mathbb{RP}^\infty$ (see Section 4 of [103]). We denote by $\bar{\lambda}$ the generator of $H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$. The mass (n -dimensional volume) of T is denoted by $M(T)$.

Let X be a finite dimensional simplicial complex. A continuous map $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is called a *p-sweepout* if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2).$$

We say X is *p-admissible* if there exists a *p-sweepout* $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ that has no concentration of mass, meaning

$$\limsup_{r \rightarrow 0} \{M(\Phi(x) \cap B_r(p)) : x \in X, p \in M\} = 0.$$

The set of all *p-sweepouts* Φ that have no concentration of mass is denoted by $\mathcal{P}_p$. Note that two maps in $\mathcal{P}_p$ can have different domains.

In [104], the first two authors defined

Definition: The p -width of (M, g) is the number

$$\omega_p(M, g) = \inf_{\Phi \in \mathcal{P}_p} \sup\{M(\Phi(x)) : x \in \text{dmn}(\Phi)\},$$

where $\text{dmn}(\Phi)$ is the domain of Φ .

The next lemma gives that the normalized p -width $p^{-\frac{1}{(n+1)}}\omega_p(M, g)$ is a Lipschitz function of the metric on sets of uniformly equivalent metrics, with a Lipschitz constant that does not depend on p .

Lemma 40. *Let $\tilde{g}$ be a C^2 Riemannian metric on M , and let $C_1 < C_2$ be positive constants. Then there exists $K = K(\tilde{g}, C_1, C_2) > 0$ such that*

$$|p^{-\frac{1}{(n+1)}}\omega_p(M, g) - p^{-\frac{1}{(n+1)}}\omega_p(M, g')| \leq K \cdot |g - g'|_{\tilde{g}}$$

for any $g, g' \in \{h \in \Gamma_2; C_1\tilde{g} \leq h \leq C_2\tilde{g}\}$ and any $p \in \mathbb{N}$.

The next lemma concerns the differentiability properties of the p -width restricted to a generic finite-dimensional family of metrics. Let $I^N = [0, 1]^N$.

Lemma 41. *Let $g : I^N \rightarrow \Gamma_q$ be a smooth embedding, $N \in \mathbb{N}$. If $q \geq N + 3$, then there exists an arbitrarily small perturbation in the C^∞ topology $g' : I^N \rightarrow \Gamma_q$ of g such that there is a subset $\mathcal{A} \subset I^N$ of full N -dimensional Lebesgue measure with the following property: for any $p \in \mathbb{N}$ and any point t of $\mathcal{A}$, the function $s \mapsto \omega_p(g'(s))$ is differentiable at t and there exists a disjoint collection $\{\Sigma_1, \dots, \Sigma_Q\}$ of closed, C^q , embedded, minimal hypersurfaces of $(M, g'(t))$ together with integers $\{m_1, \dots, m_Q\} \subset \mathbb{N}$ so that*

$$\omega_p(g'(t)) = \sum_{j=1}^Q m_j \text{vol}_{g'(t)}(\Sigma_j), \quad \sum_{j=1}^Q \text{index}(\Sigma_j) \leq p,$$

and

$$\begin{aligned} \frac{\partial}{\partial v}(\omega_p \circ g')|_{s=t} &= \frac{\partial}{\partial v} \left(\sum_{j=1}^Q m_j \text{vol}_{g'(s)}(\Sigma_j) \right) \Big|_{s=t} \\ &= \sum_{j=1}^Q m_j \int_{\Sigma_j} \frac{1}{2} \text{Tr}_{\Sigma_j, g'(t)} \left(\frac{\partial g'}{\partial v} \Big|_{s=t} \right) d\Sigma_j \end{aligned}$$

for every $v \in \mathbb{R}^N$.

Proof. Let $g : I^N \rightarrow \Gamma_q$ be a smooth embedding. Consider a sequence $\{S_i\}_i$ that enumerates all the diffeomorphism types of n -dimensional closed manifolds, and let $\mathfrak{M}(S_i)$ be the Banach manifold of pairs $(\gamma, [u])$ as in the Structure Theorem of White [149] (Theorem 2.1), where γ is a C^q Riemannian metric and $u : S_i \rightarrow M$ is a $C^{2,\alpha}$ embedding that is minimal with respect to γ . Define $\mathfrak{M} := \bigcup_i \mathfrak{M}(S_i)$ and the projection $\Pi : \mathfrak{M} \rightarrow \Gamma_q$ which sends $(\gamma, [u])$ to γ . Theorem 2.1 of [149] (see also [153]) gives that $\mathfrak{M}$ is a separable C^{q-2} Banach manifold and that Π is a C^{q-2} Fredholm map with Fredholm index zero. The pair $(\gamma, [u])$ is a critical point of Π if and only if u admits a nontrivial Jacobi field with respect to the metric γ .

We can perturb $g : I^N \rightarrow \Gamma_q$ slightly in the C^∞ topology to a C^∞ embedding $g' : I^N \rightarrow \Gamma_q$ that is transversal to $\Pi : \mathfrak{M} \rightarrow \Gamma_q$ by Smale's Transversality Theorem (Theorem 3.1 of [139]). Transversality implies $\tilde{I}^N = \Pi^{-1}(g'(I^N))$ is an N -dimensional submanifold of $\mathfrak{M}$ (Theorem 3.3 of [139]). Let $\pi = (g')^{-1} \circ \Pi|_{\tilde{I}^N}$, so $\pi : \tilde{I}^N \rightarrow I^N$. Let $\mathcal{A}'$ be the subset of points $t' \in I^N$ which are regular values of π and such that the Lipschitz function $t \mapsto \omega_p(t) := \omega_p(M, g'(t))$ is differentiable at t' for all p . This subset is of full Lebesgue measure in I^N by Rademacher's Theorem and Sard's Theorem. Note that if $t' \in \mathcal{A}'$, then $g'(t')$ is a regular value of Π .

Consider

$$\begin{aligned} \mathcal{S}'_{\kappa,c}(p) := & \left\{ t \in I^N; \exists V \in \mathcal{V}(g'(t)), \|V\|_{g'(t)}(M) = \omega_p(t), \right. \\ & \text{index}(\text{spt}(V)) \leq p, \max_{\text{spt}(V)} |A| \leq \kappa, \\ & \left. \text{and } V \text{ satisfies } (\star)_{c,\kappa,\sup_{s \in I^N} \omega_p(s)} \right\} \end{aligned}$$

for any $\kappa > 0$ and $c > 0$, where

$(\star)_{c,\kappa,a}$: every two-sided, connected component of $\text{spt } V$, has varifold distance (in the metric $g'(0)$) at least c of any varifold 2Σ , where Σ is a one-sided, embedded, connected minimal hypersurface in $g'(s)$ with $|A| \leq \kappa$ and $\text{vol}_{g'(s)}(\Sigma) \leq a$ for some $s \in I^N$.

Each $\mathcal{S}'_{\kappa,c}(p)$ is a closed set, by convergence properties of minimal hypersurfaces. Proposition 2.2 of [75] (which uses the index estimates of [102]) also holds for C^q metrics

if we allow the minimal hypersurfaces to be C^2 . This follows, for instance, by approximating the C^q metric by C^∞ metrics, applying Proposition 2.2 of [75] to these metrics and using Sharp's Compactness Theorem ([136]). It implies

$$\bigcup_{\kappa, c \in \mathbb{Q}_+} \mathcal{S}'_{\kappa, c}(p) = I^N$$

for every $p \in \mathbb{N}$.

For $\kappa > 0$ and $c > 0$, define $\mathcal{S}_{\kappa, c}(p)$ to be the set of points where the Lebesgue density of $\mathcal{S}'_{\kappa, c}(p)$ is one. By the Lebesgue density theorem, $\mathcal{S}'_{\kappa, c}(p) \setminus \mathcal{S}_{\kappa, c}(p)$ has measure zero. Finally, define the full Lebesgue measure set

$$\mathcal{A} := \mathcal{A}' \cap \bigcap_p \bigcup_{\kappa, c} \mathcal{S}_{\kappa, c}(p).$$

Fix $p \in \mathbb{N}$, and let $t \in \mathcal{A}$. There exist $\kappa > 0$ and $c > 0$ such that $t \in \mathcal{S}_{\kappa, c}(p)$. Since the Lebesgue density of $\mathcal{S}'_{\kappa, c}(p)$ at t is one, we have that for any unit direction v , there is a sequence $\{t_m(v)\}_m \subset \mathcal{S}'_{\kappa, c}(p)$ converging to t with $\frac{t_m(v) - t}{|t_m(v) - t|}$ converging to v , so that

$$\lim_{m \rightarrow \infty} \frac{\omega_p(t_m(v)) - \omega_p(t)}{|t_m(v) - t|} = \frac{\partial}{\partial v} \omega_p(t). \quad (5.3)$$

Fix v and a corresponding sequence $\{t_m(v)\}_m$. By construction, for each m there is a $V_m \in \mathcal{V}(g'(t_m(v)))$ with mass $\omega_p(t_m(v))$, with $\text{index}(\text{spt}(V_m)) \leq p$, whose support has second fundamental form bounded by κ (which is independent of m) and such that every two-sided, connected component of $\text{spt } V_m$ has varifold distance (in the metric $g'(0)$) at least c (also independent of m) from any varifold 2Σ , where Σ is a one-sided, connected minimal hypersurface in $g'(s)$ with $|A| \leq \kappa$ and $\text{vol}_{g'(s)}(\Sigma) \leq \sup_{s \in I^N} \omega_p(s)$ for some $s \in I^N$. This implies no two-sided component of $\text{spt } V_m$ can collapse, after maybe passing to a subsequence, to a one-sided component with multiplicity two. Choosing a subsequence and renumbering if necessary, V_m converges to a varifold $V \in \mathcal{V}(g'(t))$ and the supports $\text{spt}(V_m)$ converge in C^2 to $\text{spt}(V)$. This convergence is with multiplicity one, because if not one could construct by a standard argument a nontrivial Jacobi field on one of the components of $\text{spt}(V)$. This is not possible, since $g'(t)$ is a regular value of Π .

Consider a sequence $\{\Sigma_m\}$ of connected components of $\text{spt}(V_m)$ that converges in C^2 to Σ . By elliptic regularity, the convergence is also in $C^{2,\alpha}$. The corresponding points

$$\tilde{z}_m = (g'(t_m(v)), [\Sigma_m]) \in \tilde{I}^N \subset \mathfrak{M}$$

converge to a point $z \in \Pi^{-1}(g'(t)) \subset \tilde{I}^N$, $z = (g'(t), [\Sigma])$. Note that since $t \in \mathcal{A}$, π is a local diffeomorphism from a neighborhood of z in $\tilde{I}^N$ to a neighborhood of t in I^N . We write $\tilde{z} = (g'(\pi(\tilde{z})), [\Sigma(\pi(\tilde{z}))])$ for any $\tilde{z}$ in this neighborhood of z . For sufficiently large m , $[\Sigma_m] = [\Sigma(t_m(v))]$. But then

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{\text{vol}_{g'(t_m(v))}(\Sigma(t_m(v))) - \text{vol}_{g'(t)}(\Sigma(t))}{|t_m(v) - t|} &= \frac{\partial}{\partial v} \text{vol}_{g'(s)}(\Sigma(s))|_{s=t} \\ &= \frac{1}{2} \int_{\Sigma} \text{Tr}_{\Sigma, g'(t)} \left(\frac{\partial g'}{\partial v}(t) \right) d\Sigma. \end{aligned}$$

Taking into account the multiplicity of each connected component of $\text{spt}(V_m)$, the limit in (5.3) becomes

$$\frac{\partial}{\partial v} \omega_p(t) = \int_V \frac{1}{2} \text{Tr}_{V, g'(t)} \left(\frac{\partial g'}{\partial v}|_{s=t} \right) d\|V\|(M),$$

where V is of the form $\sum_{i=1}^Q m_i \tilde{\Sigma}_i$, with $\{\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_Q\}$ a disjoint collection of closed, $C^{2,\alpha}$, embedded, minimal hypersurfaces in $(M, g'(t))$ and $\{m_1, \dots, m_Q\} \subset \mathbb{N}$, $\|V\|(M) = \omega_p(t)$, $\sum_{i=1}^Q \text{index}(\tilde{\Sigma}_i) \leq p$, $\max_{\text{spt}(V)} |A| \leq \kappa$ and V satisfies $(\star_{\kappa, c, \sup_{s \in I^N} \omega_p(s)})$. By elliptic regularity, each $\tilde{\Sigma}_i$ is of class C^q . Since t is a regular value of π , every embedded minimal hypersurface of $(M, g'(t))$ is non-degenerate. Because convergence of the supports can only happen with multiplicity one, there are only finitely many V 's as above, say $\{V^{(1)}, \dots, V^{(P)}\}$. For any unit direction $v \in \mathbb{R}^N$, one has

$$\frac{\partial}{\partial v} \omega_p(t) = \int_{V^{(l)}} \frac{1}{2} \text{Tr}_{V^{(l)}, g'(t)} \left(\frac{\partial g'}{\partial v}|_{s=t} \right) d\|V^{(l)}\|(M)$$

for some $1 \leq l \leq P$. This means that there will be a single $1 \leq l \leq P$ such that the above formula is true for a linearly independent set $\{v_1, \dots, v_N\}$, and hence for every v by linearity. This finishes the proof. $\square$

The next lemma concerns the gradient of Lipschitz functions that are almost constant.

The convex hull of a set $K \subset \mathbb{R}^N$ is denoted by $\text{Conv}(K)$.

Lemma 42. *Given $\delta > 0$ and $N \in \mathbb{N}$, there exists $\varepsilon > 0$ depending on δ and N such that the following is true: for any Lipschitz function $f : I^N \rightarrow \mathbb{R}$ satisfying*

$$|f(x) - f(y)| \leq 2\varepsilon$$

for every $x, y \in I^N$, and for any subset $\mathcal{A}$ of I^N of full measure, there exist $N + 1$ sequences of points $\{y_{1,m}\}_m, \dots, \{y_{N+1,m}\}_m$ contained in $\mathcal{A}$ and converging to a common limit $y \in (0, 1)^N$ such that:

- *f is differentiable at each $y_{i,m}$,*
- *the gradients $\nabla f(y_{i,m})$ converge to $N + 1$ vectors $v_1, \dots, v_{N+1}$ with*

$$d_{\mathbb{R}^N}(0, \text{Conv}(v_1, \dots, v_{N+1})) < \delta,$$

The last lemma shows that one can make finitely many closed, embedded, minimal hypersurfaces nondegenerate by an arbitrarily small conformal change of the metric. This generalizes Proposition 2.3 of [75].

Lemma 43. *Suppose $g \in \Gamma_q$, $q \geq 2$. Let $\{\Sigma_1, \dots, \Sigma_L\}$ be a finite collection of closed, embedded, connected, C^2 minimal hypersurfaces in (M, g) . Then there exists a sequence of metrics $g_i \in \Gamma_q$, $i \in \mathbb{N}$, converging to g in the C^q topology such that Σ_j is a nondegenerate minimal hypersurface in (M, g_i) for all $j = 1, \dots, L$ and $i \in \mathbb{N}$.*

5.3 Proof of the Main Theorem

Let g be a smooth Riemannian metric on M , K be an integer and $\epsilon_1 > 0$ be a positive constant smaller than the injectivity radius of g . Let $\hat{B}_1, \dots, \hat{B}_K$ be disjoint domains in M , with piecewise smooth boundary, such that the union of their closures covers M .

Let B_k be some neighborhood of $\hat{B}_k$. We suppose that each B_k is contained in a geodesic ball of radius ϵ_1 . Choose also a smooth function $0 \leq \phi_k \leq 1$ that is equal to 1 on $\hat{B}_k$ and with $\text{spt}(\phi_k) \subset B_k$, and a point $q_k \in \hat{B}_k$ for each k . We can also suppose

that $q_k \notin B_l$ if $l \neq k$. Define the partition of unity $\psi_k = \frac{\phi_k}{\sum_q \phi_q}$. Hence $\psi_k(q_k) = 1$ and $\psi_k(q_l) = 0$ for $l \neq k$.

For a fixed k , let e be a unit vector in the tangent space of M at q_k . It determines by parallel transport along geodesics starting at q_k a unit vector field in B_k still denoted by e . We define a nonnegative symmetric $(0, 2)$ -tensor $h(e)$ on B_k as follows: $h(e)(v, w) = \langle v, e \rangle_g \langle w, e \rangle_g$.

Now consider the space $\mathcal{B}_k$ of orthonormal bases at q_k ; these $\mathcal{B}_k$ are endowed with a natural metric determined by g and of course are isometric to each other. For each k , pick L points $x_1^k, \dots, x_L^k \in \mathcal{B}_k$ such that any point in $\mathcal{B}_k$ is at distance less than ϵ_1 to one of the x_l^k . Each x_l^k is an orthonormal basis $(x_{l,1}^k, \dots, x_{l,n+1}^k)$ at q_k and so we can consider the family of symmetric $(0, 2)$ -tensors $h_{l,j}^k = h(x_{l,j}^k)$. Note that by construction, in B_k , for any l the sum $\sum_{j=1}^{n+1} h_{l,j}^k$ is the metric g .

We denote by $\mathcal{C}_{g, \tilde{K}, \epsilon_1}$ the set of all possible choices

$$(K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\})$$

as above, with $K \geq \tilde{K}$. The set $\mathcal{C}_{g, \tilde{K}, \epsilon_1}$ is non-empty, as can be seen by taking a sufficiently fine triangulation of M .

Recall that $\mathcal{V}(g)$ denotes the set of stationary integral varifolds in (M, g) whose support is an embedded minimal hypersurface. We claim that in order to show the main theorem, it suffices to prove the following property.

(P): For any metric g , for every $\epsilon_1 > 0$, $\tilde{K} > 0$ and any choice of

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g, \tilde{K}, \epsilon_1},$$

there is a metric $\tilde{g}$ arbitrarily close to g in the C^∞ topology such that there are varifolds $V_1, \dots, V_J$ of $\mathcal{V}(\tilde{g})$ whose support $\text{spt}(V_j)$ are nondegenerate, and coefficients $\alpha_1, \dots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$ satisfying

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}_{\tilde{g}}(M)} \int_M \psi_k dv_{\tilde{g}} \right| < \epsilon_1/K, \quad (5.4)$$

where the terms of the sum are computed for the metric $\tilde{g}$. Here

$$V(h) = \int_{G_n(M)} h(\nu, \nu) dV(p, \pi),$$

where $G_n(M)$ denotes the Grassmannian of n -dimensional planes of M and ν is a unit normal to the n -plane $\pi \subset T_p M$.

Indeed, let us explain why Property **(P)** implies the main theorem. We denote by $\mathcal{M}(g, \epsilon_1, \tilde{K}, S)$, with

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g, \tilde{K}, \epsilon_1},$$

the family of metrics $\tilde{g} \in \Gamma_\infty$ at distance less than ϵ_1/K to g (computed with respect to g) in the C^K topology such that there are $\{V_1, \dots, V_J\} \subset \mathcal{V}(\tilde{g})$ whose supports are nondegenerate, $\alpha_1, \dots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$, which satisfy (5.4) for all k, l, j . If $g' \in \mathcal{M}(g, \epsilon_1, \tilde{K}, S)$, and $\{\Sigma'_1, \dots, \Sigma'_Q\}$ is any finite collection of nondegenerate minimal hypersurfaces in (M, g') , then for every metric $\tilde{g}$ that is sufficiently close to g' , there is a unique collection $\{\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_Q\}$ of nondegenerate minimal hypersurfaces in $(M, \tilde{g})$ such that $\tilde{\Sigma}_i$ is close to Σ'_i , $i = 1, \dots, Q$. Moreover, $\tilde{\Sigma}_i$ converges smoothly to Σ'_i as $\tilde{g}$ converges to g' . This implies that $\mathcal{M}(g, \epsilon_1, \tilde{K}, S)$ is open in the C^∞ topology.

Define

$$\mathcal{M}(\epsilon_1, \tilde{K}) := \bigcup_{g \in \Gamma_\infty} \bigcup_{S \in \mathcal{C}_{g, \tilde{K}, \epsilon_1}} \mathcal{M}(g, \epsilon_1, \tilde{K}, S).$$

It is clearly open. Given an arbitrary metric $g \in \Gamma_\infty$, we can choose $S \in \mathcal{C}_{g, \tilde{K}, \epsilon_1}$. Property **(P)** implies that the metric g is a limit of metrics in $\mathcal{M}(g, \epsilon_1, \tilde{K}, S)$. This shows that $\mathcal{M}(\epsilon_1, \tilde{K})$ is also dense.

Define

$$\mathcal{M} := \bigcap_{m \in \mathbb{N}} \mathcal{M}(1/m, m).$$

Since each $\mathcal{M}(1/m, m)$ is open and dense, the intersection $\mathcal{M}$ is a residual subset (in the Baire sense) of the set of metrics. We will show that for any metric in $\mathcal{M}$, one can find sequences of minimal hypersurfaces like in the Main Theorem. For any metric, a symmetric $(0, 2)$ -tensor h is diagonalizable at every point. The idea is to find a fine

subdivision of M in domains B_k where h is approximately diagonal when expressed in the basis $x_{l(k)}^k$ for a certain $l(k) \in \{1, \dots, L\}$.

Let $\tilde{g} \in \mathcal{M}$. Then $\tilde{g} \in \mathcal{M}(1/m, m)$ for every $m \in \mathbb{N}$. Fix m . Then by construction there exists a metric g such that $\tilde{g} \in \mathcal{M}(g, 1/m, m, S)$ for some choice of

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g, m, 1/m}.$$

In particular, g belongs to a $1/(mK)$ -neighborhood of $\tilde{g}$ in the C^K topology. We also have $\{V_1, \dots, V_J\} \subset \mathcal{V}(\tilde{g})$, $\alpha_1, \dots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$, which satisfy

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}_{\tilde{g}}(M)} \int_M \psi_k dv_{\tilde{g}} \right| < 1/(mK). \quad (5.5)$$

Note that $g, S, J, \{V_j\}, \{\alpha_j\}$ all depend on m .

Let h be a symmetric $(0, 2)$ -tensor on M . The following computations are done with respect to the metric $\tilde{g}$, unless otherwise specified. We start by writing

$$\int_M \text{Tr}(h) = \sum_k \int_{B_k} \psi_k \text{Tr}(h),$$

and

$$\sum_i \alpha_i \frac{V_i(h)}{\|V_i\|(M)} = \sum_k \sum_i \alpha_i \frac{V_i(\psi_k h)}{\|V_i\|(M)}.$$

At each $q_k \in \hat{B}_k$, h is diagonalizable for the metric g in a g -orthonormal basis

$$u^k = (u_1^k, \dots, u_{n+1}^k) \in \mathcal{B}_k$$

with eigenvalues $\lambda_1(k), \dots, \lambda_{n+1}(k)$ and we note that $\sum_j \lambda_j(k)$ is the trace of h at q_k for the metric g . Let $l(k)$ be such that $x_{l(k)}^k$ is at distance less than $1/m$ from u^k in $\mathcal{B}_k$. We get on B_k (which are contained in balls of radius $1/m$) the following estimates with the metric $\tilde{g}$:

$$\left| h - \sum_{j=1}^{n+1} \lambda_j(k) h(u_j^k) \right|_{\tilde{g}} < \frac{C}{m}, \quad (5.6)$$

$$\left| \sum_{j=1}^{n+1} \lambda_j(k) h(u_j^k) - \sum_{j=1}^{n+1} \lambda_j(k) h_{l(k),j}^k \right|_{\tilde{g}} < \frac{C}{m}. \quad (5.7)$$

Here C depends only on $\tilde{g}$ and h , and might be different from line to line.

We have, by (5.5), that

$$\forall k \quad \left| \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}(M)} \int_M (\sum_j \lambda_j(k) \psi_k) \right| < C/(mK).$$

Hence

$$\sum_k \left| \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}(M)} \int_M (\sum_j \lambda_j(k) \psi_k) \right| < C/m,$$

and since $|\text{Tr}_g h - \text{Tr}_{\tilde{g}} h| < C/m$, we obtain readily

$$\sum_k \left| \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}(M)} \int_M \psi_k \text{Tr } h \right| < C/m.$$

Therefore

$$\left| \sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{\|V_i\|(M)} - \frac{\int_M \text{Tr } h}{(n+1)\text{vol}(M)} \right| < C/m.$$

But we also have by (5.6) and (5.7) that

$$\left| \sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{\|V_i\|(M)} - \sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h(u_j^k))}{\|V_i\|(M)} \right| < C/m,$$

and

$$\left| \sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h(u_j^k))}{\|V_i\|(M)} - \sum_i \alpha_i \frac{V_i(h)}{\|V_i\|(M)} \right| < C/m,$$

so we conclude

$$\left| \sum_i \alpha_i \frac{V_i(h)}{\|V_i\|(M)} - \frac{\int_M \text{Tr } h}{(n+1)\text{vol}(M)} \right| < C/m.$$

In the paragraph that follows, all the integrals, traces and varifold values are computed with $\tilde{g}$. Each $V_i = V_{m,i}$, $i = 1, \dots, J_m = J$, is of the form

$$V_i = \sum_{q=1}^{R_{m,i}} \Sigma_{m,i,q},$$

with $R_{m,i} \in \mathbb{N}$, $\Sigma_{m,i,q}$ a connected, closed, smooth, embedded, minimal hypersurface of $(M, \tilde{g})$. Choose integers $c_{m,i}, d_m \in \mathbb{N}$ such that $\alpha_i = \alpha_{m,i}$ satisfies

$$\left| \frac{\alpha_{m,i}}{\|V_{m,i}\|(M)} - \frac{c_{m,i}}{d_m} \right| < \frac{1}{m J_m \|V_{m,i}\|(M)}.$$

In particular, $|1 - \sum_{i=1}^{J_m} \frac{c_{m,i} \|V_{m,i}\|(M)}{d_m}| < 1/m$ and

$$\left| \sum_i \frac{c_{m,i}}{d_m} V_{m,i}(h) - \frac{\int_M \text{Tr } h}{(n+1)\text{vol}(M)} \right| < C/m.$$

Hence

$$\lim_{m \rightarrow \infty} \frac{\sum_{i=1}^{J_m} c_{m,i} V_{m,i}(h)}{\sum_{i=1}^{J_m} c_{m,i} \|V_{m,i}\|(M)} = \frac{\int_M \text{Tr } h}{(n+1)\text{vol}(M)} \quad (5.8)$$

for any symmetric $(0,2)$ -tensor h . If we choose $h = f \cdot \tilde{g}$, with $f \in C^\infty(M)$, we get

$$\lim_{m \rightarrow \infty} \frac{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \int_{\Sigma_{m,i,q}} f}{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \text{vol}(\Sigma_{m,i,q})} = \frac{\int_M f}{\text{vol}(M)}. \quad (5.9)$$

Because

$$V_{m,i}(h) = \sum_{q=1}^{R_{m,i}} \int_{\Sigma_{m,i,q}} h(\nu, \nu) = \sum_{q=1}^{R_{m,i}} \int_{\Sigma_{m,i,q}} (\text{Tr } h - \text{Tr}_{\Sigma_{m,i,q}} h),$$

we can combine (5.8) with (5.9) to conclude

$$\lim_{m \rightarrow \infty} \frac{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \int_{\Sigma_{m,i,q}} \text{Tr}_{\Sigma_{m,i,q}} h}{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \text{vol}(\Sigma_{m,i,q})} = \frac{n \int_M \text{Tr } h}{(n+1)\text{vol}(M)}.$$

In other words, we just proved that, assuming Property **(P)**, one can find for a generic metric a sequence of finite lists of closed, embedded, connected minimal hypersurfaces $\{\Sigma_{N,1}, \dots, \Sigma_{N,P_N}\}_{N \in \mathbb{N}}$ such that the following is true: if we denote $\int_{\Sigma_{N,i}} \text{Tr}_{\Sigma_{N,i}}(h) d\Sigma_{N,i}$ (resp. $\text{vol}(\Sigma_{N,i})$) by $X_{N,i}$ (resp. $\bar{X}_{N,i}$), then

$$\left| \frac{\sum_{i=1}^{P_N} X_{N,i}}{\sum_{i=1}^{P_N} \bar{X}_{N,i}} - \alpha \right| \leq \varepsilon_N, \quad (5.10)$$

where $\alpha = \frac{1}{\text{vol}(M)} \int_M \frac{n \text{Tr}_M h}{n+1} dM$ and $\lim_{N \rightarrow \infty} \varepsilon_N = 0$. From the numbers $X_{N,i}$, $\bar{X}_{N,i}$, we want to construct two sequences $\{Y_j\}_{j \in \mathbb{N}}$ and $\{\bar{Y}_j\}_{j \in \mathbb{N}}$ such that

- for all j , there exist integers $N(j)$, $i(j)$ (chosen independently of h) with $Y_j =$

$$X_{N(j),i(j)} \text{ and } \bar{Y}_j = \bar{X}_{N(j),i(j)},$$

- moreover

$$\lim_{q \rightarrow \infty} \frac{\sum_{j=1}^q Y_j}{\sum_{j=1}^q \bar{Y}_j} = \alpha.$$

Note first that all the $\bar{X}_{N,i}$ are bounded below by a uniform positive constant v , according to the monotonicity formula, and that $|X_{N,i}| \leq C(h)\bar{X}_{N,i}$ where $C(h)$ is the maximum value that the absolute value of the trace of h can take over the Grassmannian $G_n(M)$.

Let $\{Q_N\}_{N \in \mathbb{N}}$ be a sequence of positive integers that will be chosen in the following order: Q_1 is chosen depending on $\{\Sigma_{1,i}\}$ and $\{\Sigma_{2,i}\}$, Q_2 is chosen depending on Q_1 , $\{\Sigma_{1,i}\}$, $\{\Sigma_{2,i}\}$, $\{\Sigma_{3,i}\}$, and similarly Q_{N_0} is chosen depending on $\{Q_1, \dots, Q_{N_0-1}\}$, $\{\Sigma_{1,i}\}$, $\{\Sigma_{2,i}\}$, $\dots$ $\{\Sigma_{N_0+1,i}\}$.

If $1 \leq j \leq Q_1 P_1$, write $j = kP_1 + l$ where $k \in \{0, \dots, Q_1 - 1\}$ and $l \in \{1, \dots, P_1\}$. Then define $Y_j = X_{1,l}$ and $\bar{Y}_j = \bar{X}_{1,l}$ accordingly. Notice that

$$\begin{aligned} & \left| \frac{\sum_{j=1}^{kP_1+l} Y_j}{\sum_{j=1}^{kP_1+l} \bar{Y}_j} - \alpha \right| \\ & \leq \left| \frac{k(\sum_{i=1}^{P_1} X_{1,i} - \alpha \sum_{i=1}^{P_1} \bar{X}_{1,i}) + (\sum_{i=1}^l X_{1,i} - \alpha \sum_{i=1}^l \bar{X}_{1,i})}{k \sum_{i=1}^{P_1} \bar{X}_{1,i} + \sum_{i=1}^l \bar{X}_{1,i}} \right| \\ & \leq \varepsilon_1 + C(h) + |\alpha|, \end{aligned}$$

while

$$\left| \frac{\sum_{j=1}^{Q_1 P_1} Y_j}{\sum_{j=1}^{Q_1 P_1} \bar{Y}_j} - \alpha \right| \leq \varepsilon_1.$$

If $Q_1 P_1 + 1 \leq j \leq Q_1 P_1 + Q_2 P_2$, we write $j = Q_1 P_1 + kP_2 + l$ where $k \in \{0, \dots, Q_2 - 1\}$

and $l \in \{1, \dots, P_2\}$. Then define $Y_j = X_{2,l}$ and $\bar{Y}_j = \bar{X}_{2,l}$ accordingly. Now

$$\begin{aligned}
& \left| \frac{\sum_{j=1}^{Q_1 P_1 + k P_2 + l} Y_j}{\sum_{j=1}^{Q_1 P_1 + k P_2 + l} \bar{Y}_j} - \alpha \right| \\
& \leq \left| \left(Q_1 \left(\sum_{i=1}^{P_1} X_{1,i} - \alpha \sum_{i=1}^{P_1} \bar{X}_{1,i} \right) + k \left(\sum_{i=1}^{P_2} X_{2,i} - \alpha \sum_{i=1}^{P_2} \bar{X}_{2,i} \right) \right. \right. \\
& \quad \left. \left. + \left(\sum_{i=1}^l X_{2,i} - \alpha \sum_{i=1}^l \bar{X}_{2,i} \right) \right) \right| \\
& \quad \cdot \frac{1}{\left(Q_1 \sum_{i=1}^{P_1} \bar{X}_{1,i} + k \sum_{i=1}^{P_2} \bar{X}_{2,i} + \sum_{i=1}^l \bar{X}_{2,i} \right)} \\
& \leq \varepsilon_1 + \varepsilon_2 + \frac{C(h) + |\alpha|}{Q_1 P_1 v} \sum_{i=1}^{P_2} \bar{X}_{2,i} \\
& \leq \varepsilon_1 + \varepsilon_2 + (C(h) + |\alpha|) \varepsilon_2,
\end{aligned}$$

if Q_1 is sufficiently large depending on $\{\Sigma_{1,i}\}$ and $\{\Sigma_{2,i}\}$, while

$$\left| \frac{\sum_{j=1}^{Q_1 P_1 + Q_2 P_2} Y_j}{\sum_{j=1}^{Q_1 P_1 + Q_2 P_2} \bar{Y}_j} - \alpha \right| \leq 2\varepsilon_2,$$

if Q_2 is sufficiently large depending on Q_1 , $\{\Sigma_{1,i}\}$ and $\{\Sigma_{2,i}\}$.

Proceeding this way we get a sequence $\{Q_N\}$ and a sequence $\{Y_j\}$ defined so that if $1 + \sum_{N=1}^{N=N_0} Q_N P_N \leq j \leq \sum_{N=1}^{N=N_0+1} Q_N P_N$, we write $j = \sum_{N=1}^{N=N_0} Q_N P_N + k P_{N_0+1} + l$, where $k \in \{0, \dots, Q_{N_0+1} - 1\}$ and $l \in \{1, \dots, P_{N_0+1}\}$, and set $Y_j = \Sigma_{N_0+1,l}$, $\bar{Y}_j = \bar{X}_{N_0+1,l}$. We will have

$$\left| \frac{\sum_{j=1}^{\sum_{N=1}^{N=N_0} Q_N P_N + k P_{N_0+1} + l} Y_j}{\sum_{j=1}^{\sum_{N=1}^{N=N_0} Q_N P_N + k P_{N_0+1} + l} \bar{Y}_j} - \alpha \right| \leq 2\varepsilon_{N_0} + \varepsilon_{N_0+1} + (C(h) + |\alpha|) \varepsilon_{N_0+1},$$

and

$$\left| \frac{\sum_{j=1}^{\sum_{N=1}^{N=N_0+1} Q_N P_N} Y_j}{\sum_{j=1}^{\sum_{N=1}^{N=N_0+1} Q_N P_N} \bar{Y}_j} - \alpha \right| \leq 2\varepsilon_{N_0+1}.$$

This implies

$$\lim_{q \rightarrow \infty} \frac{\sum_{j=1}^q Y_j}{\sum_{j=1}^q \bar{Y}_j} = \alpha$$

for any h , and we are done.

Proof of the Property (P): Let g be a smooth Riemannian metric, $\epsilon_1 > 0$ and $\tilde{K} > 0$ be constants, and choose

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g, \tilde{K}, \epsilon_1}.$$

Let $\mathcal{U}$ be a C^∞ neighborhood of g . Let $N = KL(n+1)$. Choose $\epsilon' > 0$ sufficiently small and $q \geq N+3$ sufficiently large so that if $g' \in \Gamma_\infty$ satisfies $\|g - g'\|_{C^q} < \epsilon'$, then $g' \in \mathcal{U}$. For each k, l, j we associate a variable $t_{k,l,j} \in [0, 1]$ and we order them by lexicographical order on the indices. We can find a smooth $(0, 2)$ -tensor $\bar{h}_{l,j}^k$ so that $\|\bar{h}_{l,j}^k - h_{l,j}^k\|_{C^q} < \epsilon'$ and such that $\{\bar{h}_{l,j}^k\}_{l,j}$ is linearly independent in a neighborhood of q_k where ϕ_k is equal to 1 and $\phi_{k'}$ is zero for $k' \neq k$.

Consider the following N -parameter family of metrics. For a $t = (t_{k,l,j}) \in [0, 1]^N$, we define

$$\hat{g}(t) = g + 2 \sum_{k,l,j} \psi_k t_{k,l,j} \bar{h}_{l,j}^k.$$

As t goes to zero, we have the following expansion

$$\begin{aligned} \text{vol}(M, \hat{g}(t))^{\frac{n}{n+1}} &= \text{vol}_g(M)^{\frac{n}{n+1}} \\ &+ \frac{n}{(n+1)} \text{vol}_g(M)^{-\frac{1}{n+1}} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g + o(\|t\|_1) + O(\epsilon' \|t\|_1), \end{aligned} \quad (5.11)$$

where $\|t\|_1 = \sum_{k,l,j} |t_{k,l,j}|$. Also

$$\frac{\partial}{\partial t_{k,l,j}} \text{vol}(M, \hat{g}(t)) = \int_M \psi_k \text{Tr}_{\hat{g}(t)}(\bar{h}_{l,j}^k) dv_{\hat{g}(t)} = \int_M \psi_k dv_g + o(1) + O(\epsilon').$$

We will say that a function $f : [0, \delta]^N \rightarrow \mathbb{R}$ is ϵ' -close to another function $g : [0, \delta]^N \rightarrow \mathbb{R}$ if, when appropriately rescaled to be functions defined on $[0, 1]^N$, they are at distance less than ϵ' in the L^∞ norm, i.e.

$$\left\| \frac{1}{\delta} f_\delta - \frac{1}{\delta} g_\delta \right\|_\infty < \epsilon'$$

with $f_\delta(s) = f(\delta s)$ and $g_\delta(s) = g(\delta s)$. By (5.11), the function

$$f_0(t) := \frac{\text{vol}(M, \hat{g}(t))^{n/(n+1)}}{\text{vol}_g(M)^{n/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g$$

is $C\epsilon'$ -close to the constant function equal to 1 on $[0, \delta]^N$, where $C = C(g)$ depends only on g and might differ from line to line, if δ is sufficiently small.

If $\delta > 0$ is sufficiently small, we also have that $\hat{g} : [0, \delta]^N \rightarrow \Gamma_q$ is an embedding and $\|\hat{g}(t) - g\|_{C^q} < \epsilon'/2$ for every $t \in [0, \delta]^N$. We can slightly perturb $\hat{g}$ in the C^∞ topology into a C^∞ map $g' : [0, \delta]^N \rightarrow \Gamma_q$ so that the conclusion of Lemma 41 is satisfied. In particular, we can assume $\|g'(t) - \hat{g}(t)\|_{C^q} < \epsilon'/4$ and $\|\frac{\partial g'}{\partial v}(t) - \frac{\partial \hat{g}}{\partial v}(t)\|_{C^q} < \epsilon'/4$ for any $t \in [0, \delta]^N$ and $v \in \mathbb{R}^N$, $|v| = 1$, and the function

$$f_1(t) := \frac{\text{vol}(M, g'(t))^{n/(n+1)}}{\text{vol}_g(M)^{n/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g$$

is $C\epsilon'$ -close to the constant function equal to 1 on $[0, \delta]^N$.

The normalized widths $p^{-\frac{1}{n+1}} \omega_p(g'(t))$ of $g'(t)$ ($t \in [0, \delta]^N$) are uniformly Lipschitz continuous on $[0, \delta]^N$ by Lemma 40. Hence, by the Weyl Law for the Volume Spectrum ([94]), the functions $t \mapsto p^{-\frac{1}{n+1}} \omega_p(g'(t))$ converge uniformly to the function $t \mapsto a(n) \text{Vol}(M, g'(t))^{\frac{n}{n+1}}$. Hence if p is sufficiently large, $|p^{-\frac{1}{n+1}} \omega_p(g'(t)) - a(n) \text{Vol}(M, g'(t))^{\frac{n}{n+1}}| < \delta\epsilon'$ and the function

$$f_2(t) := \frac{\omega_p(g'(t))}{a(n) \text{vol}_g(M)^{n/(n+1)} p^{1/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g$$

is $C\epsilon'$ -close to the constant function equal to 1 on $[0, \delta]^N$.

Then at each $t \in \mathcal{A}$ (where $\mathcal{A}$ is given by Lemma 41), there is a varifold $V \in \mathcal{V}(g'(t))$ with support a minimal hypersurface Σ such that

$$\begin{aligned} \frac{\partial}{\partial t_{k,l,j}} p^{-\frac{1}{n+1}} \omega_p(g'(t)) &= p^{-\frac{1}{n+1}} \frac{\partial}{\partial t_{k,l,j}} \|V\|(M, g'(t)) \\ &= p^{-\frac{1}{n+1}} \|V\|(\psi_k \text{Tr}_\Sigma(\bar{h}_{l,j}^k)) + O(\epsilon') \\ &= p^{-\frac{1}{n+1}} (\|V\|(\psi_k \text{Tr}_{M, g'(t)} \bar{h}_{l,j}^k) - V(\psi_k \bar{h}_{l,j}^k)) + O(\epsilon') \\ &= p^{-\frac{1}{n+1}} (\|V\|(\psi_k) - V(\psi_k h_{l,j}^k)) + O(\epsilon'), \end{aligned} \tag{5.12}$$

where $\|V\|(\cdot)$, $V(\cdot)$ and the traces are computed with respect to $g'(t)$.

Given $\eta > 0$, we can choose $0 < \epsilon' < \eta$ sufficiently small compared to $C = C(g)$ so that we can apply Lemma 42 to f_2 and to $\mathcal{A}$. We get sequences of points $\{y_{1,m}\}_m, \dots, \{y_{N+1,m}\}_m$ in $\mathcal{A}$ converging to a common limit $y \in (0, \delta)^N$ such that the gradients $\nabla f_2(y_{i,m})$ converge to $N + 1$ vectors $v_1, \dots, v_{N+1}$ with

$$d_{\mathbb{R}^N}(0, \text{Conv}(v_1, \dots, v_{N+1})) < \eta.$$

Let $\{\alpha_1, \dots, \alpha_{N+1}\} \subset [0, 1]$ with $\sum_{i=1}^{N+1} \alpha_i = 1$ such that $|\alpha_1 v_1 + \dots + \alpha_{N+1} v_{N+1}| < \eta$. Then for sufficiently large m , we have

$$|\alpha_1 \nabla f_2(y_{1,m}) + \dots + \alpha_{N+1} \nabla f_2(y_{N+1,m})| < \eta,$$

and hence

$$|\alpha_1 \frac{\partial f_2}{\partial t_{k,l,j}}(y_{1,m}) + \dots + \alpha_{N+1} \frac{\partial f_2}{\partial t_{k,l,j}}(y_{N+1,m})| < \eta \quad (5.13)$$

for all k, l, j .

According to Lemma 41, each gradient based at $y_{i,m}$ corresponds to a varifold of mass $\omega_p(g'(y_{i,m}))$ whose support is a minimal hypersurface in $(M, g'(y_{i,m}))$ of index bounded by p . Hence for all i , by Sharp's Compactness Theorem ([136]) a subsequence in m of these varifolds converges to a varifold of $\mathcal{V}(g'(y))$ whose mass is $\omega_p(g'(y))$. By (5.12) and (5.13), we have $N + 1$ varifolds V_i in $\mathcal{V}(g'(y))$ such that

$$\left| \sum_i \alpha_i \frac{\|V_i\|(\psi_k) - V_i(\psi_k h_{l,j}^k)}{a(n) \text{vol}_g(M)^{n/(n+1)} p^{1/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \int_M \psi_k dv_g \right| < C\eta$$

for all k, l, j , where $\|V_i\|(\cdot)$ and $V_i(\cdot)$ are computed with respect to $g'(y)$. This implies

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{\|V_i\|(\psi_k) - V_i(\psi_k h_{l,j}^k)}{\|V_i\|(M)} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \int_M \psi_k dv_g \right| < C\eta.$$

We also have that for all i, k, l , one has

$$\left| \sum_{j=1}^{n+1} V_i(\psi_k h_{l,j}^k) - \|V_i\|(\psi_k) \right| = \left| V_i(\psi_k g) - \|V_i\|(\psi_k) \right| < C\eta \|V_i\|(M),$$

$$\text{and } \left| \frac{1}{(n+1)} \frac{1}{\text{vol}_g(M)} \int_M \psi_k dv_g - \frac{1}{(n+1)} \frac{1}{\text{vol}_{g'(y)}(M)} \int_M \psi_k dv_{g'(y)} \right| < C\eta.$$

We deduce the following:

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}_{g'(y)}(M)} \int_M \psi_k dv_{g'(y)} \right| < C\eta. \quad (5.14)$$

The metric $g'(y) \in \Gamma_q$ satisfies $\|g'(y) - g\|_{C^q} < 3\epsilon'/4$. We apply Lemma 43 to $\bigcup_{i=1}^{N+1} \text{spt}(V_i)$ and find a C^q metric $\bar{g}$ such that $\|\bar{g} - g\|_{C^q} < 4\epsilon'/5$, each $\text{spt}(V_i)$ is non-degenerate minimal with respect to $\bar{g}$ and (5.14) is still valid with $g'(y)$ replaced by $\bar{g}$. If $\tilde{g}$ is a C^∞ metric that is sufficiently close to $\bar{g}$ in the C^q topology, then $\tilde{g} \in \mathcal{U}$. Because of the nondegeneracy of $\text{spt } V_i$ with respect to $\bar{g}$ and the Implicit Function Theorem, we can also assure that there are varifolds $V_1, \dots, V_J$ of $\mathcal{V}(\tilde{g})$ whose support $\text{spt}(V_j)$ are nondegenerate, and coefficients $\alpha_1, \dots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$ satisfying

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}_{\tilde{g}}(M)} \int_M \psi_k dv_{\tilde{g}} \right| < C\eta, \quad (5.15)$$

where the terms of the sum are computed for the metric $\tilde{g}$. Since η is arbitrarily small, we have proved Property **(P)**.

Chapter 6

Realizing Heegaard splittings by minimal surfaces

Min-max theory is a good example of interplay between geometry and topology. Heegaard splittings, an essential concept to study closed 3-manifolds, give rise naturally to certain sweepouts and thus are related to min-max theory. H. J. Rubinstein conjectured that some Heegaard splittings can be isotoped to minimal surfaces. In this chapter, we confirm his conjecture. Most material here are coming from [145], and will be included in an upcoming joint work with D. Ketover and Y. Liokumovich [83], rounding up both [145] and [82].

6.1 Introduction

When studying the topology of 3-manifolds, it can be useful to realize a Heegaard splitting as a minimal surface, since the geometric nature of minimal surfaces simplifies comparisons between splittings and helps in classification or counting problems. Such a result was announced by Rubinstein in [128, Theorem 1.8], where he sketches a proof, and it was used for instance by Colding, Gabai [30] and Colding, Gabai, Ketover [31] to prove an effective version of the classification of irreducible Heegaard splittings in non-Haken hyperbolic 3-manifolds first proved by Tao Li with other methods [91] [90] [92]. Apparently a complete proof of [128, Theorem 1.8] was not published, so we state it as a conjecture:

Conjecture 44 (Rubinstein [128]). *Let (M, g) be a closed oriented Riemannian 3-manifold not diffeomorphic to the 3-sphere and suppose that there is a strongly irreducible Heegaard splitting H . Then H is either isotopic to a minimal surface or isotopic to the oriented double cover of a non-orientable minimal surface with a vertical handle attached.*

Since a Heegaard splitting determines a continuous family of surfaces foliating the manifold and whose two ends are graphs (the spines of the two handlebodies), a natural direction to prove such a result is the min-max theory for minimal surfaces. Its principle can be roughly described as follows: taking a sequence of 1-parameter smooth families of surfaces $\{\Sigma_t^i\}_{t \in [0,1]}$ sweeping out the manifold, which are tighter and tighter in the sense that $\max_t \mathcal{H}^2(\Sigma_t^i)$ converges to the infimum possible among sweepouts in a same homotopy class, there exists a subsequence of surfaces $\Sigma_{t_j}^j$ converging to a minimal surface.

Based on this observation, Rubinstein outlined a proof that in the case when the Heegaard splitting H is strongly irreducible then the min-max theory should create a minimal surface isotopic to H or whose double cover plus a vertical neck is isotopic to H . Since the minimal surface obtained by min-max could be broken into several components and thus not isotopic to H , his idea was to iterate the min-max procedure until the desired situation occurred. The proof sketched by Rubinstein, while insightful and natural, remains vague or incomplete on essentially two points. Based on another unpublished announced result of Pitts and Rubinstein [124], he assumes that the min-max surface essentially contains the topological information of the sweepouts, and this was proved later by Ketover [81]. Moreover he suggests to apply the min-max theorem but taking into account only a subdomain of the manifold, and he claims that one gets a minimal surface inside the interior of such a subdomain. This needs to be defined and justified (see Theorem 51). Besides, in [128, Corollary 1.9] another minor point is that he takes the limit of min-max surfaces by stating that their areas are bounded, which is unclear since the min-max widths he just considered are attached to subdomains and could a priori be unbounded, even if the metrics converge.

In [81], Ketover analyzed how the subsequence $\Sigma_{t_j}^j$ converges to a min-max limit surface Σ^∞ . He deduced a genus bound for Σ^∞ which was conjectured by Pitts and Rubinstein. He also made an important step towards a proof of Rubinstein's claimed result, by proving that either it was true, or Σ^∞ had several components and was obtained

by some surgeries. The problem left to get the final result was that Σ^∞ may be broken into several components.

In this chapter, we prove the conjecture thanks to a local version of the min-max theorem. In fact, the issue appears only in the case the min-max limit has more than one non-sphere component and since they bound handlebodies under our assumptions, we want to get rid of such possible handlebodies and 3-balls with stable minimal boundaries of M . Thus we reformulate the strategy of [128] as follows: instead of iterating the min-max procedure we remove a maximal disjoint union of undesired handlebodies that could appear after min-max, we get a compact 3-manifold N with boundary called core (see Section 6.2), and we apply the local min-max to N . The min-max surface, which we prove is not included in the boundary of N , cannot be non-trivially broken into several components by construction of N and the conjecture is proved. We insist that the crucial step (and main result of this chapter) is to show that we obtain a minimal surface that is not entirely included in the boundary ∂N .

Theorem 45. *Let (M, g) be a closed oriented 3-manifold. Suppose that there is a strongly irreducible Heegaard splitting H of M which, in the case where M is a 3-sphere, is supposed to be a 2-sphere. Then either H is isotopic to a minimal surface of index at most one, or isotopic to the stable minimal oriented double cover of a non-orientable minimal surface Σ_2 with a vertical handle attached.*

This statement is actually slightly more precise than the original conjecture because if H is not isotopic to a minimal surface of index at most one, then it is isotopic to the *stable* oriented double cover of a non-orientable surface with a vertical handle attached. For this precise improvement, the catenoid estimate of Ketover, Marques and Neves [84] will be useful. Notice that the proof using min-max does not work for the torus splitting of the 3-sphere: in sufficiently round spheres, starting from a strongly Heegaard splitting of genus one, min-max can only give minimal spheres and not tori. The technical reason is that such a torus in S^3 is stabilized, even though it is irreducible.

We note that in [113] Montezuma proved another kind of localized min-max theorem for a domain Ω , but it differs from our version in the following way. In [113], Ω is mean-concave, the surfaces sweep out the entire manifold and only the ones touching Ω are considered when defining the partial min-max width. In our version Ω has minimal

or mean-convex boundary, and the surfaces only sweep out the domain Ω . Moreover, our version uses the Simon-Smith setting so the analysis of [81] can be applied, whereas in [113], Theorem B is proved in the Almgren-Pitts setting and no topological information can be deduced for the min-max minimal surface (though a smooth version of it is expected).

As a corollary of the proof of Rubinstein's conjecture, we obtain:

Corollary 46. 1. *Any lens space not diffeomorphic to S^3 or $\mathbb{R}P^3$ contains a minimal torus with index at most one.*

2. *Any $\mathbb{R}P^3$ contains either a minimal torus of index at most one or a minimal projective plane with stable universal cover. In particular if the metric is bumpy, then either there is an index one minimal torus or there is an index one minimal sphere.*

Ketover was also aware of these applications of the conjecture and the second part of the second item was observed to me by Marques. The mapping degree method of White [149] gives the existence of a minimal torus in every $\mathbb{R}P^3$ with positive Ricci curvature. This result was improved in [84, Theorem 3.3] to obtain an index one minimal torus in such a manifold. The previous corollary can be thought of as an extension of these results in the direction of finding index at most one minimal tori. We will give an example showing that the second item in Corollary 46 is optimal for index at most one minimal tori not included in a 3-ball.

Specializing these results to the case of positive scalar curvature R , we first prove:

Theorem 47. *In a spherical space form not diffeomorphic to S^3 or $\mathbb{R}P^3$ with positive scalar curvature, any strongly irreducible Heegaard splitting admits an index one minimal representative in its isotopy class.*

When $R \geq 6$, the Hersch trick for index one oriented minimal surfaces only gives $16\pi/3$ as an upper area bound when the genus is odd. However when $\text{Ric} > 0$, Marques and Neves proved in [100] that in many cases there is a Heegaard splitting of index one and area less than 4π . By extending our method in [142], where it is proved that there always exists a minimal surface of area at most 4π when $R \geq 6$, we note the following generalization:

Theorem 48. *Any spherical space form not diffeomorphic to S^3 or $\mathbb{R}P^3$ with $R \geq 6$ admits an index one minimal Heegaard splitting of area less than 4π . In an $\mathbb{R}P^3$ with $R \geq 6$, either there is an index one minimal Heegaard splitting of genus 1 and with area less than 4π or there is a minimal $\mathbb{R}P^2$ with stable universal cover and with area less than 2π .*

The proof uses the local min-max theorem applied to the lift of the manifold to S^3 . Notice that in the two previous theorems, the first one gives a minimal genus Heegaard splitting while the second one gives an improved area bound for a (not necessarily irreducible) Heegaard splitting. It is tempting to can ask whether these two results can be combined (see Remark 6.5.1). It can also be interesting to compare these results with [112], where Montezuma constructs metrics with positive scalar curvature and unbounded min-max widths.

Since all the min-max constructions are localized, versions of the previous results hold for non-prime 3-manifolds, and manifolds with boundary.

6.2 Topological preliminaries and some notations

6.2.1 Generalized Heegaard splittings

We begin some basic definitions and notations in topology.

A 3-manifold W is a *compression body* if there is a connected closed oriented surface S such that W is obtained from $S \times [0, 1]$ by attaching 2-handles along mutually disjoint loops in $S \times \{1\}$ and filling in some resulting 2-sphere boundary components with 3-handles. We do not require to fill in all the 2-sphere boundary components. Denote $S \times \{0\}$ by $\partial_+ W$ and $\partial W \setminus \partial_+ W$ by $\partial_- W$. A compression body is called a *handlebody* if $\partial_- W = \emptyset$. It is said to be *trivial* if $W \approx \partial_+ W \times [0, 1]$.

Let M be a connected compact oriented 3-manifold. When M is closed, an embedded connected orientable surface H is a *Heegaard splitting* if $M \setminus H$ has two connected components each diffeomorphic to a handlebody. This notion can be generalized to the case when $\partial M \neq \emptyset$ as follows. Let $(\partial_0 M, \partial_1 M)$ be a partition of the boundary components of M . A triplet (W_0, W_1, H) is called a *generalized Heegaard splitting* of $(M, \partial_0 M, \partial_1 M)$

if W_0, W_1 are compression bodies with

$$W_0 \cup W_1 = M, \quad \partial_- W_0 = \partial_0 M, \quad \partial_- W_1 = \partial_1 M$$

$$\text{and } W_0 \cap W_1 = \partial_+ W_0 = \partial_+ W_1 = H.$$

The surface H is then also called a *generalized Heegaard splitting* of M . The *Heegaard genus* of M is the lowest possible genus of a generalized Heegaard splitting of M . Recall that any triplet $(\partial_0 M, \partial_1 M)$ as above admits a generalized Heegaard splitting.

Let S be an orientable closed surface embedded in the interior of M as above. An *essential disk* in (M, S) is a disk D embedded in M such that $D \cap S = \partial D$ and ∂D is an essential curve in S .

A generalized Heegaard splitting (W_0, W_1, H) is called *irreducible* when there are no essential disks $(D_0, \partial D_0) \subset (W_0, H)$, $(D_1, \partial D_1) \subset (W_1, H)$ such that $\partial D_0 = \partial D_1$. We say that (W_0, W_1, H) is *strongly irreducible* if there are no essential disks $(D_0, \partial D_0) \subset (W_0, H)$, $(D_1, \partial D_1) \subset (W_1, H)$ such that $\partial D_0 \cap \partial D_1 = \emptyset$.

6.2.2 H-compatible splittings and H-cores

Let (M, g) be a connected compact oriented 3-manifold whose boundary, if non-empty, is a stable minimal surface. In this subsection, we study generalized Heegaard splittings of some compact 3-manifolds N included in (M, g) , and whose boundary is also a stable minimal surface.

In this subsection, we will consider (W_0, W_1, H) a generalized Heegaard splitting of $(M, \partial_0 M, \partial_1 M)$ where $(\partial_0 M, \partial_1 M)$ is a given partition of the components of ∂M .

Consider a triplet (W'_0, W'_1, H') such that the following holds:

- the surface H' is embedded inside the interior of M and is obtained from H by an isotopy of M leaving ∂M unchanged,
- W'_0, W'_1 are compression bodies included in M so that (W'_0, W'_1, H') is a generalized Heegaard splitting of

$$(W'_0 \cup W'_1, \partial_- W'_0, \partial_- W'_1),$$

- $\partial_- W'_0$ and $\partial_- W'_1$ are two (not necessarily connected, possibly empty) oriented stable

minimal surfaces.

Such a triplet (W'_0, W'_1, H') is then called an H -compatible splitting. Note that (W_0, W_1, H) is itself an H -compatible splitting. By the maximum principle, each component of $\partial_- W'_0 \cup \partial_- W'_1$ is either disjoint from ∂M or entirely contained in it.

Given a metric g on M , let $\mathcal{S}_g(H)$ be the set of subsets $N \subset M$ such that $N = W'_0 \cup W'_1$ for a certain H -compatible splittings (W'_0, W'_1, H') . The inclusion relation $\subset$ gives a partial order on $\mathcal{S}_g(H)$. An element N is then minimal in the sense of this partial order (not to be confused with the minimality of the boundary ∂N) if there is no element $\tilde{N}$ such that $\tilde{N} \subset N$.

Lemma 49. *Let (M, g) be as above and suppose that H is an irreducible generalized Heegaard splitting. Then*

1. *either H is isotopic to a stable minimal surface,*
2. *or H is isotopic to the stable oriented double cover of a non-orientable minimal surface $\Sigma \subset M$ with a vertical handle attached,*
3. *or for any $N \in \mathcal{S}_g(H)$, there exists a minimal element $N_{min} \in \mathcal{S}_g(H)$ such that*

$$N_{min} \subset N.$$

It will be convenient to introduce this definition:

Definition 6.2.1. *Let M be as above and H be a Heegaard splitting. An H -core of M is a minimal element $\mathfrak{C}$ of $\mathcal{S}_g(H)$.*

Let C be a positive real number. A C -bounded H -core of M is an element $\mathfrak{C}$ of $\mathcal{S}_g(H)$ such that $\mathcal{H}^2(\partial \mathfrak{C}) \leq C$, and which is minimal among elements N of $\mathcal{S}_g(H)$ satisfying $\mathcal{H}^2(\partial N) \leq C$.

We can reformulate Lemma 49 as follows:

Corollary 50. *Let (M, g) be as above, H is an irreducible Heegaard splitting. Suppose that H is neither isotopic to a stable minimal surface nor isotopic to the stable oriented double cover of a non-orientable minimal surface $\Sigma \subset M$ with a vertical handle attached. Then there exists an H -core $\mathfrak{C}$ of M .*

For all $C > \mathcal{H}^2(\partial M)$, the above statement is true when “ H -core” is replaced with “ C -bounded H -core”.

6.3 Local min-max theory

6.3.1 Min-max theory in the Simon-Smith setting

Let N be an oriented connected compact 3-manifold possibly with boundary, subset of a closed oriented manifold (M, g) . The surfaces considered in this subsection are all embedded. Since M is oriented, we will only consider smooth sweepouts $\{\Sigma_t\}$ where all the slices are oriented. If Σ is a surface, we denote by $|\Sigma|$ the varifold associated to Σ with multiplicity one.

Definition 6.3.1. Let $\{\Sigma_t\}_{t \in [a, b]}$ be a family of oriented closed surfaces in N . We say that $\{\Sigma_t\}$ is a smooth sweepout if

1. for all $t \in [a, b]$, Σ_t is a smooth surface in the interior of N ,
2. Σ_t varies smoothly in $t \in (a, b)$,
3. there is a partition (A, B) of the components of ∂N such that, $\Sigma_a = A$, $\Sigma_b = B$, and $|\Sigma_t|$ converges to $|\Sigma_a|$ (resp. $|\Sigma_b|$) in the $\mathbf{F}$ -norm as $t \rightarrow a$ (resp. b).

Let Π be a collection of smooth sweepouts parametrized by $[0, 1]$. Denote by Diff_0 the set of diffeomorphisms of N isotopic to the identity map and leaving the boundary fixed. The set Π is saturated if for any map $\psi \in C^\infty([0, 1] \times N, N)$ such that $\psi(t, \cdot) \in \text{Diff}_0$ for all t and $\psi(0, \cdot) = \psi(1, \cdot) = \text{Id}$, and for any $\{\Sigma_t\}_{t \in [0, 1]} \in \Pi$, we have $\{\psi(t, \cdot)(\Sigma_t)\}_{t \in [0, 1]} \in \Pi$. We say that Π is generated by a family of smooth sweepouts $\{\{\Sigma_t^j\}_{t \in [0, 1]}\}_{j \in J}$ if Π is the smallest saturated set containing $\{\{\Sigma_t^j\}_{t \in [0, 1]}\}_{j \in J}$. The *width of N associated with Π* in the sense of Simon-Smith is defined to be

$$W(N, \Pi) = \inf_{\{\Sigma_t\} \in \Pi} \sup_{t \in [0, 1]} \mathcal{H}^2(\Sigma_t).$$

Given a sequence of smooth sweepouts $\{\{\Sigma_t^i\}\}_i \subset \Pi$, denote by $\mathbf{\Lambda}(\{\{\Sigma_t^i\}\}_i)$ the set

$$\begin{aligned} &\{V \in \mathcal{V}_2(M); \quad \exists \{i_j\} \rightarrow \infty, t_{i_j} \in [0, 1] \\ &\text{such that } \lim_{j \rightarrow \infty} \mathbf{F}(|\Sigma_{t_{i_j}}^{i_j}|, V) = 0\}. \end{aligned}$$

The sequence $\{\Sigma_t^i\} \in \Pi$ is *minimizing* if $\lim_{i \rightarrow \infty} \max_{t \in [0, 1]} \mathcal{H}^2(\Sigma_t^i) = W(N, \Pi)$, it is *pulled-tight* if moreover any element $V \in \mathbf{\Lambda}(\{\{\Sigma_t^i\}\}_i)$ with $\|V\|(M) = W(N, \Pi)$ is stationary. By [29, Proposition 4.1], it is always possible to deform $\{\Sigma_t^i\} \in \Pi$ into a pulled-tight sequence.

The Almgren map $\mathcal{A}$ defined in the previous subsection can also be defined on the family of smooth sweepouts of M . The smooth sweepouts we consider have oriented slices so by choosing a continuous orientation, it determines a family of integral cycles continuous in the flat topology. Then by [2], one can associate to it a 3-dimensional integral current $\mathcal{A}(\{\Sigma_t\})$.

6.3.2 The local smooth min-max theorem

The following theorem will be crucial for proving Rubinstein's conjecture. The point is to show that in the setting of Simon-Smith, under certain conditions, local min-max gives a minimal surface *inside the interior* of the connected domain N .

Theorem 51. *Consider an oriented connected compact 3-manifold $N \subset (M, g)$ possibly with boundary. Let (W_0, W_1, H) be a generalized Heegaard splitting of $(M, \partial_0 M, \partial_1 M)$, where $(\partial_0 M, \partial_1 M)$ is a partition of the components of ∂M . Suppose that (W'_0, W'_1, H') is an H -compatible splitting with $N = W'_0 \cup W'_1$. We write $\partial_- W'_0 = \Gamma_0$ and $\partial_+ W'_1 = \Gamma_1$. Let Π be a saturated set generated by all smooth sweepouts $\{\Sigma_t\}$ such that*

- $\mathcal{A}(\{\Sigma_t\}) = [N]$,
- for $i \in \{0, 1\}$, $\Sigma_i = \Gamma_i$,
- for all $t \in (0, 1)$, the surface Σ_t is isotopic to H' inside N .

Suppose that the metric g is bumpy, that (W'_0, W'_1, H') is a strongly irreducible Heegaard splitting and that $\max\{\mathcal{H}^2(\Gamma_0), \mathcal{H}^2(\Gamma_1)\} < W(N, \Pi)$.

Then there exists a min-max sequence $\Sigma_{t_j}^j$ converging to $\sum_{i=1}^k m_i \Sigma_i^\infty$ as varifolds, where $\Sigma_i^\infty \subset N$ are disjoint closed embedded connected minimal surfaces such that

$$\sum_{i=1}^k m_i \mathcal{H}^2(\Sigma_i^\infty) = W(N, \Pi),$$

the index of $\bigcup_{i=1}^k \Sigma_i^\infty$ is bounded by one and there is an i_0 such that

$$\Sigma_{i_0}^\infty \subset \text{Int}(N).$$

Moreover, the genus bound and the analysis of [81] hold true.

Proof. By definition of "H-compatible" (see Section 6.2), the boundary $\partial N = \Gamma_0 \cup \Gamma_1$ is a stable minimal surface. Since the metric is bumpy, ∂N is actually strictly stable and

$$\max\{\text{Area}(\Gamma_0), \text{Area}(\Gamma_1)\} < W(N, \Pi). \quad (6.1)$$

This follows from general fact that by discretizing a smooth sweepout, the Almgren-Pitts width is at most the Simon-Smith width.

We can also find a small $\hat{\delta} > 0$ so that

$$N_{\hat{\delta}} := (N \cup \{x \in M; d(x, \partial N) \leq \hat{\delta}\})$$

is a strictly mean convex domain and if a closed minimal surface is contained in $N_{\hat{\delta}}$ then it is contained in N . The saturated set Π naturally induces a saturated set $\Pi_{\hat{\delta}}$ associated with $N_{\hat{\delta}}$. It is then not difficult to check that for $\hat{\delta}$ small, $W(N_{\hat{\delta}}, \Pi_{\hat{\delta}}) = W(N, \Pi)$. If $\hat{\delta}$ is chosen small enough, by (6.1), we can apply the version of the Simon-Smith theorem proved in [100, Theorem 2.1] to get the existence of the varifold $V = \sum_{i=1}^k m_i \Sigma_i^\infty$, then the genus bound and the nature of convergence follow from [81]. The index of the union $\bigcup_{i=1}^k \Sigma_i^\infty$ is bounded by one according to Theorem 6.1 and paragraph 1.3 in [102].

The goal of the remaining of the proof is to show the existence of a component $\Sigma_{i_0}^\infty$ inside the interior $\text{Int}(N)$. The arguments will share similarities with [102, Deformation Theorem C] (in particular the several constructions in its proof), however we have to deal with smooth isotopies. On the other hand, in our case we only need to rule out the case

where the whole min-max surface is included in the boundary. Let $\{\{\Sigma_t^i\}\}_i$ be a pulled-tight minimizing sequence and suppose by contradiction that for all $V \in \mathbf{A}(\{\{\Sigma_t^i\}\}_i)$ with smooth support and mass $W(N, \Pi)$, $\text{spt}(V)$ is included in ∂N . Given a sweepout in Π , we orientate Σ_t with the unit normal ν pointing towards Γ_1 . In N , each Σ_t hence bounds a manifold with boundary $B(\Sigma_t)$ such that ν is the outward normal.

If S is a surface, let $|S|$ be the varifold it determines with multiplicity one. Denote by $S_1, \dots, S_p$ the stable minimal components of $\Gamma_0 \cup \Gamma_1$. Let V be a varifold with mass $W(N, \Pi)$, of the form:

$$V = m_1|S_1| + \dots m_p|S_p|, \quad (6.2)$$

where m_i are nonnegative integers. We call the finite family of such V by $\hat{\mathcal{V}}$. Let $\{\Sigma_t\}_{t \in [0,1]}$ be a smooth sweepout in Π . We are applying the following discussion to the pulled-tight minimizing sequence $\{\{\Sigma_t^i\}\}_i$ so we are assuming $\{\Sigma_t\}$ to be one of these sweepouts. We can in particular make $\max_t \mathcal{H}^2(\Sigma_t) - W(N, \Pi)$ arbitrarily small. Given $\alpha > 0$, consider $\mathbf{V}_\alpha$ the subset

$$\mathbf{V}_\alpha := \{t \in [0, 1]; \quad \exists V \in \hat{\mathcal{V}}, \quad \mathbf{F}(|\Sigma_t|, V) \leq \alpha\}.$$

The idea of the end of the proof is to construct from part of the sweepout $\{\Sigma_t\}$ another sweepout $\{\hat{\Sigma}_t\}$ (not necessarily homotopic to $\{\Sigma_t\}$) for which the corresponding $\mathbf{V}_\eta$ is empty, where $\eta > 0$ is a constant depending on α and N . A technical point which is already in Claims 1-4 of the proof of [102, Deformation Theorem C] is that we will make sure that the surfaces $\hat{\Sigma}_t$ are not close to any stationary integral varifold which was far from the surfaces Σ_t . We suppose that $\mathbf{V}_\alpha$ is non-empty (otherwise there is nothing to prove) and is a finite union of closed intervals. If α is sufficiently small, then for any $t \in \mathbf{V}_\alpha$, Σ_t bounds $B(\Sigma_t)$ which has volume either close to 0 or close to $\text{Vol}(N)$. Let $[a_1, b_1], \dots, [a_q, b_q]$ be the intervals in $\mathbf{V}_\alpha$ such that $B(\Sigma_t)$ has volume close to 0, where $a_1 \leq b_1 < \dots < a_q \leq b_q$. We have $b_q < 1$ since α is small enough. Similarly let $[c_1, d_1], \dots, [c_{q'}, d_{q'}]$ be the intervals composing $\mathbf{V}_\alpha \cap (b_q, 1]$. On these intervals the volume of $B(\Sigma_t)$ is close to $\text{Vol}(N)$. Then $\{\hat{\Sigma}_t\}$ will be constructed from the restriction $\{\Sigma_t\}_{t \in [b_q, c_1]}$ by appropriately closing its ends, i.e. by deforming Σ_{b_q} and Σ_{c_1} respectively to Γ_0 and Γ_1 . This new sweepout $\{\hat{\Sigma}_t\}$ will not necessarily be homotopic to $\{\Sigma_t\}$ (contrarily to the analogue situation in [102, Deformation Theorem C]).

We recall the following version “with constraints” of the γ -reduction of [110] used in the min-max setting of Simon-Smith [140] (see [29, Section 7]). Consider $\tilde{\Sigma}$ a surface embedded in $\text{Int}(N) \cap N_\delta$. Let U be an open set included in N . Let $\mathfrak{Is}(U)$ be the set of isotopies of N_δ fixing $N_\delta \setminus U$, with parameter in $[0, 1]$, and for $\mu > 0$ define

$$\mathfrak{Is}_\mu(U) = \{\psi \in \mathfrak{Is}(U); \mathcal{H}^2(\psi(\tau, \tilde{\Sigma})) \leq \mathcal{H}^2(\tilde{\Sigma}) + \mu \text{ for all } \tau \in [0, 1]\}.$$

An element of the above set is called a μ -isotopy. Suppose that the sequence $\{\psi^k\} \subset \mathfrak{Is}_\mu(U)$ is such that

$$\lim_{k \rightarrow \infty} \mathcal{H}^2(\psi^k(1, \tilde{\Sigma})) = \inf_{\psi \in \mathfrak{Is}_\mu(U)} \mathcal{H}^2(\psi(1, \tilde{\Sigma})).$$

Then in U , $\psi^k(1, \tilde{\Sigma})$ subsequently converges in the varifold sense to a smooth minimal surface $\tilde{\Gamma}$.

Let Ω_r be an r -neighborhood of $S_1 \cup \dots \cup S_p \subset \Gamma_0 \cup \Gamma_1$ so that there is a family of area-decreasing maps $\{P_t\}_{t \in [0, 1]}$ as in [102, Proposition 5.7]. Let $\alpha_1 > 0$ be such that if for $V \in \hat{\mathcal{V}}$, a stationary integral varifold Z has $\text{spt}(Z) \subset \text{spt}(V)$, $\mathbf{M}(Z) = \mathbf{M}(V)$ but $Z \neq V$, then there is a connected component Ω^Z of Ω_{r_0} so that $\|Z\|(\Omega^Z) > \|V\|(\Omega^Z) + \alpha_1$. The α considered in the previous paragraph will then be taken independent of $\{\Sigma_t\}$ and at least small enough so that the following holds: if for $V \in \hat{\mathcal{V}}$, a varifold $Z' \in \mathcal{V}_2(M)$ (not necessarily stationary or integral) has $\|Z'\|(\Omega^{Z'}) > \|V\|(\Omega^{Z'}) + \alpha_1$ for a connected component $\Omega^{Z'}$ of Ω_{r_0} , then $\mathbf{F}(Z, V) > 2\alpha$ (see [102, Subsections 5.11 and 5.14]). Let $\mu > 0$ be a small positive constant which depends on $\{\Sigma_t\}$ and which will be determined later.

Notice (see Remark 6.3.1 after this proof) that by strong irreducibility of Σ_{b_q} in N , during a γ -reduction, there are a finite number of surgeries along curves and at each step, if it is done along an essential curve (in the surface of this step) then the surgery disk is on one well-determined side of Σ_{b_q} independent of the surgery (the side of Γ_0 in the case when $B(\Sigma_{b_q})$ is close to 0). Surgeries along non-essential curves can occur on both sides and split off spheres. Moreover the non-sphere components have multiplicity one. Let us apply the bounded path version of the γ -reduction to Σ_{b_q} , with $U = \text{Int}(N_\delta) \setminus \bar{\Omega}_{r/2}$, with μ small to be determined later. If α is small enough, then by the monotonicity formula the

resulting limit varifold V_1 has support in Ω_r . Hence Σ_{b_q} can be deformed by a μ -isotopy to a surface A , obtained by attaching to a closed surface $B \subset \Omega_r$ some thin handles (diffeomorphic to $S^1 \times [0, 1]$) and caps (diffeomorphic to disks), with $\mathcal{H}^2((A \setminus B) \cup (B \setminus A))$ arbitrarily small (see [113, Lemma 7.1] an analogue lemma in the context of Almgren-Pitts theory). We push B towards Γ_0 with P_t , $0 \leq t \leq t'$ with t' very close to 1. Let C be a surface obtained by A from a μ -isotopy, close to $P_{t'}(B)$ as above in the varifold sense.

Let $\{X_t\}_{t \in [0,1]}$ denote the surfaces constructed by isotopies as above, with $X_0 = \Sigma_{b_q}$ and $X_1 = C$. Similarly to Claims 1 and 2 in the proof of [102, Deformation Theorem C], one can choose μ and $\{X_t\}$ so that the following holds: there is a positive η independent of the index i of $\{\Sigma_t^i\}$ and μ when they are respectively large and small enough, such that

$$\forall t \in [0, 1] \quad \mathbf{F}(|X_t|, V') > \eta$$

for all stationary integral varifold V' with mass $W(N, \Pi)$.

We can reapply γ -reduction to $P_{t'}(B)$ in each connected component of Ω_r . We get a limit varifold V_2 of the form (6.2). Let B^* (resp. D) be a surface deformed from $P_{t'}(B)$ (resp. C) by a μ -isotopy, arbitrarily close in the varifold sense to V_2 . Now write

$$V_2 = m_{2,1}|S_1| + \dots m_{2,p}|S_p|.$$

Suppose that some of the coefficients are strictly larger than one, say $m_{2,j} > 1$ for j in a subset $\mathcal{J}$ of $\{1, \dots, p\}$. The surface S_j is a 2-sphere for any $j \in \mathcal{J}$. We are going to further deform D , by a μ -isotopy, into a surface E arbitrarily close in the varifold sense to

$$V_3 = m_{3,1}|S_1| + m_{3,2}|S_2| + \dots m_{3,p}|S_p|, \tag{6.3}$$

where each $m_{3,i}$ is either 0 or 1. To begin this μ -isotopy, we need to prepare the surface D . Note that by Remark 6.3.2, B^* is isotopic inside Ω_r to the union of $\tilde{S}_i$ and some small area spheres bounding balls in Ω_r , where

$$\tilde{S}_i := \bigcup_{r=1}^{m_{2,i}} \{x \in N; \text{dist}(x, S_i) = \bar{\delta} \frac{r}{m_{2,i}}\},$$

where $\bar{\delta} > 0$ is arbitrarily small so that

$$\Gamma_0 \times [0, 2\bar{\delta}) \rightarrow N$$

$$(x, s) \rightarrow \exp_x(s\nu)$$

is a diffeomorphism on its image supposed to be contained in $\Omega_{r/2}$, with ν the inward unit normal of Γ_0 (recall that $\Gamma_0 \subset \partial N$). The surface D is obtained from B^* with some thin handles and caps attached. By "thin handles", we mean that they are close to curves with endpoints in B^* in the Hausdorff distance, say. Note that we could prove that B^* is μ -isotopic to $\bigcup_i \tilde{S}_i$ plus some small area spheres for μ small by adapting the proof of [110] but for simplicity here we will only need that it is isotopic to the union of $\bigcup_i \tilde{S}_i$ with small area spheres. Let φ be that isotopy.

By the local version of Remark 3.27 in [110] shown in [29, Section 7] with the blow down/blow up argument, there is a radius r_0 depending only on Γ_0 so that B^* is μ -isotopic to a surface which, up to some small area spheres bounding balls in Ω_r , coincides with B^* outside of a ball b_{2r_0} of radius $2r_0$ centered at a point of S_j , and inside the concentric ball of half radius b_{r_0} , the surface is made of a family of embedded disks $\{d_1, d_2, \dots, d_{l_0}\}$ ($l_0 > 1$) parallel to one another with area close to πr_0^2 . Using P_t we can also make these disks as close to each other as wanted. By attaching the thin handles, we get a resulting surface μ -isotopic to D . We will still denote this prepared surface D . Our next goal is to find a small handle in D and move it by isotopy to a vertical neck inside b_{r_0} between to consecutive disks (which are components of $D \cap b_{r_0}$). This is because then it is easy to decrease the area of D (without increasing it by more than μ) by a uniform amount comparable to $2\pi r_0^2$ depending only on Γ_0 .

We attach the handles one by one to B^* . When attaching these handles, we stop at the first one that makes one of the layers

$$\varphi^{-1}(\{x \in N; \text{dist}(x, S_i) = \bar{\delta}\})$$

and a layer

$$\varphi^{-1}(\{x \in N; \text{dist}(x, S_{j_0}) = \delta \frac{m_{2,1} - 1}{m_{2,1}}\}) \quad \text{for some } j_0 \in \mathcal{J}$$

into the same connected component; this handle exists because D is connected. We can deform with a μ -isotopy this handle into a handle joining two consecutive disks, say d_1 and d_2 (renumber the disks d_i if necessary). and close to a straight curve joining d_1 and d_2 . This is possible due to the following fact. The surgery corresponding to this handle is on the side of Γ_1 and in a compression body W , any two curves linking $\partial_+ W$ to a sphere component of $\partial_- W$ are isotopic inside W : indeed we can almost project each of these curves to $\partial_+ W$ and by moving the roots of the curve, plus the fact the one can inverse the self-crossings by isotopy, we deform the curve to a chosen fixed one.

After having deformed the critical handle to a vertical handle, we continue to add the remaining handles and get a surface D' μ -isotopic to D . Then by "opening up" the special handle, one decreases by a uniform amount (depending only on Γ_0) the area of D' by a μ -isotopy. This means that thanks to γ -reductions again in each connected of Ω_r we can deform by a μ -isotopy D' into D'' arbitrarily close to $V'_2 = m'_{2,1}|S_1| + \dots m_{2,p}|S_p|$, where $m'_{2,j} < m_{2,j}$. We continue until we get E from D by a μ -isotopy, such that E is close to V_3 of the form (6.3). Since Σ_{b_q} separates and $B(\Sigma_{b_q})$ has volume close to 0, this means that E is very close to Γ_0 in the varifold sense. By repeating the process described above using a sequence of smaller and smaller μ' going to zero and using the basin structure around Γ_0 , we can join E to Γ_0 (in the sense of Definition 6.3.1) by a μ -isotopy. To sum up what we just did: we deformed Σ_{b_q} to C , then from C we got a surface E which we join to Γ_0 , all by μ -isotopies. Moreover, if $\{Y_t\}_{t \in [0,1]}$ denote the surfaces constructed by isotopies as above from C to E , arguing as in Claims 3 and 4 in the proof of [102, Deformation Theorem C], one can choose μ and $\{Y_t\}$ so that for a positive η independent of the index i of $\{\Sigma_t^i\}$ and μ when they are respectively large and small enough,

$$\forall t \in [0, 1], \quad \mathbf{F}(|Y_t|, V') > \eta$$

for all stationary integral varifold V' with mass $W(N, \Pi)$. By reversing the directions of the isotopies, we get an isotopy joining Γ_0 to Σ_{b_q} , and we glue this isotopy to $\{\Sigma_t\}_{t \in [b_q, 1]}$.

The above procedure can be realized in a symmetric way for $t = c_1$ (remember that $[c_1, d_1], \dots, [c_{q'}, d_{q'}]$ are the intervals composing $\mathbf{V}_\alpha \cap (b_q, 1]$). As a result, we get a new family $\{\hat{\Sigma}_t\}_{t \in [b', a']}$, where $0 \leq b' < a' \leq 1$. After reparametrizing, we obtain a family still denoted by $\{\hat{\Sigma}_t\}_{t \in [0, 1]}$, satisfying the following properties for some η small enough and for μ arbitrarily small:

- $\{\hat{\Sigma}_t\} \in \Pi$,
- $\max_t \text{Area}(\hat{\Sigma}_t) \leq \max_t \text{Area}(\Sigma_t) + \mu$,
- $\mathbf{F}(|\hat{\Sigma}_t|, V) > \eta$ for all $t \in [0, 1]$ and all V of the form (6.2) with $\mathbf{M}(V) = W(N, \Pi)$,
- $\mathbf{F}(|\hat{\Sigma}_t|, V') > \eta$ for all $t \in [0, 1]$ and all stationary integral varifold V' with $\mathbf{M}(V') = W(N, \Pi)$ which are not in $\Lambda(\{\{\Sigma_t^i\}\}_i)$.

We emphasize that this sweepout $\{\hat{\Sigma}_t\}$ is not a priori homotopic to the original sweepout $\{\Sigma_t\}_t$.

Now we can conclude the proof. Let $\{\{\Sigma_t^i\}\}_i$ be a pulled-tight minimizing sequence and μ_i a sequence going to 0. Transforming each sweepout of this sequence as above with parameter μ_i instead of μ (but keeping $\alpha > 0$ fixed) produces a new pulled-tight minimizing sequence $\{\{\hat{\Sigma}_t^i\}\}_i \subset \Pi$. Note similarly to [102, Deformation Theorem C] that by construction:

$$\begin{aligned} & \Lambda(\{\{\hat{\Sigma}_t^i\}\}_i) \cap \{\text{stationary integral varifolds of mass } W(N, \Pi)\} \\ & \subset \Lambda(\{\{\Sigma_t^i\}\}_i) \cap \{\text{stationary integral varifolds of mass } W(N, \Pi)\}. \end{aligned} \tag{6.4}$$

But by construction of $\{\{\hat{\Sigma}_t^i\}\}_i$, any varifold $V' \in \Lambda(\{\{\hat{\Sigma}_t^i\}\}_i)$ with $\|V'\|(N) = W(N, \Pi)$ is η -far from any varifold of the form (6.2). So the usual min-max theorem (see [29]) would produce a varifold in the first intersection in (6.4) with smooth support, i.e. a minimal surface whose area counted with multiplicity is $W(N, \Pi)$, and which is not entirely contained in the boundary ∂N . This contradicts our assumption on $\Lambda(\{\{\Sigma_t^i\}\}_i)$ so the theorem is proved.

□

It is expected more generally that, similarly to the Almgren-Pitts setting (see [102]), oriented local min-max surfaces cannot be stable for bumpy metrics.

Remark 6.3.1. *Strong irreducibility has consequences on how surgeries can be performed. The following observation was used in previous works but not really explained. Suppose that $S = S_1$ is strongly irreducible in N and separates N into W, W' . Suppose also that $S_2, \dots, S_K$ are successively obtained by surgery from the previous one (for instance along a γ -reduction [110] or a smooth min-max procedure [81]). We can assume the surgery curves α_i ($i \in \{1, \dots, K\}$) to be disjoint and contained in S_1 . Then by switching W and W' if necessary, for all $i \in \{1, \dots, K\}$, if the surgery is performed along a curve α_i essential in S_i , α_i bounds a disk in W . This follows from [130, Theorem 2.1].*

From this "one-sided surgeries" property, we deduce two useful facts [128], [84, Theorem 3.3]. Firstly if the surfaces of a smooth sweepout are isotopic to a strongly irreducible Heegaard splitting H as in Theorem 51 and a min-max surface is non-orientable, then there is only one non-sphere component, which has multiplicity two when not an $\mathbb{RP}^2$, and its double cover plus a vertical handle is isotopic to H (see [67, Corollary 1.6]). Secondly if the min-max surface is orientable, then the non-sphere components have multiplicity one (this can be checked using [131, Theorem 2.11]).

Remark 6.3.2. *For a $j \in \{1, \dots, p\}$, let B' be the intersection of B^* and the r -neighborhood of $S_{2,j}$. By strong irreducibility of the Heegaard splitting H , it suffices to prove that the number of components of B' projecting with degree ± 1 on $S_{2,j}$ with P_1 is exactly $m_{2,j}$. Indeed the connected components project each with degree either ± 1 or 0 by embeddedness and orientability. The components of degree 0 bound a handlebody inside the r -neighborhood so they are spheres bounding balls included in the r -neighborhood. These spheres have small area since otherwise one could use P_t and reduce their area by a uniform amount with a μ -isotopy inside the r -neighborhood, which would contradict the fact that V_2 (which is close to B^*) was obtained by γ -reduction from $P_t(B)$. The number of components with degree ± 1 cannot exceed $m_{2,j}$ for B^* has area close to the mass of V_2 . Now if the number of components with degree ± 1 is less than $m_{2,j}$ then the area of one of these components is larger than say $(1 + \frac{1}{m_{2,j}})\mathcal{H}^2(S_{2,j})$ (B^* being very close to V_2 as varifolds). But using P_t (t close to 1) one can decrease the area of such a component by a uniform amount without increasing it (one may move the other components as well during the process). That contradicts the fact that V_2 is the limit obtained by γ -reduction with μ -isotopies. Thus the number of components with degree ± 1 is exactly $m_{2,j}$, and by*

strong irreducibility there are isotopic together to graphs over $S_{2,j}$.

6.4 Minimal Heegaard splittings in orientable 3-manifolds

In this section, M is an irreducible oriented closed 3-manifold not diffeomorphic to S^3 . All surfaces considered are closed embedded. For simplicity, we will use "minimal Heegaard splitting" to denote a closed connected embedded minimal surface which is a Heegaard splitting.

Main theorem

We now state the main theorem.

Theorem 52. *Let (M, g) be a closed oriented 3-manifold not diffeomorphic to the 3-sphere. Suppose that there is a strongly irreducible Heegaard splitting H . Then either H is isotopic to a minimal surface of index at most one, or isotopic to the stable minimal oriented double cover of a non-orientable minimal surface with a vertical handle attached.*

Of course for 3-spheres, the theorem of Simon-Smith gives the existence of a minimal 2-sphere of index at most one [140] [29]. The existence of a strongly irreducible Heegaard splitting forces M to be irreducible.

Notice the following. If h is the Heegaard genus of M and if M contains an incompressible surface of positive genus k less than h , then by [50]

- there is an oriented area-minimizing surface of genus k ,
- or there is a non-orientable minimal surface whose oriented double cover is area minimizing and of genus k .

On the other hand, if M is irreducible and does not contain any such surfaces (for instance when M is non-Haken), then there is a strongly irreducible Heegaard splitting so Theorem 52 applies. This follows from [20, Theorem 3.1], whose proof implies the following: if H is a Heegaard splitting of genus h' which is not strongly irreducible, then either it is reducible or M contains an incompressible surface of positive genus less than h' .

We list a few corollaries of Theorem 52.

Corollary 53. *The conjecture of Rubinstein is true.*

Corollary 54. *Any lens space not diffeomorphic to S^3 or $\mathbb{R}P^3$ contains a minimal Heegaard splitting of genus one with index at most one.*

Corollary 55. *Any $(\mathbb{R}P^3, g)$ contains*

- *either a Heegaard splitting of genus one and index at most one,*
- *or a minimal $\mathbb{R}P^2$ with stable oriented double cover.*

If the metric g is bumpy, then

- *either there is an index one Heegaard splitting of genus one,*
- *or there is an index one minimal sphere.*

Example: This corollary is optimal if we focus on index at most one tori not included in a 3-ball, as shown by the following example. Consider a long cylindrical piece $[0, 1] \times S^2$, cap it on one side (say $\{0\} \times S^2$) with a half-sphere, then take the quotient on the other side ($\{1\} \times S^2$) by the usual $\mathbb{Z}_2$ -action to get an $\mathbb{R}P^3$ with positive scalar curvature. If the cylindrical piece is long enough with a metric near the product metric, and if the spheres $\{t\} \times S^2$ constitute a mean convex foliation with mean curvature vector pointing towards $\{1\} \times S^2$, then by the maximum principle, the monotonicity formula and the area bound [100, Proposition A.1 (i)], there is no index one minimal torus intersecting the projective plane $(\{1\} \times S^2)/\mathbb{Z}_2$.

Before proving Theorem 52, we need two lemmas. Recall that (M, g) is closed irreducible oriented and not diffeomorphic to the 3-sphere.

Lemma 56. *Let (M, g) be as above, let H be an irreducible Heegaard splitting. Suppose that H is not isotopic to the stable oriented double cover of a non-orientable minimal surface with a vertical handle attached. Let $\mathfrak{C}$ be a core. Then for all constant $C > 0$ large enough, there is a sequence of bumpy metrics g_m converging smoothly to g and C -cores $\mathfrak{C}_m$ with respect to g_m , such that $\partial\mathfrak{C}_m$ converges smoothly to $\partial\mathfrak{C}$ (with respect to g).*

Lemma 57. *Suppose that (N, g) is a compression body endowed with g a bumpy metric, such that $\partial_+N = \Gamma$ is mean-convex and $\partial_-N = \Gamma'$ is a disjoint union of stable minimal surfaces. Then one of the following cases occurs:*

- either there is a compression body $\tilde{N}$ such that the hypotheses above are true with Γ' (resp. N) replaced by $\tilde{\Gamma}' := \partial_- \tilde{N}$ (resp. $\tilde{N}$), and moreover $\tilde{N}$ is strictly included in N ,
- or there is a smooth sweepout $\{\hat{\Sigma}_t\}_{t \in [0,1]}$ of N with $\mathcal{A}(\{\hat{\Sigma}_t\}) = [|N|]$, $\hat{\Sigma}_0 = \Gamma$, $\hat{\Sigma}_1 = \Gamma'$, $\hat{\Sigma}_t$ is isotopic to Γ for $t \in [0, 1)$, and moreover

$$\max_{t \in [0,1]} \mathcal{H}^2(\hat{\Sigma}_t) = \mathcal{H}^2(\Gamma).$$

Proof of Theorem 52. The surfaces considered are closed and embedded. Let H be a strongly irreducible Heegaard splitting of M . First suppose that the metric is bumpy. We can assume that H is not isotopic to a stable minimal surface or to the stable oriented double cover of a non-orientable minimal surface $\Sigma \subset U$ with a vertical handle attached. So in particular there is an H -core $\mathfrak{C}$ by Corollary 50. Let $\Gamma_0 \cup \Gamma_1 = \partial \mathfrak{C}$ be a boundary decomposition with $\Gamma_i = \partial_- W'_i$, where (W'_0, W'_1, H') is an H -compatible splitting. Let Π be the saturated set generated by all the smooth sweepouts $\{\Sigma_t\}$ of $\mathfrak{C}$, such that $\mathcal{A}(\{\Sigma_t\}) = [| \mathfrak{C} |]$, such that Σ_0 (resp. Σ_1) is Γ_0 (resp. Γ_1) such that the slices Σ_t ($0 < t < 1$) are smooth connected orientable and are isotopic to H . We can apply Theorem 51 to the core $\mathfrak{C}$ and we get an embedded minimal surface Σ with area $W(\mathfrak{C}, \Pi)$ (taking into account multiplicities).

If Σ is oriented, it is obtained by surgeries of H . By strong irreducibility (see Remark 6.3.1) every time a surgery occurs along an essential curve, the surgery disk is on one well-defined side of H independent of the surgery. Surgeries along non-essential curves can happen on both sides and split off spheres. Notice that we can suppose there is no minimal sphere in the interior of the core. Otherwise we can minimize its area on the non-trivial side: either we get an $\mathbb{R}P^2$ component with stable universal cover and so M is actually an $\mathbb{R}P^3$, or we get some stable spheres in $\text{Int}(\mathfrak{C})$ and we could have added these spheres to Γ_0 and get a smaller H -compatible couple, contradicting the definition of the H -core $\mathfrak{C}$ (here we are using that M is not a 3-sphere). The components of Σ bound compression bodies or are included in the boundary of $\mathfrak{C}$, and none of the latter components which is not a sphere is included in one of the compression bodies previously mentioned. Since H is strongly irreducible, the multiplicity of the non-sphere components

is one (see Remark 6.3.1). By Theorem 51, one of the component Σ' is contained in the interior $\text{Int}(\mathfrak{C})$, has positive genus, has multiplicity one, and is unstable by definition of a core again. There is exactly one such unstable component Σ' by [102]. Suppose H is not isotopic to Σ' . We minimize its area on the side $M' \subset \text{Int}(\mathfrak{C})$ which is not a compression body (say the side of Γ_1). One checks that Σ' is incompressible inside M' as follows: suppose that an essential curve $\tilde{\gamma}$ on Σ' bounds a disk on one side, it has to be the side of M' , but since H is not isotopic to Σ' but isotopic to Σ' with necks linking to some other components and some handles added to them (the necks and handles correspond to surgeries), there is an essential curve on one of the small necks or handles, disjoint from the disk bounded by $\tilde{\gamma}$ and bounding a disk on the other side of H , contradicting the strong irreducibility. So Σ' is incompressible inside M' and by γ -reduction one gets a stable minimal surface $\Sigma'' \subset M'$. Either a component of Σ'' is non-orientable, then its oriented double cover is stable and by adding a vertical handle we get a surface isotopic to H (by strong irreducibility of H): this case cannot happen by our assumption in the beginning of the proof. Or Σ'' is oriented: then replacing Γ_0 by $(\Gamma_0 \cup \Sigma'') \setminus \Sigma'$ (and removing the spheres inside compression bodies if necessary), we get a contradiction for the definition of the core $\mathfrak{C}$. So it means that the surface Σ has one component which is isotopic to H .

If Σ is non-orientable, then by the topological claim in the proof of [84, Theorem 3.3] and [81] or Remark 6.3.1, there is a unique component Σ' of Σ in the interior $\text{Int}(\mathfrak{C})$. It is non-orientable, has multiplicity at least two and for $\epsilon > 0$ small enough, each Γ_t is isotopic to $\partial N_\epsilon(\Sigma')$ with a vertical handle attached. Remember that the oriented double cover $\tilde{\Sigma}'$ of Σ' is unstable by our assumption in the beginning of the proof. We now apply Lemma 57 to the complementary inside the core $\mathfrak{C}$ of a small neighborhood of Σ' . By definition of a core, only the second item of this lemma is valid. Hence as in [84, Theorem 3.3], the unstability of $\tilde{\Sigma}'$ gives rise to a smooth sweepout $\{\tilde{\Gamma}_t\} \in \Pi$, such that

$$W(\mathfrak{C}, \Pi) \leq \max_{t \in [0,1]} \mathcal{H}^2(\Gamma_t) < 2\mathcal{H}^2(\Sigma').$$

Since $2\mathcal{H}^2(\Sigma') \leq W(\mathfrak{C}, \Pi)$, it is a contradiction. So this non-orientable case cannot happen, and the first part of the theorem is proved in the case of bumpy metrics.

Finally if the metric is not bumpy, we use Lemma 56 to approximate g by bumpy

metrics g_m . For each m , there is a C -bounded H -core $\mathfrak{C}_m$ and $W(\mathfrak{C}_m, \Pi)$ converges to $W(\mathfrak{C}, \Pi)$. If C is chosen bigger than $2W(\mathfrak{C}, \Pi)$, then we check without difficulty that the above arguments hold for a C -bounded H -core instead of an H -core for large m , since $\lim_m W(\mathfrak{C}_m, \Pi) = W(\mathfrak{C}, \Pi)$. We get for each m large enough

- either a minimal surface of at most index one isotopic to H ,
- or the stable minimal oriented double cover of a non-orientable minimal surface such that when we attach a vertical handle, we get a surface isotopic to H .

Subsequently this sequence converges by [136] to a minimal surface Σ^* . By strong irreducibility, either the limit is two-sided and the convergence is smooth, or the limit is one-sided, the oriented double cover of Σ^* is stable (by a Jacobi field argument, see [136] for instance) and H is isotopic to it with a vertical handle attached. So Σ^* is as in the statement of Theorem 52.

□

Case of non-prime oriented 3-manifolds

When an oriented 3-manifold is not prime, we can cut it along area-minimizing 2-spheres obtained by minimizing the area of the separating essential spheres. They are either embedded or the double-cover of a projective plane. Let C_0 be one of the component and denote by $\hat{C}_0$ the manifold obtained by closing with 3-handles (i.e. gluing 3-balls). If $\hat{C}_0$ has a strongly irreducible Heegaard splitting H , which we can suppose included in C_0 , then Theorem 52 applies and we obtain:

Theorem 58. *Let $(\bar{M}, g)$ be an oriented 3-manifold and $\hat{C}_0$ as above with a strongly irreducible Heegaard splitting $H \subset C_0$. Then either H is isotopic to a minimal surface of index at most one, or isotopic to the stable minimal oriented double cover of a non-orientable minimal surface with a vertical handle attached.*

6.5 Minimal Heegaard splittings in orientable 3-manifolds with positive scalar curvature

In this section, we specialize to the case of positive scalar curvature. We suppose that M is a spherical space form S^3/Γ not diffeomorphic to S^3 . Assume that M is endowed with

a metric g with scalar curvature at least 6, then any embedded orientable stable surface is a sphere (see [134]) of area at most $4\pi/3$, and index one minimal surfaces also have an area bound [100, Proposition A.1 (i)].

Existence of index one minimal Heegaard splitting with minimal genus

A corollary of Theorem 52 is the following existence theorem for irreducible minimal Heegaard splittings in spherical space forms with positive scalar curvature.

Theorem 59. *1. Let (M, g) be an $\mathbb{R}P^3$ which has scalar curvature at least 6. Then*

- *either M contains an index one Heegaard splitting of genus one which has area less than 4π ,*
- *or M contains a minimal $\mathbb{R}P^2$ with stable oriented double cover and area less than 2π .*

2. Let (M, g) be a spherical space form S^3/Γ not diffeomorphic to S^3 or $\mathbb{R}P^3$, and with positive scalar curvature. Let H be a strongly irreducible Heegaard splitting of M . Then M contains an index one minimal surface isotopic to H .

Note that, since a spherical space form admits a positive curvature metric, it does not contain incompressible surfaces so any Heegaard splitting of minimal genus is strongly irreducible by [20].

Existence of index one minimal Heegaard splitting with area less than 4π

The following result combined with Theorem 59 (1) can be thought of as an extension of Theorem 23 in [142] in the orientable case.

Theorem 60. *Let (M, g) be a spherical space form S^3/Γ not diffeomorphic to S^3 or $\mathbb{R}P^3$, and with scalar curvature at least 6. Then M contains an index one minimal Heegaard splitting with area less than 4π .*

Remark 6.5.1. • *Here is an observation following from [110, Proposition 1]: if M is a spherical space form with Heegaard genus two and positive scalar curvature, then any minimal torus is included in a 3-ball.*

- *The above theorem does not provide information on the genus of the Heegaard splitting with area less than 4π since we are using a projection argument and the min-max theory of Almgren-Pitts. For M as in the theorem, is there a Heegaard splitting with*

area less than 4π , which has minimal genus? In the special case $\text{Ric} > 0$, this is true by combining [100] and [84] but it seems quite arduous to extend the method to the case $R > 0$ (see [144]).

Case of non-prime closed oriented 3-manifolds with positive scalar curvature

An oriented closed manifold with positive scalar curvature is diffeomorphic to the connected sum of spherical space forms and some $S^1 \times S^2$'s [134, 57, 120, 122, 121] (see also [114, Corollary 0.5, Chapter 15], [99, Theorem 7.1, b])). When M is not a 3-sphere, write

$$M \approx a.\mathbb{R}P^3 \# S^3/\Gamma_1 \# \dots \# S^3/\Gamma_b \tag{6.5}$$

$$\# S^3/\tilde{\Gamma}_1 \# \dots \# S^3/\tilde{\Gamma}_c \# d.S^1 \times S^2$$

where the factors S^3/Γ_i are not diffeomorphic to $\mathbb{R}P^3$ and do not contain embedded non-orientable surfaces, the factors $S^3/\tilde{\Gamma}_i$ are not diffeomorphic to $\mathbb{R}P^3$ and contain an embedded non-orientable surface. Here a (resp. d) is the number of $\mathbb{R}P^3$ (resp. $S^1 \times S^2$) in the prime decomposition.

Recall that $S^1 \times S^2$ is the only orientable non-irreducible prime 3-manifold. As in the end of the previous section, everything extends to the non prime case.

Theorem 61. *Let $(\bar{M}, g)$ be an oriented 3-manifold not diffeomorphic to S^3 , with scalar curvature at least 6, and let $\hat{C}_0$ be as in Theorem 58. Suppose that $H \subset C_0$ is a strongly irreducible Heegaard splitting of $\hat{C}_0$.*

- *When $\hat{C}_0$ is not an $\mathbb{R}P^3$, then H is isotopic to a minimal surface of index one, and there is an index one minimal surface $\Sigma \subset C_0$ of area less than 4π , such that Σ is a Heegaard splitting in $\hat{C}_0$.*
- *When $\hat{C}_0$ is an $\mathbb{R}P^3$, either there is an index one minimal torus isotopic to H of area less than 4π or a minimal $\mathbb{R}P^2$ of area less than 2π with stable universal cover $\tilde{S}$, and H is isotopic to $\tilde{S}$ plus a vertical handle attached.*

Let $\mathcal{I}(I)$ be the family of minimal surfaces with index at most I . When the scalar curvature is positive, Theorem 1.5 in [19] shows that generically $\mathcal{I}(I)$ is finite for all

integer I . A corollary of our previous theorem is a lower bound for the cardinal of $\mathcal{I}(1)$ for any metric:

Corollary 62. *Let $(\bar{M}, g)$ be as in (6.5) and with positive scalar curvature. Then $\mathcal{I}(1)$ has at least $a + 2b + 3c + 2d - 1$ elements.*

Chapter 7

Stable spheres can appear along the Ricci flow

When the scalar curvature is positive in a closed 3-dimensional manifold, can stable minimal spheres appear along the Ricci flow? In this chapter we give an example where that occurs, hence answering a question of I. Agol.

7.1 Introduction

For quotients of the spheres of dimension 2 and 3, endowed with an arbitrary metric, the Ricci flow eventually makes the metric converge to a round metric. In dimension 2, it was proved by Hamilton [63] and B. Chow [28] that starting at any metric, the latter evolves smoothly under the Ricci flow until a trivial singularity, where the whole surface disappears at a point and after rescaling becomes asymptotically round. The situation is way more complicated in dimension 3 because non-trivial singularities can occur. In a serie of papers [120] [122] [121], Perelman was able to analyse and control the singularities by a surgery process initially proposed by Hamilton, which enables to continue the flow. One simple consequence of this breakthrough is that for quotients of the 3-sphere, after a finite number of surgeries, the manifold disappears in finite time, and also becomes asymptotically round. From a related point of view, if the initial metric is already known to be round enough, then it becomes even more so during the flow: this is Hamilton's

theorem [62] which states that if a closed 3-manifold has positive Ricci curvature then this property is preserved and after rescaling the metric converges smoothly to a round metric. Besides, it is well known that the positivity condition $\text{Ric} > 0$ prevents the existence of two-sided closed stable minimal surfaces. Hence it seems natural to expect that the absence of such stable minimal surfaces would also be preserved along the flow.

The study of stable minimal surfaces in the context of the Ricci flow is relevant for understanding singularity formation. For instance the heuristic picture for the Ricci flow on a 3-sphere is that a non-degenerate singularity which is non-trivial should be a neckpinching, thus there should be small stable minimal spheres just before the singularity time. To our knowledge, the only rigourously proved examples of initial metrics on the 3-sphere eventually producing a non-trivial singularity [5] [6], contain a stable minimal sphere. Thus one might hope to avoid non-trivial singularities if the initial metric does not contain stable minimal surfaces.

It will be enough for us to focus on the case where the scalar curvature is positive. This condition $R > 0$ is considerably weaker than $\text{Ric} > 0$ but nevertheless conveys an idea of roundness and is preserved along the flow. Notice that if $R > 0$, any two-sided oriented closed stable minimal surface is a 2-sphere. Let us reformulate the two previous questions:

Q1: Suppose that (M, g) is a closed oriented 3-manifold with positive scalar curvature and containing no stable minimal spheres. Can a stable minimal sphere appear along the Ricci flow starting at (M, g) ?

Q2: Let (M, g) be as in **Q1**. Can a non-trivial singularity occur along the Ricci flow starting at (M, g) ?

It turns out that the answer to both questions is yes and the counterexamples are the object of our main theorem (see Theorems 73 and 76):

Theorem 63. *There exists a metric g on S^3 with positive scalar curvature such that*

1. *(S^3, g) contains no stable minimal 2-spheres,*
2. *a stable minimal 2-sphere appears along the Ricci flow starting at (S^3, g) ,*

3. *a non-trivial singularity occurs in finite time.*

These counterexamples show that, even when $R > 0$, the absence of stable spheres at the beginning cannot prevent non-trivial singularities. It means that the analysis of Bamler in [12, 8, 9, 10, 11] to get finitely many surgeries is necessary and cannot be replaced by a geometric argument using stable spheres. Actually the appearance of stable geodesics is also true for some 2-spheres (see Theorem 72), but of course we cannot impose a curvature positivity condition in that case since it would be preserved by the flow and this would prevent the existence of stable closed geodesics.

Several authors previously studied the evolution of minimal surfaces along the Ricci flow in different contexts. Hamilton was the first to use that interaction for stable geodesics or minimal surfaces [64] [28], Colding-Minicozzi [34] exploited it with branched immersed minimal spheres for bounding the extinction time of some 3-manifolds, and Marques-Neves [100] proved rigidity results for min-max minimal surfaces in some 3-manifolds using similar methods. Let us point out how Question **Q1** is related to [100, Theorem 1.3]. Assuming $\text{Ric} > 0$, $R \geq 6$ on a quotient S^3/Γ and by controlling the evolution of a min-max width along the Ricci flow, Marques-Neves are able to produce a small area minimal surface in the initial metric. While in [142] we proved using a different method that this result remains true without assumption on the Ricci curvature, it would be desirable to understand to what extent their combination of Ricci flow and min-max theory can be realized when Ric is not necessarily positive. The reason of this assumption $\text{Ric} > 0$ in [100] is twofold. First they are making use of Hamilton's theorem so that they do not have to deal with surgeries. The second and most serious reason for this assumption is the following: as recalled previously, it excludes the existence of two-sided stable minimal surfaces so in particular it enables the construction of optimal sweepouts from a given unstable two-sided minimal surface. The examples that we construct to answer **Q1** suggest that combining min-max theory with the Ricci flow when $R > 0$ is not as natural as in the more restrictive case $\text{Ric} > 0$.

After answering Questions **Q1** and **Q2**, we clarify the link between small stable spheres and Type I singularities in dimension 3, without curvature assumptions (see Theorem 80):

Theorem 64. *If a non-trivial Type I singularity occurs at time T , then there are stable immersed minimal spheres with embedded image near time T whose area decreases linearly*

to zero, and a local converse holds true.

Moreover, we construct in Proposition 79 general examples of Type I neckpinching by joining any two closed 3-manifolds with a sufficiently thin neck, which generalizes in dimension 3 the rotationally symmetric metrics on S^{n+1} constructed by Angenent and Knopf [5].

Finally, one can ensure that no stable spheres appear if the initial metric is symmetric enough (see Theorem 81):

Theorem 65. *Let M be a closed connected oriented 3-manifold and a d -dimensional Lie group of isometries acting on M such that $d > 1$, or $d = 1$ and the action is free in that case. Then along the Ricci flow, no stable immersed minimal spheres with embedded image can appear if there were none at the beginning.*

In fact, there is a non-free S^1 action on the examples constructed previously to answer **Q1**. Hence these examples essentially have a maximal amount of symmetry among 3-manifolds such that new stable spheres appear along the Ricci flow. Since we have seen that stable spheres are linked to non-trivial Type I singularities, we get as a corollary (see Corollary 83):

Corollary 66. *When M as in the previous theorem is not rotationally symmetric and if a singularity occurs along the Ricci flow, then it is a Type I trivial singularity.*

The paper is organized as follows. In Section 1, some preliminaries on the Ricci flow and min-max theory are presented. Section 2 constitutes the main part of the article, certain "thin hooks" are constructed, and this enables to give explicit examples answering at the same time **Q1** and **Q2**. In Section 3, after showing a general procedure to get Type I neckpinching, we prove a relation between stable spheres and non-trivial Type I singularities. Finally, we propose in Section 4 optimal symmetry assumptions preventing the appearance of new stable spheres and a corollary concerning singularities which can possibly occur is derived.

7.2 Preliminaries

7.2.1 Ricci flow, singularities, canonical neighborhoods and geometric limits

Let (M, g) be a closed oriented Riemannian 3-manifold. A standard Ricci flow starting at (M, g) , defined on $[0, T)$, is a smooth solution of

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)}, & t \in [0, T) \\ g(0) = g, \end{cases}$$

The flow is said to develop a singularity at time T if the norm of the curvature tensor becomes unbounded as $t \rightarrow T$. The singularity is said to be

- *trivial* when

$$\{x \in M; \lim_{t \rightarrow T} |Rm(x, t)| = \infty\} = M,$$

- a *Type I* singularity when there is a constant $\bar{C}$ such that for all $t \in [0, T)$,

$$\sup_M |Rm_{g(t)}| \leq \frac{\bar{C}}{T - t},$$

- a *Type II* singularity when

$$\limsup_{t \uparrow T} \sup_M |Rm_{g(t)}|(T - t) = \infty.$$

As Perelman showed, the regions where the scalar curvature is large are modelled by the so-called canonical neighborhoods. In this subsection we will explain some of their properties. We refer the reader to B. Kleiner and J. Lott [87], J. Morgan and G. Tian [114], H-D. Cao and X-P. Zhu [18] for more details. Our presentation will follow [114]. First let us recall the definition of (C, ϵ) -canonical neighborhoods. Fix C and ϵ two positive constants. An open neighborhood U of $x \in (M, g(t))$ is a strong (C, ϵ) -canonical neighborhood if one of the following holds (see [114, Section 8 in Chapter 9 and Definition 14.18]):

1. U is a strong ϵ -neck in (M, g) centered at x ,

2. U is a (C, ϵ) -cap in (M, g) whose core contains x ,
3. U is a C -component of (M, g) satisfying Condition (8) of [114, Definition 9.72],
4. U is an ϵ -round component of (M, g) .

A strong ϵ -neck centered at $x \in (M, g(t))$ is a submanifold $N \subset M$ and a diffeomorphism $\bar{\psi}_N : S^2 \times (-1/\epsilon, 1/\epsilon) \rightarrow N$ with $x \in \bar{\psi}_N(S^2 \times \{0\})$ such that $t - R(x, t)^{-1} \geq 0$ and the evolving metric $R(x, t)\bar{\psi}^*(g(t + s/R(x, t)))$, $-1 < s \leq 0$, is ϵ -close in the $C^{[1/\epsilon]}$ -topology to the evolving cylindrical metric $ds^2 + d\theta^2$, $-1 < s \leq 0$, where $d\theta^2$ denotes the round metric of scalar curvature $1/(1 - s)$ on S^2 . A (C, ϵ) -cap is a noncompact submanifold $\mathcal{C} \subset M$ diffeomorphic to a 3-ball or $\mathbb{R}P^3$ minus a ball, with a neck $N \subset \mathcal{C}$ such that $\bar{Y} = \mathcal{C} \setminus N$ is a compact submanifold called core. The boundary $\partial \bar{Y}$ of the so-called core (the interior of $\mathcal{C} \setminus N$) is required to be the central sphere of a strong ϵ -neck in $\mathcal{C}$. After rescaling the metric to have $R(x) = 1$ at some point x in the cap, the diameter, volume and scalar curvature ratios at any two points are bounded by C . A C -component is a compact manifold diffeomorphic to S^3 or $\mathbb{R}P^3$, of positive sectional curvature and of bounded geometry controlled by C after rescaling (we added Condition (8) of [114, Definition 9.72] since the next theorem is actually true with this definition). An ϵ -round component is a compact connected manifold such that, after rescaling to make $R(x) = 1$ at some point $x \in M$, the metric is close in the $C^{[1/\epsilon]}$ -topology to a round metric. The definition of strong (C, ϵ) -canonical neighborhoods is hence scale invariant.

We say that a (standard) Ricci flow $(M, g(t))_{t \in [a, b]}$ satisfies the (C, ϵ) -canonical neighborhood assumption with parameter r if every point $(x, t) \in M \times [a, b]$ with $R_{g(t)}(x) \geq r^{-2}$ has a (C, ϵ) -canonical neighborhood. When ϵ and $1/C$ are small enough, one has the following canonical neighborhood theorem (see for instance [114, Chapter 9, Chapter 17 and Theorem 15.9]):

Theorem 67. *Let $T > 0$. Then there exists an $r_0 > 0$ depending only on T such that the following holds. Suppose that (M, g) is a closed oriented 3-manifold endowed with a normalized metric, i.e. for all $x \in M$:*

$$\max_M |Rm(x, 0)|_{g(0)} \leq 1 \text{ and } \text{vol}_{g(0)} B(x, 0, 1) \geq \omega/2,$$

where ω is the volume of the unit ball in $\mathbb{R}^3$. Assume that the Ricci flow $(M_t, g(t))_{t \in [0, t_1]}$

is well-defined until a time $t_1 \leq T$. Then $(M_t, g(t))_{t \in [0, t_1]}$ satisfies the strong (C, ϵ) -canonical neighborhood assumption with parameter r_0 .

The relationship between the Type I/II classification and the canonical neighborhoods was given in [44]: a singularity at time T is of Type II if and only if there is a sequence (x_k, t_k) with $x_k \in M$, $t_k \rightarrow T$ such that the scalar curvature at (x_k, t_k) goes to infinity and (x_k, t_k) is contained in a (C, ϵ) -cap diffeomorphic to a 3-ball (it corresponds to *iv* in [44, Proposition 1.4]). This geometric characterization of Type II singularities will be useful. We note that the other kind of (C, ϵ) -caps, those diffeomorphic to $\mathbb{R}P^3$ minus a point, have a double cover which is a strong ϵ -neck.

The scalar curvature evolves according to

$$\frac{\partial R}{\partial t} = \Delta R + 2|\text{Ric}|^2.$$

Thus when x is in a strong ϵ -neck at time t or a (C, ϵ) -cap diffeomorphic to $\mathbb{R}P^3$ minus a point, there is a positive constant C_1 so that

$$\frac{1}{C_1} R(x, t)^2 \leq \frac{\partial R(x, t)}{\partial t} \leq C_1 R(x, t)^2. \quad (7.1)$$

and there is a positive constant C_2 such that whenever x is in a strong canonical neighborhood,

$$\left| \frac{\partial R(x, t)}{\partial t} \right| \leq C_2 R(x, t)^2. \quad (7.2)$$

Along the Ricci flow, as the scalar curvature gets large it controls the whole curvature tensor. Let $(M, g(t))_{t \in [0, T]}$ be a Ricci flow such that for all $x \in M$ the smallest eigenvalue of $Rm(x, 0)$, denoted by $\nu(x, 0)$, is at least -1 . Set $X(x, t) = \max(-\nu(x, t), 0)$. Then Ivey [76] and Hamilton [65] showed the following "pinching towards positive" property:

Theorem 68. *We have the following properties.*

1. $R(x, t) \geq \frac{-6}{4t+1}$ and
2. for all (x, t) for which $0 < X(x, t)$,

$$R(x, t) \geq 2X(x, t)(\log X(x, t) + \log(1+t) - 3).$$

The notion of geometric convergence [114, Chapter 5] describes the convergence of based Ricci flows, and can be extended to any time interval (i.e. to intervals not of the form $(-T, 0]$). We will need the following convergence property of the Ricci flow.

Lemma 69. *Let $T > 0$, let (M_k, g_k) be a sequence of closed normalized 3-manifolds. Suppose that for any sequence of points $\mathfrak{s} = \{x_k\}_k$ where $x_k \in M_k$, the following holds. Subsequently the sequence of based manifolds (M_k, g_k, x_k) converges geometrically to a complete based manifold $(M_\infty^\mathfrak{s}, g_\infty^\mathfrak{s}, x_\infty)$ such that*

1. *the Ricci flow $(M_\infty^\mathfrak{s}, g_\infty^\mathfrak{s}(t))$ with initial metric $g_\infty^\mathfrak{s}(0) = g_\infty^\mathfrak{s}$ exists, is unique, and defined for $0 \leq t < T$,*
2. *for all $t < T$ there is a constant $C_0 = C_0(t)$ independent of the sequence $\mathfrak{s}$ so that the norm of the curvature tensor of $(M_\infty^\mathfrak{s}, g_\infty^\mathfrak{s}(t'))$ is bounded by C_0 for all $t' \leq t$.*

Then for any sequence $\mathfrak{s} = \{x_k\}$ with $x_k \in M_k$, the sequence of based Ricci flows $(M_k, g_k(t), x_k)$ starting at $g_k(0) = g_k$ subsequently converges geometrically to $(M_\infty^\mathfrak{s}, g_\infty^\mathfrak{s}(t), x_\infty)$ on $[0, T)$.

7.2.2 Some min-max theory

In this subsection, we present a variation of the min-max theorem in the continuous setting as described by De Lellis and Tasnady in [42] (see Chapter 2). For new notations, please see [144].

The following theorem is a slight extension of [156, Theorem 2.7]. It roughly says that if the starting slice of sweepouts belonging to a homotopically closed family Λ has n -volume less than the width of Λ , then the min-max theorem still holds as long as all the sweepouts are contained in an open set with mean convex boundary. Note that "mean convex" can be generalized to "piecewise smooth mean convex" (see [142]).

Theorem 70. *Let (M, g) be a closed $(n + 1)$ -manifold with $2 \leq n \leq 6$, and N, X open subsets of M . Suppose that $\partial X \neq \emptyset$ and that $(X \cup \partial X) \subset N$. When $\partial N \neq \emptyset$, assume that ∂N is mean convex. Then for any homotopically closed family Λ of sweepouts in N associated with X such that*

$$W(N, \partial N, \Lambda) > \mathcal{H}^n(\partial X),$$

there exists a min-max sequence $\{\Sigma_{t_n}^n\}$ of Λ converging in the varifold sense to an embedded minimal hypersurface Σ (possibly disconnected), contained in N . Moreover the n -volume of Σ , if counted with multiplicities, is equal to $W(N, \partial N, \Lambda)$.

Remark 7.2.1. If $n = 1$, the following elementary version of Theorem 70 will be useful. Suppose that N and X are diffeomorphic to the unit disk D in $\mathbb{R}^2$, define $\{c_t\}_{t \in [0,1]}$ as the smooth sweepout of X obtained by the foliation $\{x \in \mathbb{R}^2; \|x\|_{\text{eucl}} = t\}_{t \in [0,1]}$ of D , where $\|\cdot\|_{\text{eucl}}$ is the Euclidean norm in $\mathbb{R}^2$. Let $\mathcal{C}$ be the space of smooth curves endowed with the C^∞ topology. Let Λ be the homotopically closed family of sweepouts $\{\tilde{c}_t\}_{t \in [0,1]} \subset \mathcal{C}$ continuously isotope to $\{c_t\}_{t \in [0,1]}$ in N and such that $\tilde{c}_0 = c_0$. Define $W(N, \partial N, \Lambda)$ as for the higher dimensional case. If ∂N is convex and

$$W(N, \partial N, \Lambda) > \mathcal{H}^1(\partial X),$$

then there is a simple closed geodesic in N of length $W(N, \partial N, \Lambda)$. This can be proved using the mean curvature flow $\{\Phi(s, \cdot)\}$ where s is the time parameter. Define $\theta : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \theta(s, c) &= \sup\{s' \in [0, s]; \Phi(s', c) \text{ has length} \\ &\text{at least } \frac{1}{2}(W(N, \partial N, \Lambda) + \mathcal{H}^1(\partial X))\}, \end{aligned}$$

where we use the convention $\sup \emptyset = 0$. Then given a minimizing sequence of sweepouts $\{\{c_t^n\}_{t \in [0,1]}\}$, we consider the new tightened sequence $\{\{\Phi(\theta(s, c_t^n), c_t^n)\}_{t \in [0,1]}\}$ for each $s \geq 0$. By the maximum principle, the new sweepouts are entirely contained in N . Letting $s \rightarrow \infty$, any min-max sequence converges subsequentially to a simple closed geodesic inside N (see [53]).

7.3 Appearance of stable spheres and non-trivial singularities

7.3.1 Construction of thin hooks

We will construct a family of $(n+1)$ -dimensional closed manifolds by defining embedded hypersurfaces in $\mathbb{R}^{n+2}$ and using the metric induced by the Euclidean metric. As we will see, they look like hook-shaped $(n+1)$ -spheres, whose one branch is slightly swollen. The properties of these hooks will be useful to prove the two appearance theorems stated in the next subsection.

Consider a curve $\mu : [0, 1] \rightarrow \mathbb{R}^2$ such that

$$\mu(s) = \begin{cases} (1, s) & \text{for } s \in [0, 1/6], \\ (\cos(s\pi), 1/6 + \sin(s\pi)) & \text{for } s \in [1/3, 2/3], \\ (-1, 1-s) & \text{for } s \in [5/6, 1], \end{cases}$$

and μ is chosen on $[1/6, 1/3] \cup [2/3, 5/6]$ so that it is a smooth curve. For all integer $L > 0$, consider the smooth curve $\gamma_L : [0, 4] \rightarrow \mathbb{R}^2$ defined by

$$\gamma_L(s) = \begin{cases} (1, (s-1)L-1) & \text{for } s \in [0, 1], \\ (1, s-2) & \text{for } s \in [1, 2], \\ \mu(s-2) & \text{for } s \in [2, 3], \\ (-1, (3-s)L) & \text{for } s \in [3, 4]. \end{cases}$$

It will be convenient to introduce the following function $f : [-1/2, 1/2] \rightarrow \mathbb{R}$:

$$f(x) = \exp\left(1 + \frac{1}{4x^2 - 1}\right) \quad \forall x \in [-1/2, 1/2].$$

Let $d_0 > 0$ be smaller than half the focal radius of the curve μ . Now, we identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^{n+2}$. At each point $p \in \gamma_L$, denote by $H[p]$ the normal hyperplane to γ_L at p . For all $L > 1$ and $\bar{\epsilon} = (\epsilon_1, \epsilon_2) \in (0, 1/4)^2$, we choose a function $\phi[L, \bar{\epsilon}] : [0, 4] \rightarrow \mathbb{R}^+$ such that:

- $\phi[L, \bar{\epsilon}](s) = 1 + \epsilon_1 f(s - 1.5)$ for $s \in [1 + \epsilon_2, 2 - \epsilon_2]$,

- $\phi[L, \bar{\epsilon}](1/L) = \phi[L, \bar{\epsilon}](4 - 1/L) = 1$,
- $\phi[L, \bar{\epsilon}]$ is increasing on $[0, 1.5]$ and decreasing on $[1.5, 4]$,
- $\phi[L, \bar{\epsilon}]$ is strictly concave on $[0, 1]$.

Define

$$\Gamma[L, \bar{\epsilon}] = \{x \in H[\gamma_L(s)]; s \in [0, 4], d(x, \gamma_L(s)) = d_0 \phi[L, \bar{\epsilon}](s)\}.$$

We can further impose that $\phi[L, \bar{\epsilon}]$ satisfies the following properties:

- $\Gamma[L, \bar{\epsilon}]$ is a smooth closed hypersurface,
- for all $L > 1$ and $\epsilon_1 \in (0, 1/4)$ being fixed, $\Gamma[L, \bar{\epsilon}]$ converges smoothly to a hypersurface $\Gamma[L, (\epsilon_1, 0)]$ and $\phi[L, \bar{\epsilon}]$ converges uniformly to a function $\phi[L, (\epsilon_1, 0)]$ when $\epsilon_2 \rightarrow 0$,
- the domains $\Delta[L, \epsilon_1] := \Gamma[L, (\epsilon_1, 0)] \cap \{(x_1, \dots, x_{n+2}); x_2 < -L\}$ are all isometric to each other for $L > 1$ and $\epsilon_1 \in (0, 1/4)$, and they have positive sectional curvature.

In the following lemma, we list some useful properties of the manifold $\Gamma[L, \bar{\epsilon}]$ for any $n \geq 1$:

- Lemma 71.** 1. *If $n > 1$ and if $d_0, \epsilon_1, \epsilon_2$ are small enough, $\Gamma[L, \bar{\epsilon}]$ has (arbitrarily large) positive scalar curvature bounded below by a positive constant independent of L .*
2. *The manifold $\Gamma[L, \bar{\epsilon}]$ has positive sectional curvature on*

$$\Gamma[L, \bar{\epsilon}] \cap \{(x_1, \dots, x_{n+2}); x_1 > 0 \text{ and } x_2 < -1\},$$

and on the open neighborhood of

$$\Gamma[L, \bar{\epsilon}] \cap \{(x_1, \dots, x_{n+2}); x_1 > 0 \text{ and } x_2 = -1/2\}$$

consisting of all points in $\Gamma[L, \bar{\epsilon}]$ at distance less than $\tilde{\delta} > 0$ from the above set, where $\tilde{\delta}$ is independent of L and $\bar{\epsilon}$.

3. Let $Z^a = \{x \in H[\gamma_L(a)]; d(x, \gamma_L(a)) = d_0\}$ and consider $Z^2, Z^{2.5}$ as hypersurfaces in $\Gamma[L, (\epsilon_1, 0)]$. Then

$$-\int_{Z^{2.5}} (R - \text{Ric}(\nu, \nu)) > -\int_{Z^2} (R - \text{Ric}(\nu, \nu)),$$

where ν denote a unit normal on these hypersurfaces and R (resp. Ric) is the scalar curvature of $\Gamma[L, (\epsilon_1, 0)]$ (resp. its Ricci curvature) endowed with the metric induced by $\mathbb{R}^{n+2}$.

7.3.2 Appearance of stable geodesics and stable spheres

In this subsection, we will use "stable sphere" (resp. "stable geodesic") to denote a closed stable embedded minimal 2-sphere (resp. a simple closed stable geodesic).

Theorem 72. *There exists a 2-sphere (M, g) such that*

1. (M, g) does not contain stable geodesics,
2. a stable geodesic appears along the Ricci flow starting at (M, g) .

Theorem 73. *Let M be a spherical space form S^3/Γ which is endowed with a metric g of positive scalar curvature. Suppose that (M, g) does not contain any stable sphere or embedded minimal $\mathbb{R}P^2$ with stable oriented double cover. Then for any point $p \in M$ and radius $r > 0$, there is a metric $\tilde{g}$ on M coinciding with g outside $B_g(p, r)$ such that*

1. $\tilde{g}$ has positive scalar curvature,
2. $(M, \tilde{g})$ does not contain any stable sphere or embedded minimal $\mathbb{R}P^2$ with stable oriented double cover,
3. a stable sphere appears along the Ricci flow starting at $(M, \tilde{g})$.

In the case where M is two-dimensional, we clearly cannot assume its Gauss curvature to be positive at time 0 since this property will be preserved along the Ricci flow and this will prevent the existence of stable geodesics.

From now on, we assume $d_0, \epsilon_1, \epsilon_2$ small enough so that by Lemma 71 (1), $R > 0$ on $\Gamma[L, \bar{\epsilon}]$ when $n = 2$. To prove Theorem 72, we will need the following lemma.

Lemma 74. *Let $n = 1$. There exists a positive constant C_0 such that if $L > C_0$ and $\epsilon_1 < 1/C_0$, then for all ϵ_2 sufficiently small the surface $\Gamma[L, \bar{\epsilon}]$ contains no stable geodesics.*

The next lemma is true for $1 \leq n \leq 6$. We fix $\epsilon_1 \in (0, 1/4)$ and $L > 1$. Let $\delta > 0$ and define

$$Y^{\epsilon_2} = \Gamma[L, (\epsilon_1, \epsilon_2)] \cap \{(x_1, \dots, x_{n+2}); x_1 < 0 \text{ or } x_2 > -\delta\}.$$

We choose $\delta \in (0, 1/2)$ so that the boundaries ∂Y^{ϵ_2} are isometric and convex for all $0 < \epsilon_2 < \delta$. Define also

$$X^{\epsilon_2} = \Gamma[L, (\epsilon_1, \epsilon_2)] \cap \{(x_1, \dots, x_{n+2}); x_1 < 0 \text{ or } x_2 > 0\}.$$

Similarly, we define Y and X by replacing $\Gamma[L, (\epsilon_1, \epsilon_2)]$ by $\Gamma[L, (\epsilon_1, 0)]$ in the above formulas and we write $Y^0 := Y$, $X^0 := X$. Now for $\epsilon_2 \in [0, \delta)$, suppose that Y^{ϵ_2} is isometrically embedded in a closed $(n+1)$ -manifold N^{ϵ_2} , in such a way that N^{ϵ_2} converges to N^0 as $\epsilon_2 \rightarrow 0$. Let $\{N_t^{\epsilon_2}\}_{t \in [0, T]}$ be a solution of the Ricci flow starting at N^{ϵ_2} defined on a time interval $[0, T)$. If V is a subset of N^{ϵ_2} , let V_t denote the Ricci flow at time t starting at V obtained by restriction of the original Ricci flow solution on N^{ϵ_2} . By abuse of notations, we view Y^{ϵ_2} as a subset of N^{ϵ_2} in Lemma 75.

In the proof, we will consider currents and varifolds in the closure $\bar{Y}$ which is isometrically embedded in $\mathbb{R}^{n+2}$. If U is an open subset of $\bar{Y}$, the corresponding $(n+1)$ -dimensional current will be called $[[U]]$ and if C is an integral current, $|C|$ will be the name of the integer rectifiable varifold it determines by forgetting its orientation. If $k \in [0, n+1]$, the Grassmannian of k -planes in $\mathbb{R}^{n+2}$ is denoted by $\mathbf{Gr}(k, n+2)$ and its restriction to Y is denoted by $\mathbf{Gr}(k, n+2, Y)$. The Hausdorff measure $\mathcal{H}^k$ of a subset of Y_t is computed using the metric on Y_t .

Lemma 75. *Suppose $1 \leq n \leq 6$ and let $\epsilon_1 \in (0, 1/4)$. There exists a positive constant C_1 such that if $L > C_1$, then the following holds. For all $\epsilon_2 \in [0, \delta)$ small enough, there is a positive time $t_0 = t_0(\epsilon_2)$ such that $Y_{t_0}^{\epsilon_2}$ contains an embedded stable minimal hypersurface. Moreover, t_0 can be chosen so that*

$$t_0(\epsilon_2) \rightarrow 0 \text{ as } \epsilon_2 \rightarrow 0.$$

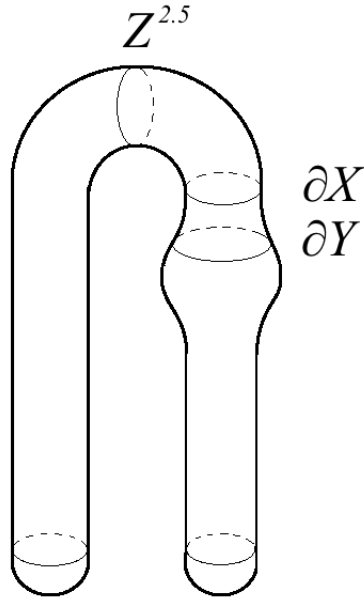


Figure 7.1: When ϵ_1 is small and L large, $\Gamma[L, (\epsilon_1, 0)]$ looks like a thin hook.

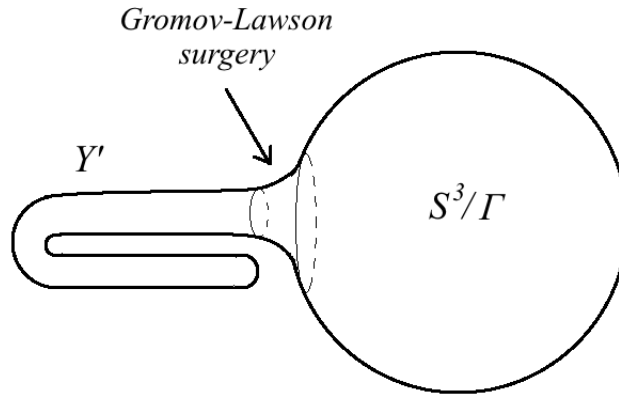


Figure 7.2: Part of a thin hook, Y' , is glued to S^3/Γ via the Gromov-Lawson procedure.

Proof of Theorem 72 and Theorem 73. Theorem 72 follows from Lemmas 74 and 75, by taking $M = \Gamma[L, (\epsilon_1, \epsilon_2)] = N^{\epsilon_2}$ with $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $1/L$ sufficiently small.

To prove Theorem 73, let (M, g) be as in the statement. Choose any $p \in M$ and $r > 0$ smaller than the injectivity radius of (M, g) so that $\partial B_g(p, s)$ is convex whenever $0 < s < r$. Let $r_0 > 0$ be smaller than r . Let ϵ_2 be positive, consider a scaled-down version of Y^{ϵ_2} (as defined just before Lemma 75) that we call Y' and glue it to M around p by applying the Gromov-Lawson construction in $B_g(p, r_0)$ (see Figure 7.2). We can modify the size of the following parameters: r_0 , δ , ϵ_2 , Y' . Analyzing how the forementioned construction is defined in [57] and taking the previous parameters small enough, we see that it can be done so as to get a new metric $\tilde{g}$ on M verifying:

- $\tilde{g}$ coincides with g outside $B_g(p, r_0)$,
- Y' is isometrically embedded in $(B_g(p, r_0), \tilde{g})$,
- $(B_g(p, r), \tilde{g})$ is foliated by convex spheres,
- the scalar curvature of $(M, \tilde{g})$ is bounded below by a positive constant independent of the parameters when the latter go to zero.

If the parameters are all small enough, then $(M, \tilde{g})$ contains no stable embedded minimal 2-sphere or minimal $\mathbb{R}P^2$ with stable oriented double cover. Otherwise we could take the limit (subsequently by [132]) and get a non-trivial oriented stable embedded minimal surface S in $(M \setminus \{p\}, g)$ with finite area by the fourth item above (see [100, Proposition A.1] for instance). By curvature estimates for stable surfaces ([132]), the embedding is then proper, S has finite Euler characteristic and the singularity at p is removable (see [88, Lemma 2.5], [25, Proposition C.1], [60] for instance). Hence the closure $\bar{S}$ is a smooth stable minimal 2-sphere or a minimal $\mathbb{R}P^2$ with stable oriented double cover in (M, g) , contradicting our assumption. Finally, Lemma 75 ensures that a stable sphere appears along the Ricci flow starting at $(M, \tilde{g})$ provided $1/L$, ϵ_2 are small enough.

□

7.3.3 Appearance of non-trivial singularities

Theorem 76. *Let M be a closed 3-manifold satisfying the hypotheses of Theorem 73. Then for any point $p \in M$ and radius $r > 0$, there is a metric $\hat{g}$ on M coinciding with g*

outside $B_g(p, r)$ such that

1. $\hat{g}$ has positive scalar curvature,
2. $(M, \hat{g})$ does not contain any stable sphere or embedded minimal $\mathbb{R}P^2$ with stable oriented double cover,
3. a non-trivial singularity occurs along the Ricci flow starting at $(M, \hat{g})$.

For the appearance of stable spheres described in the previous subsection, a first order argument on the evolution of the thin hooks was enough. However we now want to study the long-time behavior and to prove that a non-trivial singularity occurs, we need to modify the metric of the hooks. We twist and stretch the bent part as follows. For all $\bar{\epsilon}$, L and for all $a \in [2, 3]$, using the previous notations we consider the subsets

$$Z^a[L, \bar{\epsilon}] = \{s \in H[\gamma_L(a)]; d(x, \gamma_L(a)) = d_0\phi[L, \bar{\epsilon}](a)\} \quad a \in [2, 3].$$

Let $\beta : [2, 3] \rightarrow [0, 1]$ be a bump function equal to zero in a neighborhood of $\{2, 3\}$ and equal to one in $[2+1/12, 3-1/12]$. Denote by g_{eucl} the Euclidean metric in $\mathbb{R}^{n+2}$. For any $x \in Z^a[L, \bar{\epsilon}]$ (with $a \in [2, 3]$), let $V(x)$ be a unit vector based at x tangent to $\Gamma[L, \bar{\epsilon}]$ but normal to the hypersurface $Z^a[L, \bar{\epsilon}]$, in the metric induced by g_{eucl} . Then for any stretching factor $L_{st} \geq 0$, we define a new metric $g(L_{st})$ on $\Gamma[L, \bar{\epsilon}]$ such that it only differs from the metric induced by g_{eucl} in $\bigcup_{a \in [2, 3]} Z^a[L, \bar{\epsilon}]$ and for all $x \in \bigcup_{a \in [2, 3]} Z^a[L, \bar{\epsilon}]$, $u, v \in T_x \Gamma[L, \bar{\epsilon}] \subset T_x \mathbb{R}^{n+2}$:

$$g(L_{st})(u, v) = g_{eucl}(u, v) + \beta(a)L_{st}g_{eucl}(u, V(x))g_{eucl}(v, V(x)).$$

Hence the modified metric $g(L_{st})$ is similar to the metric induced by g_{eucl} , but strongly twisted and stretched between Z^2 and Z^3 when L_{st} is large. The choice of β guarantees that when L_{st} goes to infinity and ϵ_2 goes to zero, the other parameters being fixed,

- $(\Gamma[L, \bar{\epsilon}], g(L_{st}))$ converges locally around $Z^{2+1/12}$ to $S^2 \times \mathbb{R}$ endowed with the product metric $d_0^2 h_1 + d\theta^2$, where h_1 is the round metric of Gauss curvature 1,
- $(\Gamma[L, \bar{\epsilon}], g(L_{st}))$ converges locally around $Z^{2.5}$ to a warped product metric g_{tw} on $S^2 \times \mathbb{R}$ different from a product metric.

Note nevertheless that in the second limit any slice $S^2 \times \{\theta\}$ is also endowed with the round metric $d_0^2 h_1$. The Ricci flow for warped product metrics on $S^2 \times \mathbb{R}$ which are $\mathbb{R}$ -invariant (and hence with base S^2) has a well-controlled behavior and was studied in [96]. For a metric g on a 3-manifold, let $T_{ext}(g) \in (0, \infty)$ be its extinction time when well-defined: when it exists it is defined as the time where a trivial singularity occurs. The following is a key lemma explaining why we consider these twisted hooks.

Lemma 77. *Let g_{inv} be an $\mathbb{R}$ -invariant warped product metric on $S^2 \times \mathbb{R}$. Suppose that g_{inv} is not a product metric and that any slice $S^2 \times \{\theta\} \subset S^2 \times \mathbb{R}$ has area $4\pi d_0^2$ computed with g_{inv} . Then*

$$T_{ext}(g_{inv}) > T_{ext}(d_0^2 h_1 + d\theta^2).$$

Heuristically, to make a singularity appear, we will choose the stretching factor L_{st} very large so that there are two regions evolving locally like two $\mathbb{R}$ -invariant $S^2 \times \mathbb{R}$, one of them being endowed with a product metric and separating the other one from a large region (to which we glued the twisted hook). Since the previous lemma suggests that the neck $S^2 \times \mathbb{R}$ with a product metric should disappear first while the other regions stay large, a non-trivial singularity should occur. Let us make this reasoning rigorous with the following lemma.

Lemma 78. *There exists a constant $\hat{C} > 0$ and a time $\hat{T} > 0$ such that the following holds. Let $(N, g(t))$, $0 \leq t \leq t_1$ be a solution of the Ricci flow, assume that the initial metric $g(0)$ is normalized and that N is a closed oriented connected 3-manifold. Suppose that at t_1 , $x \in N$ is in the center of a strong ϵ -neck. Suppose that the center sphere of this strong ϵ -neck separates N into two components N_1, N_2 such that there are $x_i \in N_i$ ($i = 1, 2$) with*

$$R(x, t_1) > \hat{C}(1 + |R(x_i, t_1)|),$$

where $R(\cdot, t_1)$ is the scalar curvature function at time t_1 . Then, along the Ricci flow starting at $(N, g(0))$, a non-trivial singularity occurs before time $t_1 + \hat{T}$.

Proof of Theorem 76. First, we glue a small twisted hook to M around a point p as in the proof of Theorem 73. If $\epsilon_1, \epsilon_2, 1/L$ and the size of the twisted hook are sufficiently small, then the new metric $\tilde{g}$ on M does not contain any stable sphere or minimal $\mathbb{R}P^2$ with stable oriented double cover and has positive scalar curvature. We take care of

rescaling the new metric so that it becomes normalized. Let (M_k, g_k) be a sequence of such rescalings, where the parameters ϵ_1 , ϵ_2 , $1/L$ and the size of the hook go to 0. It is also possible to guarantee that for any sequence $\mathfrak{s} = \{x_k\}$ with $x_k \in M_k$, the based manifolds (M_k, g_k, x_k) converge to one of the following geometric limits:

- (a) the flat $\mathbb{R}^3$ (corresponding to points x_k not near the hook),
- (b) a rotationally symmetric non-compact 3-manifold with two ends, one being a standard product metric on $S^2 \times [0, \infty)$ with scalar curvature 1 and the other one being a flat $\mathbb{R}^3 \setminus B(0, 1)$ (correspond to x_k near the part where the hook is glued),
- (c) a warped product on $S^2 \times \mathbb{R}$ with base a round S^2 with scalar curvature 1 (corresponding to x_k inside the hook far from the tip),
- (d) an $\mathbb{R}^3$ endowed with the standard initial metric (see [114, Chapter 12]) (corresponding to x_k near the tip of the hook).

Let T be the maximum of the maximal times for which the Ricci flows starting at one of these four metrics are smoothly defined. By [22] and [137], the hypotheses of Lemma 69 are satisfied. Notice that if $x_k \in Z^{2+1/12}$ (resp. $Z^{2.5}$) for all k then the geometric limit is a product metric (resp. non-trivial warped product metric) on $S^2 \times \mathbb{R}$, whose life span under the Ricci flow is equal (resp. strictly longer) than that of the standard initial metric by [114, Theorem 12.5] (resp. Lemma 77).

Suppose by contradiction that no non-trivial singularity occurs along the Ricci flow starting at $(M, \tilde{g})$. Two cases are a priori possible: $T = 1$ the life span of the standard initial metric, or $T < 1$. The latter situation corresponds to the maximum of the scalar curvature being reached around the gluing part near T , namely it means that T is the maximal existence time for the second geometric limit in the previous list. Note that this Ricci flow being rotationally symmetric with two ends, the only canonical neighborhood that can appear is a strong ϵ -neck. By the above remarks and Lemma 69, in both cases one finds $\delta > 0$ so that for all k large, the Ricci flows $(M_k, g_k(t))$ have no singularity until at least $\hat{t} := T - \delta$, time at which for some $q, q_1 \in M_k$, and for any $q_2 \in Z^{2.5}$:

- for $i = 1, 2$, $R(q, \hat{t}) > \hat{C}(1 + R(q_i, \hat{t}))$ ($\hat{C}$ being the constant in Lemma 78),
- q is in a strong ϵ -neck whose central sphere separates q_1, q_2 .

Actually, q_1 is chosen to be a point of M far from p where the gluing is realized in the original metric g . The hypothesis of Lemma 78 are satisfied and a non-trivial singularity occurs, which contradicts our assumption that only a trivial singularity occurs.

□

Remark 7.3.1. *1. In the proof of Theorem 73, we used hooks with a stretching factor $L_{st} = 0$ for simplicity. However, it is not difficult to check that $\bar{\epsilon}$ and L can be chosen so that for any stretching factor L_{st} , Lemma 75 remains true. Hence putting Theorems 73 and 76 together, we conclude that there are 3-manifolds with positive scalar curvature such that along the Ricci flow, a stable sphere appears and some time later, a non-trivial singularity occurs. In particular Theorem 63 is proved.*

2. Although according to Theorem 76, a non-trivial singularity occurs in certain cases, it does not provide information on where it happens: intuitively one expects the singularity to occur at the neck with a product metric or at the tip of the twisted hook or at both places, depending on the shape of the tip.

7.4 Stable spheres and Type I singularities

In [5], examples of rotationally symmetric S^{n+1} developing a Type I neckpinching are constructed. Actually in dimension 3, this is part of a much more general fact. By joining any two oriented 3-manifolds with a thin neck, we obtain an initial data which will produce a non-trivial Type I singularity under the Ricci flow.

Proposition 79. *Let (M_1, g_1) , (M_2, g_2) be two closed oriented 3-manifolds. For any pair of points $p_i \in M_i$ ($i = 1, 2$), radius $\hat{r} > 0$ small enough, length $l \geq 0$ and $\delta > 0$, there exists a metric g on the connected sum $M = M_1 \# M_2$ such that:*

- 1. there is a subset $N \subset M$ diffeomorphic to $S^2 \times (0, 1)$ so that $M \setminus N$ is isometric to $(M_1 \setminus B(p_1, \hat{r})) \cup (M_2 \setminus B(p_2, \hat{r}))$,*
- 2. M is δ -close in the Hausdorff-Gromov distance to the union of M_1 , M_2 and a curve of length l joining p_1 to p_2 ,*
- 3. a non-trivial singularity of Type I occurs along the Ricci flow starting at (M, g) .*

From the proof of Proposition 79, it can be shown for the examples where a Type I singularity appears at some time t_1 that for all $t \in [0, t_1)$ there is an embedded stable minimal sphere $S(t)$ whose area goes to 0 as $t \rightarrow t_1$. These spheres correspond to the neckpinching. One can wonder if this is a general phenomenon. The next theorem confirms that indeed small stable spheres or $\mathbb{R}P^2$ with stable oriented double cover are closely related to Type I singularities. When a minimal surface is an embedded stable sphere or an embedded $\mathbb{R}P^2$ with stable oriented double cover, we will call it a *stable immersed sphere with embedded image*.

Theorem 80. *Let M be an oriented closed connected 3-manifold. Consider a Ricci flow $(M, g(t))$, $0 \leq t < T$ and suppose that there is a non-trivial Type I singularity at time T . Then for all time t close to T , $(M, g(t))$ contains a stable immersed sphere with embedded image $S(t)$ such that*

$$C'(T - t) \leq \mathcal{H}^2(S(t)) \leq C''(T - t),$$

where C', C'' are constants independent of t .

Conversely, suppose that there is a sequence of times s_k converging to T and a sequence of stable immersed spheres with embedded image S_k in $(M, g(s_k))$. Suppose also that the area of S_k goes to zero and

$$\mathcal{H}^2(S_k) \geq C'(T - s_k),$$

where C' is a constant independent of k . Then there is a singularity at time T and it is locally of Type I in the following sense:

$$\forall A > 0, \exists \bar{C} = \bar{C}(C', A, C, \epsilon), \forall k > \bar{C},$$

$$\sup\{|Rm(x, s_k)|; (\max_{S_k} R) \cdot d(x, S_k)^2 \leq A\} \leq \frac{\bar{C}}{T - s_k}.$$

Remark 7.4.1. *From the proof of the previous theorem, it becomes clear that when there is a sequence of stable immersed spheres with embedded image S_k at times s_k going to T , with area converging to 0, then the minimum of the scalar curvature on these spheres, $\min_{S_k} R$, goes to infinity and $\max_{S_k} R / \min_{S_k} R$ is bounded.*

7.5 Symmetry and non-appearance of stable spheres

In this section, we study under which symmetry assumptions one can rule out the appearance of stable immersed spheres with embedded image along the Ricci flow. In the case of a finite group G acting effectively by isometries on a 3-manifold, there is a point p which is fixed only by the identity and one can glue disjoint thin hooks at the images of p under the elements of G , in an equivariant way. This gives a G -invariant metric for which stable spheres appear along the Ricci flow. Hence, we will only focus on positive dimensional compact Lie groups. Consider (M, g) , an oriented connected closed 3-manifold on which a d -dimensional compact Lie group G of isometries acts effectively. Assume that

1. either $d > 1$,
2. or $d = 1$ and the action is free.

We will say that (M, g) (as above) is rotationally symmetric if a subgroup G_0 of G is isomorphic to $SO(3)$ and there is a G_0 invariant 2-sphere or $\mathbb{R}P^2$ embedded in (M, g) . This amounts to saying that a cover of (M, g) is a warped product $I \times S^2$ with fiber S^2 , where $I = \mathbb{R}$ or $I = [0, 1]$ (the warped product is then degenerate at 0, 1). In that case, M is diffeomorphic to S^3 , $\mathbb{R}P^3$, $\mathbb{R}P^3 \# \mathbb{R}P^3$ or $S^2 \times S^1$.

Theorem 81. *Let (M, g) be as above. Suppose that it contains no stable immersed spheres with embedded image. Then, along the Ricci flow starting at (M, g) , stable immersed spheres with embedded image cannot appear.*

Remark 7.5.1. *1. Note that the 3-dimensional (twisted) hooks defined in Section 7.3 have an effective S^1 -action which is not free, so according to Theorem 81 these examples where stable spheres appear have in some sense a maximal amount of symmetry.*

2. A byproduct of the proof of Theorem 81 is that if (M, g) (as above) contains a stable immersed sphere with embedded image S and if (M, g) is not rotationally symmetric, then it is an $S^2 \times S^1$ foliated by stable spheres which are images of S under a family of isometries.

Lemma 82. *Let (M, g) be as above. If a Type II singularity occurs then (M, g) is a rotationally symmetric sphere or $\mathbb{R}P^3$.*

Because of the link between Type I singularities and stable spheres described in Section 7.4, we readily obtain the following corollary.

Corollary 83. *Let (M, g) be as above. The following holds along the Ricci flow.*

1. *When M is a rotationally symmetric 3-sphere and does not contain stable spheres, then no non-trivial Type I singularity occurs.*
2. *When M is a rotationally symmetric $\mathbb{R}P^3$ and does not contain stable immersed spheres with embedded image, then no non-trivial Type I singularity occurs.*
3. *When M is rotationally symmetric and neither a 3-sphere nor an $\mathbb{R}P^3$, no Type II singularity occurs.*
4. *When M is not rotationally symmetric and if a singularity occurs, then it is a Type I trivial singularity.*

A question still left unanswered is whether a Type II singularity can appear in the case of item (1). In item (3), the other kinds of singularities can occur. The first item was proved in [58] for all dimensions. In the case of a free S^1 action, it was already suggested in [96, Remark 2.6] to combine the singularity analysis with the symmetry.

Chapter 8

Embeddedness of least area minimal hypersurfaces

A few decades ago, Calabi-Cao showed that on a convex sphere, the shortest closed geodesics are simple. In dimension 3, Freedmann-Hass-Scott showed that in a homotopy class, a least area immersed surface minimizes self-intersection in a certain sense. In this chapter, we prove an analogue of Calabi-Cao's theorem for higher dimensions, without curvature assumptions: least area immersed minimal hypersurfaces are embedded.

8.1 Introduction

Let (M^{n+1}, g) be a closed $(n+1)$ -manifold with $2 \leq n \leq 6$. Consider a closed n -manifold Γ and an immersion $\psi : \Gamma \rightarrow M$. If ψ is minimal then $\psi(\Gamma) = \Sigma$ is called a minimal hypersurface and Σ is said to be a minimal hypersurface of least area if for every minimal hypersurface $\Sigma' \subset M$,

$$\mathcal{H}^n(\Sigma) \leq \mathcal{H}^n(\Sigma').$$

We will only consider closed minimal hypersurfaces. Minimal hypersurfaces which minimize the area among a family of *embedded* minimal hypersurfaces have been previously examined in some situations related to min-max theory. In [100], Marques and Neves initiated the characterisation of the area and the Morse index of surfaces produced by

min-max theory in three-manifolds: among other things, they showed that in a closed oriented three-manifold with positive Ricci curvature, with Heegaard genus h and which does not contain any embedded non-orientable surface, there is an index one minimal surface Σ of genus h produced by the smooth min-max theory (see [29]) such that

$$\mathcal{H}^2(\Sigma) = \inf_{S \in \mathcal{E}_h} \mathcal{H}^2(S),$$

where $\mathcal{E}_h$ denotes the collection of all connected embedded minimal surfaces of genus not larger than h . This result was later extended to higher dimensions by Zhou [156] in the following way. Let M^{n+1} be a closed oriented manifold with $2 \leq n \leq 6$. Let $\mathcal{O}$ be the collection of all embedded orientable closed minimal hypersurfaces in M and $\mathcal{U}$ be the collection of the non orientable ones. Zhou studied the minimal hypersurfaces produced by Almgren-Pitts' theory applied to the fundamental class $[M]$, under the assumption of positive Ricci curvature. It turns out in particular that their area counted with multiplicity is

$$\mathcal{A}_1(M) = \inf(\{\mathcal{H}^n(\Sigma); \Sigma \in \mathcal{O}\} \cup \{2\mathcal{H}^n(\Sigma); \Sigma \in \mathcal{U}\}).$$

Recently, Mazet and Rosenberg [107] proved that, without assumptions on the Ricci curvature, this area $\mathcal{A}_1(M)$ is realized by a hypersurface and classified all such hypersurfaces: either they are stable or they are of index one and produced by min-max constructions. The forementioned authors defined the quantity $\mathcal{A}_1(M)$ by separating orientable and non-orientable minimal hypersurfaces. This is due to technical reasons coming from their use of Almgren-Pitts' min-max theory. In this chapter we do not need to make this distinction and we are interested in least area minimal hypersurfaces, where "least area" is taken in the geometric sense. The aim is to prove by min-max methods that in M , any least area minimal hypersurface is actually embedded. It extends a result of Calabi and Cao [17].

Theorem 84. *Let (M^{n+1}, g) be a closed $(n+1)$ -manifold with $2 \leq n \leq 6$. Then there exists a least area closed minimal hypersurface $\Sigma \subset M$ and any such Σ is embedded.*

We state in Section 8.3 a more detailed version, which precises that, similarly to [107], the least area minimal hypersurfaces are either embedded stable hypersurfaces or given

by Almgren-Pitts' min-max theory applied to the fundamental class $[M]$. Interestingly, no assumptions on the curvature are needed here whereas in dimension $n = 1$ there are counterexamples if the curvature of the sphere is not nonnegative. This is related to the fact that min-max theory produces embedded hypersurfaces for $n = 2, \dots, 6$ but only immersed geodesics for $n = 1$.

In [17], Calabi and Cao proved that on a Riemannian sphere $\mathcal{S}$ with nonnegative curvature, any non-trivial closed geodesic of the shortest length is simple. Their proof is based on the fact that if $\gamma \subset \mathcal{S}$ is a closed geodesic which is not simple, then there is a nontrivial loop $\{\sigma_t\}_{t \in [0,1]}$ in the space of one-dimensional integral currents without boundary such that each σ_t has mass less than the length of γ . Then by the min-max theory of Almgren and Pitts, there exists a geodesic strictly shorter than γ . Actually they noted that, in dimension one, the min-max principle can be shown simply without Geometric Measure Theory. Their construction of the "sweepout" family $\{\sigma_t\}$ relies in an essential way on the dimension of the problem: indeed for curves, there is an ad hoc shortening procedure known as the Birkhoff curve shortening process. It turns out that there is already no equivalent process for surfaces. The proof of Theorem 84 will retain the main structure of the proof of Calabi and Cao, but we have to replace the Birkhoff process by a method developed by Marques and Neves in [100] when $n = 2$ and extended to $2 \leq n \leq 6$ by Zhou in [156]. The restriction on the dimension is a classical one due to the regularity results of Schoen and Simon [133]. The structure of an immersed minimal hypersurface ($n \geq 2$) is less understood than the case of geodesics, and it will add some technical issues.

As a corollary of Theorem 84, we can prove the following result conjectured by Marques and Neves in [100]: it asserts that a positive lower bound on the scalar curvature of a closed three-manifold M gives a rigid upper bound on the area of the smallest embedded closed minimal surface in M .

Theorem 85. *Let M^3 be a closed three-manifold with scalar curvature R at least 6, not isometric to the round unit three-sphere S^3 . Then there exists a closed embedded minimal surface Σ of index zero or one such that*

$$\mathcal{H}^2(\Sigma) < 4\pi.$$

Marques and Neves already proved this theorem with the additional assumption that the Ricci curvature is positive. In their proof, they use the min-max theory of Simon-Smith and Hamilton's article on the Ricci flow for three-manifolds with positive Ricci curvature. Our proof is based on the theory of Almgren-Pitts and only uses the short-time existence theorem for the Ricci flow.

This chapter is organized as follows: in Section 8.2 are reviewed the basic notions of Almgren-Pitts' theory as well as the continuous min-max theory, then some min-max constructions are presented and an outline of the principal proof sums up the strategy and the differences with [17]. The proof of Theorem 84 is given in Section 8.3 and we explain in Section 8.4 how Theorem 85 is a consequence of the main theorem.

8.2 Preliminaries

In this section, we present some useful min-max results. Afterwards, a heuristic description of the main steps in the proof of Theorem 84 is given as well.

8.2.1 Some min-max constructions

Let (M, g) be a closed $(n + 1)$ -manifold with $2 \leq n \leq 6$. For a two-sided hypersurface Σ , the convention is that the mean curvature vector is $-H(\Sigma)\nu$, where ν is a continuous choice of unit normal vector of Σ , called outward unit normal. A hypersurface with boundary will be said to be generically immersed if it is the image of an immersion with normal crossings, in other words a "generic" immersion (see Definition 3.1 in [52]).

Definition 8.2.1. *Let N be an open subset of M and suppose that ∂N is a non-empty rectifiable hypersurface. The boundary ∂N is said to be piecewise smooth mean convex if it satisfies the following property:*

- (i) *there is a generically immersed compact two-sided n -submanifold F with smooth boundary such that N is a connected component of $M \setminus F$ and*

$$\partial N \cap \partial F = \emptyset,$$

- (ii) *F has positive mean curvature at every point of ∂N with respect to any outward*

unit normal determined by N .

More generally, a rectifiable hypersurface A is said to be piecewise smooth mean convex if there is an open set $N \subset M$ such that A is an open subset of ∂N , the first point (i) is true and the second point (ii) is satisfied for every point of A .

We emphasize that (piecewise smooth) mean convexity means that the mean curvature is strictly positive. The first property of a piecewise smooth mean convex boundary is that it acts as a barrier for area minimizing problems.

Proposition 86. *Let (M, g) be a closed $(n + 1)$ -manifold and N an open set of M such that the boundary ∂N is non-empty and piecewise smooth mean convex. Then there is a positive number $\delta > 0$ and a vector field $\mathbf{V}$ in N such that*

(i) $\sum_{i=1}^n \langle \nabla_{e_i} \mathbf{V}, e_i \rangle \leq 0$ for every orthonormal family $\{e_1, \dots, e_n\}$ of vectors whose base point is in N ,

(ii) if $p \in N$ and $d(p, \partial N) = d \leq \delta$, then

$$\langle \mathbf{V}(p), \frac{\partial}{\partial s} \Big|_{s=d} \exp_q(-s\nu) \rangle > 0,$$

where q is any point of ∂N such that $d(p, q) = d$ and ν is the outward unit normal of ∂N at q (which is well defined).

Consequently, if Σ is a hypersurface in N , then there is a diffeomorphism Ψ of N such that $\mathcal{H}^n(\Psi(\Sigma)) \leq \mathcal{H}^n(\Sigma)$ and $d(\Psi(\Sigma), \partial N) \geq \delta/2$.

Thanks to the previous proposition, we can check the following version of the min-max theorem.

Theorem 87. *Let (M, g) be a closed $(n + 1)$ -manifold with $2 \leq n \leq 6$, and N an open set of M possibly with a non empty topological boundary. When $\partial N \neq \emptyset$, assume that ∂N is piecewise smooth mean convex. Then for any homotopically closed family Λ of sweepouts of N , with*

$$W(N, \partial N, \Lambda) > \mathcal{H}^n(\partial N)$$

when $\partial N \neq \emptyset$, there exists a min-max sequence $\{\Sigma_{t_k}^k\}$ of Λ converging in the varifold

sense to an embedded minimal hypersurface Σ (possibly disconnected), contained in N . Moreover the n -volume of Σ , if counted with multiplicities, is equal to $W(N, \partial N, \Lambda)$.

We will say that a hypersurface Σ is produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ if the varifold $|\Sigma|$ belongs to the critical set $\mathbf{C}(S)$ of a sequence $S \in \Pi$, where $\Pi \in \pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ corresponds to the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$ by the isomorphism (2.1) and Remark 2.1.1. The following proposition will be useful.

Proposition 88. *Let Σ be an embedded connected unstable minimal hypersurface in M and suppose that there is no embedded stable minimal hypersurface S with $\mathcal{H}^n(S) < \mathcal{H}^n(\Sigma)$. Then there is a minimal hypersurface Γ produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$ such that*

$$\mathcal{H}^n(\Gamma) \leq \mathcal{H}^n(\Sigma)$$

and if equality holds, then Σ itself is produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$.

8.2.2 Outline of the proof of Theorem 84

M is a closed $(n+1)$ -manifold M with $2 \leq n \leq 6$. The main step of the proof is to show that if a least area hypersurface exists, then it is necessarily embedded (Proposition 94). This step is essentially an extension of the idea in [17] to higher dimensions. Given a minimal hypersurface Σ which is not embedded and with no stable embedded minimal hypersurface of less area, we construct an "optimal" non-trivial sweepout corresponding to Σ in the sense that Σ is the middle slice of a mapping $A : I \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology and other slices have area strictly less than $\mathcal{H}^n(\Sigma)$. This way of thinking Σ as the middle slice of an optimal sweepout in order to compare it with other sweepouts and deduce properties about the embeddedness, the index, the area or the multiplicity of Σ appears in [17], [100], [156] and [107]. We will first show that there is a partition of the family of connected components of $M \setminus \Sigma$ into two classes $\mathcal{C}_1$ and $\mathcal{C}_2$, such that

$$\Sigma = \bigcup_{c \in \mathcal{C}_i} \partial c \text{ for } i \in \{1, 2\}.$$

Roughly speaking, A is then obtained by retracting Σ to $\emptyset$ in two different ways corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$. The construction of this 1-parameter family of currents is based on Theorem 87. Then we use the interpolation proved by Marques and Neves in [101] for $n = 2$, later checked for higher dimensions by Zhou [156]. It enables to obtain, from a family of currents continuous in the flat topology, a homotopy sequence of mappings, to which one can apply Pitts' theory with some precise control on the obtained discrete slices. This interpolation theorem applied to A will give the wanted S . The latter belongs to a homotopy class of mappings into $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ called Π such that

$$0 < \mathbf{L}(\Pi) < \mathcal{H}^2(\Sigma).$$

The theory of Almgren and Pitts then produces an embedded minimal hypersurface whose area is strictly less than $\mathcal{H}^n(\Sigma)$, which completes the main step. In these arguments, we use flat chains modulo 2 (see Remark 2.1.2) because we work with non necessarily orientable submanifolds.

To illustrate what are the two ways of retracting Σ , let's consider the following 1-parameter family of currents $\varphi : [-1, 1] \rightarrow \mathcal{Z}_n(\mathbb{R}^{n+1}, \mathbb{Z}_2)$:

$$\varphi(t) = \begin{cases} \partial\{x_1 \leq t, x_2 \leq t\} \cup \partial\{x_1 \geq -t, x_2 \geq -t\} & \text{for } t < 0 \\ \partial\{x_1 \geq t, x_2 \leq -t\} \cup \partial\{x_1 \leq -t, x_2 \geq t\} & \text{for } t \geq 0 \end{cases}$$

where $x_1, x_2, \dots, x_{n+1}$ are the coordinate functions of $\mathbb{R}^{n+1}$. Let's describe what is happening in the plane generated by x_1 and x_2 . During the first half $[-1, 0[$, the boundaries of the South-West and North-East corners get closer until they meet along the "edge" $\{x = 0, y = 0\}$ and during the second half $[0, 1]$, the boundaries of the North-West and South-East corners move away one from another. This family of currents will be the local model for the construction of the deformation described previously.

When one tries to extend the proof in [17] to $n \geq 2$, several technical issues arise in higher dimension. Firstly, Calabi and Cao only had to deal with curves which form an eight, whereas here Σ is not a priori so nicely immersed and we have to understand why there is still a good partition of the components of $M \setminus \Sigma$ into two classes (see Lemma 90). Secondly in order to construct the optimal sweepout, we have to use the technics in [100]

and [156], but they are only developed for the smooth embedded case and here, we have to work with components of $M \setminus \Sigma$ whose boundaries are only rectifiable (see Theorem 87 and Proposition 93).

8.3 Proof of Theorem 84

All minimal hypersurfaces considered here are closed. Define the following quantities:

$$\mathfrak{A}(M) = \inf \{ \mathcal{H}^n(\Sigma'); \Sigma' \subset M \text{ is a minimal hypersurface} \},$$

$$\mathfrak{A}_S(M) = \inf \{ \mathcal{H}^n(\Sigma'); \Sigma' \subset M \text{ is minimal, stable and embedded} \}.$$

By definition, $\mathfrak{A}_S(M) \geq \mathfrak{A}(M)$. In contrast to [156] and [107], we will not take care of the orientability, thus here "of least area" is to be understood in the geometric sense. We will prove the following more precise version of Theorem 84:

Theorem 89. *Let M be a closed $(n+1)$ -manifold where $2 \leq n \leq 6$. Then there exists a least area minimal hypersurface Σ_0 , i.e.*

$$\mathcal{H}^n(\Sigma_0) = \mathfrak{A}(M).$$

Such a hypersurface is either an embedded stable minimal hypersurface or a two-sided hypersurface of index one produced by the min-max theory of Almgren-Pitts with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$. In particular it is always embedded.

Let M be a closed $(n+1)$ -manifold where $2 \leq n \leq 6$. Let $\Sigma = \psi(\Gamma)$ be a minimal hypersurface, image of a closed n -manifold Γ by the immersion ψ . The latter will be supposed to be without "double cover", i.e. there is not a pair (U_1, U_2) of disjoint open sets in Γ such that $\psi(U_1) = \psi(U_2)$; since Σ is a minimal hypersurface, it is always possible to choose ψ without double cover. Denote by $\mathcal{C}_\Sigma$ the set of connected components of $M \setminus \Sigma$. The complement in M of an immersed hypersurface may be quite complicated in general but it is well described if the immersion is a map with normal crossings (see [52, Definition 3.1]), that is, a "generic" immersion.

Define the closed set

$$E = \{x \in \Sigma; \psi^{-1}(\{x\}) \text{ has at least two distinct elements}\} \quad (8.1)$$

to be the set along which Σ self-intersects. Two minimal hypersurfaces tangentially intersecting at a point can be locally written as graphs of two functions whose difference satisfies a homogeneous elliptic equation. The explicit equation is derived in [35, Chapter 7, §1] for instance (the authors do it for $n + 1 = 3$ but an analogue equation is clearly true for higher dimensions). By the description of nodal sets for elliptic equations in [7] or [66, Theorem 1.10], it is known that E is an $(n - 1)$ -rectifiable set: more precisely, the set where Σ intersects itself tangentially is an $(n - 2)$ -rectifiable set.

Lemma 90. *Let Σ be a non-embedded minimal hypersurface in M such that $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$. Then there is a partition of $\mathcal{C}_\Sigma$ into 2 classes $\mathcal{C}_1$ and $\mathcal{C}_2$ such that if c_1, c_2 are in $\mathcal{C}_\Sigma$, if for $p \in \Sigma \setminus E$ and $r > 0$:*

$$(B(p, r) \setminus \Sigma) \subset (c_1 \cup c_2),$$

then c_1 and c_2 are not in the same class. In particular, for $i \in \{1, 2\}$,

$$\mathcal{H}^n(\Sigma) = \sum_{c \in \mathcal{C}_i} \mathcal{H}^n(\partial c).$$

Moreover, such a partition is uniquely determined.

If $\psi : \Gamma \rightarrow M$ is a minimal immersion of an n -dimensional manifold into (M, g) such that $\psi(\Gamma)$ is two-sided, the Jacobi operator is given by

$$L\phi = \Delta\phi + |\mathbf{A}|^2\phi + \text{Ric}_g(\nu, \nu)\phi,$$

where $\phi \in C^\infty(\Gamma)$, $\mathbf{A}$ is the second fundamental form, and ν is a choice of outward unit normal of $\psi(\Gamma)$ defined on Γ . We will adopt the convention that λ is an eigenvalue of L if there exists a non-zero function ϕ such that $L\phi + \lambda\phi = 0$. Moreover, if $f \in C^\infty(\Gamma)$, define the map

$$\text{exp}_{\psi, f} : \Gamma \rightarrow M$$

$$\text{e}\tilde{\text{x}}\text{p}_{\psi,f}(x) = \exp_{\psi(x)}(f(x)\nu(x)).$$

When $\|f\|_\infty$ is small, $\text{e}\tilde{\text{x}}\text{p}_{\psi,f}$ is an immersion. Besides there is a unique choice of unit normal on

$$\text{e}\tilde{\text{x}}\text{p}_{\psi,r f}(\Gamma) \quad r \in [0, 1]$$

which is continuous with respect to $r \in [0, 1]$ and coinciding with ν at $r = 0$. Thus $\text{e}\tilde{\text{x}}\text{p}_{\psi,f}(\Gamma)$ is endowed with a natural choice of outward unit normal coming from ν and still called ν . It is known that at each point of Γ :

$$\left. \frac{\partial}{\partial r} \right|_{r=0} \langle \vec{H}(\text{e}\tilde{\text{x}}\text{p}_{\psi,r f}(\Gamma)), \nu \rangle = L(f), \quad (8.2)$$

where ν denote the natural choice of outward unit normal on $\text{e}\tilde{\text{x}}\text{p}_{\psi,r f}(\Gamma)$ coming from ν .

Let $\psi : \Gamma \rightarrow M$ be an immersion of a connected n -dimensional manifold Γ into (M, g) . If $\psi(\Gamma)$ is two-sided, then let $\tilde{\Gamma} = \Gamma_1 \cup \Gamma_2$ be two copies of Γ and define $\nu : \tilde{\Gamma} \rightarrow TM$ to be a continuous choice of an outward unit normal such that ν restricted to Γ_1 gives the opposite choice of ν restricted to Γ_2 . If $\psi(\Gamma)$ is one-sided, then let $\tilde{\Gamma}$ be a connected double cover of Γ such that there exists a continuous choice of outward unit normal $\nu : \tilde{\Gamma} \rightarrow TM$. Denote by $\pi : \tilde{\Gamma} \rightarrow \Gamma$ the canonical projection. A function ϕ defined on M (or Γ , or $\otimes^p T\Gamma$) lifts to a function on $\tilde{\Gamma}$ (or $\tilde{\Gamma}$, or $\otimes^p T\tilde{\Gamma}$) still denoted by ϕ . Consider the immersion $\tilde{\psi} = \psi \circ \pi$.

Usually if one considers a smooth two-sided unstable minimal hypersurface, one can push it using the first eigenvalue of the Jacobi operator to get a mean convex boundary which acts as a barrier for the Plateau problem for instance. Now if Σ is a non-embedded minimal surface, and $c \in \mathcal{C}_\Sigma$, then constructing such a barrier "approximating" ∂c inside c is still possible. To make this statement rigorous, we begin with the following lemma which is a trick showing the existence of a hypersurface being "close" to Σ and mean convex except perhaps in a small ball.

Lemma 91. *Consider Γ a compact connected n -dimensional manifold with a possibly non-empty smooth boundary and let $\psi : \Gamma \rightarrow M$ be a minimal immersion into (M, g) . Let $p \in \psi(\Gamma)$ be a point such that $\psi(\Gamma)$ is an embedded hypersurface in a neighborhood U of p . Then, using the notations previously defined, there is a metric h equal to g in $M \setminus U$*

and a function

$$f \in C^\infty(\tilde{\Gamma} \cup \partial\tilde{\Gamma})$$

positive on $\tilde{\Gamma}$ and vanishing on $\partial\tilde{\Gamma}$ such that for all $s \in (0, 1]$, $\text{exp}_{\tilde{\psi}, sf}$ is an immersion and the mean curvature of $\text{exp}_{\tilde{\psi}, sf}(\tilde{\Gamma})$, endowed with the natural choice of outward unit normal, is negative with respect to h .

Recall the following definition, introduced in [101]. If $\Phi : [a, b] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ is continuous in the flat topology, we define

$$\mathbf{m}(\Phi, r) = \sup\{|\Phi(x)|(B_r(p)); x \in [a, b], p \in M\}.$$

We will say that Φ satisfies the technical condition $[*]$ if

$$\mathbf{m}(\Phi, r) \rightarrow 0 \text{ as } r \rightarrow 0.$$

In what follows, we consider an immersed minimal hypersurface $\Sigma = \psi(\Gamma)$, where Γ is a closed n -manifold. Σ will be said to be connected if Γ is connected. From the earlier description of the set E (see (8.1)), we know that the boundary of a component $c \in \mathcal{C}_\Sigma$ has positive finite n -dimensional Hausdorff measure and is $\mathcal{H}^n$ -almost everywhere locally an embedded hypersurface. We will restrict our attention to components $c \in \mathcal{C}_\Sigma$ verifying the following "local separation" property.

$$(\mathbf{LS}) \quad \text{For any } p \in \partial c \setminus E \text{ and } r > 0, B(p, r) \setminus \Sigma \text{ is not included in } c.$$

Note that if Σ is non-embedded and $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$, then any $c \in \mathcal{C}_\Sigma$ automatically satisfies condition **(LS)** by Lemma 90. For such a component, the next lemma enables to get rid of a certain subset of c and decrease the area of ∂c inside c . It is essential that Σ is not embedded. We say that $x \in \partial c$ has a *local roof structure* if there is a ball $B(x, r_x)$ and a diffeomorphism $\mathcal{D} : \mathbb{R}^{n+1} \rightarrow B(x, r_x)$ such that

$$\mathcal{D}(\{x_i = 0\}) \subset \Sigma \quad \forall i \in \{1, 2\}$$

$$\text{and } \mathcal{D}(\{x_1 > 0, x_2 > 0\}) \subset c.$$

Lemma 92. *Suppose that Σ is a non-embedded connected minimal hypersurface in M . Consider a component $c \in \mathcal{C}_\Sigma$ satisfying condition **(LS)**. Then there is a point $x \in \partial c$ having a local roof structure.*

Moreover there is a map $\xi : [0, 1] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology such that for all $s \in [0, 1]$, $\xi(s) = \partial[|G_s|]$, where G_s are open sets with the following properties:

- (i) $G_0 = c$ and $\forall s \in [0, 1]$ $G_s \subset c$, $\mathbf{M}(\xi(s)) = \mathcal{H}^n(\partial G_s)$,
- (ii) $\partial G_1 \cap c$ separates c and is piecewise smooth mean convex with respect to the normal pointing outside G_1 ,
- (iii) $\forall s \in (0, 1]$, $\mathcal{H}^n(\partial G_s) < \mathcal{H}^n(\partial c)$,
- (iv) ξ satisfies $[\ast]$.

Equipped with the previous two lemmas, we are now able to prove the following result: if Σ is connected and non-embedded then for any $c \in \mathcal{C}_\Sigma$ satisfying **(LS)**, the boundary ∂c can be approximated by a piecewise smooth mean convex hypersurface with less area.

Proposition 93. *Let Σ be a non-embedded connected minimal hypersurface in M . Consider a component $c \in \mathcal{C}_\Sigma$ satisfying condition **(LS)**. Then there exists a map $\theta : [0, 1] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology such that for all $s \in [0, 1]$, $\theta(s) = \partial[|\rho_s|]$, where ρ_s are open sets with the following properties:*

- (i) $\rho_0 = c$, $\forall s$ $\rho_s \subset c$ and $\mathbf{M}(\theta(s)) = \mathcal{H}^n(\partial \rho_s)$,
- (ii) $\partial \rho_1$ is piecewise smooth mean convex,
- (iii) $\forall s \in (0, 1]$, $\mathcal{H}^n(\partial \rho_s) < \mathcal{H}^n(\partial c)$,
- (iv) there exists a homotopically closed set of continuous sweepouts of ρ_1 , called Λ ,
- (v) θ satisfies $[\ast]$.

Remark 8.3.1. *It should be possible to prove that, with the notations in the proof of Proposition 93, $s \mapsto \mathcal{H}^n(\partial W_s)$ is decreasing if $\tilde{f}$ is sufficiently small. But even if it assumed to be true, we need Lemma 92 to get rid of a part of c , deform the metric g in this part and get a hypersurface being mean convex for the original metric g .*

We are ready to show that a least area minimal hypersurface is necessarily embedded. The regularity result [125] [Theorem 7.12] will be used in the proof. Even though it is shown by Pitts only for $2 \leq n \leq 5$, the result is still true for $n = 6$ thanks to the curvature estimates of Schoen and Simon (see [133, Section 7]). Besides each time Theorem 7.12 of [125] and Theorem 5.5 of [156] are invoked, we actually apply their "modulo 2" versions (see Remark 2.1.2).

Proposition 94. *Suppose that Σ is a non-embedded connected minimal hypersurface and that $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$, then there exists a connected unstable minimal hypersurface Σ_0 such that:*

- (i) Σ_0 is embedded,
- (ii) $\mathcal{H}^n(\Sigma_0) = \mathbf{L}(\Pi) < \mathcal{H}^n(\Sigma)$, where Π is the homotopy class of mappings in $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ corresponding to the fundamental class in $H_{n+1}(M, \mathbb{Z}_2)$.

We can now finish the proof of the main theorem.

Proof of Theorem 84 and Theorem 89. Let's first check the existence of a least area minimal hypersurface, using arguments appearing in [107]. By the compactness result in [133], $\mathfrak{A}_S(M)$ is achieved (take a minimizing sequence and apply the compactness theorem in balls of radius smaller than the injectivity radius of M). Thus if

$$\mathfrak{A}_S(M) = \mathfrak{A}(M),$$

the existence of a minimizer is proved. Suppose on the contrary that

$$\mathfrak{A}_S(M) > \mathfrak{A}(M),$$

and take a sequence $\{\Sigma_i\}_i$ of connected minimal hypersurfaces such that $\mathcal{H}^n(\Sigma_i) < \mathfrak{A}_S(M)$ and $\lim_{i \rightarrow \infty} \mathcal{H}^n(\Sigma_i) = \mathfrak{A}(M)$. By Proposition 88 and Proposition 94, each hypersurface Σ_i has an area bigger than or equal to that of an embedded hypersurface, whose area is $\mathbf{L}(\Pi)$ where Π is the homotopy class of mappings in $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ corresponding to the fundamental class in $H_{n+1}(M, \mathbb{Z}_2)$. Hence any minimal hypersurface produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$ has area $\mathfrak{A}(M)$ in the case $\mathfrak{A}_S(M) > \mathfrak{A}(M)$.

Then Proposition 94 implies that any least area minimal hypersurface is embedded. Moreover we can use Proposition 88 to show that if Σ_0 is a least area minimal hypersurface, it is either stable or coming from Almgren-Pitts' min-max theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$. Finally, if Σ_0 is not stable, we have seen in the proof of Proposition 88 that Σ_0 is two-sided and by reasoning along the lines of [100, Proposition 3.1] (see also [156], [107]), Σ_0 is indeed of index one. This finishes the proof of Theorem 89.

□

8.4 Area rigidity of minimal surfaces in three-manifolds of positive scalar curvature

As an application of Theorem 89, we give a short proof of a conjecture of Marques and Neves (see Theorem 1.3 and below in [100]). Note that the proof only uses min-max methods and the short-time existence theorem for Hamilton's Ricci flow.

Theorem 95. *Let M^3 be a closed three-manifold with scalar curvature R at least 6, not isometric to the round unit three-sphere S^3 . Then there exists a closed embedded minimal surface Σ of index zero or one such that*

$$\mathcal{H}^2(\Sigma) < 4\pi.$$

Moreover, Σ can be chosen so that if it is not stable, then it is two-sided and has area equal to the width of the fundamental class of M , in the sense of Almgren-Pitts.

Proof. By [100, Theorem 1.2] and Theorem 89, the theorem is true for M diffeomorphic to S^3 . In the general case, it is enough to find a finite Riemannian covering $\tilde{M}$ of M which contains a minimal surface $\tilde{\Sigma}$ of area less than 4π . Indeed, if p denotes the natural projection from $\tilde{M}$ to M , then $p(\tilde{\Sigma})$ is an immersed minimal surface of area less than 4π and the result follows readily from Theorem 89. By [114, Corollary 0.5, Chapter 15] and [99, Theorem 7.1, b)], an oriented cover M_{or} of M is a connected sum of spherical space forms and finitely many copies of $S^2 \times S^1$. When this cover M_{or} is a quotient of the three-sphere, we can just take $\tilde{M} = S^3$. Otherwise M_{or} contains an essential two-sphere

S . From Lemma 1 and Theorem 1 in [110], we can minimize the area of S in its isotopy class by γ -reduction and get a non-trivial stable embedded minimal surface diffeomorphic to S^2 or $\mathbb{RP}^2$. By [100, Proposition A.1, (i), (ii)], it has area bounded by $4\pi/3$ or 2π . Thus in this case we can take $\tilde{M} = M_{or}$.

□

Chapter 9

A maximum principle for self-shrinkers of the mean curvature flow

In this final chapter, we are interested in the asymptotic behavior of non-compact self-shrinking surfaces embedded in $\mathbb{R}^3$. Using a graphical maximum principal, we prove a linear upper bound for the second fundamental form, which was later used by L. Wang in her description of ends of non-compact self-shrinkers [148].

9.1 Introduction

A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker for the mean curvature flow, normalized so that it shrinks in unit time to the origin $0 \in \mathbb{R}^{n+1}$, if it satisfies the following equation:

$$H = \frac{\langle x, \nu \rangle}{2} \tag{9.1}$$

where ν is the outer normal unit vector, $H = \div(\nu)$ denotes the mean curvature and x is the position vector. Such a surface will be called more briefly a self-shrinker. Equivalently, one can say that the set of hypersurfaces $\{\sqrt{-t}\Sigma; t < 0\}$ is a solution of the mean curvature

flow, i.e. verifies the equation :

$$(\partial_t x)^\perp = -H\nu. \quad (9.2)$$

Self-shrinkers provide models for blow-ups at singularities of mean curvature flow: consider a family of hypersurfaces evolving by the mean curvature flow and starting from a closed embedded hypersurface and focus on a point of singularity, then rescalings yield a subsequence converging weakly to a "tangent flow", which satisfies (9.1) (see [72], [150], [36]). The classification of embedded self-shrinkers proved to be a difficult problem. The simplest examples are given by cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$ where $\mathbb{S}^k$ is the k -sphere of radius $\sqrt{2k}$. If $n \geq 2$, Huisken, then Colding and Minicozzi showed that those hypersurfaces were the only ones with polynomial volume growth and whose mean curvature H is nonnegative ([69], [70], [36] Theorem 0.17). When $n = 1$, straight lines passing through the origin and the circle of radius $\sqrt{2}$ are the only self-shrinking embedded curves [1], but as soon as the dimension is greater than or equal to 2, there are non trivial self-shrinkers, as the Angenent torus [4]. See also [116] and [117] for construction of complete embedded self-shrinkers of high genus.

The aim of this note is to present some applications of a maximum principle adapted to self-shrinking hypersurfaces viewed locally as graphs (Proposition 101, Corollary 102). It will be used to rule out some hypersurfaces from the set of self-shrinkers. The usual maximum principle states for example that the distance between two compact hypersurfaces moving by the mean curvature flow is non-decreasing in time. Here, we prove a maximum principle for graphs, the advantage being that within a hypersurface, sometimes one can find two subsets forming graphs of two functions whose difference achieves a minimum, whereas a minimal distance (even a local one) between the two subsets is not achieved.

Using this remark (see Proposition 101, Corollary 102), we prove that the only embedded rotationally symmetric compact self-shrinkers are either a $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ or a round sphere with the appropriate radius (this fact is part of a more general statement covering the non compact case proven by Kleene and Møller [86]):

Theorem 96 ([86]). *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a compact embedded rotationally symmetric hypersurface. If Σ is a self-shrinker then either it is the sphere of radius $\sqrt{2n}$ centered at the origin or a $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.*

Note that from Brendle [16], it is now known that any closed embedded self-shrinking surface of genus zero in $\mathbb{R}^3$ is the round sphere.

Then, with the same argument, we show that self-shrinkers are weakly asymptotic to cones, a result known by Ilmanen ([73] p.8, though I didn't find his proof). In the two-dimensional case, some additional information is given.

Theorem 97 ([73]). *If $\Sigma \subset \mathbb{R}^{n+1}$ is a complete properly immersed self-shrinker, then there exists a cone $\mathcal{C}$ such that*

$$\lambda\Sigma \rightarrow \mathcal{C} \text{ when } \lambda \rightarrow 0^+$$

locally in the Hausdorff metric on closed sets.

In the case $n = 2$, if the number of connected components of $S(0, r) \cap \Sigma$ is bounded above, then $\mathcal{C} \cap \mathbb{S}^2$ has 2-dimensional Lebesgue measure 0.

Finally in the case of surfaces, our result in Section 9.4 gives a linear bound for the norm of the second fundamental form $|A|$ of self-shrinking surfaces, under the assumption that the "curvature concentration" is bounded, i.e. that these surfaces satisfy a local integral bound for $|A|$ (see Definition 9.4.1). This is a natural assumption in the context of weak blowups (see Proposition 108). It is proved that:

Theorem 98. *If $\Sigma \subset \mathbb{R}^3$ is a complete properly embedded self-shrinker of finite genus g such that "the curvature concentration is bounded by κ ", then*

$$\exists C, \forall x \in \Sigma, |A(x)| \leq C(1 + |x|).$$

Moreover for such a surface with ends that are " δ -separated at infinity", the constant C only depends on g , δ , κ and on a "bound for the topology of Σ ".

For more precise statements of this result, see Definition 9.2.1, Definition 9.4.2, Theorem 109 and Theorem 114.

As in [37], the surfaces in the second part of the theorem are supposed to be homeomorphic to closed surfaces with finitely many disjoint disks removed. The proof is based on a blow-up argument, the compactness for self-shrinkers [37], the maximum principle and some results about classical minimal surfaces. Consequently, it is pointed out that

translating solitons of the mean curvature flow can model regions of Σ far from the origin (Corollary 113).

9.2 A maximum principle for embedded self-shrinkers

The mean curvature flow equation for graphs is given by ([97] p.10):

Lemma 99. *Suppose that $\phi_t : M \rightarrow \mathbb{R}^{n+1}$ are smooth hypersurfaces moving by mean curvature and are graphs on the open subset Ω of the hyperplane $\langle e_1, \dots, e_n \rangle \subset \mathbb{R}^{n+1}$, that is, there exists a smooth function $f : \Omega \times [0, T) \rightarrow \mathbb{R}$ with*

$$\phi_t(p) = (x_1(p), \dots, x_n(p), f(x_1(p), \dots, x_n(p), t)),$$

then

$$\partial_t f = \Delta f - \frac{\text{Hess } f(\nabla f, \nabla f)}{1 + |\nabla f|^2} = \sqrt{1 + |\nabla f|^2} \div \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right). \quad (9.3)$$

Conversely, if f satisfies this equation, then it corresponds to hypersurfaces moving by mean curvature.

Next we give the maximum principle for self-shrinkers:

Lemma 100. *Let $f, g : \Omega \times [0, T) \subset \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be two functions satisfying (9.3), where Ω is an open subset of $\mathbb{R}^n$. Suppose that there exists a compact set $K \subset \Omega$ such that for all $t' \in [0, T)$, the minimum of $(f - g)(\cdot, t')$ is attained at least at one point of K .*

Define $u(t) = \min_{p \in \Omega} (f - g)(p, t)$. Then u is a locally Lipschitz function, hence differentiable almost everywhere and if it exists, the differential is nonnegative.

Proof. By Hamilton's trick ([97] p.26), u is a locally Lipschitz function, hence differentiable almost everywhere and where it makes sense:

$$\frac{du(t)}{dt} = \frac{\partial(f - g)(p, t)}{\partial t},$$

$p \in K$ being a point where the minimum of $(f - g)$ is attained. But at such a point p ,

$\nabla f = \nabla g$, the Hessian of $f - g$ is nonnegative and $\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$ is a vector whose Euclidian norm is less than 1, hence the lemma. $\square$

To show that a hypersurface $\mathcal{S}$ is not a self-shrinker, one can thus try to apply this maximum principle to two graphs given by two parts of $\mathcal{S}$. Next we give a proposition implementing this strategy and which will be used in various forms throughout this note:

Proposition 101. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ two disjoint complete hypersurfaces of $\mathbb{R}^{n+1}$, which can be noncompact and have boundaries. Define the function $h : \mathcal{S}_1 \rightarrow \mathbb{R} \cup \{\infty\}$:*

$$\begin{aligned} h(a) &= \min\{a_{n+1} - b_{n+1}; (a_1, \dots, a_n, b_{n+1}) \in \mathcal{S}_2 \text{ and } b_{n+1} < a_{n+1}\} \\ &\quad \text{if } \{b_{n+1}; b_{n+1} < a_{n+1} \text{ and } (a_1, \dots, a_n, b_{n+1}) \in \mathcal{S}_2\} \neq \emptyset \\ h(a) &= \infty \text{ otherwise.} \end{aligned}$$

If h achieves a local finite minimum at $a \in \mathcal{S}_1$ and $b \in \mathcal{S}_2$, two points not in the respective boundaries, and if $\langle \nu_a, e_{n+1} \rangle \neq 0$ where ν is the outward normal vector of $\mathcal{S}$, then $\mathcal{S}_1$ or $\mathcal{S}_2$ is not a part of a self-shrinker.

Proof. Suppose that the two hypersurfaces are self-shrinkers. Under these hypotheses, we can locally write $\mathcal{S}_1$ and $\mathcal{S}_2$ as graphs of f and g which satisfy the hypotheses of lemma 100. If $u(t) = \min(f - g)(p, t)$ then $\frac{du_{\min}(t)}{dt} \geq 0$ almost everywhere. On the other hand, by the definition of self-shrinker, the solution $\mathcal{S}_i(t)$ ($i = 1, 2$) of the mean curvature flow with $\mathcal{S}_i(0) = \mathcal{S}_i$ is given by $\mathcal{S}_i(t) = \sqrt{-t}\mathcal{S}_i$ ($-1 \leq t \leq 0$), which would mean $\frac{du_{\min}(t)}{dt} < 0$. This is the desired contradiction. $\square$

Next, we give a specialized form of this proposition, but before let's define hypersurfaces with δ -separated ends.

Definition 9.2.1. *Let $\delta > 0$. $\mathcal{S} \subset \mathbb{R}^{n+1}$ is a properly immersed hypersurface. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two disjoint open connected subsets of $\mathcal{S}$ with $\overline{\mathcal{S}_1} \cup \overline{\mathcal{S}_2} = \mathcal{S}$ ($\overline{\mathcal{S}_i}$ being the closure of $\mathcal{S}_i$ in $\mathcal{S}$) and $\partial\overline{\mathcal{S}_1} = \partial\overline{\mathcal{S}_2}$. Suppose that this boundary $\partial\overline{\mathcal{S}_1}$ is bounded.*

If for every such partition $\overline{\mathcal{S}_1} \cup \overline{\mathcal{S}_2} = \mathcal{S}$, for r sufficiently large, one of the $(\frac{1}{r}\mathcal{S}_i) \setminus B(0, 1)$ is empty or

$$\inf\{\|x_1 - x_2\|; x_i \in (\frac{1}{r}\mathcal{S}_i) \setminus B(0, 1)\} \geq \delta$$

then $\mathcal{S}$ is said to have δ -separated ends.

Corollary 102. *Let $\delta > 0$. Let $\mathcal{S} \subset \mathbb{R}^{n+1}$ be a complete properly immersed hypersurface with δ -separated ends.*

Suppose that there exists an embedding $\alpha : \mathbb{S}^{n-1} \rightarrow \mathcal{S}$ such that the two disjoint connected components of $\mathcal{S} \setminus \alpha(\mathbb{S}^{n-1})$ are called $\mathcal{S}^+$ and $\mathcal{S}^-$, and a set $A \subset \mathbb{S}^n$ of non zero n -dimensional Lebesgue measure with the following properties: if $\tilde{e} = (e_1, \dots, e_{n+1})$ is an orthonormal base of $\mathbb{R}^{n+1}$ with $e_{n+1} \in A$ then in the coordinates determined by $\tilde{e}$

- 1. there exists $x = (x_1, \dots, x_n, x_{n+1}) \in \mathcal{S}^-$ and $y = (x_1, \dots, x_n, y_{n+1}) \in \mathcal{S}^+$ such that $y_{n+1} < x_{n+1}$,*
- 2. there exists a neighborhood $\mathcal{V}$ of $\alpha(\mathbb{S}^{n-1})$ in the closure of $\mathcal{S}^-$ verifying that if $a = (z_1, \dots, z_n, a_{n+1}) \in \mathcal{V}$ then there is a point $u = (z_1, \dots, z_n, u_{n+1}) \in \mathcal{S}^-$ with $a_{n+1} > u_{n+1}$ such that there is no $v = (z_1, \dots, z_n, v_{n+1}) \in \mathcal{S}^+$ with $a_{n+1} > v_{n+1} > u_{n+1}$,*
- 3. if $z = (z_1, \dots, z_{n+1}) \in \alpha(\mathbb{S}^{n-1})$ and $b = (z_1, \dots, z_n, b_{n+1}) \in \mathcal{S}^-$ satisfy $b_{n+1} > z_{n+1}$, then there is a $w = (z_1, \dots, z_n, w_{n+1}) \in \mathcal{S}^+$ such that $b_{n+1} > w_{n+1} > z_{n+1}$.*

Then $\mathcal{S}$ is not a self-shrinker.

Remark 9.2.1. *This corollary covers the case where $\mathcal{S}$ is compact (see Figure 9.1) because such a surface automatically has δ -separated ends. The existence of A will always be easy to check in our applications. Besides, although we will only need the situation of the corollary, where the common boundary is given by an $(n-1)$ -sphere $\alpha(\mathbb{S}^{n-1})$, one can actually consider more general boundaries.*

9.3 Some applications

By "maximum principle for self-shrinkers", one usually means that the distance between two hypersurfaces moving by mean curvature is non-decreasing. Here, Corollary 102 gives a maximum principle for graphs and one has to choose the axis $\mathbb{R}e_{n+1}$: this non canonical choice enables more flexibility, as we will see with the following paragraphs.

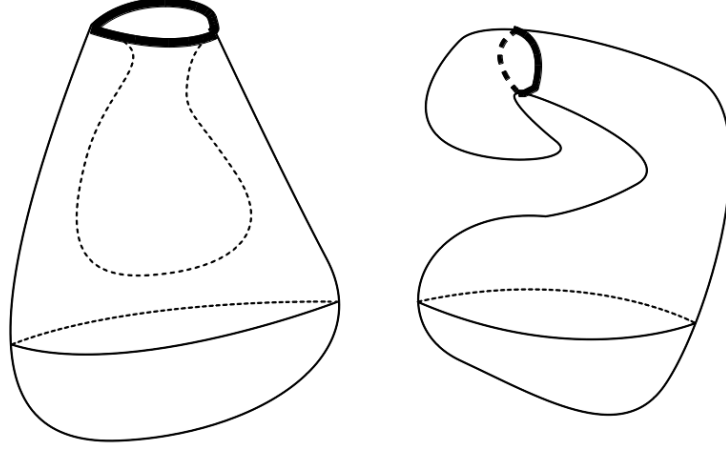


Figure 9.1: Some typical configurations where Corollary 102 applies, but where the usual maximum principle is inefficient. Embeddings of $\mathbb{S}^{n-1}$ are represented in bold line.

9.3.1 Rotationally symmetric compact self-shrinkers

Let u be a vector of $\mathbb{R}^{n+1}$, $\mathbb{S}^{n-1}$ is identified with the unit sphere of the hyperplane orthogonal to u . Consider a simple curve $\gamma : [a, b] \rightarrow \mathbb{R} \times \mathbb{R}^+$. Let Σ_γ be the image of an embedding $\varphi : \mathbb{S}^{n-1} \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ which can be written as $\varphi(\omega, s) = x(s)u + r(s)\omega$, where $s \mapsto (x(s), r(s))$ is a parametrization of γ . Σ_γ is then said to be the rotationally symmetric hypersurface generated by γ : it is obtained by rotating γ around the axis $\mathbb{R}u$. Here is a result proved in [86] (see also [45]).

Theorem 103 ([86]). *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is an embedded compact rotationally symmetric self-shrinker, then Σ^n is:*

1. *either the sphere $\mathbb{S}^n$ of radius $\sqrt{2n}$ centered at the origin,*
2. *or an embedded $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.*

9.3.2 Asymptotic behavior and cones

We now apply Proposition 101 to get a simple proof of a result of Ilmanen (see [73] Lecture 2, B , remark on p.8).

Definition 9.3.1. *Let K be a compact subset of $\mathbb{S}^n$. The set*

$$\{rx | r > 0 \quad x \in K\} \cup \{0\} \subset \mathbb{R}^{n+1}$$

is called the cone generated by K .

As usual, $S(0, r)$ (resp. $B(0, r)$) denotes the sphere (resp. the ball) of radius r centered at the origin.

Theorem 104 ([73]). *Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a complete properly immersed self-shrinker. Then there exists a cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ generated by a compact set $K \subset \mathbb{S}^n$ such that:*

$$\lambda\Sigma \rightarrow \mathcal{C} \text{ as } \lambda \rightarrow 0^+,$$

locally for the Hausdorff metric.

Moreover in the case $n = 2$, if the number of connected components of $S(0, r) \cap \Sigma$ is bounded when $r \rightarrow \infty$, then the compact K is of 2-dimensional Lebesgue measure 0.

Proof. Denote by d_H the Hausdorff distance for non-empty compact sets in $\mathbb{R}^{n+1}$ and $\Sigma_\lambda = \lambda\Sigma$ for $\lambda > 0$. Let $r > 0$ fixed. Define

$$K_r^\lambda = \Sigma_\lambda \cap S(0, r).$$

It is sufficient to show that K_r^λ converges in the Hausdorff metric to a compact set $K_r \subset S(0, r)$ when λ goes to 0.

Let's introduce some other notations: if $0 < \lambda < r/\sqrt{2n}$ we write

$$L_r^\lambda = \{p \in S(0, r); d(p, K_r^\lambda) \leq \sqrt{2n}\lambda\}$$

where d is the distance between a point and a closed set,

$$M_r^\lambda = \cap_{\lambda \leq \mu \leq \frac{r}{\sqrt{2n}}} L_r^\mu,$$

$$Y_r = \Sigma \setminus B(0, r)$$

and

$$Z_r^\lambda = \{p \in \mathbb{R}^{n+1}; \|p\| \geq r \text{ and } r \frac{p}{\|p\|} \in L_r^\lambda\}.$$

Firstly, let's show a kind of monotonicity relation:

$$\forall \lambda, \mu \in]0, \frac{r}{\sqrt{2n}}[\quad \mu \leq \lambda \Rightarrow K_r^\mu \subset L_r^\lambda. \quad (9.4)$$

Note that if $\lambda = \frac{r}{\sqrt{2n}}$, then $K_r^\lambda \neq \emptyset$. Indeed, by homogeneity, this is tantamount to saying that $K_{\sqrt{2n}}^1 \neq \emptyset$, which is true by the Maximum Principle. Likewise, to prove 9.4, we need to show

$$\forall r > \sqrt{2n} \quad Y_r \subset Z_r^1.$$

To prove this, suppose that there is $a \in S(0, r) \setminus L_r^1$ and $b \in Y_r$ such that $r \frac{b}{\|b\|} = a$. Choose e_{n+1} "very close" to $-\frac{a}{\|a\|}$, in a sense precised below. Consider then the two following hypersurfaces $\mathcal{S}_1$ and $\mathcal{S}_2$: $\mathcal{S}_1$ is the spherical self-shrinker of radius $\sqrt{2n}$ centered at the origin and $\mathcal{S}_2 = Y_r$. Now, for e_{n+1} well chosen near $-\frac{a}{\|a\|}$, the function $h : \mathcal{S}_1 \rightarrow \mathbb{R} \cup \{\infty\}$ is not ∞ everywhere and attains its minimum on points not on the boundary of $\mathcal{S}_2$ because $d(a, K_r^1) > \sqrt{2n}$. We conclude with Proposition 101. In particular, M_r^λ is not empty if K_r^λ is not.

Suppose now that Σ is unbounded, otherwise there is nothing to prove. This means that $K_r^\lambda \neq \emptyset$ for $\lambda \leq \frac{r}{\sqrt{2n}}$. Define

$$K_r = \bigcap_{\lambda \leq \frac{r}{\sqrt{2n}}} L_r^\lambda.$$

The M_r^λ being non empty for $\lambda \leq \frac{r}{\sqrt{2n}}$ and included one in the other, K_r is a non empty compact set. Let's show that K_r^λ converges to K_r . As $K_r^\lambda \subset M_r^\lambda$,

$$\sup_{x \in K_r^\lambda} d(x, M_r^\lambda) = 0.$$

Then by the definition of M_r^λ , if $y \in M_r^\lambda$, there exists $s \in K_r^\lambda$ such that $d(x, y) \leq \sqrt{2n}\lambda$,

so

$$\sup_{y \in M_r^\lambda} d(y, K_r^\lambda) \leq \sqrt{2n\lambda}.$$

Now, K_r is the intersection of the M_r^λ which constitute a decreasing nested sequence of non empty compact sets so M_r^λ converge to K_r . Finally,

$$\begin{aligned} d_H(K_r^\lambda, K_r) &\leq d_H(K_r^\lambda, M_r^\lambda) + d_H(M_r^\lambda, K_r) \\ &\rightarrow_{\lambda \rightarrow 0} 0, \end{aligned}$$

which is the desired convergence.

In what follows, suppose that $n = 2$, that Σ is properly immersed and that the number of connected components of $S(0, r) \cap \Sigma$ is bounded when $r \rightarrow \infty$. The d -dimensional Lebesgue measure of a submanifold of Σ its d -volume, denoted by $\text{Vol}_d(M)$. We have

$$\text{Vol}_2(L_r^\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+. \quad (9.5)$$

To show this, we use the Euclidean volume growth for properly immersed self-shrinkers ([43]). Because K_r is the intersection of the L_r^λ , it is sufficient to prove that there is a sequence $(\lambda_k)_{k \in \mathbb{N}}$ converging to 0 with $\text{Vol}_2(L_r^{\lambda_k}) \rightarrow 0$. The Sard Lemma implies that for almost all $\lambda > 0$, K_r^λ is a 1-dimensional submanifold of $S(0, r)$. Let Λ be the set of those λ . Suppose that there is an $\epsilon > 0$ and a $\lambda_0 > 0$ such that

$$\forall \lambda < \lambda_0, \text{Vol}_2(L_r^\lambda) > \epsilon, \quad (9.6)$$

we want to find a contradiction.

Lemma 105. *Take $r > 0$, $0 < \eta < 1$. Suppose that $\gamma \subset S(0, r) \subset \mathbb{R}^3$ is an immersed closed curve of length l . Then there exist C_1, C_2 two constants independent of γ, r and η such that*

$$\text{Vol}_2(\{x \in S(0, r); d(x, \gamma) \leq \eta\}) \leq \eta \cdot (C_1 \cdot l + C_2 \cdot \eta).$$

This lemma and the assumption (9.6) imply that

$$\text{Vol}_1(K_r^\lambda) \rightarrow \infty$$

when $\lambda \in \Lambda$ goes to 0. Equivalently, one can write that

$$\text{Vol}_1(K_r^1) = \sigma(r)r,$$

where σ is a stricly positive function defined for almost all r and converging to ∞ . But, if N is the norm function of $\mathbb{R}^{n+1}$, the co-area formula gives:

$$\begin{aligned} \text{Vol}_2(\Sigma \cap \{\sqrt{2.2} \leq N \leq R\}) &= \int_{\Sigma \cap \{2 \leq N \leq R\}} d\text{Vol}_2 \\ &\geq \int_{\Sigma \cap \{2 \leq N \leq R\}} \|\nabla N\| d\text{Vol}_2 \\ &= \int_2^R \int_{K_r^1} d\text{Vol}_1 dr \\ &= \int_2^R \sigma(r)r dr \\ &\geq \frac{1}{2}(R^2 - R_0^2) \min_{[R_0, R]} \sigma \end{aligned}$$

where $R > R_0$ are two real numbers greater than 2, and $\{2 \leq N \leq R\}$ denotes the set $B(0, R) \setminus B(0, 2)$. As $\sigma \rightarrow \infty$, this computation contradicts the quadratic volume growth of Σ , so in fact $\text{Vol}_2(K_r) = 0$.

□

Next, we give two propositions which are based on a refined form of the argument used previously to show the monotonicity relation, which was the key argument for proving Theorem 104.

For a point $q \in \mathbb{R}^{n+1}$ and a vector $v \in \mathbb{R}^{n+1}$, we denote by $L(q, v)$ the half-line beginning at q and whose direction is given by v . Recall also the notation $K_r^1 = \Sigma \cap S(0, r)$ where Σ is a complete properly immersed self-shrinker.

Proposition 106. *Let $r > 0$ and p_0 be a point in $S(0, r)$ (also considered as a vector in $\mathbb{R}^{n+1}$). Define*

$$U_r(p_0) = \{q \in \mathbb{R}^{n+1} \setminus B(0, r); L(q, -p_0) \cap S(0, \sqrt{2n}) \neq \emptyset\},$$

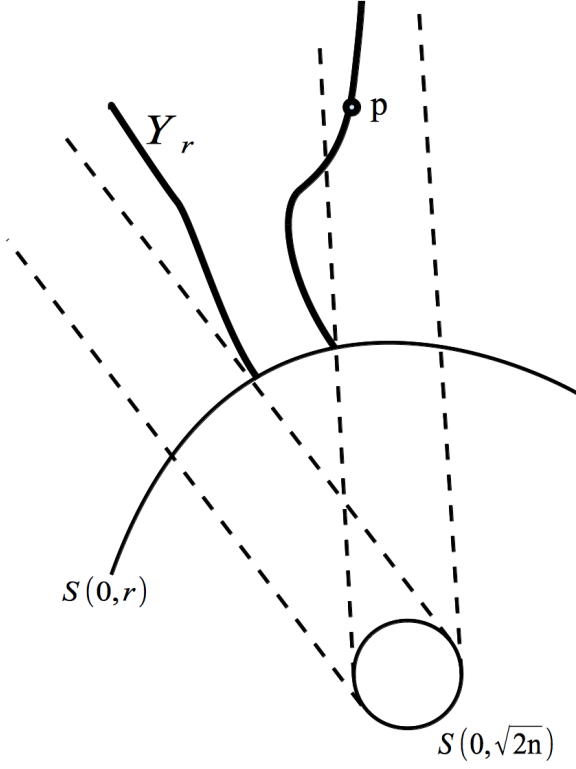


Figure 9.2: In this figure, Σ cannot be a self-shrinker because the point p is out of X_r .

$$V_r = \bigcup_{p \in S(0, r) \text{ and } U_r(p) \cap K_r^1 = \emptyset} U_r(p).$$

Consider the sets

$$Y_r = \Sigma \setminus B(0, r),$$

$$X_r = \mathbb{R}^{n+1} \setminus (B(0, r) \cup V_r).$$

Then

$$\forall r > \sqrt{2n} \quad Y_r \subset X_r.$$

The following proposition illustrates the fact that some bound on the curvature of the trace $K_r^1 = \Sigma \cap S(0, r)$ gives a bound on the mean curvature of Σ in the case where $n = 2$.

Proposition 107. *Suppose $n = 2$ and let Σ_0 be an end of $\Sigma \subset \mathbb{R}^3$, i.e. a connected component of $\Sigma \setminus B(0, r_0)$ for a $r_0 > 0$. Let $\epsilon > 0$. If Σ_0 intersects $S(0, r)$ transversally for all $r > r_0$ (in particular $\Sigma_0 \cap S(0, r)$ a union of closed simple curves) and if the*

curvature of $\Sigma_0 \cap S(0, r)$ is bounded by $\frac{1}{2} - \epsilon$ then the mean curvature of Σ_0 is bounded by 2 if r_0 is chosen large enough.

9.4 A linear bound for the second fundamental form of some self-shrinkring surfaces

Let's define the curvature concentration:

Definition 9.4.1. *Let $\kappa > 0$. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$. The curvature concentration of $\mathcal{S}$ is bounded by κ if*

$$\forall x \in \mathcal{S} \setminus B(0, 1), \quad \int_{\mathcal{S} \cap B(x, \rho(x))} |A|^2 < \kappa,$$

where

$$\rho(x) = \frac{1}{2\|x\|}.$$

Requiring that a surface has bounded curvature concentration seems quite restrictive, but it is in fact natural in the context of Brakke limit flows:

Proposition 108. *Let $\{M\}_{t \in [0, T)}$ be a family of embedded surfaces flowing smoothly in $\mathbb{R}^3$ and beginning at a closed surface M_0 . If $\mathcal{S}$ is a self-shrinker produced by a weak blowup at T then there exists a constant $\kappa(M_0) > 0$ depending only on M_0 , such that the curvature concentration of $\mathcal{S}$ is bounded by $\kappa(M_0)$.*

The main theorem of this section gives a linear bound for $|A|$:

Theorem 109. *If $\Sigma \subset \mathbb{R}^3$ is a complete properly embedded self-shrinker of finite genus g such that the curvature concentration is bounded by κ , then*

$$\exists C = C(\Sigma), \forall x \in \Sigma, |A(x)| \leq C(1 + |x|).$$

Remark 9.4.1. *Self-shrinkers can be viewed as minimal surfaces under a conformal change of metric but, as noted in [37], this kind of result does not follow for instance*

from Choi-Schoen's compactness theorem [27] mainly because the new metric can not even be extended to a complete metric.

Proof. Let $K = (H^2 - |A|^2)/2$ denote the Gauss curvature. Suppose that the conclusion is not verified along a sequence of points $p_k \in \Sigma$. Define $\mu_k = |A(p_k)|$. Then, because of the self-shrinker equation (9.1):

$$\forall \alpha > 0 \quad \max_{y \in B(p_k, \alpha/\mu_k)} H(y)/\mu_k \rightarrow 0. \quad (9.7)$$

Moreover, $-K(p_k)/|A(p_k)|^2 \rightarrow 1/2$. By modifying the p_k if necessary, $\mu_k \cdot (\Sigma - p_k)$ is a sequence of surfaces whose second fundamental form is locally uniformly bounded and whose mean curvature goes to zero. Indeed, we have

Lemma 110. *Under the hypotheses of the theorem, there is a sequence p_k such that*

$$\forall \alpha > 0, \quad |A(p_k)|/(1 + ||p_k||) \rightarrow \infty \text{ and}$$

$$\max\{|A(q)|/\mu_k; q \in \Sigma \cap B(p_k, \alpha/\mu_k)\} \leq 2 \text{ for large } k.$$

Define now $\mathbf{S}_k = \mu_k \cdot (\Sigma - p_k)$. The previous lemma shows that $\mathbf{S}_k$ is a sequence of surfaces such that at the origin, the Gauss curvature is $-1/2$ and the second fundamental form is locally uniformly bounded. Besides, (9.7) implies that for all $a > 0$ and all $x_k \in \Sigma \cap B(p_k, a/\mu_k)$, the quantity $H(x_k)/\mu_k$ goes to 0. In other words, the mean curvature of $\mathbf{S}_k$ converges locally uniformly to 0. The local uniform bound on $|A|$ means that in any ball of $\mathbb{R}^3$ small enough (the radius depends only on this bound), for k large, $\mathbf{S}_k$ is the graph of a function with bounded gradient and Hessian. Consequently, these functions satisfy uniformly elliptic equations with uniformly controlled coefficients. Thus by Schauder estimates and standard elliptic theory, for all $r > 0$, the intrinsic balls $B^{\mathbf{S}_k}(0, r)$ converge subsequently in the C^m topology to a smooth embedded surface with boundary called $\mathbf{T}_r$, which is in fact minimal. Define $\mathbf{S}$ to be the union of the $\mathbf{T}_r$ for $r > 0$. It's an embedded complete minimal surface. Here the surfaces considered are all oriented embedded, the integral of $|A|^2$ is bounded uniformly and $\mathbf{S}$ is non flat. Consequently the convergence has in fact multiplicity one, i.e. $B^{\mathbf{S}_k}(0, r)$ converge smoothly with multiplicity one to $B^{\mathbf{S}}(0, r) = \mathbf{T}_r$. Besides, note that

$$\forall r > 0 \quad \int_{\mathbf{S} \cap B(0,r)} |A|^2 = 2 \int_{\mathbf{S} \cap B(0,r)} |K| \leq \limsup_{k \rightarrow \infty} \int_{\mathbf{S}_k \cap B(0,r)} |A|^2 \leq \kappa. \quad (9.8)$$

Now, let's cite two results which will imply that $\mathbf{S}$ is necessarily a catenoid.

The first well-known theorem of Osserman describes minimal surfaces with finite total curvature.

Theorem 111. [118] *Let $M \subset \mathbb{R}^3$ be a complete oriented minimal surface with Gauss curvature K such that $\int_M |K| < \infty$. Then there exists a closed Riemann surface $\tilde{M}$, a finite set of points $\{p_1, \dots, p_k\} \in \tilde{M}$ and a conformal diffeomorphism between M and $\tilde{M} \setminus \{p_1, \dots, p_k\}$. Moreover, the Gauss map extends meromorphically across the punctures.*

The following theorem, proved by López and Ros, then Meeks and Rosenberg, classifies the properly embedded minimal surfaces with finite topology and genus 0:

Theorem 112 ([95], [111]). *The only properly embedded minimal surfaces of $\mathbb{R}^3$, with finite topology and genus 0 are planes, helicoids and catenoids.*

By invariance under dilatations of integrals like $\int_{\mathbf{S}_k \cap B(0,r)} |A|^2$, the first of the theorems above, the assumption on the curvature concentration and (9.8) show that $\mathbf{S}$ has finite topology. Note that moreover, the normal vectors are well defined at each end, which implies that $\mathbf{S}$ is proper. Because the norm of the second fundamental form of Σ at p_k goes to infinity, $\|p_k\| \rightarrow \infty$. Hence, $\mathbf{S}$ has genus 0 (the convergence has multiplicity one). The second theorem, plus the fact that the Gauss curvature of $\mathbf{S}$ at the origin is $-1/2$, imply that $\mathbf{S}$ is necessarily a catenoid. Note that, as the multiplicity of the convergence is one, for k large Σ_k really looks like a small catenoid near p_k .

The end of the proof consists in finding a contradiction with the fact that $\mathbf{S}$ is a catenoid. Note that a posteriori, we could have chosen p_k to have the origin on the closed simple geodesic of the catenoid $\mathbf{S}$, and also to make $p_k/\|p_k\|$ converge to a vector, say v . Let (e'_1, e'_2, e'_3) be a basis of $\mathbb{R}^3$ such that e'_3 gives the rotation axis of $\mathbf{S}$.

Let $c \subset \mathbf{S}$ be the closed simple geodesic which encircles the neck of $\mathbf{S}$. Let (c_k) be a sequence of curves in Σ corresponding to embedded curves in $\mathbf{S}_k$ converging smoothly to $c \subset \mathbf{S}$. Let Σ_k^+ and Σ_k^- be the two open connected components separated by c_k , Σ_k^+

being the component whose rescaling converge to $\{x \in \mathbf{S}; \langle x, e'_3 \rangle > 0\}$ (we can suppose that there are two distinct components because the genus of Σ is finite and $\|p_k\| \rightarrow \infty$). Note that seen from afar, a catenoid is like two superposed planes. So for k large, Σ_k^+ and Σ_k^- are two surfaces nearly parallel and flat around p_k and glued together along c_k .

There are two cases: either v is parallel to e'_3 or it isn't.

Suppose that v is parallel to e'_3 , say $v = e'_3$. Then by applying Proposition 106, we see that, for k large, Σ_k^+ should be contained in a set looking like a long tube with boundary c_k going to infinity and as narrow as c_k near this little neck. But Σ_k^+ is more like a plane orthogonal to $e'_3 = p_k/\|p_k\|$ with a small half neck at p_k . This is the desired contradiction.

Suppose otherwise that v is not parallel to e'_3 . The latter can be chosen so that $\langle e'_3, v \rangle > 0$. In this case, by the usual maximum principal, for all k large

$$\Sigma_k^\pm \cap S(0, 2) \neq \emptyset.$$

This observation plus the fact that the genus of Σ is finite would imply that Σ is not properly embedded, which is absurd.

□

This theorem implies that translating solitons can help to understand regions of Σ far from the origin:

Corollary 113. *Let $\Sigma \subset \mathbb{R}^3$ be as in the previous theorem. Consider a sequence $x_k \in \Sigma$ of points whose norm goes to infinity. Define $\mathbf{S}_k = \|x_k\| \cdot (\Sigma - x_k)$. Then subsequently, for all $d > 0$, $B^{\mathbf{S}_k}(0, d)$ converge smoothly with multiplicity one to a surface with boundary T_d and the union $T = \cup_d T_d$ is a complete embedded translating soliton of the mean curvature flow with genus 0.*

For the end of this section, we give a result analogous to Theorem 109, but for surfaces with δ -separated ends, so that the constant this time won't depend on the surface itself but on some geometric parameters mentioned in the following definitions:

Definition 9.4.2. *Let $R > 0$. A complete properly embedded surface $\mathcal{S} \subset \mathbb{R}^3$ with finite genus $g(\mathcal{S})$ is said to have topology bounded by R if the genus of $B(0, R) \cap \mathcal{S}$ is equal to $g(\mathcal{S})$.*

Definition 9.4.3. Define $\mathfrak{F}_{g,R}^{\delta,\kappa}$ the family of complete self-shrinking surfaces properly embedded in $\mathbb{R}^3$ with finite topology such that:

1. the genus is less than g ,
2. the topology is bounded by R ,
3. the curvature concentration is bounded by κ ,
4. the ends are δ -separated.

The proof of the following theorem is similar to the previous one but the new point is that the assumption on the ends of the self-shrinking surfaces enables us to use Corollary 102. More precisely, small catenoid-like necks on self-shrinking surfaces with δ -separated ends are ruled out by this corollary.

Theorem 114. *There exists a constant $C = C(\delta, \kappa, g, R)$ depending only on, δ , κ , g and R such that*

$$\forall \mathcal{S} \in \mathfrak{F}_{g,R}^{\delta,\kappa}, \quad \forall x \in \mathcal{S}, \quad |A(x)| \leq C(1 + \|x\|).$$

Proof. The beginning of the proof is nearly identical to the previous one, except that this time we consider $p_k \in \mathcal{S}_k \in \mathfrak{F}_{g,R}^{\delta,\kappa}$ hypothetically contradicting the theorem, define $\mathbf{S}_k = \mu_k \cdot (\mathcal{S}_k - p_k)$ and use this lemma:

Lemma 115. *Under the hypotheses of the theorem, there is a sequence (p_k) with $p_k \in \mathcal{S}_k \in \mathfrak{F}_{g,R}^{\delta,\kappa}$ such that*

$$\forall \alpha > 0, \quad |A(p_k)|/(1 + \|p_k\|) \rightarrow \infty \text{ and}$$

$$\max\{|A(q)|/\mu_k; q \in \mathcal{S}_k \cap B(p_k, \alpha/\mu_k)\} \leq 2 \text{ for large } k.$$

As before, we get a limit $\mathbf{S}$. To prove that it is once again a catenoid, the same arguments work if we can prove that the genus of $\mathbf{S}$ is 0, which was clear when only one surface was considered. We use a theorem of compactness for self-shrinkers by Colding and Minicozzi.

Theorem 116 ([37]). *Let g be a integer. The space $\mathcal{F}_g$ of properly embedded self-shrinkers without boundary, with finite topology and genus less than g is compact for the topology*

of C^m convergence on compacts.

Remark 9.4.2. *Colding and Minicozzi initially supposed that the volume growth was quadratic, with is a natural assumption in the context of limit surfaces for mean curvature flow flowing from embedded closed surface [36]. This hypothesis is in fact always true as long as the self-shrinker is proper, as shown by Ding and Xin [43].*

From this theorem and the fact that $\mathfrak{F}_{g,R}^{\delta,\kappa} \subset \mathcal{F}_g$, we deduce that $\|p_k\| \rightarrow \infty$. The topology being uniformly bounded by R , $\mathbf{S}$ has indeed genus 0 (remind that the convergence of $B^{\mathbf{S}_k}(0, r)$ to $B^{\mathbf{S}}(0, r)$ has multiplicity one).

The end of the proof is based on Corollary 102, as explained previously. A posteriori, we could have chosen p_k to have the origin on the closed simple geodesic of the catenoid $\mathbf{S}$, and also to make $p_k/\|p_k\|$ converge to a vector, say v . Let (e'_1, e'_2, e'_3) be a basis of $\mathbb{R}^3$ such that e'_3 gives the rotation axis of $\mathbf{S}$. We denote by P the plane containing e'_1 and e'_2 .

We will distinguish two cases: either the limit v is not in the plane P , or v is in P (i.e. v is orthogonal to e'_3).

Case 1.

Suppose that v is not in the plane P . Let $c \subset \mathbf{S}$ be the closed simple geodesic which encircles the neck of $\mathbf{S}$. Note that because of the choice of p_k in Lemma 115 and because $\mathbf{S}$ and $\tilde{\mathbf{S}} \subset \mathbb{R}^3$ necessarily intersect if $\tilde{\mathbf{S}}$ is a plane or a catenoid, one can suppose that $B(0, k) \cap \mathbf{S}_k$ has only one connected component for all k . Let (c_k) be a sequence of curves in $\mathbf{S}_k$ corresponding to embedded curves in $\mathbf{S}_k$ converging smoothly to $c \subset \mathbf{S}$. We can suppose that $\langle e'_3, v \rangle > 0$. Let $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ be the two open connected components separated by c_k , $\mathcal{S}_k^+$ being the component whose rescaling converge to $\{x \in \mathbf{S}; \langle x, e'_3 \rangle > 0\}$ (we can suppose that there are two distinct components because the topology of the $\mathbf{S}_k$ is bounded and $\|p_k\| \rightarrow \infty$). Note that seen from afar, a catenoid is like two superposed planes. So for k large, $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ are two surfaces nearly parallel and flat around p_k and glued together along c_k .

Suppose first that $\mathcal{S}_k^+$ intersects the self-shrinking sphere $S(0, 2)$: this is in fact always the case if $v \neq e'_3$ by the usual maximum principle, because then for k large there are points in $\mathcal{S}_k^+$ which are closer to the origin than any points on c_k . Then, we can use Corollary 102. Indeed, take $y \in \mathcal{S}_k^+ \cap S(0, 2)$. By changing the canonical basis if necessary, we can suppose that $e_3 = (p_k - y)/\|p_k - y\|$. Let's check for instance the second hypothesis of

Corollary 102. For every point q in a tubular neighborhood of c_k , for k large, the half-line $[q, y)$ first touches $\mathcal{S}_k^-$ before arriving at y (recall that $B(0, k) \cap \mathbf{S}_k$ has only one connected component for all k). By the same kind of arguments, the hypothesis of Corollary 102 are all verified. So $\mathcal{S}_k$ is not a self-shrinker, which is absurd.

Suppose now that $\mathcal{S}_k^+$ does not intersect the self-shrinking sphere $S(0, 2)$: necessarily $v = e'_3$. By applying Proposition 106, we see that, for k large, $\mathcal{S}_k^+$ should be contained in a set looking like a long tube with boundary c_k going to infinity and as narrow as c_k near this little neck. But $\mathcal{S}_k^+$ is more like a plane orthogonal to $e'_3 = p_k / \|p_k\|$ with a small half neck at p_k . This is the desired contradiction.

Case 2.

Suppose that $v \in P$. Choose a sequence of embedded curve $(c_k) \subset \mathcal{S}_k$ as before, let $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ be the two open connected components separated by c_k , $\mathcal{S}_k^+$ being the component whose rescaling converge to $\{x \in \mathbf{S}; \langle x, e'_3 \rangle > 0\}$. By Theorem 104, we know that $\lambda \mathcal{S}_k^+$ (resp. $\lambda \mathcal{S}_k^-$) converge (when $\lambda \rightarrow 0$) to a cone whose intersection with $S(0, 1)$ is called K_k^+ (resp. K_k^-). By δ -separation of the ends of $\mathcal{S}_k$, $d(K_k^+, K_k^-) \geq \delta$. Define $\Theta_{e, \theta}$ the linear rotation with axis $\mathbb{R}e$ and angle θ . Consider the axis $e_0 = v \wedge e'_3$. Because $\mathcal{S}_k$, $d(K_k^+, K_k^-) \geq \delta$, for θ sufficiently small independent of k , there is a $\bar{r}$ depending on k such that if $r > \bar{r}$,

$$\Theta_{e, \theta}(\mathcal{S}_k^+) \setminus B(0, r) \cap \mathcal{S}_k^- \setminus B(0, r) = \emptyset.$$

Now, fix an angle $0 < \theta_0 < \pi$ sufficiently small. With these choices,

$$\forall k \text{ large, } \Theta_{e_0, \theta_0}(\mathcal{S}_k^+) \cap \mathcal{S}_k^- = \emptyset.$$

Indeed, if it is not true for k' , then consider the first θ between 0 and θ_0 such that $\Theta_{e, \theta}(\mathcal{S}_{k'}^+)$ touches $\mathcal{S}_{k'}^-$. Then these surfaces are in contact at a point in the ball $B(0, \bar{r})$ and not outside. But that contradicts the usual maximum principle. We can finally conclude, because for k large,

$$\inf\{\|x - y\|; x \in \Theta_{e_0, \theta_0}(\mathcal{S}_k^+), y \in \mathcal{S}_k^-\}$$

will be achieved since θ_0 is independent of k : once again the usual maximum principle is violated so $\mathcal{S}_k$ can't be self-shrinkers for k large, which is absurd.

□

Chapter 10

Bibliography

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