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A New Northcott Property for Faltings Height

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A NEW NORTHCOTT PROPERTY FOR FALTINGS HEIGHT

LUCIA MOCZ

A DISSERTATION
PRESENTED TO THE FACULTY
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Abstract

In this work we prove a new Northcott property for the Faltings height. Namely we show, assuming the Colmez Conjecture and the Artin Conjecture, that there are finitely many CM abelian varieties over the complex numbers of a fixed dimension which have bounded Faltings height. The technique developed uses new tools from integral p -adic Hodge theory to study the variation of Faltings height within an isogeny class of CM abelian varieties. In special cases, we are moreover able to use the technique to develop new Colmez-type formulas for the Faltings height.

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To my family.

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Before unearthing this letter, I had questioned myself about the ways in which a book can be infinite. I could think of nothing other than a cyclic volume, a circular one. A book whose last page was identical with the first, a book which had the possibility of continuing indefinitely.

José Luis Borges, **Labyrinths**

1

Overview

1.1 Goals

We develop in this work an explicit new technique to study the change in Faltings height within an isogeny class of CM abelian varieties over $\mathbb{C}$. Most importantly, we obtain a new formula for computing this change for certain “fundamental” isogenies between CM abelian varieties. Assuming the Colmez conjecture and the Artin conjecture, we are able to deduce from these formulas a new Northcott property for the Faltings heights of CM abelian varieties, namely that there are finitely many isomorphism classes of CM abelian varieties over $\mathbb{C}$ of a fixed dimension g which have

bounded Faltings height. In certain special small dimension cases, these computations allow us to write new Colmez-type formulas for the Faltings height.

The results presented here are new, although the question of computing the Faltings height of CM elliptic curves with non-maximal CM order was previously considered in a paper by Nakajima and Taguchi [31], and the variation of the Faltings height for non-CM elliptic curves was studied by Szpiro and Ullmo in [40]. Our techniques, which are also new to questions of this type, are more powerful than those considered in [31] and [40], both of which are not easily seen to be generalizable to abelian varieties of dimension greater than 1. The main input for our formulas comes from integral p -adic Hodge theory, namely to use Kisin modules to analyze the ramification behavior of isogenies between CM abelian varieties. These new tools may be introduced to reinterpret some classical results of Faltings [15] and Raynaud [36], but we remark that this would not yield a strengthening of their results. The technique generalizes to study isogenies between CM motives with an appropriate formulation of motivic Faltings height. Work on motivic Faltings height has been done by Kato [20] and Koshikawa [24], and a recent preprint by Johannes Anschütz [2] shows how to define CM objects in any rigid Tannakian category, which gives the appropriate setting for posing the question on CM motives.

To define the Faltings height of an abelian variety, we let K be a number field and A denote an abelian variety of dimension $g \geq 1$ defined over the field K and we assume it to have semi-stable reduction everywhere. Let $\mathcal{A}/\mathcal{O}_K$ denote the corresponding Néron model with identity section $s : \mathrm{Spec}(\mathcal{O}_K) \rightarrow \mathcal{A}$ extending the identity section on A . Denote by $\omega_{\mathcal{A}} := s^* \bigwedge^g \Omega_{\mathcal{A}/\mathcal{O}_K}^1$ the Hodge bundle on $\mathcal{A}$, and $\overline{\omega}_{\mathcal{A}}$ the line bundle $\omega_{\mathcal{A}}$ endowed with Hermitian metrics at all infinite places (defined in §2.1). The Hermitian structure allows us to use an Arakelov-type intersection theory, and more precisely an Arakelov-degree of line bundles. The Arakelov degree of $\overline{\omega}_{\mathcal{A}}$ is a useful invariant for studying questions of diophantine approximations (e.g., the classical

work of Faltings [15] or the more recent work of Tsimerman [42]), and goes under the name of *Faltings height*. The Faltings height may also be written by the explicit formula

$$[K : \mathbb{Q}]h_{\text{Fal}}(A) := \widehat{\deg}(\overline{\omega}_{\mathcal{A}}) = \log(\#s^*(\omega_{\mathcal{A}})/\mathcal{O}_K \cdot \alpha) - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \frac{1}{2} \log \int_{A^{\sigma}(\mathbb{C})}^{\text{norm}} \alpha^{\sigma} \wedge \overline{\alpha}^{\sigma}$$

where $\alpha \in \omega_{\mathcal{A}}$ and the integral has an appropriate normalization constant (see [35] for a discussion on various normalizations). We note that this explicit formula is base-change invariant and independent of the choice of $\alpha \in \omega_{\mathcal{A}}$ by the adelic product formula.

The original Northcott property for Faltings height shown by Faltings in [15] states that for a fixed positive integer d and fixed positive real number C , there exist finitely many abelian varieties A defined over a number field K such that $[K : \mathbb{Q}] \leq d$ and $h_{\text{Fal}}(A) < C$. This property is one of the key propositions in Faltings' proof of the Shafarevich and Mordell conjectures [15].

We assume now that all our abelian varieties have complex multiplication (see §3 for a definition). *A priori*, a CM abelian variety is a complex point on a Siegel moduli space, but the theorem of Shimura–Taniyama asserts that every CM abelian variety descends to a number field and moreover has a model over a number field over which it has good reduction everywhere. Both of these assertions are key to transforming the abstract geometric question of counting certain complex points on the Siegel moduli space exhibiting extra symmetric behavior to a question in arithmetic.

The first main theorem we prove is the following.

Theorem 1.1.1 (CM Northcott Property for Isogeny Classes). *Let $C > 0$ be a constant and $g > 0$ be a fixed integer. Then the number of isomorphism classes of CM abelian varieties A of dimension g with $h_{\text{Fal}}(A) < C$ is finite within isogeny classes.*

This theorem is proved quantitatively: the change in Faltings height is numerically computed based on the combinatorics of prime splitting in the CM field, or more generally a CM algebra, which geometrically corresponds to a canonical decomposition of the isogeny. At each prime-power-degree isogeny, the key computation is that of the relative Hodge bundle, which is done locally using Kisin modules.

Since our computations are precise, a closed formula for the change in Faltings height between any two isogenous CM abelian varieties may be obtained in any case where the splitting behavior of all primes dividing the degree of the isogeny is entirely understood. We compute one example of this formula, which is new to the literature, namely the Faltings height variation for simple CM abelian surfaces whose CM endomorphism ring is Galois. Namely, we let A be a simple CM abelian surface over a number field K with CM by $\mathcal{O}_E$, where $E/\mathbb{Q}$ is a cyclic, Galois CM field of degree 4 and $F \subseteq E$ its totally real subfield. After enlarging K if necessary, let A_{p^n} be a simple CM abelian surface isogenous to A over K with CM by $\mathcal{O}$, where $\mathcal{O} \subseteq \mathcal{O}_E$ is a non-maximal order with index of index p^n in $\mathcal{O}_E$. Finally, let $\mathcal{G}$ denote the kernel of the isogeny of minimal degree between the Néron models of the previous two abelian varieties; this is, in particular, a finite flat group scheme over $\mathcal{O}_K$ of order p^n .

Theorem 1.1.2. *Let $\Delta_{p^n} = h_{\text{Fal}}(A_{p^n}) - h_{\text{Fal}}(A)$.*

1. *If $p = \mathfrak{p}^2$ in E is such that p ramifies in $F/\mathbb{Q}$ and $\mathfrak{p}$ remains inert in E/F ,*

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{2}{p+1} \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \right] \log p.$$

2. *If $p = \mathfrak{p}$ is inert in E , let $(\lambda_1, \lambda_2, \lambda_3)$ be a non-increasing tuple characterizing $\mathcal{G}$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = n$, $\mathcal{G}$ is p^{λ_1} -torsion, and the p -height $\iota \in \{1, 2, 3\}$ of $\mathcal{G}$ is the largest integer such that $\lambda_i \neq 0$. Then*

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{p-1}{p^4-1} \left(\sum_{i=\lambda_2+1}^{\lambda_1} \frac{p+1}{p^i} \right) \left(\sum_{j=\lambda_3+1}^{\lambda_2} \frac{(p+1)^2}{p^j} \right) \left(\sum_{k=1}^{\lambda_3} \frac{(p^2+2)(p+1)}{p^k} \right) \right] \log p$$

where the sums are evaluated if and only if the difference in bounds is at least zero.

Such formulas for $h_{\text{Fal}}(A_{p^n})$ for all 9 possibilities of the splitting behavior for p in E are given in Theorem 8.2.2. This theorem answers a question posed by Habegger and Pazuki in [19]. In their paper, they ask whether there are finitely many curves C of genus 2 defined over $\overline{\mathbb{Q}}$ with good reduction at all but a given finite set of places and for which $\text{Jac}(C)$ has complex multiplication. They show this to be true when the curve has good reduction everywhere and a simple CM Jacobian with CM by a maximal order. To extend their result to a wider class of CM Jacobians, the problem is reduced to the stated Northcott property in the theorem above. Note that while the Northcott property is general, the result by Habegger and Pazuki and stated application is specific to genus 2 curves, and highlights a unique property that may be shared only by curves of low genus.

We establish conditionally the CM Northcott property for isomorphism classes of CM abelian varieties by invoking a famous conjecture due to Colmez [?].

Conjecture 1.1.3 (Colmez Conjecture, [12]). *Let A be an abelian variety over K of dimension g , and suppose that A has CM by the ring of integers $\mathcal{O}_E \subseteq E$, where E is a CM field and $[E : \mathbb{Q}] = 2g$. Then:*

$$h_{\text{Fal}}(A) = - \sum_i c_i \left(\frac{L'(\chi_i, 0)}{L(\chi_i, 0)} + \frac{1}{2} \log f_{\chi_i} \right) + \frac{g}{2} \log 2\pi$$

where the χ_i are a finite collection of irreducible Artin characters of $\text{Gal}(K/\mathbb{Q})$ with Artin conductor f_{χ_i} , and the c_i are positive constants (see §2.2 for details).

In [14], Colmez develops a lower bound on the Faltings height by invoking the Artin conjecture to bound the (logarithmic derivatives of) Artin L -functions in the formula. This lower bound is expressed in terms of Artin conductors of Galois representations, and together with Theorem 1.1.1 is sufficient to establish the following theorem.

Theorem 1.1.4 (CM Northcott Property). *Let $C > 0$ be a fixed constant and $g > 0$ be a fixed integer. Then assuming Conjecture 1.1.3 and the Artin conjecture, the number of $\overline{\mathbb{Q}}$ -isomorphism classes of CM abelian varieties A of dimension g and with $h_{\text{Fal}}(A) < C$ is finite.*

Cases where the Colmez conjecture are known to be true include abelian CM fields, which was shown up to a factor of $\log 2$ by Colmez in [12] and later cleared up by Obus in [34], and many classes of abelian surfaces shown by Yang [43]. An average version has also been shown independently by Yuan–Zhang [44] and Andreatta–Goren–Howard–Madhapusi–Pera [1]. We note that the average formula is not strong enough to establish our results unless all abelian varieties in computing the average have the same Faltings height.

The quantitative versions of Theorem 1.1.1 and Theorem 1.1.4 have applications to future questions requiring an understanding of the distribution of (CM) points on various moduli spaces of abelian varieties. For instance, it recovers the average growth of the Faltings height of the Hecke orbit of a CM abelian variety, a formula computed for some low-dimensional abelian varieties by Autissier in [3]. This Autissier growth formula was the key input to [11] and [38] to construct thin sets of primes of certain reduction types on elliptic curves and abelian surfaces. In [38], the authors show this growth formula computed on a CM abelian surface provides the growth formula for all abelian surfaces. This lemma is likely true in general, and these formulas then extend the Autissier growth formula to all dimensions, providing means to extend [11] and [38] even further.

1.2 Compass and Map

The organization of this work is as follows. In Chapter 2, we introduce the Faltings height and cite the key theorems and conjectures: the Faltings Isogeny Lemma, the

Colmez Conjecture, and the lower bound on Faltings height from the Colmez Conjecture and the Artin Conjecture. In Chapter 3, we review the theory of complex multiplication. The theory we require goes well beyond Shimura and Taniyama’s introduction of the subject, since we require the CM theory of Néron models of abelian varieties and p -divisible groups, and is treated in vast amount of detail by Chai–Conrad–Oort [10]. Chapter 4 is devoted to the relevant recent developments in (integral) p -adic Hodge theory we need to compute the (relative) Hodge bundle in the change in Faltings height, most namely the development revolving around the Kisin modules originally introduced by Kisin in [21]. The brief Chapter 5 reviews the arithmetic and group theoretic properties encoded by Kisin modules, such as a characterization of finite flat subgroup schemes and the theory of their Harder-Narasimhan (HN) filtration.

Chapters 6–8 are the heart of this work and devoted to the proof of the CM Northcott Property and the application to Colmez-type formulas. In Chapter 6, we introduce the notion of an $\mathcal{O}_E$ -linear CM Kisin module and deduce decomposition theorems of their structure, including a general theorem on a weak characterization of unstable submodules of $\mathcal{O}_E$ -linear CM Kisin torsion modules. The work in this section inspired the recent development of CM Breuil-Kisin-Fargues (BKF) modules introduced by Johannes Anschütz in [2], although he works with a certain “isogeny” category of BKF modules and does not recover (as we do) information on individual elements in the “isogeny classes”. There is current work in progress by the present author to extend the present computations to his setting. In Chapter 7, we compute precisely the HN slopes of unstable submodules of $\mathcal{O}_E$ -linear torsion CM Kisin modules using explicit CM and Lubin-Tate theory. We emphasize that one may deduce from these computations the *exact* change in Faltings height between any given two isogeneous CM abelian varieties. This is not yet possible to do without the CM hypothesis with current technology, but it is future work to investigate this question.

Finally, Chapter 8 is devoted to proving Theorems 1.1.1, 1.1.2, and 1.1.4 from the earlier developments.

Like all walls it was ambiguous, two faced. What was inside it and what was outside it depended upon which side you were on.

Ursula Le Guin, **The Dispossessed**

2

Faltings Height

2.1 Semi-stable Faltings height

Let K be a number field and $\mathcal{O}_K \subseteq K$ be its ring of integers. We consider abelian varieties A of dimension $g \geq 1$ defined over the field K with semi-stable reduction everywhere and write $\mathcal{A}/\mathcal{O}_K$ for their Néron model. Denote by $s : \mathrm{Spec}(\mathcal{O}_K) \rightarrow \mathcal{A}$ the extension of the zero section, and by $\omega_{\mathcal{A}} := s^*\Omega_{\mathcal{A}/\mathcal{O}_K}^g$ the canonical line bundle pulled back along the zero section to a line bundle on $\mathrm{Spec}(\mathcal{O}_K)$.

Recall the following definition from Arakelov theory.

Definition 2.1.1. A **Hermitian line bundle** on $\mathrm{Spec}(\mathcal{O}_K)$ is a pair $(\mathcal{L}, \{||\cdot||_{\sigma}\}_{\sigma:K \hookrightarrow \mathbb{C}})$

consisting of an invertible sheaf $\mathcal{L}$ on $\mathrm{Spec}(\mathcal{O}_K)$ and a collection of Hermitian metrics $\|\cdot\|_\sigma$ on $\mathcal{L}^{\mathrm{an}} \otimes_\sigma \mathbb{C}$ ranging over the complex embeddings $\sigma : K \hookrightarrow \mathbb{C}$ such that $\|s^c\|_{\sigma^c} = \|s\|_\sigma$, where c denotes complex conjugation.

The line bundle $\omega_{\mathcal{A}}$ can be equipped with a collection of hermitian metrics to give it the structure of a hermitian line bundle. Define the hermitian norm $\|\cdot\|_\sigma$ for each embedding $\sigma : K \hookrightarrow \mathbb{C}$ by the formula

$$\|\alpha\|_\sigma^2 = \frac{1}{(2\pi)^g} \left| \int_{\mathcal{A}_\sigma(\mathbb{C})} \alpha \wedge \bar{\alpha} \right|$$

where $\mathcal{A}_\sigma$ denotes the “completion” of the Néron model at the archimedean fiber, i.e., a complex manifold which descends to an algebraic model A/K . Here we consider $\alpha \in H^0(\mathcal{A}_\sigma, \Omega_{\mathcal{A}_\sigma}^g)$ via the isomorphism $\omega_{\mathcal{A}} \otimes_\sigma \mathbb{C} \simeq H^0(\mathcal{A}_\sigma, \Omega_{\mathcal{A}_\sigma}^g)$. Then the hermitian line bundle $\bar{\omega}_{\mathcal{A}}$ will denote the pair $(\omega_{\mathcal{A}}, \{\|\cdot\|_\sigma\}_{\sigma:K \hookrightarrow \mathbb{C}})$.

Recall as well the following definition from Arakelov theory.

Definition 2.1.2. The **Arakelov degree** of a Hermitian line bundle $(\mathcal{L}, \{\|\cdot\|_\sigma\}_{\sigma:K \hookrightarrow \mathbb{C}})$ on $\mathrm{Spec}(\mathcal{O}_K)$ is the value

$$\widehat{\deg}(\bar{\mathcal{L}}) = \log \#(\mathcal{L}/\alpha) - \sum_{\sigma:K \hookrightarrow \mathbb{C}} \log \|\alpha\|_\sigma$$

where α is a non-zero global section of $\mathcal{L}$.

Remark 2.1.3. By the product formula, this is independent of the choice of α , so the Arakelov degree is a well-defined invariant.

Using the Arakelov degree, we define the Faltings height as follows. Note that the Faltings height is an arithmetic invariant on the isomorphism classes of abelian varieties.

Definition 2.1.4. The (stable) **Faltings height** of A is

$$h_{\text{Fal}}(A) := \frac{1}{[K : \mathbb{Q}]} \widehat{\deg}(\overline{\omega}_{\mathcal{A}}).$$

This is computed explicitly by the formula

$$\widehat{\deg}(\overline{\omega}_{\mathcal{A}}) = \log \#(\omega_{\mathcal{A}}/\mathcal{O}_K \alpha) - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|\alpha\|_{\sigma}.$$

The invariant receives its name as a height due to the fact it satisfies the following celebrated Northcott property demonstrated by Faltings in [15]. This Northcott property was a key insight in Faltings' proof of the Mordell conjecture.

Theorem 2.1.5 (Faltings' Northcott Property, [15]). *Let C and d be fixed positive constants and $g \geq 1$ be an integer. Then the set of isomorphism classes of abelian varieties A/K of dimension g with $[K : \mathbb{Q}] < d$ and $h_{\text{Fal}}(A) < C$ is finite.*

The Faltings height is not an isogeny-invariant. This very property of variation of the Faltings height in isogeny classes was exploited heavily by Faltings in [15], and is measured by the following formula.

Lemma 2.1.6 (Faltings Isogeny Lemma, [15]). *Let $\phi : A_1 \rightarrow A_2$ be an isogeny over K , and let $\mathcal{G}$ be the finite flat group scheme kernel of the extension $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ over $\mathcal{O}_K$, where $\mathcal{A}_i$ is the Néron model corresponding to A_i for $i \in \{1, 2\}$. Then*

$$h_{\text{Fal}}(A_2) = h_{\text{Fal}}(A_1) + \frac{1}{2} \log(\deg(\phi)) - \frac{1}{[K : \mathbb{Q}]} \log(\#s^* \Omega_{\mathcal{G}/\mathcal{O}_K}^1).$$

The techniques to be developed in this paper aim to compute the ramification term on the right side of this formula and show its contribution is, in all relevant cases, negligible compared to the term provided by the degree of the isogeny.

2.2 Colmez Conjecture

We state here the Colmez Conjecture for the Faltings height of CM abelian varieties. The theory of complex multiplication for abelian varieties and p -divisible groups will be reviewed in §3, so we invite the reader to briefly review the definitions if she wants first to familiarize herself with the subject. Apart from the basic definitions, however, familiarity with the theory discussed in the section will not be relevant to state the conjecture.

Let (E, Φ) be a CM pair where E is a CM field such that $[E : \mathbb{Q}] = 2g$ and $\Phi = \{\phi_1, \dots, \phi_g\}$ is a CM type of E . For $\phi : E \rightarrow \overline{\mathbb{Q}}$, define the function $a_{E, \phi, \Phi}$ on $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by

$$a_{E, \phi, \Phi}(g) = \begin{cases} 1 & \text{if } g\phi \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_{E, \Phi} = \sum_{\phi \in \Phi} a_{E, \phi, \Phi}$ and construct the class function

$$A_{E, \Phi}^0(g) := \frac{1}{|G_{\mathbb{Q}}/G_E|} \sum_{h \in G_{\mathbb{Q}}/G_E} A_{E, \Phi}(hgh^{-1})$$

where $G_{\mathbb{Q}}$ and G_E denote the absolute Galois groups of $\mathbb{Q}$ and E , respectively. Then, letting E^* denote the reflex field of (E, Φ) , this class function satisfies $A_{E, \Phi}^0(g) = A_{E, \Phi}^0(1)$ if and only if $g \in \text{Gal}(\overline{\mathbb{Q}}/E^*)$. Since irreducible Artin characters form a basis for the space of locally constant class functions on $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $A_{E, \Phi}^0$ has a representation

$$A_{E, \Phi}^0 = \sum_i c_i \chi_i$$

where the sum runs through all irreducible characters on $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and c_i denotes the multiplicity by which they occur. As demonstrated in [14], $c_i = 0$ for all but finitely many characters, and otherwise $c_i > 0$. Moreover, $c_i > 0$ only when χ_i is odd, and therefore $L(\chi_i, 0) \neq 0$ for the characters that appear with a positive constant.

Conjecture 2.2.1 (Colmez Conjecture, [12]). *Let A be an abelian variety of dimension g over a number field K , and suppose that A has CM by the ring of integers $\mathcal{O}_E \subseteq E$, where E is a CM field such that $[E : \mathbb{Q}] = 2g$ and Φ is a CM type of E . Let $A_{E,\Phi}^0 = \sum_i c_i \chi_i$ be as defined above, where the sum is over all irreducible characters $\chi_i \in \text{Hom}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{C})$, each with corresponding (unique) Artin conductor f_{χ_i} . Then:*

$$h_{\text{Fal}}(A) = - \sum_i c_i \left(\frac{L'(\chi_i, 0)}{L(\chi_i, 0)} + \frac{1}{2} \log f_{\chi_i} \right) + \frac{g}{2} \log 2\pi.$$

In the original paper [12], Colmez proves this conjecture for abelian CM fields satisfying a certain ramification condition above the prime 2. Obus [34] was later able to drop the ramification condition and generalize the result to all abelian CM fields. Yang [43] through independent methods verified the conjecture for abelian surfaces whose CM field is not Galois over $\mathbb{Q}$. In two different approaches towards this conjecture, Yuan–Zhang [44] and Andreatta–Goren–Howard–Madapusi-Pera[1] proved an average version (where the average is over the CM types) of this is true.

Our main interest in the Colmez conjecture is to bound the Faltings height from below. In a separate work by Colmez [14] the following bound was obtained by assuming the Artin Conjecture that Artin L -functions are holomorphic over $\mathbb{C}$.

Theorem 2.2.2 (Lower Bound on Faltings height, [14]). *Assuming Conjecture 2.2.1 and the Artin Conjecture, there exists an effective constant $C > 0$ such that*

$$h_{\text{Fal}}(A) \geq \frac{1}{2} \left(-\log 2\pi - \log \pi + \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \right) + C \mu_{\text{Art}}(A_{E,\Phi}^0)$$

where $\mu_{\text{Art}}(A_{E,\Phi}^0) = \sum_{\chi} \langle A_{E,\Phi}^0, \chi \rangle \log f_{\chi}$ and f_{χ} is the Artin conductor corresponding to the character χ .

The lower bound from this theorem immediately gives the following (primitive) CM Northcott property.

Corollary 2.2.3. *Let C be a fixed positive constant and $g \geq 1$ an integer. Then assuming Conjecture 2.2.1 and the Artin Conjecture, the set of isomorphism classes of simple CM abelian varieties of dimension g with CM by a maximal order $\mathcal{O}_E \subseteq E$ with $[E : \mathbb{Q}] = g$ and $h_{\text{Fal}}(A) < C$ is finite.*

Symmetry is tedious, and tedium is the very basis of mourning.
Despair yawns.

Victor Hugo, **Les Misérables**

3

Complex Multiplication

3.1 CM Abelian Varieties

Definitions

We give here a brief survey of the relevant theory of complex multiplication for abelian varieties. This theory is well-known, so the familiar reader can skip this section as it is entirely self-contained. For a comprehensive treatment of the subject, see Chapter 1 in [10]. We note that we need more than just the theory introduced in [39], which does not develop the theory on integral models of CM abelian varieties and CM types valued in arbitrary algebraically closed fields.

Definition 3.1.1. A **CM pair** (E, Φ_L) consists of:

- A number field E , called the **CM field**, having a non-trivial involution $\tau \in \text{Aut}(E)$ such that every embedding $i : E \hookrightarrow \mathbb{C}$ satisfies $i(\tau(x)) = i(x)^c$ for all $x \in E$. Here c denotes complex conjugation in $\mathbb{C}$.
- A subset $\Phi_L \subseteq \text{Hom}_{\mathbb{Q}\text{-alg}}(E, L)$, called a **CM type** valued in an algebraically-closed field L of characteristic zero, such that $\text{Hom}_{\mathbb{Q}\text{-alg}}(E, L) = \Phi_L \sqcup \Phi_L \circ \tau$.

A **CM algebra** P is a finite product of CM fields $\prod_i E_i$. We define on it a CM type by $\Phi_L = \coprod_i \Phi_{L,i}$ where the $\Phi_{L,i}$ is a CM type on E_i .

Remark 3.1.2. When $L = \mathbb{C}$ in the definition above, or the algebraically-closed field L is otherwise evident, we denote the CM type by Φ .

The element τ in the definition is necessarily unique and its fixed field is a totally real subfield $F \subseteq E$ which satisfies $[E : F] = 2$. We call F the *maximal totally real subfield* of E .

Definition 3.1.3. An abelian variety A of dimension g defined over $\mathbb{C}$ **has CM by** (P, Φ) if:

- There is an injective $\mathbb{Q}$ -algebra homomorphism

$$i : P \hookrightarrow \text{End}^0(A) := \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}(A)$$

of a CM algebra P such that $[P : \mathbb{Q}] = 2g$.

- The $P \otimes_{\mathbb{Q}} \mathbb{C}$ -module structure of $\text{Lie}(A)$ is given by the CM type Φ valued in $\mathbb{C}$.

By the Poincaré reducibility theorem (see, e.g., Theorem 1.2.1.3 in [10]), every abelian variety is isogeneous to a product of its distinct isotypic pieces. Hence, if an

abelian variety A has CM by (P, Φ) , then each of the isotypic factors A_i have CM by a sub-algebra $P_i \subseteq P$ such that $P = \prod_i P_i$. Moreover, each P_i is a simple algebra of finite dimension over $\mathbb{Q}$, and hence a CM field. We will not lose any content for establishing the remainder of our theory if we assume that $P = E$ is a CM field and even if we assume that A is simple, so we will use the term *CM abelian variety* in the remainder of this section to refer to this case.

Example 3.1.4. When $A/\mathbb{C}$ has complex multiplication by (E, Φ) , where E is a CM field, then necessarily $\text{End}(A) = \mathcal{O}$ for some (not necessarily maximal) order $\mathcal{O} \subseteq E$. Its dual abelian variety has complex multiplication by $(E, \Phi \circ \tau)$, where the E -action on A is induced by the composition of the dual action with complex conjugation.

Remark 3.1.5. When we wish to specify the endomorphism order of a CM abelian variety, we say it has CM by $(\mathcal{O}, \Phi)$. Note that when $\mathcal{O} = \mathcal{O}_E$, Φ gives a type on the integral structure $\mathcal{O}_E$.

The starting point for questions of arithmetic interest involving CM theory is the following theorem.

Theorem 3.1.6 (Shimura–Taniyama, [10], [39]). *Let $A/\mathbb{C}$ be a CM abelian variety. Then there exists a number field K such that A descends to K . Moreover, there exists a finite extension K'/K such that $A_{K'} := A \otimes_K K'$ has good reduction at all the finite places of K' .*

By this theorem, one may construct appropriate integral models of CM abelian varieties. Moreover, since CM abelian varieties have good reduction everywhere over some finite extension K'/K , their abelian group structure extends to the entire integral model (see [6]). The CM structure is also preserved on the integral model, i.e., complex multiplication is a theory on the generic fiber. This last fact is shown in the following theorem.

Proposition 3.1.7. *Let A/K be an abelian variety with CM by $(\mathcal{O}, \Phi)$ and let $\mathcal{A}/\mathcal{O}_K$ denote its Néron model. Then $\text{End}(\mathcal{A}) = \text{End}(A)$ and $\text{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_K} \mathbb{C}$ has the structure of an $\mathcal{O} \otimes \mathbb{C}$ -module under Φ .*

Proof. The map $\text{End}(\mathcal{A}) \rightarrow \text{End}(A)$ always exists by restricting endomorphisms on the integral model to the generic fiber, and is therefore also always injective. Surjectivity follows from the universal mapping property of Néron models (see [6]). By the isomorphisms

$$\text{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_K} \mathbb{C} \simeq (\text{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_K} K) \otimes_K \mathbb{C} \simeq \text{Lie}(A) \otimes_K \mathbb{C},$$

$\text{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_K} \mathbb{C}$ has an $\mathcal{O} \otimes \mathbb{C}$ -module structure under Φ inherited from the $\mathcal{O} \otimes \mathbb{C}$ -module structure of $\text{Lie}(A) \otimes_K \mathbb{C}$. □

Theory of the Reflex Norm

Theorem 3.1.6 has a stronger form which provides more precise information on the field of definition K to which a CM abelian variety descends. This is contingent on the *theory of the reflex norm*, the celebrated player of CM theory. We introduce this theory starting from the following definition.

Definition 3.1.8. Let

$$h_{\Phi, E} = \prod_{i \in \text{Hom}(E, \mathbb{C})} h_{i, E} : \mathbb{G}_m \rightarrow \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)_{\mathbb{C}} = \prod_{i \in \text{Hom}(E, \mathbb{C})} \mathbb{G}_{m, i}$$

be a cocharacter where each $h_{i, E}$ is defined such that

$$h_{i, E} : t \mapsto \begin{cases} t & \text{if } i \in \Phi, \\ 1 & \text{if } i \notin \Phi. \end{cases}$$

The minimal field of definition for $h_{\Phi, E}$ is the **reflex field** E^* of E .

The reader can check that the minimal field of definition of $h_{\Phi,E}$ over $\mathbb{Q}$ is stabilized by those elements of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which fix the CM type Φ (this, in fact, was the classical definition of the reflex field in [39]). Note that the reflex field is always itself a CM field.

Definition 3.1.9. Let (E, Φ) be a CM field together with a $\overline{\mathbb{Q}}$ -valued CM type and let E^* be its reflex field. For any finite extension K/E^* , the **reflex norm** $N_{\Phi,K}$ is the composite

$$\text{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \xrightarrow{\text{Res}_{K/\mathbb{Q}}(h_{\Phi,E})} \text{Res}_{K/\mathbb{Q}}(\text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)_K) \xrightarrow{\text{Nm}_{K/\mathbb{Q}}} \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m).$$

Remark 3.1.10. For any $\mathbb{Q}$ -algebra R , the reflex norm induces a map

$$N_{\Phi,K \otimes R} : (K \otimes R)^\times \rightarrow (E \otimes R)^\times$$

$$x \mapsto \det_E(x|_{V_{\Phi,K \otimes \mathbb{Q}} R})$$

where $V_{\Phi,K}$ is any $E \otimes_{\mathbb{Q}} K$ -module satisfying $V_{\Phi,K} \otimes_K \overline{\mathbb{Q}} \simeq \prod_{i \in \Phi} \overline{\mathbb{Q}}$.

The stronger form of Theorem 3.1.6 can now be stated as follows. We will let below $r_K^{\text{Art}} : \mathbb{A}_K^\times / K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$ be the global Artin reciprocity map where K^{ab} is the standard notation for the maximal abelian extension of K . We let K_v denote the localization of K at a place v with ring of integral elements $\mathcal{O}_{K_v}$ and a specified uniformizer π_v , and $k_v = \mathcal{O}_{K_v} / \pi_v \mathcal{O}_{K_v}$ be the residue field at v of size q_v .

Theorem 3.1.11 (Main Theorem of Complex Multiplication, [10]). *Let A be an abelian variety with CM by (E, Φ) defined over a number field K . Then the following hold true:*

1. *The reflex field E^* is contained in K .*

2. *There exists a unique algebraic Hecke character*

$$\epsilon : \mathbb{A}_K^\times \rightarrow E^\times$$

such that at each prime ℓ , the continuous homomorphism

$$\begin{aligned} \phi_\ell : \text{Gal}(K^{\text{ab}}/K) &\rightarrow E_\ell^\times \\ r_K^{\text{Art}}(a) &\mapsto \epsilon(a) N_{\Phi, K}(a_\ell^{-1}) \end{aligned}$$

is equal to the ℓ -adic representation of $\text{Gal}(K^{\text{ab}}/K)$ on the ℓ -adic Tate module $V_\ell(A)$.

3. *A has good reduction at a finite place v of K if and only if $\epsilon|_{\mathcal{O}_{K_v}^\times} = 1$. In this case, for any choice of uniformizer $\pi_v \in \mathcal{O}_{K_v}$, $\epsilon(\pi_v) = \text{Fr}_v^{q_v}$, where $\text{Fr}_v \in E$ denotes the endomorphism lifting the relative Frobenius on the reduction A_v of A defined over the finite field k_v .*

3.2 CM p -divisible Groups

Definitions

The theory of complex multiplication for p -divisible groups arises naturally from the integral theory of complex multiplication for abelian varieties. Throughout this section, K will be a local field of characteristic 0 with finite residue field k of characteristic $p > 0$ and $\mathcal{O}_K \subseteq K$ will be its ring of integers. Then let A/K be a CM abelian variety, $\mathcal{A}/\mathcal{O}_K$ be a corresponding local Néron model, and $\mathcal{A}[p^\infty]/\mathcal{O}_K$ a p -divisible group. Since $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow \text{End}(\mathcal{A}[p^\infty])$, one expects a theory of complex multiplication for p -divisible groups parallel to the theory for abelian varieties. We refer the reader to Chapter 3 of [10] for a comprehensive treatment of such a theory; below we review

what to us are relevant statements.

Recall first that for any p -divisible group $\mathcal{G}$ defined over $\mathcal{O}_K$, the endomorphism ring $\text{End}(\mathcal{G})$ is a finite free $\mathbb{Z}_p$ -module which is necessarily a $\mathbb{Z}_p$ -subalgebra of $\text{End}(T_p\mathcal{G})$. Each p -divisible group also carries with it an invariant h called the *height*, which is an integer always bounded below by (and sometimes equal to) the dimension d of the p -divisible group.

Definition 3.2.1. A p -divisible group $\mathcal{G}/\mathcal{O}_K$ of dimension d and height h **has** (p -**adic**) **CM by** (E, Φ) if:

- There is an injective $\mathbb{Q}_p$ -algebra homomorphism

$$i : E \hookrightarrow \text{End}^0(\mathcal{G}) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{End}(\mathcal{G})$$

where $E/\mathbb{Q}_p$, called the (p -**adic**) **CM algebra**, is a commutative, semisimple $\mathbb{Q}_p$ -algebra such that $[E : \mathbb{Q}_p] = h$.

- The $E \otimes_{\mathbb{Q}_p} K$ -module structure of $\text{Lie}(\mathcal{G}) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}_p}$ is given by a subset $\Phi \subseteq \text{Hom}_{\mathbb{Q}_p\text{-alg}}(E, \overline{\mathbb{Q}_p})$, called the (p -**adic**) **CM type**, such that $\#\Phi = d$.

Example 3.2.2. Let $\mathcal{A}/\mathcal{O}_K$ be a local integral model of an abelian variety A/K which we suppose has CM by (E, Φ) . Then $\mathcal{A}[p^\infty]$ has CM by (E_p, Φ_p) , where $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E$ and $\Phi_p := \Phi_{\overline{\mathbb{Q}}} \cap \text{Hom}_{\mathbb{Q}_p\text{-alg}}(E_p, \overline{\mathbb{Q}_p})$. The intersection in the definition of Φ_p means the set of $E \otimes \mathbb{Q}_p \rightarrow \overline{\mathbb{Q}_p}$ that induce something in $\Phi_{\mathbb{Q}}$ by restriction to E , which we obtain after fixing an inclusion $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ (and does not depend on this choice). In fact, if $\text{End}(A) = \mathcal{O}$, then $\text{End}(\mathcal{A}[p^\infty]) = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

A good class of examples of CM p -divisible groups are those that arise from CM abelian varieties, as demonstrated above. We note that although a CM abelian variety might be simple, its p -divisible group $\mathcal{A}[p^\infty]$ may not be so. Analogous to CM abelian varieties, each CM p -divisible group is isogenous to one that isotypically decomposes

along the simple subalgebras of its CM algebra. We will assume for the remainder of this section that all CM p -divisible groups that we consider are simple.

Example 3.2.3. Let $\mathcal{G}/\mathcal{O}_K$ have $(p$ -adic) CM by (E, Φ) . Then its Cartier dual $\mathcal{G}^*$ has a complementary structure Φ^c such that $\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(E, \overline{\mathbb{Q}_p}) = \Phi \coprod \Phi^c$. Moreover, when $\mathcal{G} = \mathcal{A}[p^\infty]$ for a CM abelian variety $\mathcal{A}/\mathcal{O}_K$, then $\mathcal{A}[p^\infty]^* = \mathcal{A}^\vee[p^\infty]$ and Φ^c agrees with the induced type from $\mathcal{A}^\vee$ as in Example 3.2.2.

Theory of the Reflex Norm

CM p -divisible groups carry an analogous theory of the reflex norm to abelian varieties. We begin with the definition of a reflex field for a CM p -divisible group and recall that we assume E to be a $(p$ -adic) CM field.

Definition 3.2.4. Let $\{\xi_i\}_{i \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(E, \overline{\mathbb{Q}_p})}$ be a $\mathbb{Z}_p$ -basis of the character group of $\mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m$. Let

$$\mu_{\Phi, E} : \mathbb{G}_m \rightarrow \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m$$

be defined such that

$$\langle \xi_i, \mu_{\Phi, E} \rangle = \begin{cases} 1 & \text{if } i \in \Phi, \\ 0 & \text{if } i \notin \Phi. \end{cases}$$

The minimal field of definition for $\mu_{\Phi, E}$ is the **$(p$ -adic) reflex field** E^* of E .

Example 3.2.5. Let $\mathcal{A}/\mathcal{O}_K$ be a local integral model of an abelian variety and suppose it has CM by the global pair (E, Φ) and let E^* be the corresponding global reflex field. Then the $(p$ -adic) reflex algebra of $\mathcal{A}[p^\infty]$ is $E_p^* := E^* \otimes_{\mathbb{Q}} \mathbb{Q}_p$. This follows from the compatibility of h_Φ with the local cocharacter μ_{Φ_p} .

Definition 3.2.6. Let K/E^* be any finite field extension of the $(p$ -adic) reflex field E^* of the $(p$ -adic) CM field (E, Φ) , and let $\mu : \mathbb{G}_m \rightarrow (\mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m)_K$ be the K -descent

of μ_Φ , as in Definition 3.2.4. Then the **(p -adic) reflex norm**

$$N\mu_\Phi : \mathrm{Res}_{K/\mathbb{Q}_p} \mathbb{G}_m \rightarrow \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m$$

is the composition

$$\mathrm{Res}_{K/\mathbb{Q}_p} \mathbb{G}_m \xrightarrow{\mathrm{Res}_{K/\mathbb{Q}_p}(\mu)} \mathrm{Res}_{K/\mathbb{Q}_p}((\mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m)_K) \xrightarrow{\mathrm{Nm}_{K/\mathbb{Q}_p}} \mathrm{Res}_{E/\mathbb{Q}_p}(\mathbb{G}_m).$$

Remark 3.2.7. For any $\mathbb{Q}_p$ -algebra R , the (p -adic) reflex norm induces a map

$$\begin{aligned} N_{\Phi, K \otimes R} : (K \otimes R)^\times &\rightarrow (E \otimes R)^\times \\ x &\mapsto \det_{E \otimes R}(x|_{V_{\Phi, K \otimes \mathbb{Q}_p} R}) \end{aligned}$$

where $V_{\Phi, K}$ is any $E \otimes_{\mathbb{Q}_p} K$ -module satisfying $V_{\Phi, K} \otimes_K \overline{\mathbb{Q}_p} \simeq \prod_{i \in \Phi} \overline{\mathbb{Q}_p}$.

We will assume in what follows that the fields E , E^* , and K are as in Definition 3.2.6. Let moreover $r_F^{\mathrm{loc}} : F^\times \rightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F)$ denote the local Artin reciprocity map from local class field theory defined on the units of a local field F , where F^{ab} is the maximal abelian extension of F . We also mention the facts, which can be found in [17] and [10], that every Galois representation attached to a p -divisible group is crystalline and also that all 1-dimensional crystalline representations are locally algebraic. These facts apply in particular to the present case in which we consider Galois representations arising from CM p -divisible groups.

Lemma 3.2.8. *Let $\rho : \mathrm{Gal}(K^{\mathrm{ab}}/K) \rightarrow E^\times$ be a crystalline (i.e., locally algebraic) representation on a 1-dimensional E -vector space, and let $\chi : \mathrm{Res}_{K/\mathbb{Q}_p} \mathbb{G}_m \rightarrow \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m$ be a $\mathbb{Q}_p$ -homomorphism. Suppose that $\rho \circ r_E^{\mathrm{loc}}$ and χ agree on a restriction to an open neighborhood of $\mathcal{O}_K^\times$. Then the two maps agree on the entirety of $\mathcal{O}_K^\times$.*

Proof. The assertion is a special case of a general result in p -adic Hodge theory that if an element in the representation space of a crystalline p -adic representation is fixed

by an open subgroup of the inertia subgroup, then it is fixed by the whole inertia subgroup (see, e.g., Theorem 3.1.1 in [4]). The open neighborhoods of $\mathcal{O}_K^\times$ correspond to open subgroups of the inertia group by local class field theory. $\square$

Corollary 3.2.9 ([10] Corollary 3.7.9). *Every crystalline (i.e., locally algebraic) representation on a 1-dimensional E -vector space arising from a CM p -divisible group is essentially given by the reflex norm. Precisely, for $L = K.W(\bar{k})$, a representation*

$$\rho : \text{Gal}(\bar{L}/L) \rightarrow E^\times$$

attached to a p -divisible group $\mathcal{G}$ with (p -adic) CM by (E, Φ) equals the composition

$$\text{Gal}(L^{\text{ab}}/L) \rightarrow I_{E^*}^{\text{ab}} \xleftarrow[\sim]{r_{E^*}^{\text{loc}}} \mathcal{O}_{E^*}^\times \xrightarrow{1/N\mu_\Phi} E^\times,$$

where $I_{E^*}^{\text{ab}}$ is the abelian part of the inertia group of E^* .

Proof. Here we lay out the ideas; for details see Proposition 3.4.3, Proposition 3.4.4, and Corollary 3.4.9 in [10]. The strategy is to construct a CM abelian variety which locally at p is isogenous to the given CM p -divisible group. Then the main theorem of complex multiplication for abelian varieties demonstrates that the restriction of ρ to the inertia subgroup $I_L^{\text{ab}} \subseteq \text{Gal}(L^{\text{ab}}/L)$ equals the reciprocal of the composition

$$I_L^{\text{ab}} \xrightarrow{\sim} I_{E^*}^{\text{ab}} \xleftarrow[\simeq]{r_{E^*}^{\text{loc}}} \mathcal{O}_{E^*}^\times \xrightarrow{N\mu_\Phi} E^\times.$$

The argument concludes by giving an isogeny between the given p -divisible group and one with an unramified twist so the image of the representation is given entirely by the inertia subgroup. Note that L in the statement may be replaced by any finite extension of $\mathbb{Q}_p$ given by taking the compositum $K.W(k')$ for a large enough finite field extension $k' \supseteq k$, so long as the appropriate unramified twist of $\mathcal{G}$ is defined over $K.W(k')$. $\square$

We conclude by remarking that Lemma 3.2.8 and Corollary 3.2.9 are the local (p -divisible group) analogues of the Main Theorem of Complex Multiplication for abelian varieties.

*No permanence is ours; we are a wave
That flows to fit whatever form it finds:
Through night or day, cathedral or the cave
We pass forever, craving form that binds.*

Hermann Hesse, **The Glass Bead Game**

4

Integral p -adic Hodge Theory

4.1 Deformation Theory of p -divisible Groups

The original approach to classify p -divisible groups by Messing in [30] was along their deformation theory. This classification recovers useful invariants pertaining to a p -divisible group, such as the Hodge bundle $\omega_{\mathcal{G}}$ and the Lie algebra $\mathrm{Lie}(\mathcal{G})$. In subsequent work, Breuil [5] developed an algebraic theory of what are now called *Breuil modules* to capture Messing’s abstract theory in computations. Here we summarize some of these developments by [30] and [5], and show how to write the Hodge bundle $\omega_{\mathcal{G}}$ using Breuil modules.

Let k be a perfect field of characteristic $p > 0$ and $W = W(k)$ be its ring of Witt vectors. Let T_0 be a W -scheme on which p is nilpotent and T be a W -scheme on which p is locally-nilpotent with a closed embedding $T_0 \hookrightarrow T$ over W . Let $\mathcal{G}_0$ to be a p -divisible group on T_0 and $\mathcal{G}$ to be a lift of $\mathcal{G}_0$ to T , i.e., $\mathcal{G}$ is a T -deformation of $\mathcal{G}_0$. We recall that such a $\mathcal{G}$ has a natural *relative Frobenius* morphism acting on it, which is the pullback of the Frobenius acting on the base.

In [30], Messing constructs a contravariant functor from p -divisible groups $\mathcal{G}$ defined over T to *deformation pairs*:

$$\mathcal{G} \mapsto (\mathbb{D}_T(\mathcal{G}), \text{Fil}^1(\mathbb{D}_T(\mathcal{G}))),$$

where the deformation pair consists of a Frobenius crystal $\mathbb{D}_T(\mathcal{G})$ on the crystalline site of T together with a filtered piece $\text{Fil}^1(\mathbb{D}_T(\mathcal{G})) \subseteq \mathbb{D}_T(\mathcal{G})$. For the definition of Frobenius crystals we encourage the reader to see [30]. We summarize the properties we need about crystals from *loc. cit.* in the following lemma.

Lemma 4.1.1 ([30]). *Let T be a W -scheme on which p is locally-nilpotent, $T_0 \hookrightarrow T$ be a closed W -subscheme on which p is nilpotent, and $T' \rightarrow T$ be any crystalline homomorphism of W -schemes.*

1. *(Crystalline Base Change). $\mathbb{D}_{T'} \simeq \mathbb{D}_T \otimes_T T'$, i.e., the Frobenius crystal is compatible with arbitrary crystalline base change.*
2. *(Frobenius Linearization). When p is nilpotent on T , the relative Frobenius ϕ on $\mathcal{G}$ induces a map*

$$\phi^*(\mathbb{D}_T(\mathcal{G})) \xrightarrow{\sim} \mathbb{D}_T(\phi^*(\mathcal{G})) \rightarrow \mathbb{D}_T(\mathcal{G}).$$

3. (Crystal Growth). If $\mathcal{G}/T$ is a lift of $\mathcal{G}_0/T_0$ then

$$\mathbb{D}_{T_0}(\mathcal{G}_0)(T) \xrightarrow{\sim} \mathbb{D}_T(\mathcal{G})(T).$$

4. (Isomorphism Invariance). If $\mathcal{G}$ and $\mathcal{G}'$ are non-isomorphic p -divisible groups over T , then a given isomorphism of crystals $\mathbb{D}_T(\mathcal{G}) \xrightarrow{\sim} \mathbb{D}_T(\mathcal{G}')$ does not map $\mathrm{Fil}^1(\mathbb{D}_T(\mathcal{G}))$ onto $\mathrm{Fil}^1(\mathbb{D}_T(\mathcal{G}'))$.

5. (Compatibility with Duality). $\mathbb{D}_T$ admits a dual $\mathbb{D}_T^*$ satisfying $\mathbb{D}_T^*(\mathcal{G}) = \mathbb{D}_T(\mathcal{G}^*)$, where $\mathcal{G}^*$ denotes the Cartier dual of $\mathcal{G}$.

The deformation exact sequence of a p -divisible group $\mathcal{G}$ defined over T is given by

$$0 \rightarrow \mathrm{Lie}(\mathcal{G})^\vee \rightarrow \mathbb{D}_T(\mathcal{G})(T) \rightarrow \mathrm{Lie}(\mathcal{G}^*) \rightarrow 0$$

where $\mathrm{Lie}(\mathcal{G})^\vee$ denotes the $\mathcal{O}_T$ -linear dual of the module $\mathrm{Lie}(\mathcal{G})$ and $\mathcal{G}^*$ denotes the Cartier dual of the group $\mathcal{G}$. By Lemma 4.1.1(3), this sequence corresponds to a deformation pair $(\mathbb{D}_{T_0}(\mathcal{G}_0)(T), \mathrm{Lie}(\mathcal{G})^\vee)$, and by Lemma 4.1.1(1) and (4), each T -deformation of $\mathcal{G}_0/T_0$ uniquely determines such a pair up to crystalline base change. It turns out that this completely classifies all T -deformations of $\mathcal{G}_0/T_0$.

Theorem 4.1.2 (Messing's Thesis [30]). *Let T be a W -scheme on which p is locally nilpotent and $T_0 \hookrightarrow T$ be a closed W -subscheme on which p is nilpotent. Then there is a one-to-one correspondence*

$$\{T\text{-deformations of } \mathcal{G}_0/T_0\} \leftrightarrow \{\text{pairs } (\mathbb{D}_{T_0}(\mathcal{G}_0)(T), L)\}$$

where L is an $\mathcal{O}_T$ -submodule such that the quotient $\mathbb{D}_{T_0}(\mathcal{G}_0)(T)/L$ is $\mathcal{O}_T$ -free. In particular, a given deformation $\mathcal{G}/T$ corresponds to the pair $(\mathbb{D}_T(\mathcal{G})(T), \mathrm{Lie}(\mathcal{G})^\vee)$.

Breuil's theory encodes the abstract data of the pairs $(\mathbb{D}_{T_0}^*(\mathcal{G})(T), \mathrm{Lie}(\mathcal{G})^\vee)$ to

computable algebraic structures now known as Breuil modules. We retain that k is a perfect field of characteristic $p > 0$ and $W = W(k)$ its ring of Witt vectors with fraction field denoted by K_0 . Fix K/K_0 to be a finite, totally ramified extension with characteristic polynomial given by $\text{Eis}(u) = u^e + a_{e-1}u^{e-1} + \dots + a_0$, where we note that $p|a_i$ for all $0 \leq i \leq e-1$ and $p^2 \nmid a_0$. We define S to be the p -adic completion of $W[u, \text{Eis}(u)^n/n!]_{n \geq 1}$. Then S has a natural Frobenius endomorphism ϕ extending the Frobenius on W by mapping $u \mapsto u^p$. The ideal $\text{Fil}^1(S) = \ker(S \xrightarrow{w \mapsto \pi} \mathcal{O}_K)$ is also naturally equipped with divided powers so that S is an object in the crystalline site of $\mathcal{O}_K$. We also define the ring $S_\infty = S \otimes_W K_0/W$.

Definition 4.1.3. Let $\mathbf{BT}_{/S}^\phi$ be the category of pairs $(\mathcal{M}, \text{Fil}^1(\mathcal{M}))$, where $\mathcal{M}$ is a finite free S -module and $\text{Fil}^1(\mathcal{M}) \subseteq \mathcal{M}$ is an S -submodule, satisfying:

1. $\text{Fil}^1(S) \cdot \mathcal{M} \subseteq \text{Fil}^1(\mathcal{M})$,
2. $\mathcal{M}/\text{Fil}^1(\mathcal{M})$ is a free $\mathcal{O}_K$ -module, and
3. there exists a ϕ semi-linear map

$$\phi_{\mathcal{M}} : \text{Fil}^1(\mathcal{M}) \rightarrow \mathcal{M}$$

such that its induced map $\phi^*(\text{Fil}^1(\mathcal{M})) \rightarrow \mathcal{M}$ is surjective.

An object in this category is called a **Breuil module** and the map $\phi_{\mathcal{M}}$ its **Frobenius**. The morphisms in this category are natural morphisms of modules which preserve the Frobenius $\phi_{\mathcal{M}}$ and filtrations.

We characterize the following objects in the category.

- An object $(\mathcal{M}, \text{Fil}^1 \mathcal{M}) \in \mathbf{BT}_{/S}^\phi$ is *connected* if

$$m \mapsto \phi_{\mathcal{M}}(\text{Eis}(u)m)$$

is topologically nilpotent on $\mathcal{M}/u\mathcal{M}$ for the p -adic topology.

- The *dual object* to a pair $(\mathcal{M}, \text{Fil}^1 \mathcal{M}) \in \mathbf{BT}_{/S}^\phi$ is defined as

$$(\mathcal{M}^*, \text{Fil}^1 \mathcal{M}^*) := (\text{Hom}_S(\mathcal{M}, S), \{f \in \mathcal{M}^* : f(\text{Fil}^1 \mathcal{M}) \subseteq \text{Fil}^1 S_\infty\})$$

and for all $f \in \text{Fil}^1 \mathcal{M}^*$, the Frobenius $\phi_{\mathcal{M}}^*$ is the unique morphism making the following diagram commute

$$\begin{array}{ccc} \text{Fil}^1 \mathcal{M} & \xrightarrow{\phi_{\mathcal{M}}} & \mathcal{M} \\ f \downarrow & & \downarrow \phi_{\mathcal{M}}^*(f) \\ \text{Fil}^1 S & \xrightarrow{\phi_S} & S \end{array} .$$

For a proof of uniqueness, we refer to Caruso [9].

In what follows, we drop the subscript on the functor $\mathbb{D}$ for clarity of notation.

Theorem 4.1.4 (Breuil's Classification, [5]). *Let $\mathcal{G}/\mathcal{O}_K$ be a p -divisible group. Then there is a functor*

$$\begin{aligned} \mathcal{M} : p\text{-div}/\mathcal{O}_K &\rightarrow \mathbf{BT}_{/S}^\phi \\ \mathcal{G} &\mapsto (\mathcal{M}(\mathcal{G}) = \mathbb{D}(\mathcal{G})(S), \text{Fil}^1(\mathcal{M}(\mathcal{G})) = \text{Fil}^1(\mathbb{D}(\mathcal{G}))(S)) \end{aligned}$$

which induces an equivalence of categories when $p > 2$. When $p = 2$, this is an equivalence on connected objects.

Note that $\mathcal{M}$ identifies connected objects in each category and is compatible with duality, i.e., $\mathcal{M}(\mathcal{G}^*) = \mathcal{M}(\mathcal{G})^*$. These assertions are shown in [9].

Examples 4.1.5. Although we will not work with Breuil modules directly, it is helpful for the reader to see the following examples (see also [8]).

1. Let $\mathcal{G} = \varinjlim p^{-n}\mathbb{Z}/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$. Then

$$\mathcal{M}(\mathcal{G}) = Se, \quad \text{Fil}^1(\mathcal{M}(\mathcal{G})) = Se, \quad \phi_{\mathcal{M}(\mathcal{G})} = \phi,$$

where e denotes a generator of the S -modules $\mathcal{M}(\mathcal{G})$.

2. Let $\mathcal{G} = \varinjlim \mu_{p^n} = \mu_{p^\infty}$. Then

$$\mathcal{M}(\mathcal{G}) = Se, \quad \text{Fil}^1(\mathcal{M}(\mathcal{G})) = \text{Fil}^1(S)e, \quad \phi_{\mathcal{M}(\mathcal{G})} = \frac{1}{p}\phi$$

where e denotes a generator of the S -module $\mathcal{M}(\mathcal{G})$.

Corollary 4.1.6. *Let $\mathcal{M} = \mathcal{M}(\mathcal{G})$ and $\mathcal{M}^* = \mathcal{M}(\mathcal{G})^*$. Then*

$$\omega_{\mathcal{G}} \simeq \text{Fil}^1\mathcal{M}/(\text{Fil}^1S)\text{Fil}^1\mathcal{M},$$

$$\text{Lie}(\mathcal{G}) \simeq \mathcal{M}^*/\text{Fil}^1\mathcal{M}^*.$$

Proof. By Theorem 4.1.4 we obtain the isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Lie}(\mathcal{G})^\vee & \longrightarrow & \mathbb{D}(\mathcal{G})(S) & \longrightarrow & \text{Lie}(\mathcal{G}^*) \longrightarrow 0 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ 0 & \longrightarrow & \text{Fil}^1\mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/\text{Fil}^1\mathcal{M} \longrightarrow 0 \end{array}$$

where we recall that we use the convention that $\text{Lie}(\mathcal{G})^\vee$ denotes the $\mathcal{O}_S$ -linear dual of the first filtered piece of $\mathbb{D}(\mathcal{G})$ (see the paragraph following Lemma 4.1.1). By Lemma 4.1.1(1), since S is in the crystalline site of $\mathcal{O}_K$, we also obtain the exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_{\mathcal{G}} & \longrightarrow & \mathbb{D}(\mathcal{G})(\mathcal{O}_K) & \longrightarrow & \text{Lie}(\mathcal{G}^*) \longrightarrow 0 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ 0 & \longrightarrow & \text{Fil}^1\mathcal{M}/(\text{Fil}^1S)\text{Fil}^1\mathcal{M} & \longrightarrow & \mathcal{M}/(\text{Fil}^1S)\mathcal{M} & \longrightarrow & \mathcal{M}/\text{Fil}^1\mathcal{M} \longrightarrow 0. \end{array}$$

The first vertical arrow is the isomorphism $\omega_{\mathcal{G}} \simeq \text{Fil}^1\mathcal{M}/(\text{Fil}^1S)\text{Fil}^1\mathcal{M}$, and, repeating the construction with $\mathcal{G}^*$ in place of $\mathcal{G}$, the last vertical arrow is the isomorphism $\text{Lie}(\mathcal{G}) \simeq \mathcal{M}^*/\text{Fil}^1\mathcal{M}^*$. $\square$

4.2 Kisin Modules

A modern approach to classifying p -divisible groups is via their *Kisin modules*. This improves on the classification by Messing and Breuil reviewed in the previous section in that it gives also a complete classification of all integral and torsion crystalline representations, not just those arising from p -divisible groups. However, to deduce the Hodge theory of p -divisible groups, one needs the correspondence between Kisin modules and Breuil modules (we do this in §4.4).

Let k be a perfect field of characteristic $p > 0$ and $W = W(k)$ be its ring of Witt vectors. Let $\mathfrak{S} = W[[u]]$ be a ring equipped with a natural Frobenius endomorphism ϕ extending the Frobenius on W by sending $u \mapsto u^p$. We also fix K to be a finite, totally ramified extension of $K_0 := W[1/p]$ with ring of integers $\mathcal{O}_K$ and we fix a choice of uniformizer π_K . Then let $\text{Eis}(u) \in W[u]$ denote the Eisenstein polynomial corresponding to the minimal polynomial of the element π_K . In particular, $\text{Eis}(u) = u^e + a_{e-1}u^{e-1} + \cdots + a_1u + a_0$ where e is the ramification index of K/K_0 , $p|a_i$ for all i , and $a_0 = cp$ for some $c \in W^\times$. We finally let $\mathcal{O}_{C^b} = \varprojlim \mathcal{O}_C/p$, where C is an algebraic closure of K and the limit is over maps $\alpha \mapsto \alpha^p$. We note that $\mathfrak{S} \hookrightarrow W(\mathcal{O}_{C^b})$ with compatible Frobenius structures.

Definition 4.2.1. Let $\mathbf{Mod}_{/\mathfrak{S}}^\phi$ be the category of finite $\mathfrak{S}$ -modules $\mathfrak{M}$ equipped with a ϕ semi-linear isomorphism

$$1 \otimes \phi_{\mathfrak{M}} : \phi^*(\mathfrak{M}) \left[\frac{1}{\text{Eis}(u)} \right] \xrightarrow{\sim} \mathfrak{M} \left[\frac{1}{\text{Eis}(u)} \right]$$

where $\phi^*(\mathfrak{M}) := \mathfrak{S} \otimes_{\phi} \mathfrak{M}$. An object in this category is called a **Kisin module** and its endomorphism $\phi_{\mathfrak{M}}$ its **Frobenius**. Morphisms in this category are the naturally-defined morphisms of modules which commute with the semi-linear Frobenius isomorphism.

We also introduce a category of modules which will behave as the “generic fiber” of

the Kisin modules defined above. We let $\mathcal{O}_{\mathcal{E}}$ denote the p -adic completion of $\mathfrak{S}[1/u]$, which inherits its Frobenius map ϕ from $\mathfrak{S}$, and $\mathcal{E}$ be its fraction field. Note that the residue field of $\mathcal{O}_{\mathcal{E}}$ is $k((u))$.

Definition 4.2.2. Let $\mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\phi}$ be the category of finite $\mathcal{O}_{\mathcal{E}}$ -modules M equipped with a Frobenius semi-linear isomorphism

$$1 \otimes \phi_M : \phi^* M \xrightarrow{\sim} M$$

where $\phi^* M := \mathcal{O}_{\mathcal{E}} \otimes_{\phi} M$. An object in this category is an **étale $\mathcal{O}_{\mathcal{E}}$ -module** and its endomorphism ϕ_M its **Frobenius**. Morphisms are morphisms of modules which commute with ϕ_M .

Remark 4.2.3. We will sometimes drop the subscript $\mathfrak{M}$ or M of $\phi_{\mathfrak{M}}$ or ϕ_M in the definitions when the Kisin module or étale $\mathcal{O}_{\mathcal{E}}$ -module is specified or where its retention clutters notation. The distinction of ϕ from $\phi_{\mathfrak{M}}$ or ϕ_M will be clear in those situations.

Theorem 4.2.4 ([21], Proposition 2.1.12). *The functor*

$$\begin{aligned} \mathrm{GF} : \mathbf{Mod}_{/\mathfrak{S}}^{\phi} &\rightarrow \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\phi} \\ \mathfrak{M} &\mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \end{aligned}$$

is fully faithful.

In the category $\mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\phi}$, we can define the dual object to an object $M \in \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\phi}$ as the module

$$M^{\vee} := \mathrm{Hom}_{\mathcal{E}, \phi}(M, \mathcal{O}_{\mathcal{E}})$$

with Frobenius endomorphism induced by

$$1 \otimes \phi_{M^{\vee}} : \phi^* M^{\vee} = (\phi^* M)^{\vee} \xrightarrow{((1 \otimes \phi_M)^{\vee})^{-1}} M^{\vee}.$$

Note that duality induces an involution on all of $\mathbf{Mod}_{/\mathcal{O}_E}^\phi$.

One cannot, however, specify an involution by duality on all of $\mathbf{Mod}_{/\mathfrak{S}}^\phi$, so we restrict to the subcategory $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, \text{fr}} \subseteq \mathbf{Mod}_{/\mathfrak{S}}^\phi$ of $\mathfrak{S}$ -free modules to make this definition. Then given $\mathfrak{M} \in \mathbf{Mod}_{/\mathfrak{S}}^{\phi, \text{fr}}$, its dual is the module

$$\mathfrak{M}^\vee := \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S})$$

with Frobenius

$$\begin{aligned} 1 \otimes \phi_{\mathfrak{M}^\vee} : \phi^* \mathfrak{M}^\vee \left[\frac{1}{\text{Eis}(u)} \right] &= \left(\phi^* \mathfrak{M} \left[\frac{1}{\text{Eis}(u)} \right] \right)^\vee \\ &\xrightarrow{((1 \otimes \phi_{\mathfrak{M}})^\vee)^{-1}} \left(\mathfrak{M} \left[\frac{1}{\text{Eis}(u)} \right] \right)^\vee = \mathfrak{M}^\vee \left[\frac{1}{\text{Eis}(u)} \right]. \end{aligned}$$

The reader can check that duality is preserved under GF.

Let now $\mathbf{Rep}_{G_K}^{\text{criso}}$ denote the category of crystalline G_K -stable $\mathbb{Z}_p$ -lattices spanning a crystalline G_K -representation.

Theorem 4.2.5 ([21],[26]). *There is a fully faithful tensor functor*

$$\mathfrak{M} : \mathbf{Rep}_{G_K}^{\text{criso}} \rightarrow \mathbf{Mod}_{/\mathfrak{S}}^\phi$$

which has the following properties. For any $L \in \mathbf{Rep}_{G_K}^{\text{criso}}$:

1. *(Preservation of Rank). $\mathfrak{M}(L)$ is finite free over $\mathfrak{S}$, of rank equal to the rank of L .*
2. *(Compatibility with Tensor Powers). There are canonical ϕ -equivariant isomorphisms*

$$\begin{aligned} \text{Sym}^n(\mathfrak{M}(L)) &\simeq \mathfrak{M}(\text{Sym}^n(L)), \\ \bigwedge^n(\mathfrak{M}(L)) &\simeq \mathfrak{M}(\bigwedge^n(L)). \end{aligned}$$

3. (Compatibility with Unramified Base Change). If k'/k is an algebraic extension of fields, then there exists a canonical ϕ -equivariant isomorphism

$$\mathfrak{M}(L|_{G_{K'}}) \xrightarrow{\sim} \mathfrak{M}(L) \otimes_{\mathfrak{S}} \mathfrak{S}'$$

where $\mathfrak{S}' = W(k')[[u]]$ and $G_{K'} = \text{Gal}(\overline{K} \cdot W(k')[1/p]/K \cdot W(k')[1/p])$. In particular,

$$\mathfrak{M}(L|_{I_K}) \xrightarrow{\sim} \mathfrak{M}(L) \otimes_{\mathfrak{S}} \mathfrak{S}^{\text{ur}}$$

where $\mathfrak{S}^{\text{ur}} := W(\overline{k})[[u]]$.

4. (Compatibility with Formation of Duals). Let $L^{\vee}$ denote the dual representation to L . Then there exists a canonical ϕ -equivariant isomorphism

$$\mathfrak{M}(L^{\vee}) \simeq \mathfrak{M}(L)^{\vee}.$$

The functor $\mathfrak{M}$ in the theorem easily extends to the category $\mathbf{Rep}_{G_K}^{\text{cris,tors}}$ of torsion semi-stable G_K -modules with non-negative Hodge-Tate weights. We note that objects in $\mathbf{Rep}_{G_K}^{\text{cris,tors}}$ have a 2-term resolution by objects in $\mathbf{Rep}_{G_K}^{\text{criso}}$. Therefore, the main content of the following theorem is that $\mathfrak{M}$ extends to a functor on the category $\mathbf{Rep}_{G_K}^{\text{cris,tors}}$ so that its image in $\mathbf{Mod}_{/\mathfrak{S}}^{\phi}$ is independent of this choice of resolution.

Theorem 4.2.6 ([21],[26]). *The functor $\mathfrak{M}$ in Theorem 4.2.5 extends to a fully faithful functor*

$$\mathfrak{M} : \mathbf{Rep}_{G_K}^{\text{cris,tors}} \rightarrow \mathbf{Mod}_{/\mathfrak{S}}^{\phi}$$

which is compatible with the formation of symmetric and tensor powers, unramified base change, and with the formation of duals.

The theory of duality on torsion Kisin in this theorem is due to Caruso [9] and Liu [28]. Objects in $\mathbf{Rep}_{G_K}^{\text{cris,tors}}$ have image in the subcategory $\mathbf{Mod}_{/\mathfrak{S}}^{\phi,\text{tors}} \subseteq \mathbf{Mod}_{/\mathfrak{S}}^{\phi}$,

consisting of p -power torsion objects which have a 2-term resolution by objects in $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, \text{fr}}$. Then given $L_1, L_2 \in \mathbf{Rep}_{G_K}^{\text{criso}}$ and $L \in \mathbf{Rep}_{G_K}^{\text{cris, tors}}$ related by the short exact sequence

$$0 \rightarrow L \rightarrow L_1 \rightarrow L_2 \rightarrow 0,$$

duality is the unique involution making the following diagram commute

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathfrak{M}(L_1^\vee) & \rightarrow & \mathfrak{M}(L_2^\vee) & \rightarrow & \mathfrak{M}(L^\vee) \rightarrow 0 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ 0 & \rightarrow & \mathfrak{M}(L_1)^\vee & \rightarrow & \mathfrak{M}(L_2)^\vee & \rightarrow & \mathfrak{M}(L)^\vee \rightarrow 0. \end{array}$$

Liu provides an intrinsic definition of duality on the category $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, \text{tors}}$ in [28], which is shown to be compatible with the duality induced on $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, \text{tors}}$ by the above manner. Liu also shows that GF restricted to this subcategory has image in the category $\mathbf{Mod}_{\mathcal{O}_{\mathfrak{E}}}^{\phi, \text{tors}}$ consisting of p -power torsion étale $\mathcal{O}_{\mathfrak{E}}$ -modules. Duality on $\mathbf{Mod}_{\mathfrak{S}}^{\phi, \text{tors}}$ commutes with duality on $\mathbf{Mod}_{\mathcal{O}_{\mathfrak{E}}}^{\phi, \text{tors}}$.

Up to this point, the definition of a Kisin module relied on a choice of uniformizer and, therefore, using compatible choices of uniformizers to define duality and base change. It is a technical theorem of Liu [27] that defining a Kisin module is independent of this choice, and therefore compatible choices do not have to be made. We state the theorem below for reference.

Theorem 4.2.7 (Theorem 1.0.1, [27]). *Let $\mathfrak{M}$ and $\mathfrak{M}'$ both be Kisin modules associated to the same object $L \in \mathbf{Rep}_{G_K}^{\text{cris, o}}$ but constructed on different choices of uniformizers π_K and π'_K of K . Then there is a canonical isomorphism*

$$W(\mathcal{O}_{C^\flat}) \otimes_{\phi, \pi_K} \mathfrak{M} \simeq W(\mathcal{O}_{C^\flat}) \otimes_{\phi, \pi'_K} \mathfrak{M}'$$

where the tensor product on each side is with respect to the embedding $\mathfrak{S} \hookrightarrow W(\mathcal{O}_{C^\flat})$ given by $u \mapsto [\pi]$ along the uniformizers $\pi = \pi_K$ and $\pi = \pi'_K$. In particular, different choices of uniformizers define isomorphic Kisin modules.

4.3 Classification of Barsotti-Tate Groups by Kisin Modules

We define the subcategories of Kisin modules which were considered in [21] to classify Barsotti-Tate groups. We retain the notation from the previous two sections.

Definition 4.3.1. Let $\mathbf{BT}_{/\mathfrak{S}}^{\phi,r}$ be the full subcategory of $\mathbf{Mod}_{/\mathfrak{S}}^{\phi}$ consisting of $\mathfrak{S}$ -free modules $\mathfrak{M}$ such that the cokernel of $1 \otimes \phi_{\mathfrak{M}} : \phi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $\mathrm{Eis}^r(u)$. Such Kisin modules are said to have **height** at most r .

A Kisin module of height r comes naturally equipped with another map $\psi_{\mathfrak{M}} : \phi^* \mathfrak{M} \rightarrow \mathfrak{M}$, called the **Verschiebung**, defined so that it satisfies

$$(1 \otimes \phi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = \mathrm{Eis}^r(u).$$

The reader can check this gives $\psi_{\mathfrak{M}}$ uniquely.

Let $p\text{-div}/\mathcal{O}_K$ denote the category of p -divisible groups over $\mathcal{O}_K$. Given an object $\mathcal{G}/\mathcal{O}_K$ in this category, we let $\mathcal{G}^*/\mathcal{O}_K$ denote its Cartier dual and $T_p \mathcal{G} := \varprojlim_n \mathcal{G}[p^n]$ denote its Tate module.

Theorem 4.3.2 ([21], [26]). *The functor $\mathfrak{M}$ of Theorem 4.2.5 induces a fully faithful contravariant equivalence*

$$\begin{aligned} \mathfrak{M} : (p\text{-div}/\mathcal{O}_K) &\rightarrow \mathbf{BT}_{/\mathfrak{S}}^{\phi,1} \\ \mathcal{G} &\mapsto \mathfrak{M}(\mathcal{G}) := \mathfrak{M}(T_p \mathcal{G}^*). \end{aligned}$$

The functor $\mathfrak{M}$ moreover admits an inverse functor $\mathcal{G}$, which gives the isomorphism of $G_{K_{\infty}}$ -modules

$$\epsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})|_{G_{K_{\infty}}} \rightarrow T_{\mathfrak{S}}^*(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, W(\mathcal{O}_{C^b}))$$

where we recall that $\mathcal{O}_{C^\flat} = \varprojlim_n \mathcal{O}_C/p^n$ and $W(\mathcal{O}_{C^\flat})$ is equipped with the Frobenius morphism coming from the Witt construction.

Examples 4.3.3. The following examples can be computed by elementary means.

1. Let $\mathcal{G} = \varinjlim p^{-n}\mathbb{Z}/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$. Then

$$\mathfrak{M}(\mathcal{G}) \simeq \mathfrak{S}e, \quad \phi_{\mathfrak{M}(\mathcal{G})}(e) = \frac{1}{c}\text{Eis}(u)e, \quad \psi_{\mathfrak{M}(\mathcal{G})}(e) = c \otimes e$$

where e denotes a generator of the $\mathfrak{S}$ -module $\mathfrak{M}$. We denote this Kisin module by $\mathfrak{S}(1)$.

2. Let $\mathcal{G} = \varinjlim \mu_{p^n} = \mu_{p^\infty}$. Then

$$\mathfrak{M}(\mathcal{G}) \simeq \mathfrak{S}e, \quad \phi_{\mathfrak{M}(\mathcal{G})}(e) = \frac{1}{c}e, \quad \psi_{\mathfrak{M}(\mathcal{G})}(e) = c \otimes \text{Eis}(u)e$$

where e denotes a generator of the $\mathfrak{S}$ -module $\mathfrak{M}$. We denote this Kisin module by $\mathfrak{S}$.

Corollary 4.3.4. *Let $\mathcal{G}$ be an object in $p\text{-div}/\mathcal{O}_K$ and $\mathfrak{M}(\mathcal{G})^*$ denote the module $\text{Hom}_{\mathfrak{S}}(\mathfrak{M}(\mathcal{G}), \mathfrak{S}(1))$.*

1. *The rank of $\mathfrak{M}(\mathcal{G})$ encodes the height of the p -divisible group $\mathcal{G}$, i.e., $\text{rk}_{\mathfrak{S}}\mathfrak{M}(\mathcal{G}) = \text{ht}(\mathcal{G})$. Moreover, $\phi_{\mathfrak{M}(\mathcal{G})}$ and $\psi_{\mathfrak{M}(\mathcal{G})}$ encode the relative Frobenius and relative Verschiebung endomorphisms on $\mathcal{G}$.*
2. *$\mathfrak{M}$ commutes with $*$ -duality, i.e., $\mathfrak{M}(\mathcal{G})^* \simeq \mathfrak{M}(\mathcal{G}^*)$. In particular, the Frobenius and Verschiebung endomorphisms on $\mathfrak{M}(\mathcal{G})^*$ are given by*

$$\phi_{\mathfrak{M}(\mathcal{G})^*}(T) := \frac{1}{c}(1 \otimes T) \circ \psi_{\mathfrak{M}(\mathcal{G})},$$

$$\psi_{\mathfrak{M}(\mathcal{G})^*}(T) := cT \circ (1 \otimes \phi_{\mathfrak{M}(\mathcal{G})}).$$

Proof. 1. Since $\mathfrak{M}$ preserves rank, and the rank of the Tate module of a p -divisible group equals the height of the group, the first part of the statement follows. The correspondence between the Frobenius and Verschiebung morphisms is implicit in Theorem 4.3.2.

2. By definition, there is a perfect pairing on the $\mathfrak{S}$ -modules

$$\langle, \rangle_{\mathfrak{M}} : \mathfrak{M}(\mathcal{G}) \times \mathfrak{M}(\mathcal{G})^* \rightarrow \mathfrak{S}(1).$$

The inverse functor $\mathcal{G}$ therefore induces a perfect pairing on the G_{K_∞} -modules

$$\langle, \rangle_{T_{\mathfrak{S}}^*(\mathfrak{M})} : T_{\mathfrak{S}}^*(\mathfrak{M}) \times T_{\mathfrak{S}}^*(\mathfrak{M}^*) \rightarrow \mathfrak{S}(1).$$

Using the isomorphisms $\epsilon_{\mathfrak{M}}$ and $\epsilon_{\mathfrak{M}^*}$ from Theorem 4.3.2, we obtain the exact diagram

$$\begin{array}{ccc} \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}^*) & \longrightarrow & \mathcal{G}(\mathfrak{S}(1)) \\ \epsilon_{\mathfrak{M}} \downarrow \simeq & & \downarrow \simeq \\ T_{\mathfrak{S}}^*(\mathfrak{M}) \times T_{\mathfrak{S}}^*(\mathfrak{M}^*) & \longrightarrow & T_{\mathfrak{S}}^*(\mathfrak{S}(1)) \end{array}$$

By this diagram, to show that $*$ -duality commutes with $\mathfrak{M}$ is equivalent to showing that $*$ -duality commutes with $\mathcal{G}$. This follows from the fact that the pairing in the top row of the diagram is identified with pairing of the p -divisible group $\mathcal{G}$ with its Cartier dual $\mathcal{G}^*$.

The Frobenius and Verschiebung morphisms are computed from these diagrams. □

Remark 4.3.5. The module $\mathfrak{M}^*$ from the above corollary relates to the Kisin module dual $\mathfrak{M}^\vee$ by a twist: $\mathfrak{M}^* \simeq \mathfrak{M}^\vee \otimes_{\mathfrak{S}} \mathfrak{S}(1)$. We prefer to introduce this notation to make it clear this module is identified with the Cartier dual of the underlying p -divisible group.

This classification extends naturally to finite flat group schemes. It is well-known (see, e.g., [4]) that any finite flat group scheme $\mathcal{G}$ defined over $\mathcal{O}_K$ has a 2-term resolution by p -divisible groups $\mathcal{G}_1$ and $\mathcal{G}_2$ each also defined over $\mathcal{O}_K$. Thus $\mathfrak{M}(\cdot)$ extends to the category of finite flat group schemes in such a way that on the exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow 0$$

it induces the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{M}(\mathcal{G}_2) & \longrightarrow & \mathfrak{M}(\mathcal{G}_1) & \longrightarrow & \mathfrak{M}(\mathcal{G}) \longrightarrow 0 \\ & & \phi_{\mathfrak{M}(\mathcal{G}_2)} \downarrow & & \phi_{\mathfrak{M}(\mathcal{G}_1)} \downarrow & & \phi_{\mathfrak{M}(\mathcal{G})} \downarrow \\ 0 & \longrightarrow & \mathfrak{M}(\mathcal{G}_2) & \longrightarrow & \mathfrak{M}(\mathcal{G}_1) & \longrightarrow & \mathfrak{M}(\mathcal{G}) \longrightarrow 0 \end{array}$$

and the construction of $\mathfrak{M}(\mathcal{G})$ is independent of the choice of p -divisible groups providing a resolution of $\mathcal{G}$. We have already seen that $\mathfrak{M}$ extends in this way in Theorem 4.2.6. Here we identify the precise subcategory of $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, \text{tors}}$ which classifies finite flat group schemes.

Definition 4.3.6. Let $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, r}$, be the full subcategory of $\mathbf{Mod}_{/\mathfrak{S}}^{\phi}$ consisting of $\mathfrak{S}$ -modules of projective dimension 1 which are p -power torsion and such that the cokernel of $1 \otimes \phi_{\mathfrak{M}} : \phi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $\text{Eis}^r(u)$. This category is often called the category of **finite Kisin modules of height at most r** .

Let $p\text{-Gr}/\mathcal{O}_K$ denote the category of finite flat group schemes over $\mathcal{O}_K$ of p -power order.

Theorem 4.3.7 ([21],[26]). *Let $0 \rightarrow \mathcal{G} \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow 0$ be any resolution of the finite flat group scheme $\mathcal{G}/\mathcal{O}_K$ by p -divisible groups $\mathcal{G}_1/\mathcal{O}_K$ and $\mathcal{G}_2/\mathcal{O}_K$. The functor $\mathfrak{M}$ of Theorem 4.2.5 extends to induce a fully faithful functor*

$$\begin{aligned} \mathfrak{M} : (p\text{-Gr}/\mathcal{O}_K) &\rightarrow \mathbf{Mod}_{/\mathfrak{S}}^{\phi, 1} \\ \mathcal{G} &\mapsto \mathfrak{M}(\mathcal{G}) := \text{coker}(\mathfrak{M}(\mathcal{G}_1) \rightarrow \mathfrak{M}(\mathcal{G}_2)) \end{aligned}$$

which is an equivalence when $p > 2$. When $p = 2$, $\mathfrak{M}$ gives an equivalence on the full subcategories of connected objects.

To study finite flat group schemes of p^n -torsion for a fixed positive integer n we introduce further subcategories of $\mathbf{Mod}_{/\mathfrak{S}}^\phi$ and modify $\mathfrak{M}$ in the following way. Recall that $\mathfrak{S} = W[[u]]$. Then $\mathfrak{S}$ projects onto the rings $\mathfrak{S}_n = W_n[[u]]$ for each integer $n > 0$, where W_n is the n th truncation of the Witt ring W . We note that $\mathfrak{S}_n$ has a Frobenius extending the Frobenius on W_n by $u \mapsto u^p$.

Definition 4.3.8. Let $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ be the subcategory of $\mathbf{Mod}_{/\mathfrak{S}}^{\phi,r}$ consisting of p^n -torsion objects for a fixed positive integer n . These modules are defined over the ring $\mathfrak{S}_n$ and are called **finite $\mathfrak{S}_n$ -modules of height r** .

For each n we construct the functor

$$\begin{aligned} - \otimes \mathfrak{S}_n : \mathbf{Mod}_{/\mathfrak{S}}^{\phi,1} &\rightarrow \mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,1} \\ \mathfrak{M} &\mapsto \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n =: \mathfrak{M}_{\mathfrak{S}_n}. \end{aligned}$$

It is clear this commutes with ϕ .

Proposition 4.3.9. *The functor $- \otimes \mathfrak{S}_n$ is full, and faithful on p^n -torsion objects in $\mathbf{Mod}_{/\mathfrak{S}}^{\phi,1}$. In particular, there is an equivalence between finite flat p^n -torsion group schemes and the category $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,1}$.*

Proof. Each part of this statement is easily checked by the compatibility of $- \otimes \mathfrak{S}_n$ with the Frobenius morphisms. The equivalence then follows from Theorem 4.3.2. $\square$

Remark 4.3.10. Every object in $\mathbf{Mod}_{/\mathfrak{S}_1}^{\phi,r}$ is $\mathfrak{S}_1$ -free. This, however, is not true of objects in $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ when $n > 1$.

Examples 4.3.11. Let $\mathrm{Eis}_n(u) \in W_n[u]$ be the projection of $\mathrm{Eis}(u) \in W[u]$.

1. Let $\mathcal{G}_n = p^{-n}\mathbb{Z}/\mathbb{Z}$. Then

$$\mathfrak{M}(\mathcal{G}_n) \simeq \mathfrak{S}_n e, \quad \phi_{\mathfrak{M}(\mathcal{G}_n)}(e) = \frac{1}{c} \text{Eis}_n(u)e, \quad \psi_{\mathfrak{M}(\mathcal{G}_n)}(e) = c \otimes e$$

where e denotes a generator of the $\mathfrak{S}_n$ -module $\mathfrak{M}(\mathcal{G}_n)$. We denote this Kisin module by $\mathfrak{S}_n(1)$.

2. Let $\mathcal{G}_n = \mu_{p^n}$. Then

$$\mathfrak{M}(\mathcal{G}_n) \simeq \mathfrak{S}_n e, \quad \phi_{\mathfrak{M}(\mathcal{G}_n)}(e) = \frac{1}{c} e, \quad \psi_{\mathfrak{M}(\mathcal{G}_n)}(e) = c \otimes \text{Eis}_n(u)e$$

where e denotes a generator of the $\mathfrak{S}_n$ -module $\mathfrak{M}(\mathcal{G}_n)$. We denote this Kisin module by $\mathfrak{S}_n$.

We note that in these examples $\mathcal{G} = \varinjlim_n \mathcal{G}_n$ is a p -divisible group, and $\mathfrak{M}(\mathcal{G}) = \varprojlim_n \mathfrak{M}(\mathcal{G}_n)$ since the $\mathfrak{M}(\mathcal{G}_n)$ have compatible Frobenius structures.

Remark 4.3.12. In light of Proposition 4.3.9, we will write $\mathfrak{M}_{\mathfrak{S}_n}(\mathcal{G}) := \mathfrak{M}(\mathcal{G}) \otimes_{\mathfrak{S}} \mathfrak{S}_n$. Then $\mathfrak{M}_{\mathfrak{S}_n}$ is a functor on any Barsotti-Tate group and its image is the Kisin module corresponding to its maximal p^n -torsion subgroup scheme.

4.4 Deformation Theory with Kisin Modules

We interpret the deformation theory of p -divisible groups in the language of Kisin modules. Recall that S is the p -adic completion of $W[u, \text{Eis}(u)^n/n!]_{n \geq 1}$ with a Frobenius extending the Frobenius on W by $u \mapsto u^p$ and a filtration given by $\text{Fil}^1(S) = \ker(S \xrightarrow{u \mapsto \pi} \mathcal{O}_K)$. We also recall that $\mathfrak{S} = W[[u]]$ with a Frobenius extending the Frobenius on W by $u \mapsto u^p$. The categories $\mathbf{BT}_{/S}^\phi$ and $\mathbf{BT}_{/\mathfrak{S}}^{\phi,1}$ denoted Breuil modules and Kisin modules of height 1, respectively, and were defined in §4.1 and §4.3.

The ring S has a natural structure of an $\mathfrak{S}$ -algebra under $u \mapsto u$, which induces

the functor

$$\begin{aligned}\mathcal{M} : \mathbf{BT}_{/\mathfrak{S}}^{\phi,1} &\rightarrow \mathbf{BT}_{/S}^{\phi} \\ \mathfrak{M} &\mapsto \mathcal{M}(\mathfrak{M}) := S \otimes_{\mathfrak{S}} \phi^* \mathfrak{M}\end{aligned}$$

where the S -module filtration $\mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}) \subseteq \mathcal{M}(\mathfrak{M})$ is defined by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}) & \hookrightarrow & \mathcal{M}(\mathfrak{M}) = S \otimes_{\mathfrak{S}} \phi^* \mathfrak{M} \\ \downarrow & & \downarrow 1 \otimes \phi_{\mathfrak{M}} \\ \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} & \hookrightarrow & S \otimes_{\mathfrak{S}} \mathfrak{M} \end{array} \quad .$$

This filtered submodule is also written as

$$\mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}) = S \cdot \mathrm{Fil}^1 \phi^*(\mathfrak{M}) + \mathrm{Fil}^1 S \cdot \mathcal{M}(\mathfrak{M})$$

where $\mathrm{Fil}^1 \phi^*(\mathfrak{M})$ is defined by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Fil}^1 \phi^*(\mathfrak{M}) & \hookrightarrow & \phi^* \mathfrak{M} = \mathfrak{S} \otimes_{\phi, \mathfrak{S}} \mathfrak{M} \\ \downarrow & & \downarrow 1 \otimes \phi_{\mathfrak{M}} \\ \mathrm{Eis}(u) \cdot \mathfrak{M} & \hookrightarrow & \mathfrak{M} \end{array} \quad .$$

Theorem 4.4.1 ([21]). *Let $\mathcal{G}/\mathcal{O}_K$ be a p -divisible group. Then there exists an isomorphism*

$$\mathbb{D}(\mathcal{G})(S) \xrightarrow{\sim} \mathcal{M}(\mathfrak{M}(\mathcal{G}))$$

which is compatible with ϕ .

Corollary 4.4.2. *There is a canonical ϕ -equivariant isomorphism*

$$\phi^*(\mathfrak{M}(\mathcal{G})/u\mathfrak{M}(\mathcal{G})) \xrightarrow{\sim} \mathbb{D}(\mathcal{G}_0)(W)$$

where $\mathcal{G}_0 = \mathcal{G} \otimes_{\mathcal{O}_K} k$. In particular, the linear Frobenius and Verschiebung endomorphisms on the Dieudonné module are given by $1 \otimes \phi_{\mathfrak{M}(\mathcal{G})} \bmod u$ and $1 \otimes \psi_{\mathfrak{M}(\mathcal{G})} \bmod u$.

Proof. The ring W has the structure of an S -algebra by mapping $u \mapsto 0$. We compute

$$\begin{aligned} \mathbb{D}(\mathcal{G}_0)(W) &\xrightarrow{\sim} \mathbb{D}(\mathcal{G})(S) \otimes_S W \xrightarrow{\sim} \mathcal{M}(\mathfrak{M}(\mathcal{G})) \otimes_S W \\ &= \phi^*(\mathfrak{M}(\mathcal{G})) \otimes_{\mathfrak{S}} W \\ &= \phi^*(\mathfrak{M}(\mathcal{G}))/u\phi^*(\mathfrak{M}(\mathcal{G})) \end{aligned}$$

where the first isomorphism follows from the crystalline base change property of the Frobenius crystal in Lemma 4.1.1 (1) and the second isomorphism follows from Theorem 4.4.1. Since each step in the computation commutes with the Frobenius operator, we conclude that the Frobenius and Verschiebung operators on the Dieudonné module are reductions of (the linearization of) the original modulo u . $\square$

Lemma 4.4.3. *Let $\mathcal{G}/\mathcal{O}_K$ be a p -divisible group. There are natural isomorphisms*

$$\begin{aligned} \omega_{\mathcal{G}} &\simeq \mathfrak{M}(\mathcal{G})/\phi_{\mathfrak{M}(\mathcal{G})}(\mathfrak{M}(\mathcal{G})), \\ \mathrm{Lie}(\mathcal{G}) &\simeq \phi^*\mathfrak{M}(\mathcal{G})^*/\mathrm{Fil}^1\phi^*\mathfrak{M}(\mathcal{G})^*. \end{aligned}$$

Proof. Let $\mathfrak{M} = \mathfrak{M}(\mathcal{G})$ and $\mathcal{M} = \mathcal{M}(\mathfrak{M})$. By Theorem 4.4.1 and Corollary 4.1.6, we have the exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_{\mathcal{G}} & \longrightarrow & \mathbb{D}(\mathcal{G})(\mathcal{O}_K) & \longrightarrow & \mathrm{Lie}(\mathcal{G}^*) \longrightarrow 0 \\ & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ 0 & \longrightarrow & \mathrm{Fil}^1\mathcal{M}/(\mathrm{Fil}^1S)\mathrm{Fil}^1\mathcal{M} & \longrightarrow & \mathcal{M}/(\mathrm{Fil}^1S)\mathcal{M} & \longrightarrow & \mathcal{M}/\mathrm{Fil}^1\mathcal{M} \longrightarrow 0. \end{array}$$

The first vertical arrow gives the isomorphisms

$$\begin{aligned}
\omega_{\mathcal{G}} &\simeq \mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}(\mathcal{G})) / (\mathrm{Fil}^1 S) \mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}(\mathcal{G})) \\
&= [S \cdot \mathrm{Fil}^1 \phi^*(\mathfrak{M}) + \mathrm{Fil}^1 S \cdot \mathcal{M}(\mathfrak{M})] / (\mathrm{Fil}^1 S) [S \cdot \mathrm{Fil}^1 \phi^*(\mathfrak{M}) + \mathrm{Fil}^1 S \cdot \mathcal{M}(\mathfrak{M})] \\
&\simeq \mathfrak{M}(\mathcal{G}) / \phi_{\mathfrak{M}(\mathcal{G})}(\mathfrak{M}(\mathcal{G})).
\end{aligned}$$

Replacing the objects in the diagram with their dual objects, the last vertical arrow gives the isomorphisms

$$\begin{aligned}
\mathrm{Lie}(\mathcal{G}) &\simeq \mathcal{M}(\mathfrak{M}(\mathcal{G}^*)) / \mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}(\mathcal{G}^*)) \\
&= [S \otimes \phi^* \mathfrak{M}(\mathcal{G}^*)] / [S \cdot \mathrm{Fil}^1 \phi^*(\mathfrak{M}(\mathcal{G}^*)) + \mathrm{Fil}^1 S \cdot (S \otimes \phi^* \mathfrak{M}(\mathcal{G}^*))] \\
&= \phi^* \mathfrak{M}(\mathcal{G}^*) / \mathrm{Fil}^1 \phi^* \mathfrak{M}(\mathcal{G}^*).
\end{aligned}$$

The isomorphism $\mathfrak{M}(\mathcal{G}^*) \simeq \mathfrak{M}(\mathcal{G})^*$ from Corollary 4.3.4 now gives the statement. $\square$

Recall now that $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,1}$ is equivalent to the category of p^n -torsion finite flat group schemes. For each n we can also define the category $\mathbf{Mod}_{/S_n}^{\phi}$ in the natural way, where S_n is the level- n quotient of S constructed in the obvious way on the n th truncation of W . Then we can construct a functor

$$\begin{aligned}
- \otimes S_n : \mathbf{Mod}_{/S}^{\phi} &\rightarrow \mathbf{Mod}_{/S_n}^{\phi} \\
\mathcal{M} &\mapsto \mathcal{M} \otimes_S S_n.
\end{aligned}$$

It is clear this commutes with ϕ and with filtrations.

Proposition 4.4.4. *The functor $\mathcal{M}$ induces a map*

$$\mathcal{M} : \mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,1} \rightarrow \mathbf{Mod}_{/S_n}^{\phi}.$$

When $n = 1$, this is an equivalence when $p > 2$ and an equivalence on connected objects when $p = 2$.

Proof. The functor $\mathcal{M}$ commutes with $-\otimes \mathfrak{S}_n$ and $-\otimes S_n$. We deduce the equivalence from Theorem 4.4.1. $\square$

Proposition 4.4.5. *Suppose $k = \mathbb{F}_q$, $K_0 = W(k)[1/p]$, and K/K_0 is a finite, totally ramified extension. For any p -torsion finite flat group scheme $\mathcal{G}/\mathcal{O}_K$,*

$$\#\omega_{\mathcal{G}} = q^{v_u(\det \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})})}.$$

Proof. By Proposition 4.4.4, the isomorphisms of Lemma 4.4.3 extend to the category $\mathbf{Mod}_{/\mathfrak{S}_1}^{\phi,1}$, so

$$\omega_{\mathcal{G}} \simeq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}) / \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}).$$

Now choose a basis $e_1, \dots, e_h$ of $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ and let $u^{s_1}, \dots, u^{s_h}$ denote the elementary divisors of the endomorphism $\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$. Then the degree of the characteristic ideal of $\omega_{\mathcal{G}}$ is $\sum_{i=1}^h s_i = v_u(\det \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})})$. The statement now follows as a comparison of modules over $\mathbb{F}_q$. $\square$

Remark 4.4.6. Recall that the category of Kisin modules is a tensor category and that $\mathfrak{M}$ commutes with symmetric and exterior powers. Therefore, we may equivalently state Proposition 4.4.5 in the following way: let $\mathfrak{L} := \bigwedge^h \mathfrak{M}(\mathcal{G})$ where h denotes the height of the p -torsion finite flat group scheme $\mathcal{G}/\mathcal{O}_K$, then

$$\#\omega_{\mathcal{G}} = q^{v_u(\phi_{\mathfrak{L}})}.$$

“What are letters?”

“Kinda like mediaglyphics except they’re all black, and they’re tiny, they don’t move, they’re old and boring and really hard to read. But you can use ‘em to make short words for long words.”

Neal Stephenson, **The Diamond Age**

5

Finite Group Theory with Kisin Modules

5.1 Classifying Finite Flat Subgroup Schemes

We restrict our attention in this section to studying the category $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ introduced in §4.3. We recall this is the category of p^n -torsion Kisin modules for a fixed positive integer n . When $r = 1$ the category is equivalent to the category of p^n -torsion finite flat group schemes by Proposition 4.3.9.

Definition 5.1.1. Let $\mathfrak{M} \in \mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$. A submodule $\mathfrak{N} \subseteq \mathfrak{M}$ is **saturated in $\mathfrak{M}$** (or just **saturated**) if $\mathfrak{N} = (\mathfrak{N} \otimes_{\mathfrak{S}} \mathfrak{S}[1/u]) \cap \mathfrak{M}$.

Lemma 5.1.2 (Proposition 2.3.2, [28]). *Let $\mathfrak{M} \in \mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ and $\mathfrak{N} \subseteq \mathfrak{M}$ be a ϕ -stable submodule. Then the following are equivalent:*

1. $\mathfrak{N}$ is saturated in $\mathfrak{M}$,
2. $\mathfrak{N}$ and $\mathfrak{M}/\mathfrak{N}$ are both objects in $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$, i.e., both have projective dimension 1,
3. $\mathrm{GF}(\mathfrak{N}) = \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is a subobject of $\mathrm{GF}(\mathfrak{M}) = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$.

The equivalence between (1), (2), and (3) relies on the following lemma.

Lemma 5.1.3 (Proposition 2.3.2, [28]). *Any object $\mathfrak{M} \in \mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ is u -torsion free.*

Proof. Each object $\mathfrak{M} \in \mathbf{Mod}_{/\mathfrak{S}}^{\phi,r}$ has projective dimension 1, hence has a resolution

$$0 \rightarrow \mathfrak{M}_2 \xrightarrow{\phi} \mathfrak{M}_1 \rightarrow \mathfrak{M} \rightarrow 0 \quad (5.1.4)$$

by finite, free $\mathfrak{S}$ -modules $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Let $\{e_1, \dots, e_d\}$, resp. $\{f_1, \dots, f_d\}$, be a choice of $\mathfrak{S}$ -basis for $\mathfrak{M}_1$, resp. $\mathfrak{M}_2$, such that $(e_1, \dots, e_d) = (f_1, \dots, f_d)A$, where A is the transition matrix corresponding to ϕ in (5.1.4). Since $\mathfrak{M}$ is killed by p^n for some positive integer n , there exists a matrix B with $AB = p^n I$.

Assume for the sake of contradiction that there exists a non-zero element $\bar{a} \in \mathfrak{M}$ which is killed by u^k for some $k \geq 1$. Let $a = \sum_{i=1}^d a_i e_i$ be a lift of $\bar{a}$ to $\mathfrak{M}_1$. Then we have

$$u^k(a_1, \dots, a_d) = (b_1, \dots, b_d)A$$

for some $b_i \in \mathfrak{S}$. Since $AB = p^n I$, we further have

$$(b_1, \dots, b_d) = u^k(p^{-n})(a_1, \dots, a_d)B.$$

Let $(c_1, \dots, c_d) = p^{-n}(a_1, \dots, a_d)B$. Then $c_i \in \mathfrak{S}$ and $(a_1, \dots, a_d) = (c_1, \dots, c_d)A$. But by the exactness of (5.1.4), this implies $\bar{a} = 0$, a contradiction.

To conclude the statement for the category $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$, we apply the functor $- \otimes \mathfrak{S}_n$ from Proposition 4.3.9. Then an object in $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ is u -torsion free if its lift in $\mathbf{Mod}_{/\mathfrak{S}}^{\phi,r}$ is u -torsion free. $\square$

Proof of Lemma 5.1.2. By Lemma 5.1.3, objects of $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,r}$ are u -torsion free, and hence GF induces the bijection

$$\{\text{saturated submodules of } \mathfrak{M}\} \xrightarrow{\sim} \{\text{subobjects of } \text{GF}(\mathfrak{M})\}$$

$$\mathfrak{N} \mapsto \mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$$

$$\mathcal{N} \cap \mathfrak{M} \otimes 1 \leftarrow \mathcal{N}.$$

This shows the equivalence between (1) and (3).

For the equivalence between (1) and (2), by the proof of Lemma 5.1.3, an object $\mathfrak{M}$ in $\mathbf{Mod}_{/\mathfrak{S}}^{\phi,r}$ has projective dimension 1 if and only if u is regular for $\mathfrak{M}$. Therefore, any submodule $\mathfrak{N} \subseteq \mathfrak{M}$ necessarily has projective dimension 1. To establish the statement, we must relate the condition that the quotient $\mathfrak{M}/\mathfrak{N}$ has projective dimension 1 to the condition that $\mathfrak{N}$ is saturated in $\mathfrak{M}$. If $\mathfrak{N}$ is saturated in $\mathfrak{M}$, it is easy to see that u is regular on $\mathfrak{M}/\mathfrak{N}$, and conversely, and so we apply Lemma 5.1.3 again. $\square$

Saturated submodules have a geometric interpretation on objects in $\mathbf{Mod}_{/\mathfrak{S}_n}^{\phi,1}$. In particular, they describe finite flat subgroup schemes under the equivalence in Theorem 4.3.7. We prove this in the following proposition.

Proposition 5.1.5. *Let $\mathcal{G}/\mathcal{O}_K$ be a finite flat group scheme of height g . There is a one-to-one correspondence between saturated submodules $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ and finite, flat subgroup schemes $\mathcal{H} \subseteq \mathcal{G}$; in particular,*

$$\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}) = \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})/\mathfrak{N}.$$

Proof. Let $\mathcal{H} \subseteq \mathcal{G}$ be a finite, flat subgroup scheme. Then we necessarily have the

short exact sequence of finite flat subgroup schemes

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H} \rightarrow 0.$$

Applying the contravariant functor $\mathfrak{M}_{\mathfrak{S}_1}$, we obtain

$$0 \rightarrow \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}/\mathcal{H}) \rightarrow \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}) \rightarrow \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}) \rightarrow 0.$$

Since $\mathfrak{M}_{\mathfrak{S}_1}$ is an exact functor and quotients exist in the category of Kisin modules, it follows that

$$\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}) \simeq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})/\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}/\mathcal{H}).$$

By Lemma 5.1.2, $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}/\mathcal{H})$ must be saturated in $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ for $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H})$ to have projective dimension 1. The converse follows from the equivalence in Theorem 4.3.7.

□

5.2 Harder-Narasimhan Theory of Finite Kisin Modules

We now introduce the Harder-Narasimhan (HN) theory for the category $\mathbf{Mod}_{/\mathfrak{S}}^{\phi,r}$. HN theory was originally developed for finite flat group schemes by Fargues [16] and extended to Kisin modules more broadly by Levin and Wang-Erickson [25]. The study of the Harder-Narasimhan filtration of finite flat group schemes, and more broadly finite Kisin modules, is intimately linked with understanding the variation of (semi-stable) Faltings height (see Remark 5.2.3).

We start with a warm-up lemma, which will be relevant to the computation of the HN slopes.

Lemma 5.2.1. *Suppose $k = \mathbb{F}_q$, $K_0 = W(k)[1/p]$, and K/K_0 is a finite, totally*

ramified extension. Let $\mathcal{G}/\mathcal{O}_K$ be a finite flat p -torsion group scheme of height g and $\mathcal{H} \subseteq \mathcal{G}$ be a flat subgroup scheme of height d . Then there is a line $\mathfrak{L} \subseteq \bigwedge^{g-d} \mathfrak{M}(\mathcal{G})$ such that

$$\#\omega_{\mathcal{H}} = q^{v_u(\det(\phi_{\mathfrak{M}(\mathcal{G})}) - v_u(\phi_{\mathfrak{L}})}.$$

Proof. As in Proposition 5.1.5, we have a short exact sequence of $\mathfrak{S}$ -modules

$$0 \rightarrow \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}/\mathcal{H}) \rightarrow \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}) \rightarrow \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}) \rightarrow 0$$

where the terms in the sequence have $\mathfrak{S}$ -rank $g-d$, g , and d , respectively, corresponding to the height of the groups. Since $\mathbf{Mod}_{/\mathfrak{S}}^{\phi}$ is a tensor category, determinants are additive on the short exact sequence. Moreover, the $\mathfrak{S}$ -linear determinant of $\mathfrak{M}(\mathcal{G}/\mathcal{H})$ will be a line $\mathfrak{L} \subseteq \bigwedge^{g-d} \mathfrak{M}(\mathcal{G})$. The statement now follows from Proposition 4.4.5. $\square$

We are now ready to introduce the Harder-Narasimhan filtration on $\mathbf{Mod}_{\mathfrak{S}_n}^{\phi, r}$.

Definition 5.2.2. The **Harder-Narasimhan slope** (or **HN slope**) of an object $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_n}^{\phi, r}$ is

$$\mu(\mathfrak{M}) := \frac{\deg(\mathfrak{M})}{\mathrm{rk}(\mathfrak{M})}$$

where

1. $\deg(\mathfrak{M})$ is the **degree** of $\mathfrak{M}$, defined as

$$\deg(\mathfrak{M}) := \frac{1}{[K : \mathbb{Q}_p]} \ell_{\mathbb{Z}}(\mathrm{coker}(\phi_{\mathfrak{M}})),$$

2. $\mathrm{rk}(\mathfrak{M})$ is the **rank** of $\mathfrak{M}$, defined as

$$\mathrm{rk}(\mathfrak{M}) = \ell_{\mathcal{O}_{\mathcal{E}}}(\mathrm{GF}(\mathfrak{M})).$$

Remark 5.2.3. When $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_n}^{\phi,1}$, it can be shown that the degree and rank in Definition 5.2.2 equal to the degree and rank defined for finite flat group schemes in [16] for objects in $\mathbf{Mod}_{\mathfrak{S}_n}^{\phi,1}$. Thus, if we let $\mathcal{A}_1/\mathcal{O}_K$ and $\mathcal{A}_2/\mathcal{O}_K$ denote Néron models of abelian varieties with potential good reduction and $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be an isogeny of p -power degree, then $\mathcal{G}/\mathcal{O}_K = \ker(f)$ is a finite flat p -group scheme and $\mu(\mathfrak{M}(\mathcal{G}^*))$ is the ratio of the terms in $\log(\deg(f))$ and $\frac{1}{[K:\mathbb{Q}]} \log(\#s^*\Omega_{\mathcal{G}/\mathcal{O}_K}^1)$ appearing in Lemma 2.1.6. Thus, the Harder-Narasimhan theory of finite flat group schemes relates in a natural way to understanding the variation of the Faltings height, an observation that has already been made in [16].

Definition 5.2.4. An object $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_n}^{\phi,r}$ is **semistable** if for all saturated submodules $\mathfrak{N} \subseteq \mathfrak{M}$ their slopes satisfy the inequality $\mu(\mathfrak{N}) \geq \mu(\mathfrak{M})$. If an object is not semistable then it is called **unstable**.

Example 5.2.5. All objects $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_1}^{\phi,1}$ of rank 1 are semistable.

Lemma 5.2.6 (Corollary 2.3.12, [25]). *Let $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_n}^{\phi,r}$ be semistable. Then a saturated submodule (resp., quotient by a saturated submodule) $\mathfrak{N} \subseteq \mathfrak{M}$ (resp., $\mathfrak{M} \twoheadrightarrow \mathfrak{N}$) such that $\mu(\mathfrak{N}) = \mu(\mathfrak{M})$ is again semistable.*

Theorem 5.2.7 (Harder-Narasimhan Filtration, [16], [25]). *Let $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_n}^{\phi,r}$. Then there exists a unique filtration*

$$0 = \mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \cdots \subseteq \mathfrak{M}_k = \mathfrak{M}$$

by saturated subobjects such that for all i , $\mathfrak{M}_{i+1}/\mathfrak{M}_i$ is semi-stable and $\mu(\mathfrak{M}_{i+1}/\mathfrak{M}_i) > \mu(\mathfrak{M}_i/\mathfrak{M}_{i-1})$. Moreover, this filtration is preserved under endomorphisms.

Definition 5.2.8. The **Harder-Narasimhan polygon** (or **HN polygon**) of an object $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}_n}^{\phi,r}$ is the convex polygon whose segments have slope $\mu(\mathfrak{M}_{i+1}/\mathfrak{M}_i)$

and length $\mathrm{rk}(\mathfrak{M}_{i+1}/\mathfrak{M}_i)$, where the modules $\mathfrak{M}_i$ are those occurring in the Harder-Narasimhan filtration. We represent the HN-polygon as a piecewise linear function

$$\mathrm{HN}(\mathfrak{M}) : [0, \mathrm{rk}(\mathfrak{M})] \rightarrow [0, \mathrm{deg}(\mathfrak{M})]$$

such that $\mathrm{HN}(\mathfrak{M})(0) = 0$ and $\mathrm{HN}(\mathfrak{M})(\mathrm{rk}(\mathfrak{M})) = \mathrm{deg}(\mathfrak{M})$.

Example 5.2.9. The Harder-Narasimhan polygon of a semistable finite Kisin module is a single line of slope $\mu(\mathfrak{M})$.

Remark 5.2.10. When $\mathfrak{M} \in \mathbf{BT}_{\mathfrak{S}}^{\phi, r}$ is *flat* over $\mathfrak{S}$, we can extend the Harder-Narasimhan theory on its projections $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n$ to $\mathfrak{M}$ as follows. We define

$$\mathrm{deg}_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{deg}(\mathfrak{M}/p^n)/n$$

$$\mu_{\mathfrak{S}}(\mathfrak{M}) := \mu(\mathfrak{M}/p^n).$$

That this is independent of n is the content of Proposition 6.3.5 in [25]. Alternatively, if we let $\mathrm{HN}(\mathfrak{M}/p^n)$ denote the HN polygon of the object $\mathfrak{M}/p^n \in \mathbf{Mod}_{/\mathfrak{S}}^{\phi, r}$, the HN polygon (and hence the HN filtration) of $\mathfrak{M}$ is obtained as the limit of the maps

$$x \mapsto \frac{1}{n} \mathrm{HN}(\mathfrak{M}/p^n)(nx)$$

which converges uniformly increasingly as $n \rightarrow \infty$ (Theorem 6.2.3, [25]). It is shown in [25] that these two definitions are equivalent and moreover the isogeneous flat Kisin modules have the same HN filtration. We leave it to the reader to check for herself the details or consult [25] for reference.

Existence is prior to essence.

Paul Sartre, **Existentialism is a Humanism**

6

CM Kisin Modules

6.1 $\mathcal{O}_E$ -Linear CM Kisin Modules

Fix E to be a (p -adic) CM field such that $[E : \mathbb{Q}_p] = h$ and let E^* be its (p -adic) reflex field. Denote by $\mathcal{O}_E \subseteq E$ the ring of integral elements. Let $K/\mathbb{Q}_p$ be a finite field extension with ring of integers $\mathcal{O}_K$, a choice of uniformizer π_K , and residue field $k = \mathcal{O}_K/\pi_K\mathcal{O}_K$, and assume that $K \supseteq E^*$. Let $W = W(k)$ and $K_0 = W[1/p] \subseteq K$ be the maximal unramified subextension of K over $\mathbb{Q}_p$. Then the characteristic polynomial of π_K over K_0 is an Eisenstein polynomial $\text{Eis}(u) \in W[u]$, i.e., $\text{Eis}(u) = u^e + a_{e-1} + \cdots + a_1u + a_0$, where e is the ramification index of K/K_0 ,

$p|a_i$ for all i , and $a_0 = cp$ for some $c \in W(k)^\times$.

In Definition 3.1.9 we introduced the notion of the reflex norm. Evaluating it on $\mathbb{Q}$ -points gives the map:

$$\begin{aligned} N_{\Phi,K} : K^\times &\rightarrow (K \otimes E)^\times \rightarrow E^\times \\ x &\mapsto \det_E(x|_{V_{\Phi,K}}). \end{aligned}$$

Here, $V_{\Phi,K}$ is a finitely-generated $K \otimes_{\mathbb{Q}_p} E$ -module and $x \in K$ induces a $K \otimes_{\mathbb{Q}_p} E$ -endomorphism on $V_{\Phi,K}$. The subscript E indicates we take the E -linear determinant of the endomorphism induced by x on $V_{\Phi,K}$ considered as an E -vector space.

For any p -adic local subfield $L \subseteq K$ of finite index, we can define a *relative reflex norm map* to be

$$\begin{aligned} N_{\Phi,K,L \otimes E} : K^\times &\rightarrow (K \otimes E)^\times \rightarrow (L \otimes E)^\times = \prod_i E_i^\times \\ x &\mapsto \det_{L \otimes E}(x|_{V_{\Phi,K,L \otimes E}}) := \prod_i \det_{E_i}(x|_{V_{\Phi,K,E_i}}), \end{aligned}$$

where $V_{\Phi,K,L \otimes E} \simeq V_{\Phi,K}$ as an $L \otimes E$ -module, V_{Φ,K,E_i} is a finitely-generated $K \otimes E_i$ -module, and $V_{\Phi,K,L \otimes E} \simeq \bigoplus_i V_{\Phi,K,E_i}$ is induced from the decomposition of $L \otimes E \simeq \prod_i E_i$ into a product of fields. Note that the E_i do not all have to be isomorphic fields, so as $\mathbb{Q}_p$ -vector spaces the V_{Φ,K,E_i} could all have different dimension. This determinant is well-defined up to permutation of the fields E_i . The following lemma provides a more intrinsic definition.

Lemma 6.1.1. *Let A be a ring and M a finite projective A -module with an endomorphism x . Let F be a finite free A -module such that $F \simeq M \oplus M'$ and I_x the extension by the identity of x to F . Then setting $\det_A(x|_M) = \det_A(I_x|_F)$ is independent of the choice of F .*

Proof. Let F_1 and F_2 be two choices of free A -modules with respective decompositions

$M \oplus M_1$ and $M \oplus M_2$ into projective A -modules. Then for $i \in \{1, 2\}$ let x extend to endomorphisms $I_{x,i}$ on F_i which acts on M by x and on M_i by 1. Construct the module $F_1 \oplus F_2 = M \oplus M_1 \oplus M \oplus M_2$, and consider the automorphism u which switches the two factors of M and behaves as the identity on the other two factors. This automorphism satisfies

$$u(I_{x,1} \oplus \text{id})u^{-1} = \text{id} \oplus I_{x,2}.$$

Taking determinants on each side of this equality shows that $\det_A(I_{x,1}|_{F_1}) = \det_A(I_{x,2}|_{F_2})$. □

Apply the lemma with $A = L \otimes E$ and $M = V_{\Phi, K, L \otimes E}$. There is a natural choice of a free $L \otimes E$ -module $F = K \otimes E$ which has $L \otimes E$ -rank $[K : L]$ and splits as

$$K \otimes E \simeq V_{\Phi, K, L \otimes E} \oplus V_{\Phi^c, K, L \otimes E}.$$

For $x \in K^\times$, define I_x as in the proof of the lemma to be the endomorphism of F which acts on $V_{\Phi, K, L \otimes E}$ by x and on $V_{\Phi^c, K, L \otimes E}$ by 1. Then by the lemma $N_{\Phi, K, L \otimes E}(x) = \det_{L \otimes E}(I_x|_{K \otimes E})$, and is easily seen to coincide with the definition above, up to a permutation of the E_i -factors. Note that $N_{\Phi^c, K, L \otimes E}(x) = \det_{L \otimes E}(I_x^c|_{K \otimes E})$ where I_x^c acts on $V_{\Phi, K, L \otimes E}$ by 1 and on $V_{\Phi^c, K, L \otimes E}$ by x . By multiplicativity of the determinant, this moreover induces the decomposition

$$\text{Nm}_{K/L}(x) = N_{\Phi, K, L \otimes E}(x) N_{\Phi^c, K, L \otimes E}(x). \quad (6.1.2)$$

When $L = \mathbb{Q}_p$, the definition easily recovers the classic definition of the reflex norm.

For each finite extension K/E^* , the relative reflex norm extends to a multiplicative

map on polynomial rings after a base extension by $\mathbb{Q}_p[u]$ defined by

$$\begin{aligned} R_{\Phi,K,L \otimes E} : K[u] &\rightarrow (L \otimes E)[u] \\ f(u) &\mapsto N_{\Phi,K,L \otimes E}(f(u)). \end{aligned}$$

The multiplicativity of this map comes from the multiplicativity of the relative reflex norm. We let $P_{\Phi,x,L \otimes E}(u)$ denote the image of the polynomial $u - x$ for $x \in K$ under this map.

Definition 6.1.3. An $\mathcal{O}_E$ -linear CM Kisin Module of type $(\mathcal{O}_E, \Phi)$ is an object of $\mathbf{BT}_{/\mathfrak{S}}^{\phi,1}$ isomorphic to a rank 1 projective $\mathfrak{S} \otimes \mathcal{O}_E$ -module with generator e and Frobenius and Verschiebung endomorphisms given by

$$\begin{aligned} \phi_{\mathfrak{M}(\mathcal{G})}(e) &= \frac{1}{c} P_{\Phi^c, \pi_K, K_0 \otimes E}(u) e, \\ \psi_{\mathfrak{M}(\mathcal{G})}(e) &= \phi^*(c P_{\Phi, \pi_K, K_0 \otimes E}(u)) e. \end{aligned}$$

In particular, the Frobenius and Verschiebung endomorphisms satisfy the relation

$$\text{Eis}(u) = P_{\Phi, \pi_K, K_0 \otimes E} \cdot P_{\Phi^c, \pi_K, K_0 \otimes E}(u).$$

Lemma 6.1.4. *Let $\mathfrak{M}$ be an $\mathcal{O}_E$ -linear CM Kisin module of type $(\mathcal{O}_E, \Phi)$. Suppose that K contains all $\overline{\mathbb{Q}_p}$ -embeddings of E , and let $E^{\text{ur}} \subseteq E$ denote the maximal unramified subfield. Then under the identification*

$$\mathfrak{M} := (\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E) e \xrightarrow{\cong} \bigoplus_{\tau \in \text{Hom}(E^{\text{ur}}, K_0)} (\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}}, \tau}} \mathcal{O}_E) e_{\tau}$$

the Frobenius and Verschiebung are explicitly

$$\phi_{\mathfrak{M}(\mathcal{G})} e_{\tau} = f_{\sigma(\tau)}(u) e_{\sigma(\tau)}$$

$$\psi_{\mathfrak{M}(\mathcal{G})}e_\tau = \phi^*(v_{\sigma^{-1}(\tau)}(u))e_{\sigma^{-1}(\tau)}$$

where

$$\begin{aligned} f_\tau(u) &= \prod_{\nu \in \Phi_\tau} h_\nu(u), & \Phi_\tau &= \{\nu \in \Phi : \nu|_{E^{\text{ur}}} = \tau\} \\ v_\tau(u) &= \prod_{\nu \in \Phi_\tau^c} h_\nu(u), & \Phi_\tau^c &= \{\nu \in \Phi^c : \nu|_{E^{\text{ur}}} = \tau\} \end{aligned}$$

and the polynomials $h_\nu(u) \in \mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}}, \nu}} \mathcal{O}_E$ satisfy $\text{Eis}(u) = \prod_{\nu \in \text{Hom}_\tau(E, K)} h_\nu(u)$.

Proof. This is a natural consequence of the definition of the map $R_{\Phi, E^*, L \otimes E}$. $\square$

Example 6.1.5. Assume that $E/\mathbb{Q}_p$ is unramified and that K_0 contains all embeddings of E into $\overline{\mathbb{Q}_p}$. Then the module $\mathfrak{M}$ from Definition 6.1.3 has an explicit $\mathfrak{S}$ -basis $\{e_1, \dots, e_h\}$ so that its Frobenius $\phi_{\mathfrak{M}}$ has the presentation

$$\phi_{\mathfrak{M}} : e_i \mapsto \begin{cases} \frac{1}{c} \text{Eis}(u) e_{\sigma(i)} & \text{if } i \in \Phi \\ e_{\sigma(i)} & \text{if } i \notin \Phi \end{cases}$$

where $\sigma = \phi|_W$. Since σ is cyclic, we can choose an ordering such that $\sigma(i) = i + \delta \pmod{h}$ for $1 \leq i \leq h$ and $1 \leq \delta \leq h$. In fact, since σ has order h on the residue field, we can take $\delta = 1$. An element $a \in \mathcal{O}_E$ acts on the vector $x = \sum_i x_i$ by $a(x) = \sum_i i(a)x_i$, where $i : \mathcal{O}_E \hookrightarrow W$. We let the reader construct the analogous explicit form of $\psi_{\mathfrak{M}}$ as an easy exercise.

Example 6.1.6. Assume that $E/\mathbb{Q}_p$ is a totally ramified Galois extension. Then we can write the (reduced) module $\mathfrak{M}_{\mathfrak{S}_1}$ from Definition 6.1.3 with an explicit basis $\{e_1, \dots, e_h\}$ as follows. We let $\pi = \pi_E$ and note that

$$\text{Eis}(u) = u^e - ph(u) = \prod_{g \in \text{Gal}(E/\mathbb{Q}_p)} (u^{\frac{e}{h}} - g(\pi)h_g(u)) = \prod_{i=1}^h (u^{\frac{e}{h}} - c_i \pi h_i(u))$$

where $g(\pi) = c_i \pi$ for some $c_i \in \mathcal{O}_E^\times$ and $h_i(u) \in W_E[[u]]$, where $W_E = W \otimes_{W \cap E} E$ and the degree of $h_i(u)$ is strictly less than e/h . Then

$$P_{K, \pi_K, K_0 \otimes E}(u)e = \prod_{i \in \Phi^c} \left(u^{\frac{e}{h}} - c_i \pi h_i(u) \right) e.$$

We let $H_n(u) = \sum_{\substack{S \subseteq \Phi^c \\ \#S=n}} \prod_{i \in S} c_i h_i(u) = \sum_{k \geq 0} \pi^k H_{n,k}(u)$, where $H_{n,k}(u)$ are defined to have unit coefficients in W_E (note, in particular, that $H_{0,0}(u) = 1$ and $H_{0,k} = 0$ for $k > 0$). Define

$$G_{n,j}(u) = \sum_{\substack{k \geq 0 \\ k \equiv j \pmod{d}}} p^{\lfloor \frac{k}{h} \rfloor} H_{n,k}(u),$$

and

$$P_k(u) = \sum_{\substack{j, n \geq 0 \\ j+n=k}} G_{n,j}(u) u^{\frac{e}{h}(h-d-n)},$$

so that

$$P_{K, \pi_K, K_0 \otimes E}(u)e = \sum_{j=0}^{d-1} \pi^j \left(\sum_{n=0}^{h-d} \pi^n G_{n,j}(u) u^{\frac{e}{h}(h-d-n)} \right) e = \sum_{k=0}^{h-1} \pi^k P_k(u)e.$$

Using the basis $e_i = \pi^{i-1}$ for $1 \leq i \leq h$, we may write

$$\phi_{\mathfrak{M}}(e_i) = \sum_{j=0}^{h-1-i} [P_j(u)] e_{j+i} + p \sum_{j=h-i}^{h-1} [P_j(u)] e_{j-(h-i-1)}.$$

Then, reducing the coefficients modulo p , $G_{n,j}(u) = H_{n,j}(u)$ in the above matrix presentation and $\phi_{\mathfrak{M}}$ is represented by an upper-triangular matrix.

Now let $\mathcal{G}/\mathcal{O}_K$ be a CM p -divisible group with (p -adic) CM by $(\mathcal{O}_E, \Phi)$, and we assume that $\mathcal{G}$ has an $\mathcal{O}_E$ -linear structure, i.e., $\mathcal{O}_E \hookrightarrow \text{End}(\mathcal{G})$. We note that $\mathcal{G}$ has height h and denote its dimension by d .

Proposition 6.1.7. *An $\mathcal{O}_E$ -linear CM Kisin module of type $(\mathcal{O}_E, \Phi)$ is isomorphic*

to $\mathfrak{M}(\mathcal{G})$ for a CM p -divisible group $\mathcal{G}$ with (p -adic) CM by $(\mathcal{O}_E, \Phi)$ and an $\mathcal{O}_E$ -linear structure.

Proof. Let $\mathfrak{M}$ be an $\mathcal{O}_E$ -linear CM Kisin module of type $(\mathcal{O}_E, \Phi)$. By Theorem 4.3.2, $\mathfrak{M} \simeq \mathfrak{M}(\mathcal{G}')$ for some p -divisible group $\mathcal{G}'$. By Lemma 4.4.3, we compute $\text{Lie}(\mathcal{G}')$ to be

$$\phi^*\mathfrak{M}(\mathcal{G}'^*)/\text{Fil}^1\phi^*\mathfrak{M}(\mathcal{G}'^*) \simeq \phi^*\mathfrak{M}^*/\text{Fil}^1\phi^*\mathfrak{M}^* \simeq \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E / (P_{\Phi, \pi_K, K_0 \otimes E}(u)).$$

This induces the E -linear isomorphism

$$\text{Lie}(\mathcal{G}') \otimes \overline{\mathbb{Q}_p} \simeq [\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E / (P_{\Phi, \pi_K, K_0 \otimes E}(u))] \otimes \overline{\mathbb{Q}_p} \simeq \prod_{i \in \Phi} (\overline{\mathbb{Q}_p})_i$$

and hence $\mathcal{G}'$ is an $\mathcal{O}_E$ -linear CM p -divisible group with CM by $(\mathcal{O}_E, \Phi)$. By Corollary 3.2.9, after passing to an appropriate unramified extension, any two $\mathcal{O}_E$ -linear CM Kisin modules of the same type are isomorphic, so $\mathcal{G}' \simeq \mathcal{G}$ and $\mathfrak{M} \simeq \mathfrak{M}(\mathcal{G})$. $\square$

The Galois representation attached to $\mathcal{G}[p^s]$ is a torsion-crystalline representation on a one-dimensional E -vector space, hence abelian. By Corollary 3.2.9, the image agrees with the map

$$N_{\Phi, K} : \mathcal{O}_K^\times \rightarrow (\mathcal{O}_E/p^s\mathcal{O}_E)^\times,$$

and by local class field theory, the representation factors through a totally ramified abelian field extension K^s/K of degree $\#(\mathcal{O}_E/p^s\mathcal{O}_E)^\times$. Let $s\text{-Eis}(u) = u^{e_s} + \cdots + c_s p$ be the characteristic polynomial of the extension K^s/K_0 with $c_s \in W^\times$.

Definition 6.1.8. The **level- s model** of an $\mathcal{O}_E$ -linear CM Kisin module $\mathfrak{M} \simeq \mathfrak{M}(\mathcal{G})$ is a rank 1 projective $\mathfrak{S} \otimes \mathcal{O}_E$ -module $\mathfrak{M}^s$ with generator e and its Frobenius and Verschiebung endomorphisms given by

$$\phi_{\mathfrak{M}^s}(e) = \frac{1}{c_s} P_{\Phi^c, \pi_{K^s}, K_0 \otimes E}(u)e,$$

$$\psi_{\mathfrak{M}^s}(e) = \phi^* (c_s P_{\Phi, \pi_{K^s}, K_0 \otimes E}(u)) e.$$

In particular,

$$s\text{-Eis}(u) = P_{\Phi, \pi_{K^s}, K_0 \otimes E} \cdot P_{\Phi^c, \pi_{K^s}, K_0 \otimes E}(u).$$

Lemma 6.1.9. *Let $\mathfrak{M} \simeq \mathfrak{M}(\mathcal{G})$ be an $\mathcal{O}_E$ -linear CM Kisin module and $\mathfrak{M}^s$ be its level- s module. Then $\mathfrak{M}^s \simeq \mathfrak{M}(\mathcal{G}_s)$ where $\mathcal{G} \otimes_{\mathcal{O}_K} \mathcal{O}_{K^s} \simeq \mathcal{G}_s$. In particular, all p^s -torsion on $\mathcal{G}$ is defined over the field K^s .*

Proof. There are $\#(\mathcal{O}_E/p^s \mathcal{O}_E) = p^{s[E:\mathbb{Q}_p]}$ different p -torsion points on $\mathcal{G}_s$. Since the height of $\mathcal{G}_s$ equals $[E : \mathbb{Q}_p]$, all the p^s -torsion points are rational over K^s . The isomorphism $\mathfrak{M}^s \simeq \mathfrak{M}(\mathcal{G}_s)$ follows from Proposition 6.1.7 and Theorem 4.2.7. $\square$

Remark 6.1.10. The abelian extensions of K can be described by Lubin-Tate theory. Specifically, given a uniformizer $\pi = \pi_E$ of E , one can define a formal group law by

$$[\pi](u) = \pi u + u^q$$

where q denotes the size of the residue field of E . Then the class extension K^s/K is defined by the Eisenstein polynomial

$$h_{\pi, s}(u) = \frac{[\pi^s](u)}{[\pi^{s-1}](u)} = \pi + ([\pi^{s-1}](u))^{q-1}$$

where we define $[\pi^0](u) = u$. This further explicates Lemma 6.1.4 by setting

$$f_\tau(u) = \prod_{g \in \Sigma_\tau} (g_* h_{\pi, s})(u)$$

$$v_\tau(u) = \prod_{g \in \Sigma_\tau^c} (g_* h_{\pi, s})(u),$$

where

$$\begin{aligned}\Sigma_\tau &= \{g \in \text{Gal}(\overline{\mathbb{Q}}_p/K_0)/\text{Gal}(\overline{\mathbb{Q}}_p/K) : g^{-1} \circ \tau \in \Phi_\tau\}, & \Phi_\tau &= \{i \in \Phi : i|_{E^{\text{ur}}} = \tau\} \\ \Sigma_\tau^c &= \{g \in \text{Gal}(\overline{\mathbb{Q}}_p/K_0)/\text{Gal}(\overline{\mathbb{Q}}_p/K) : g^{-1} \circ \tau \in \Phi_\tau^c\}, & \Phi_\tau^c &= \{i \in \Phi^c : i|_{E^{\text{ur}}} = \tau\}.\end{aligned}$$

6.2 Quasi-Kisin Modules

We introduce the notion of a *quasi-Kisin module* to simplify computations involving $\mathcal{O}_E$ -linear CM Kisin modules, which were introduced in Proposition 6.1.7 and Lemma 6.1.4. The target computation is that of the relative Hodge bundle in Lemma 2.1.6, related to Kisin modules by Lemma 4.4.3 and Proposition 4.4.5.

Fix k to be a finite field containing the subfield $\mathbb{F}_q$ where $q = p^f$. Define $\mathfrak{S}^{(q)}$ to be the module $W(k)[[u]]$ with a q -Frobenius, i.e., $\sigma^{(q)}|_{W(k)} : \alpha \mapsto \alpha^q$ and extends to $u \mapsto u^q$ on $W(k)[[u]]$.

Definition 6.2.1. A **quasi-Kisin module** (or **q-Kisin Module**) is a finite $\mathfrak{S}^{(q)}$ -module $\mathfrak{M}^{(q)}$ equipped with the q -Frobenius semi-linear isomorphism

$$1 \otimes \phi_{\mathfrak{M}}^{(q)} : \phi^{(q),*}(\mathfrak{M}^{(q)})[1/\text{Eis}^{(q)}(u)] \xrightarrow{\sim} \mathfrak{M}^{(q)}[1/\text{Eis}^{(q)}(u)]$$

where $\phi^{(q),*}(\mathfrak{M}) := \mathfrak{S}^{(q)} \otimes_{\phi^{(q)}} \mathfrak{M}$ and $\text{Eis}^{(q)}(u) = \phi^f(\text{Eis}(u))$.

There is no correspondence such as Theorem 4.3.2 for quasi-Kisin modules, so they do not describe geometric objects in the way the category of Kisin modules does. However, one can show that any $\mathcal{O}_E$ -linear CM Kisin module is composed of quasi-Kisin pieces.

Fix E to be a (p -adic) CM field such that $E/\mathbb{Q}_p$ has ramification degree $r \geq 1$ and inertia degree f , and let $\mathcal{O}_E \subseteq E$ be its ring of integers with choice of uniformizer π_E . Denote by $\mathcal{G}/\mathcal{O}_K$ an $\mathcal{O}_E$ -linear CM p -divisible group with (p -adic) CM type $(\mathcal{O}_E, \Phi)$

and let $h = [E : \mathbb{Q}_p]$ be its height and d be its dimension. Let $I := \text{Hom}(E^{\text{ur}}, K_0)$. Then recall by Lemma 6.1.4 that, after enlarging K if necessary, we have the decomposition

$$\mathfrak{M}(\mathcal{G}) = (\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E)e \xrightarrow{\sim} \bigoplus_{i \in I} (\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}}, i}} \mathcal{O}_E)e_i$$

into projective $\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}}, i}} \mathcal{O}_E$ -modules of rank 1. We will write

$$\mathfrak{M}(\mathcal{G})_i := \mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}}, i}} \mathcal{O}_E$$

and use the subscript i to denote the i -isotypic piece in this decomposition.

Lemma 6.2.2. *Let $E/\mathbb{Q}_p$ be a (p -adic) CM field, and assume that $\mathcal{G}/\mathcal{O}_K$ is a CM p -divisible group with (p -adic) CM by $(\mathcal{O}_E, \Phi)$. If we let $\mathfrak{M}^{(q)} \simeq \mathfrak{M}(\mathcal{G}) \otimes_{\mathfrak{S}} \mathfrak{S}^{(q)}$ as modules be equipped with a $\phi_{\mathfrak{M}}^{(q)}$ -Frobenius $\phi_{\mathfrak{M}^{(q)}}^{(q)} = \phi_{\mathfrak{M}}^f$, then there is a decomposition into $\mathfrak{S}^{(q)}$ -modules*

$$\mathfrak{M}^{(q)} \simeq \bigoplus_{i \in I} \mathfrak{M}_i^{(q)}$$

where $\phi_{\mathfrak{M}^{(q)}}^{(q)} = \bigoplus_{i \in I} \phi_{\mathfrak{M}_i^{(q)}}^{(q)}$. If $E/\mathbb{Q}_p$ is Galois, then the $\mathfrak{M}_i^{(q)}$ all have the same $\mathfrak{S}^{(q)}$ -rank.

Proof. This decomposition is the quasi-Kisin analogue of Lemma 6.1.4. In particular, in *loc. cit.*, the Frobenius acts cyclically of order r on elements of I . The statement on the rank is a basic fact from Galois theory. $\square$

We will refer to the decomposition in Lemma 6.2.2 as the **quasi-Kisin decomposition** of the Kisin module $\mathfrak{M}$.

Corollary 6.2.3. *If $\mathfrak{N} \subseteq \mathfrak{M}$ is a saturated $\mathfrak{S}$ -submodule, then there exists a saturated $\mathfrak{S}^{(q)}$ -submodule $\mathfrak{N}^{(q)} \subseteq \mathfrak{M}^{(q)}$ with a decomposition*

$$\mathfrak{N}^{(q)} \simeq \bigoplus_{i \in I} \mathfrak{N}_i^{(q)}$$

such that $\phi_{\mathfrak{N}^{(q)}}^{(q)} = \bigoplus_{i \in I} \phi_{\mathfrak{N}_i^{(q)}}^{(q)}$ and

$$v_u(\det \phi_{\mathfrak{N}}) = \left(\frac{p-1}{p^f-1} \right) v_u(\det \phi_{\mathfrak{N}^{(q)}}) = \left(\frac{p-1}{p^f-1} \right) \sum_{i \in I} v_u(\phi_{\mathfrak{N}_i^{(q)}}^{(q)})$$

where f denotes the inertia degree of $E/\mathbb{Q}_p$. Conversely, a saturated $\mathfrak{S}^{(q)}$ -module $\mathfrak{N}^{(q)} \subseteq \mathfrak{M}^{(q)}$ corresponds to a saturated $\mathfrak{S}$ -module $\mathfrak{N} \subseteq \mathfrak{M}$ if its Frobenius respects the quasi-Kisin decomposition of $\mathfrak{M}$.

Proof. We construct the $\mathfrak{S}^{(q)}$ module $\mathfrak{N}^{(q)}$ from $\mathfrak{N}$ as we constructed $\mathfrak{M}^{(q)}$ from $\mathfrak{M}$ in Lemma 6.2.2. Then $\mathfrak{N}^{(q)}$ is necessarily a submodule of $\mathfrak{M}^{(q)}$ as $\mathfrak{N}$ is a submodule of $\mathfrak{M}$, and saturation also follows. We moreover obtain the decomposition on $\mathfrak{N}^{(q)}$ from the decomposition on $\mathfrak{M}^{(q)}$, and the statement on the determinants follows by the definition $\phi_{\mathfrak{N}^{(q)}}^{(q)} = \phi_{\mathfrak{N}}^f$ and following its effect on the leading coefficient of the Eisenstein polynomial at each iteration. Finally, the converse part of the corollary is clear by reversing the steps in the above construction. $\square$

We say that a (p -adic) CM field E is *ramified along a CM type Φ* if for any two elements $\tau_1, \tau_2 \in I$, there is an element $g \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that $g\Phi_{\tau_1} = \Phi_{\tau_2}$, where Φ_{τ_i} are as defined in Lemma 6.1.4. Otherwise, it is *unramified along the CM type Φ* .

Corollary 6.2.4. *If $E/\mathbb{Q}_p$ is Galois and ramified along the CM type Φ , then the $\mathfrak{M}_i^{(q)}$ are all isomorphic as $\mathfrak{S}^{(q)}$ -modules.*

Proof. Since $E/\mathbb{Q}_p$ is Galois, each of the $\mathfrak{M}_i^{(q)}$ have the same rank. Therefore, one needs to check their Frobenius endomorphisms $\phi_{\mathfrak{M}_i^{(q)}}^{(q)}$ induce isomorphic $\mathfrak{S}^{(q)}$ -modules. By Lemma 6.1.4, $\phi_{\mathfrak{M}_i^{(q)}}^{(q)} = f_i(u)^{\phi^f}$, and since $E/\mathbb{Q}_p$ ramifies along the CM type, the $f_i(u)$ are all E^{ur} -conjugate to each other. $\square$

We use the quasi-Kisin decomposition to reduce the computation of the relative Hodge bundle to either the case where the (p -adic) CM field $E/\mathbb{Q}_p$ is unramified or where it is totally ramified and Galois. In each instance we need a form of the Serre

tensor construction on CM p -divisible groups. A general introduction to the Serre tensor construction is given in [10].

Proposition 6.2.5. *Let E'/E be a totally ramified extension of degree ρ between (p -adic) CM fields, and choose a uniformizer $\pi_{E'}$ for E' and a uniformizer π_E for E such that $\text{Nm}_{E'/E}(\pi_{E'}) = \pi_E$. Let $\mathcal{G}$ be an $\mathcal{O}_E$ -linear CM p -divisible group and $\mathcal{G}' := \mathcal{G} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$ be the Serre tensor construction. Denote by $\mathcal{G}_j$ the image of $\mathcal{G}$ under $\pi_{E'}^j$ for $j = 0, \dots, \rho - 1$. Then every $\mathcal{O}_{E'}$ -stable finite subgroup $\mathcal{H}' \subseteq \mathcal{G}'$ has a decomposition $\mathcal{H}' = \prod_{j=0}^{\rho-1} \mathcal{H}_j$ where $\mathcal{H}_j \subseteq \mathcal{G}_j$ is an $\mathcal{O}_E$ -stable finite subgroup. In particular,*

$$v_u(\det \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}')}) = \sum_j v_u(\det \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}_j)}).$$

Moreover, when $\mathcal{H}' \cap \mathcal{G}'[\pi_{E'}^r]$ has the same height for all $1 \leq r < \rho$,

$$v_u(\det \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}')}) = \rho v_u(\det \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H}_0)}).$$

Proof. Enlarge K so that it contains all embeddings of E' . Define

$$\begin{aligned} \mathfrak{M}(\mathcal{G})_i &:= \mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}},i}} \mathcal{O}_E, \\ \mathfrak{M}(\mathcal{G}')_i &:= (\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}},i}} \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_{E'} = \mathfrak{M}(\mathcal{G})_i \otimes_{\mathcal{O}_{E^{\text{ur}}}} \mathcal{O}_{E'}. \end{aligned}$$

By Lemma 6.1.4,

$$\mathfrak{M}(\mathcal{G}') = ((\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}) e \xrightarrow{\sim} \bigoplus_{i \in I} ((\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}},i}} \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}) e_i = \bigoplus_i \mathfrak{M}(\mathcal{G}')_i.$$

Moreover,

$$\mathfrak{M}(\mathcal{G}_j) = (\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E \pi_{E'}^j) e \xrightarrow{\sim} \bigoplus_{i \in I} (\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}},i}} \mathcal{O}_E \pi_{E'}^j) e_i.$$

Since $\mathcal{H}'$ is $\mathcal{O}_{E'}$ -stable by hypothesis, its Kisin module is of the form

$$\mathfrak{M}(\mathcal{H}') = \bigoplus_{i \in I} (\mathfrak{S} \otimes_{\mathcal{O}_{E^{\text{ur}}, i}} \pi_{E'}^{-k_i} \mathcal{O}_{E'} / \mathcal{O}_{E'}) e_i \simeq \bigoplus_{i \in I} \pi_{E'}^{-k_i} \mathfrak{M}(\mathcal{G}')_i / \mathfrak{M}(\mathcal{G}')_i$$

for some collection of non-negative integers k_i . This induces the module isomorphism

$$\mathfrak{M}(\mathcal{H}')_i := \pi_{E'}^{-k_i} \mathfrak{M}(\mathcal{G}')_i / \mathfrak{M}(\mathcal{G}')_i \simeq \bigoplus_{j=0}^{\rho-1} \pi_E^{-\lfloor \frac{k_i+j}{\rho} \rfloor} \mathfrak{M}(\mathcal{G}_j)_i / \mathfrak{M}(\mathcal{G}_j)_i.$$

We wish to show that

$$\mathfrak{M}_j := \bigoplus_{i \in I} \pi_E^{-\lfloor \frac{k_i+j}{\rho} \rfloor} \mathfrak{M}(\mathcal{G}_j)_i / \mathfrak{M}(\mathcal{G}_j)_i$$

is a finite $\mathcal{O}_E$ -stable Kisin module so that it corresponds to an $\mathcal{O}_E$ -stable finite subgroup $\mathcal{H}_j \subseteq \mathcal{G}_j$ under Theorem 4.4.1.

Since $\mathfrak{M}(\mathcal{G}_j)$ is $\mathcal{O}_E$ -linear, there exist non-negative integers ℓ_i such that

$$\phi_{\mathfrak{M}(\mathcal{G}_j)} : \mathfrak{M}(\mathcal{G}_j)_i \mapsto \pi_E^{\ell_i} \mathfrak{M}(\mathcal{G}_j)_{\sigma(i)}.$$

Then

$$\phi_{\mathfrak{M}(\mathcal{G}')} : \mathfrak{M}(\mathcal{G}')_i \mapsto \pi_{E'}^{\rho \ell_i} \mathfrak{M}(\mathcal{G}')_{\sigma(i)}$$

and since $\mathfrak{M}(\mathcal{H}')$ is $\mathcal{O}_{E'}$ -stable,

$$k_i - \rho \ell_i \leq k_{\sigma(i)}.$$

But this implies that

$$\lfloor \frac{k_i + j}{\rho} \rfloor - \ell_i \leq \lfloor \frac{k_{\sigma(i)} + j}{\rho} \rfloor$$

so that

$$\phi_{\mathfrak{M}(\mathcal{G}_j)} : \pi_E^{-\lfloor \frac{k_i+j}{\rho} \rfloor} \mathfrak{M}(\mathcal{G}_j)_i / \mathfrak{M}(\mathcal{G}_j)_i \mapsto \pi_E^{-\lfloor \frac{k_i+j}{\ell} \rfloor + \ell_i} \mathfrak{M}(\mathcal{G}_j)_{\sigma(i)} / \mathfrak{M}(\mathcal{G}_j)_{\sigma(i)}$$

and the image is contained in $\pi_E^{-\lfloor \frac{k_{\sigma(i)}+j}{\rho} \rfloor} \mathfrak{M}(\mathcal{G}_j)_{\sigma(i)} / \mathfrak{M}(\mathcal{G}_j)_{\sigma(i)}$, which demonstrates stability by the Frobenius endomorphism. Thus, by Theorem 4.3.2, $\mathfrak{M}_j \simeq \mathfrak{M}(\mathcal{H}_j)$ for some $\mathcal{O}_E$ -stable finite subgroup $\mathcal{H}_j$ of $\mathcal{G}_j$. The rest of the proposition now follows from the multiplicativity of the determinant. $\square$

Corollary 6.2.6. *Let (E, Φ) be a $(p$ -adic) CM type and $\mathcal{G}/\mathcal{O}_K$ a CM p -divisible group with $(p$ -adic) CM by $(\mathcal{O}_E, \Phi)$. Assume that $E/\mathbb{Q}_p$ does not ramify along the CM type Φ . Define $\rho = [E : E^{\text{ur}}]$ and let π_E be a choice of uniformizer for E . Then $\mathcal{G} \simeq \mathcal{G}^{\text{ur}} \otimes_{\mathcal{O}_{E^{\text{ur}}}} \mathcal{O}_E$, and for any saturated submodule $\mathfrak{N} \subseteq \mathfrak{M}(\mathcal{G}[p^n])$ corresponding to a group scheme $\mathcal{H} \subseteq \mathcal{G}[p^n]$ such that $\mathcal{H} \cap \mathcal{G}[\pi_E^r]$ has the same height for all $1 \leq r < \rho$, there exists a saturated submodule $\mathfrak{N}_0 \subseteq \mathfrak{M}(\mathcal{G}^{\text{ur}})$ such that*

$$v_u(\det \phi_{\mathfrak{N}}) = \rho v_u(\det \phi_{\mathfrak{N}_0}) = \rho \left(\frac{p-1}{p^f-1} \right) \sum_{i \in I} v_u \left(\phi_{\mathfrak{N}_{0,i}^{(q)}} \right)$$

where $\bigoplus_{i \in I} \mathfrak{N}_{0,i}^{(q)}$ is the quasi-Kisin decomposition of $\mathfrak{N}_0$.

Proof. By the structure theorem for local fields and the hypothesis that $E/\mathbb{Q}_p$ does not ramify along the CM type Φ , $\mathcal{G}$ is the Serre tensor construction of a p -divisible group $\mathcal{G}^{\text{ur}}$ with $(p$ -adic) CM by $(\mathcal{O}_{E^{\text{ur}}}, \Phi^{\text{ur}} = \Phi|_{E^{\text{ur}}})$. By Proposition 6.2.5 and Proposition 5.1.5, we then obtain the existence of $\mathfrak{N}_0$ and are able to compute the constant ρ appearing in the formula. Finally, the decomposition of $\mathfrak{N}_0$ is an immediate application of Lemma 6.2.2. $\square$

Proposition 6.2.7. *Let $\tilde{E}/E$ denote the Galois closure of a $(p$ -adic) CM field E . Let $\mathcal{G}$ be an $\mathcal{O}_E$ -linear CM p -divisible group and $\tilde{\mathcal{G}} := \mathcal{G} \otimes_{\mathcal{O}_E} \mathcal{O}_{\tilde{E}}$ be the Serre tensor construction. Then for every subgroup scheme $\mathcal{H} \subseteq \mathcal{G}$, there is a subgroup scheme*

$\tilde{\mathcal{H}} \subseteq \tilde{\mathcal{G}}$ which pulls back to $\mathcal{H}$ under the embedding $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ and

$$v_u(\det \phi_{\mathfrak{M}(\tilde{\mathcal{H}})}) = \rho v_u(\det \phi_{\mathfrak{M}(\mathcal{H})})$$

where $\rho = [\tilde{E} : E]$.

Proof. The field extension $\tilde{E}/E$ is necessarily totally ramified. Moreover, $\mathcal{O}_{\tilde{E}}$ is a free $\mathcal{O}_E$ -module of rank ρ , and thus as a p -divisible group $\tilde{\mathcal{G}}$ is isomorphic to $\mathcal{G}^\rho$. We can even further assume that the closed immersion $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ is just the inclusion of the first factor. Then $\tilde{\mathcal{H}} = \mathcal{H}^\rho$ is a subgroup scheme of $\tilde{\mathcal{G}}$ which pulls back to $\mathcal{H}$ with the desired property. $\square$

Corollary 6.2.8. *Let $E/\mathbb{Q}_p$ be a (p -adic) CM field and $\tilde{E}/\mathbb{Q}_p$ denote its Galois closure. Let $\mathcal{G}/\mathcal{O}_K$ be a CM p -divisible group with (p -adic) CM by $(\mathcal{O}_E, \Phi)$ and assume that E ramifies along Φ . We further denote by $\tilde{\mathcal{G}}$ the Serre tensor construction $\mathcal{G} \otimes_{\mathcal{O}_E} \mathcal{O}_{\tilde{E}}$. Then for any saturated submodule $\mathfrak{N} \subseteq \mathfrak{M}(\mathcal{G})$ there exists a saturated submodule $\tilde{\mathfrak{N}} \subseteq \mathfrak{M}(\tilde{\mathcal{G}})$ such that*

$$v_u(\det \phi_{\mathfrak{N}}) = \frac{1}{\rho} v_u(\det \phi_{\tilde{\mathfrak{N}}}) = \frac{1}{\rho} \sum_{i \in I} v_u(\det \phi_{\tilde{\mathfrak{N}}_i}).$$

Proof. By Proposition 6.2.7 and Proposition 5.1.5, it immediately follows that there exists a saturated submodule $\tilde{\mathfrak{N}} \subseteq \mathfrak{M}(\tilde{\mathcal{G}})$ such that $v_u(\det \phi_{\tilde{\mathfrak{N}}}) = \rho(\det \phi_{\mathfrak{N}})$. The decomposition $\tilde{\mathfrak{N}} = \bigoplus_{i \in I} \tilde{\mathfrak{N}}_i$ is likewise an immediate application of Lemma 6.2.2. $\square$

Remark 6.2.9. By Corollary 6.2.4 and Corollary 6.2.8, if $E/\mathbb{Q}_p$ is ramified along the CM type, then the computation of the relative Hodge bundle at each piece in the decomposition is completely analogous to a case when the (p -adic) CM field is totally ramified and Galois. We therefore consider, somewhat abusively, the case of a totally ramified and Galois (p -adic) CM field in place of being ramified along the CM type for computing the relative Hodge bundle at each instance it occurs.

6.3 HN Theory of CM Torsion Modules

Harder-Narasimhan theory (reviewed in §5.2) allows us to identify subgroups of $\mathcal{G}[p^n]$ where $\mathcal{G}/\mathcal{O}_K$ is an $\mathcal{O}_E$ -linear CM p -divisible group. The results here are very specific to the case when $\mathcal{G}$ is a CM p -divisible group, as its isogeny class, and therefore its Harder-Narasimhan theory, is determined by the rational endomorphism algebra.

We first demonstrate that $\mathcal{G}[p^n]$ is semistable for all n .

Lemma 6.3.1. *Let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group of dimension d and height h . Then $\mathfrak{M}(\mathcal{G}[p^n])$ is semistable for all $n \geq 1$. Moreover,*

$$\mu(\mathfrak{M}(\mathcal{G}[p^n])) = \frac{h-d}{h}.$$

Proof. By Proposition 6.1.7, $\mathfrak{M}(\mathcal{G}[p^n])$ is $\mathfrak{S}_n \otimes \mathcal{O}_E$ -projective of rank 1. Then since $\mathcal{O}_E$ -multiplication preserves the Harder-Narasimhan filtration, any submodule of $\mathfrak{M}(\mathcal{G}[p^n])$ in the filtration must be $\mathfrak{S}_n \otimes \mathcal{O}_E$ -projective of rank 1 as well. The inclusion of such a submodule is an isomorphism under the functor GF , and therefore itself an isomorphism. This shows $\mathfrak{M}(\mathcal{G}[p^n])$ is semistable.

For the latter claim, we note that $\#\mathcal{G}[p^n] = p^{nh}$, so $\mathrm{rk}(\mathfrak{M}(\mathcal{G}[p^n])) = nh$. Moreover, since $\mathcal{G}$ has dimension d , $v_u(\det \phi_{\mathfrak{M}(\mathcal{G}[p^n])}) = (h-d)e$, and hence by Proposition 4.4.5, $\deg(\mathfrak{M}(\mathcal{G}[p^n])) = (h-d)ne$. $\square$

Using the fact now that $\mathcal{G}[p^n]$ is semistable and has an $\mathcal{O}_E$ -linear structure, we can characterize subgroups in the following special case.

Lemma 6.3.2. *Suppose that $E/\mathbb{Q}_p$ is totally ramified, and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group of dimension d and height h . Let $\pi = \pi_E$ denote a uniformizer of $\mathcal{O}_E$ and let $\mathfrak{N} \subseteq \mathfrak{M}(\mathcal{G}[\pi^r]) =: \mathfrak{M}$ be a saturated submodule. Then $\mu(\mathfrak{N}) = \mu(\mathcal{G}[\pi^r])$ if and only if $\mathfrak{M}/\mathfrak{N} = \mathfrak{M}(\mathcal{G}[\pi^k])$ for some integer $0 \leq k \leq r$.*

Proof. We proceed by induction on the rank of $\mathfrak{N}$. If $\mathfrak{N} = 0$, then take $k = r$. Otherwise there is a unique integer $k < r$ such that $\mathfrak{N}$ is contained in $\pi^k \mathfrak{M} / \pi^r \mathfrak{M}$ but not in $\pi^{k+1} \mathfrak{M} / \pi^r \mathfrak{M}$. This induces a projection $\mathfrak{N} \rightarrow \pi^k \mathfrak{M} / \pi^{k+1} \mathfrak{M}$ with kernel $\mathfrak{N}_1$ and image $\mathfrak{N}_2$, each of which are objects in $\mathbf{Mod}_{/\mathfrak{S}}^{\phi, r}$. If we show that

$$\mathfrak{N}_1 = \pi^{k+1} \mathfrak{M} / \pi^r \mathfrak{M}$$

$$\mathfrak{N}_2 = \pi^k \mathfrak{M} / \pi^{k+1} \mathfrak{M}$$

this will imply $\mathfrak{N} = \pi^k \mathfrak{M} / \pi^r \mathfrak{M}$, and thus the statement.

Note that $\mathfrak{N}_2$ is a non-zero Kisin submodule of $\pi^k \mathfrak{M} / \pi^{k+1} \mathfrak{M} = \mathfrak{M} / \pi \mathfrak{M}$. The latter is of rank 1 and degree $\frac{h-d}{h}$, hence semistable. Thus $\mathfrak{N}_2$ has rank 1 and its slope is at least $\frac{h-d}{h}$. However, $\mathfrak{N}_2$ is a quotient of $\mathfrak{N}$, which is semistable of slope $\frac{h-d}{h}$, and thus the slope of $\mathfrak{N}_2$ is at most and so equal to $\frac{h-d}{h}$. It follows $\mathfrak{N}_2 = \pi^k \mathfrak{M} / \pi^{k+1} \mathfrak{M}$ as a submodule of the same degree and the same rank.

By induction, $\mathfrak{N}_1 = \pi^\ell \mathfrak{M} / \pi^r \mathfrak{M}$ for some integer $k+1 \leq \ell \leq r$. Since $\mathfrak{N}_2$ is π -torsion, $\pi \mathfrak{N} \subseteq \mathfrak{N}_1$. On the other hand, since $\mathfrak{N}_1$ is $\pi^{r-\ell}$ -torsion, $\mathfrak{N}$ is $\pi^{r-\ell+1}$ -torsion. Therefore, $\mathfrak{N} \subseteq \pi^{\ell-1} \mathfrak{M} / \pi^r \mathfrak{M}$. By the definition of k , $\ell-1 \leq k$, but since we also have $k+1 \leq \ell$, $\ell = k+1$. Hence, $\mathfrak{N}_1 = \pi^{k+1} \mathfrak{M} / \pi^r \mathfrak{M}$, which demonstrates the statement. $\square$

Using these lemmas, we identify the torsion quotients $\mathfrak{M}$ of $\mathfrak{M}(\mathcal{G})$ corresponding to the exact sequence

$$0 \rightarrow \mathfrak{M}(\mathcal{G}') \rightarrow \mathfrak{M}(\mathcal{G}) \rightarrow \mathfrak{M} \rightarrow 0 \quad (6.3.3)$$

where $\mathcal{G}'$ is a CM p -divisible group with CM by $(\mathcal{O}, \Phi)$ for some non-maximal order $\mathcal{O} \subseteq E$ as those saturated Kisin modules with strictly smaller slope than that of $\mathfrak{M}(\mathcal{G})$. Since it is very important, we spell it out explicitly below.

Lemma 6.3.4. *Let $\mathcal{G} / \mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group and $\mathfrak{N} \subseteq \mathfrak{M}(\mathcal{G}[p^n])$*

be a saturated submodule such that $\mathfrak{M}(\mathcal{G}[p^n])/\mathfrak{N}$ is a quotient of the form (6.3.3). Then $\mu(\mathfrak{N}) < \mu(\mathfrak{M})$.

Proof. By Corollary 6.2.6 and Corollary 6.2.8, we may assume that $E/\mathbb{Q}_p$ is either unramified or totally ramified (and Galois). The latter case follows by Lemma 6.3.2 and the semistability of $\mathcal{G}[p^n]$ proved in Lemma 6.3.1. In the former case, the argument follows similarly: since $\mathfrak{N}$ is not $\mathcal{O}_E$ -stable, the proof of Lemma 6.3.1 shows it cannot lie in the Harder-Narasimhan filtration of $\mathfrak{M}(\mathcal{G}[p^n])$, and a modification of the argument in Lemma 6.3.2 demonstrates that only the $\mathcal{O}_E$ -stable submodules preserve the HN slope. Since $\mathcal{G}[p^n]$ is semistable by Lemma 6.3.1, the statement follows. $\square$

The corollary combined with the preceding two lemmas show *a priori* that the change in Faltings height in the formula given by Lemma 2.1.6 must be *positive* when A_1 is taken to have CM by a maximal order. Positivity of the Faltings height variation is not strong enough to prove a Northcott property, since the contribution by the relative Hodge bundle to the variation must be shown to not compete with the degree term. However, it identifies a “minimal” element within an isogeny class of CM abelian varieties, namely the CM abelian variety corresponding to the maximal order (or the CM abelian variety over the minimal field of definition), which is a result of intrinsic interest.

If you are under the impression you have already perfected yourself, you will never rise to the heights you are no doubt capable of.

Kazuo Ishiguro, **The Remains of the Day**

7

Differential Computations

7.1 Unramified Case

Fix E to be a (p -adic) CM field such that $E/\mathbb{Q}_p$ is unramified, and let $\mathcal{O}_E \subseteq E$ be its ring of integers with choice of uniformizer π_E . We bound slopes of submodules of $\mathfrak{M}(\mathcal{G}[p^n])$, where $\mathcal{G}/\mathcal{O}_K$ is an $\mathcal{O}_E$ -linear CM p -divisible group with (p -adic) CM type $(\mathcal{O}_E, \Phi)$. We denote by $h = [E : \mathbb{Q}_p]$ the height of $\mathcal{G}$ and by d its dimension.

We first compute saturated submodules of $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}[p])$. By Proposition 5.1.5, these correspond to finite flat subgroup schemes $\mathcal{H} \subseteq \mathcal{G}[p]$ of order p^k where $k < h$. Since $p \in \mathcal{O}_E$ is prime, we have *a priori* that all such saturated submodules correspond to

torsion quotients of the form in (6.3.3). It will be evident from the computations that this is the minimal degree for an isogeny to such a CM p -divisible group $\mathcal{G}'$.

Recall the presentation in Example 6.1.5 for $\mathfrak{M}(\mathcal{G})$. Here we assumed that K_0 contains all embeddings of E into $\overline{\mathbb{Q}}_p$ so we can choose an $\mathfrak{S}$ -basis $\{e_1, \dots, e_h\}$ such that $\phi_{\mathfrak{M}}$ has the presentation

$$\phi_{\mathfrak{M}} : e_i \mapsto \begin{cases} \frac{1}{c} \text{Eis}(u) e_{i+1} & \text{if } i \in \Phi \\ e_{i+1} & \text{if } i \notin \Phi \end{cases} \quad (7.1.1)$$

where addition on the indexing set is performed modulo h . We consider the set $\{1, \dots, h\}$ to be a torsor under the action of $\text{Gal}(E/\mathbb{Q}_p)$, where the action is by addition modulo h and induced from the corresponding cyclic permutation of the basis $(e_1, \dots, e_h)$.

Lemma 7.1.2. *Let $\mathfrak{M} = \mathfrak{M}(\mathcal{G})$ where $\mathcal{G}$ has CM type $\Phi = \{\alpha_1, \dots, \alpha_{h-d}\} \subseteq \{1, \dots, h\}$ as an ordered set. If we let $\mathfrak{M}^{(q)} \simeq \mathfrak{M}$ as modules with a $\phi^{(q)}$ -Frobenius $\phi_{\mathfrak{M}^{(q)}}^{(q)} = \phi_{\mathfrak{M}}^q$, then*

$$\mathfrak{M}^{(q)} \simeq \bigoplus_{\tau \in \text{Hom}(E^{\text{ur}}, K_0)} \mathfrak{M}_{\tau}^{(q)}$$

where each $\mathfrak{M}_{\tau}^{(q)}$ is a $\mathfrak{S}^{(q)}$ -module of rank 1 with $\phi^{(q)}$ -Frobenius

$$\phi_{\mathfrak{M}_{\tau}^{(q)}}^{(q)}(x) = \left(u^{e \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} + p g_{\tau}(u) \right) \cdot x$$

for some polynomial $g_{\tau}(u) \in \mathfrak{S}^{(q)}$ with degree strictly less than $e \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}$.

Proof. We apply the quasi-Kisin decomposition in Lemma 6.2.2. The precise form follows from the presentation (7.1.1) and the theory of Eisenstein polynomials. $\square$

Let $I = \text{Hom}(E^{\text{ur}}, K_0)$ and $S_r \subseteq I$ denote a subset of size r .

Proposition 7.1.3. *Let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear p -divisible group of type $(\mathcal{O}_E, \Phi)$ and assume that $E/\mathbb{Q}_p$ is unramified. Then for any subgroup $\mathcal{H} \subseteq \mathcal{G}[p]$ of height $k < h$,*

$$v_u(\det(\phi_{\mathfrak{M}(\mathcal{H})})) = \min_{S_k \subseteq I} \left\{ \sum_{\tau \in S_k} \left(\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)} \right) \right\} \cdot \left(\frac{p-1}{p^h-1} \right) e.$$

Proof. Let $\mathfrak{M}^{(q)}$ denote the quasi-Kisin module corresponding to $\mathfrak{M}(\mathcal{G}[p])$ by Lemma 6.2.2. Then by Proposition 5.1.5 and Corollary 6.2.3, there exists a saturated quasi-Kisin submodule $\mathfrak{N}^{(q)} \subseteq \mathfrak{M}^{(q)}$ of rank $h-k$ corresponding to the subgroup $\mathcal{H} \subseteq \mathcal{G}[p]$ and by which one can compute $v_u(\det(\phi_{\mathfrak{M}(\mathcal{H})}))$. Then using the presentation in Lemma 7.1.2 and again Corollary 6.2.3, a submodule of rank $h-k$ is saturated whenever the iterated sum

$$\sum_{\tau \in S_{h-k}} \left(\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)} \right)$$

is maximized over different sets $S_{h-k} \subseteq I$. $\square$

We now compute the relative Hodge bundle for saturated submodules of $\mathfrak{M}(\mathcal{G}[p^n])$. Any p^n -torsion finite flat subgroup scheme $\mathcal{H} \subseteq \mathcal{G}[p^n]$ is characterized by the non-increasing tuple $(\lambda_1, \dots, \lambda_h)$ where $0 \leq \lambda_i \leq n$ are all integers, $\lambda_1 = n$, and $\lambda_{k+1}, \dots, \lambda_h = 0$ for some $k < h$ determined by $\mathcal{H}$ (this corresponds to the assumption that $\mathcal{H} \cap \mathcal{G}[p] = 0$, for otherwise we get the same quotient p -divisible group). Let $T \subseteq \{0, \dots, h-1\}$ consist of those elements i such that $\lambda_{k-i} - \lambda_{k-i+1} > 0$ (in particular, $0 \in T$).

Proposition 7.1.4. *Let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group with (p -adic) CM type $(\mathcal{O}_E, \Phi)$, and assume that $E/\mathbb{Q}_p$ is unramified. Let $\mathcal{H} \subseteq \mathcal{G}[p^n]$ be of type $(\lambda_1, \dots, \lambda_h)$ and have p -height $k < h$. Then letting*

$$v_u(\det(\phi_{\mathfrak{M}(\mathcal{H})})) = \sum_{i \in T} \sum_{j=\lambda_{k-i+1}+1}^{\lambda_{k-i}} \left(\min_{S_{d(i)} \subseteq I} \left\{ \sum_{\tau \in S_{d(i)}} \left(\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)} \right) \right\} \cdot \left(\frac{p-1}{p^j(p^h-1)} \right) \right) e$$

where $d(i) = \max\{j : \lambda_j = \lambda_{k-i}\}$ (in particular, $d(0) = k$).

Proof. We assume at first that $\lambda_1 = \lambda_k = n$ so that T consists of a single element and $d(i) = k$. Then the formula simplifies to the inner sum and we iterate on n .

When $n = 1$ the statement reduces to Proposition 7.1.3. When $n > 1$, for any integer $0 \leq r \leq n$ we have a factorization of the finite flat group scheme $\mathcal{H}$ into a short exact sequence of finite flat group schemes

$$0 \rightarrow \mathcal{H}_r \rightarrow \mathcal{H} \rightarrow \mathcal{H}_{n-r} \rightarrow 0 \quad (7.1.5)$$

where $\mathcal{H}_r = \mathcal{H} \cap \mathcal{G}[p^r]$. Geometrically, this means the isogeny $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ factors as a sequence of isogenies $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}_r \rightarrow \mathcal{G}/\mathcal{H}$. Iterating on r gives a sequence of length n where each isogeny in the sequence has p -torsion kernel.

Let $r = 1$. Then (7.1.5) transforms under $\mathfrak{M}$ to the short exact sequence of Kisin modules

$$0 \rightarrow \mathfrak{M}(\mathcal{G}_1) \rightarrow \mathfrak{M}(\mathcal{G}) \rightarrow \mathfrak{M}(\mathcal{H}_1) \rightarrow 0$$

which by Lemma 6.2.2 corresponds to a short exact sequence of quasi-Kisin modules

$$0 \rightarrow \mathfrak{M}^{(q)}(\mathcal{G}_1) \rightarrow \mathfrak{M}^{(q)}(\mathcal{G}) \rightarrow \mathfrak{M}^{(q)}(\mathcal{H}_1) \rightarrow 0.$$

By Lemma 7.1.2 and the proof of Proposition 7.1.3, we can write

$$\phi_{\mathfrak{M}^{(q)}(\mathcal{H}_1)}^{(q)} = \bigoplus_{i \in \mathcal{S}_k} \phi_{\mathfrak{M}_i^{(q)}}^{(q)},$$

where $\mathcal{S}_k \subseteq I$ is a subset of size k which minimizes the sum $\sum_{\tau \in \mathcal{S}_k} \left(\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)} \right)$. Then since $\mathfrak{M}(\mathcal{H}_1)$ is the Kisin module of a p -torsion group of height k , $\mathfrak{M}^{(q)}(\mathcal{G}_1)$ must have index p in $\mathfrak{M}^{(q)}(\mathcal{G})$ on k generators. In other words, there is a presentation of

$\mathfrak{M}^{(q)}(\mathcal{G}_1)$ such that

$$\phi_{\mathfrak{M}^{(q)}(\mathcal{G}_1)}^{(q)} \equiv \bigoplus_{\tau \in \mathcal{S}_k} u^{\frac{1}{p} \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} \oplus \bigoplus_{\tau \notin \mathcal{S}_k} u^{\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} \pmod{p}.$$

Iterating on r , this provides a presentation of $\mathfrak{M}^{(q)}(\mathcal{G}_r)$ such that

$$\phi_{\mathfrak{M}^{(q)}(\mathcal{G}_r)}^{(q)} \equiv \bigoplus_{\tau \in \mathcal{S}_k} u^{\frac{1}{p^r} \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} \oplus \bigoplus_{\tau \notin \mathcal{S}_k} u^{\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} \pmod{p}.$$

Replacing $\mathcal{G}$ by $\mathcal{G}_r$ in Proposition 7.1.3 allows us to compute the relative Hodge bundle on $\mathcal{H}/\mathcal{H}_r \cap \mathcal{G}_r[p]$. Then the formula follows from the additivity of the relative Hodge bundle on exact sequences of finite flat group schemes.

When $\lambda_1 > \lambda_k$ we argue in the same way, only at each step of the iteration on r the height of $\mathcal{H}/\mathcal{H}_r \cap \mathcal{G}_r[p]$ can change. We account for it by constructing the appropriate set T in the statement. $\square$

Theorem 7.1.6. *Let $\mathcal{G}$ be an $\mathcal{O}_E$ -linear CM p -divisible group of type $(\mathcal{O}_E, \Phi)$ and assume that $E/\mathbb{Q}_p$ is unramified. Then for any subgroup $\mathcal{H} \subseteq \mathcal{G}[p^n]$ of p -height $k < h$,*

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^k \leq \left(\frac{p-1}{p^\delta - 1} \right) \left(\frac{1-p^{-k}}{1-p^{-1}} \right) \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \log p$$

where $\delta = \frac{h}{h-d} > 1$.

Proof. By Proposition 4.4.5, the inequality is obtained by computing upper bounds on the formula for $v_u(\det \phi_{\mathfrak{M}(\mathcal{H})})$ in Proposition 7.1.4. First, the quantity

$$\min_{S_{d(i)} \subseteq I} \left\{ \sum_{\tau \in S_{d(i)}} \left(\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)} \right) \right\}$$

is maximal when the CM type is $\{1, 1 + \delta, \dots, 1 + \delta(h-d-1)\}$, where $\delta = \frac{h}{h-d}$. For

a given $d(i)$, this maximal value is

$$M(d(i)) = \sum_{\ell=0}^{d(i)-1} \left(\sum_{s=1}^{h-d} p^{h-\delta s-\ell} \right) = \left(\frac{p^h - 1}{p^\delta - 1} \right) \left(\frac{1 - p^{-d(i)}}{1 - p^{-1}} \right).$$

The inner sum that iterates this quantity in Proposition 7.1.4 is maximal when $d(i) = k$ for all i . Then

$$\begin{aligned} v_u(\det(\phi_{\mathfrak{M}(\mathcal{H})})) &\leq M(k) \left(\frac{p-1}{p^h-1} \right) \left(\frac{1-p^{-n}}{1-p^{-1}} \right) e \\ &= \left(\frac{p-1}{p^\delta-1} \right) \left(\frac{1-p^{-k}}{1-p^{-1}} \right) \left(\frac{1-p^{-n}}{1-p^{-1}} \right) e. \end{aligned}$$

□

7.2 Totally Ramified Case

Fix E to be a (p -adic) CM field such that $E/\mathbb{Q}_p$ is totally ramified, and let $\mathcal{O}_E \subseteq E$ be its ring of integers with choice of uniformizer π_E . As in §7.1, we bound slopes of saturated Kisin submodules of $\mathfrak{M} = \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}[p])$, where $\mathcal{G}/\mathcal{O}_K$ is an $\mathcal{O}_E$ -linear CM p -divisible group with (p -adic) CM type $(\mathcal{O}_E, \Phi)$. We denote by $h = [E : \mathbb{Q}_p]$ the height of $\mathcal{G}$ and by d its dimension.

We first make the computation explicit in the case $h = 2$.

Proposition 7.2.1. *Assume that $[E : \mathbb{Q}_p] = 2$ and p ramifies in E , and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group. Denote by $\mathfrak{M} := \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}[p])$. Then a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \mathfrak{M}$ has either*

$$v_u(\det(\phi_{\mathfrak{L}})) = \begin{cases} \frac{e}{2} & \text{if } \mathfrak{M}/\mathfrak{L} \simeq \mathfrak{M}(\mathcal{G}[\pi_E]), \\ \left(\frac{2p-1}{p} \right) \frac{e}{2} & \text{otherwise.} \end{cases}$$

Proof. We let $\pi = \pi_E$ and use Lemma 6.1.9 and Remark 6.1.10 to construct the

level-2 model $\mathfrak{M}^2(\mathcal{G})$ using the Eisenstein polynomial

$$2\text{-Eis}(u) = (\pi + (\pi u + u^p)^{p-1})(\bar{\pi} + (\bar{\pi} u + u^p)^{p-1}) = u^e + ph(u)$$

so that

$$P_{K, \pi_K, K_0 \otimes E}(u) = \pi + (\pi u + u^p)^{p-1} = u^{\frac{e}{2}} + \pi h_2(u).$$

Then on the $\mathfrak{S}$ -basis $\{1, \pi\}$, the Frobenius of $\mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p])$ has the explicit presentation

$$\phi_{\mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p])} = \begin{pmatrix} u^{\frac{e}{2}} & -u^{\left(\frac{p-1}{p}\right)\frac{e}{2}} + 1 \\ 0 & u^{\frac{e}{2}} \end{pmatrix}.$$

Since $\mathfrak{L}$ has dimension 1, $\phi_{\mathfrak{L}}$ acts as semi-linear multiplication by a monomial u^μ on a choice of generating element e of $\mathfrak{L}$. By the commutation of the diagram

$$\begin{array}{ccc} \mathfrak{L} & \hookrightarrow & \mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p]) \\ u^\mu = \phi_{\mathfrak{L}} \downarrow & & \downarrow \phi_{\mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p])} \\ \mathfrak{L} & \hookrightarrow & \mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p]) \end{array}$$

an element $v = (f_1, f_2) \in \mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p])$ lies in $\mathfrak{L}$ if and only if

$$\begin{aligned} u^\mu f_1 &= u^{\frac{e}{2}} f_1^p \\ u^\mu f_2 &= -u^{\left(\frac{p-1}{p}\right)\frac{e}{2}} f_1^p + f_1^p + u^{\frac{e}{2}} f_2^p. \end{aligned}$$

There are exactly two solutions to this system of equations, which may be found and verified empirically:

1. $f_1 = 0$ and $f_2 = 1$: this satisfies $\mu = \frac{e}{2}$ and corresponds to the group scheme $\mathcal{G}[\pi]$ (the latter conclusion is also seen from Lemma 6.3.2).
2. $f_1 = -u^{\frac{e}{2p}}$ and $f_2 = 1$: this satisfies $\mu = \left(\frac{2p-1}{p}\right)\frac{e}{2}$ and by Lemma 6.3.2 is a solution to (6.3.3).

□

Theorem 7.2.2. *Assume that $[E : \mathbb{Q}_p] = 2$ and that p ramifies in E , and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group. Then for any subgroup $\mathcal{H} \subseteq \mathcal{G}[p^n]$ such that $\mathcal{H} \cap \mathcal{G}[\pi_E^r] = 1$ for all $r \geq 0$,*

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^2 = \frac{1}{2p} \left(\frac{1 - p^{-n}}{1 - p^{-1}} \right) \log p.$$

Proof. When $n = 1$, by our hypothesis on $\mathcal{H}$ the equality is just a translation of Proposition 7.2.1 by using Proposition 4.4.5.

When $n > 1$, for any $0 \leq r \leq n$, $\mathcal{H}$ factors into the short exact sequence of finite flat group schemes

$$0 \rightarrow \mathcal{H}_r \rightarrow \mathcal{H} \rightarrow \mathcal{H}_{n-r} \rightarrow 0$$

where $\mathcal{H}_r = \mathcal{H} \cap \mathcal{G}[p^r]$. Iterating this factorization, $\mathcal{H}$ is constructed as an extension of n groups each of p -torsion.

Let $r = 1$. Then the following resolution on $\mathcal{H}_1$

$$0 \rightarrow \mathcal{H}_1 \rightarrow \mathcal{G} \rightarrow \mathcal{G}_1 \rightarrow 0$$

induces the exact sequence of Kisin modules

$$\mathfrak{M}(\mathcal{G}_1) \rightarrow \mathfrak{M}(\mathcal{G}) \rightarrow \mathfrak{M}(\mathcal{H}_1) \rightarrow 0.$$

By Proposition 5.1.5, and Lemma 6.1.9 and Remark 6.1.10 to construct the level-2 model $\mathfrak{M}^2(\mathcal{G})$, there exists a saturated line $\mathfrak{L} \subseteq \mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G})$ so that $\mathfrak{M}^2(\mathcal{H}_1) =$

$\mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G}[p])/\mathfrak{L}$ and which satisfies

$$\phi_{\mathfrak{L}} \equiv u^{\left(\frac{2p-1}{p}\right)\frac{e}{2}} \pmod{p}$$

$$\phi_{\mathfrak{M}(\mathcal{H}_1)} \equiv u^{\frac{e}{2p}} \pmod{p}.$$

Choosing the $\mathfrak{S}$ -basis $\{1, -u^{\frac{e}{2p}} + \pi\}$, we may then write

$$\phi_{\mathfrak{M}_{\mathfrak{S}_1}^2(\mathcal{G})} \equiv \begin{pmatrix} u^{\frac{e}{2p}} & -u^{\left(\frac{p-1}{p}\right)\frac{e}{2}} + 1 \\ 0 & u^{\frac{2p-1}{2p}e} \end{pmatrix} \pmod{p}.$$

Since $\mathcal{H}_1$ has p -torsion, the image of $\mathfrak{M}(\mathcal{G}_1)$ in $\mathfrak{M}^2(\mathcal{G})$ has index p , so that we may write

$$\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_1)} \equiv \begin{pmatrix} u^{\frac{e}{2p}} & -u^{\left(\frac{p-1}{p^2}\right)\frac{e}{2}} + 1 \\ 0 & u^{\frac{2p-1}{2p^2}e} \end{pmatrix} \pmod{p}.$$

Iterating on r ,

$$\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_r)} \equiv \begin{pmatrix} u^{\frac{e}{2p}} & -u^{\left(\frac{p-1}{p^{r+1}}\right)\frac{e}{2}} + 1 \\ 0 & u^{\frac{2p-1}{2p^{r+1}}e} \end{pmatrix} \pmod{p}.$$

Replacing $\mathfrak{M}^2(\mathcal{G})$ by $\mathfrak{M}(\mathcal{G}_r)$ and $\mathcal{H}_1$ by $\mathcal{H}_r/\mathcal{H}_{r-1}$ in Proposition 7.2.1, the result follows by the additivity of the relative Hodge bundle on short exact sequences of finite flat group schemes. $\square$

Consider now $E/\mathbb{Q}_p$ totally ramified and Galois, and assume that K contains all embeddings of E into $\overline{\mathbb{Q}_p}$. Then there exists a factorization

$$P_{K, \pi_K, K_0 \otimes E}(u)e = \prod_{i \in \Phi^c} (u^{\frac{e}{h}} - c_i \pi h_i(u))e$$

such that $c_i \in \mathcal{O}_E^\times$, $\pi = \pi_E$, and $h_i(u) \in W_E[[u]]$ is a polynomial of degree strictly smaller than e/h , where $W_E = W \otimes_{W \cap E} E$. Then as in Example 6.1.6 we can choose

the $\mathfrak{S}$ -basis $\{e_1, \dots, e_h\}$ with $e_i = \pi_E^{i-1}$ so that $\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}[p])}$ has the presentation

$$\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}[p])}(e_i) = \sum_{j=0}^{h-1-i} [P_j(u)] e_{j+i}$$

where the polynomials $P_j(u)$ are defined in Example 6.1.6. Note that, by the condition on the degree of $h_i(u)$, the degree of $P_j(u)$ is strictly dominated by the degree of $P_1(u)$ for each $j > 1$.

For a general saturated submodule, we compute only the following (weak) bound which will be sufficient for establishing the Northcott property, as within any isogeny class there will always only be finitely many primes exhibiting this ramification behavior. Precise formulas are computed in §8.2 for the special case of a totally ramified, cyclic extension $E/\mathbb{Q}_p$ of degree 4.

Theorem 7.2.3. *Let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group and assume that $E/\mathbb{Q}_p$ is totally ramified. Then for any subgroup $\mathcal{H} \subseteq \mathcal{G}[p^n]$ of p -height k such that $\mathcal{H} \cap \mathcal{G}[\pi_E^r] = 1$ for all $r \geq 0$,*

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^k < k \left(\frac{h-d}{h} \right) \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \log p.$$

Proof. By Lemma 6.3.4, *a priori* one has $v_u(\det(\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{H})})) < ek \left(\frac{h-d}{h} \right)$ by our hypothesis on $\mathcal{H}$. To deduce the stronger bound, we lose nothing to assume that $\mathcal{H}$ has constant p^r -height k for all $1 \leq r \leq n$. Saturated rank $h-k$ submodules of $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ have a corresponding saturated rank 1 submodule $\mathfrak{L} \subseteq \bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ with Frobenius

$$\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})} = \bigwedge^{h-k} \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}.$$

Since $\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$ has a representation by an upper triangular matrix as in Example 6.1.6, $\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$ also has a representation by an upper triangular matrix by taking the appropriate tensor power. Moreover, the degree of the polynomial entries in the

resulting matrix are strictly bounded by the degree of the entries on the diagonal, a condition which is imposed from Example 6.1.6.

Let $v = (f_1, \dots, f_{\binom{h}{h-k}}) \in \mathfrak{L} \subseteq \bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ generate a saturated $\mathfrak{S}_1$ -line. Then we may assume $f_\lambda = 1$ for some $1 \leq \lambda \leq \binom{h}{h-k}$. By Proposition 5.1.5, $\mathfrak{L}$ corresponds to a subgroup $\mathcal{H} \subseteq \mathcal{G}[p^n]$ of rank k , and by the degree of the entries of $\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$, we are guaranteed it satisfies our hypothesis as long as $\deg_u(f_i) > 0$ for some $1 \leq i \leq \binom{h}{h-k}$.

Replace the λ th basis element representing $\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$ by v for the above choice of λ . Then we may perform the same devissage argument in Theorem 7.2.2 on the module $\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ using this new basis under the assumption that $\mathcal{H}$ has constant p^r -height k . On applying Proposition 4.4.5, this gives the bound. $\square$

7.3 Subgroups of Products

Fix $P = \prod_{i=1}^N E_i$ to be a (p -adic) CM algebra, and let $\mathcal{O}_{E_i} \subseteq E_i$ be the ring of integers of the CM field E_i with choice of uniformizer π_{E_i} . Define $\mathcal{G}/\mathcal{O}_K$ to be a CM p -divisible group with (p -adic) CM by $(P = \prod_i \mathcal{O}_{E_i}, \Phi_P = \prod_i \Phi_i)$. Then $\mathcal{G} \simeq \prod_i \mathcal{G}_i$, where each $\mathcal{G}_i$ is a (p -adic) CM p -divisible group of dimension d_i with (p -adic) CM by $(\mathcal{O}_{E_i}, \Phi_i)$ corresponding precisely to those pairs found in the decomposition of (P, Φ_P) . We let $h_i = [E_i : \mathbb{Q}_p]$ denote the height of each group and $h_P = \sum_i h_i$ denote the height of $\mathcal{G}$. The goal is to extend the differential computations in §7.1 and §7.2 to subgroups of $\mathcal{G}/\mathcal{O}_K$. Note that those sections concern the case when $\mathcal{G} = \mathcal{G}_i$, and classification of their subgroups is not sufficient as not all subgroups of $\mathcal{G}[p^n]$ decompose into a product of the subgroups of $\mathcal{G}_i$.

We first demonstrate that this problem can be formulated up to isotypicity, and separate the cases when $E/\mathbb{Q}_p$ is unramified and when $E/\mathbb{Q}_p$ is totally ramified and Galois to do so.

Proposition 7.3.1. *Let $\mathcal{G} = \prod_i \mathcal{G}_i$ be a CM p -divisible group having (p -adic) CM*

by $(\prod_{i=1}^N \mathcal{O}_{E_i}, \coprod_i \Phi_i)$ such that $(E_i, \Phi_i) = (E, \Phi)$ for all $1 \leq i \leq N$, where $E/\mathbb{Q}_p$ is unramified and $\Phi = \{\alpha_1, \dots, \alpha_{h-d}\} \subseteq \text{Hom}(E, \overline{\mathbb{Q}_p})$ is a (p -adic) CM type on E . Then if $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ is a simple, saturated submodule of rank $h-k$, there exists a saturated submodule $\mathfrak{N}_1 \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_1)$ of rank $h-k$ such that

$$v_u(\det(\phi_{\mathfrak{N}})) = v_u(\det(\phi_{\mathfrak{N}_1})).$$

Proof. By Theorem 4.3.2, $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}) = \prod_{i=1}^N \mathfrak{M}(\mathcal{G}_i)$ so that $\phi_{\mathfrak{M}(\mathcal{G})} = \prod_{i=1}^N \phi_{\mathfrak{M}(\mathcal{G}_i)}$. Then $\mathfrak{M}(\mathcal{G})$ has a quasi-Kisin decomposition induced by the quasi-Kisin decomposition of each $\mathfrak{M}(\mathcal{G}_i)$ from Lemma 7.1.2 as

$$\mathfrak{M}^{(q)}(\mathcal{G}) = \bigoplus_{\tau \in \text{Hom}(E^{\text{ur}}, K_0)} \mathfrak{M}_{N,\tau}^{(q)}$$

where each $\mathfrak{M}_{N,\tau}^{(q)}$ is a $\mathfrak{S}^{(q)}$ -module of rank N with $\phi^{(q)}$ -Frobenius

$$\phi_{\mathfrak{M}_{N,\tau}^{(q)}}^{(q)}(f_1, \dots, f_N) = (f_1^q, \dots, f_N^q) \left(u^{e \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} + pg_{\tau}(u) \right) \text{Id}_N$$

for some polynomials $g_{\tau}(u) \in \mathfrak{S}^{(q)}$ with degree strictly less than $e \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}$ and Id_N denoting the identity matrix of rank N .

Let $\mathfrak{M}_{\lambda,\iota,\tau}^{(q)} \subseteq \mathfrak{M}_{N,\tau}^{(q)}$ be the quasi-Kisin submodule of rank λ defined so that

$$\phi_{\mathfrak{M}_{\lambda,\iota,\tau}^{(q)}}^{(q)}(f_1, \dots, f_{\lambda}) = (f_1^q, \dots, f_{\lambda}^q) \left(u^{e \sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)}} + pg_{\tau}(u) \right) \text{Id}_{\lambda,\iota}$$

where $\text{Id}_{\lambda,\iota}$ is the identity matrix of rank λ together with an embedding (i.e., a specified selection of basis elements) $\iota : \text{Id}_{\lambda} \hookrightarrow \text{Id}_N$. Then by a slight modification of Corollary 6.2.3 and Proposition 7.1.3, every simple, saturated quasi-Kisin submodule corresponding to a saturated Kisin submodule $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ of rank $h-k$ takes the

form

$$\bigoplus_{\tau \in S_{h-k, \max}} \mathfrak{M}_{\tau}^{(q)} \xrightarrow{\prod_{\tau \in S_{h-k, \max}} \Delta_{\lambda, \iota, \tau}} \bigoplus_{\tau \in S_{h-k, \max}} \mathfrak{M}_{\lambda, \iota, \tau}^{(q)} \subseteq \mathfrak{M}^{(q)}(\mathcal{G})$$

for some collection of pairs (λ, ι) , where $\Delta_{\lambda, \iota, \tau} : \mathfrak{M}_{\tau}^{(q)} \hookrightarrow \mathfrak{M}_{\lambda, \iota, \tau}^{(q)} \subseteq \mathfrak{M}^{(q)}(\mathcal{G})$ denotes the diagonal embedding and $S_{h-k, \max} \subseteq \text{Hom}(E, \overline{\mathbb{Q}}_p)$ is the subset of size $h-k$ which maximizes the sum

$$\sum_{\tau \in S_{h-k}} \left(\sum_{s=1}^{h-d} p^{h-\tau^{-1}(\alpha_s)} \right)$$

over all subsets $S_{h-k} \subseteq \text{Hom}(E, \overline{\mathbb{Q}}_p)$ of size $h-k$. The statement now follows as the determinant for every collection of pairs (λ, ι) is computed by $\bigoplus_{\tau \in S_{h-k, \max}} \mathfrak{M}_{\tau}^{(q)}$. $\square$

Theorem 7.3.2. *Let $\mathcal{G} = \mathcal{G}_1^N$ be a CM p -divisible group having $(p$ -adic) CM by $(\mathcal{O}_E^N, \Phi^N)$, where $E/\mathbb{Q}_p$ is unramified of degree h and $\Phi \subseteq \text{Hom}(E, \overline{\mathbb{Q}}_p)$ is a $(p$ -adic) CM type on E with $h-d$ elements. Let $\mathcal{H} \subseteq \mathcal{G}[p^n]$ be an irreducible subgroup with p -height $k < h$. Then*

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^k \leq \left(\frac{p-1}{p^\delta - 1} \right) \left(\frac{1-p^{-k}}{1-p^{-1}} \right) \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \log p$$

where $\delta = \frac{h}{h-d}$.

Proof. By Theorem 4.3.2 and Proposition 4.4.5, we translate the problem to computing the Harder-Narasimhan slope of irreducible, saturated submodules $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_n}(\mathcal{G}[p^n])$. Performing a devissage on $\mathfrak{N}$ to write it as a composition series of $\mathfrak{S}_1$ -modules, we may apply Proposition 7.3.1 and subsequently Theorem 7.1.6 for the relevant computation. $\square$

Proposition 7.3.3. *Let $\mathcal{G} = \prod_i \mathcal{G}_i$ be a CM p -divisible group having $(p$ -adic) CM by $(\prod_{i=1}^N \mathcal{O}_{E_i}, \prod_i \Phi_i)$ such that $(E_i, \Phi_i) = (E, \Phi)$ for all $1 \leq i \leq N$, where $E/\mathbb{Q}_p$ is totally ramified and Galois and $\Phi \subseteq \text{Hom}(E, \overline{\mathbb{Q}}_p)$ is a $(p$ -adic) CM type on E . Then if $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ is a simple, saturated submodule of rank $h-k$, there exists a saturated*

submodule $\mathfrak{N}_1 \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_1)$ of rank $h - k$ such that

$$v_u(\det(\phi_{\mathfrak{N}})) = v_u(\det(\phi_{\mathfrak{N}_1})).$$

Proof. By Theorem 4.3.2, $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}) = \prod_{i=1}^N \mathfrak{M}(\mathcal{G}_i)$ so that $\phi_{\mathfrak{M}(\mathcal{G})} = \prod_{i=1}^N \phi_{\mathfrak{M}(\mathcal{G}_i)}$. Then $\phi_{\mathfrak{M}(\mathcal{G})}$ has a representation by a block upper triangular matrix where the blocks correspond to $\phi_{\mathfrak{M}(\mathcal{G}_i)}$ by using the matrix representation of Example 6.1.6.

Saturated rank $h - k$ submodules of $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ correspond to saturated $\mathfrak{S}_1$ -lines $\mathfrak{L} \subseteq \bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ with Frobenius

$$\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})} = \bigwedge^{h-k} \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}.$$

Thus, $\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$ has a matrix representation by a block upper triangular matrix where each block representing $\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_i)}$ occurs with N^2 multiplicity (and there are additional blocks). Define $\mathfrak{M}_{N^2,k} \subseteq \bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ to be the saturated $\mathfrak{S}_1$ -submodule of rank N^2 with Frobenius

$$\phi_{\mathfrak{M}_{N^2,k}} = \prod_{i=1}^{N^2} \phi_{\bigwedge^{h-k} \mathfrak{M}(\mathcal{G}_1)}.$$

Let $\mathfrak{M}_{\lambda,\iota,k} \subseteq \mathfrak{M}_{N^2,k}$ be the $\mathfrak{S}_1$ -submodule of rank λ with Frobenius

$$\phi_{\mathfrak{M}_{\lambda,\iota,k}} = \prod_{i=\iota(1)}^{\iota(\lambda)} \phi_{\bigwedge^{h-k} \mathfrak{M}(\mathcal{G}_1)}$$

where $\iota : \{1, \dots, \lambda\} \hookrightarrow \{1, \dots, N^2\}$ represents the embedding $\mathfrak{M}_{\lambda,\iota,k} \hookrightarrow \mathfrak{M}_{N^2,k}$. Then by the proof of Theorem 7.2.3, every simple, saturated Kisin submodule of $\mathfrak{M}(\mathcal{G})$ of rank $h - k$ corresponds to a module of the form

$$\mathfrak{L} \subseteq \bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_1) \xrightarrow{\Delta_{\lambda,\iota,k}} \mathfrak{M}_{\lambda,\iota,k} \subseteq \bigwedge^{h-k} \mathfrak{M}(\mathcal{G})$$

for some pair (λ, ι) , where $\Delta_{\lambda, \iota, k} : \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_1) \hookrightarrow \mathfrak{M}_{\lambda, \iota, k}$ denotes the diagonal embedding and $\mathfrak{L}$ is saturated of rank 1. The statement now follows as the determinant for every pair (λ, ι) is computed by $\mathfrak{L}$. $\square$

Theorem 7.3.4. *Let $\mathcal{G} = \mathcal{G}_1^N$ be a CM p -divisible group having $(p$ -adic) CM by $(\mathcal{O}_E^N, \Phi^N)$, where $E/\mathbb{Q}_p$ is totally ramified and Galois of degree h and $\Phi \subseteq \text{Hom}(E, \overline{\mathbb{Q}_p})$ is a $(p$ -adic) CM type on E with $h - d$ elements. Let $\mathcal{H} \subseteq \mathcal{G}[p^n]$ be an irreducible subgroup with p -height $k < h$ such that $\mathcal{H} \cap \prod_i' \mathcal{G}_1[\pi_E^{r_i}] = 1$ for all $r_i \geq 0$ and subproducts $\prod_i' \mathcal{G}_1[\pi_E^{r_i}] \subseteq \mathcal{G}_1[\pi_E^{ne}]^N$. Then*

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^k < k \left(\frac{h - d}{h} \right) \left(\frac{1 - p^{-n}}{1 - p^{-1}} \right) \log p.$$

Proof. By Theorem 4.3.2 and Proposition 4.4.5, we translate the problem to computing the Harder-Narasimhan slope of irreducible, saturated submodules $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_n}(\mathcal{G}[p^n])$. Performing a devissage on $\mathfrak{N}$ to write it as a composition series of $\mathfrak{S}_1$ -modules, we may apply Proposition 7.3.3 and subsequently Theorem 7.2.3 for the relevant computation. $\square$

Now we consider more generally when the E_i in the decomposition of P are allowed to be either unramified or totally ramified and Galois. We further let $I_{\text{ur}, k}$, resp., $I_{\text{ram}, k}$, to denote the set of indices i for which $E_i/\mathbb{Q}_p$ are unramified, resp., ramified, and $[E_i : \mathbb{Q}_p] > h - k$.

Proposition 7.3.5. *Let $\mathcal{G} = \prod_i \mathcal{G}_i$ be a CM p -divisible group having $(p$ -adic) CM by $(\prod_{i=1}^N \mathcal{O}_{E_i}, \prod_i \Phi_i)$ such that each pair (E_i, Φ_i) is distinct and $E_i/\mathbb{Q}_p$ is either unramified or totally ramified and Galois. Then if $\mathfrak{N} \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ is a simple, saturated submodule of rank $h - k$, there exists a saturated submodule $\mathfrak{N}_i \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_i)$ of rank $h - k$ for some i such that*

$$v_u(\det(\phi_{\mathfrak{N}})) = v_u(\det(\phi_{\mathfrak{N}_i})).$$

Proof. By Theorem 4.3.2, $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}) = \prod_{i=1}^N \mathfrak{M}(\mathcal{G}_i)$ so that $\phi_{\mathfrak{M}(\mathcal{G})} = \prod_{i=1}^N \phi_{\mathfrak{M}(\mathcal{G}_i)}$. Then $\phi_{\mathfrak{M}(\mathcal{G})}$ has a representation as a block diagonal matrix. We may assume that $[E_i : \mathbb{Q}_p] > h - k$ for every index i and write by I_{ur} and I_{ram} the set of indices for which correspond to those $i \in I$ for which $E_i/\mathbb{Q}_p$ is unramified or ramified, respectively.

Suppose first that I_{ram} is empty. Then for $q = \text{lcm}_{i \in I_{\text{ur}}} \{p^{h_i}\}$, $\mathfrak{M}(\mathcal{G})$ has a quasi-Kisin decomposition induced by the quasi-Kisin decomposition of each $\mathfrak{M}(\mathcal{G}_i)$ for $i \in I_{\text{ur}}$ from Lemma 7.1.2 as

$$\mathfrak{M}^{(q)}(\mathcal{G}) = \bigoplus_{i \in I_{\text{ur}}} \bigoplus_{\tau \in \text{Hom}(E_i, K_0)} \mathfrak{M}_{i, \tau}^{(q)}(\mathcal{G}_i).$$

Whenever $\mathcal{G}_i$ is not a Serre tensor construction of $\mathcal{G}_j$ for any distinct pair of indices i and j , the slopes in the decomposition are all unique, and a modification of Corollary 6.2.3 gives $\mathfrak{N} = \mathfrak{N}_i \subseteq \mathfrak{M}(\mathcal{G}_i)$ for some saturated submodule. Otherwise, the method of considering diagonal submodules for Serre tensor constructions works identically to isotypicity in Proposition 7.3.1. Then $\mathfrak{N}_i$ is the projection to a factor in a diagonal embedding if necessary.

If I_{ur} is empty, then $\mathfrak{N}$ corresponds to a saturated line in $\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ with Frobenius

$$\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})} = \bigwedge^{h-k} \phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}.$$

Thus $\phi_{\bigwedge^{h-k} \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$ has a matrix representation by a block upper triangular matrix. Again whenever $\mathcal{G}_i$ is not a Serre tensor construction of $\mathcal{G}_j$ for any distinct pair of indices i and j , all potential semi-linear eigenvalues are unique and $\mathfrak{N} = \mathfrak{N}_i$ for some i . Otherwise, the method of considering diagonal submodules for Serre tensor constructions works identically to isotypicity in Proposition 7.3.3. Then $\mathfrak{N}_i$ is the projection to a factor in a diagonal embedding if necessary.

In general, the valuation of $\mathfrak{N}$ can be computed by one of these two instances. $\square$

Theorem 7.3.6. *Let $\mathcal{G} \simeq \prod_{i=1}^N \mathcal{G}_i$ where $\mathcal{G}$ has (p -adic) CM by (E, Φ) and $\mathcal{G}_i$ are each*

$\mathcal{O}_{E_i}$ -linear CM p -divisible groups of type $(\mathcal{O}_{E_i}, \Phi_i)$, where each E_i is either unramified or totally ramified and Galois. Then for any simple subgroup $\mathcal{H} \subseteq \mathcal{G}[p^n]$ of p -height $k < h$, such that $\mathcal{H} \cap \prod' \mathcal{G}_i[\pi_{E_i}^{r_i}] = 0$ for all $1 \leq r_i \leq n$ and subproducts $\prod' \mathcal{G}_i[\pi_{E_i}^{r_i}] \subseteq \prod_i \mathcal{G}_i[\pi_{E_i}^{r_i}]$,

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^k = \# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}_i/\mathcal{O}_K}^k.$$

Proof. We may assume that each pair (E_i, Φ) in the decomposition of the (p -adic) CM pair (P, Φ_P) is distinct (so the fields E_i can repeat, but then their CM types must differ) by Proposition 7.3.1 and Proposition 7.3.3. Then by Proposition 7.3.5, there is a saturated submodule $\mathfrak{N}_i \subseteq \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}_i)$ of rank $h_i - k$ by which we can compute the determinant. This module corresponds to a subgroup $\mathcal{H}_i \subseteq \mathcal{G}_i$ for some i by Theorem 4.3.2, which gives the statement. $\square$

You have wakened not out of sleep, but into a prior dream, and that dream lies within another, and so on, to infinity, which is the number of grains of sand. The path that you are to take is endless, and you will die before you have truly awakened.

José Luis Borges, **Labyrinths**

8

Main Theorems

8.1 CM Northcott Property

We prove the two versions of the CM Northcott property stated as Theorem 1.1.1 and Theorem 1.1.4 in the introduction.

Fix a CM field E with $[E : \mathbb{Q}] = 2g$ and let $F \subseteq E$ be its maximal totally real subfield. Denote by $\mathcal{O}_E \subseteq E$ the corresponding ring of integers and $\mathcal{O}_n \subseteq \mathcal{O}_E$ a non-maximal order of index n in $\mathcal{O}_E$. Let A and A_n be abelian varieties defined over a number field K both of dimension g having, respectively, CM by $(\mathcal{O}_E, \Phi)$ and $(\mathcal{O}_n, \Phi)$, and let $\mathcal{A}$ and $\mathcal{A}_n$ denote their corresponding Néron models over $\mathcal{O}_K$. Then

one may construct an isogeny $\phi : \mathcal{A} \rightarrow \mathcal{A}_n$ whose kernel is a finite flat group scheme $\mathcal{G}_n$ defined over $\mathcal{O}_K$ having size n .

Lemma 8.1.1. *Let A and A_{p^r} be abelian varieties over a number field K as above, and let $\phi : A \rightarrow A_{p^r}$ be an isogeny of degree p^r over K whose kernel is p^n -torsion (so $r \geq n$). Assume moreover that p factors in E as a finite product $p = \prod_i \mathfrak{p}_i^{\nu_i} \subseteq E$ with $\nu_i > 0$ and $\text{Nm}(\mathfrak{p}_i) = p^{f_i}$.*

1. *If p splits completely in E , i.e., $\nu_i = 1$ and $f_i = 1$ for each i , then*

$$h_{\text{Fal}}(A_{p^r}) - h_{\text{Fal}}(A) = \frac{r}{2} \log p.$$

2. *If all primes above p are unramified along the CM extension E/F , then*

$$h_{\text{Fal}}(A_{p^r}) - h_{\text{Fal}}(A) \geq \left[\frac{r}{2} - \sum_{i: f_i > 1} \nu_i \left(\frac{p-1}{p^{f_i}-1} \right) \left(\frac{1-p^{-k_i}}{1-p^{-1}} \right) \left(\frac{1-p^{-n_i}}{1-p^{-1}} \right) \right] \log p$$

where $n = \sum_i n_i$ and $k = \sum_i k_i$ are determined by the decomposition $A[p^n] \simeq \prod_i A[\mathfrak{p}_i^{\nu_i n}]$.

3. *If all primes above p are ramified along the CM extension E/F , and we define the Galois closure $\tilde{E}_{\mathfrak{p}_i}$ of $E_{\mathfrak{p}_i}$ to have relative degree $\rho_i = [\tilde{E}_{\mathfrak{p}_i} : E_{\mathfrak{p}_i}]$, then*

$$h_{\text{Fal}}(A_{p^r}) - h_{\text{Fal}}(A) > \left[\frac{r}{2} - \sum_i \frac{1}{\rho_i} \left(\frac{p-1}{p^{f_i}-1} \right) \left(\frac{h_i - d_i}{h_i} \right) \left(\frac{1-p^{-n_i}}{1-p^{-1}} \right) \right] \log p$$

where $n = \sum_i n_i$, $g = \sum_i d_i$, and $h = \sum_i h_i$ are determined by the decomposition $A[p^n] \simeq \prod_i A[\mathfrak{p}_i^{\nu_i n}]$.

4. *For a general prime p , let I_2 (resp., I_3) denote the set of indices i such that $\mathfrak{p}_i$ lies over a prime that is unramified (resp., ramified) along the CM extension*

E/F . Then, retaining the notation from cases (2) and (3),

$$h_{\text{Fal}}(A_{p^r}) - h_{\text{Fal}}(A) > \frac{r}{2} - \left[\sum_{i \in I_2: f_i > 1} \nu_i \left(\frac{p-1}{p^{f_i}-1} \right) \left(\frac{1-p^{-k_i}}{1-p^{-1}} \right) \left(\frac{1-p^{-n_i}}{1-p^{-1}} \right) \right. \\ \left. + \sum_{i \in I_3} \frac{1}{\rho_i} \left(\frac{p-1}{p^{f_i}-1} \right) \left(\frac{h_i-d_i}{h_i} \right) \left(\frac{1-p^{-n_i}}{1-p^{-1}} \right) \right] \log p$$

where the terms are determined by the decomposition $A[p^n] \simeq \prod_i A[\mathfrak{p}_i^{\nu_i n}]$ as in statements (2) and (3).

Proof. After a reduction to a local problem, the proof is an exercise in piecing together the decomposition theorems in §6.2 and the computations in §7. First, ϕ decomposes along the factorization $p = \prod_i \mathfrak{p}_i^{\nu_i}$ into a composition of isogenies ϕ_i each of degree $p^{f_i \nu_i}$. Since the $\mathfrak{p}_i$ in this decomposition are relatively prime, there is no canonical sequence for this decomposition, so the order may be permuted arbitrarily and we get a composition series for the finite flat group scheme kernel of ϕ . We lose nothing in this computation to assume that the kernel of ϕ splits along the decomposition of p , and otherwise apply Theorem 7.3.6, which allows us to project along this factorization and compute based on an appropriate projection. Then, as Lemma 2.1.6 is additive on exact sequences of finite flat group schemes, we may compute everything for the finite flat subgroup schemes of each of the finite flat group schemes $\mathcal{A}[\mathfrak{p}_i^{\nu_i n}]$.

Suppose first that p splits completely as in statement (1). Then by Lemma 6.3.4, the isogeny ϕ must be étale for otherwise we contradict our hypothesis on A_{p^n} . Thus, the relative Hodge bundle is trivial and has no contribution to the variation of the Faltings height.

Assume now that no prime above p ramifies in the CM extension E/F as in statement (2). Then the relative Hodge bundle at each ϕ_i may be computed by the general formula in Corollary 6.2.6, which demonstrates how to incorporate the ramification degree, and the bound in Theorem 7.1.6.

Now assume that all primes above p do ramify in the CM extension E/F as in statement (3). Then the relative Hodge bundle at each ϕ_i may be computed by the general formula in Corollary 6.2.8, which demonstrates how to incorporate the degree of the Galois extension, Corollary 6.2.4, which incorporates the inertia degree, and the bound in Theorem 7.2.3.

For a general p as in statement (4) we combine preceding cases. $\square$

We first prove Theorem 1.1.1.

Theorem 8.1.2 (CM Northcott Property for Isogeny Classes). *Let C be a fixed positive constant and $g \geq 1$ be a fixed integer. Then the number of isomorphism classes of CM abelian varieties A of dimension g with $h_{\text{Fal}}(A) < C$ within an isogeny class is finite.*

Proof. By the Poincaré reducibility theorem, every CM abelian variety is isogenous to one that is a product of simple abelian varieties each with CM by $(\mathcal{O}_{E_i}, \Phi_i)$ for some CM field $E_i/\mathbb{Q}$. Therefore we fix $A \simeq \prod_i A_i$ where each factor A_i is simple with CM by $(\mathcal{O}_{E_i}, \Phi_i)$. By Lemma 6.3.2, the Faltings height variation is zero precisely when the group scheme kernel of the isogeny is $\prod_i \mathcal{O}_{E_i}$ -stable, i.e., when the isogeny is between two CM abelian varieties both with CM by $(\prod_i \mathcal{O}_{E_i}, \prod_i \Phi_i)$. Since there are finitely many such isomorphism classes of CM abelian varieties by a theorem of Brauer–Siegel, the theorem reduces to counting pairs $(A, \mathcal{G}_n)$ where $\mathcal{G}_n \subseteq \mathcal{A}$ is a finite flat subgroup scheme which is not $\mathcal{O}_E$ -stable and has prescribed order n .

The computations in §7.1, §7.2, and §7.3 demonstrate the theorem. Indeed, since the Hodge bundle is additive on short exact sequences and the isogeny decomposes along the primary decomposition of n in $\prod_i E_i$, we may always assume that $n = p^s$ for some prime p and $s \geq 1$. Lemma 8.1.1, which may be strengthened to all subgroups of products of simple abelian varieties by Theorem 7.3.6 and the reductions in §6.2 to the cases of totally ramified or totally unramified (p -adic) CM fields, summarizes

the computations within each p -isogeny class and allows us to conclude. $\square$

We now prove the (conditional) Theorem 1.1.4.

Theorem 8.1.3 (CM Northcott Property). *Let C be a fixed positive constant and $g \geq 1$ be a fixed integer. Then assuming Conjecture 2.2.1 and the Artin Conjecture, the number of CM abelian varieties of dimension g with $h_{\text{Fal}}(A) < C$ is finite.*

Proof. Theorem 8.1.2 demonstrates finiteness within an isogeny class. By the Poincaré reducibility theorem, every isogeny class contains an abelian variety $A \simeq \prod_i A_i$ unique up to isomorphism where each factor A_i is simple with CM by $(\mathcal{O}_{E_i}, \Phi_i)$. Since the Faltings height is additive on products of abelian varieties, we conclude by Corollary 2.2.3, which relies on the conjectures of Colmez and Artin. $\square$

Remark 8.1.4. When we consider only the category of *simple* CM abelian varieties, we can describe this growth in terms of the *conductor* of the order. We recall that the conductor is the smallest ideal that is an ideal of both $\mathcal{O}_E$ and $\mathcal{O}$. In particular, the conductor always divides the degree, so to count along the conductor, we can count along those finite flat group schemes whose order divides the conductor with the smallest HN slope.

8.2 Colmez-Type Formulas

We make the computations precise to give a few intrinsic cases of Colmez-type formulas for abelian varieties of low dimension. Our first result is to recover the formula of Nakkajima-Taguchi [31] in the case of elliptic curves. Here we are once again in a global setting, and will let E denote a CM field with ring of integers $\mathcal{O}_E$ and order $\mathcal{O}_n \subseteq \mathcal{O}_E$ of index n such that $\mathcal{O}_E/\mathcal{O}_n$ has no $\mathcal{O}_E$ -stable fixed piece.

Theorem 8.2.1 (Colmez Formula for CM Elliptic Curves). *Let A_n be an elliptic curve which has CM by $(\mathcal{O}_n, \Phi)$ and let A be an elliptic curve which has CM by*

$(\mathcal{O}_E, \Phi)$. Then

$$h_{\text{Fal}}(A_n) = h_{\text{Fal}}(A) + \frac{1}{2} \left(\sum_{p|n} \log p \left(r_p - \left(\frac{1 - \left(\frac{p}{d}\right)}{p - \left(\frac{p}{d}\right)} \right) \left(\frac{1 - p^{-r_p}}{1 - p^{-1}} \right) \right) \right)$$

where r_p is the largest exponent such that $p^{r_p} | n$.

Combining this with Conjecture 2.2.1 (a theorem in the case of elliptic curves), we obtain the formula

$$\begin{aligned} h_{\text{Fal}}(A_n) = & -\frac{L'(\chi_{-d}, 0)}{L(\chi_{-d}, 0)} - \frac{1}{2} \log d + \frac{1}{2} \log 2\pi \\ & + \frac{1}{2} \left(\sum_{p|n} \log p \left(r_p - \left(\frac{1 - \left(\frac{p}{d}\right)}{p - \left(\frac{p}{d}\right)} \right) \left(\frac{1 - p^{-r_p}}{1 - p^{-1}} \right) \right) \right) \end{aligned}$$

where d is the discriminant of E and χ_{-d} the corresponding quadratic Artin character.

Proof. By the same argument as in Theorem 8.1.3, we may assume that $n = p^{r_p}$ and study the problem locally. Since A is an elliptic curve, the trichotomy of the splitting behavior of p in the CM field E (encoded by the quadratic residue character) determines the structure of the finite flat group scheme kernel for the minimal degree isogeny between A and A_n , and therefore the Faltings height.

When p splits, Deuring's theorem shows that the isogeny is necessarily étale and the only contribution to the Faltings height variation is the degree of the isogeny. When p is inert, the differential in the calculation of this variation necessarily attains the maximum along the bound in Theorem 7.1.6 with $k = 1$ and $\delta = 2$, since A is an elliptic curve and, up to complex conjugation, admits a unique CM type. When p is ramified, the computation has already been done in Theorem 7.2.2. $\square$

We extend this formula to simple abelian surfaces for which $E/\mathbb{Q}$ is Galois. We note that by this hypothesis $E/\mathbb{Q}$ must be cyclic, and there can even only be one CM type (up to complex conjugation) on these abelian surfaces, given by $\Phi = \{1, 2\}$.

These assertions are checked explicitly in [19]. The local computations for this theorem are found in §7 and Appendix A.

Theorem 8.2.2 (Colmez Formula for Cyclic CM Abelian Surfaces). *Let A be a simple CM abelian surface over a number field K with CM by $(\mathcal{O}_E, \Phi)$ where $E/\mathbb{Q}$ is a cyclic, Galois CM field of degree 4 and $F \subseteq E$ its totally real subfield. Let A_{p^n}/K be a simple CM abelian surface with CM by $(\mathcal{O}, \Phi)$ where $\mathcal{O} \subseteq \mathcal{O}_E$ is a non-maximal order with conductor divisible by p , and an isogeny $\phi : A \rightarrow A_{p^n}$ has minimal degree p^n . Let $\Delta_{p^n} = h_{\text{Fal}}(A_{p^n}) - h_{\text{Fal}}(A)$. There are nine cases:*

1. *If $p = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3 \mathfrak{p}_4$ is totally split in $\mathcal{O}_E$, then*

$$\Delta_{p^n} = \frac{n}{2} \log p.$$

2. *If $p = \mathfrak{p}_1 \mathfrak{p}_2$ in $\mathcal{O}_E$ is such that p remains inert in $F/\mathbb{Q}$ and the prime above p splits in E/F , then*

$$\Delta_{p^n} = \frac{n}{2} \log p.$$

3. *$p = \mathfrak{p}_1^2 \mathfrak{p}_2^2$ in $\mathcal{O}_E$ is such that p ramifies in $F/\mathbb{Q}$ and the prime above p splits in E/F , then*

$$\Delta_{p^n} = \frac{n}{2} \log p.$$

4. *If $p = \mathfrak{p}_1 \mathfrak{p}_2$ in $\mathcal{O}_E$ is such that p splits in $F/\mathbb{Q}$ and each $\mathfrak{p}_i$ remains inert in E/F ,*

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{1}{p+1} \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \right] \log p.$$

5. *If $p = \mathfrak{p}_1^2 \mathfrak{p}_2^2$ in $\mathcal{O}_E$ is such that p splits in $F/\mathbb{Q}$ and each $\mathfrak{p}_i$ ramifies in E/F , then*

$$h_{\text{Fal}}(A_{p^n}) = h_{\text{Fal}}(A) + \left[\frac{n}{2} - \frac{1}{2p} \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \right] \log p.$$

6. If $p = \mathfrak{p}^2$ in $\mathcal{O}_E$ is such that p ramifies in $F/\mathbb{Q}$ and $\mathfrak{p}$ remains inert in E/F ,

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{2}{p+1} \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \right] \log p.$$

7. If $p = \mathfrak{p}^2$ in $\mathcal{O}_E$ such that p remains inert in $F/\mathbb{Q}$ and the prime above p ramifies in E/F ,

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{1}{p(p+1)} \left(\frac{1-p^{-n}}{1-p^{-1}} \right) \right] \log p.$$

8. If $p = \mathfrak{p}$ is inert in $\mathcal{O}_E$, let $(\lambda_1, \lambda_2, \lambda_3)$ be a non-increasing tuple characterizing $\mathcal{G}$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = n$, $\mathcal{G}$ is p^{λ_1} -torsion, and the p -height $\iota \in \{1, 2, 3\}$ of $\mathcal{G}$ is the largest integer such that $\lambda_\iota \neq 0$. Then

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{p-1}{p^4-1} \left(\sum_{i=\lambda_2+1}^{\lambda_1} \frac{p+1}{p^i} \right) \left(\sum_{j=\lambda_3+1}^{\lambda_2} \frac{(p+1)^2}{p^j} \right) \left(\sum_{k=1}^{\lambda_3} \frac{(p^2+2)(p+1)}{p^k} \right) \right] \log p$$

where the sums are evaluated if and only if the difference in bounds is at least zero.

9. If $p = \mathfrak{p}^4$ is totally ramified in $\mathcal{O}_E$, let $(\lambda_1, \lambda_2, \lambda_3)$ be a non-increasing tuple characterizing $\mathcal{G}$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = n$, $\mathcal{G}$ is p^{λ_1} -torsion, and the p -height $\iota \in \{1, 2, 3\}$ of $\mathcal{G}$ is the largest integer such that $\lambda_\iota \neq 0$. Then

$$\Delta_{p^n} = \left[\frac{n}{2} - \frac{1}{4(p^4-p^3)} \left(\sum_{i=\lambda_2+1}^{\lambda_1} \frac{R_1(p)}{p^i} \right) \left(\sum_{j=\lambda_3+1}^{\lambda_2} \frac{R_2(p)}{p^j} \right) \left(\sum_{k=1}^{\lambda_3} \frac{R_3(p)}{p^k} \right) \right] \log p$$

where the sums are evaluated if and only if the difference in bounds is at least zero, and $R_i(p)$ are defined in Theorem A.0.10.

Proof. We will denote by $\mathcal{A}/\mathcal{O}_K$ the Néron model of A/K , and, for any prime $\mathfrak{p} \subseteq \mathcal{O}_E$, $\mathcal{G}_{\mathfrak{p}} = \mathcal{A}[\mathfrak{p}^\infty]$ the corresponding CM p -divisible group with (p -adic) CM type $(E_{\mathfrak{p}}, \Phi_{\mathfrak{p}})$. We will let $\mathcal{G}$ denote the finite flat group scheme kernel of $\phi : \mathcal{A} \rightarrow \mathcal{A}_{p^n}$ and

$\omega_{\mathcal{G}} = s^* \Omega_{\mathcal{G}/\mathcal{O}_K}^1$ denote the relative Hodge bundle. By Lemma 2.1.6, we then obtain the change in Faltings height as a computation of $\omega_{\mathcal{G}}$.

In (1), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}_1} \times \mathcal{G}_{\mathfrak{p}_2} \times \mathcal{G}_{\mathfrak{p}_3} \times \mathcal{G}_{\mathfrak{p}_4}$ and $\mathcal{G}_{\mathfrak{p}_1} \simeq \mathcal{G}_{\mathfrak{p}_2}^\vee$ and $\mathcal{G}_{\mathfrak{p}_3} \simeq \mathcal{G}_{\mathfrak{p}_4}^\vee$. Since p splits in E/F , we may suppose $\mathcal{G}_{\mathfrak{p}_1}$ and $\mathcal{G}_{\mathfrak{p}_3}$ are CM p -divisible groups with $d = 0$ and $h = 1$, and $\mathcal{G}_{\mathfrak{p}_2}$ and $\mathcal{G}_{\mathfrak{p}_4}$ are CM p -divisible groups with $d = 1$ and $h = 1$. Then $\mathcal{G}$ must be a subgroup of $\mathcal{G}_{\mathfrak{p}_1}[p^n] \times \mathcal{G}_{\mathfrak{p}_3}[p^n]$, otherwise by Lemma 6.3.4 A_{p^n} cannot have CM by an order whose conductor divides p . Since $\mathcal{G}_{\mathfrak{p}_1}$ and $\mathcal{G}_{\mathfrak{p}_3}$ are étale, $\omega_{\mathcal{G}} = 0$.

In (2), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}_1} \times \mathcal{G}_{\mathfrak{p}_2}$ and $\mathcal{G}_{\mathfrak{p}_1} \simeq \mathcal{G}_{\mathfrak{p}_2}^\vee$. Since p is inert in $F/\mathbb{Q}$, $E_{\mathfrak{p}_i}/\mathbb{Q}_p$ is unramified for each $\mathfrak{p}_i$. However, since the prime above p splits in E/F , we may suppose $\mathcal{G}_{\mathfrak{p}_1}$ is a CM p -divisible group with $d = 0$ and $h = 2$, and $\mathcal{G}_{\mathfrak{p}_2}$ is a CM p -divisible group with $d = 2$ and $h = 2$. Then $\mathcal{G}$ must be a subgroup of $\mathcal{G}_{\mathfrak{p}_1}[p^n]$, otherwise by Lemma 6.3.4 A_{p^n} cannot have CM by an order whose conductor divides p . Since $\mathcal{G}_{\mathfrak{p}_1}$ is étale, $\omega_{\mathcal{G}} = 0$.

In (3), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}_1} \times \mathcal{G}_{\mathfrak{p}_2}$ and $\mathcal{G}_{\mathfrak{p}_1} \simeq \mathcal{G}_{\mathfrak{p}_2}^\vee$. Since p is ramified in $F/\mathbb{Q}$, $E_{\mathfrak{p}_i}/\mathbb{Q}_p$ is totally ramified and Galois for each $\mathfrak{p}_i$. However, since the prime above p splits in E/F , we may suppose $\mathcal{G}_{\mathfrak{p}_1}$ is a CM p -divisible group with $d = 0$ and $h = 2$, and $\mathcal{G}_{\mathfrak{p}_2}$ is a CM p -divisible group with $d = 2$ and $h = 2$. Then $\mathcal{G}$ must be a subgroup of $\mathcal{G}_{\mathfrak{p}_1}[p^n]$, otherwise by Lemma 6.3.4 A_{p^n} cannot have CM by an order whose conductor divides p . Since $\mathcal{G}_{\mathfrak{p}_1}$ is étale, $\omega_{\mathcal{G}} = 0$.

In (4), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}_1} \times \mathcal{G}_{\mathfrak{p}_2}$ and $\mathcal{G}_{\mathfrak{p}_1} \simeq \mathcal{G}_{\mathfrak{p}_2}$. Since the primes above p are inert in E/F , $E_{\mathfrak{p}_i}/\mathbb{Q}_p$ is unramified for each $\mathfrak{p}_i$, and $\mathcal{G}_{\mathfrak{p}_i}$ are each CM p -divisible groups with $d = 1$ and $h = 2$. By Proposition 7.3.1, we lose nothing to assume $\mathcal{G} \subseteq \mathcal{G}_{\mathfrak{p}_1}[p^n]$ in order to compute $\omega_{\mathcal{G}}$, and thus may directly apply Theorem 8.2.1.

In (5), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}_1} \times \mathcal{G}_{\mathfrak{p}_2}$ and $\mathcal{G}_{\mathfrak{p}_1} \simeq \mathcal{G}_{\mathfrak{p}_2}$. Since the primes above p are ramified in E/F , $E_{\mathfrak{p}_i}/\mathbb{Q}_p$ is totally ramified and Galois for each $\mathfrak{p}_i$, and $\mathcal{G}_{\mathfrak{p}_i}$ are each CM p -divisible groups with $d = 1$ and $h = 2$. By Proposition 7.3.3, we lose nothing to assume $\mathcal{G} \subseteq \mathcal{G}_{\mathfrak{p}_1}[p^n]$, and thus may directly apply Theorem 8.2.1.

In (6), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}}$. Since the prime above p is inert in E/F and p is ramified in $F/\mathbb{Q}$, there exists a degree 2 unramified subfield $E^{\text{ur}} \subseteq E_{\mathfrak{p}}$ such that $\mathcal{G}_{\mathfrak{p}} \simeq \mathcal{G}_{\mathfrak{p}}^{\text{ur}} \otimes_{\mathcal{O}_{E^{\text{ur}}}} \mathcal{O}_{E_{\mathfrak{p}}}$ and $\mathcal{G}_{\mathfrak{p}}^{\text{ur}}$ is a CM p -divisible group with $d = 1$ and $h = 2$. Then by Corollary 6.2.6, we lose nothing to study subgroups of $\mathcal{G}_{\mathfrak{p}}^{\text{ur}}$ in order to compute $\omega_{\mathcal{G}}$, and thus may apply Theorem 8.2.1 in conjunction with Corollary 6.2.6.

In (7), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}}$. Since p is ramified in E/F and inert in $F/\mathbb{Q}$, by Corollary 6.2.4 we lose nothing to study subgroups of a CM p -divisible group $\mathcal{G}'_{\mathfrak{p}}$ with $d = 1$ and $h = 2$ having (p -adic) CM by a totally ramified degree 2 extension of $\mathbb{Q}_p$, as by Corollary 6.2.3 we may recover this information to compute $\omega_{\mathcal{G}}$. We thus apply Theorem 8.2.1 to study the subgroups of $\mathcal{G}'_{\mathfrak{p}}[p^n]$.

In (8), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}}$. Since p remains inert in $E/\mathbb{Q}$, $E_{\mathfrak{p}}/\mathbb{Q}_p$ is unramified and $\mathcal{G}_{\mathfrak{p}}$ is a CM p -divisible group with $d = 2$ and $h = 4$. By the assumption that $E/\mathbb{Q}$ is cyclic, we may further deduce that $\Phi = \{1, 2\}$ as the decomposition group embeds into the Galois group, and the unique involutive element of $\text{Gal}(E/\mathbb{Q})$ must conjugate Φ to the opposite set of embeddings in $\text{Hom}(E, \overline{\mathbb{Q}})$. Then we can directly apply Proposition 7.1.4 using $(E_{\mathfrak{p}}, \Phi_{\mathfrak{p}} = \{1, 2\})$.

In (9), $\mathcal{A}[p^\infty] \simeq \mathcal{G}_{\mathfrak{p}}$. Since p is totally ramified in E , $E_{\mathfrak{p}}/\mathbb{Q}_p$ is totally ramified and Galois, and $\mathcal{G}_{\mathfrak{p}}$ is a CM p -divisible group with $d = 2$ and $h = 4$. By the assumption that $E/\mathbb{Q}$ is cyclic, we may further deduce that $\Phi_{\mathfrak{p}} = \{1, 2\}$. Then this computation is done in Theorem A.0.10. □



Ramification Computations for Cyclic Abelian Surfaces

The computations presented here provide the analogue of Theorem 7.2.2 for certain $\mathcal{O}_E$ -linear Kisin modules that correspond to abelian surfaces.

Here we let k be a finite field of characteristic $p > 0$ and $W = W(k)$ be its ring of Witt vectors. We construct the ring $\mathfrak{S} = W[[u]]$ which has a natural Frobenius homomorphism ϕ extending the Frobenius on W by sending $u \mapsto u^p$. We also let $\mathfrak{S}_n = W_n[[u]]$, where W_n is the n th truncation of the Witt construction of W ; $\mathfrak{S}_n$ is likewise equipped with a Frobenius which we also denote by ϕ extending the Frobenius

on W_n by $u \mapsto u^p$. We refer the reader to Definition 4.3.1 for the definition of an $\mathfrak{S}$ -module and to Proposition 4.3.9 for their reduction to $\mathfrak{S}_n$ -modules.

Let (E, Φ) be a $(p$ -adic) CM pair with $E/\mathbb{Q}_p$ a totally ramified, Galois field of degree h and Φ a $(p$ -adic) CM type. In most of this section we will let $h = 4$ and $E/\mathbb{Q}_p$ be cyclic with $(p$ -adic) CM type $\{1, 2\}$. In Definition 6.1.3 we introduced the notion of an $\mathcal{O}_E$ -linear CM Kisin module, where $\mathcal{O}_E \subseteq E$ is the ring of integers, a definition which is fundamental to the computations in this appendix. We invite the reader to first look at the fundamental structure theorems regarding these modules, most namely Proposition 6.1.7 and Lemma 6.1.9.

The following lemma is fundamental.

Lemma A.0.1. $\mathbb{Q}_p$ admits a totally ramified cyclic extension of degree 4 only if $p \equiv 1 \pmod{4}$ or $p = 2$.

Proof. The Galois group of a finite, ramified, cyclic extension of $\mathbb{Q}_p$ is isomorphic to a finite quotient of $\mathbb{Z}_p^\times$. If $p \neq 2$, $\mathbb{Z}_p^\times \simeq \mu_{p-1} \times 1 + p\mathbb{Z}_p$, and as $(4, p) = 1$, it follows that this quotient must be a quotient of μ_{p-1} . $\square$

Lemma A.0.2. Let $\mathfrak{M}$ be an $\mathfrak{S}_1$ -module of rank h and $\{e_1, \dots, e_h\}$ be a choice of basis such that

$$\phi_{\mathfrak{M}}(f_1, \dots, f_h) = (f_1^p, \dots, f_h^p) \begin{pmatrix} a_1 & a_2 & \cdots & a_{h-1} & a_h \\ 0 & a_1 & \cdots & a_{h-2} & a_{h-1} \\ & & \ddots & \vdots & \vdots \\ \vdots & & & a_1 & a_2 \\ 0 & \cdots & 0 & a_1 & \end{pmatrix}$$

where the entries satisfy the inequalities

$$\begin{aligned} \deg_u(a_i) &> \deg_u(a_j), \\ \deg_u(a_i^p a_{k-i+1}) &> \deg_u(a_j^p a_{k-j+1}), \end{aligned}$$

for all $1 \leq i < j \leq k \leq h$. Then, up to $\mathfrak{S}_1^\times$ -multiples, there are at most h distinct saturated lines $\mathfrak{L} \subseteq \mathfrak{M}$ with

$$v_u(\det(\phi_{\mathfrak{L}})) \in \left\{ \deg_u(a_1) + (p-1)\deg_u\left(\frac{a_1}{a_k}\right) \right\}_{1 \leq k \leq h}.$$

Proof. Let $\mathfrak{L} \subseteq \mathfrak{M}$ be a saturated line with Frobenius $\phi_{\mathfrak{L}} = u^\mu$. By the commutation of the diagram

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\quad} & \mathfrak{M} \\ u^\mu = \phi_{\mathfrak{L}} \downarrow & & \downarrow \phi_{\mathfrak{M}} \\ \mathfrak{L} & \xrightarrow{\quad} & \mathfrak{M} \end{array}$$

an element $v = (f_1, \dots, f_h) \in \mathfrak{M}$ is also in $\mathfrak{L}$ if and only if it satisfies the system of equations

$$a_k f_1^p + \dots + a_1 f_k^p = u^\mu f_k \quad \text{for } 1 \leq k \leq h. \quad (\text{A.0.3})$$

We may assume $f_1 \neq 0$ (otherwise let $1 \leq s \leq h$ be the smallest integer such that $f_s \neq 0$ and rephrase the problem with h replaced by $h - s + 1$) so that $u^\mu = a_1 f_1^{p-1}$.

We proceed by induction and show at each step that the following three conditions are met:

1. If $v = (f_1, \dots, f_h) \in \mathfrak{L}$ then if $f_j = 0$ for some $1 \leq j \leq h$, then $f_i = 0$ for all $1 \leq i < j$.
2. If $v = (f_1, \dots, f_h) \in \mathfrak{L}$ then $\deg_u(f_i) \geq \deg_u(f_j)$ for $1 \leq i < j \leq h$. In particular, since $\mathfrak{L} \subseteq \mathfrak{M}$ is a saturated submodule, we may assume that $f_h = 1$.

3. Compute $v_u(\det(\phi_{\mathfrak{L}}))$ for all potential solutions to the problems above.

For the base case $h = 2$, (1)–(3) have already been verified explicitly in Proposition 7.2.1.

To show the induction step for (1), we assume that $f_h = 0$ and by the induction hypothesis that $f_{h-1} \neq 0$, otherwise $f_k = 0$ for all $1 \leq k \leq h-1$. A solution to the equation

$$a_h f_1^p + \cdots + a_2 f_{h-1}^p = 0,$$

which is satisfied since $f_h = 0$, then exists if and only if

$$\deg_u(a_i f_{h-i+1}^p) = \deg_u(a_j f_{h-j+1}^p)$$

for some pairs $i > j$. Since $f_{h-1} \neq 0$ by hypothesis, $f_k \neq 0$ for some $1 \leq k < h-1$, and hence for all f_i for $k \leq i \leq h-1$ by the induction hypothesis. We may also safely assume $k = 1$, for otherwise replace the h equations by $h-k$ equations in (A.0.4). But then by the induction hypothesis, the equality of degrees above becomes either

$$\deg_u(a_i a_{h-i+1}^p) = \deg_u(a_j a_{h-j+1}^p)$$

for some pairs $i > j$. This contradicts our inequality hypotheses, and hence the only solution exists when $f_i = 0$ for all $1 \leq i \leq h$.

Similarly to show the induction step for (2), suppose the statement holds for $h-1$ and yet fails at h , so that, in particular, $\deg_u(f_h) > \deg_u(f_{h-1})$. By the induction hypothesis, we may thus assume that $f_{h-1} = 1$, so that we obtain the equation

$$a_1 f_h^p + \cdots + a_h f_1^p = (a_{h-1} f_1^p + \cdots + a_2 f_{h-2}^p + a_1) f_h.$$

We may moreover assume by the induction hypothesis that

$$\deg_u(f_1) - \deg_u(f_k) = \deg_u(a_1) - \deg_u(a_k)$$

for all $1 \leq k \leq h-1$ so that there exists a solution to the equation if and only if

$$\deg_u(a_1 f_h^p) = \deg_u(a_{h-1} f_1^p f_h).$$

This last assertion follows from the inequalities of our hypothesis, which show that $\deg_u(a_{h-1} f_1^p f_h)$ dominates all the excluded terms. Then by our induction hypothesis, we may conclude from this that $\deg_u(f_h) = \deg_u(f_1)$. But then $a_h f_1^p + \dots + a_2 f_{h-1}^p = 0$, which by the induction hypothesis has a solution if and only if $f_i = 0$ for all $1 \leq i \leq h-1$.

Finally, for (3) we may now *a priori* assume by (1) and (2) that $f_k \neq 0$ for all $1 \leq k \leq h$ and $\deg_u(f_i) \geq \deg_u(f_j)$ for $i < j$. We can then rewrite the system (A.0.3) as

$$\left(\frac{f_1}{f_k}\right)^p + \frac{a_{k-1}}{a_k} \left(\frac{f_2}{f_k}\right)^p + \dots + \frac{a_1}{a_k} = \frac{a_1}{a_k} \left(\frac{f_1}{f_k}\right)^{p-1} \quad \text{for } 1 \leq k \leq h \quad (\text{A.0.4})$$

and note that the solutions for the first k set of equations for $1 \leq k \leq h-1$ is given by the induction hypothesis. Thus, for all $1 \leq k \leq h-1$,

$$\deg_u(f_k) = \deg_u\left(\frac{f_k}{f_{h-1}}\right) + \deg_u(f_{h-1})$$

and $v = \left(\frac{f_1}{f_{h-1}}, \dots, \frac{f_{h-2}}{f_{h-1}}\right)$ is a solution for the system (A.0.4) on $h-1$ equations when for all $1 \leq k \leq h-1$,

$$\deg_u\left(\frac{f_k}{f_{h-1}}\right) = \deg_u\left(\frac{a_k}{a_{h-1}}\right).$$

This has a solution if and only if

$$\max_{1 \leq k \leq h} \left\{ \deg_u \left(\frac{a_{h-k+1}}{a_h} \right) \left(\frac{a_k}{a_{h-1}} \right)^p + \deg_u \left(\frac{f_{h-1}}{f_h} \right)^p \right\} = \deg_u \left[\frac{a_1}{a_h} \left(\frac{f_1}{f_h} \right)^{p-1} \right].$$

By our hypothesis that $\deg_u(a_i^p a_{h-i+1}) > \deg_u(a_j^p a_{h-j+1})$ for $1 \leq i < j \leq h$, the unique solution (if it exists) is given precisely when the maximum on the left is attained by $k = 1$. Since $\mathfrak{L}$ is saturated, we may assume by (2) that $f_h = 1$. This provides solutions for f_k and μ uniquely, for all $1 \leq k \leq h$. $\square$

We recall here Example 6.1.6, which provides for $E/\mathbb{Q}_p$ any Galois and totally ramified field of degree h the representation of the (reduced) Kisin module associated to a CM p -divisible group $\mathcal{G}$ with (p -adic) CM by $(\mathcal{O}_E, \Phi)$. For this, we choose the $\mathfrak{S}_1$ -basis $\{\pi^0, \dots, \pi^{h-1}\}$, where π is a uniformizer for E , so that

$$\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G}[p])}(e_i) = \sum_{j=0}^{h-1-i} [P_j(u)] e_{j+i}$$

where the polynomials $P_j(u)$ are precisely defined in Example 6.1.6. We note the important property that the degree of $P_j(u)$ for $j > 1$ is strictly less than the degree of $P_1(u)$.

Proposition A.0.5. *Assume that $p \geq h$ and $E/\mathbb{Q}_p$ is totally ramified and cyclic of degree h , and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group of type $(\mathcal{O}_E, \Phi)$. Denote by $\mathfrak{M} := \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ and assume that $\sum_{i \in \Phi^c} c_i \neq 0$ where c_i are the units in Example 6.1.6. Then for a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \mathfrak{M}$,*

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{[(h-d+p-1)(p^h - p^{h-1}) - (p-1)(p^h - 2p^{h-1} + p^{h-k})]e\}.$$

Proof. This follows if we can apply Lemma A.0.2 to the presentation of $\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ of Example 6.1.6. We therefore check the conditions of the hypotheses, namely the

conditions on the degree of the polynomials which appear in the entries of $\phi_{\mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})}$.

Recall that by Lemma 6.1.9 and Remark 6.1.10 we can express the polynomials $u^{\frac{e}{h}} + \pi h_i(u)$ (here we absorb the unit c_i into our choice of uniformizer π) by using the h th iteration of the Lubin-Tate polynomial. By the hypothesis that $p > h$, we may therefore write

$$\begin{aligned} u^{\frac{e}{h}} + \pi h_i(u) &\equiv \pi + (\pi^{h-1}u + \pi^{h-2}u^p + \cdots + u^{p^{h-1}})^{p-1} \pmod{p} \\ &\equiv \pi + \sum_{i_1=0}^{p-1} \left(\pi^{i_1} (u^{p^{h-1}})^{p-1-i_1} \cdot \sum_{i_2=0}^{i_1} \left(\pi^{i_2} (u^{p^{h-2}})^{i_1-i_2} \cdot \left(\sum_{i_3=0}^{i_2} \pi^{i_3} (u^{p^{h-3}})^{i_2-i_3} \cdot \right. \right. \right. \\ &\quad \left. \left. \left. \cdots \sum_{i_{h-1}=0}^{i_h} \pi^{i_{h-1}} (u^p)^{i_{h-2}-i_{h-1}} u^{i_{h-1}} \right) \right) \right) \pmod{\pi^h}. \end{aligned}$$

Each monomial comprising $u^{\frac{e}{h}} + \pi h_i(u)$ is thus identified by a tuple $(i_1, \dots, i_{h-1})$ such that $i_1 \geq i_2 \geq \cdots \geq i_{h-1}$. Moreover, if $\sum_{j=1}^{h-1} i_j = k$, the coefficient of the corresponding monomial (up to unit multiplication) is π^k . One readily checks that the maximal degree a monomial with coefficient π^k takes is $p^h - 2p^{h-1} + p^{h-k}$, which occurs precisely when $i_1 = i_k = 1$ and $i_{k+1} = i_{h-1} = 0$.

By the hypothesis that $\sum_{i \in \Phi^c} c_i \neq 0$, the previous computation gives

$$\deg_u(a_k) = (p^h - 2p^{h-1} + p^{h-k}) + (h - d - 1)(p^h - p^{h-1}).$$

Then both hypotheses on the degree of the entries a_k in Lemma A.0.2 are satisfied, and we compute the potential degrees from these values for $\deg_u(a_k)$. $\square$

When $h = 4$, the case of interest for an abelian surface, this proposition gives for all $p > 4$,

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{2p^4 - 2p^3, 3p^4 - 4p^3 + p^2, 3p^4 - 3p^3 - p^2 + p, 3p^4 - 3p^3 - p + 1\}.$$

By Lemma A.0.1 this covers solutions for all potentially ramified primes except $p = 2$. We consider this separately below.

Lemma A.0.6. *Let $E/\mathbb{Q}_2$ be totally ramified and cyclic of degree $h = 4$, and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group of type $(\mathcal{O}_E, \Phi)$ with $\Phi = \{1, 2\}$. Denote by $\mathfrak{M} := \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$. Then for a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \mathfrak{M}$,*

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{2(2^4 - 2^3), 2(2^4 - 2^3) + 2^3\}.$$

Proof. By Example 6.1.6, we can choose in each case the basis $(1, \pi, \pi^2, \pi^3)$ for a choice of uniformizer π of E such that

$$\phi_{\mathfrak{M}}(f_1, f_2, f_3, f_4) = (f_1^p, f_2^p, f_3^p, f_4^p) \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ & a_1 & a_2 & a_3 \\ & & a_1 & a_2 \\ & & & a_1 \end{pmatrix}.$$

By Lemma 6.1.9 and Remark 6.1.10, we can express the polynomials $u^{\frac{e}{4}} + c_i \pi h_i(u)$ as

$$u^{\frac{e}{4}} + c_i \pi h_i(u) \equiv u^8 + c_i \pi(u^4 + 1) + c_i^2 \pi^2(u^4 + u^2) + c_i^3 \pi^3(u^2 + u) \pmod{\pi^4}$$

so that

$$a_1 = u^{16}, \quad a_2 = 0, \quad a_3 = u^8 + 1, \quad a_4 = 0.$$

We then compute that the only solutions to this system of equations are

$$\begin{aligned} (0, 0, x, y) & \quad \phi_{\mathfrak{L}} = u^{16}, \\ (u^8 x, u^8 y, x, y) & \quad \phi_{\mathfrak{L}} = u^{24}, \end{aligned}$$

where $x, y \in \mathbb{F}_2$ are not both 0. □

Now we consider the situation of submodules of higher rank. Saturated submodules of rank $h - k$ in a Kisin module $\mathfrak{M}$ of rank h correspond to saturated lines in $\bigwedge^{h-k} \mathfrak{M}$. Moreover, since $\phi_{\bigwedge^{h-k} \mathfrak{M}} = \bigwedge^{h-k} \phi_{\mathfrak{M}}$, we can compute these modules by taking partial determinants.

Specializing to $h = 4$, we invoke Example 6.1.6 to write a representation of the module $\mathfrak{M} := \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ on the explicit basis $\{1, \pi, \pi^2, \pi^3\}$ for a choice of uniformizer π of E . We make the assumption that $E/\mathbb{Q}_p$ is cyclic and the (p -adic) CM type is $\Phi = \{1, 2\}$. Then

$$\phi_{\bigwedge^2 \mathfrak{M}} = \begin{pmatrix} a_1^2 & a_1 a_2 & a_2^2 - a_1 a_3 & a_1 a_3 & a_2 a_3 - a_1 a_4 & a_3^2 - a_2 a_4 \\ & a_1^2 & a_1 a_2 & a_1 a_2 & a_2^2 & a_2 a_3 - a_1 a_4 \\ & & a_1^2 & 0 & a_1 a_2 & a_2^2 - a_1 a_3 \\ & & & a_1^2 & a_1 a_2 & a_1 a_3 \\ & & & & a_1^2 & a_1 a_2 \\ & & & & & a_1^2 \end{pmatrix}$$

and

$$\phi_{\bigwedge^3 \mathfrak{M}} = \begin{pmatrix} a_1^3 & a_1^2 a_2 & a_1 a_2^2 - a_1^2 a_3 & a_2^3 - 2a_1 a_2 a_3 + a_1^2 a_4 \\ & a_1^3 & a_1^2 a_2 & a_1 a_2^2 - a_1^2 a_3 \\ & & a_1^3 & a_1^2 a_2 \\ & & & a_1^3 \end{pmatrix},$$

where $a_i = P_i(u)$ (defined in Example 6.1.6) and we recall from Proposition A.0.5 that

$$\deg_u(P_i(u)) = 2p^4 - 3p^3 + p^{4-i}$$

when $p > 4$.

Proposition A.0.7. *Assume that $p > 4$ and $E/\mathbb{Q}_p$ is totally ramified and cyclic of degree 4, and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group of type $(\mathcal{O}_E, \Phi)$. Denote by $\mathfrak{M} := \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$ and assume that $\sum_{i \in \Phi^c} c_i \neq 0$ where c_i are the units in Example 6.1.6. Then:*

1. *For a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \bigwedge^2 \mathfrak{M}$,*

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{4p^4 - 4p^3, 5p^4 - 6p^3 + p^2, 5p^4 - 5p^3 - p^2 + p, 5p^4 - 5p^3 - p + 1, \\ 6p^4 - 8p^3 + 2p^2, 6p^4 - 7p^3 + p^2 - p + 1\}.$$

2. *For a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \bigwedge^3 \mathfrak{M}$,*

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{6p^4 - 6p^3, 7p^4 - 8p^3 + p^2, 7p^4 - 7p^3 - p^2 + p, 7p^4 - 7p^3 - p + 1\}.$$

Proof. For (1), let $v = (f_1, \dots, f_6)$ generate $\mathfrak{L}$. Then it satisfies the system of equations

$$\begin{aligned} a_1^2 f_1^p &= u^\mu f_1, \\ a_1 a_2 f_1^p + a_1^2 f_2^p &= u^\mu f_2, \\ (a_2^2 - a_1 a_3) f_1^p + a_1 a_2 f_2^p + a_1^2 f_3^p &= u^\mu f_3, \\ a_1 a_3 f_1^p + a_1 a_2 f_2^p + a_1^2 f_4^p &= u^\mu f_4, \\ (a_2 a_3 - a_1 a_4) f_1^p + a_2^2 f_2^p + a_1 a_2 f_3^p + a_1 a_2 f_4^p + a_1^2 f_5^p &= u^\mu f_5, \\ (a_3^2 - a_2 a_4) f_1^p + (a_2 a_3 - a_1 a_4) f_2^p + (a_2^2 - a_1 a_3) f_3^p + a_1 a_3 f_4^p + a_1 a_2 f_5^p + a_1^2 f_6^p &= u^\mu f_6. \end{aligned} \tag{A.0.8}$$

If $f_1 = f_2 = f_3 = 0$ or $f_1 = f_2 = f_4 = 0$, the argument from Proposition A.0.5 shows that the resulting system of equations satisfies the conditions of Lemma A.0.2

and we obtain a corresponding set of solutions

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{2\deg_u(a_1) + (p-1)(\deg_u(a_1) - \deg_u(a_k))\}_{1 \leq k \leq 3}.$$

If $f_1 = f_2 = 0$ and $f_3, f_4 \neq 0$, then by the third and fourth equations in the system (A.0.8) we have $f_3 = f_4$, and making this substitution the remaining system satisfies the conditions of Lemma A.0.2. This gives the corresponding set of solutions

$$\begin{aligned} v_u(\det(\phi_{\mathfrak{L}})) \in \{ & 2\deg_u(a_1), 2\deg_u(a_1) + (p-1)(\deg_u(a_1) - \deg_u(a_2)), \\ & 2\deg_u(a_1) + 2(p-1)(\deg_u(a_1) - \deg_u(a_2)) \}. \end{aligned}$$

If $f_1 = 0$ and $f_2 \neq 0$, then again $f_3 = f_4$ by the third and fourth equations in the system (A.0.8). However, the remaining system does not satisfy Lemma A.0.2, as $\deg_u(P_1(u)P_4(u)) > \deg_u(P_2^2(u))$. We note that if $f_2 \neq 0$ then $f_i \neq 0$ for $i > 2$. Moreover, using the second through fifth equations of the system (A.0.8), the proof of Lemma A.0.2 applies to show that $\deg_u(f_2) > \deg_u(f_3) > \deg_u(f_4) > \deg_u(f_5)$. Hence, up to multiplication by a unit, either $f_5 = 1$ or $f_6 = 1$. These correspond to the solutions

$$\begin{aligned} v_u(\det(\phi_{\mathfrak{L}})) \in \{ & 2\deg_u(a_1) + 2(p-1)(\deg_u(a_1) - \deg_u(a_2)), \\ & 2\deg_u(a_1) + (p-1)(\deg_u(a_1) - \deg_u(a_4)) \}. \end{aligned}$$

Finally, if $f_1 \neq 0$, then it still follows that $\deg_u(f_3) = \deg_u(f_4)$ by the third equation in the system (A.0.8). Moreover, $f_i \neq 0$ for all $1 \leq i \leq 6$. While again (A.0.8) does not satisfy Lemma A.0.2, the proof of *loc. cit.* demonstrates that $\deg_u(f_1) > \deg_u(f_2) > \deg_u(f_3) > \deg_u(f_4) > \deg_u(f_5)$ by analyzing the first five

equations in (A.0.8). Hence either $f_5 = 1$ or $f_6 = 1$, which correspond to the solutions

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{2\deg_u(a_1) + (p-1)(\deg_u(a_1) - \deg_u(a_4)), \\ 2\deg_u(a_1) + (p-1)(2\deg_u(a_1) - \deg_u(a_2) - \deg_u(a_4))\}.$$

For (2), the argument from Proposition A.0.5 on the degrees of $a_i = P_i(u)$ shows that the entries of $\phi_{\wedge^3 \mathfrak{M}}$ satisfy Lemma A.0.2. Thus, after a simplification using the known degrees of the entries $a_i = P_i(u)$, we obtain

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{3\deg_u(a_1) + (p-1)(\deg_u(a_1) - \deg_u(a_k))\}_{1 \leq k \leq 4}.$$

□

By Lemma A.0.1, we again cover by this proposition all potentially ramified primes except $p = 2$. We again consider this separately below.

Lemma A.0.9. *Let $E/\mathbb{Q}_2$ be totally ramified and cyclic of degree $h = 4$, and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group of type $(\mathcal{O}_E, \Phi)$. Denote by $\mathfrak{M} := \mathfrak{M}_{\mathfrak{S}_1}(\mathcal{G})$. Then:*

1. *For a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \wedge^2 \mathfrak{M}$,*

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{4(2^4 - 2^3), 4(2^4 - 2^3) + 2^3\}.$$

2. *For a saturated $\mathfrak{S}_1$ -line $\mathfrak{L} \subseteq \wedge^3 \mathfrak{M}$,*

$$v_u(\det(\phi_{\mathfrak{L}})) \in \{6(2^4 - 2^3), 6(2^4 - 2^3) + 2^3\}.$$

Proof. By the computation of $P_i(u)$ in the proof of Lemma A.0.6,

$$\phi_{\wedge^2 \mathfrak{M}} = \begin{pmatrix} u^{32} & 0 & u^{16}(u^8 + 1) & u^{16}(u^8 + 1) & 0 & u^{16} + 1 \\ & u^{32} & 0 & 0 & 0 & 0 \\ & & u^{32} & 0 & 0 & u^{16}(u^8 + 1) \\ & & & u^{32} & 0 & u^{16}(u^8 + 1) \\ & & & & u^{32} & 0 \\ & & & & & u^{32} \end{pmatrix}$$

and

$$\phi_{\wedge^3 \mathfrak{M}} = \begin{pmatrix} u^{48} & 0 & u^{32}(u^8 + 1) & 0 \\ & u^{48} & 0 & u^{32}(u^8 + 1) \\ & & u^{48} & 0 \\ & & & u^{48} \end{pmatrix}.$$

For (1), let $v = (f_1, \dots, f_6) \in \mathfrak{M}$ generate $\mathfrak{L}$. Then we easily compute that all saturated lines are of the form

$$\begin{aligned} (0, 0, u^8, 0, 0, 1), (0, 0, 0, u^8, 0, 1) & \quad \phi_{\mathfrak{L}} = u^{40} \\ (0, x, w, w, y, z) & \quad \phi_{\mathfrak{L}} = u^{32}, \end{aligned}$$

where $x, y, z, w \in \mathbb{F}_2$ are not all 0.

For (2), let $v = (f_1, \dots, f_4)$ generate $\mathfrak{L}$. Then again we easily compute that all saturated lines are of the form

$$\begin{aligned} (0, 0, x, y) & \quad \phi_{\mathfrak{L}} = u^{48} \\ (u^8 x, u^8 y, x, y) & \quad \phi_{\mathfrak{L}} = u^{56}, \end{aligned}$$

where $x, y \in \mathbb{F}_2$ are not both 0. □

Theorem A.0.10. *Assume that $[E : \mathbb{Q}_p] = 4$ and that p is totally ramified in E , and let $\mathcal{G}/\mathcal{O}_K$ be an $\mathcal{O}_E$ -linear CM p -divisible group. Let $\mathcal{H} \subseteq \mathcal{G}[p^n]$ be a subgroup such that $\mathcal{H} \cap \mathcal{G}[\pi_E^r] = 1$ for all $r \geq 0$, and characterized by the non-increasing tuple $(\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = \log \# \mathcal{H}$, $\mathcal{H}$ is p^{λ_1} -torsion, and the p -height $\iota \in \{1, 2, 3\}$ of $\mathcal{H}$ is the largest integer such that $\lambda_\iota \neq 0$. Then*

$$\# \frac{1}{[K : \mathbb{Q}_p]} \log s^* \Omega_{\mathcal{H}/\mathcal{O}_K}^4 = \frac{1}{4p^4 - 4p^3} \left(\sum_{i=\lambda_2+1}^{\lambda_1} \frac{R_1(p)}{p^i} \right) \left(\sum_{j=\lambda_3+1}^{\lambda_2} \frac{R_2(p)}{p^j} \right) \left(\sum_{k=1}^{\lambda_3} \frac{R_3(p)}{p^k} \right) \log p,$$

where the sums are evaluated if and only if the difference in bounds is at least zero, and for $p \equiv 1 \pmod{4}$,

$$R_1(p) = \begin{cases} p^4 - p^2 \\ p^4 - p^3 + p^2 - p \\ p^4 - p^3 + p - 1 \end{cases}$$

$$R_2(p) = \begin{cases} 3p^4 - 2p^3 - p^2 \\ 3p^4 - 3p^3 + p^2 - p \\ 3p^4 - 3p^3 + p - 1 \\ 2p^4 - 2p^2 \\ 2p^4 - p^3 - p^2 + p - 1 \end{cases}$$

$$R_3(p) = \begin{cases} 5p^4 - 4p^3 - p^2 \\ 5p^4 - 5p^3 + p^2 - p \\ 5p^4 - 5p^3 + p - 1 \end{cases}$$

and for $p = 2$,

$$R_1(2) = 2(2^4 - 2^3) - 2^3$$

$$R_2(2) = 4(2^4 - 2^3) - 2^3$$

$$R_3(2) = 6(2^4 - 2^3) - 2^3.$$

Proof. This compiles the computations in Proposition A.0.5, Lemma A.0.6, Proposition A.0.7, and Lemma A.0.9 using Proposition 4.4.5. □

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