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Min-max minimal hypersurfaces in higher dimensions

Li, Yangyang

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Min-max minimal hypersurfaces in higher dimensions

Yangyang Li

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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by the Department of
Mathematics
Adviser: Fernando Codá Marques

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Abstract

In the recent decade, the Almgren-Pitts min-max theory has advanced the existence theory of minimal hypersurfaces in a closed Riemannian manifold (M^{n+1}, g). When $3 \leq n+1 \leq 7$, many properties of these minimal hypersurfaces, such as areas, Morse indices, multiplicities, and spatial distributions, have been well studied. However, in higher dimensions ($n+1 \geq 8$), min-max minimal hypersurfaces may contain singularities. This phenomenon invalidates many helpful techniques in the low dimensions to investigate these geometric objects.

I will show how one can utilize various deformation arguments to overcome the obstacles and prove generic abundance, index estimates, and most of the geometric properties of min-max minimal hypersurfaces. In particular, in dimension eight, en route to obtaining generic results, in joint work with Zhihan Wang, we prove generic regularity of minimal hypersurfaces.

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To my parents and my love.

Contents

Abstract iii

Acknowledgements iv

1 Introduction **1**

1.1 Almgren-Pitts min-max theory 3

1.2 Geometric Properties of min-max minimal hypersurfaces 4

1.2.1 Area growth 5

1.2.2 Abundance 5

1.2.3 Multiplicity one conjecture 6

1.2.4 Morse index 6

1.2.5 Geometric complications 7

1.2.6 Spatial distributions 8

1.3 Main goal and organization of the thesis 9

1.3.1 Main goal 10

1.3.2 Organization 10

2 Preliminaries on min-max theory **15**

3 Generic abundance **24**

3.1 Preliminaries 25

3.1.1 Basic Notations in the Almgren-Pitts Min-max Theory 25

3.1.2 Singular Minimal Hypersurfaces with Optimal Regularity and Compactness with Stability Condition 28
3.2 Minimal Hypersurfaces from Almgren-Pitts Min-max Construction . 29
3.2.1 Almgren-Pitts Realizations of m-parameter Homotopy Families and Their Compactness 29
3.2.2 Almgren-Pitts Realizations of (p, m)-width and p-width and Their Compactness 33
3.3 Proof of Main Theorem 37

4 Morse index and multiplicity estimates 41
4.1 Introduction 41
4.2 Restrictive min-max theories 43
4.3 k-unstability and Morse index 45
4.3.1 k-unstability 45
4.3.2 Almgren-Pitts realizations and Morse index 50
4.4 Deformation theorems and Morse index upper bounds 51
4.5 Construction of c-CMC hypersurfaces and improved Morse Index Bound 71
4.5.1 Approximation by c-CMC hypersurfaces 71
4.5.2 Approximation by c-CMC hypersurfaces 75
4.5.3 Proof of the Improved Morse Index Upper Bound 78

5 Smooth minimal hypersurfaces in dimension 8 80
5.1 Introduction 80
5.1.1 Some conventions 83
5.1.2 Sketch of Proof 83
5.1.3 Organization of the Chapter 85
5.2 Preliminaries 85
5.2.1 1-parameter Min-max Theory 86

5.2.2 Surgery Procedure à la Chodosh-Liokumovich-Spolaor 89
5.2.3 Global perturbation à la Wang 90
5.3 Uniqueness of Min-max Minimal Hypersurfaces 90
5.4 Singular Capacity of Minimal Hypersurfaces 92

6 Full generic regularity in dimension 8 **94**
6.1 Introduction 94
6.2 Preliminaries 96
6.2.1 Stable Minimal Hypercone 98
6.2.2 Singularities of LSMH 100
6.2.3 Singular Capacity 102
6.3 Canonical Neighborhoods 104
6.4 Proof of Main Theorem 108

A Generic geometric complication of smooth minimal hypersurfaces **115**
A.1 introduction 115
A.2 Multiplicity and Morse Index for Confined Min-max Closed Minimal Hypersurfaces 118
A.2.1 A. Song's Construction of a Non-compact Manifold with Cylindrical Ends 119
A.2.2 Improved Confined Min-max Theory 122
A.3 Proof of Main Theorem 126
A.A X. Zhou's Multiplicity One Theorem 132

Chapter 1

Introduction

In Riemannian geometry, *minimal surfaces* (or generally, *minimal hypersurfaces*) are the submanifolds extending the notion of geodesics, which can be described as critical points of the area functional, or more intuitively, which locally minimizes their areas. They are natural models for plenty of physical phenomena, including soap films, black holes, and protein folding. The study of minimal surfaces has attracted more and more mathematicians. It also has remarkable applications in general relativity. These include the unexpected proofs of the positive mass theorem [66, 67], which, generally speaking, indicates that the gravitational energy of an isolated system is nonnegative, and the proposal of the famous Penrose inequality [10, 37] estimating the mass of spacetime in terms of the minimal surface modeling the apparent horizon of black holes.

The theory of minimal surfaces dates back to the works of Joseph-Louis Lagrange in the 1760s, who utilized *calculus of variations* to study minimal surfaces as critical points of the area functional. More precisely, he searched for the surface $z = z(x, y)$ of the least area spanned a given closed contour, and derived the corresponding Euler-Lagrange equation

$$\operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = 0\,. \tag{1.0.1}$$

This problem can be formulated in an unparameterized form, i.e., the existence of a minimal surface with a given boundary. However, it is named as *Plateau's problem* after Joseph Plateau, a Belgian physicist who conducted extensive experiments with soap films and formulated the structure of soap films (Plateau's laws) [61]. Plateau's problem was solved independently by T. Radó [62] and J. Douglas [24] around the 1930s.

The extension of the problem to higher dimensions, though much more difficult to study, was finally solved by the pioneer mathematicians E. De Giorgi [20], H. Federer and W. H. Fleming [28] around the 1960s. En route, mathematicians invented and developed geometric measure theory (GMT), which studies the geometric properties of sets via measure theory. The theory solves Plateau's problem within a nontrivial homology class if one accepts that solutions are not necessarily smooth submanifolds, but "area-minimizing rectifiable currents" or "stationary integral varifolds". Moreover, the regularity theory is a central topic in the study of geometric measure theory, which dates back to around the 1970s De Giorgi's estimates [20] and Allard's regularity theorem [1, 2]. The sharp regularity results therein are heavily relied on an epoch-making paper by J. Simons [72] on the nonexistence of stable cones in $\mathbb{R}^{n+1}$ for $2 \leq n \leq 6$. In particular, in a Riemannian manifold M^{n+1} with $2 \leq n \leq 6$, an area-minimizing integral n-cycle must be a smooth minimal hypersurface. However, even when $n = 7$, Bombieri-De Giorgi-Giusti [9] has proved that the famous Simons cone

$$C = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\}, \tag{1.0.2}$$

is an area-minimizing hypercone in $\mathbb{R}^8$, which has a singular point at the origin. A similar phenomenon occurs in the stable minimal hypersurface (varifold) setting with natural a priori assumption on the singular set, which was originally proved by Schoen-Simon [65] and later improved by N. Wickramasekara [82]. More quantitative arguments can refer to [14, 58]. These works can also apply to a closed Riemannian

manifold with a nontrivial codimension one homology group to obtain the existence and regularity of minimal hypersurfaces.

Nevertheless, in a more general setting, the existence of (immersed or embedded) closed minimal hypersurfaces couldn't form a minimizing sequence, since there might not be area-minimizing hypersurfaces but critical solutions to the area functional. For example, in the surface case, i.e., $\mathbb{S}^2$ doesn't admit any closed geodesic minimizing in its homotopy class. G.D. Birkhoff [8] managed to show the existence of a closed geodesic using a min-max theory for sweepouts over the sphere. For minimal (hyper)surfaces, the min-max theory is much more involved as the regularity turns out to be the main obstacle. Fredricks J. Almgren and Jon Pitts managed to conquer it and propose a groundbreaking min-max scheme, now known as *Almgren-Pitts min-max theory*.

1.1 Almgren-Pitts min-max theory

In 1962, in his Ph.D. thesis, F.J. Almgren [4] proved the isomorphism between the homology group of a closed Riemannian manifold and the homotopy group of its cycle space, i.e.,

$$\pi_k(\mathcal{Z}_m(M,G);0) \equiv H_{k+m}(M;G)\,, \tag{1.1.1}$$

for some group $G = \mathbb{Z}$ or $\mathbb{Z}_2$. In particular, Since the top $\mathbb{Z}_2$-coefficient homology group is nontrivial for any closed Riemannian manifold, so is the π_1 group of the codimension one cycle space.

This fact motivates F. Almgren [3] and J. Pitts [59] during the 1960s-1970s to propose a Morse theoretic method generating minimal hypersurfaces realizing the min-max volume width, i.e.,

$$\omega = \inf_{\Phi\subset\Pi}\max_{t\in[0,1]}\mathbf{M}(\Phi(t))$$

where Π is the set consisting of all non-trivial sweepouts $\Phi : [0,1] \to \mathcal{Z}_n(M,G)$ with $\Phi(0) = \Phi(1) = \emptyset$.

Together with a hard regularity theory by Schoen-Simon [65], the theory successfully produces a minimal hypersurface in any closed Riemannian manifold with a singular set of codimension at least codimension 7. In particular, when the ambient manifold has dimension between 3 and 7, the min-max minimal hypersurface is indeed a disjoint union of smooth minimal hypersurfaces.

Moreover, starting from the work of Marques-Neves [51], we can find more minimal hypersurfaces from this scheme. Roughly speaking, we can generate a sequence of minimal hypersurfaces indexed by positive natural numbers (See its precise statement in Theorem 2.0.4).

Theorem 1.1.1 (Min-max theorem for p-width)**.** *Given a Riemannian manifold (M^{n+1}, g), for each $p \in \mathbb{N}$ and any $p \geq 1$, there exists a minimal hypersurfaces with optimal regularity Σ_p with $\omega_p = \|\Sigma_p\|$ for some min-max width (critical value) ω_p.*

1.2 Geometric Properties of min-max minimal hypersurfaces

As we have seen above, Almgren-Pitts min-max theory is a powerful machinery to generate a variety of minimal hypersurfaces indexed by an integer p. It is also interesting to study their shapes, i.e., their geometric properties, with respect to their ambient manifold (M^{n+1}, g) and the index p.

1.2.1 Area growth

In [31] (See also [33]), Mikhael Gromov has shown that the p-width ω_p grows sub-linearly with respect to p. He also conjectured a Weyl law for these volume spectra. This conjecture is confirmed by Liokumovich-Marques-Neves.

Theorem 1.2.1 ([48]). *There exists a positive constant a_n only depending on the dimension, such that*

$$\lim_{p\to\infty} p^{-1/(n+1)}\omega_p(M,g) = a_n \operatorname{vol}(M,g)^{n/(n+1)} \tag{1.2.1}$$

Note, however, that the multiplicity of each Σ_p is not necessarily one, this doesn't directly imply infinitely many geometrically distinct minimal hypersurfaces in (M^{n+1}, g).

1.2.2 Abundance

In 1982, S.-T. Yau conjectured the abundance of minimal surfaces as follows.

Conjecture 1.2.2 (Yau's conjecture, [84]). Any closed three-dimensional manifold must contain an infinite number of immersed minimal surfaces.

In [39], K. Irie, F.C. Marques and A. Neves, using the Weyl law [48] for volume spectra by Y. Liokumovich and the last two named authors, proved a stronger version of Yau's conjecture in the generic case.

Later, in [56], F.C. Marques, A. Neves and A. Song gave a quantitative description of the density, i.e., the equidistribution of a sequence of minimal hypersurfaces under the same condition.

The Yau's conjecture for $2 \leq n \leq 6$ with arbitrary C^∞ metrics was finally resolved by A. Song [76] using the methods developed by F.C. Marques and A. Neves in [54].

1.2.3 Multiplicity one conjecture

In [52], Marques-Neves conjectured that for *bumpy* metrics, min-max minimal hypersurfaces should be of multiplicity one at least when $2 \leq n \leq 6$. They confirmed this for the first width therein. Here, a closed Riemannian manifold (M^{n+1}, g) is called *bumpy*, if any immersed closed minimal hypersurface is non-degenerate, i.e., has no non-trivial Jacobi field. B. White [81] (See also Theorem 9 in [5]) has shown that the space of bumpy metrics on M is a residual subset of the space of C^∞ Riemannian metrics on M.

Later, in the Allen-Cahn min-max setting, Chodosh-Mantoulidish [17] affirmed the conjecture for dimension $n+1 = 3$. And the conjecture was fully confirmed by X. Zhou in 2019.

Theorem 1.2.3 ([86]). *Given a closed manifold M $n+1$ of dimension $3 \leq n+1 \leq 7$ with a bumpy metric g, the min-max minimal hypersurfaces $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ associated with volume spectrum are all two-sided and have multiplicity one and index bounded by k.*

1.2.4 Morse index

In [52], Marques-Neves have shown that the index of the support of a min-max minimal hypersurface is bounded above by the number of parameters when $3 \leq n+1 \leq 7$. Later, combining with Zhou's multiplicity one theorem, they were able to improve the inequality to an identity for bumpy metrics.

Theorem 1.2.4 ([55]). *On a bumpy closed Riemannian manifold (M^{n+1}, g) $(3 \leq n+1 \leq 7)$, for each $p \in \mathbb{N}^+$, there exists a smooth, closed, embedded, multiplicity one, two-sided, minimal hypersurface Σ_p such that*

$$\omega_p(M, g) = \mathrm{Area}_g(\Sigma_p)\,, \quad \mathrm{index}(\Sigma_p) = p\,, \tag{1.2.2}$$

and

$$\lim_{p\to\infty}\frac{\mathrm{Area}_g(\Sigma_p)}{p^{\frac{1}{n+1}}}=c(n)\mathrm{vol}(M,g)^{\frac{n}{n+1}}\,, \tag{1.2.3}$$

where $\omega_p(M,g)$ *is the volume spectrum, more precisely, the min-max p-width, and* $c(n)>0$ *is a dimensional constant from the Weyl law in [48].*

In fact, for general metrics, one can also get a better estimate, since components with multiplicity higher than one are weakly stable.

Theorem 1.2.5 ([86, Theorem C])**.** *Given a closed manifold* M^{n+1} $(3\le n+1\le 7)$*, there exists a sequence of integral varifolds* V_p *each of whose support is a disjoint union of smooth, connected, closed, embedded, minimal hypersurfaces* $\{\Sigma_1^p,\cdots,\Sigma_{l_p}^p\}$ *with multiplicities* $\{m_i^p\}\subset\mathbb{N}^+$*, such that*

$$\omega_p(M,g)=\sum_{i=1}^{l_p}m_i^p\cdot\mathrm{area}_g(\Sigma_i^p), \tag{1.2.4}$$

is the volume spectrum with sublinear growth, and the components with multiplicity $m_i^p>1$ *are degenerately stable, while*

$$\sum_{i:m_i^p=1}\mathrm{index}(\Sigma_i^p)\le p. \tag{1.2.5}$$

1.2.5 Geometric complications

Σ_p in Theorem 1.2.4 has large Morse index and large area, but it might have multiple disjoint components. Therefore, each connected component might have relatively small Morse index or small area.

In 2019, O. Chodosh and C. Mantoulidis [16] showed that there exists a sequence of connected minimal hypersurfaces whose areas tend to infinity.

Theorem 1.2.6 ([16, Theorem 1.4])**.** *On a bumpy closed Riemannian manifold* (M^{n+1},g) $(3\le n+1\le 7)$*, either:*

- *there exists a sequence of connected closed embedded stable minimal hypersurfaces* Σ_j *with* $\mathrm{Area}(\Sigma_j) \to \infty$ *, or,*

- *the hypersurfaces* Σ_p *from Theorem A.1.1 have at least one connected component* Σ'_p *with* $\mathrm{Area}(\Sigma'_p) \geq Cp^{1/(n+1)}$ *for some* $C = C(M, g) > 0$ *independent of* p *.*

In spired by their work, we [44] were able to show the geometric complexity of connected minimal hypersurfaces with respect to both area and Morse index.

Theorem 1.2.7. *Given any bumpy closed Riemannian manifold* (M^{n+1}, g) $(3 \leq n+1 \leq 7)$*, there exists a sequence of connected closed embedded two-sided minimal hypersurfaces* $\{\Gamma_i\}$ *such that*

$$\lim_{i\to\infty} \mathrm{Area}(\Gamma_i) = +\infty\,, \quad \lim_{i\to\infty} \mathrm{index}(\Gamma_i) = +\infty\,. \tag{1.2.6}$$

1.2.6 Spatial distributions

The result by Irie-Marques-Neves mentioned above is in fact stronger than the Yau's conjecture.

Theorem 1.2.8 ([39])**.** *Let* M^{n+1} *be a closed manifold of dimension* $n+1$*, with* $3 \leq n+1 \leq 7$*. Then for a* C^∞*-generic Riemannian metric* g *on* M*, the union of all closed, smooth, embedded minimal hypersurfaces is dense.*

Later, A. Song and the last two named authors obtained a quantitative version of the density, i.e., the distributions of a sequence of minimal hypersurfaces. Note, however, that this sequence is not necessarily the min-max minimal hypersurfaces.

Theorem 1.2.9 ([56])**.** *Let* M^{n+1} *be a closed manifold of dimension* $n+1$*, with* $3 \leq n+1 \leq 7$*. Then for a* C^∞*-generic Riemannian metric* g *on* M*, there exists a sequence* $\{\Sigma_j\}_{j\in\mathbb{N}}$ *of closed, smooth, embedded, connected minimal hypersurfaces that*

is equidistributed in M: for any $f \in C^\infty(M)$ we have

$$\lim_{q\to\infty} \frac{1}{\sum_{i=1}^q \mathrm{vol}_g(S_j)} \sum_{i=1}^q \int_{\Sigma_j} f \mathrm{d}\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M f \mathrm{d}M\,.$$

Even more, for any symmetric $(0,2)$-tensor h on M we have:

$$\lim_{q\to\infty} \frac{1}{\sum_{i=1}^q \mathrm{vol}_g(S_j)} \sum_{i=1}^q \int_{\Sigma_j} \mathrm{Tr}_{\Sigma_j}(h) \mathrm{d}\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M \frac{n\mathrm{Tr}_M h}{n+1} \mathrm{d}M\,.$$

1.3 Main goal and organization of the thesis

The generic properties of min-max minimal hypersurfaces heavily rely on the following structure theorem of minimal hypersurfaces by B. White.

Theorem 1.3.1 (White's Structure theorem [79, 81])**.** *Given a closed Riemannian manifold M, an integer $k \geq 4$, and $\alpha \in (0,1)$, let*

- *$\mathcal{G}^{k,\alpha}(M)$ be the space of $C^{k,\alpha}$ Riemannian metrics on M;*
- *$\mathcal{M}^{k,\alpha}(M)$ be the space of pairs (g,Σ), where $g \in \mathcal{G}^{k,\alpha}(M)$ and Σ is a smooth minimal hypersurface in (M,g).*

Then the projection map $\Pi : \mathcal{M}^{k,\alpha}(M) \to \mathcal{G}^{k,\alpha}(M)$ to the first factor is a Fredholm map with Fredholm zero.

*In particular, the regular values of Π, called **bumpy metrics**, form a generic subset $\mathcal{G}^{k,\alpha}_{\mathrm{bumpy}}(M)$ of $\mathcal{G}^{k,\alpha}(M)$. For any $g \in \mathcal{G}^{k,\alpha}_{\mathrm{bumpy}}(M)$, if Σ is a smooth g-minimal hypersurface, then Σ is non-degenerate.*

Intuitively, for bumpy metrics, the area functional looks like a Morse function, and the critical points of the area functional are "discrete", if there are "only" smooth minimal hypersurfaces. Therefore, in higher dimensions ($n+1 \geq 8$), the lack of generalized bumpy metrics for singular minimal hypersurfaces is the main obstruction to extend the arguments for the geometric properties of min-max minimal hypersurfaces.

1.3.1 Main goal

The goal of this dissertation is to study the geometric properties of minimal hypersurfaces generated from the Almgren-Pitts min-max theory. We would like to know whether the results mentioned in the previous section can be generalized to higher dimensions ($n+1 \geq 8$) and to what extent.

1.3.2 Organization

In Chapter 3 [42], via a compactness argument, we adapt Irie-Marques-Neves technique to prove the generic Yau's conjecture in higher dimensions. Moreover, for $2 \leq n \leq 6$, our argument also implies the denseness of the minimal hypersurfaces realizing min-max widths for generic metrics. This partially supports the equidistribution of the minimal hypersurfaces realizing min-max widths conjectured by F.C. Marques, A. Neves, and A. Song in [56].

In Chapter 4 [43], we give an improved Morse index bound of minimal hypersurfaces from Almgren-Pitts min-max construction in any closed Riemannian manifold M^{n+1} ($n+1 \geq 3$), which generalizes a result by X. Zhou [86] for $3 \leq n+1 \leq 7$. The novel techniques are the construction of hierarchical deformations and the restrictive min-max theory. These techniques do not rely on bumpy metrics, and thus could be adapted to many other min-max settings.

Chapter 5 is based on a joint work with Zhihan Wang [45]. We show that every 8-dimensional closed Riemmanian manifold with C^∞-generic metrics admits a smooth minimal hypersurface. This generalized previous results by N. Smale [74] and Chodosh-Liokumovich-Spolaor [15]. Different from their local perturbation techniques, our construction is based on a global perturbation argument in [78] and a novel geometric invariant which counts singular points with suitable weights.

In Chapter 6, we explain the proof of another joint work [46] that in an 8-dimensional closed Riemmanian manifold with C^∞-generic metrics, every minimal

hypersurface is smooth and nondegenerate. This confirms a full generic regularity conjecture of minimal hypersurfaces in dimension eight. This also enables us to generalize many generic geometric properties of (Almgren-Pitts) min-max minimal hypersurfaces, previously only known in low dimensions, to dimension eight.

For completeness, we also include the proof of geometric complication of min-max minimal hypersurfaces in low dimensions in the appendix A. This can be extended to dimension 8 with the aid of the full generic regularity results in Chapter 6.

Notations

We list some notations in geometric measure theory and Almgren-Pitts' min-max theory. Interested readers could refer to [70] and [59].

- $\mathcal{H}^n$: n-dimensional Housdorff measure;
- $\mathbf{M}_U$: the mass norm in an open subset U on the flat chain space ($\mathbf{M}_M$ abbreviated to $\mathbf{M}$);
- $\mathcal{F}$: the flat metric on the flat chain space defined by

$$\mathcal{F}(S,T) = \inf\{\mathbf{M}(R) + \mathbf{M}(P) : S - T = \partial R + P\}\,,$$

  where $S, T, P \in \mathbf{I}_k(M;\mathbb{Z}_2)$ and $R \in \mathbf{I}_{k+1}(M;\mathbb{Z}_2)$;
- $\mathbf{I}_k(M^{n+1};\nu;\mathbb{Z}_2)$: the space of k-dimensional modulo 2 flat chains in M with metric ν. If ν is $\mathcal{F}$, we will simply put it as $\mathbf{I}_k(M^{n+1};\mathbb{Z}_2)$.
- ν: a metric on the flat chain space, and usually, we take $\nu = \mathcal{F}$, $\mathbf{F}$ or $\mathbf{M}$ (See the definitions in [59]);
- $\mathcal{Z}_k(M^{n+1};\nu;\mathbb{Z}_2)$: the connected component containing 0 of the space of k-dimensional modulo 2 flat cycles in M with ν metric. If ν is $\mathcal{F}$, we will simply put it as $\mathcal{Z}_k(M^{n+1};\mathbb{Z}_2)$;
- $\mathcal{Z}^a$: the subspace $\{T \in \mathcal{Z}_n(M^{n+1};\mathbf{F};\mathbb{Z}_2) : \mathbf{M}(T) < a\}$;

- $\mathcal{V}_n(M)$: the space of n-dimensional varifolds in M;
- $\mathcal{IV}_n(M)$: the space of n-dimensional integral varifolds in M;
- $\mathcal{V}_n(M)$: the closure of $\mathcal{IV}_n(M)$ in the whole space of varifolds;
- $\mathcal{R}(M)$: the space of n-dimensional integral varifold in M whose support is regular away from a closed singular set of Hausdorff dimension $\leq n-7$;
- $|T|$: the associated integral varifold for $T \in \mathbf{I}_n(M;\mathbb{Z}_2)$;
- $\|T\|$: the associated Radon measure for $T \in \mathbf{I}_n(M;\mathbb{Z}_2)$;
- $\mathbf{F}$: the metric on $\mathcal{V}_n(M)$ defined by

$$\mathbf{F}(V,W) = \sup\{\|V\|(f) - \|W\|(f) : f \in C^1(M), |f| \leq 1, |Df| \leq 1\}\,.$$

  Moreover, $\mathbf{F}_U(V,W) := \mathbf{F}(V_1 \llcorner U, V_2 \llcorner U)$ for any open subset U;
- $\|V\|$: the associated Radon measure for $V \in \mathcal{IV}_n(M)$;
- $\theta_V(a,r)$: the density ratio of V in $B(a,r)$, and $\theta_V(a) := \theta_V(a,0_+)$ if exists;
- $\mathcal{C}(M)$: the space of Cacciopoli sets in M, i.e., subsets of M with finite perimeter. The metric on $\mathcal{C}(M)$ is the metric induced by the symmetric difference, i.e., for a pair of sets $X, Y \in \mathcal{C}(M)$,

$$d_{\mathcal{C}(M)}(X,Y) = \mathcal{H}^{n+1}(X \Delta Y)\,.$$

- $I(1,n)$: the cell complex on the unit interval I whose 1-cells are the intervals $[0, 1\cdot 3^{-n}], [1\cdot 3^{-n}, 2\cdot 3^{-n}], \cdots, [1-3^{-n}, 1]$, and whose 0-cells are the endpoints $[0], [3^{-n}], [2\cdot 3^{-n}], \cdots, [1]$;
- $I(m,n)$: the cell complex on I^m, i.e., $I(1,n)^{m\otimes} = I(1,n)\otimes I(1,n)\otimes\cdots\otimes I(1,n)$;

- $I(m,n)_0$: the set of 0-cells in $I(m,n)$;

Chapter 2

Preliminaries on min-max theory

We shall start with the Almgren-Pitts min-max theory with multiple parameters.

For a closed Riemannian manifold (M^{n+1}, g), if we fix $G = \mathbb{Z}_2$ and $m = n$ in Almgren's isomorphism (1.1.1),

$$\pi_k(\mathcal{Z}_n(M^{n+1}; \mathbb{Z}_2), 0) = \begin{cases} \mathbb{Z}_2, k = 1 \\ \mathbb{Z}_2, k > 1 \end{cases} \tag{2.0.1}$$

and this implies that $\mathcal{Z}_n(M^{n+1}; \mathbb{Z}_2)$ (for simplicity, we only consider the connected component containing 0) is weakly homotopy equivalent to $\mathbb{RP}^\infty$ ([53], Section 4). Hence, the space has nontrivial topology, which makes the Morse theory possible. Let's denote the generator of $H^1(\mathcal{Z}_n(M^{n+1}; \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$ by $\bar{\lambda}$.

Definition 2.0.1 ([59], [42])**.** Given a Riemannian manifold (M^{n+1}, g), a continuous map $\Phi : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; \mathbb{Z}_2)$ is called a ***p*-sweepout**, provided that X is a cubic subcomplex and $\Phi^*(\bar{\lambda}^p) \neq 0$. In addition, the set of all p-sweepouts is called the ***p*-admissible set** $\mathcal{P}_p = \mathcal{P}_p(M, g)$ on M.

Definition 2.0.2. The (min-max) ***p*-width** $\omega_p(M,g)$ is

$$\omega_p(M,g) := \inf_{\Phi\in\mathcal{P}_p}\sup_x \mathbf{M}(\Phi(x)). \tag{2.0.2}$$

Definition 2.0.3. A **min-max sequence** of $\mathcal{P}_p$ is a sequence of p-sweepouts

$$S = \{\Phi_i\}_{i=1}^{\infty} \subset \mathcal{P}_p, \tag{2.0.3}$$

with

$$\lim_{i\to\infty}\sup_x \mathbf{M}(\Phi_i(x)) = \omega_p(M,g). \tag{2.0.4}$$

Its **critical set** is

$$\mathbf{C}(S) = \{V\in\mathcal{V}_n(M) | \|V\|(M) = \omega_p(M,g) \text{ and } V = \lim_j |\Phi_{i_j}(x_j)|\}. \tag{2.0.5}$$

Note that the definition of $\mathcal{P}_p$ seems more restrictive than those in the literatures [31], [33], [54], but in fact, one can prove that the ω_p defined here coincide with those in the literatures (See Chapter 3).

Theorem 2.0.4 (Min-max theorem for p-width, *c.f.* Corollary 3.2.7)**.** *Given a Riemannian manifold (M^{n+1},g), for each $p\in\mathbb{N}$ and any min-max sequence S of $\mathcal{P}_p$, there exists a minimal hypersurfaces with optimal regularity $V\in\mathbf{C}(S)$, with the property $(2p+1)$ and $\|V\|(M) = \omega_p(M,g)$.*

Remark 2.0.5. A minimal hypyersurface with optimal regularity here means that it is a n-dimensional stationary varifold whose singular set has Hausdorff codimension no less than 7. In particular, for $3\le n+1\le 7$, its support is a union of smooth minimal hypersurfaces.

Here, the property (m) is taken from the proof in Pitts' combinatorial arguments [59, Proposition 4.9], which turns out to be helpful in proving the compactness [42].

Definition 2.0.6. A varifold V is said to have the **property ($\boldsymbol{m}$)**, provided that for any point $p \in M$ and $I_m = 3^{2m}$ concentric annuli $\{A_0(p, r_i - s_i, r_i + s_i)\}$ where $\{r_i\}$ and $\{s_i\}$ satisfy

$$r_i - 2s_i > 10(r_{i+1} + 2s_{i+1}), i = 1, \cdots, I_m - 1 \tag{2.0.6}$$

$$r_{I_m} - 2s_{I_m} > 0, \tag{2.0.7}$$

V is almost minimizing [59, Definition 3.1] in at least one of $\{A_0(p, r_i - s_i, r_i + s_i)\}$. In particular, the support of V has optimal regularity.

In general, we can define an m-parameter $\mathbf{F}$-homotopy family Π whose elements are maps with domains as a subcomplex of $I(m, k)$.

Definition 2.0.7. We call a set Π of continuous maps from finite dimensional simplicial complexes to $\mathcal{Z}_n(M^{n+1}; \mathbf{F}; \mathbb{Z}_2)$ an ***m*-parameter F-homotopy family** if the following properties hold.

- For any $\Phi \in \Pi$, $X = \mathrm{dmn}(\Phi)$ is a subcomplex of $I(m, k)$ for some $k \in \mathbb{N}^+$.
- For any $\Phi \in \Pi$, every continuous $\Phi' : \mathrm{dmn}(\Phi) \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; \mathbb{Z}_2)$ homotopic to Φ in the $\mathbf{F}$ topology also lies in Π.

Definition 2.0.8. The **min-max width** of Π is

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in \mathrm{dmn}(\Phi)} \mathbf{M}(\Phi(x)) \,. \tag{2.0.8}$$

Definition 2.0.9. A **min-max sequence** of Π is a sequence

$$S = \{\Phi_i\}_{i=1}^{\infty} \subset \Pi \,, \tag{2.0.9}$$

with

$$\lim_{i \to \infty} \sup_x \mathbf{M}(\Phi_i(x)) = \mathbf{L}(\Pi) \,. \tag{2.0.10}$$

Its **critical set** is

$$\mathbf{C}(S) = \{V \in \mathcal{V}_n(M) | \|V\|(M) = \mathbf{L}(\Pi) \text{ and } V = \lim_j |\Phi_{i_j}(x_j)|\}\,. \tag{2.0.11}$$

Theorem 2.0.10 (Min-max theorem for Π)**.** *If Π is a m-parameter $\mathbf{F}$-homotopy family, $\mathbf{L}(\Pi) > 0$, and S is a min-max sequence of Π, then there exists a minimal hypersurfaces with optimal regularity $V \in \mathbf{C}(S)$ with the property (m) and $\|V\|(M) = \mathbf{L}(\Pi)$.*

For completeness, we sketch the proof for these Min-max theorems. The details can be found in [59] and [55].

Sketch of proof. Let $S = \{\Phi_i\}$. All the varifolds in the critical set $\mathbf{C}(S)$ have the same volume, i.e., the min-max width. However, some varifolds might not be stationary or with optimal regularity. Thus, the main goal is to perturb away the "bad" varifolds by constructing homotopic deformations.

Step 1: Pull tight.

For each non-stationary varifold $V \in \mathcal{V}_n(M)$, we can choose the dual vector field of δV which will decrease its volume rapidly. If we "combine" these vector fields on $\mathcal{V}_n(M)$ using a partition of unitiy, we will obtain a continuous deformation as the induced flow on $\mathcal{V}_n(M)$ in the $\mathbf{F}$-topology, which can be pulled back to the cycle space with $\mathbf{F}$ metric under the map $T \to |T|$. For any $\Phi_i \in S$, the deformation induces a homotopy map

$$H_i^{(1)} : [0,1] \times \mathrm{dmn}(\Phi_i) \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)\,, \tag{2.0.12}$$

such that $H_i^{(1)}(0) = \Phi_i$.

Let $S^* = \{\Psi_i := H_i^{(1)}(1)\}$. We can check that it is a min-max sequence with $\mathbf{C}(S^*) \subset \mathbf{C}(S)$ and every $V \in \mathbf{C}(S^*)$ is stationary. It is also worthy to note that

$\sup_{t,x} \mathbf{M}(H_i^{(1)}(t,x)) \leq \sup_x \mathbf{M}(\Phi_i(x))$.

Step 2: Discretization.

By [55, Proposition 3.7], for each $\Psi_i : X_i \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ and any $\delta_i > 0$, we can deform Ψ_i to a continuous map $\Psi_i' : X_i \to \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$ along a homotopy

$$H_i^{(2)} : [0,1] \times X_i \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)\,, \tag{2.0.13}$$

with $H_i^{(2)}(0) = \Psi_i$ and $H_i^{(2)}(1) = \Psi_i'$, such that

$$\sup_{t,x} \mathbf{F}(H_i^{(2)}(t,x), \Psi_i(x)) < \delta_i. \tag{2.0.14}$$

Thus, we always have $\sup_{t,x} \mathbf{M}(H_i^{(2)}(t,x)) \leq \sup_x \mathbf{M}(\Psi_i(x)) + \delta_i$. If $\delta_i \to 0$, it follows immediately that $S' := \{\Psi_i'\}$ is a min-max sequence with

$$\mathbf{C}(S') = \mathbf{C}(S^*). \tag{2.0.15}$$

Following the proof of [52, Theorem 3.8] (See also 3.2.2), we can find a family of discrete sweepouts to approximate S'. More precisely, for each $\Psi_i' \in S'$ with $X_i := \mathrm{dmn}(\Psi_i') \subset I(m, n_i)$, we can take $N_i > n_i$ large enough and define a discrete sweepout

$$\psi_i := \Psi_i'|_{X_i \cap I(m,N_i)_0}, \tag{2.0.16}$$

such that the fineness

$$\mathbf{f}(\psi_i) := \sup_{x,y \in I(m,N_i)_0} |\psi_i(x) - \psi_i(y)| \cdot 3^{N_i} \cdot d_0(x,y) < (\delta_i)^2\,, \tag{2.0.17}$$

where d_0 is the Manhattan metric.

Now, since $\mathbf{C}(\{\psi_i\}) \subset \mathbf{C}(S')$, it suffices to show that there exists V in $\mathbf{C}(\{\psi_i\})$

with optimal regularity and property (m).

Step 3: Interpolation.

We will only show the property (m), since the optimal regularity can be proved similarly using compactness theorems in [65].

Let's argue by contradiction and suppose that there exists $\varepsilon_0 > 0$ with the following property.

For any V in $\mathbf{C}(S)$ and any $\delta > 0$, there exists $I_m = 3^{2m}$ concentric annuli, such that any current whose associated varifold is in $\mathbf{B}^{\mathbf{F}}(V, \varepsilon_0)$, has an (ε_0, δ)-deformation [55, Definition 4.1] in each concentric annuli. Roughly speaking, an (ε_0, δ)-deformation means that the mass increases by at most δ along a (discrete) homotopy, but decreases by at least ε_0 eventually.

We've chosen I_m large enough with the following combinatorial property. For each $x \in X_i \cap I(m, N_i)_0$, we can associate it with an annulus in the concentric annuli for $\psi_i(x)$, such that, for any nearby x, y, the two annuli chosen for $\psi_i(x)$ and $\psi_i(y)$ respectively will not intersect. Thus, we could construct a deformation h_i along which $\sup \mathbf{M}(\psi_i(x))$ increases by at most $(\delta_i)^2$, but eventually decreases by at least $3\varepsilon_0/4$.

Now, it follows from [54, Theorem 3.10] that for i large enough, we can apply the Almgren's interpolation to ψ_i and the deformation h_i we just constructed, to obtain a continuous sweepout Ψ_i'' and a homotopy $H_i^{(4)}$, where Ψ_i'' can be deformed from Ψ_i' via a homotopy $H_i^{(3)}$. It's worthy to note that for i large enough, i.e., δ_i small enough, $H_i^{(3)}$ and $H_i^{(4)}$ will not increase $\sup_x \mathbf{M}(\Psi_i'(x))$ by more than δ_i.

In summary, when δ_i is chosen even smaller (e.g. $100\delta_i < \varepsilon_0$), we can concatenate $H_i^{(1)}$, $H_i^{(2)}$, $H_i^{(3)}$ and $H_i^{(4)}$, and obtain a new sweepout denoted by $\tilde{\Psi}_i$ homotopic to Φ_i with

$$\lim_{i\to\infty} \sup_x \mathbf{M}(\tilde{\Psi}_i(x)) < \lim_{i\to\infty} \sup_x \mathbf{M}(\Phi_i(x)) - \varepsilon_0/2\,, \tag{2.0.18}$$

contradicting to the fact that $\{\Phi_i\}$ is a min-max sequence. Hence, there exits a V

with the property (m). $\square$

Another variant min-max setting that we shall be interested in is based on the (X, Z)-homotopy class introduced by X. Zhou in [86].

On $\mathcal{C}(M)$, we will define the $\mathbf{F}$-metric to be

$$\mathbf{F}(\Omega_1, \Omega_2) = \mathcal{F}(\Omega_1 - \Omega_2) + \mathbf{F}(|\partial\Omega_1|, |\partial\Omega_2|)\,. \tag{2.0.19}$$

For a constant $c > 0$, let's define the weighted area functional $\mathcal{A}^c : \mathcal{C}(M) \to \mathbb{R}$ to be

$$\mathcal{A}^c(\Omega) = \mathcal{H}^n(\partial\Omega) - c\mathcal{H}^{n+1}(\Omega)\,. \tag{2.0.20}$$

The first variational formula of $\mathcal{A}^c$ along a smooth vector field $Y \in \mathfrak{X}(M)$ is

$$\delta\mathcal{A}^c(\Omega)(Y) = \int_{\partial\Omega} \mathrm{div}_{\partial\Omega} Y \,\mathrm{d}\mu_{\partial\Omega} - \int_{\partial\Omega} c\langle Y, \nu\rangle \mathrm{d}\mu_{\partial\Omega}, \tag{2.0.21}$$

where $\nu = \nu_{\partial\Omega}$ is the outward unit normal on $\partial\Omega$. It follows that the smooth part of the boundary of a critical point is in fact a hypersurface with constant mean curvature c, i.e., c-CMC hypersurface.

Its second variational formula for Ω along $Y = \varphi\nu$ is the same as that of the usual area functional for $\Sigma = \partial\Omega$, i.e.,

$$\delta^2\mathcal{A}^c(\Omega)(Y, Y) = II_\Sigma(\varphi, \varphi) = \int_\Sigma \left(|\nabla\varphi|^2 - \left(\mathrm{Ric}^M(\nu, \nu) + |A_\Sigma|^2\right)\varphi^2\right). \tag{2.0.22}$$

Definition 2.0.11 ([86, Definition 1.1])**.** Given a pair (X, Z), where Z is a subcomplex of X, and $\Phi_0 : X \to (\mathcal{C}(M), \mathbf{F})$, we define the **$(X, Z)$-homotopy class of Φ_0**, denoted by $\Pi(\Phi_0)$, to be the set of all sequences of maps $\{\Phi_i : X \to (\mathcal{C}(M), \mathbf{F})\}$ satisfying the following conditions:

1. Each Φ_i is homotopic to Φ_0 in the $\mathbf{F}$-topology on $\mathcal{C}(M)$;

2. The homotopy $H_i : [0,1] \times X \to (\mathcal{C}(M), \mathbf{F})$ associated with Φ_i satisfies

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(H(t,x), \Phi_0(x)) : t \in [0,1], x \in Z\} = 0\,. \tag{2.0.23}$$

Here, each $\{\Phi_i\}_{i\in\mathbb{N}}$ is called a **(X, Z)-homotopy sequence of mappings into $\mathcal{C}(M)$**.

Definition 2.0.12. The ***c*-width** of $\Pi(\Phi_0)$ is defined by:

$$\mathbf{L}^c(\Pi(\Phi_0)) = \inf_{\{\Phi_i\}\in\Pi(\Phi_0)} \limsup_{i\to\infty} \sup_{x\in X}\{\mathcal{A}^c(\Phi_i(x))\}\,. \tag{2.0.24}$$

Definition 2.0.13. A **min-max sequence** of $\Pi(\Phi_0)$ is a sequence

$$S = \{\Phi_i\}_{i=1}^{\infty} \in \Pi(\Phi_0), \tag{2.0.25}$$

with

$$\lim_{i\to\infty} \mathbf{L}^c(\Phi_i) = \mathbf{L}^c(\Pi(\Phi_0)). \tag{2.0.26}$$

Its **critical set** is

$$\mathbf{C}(S) = \{V \in \mathcal{V}_n(M) | V = \lim_j |\Phi_{i_j}(x_j)| \text{ and } \lim_j \mathcal{A}^c(\Phi_{i_j}(x_j)) = \mathbf{L}^c(\Pi(\Phi_0))\}. \tag{2.0.27}$$

Following a similar proof as sketched above, one can obtain the min-max theorem for $\Pi(\Phi_0)$.

Theorem 2.0.14 (Min-max Theorem for $\Pi(\Phi_0)$, [87, Theorem 0.3] [88, Theorem 0.3] [86, Theorem 1.7] [21, Theorem 3.4])**.** *Given a closed Riemannian manifold* (M^{n+1}, g) $(n+1 \geq 3)$ *and* $c > 0$*, if the* (X, Z)*-homotopy class* $\Pi(\Phi_0)$ *of* $\Phi_0 : X \to \mathcal{C}(M)$ *satisfies*

$$\mathbf{L}^c(\Pi(\Phi_0)) > \sup_{x\in Z} \mathcal{A}^c(\Phi_0(x))\,, \tag{2.0.28}$$

then there exists a $\Omega \in \mathcal{C}(M)$ *such that* $\mathcal{A}^c(\Omega) = \mathbf{L}^c(\Pi(\Phi_0))$ *and* $\partial\Omega$ *is a c-CMC hypersurface with optimal regularity.*

Chapter 3

Generic abundance

In this chapter ([42]), we will confirm Yau's conjecture 1.2.2 in higher dimension ($n \geq 7$) for a closed manifold M with a C^∞-generic metric g. Due to the existence of nontrivial singularities in area-minimizing currents in higher dimension, we can at best expect that the minimal hypersurfaces have optimal regularity, i.e., codim(Sing) ≥ 7.

Theorem 3.0.1 (Main Theorem)**.** *Given a closed manifold $M^{n+1}(n \geq 7)$, there exists a (Baire sense) generic subset of C^∞ metrics such that M endowed with any one of those metrics contains infinitely many singular minimal hypersurfaces with optimal regularity.*

Here is the outline of the proof of our main theorem.

First, we establish the compactness for the Almgren-Pitts Realization $\mathcal{APR}(\Pi)$ of the min-max width of an m-parameter homotopy family Π, where $\mathcal{APR}(\Pi)$ is a nonempty subset of minimal hypersurfaces with optimal regularity, volume $\mathbf{L}(\Pi)$ and certain stability property.

Then, by showing that the p-width could be achieved by the min-max width of an m-parameter homotopy family $\mathcal{P}_{p,m}$, we also have the compactness for the Almgren-Pitts Realization $\mathcal{APR}_p$ of p-width.

Finally, the proof of the main theorem follows the idea of Proposition 3.1 in [39]

with a tricky modification. Roughly speaking, to overcome the difficulty of the lack of bumpy metrics, we will apply the compactness results to obtain the openness of "good" metrics and use $\mathcal{M}_f$ (the set of metrics where Yau's conjecture fails) to be the starting point of metric perturbation. If we divide the metric space into two parts $\mathcal{O} = \mathrm{Int}(\overline{\mathcal{M}_f})$ and $\mathcal{O}^c$, then on the one hand, in $\mathcal{O}$, with two ingredients mentioned above, we can show that $\mathcal{M}_f \cap \mathcal{O}$ is meagre. On the other hand, by definition, it is clear that $\mathcal{M}_f \cap \mathcal{O}^c$ is nowhere dense. In summary, $\mathcal{M}_f$ is meagre.

In addition, together with White's structure theorem on minimal submanifolds [81], our proof above also implies the generic denseness of min-max minimal hypersurfaces.

Theorem 3.0.2 (a.k.a. Corollary 3.3.5)**.** *For a closed manifold $M^{n+1}(2 \leq n \leq 6)$ with a C^∞-generic metric g, min-max minimal hypersurfaces are dense.*

3.1 Preliminaries

3.1.1 Basic Notations in the Almgren-Pitts Min-max Theory

In the following, we will use $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ to denote the space of modulo two n-dimensional flat cycles endowed with the flat topology. In Almgren's thesis [4], he gave a natural isomorphism

$$\pi_k(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2), 0) = H_{n+k}(M^{n+1}, \mathbb{Z}_2), \tag{3.1.1}$$

and later, it was shown that $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$ ([53], Section 4). We denote the generator of $H^1(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$ by $\bar{\lambda}$.

In the cycles space, we can define $\mathbf{M}$ to be the mass norm or mass functional (See [27], Section 4.2). The nontrivial topology of $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ indicates that the min-max theory for the mass functional can be developed in it.

Let X be a finite dimensional simplicial complex and we call a map Φ a **p-sweepout** if it is a continuous map $\Phi : X \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ such that $\Phi^*(\bar{\lambda}^p) \neq 0$. The **$p$-admissible set** $\mathcal{P}_p = \mathcal{P}_p(M, g)$ is the set of all p-sweepouts Φ that have **no mass concentration** (See [51] Section 4), i.e.

$$\lim_{r\to 0} \sup\{\mathbf{M}(\Phi(x) \cap B_r(y)) : x \in X, y \in M\} = 0. \tag{3.1.2}$$

Now the **p-width** $\omega_p(M, g)$ can be defined by $\inf_{\Phi\in\mathcal{P}_p} \sup\{\mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\}$ ([31], [33], [54], [48]).

We also define a **min-max sequence** $S = \{\Phi_i\}_{i\in\mathbb{N}}$ to be a sequence in $\mathcal{P}_p$ satisfying

$$\lim_{i\to\infty} \sup_{x\in X_i} \mathbf{M}(\Phi_i(x)) = \omega_p. \tag{3.1.3}$$

The critical set of S is

$$\mathbf{C}(S) = \{V | \|V\|(M) = \omega_p \text{ and } V = \lim_j |\Phi_{i_j}(x_j)| \text{ for a sequence } \{x_j \in X_{i_j}\}\}. \tag{3.1.4}$$

Note that the domains of maps in $\mathcal{P}_p$ could be different. In order to get the compactness, we need to make some restriction on the domains.

For convenience, we shall use some notions of cell complexes from [59].

Definition 3.1.1 ([59],4.1(1))**.**

- For $n \in \mathbb{N}^+$, let $I(1, n)$ denote the cell complex on the unit interval I whose 1-cells are the intervals $[0, 1 \cdot 3^{-n}], [1 \cdot 3^{-n}, 2 \cdot 3^{-n}], \cdots, [1 - 3^{-n}, 1]$, and whose 0-cells are the endpoints $[0], [3^{-n}], [2 \cdot 3^{-n}], \cdots, [1]$.

- For $n, m \in \mathbb{N}^+$, $I(m, n) = I(1, n)^{m\otimes} = I(1, n) \otimes I(1, n) \otimes \cdots \otimes I(1, n)$ is the cell complex on I^m. In addition, $I(m, n)_0$ denote the set of all 0-cells in $I(m, n)$.

Now we can give a definition of an m-parameter homotopy family in general,

which need not be a subset of some $\mathcal{P}_p$.

Definition 3.1.2. We call a subset Π of continuous maps from finite dimensional simplicial complexes to $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ a **(continuous) m-parameter homotopy family** if the following properties hold.

- For any $\Phi \in \Pi$, $X = \mathrm{dmn}(\Phi)$ is a subcomplex of $I(m, k)$ for some $k \in \mathbb{N}^+$ and Φ has no mass concentration.
- For any $\Phi \in \Pi$, every continuous $\Phi' : \mathrm{dmn}(\Phi) \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ homotopic to Φ in the flat topology also lies in Π, provided that Φ' has no mass concentration.

Remark 3.1.3. In Pitts' original proof [59], he considered discrete sweepouts defined in $[I^m, \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)]^{\#}$. Fortunately, due to the remarkable interpolation results by F.C. Marques and A. Neves ([51], Section 13 and 14), the discrete settings and the continuous settings are interchangeable in some sense. Thus, the m-parameter homotopy family defined here will preserve most of the properties that the discrete homotopy families have.

Similar to p-width and the min-max sequence for p-width, we can also define a min-max invariant for the m-parameter homotopy family Π,

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{\mathbf{M}(\Phi(x))\}, \tag{3.1.5}$$

and a min-max sequence $S = \{\Phi_i\} \subset \Pi$,

$$\limsup_{i \to \infty} \sup_{x \in X_i = \mathrm{dmn}(\Phi_i)} \mathbf{M}(\Phi_i(x)) = \mathbf{L}(\Pi). \tag{3.1.6}$$

Now the critical set of S is

$$\mathbf{C}(S) = \{V | \|V\|(M) = \mathbf{L}(\Pi) \text{ and } V = \lim_{j \to \infty} |\Phi_{i_j}(x_j)| \text{ for a sequence } \{x_j \in X_{i_j}\}\}. \tag{3.1.7}$$

3.1.2 Singular Minimal Hypersurfaces with Optimal Regularity and Compactness with Stability Condition

Definition 3.1.4. We call a varifold V in M^{n+1} a **singular minimal hypersurface with optimal regularity**, if it is an n-dimensional stationary integral varifold and its singular part has dimension $\dim(\mathrm{Sing}(V)) \leq n-7$.

By Allard compactness (See [1], [70]), a sequence $\{V_j\}$ of singular minimal hypersurfaces with optimal regularity with bounded volume (up to a subsequence) will converge to an n-dimensional stationary integral varifold in the varifold sense. However, without extra information of the sequence, we could not obtain further regularity of the limit varifold.

If we further assume that $\{V_j\}$ is stable in some open subset U, then from the result of [65] (See also [82]), we have the following compactness consequence.

Proposition 3.1.5. *Given a sequence of singular minimal hypersurfaces with optimal regularity $\{V_j\}$ in a smooth closed manifold (M, g) with uniformly bounded volume and converging to an n-dimensional stationary integral varifold V in the varifold sense, suppose that each V_j is stable (See the definition in [65]) in an open subset U of M, then V is stable in U and of optimal regularity in U.*

Remark 3.1.6. This is also true when V_j is defined on (M, g_j) and g_j converges to g in C^3 ([65], Theorem 2).

Recently, A. Dey [23] generalized this result with the assumption that the p-th eigenvalue is uniformly bounded from below.

3.2 Minimal Hypersurfaces from Almgren-Pitts Min-max Construction

In this section, we will prove some properties of singular minimal hypersurfaces obtained from Almgren-Pitts Min-max construction, especially realizations of p-widths and their compactness.

3.2.1 Almgren-Pitts Realizations of m-parameter Homotopy Families and Their Compactness

One of the novelties in Pitts' monograph is the application of the combinatorial arguments, Proposition 4.9 and Theorem 4.10 in [59], from which he could prove the regularity of varifolds in a critical set. Here, we shall show that these arguments could imply more properties of varifolds in a critical set.

First, we adapt Proposition 4.9 in [59] to our current setup and improve the constant slightly.

In the following, $A(p, s, r)$ denotes an open annulus in M centered at p with inner radius s and outer radius r, provided that $s < r$ and r is no greater than the injective radius of M at p. $\bar{A}(p, s, r)$ is simply its closure.

Lemma 3.2.1. *Suppose that X is a subcomplex of some $I(m, n)$. If every cell σ of X is associated with a point $p_\sigma \in M$ and a finite set $A(\sigma) = \{\bar{A}(p_\sigma, s_j, r_j)\}_{j=1,2,\cdots,5^m}$, where $r_j > s_j > 10r_{j+1} > 0$ ($r_1 < \mathrm{inj}_M/2$), then we can find a function defined on the set of cells of X (denoted by $X(n)$),*

$$\alpha : X(n) \to \bigcup_{\sigma \in X(n)} A(\sigma), \tag{3.2.1}$$

such that $\alpha(\sigma) \in A(\sigma)$ and $\alpha(\sigma) \cap \alpha(\tau) = \emptyset$ whenver $\sigma \neq \tau$ and $\sigma, \tau < \gamma$ for some cell $\gamma \in X(n)$.

Proof. We shall define α inductively.

Let $\mathcal{D} \subset X(n)$ where α is already defined. In addition, by reducing A when $\mathcal{D}$ gets larger, we always assume that the finite but nonempty sets $A(\sigma)$'s satisfy the following conditions.

- $\forall \sigma_1 \in \mathcal{D}, \sigma_2 \in X(n)\backslash\mathcal{D}$ with $\sigma_1, \sigma_2 < \gamma$ for some $\gamma \in X(n)$,

$$\alpha(\sigma_1) \cap \bar{A}(p_{\sigma_2}, s, r) = \emptyset, \quad \forall \bar{A}(p_{\sigma_2}, s, r) \in A(\sigma_2)\,. \tag{3.2.2}$$

At the beginning, $\mathcal{D} = \emptyset$ so the assumptions above hold trivially.

Whenever $\mathcal{D} \neq X(n)$, we can take an annulus with the smallest outer radius in $\bigcup_{\sigma \in X(n)\backslash\mathcal{D}} A(\sigma)$, say, $\bar{A}(p_{\sigma_0}, s, r) \in A(\sigma_0)$ for some $\sigma_0 \in X(n)\backslash\mathcal{D}$. Now, σ_0 can be added into $\mathcal{D}$ and let $\alpha(\sigma_0) := \bar{A}(p_{\sigma_0}, s, r)$. Note that α still satisfies the pairwisely disjoint conditions on $\mathcal{D}$ due to relations (3.2.2) in the inductive assumptions.

Then, we will modify A so that the inductive assumptions hold for this new $\mathcal{D}$. For each $\tau \in X(n)\backslash\mathcal{D}$ which is a face of some cell $\gamma \in X(n)$ with $\sigma_0 < \gamma$, we remove $\bar{A}(p_\tau, \tilde{s}, \tilde{r}) \in A(\tau)$ from the set $A(\tau)$ if $\bar{A}(p_\tau, \tilde{s}, \tilde{r}) \cap \alpha(\sigma_0) \neq \emptyset$, which obviously leads to relations (3.2.2). Now it suffices to verify the nonemptiness. Since $\bar{A}(p_{\sigma_0}, s, r)$ is of the smallest outer radius, by the radius relations of the annuli in $A(\tau)$, it is easy to check that we remove at most one annulus from each $A(\tau)$. Note that X is a subcomplex of $I(m, n)$, so for any cell $\tau \in X(n)$, we can find at most $5^m - 1$ other cells such that any one of them and τ are both faces of some cell $\gamma \in X(n)$. In other words, there will be at most $5^m - 1$ annuli being removed from each $A(\tau)$ and therefore, $A(\tau)$ will never be empty, which conclude the inductive assumptions.

It follows from the finiteness of the size of $X(n)$ that the inductive procedure above could give the definition of α on the whole $X(n)$ as desired. □

Now, we can state our main lemma on the properties of varifolds in a critical set.

Lemma 3.2.2. *Suppose $S = \{\Phi_i\}$ is a pulled-tight critical sequence (See Section 15 in [51]) for an m-parameter homotopy family Π with $\mathbf{L}(\Pi) > 0$. Then there exists a varifold $V \in \mathbf{C}(S)$ such that for any $I_m = 5^m$ concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with $r_j > 0$, $s_j > 0$, $r_j - 2s_j > 10(r_{j+1} + 2s_{j+1})$, $r_{I_m} - 2s_{I_m} > 0$ and $r_1 < \mathrm{inj}_M/2$, V is almost minimizing (See Definitions 3.1 in [59]) in at least one of the annuli. Therefore, by Theorem 3.3 in [59], V has **property (m)**:*

For any I_m concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with $r_j > 0$, $s_j > 0$, $r_j - 2s_j > 10(r_{j+1} + 2s_{j+1})$, $r_{I_m} - 2s_{I_m} > 0$ and $r_1 < \mathrm{inj}_M/2$, V is stable in at least one of the annuli.

Proof. Suppose not, and then for each V in $\mathbf{C}(S)$, there is $p^V \in M$ such that there exist 5^m concentric annuli $A(p^V, r_j^V - s_j^V, r_j^V + s_j^V)$ with the conditions mentioned above and V is not almost minimizing in any one of the annuli.

As explained in Remark 3.1.3, one could use a family of discrete sweepouts to approximate S. Roughly speaking, for each $\Phi_i \in S$ with $X_i = \mathrm{dmn}(\Phi_i)$ where $X_i \subset I(m, n_i)$ is a subcomplex, we can take $N_i > n_i$ large enough and define

$$\psi_i : X_i \cap I(m, N_i)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2), \tag{3.2.3}$$

which approximates Φ_i. Note that the discrete sweepouts $S_\# = \{\psi_i\}$ satisfy that $\mathbf{C}(S_\#) = \mathbf{C}(S)$ and the Almgren extension Ψ_i of ψ_i is homotopic to Φ_i in the flat topology. Interested readers might refer to the proof for Theorem 3.8 in [52] for more details. Although they require that each Φ_i is continuous with respect to the $\mathbf{F}$ norm (See the definition in [59] 2.1(20)), the same proof works in our setup as well.

For convenience, in the following, we may assume that X_i itself is a subcomplex of $I(m, N_i)$, since one could always refine X_i in a canonical way.

Now following the original proof of Theorem 4.10 in [59], for i large enough, we can assign to each face $\sigma \in X_i(N_i)$ a set $A(\sigma) = \{A(p^{V_\sigma}, r_j^{V_\sigma} - s_j^{V_\sigma}, r_j^{V_\sigma} + s_j^{V_\sigma})\}_{j=1,\cdots,5^m}$

for some V_σ associated to σ, where V_σ is one of the $\{V_1, \dots, V_\nu\}$ therein. By Lemma 3.2.1, we could define α on $X_i(N_i)$. Therefore, the existsence of a homotopic family of discrete sweepouts $S^*_\# = \{\psi^*_i\}$ such that $\mathbf{L}(S^*_\#) < \mathbf{L}(S_\#) = \mathbf{L}(S)$ is just verbatim.

The Almgren extension Ψ^*_i of ψ^*_i is homotopic to Ψ_i and thus to Φ_i which implies that $\{\Psi^*_i\} \subset \Pi$. As long as each N_i is chosen large enough, $\mathbf{L}(\{\Psi^*_i\}) = \mathbf{L}(S^*_\#) < \mathbf{L}(S)$, which gives a contradiction to the choice of S. Hence, the conclusion holds. □

By applying Theorem 4 in [65], the varifold obtained in the lemma above is indeed a singular minimal hypersurface with optimal regularity.

Definition 3.2.3. We define **Almgren-Pitts Realization** of the m-parameter homotopy family Π, denoted by $\mathcal{APR}(\Pi)$, to be the nonempty set of varifolds V satisfying

- $\|V\|(M) = \mathbf{L}(\Pi)$;
- V is a singular minimal hypersurface with optimal regularity;
- V has property (m).

Now, we can show that $\mathcal{APR}$ has following compactness property.

Proposition 3.2.4 (Compactness of $\mathcal{APR}$)**.** *Given a sequence of m-parameter homotopy families Π_i with $0 < \inf \mathbf{L}(\Pi_i) \leq \sup \mathbf{L}(\Pi_i) < \infty$ and $V_i \in \mathcal{APR}(\Pi_i)$, there is a subsequence (still denoted using index i) Π_i such that*

$$\lim_{i\to\infty} \mathbf{L}(\Pi_i) = L \in \mathbb{R}^+, \tag{3.2.4}$$

and

$$V_i \rightharpoonup V, \tag{3.2.5}$$

in the varifold sense. Moreover, $\mathrm{spt}(V)$ *is also a singular minimal hypersurface with optimal regularity and property (m).*

Proof. By Allard compactness, we only need to check that V has property (m) and optimal regularity.

Firstly, we show that V has property (m).

Suppose not, and then we can find a set of I_m concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with the conditions mentioned in property (m), such that V is not stable in any annulus. By Proposition 3.1.5, we know that for each j, there is a positive integer $i_j > 0$ such that when $i > i_j$, V_i is not stable in each annulus. If we take $i_0 = \max_{j=1,\dots,I_m}\{i_j\}$, then V_i is not stable in any one of $\{A(p, r_j - s_j, r_j + s_j)\}$ provided that $i > i_0$.

This contradicts to the definition of $\mathcal{APR}(\Pi_i)$ that V_i has property (m).

Next, to show the optimal regularity of V, note that the same argument above together with Proposition 3.1.5 also implies that for any $p \in M$ and any I_m concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with the same properties mentioned above, V is stable and of optimal regularity inside at least one of them. Thus, there exists a constant $r_p \in (0, \mathrm{inj}_M/2)$ depending only on p such that V is stable and of optimal regularity in $A(p, s, r_p)$ for any $s \in (0, r_p)$ (See Theorem 4.10 in [59]).

By Theorem 3.1 in [82] and the Remark (3) before it, we know that V is of optimal regularity in the open ball $B(p, r_p)$. Since M is compact, taking a finite open cover, one can easily show that V is of optimal regularity in M. □

3.2.2 Almgren-Pitts Realizations of (p, m)-width and p-width and Their Compactness

In [83], G. Xu defines (p, m)-width to be

$$\omega_{p,m}(M, g) = \inf_{\Phi \in \mathcal{P}_{p,m}} \max_{x \in \mathrm{dmn}(\Phi)} \mathbf{M}(\Phi(x)), \tag{3.2.6}$$

where $\mathcal{P}_{p,m}$ is the set of mass-concentration-free sweepouts from a subcomplex of some $I(m,k)$ into $\mathcal{Z}_n(M;\mathbb{Z}_2)$ detecting $\bar{\lambda}^p$. And he also proved that when $m \geq 2p+1$, $\mathcal{P}_{p,m} \neq \emptyset$ and the (p,m)-width can be realized by a singular minimal hypersurface with optimal regularity.

Note that the only difference between p-width and (p,m)-width is the domain of the sweepouts. Since $\mathcal{P}_{p,m}$ is an m-parameter homotopy family, the realization of (p,m)-width is just a corollary of the compactness property (Proposition 3.2.4).

Corollary 3.2.5 ([83], Theorem 1.12)**.** *For $m \geq 2p+1$, there is a varifold V such that $\|V\|(M) = \omega_{p,m}(M)$ and $\mathrm{spt}(V)$ is a singular minimal hypersurface with optimal regularity.*

It is obvious that

$$\omega_{p,2p+1}(M) \geq \omega_{p,2p+2}(M) \geq \cdots \geq \omega_{p,m}(M) \geq \cdots \geq \omega_p(M). \tag{3.2.7}$$

The question is whether for some $m \geq 2p+1$, we can have the equality between $\omega_{p,m}(M)$ and $\omega_p(M)$. Here, we would like to confirm this by a simple argument.

Proposition 3.2.6. $\omega_{p,2p+1}(M,g) = \omega_p(M,g)$.

Proof. Given a min-max sequence $S = \{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_p$ for ω_p, denote $X_i^{(p)}$ as the p-dimensional skeleton of $X_i = \mathrm{dmn}(\Phi_i)$, and following the proof in Proposition 2.2 in [39], we have

$$H^l(X_i, X_i^{(p)}; \mathbb{Z}_2) = 0 \ (l \leq p), \tag{3.2.8}$$

and the exact sequence

$$\cdots \to H^p(X_i, X_i^{(p)}; \mathbb{Z}_2) \to H^p(X_i; \mathbb{Z}_2) \to H^p(X_i^{(p)}; \mathbb{Z}_2) \to H^{p+1}(X_i, X_i^{(p)}; \mathbb{Z}_2) \to \cdots . \tag{3.2.9}$$

The pullback map from $H^p(X_i;\mathbb{Z}_2)$ to $H^p(X_i^{(p)};\mathbb{Z}_2)$ is injective so $\Phi_i|_{X_i^{(p)}} \in \mathcal{P}_p$.

Since

$$\begin{aligned}\omega_p(M,g) &\leq \limsup_{i\to\infty}\{\mathbf{M}(\Phi_i(x))|x\in X_i^{(p)}\}\\ &\leq \limsup_{i\to\infty}\{\mathbf{M}(\Phi_i(x))|x\in X_i\}=\omega_p(M,g),\end{aligned} \tag{3.2.10}$$

we also have that

$$\limsup_{i\to\infty}\{\mathbf{M}(\Phi_i(x))|x\in X_i^{(p)}\}=\omega_p(M,g). \tag{3.2.11}$$

As a consequence, we may assume that in the min-max sequence S, $\mathrm{dmn}(\Phi_i)=X_i^{(p)}$.

Now that $X_i^{(p)}$ is a finite p-dimensional simplicial complex, and thus is homeomorphic to the support of some cubical subcomplex of some N-dimensional cube I^N (Chapter 4 [11]). And we can take a canonical projection (a closed map) mapping $X_i^{(p)}$ into I^{2p+1} within $1/2$ of the interior. By the general position theorem for maps, Theorem 5.3 in [64], with $P_0=\emptyset$ and $P=X_i^{(p)}$ therein, there exists a piecewise-linear embedding map from $X_i^{(p)}$ to some triangulation T of I^{2p+1} whose image has a distance, say, at least $1/6$ to ∂I^{2p+1}.

Next, we can "thicken" $X_i^{(p)}$ in I^{2p+1} to obtain a subcomplex Y_i of some $I(2p+1,k)$ such that $X_i^{(p)}$ is a retract from Y_i, which induces a map $\Psi_i: Y_i\to\mathcal{Z}_n(M;\mathbb{Z}_2)$ by $\Psi_i(y):=\Phi_i(r(y))$. Moreover, $(\Psi_i)^*(\bar{\lambda}^p)=r^*\circ(\Phi_i)^*(\bar{\lambda}^p)\neq 0$, since $i^*\circ r^*=id^*$ implies that r^* is an injection. It is also easy to see that $\max_{y\in Y_i}\{\mathbf{M}(\Psi_i(y))\}=\max_{x\in X_i}\{\mathbf{M}(\Phi_i(x))\}$.

Indeed, there exists a regular neighborhood Z_i of $X_i^{(p)}$ in T', a refinement of T ([64] P.33). Moreover, by Corollary 3.30 in [64], we also know that $X_i^{(p)}$ is a deformation retract of Z_i, and thus, we have a retraction $r: Z_i\to X_i^{(p)}$. Now, since $d=\mathrm{dist}(X_i^{(p)},\partial Z_i)>0$, we can take a large enough interger $k=k(p,d)$ such that the subcomplex $Y_i\subset I(2p+1,k)$ with

$$Y_i(k)=\{\alpha\in I(2p+1,k)|\exists\beta\in I(2p+1,k),\ s.t.\ \alpha<\beta,\ \beta\cap X_i^{(p)}\neq\emptyset\}, \tag{3.2.12}$$

satisfies that $X_i^{(p)} \subset Y_i \subset Z_i$. Apparently, $r|_{Y_i}$ is also a retraction from Y_i to $X_i^{(p)}$, which confirms our assertion.

In summary, $\omega_p(M,g)$ can be achieved by the sequence $\{\Psi_i\} \subset \mathcal{P}_{p,2p+1}$, so we have $\omega_p(M,g) \geq \omega_{p,2p+1}(M,g)$ and hence $\omega_p(M,g) = \omega_{p,2p+1}(M,g)$. □

Corollary 3.2.7 (Realization of p-width)**.** *Each p-width can be realized by a singular minimal hypersurface with optimal regularity and moreover, with property $(2p+1)$.*

Remark 3.2.8. When $2 \leq n \leq 6$, this has been proved ([39], Proposition 2.2), where they used the upper bound of Morse index of minimal hypersurfaces from min-max construction [52]. However, when $n \geq 7$, without an adequate alternative of bumpy metrics defined in the singular setting, the technique in [52] to make a critical sequence bypass all minimal hypersurface with large Morse index could not be applied directly. Thus, the compactness result using [68] is still open.

Definition 3.2.9. We define the **Almgren-Pitts realizations of p-width** $\mathcal{APR}_p(M,g)$ to be $\mathcal{APR}(\mathcal{P}_{p,2p+1})$.

Now we have the following compactness for varying metrics.

Proposition 3.2.10 (Compactness of $\mathcal{APR}_p$ for Varying Metrics)**.** *Given a smooth closed manifold M and C^∞ metrics $\{g_i\}$ and g such that*

$$g_i \xrightarrow{C^3} g, \tag{3.2.13}$$

and $V_i \in \mathcal{APR}_p(M,g_i)$ for some $p > 0$, there is a subsequence of $\{V_i\}$ (still denoted by $\{V_i\}$) such that

$$V_i \rightharpoonup V, \tag{3.2.14}$$

in the varifold sense. Moreover, $V \in \mathcal{APR}_p(M,g)$.

Proof. From the continuity of p-width ([39], Lemma 2.1), we have that $\|V\|(M) =$

$\omega_p(M,g)$. With Remark 3.1.6, the proof that V has optimal regularity and property $(2p+1)$ is simply verbatim of the proof of Proposition 3.2.4. □

3.3 Proof of Main Theorem

For any open subset $U \subset M$, we define

$$\mathcal{M}_{U,p} := \{g \in \Gamma_\infty(M) | \forall V \in \mathcal{APR}_p(M,g),\ \|V\|(U) > 0\}, \tag{3.3.1}$$

where $\Gamma_\infty(M)$ is the set of all smooth Riemannian metrics on M and let

$$\mathcal{M}_U := \bigcup_{p=1}^{\infty} \mathcal{M}_{U,p}. \tag{3.3.2}$$

Proposition 3.3.1. *$\mathcal{M}_{U,p}$ is an open subset of $\Gamma_\infty(M)$ for any open subset U, and so is $\mathcal{M}_U$.*

Proof. Given $g_0 \in \mathcal{M}_{U,p}$, we would like to show that there is an $\delta > 0$ such that $B_\delta(g_0, C^3) \cap \Gamma_\infty(M) \subset \mathcal{M}_{U,p}$.

Suppose not, there will be a sequence $g_i \in \Gamma_\infty(M)$ such that $g_i \xrightarrow{C^3} g_0$ but $g_i \notin \mathcal{M}_{U,p}$. Therefore, we can choose a sequence $\{V_i\}$ such that $V_i \in \mathcal{APR}_p(M,g_i)$ but $V_i(U) = 0$. From Proposition 3.2.10, up to a subsequence,

$$V_i \rightharpoonup V, \tag{3.3.3}$$

where $V \in \mathcal{APR}_p(M,g_0)$.

Since U is open, $\|V\|(U) \leq \lim_{i\to\infty} \|V_i\|(U) = 0$ which gives a contradiction. □

Now we define $\mathcal{M}_f$ to be the set of metrics on M where there are only finitely many singular minimal hypersurfaces of optimal regularity w.r.t. that metric.

Lemma 3.3.2 (Key Lemma)**.** *For any open subset $\mathcal{O}$ of Γ_∞, if $\mathcal{M}_f$ is dense in $\mathcal{O}$, then $\mathcal{M}_U$ is both open and dense in $\mathcal{O}$ as well. Thus, $\mathcal{M}_f \cap \mathcal{O}$ is a meagre set inside $\mathcal{O}$.*

Remark 3.3.3. As we will see in the proof, $\mathcal{M}_f$ plays the same role as bumpy metrics in the proof of Proposition 3.1 in [39].

Proof. Fix U as an open subset of M. For any g in $\mathcal{O}$, from the denseness of $\mathcal{M}_f$, there is a $g' \in \mathcal{M}_f$ such that g' is arbitrarily close to g. Now, if $g' \in \mathcal{M}_U$ then we are done.

Suppose that $g' \notin \mathcal{M}_U$. We can follow the proof of Proposition 3.1 in [39], since now the set

$$\begin{aligned}\mathcal{C} =&\{\sum_{j=1}^{N} m_j \mathrm{vol}_{g'}(\Sigma_j) : N \in \mathbb{N}, \{m_j\}_{j=1}^N \subset \mathbb{N}, \{\Sigma_j\}_{j=1}^N \\ &\text{singular minimal hypersurfaces with optimal regularity}\},\end{aligned} \tag{3.3.4}$$

is countable and thus has empty interior.

Let h be a smooth nonnegative function with $\mathrm{spt}(h) \subset U$ and $h(x) > 0$ for some $x \in U$. Let $g'(t) = (1 + th)g'$. Since $\mathcal{O}$ is open, there is a $t_0 > 0$ such that $\{g'(t) | 0 \leq t \leq t_0\} \subset \mathcal{O}$. Moreover, using the same argument in Proposition 3.1 in [39], there exists a $t_1 \in (0, t_0)$ arbitrarily small and $p = p(t_0) \in \mathbb{N}$ such that $\omega_p(M, g'(t_1)) > \omega_p(M, g')$ and $\omega_p(M, g'(t_1)) \notin \mathcal{C}$. Now it suffices to show that $g'(t_1) \in \mathcal{M}_{U,p}$.

Suppose not, we can find $V \subset \mathcal{APR}_p(M, g'(t_1))$ such that $\|V\|(U) = 0$. Note that $g'(t_1) = g'$ outside U so we have

$$\|V\|(M) = \sum_{j=1}^{N} m_j \mathrm{vol}_{g'(t_1)}(\Sigma_j) = \sum_{j=1}^{N} m_j \mathrm{vol}_{g'}(\Sigma_j) \in \mathcal{C}, \tag{3.3.5}$$

where $\{\Sigma_j\}$ is a finite set of singular minimal hypersurfaces with optimal regularity

with respect to both $g'(t_1)$ and g'. This gives a contradiction.

Let $\{U_i\}$ be a countable basis of M, then $\mathcal{M} = \bigcap \mathcal{M}_{U_i}$ is of second Baire category in $\mathcal{O}$ so $\mathcal{M}_f \cap \mathcal{O} \subset \mathcal{M}^C \cap \mathcal{O}$ is a meagre set. □

Proof of Main Theorem. Let $\mathcal{O} = \mathrm{Int}(\overline{\mathcal{M}_f})$ and it is easy to see that $\mathcal{M}_f \subset (\mathcal{M}_f \cap \mathcal{O}) \bigcup \partial(\overline{\mathcal{M}_f})$. From Lemma 3.3.2, we know that $\mathcal{M}_f \cap \mathcal{O}$ is meagre. Since $\partial(\overline{\mathcal{M}_f})$ is nowhere dense, $\mathcal{M}_f$ is also meagre. □

Remark 3.3.4. In the Key Lemma 3.3.2, we only use the fact that $\mathcal{C}$ is a set with empty interior. Thus, if $\mathcal{M}_{ei}$ is the set of metrics where $\mathcal{C}$ has empty interior and $\mathcal{O}_{ei} = \mathrm{Int}\overline{\mathcal{M}_{ei}}$, we also have that $\mathcal{M}$ is of second Baire category in $\mathcal{O}_{ei}$. As a consequence, we have the denseness of singular minimal hypersurfaces with optimal regularity in generic metrics inside $\mathcal{O}_{ei}$.

Corollary 3.3.5. *For a closed manifold $M^{n+1}(2 \leq n \leq 6)$ with a C^∞-generic metric g, the union of minimal hypersurfaces in $\mathcal{APR}(M,g) := \bigcup_{p=1}^{\infty} \mathcal{APR}_p(M,g)$, i.e., the minimal hypersurfaces realizing min-max widths, is a dense subset of M.*

Proof. By Theorem 2.7 in [81] (an analogue could be refered to Theorem 9 in [5]), the set of bumpy metrics is generic in $\Gamma_\infty(M)$. It follows from Sharp's compactness theorem ([68] Theorem 2.3 and Remark 2.4) that the bumpy metric belongs to $\mathcal{M}_{ei}$ and therefore, $\mathcal{M}_{ei}$ is also generic and thus dense in $\Gamma_\infty(M)$. From the remark above, we know that the Key Lemma leads to the conclusion. □

Parallel to the existence of bumpy metrics when $2 \leq n \leq 6$, we make the following conjecture.

Conjecture 3.3.6. $\mathcal{M}_{ei}$ is dense in $\Gamma_\infty(M)$.

Corollary 3.3.7. *If Conjecture 3.3.6 above holds, singular minimal hypersurfaces with optimal regularity in M with generic metrics are dense.*

In particular, let $\mathcal{M}_c$ be the set of the metrics where there are only countably many singular minimal hypersurfaces of optimal regularity w.r.t. that metric and then the following conjecture would imply Conjecture 3.3.6.

Conjecture 3.3.8. $\mathcal{M}_c$ is dense in $\Gamma_\infty(M)$.

Morally speaking, Conjecture 3.3.8 can even lead to upper bounds of Morse index following the techniques in [52].

Chapter 4

Morse index and multiplicity estimates

4.1 Introduction

The Morse index results in Chapter 1 relies on the application of B. White's bumpy metrics [79] in order to bypass all the minimal hypersurfaces with large Morse index, so the lack of desirable bumpy metrics could invalidate the direct application of the existing methods. In a recent paper [77], A. Song overcame this difficulty for 1-parameter sweepouts in non-compact setting by constructing 2-step deformations to establish Morse index estimates and the multiplicity one theorem. Inspired by his result, we will construct hierarchical deformations for multi-parameter sweepouts, which could be viewed as an adaption of [52, Deformation Theorem A] to the general metric setting (*c.f.* Theorem 4.4.4). With these deformations, we are able to prove the Morse index bound in higher dimensions.

Theorem 4.1.1 (Morse Index Bound)**.** *Suppose that* (M^{n+1}, g) *is a closed Riemannian manifold with* $n+1 \geq 3$*. Then for any* $p \in \mathbb{N}^+$*, there exists a stationary, integral varifold* V *with* $\mathrm{spt}(V) = \Sigma$ *such that*

- $\|V\|(M) = \omega_p(M, g)$;
- $\mathrm{index}(\Sigma) \leq p$;
- Σ *is a minimal hypersurface with optimal regularity, i.e.,* $\mathcal{H}^s(\mathrm{sing}(\Sigma)) = 0, \forall s > n-7$. *In particular, when* $n \leq 6$, Σ *is smooth.*

Remark 4.1.2. An alternative proof was obtained concurrently by A. Dey in [22], where he made a comparison of the Almgren-Pitts min-max theory and the Allen-Cahn min-max theory. Using the existing upper bound in the Allen-Cahn setting [29], he was also able to conclude the same result in the Almgren-Pitts setting.

Remark 4.1.3. As one will see in the following, our method does not rely on the existence of bumpy metrics, or even the objects in search being minimal hypersurfaces. In principle, with a little modification, the same Morse index upper bound could be obtained for CMC hypersurfaces, PMC hypersurfaces, free-boundary minimal hypersurfaces, non-compact ambient manifold, etc.

In our first Morse index bound theorem, the multiplicities do not play a role in the estimates. To capture this information, we will construct a sequence of c-CMC hypersurfaces to approximate a minimal hypersurface realizing the volume spectrum. Hence, we will introduce a restrictive min-max theory to make such a construction possible.

Theorem 4.1.4 (Improved Morse Index Bound)**.** *Suppose that* (M^{n+1}, g) *is a closed Riemannian manifold with* $n + 1 \geq 3$. *Then for any* $p \in \mathbb{N}^+$, *there exists a stationary, integral varifold* V, *whose support is a disjoint union of connected minimal hypersurfaces with optimal regularity* $\{\Sigma_i\}_{i=1,\cdots,l_p}$ *with multiplicities* $\{m_i\}$ *such that*

- $\|V\|(M) = \sum_{i=1}^{l_p} m_i \cdot \mathrm{area}_g(\Sigma_i) = \omega_p(M, g)$;
- *Every* Σ_i *with* $m_i \geq 3$ *is stable;*

- $\sum_{m_i \leq 2} \text{index}(\Sigma_i) \leq p$.

Remark 4.1.5. The last two bullets do not seem sharp. One would expect that the multiplicity 2 minimal hypersurfaces should also be stable as well.

Corollary 4.1.6 (Multiplicity 2 for positive Ricci curvature)**.** *Suppose that* (M^{n+1}, g) $(n+1 \geq 3)$ *is a closed Riemannian manifold with positive Ricci curvature. Then for any* $p \in \mathbb{N}^+$*, there exists a stationary, integral varifold* V*, whose support is a connected minimal hypersurfaces with optimal regularity* Σ *with multiplicities* $m \leq 2$*, such that*

- $\|V\|(M) = m \cdot \text{area}_g(\Sigma) = \omega_p(M, g)$*;*
- $\text{index}(\Sigma_i) \leq p$.

Remark 4.1.7. When $n+1 \leq 7$ or $p = 1$, the multiplicity has been proved to be 1 (See [86], [63], [6]). We also expect this holds in the general case as well.

4.2 Restrictive min-max theories

In the proof of the min-max theorems (even in X. Zhou's setting), an important observation is that for i large enough, along the homotopy maps $H_i^{(1)}, H_i^{(2)}, H_i^{(3)}$ and $H_i^{(4)}$, the mass will only increase by at most $3\delta_i$ in total. In fact, the homotopy maps in Theorem 4.4.4 that we will prove later also satisfies this property. Hence, this indicates that, in the definition of homotopy classes, we can set an upper bound of the (c-)area functional along the homotopy maps, and the Almgren-Pitts min-max scheme still works for such a restrictive homotopy classes.

Definition 4.2.1. Given a pair (X, Z), where Z is a subcomplex of X, a constant $c > 0$, and $\Phi_0 : X \to (\mathcal{C}(M), \mathbf{F})$, we define the **restrictive (X, Z)-homotopy class of Φ_0 for $\mathcal{A}^c$ with an upper bound δ**, denoted by $\Pi_c^\delta(\Phi_0)$, to be the set of all sequences of $\mathbf{F}$-continuous maps $\{\Phi_i : X \to (\mathcal{C}(M), \mathbf{F})\}$ satisfying the following conditions:

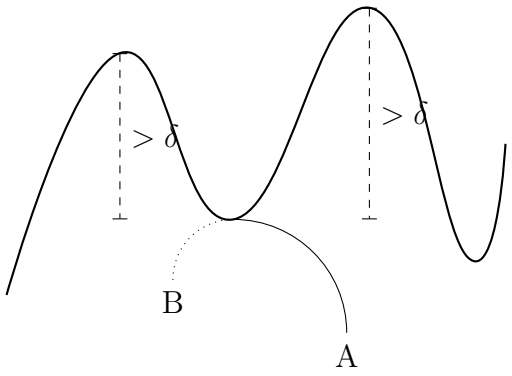

Figure 4.1: Restrictive Mountain Pass with an upper bound δ

1. Each Φ_i is homotopic to Φ_0 in the $\mathbf{F}$-topology on $\mathcal{C}(M)$, and

2. The homotopy $H_i : [0,1] \times X \to (\mathcal{C}(M), \mathbf{F})$ associated with Φ_i satisfies

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(H_i(t,x), \Phi_0(x)) : t \in [0,1], x \in Z\} = 0\,, \tag{4.2.1}$$

and

$$\sup_{t,x} \mathcal{A}^c(H_i(t,x)) < \sup_x \mathcal{A}^c(\Phi_0(x)) + \delta\,. \tag{4.2.2}$$

Here, each $\{\Phi_i\}_{i\in\mathbb{N}}$ is called a **restrictive (X, Z)-homotopy sequence of mappings into $\mathcal{C}(M)$**.

Similarly, one can define the widths $\mathbf{L}(\Pi^\delta)$, $\mathbf{L}^c(\Pi_c^\delta(\Phi_0))$, min-max sequences, critical sets, etc. And the min-max theorems in this new setting can be proved analogously.

Theorem 4.2.2 (Min-max theorem for Π^δ)**.** *If Π^δ is a m-parameter $\mathbf{F}$-homotopy family with an upper bound δ, $\mathbf{L}(\Pi^\delta) > 0$ and S is a min-max sequence of Π^δ, there exists a minimal hypersurfaces with optimal regularity $V \in \mathbf{C}(S)$ with the property (m) and $\|V\|(M) = \mathbf{L}(\Pi^\delta)$.*

Theorem 4.2.3 (Min-max Theorem for $\Pi_c^\delta(\Phi_0)$)**.** *Given a closed Riemannian manifold $(M^{n+1}, g)\,(n \geq 2)$ and $c > 0$ and $\delta > 0$, if the restrictive (X, Z)-homotopy class*

$\Pi_c^\delta(\Phi_0)$ *of* $\Phi_0 : X \to \mathcal{C}(M)$ *for* $\mathcal{A}^c$ *with an upper bound* δ *satisfies*

$$\mathbf{L}^c(\Pi_c^\delta(\Phi_0)) > \sup_{x\in Z} \mathcal{A}^c(\Phi_0(x))\,, \tag{4.2.3}$$

then there exists a $\Omega \in \mathcal{C}(M)$ *such that* $\mathcal{A}^c(\Omega) = \mathbf{L}^c(\Pi_c^\delta(\Phi_0))$ *and* $\partial\Omega$ *is a c-CMC hypersurface with optimal regularity.*

Remark 4.2.4. To prove the Morse inequalities, the authors of [50] consider the homology min-max for $H_k(\mathcal{Z}^b, \mathcal{Z}^a)$, where the homotopy class Π constructed therein is essentiall the same as the restrictive homotopy class with $Z = \emptyset$ and $c = 0$.

4.3 k-unstability and Morse index

In the following, (M^{n+1}, g) denotes a smooth and closed Riemannian manifold and $\omega_p = \omega_p(M, g)$ for convenience.

4.3.1 k-unstability

Definition 4.3.1 ([52])**.** A stationary varifold $\Sigma \in \mathcal{V}_n(M)$ is called ***k*-unstable**, provided that there exists $\varepsilon > 0$, $0 < c_0 < 1$ and a smooth family $\{F_v\}_{v\in\overline{B}^k} \subset \mathrm{Diff}(M)$ with

$$F_0 = \mathrm{Id}, F_{-v} = F_v^{-1}\,, \tag{4.3.1}$$

for all $v \in \overline{B}^k$ (a unit ball) such that, for any $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$, the smooth function

$$A^V : \overline{B}^k \to [0, \infty), \qquad v \mapsto \|(F_v)_\# V\|(M)\,, \tag{4.3.2}$$

satisfies

1. A^V has a unique maximum at $m(V) \in B^k_{c_0/\sqrt{10}}(0)$;
2. $-\frac{1}{c_0}\mathrm{Id} \le D^2A^V(u) \le -c_0\mathrm{Id}$ for all $u \in \overline{B}^k$.

Lemma 4.3.2. *Given a k-unstable Σ with a quadruple $(\varepsilon, c_0, \{F_v\}, m)$, there exists $\varepsilon_0 \in (0, \varepsilon)$ small enough, such that for any $\tilde{\varepsilon} \in (0, \varepsilon_0]$ and $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\tilde{\varepsilon}}(\Sigma)$,*

$$(F_{S^{k-1}})_{\#}(V) \cap \overline{\mathbf{B}}^{\mathbf{F}}_{2\tilde{\varepsilon}}(\Sigma) = \emptyset. \tag{4.3.3}$$

Proof. Suppose not and there exists $V_i \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$, $\varepsilon_i \to 0$, and $v_i \in S^k$, such that

1. $\mathbf{F}(V_i, \Sigma) \leq 2\varepsilon_i$.
2. $(F_{v_i})_{\#}(V_i) \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon_i}(\Sigma)$.

By the compactness of S^k, up to a subsequence, we may assume that $v_i \to v \in S^k$. Since F_v is smooth, we can take a limit as $i \to \infty$ and obtain

$$(F_v)_{\#}(\Sigma) = \Sigma\,. \tag{4.3.4}$$

However, by calculating the area of both side, we have the following inequality

$$\begin{aligned}
\|\Sigma\|(M) - \|(F_v)_{\#}(\Sigma)\|(M) &= A^{\Sigma}(0) - A^{\Sigma}(v) \\
&= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} A^{\Sigma}(tv)\mathrm{d}t \\
&= -\int_0^1 \int_0^t \frac{\mathrm{d}^2}{\mathrm{d}^2 s} A^{\Sigma}(sv)\mathrm{d}s\mathrm{d}t \\
&= -\int_0^1 \int_0^t D^2 A^{\Sigma}(sv)(v,v)\mathrm{d}s\mathrm{d}t \\
&\geq \int_0^1 \int_0^t c_0 \mathrm{d}s\mathrm{d}t \\
&= \frac{c_0}{2} > 0\,,
\end{aligned} \tag{4.3.5}$$

which implies that $(F_v)_{\#}(\Sigma) \neq \Sigma$, giving a contradiction. □

In the following, for technical reason, we always take $\varepsilon = \varepsilon_0$ in the definition of k-unstability, and it is easy to check that the varifold is still k-unstable with the new ε.

Lemma 4.3.3. *Given a k-unstable varifold Σ with a quadruple $(\varepsilon, c_0, \{F_v\}, m)$ and any fixed $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$, if an integral curve $x : [0, L] \to \overline{B}^k$ disjoint from $m(V)$ is defined by*

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -f(t)\nabla A^V(x(t))\,, \tag{4.3.6}$$

where $f(t)$ is a positive continuous function, then we have

$$A^V(x(L)) - A^V(x(0)) \leq -\frac{c_0}{2}|x(L) - x(0)|^2 \tag{4.3.7}$$

Proof. By the fundamental theorem of calculus, we have

$$\begin{aligned}
&A^V(x(L)) - A^V(x(0)) \\
=&\int_0^L \nabla A^V(x(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}x(t)\mathrm{d}t \\
=&-\int_0^L f(t)|\nabla A^V(x(t))|^2\mathrm{d}t \\
\leq&-\int_0^L f(t)|\nabla A^V(x(t))|\left(\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}|\nabla A^V(x(s))|\mathrm{d}s\right)\mathrm{d}t \\
=&-\int_0^L f(t)|\nabla A^V(x(t))|\int_0^t \frac{\nabla^2 A^V(x(s))\left(\nabla A^V(x(s)), \frac{\mathrm{d}}{\mathrm{d}s}x(s)\right)}{|\nabla A^V(x(s))|}\mathrm{d}s\mathrm{d}t \\
=&\int_0^L f(t)|\nabla A^V(x(t))|\int_0^t f(s)\frac{\nabla^2 A^V(x(s))\left(\nabla A^V(x(s)), \nabla A^V(x(s))\right)}{|\nabla A^V(x(s))|}\mathrm{d}s\mathrm{d}t \\
\leq&-c_0\int_0^L f(t)|\nabla A^V(x(t))|\left(\int_0^t f(s)|\nabla A^V(x(s))|\mathrm{d}s\right)\mathrm{d}t\,.
\end{aligned} \tag{4.3.8}$$

Here, we use the fact that $\nabla A^V(v) \neq 0$ for any $v \neq m(V)$. Indeed, this follows from

$$\begin{aligned}
\left\langle \nabla A^V(v), v - m(V)\right\rangle &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla A^V(v), v - m(V)\right\rangle \mathrm{d}t \\
&= \int_0^1 \nabla^2 A^V(t(v - m(V)) + m(V))\,(v - m(V), v - m(V))\,\mathrm{d}t \\
&\leq -c_0|v - m(V)|^2 < 0\,.
\end{aligned} \tag{4.3.9}$$

Now, let's consider the length function l satisfying

$$l'(t) = \left| \frac{d}{dt} x(t) \right| = f(t) |\nabla A^V(x(t))| \,, \tag{4.3.10}$$

with $l(0) = 0$. Apparently, the length of the curve $l(L)$ is no less than $|x(L) - x(0)|$.

Therefore, we have

$$\begin{aligned} & A^V(x(L)) - A^V(x(0)) \\ \leq & - c_0 \int_0^L l'(t) \int_0^t l'(s) \mathrm{d}u \mathrm{d}t \\ = & - c_0 \int_0^L l'(t) l(t) \mathrm{d}t \\ = & - \frac{c_0}{2} l(L)^2 \\ \leq & - \frac{c_0}{2} |x(L) - x(0)|^2 \end{aligned} \tag{4.3.11}$$

□

Lemma 4.3.4. *For every k-unstable varifold $\Sigma \in \mathcal{V}_n(M)$, there exists an increasing function*

$$h_\Sigma : \mathbb{R}^+ \to \mathbb{R}^+, \tag{4.3.12}$$

such that for any $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$, and $v, w \in \overline{B}^k$,

$$|v - w| \geq h_\Sigma \left(\mathbf{F} \left((F_v^\Sigma)_\# (V), (F_w^\Sigma)_\# (V) \right) \right) \,. \tag{4.3.13}$$

Proof. It suffices to show that for every $c > 0$, if

$$\{(v, w) | \exists V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma),\, \mathbf{F}((F_v^\Sigma)_\# (V), (F_w^\Sigma)_\# (V)) = c\} \neq \emptyset \,, \tag{4.3.14}$$

then

$$\{|v - w| | \exists V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma),\, \mathbf{F}((F_v^\Sigma)_\# (V), (F_w^\Sigma)_\# (V)) = c\} \,, \tag{4.3.15}$$

has a positive lower bound.

Let's argue by contradiction. Suppose that there exists a sequence $V_i \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$ and $v_i, w_i \in \overline{B}^k$ with $\mathbf{F}((F^\Sigma_{v_i})_\#(V_i), (F^\Sigma_{w_i})_\#(V_i)) = c$ but $|v_i - w_i| \to 0$. By compactness, up to a subsequence, we may assume that

$$V_i \to V, v_i \to v, w_i \to v\,, \tag{4.3.16}$$

and $\mathbf{F}((F^\Sigma_v)_\#(V), (F^\Sigma_v)_\#(V)) = c$ gives a contradiction. □

Definition 4.3.5. For a stationary varifold $\Sigma \in \mathcal{V}_n(M)$, we define its **index** to be

$$\mathrm{index}(\Sigma) = k, \tag{4.3.17}$$

provided that it is k-unstable but not $(k+1)$-unstable. In particular, Σ is called **stable** if it is not 1-unstable.

Remark 4.3.6. This definition is compatible with the usual definition of the Morse index for a smooth minimal hypersurface (See [52]).

Similarly, we can also adopt the weak index notion to critical points of the $\mathcal{A}^c$ functional.

Definition 4.3.7 ([86, (modified) Definition 2.1])**.** A critical point $\Omega \in \mathcal{C}(M)$ of $\mathcal{A}^c$ is called ***k*-unstable**, provided that there exists $\varepsilon > 0$, $0 < c_0 < 1$ and a smooth family $\{F_v\}_{v \in \overline{B}^k} \subset \mathrm{Diff}(M)$ with $F_0 = \mathrm{Id}, F_{-v} = F_v^{-1}$ for all $v \in \overline{B}^k$ such that, for any $\Omega' \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega)$, the smooth function

$$A^{c,\Omega'} : \overline{B}^k \to [0, \infty), \qquad A^{c,\Omega'}(v) = \mathcal{A}^c(F_v(\Omega')), \tag{4.3.18}$$

satisfies

1. $A^{c,\Omega'}$ has a unique maximum at $m(\Omega') \in B^k_{c_0/\sqrt{10}}(0)$;
2. $-\frac{1}{c_0}\mathrm{Id} \leq D^2 A^{\Omega'}_c(u) \leq -c_0 \mathrm{Id}$ for all $u \in \overline{B}^k$.

In addition, by taking ε small enough, we also assume that for any $\Omega' \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega)$,

$$(F_{S^{k-1}})_{\#}(\Omega') \cap \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Omega) = \emptyset. \tag{4.3.19}$$

Definition 4.3.8. For a critical point $\Omega \in \mathcal{C}(M)$ of $\mathcal{A}^c$, we define its **index** to be

$$\text{index}(\Omega) = k, \tag{4.3.20}$$

provided that Ω is k-unstable but not $(k+1)$-unstable. In particular, Ω is called **stable** if it is not 1-unstable.

4.3.2 Almgren-Pitts realizations and Morse index

Definition 4.3.9. Let $\mathcal{U}(\omega_p)$ be the subset of $\mathcal{APR}_p$ consisting of all the $(p+1)$-unstable minimal hypersurfaces, and $\mathcal{S}(\omega_p)$ the complement of $\mathcal{U}(\omega_p)$ in $\mathcal{APR}_p$, which is the subset of minimal hypersurfaces with a Morse index upper bound p.

Remark 4.3.10. If $\mathcal{U}(\omega_p) = \emptyset$, the main theorems are trivially true. Thus, we may assume that $\mathcal{U}(\omega_p) \neq \emptyset$.

Lemma 4.3.11. *$\mathcal{U}(\omega_p)$ is a countable union of compact sets, i.e.,*

$$\mathcal{U}(\omega_p) = \bigcup_{i=1}^{\infty} \mathcal{U}_i(\omega_p)\,, \tag{4.3.21}$$

where $\mathcal{U}_i(\omega_p)$ is compact.

Proof. By [42, Proposition 3.1], $\mathcal{APR}_p$ is a compact set. By the compactness result in [23], we also know that $\mathcal{S}(\omega_p)$ is also compact.

If $\mathcal{S}(\omega_p) = \emptyset$, then define $\mathcal{U}_i(\omega_p) := \mathcal{U}(\omega_p)$ for all i, which is compact.

If $\mathcal{S}(\omega_p) \neq \emptyset$, we can define

$$\mathcal{U}_i(\omega_p) := \{V \in \mathcal{U}(\omega_p) | \mathbf{F}(V, \mathcal{S}(\omega_p)) \geq \frac{1}{i}\}\,. \tag{4.3.22}$$

Since the $\mathbf{F}$-topology is equivalent to the varifold topology on $\mathcal{APR}_p$, $\mathcal{U}_i(\omega_p)$ is a compact set and $\bigcup_i \mathcal{U}_i(\omega_p)$ is exactly the complement of $\mathcal{S}(\omega_p)$.

In summary, we have

$$\mathcal{U}(\omega_p) = \bigcup_{i=1}^{\infty} \mathcal{U}_i(\omega_p)\,, \tag{4.3.23}$$

which leads to the conclusion. □

4.4 Deformation theorems and Morse index upper bounds

To begin with, we introduce some notations for a fixed sequence of varifolds $\{\Sigma_k\}_{k=1}^{\infty} \subset \mathcal{U}(\omega_p)$. By the definition of $(p+1)$-unstability, each one is associated with a quadruple $(\varepsilon_k, c_{0,k}, \{F_v^k\}, m_k)$.

For a nonempty set $K \subset \mathbb{N}^+$, we define

$$\mathbf{B}_K^{\mathbf{F}} := \mathbf{B}_K^{\mathbf{F}}(\{\Sigma_k\}, \{\varepsilon_k\}) = \bigcap_{k \in K} \mathbf{B}_{\varepsilon_k}^{\mathbf{F}}(\Sigma_k)\,, \tag{4.4.1}$$

and

$$\bar{k}(K) := \min\{k \in K\}\,. \tag{4.4.2}$$

For $\lambda \geq 0$, we denote by $\mathbf{B}_{\lambda,K}^{\mathbf{F}}$ the set $\bigcap_{k\in K} \mathbf{B}_{(2-2^{-\lambda})\varepsilon_k}^{\mathbf{F}}(\Sigma_k)$. In particular, we have for any $0 \leq \lambda_1 \leq \lambda_2$,

$$\mathbf{B}_K^{\mathbf{F}} = \mathbf{B}_{0,K}^{\mathbf{F}} \subset \mathbf{B}_{\lambda_1,K}^{\mathbf{F}} \subset \mathbf{B}_{\lambda_2,K}^{\mathbf{F}} \subset \bigcap_{k\in K} \mathbf{B}_{2\varepsilon_k}^{\mathbf{F}}(\Sigma_k)\,. \tag{4.4.3}$$

Also note that by Lemma 4.3.2, for any $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k)$,

$$(F^k_{S^p})_\#(V) \cap \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) = \emptyset\,. \tag{4.4.4}$$

For any nonempty finite subset $K \subset \mathbb{N}$ and $\lambda \geq 0$, if $\mathbf{B}^{\mathbf{F}}_{\lambda,K} \neq \emptyset$, then for every

$$V \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K} := \bigcap_{\lambda'>\lambda} (\mathbf{B}^{\mathbf{F}}_{\lambda',K})\,,$$

we can define the associated gradient flow $\{\phi^V_{\lambda,K}(\cdot,t)\}_{t\geq 0} \subset \mathrm{Diff}(\overline{B}^{p+1})$ generated by the vector field

$$u \mapsto -\mathbf{F}\left((F^{\bar{k}(K)}_u)_\#(V), (\mathbf{B}^{\mathbf{F}}_{\lambda+1,K})^c\right) \nabla A^V_{\bar{k}(K)}(u)\,, \tag{4.4.5}$$

where $A^V_{\bar{k}(K)}(u) := \|(F^{\bar{k}(K)}_u)_\#(V)\|(M)$ for any $u \in \overline{B}^{p+1}$.

Lemma 4.4.1. *Given any $\lambda > 0$, a nonempty finite set $K \subset \mathbb{N}^+$ and small $\eta > 0$, there exist $c(\lambda, K) > 0$ and $T = T(\eta, K, \lambda) > 0$ satisfying the following property.*

For any $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$ and any $v \in \overline{B}^{p+1}$ with $|v - m_{\bar{k}(K)}(V)| \geq \eta$ and

$$(F^{\bar{k}(K)}_v)_\#(V) \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K}, \tag{4.4.6}$$

then we have

$$A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v,T)) < A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v,0)) - c(\lambda,K). \tag{4.4.7}$$

Remark 4.4.2. For any $\Sigma \in \mathcal{U}(\omega_p)$, we can take $\Sigma_k \equiv \Sigma$ in the lemma above. For technical reasons, we will take the corresponding ε even smaller, such that for $K = \{k\}$ and any $\lambda = \{0, 1, 2, \cdots, p+1\}$, we also have

$$A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v,T)) < \omega_p - c(\lambda,K). \tag{4.4.8}$$

Proof. If $\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K} = \emptyset$, then the conclusion holds trivially, so in the following, we assume that $\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K} \neq \emptyset$.

For each $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$, we define

$$D_{small}(V) := \{v \in \overline{B}^{p+1} | (F_v^{\bar{k}(K)})_{\#}(V) \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K}\}, \tag{4.4.9}$$

and

$$D_{large}(V) := \{v \in \overline{B}^{p+1} | (F_v^{\bar{k}(K)})_{\#}(V) \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+1,K}\}. \tag{4.4.10}$$

We define $d(V)$ to be the Euclidean distance between $D_{small}(V)$ and $\partial D_{large}(V)$,

$$d(V) := \text{dist}(D_{small}(V), \partial D_{large}(V)). \tag{4.4.11}$$

Claim 1. $d(V) > 0$.

Proof. Indeed, by the continuity of $(F_{\cdot}^{\bar{k}(K)})_{\#}(V)$, if $w \in \partial D_{large}(V)$, by (4.4.4), then either $w \in S^p$, or $(F_v^{\bar{k}(K)})_{\#}(V) \in \partial \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+1,K}$. However, the former case is excluded by (4.4.4). In the latter case, by the continuity of $(F_{\cdot}^{\bar{k}(K)})_{\#}(V)$ again and the fact that $\partial \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+1,K} \cap \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K} = \emptyset$, we can conclude that $w \notin D_{small}(V)$. In other words, $\partial D_{large}(V) \cap D_{small}(V) = \emptyset$, and thus, $d(V) > 0$ since these two sets are compact. ■

Claim 2. *$d(\cdot)$ has a uniform positive lower bound $\tilde{d} = \tilde{d}(\lambda, K) > 0$ on $\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$.*

Proof. Suppose for contradiction that there exists a sequence $V_i \subset \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$, such that

$$d(V_i) \to 0.$$

Since $\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$ is a compact set, we may assume that $V_i \to V \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$. Since $\partial D_{large}(V_i)$ and $D_{small}(V_i)$ are compact, there exist $v_i^1 \in D_{small}(V_i)$ and $v_i^2 \in \partial D_{larget}(V_i)$ such

that

$$d(V_i) = \mathrm{dist}(v_i^1, v_i^2). \tag{4.4.12}$$

By the compactness of $\overline{B}^{p+1}$ and $d(V_i) \to 0$, up to subsequence, there is $v \in \overline{B}^{p+1}$, so that

$$v_i^1 \to v, \quad v_i^2 \to v\,.$$

It follows from the closedness of $\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K}$ and $\partial\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+1,K}$ that $v \in D_{small}(V) \cap \partial D_{large}(V)$, which contradicts to Claim 1. Thus, d has a uniform positive lower bound independent of V. ■

Now, let's consider the set of pairs

$$P := \{(V, v) \in \overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K} \times \overline{B}^{p+1} | v \neq m_{\bar{k}(K)}(V), v \in D_{small}(V)\}. \tag{4.4.13}$$

For any $(V, v) \in P$, $A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v, t))$ is strictly decreasing with respect to $t > 0$, since $v \neq m_{\bar{k}(K)}(V)$. Therefore, we have

$$\delta A_{\lambda,K}(V, v) := \lim_{t\to\infty} (A^V_{\bar{k}}(\phi^V_{\lambda,K}(v, 0)) - A^V_{\bar{k}}(\phi^V_{\lambda,K}(v, t))) > 0 \tag{4.4.14}$$

Claim 3. *$\delta A_{\lambda,K}(V, v)$ has a uniform positive lower bound $2c(\lambda, K) > 0$ on P.*

Proof. Indeed, by the definition of $\phi^V_{\lambda,K}(\cdot, t)$, we know that,

$$\mathbf{F}(\phi^V_{\lambda,K}(v, t), \partial D_{large}(V)) \to 0, \tag{4.4.15}$$

as $t \to \infty$. By Lemma 4.3.3 and Claim 2, we obtain for $\delta A_{\lambda,K}$ a uniform lower bound

$$2c(\lambda, K) := \frac{c_{0,\bar{k}(K)}}{2} \tilde{d}(\lambda, K)^2\,. \tag{4.4.16}$$

■

Finally, to conclude the lemma, it suffices to find $T = T(\eta, K, \lambda) > 0$ such that for all $(V, v) \in P$ and $|v - m_{\bar{k}(K)}(V)| \geq \eta$,

$$A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v, T)) < A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v, 0)) - c(\lambda, K)\,. \tag{4.4.17}$$

Following the proof of [52, Lemma 4.5], we argue by contradiction. Suppose there exists a sequence $(V_i, v_i) \in P$ and $|v_i - m_{\bar{k}(K)}(V_i)| \geq \eta$, such that

$$A^{V_i}_{\bar{k}(K)}(\phi^{V_i}_{\lambda,K}(v_i, t)) \geq A^{V_i}_{\bar{k}(K)}(\phi^{V_i}_{\lambda,K}(v_i, 0)) - c(\lambda, K). \tag{4.4.18}$$

for any $t \in [0, i]$. By compactness, we may assume that $V_i \to V$ and $v_i \to v$ with $(V, v) \in P$ and $|v - m_{\bar{k}(K)}(V)| \geq \eta$. Hence, for any $t \geq 0$,

$$A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v, t)) \geq A^V_{\bar{k}(K)}(\phi^V_{\lambda,K}(v, 0)) - c(\lambda, K), \tag{4.4.19}$$

contradicting Claim 3. □

Lemma 4.4.3. *Given a constant $\lambda > 0$, a nonempty finite subset $K \subset \mathbb{N}^+$ with $\mathbf{B}^{\mathbf{F}}_{\lambda,K} \neq \emptyset$, and $c(\lambda, K)$ the constant in Lemma 4.4.1, then for any $\delta \in (0, c(\lambda, K)/4)$, any $l \in \{0, 1, \cdots, p\}$ and any continuous map*

$$\Theta : X^l \to \mathbf{B}^{\mathbf{F}}_{\lambda,K}, \tag{4.4.20}$$

where X^l is an l-dimensional finite simplicial complex, there exists a homotopy map

$$H^{\Theta,\delta}_{\lambda,K} : X^l \times [0, 1] \to \mathbf{B}^{\mathbf{F}}_{\lambda+1,K}, \tag{4.4.21}$$

such that

1. $H^{\Theta,\delta}_{\lambda,K}(\cdot,0)=\Theta(\cdot)$;

2. $\forall x\in X^l$, $\|H^{\Theta,\delta}_{\lambda,K}(x,1)\|(M)\leq\|\Theta(x)\|(M)-\tilde{c}(\lambda,K)$;

3. $\forall x\in X^l, t\in[0,1]$, $\|H^{\Theta,\delta}_{\lambda,K}(x,t)\|(M)-\|\Theta(x)\|(M)\leq\delta$.

Here, $\tilde{c}(\lambda,K)=c(\lambda,K)/2$. *More precisely, we have*

- *For* $t\in[0,1/2]$, $\mathbf{F}(H^{\Theta,\delta}_{\lambda,K}(x,t),\Theta(x))\leq\delta$;
- *For* $t\in[1/2,1]$, *there exists* $v(x)\in\overline{B}^{p+1}$, *such that*

$$H^{\Theta,\delta}_{\lambda,K}(x,t)=\phi^{\Theta(x)}_{\lambda,K}(v(x),(2t-1)T)\,, \tag{4.4.22}$$

where $T=T(\eta,K,\lambda)$ *is a constant as in Lemma 4.4.1 for some* $\eta=\eta(\delta,K,\Theta,\lambda)$.

Proof. Firstly, since $\mathbf{F}(\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K},\partial\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K})>0$, there exists $\delta'\in(0,\delta)$ such that for any $V\in\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$, if a varifold W satisfies that $\mathbf{F}(V,W)\leq\delta'$, then

$$\begin{aligned}\|W\|(M)&\leq\|V\|(M)+\delta\,,\\ W&\in\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K}\,.\end{aligned} \tag{4.4.23}$$

By the compactness of $\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}$ and $\overline{B}^{p+1}$, We can also choose $\delta''>0$ such that

$$\mathbf{F}((F^{\bar{k}(K)}_v)_\#(V),V)\leq\delta'<\delta, \tag{4.4.24}$$

for any $V\in\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda,K}, v\in\overline{B}^{p+1}(\delta'')$.

Since $\Sigma_{\bar{k}(K)}$ is $(p+1)$-unstable, $l<p+1$, and $\Theta(X^l)\subset\overline{\mathbf{B}}^{\mathbf{F}}_{\lambda+0.5,K}\subset\overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon_{\bar{k}(K)}}(\Sigma_{\bar{k}(K)})$, we can follow the construction of $\hat{H}^i$ in the proof of [52, Theorem 5.1] and obtain a homotopy map

$$\hat{H}:X^l\times[0,1]\to\overline{B}^{p+1}(\delta''), \tag{4.4.25}$$

such that $\hat{H}(\cdot,0)=0$ and

$$\inf_{x\in X^l}|m_{\bar{k}(K)}(\Theta(x))-\hat{H}(x,1)|\geq\eta>0\,, \tag{4.4.26}$$

for some $\eta=\eta(\delta,K,\Theta,\lambda)>0$.

Finally, applying Lemma 4.4.1 with $v=\hat{H}(x,1)$ for $\Theta(x)$, we can obtain

$$H^{\Theta,\delta}_{\lambda,K}(x,t)=\begin{cases}(F^{\bar{k}}_{\hat{H}(x,2t)})_{\#}\Theta(x) & t\in[0,1/2]\\ \phi^{\Theta(x)}_{\lambda,K}(\hat{H}(x,1),(2t-1)T) & t\in(1/2,1]\,,\end{cases} \tag{4.4.27}$$

which fulfills all the conditions. □

Theorem 4.4.4 (Deformation Theorem A)**.** *Given a min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}}$ for ω_p, where $X_i=\mathrm{dmn}(\Phi_i)$ has dimension no greater than p, then up to a subsequence, there exists another sequence $\{\Psi_i\}$ such that*

1. *Ψ_i is homotopic to Φ_i in the $\mathbf{F}$-topology;*
2. *$\mathbf{L}(\{\Psi_i\}_{i\in\mathbb{N}})=\omega_p$;*
3. *There exists a positive function $\bar{\varepsilon}:\mathcal{U}(\omega_p)\to\mathbb{R}^+$ satisfying the following property. For any $\Sigma\in\mathcal{U}(\omega_p)$, there exists $i_0\in\mathbb{N}$ such that for all $i\geq i_0$,*

$$|\Psi_i|(X_i)\cap\mathbf{B}^{\mathbf{F}}_{\bar{\varepsilon}(\Sigma)}(\Sigma)=\emptyset\,. \tag{4.4.28}$$

Hence, $\mathbf{C}(\{\Psi_i\})\cap\mathcal{APR}_p\subset\mathcal{S}(\omega_p)$.

Moreover, for any given $\delta>0$, in the first bullet, we can always assume that the homotopy $H_i:X_i\times[0,1]\to\mathcal{Z}(M;\mathbf{F};\mathbb{Z}_2)$ satisfies

$$\sup_{t,x}\mathbf{M}(H_i(t,x))<\sup_{x}\mathbf{M}(\Phi_i(x))+\delta\,. \tag{4.4.29}$$

Proof. The proof consists of 4 steps.

Step 1. We construct a (finite or countable) sequence of $(p+1)$-unstable minimal hypersurfaces $\{\Sigma_k\} \subset \mathcal{U}(\omega_p)$ and their associated quadruples $\{(\varepsilon_k, c_{0,k}, \{F_v^k\}, m_k)\}$ such that the following conditions hold.

- $\forall k_1 \neq k_2, \mathbf{B}^{\mathbf{F}}_{\varepsilon_{k_1}}(\Sigma_{k_1}) \not\subset \mathbf{B}^{\mathbf{F}}_{\varepsilon_{k_2}}(\Sigma_{k_2})$;
- $\mathcal{U}(\omega_p) \subset \bigcup_{k=1}^{\infty} \mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$ and moreover, $\forall m \in \mathbb{N}^+, \exists k_m \in \mathbb{N}^+$,

$$\mathcal{U}_m(\omega_p) \subset \bigcup_{i=1}^{k_m} \mathbf{B}^{\mathbf{F}}_{\varepsilon_i}(\Sigma_i)\,, \tag{4.4.30}$$

  where $\mathcal{U}_m(\omega_p)$ is defined in the proof of Lemma 4.3.11;
- For any fixed k, $\bar{K}(\Sigma_k) := \{k' \in \mathbb{N}^+ : \ \mathbf{B}^{\mathbf{F}}_{2\varepsilon_{k'}}(\Sigma_{k'}) \cap \mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) \neq \emptyset\}$ has

$$\#\bar{K}(\Sigma_k) < \infty\,. \tag{4.4.31}$$

For each $\Sigma \in \mathcal{U}(\omega_p)$, since it is $(p+1)$-unstable, we can associate it with a quadruple $(\varepsilon_\Sigma, c_{0,\Sigma}, \{F_v^\Sigma\}, m_\Sigma)$ and then $\{\mathbf{B}^{\mathbf{F}}_{\varepsilon_\Sigma}(\Sigma)\}$ covers $\mathcal{U}(\omega_p)$. Note that we can always take a smaller ε_Σ such that Σ is still $(p+1)$-unstable with the new quadruple.

If $\mathcal{U}(\omega_p)$ itself is compact, we can find a finite cover $\{\Sigma_k\}$ with the quadruples above satisfying all the conditions.

Otherwise, by Lemma 4.3.11, $\mathcal{U}(\omega_p) = \bigcup_m \mathcal{U}_m(\omega_p)$, where $\mathcal{U}_m(\omega_p)$ is compact. Let's construct the sequence inductively in m.

- For $m = 1$, for each $\Sigma \in \mathcal{U}_1(\omega_p)$, we take ε_Σ to be smaller than $\frac{1}{2}$, and then take

a finite cover $\{\Sigma_k\}_{k=1}^{k_1}$ of $\mathcal{U}_1(\omega_p)$ with the smallest k_1. Moreover, we also have

$$\bigcup_{k=1}^{k_1} \mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) \subset \mathcal{U}_2(\omega_p)\,. \tag{4.4.32}$$

- Inductively, for $m > 1$, $\mathcal{U}_{m-1}(\omega_p)$ is covered by $\{\Sigma_k\}_{k=1}^{k_{m-1}}$, so we only need to consider the compact set $\mathcal{R}_m = \mathcal{U}_m(\omega_p) \setminus \bigcup_{k=1}^{k_{m-1}} \mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$. For each $\Sigma \in \mathcal{R}_m$, we take ε_Σ to be smaller than $\frac{1}{m(m+1)}$, and then take a finite cover $\{\Sigma_k\}_{k=1+k_{m-1}}^{k_m}$ with the smallest k_m. Note that we still have

$$\bigcup_{k=1}^{k_m} \mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) \subset \mathcal{U}_{m+1}(\omega_p)\,. \tag{4.4.33}$$

  It follows that this procedure generates a sequence $\{\Sigma_k\}_{k\in\mathbb{N}}$ fulfilling all the conditions.

Step 2. We construct a function $\eta : \mathcal{U}(\omega_p) \to (0, 1]$, such that

- $\mathcal{N}_\eta := \bigcup_{\Sigma\in\mathcal{U}(\omega_p)} \mathbf{B}^{\mathbf{F}}_{2\eta(\Sigma)}(\Sigma) \subset \bigcup_k \mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$.
- $\mathcal{N}_{\eta,m} := \bigcup_{\Sigma\in\mathcal{U}_m(\omega_p)} \mathbf{B}^{\mathbf{F}}_{2\eta(\Sigma)}(\Sigma) \subset \bigcup_{k=1}^{k_m} \mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$.
- On each $\mathcal{U}_m(\omega_p)$, η has a positive lower bound $\tilde{\eta}_m > 0$.
- For any nonempty subset $\tilde{K} \subset \{k \mid \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \cap \mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) \neq \emptyset\}$.

$$\forall V \in \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma),\ \lambda \in \{0, 1, \cdots, p\},\ |\omega_p - \|V\|(M)| \leq \tilde{c}(\lambda, \tilde{K})/10, \tag{4.4.34}$$

- For any $\lambda \in \{0, 1, \cdots, p\}$, $l \in \{0, 1, \cdots, p\}$, a finite subset $K \subset \mathbb{N}$ with $\mathcal{N}_\eta \cap \mathbf{B}^{\mathbf{F}}_{\lambda,K} \neq \emptyset$, $\delta \in (0, c(\lambda, K)/4)$ and any continuous map $\Theta : X^l \to \mathcal{N}_\eta \cap \mathbf{B}^{\mathbf{F}}_{\lambda,K}$, there exists a homotopy map $H^{\Theta,\delta}_{\lambda,K} :\ X^l \times [0, 1] \to \mathbf{B}^{\mathbf{F}}_{\lambda+1,K}$, such that

1. $H^{\Theta,\delta}_{\lambda,K}(\cdot,0)=\Theta(\cdot)$;

2. $\|H^{\Theta,\delta}_{\lambda,K}(x,t)\|(M)-\|\Theta(x)\|(M)\le\delta,\ \forall t\in[0,1]$;

3. For $t\in[0,1/2]$, $\mathbf{F}(H^{\Theta,\delta}_{\lambda,K}(x,t),\Theta(x))\le\delta$;

4. For $t\in[1/2,1]$, there exists $v(x)\in\overline{B}^{p+1}$, such that for some $T>0$,

$$H^{\Theta,\delta}_{\lambda,K}(x,t)=\phi^{\Theta(x)}_{\lambda,K}(v(x),(2t-1)T)\,; \tag{4.4.35}$$

5. $\forall x\in X^l,\ H^{\Theta,\delta}_{\lambda,K}(x,1)\notin\mathcal{N}_\eta$.

Claim 4. *There exists a continuous positive function $\eta_1:\mathcal{U}(\omega_p)\to\mathbb{R}^+$ such that for any $m\in\mathbb{N},\Sigma\in\mathcal{U}_m(\omega_p)$,*

$$\mathbf{B}^{\mathbf{F}}_{2\eta_1(\Sigma)}(\Sigma)\subset\bigcup_{k=1}^{k_m}\mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)\,. \tag{4.4.36}$$

Proof. We can define η_1 inductively.

If $\Sigma\in\mathcal{U}_1(\omega_p)$, we define

$$\eta_1(\Sigma):=\max\{r>0\mid\mathbf{B}^{\mathbf{F}}_{2r}(\Sigma)\subset\bigcup_{k=1}^{k_1}\mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)\}\,. \tag{4.4.37}$$

If $m\ge2$ is the least number such that $\Sigma\in\mathcal{U}_m(\omega_p)\setminus\mathcal{U}_{m-1}(\omega_p)$, we define

$$\eta_1(\Sigma):=\min\left\{\begin{array}{l}\max\left\{r>0\mid\mathbf{B}^{\mathbf{F}}_{2r}(\Sigma)\subset\bigcup_{k=1}^{k_m}\mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)\right\},\\ \min\left\{\eta_1(\Sigma')+\mathbf{F}(\Sigma,\Sigma')\mid\Sigma'\in\mathcal{U}_{m-1}(\omega_p)\right\}\end{array}\right\}>0\,. \tag{4.4.38}$$

One can verify that η_1 is continuous, and thus, has a positive lower bound on each compact set $\mathcal{U}_m(\omega_p)$. ■

We define $\eta(\Sigma)$ to be the largest number in $(0,\min(\eta_1(\Sigma),1)]$ such that

$$\forall V\in\mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma),\ \lambda\in\{0,1,\cdots,p\},\ |\omega_p-\|V\|(M)|\le\tilde{c}(\lambda,\tilde{K})/10, \tag{4.4.39}$$

for any nonempty subset $\tilde{K} \subset \{k \mid \mathbf{B}^{\mathbf{F}}_{\eta_1(\Sigma)}(\Sigma) \cap \mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) \neq \emptyset\}$. Here $\tilde{c}(\lambda, \tilde{K})$ is defined in Lemma 4.4.3.

To show that η has a positive lower bound $\tilde{\eta}_m > 0$ on each $\mathcal{U}_m(\omega_p)$, we first observe that for any $\Sigma \in \mathcal{U}_m(\omega_p)$, $\{k \mid \mathbf{B}^{\mathbf{F}}_{\eta_1(\Sigma)}(\Sigma) \cap \mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k) \neq \emptyset\}$ is a subset of

$$K_m = \bigcup_{k=1}^{k_m} \bar{K}(\Sigma_k)\,, \tag{4.4.40}$$

since $\bigcup_{\Sigma \in \mathcal{U}_m(\omega_p)} \mathbf{B}^{\mathbf{F}}_{2\eta_1(\Sigma)}(\Sigma) \subset \bigcup_{k=1}^{k_m} \mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$. By the third bullet in **Step 1**, K_m is also a finite set. It follows from the compactness of $\mathcal{U}_m(\omega_p)$ that there exists a positive number $\tilde{\eta}_m$, such that for any $\Sigma \in \mathcal{U}_m(\omega_p)$ and any $V \in \mathbf{B}^{\mathbf{F}}_{\tilde{\eta}_m}(\Sigma)$,

$$|\omega_p - \|V\|(M)| \leq \min_{K \subset K_m, \lambda \in \{0,1,2,\cdots,p\}} \tilde{c}(\lambda, K)/3, \tag{4.4.41}$$

and thus, $\eta(\Sigma) \geq \tilde{\eta}_m > 0$ on $\mathcal{U}_m(\omega_p)$.

To prove the fourth bullet, we apply Lemma 4.4.3 to construct the homotopy map $H^{\Theta,\delta}_{\lambda,K}$. It follows immediately that conditions (1), (2), (3) and (4) are satisfied. Suppose that condition (5) doesn't hold, and then there exist some $x \in X^l$ and some $\Sigma \in \mathcal{U}(\omega_p)$, such that $H^{\Theta,\delta}_{\lambda,K}(x,1) \in \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \cap \mathbf{B}^{\mathbf{F}}_{\lambda+1,K}$. Hence, by the definition of η, we have

$$\omega_p - \|H^{\Theta,\delta}_{\lambda,K}(x,1)\|(M) < \tilde{c}(\lambda, K)/2\,. \tag{4.4.42}$$

On the other hand, we also have $H^{\Theta,\delta}_{\lambda,K}(x,0) = \Theta(x) \in \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \cap \mathbf{B}^{\mathbf{F}}_{\lambda+1,K}$, and thus,

$$\|\Theta(x)\|(M) - \omega_p < \tilde{c}(\lambda, K)/2\,, \tag{4.4.43}$$

so $\|\Theta(x)\|(M) - \|H^{\Theta,\delta}_{\lambda,K}(x,1)\|(M) < \tilde{c}(\lambda, K)$, which gives a contradiction to (2) of Lemma 4.4.3.

Step 3. We show that the homotopy map defined in **Step 2** would not push a varifold too close to $\mathcal{U}(\omega_p)$, provided that it is far away from $\mathcal{U}(\omega_p)$ initially.

More precisely, we would like to show that for any $\Sigma \in \mathcal{U}(\omega_p)$, there exist two sequences $\{\varepsilon_q(\Sigma)\}_{q=1}^p \subset \mathbb{R}^+$ and $\{a_q(\Sigma)\}_{q=1}^{p+1}$ with $a_1(\Sigma) = 2$ and $a_q(\Sigma) \geq 2^q$ satisfying the following property. In the fifth bullet of **Step 2**, for any $q \in \{1, \ldots, p\}$, $\Sigma \in \mathcal{U}(\omega_p)$ with $\mathbf{B}^{\mathbf{F}}_{\lambda+1,K} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset$, and $x \in X^l$ with $\Theta(x) \notin \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)/a_q(\Sigma)}(\Sigma)$ as well as $\|\Theta(x)\|(M) - \omega_p \leq \varepsilon_q(\Sigma)$, we have for all $t \in [0,1]$ and $\delta \in \left(0, \min\left(\frac{\eta(\Sigma)}{4a_q(\Sigma)}, \varepsilon_q(\Sigma)\right)\right)$,

$$H^{\Theta,\delta}_{\lambda,K}(x,t) \notin \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)/a_{q+1}(\Sigma)}(\Sigma)\,. \tag{4.4.44}$$

Moreover, in each $\mathcal{U}_m(\omega_p)$, $\varepsilon_q(\Sigma)$ has a positive lower bounds $\tilde{\varepsilon}^m_q$ and $a_q(\Sigma)$ has a uniform upper bounds $\tilde{a}^m_q$.

Let's construct $a_q(\Sigma)$ and $\varepsilon_q(\Sigma)$ inductively. Note that at the beginning, we have $a_1(\Sigma) = 2$.

For $\Sigma \in \mathcal{U}(\omega_p)$, suppose that we have defined $a_q(\Sigma)$ and, if $q > 1$, $\varepsilon_{q-1}(\Sigma)$. For any natural number k with $\overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset$, and $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}}$, if $\left((F^k_\cdot)_\#(V)\right)^{-1} \overline{\mathbf{B}}^{\mathbf{F}}_{\eta(\Sigma)/(2a_q(\Sigma))}(\Sigma) \neq \emptyset$ and $\left((F^k_\cdot)_\#(V)\right)^{-1} \partial\mathbf{B}^{\mathbf{F}}_{3\eta(\Sigma)/(4a_q(\Sigma))}(\Sigma) \neq \emptyset$, we define

$$\begin{aligned} d_{q,\Sigma,k}(V) := \operatorname{dist}\Big(&\left((F^k_\cdot)_\#(V)\right)^{-1} \partial\mathbf{B}^{\mathbf{F}}_{3\eta(\Sigma)/(4a_q(\Sigma))}(\Sigma), \\ &\left((F^k_\cdot)_\#(V)\right)^{-1} \overline{\mathbf{B}}^{\mathbf{F}}_{\eta(\Sigma)/(2a_q(\Sigma))}(\Sigma)\Big)\,, \end{aligned} \tag{4.4.45}$$

and otherwise, $d_{q,\Sigma,k}(V) := +\infty$. Note that by (4.4.8), (4.4.34) and the triangle inequality for the volume, we obtain

$$\overline{\mathbf{B}}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \cap (F^k_{S^p})_\#(V) = \emptyset\,. \tag{4.4.46}$$

Hence, by Lemma 4.3.4, we have

$$d_{q,\Sigma,k}(V) \geq h_{\Sigma_k}(\eta(\Sigma)/(4a_q(\Sigma))). \tag{4.4.47}$$

We also observe that any k with $\overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset$ is inside some finite subset K_m as in (4.4.40). Hence, we can define

$$\varepsilon_q(\Sigma) := \min_{\substack{k, \\ \overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset}} \frac{c_{0,k}}{16} \left(h_{\Sigma_k}(\eta(\Sigma)/(4a_q(\Sigma)))\right)^2 . \tag{4.4.48}$$

Claim 5. *There exists $\tilde{\varepsilon}^m_q > 0$, such that $\inf_{\Sigma \in \mathcal{U}_m(\omega_p)} \varepsilon_q(\Sigma) \geq \tilde{\varepsilon}^m_q$.*

Proof. For any fixed $\mathcal{U}_m(\omega_p)$, by the first bullet of **Step 2**, we know that

$$K'_m = \{k | \exists \Sigma \in \mathcal{U}_m(\omega_p), \overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset\} \subset \bigcup_{k=1}^{k_m} \bar{K}(\Sigma_k) , \tag{4.4.49}$$

which implies that $\# K'_m < \infty$.

By our inductive assumption and the definition of $\eta(\Sigma)$, $\eta(\Sigma)/a_q(\Sigma)$ has a positive lower bound, say $c'_m > 0$ among all $\Sigma \in \mathcal{U}_m(\omega_p)$. Hence,

$$\begin{aligned}
\inf_{\Sigma \in \mathcal{U}_m(\omega_p)} \varepsilon_q(\Sigma) &= \inf_{\Sigma \in \mathcal{U}_m(\omega_p)} \min_{\substack{k, \\ \overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset}} \frac{c_{0,k}}{16} \left(h_{\Sigma_k}(\eta(\Sigma)/(4a_q(\Sigma)))\right)^2 \\
&= \min_{k \in K'_m} \inf_{\substack{\Sigma \in \mathcal{U}_m(\omega_p), \\ \overline{\mathbf{B}}^{\mathbf{F}}_{p+1,\{k\}} \cap \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)}(\Sigma) \neq \emptyset}} \frac{c_{0,k}}{16} \left(h_{\Sigma_k}(\eta(\Sigma)/(4a_q(\Sigma)))\right)^2 \\
&\geq \min_{k \in K'_m} \frac{c_{0,k}}{16} h_{\Sigma_k}(c'_m)^2 > 0 .
\end{aligned} \tag{4.4.50}$$

■

Now given a_q and ε_q, we are going to define a_{q+1}. Let's take $a_{q+1}(\Sigma) \geq 2a_q(\Sigma) + 1$ to be the smallest number such that for any $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{\eta(\Sigma)/a_{q+1}(\Sigma)}(\Sigma)$,

$$\|V\|(M) \geq \omega_p - \varepsilon_q(\Sigma) . \tag{4.4.51}$$

Claim 6. *There exists $\tilde{a}^m_{q+1} > 0$, such that $\sup_{\Sigma \in \mathcal{U}_m(\omega_p)} a_{q+1}(\Sigma) \leq \tilde{a}^m_{q+1}$.*

Proof. Since $\eta(\Sigma) \leq 1$, it suffices to there exists a constant $c''_m > 0$ such that for any $\Sigma \in \mathcal{U}_m(\omega_p)$ and $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{c''_m}(\Sigma)$,

$$\|V\|(M) \geq \omega_p - \tilde{\varepsilon}^m_q \,. \tag{4.4.52}$$

Then, we have $a_{q+1}(\Sigma) \leq 1/c''_m$ for any $\Sigma \in \mathcal{U}_m(\omega_p)$.

Suppose for contradiction that there exists two sequences $\{\Sigma_i\} \subset \mathcal{U}_m(\omega_p)$ and $\{V_i \in \overline{\mathbf{B}}^{\mathbf{F}}_{1/i}(\Sigma_i)\}$ such that

$$\|V_i\|(M) < \omega_p - \tilde{\varepsilon}^m_q \,. \tag{4.4.53}$$

By the compactness of $\mathcal{U}_m(\omega_p)$, up to a subsequence, we may assume that $\Sigma_i \to \Sigma$ and $V_i \to \Sigma$. However, $\|\Sigma\|(M) = \omega(p)$, and thus,

$$\omega_p < \omega_p - \tilde{\varepsilon}^m_q \,, \tag{4.4.54}$$

giving a contradiction. ■

Claim 7. *$H^{\Theta,\delta}_{\lambda,K}(x,t) \notin \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)/a_{q+1}(\Sigma)}(\Sigma)$ provided that $\Theta(x) \notin \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)/a_q(\Sigma)}(\Sigma)$, $\|\Theta(x)\|(M) - \omega_p \leq \varepsilon_q(\Sigma)$ and $\delta < \min\left(\frac{\eta(\Sigma)}{4a_q(\Sigma)}, \varepsilon_q(\Sigma)\right)$.*

Proof. Indeed, since $\delta < \eta(\Sigma)/(4a_q(\Sigma))$ and $\Theta(x) \notin \mathbf{B}^{\mathbf{F}}_{\eta(\Sigma)/a_q}(\Sigma)$, by (2) of the fifth bullet point of **Step 2**, we have for $t \in [0, 1/2]$,

$$\mathbf{F}(H^{\Theta,\delta}_{\lambda,K}(x,t), \Sigma) \geq 3\eta(\Sigma)/(4 \cdot a_q) \geq \eta(\Sigma)/a_{q+1}(\Sigma) \,. \tag{4.4.55}$$

To show that $\mathbf{F}(H^{\Theta,\delta}_{\lambda,K}(x,t), \Sigma) \geq \eta(\Sigma)/a_{q+1}(\Sigma)$ for $t \in (1/2, 1]$, we first note that $\delta < \varepsilon_q(\Sigma)$, and thus, $\|H^{\Theta,\delta}_{\lambda,K}(x, 1/2)\|(M) \leq \omega_p + 2\varepsilon_q(\Sigma)$. Let's argue by contradiction and suppose there exists a $t_1 \in (1/2, 1]$ such that $H^{\Theta,\delta}_{\lambda,K}(x,t_1) \in \overline{\mathbf{B}}^{\mathbf{F}}_{\eta(\Sigma)/a_{q+1}(\Sigma)}(\Sigma)$. In

this case, let $k = \bar{k}(K)$ and

$$\begin{aligned}
&\operatorname{dist}\left(\left((F^k_\cdot)_\#(\Theta(x))\right)^{-1}(H^{\Theta,\delta}_{\lambda,K}(x,t_1)), \left((F^k_\cdot)_\#(\Theta(x))\right)^{-1}(H^{\Theta,\delta}_{\lambda,K}(x,1/2))\right)\\
\geq&\operatorname{dist}\left(\left((F^k_\cdot)_\#(\Theta(x))\right)^{-1}(\overline{\mathbf{B}}^{\mathbf{F}}_{\eta(\Sigma)/a_{q+1}(\Sigma)}(\Sigma)), \left((F^k_\cdot)_\#(\Theta(x))\right)^{-1}(\partial\mathbf{B}^{\mathbf{F}}_{3\eta(\Sigma)/(4a_q)(\Sigma)}(\Sigma))\right)\\
\geq& d_{q,\Sigma,k}(\Theta(x)) \geq \tilde{d}_{q,\Sigma,k} := h_{\Sigma_k}(\eta(\Sigma)/(4a_q(\Sigma))) > 0\,.
\end{aligned} \tag{4.4.56}$$

It follows from Lemma 4.3.3 that

$$\begin{aligned}
\|H^{\Theta,\delta}_{\lambda,K}(x,t_1)\|(M) &\leq \|H^{\Theta,\delta}_{\lambda,K}(x,1/2)\|(M) - \frac{c_{0,k}}{2}(\tilde{d}_{q,\Sigma,k})^2\\
&\leq \omega_p + 2\varepsilon_q(\Sigma) - 8\varepsilon_q(\Sigma)\\
&= \omega_p - 6\varepsilon_q(\Sigma)\,,
\end{aligned} \tag{4.4.57}$$

contradicting to the definition of $a_{q+1}(\Sigma)$. ■

Before the end of this step, we also define

$$\mu_m := \min_{q\in\{1,2,\cdots,p\}} \inf_{\Sigma\in\mathcal{U}_m(\omega_p)} \min\left(\eta(\Sigma)/(4a_q(\Sigma)), \varepsilon_q(\Sigma)\right) > 0\,. \tag{4.4.58}$$

Step 4 [Hierarchical Deformations] Up to a subsequence, we construct the homotopy map $H_i : X \times [0,1] \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$, such that $H_i(\cdot,0) = \Phi_i$ and $H_i(\cdot,1) = \Psi_i$ with all the required properties.

We choose a subsequence $\{\Phi_{j_i}\}$ such that $\sup\{\mathbf{M}(\Phi_{j_i}(x))\} \leq \omega_p + \mu_i/2$ where μ_i is defined in the end of **Step 3**. For simplicity, we still denote the sequence by $\{\Phi_i\}$.

For a fixed $\Phi_i : X_i \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ and a positive function $\psi : \mathcal{U}_i(\omega_p) \to \mathbb{R}^+$, we

define

$$\mathcal{N}_\psi^i := \bigcup_{\Sigma\in\mathcal{U}_i(\omega_p)} \mathbf{B}^{\mathbf{F}}_{2\psi(\Sigma)}(\Sigma)\,, \tag{4.4.59}$$

and then it suffices to deform Φ_i to Ψ_i such that $|\Psi_i|\cap\mathcal{N}^i_{\eta/(2a_{p+1})}=\emptyset$.

To do so, we start with a finer subdivision on X_i such that for each (closed) face F of X_i,

1. If $|\Phi_i|(F)\cap\mathcal{N}^i_{\eta/2}\neq\emptyset$, then $|\Phi_i|(F)\subset\mathcal{N}^i_\eta$. The set consisting of all such faces will be denoted by $\mathcal{B}$.

2. For $F\in\mathcal{B}$, we can assign a nonempty finite set $K(F)\subset\mathbb{N}$, such that $|\Phi_i|(F)\subset\mathbf{B}^{\mathbf{F}}_{K(F)}$, and if $F\subset F'$ both in $\mathcal{B}$, then we have $K(F)\supset K(F')$.

Indeed, this is possible, since η has a uniform lower bound on $\mathcal{U}_i(\omega_p)$, which implies $\mathbf{F}(\partial\mathcal{N}^i_\eta,\mathcal{N}^i_{\eta/2})\geq c_i>0$. Hence, (1) can be satisfied by barycentric subdivision finitely many times, till the variation of $|\Phi_i|$ with respect to the $\mathbf{F}$-norm over any face is bounded by $c_i/2$. Moreover, by the second bullet of **Step 2**, each $|\Phi_i|(F)$ with $F\in\mathcal{B}$ can be covered by finitely many $\mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$'s. It follows from Lebesgue's number lemma that by further barycentric subdivision finitely many times, each $|\Phi_i|(F)$ with $F\in\mathcal{B}$ can be covered by one of $\{\mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)\}$. Now, for each $F\in\mathcal{B}$ such that no $F'\in\mathcal{B}$ has $F\subset F'$, we can define $K(F)$ to be the singleton $\{k\}$ such that $|\Phi_i|(F)\subset\mathbf{B}^{\mathbf{F}}_{\varepsilon_k}(\Sigma_k)$. Then, (2) can be defined inductively on the decreasing dimensions by taking union of sets.

With these finely subdivided simplicial complex X_i, we would like to construct the homotopy map H_i inductively on its k-skeleton, i.e., $X_i^{(k)}$.

For $k=0$, we apply **Step 2** with $\lambda=0$ and $\delta<\mu_i/4$ to all the 0-cells in $\mathcal{B}$. For all the 0-cells outside $\mathcal{B}$, we simply construct a constant homotopy map. Thus, we obtain a map $H_i^{(0)}$ defined on $X_i^{(0)}\times[0,1]$. Moreover, $|H_i^{(0)}|(X_i^{(0)}\times 1)\cap\mathcal{N}^i_{\eta/a_1}=\emptyset$ and $|H_i^{(0)}|(x,[0,1])\subset\mathbf{B}^{\mathbf{F}}_{1,K(x)}$ for any $x\in\mathcal{B}^{(0)}$.

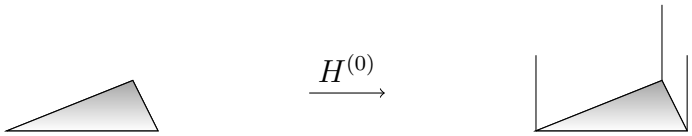

Figure 4.2: Deformation $H^{(0)}$

Inductively, suppose that we have constructed $H_i^{(k-1)}$ on $X_i^{(k-1)}$ ($k \geq 1$), and we shall consider the k-cells in X_i. Fixing $F_k \in X_i^{(k)} \setminus X_i^{(k-1)}$, then $\partial F_k \in X_i^{(k-1)}$. Note that $F_k' := F_k \cup (\partial F_k \times [0,1])$ is homeomorphic to F_k ($\cong D^k$), so we can define Θ on F_k' by concatenating $H_i^{(k-1)}$ on $\partial F_k \times [0,1]$ and Φ_i on F_k. Now, we shall have two ways to construct a homotopy map $\tilde{H}_i^{(k)}$ with initial data Θ depending on whether F_k is inside $\mathcal{B}$ or not.

If $F_k \notin \mathcal{B}$, then according to the definition of $\mathcal{B}$, no cell in ∂F_k belongs to $\mathcal{B}$ either. Thus, the homotopy map $\tilde{H}_i^{(k)}$ on F_k' in this case can be defined as a constant homotopy. It's worthy to note that

$$|\tilde{H}_i^{(k)}(F_k' \times 1 \cup \partial F_k' \times [0,1])| \cap \mathcal{N}_{\eta/a_{k+1}}^i = \emptyset\,. \tag{4.4.60}$$

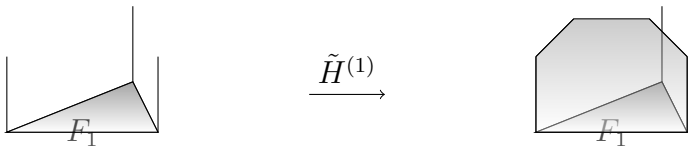

Figure 4.3: Deformation $\tilde{H}^{(1)}$

If $F_k \in \mathcal{B}$, by the second condition on the subdivision and the construction of $H_i^{(k-1)}$, we know that $|\Theta|(F_k') \subset \mathbf{B}^{\mathbf{F}}_{k,K(F_k)}$. However, $|\Theta|(F_k')$ might not be contained in $\mathcal{N}_\eta^i$, so we couldn't apply **Step 2** directly here. Instead, similar to the refinement mentioned at the beginning, we can make a subdivision on F_k' such that, each (closed) k-dimensional face $\tilde{F}$ with $|\Theta|(\tilde{F}) \cap \mathcal{N}_{\eta/a_k}^i \neq \emptyset$ must have $|\Theta|(\tilde{F}) \subset \mathcal{N}_\eta^i$. The union of all such k-dimensional faces now will be denoted by Y. Note that $|\Theta|(\partial Y) \cap \mathcal{N}_{\eta/a_k}^i = \emptyset$, since $|\Theta|(\partial F_k') \cap \mathcal{N}_{\eta/a_k}^i = \emptyset$ by induction. With this new subdivision, we can define a

map

$$\hat{H} : F_k' \times [0,1] \cup Y \times [1,2] \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2) \tag{4.4.61}$$

such that $\hat{H}(\cdot, t) := \Theta(\cdot)$ for $t \in [0,1]$, and $\hat{H}(\cdot, t+1) = H^{|\Theta|,\delta}_{k,K(F_k)}(\cdot, t)$ for $t \in [0,1]$ in **Step 2** with $\delta < \varepsilon_i$. By the construction of Y, we have

$$|\hat{H}((F_k' - Y) \times 1 \cup \partial F_k' \times [0,1])| \cap \mathcal{N}^i_{\eta/a_{k+1}} = \emptyset\,. \tag{4.4.62}$$

By (5) in the fifth bullet of **Step 2**,

$$|\hat{H}(Y \times 2)| \cap \mathcal{N}^i_{\eta/a_{k+1}} = \emptyset\,. \tag{4.4.63}$$

By **Step 3**,

$$|\hat{H}(\partial Y \times [1,2])| \cap \mathcal{N}^i_{\eta/a_{k+1}} = \emptyset\,. \tag{4.4.64}$$

We can derive $\tilde{H}^{(k)}_i : F_k' \times [0,1] \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ from $\hat{H}$ induced by the homeomorphism

$$\begin{aligned}(F_k' \times [0,1], \partial F_k' \times [0,1], F_k' \times 1) \cong (&F_k' \times [0,1] \cup Y \times [1,2], \\ &\partial F_k' \times [0,1], (F_k' \setminus Y) \times 1) \cup (Y \times 2) \cup (\partial Y \times [1,2]))\,,\end{aligned} \tag{4.4.65}$$

By the property of $\hat{H}$, we also have

$$|\tilde{H}^{(k)}_i(F_k' \times 1 \cup \partial F_k' \times [0,1])| \cap \mathcal{N}^i_{\eta/a_{k+1}} = \emptyset\,. \tag{4.4.66}$$

For both cases, we can derive $H^{(k)}_i : F_k \times [0,1] \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ from $\tilde{H}^{(k)}_i$ induced by the homeomorphism

$$(F_k \times [0,1], F_k \times 1) \cong (F_k' \times [0,1], F_k' \times 1 \cup \partial F_k' \times [0,1]), \tag{4.4.67}$$

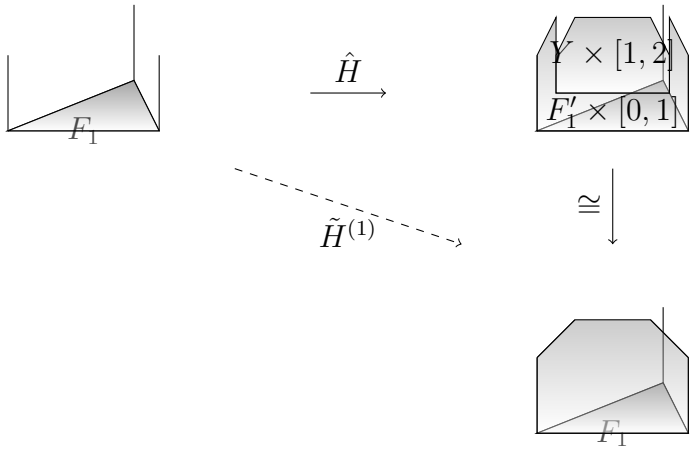

Figure 4.4: Deformations $\hat{H}$ and $\tilde{H}^{(1)}$

satisfying that $H_i^{(k)}|_{\partial F_k \times [0,1]} = H_i^{(k-1)}|_{\partial F_k \times [0,1]}$ and $H_i^{(k)}|_{F_k}(\cdot, 0) = \Phi_i|_{F_k}$. Note that,

$$|H_i^{(k)}(F_k \times 1)| = |\tilde{H}_i^{(k)}(F_k' \times 1 \cup \partial F_k' \times [0,1])|\,, \tag{4.4.68}$$

Therefore, we can concatenate all the $H_i^{(k)}$'s defined on $F_k \times [0,1]$'s and obtain

$$H_i^{(k)} : X_i^{(k)} \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2). \tag{4.4.69}$$

Moreover, we have

$$|H_i^{(k)}|(X_i^{(k)} \times 1) \cap \mathcal{N}^i_{\eta/a_{k+1}} = \emptyset\,. \tag{4.4.70}$$

In summary, we can take $\Psi_i := H_i^{(p)}(\cdot, 1)$ and $\bar{\varepsilon}(\Sigma) := \eta(\Sigma)/(2 \cdot a_{p+1}(\Sigma))$. It follows immediately that all the conditions hold. □

Now, with the deformation theorem above, we are able to prove our first Morse index upper bound theorem.

Proof of Theorem 4.1.1. For any min-max sequence $\{\Phi_i\}$ for p-width, by Theorem 4.4.4, there exists a new min-max sequence $\{\Psi_i\}$ such that $\mathbf{C}(\{\Psi_i\}) \cap \mathcal{APR}_p \subset \mathcal{S}(\omega_p)$. Therefore, the Almgren-Pitts theory generates a minimal hypersurface with optimal

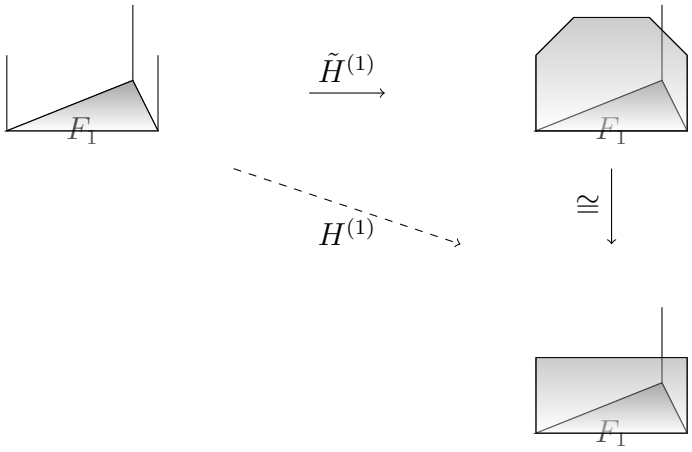

Figure 4.5: Deformations $\tilde{H}^{(1)}$ and $H^{(1)}$

regularity $V \in \mathbf{C}(\{\Psi_i\}) \cap \mathcal{APR}_p \subset \mathcal{S}(\omega_p)$, i.e.,

$$\text{index}(\text{spt}(V)) \leq p\,.$$

□

In addition, combined with [55, Theorem 4.7], we can obtain a stronger result, i.e., the existence of an optimal min-max sequence.

Theorem 4.4.5. *Suppose that (M^{n+1}, g) is a closed Riemannian manifold with $n+1 \geq 3$, then for any $p \in \mathbb{N}^+$, there exists a min-max sequence $\{\Phi_i\} \subset \mathcal{P}_p$ such that*

$$\mathbf{C}\{\Phi_i\} \subset \mathcal{S}(\omega_p)\,. \tag{4.4.71}$$

Proof. By the Deformation Theorem A (Theorem 4.4.4) above, it suffices to show that given any pulled-tight min-max sequence $\{\Phi_i\}$ for p-width, we can obtain a homotopic min-max sequence $\{\Psi_i\}$ such that $\mathbf{L}(\{\Psi_i\}) = \mathbf{L}(\{\Phi_i\})$, and $\mathbf{C}(\{\Psi_i\}) \subset \mathbf{C}(\{\Phi_i\}) \cap \mathcal{APR}_p$.

In [55, Theorem 4.7], for any fixed j, we can choose $R, \delta, \overline{\delta}$ to be $R_j = \delta_j = \overline{\delta}_j = 4^{-j}$, and then we could always find a sequence $\{\Psi_i^j\}$, such that the sequence $\{\Psi_i^j\}$

satisfies all the conclusions with $\mathcal{APR}_p$ in place of $\mathcal{W}_L$, since $\mathcal{APR}_p$ is compact. Applying the diagonal method, we can obtain a desired $\{\Psi_i\}$ from $\{\Psi_i^j\}$. □

Analogously, we can obtain a Morse index upper bound for the $\mathcal{A}^c$ functional.

Theorem 4.4.6. *Given a closed Riemannian manifold (M^{n+1}, g) $(n \geq 2)$, $c > 0$, and $\delta > 0$, if the (X, Z)-homotopy class $\Pi(\Phi_0)$ of $\Phi_0 : X^k \to \mathcal{C}(M)$ satisfies*

$$\mathbf{L}^c(\Pi(\Phi_0)) > \sup_{x \in Z} \mathcal{A}^c(\Phi_0(x)) \,, \tag{4.4.72}$$

then there exists an $\Omega \in \mathcal{C}(M)$ such that $\mathcal{A}^c(\Omega) = \mathbf{L}^c(\Pi(\Phi_0))$ and $\partial\Omega$ is a c-CMC hypersurface with optimal regularity and Morse index upper bound k.

In addition, the same conclusion also holds for the restrictive (X, Z)-homotopy class $\Pi_c^\delta(\Phi_0)$ of Φ_0 for $\mathcal{A}^c$ with an upper bound δ, provided that

$$\mathbf{L}^c(\Pi_c^\delta(\Phi_0)) > \sup_{x \in Z} \mathcal{A}^c(\Phi_0(x)) \,. \tag{4.4.73}$$

4.5 Construction of c-CMC hypersurfaces and improved Morse Index Bound

4.5.1 Approximation by c-CMC hypersurfaces

It is well-known that a smooth minimal hypersurface with a positive Jacobi field must be stable. In fact, it is true even for a minimal hypersurface with optimal regularity.

Lemma 4.5.1. *Suppose that $\Sigma^n \subset (M^{n+1}, g)$ is an oriented minimal hypersurface with optimal regularity, whose unit normal in the regular part is chosen as ν. If there exists a smooth positive solution u to the Jacobi equation defined on $\Sigma \setminus S$, where*

$S \supset \mathrm{Sing}(\Sigma)$ *is a closed subset of codimension no less than* 7*, i.e.,*

$$L_{\Sigma^n} u := -\Delta u - \left(\mathrm{Ric}^M(\nu,\nu) + |A_\Sigma|^2\right) u = 0 \quad \textit{in } \Sigma \setminus S\,, \tag{4.5.1}$$

then Σ *is stable.*

Remark 4.5.2. Although u is smooth in $\Sigma \setminus S$, u might be unbounded near S.

Proof. Suppose for contradiction that Σ is unstable. Since $\dim S \leq n-7$, there exists a smooth function f compactly supported in $\Sigma \setminus S$ satisfying the following unstability inequality,

$$\int_\Sigma |\nabla f|^2 - \left(\mathrm{Ric}^M(\nu,\nu) + |A_\Sigma|^2\right) f^2 < 0\,. \tag{4.5.2}$$

Note that u is smooth, bounded and positive in $\mathrm{spt}\, f$, we may write f as $g \cdot u$, where g is compactly supported in $\Sigma \setminus S$ and thus obviously non-constant. Therefore, the left hand side is

$$\begin{aligned}
&\int_\Sigma |\nabla(gu)|^2 - \left(\mathrm{Ric}^M(\nu,\nu) + |A_\Sigma|^2\right)(gu)^2 \\
=&\int_\Sigma -\Delta(gu)\cdot gu - \left(\mathrm{Ric}^M(\nu,\nu) + |A_\Sigma|^2\right)(gu)^2 \\
=&\int_\Sigma -\Delta g \cdot gu^2 - 2\nabla g\cdot\nabla u\cdot gu - gu^2\left(-\Delta u - \left(\mathrm{Ric}^M(\nu,\nu) + |A_\Sigma|^2\right)u\right) \\
=&\int_\Sigma |\nabla g|^2\cdot u^2 > 0\,,
\end{aligned} \tag{4.5.3}$$

which gives a contradiction to the inequality (4.5.2) above. □

In X. Zhou's multiplicity one theorem for sweepouts of boundaries ([86, Theorem 4.1]), he used two-sided PMC hypersurfaces to approximate minimal hypersurfaces to exclude higher-multiplicity components. Here, we will approximate minimal hypersurfaces by c-CMC hypersurfaces instead.

Proposition 4.5.3. *Given a sequence of two-sided, multiplicity one* c_i*-CMC hypersurfaces* V^i *as the boundary of some Cacciopoli set with optimal regularity and Morse*

index upper bound k in (M, g), if $c_i \searrow 0$ and $\|V^i\|(M) \to A \in (0, \infty)$, then there exists a minimal hypersurface V with optimal regulariy, whose support is a disjoint union of connected minimal hypersurfaces $\{\Sigma_j\}_{j=1,\cdots,l_k}$ with multiplicities $\{m_j\}$, such that, up to a subsequence,

$$V^i \rightharpoonup V\,, \tag{4.5.4}$$

and thus $\|V\|(M) = A$.

Moreover, every component of V with multiplicity greater than 2 is stable and

$$\sum_{m_j \leq 2} \text{index}(\Sigma_j) \leq k\,. \tag{4.5.5}$$

Proof. By the compactness of CMC hypersurfaces (See [7], [68]), we know that the limit is a minimal hypersurface V with optimal regularity, and the support has Morse index upper bound k.

It suffices to show that if $m_j \geq 3$, then the corresponding component Σ_j is stable. For such Σ_j, we will consider two cases, either $\text{reg}(\Sigma_j)$ is 2-sided or 1-sided.

Case 1 If Σ_j is 2-sided, by the sheeting theorem in [7] and [68], we know that outside at most k points and the singular set of Σ_j, there exists an exhaustion by compact domains $\{U_i \subset \Sigma_j\}$ and small neighborhoods $\tilde{U}_i$ of U_i in the image of the exponential map of its normal bundle with a fixed unit normal satisfying the following conditions. Up to a subsequence, the inverse image of $\text{spt}(V^i) \cap \tilde{U}_i$ under the exponential map is smooth multi-graphs graphically converging to Σ_j. Note that the number of the graphs is determined by the multiplicity m_j. Therefore, $\text{spt}(V^i) \cap \tilde{U}_i$ can be written as a set of m_j-normal graphs

$$\{u_i^1, u_i^2, \cdots, u_i^{m_j} : u_i^{\cdot} \in C^\infty(U_i)\}\,,$$

with

$$u_i^1 \leq u_i^2 \leq \cdots \leq u_i^{m_j},$$

where $u_i^{\cdot} \to 0$ in the smooth topology as $i \to \infty$.

Since V_i is the boundary of some Cacciopoli set, it follows immediately from the constancy theorem that these graphs have alternate unit outer normal. In particular, $m_j \geq 3$ implies that there exists two graphs u_i^1 and u_i^3 whose unit outer normal both pointing upwards or downwards w.r.t. the unit normal over U_i.

Following the proof of [86, Theorem 4.1], we can obtain a similar equation as (4.3) therein, i.e.,

$$L_{\Sigma_j}(u_k^3 - u_k^1) + o(u_k^3 - u_k^1) = 0. \tag{4.5.6}$$

Fix a point $p \in U_1$ and the strong maximum principle [86, Lemma 3.12] indicates that $u_k^3 - u_k^1 > 0$. Thus, if we define $h^k(x) := (u_k^3(x) - u_k^1(x))/(u_k^3(p) - u_k^1(p))$, a standard Harnack inequality will lead to the limit $h(x) > 0$ defined over Σ_j outside at most k points and the singular set of Σ_j, where the convergence is taken in the smooth topology.

In particular, $h(x)$ is a positive "Jacobi field" as in Lemma 4.5.1. It follows immediately that Σ_j is stable.

Case 2 If Σ_j is 1-sided, we can follow part 8 in the proof of [86, Theorem 4.1] by considering a 2-sided double cover $\tilde{\Sigma}_j$ of Σ_j. The proof in **Case 1** applied to $\tilde{\Sigma}_j$ implies that $\tilde{\Sigma}_j$ is stable, and therefore Σ_j is stable by simple lifting of vector fields. □

4.5.2 Approximation by c-CMC hypersurfaces

For each fixed $p \in \mathbb{N}^+$, let's fix a pulled-tight min-max sequence $S = \{\Phi_i\} \subset \mathcal{P}_p$, satisfying that

$$\lim_{i\to\infty} \sup_x \{\mathbf{M}(\Phi_i(x))\} = \omega_p, \tag{4.5.7}$$

and $\sup_x\{\mathbf{M}(\Phi_i(x))\} \leq \omega_p + 1/i$.

Without loss of generality, we can always assume that $X_i := \mathrm{dmn}(\Phi_i)$ is of dimension p, since otherwise we can simply restrict Φ_i to its p-skeleton. Moreover, by Theorem 4.4.5, we can also assume that any $V \in \mathbf{C}(S)$ is a minimal hypersurface with optimal regularity.

Construction of Restrictive F-homotopy Families

For each fixed $i \in \mathbb{N}^+, l \in \{0, 1, \cdots, p\}$, let's define $X_i^{(l)}$ to be the l-skeleton of X_i and $\Phi_i^{(l)} := \Phi_i|_{X_i^{(l)}}$. The corresponding **restrictive F-homotopy family** $\Pi_i^{(l)}$ for $\Phi_i^{(l)}$ is defined as

$$\begin{aligned}\Pi_i^{(l)} = \{\Psi : X_i^{(l)} \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2) | \exists H : X_i^{(l)} \times [0,1] \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)\,,\\ H(\cdot,0) = \Phi_i^{(l)}\,, H(\cdot,1) = \Psi\,,\\ \sup_{x,t}\{\mathbf{M}(H(x,t))\} \leq \sup_x \mathbf{M}(\Phi_i(x)) + \frac{l}{i}\}\,.\end{aligned} \tag{4.5.8}$$

Thus, we can still define a min-max value $\mathbf{L}_i^{(l)}$ for each $\Pi_i^{(l)}$ by

$$\mathbf{L}_i^{(l)} := \inf_{\Psi\in\Pi_i^{(l)}} \sup_x \mathbf{M}(\Psi(x)) \leq \sup_x \mathbf{M}(\Phi_i(x)). \tag{4.5.9}$$

Since $\mathbf{L}_i^{(l)}$ is well-defined for each i and l, one can also define

$$\mathbf{L}^{(l)} = \lim_i \inf \mathbf{L}_i^{(l)}, \tag{4.5.10}$$

and it is easy to see that for each l, $\mathbf{L}^{(l)} \leq \omega_p$ and $\mathbf{L}^{(p)} = \omega_p$.

Lemma 4.5.4. *$\mathbf{L}^{(0)} < \omega_p$, provided that every minimal hypersurface with optimal regularity in $\mathcal{APR}_p$ is 1-unstable.*

Proof of Lemma 4.5.4. Since $S = \{\Phi_i\}$ is pulled-tight and the compact set $\mathbf{C}(S)$ consists of 1-unstable minimal hypersurfaces with optimal regularity, we can find a finite cover $\{\mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k)\}$ from the cover $\{\mathbf{B}^{\mathbf{F}}_{2\varepsilon_\Sigma}(\Sigma)\}_{\Sigma\in\mathbf{C}(S)}$.

Hence, for $\varepsilon = \min_k\{\varepsilon_k\}$, i large enough and any $x \in X_i^{(0)}$, either $\mathbf{M}(\Phi_i(x)) < \omega_p - \varepsilon$, or $\Phi_i(x)$ is in $\mathbf{B}^{\mathbf{F}}_{2\varepsilon_k}(\Sigma_k)$ for some k. In the latter case, as in Lemma 4.4.3, we can decrease $\mathbf{M}(\Phi_i(x))$ by $c_{0,k}/2 > 0$ via a homotopy map increasing the area at most $1/i$.

In summary, $\mathbf{L}^{(0)} \leq \omega_p - \min(\varepsilon, \min_k(c_{0,k}))$. □

Construction of Restrictive (X, Z)-homotopy Classes

In this part, we shall assume that every minimal hypersurface in $\mathcal{APR}_p$ is 1-unstable, and let $l < p$ be the largest number such that $\mathbf{L}^{(l)} < \omega_p$. Under these assumptions, we are going to construct c-CMC hypersurfaces approximating a p-width minimal hypersurface.

In fact, it follows immediately from the assumptions that we can take a subsequence $\{\mathbf{L}_{i_j}^{(l)}\}$ such that $\lim_{j\to\infty} \mathbf{L}_{i_j}^{(l)} = \omega_p - c_1$ for some positive c_1 and in addition, $\mathbf{L}_{i_j}^{(l+1)} \geq \omega_p - 1/j$. For i_j large enough, we have $\mathbf{L}_{i_j}^{(l)} < \omega_p - \frac{1}{2}c_1$, i.e., there exists a $\Psi_{i_j} \in \Pi_{i_j}^{(l)}$ with a homotopy map H_{i_j}, such that $\sup_x \Psi_{i_j}(x) < \omega_p - \frac{1}{2}c_1$.

Now, for each i_j with $j \geq 100/c_1$, we are going to modify $\Phi_{i_j}^{(l+1)}$ such that we can construct a restrictive (X, Z)-homotopy class from it. The modified map $\Psi_{i_j}^{(l+1)}$ of

$\Phi_{i_j}^{(l+1)}$ is defined as follows.

Firstly, on the l-skeleton, $\Psi_{i_j}^{(l+1)}|_{X_{i_j}^{(l)}} := \Psi_{i_j}$. Then, each $(l+1)$-cell is a $(l+1)$-dimensional cube, and thus, could be parametrized by $[-1,1]^{l+1}$. In the interior, we define

$$\Psi_{i_j}^{(l+1)}(\frac{x}{2}) := \Phi_{i_j}^{(l+1)}(x), \quad \forall x \in [-1,1]^{l+1}\,.$$

Near the boundary, we define

$$\Psi_{i_j}^{(l+1)}(\frac{t+1}{2}x) := H_{i_j}(x,t), \quad \forall x \in \partial[-1,1]^{l+1},\ t \in [0,1]\,.$$

One could easily check that $\Psi_{i_j}^{(l+1)} \in \Pi_{i_j}^{(l+1)}$ and moreover, we have

$$\omega_p - 1/j \leq \sup_x \mathbf{M}(\Psi_{i_j}^{(l+1)}(x)) \leq \sup_x \mathbf{M}(\Phi_{i_j}(x)) + \frac{l}{i_j}. \tag{4.5.11}$$

Then, since $(\Psi_{i_j}^{(l+1)})^*(\bar{\lambda}) = (\Phi_{i_j}^{(l+1)})^*(\bar{\lambda}) \neq 0$, we lift the sweepout Ψ_{i_j} to the double cover $\partial : \mathcal{C}(M) \to \mathcal{Z}(M, \mathbb{Z}_2)$. We obtain a double cover $\pi : \tilde{X}_{i_j}^{(l)} \to X_{i_j}^{(l)}$ and the lifting map:

$$\tilde{\Psi}_{i_j}^{(l+1)} : \tilde{X}_{i_j}^{(l)} \to (\mathcal{C}(M), \mathbf{F})\,, \tag{4.5.12}$$

satisfying $\partial\tilde{\Psi}_{i_j}^{(l+1)} = \Psi_{i_j}^{(l+1)} \circ \pi$.

Since $\mathbf{L}_{i_j}^{(l+1)} \geq \omega_p - 1/j \geq \sup \mathbf{M}(\Psi_{i_j})|_{X_{i_j}^{(l)}} + c_1/3$ and $X_{i_j}^{(l+1)} \setminus X_{i_j}^{(l)}$ is a union of disjoint $(l+1)$-cells, there exists a $(l+1)$-cell $\tilde{C}_{i_j}^{(l+1)}$ in $\tilde{X}_{i_j}^{(l+1)}$ satisfying the following property. Let $\Pi_c^\delta(\tilde{\Psi}_{i_j}^{(l+1)}|_{\tilde{C}_{i_j}^{(l+1)}})$ be the associated restrictive $(\tilde{C}_{i_j}^{(l+1)}, \partial\tilde{C}_{i_j}^{(l+1)})$-homotopy class, then we have

$$\begin{aligned} \mathbf{L}_{i_j}^{(l+1)} \geq \mathbf{L}^c(\Pi_c^\delta(\tilde{\Psi}_{i_j}^{(l+1)}|_{\tilde{C}_{i_j}^{(l+1)}})) &\geq \mathbf{L}_{i_j}^{(l+1)} - c \cdot \mathrm{Vol}(M) \\ &> \sup \mathcal{A}^c(\tilde{\Psi}_{i_j}^{(l+1)})|_{\partial\tilde{C}_{i_j}^{(l+1)}}\,, \end{aligned} \tag{4.5.13}$$

provided that $c < \frac{c_1}{10000 \cdot i_j \cdot \mathrm{Vol}(M)}$ and $\delta < \frac{1}{i_j}$. Indeed, if this not true, one can construct

a new map $\Psi' \in \Pi^{(l+1)}_{i_j}$ such that $\sup \mathbf{M}(\Psi) < \mathbf{L}^{(l+1)}_{i_j}$, which gives a contradiction to the definition of $\mathbf{L}^{(l+1)}_{i_j}$.

Construction of c-CMC hypersurfaces

For each i_j with $j \geq 100/c_1$, applying Theorem 4.4.6 on $\Pi^{\delta}_{c}(\tilde{\Psi}^{(l+1)}_{i_j}|_{\tilde{C}^{(l+1)}_{i_j}})$, we obtain a c-CMC minimal hypersurface V_{i_j} with Morse index no greater than p with $c = \frac{c_1}{100000 \cdot i_j \cdot \mathrm{Vol}(M)}$ and $\delta = \frac{1}{10 \cdot i_j}$. Moreover,

$$\mathcal{A}^c(V_{i_j}) \leq \|V_{i_j}\|(M) \leq \mathcal{A}^c(V_{i_j}) + c \cdot \mathrm{Vol}(M)\,. \tag{4.5.14}$$

and thus, by the assumption of c, we have

$$\mathbf{L}^{(l+1)}_{i_j} - \frac{c_1}{100000 \cdot i_j} \leq \|V_{i_j}\|(M) \leq \mathbf{L}^{(l+1)}_{i_j} + \frac{c_1}{100000 \cdot i_j}\,. \tag{4.5.15}$$

4.5.3 Proof of the Improved Morse Index Upper Bound

If there exists a stable minimal hypersurface with optimal regularity in $\mathcal{APR}_p$, then the conclusion holds apparently.

Otherwise, by the construciton in the previous subsection, we know that there exists a sequence of c_i-CMC hypersurfaces V_i with optimal regularity and Morse index upper bound p satisfying that

$$\|V_i\|(M) \to \omega_p\,, \tag{4.5.16}$$

$$c_i \searrow 0\,. \tag{4.5.17}$$

It follows immediately from Proposition 4.5.3 that there exists a p-width minimal hypersurface V with optimal regularity. Furthermore, its support is a disjoint union of connected minimal hypersurfaces $\{\Sigma_j\}_{j=1,\cdots,l_k}$ with multiplicities $\{m_j\}$ such that every component of V with multiplicity greater than 2 is stable and

$$\sum_{m_j \leq 2} \text{index}(\Sigma_j) \leq p\,. \tag{4.5.18}$$

In summary, Theorem 4.1.4 holds.

Chapter 5

Smooth minimal hypersurfaces in dimension 8

5.1 Introduction

As mentioned in the introduction chapter 1, the existence of Simons cone has shown that even in dimension 8, a minimal hypersurface can have singular set. In fact, N. Smale [73] has constructed explicit examples of minimal hypersurfaces with boundary containing arbitrarily many singular points virtually by applying bridge principle on stable cones. Later, in [75], he also constructed a closed Riemannian manifold admitting a unique area-minimizer in a homology class, which contains 2 singular points. [71] shown that any closed subset in $\mathbb{R}^{n-7}$ can be realized as singular set of some stable minimal hypersurface in $(\mathbb{R}^{n+1}, g)$ for some smooth metric g arbitrarily close to the Euclidean one, making the rectifiability result of singular sets by [58] sharp in some way.

In this chapter, we discuss a joint work [45] with Zhihan Wang concerning the resolution of singularities.

Q 1: *Is it possible to resolve singularities of a minimal hypersurface by a small perturbation of the ambient metric?*

The same question on area-minimizing hypersurfaces has been raised in [84, 74]. An affirmative answer to this question together with the construction of minimal hypersurfaces from Almgren-Pitts min-max theory would directly lead to the generic existence of smooth minimal hypersurfaces.

Q 2: *Does there exist a smooth minimal hypersurface in a closed Riemannian manifold with generic metrics?*

In a closed manifold M of dimension 8, N. Smale [74] confirmed the existence of such a small perturbation for an area minimizer in its homology classes and thus, the generic existence of smooth minimal hypersurfaces. Recently, Chodosh-Liokumovich-Spolaor [15] justified the generic existence in the positive Ricci curvature case by constructing an optimal nested volume parametrized sweepout for 1-parameter min-max minimal hypersurfaces. Both of their arguments essentially employed the local analysis of isolated singular points by Hardt-Simon [35], which produces unique Hardt-Simon foliations of smooth minimal hypersurfaces in either side of a regular area-minimizing cone. Similar results were obtained for regular one-sided area-minimizing cone by F. Lin [47] and Z. Liu [49]. Nevertheless, by a calibration argument [9, 40], the existence of such a smooth foliations implies that the regular minimal cone should be at least one-sided area-minimizing, and thus, it seems impossible to apply this local analysis directly on a regular stable minimal cone which is not area-minimizing one either side. To the best of the authors' knowledge, the existence of such a cone is still widely open, and a negative answer to this will significantly simplify our arguments and other important analysis in the literature.

To conquer the absence of smooth foliations, we utilize a global perturbation argument from [78] (See Lemma 5.2.10) and a novel geometric invariant, "singular

capacity" (See Definition 5.4.1) to resolve singular points by *small* but *global* perturbations. As an application, we are able to settle the second question and prove the generic existence of a smooth minimal hypersurface in a given closed Riemannian manifold.

Theorem 5.1.1 (Main Theorem)**.** *Let M be an 8-dimensional closed Rimeannian manifold with $H_7(M, \mathbb{Z}_2) = 0$. Then for any C^k-generic ($k \geq 4$ or $k = \infty$) Riemannian metric g, there exists a closed embedded smooth minimal hypersurface Σ in (M, g).*

Theorem 5.1.2. *Every 8-dimensional closed Riemannian manifold M with any C^∞-generic metric admits a closed smooth minimal hypersurface Σ.*

Proof. If $H_7(M, \mathbb{Z}_2) \neq 0$, then we can take a nontrivial homology class $\alpha \in H_7(M, \mathbb{Z}_2)$. The standard Federer-Fleming minimizing process [28] implies the existence of an area-minimizing multiplicity one minimal hypersurface $\Sigma \subset (M, g)$ with $[\Sigma] = \alpha$. In this case, by Lemma 5.2.9 below, one can show that there exists a metric g' arbitrarily close to g such that (M, g') admits a smooth non-degenerate minimal hypersurface Σ' close to Σ and $[\Sigma'] = \alpha$ as well. The openness of the set of such g''s follows immediately from the non-degenerateness and B. White's structure theorem [79, 81].

The $H_7(M, \mathbb{Z}_2) = 0$ case is exactly the main theorem. □

For simplicity, we shall only prove the main theorem for the open subset $\mathrm{int}(\mathscr{G}_F(M))$ where

$$\mathscr{G}(M) := \{C^k \text{ Riemannian metrics on } M\}$$
$$\mathscr{G}_F(M) := \{g \in \mathscr{G} | (M, g) \text{ has Frankel property}\}.$$

In fact, this is the most difficult region to handle. The complete proof can be found in the original paper.

5.1.1 Some conventions

Unless otherwise stated, a minimal hypersurface Σ in an $(n+1)$-dimensional closed manifold M is referred to *a smooth, locally stable n-dimensional submanifold with locally bounded area and optimal regularity*, i.e. $\mathcal{H}^n(\Sigma \cap K) < +\infty$ for every compact subset $K \subset M$, and $\mathcal{H}^{n-2}(\overline{\Sigma} \setminus \Sigma) = 0$. These are exactly the minimal hypersurfaces generated from area minimizing arguments or Almgren-Pitts min-max theory (See [59, 65, 42]). We always identify Σ with the regular part of $\overline{\Sigma}$, and denote $\overline{\Sigma} \setminus \Sigma$ by $\mathrm{Sing}(\Sigma)$.

For a Caccioppoli set A, we shall identify A with $\mathcal{H}^{n+1}$-density 1 part of A. We will use $\partial^t A$ to denote its topological boundary to avoid confusion with the reduced boundary.

5.1.2 Sketch of Proof

The openness follows directly from White's structure theorems [79, 81].

To prove this density, we need to introduce in general a geometric invariant, **singular capacity**, denoted by **SCap**, on the space of 8-dimensional Riemannian manifolds paired with a locally stable minimal hypersurface. Roughly speaking, this invariant counts how many singularities "potentially" a minimal hypersurface contains. The key of this invariant is its upper semi-continuity:

$$\mathbf{SCap}(\Sigma; U, g) \geq \limsup_{j\to\infty} \mathbf{SCap}(\Sigma_j; U, g_j)$$

where $g_j \to g$ in C^k_{loc} and $\Sigma_j \to \Sigma$ in the varifold sense with some technical assumptions.

With the notion of singular capacity, the proof can be decomposed into three steps.

Step 1 (Uniqueness) For every $g \in \mathrm{int}(\mathscr{G}_F)$, let $V = \kappa|\Sigma| \in \mathcal{R}$ be a 1-width min-max integral varifold generated from an ONVP sweepout. One can observe the following dichotomy:

- either $\mathfrak{h}_{nm}(\Sigma) = \emptyset$;
- or $V = |\Sigma|$ and Σ is connected, two-sided and separating with isolated singular points $\mathrm{Sing}(\Sigma) = \{p_0, p_1, ..., p_k\}$.

Let's consider the latter case. By constructing a metric perturbation by hand, we may further assume that Σ is the unique and non-degenerate one realizing 1-width provided that $\mathcal{H}^0(\mathfrak{h}_{nm}(\Sigma)) = 1$. Let ν denote a unit normal on Σ.

Step 2 (Singular capacity) Let $f \in \mathscr{F} \subset C_c^\infty(M - \mathrm{Sing}(\Sigma))$ be a generically chosen function with $\nu(f)|_\Sigma \not\equiv 0$. Let the metric perturbation be $g_t := g(1 + tf)$ and Σ_t be a 1-width min-max minimal hypersurfaces in (M, g_t).

It follows immediately from the uniqueness of Σ that $\Sigma_t \to \Sigma$ in the varifold sense. By analyzing the associated Jacobi field $u \cdot \nu$ on Σ generated by $\{\Sigma_t\}$ as in [78, Section 4], we can show that at some $p \in \mathrm{Sing}(\Sigma)$, Σ_t is smooth in a small neighborhood U_p of p for t close to 0.

However, a naive induction argument based on the $\mathcal{H}^0(\mathrm{Sing})$ would not work, because in a neighborhood $U_{p'}$ of another singular point $p' \in \overline{\Sigma}$, Σ_t might have more than 1 singular point inside (See the figure below). Fortunately, by upper semi-continuity of the singular capacity, we can conclude that

$$\mathbf{SCap}(\Sigma_{t_j}; M, g_{t_j}) \leq \mathbf{SCap}(\Sigma; M, g) - 1\,.$$

Step 3 Iterate Step 1 and Step 2, and then a backward induction argument would lead to the following dichotomy. For some $\tilde{g} \in \mathrm{int}(\mathscr{G}_F)$ arbitrarily close to g, we have

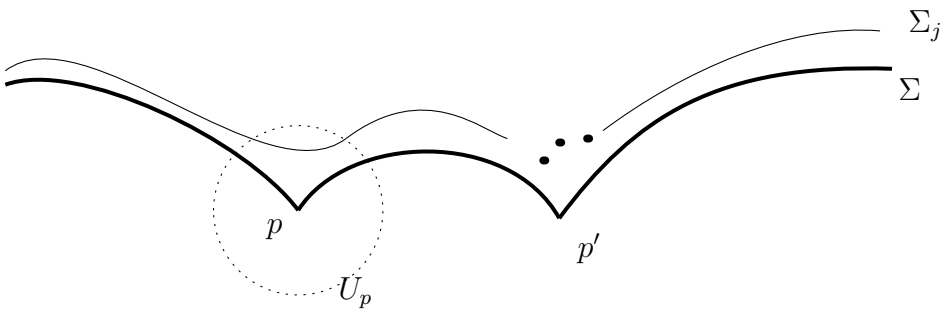

Figure 5.1: Resolution of the singular point p

a minimal hypersurface $\Sigma \subset (M, \tilde{g})$ satisfying

- either $\mathfrak{h}_{nm}(\Sigma) = \emptyset$;
- or $\mathbf{SCap}(\Sigma) = 0$.

In the former case, Hardt-Simon type metric perturbation would lead to a smooth minimal hypersurface (See Subsection 5.2.2).In the latter case, by definition, Σ itself is smooth.

In summary, we conclude the main theorem in $\mathrm{int}(\mathscr{G}_F)$.

5.1.3 Organization of the Chapter

In Section 5.2, we first recall a modified version of 1-parameter min-max theory and a Hardt-Simon type surgery procedure from [15], as well as a global perturbation argument from [78].

With these ingredients, we shall supplement the details of **Step 1** in Section 5.3 and **Step 2** in Section 5.4.

5.2 Preliminaries

We mainly focus on a closed Riemannian manifold (M^{n+1}, g) with $H_n(M, \mathbb{Z}_2) = 0$. In this case, by Poincaré duality, its first cohomology group with $\mathbb{Z}_2$ coefficients

$H^1(M, \mathbb{Z}_2) = 0$. Therefore, the Stiefel-Whitney class $\omega_1(M)$ vanishes and M is orientable.

5.2.1 1-parameter Min-max Theory

Let's consider the boundary map $\partial : \mathcal{C}(M) \to \mathcal{Z}_n(M; \mathbb{Z}_2)$: More precisely, for any $A \in \mathcal{C}(M)$, we use ∂A to denote the mod 2 flat cycle $\partial(\mathbf{E}^{n+1} \llcorner A)$, where $\mathbf{E}^{n+1}$ is defined in the previous subsection. It's easy to see that

$$P(A) = \mathbf{M}(\partial A)\,. \tag{5.2.1}$$

Since we assume that $H_n(M, \mathbb{Z}_2) = 0$, the boundary map is surjective. By Constancy Theorem, this map is in fact a double cover. With this observation, we can utilise Caccioppoli sets instead of mod 2 flat cycles to define sweepouts on M. This has been used in [85, 87, 88, 86, 15].

Definition 5.2.1. A (1-parameter) **sweepout** on M is a continuous map $\Phi : [0, 1] \to \mathcal{C}(M)$ with $\Phi(0) = 0$ and $\Phi(1) = M$. The collection of all sweepouts is denoted by $\mathcal{P}$.

Definition 5.2.2. The **min-max width** on M is defined to be

$$\mathcal{W} = \inf_{\Phi \in \mathcal{P}} \sup_{t \in [0,1]} \mathbf{M}(\partial \Phi(t)) > 0\,.$$

The Almgren-Pitts theory [3, 59, 65] has shown that the width $\mathcal{W}$ could always be realized by a minimal hypersurface (possibly with multiplicities). Recently, O. Chodosh, Y. Liokumovich and L. Spolaor [15] gave a refined description of such a minimal hypersurface in terms of its singularities and Morse index. Here, we adapt their results to our setting.

Definition 5.2.3 (ONVP sweepouts)**.** A sweepout Φ is called an **optimal nested volume parametrized (ONVP) sweepout** if it satisfies the following conditions.

1. **optimal** $\sup_{x\in[0,1]} \mathbf{M}(\partial\Phi(x)) = \mathcal{W}$;

2. **nested** $\Phi(x_1) \subset \Phi(x_2)$, for all $0 \leq x_1 \leq x_2 \leq 1$;

3. **volume parametrized** $\mathrm{Vol}(\Phi(x)) = x \cdot \mathrm{Vol}(\mathrm{M}, \mathrm{g})$, for every $x \in [0,1]$.

The **critical domain** of Φ is the set

$$\mathbf{m}(\Phi) = \{x \in [0,1] : \limsup_{y\to x} \mathbf{M}(\partial\Phi(y)) = \mathcal{W}\}\,. \tag{5.2.2}$$

Similarly, the **left** (resp. **right**) **critical domain** of Φ can be defined as the set $\mathbf{m}_L(\Phi)$ (resp. $\mathbf{m}_R(\Phi)$) only involving $y \nearrow x$ (resp. $y \searrow x$). Apparently, $\mathbf{m}(\Phi) = \mathbf{m}_L(\Phi) \bigcup \mathbf{m}_R(\Phi)$.

The **critical set** of Φ is

$$\mathbf{C}(\Phi) := \{V \in \mathcal{V}_n(M) : V = \lim_{j\to\infty} |\partial\Phi(x_j)|,\ x_j \in [0,1];\ \|V\|(M) = \mathcal{W}\}. \tag{5.2.3}$$

Definition 5.2.4 (Excessive points)**.** A point $x_0 \in [0,1]$ is called **left** (resp. **right**) **excessive** for a sweepout Φ, if there exists a constant $\varepsilon > 0$ and an interval $I = [a,b]$, $[a,b)$, $(a,b]$ or (a,b) with $(x_0 - \varepsilon, x_0] \subset I$ (resp. $[x_0, x_0 + \varepsilon) \subset I$), satisfying the following replacement condition.

We can find a continuous map on I, $\{\Phi^I(x)\}_{x\in I}$, such that $\Phi^I(a) = \Phi(a)$ and $\Phi^I(b) = \Phi(b)$ but for all $x \in I$,

$$\limsup_{I\ni y\to x} \mathbf{M}(\partial\Phi^I(y)) < W\,. \tag{5.2.4}$$

Theorem 5.2.5 ([15, Theorem 19])**.** *For any closed Riemannian manifold (M,g), there exists an (ONVP) sweepout Ψ such that every $x \in \mathbf{m}_L(\Psi)$ is not left-excessive and every $x \in \mathbf{m}_R(\Psi)$ is not right excessive. The Almgren-Pitts min-max theory implies that there exists a stationary integral varifold $V \in \mathbf{C}(\Psi)$ whose support is the*

closure of a minimal hypersurface Σ*, and* $\|V\|(M) = \mathcal{W}$*.*

Definition 5.2.6 (One-sided homotopy area-minimizing)**.** Given a minimal hypersurface $\Sigma \subset (M, g)$, $p \in \overline{\Sigma}$ and $r > 0$ small enough such that $\overline{\Sigma} \cap B_r(p)$ separates the open ball $B_r(p)$ into two open connected components E_+ and E_-. Σ is said to be **one-sided homotopy area-minimizing (OSHAM)** in $B_r(p)$, if there does not exist a deformation $\{\Omega(t) \subset \mathcal{C}(M)\}_{t\in[0,1]}$ satisfying the following conditions.

1. $\Omega(0) \cap B_r(p) = E_\pm$ and $\Omega(t) \subset \Omega(s)$ for any $t \geq s$;
2. $\partial^*\Omega(t)\Delta\partial^* E_\pm \subset B_r(p)$;
3. $\mathbf{M}_{B_r(p)}(\partial\Omega(t)) \leq \mathbf{M}_{B_r(p)}(\partial E_\pm)$ and $\mathbf{M}_{B_r(p)}(\partial\Omega(1)) < \mathbf{M}_{B_r(p)}(\partial E_\pm)$.

We define

$$\mathfrak{h}_{nm}(\Sigma) := \{p \in \overline{\Sigma} : \forall r > 0 \text{ small, } \Sigma \cap B_r(p) \text{ is not (OSHAM) in } B_r(p)\}\,. \tag{5.2.5}$$

Theorem 5.2.7 ([15, Theorem 4, Proposition 29, Lemma 30])**.** *Given a closed Riemannian manifold* (M^8, g) *and a sweepout* Ψ *in Theorem 5.2.5, by possibly replacing* $\Phi(x)$ *by* $M \setminus \Phi(1-x)$*, there exists a sequence* $x_i \nearrow x_0 \in \mathbf{m}_L(\Phi)$ *such that*

$$\lim_i |\partial\Phi(i)| \to V = \sum_i \kappa_i |\Sigma_i| \in \mathbf{C}(\Phi)\,,$$

where Σ_i*'s are pairwise disjoint minimal hypersurfaces with* $\kappa_i \in \{1, 2\}$ *and*

$$\sum_i \mathcal{H}^0(\mathfrak{h}_{nm}(\Sigma_i)) + \sum_i \mathrm{Index}(\Sigma_i) \leq 1\,. \tag{5.2.6}$$

Moreover, if there exists a $\kappa_i = 2$*, then*

$$\mathfrak{h}_{nm}(\Sigma) = \emptyset\,. \tag{5.2.7}$$

Otherwise, V is of multiplicity one and $V = |\partial\Phi(x_0)|$.

One essential ingredient of the proof is the following interpolation lemma which follows from a result of K.J. Falconer [26] (See also [34, Appendix 6], [13, Lemma 5.3]).

Lemma 5.2.8 ([15, Lemma 16])**.** *On (M^{n+1}, g), for every $L, \varepsilon > 0$, there exists a $\delta > 0$ satisfying the following property.*

For any $\Omega_0, \Omega_1 \in \mathcal{C}(M)$ with $\Omega_0 \subset \Omega_1$, $P(\Omega_i) \leq L$ and $\mathrm{Vol}(\Omega_1 \setminus \Omega_0) \leq \delta$, there exists a nested continuous family $\{\Omega_t\}$ with

$$P(\Omega_t) \leq \max\{P(\Omega_1), P(\Omega_0)\} + \varepsilon\,, \tag{5.2.8}$$

for all $t \in [0, 1]$.

5.2.2 Surgery Procedure à la Chodosh-Liokumovich-Spolaor

In this subsection, we will recall the surgery procedure in [15] to perturb away singularities locally when $\mathfrak{h}_{nm}(\Sigma) = \emptyset$ for a minimal hypersurface $\Sigma^7 \subset (M^8, g)$. Since the surgery was done locally, from now on, we focus on a singular point $p \in \mathrm{Sing}(\Sigma)$. WLOG, let's assume that for $\varepsilon_0 > 0$, Σ has only one singular point and is OSHAM in $B_{2\varepsilon_0}(p)$.

Lemma 5.2.9 (Surgery Procedure [15, Proposition 31])**.** *Let M, g, Σ, p and ε_0 be as above. For every $k \geq 4$, $\delta > 0$, there exists a Riemmanian metric g' and a minimal hypersurface $\Sigma' \subset (M, g')$ satisfying the following conditions.*

1. *$\|g - g'\|_{C^k} < \delta$;*
2. *$g = g'$ and $\Sigma = \Sigma'$ outside $B_{2\varepsilon_0}(p)$;*
3. *$\mathrm{Sing}(\Sigma') \cap B_{2\varepsilon_0} = \emptyset$.*

5.2.3 Global perturbation à la Wang

In [78], via analysis of Jacobi fields over a minimal hypersurface, Z. Wang found that the Jacobi field generated from $\Sigma_j \to \Sigma$ could help us understand the behavior of Σ_j near Σ. In particular, it depicts a picture where generically, one of the singular points of Σ can be perturbed away as Σ_j.

Lemma 5.2.10 ([45, Lemma 2.24])**.** *Assume that $\Sigma \subset (M^8, g)$ is a nondegenerate minimal hypersurface, then there exists an open dense subset $\mathscr{F} \subset C_c^\infty(M \setminus \mathrm{Sing}(\Sigma))$ depending only on M, g and Σ with the following property.*

For any $f \in \mathscr{F}$, $\{f_j\}$ a sequence of smooth functions with $f_j \to f$ in C^4, and any sequence $c_j \to 0_+$, if every $(M, g_j := (1 + c_j f_j)g)$ admits a minimal hypersurface Σ_j such that

- $\mathrm{Index}(\Sigma_j) = \mathrm{Index}(\Sigma)$*, and*
- $\Sigma_j \to \Sigma$ *in the varifold sense with multiplicity* 1*,*

then there exists a small neighborhood $U_p \subset M$ of some $p \in \mathrm{Sing}(\Sigma)$ such that $\mathrm{Sing}(\Sigma_j) \cap U_p = \emptyset$ for infinitely many j.

5.3 Uniqueness of Min-max Minimal Hypersurfaces

In this section, we shall discuss how to generate a candidate minimal hypersurface in a given Riemannian manifold (M^{n+1}, g) with $H_n(M, \mathbb{Z}_2) = 0$. For simplicity, whenever it is clear, we shall abuse the use of set relations $=$ and $\subset$ for Caccioppoli sets in the measure-theoretic sense, i.e., up to a measure zero set. As mentioned at the beginning, we focus on metrics inside $\mathrm{int}\mathscr{G}_F(M)$.

Definition 5.3.1. A closed Riemannian manifold (M^{n+1}, g) is said to have **Frankel property**, if any pair of minimal hypersurfaces has nonempty intersections.

Theorem 5.2.5 and Theorem 5.2.7 together imply the existence of an (ONVP) sweepout Φ and a sequence $x_i \nearrow x_0 \in \mathbf{m}_L(\Phi)$ such that $|\partial\Phi(x_i)| \to V \in \mathcal{R}$, where $V = \sum_i \kappa_i |\Sigma_i|$ for some pairwise disjoint minimal hypersurfaces Σ_i, where $\kappa_i \in \{1, 2\}$.

If the ambient manifold (M^8, g) has Frankel property, then $V = \kappa|\Sigma|$ for $\kappa \in \{1, 2\}$. Moreover, one of the following conditions holds:

1. either $\mathfrak{h}_{nm}(\Sigma) = \emptyset$;
2. or $\mathfrak{h}_{nm}(\Sigma) \neq \emptyset$, $\kappa = 1$, and $\Sigma = \partial^*\Phi(x_0)$ is stable.

In the first case, we can simply apply the surgery procedure (Subsection 5.2.2) to resolve all singularities

In the second case, we will perturb the Riemannian metric such that the min-max width is realized uniquely by Σ.

Lemma 5.3.2 (Unique realization of min-max width, [45, Lemma 3.4])**.** *Given a closed Riemannian manifold M^8 with $H_7(M, \mathbb{Z}_2) = 0$, let g be a metric inside $\mathrm{int}(\mathscr{G}_F)$, and Σ be a minimal hypersurface realizing the min-max width $\mathcal{W}$ generated from an (ONVP) sweepout as in Theorem 5.2.7.*

If $\mathfrak{h}_{nm}(\Sigma) = \{p\}$, then $\forall \varepsilon > 0$, there exists a metric $g' \in \mathrm{int}(\mathscr{G}_F)$ with $\|g - g'\|_{C^k} < \varepsilon$ satisfying that $\mathcal{W}(M, g')$ is uniquely realized by Σ if generated from an (ONVP) sweepout. Furthermore, Σ can be taken nondegenerate in (M, g').

This is possible, since the existence of $\mathfrak{h}_{nm}(\Sigma)$ enables us to modify the sweepout such that near Σ, the mass of each slice is stricly smaller than $\mathrm{Area}(\Sigma)$, which provides a room for us to perturb the ambient metric without breaking the optimality of the sweepout.

Lemma 5.3.3 ([45, Lemma 3.2])**.** *For a closed ambient manifold (M^8, g) with Frankel property, let Σ be a minimal hypersurface generated from an (ONVP) sweepout Φ via $x_i \nearrow x_0 \in \mathbf{m}_L(\Phi)$. If $\mathfrak{h}_{nm}(\Sigma) = \{p\}$, we can construct a new (ONVP) sweepout Ψ satisfying the following property.*

For any $r > 0$ small enough, there exists $\varepsilon_0 > 0$, an open set $U \supset \overline{\Sigma}$, and a compact set $K \subset U \cap B_r(p)$ containing p such that for any $x \in [0,1]$, we have

$$\mathbf{M}(\partial\Psi(x)) < \mathrm{Area}(\Sigma) - \varepsilon_0\,, \tag{5.3.1}$$

provided that $\partial^*\Psi(x) \cap U \setminus (K \cup \overline{\Sigma}) \neq \emptyset$.

5.4 Singular Capacity of Minimal Hypersurfaces

Let $\mathscr{C}$ be the space of stable minimal hypercones in $\mathbb{R}^8$. By the standard dimension reduction argument, every cone in $\mathscr{C}$ has a smooth cross section with $S^7(1)$.

Let $\mathscr{M}$ be the space of triples $(\Sigma; M, g)$, where (M, g) is an open subset of a Riemannian manifold and Σ is a minimal hypersurface in (M, g) with finitely many singular points. The topology on $\mathscr{M}$ is induced by C^4_{loc} convergence in g with fixed M and multiplicity one varifold convergence in Σ.

Definition 5.4.1. A map $\mathbf{SCap} : \mathscr{M} \to \mathbb{N}$ is called a **singular capacity**, if

(i) For every nontrivial $C \in \mathscr{C}$ and every open subset $U \subset \mathbb{R}^8$ containing the origin, we have

$$1 \leq \mathbf{SCap}(C; \mathbb{R}^8, g_{Euc}) = \mathbf{SCap}(C; U, g_{Euc}) < +\infty;$$

where g_{Euc} is the Euclidean metric. We abbreviate for simplicity $\mathbf{SCap}(C; \mathbb{R}^8, g_{Euc})$ to $\mathbf{SCap}(C)$;

(ii) For every $(\Sigma; M, g) \in \mathscr{M}$,

$$\mathbf{SCap}(\Sigma; M, g) := \sum_{p \in \mathrm{Sing}(\Sigma)} \mathbf{SCap}(C_p)\,,$$

where C_p is the unique tangent cone of Σ at p (conventionally, $\mathbf{SCap}(\Sigma; M, g) := 0$ if Σ is smooth);

(iii) If $(\Sigma_j; M, g_j) \to (\Sigma; M, g)$ in $\mathscr{M}$ and Σ_j is stable in (M, g_j), then for every open subset $U \subset\subset M$ with $\partial U \cap \mathrm{Sing}(\Sigma) = \emptyset$,

$$\mathbf{SCap}(\Sigma; U, g) \geq \limsup_{j\to\infty} \mathbf{SCap}(\Sigma_j; U, g_j)\,.$$

In [45, Section 4], we prove the existence of singular capacity on $\mathscr{M}$. The proof is based on an observation of variations of volume ratios, which was inspired by the work of Cheeger-Naber [14].

Theorem 5.4.2 ([45, Theorem 4.2])**.** *There exists a singular capacity* **SCap** *on* $\mathscr{M}$ *satisfying the following condition.*

For every $\Lambda \geq 1$*, there exists* $N(\Lambda) \in \mathbb{N}$ *such that*

$$\mathbf{SCap}(C) \leq N(\Lambda)$$

for every $C \in \mathscr{C}$ *with density at* 0 *less than or equal to* Λ*.*

Chapter 6

Full generic regularity in dimension 8

6.1 Introduction

Continuing the study in the previous chapter, in this project [46], we are interested in the question of generic regularity. In [84], Yau has conjectured that a perturbation of the metric can eliminate the singularities of a given area minimizing hypersurface in its homology class. A more optimistic conjecture is the generic regularity in full as following.

Conjecture (Full generic regularity I)**.** For a closed smooth manifold M^{n+1} endowed with a (Baire) C^∞-generic Riemannian metric g, every g-area homology minimizer is smooth.

Since the min-max construction doesn't produce area minimizers generally, it is natural to extend this conjecture to min-max minimal hypersurfaces, i.e., those generated from the min-max theory.

Conjecture (Full generic regularity II)**.** For a closed smooth manifold M^{n+1} endowed with a (Baire) generic Riemannian metric g, every min-max g-minimal hypersurface

is smooth.

In the following, we always assume that M^8 is a closed smooth manifold of dimension 8. Given a Riemannian metric g, an embedded smooth hypersurface $\Sigma \subset (M^{n+1}, g)$ is called a **locally stable minimal hypersurface (LSMH)**, if

$$\mathscr{H}^n(\Sigma) < +\infty\,, \quad \mathscr{H}^{n-2}(\bar{\Sigma} \setminus \Sigma) = 0\,,$$

and $\forall p \in M$, there's a neighborhood U_p in which Σ is stable.

The goal of this chapter is to prove the following theorem.

Theorem 6.1.1. *Let M be a closed smooth manifold of dimension 8, $k \geq 4$, $\alpha \in (0,1)$. Then for $C^{k,\alpha}$ generic metric g on M, every embedded locally stable minimal hypersurface is regular and nondegenerate.*

Remark 6.1.2. This can be viewed as a weak generalization of B. White's structure theorem.

Note that every minimal hypersurfaces obtained by Almgren-Pitts min-max theory (or Allen-Cahn min-max theory) is locally stable. Therefore, we confirm the full generic regularity conjecture II in dimension eight.

Corollary 6.1.3. *Let M be a closed smooth manifold of dimension 8, $k \geq 4$. Then for $C^{k,\alpha}$ generic metric g on M, every minimal hypersurface generated from Almgren-Pitts min-max theory (or Allen-Cahn min-max theory) is regular and nondegenerate.*

Remark 6.1.1. As an application, this enables us to extend most of the generic geometric properties mentioned in Section 1.2 to a closed eight-dimensional Riemannian manifold. Interested readers may refer to our original paper [46].

Historically, in dimension 8, Yau's generic smoothing conjecture of area minimizing hypersurface was affirmed by N. Smale [73], using the Hardt-Simon analysis of

area-minimizing hypercones in $\mathbb{R}^8$ [35] in 1993 (Mazzeo-Smale [57] also extended this to higher dimensions with technical assumptions on the singularities in 1994). Recently, Chodosh-Liokumovich-Spolaor [15] showed that in the case of positive Ricci curvature, the first-width min-max minimal hypersurfaces can be smoothened via a metric perturbation as well. These have implied that in a generic closed 8-dimensional Riemannain manifold with suitable topological assumption ($H_7(M^8) \neq 0$) or geometric assumption ($Ric(M^8) > 0$) as above, there exists at least one smooth minimal hypersurface. Inspired by their work, we [45] removed those assumptions and proved the generic existence of a smooth minimal hypersurface in M^8. The main theorem herein genuinely improves our previous work on the generic existence.

6.2 Preliminaries

Let $\Sigma \subset (M^{n+1}, g)$ be a C^2 hypersurface (need not be two-sided). Throughout this article, every such hypersurface Σ is assumed to have

$$\mathscr{H}^n(\Sigma) < +\infty\,, \quad \mathscr{H}^{n-2}(\overline{\Sigma} \setminus \Sigma) = 0\,.$$

Let

- $\mathrm{Reg}(\Sigma) := \{x \in \overline{\Sigma} : \overline{\Sigma}$ is an embedded C^2 hypersurface near $x\}$;
- $\mathrm{Sing}(\Sigma) := \overline{\Sigma} \setminus \mathrm{Reg}(\Sigma)$.

We may assume without loss of generality that $\Sigma = \mathrm{Reg}(\Sigma)$. Σ is said to be **regular** if $\mathrm{Sing}(\Sigma) = \emptyset$.

The Riemannian metric on Σ is induced from (M, g). Since we may deal with families of minimal hypersurfaces and metrics, we may use subscript $\nabla_{\Sigma,g}$, $\Delta_{\Sigma,g}$ etc. to emphasis that they are operators with respect to Σ, g, and may omit the subscript

if there's no confusion. At least locally, by choosing an orthonormal frame $\{e_i\}$ of Σ, we can let

- $(\vec{A}_{\Sigma,g})_{ij} := (e_i \cdot \nabla^g e_j)^\perp$ be the second fundamental form of Σ;
- $\vec{H}_{\Sigma,g} := (\sum_i e_i \cdot \nabla^g e_i)^\perp$ be the mean curvature vector of Σ.

Recall that Σ is minimal in (M, g) if and only if $\vec{H}_{\Sigma,g} = 0$.

Following [65], a minimal hypersurface $\Sigma \subset (M, g)$ is called **stable** in an open subset $U \subset M$, if for every C^2 family of diffeomorphism $\{\phi_t : U \to U\}_{t\in(-1,1)}$ such that $\phi_t = id_U$ near ∂U, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{H}^n(\phi_t(\Sigma)) \geq 0.$$

Since $n+1 = 8$, a stable minimal hypersurfaces in U has isolated singular set; And if Σ has a global unit normal field $\nu = \nu_{\Sigma,g}$ in U, then Σ being stable in U is equivalent to that

$$Q_{\Sigma,g}(\varphi, \varphi) := \int_\Sigma |\nabla_{\Sigma,g}\varphi|^2 - (|A_{\Sigma,g}|^2 + Ric_g(\nu, \nu))\varphi^2 \geq 0, \quad \forall \varphi \in C_c^2(\Sigma \cap U). \tag{6.2.1}$$

In this case, let

$$L_{\Sigma,g} := \Delta_{\Sigma,g} + |A_{\Sigma,g}|^2 + Ric_g(\nu, \nu), \tag{6.2.2}$$

be the Euler-Lagrange operator associated to $Q_{\Sigma,g}$, called **Jacobi operator**. Any solution $u \in C^2_{\text{loc}}(\Sigma)$ to $L_{\Sigma,g}u = 0$ is called a **Jacobi field**.

Definition 6.2.1. A C^2 hypersurface Σ is a **locally stable minimal hypersurface (LSMH)** if for every $p \in M$, there exists some neighborhood $U_p \ni p$ in which Σ is stable.

For an integer $k \geq 4$ and a real number $\alpha \in (0,1)$, let

- $\mathcal{G}^{k,\alpha}(M)$ be the space of $C^{k,\alpha}$-Riemannian metrics on M, which is naturally a Banach manifold;
- $\mathcal{M}^{k,\alpha}(M)$ be the space of pairs (g,Σ), where $g \in \mathcal{G}^{k,\alpha}(M)$ and Σ is a connected LSMH, two-sided or one-sided, in (M,g); This space is endowed with the topology induced by $C^{k,\alpha}$-convergence in the g factor and multiplicity 1 $\mathbf{F}$-convergence in the Σ factor;
- $\Pi : \mathcal{M}^{k,\alpha}(M) \to \mathcal{G}^{k,\alpha}(M)$ be the projection onto the first factor.

Following [52], the Morse index of a two-sided LSMH $\Sigma \subset (M^{n+1}, g)$ is defined by

$$\begin{aligned}\mathrm{Index}(\Sigma) := \sup\Big\{ \dim \mathcal{L} : \mathcal{L} \subset \mathscr{X}_c(M) \text{ is a linear subspace such that} \\ \frac{d^2}{dt^2}\Big|_{t=0} \mathscr{H}^n(e^{tX}(\Sigma)) < 0,\ \forall \mathbf{0} \neq X \in \mathcal{L}\Big\}.\end{aligned} \tag{6.2.3}$$

A two-sided LSMH in a closed manifold always has finite index, and coincides with the index of its Jacobi operator. By convention, we define the index of a one-sided LSMH to be the index of its two-sided double cover.

6.2.1 Stable Minimal Hypercone

Let $\mathbf{C} \subset \mathbb{R}^{n+1}$ be a stable minimal hypercone, with normal field ν. Since $n+1 = 8$, $\mathbf{C}$ has isolated singularity at $\mathbf{0}$, or equivalently, $S := \mathbf{C} \cap \mathbb{S}^n$ is a regular minimal hypersurface in $\mathbb{S}^n$. Such a cone with only isolated singularity at $\mathbf{0}$ is called a regular cone in higher dimensions. By Łojasiewicz inequality [69], the density of stable minimal hypercones in $\mathbb{R}^8$ is a discrete set,

$$\{\theta(\mathbf{0};\mathbf{C}) : \mathbf{C} \subset \mathbb{R}^8 \text{ stable minimal hypercone}\} = \{1 = \theta_0 < \theta_1 < \theta_2 < \cdots \nearrow +\infty\}. \tag{6.2.4}$$

We can parametrize $\mathbf{C}$ by

$$(0,+\infty)\times S\to \mathbf{C},\quad (r,\omega)\mapsto x=r\omega. \tag{6.2.5}$$

Recall by [Simons '68], the Jacobi operator of $\mathbf{C}$ is decomposed under this parametrization as

$$L_{\mathbf{C}}=\partial_r^2+\frac{n-1}{r}\partial_r+\frac{1}{r^2}(\Delta_S+|A_S|^2)\,,$$

where A_S is the second fundamental form of S in $\mathbb{S}^n$.

To describe the space of Jacobi field on $\mathbf{C}$, let

$$\mu_1<\mu_2\leq ...\nearrow+\infty$$

be the eigenvalues of $-(\Delta_S+|A_S|^2)$, and

$$\varphi_1,\varphi_2,...$$

be a corresponding $L^2(S)$-orthonormal eigenfunctions, with $\varphi_1>0$. By [72], the stability of $\mathbf{C}$ is equivalent to that $\mu_1\geq -(n-2)^2/4$. By [12], a general Jacobi field $v\in C^\infty(\mathbf{C})$ is given by sum of homogeneous Jacobi fields,

$$v(r,\omega)=\sum_{j\geq 1}(v_j^+(r)+v_j^-(r))\varphi_j(\omega), \tag{6.2.6}$$

where

$$\begin{aligned} v_j^+(r)&=c_j^+\cdot r^{\gamma_j^+};\\ v_j^-(r)&=\begin{cases} c_j^-\cdot r^{\gamma_j^-}\,, & \text{if } \mu_j>-\frac{(n-2)^2}{4};\\ c_j^-\cdot r^{\gamma_j^-}\log r, & \text{if } \mu_j=-\frac{(n-2)^2}{4}. \end{cases} \end{aligned} \tag{6.2.7}$$

for some $c_j^{\pm} \in \mathbb{R}$ and

$$\gamma_j^{\pm} = \gamma_j^{\pm}(\mathbf{C}) := -\frac{n-2}{2} \pm \sqrt{\mu_j + \frac{(n-2)^2}{4}}. \tag{6.2.8}$$

Note that $\mu_j = -(n-2)^2/4$ if and only if $j = 1$ and $\gamma_1^+ = \gamma_1^- = -(n-2)/2$. Clearly, if $\{\mathbf{C}_k\}_{1 \le k \le \infty}$ is a family of regular cone such that

$$\mathbf{F}(|\mathbf{C}_k|, |\mathbf{C}_\infty|) \to 0$$

then the asymptotic spectrum also converges: For every $j \ge 1$, as $k \to \infty$,

$$\mu_j(\mathbf{C}_k) \to \mu_j(\mathbf{C}_\infty)\,.$$

6.2.2 Singularities of LSMH

It is known from [65, 38] that near each singular point p, the LSMH $\Sigma \subset M^{n+1}$ is modeled on stable minimal hypercones. More precisely, for every $r_j \searrow 0$, the blow up sequence $(\Sigma - p)/r_j$ **F**-subconverges to some stable minimal hyercone $\mathbf{C} \subset \mathbb{R}^{n+1}$ with multiplicity 1, which is called a tangent cone of Σ at p. When $n+1 = 8$, since every stable minimal hypercone is regular, by [70], the tangent cone of Σ at p is independent of the choice of blow-up radii $\{r_j\}$, and can be denoted by $\mathbf{C}_p\Sigma$ here.

Let Σ be a LSMH in a closed manifold (M^8, g) with a singularity p. For a L^2_{loc}-function v defined near p, [78] introduced the notion of **asymptotic rate** of v at p, defined to be

$$\mathcal{AR}_p(v) := \sup\{\gamma : \lim_{s \searrow 0} \int_{A^g(p;s,2s)} v^2 \cdot \rho^{-n-2\gamma}\, d\|\Sigma\| = 0\},$$

where $\rho(x) := \mathrm{dist}_g(x, \mathrm{Sing}(\Sigma))$ is the distance function to the singular set, and $A^g(p; s, t)$ is the geodesic annulus centered at p with inner radius s and outer radius

t. Heuristically, if v grows like ρ^γ near p, then $\mathcal{AR}_p(v) = \gamma$.

Definition 6.2.2 (Slower growth)**.** A function $u \in L^2_{loc}(\Sigma)$ is a **function of slower growth**, if $\forall p \in \mathrm{Sing}(\Sigma)$,

$$\mathcal{AR}_p(u) \geq \gamma_2^+(\mathbf{C}_p\Sigma)\,.$$

In particular, if $\mathrm{Sing}(\Sigma) = \emptyset$, the condition is always satisfied for any L^2_{loc} function.

Similarly, when Σ is two-sided, a Jacobi field on Σ is a **Jacobi field of slower growth** if it is a function of slower growth. We denote by $\mathrm{Ker}^+ L_{\Sigma,g}$ the space of all the Jacobi fields of slower growth on $\Sigma \subset (M, g)$.

A two-sided LSMH Σ is called **semi-nondegenerate**, if

$$\mathrm{Ker}^+ L_{\Sigma,g} = \{0\}\,.$$

A one-sided connected LSMH Σ is called **semi-nondegenerate**, if its connected two-sided double cover $\hat{\Sigma}$ has

$$(\mathrm{Ker}^+ L_{\hat{\Sigma},\hat{g}})^{\mathbb{Z}_2} = \{0\}\,.$$

where $(\mathrm{Ker}^+ L_{\hat{\Sigma},\hat{g}})^{\mathbb{Z}_2}$ denotes the set of functions in $\mathrm{Ker}^+ L_{\hat{\Sigma},\hat{g}}$ can descend to Σ.

Remark 6.2.3. By definition, if $\mathrm{Sing}(\Sigma) = \emptyset$, then Σ is semi-nondegenerate if and only if Σ is nondegenerate.

The following Theorem generalizes [45] to semi-nondegenerate or one-sided LSMH, and plays an important role in the proof of Theorem 6.1.1.

Theorem 6.2.4. *Let $k \geq 4$ be an integer, $\alpha \in (0,1)$, and $\Sigma \subset (M^8, g)$ be a semi-nondegenerate LSMH with $\mathrm{Sing}(\Sigma) \neq \emptyset$. Then there exists an open dense subset $\mathscr{F}^{k,\alpha} \subset C^{k,\alpha}(M)$ depending on Σ, M, g with the following property. For any $f \in \mathscr{F}^{k,\alpha}$, and $j \geq 1$, Suppose that*

(a) $g_j = (1 + t_j f_j)g$ *is a family of metrics, where either* $\mathbb{R} \ni t_j \searrow 0$ *and* $f_j \to f$ *in* $C^4(M)$ *as* $j \to \infty$*, or* $t_j \equiv 0$ *for all* j*.*

(b) $\Sigma_j \subset (M, g_j)$ *is an LSMH with*

$$\mathrm{Index}(\Sigma_j) = \mathrm{Index}(\Sigma)$$

and

$$\mathbf{F}(|\Sigma_j|_{g_j}, |\Sigma|_g) \to 0$$

as $j \to \infty$*.*

Then there exists $p \in \mathrm{Sing}(\Sigma)$ *and some neighborhood* $U_p \subset M$ *both depending on the sequence* $\{(g_j, \Sigma_j)\}$ *such that for infinitely many* $j >> 1$*,*

$$\mathrm{Sing}(\Sigma_j) \cap U_p = \emptyset.$$

6.2.3 Singular Capacity

Let $\mathscr{M}$ be the space of triples (Σ, N, g), where (N, g) is a Riemannian manifold of dimension 8, not necessarily complete; $\Sigma \subset (N, g)$ be a connected LSMH with finitely many singularities. Let $\mathscr{M}^{\text{2-sided}}$ be the space of triples $(\Sigma, N, g) \in \mathscr{M}$ such that Σ is two-sided. As before, we endow topology on $\mathscr{M}$ and $\mathscr{M}^{\text{2-sided}}$ by C^4_{loc} convergence in (N, g) and multiplicity 1 **F**-convergence in Σ.

In the previous chapter, we have introduced the following functional **SCap** on $\mathscr{M}^{\text{2-sided}}$, which serves as a way to count the number of singularities with weight such that it is upper semi-continuous under convergence.

Theorem 6.2.5 ([45, Theorem 4.2])**.** *There exists a functional* $\mathbf{SCap} : \mathscr{M}^{\text{2-sided}} \to \mathbb{N}$*, called* ***singular capacity****, such that*

(i) For every nontrivial stable minimal hypercone $\mathbf{C} \subset \mathbb{R}^8$*, we have*

$$1 \leq \mathbf{SCap}(\mathbf{C}, \mathbb{R}^8, g_{\text{eucl}}) < +\infty.$$

Denote for simplicity that $\mathbf{SCap}(\mathbf{C}) := \mathbf{SCap}(\mathbf{C}; \mathbb{R}^8, g_{\text{eucl}})$*;*

(ii) For every triple $(\Sigma, M, g) \in \mathscr{M}^{2\text{-}sided}$*,*

$$\mathbf{SCap}(\Sigma, M, g) = \sum_{p \in \text{Sing}(\Sigma)} \mathbf{SCap}(\mathbf{C}_p(\Sigma)),$$

where $\mathbf{C}_p(\Sigma)$ *denotes the tangent cone of* Σ *at* p*. Conventionally,* $\mathbf{SCap}(\Sigma, M, g) = 0$ *if* $\text{Sing}(\Sigma) = \emptyset$*;*

(iii) If $(\Sigma_j, M, g_j) \to (\Sigma_\infty, M, g_\infty)$ *in* $\mathscr{M}^{2\text{-}sided}$*, and* $\forall 1 \leq j \leq +\infty$*,* Σ_j*'s are stable, then for every open subset* $U \subset M$ *with* $\partial U \cap \text{Sing}(\Sigma) = \emptyset$*, we have*

$$\limsup_{j \to \infty} \mathbf{SCap}(\Sigma_j, U, g_j) \leq \mathbf{SCap}(\Sigma_\infty, U, g_\infty).$$

Here, we shall need a modified singular capacity, which is still upper semi-continuous under convergence with multiplicities. Moreover, it can deal with one-sided minimal hypersurfaces as well.

Theorem 6.2.6. *There exists a functional* $\mathbf{SCap}^* : \mathscr{M} \to \mathbb{N}$ *with the following properties.*

(i) For every nontrivial stable minimal hypercone $\mathbf{C} \subset \mathbb{R}^8$*, we have*

$$1 \leq \mathbf{SCap}^*(\mathbf{C}, \mathbb{R}^8, g_{\text{eucl}}) < +\infty.$$

We denote for simplicity $\mathbf{SCap}^*(\mathbf{C}) := \mathbf{SCap}^*(\mathbf{C}; \mathbb{R}^8, g_{\text{eucl}})$*;*

(ii) For every triple $(\Sigma, N, g) \in \mathscr{M}$*,*

$$\mathbf{SCap}^*(\Sigma, N, g) := \begin{cases} \sum_{p \in \mathrm{Sing}(\Sigma)} \mathbf{SCap}^*(\mathbf{C}_p(\Sigma)) & \text{if } \Sigma \text{ is two-sided;} \\ 2 \sum_{p \in \mathrm{Sing}(\Sigma)} \mathbf{SCap}^*(\mathbf{C}_p(\Sigma)) & \text{if } \Sigma \text{ is one-sided.} \end{cases} \tag{6.2.9}$$

Conventionally, $\mathbf{SCap}^*(\Sigma, N, g) = 0$ *if* $\mathrm{Sing}(\Sigma) = \emptyset$*;*

(iii) Given an integer $I \geq 0$*, Suppose that* $(\Sigma_j, N, g_j) \to (\Sigma_\infty, N, g_\infty)$ *in* $\mathscr{M}$*, where* $\forall 1 \leq j \leq +\infty$*,* $\mathrm{Index}(\Sigma_j) \equiv I$*, and either all two-sided or all one-sided.*

Then for every open subset $U \subset N$ *with* $\partial U \cap \mathrm{Sing}(\Sigma) = \emptyset$*, we have*

$$\limsup_{j \to \infty} \mathbf{SCap}^*(\Sigma_j, U, g_j) \leq \mathbf{SCap}^*(\Sigma_\infty, U, g_\infty). \tag{6.2.10}$$

(iv) Suppose N *is a closed eight-manifold. Let* $\{g_j\}$ *be a family of* C^4 *metrics on* N*, and* $\Sigma_j \subset (N, g_j)$ *be connected stable minimal hypersurfaces,* $1 \leq j \leq \infty$*, such that when* $j \to \infty$*,* $g_j \to g_\infty$ *in* C^4*,* $|\Sigma_j|$ $\mathbf{F}$*-converges to* $m|\Sigma_\infty|$ *for some integer* $m \geq 2$*. If* $\mathrm{Sing}(\Sigma_\infty) \neq \emptyset$*, then we have,*

$$\limsup_{j \to \infty} \mathbf{SCap}^*(\Sigma_j, N, g_j) \leq \mathbf{SCap}^*(\Sigma_\infty, N, g_\infty) - 1. \tag{6.2.11}$$

6.3 Canonical Neighborhoods

Throughout this section, let M be a closed smooth manifold of dimension 8, $k \geq 4$ be an integer, and $\alpha \in (0, 1)$.

Definition 6.3.1. Let g be a $C^{k,\alpha}$-metric on M, $\Sigma \subset (M^8, g)$ be a LSMH, $\Lambda \geq 1$, and $\delta > 0$. We denote

$$\mathrm{injrad}(g, \Sigma) := \min\{\mathrm{injrad}(M, g), \min\{\mathrm{dist}(p, p')/2 : p \neq p' \in \mathrm{Sing}(\Sigma)\}\}.$$

We define $\mathcal{L}^{k,\alpha}(\Sigma, g; \Lambda, \delta)$ to be the space of all pairs $(g', \Sigma') \in \mathcal{M}^{k,\alpha}(M)$ such that,

- g' is a $C^{k,\alpha}$ metric on M with $\|g'\|_{C^{k,\alpha}} \leq \Lambda$ and $\|g' - g\|_{C^{k-1,\alpha}} \leq \delta$;
- Σ' is a LSMH in (M, g') satisfying

$$\mathbf{F}(|\Sigma|_g, |\Sigma'|_{g'}) \leq \delta, \quad \mathrm{Index}(\Sigma'; g') \leq \Lambda\,;$$

- For every $p \in \mathrm{Sing}(\Sigma)$, there exists $p' \in \mathrm{Sing}(\Sigma') \cap B^g_{\mathrm{injrad}(g,\Sigma)}(p)$ such that

$$\theta(p'; \|\Sigma'\|_g) = \theta(p; \|\Sigma\|)$$

$\mathcal{L}^{k,\alpha}(\Sigma, g; \Lambda, \delta)$ is said to be a **canonical (pseudo-)neighborhood** of (g, Σ) in $\mathcal{M}^{k,\alpha}(M)$, and its topology is induced from $\mathcal{M}^{k,\alpha}(M)$.

Clearly, when g, Σ is fixed, $\mathcal{L}^{k,\alpha}(\Sigma, g; \Lambda, \delta)$ is monotonely increasing in Λ, δ. Also note that in general, $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ is NOT a genuine *neighborhood* of (g, Σ) in $\mathcal{M}^{k,\alpha}(M)$. In other words, there might be $(g', \Sigma') \in \mathcal{M}^{k,\alpha}(M)$ arbitrarily close to (g, Σ) but not contained in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$.

Lemma 6.3.2 (Compactness of $\mathcal{L}^{k,\alpha}$)**.** *For every $(g, \Sigma) \in \mathcal{M}^{k,\alpha}(M)$ and every $\Lambda > 1$, there exists $\delta_0(g, \Sigma, \Lambda, k, \alpha) \in (0, 1)$ with the following property. For every $\delta \in (0, \delta_0)$, $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ is compact under $C^{k-1,\alpha}$-norm in metric and $\mathbf{F}$-metric in minimal hypersurfaces.*

The main reason to consider canonical neighborhoods is that the following Sard-Smale-type Theorem holds.

Theorem 6.3.3. *For every $(g, \Sigma) \in \mathcal{M}^{k,\alpha}(M)$ and every $\Lambda > 0$, there exists $\kappa_0 =$*

$\kappa_0(g, \Sigma, \Lambda) > 0$ *such that*

$$\mathcal{G}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0) := \{g' \in \mathcal{G}^{k,\alpha}(M) : \text{ either } g' \notin \Pi(\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)), \text{ or every } \Sigma' \text{ with } (g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0) \text{ is semi-nondegenerate}\}.$$

is an open dense subset of $\mathcal{G}^{k,\alpha}(M)$.

Heuristically, one can view $\mathcal{G}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$ as the set of regular values of $\Pi|_{\mathcal{L}^{k,\alpha}(g,\Sigma;\Lambda,\kappa_0)}$. When we only focus on regular minimal hypersurfaces, such a Sard-Smale Theorem was proved in [79, 81] by showing that $\mathcal{M}^{k,\alpha}$ is a Banach manifold, and Π is a Fredholm map with Fredholm index 0. Here, however, it's hard to expect any Banach manifold structure on $\mathcal{L}^k(g, \Sigma; \Lambda, \delta)$. The way we prove Theorem 6.3.3 is to unwrap the proof of infinite dimensional Sard-Smale Theorem by hand. More precisely, we need

(i) A controlled behavior of Jacobi fields of slower growth under convergence in $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$;

(ii) A good way to slice $\mathcal{G}^{k,\alpha}(M)$ into union of finite dimensional subspaces $\{\mathcal{F}_{g',\Sigma'}\}$ such that each $\Pi^{-1}(\mathcal{F}_{g',\Sigma'}) \cap \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$ is bi-Lipschitz to a compact subset in the finite dimensional vector space $\mathrm{Ker}^+ L_{\Sigma',g'}$.

Both of these rely on a good control on growth rate of certain functions (Jacobi fields or graphical functions of nearby LSMH) on $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$, where we make full use of the technical definition of $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ and Jacobi fields of slower growth.

To derive a Sard-Smale type theorem on $\mathcal{M}^{k,\alpha}(M)$, it suffices to show that it can be covered by countably many canonical neighborhoods.

Theorem 6.3.4. *Let* $\kappa : \mathcal{M}^{k,\alpha}(M) \times \mathbb{R}_+ \to (0, +\infty)$ *be a positive function, not necessarily continuous. Then there exists a countable number of triples* $\{(g_j, \Sigma_j; \Lambda_j)\} \in$

$\mathcal{M}^{k,\alpha}(M) \times \mathbb{R}_+$ *such that*

$$\mathcal{M}^{k,\alpha}(M) = \bigcup_{j\geq 1} \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

where $\kappa_j := \kappa(g_j, \Sigma_j; \Lambda_j)$.

Theorem 6.3.4 is proved by parameterizing all minimal hypersurfaces inspired by the work [25].

A direct corollary of Theorem 6.3.3 and 6.3.4 is the genericity of semi-nondegeneracy.

Corollary 6.3.5. *Given an integer* $k \geq 4$, $\alpha \in (0,1)$ *and a eight-dimensional closed smooth manifold* M^8, *let* $\mathcal{G}_0^{k,\alpha}(M)$ *be the space of metrics* $g \in \mathcal{G}^{k,\alpha}(M)$ *such that*

- *every regular connected minimal hypersurface* Σ *under* g *is* ***strongly nondegenerate****, i.e. either* Σ *is two-sided and nondegenerate, or* Σ *is one-sided and has a non-degenerate double cover;*
- *every connected LSMH under* g *is semi-nondegenerate.*

Then $\mathcal{G}_0^{k,\alpha}(M)$ *is a generic subset of* $\mathcal{G}^{k,\alpha}(M)$ *in the Baire Category sense, i.e.* $\mathcal{G}_0^{k,\alpha}(M)$ *contains a countable intersection of open dense subsets in* $\mathcal{G}^{k,\alpha}(M)$.

Proof. In Theorem 6.3.4, by choosing κ to be κ_0 in Theorem 6.3.3, we have

$$\mathcal{M}^{k,\alpha}(M) = \bigcup_{j\geq 1} \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

where $\kappa_j = \kappa_0(g_j, \Sigma_j; \Lambda_j)$. For each $\mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$, Theorem 6.3.3 generates an open dense subset $\mathcal{G}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$, and we can define

$$\tilde{\mathcal{G}}^{k,\alpha}(M) := \bigcap_{j=1}^{\infty} \mathcal{G}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

which by definition is clearly generic.

By White's structure theorem [79, 81], there exists a generic subset $\mathcal{G}^{k,\alpha}_{\text{bumpy}}(M)$ of $\mathcal{G}^{k,\alpha}(M)$, such that every regular connected minimal hypersurface Σ under g is strongly nondegenerate.

It suffices to show that $\tilde{\mathcal{G}}^{k,\alpha}(M) \cap \mathcal{G}^{k,\alpha}_{\text{bumpy}}(M) \subset \mathcal{G}^{k,\alpha}_0(M)$. Indeed, for any $g \in \tilde{\mathcal{G}}^{k,\alpha}(M) \cap \mathcal{G}^{k,\alpha}_{\text{bumpy}}(M) \subset \mathcal{G}^{k,\alpha}_0(M)$, and any LSMH $\Sigma \subset (M, g)$, there exists a $j \in \mathbb{N}^+$ such that $(g, \Sigma) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$. Hence, by the fact that $g \in \tilde{\mathcal{G}}^{k,\alpha}(M) \subset \mathcal{G}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$, we have Σ is semi-nondegenerate. If further Σ is regular, then by the definition of $\mathcal{G}^{k,\alpha}_{\text{bumpy}}(M)$, it is also strongly non-degenerate. In summary, the inclusion holds and $\mathcal{G}^{k,\alpha}_0(M)$ is also generic. □

6.4 Proof of Main Theorem

The goal of this section is to prove Theorem 6.1.1. Recall that $\mathcal{G}^{k,\alpha}_0(M)$ defined in Corollary 6.3.5 is shown to be a generic subset of $\mathcal{G}^{k,\alpha}(M)$.

Now for $\Lambda > 0$ and integer $I \geq 1$, we can define

$$\mathcal{G}^{k,\alpha}_{\Lambda,I}(M) := \big\{ g \in \mathcal{G}^{k,\alpha}_0(M) : \text{every } \Sigma \text{ with } (g, \Sigma) \in \mathcal{M}^{k,\alpha}(M), \text{Index}(\Sigma) \leq I - 1 \\ \text{and } \|\Sigma\|_g(M) \leq \Lambda \text{ is regular} \big\}. \tag{6.4.1}$$

For simplicity, we may omit the superscript k, α. For any $\Lambda > 0$, we also write

$$\mathcal{G}_{\Lambda,0}(M) := \mathcal{G}^{k,\alpha}_0(M). \tag{6.4.2}$$

Lemma 6.4.1. *For every $\Lambda > 0$ and integer $I \geq 0$, $\mathcal{G}_{\Lambda,I}(M)$ is relatively open and dense in $\mathcal{G}^{k,\alpha}_0(M)$.*

Given $g \in \mathcal{G}_0^{k,\alpha}(M)$, $\Lambda > 0$ and integer $I \geq 0$, let

$$\mathcal{M}(g;\Lambda,I) := \{(g,\Sigma) \in \mathcal{M}^{k,\alpha}(M) : \|\Sigma\|_g(M) \leq \Lambda, \mathrm{Index}(\Sigma) = I\};$$

$$\mathbf{SCap}^*(g;\Lambda,I) := \begin{cases} \sup\{\mathbf{SCap}^*(\Sigma,M,g) : (g,\Sigma) \in \mathcal{M}(g;\Lambda,I)\}, & \text{if } \mathcal{M}(g;\Lambda,I) \neq \emptyset; \\ 0, & \text{if } \mathcal{M}(g;\Lambda,I) = \emptyset; \end{cases}$$

$$\mathcal{M}^{max}(g;\Lambda,I) := \{(g,\Sigma) \in \mathcal{M}(g;\Lambda,I) : \mathbf{SCap}^*(\Sigma,M,g) = \mathbf{SCap}^*(g;\Lambda,I)\}.$$

Note that by definition, for every $g \in \mathcal{G}_{\Lambda,I}(M)$,

$$\mathbf{SCap}^*(g;\Lambda,I) = 0 \text{ if and only if } g \in \mathcal{G}_{\Lambda,I+1}(M). \tag{6.4.3}$$

Lemma 6.4.2. *Let $\Lambda > 0$, integer $I \geq 0$ and $g \in \mathcal{G}_{\Lambda,I}(M)$. If $\{(g,\Sigma_j)\}_{j\geq 1}$ is a sequence in $\mathcal{M}(g;\Lambda,I)$, then $|\Sigma_j|$ subconverges to $m|\Sigma_\infty|$, where*

(i) $(g,\Sigma_\infty) \in \mathcal{M}(g;\Lambda,I)$;

(ii) Either $m = 1$, or $m \geq 2$ and $I = 0$.

In particular, $\mathbf{SCap}^*(g;\Lambda,I) < +\infty$.

Proof. By [65] and connectedness of Σ_j, $|\Sigma_j|$ **F**-subconverges to $m|\Sigma_\infty|$ for some connected LSMH $\Sigma_\infty \subset (M,g)$ and integer $m \geq 1$ with

$$m\|\Sigma_\infty\|_g(M) \leq \Lambda, \quad m\,\mathrm{Index}(\Sigma_\infty) \leq I\,.$$

To see (i), suppose for contradiction $\mathrm{Index}(\Sigma_\infty) \leq I - 1$, and hence $I \geq 1$. Since $g \in \mathcal{G}_{\Lambda,I}(M)$, we know from definition that Σ_∞ is regular and strongly nondegenerate. Hence by [68], $m = 1$ and $\Sigma_j \equiv \Sigma_\infty$ for $j >> 1$. This contradicts to the fact that $\mathrm{Index}(\Sigma_j) = I$. Therefore,

$$\mathrm{Index}(\Sigma_\infty) = I\,.$$

(ii) also follows from [68].

By Theorem 6.2.6, for any $\{(g, \Sigma_j)\}_{j\geq 1} \subset \mathcal{M}(g; \Lambda, I)$, up to a subsequence,

$$\limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g) \leq \mathbf{SCap}^*(\Sigma_\infty, M, g) < +\infty\,.$$

Hence, we have $\mathbf{SCap}^*(g; \Lambda, I) < +\infty$. □

Lemma 6.4.3. *For every $\Lambda > 0$, integer $I \geq 0$ and every $g \in \mathcal{G}_{\Lambda,I}(M)$, $\mathcal{M}^{max}(g; \Lambda, I)$ is a finite set.*

Proof. Suppose otherwise, there exists a pairwise distinct sequence

$$\{(g, \Sigma_j)\}_{j\geq 1} \subset \mathcal{M}^{max}(g; \Lambda, I)\,.$$

By [65] and Lemma 6.4.2, $|\Sigma_j|$ $\mathbf{F}$-subconverges to $m|\Sigma_\infty|$ for some $(g, \Sigma_\infty) \in \mathcal{M}(g; \Lambda, I)$. And by the definition of $\mathcal{M}^{max}$, we have

$$\mathbf{SCap}^*(\Sigma_\infty, M, g) \leq \mathbf{SCap}^*(g; \Lambda, I) = \mathbf{SCap}^*(\Sigma_j, M, g). \tag{6.4.4}$$

Then by Theorem 6.2.6, either $m = 1$ or $\mathrm{Sing}(\Sigma_\infty) = \emptyset$. Since $g \in \mathcal{G}_{\Lambda,I}(M) \subset \mathcal{G}_0^{k,\alpha}(M)$, the latter case can't happen by the strong nondegeneracy of Σ_∞. In other words, we have $m = 1$.

Applying Theorem 6.2.4 to $\Sigma_j \to \Sigma_\infty$, we know that $\exists\ p \in \mathrm{Sing}(\Sigma_\infty)$ near which $\mathrm{Sing}(\Sigma_j)$ has no singularity for infinitely many $j >> 1$. Then by Theorem 6.2.6,

$$\liminf_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g) \leq \mathbf{SCap}^*(\Sigma_\infty, M, g) - 1,$$

which contradicts to (6.4.4). □

Lemma 6.4.4. *For every $\Lambda > 0$, integer $I \geq 0$ and every $g \in \mathcal{G}_{\Lambda,I}(M) \cap \mathrm{Int}(\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)))$ such that $\mathbf{SCap}^*(g; \Lambda, I) \geq 1$, there exists a sequence of metrics $g_j \in \mathcal{G}_{\Lambda,I}(M) \cap$*

$\mathrm{Int}(\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)))$, $j \geq 1$, *such that* $g_j \to g$ *in* $C^{k,\alpha}$ *and*

$$\mathbf{SCap}^*(g_j;\Lambda,I) \leq \mathbf{SCap}^*(g;\Lambda,I) - 1. \tag{6.4.5}$$

Proof. By Lemma 6.4.3, we can write

$$\mathcal{M}^{max}(g;\Lambda,I) =: \{(g,\Sigma^{(1)}),(g,\Sigma^{(2)}),\ldots,(g,\Sigma^{(q)})\}\,.$$

For each $1 \leq l \leq q$, let $\mathscr{F}^{(l)} \subset C^{k,\alpha}(M)$ be the open dense subset associated to $(g,\Sigma^{(l)})$ given by Theorem 6.2.4. Thus $\mathscr{F} := \bigcap_{l=1}^{q} \mathscr{F}^{(l)} \neq \emptyset$.

Take $f \in \mathscr{F}$, and there exists a sequence of metrics

$$g_j = (1 + f_j/j)g \in \mathcal{G}_{\Lambda,I} \cap \mathrm{Int}(\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)))$$

such that $f_j \to f$ in $C^{k,\alpha}$ when $j \to \infty$. We now claim that by passing to a subsequence, g_j satisfies (6.4.5).

To see this, suppose for contradiction that when $j >> 1$, there exists $(g_j,\Sigma_j) \in \mathcal{M}(g_j;\Lambda,I)$ such that

$$\mathbf{SCap}^*(\Sigma_j,M,g_j) \geq \mathbf{SCap}^*(g;\Lambda,I) \geq 1\,. \tag{6.4.6}$$

By [65] and the connectedness of Σ_j, $|\Sigma_j|$ **F**-subconverges to $m|\Sigma_\infty|$ for some connected LSMH $\Sigma_\infty \subset (M,g)$ and integer $m \geq 1$ with $m\|\Sigma_\infty\|_g(M) \leq \Lambda$ and $m\,\mathrm{Index}(\Sigma_\infty) \leq I$.

We shall deal with the following cases separately.

Case I $\mathrm{Index}(\Sigma_\infty) = I$ and $m = 1$. Then $(g, \Sigma_\infty) \in \mathcal{M}(g; \Lambda, I)$, and by Theorem 6.2.6 (iii) and (6.4.6),

$$\mathbf{SCap}^*(\Sigma_\infty, M, g) \geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g_j) \geq \mathbf{SCap}^*(g; \Lambda, I).$$

This means $(g, \Sigma_\infty) \in \mathcal{M}^{max}(g; \Lambda, I)$. WLOG $\Sigma_\infty = \Sigma^{(1)}$. But since $f \in \mathscr{F} \subset \mathscr{F}^{(1)}$, by the choice of g_j and Theorem 6.2.4 we have,

$$\begin{aligned}\mathbf{SCap}^*(\Sigma_\infty, M, g) - 1 &\geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g_j) \\ &= \mathbf{SCap}^*(g; \Lambda, I) = \mathbf{SCap}^*(\Sigma_\infty, M, g)\,,\end{aligned}$$

which is a contradiction.

Case II $\mathrm{Index}(\Sigma_\infty) = I$ and $m \geq 2$. Then $I = 0$ and $(g, \Sigma_\infty) \in \mathcal{M}(g; \Lambda, I)$.

- *Case II (1)*. $\mathrm{Sing}(\Sigma_\infty) = \emptyset$. Since Σ_j's are stable minimal hypersurfaces, by [65], for large enough j, Σ_j is a multi-graph over Σ_∞, and thus,

  $$\mathrm{Sing}(\Sigma_j) = \emptyset\,,$$

  contradicting (6.4.6).

- *Case II (2)*. $\mathrm{Sing}(\Sigma_\infty) \neq \emptyset$. By Theorem 6.2.6 (iv) and (6.4.6) we have,

  $$\begin{aligned}\mathbf{SCap}^*(\Sigma_\infty, M, g) &\geq \limsup_{j\to\infty} \mathbf{SCap}^*(\Sigma_j, M, g_j) + 1 \\ &\geq \mathbf{SCap}^*(g; \Lambda, I) + 1 \geq \mathbf{SCap}^*(\Sigma_\infty, M, g) + 1\,,\end{aligned}$$

  which is a contradiction.

Case III Index$(\Sigma_\infty) \leq I-1$ and $m=1$. Then $I \geq 1$ and by the definition of $g \in \mathcal{G}_{\Lambda,I}(M)$, Σ_∞ is both regular and strongly nondegenerate. By Allard's Regularity, Sing$(\Sigma_j) = \emptyset$ for $j >> 1$, contradicting (6.4.6).

Case IV Index$(\Sigma_\infty, g) \leq I-1$ and $m \geq 2$. Then again $I \geq 1$ and Σ_∞ is regular and strongly nondegenerate. By Sharp's compactness theorem [68], either Σ_∞ is two-sided and $\{\Sigma_j\}$ induces a bounded positive Jacobi field on Σ_∞, or Σ_∞ is one-sided and after passing to double covers, $\{\Sigma_j\}$ induces a bounded positive Jacobi field on the two-sided double cover $\hat{\Sigma}_\infty$ of Σ_∞. Both violate the strong nondegeneracy of Σ_∞. □

Proof of Lemma 6.4.1. To prove the relative openness, let's consider an arbitrary family of metrics $\{g_j\}_j \subset \mathcal{G}_0^{k,\alpha}(M) \setminus \mathcal{G}_{\Lambda,I}(M)$ such that

$$g_j \to g_\infty \in \mathcal{G}_0^{k,\alpha}(M) \text{ in } C^{k,\alpha}\,.$$

The choice of g_j guarantees that there exists an LSMH $\Sigma_j \subset (M, g_j)$ with Index$(\Sigma_j) \leq I-1$, vol$(\Sigma_j) \leq \Lambda$ and Sing$(\Sigma_j) \neq \emptyset$. By Sharp's compactness again, $|\Sigma_j|$ **F**-subconverges to some LSMH with multiplicity $m|\Sigma_\infty|$ with area $\leq \Lambda$ and index $\leq I-1$. We now assert that $g_\infty \notin \mathcal{G}_{\Lambda,I}(M)$, to conclude the proof of relative openness. If $g_\infty \in \mathcal{G}_{\Lambda,I}(M)$, by definition, Σ_∞ is both regular and strongly nondegenerate. Hence $m=1$ and then by Allard Regularity, for sufficiently large j, Sing$(\Sigma_j) = \emptyset$, which contradicts the choice of Σ_j.

To prove denseness, we shall do the induction in $I \geq 0$.

- Case $I=0$ follows directly from (6.4.2).
- Now suppose $I \geq 0$ such that $\mathcal{G}_{\Lambda,I}(M)$ is dense in $\mathcal{G}_0^{k,\alpha}(M)$, and then by Corollary 6.3.5, it's also dense in $\mathcal{G}^{k,\alpha}(M)$. To prove the denseness of $\mathcal{G}_{\Lambda,I+1}(M)$, it

suffices to show that every metric $g \in \mathcal{G}_{\Lambda,I}(M)$ can be approximated by metrics in $\mathcal{G}_{\Lambda,I+1}(M)$.

Note that by inductive assumption, $\mathrm{Clos}(\mathcal{G}_{\Lambda,I}(M)) = \mathcal{G}^{k,\alpha}(M)$. By Lemma 6.4.2, $\mathbf{SCap}^*(g;\Lambda,I) < \infty$. For any $\epsilon > 0$, we can apply Lemma 6.4.4 finitely many times to find a metric $\bar{g}_\epsilon \in \mathcal{G}_{\Lambda,I}(M)$ such that $\|g - g_j\|_{C^{k,\alpha}} < \epsilon$ and $\mathbf{SCap}(\bar{g}_j;\Lambda,I) = 0$. By (6.4.3), $\bar{g}_\epsilon \in \mathcal{G}_{\Lambda,I+1}(M)$. In conclusion, the arbitrariness of ϵ implies that $\mathcal{G}_{\Lambda,I+1}(M)$ is dense in $\mathcal{G}_{\Lambda,I}(M)$ and thus dense in $\mathcal{G}_0^{k,\alpha}(M)$.

□

Proof of Theorem 6.1.1. By Corollary 6.3.5 and Lemma 6.4.1, we know that

$$\mathcal{G}_\infty^{k,\alpha}(M) := \bigcap_{\text{integer } \Lambda>0, I\geq 0} \mathcal{G}_{\Lambda,I}(M),$$

is a generic subset of $\mathcal{G}^{k,\alpha}(M)$ in the Baire category sense. By (6.4.1), (6.4.2) and the definition of $\mathcal{G}_0^{k,\alpha}(M)$ in Corollary 6.3.5, for every $g \in \mathcal{G}_\infty^{k,\alpha}(M)$, every LSMH $\Sigma \subset (M,g)$ is regular and strongly non-degenerate. □

Appendix A

Generic geometric complication of smooth minimal hypersurfaces

A.1 introduction

In this part, we shall focus on the generic case, more precisely, the bumpy metric case. Recall that a closed Riemannian manifold (M^{n+1}, g) is called *bumpy*, if any immersed closed minimal hypersurface is non-degenerate, i.e., has no non-trivial Jacobi field. B. White [81] (See also Theorem 9 in [5]) has shown that the space of bumpy metrics on M is a residual subset of the space of C^∞ Riemannian metrics on M.

In a bumpy Riemannian manifold, due to non-degeneracy, minimal hypersurfaces from min-max theory are expected to have better properties. More precisely, in Allen-Cahn min-max setting (proposed by M. Guaraco [32], developed in Gaspar-Guaraco [30]), O. Chodosh and C. Mantoulidis [17] proved multiplicity one conjecture ($n+1 = 3$), and more recently, in Almgren-Pitts setting, X. Zhou [86], using the prescribed mean curvature min-max theory developed in Zhou-Zhu [88], confirmed Marques-Neves multiplicity one conjecture ($3 \leq n+1 \leq 7$). These results combined with the work of Marques-Neves [55] on the Morse index of p-width minimal hypersurfaces

lead to:

Theorem A.1.1 ([55], Theorem 8.4)**.** *On a bumpy closed Riemannian manifold* (M^{n+1}, g) $(3 \leq n+1 \leq 7)$*, for each* $p \in \mathbb{N}^+$*, there exists a smooth, closed, embedded, multiplicity one, two-sided, minimal hypersurface* Σ_p *such that*

$$\omega_p(M, g) = \text{Area}_g(\Sigma_p)\,, \quad \text{index}(\Sigma_p) = p\,, \tag{A.1.1}$$

and

$$\lim_{p\to\infty} \frac{\text{Area}_g(\Sigma_p)}{p^{\frac{1}{n+1}}} = c(n)\text{vol}(M, g)^{\frac{n}{n+1}}\,, \tag{A.1.2}$$

where $\omega_p(M, g)$ *is the volume spectrum, more precisely, the min-max p-width, and* $c(n) > 0$ *is a dimensional constant from the Weyl law in [48].*

However, Σ_p here might have multiple disjoint components, so it is not obvious whether or not one could find a connected minimal hypersurface with either large Morse index or large area from the theorem. Note that in a manifold with Frankel property, for example, with positive Ricci curvature, Σ_p will have only one connected component, so the answer will be trivially yes.

On the Morse index side, by monotonicity formula, we know that each component has a uniform area lower bound and therefore, the pigeon hole principle will directly imply that for each Σ_p , there exists at least one component with Morse index at least proportional to $p^{n/(n+1)}$. Nevertheless, such a component might not have large area.

On the area side, O. Chodosh and C. Mantoulidis [16] showed that there exists a sequence of connected minimal hypersurfaces whose areas tend to infinity.

Theorem A.1.2 (Theorem 1.4, [16])**.** *On a bumpy closed Riemannian manifold* (M^{n+1}, g) $(3 \leq n+1 \leq 7)$ *, either:*

- *there exists a sequence of connected closed embedded stable minimal hypersurfaces* Σ_j *with* $\text{Area}(\Sigma_j) \to \infty$ *, or,*

- *the hypersurfaces Σ_p from Theorem A.1.1 have at least one connected component Σ'_p with* $\mathrm{Area}(\Sigma'_p) \geq Cp^{1/(n+1)}$ *for some $C = C(M, g) > 0$ independent of p.*

Nonetheless, even in the latter case, they couldn't get a lower bound on the Morse index on these components. Hence, it is natural to raise the following question:

Q: *Does any bumpy closed Riemannian manifold admit a connected minimal hypersurface with both arbitrarily large area and Morse index?*

Here, we shall give a positive answer to this question.

Theorem A.1.3 (Main Theorem)**.** *Given any bumpy closed Riemannian manifold (M^{n+1}, g) $(3 \leq n+1 \leq 7)$, there exists a sequence of connected closed embedded two-sided minimal hypersurfaces $\{\Gamma_i\}$ such that*

$$\lim_{i\to\infty} \mathrm{Area}(\Gamma_i) = +\infty\,, \quad \lim_{i\to\infty} \mathrm{index}(\Gamma_i) = +\infty\,. \tag{A.1.3}$$

Remark A.1.4. When $n+1 = 3$, many examples of minimal surfaces with bounded topology but without Morse index bounds or area bounds have been constructed in pursuit of answering a question raised by J. Pitts and J. Rubinstein [60], i.e., whether embedded minimal surfaces of fixed genus in a three-manifold have bounded Morse index. For any closed 3-manifold, T. Colding and W. Minicozzi [19] constructed a (bumpy) metric admitting stable minimal tori without area bounds. Later, Colding-Hingston [18] and Hass-Norbury-Rubinstein [36] constructed metrics admitting minimal tori and minimal spheres without Morse index bounds, respectively.

Moreover, the same proof also gives a better description on the growth rate of areas and Morse indices in the latter case of Theorem A.1.2, i.e.,

Theorem A.1.5. *In a bumpy closed Riemannian manifold (M^{n+1}, g), if there are at most finitely many stable minimal hypersurfaces, then there exists a sequence of*

connected minimal hypersurfaces Γ_p *such that*

$$\text{index}(\Gamma_p) = p\,, \tag{A.1.4}$$

and their areas satisfy either

$$\lim_{p\to\infty} \frac{\text{Area}(\Gamma_p)}{p^{1/(n+1)}} = c(n)\,, \tag{A.1.5}$$

or

$$\lim_{p\to\infty} \frac{\text{Area}(\Gamma_p)}{p} = c_0(M,g) > 0\,. \tag{A.1.6}$$

The idea is based on an adaptation of X. Zhou's multiplicity one theorem [86] and Marques-Neves Morse index theorem [55] to a "core" manifold introduced by A. Song [76] in the bumpy metric case, which will be described in Section 2. In Section 3, we will divide the proof into three cases based on the number of closed two-sided stable minimal hypersurfaces (none, finitely many and infinitely many), and then apply the adaptive theorems case by case, combined with several counting arguments, to conclude our results.

Notation: In the following, we shall use both Σ and Γ to denote minimal hypersurfaces, where Σ is usually a boundary component of a "core" manifold while Γ lies inside the interior.

A.2 Multiplicity and Morse Index for Confined Min-max Closed Minimal Hypersurfaces

In this section, we will only consider a Riemannian manifold with boundary $(N, \partial N)$, which is a submanifold of the bumpy manifold (M, g), where its boundary $\partial N =$

$\bigcup_{i=1}^{m} \Sigma_i$ is a finite union of connected closed embedded two-sided stable minimal hypersurfaces in M.

A.2.1 A. Song's Construction of a Non-compact Manifold with Cylindrical Ends

In this subsection, for completeness, we give a brief description of cylindrical ends attached to the minimal hypersurface boundary and their properties (Subection 2.2 in [76]).

Since the metric of M is bumpy, we know that Σ_i's are non-degenerately stable and have a contracting neighborhood in M, i.e., there exists a diffeomorphism

$$\Phi : \partial N \times [0, \hat{t}] \to N \,, \tag{A.2.1}$$

where $\Phi(\partial N \times \{0\}) = \partial N$ is the identity map and for all $t \in [0, \hat{t}]$, $\Phi(\partial N \times \{t\})$ has non-zero mean curvature vector pointing towards ∂N, i.e., has positive mean curvature w.r.t. the direction pointing outwards ∂N.

As in A. Song's paper [76], we define the corresponding non-compact manifold with cylindrical ends:

$$\mathcal{C}(N) := N \cup_{\varphi} (\partial N \times [0, \infty)) \,, \tag{A.2.2}$$

where $\varphi : \partial N \times \{0\} \to \partial N$ is the canonical identification. The Lipschitz continuous metric h on $\mathcal{C}(N)$ will be given by

$$h|_N = g|_N \,, \quad h|_{\partial N \times [0,\infty)} = g|_{\partial N} \oplus \mathrm{d}s^2 \,. \tag{A.2.3}$$

Now, for $\varepsilon \in (0, \hat{t})$, let $N_\varepsilon = N \backslash \Phi(\partial N \times [0, \varepsilon))$ whose boundary is $\partial N_\varepsilon = \Phi(\partial N \times \{\varepsilon\})$. Due to compactness, there exists a positive number $\hat{d} > 0$ independent of ε such that we can define the Fermi coordinates on a $\hat{d}$-neighborhood of ∂N_ε as

follows:

$$\begin{aligned} \gamma_\varepsilon : \partial N_\varepsilon \times [-\hat{d}, \hat{d}] &\to M \,, \\ \gamma_\varepsilon(x,t) &= \exp(x, t\nu) \,, \end{aligned} \tag{A.2.4}$$

where exp is the exponential map and ν is the inward unit normal of ∂N_ε. In this Fermi coordinate, g can be written as $g_t^\varepsilon \oplus \mathrm{d}t^2$.

One could choose $\delta_\varepsilon > 0$ small enough so that

- $\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$;
- $\gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon]) \subset \Phi(\partial N \times [\varepsilon, \hat{t}])$;
- for all $t \in [0, \delta_\varepsilon]$, $\gamma_\varepsilon(\partial N_\varepsilon \times \{t\})$ has positive mean curvature as well.

One could also find a smooth function $\vartheta_\varepsilon : [0, \delta_\varepsilon] \to \mathbb{R}$ and a constant $z_\varepsilon \in (0, \delta_\varepsilon)$ satisfying:

- $\vartheta_\varepsilon \geq 1$ and $\frac{\partial}{\partial t}\vartheta_\varepsilon \leq 0$;
- $\vartheta_\varepsilon \equiv 1$ in a neighborhood of δ_ε;
- ϑ_ε is a constant on $[0, z_\varepsilon]$;
- $\lim_{\varepsilon\to 0} \int_{[0,\delta_\varepsilon]} \vartheta_\varepsilon = \infty$;
- $\lim_{\varepsilon\to 0} \int_{[z_\varepsilon,\delta_\varepsilon]} \vartheta_\varepsilon = 0$.

Abusing the notation, on N_ε , let $\vartheta_\varepsilon(\gamma_\varepsilon(x,t)) := \vartheta_\varepsilon(t)$ for all $(x,t) \in \partial N_\varepsilon \times [0, \delta_\varepsilon]$.

Now, one could define a new metric h_ε on N_ε :

$$h_\varepsilon(q) = \begin{cases} g_t(q) \oplus (\vartheta_\varepsilon(q)\mathrm{d}t)^2 & q \in \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon]) \\ g(q) & q \in N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon]) \end{cases} . \tag{A.2.5}$$

In the following, we always assume that N_ε is endowed with the metric h_ε .

Here are some properties on the convergence from $(N_\varepsilon, h_\varepsilon)$ to $(\mathcal{C}(N), h)$.

Lemma A.2.1 (Lemma 4, [76]). *For $t \in [0, \delta_\varepsilon]$, the slices $\gamma_\varepsilon(\partial N_\varepsilon \times \{t\})$ satisfy*

1. *they have non-zero mean curvature vector pointing in the direction of $-\gamma_{\varepsilon *}\left(\frac{\partial}{\partial t}\right)$;*
2. *their mean curvature goes uniformly to 0 as ε converges to 0;*
3. *their second fundamental form is bounded by a constant C independent of ε.*

Lemma A.2.2 (Lemma 5, [76]). *Let $q \in N \backslash \partial N$ such that for small ε, $q \in N_\varepsilon$. Then $(N_\varepsilon, h_\varepsilon, q)$ converges geometrically to $(\mathcal{C}(N), h, q)$ in the C^0 topology. Moreover, the geometric convergence is smooth outside of $\partial N \subset \mathcal{C}(N)$ in the following sense.*

1. *Let $q \in N \backslash \partial N$ such that for small ε, $q \in N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon))$. Then $(N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon)), h_\varepsilon, q)$ converges geometrically to $(N \backslash \partial N, g, q)$ in the C^∞ topology.*
2. *Fix any connected component C of ∂N; for ε small enough, we can choose a component C_ε of ∂N_ε so that C_ε converges to C as $\varepsilon \to 0$. Furthermore, if we let $\{\varepsilon_k\}$ be a sequence converging to 0, and let $q_k \in \gamma_{\varepsilon_k}(C_{\varepsilon_k} \times [0, \delta_{\varepsilon_k}))$ be a point at any fixed distance $\hat{d} > 0$ from $\gamma_{\varepsilon_k}(C_{\varepsilon_k} \times \{\delta_{\varepsilon_k}\})$ for the metric h_{ε_k}, $\hat{d}$ being independent of k. Then subsequentially,*

$$(\gamma_{\varepsilon_k}(C_{\varepsilon_k} \times [0, \delta_{\varepsilon_k})), h_{\varepsilon_k}, q_k) \to (C \times (0, \infty), h, q_\infty), \tag{A.2.6}$$

geometrically in the C^∞ topology where $q_\infty \in C \times (0, \infty)$ at a distance $\hat{d}$ from $C \times \{0\}$.

Lemma A.2.3 (Lemma 6, [76]). *For any $\mathfrak{d}_1 \in (0, \hat{d})$, there exists $\eta > 0$ such that for all $\varepsilon > 0$ small, there is an embedding $\theta_\varepsilon : \partial N \times [-\mathfrak{d}_1, \mathfrak{d}_1] \to N_\varepsilon$ satisfying the following properties:*

1. *$\theta_\varepsilon(\partial N \times \{0\}) = \gamma_\varepsilon(\partial N_\varepsilon \times \{\delta_\varepsilon\})$ and*

$$\theta_\varepsilon(\partial N \times [-\mathfrak{d}_1, \mathfrak{d}_1]) = \{q \in N_\varepsilon : d_{h_\varepsilon}(q, \gamma_\varepsilon(\partial N_\varepsilon \times \{\delta_\varepsilon\})) \leq \mathfrak{d}_1\}; \tag{A.2.7}$$

2. $\|\theta_\varepsilon^* h_\varepsilon\|_{C^1(\partial N\times[-\mathfrak{d}_1,\mathfrak{d}_1])} \leq \eta$ *where* $\|\cdot\|_{C^1(\partial N\times[-\mathfrak{d}_1,\mathfrak{d}_1])}$ *is computed w.r.t. the product metric* $h' := g_{\llcorner\partial N} \oplus \mathrm{d}s^2$*;*

3. *the metrics* $\theta_\varepsilon^* h_\varepsilon$ *converge in the* C^0 *topology to a Lipschitz continuous metric which is smooth outside of* $\partial N \times \{0\}$ *and*

$$\lim_{\varepsilon\to 0} \|\theta_\varepsilon^* h_{\varepsilon\llcorner}(\partial N \times [0,\mathfrak{d}_1]) - \gamma_0^* g_{\llcorner}(\partial N \times [0,\mathfrak{d}_1])\|_{C^0(\partial N\times[0,\mathfrak{d}_1])} = 0. \tag{A.2.8}$$

A.2.2 Improved Confined Min-max Theory

In $\mathcal{C}(N)$, one is able to define the min-max p-width $\omega_p(\mathcal{C}(N))$ with $K_1 \subset K_2 \subset \cdots \subset \mathcal{C}(N)$ an exhaustion of $\mathcal{C}(N)$ by:

$$\omega_p(\mathcal{C}(N)) = \omega_p(\mathcal{C}(N), h) := \lim_{i\to\infty} \omega_p(K_i, h) \in [0,\infty]\,. \tag{A.2.9}$$

By Theorem 8 in [76], suppose that Σ_1 has the largest area among all the boundary components and we know that

$$p\cdot \mathrm{Area}(\Sigma_1) \leq \omega_p(\mathcal{C}(N)) \leq p\cdot \mathrm{Area}(\Sigma_1) + \hat{C} p^{\frac{1}{n+1}}\,, \tag{A.2.10}$$

where $\hat{C} \in (0,\infty)$ depends on h. In addition, By Lemma A.2.2 and Lemma A.2.3,

$$\lim_{\varepsilon\to 0} \omega_p(N_\varepsilon, h_\varepsilon) = \omega_p(\mathcal{C}(N))\,. \tag{A.2.11}$$

Theorem 9 in [76] asserts that such a p-width could be realized by a union of minimal hypersurfaces in the interior of N. Since the metric of M is bumpy, we could improve it using the multiplicity one theorem by X. Zhou [86] and the Morse index theorem by F. Marques and A. Neves [55].

Theorem A.2.4. *In* $(\mathcal{C}(N), h)$*, for any* $p \in \mathbb{N}^+$*, there exists a pairwise disjoint union*

of connected, smooth, closed, embedded, two-sided, minimal hypersurfaces $\Gamma_1, \cdots, \Gamma_l$ *contained in* $N \backslash \partial N$ *such that*

$$\begin{aligned} \omega_p(\mathcal{C}(N)) &= \sum_{j=1}^{l} \mathrm{Area}(\Gamma_j)\,, \\ p &= \sum_{j=1}^{l} \mathrm{index}(\Gamma_j)\,. \end{aligned} \tag{A.2.12}$$

Proof. The proof will consist of three steps.

Step 1 We are going to show that $\exists \varepsilon_0 > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$, any connected embedded minimal hypersurface in N_ε with area bounded by $(\omega_p(\mathcal{C}(N)) + 1)$ from above and index bounded by p from above is inside $N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon])$. In fact, this essentially follows the proof for Theorem 9 in [76] so we only sketch it here.

Note that the metric h_ε defined in (A.2.5) indicates that the region $\gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon])$ looks more and more cylindrical, and each slice has non-zero mean curvature by Lemma A.2.1 (1). If we fixed a point $q \in N \backslash \partial N$, then for $\varepsilon > 0$ small enough, then we know that $q \in N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon])$.

First, the monotonicity formula together with the maximum principle implies that there exists an $\tilde{R} > 0$ independent of ε such that any free boundary (possibly closed) embedded minimal hypersurface Γ in N_ε with area bounded by $(\omega_p(\mathcal{C}(N)) + 1)$ is inside $B_{h_\varepsilon}(q, \tilde{R})$ as long as $\varepsilon > 0$ is small enough. Thus, Γ must be closed.

Second, for any sequence of embedded minimal hypersurfaces $\{\Gamma_{\varepsilon_k} \subset N_{\varepsilon_k}\}$ with area bounded by $(\omega_p(\mathcal{C}(N), h) + 1)$ and Morse index bounded by p, if $\varepsilon_k \to 0$, then by Lemma A.2.2 and Lemma A.2.3, we have a limiting varifold V_∞ in $\mathcal{C}(N)$ from a subsequence of $\{\|\Gamma_{\varepsilon_k}\|\}$ with total volume $\limsup_{k\to\infty} \mathrm{Area}(\Gamma_{\varepsilon_k})$ bounded by $(\omega_p(\mathcal{C}(N), h) + 1)$. By Sharp's compactness [68], $\mathrm{spt}(V_\infty)$ is smooth outside $\partial N \times \{0\}$. The maximum principle implies that if $\mathrm{spt}(V_\infty) \cap (\mathcal{C}(N) \backslash \mathrm{clos}(N)) \neq \emptyset$, $\mathrm{spt}(V_\infty)$ would be a connected component of some slice $\partial N \times \{\delta\}$. However, due to the mean con-

caveness of the foliation, Γ_{ε_k} for k large intersects $\Phi(\partial N \times \{\hat{d}\})$ and so does $\mathrm{spt}(V_\infty)$, giving a contradiction.

As a consequence, $\mathrm{spt}(V_\infty)$ is contained in (N, g). A delicate and direct computation in the proof for Theorem 9 in [76] shows that V_∞ is in fact g-stationary, and the maximum principle by B. White [80] implies that V_∞ is contained in the interior of N. And therefore, $\Sigma_\infty = \mathrm{spt}(V_\infty)$ is a smooth minimal hypersurface embedded in N with area bounded by $(\omega_p(\mathcal{C}(N), h) + 1)$ and Morse index bounded by p.

Finally, since ∂N has no Jacobi field due to bumpiness, by Sharp's compactness theorem again, $\exists \mathfrak{d}_2 > 0$ s.t. any closed embedded minimal hypersurface in the interior of N with area bounded by $(\omega_p(\mathcal{C}(N), h) + 1)$ and Morse index bounded by p has a distance at least $\mathfrak{d}_2$ from ∂N. Because the convergence above holds for any sequence with $\varepsilon_k \to 0$, and $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, the monotonicity formula and the maximum principle together imply that $\exists \varepsilon_0 > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$, any connected embedded minimal hypersurface in N_ε with area bounded by $(\omega_p(\mathcal{C}(N), g) + 1)$ from above and index bounded by p from above has a distance at least $\mathfrak{d}_2/2 > 0$ away from $\gamma_\varepsilon(\partial N_\varepsilon \times \{\delta_\varepsilon\})$. Hence, the minimal hypersurface is inside $N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon])$.

A byproduct is that all these minimal hypersurfaces in $(N_\varepsilon, h_\varepsilon)$ with bounded area and bounded Morse index are in fact minimal hypersurfaces in the interior of (N, g) so non-degenerate and only finitely many.

Step 2 Fix $\varepsilon \in (0, \varepsilon_0)$ small enough such that $\omega_p(N_\varepsilon, h_\varepsilon) < \omega_p(\mathcal{C}(N)) + 1$. Following the proof for Theorem 8.3 in [55], we can perturb metric h_ε as in Proposition 8.6 therein to get h_ε^i, and prove for each h_ε^i the existence of a sequence of two-sided p-width minimal hypersurfaces with multiplicity one and index p. Then, taking the limit and using the nondegeneracy, we obtain a two-sided minimal hypersurface with multiplicity one, index p and area $\omega_p(N_\varepsilon, h_\varepsilon)$.

To be more precise, let $\mathcal{M}_p(h_\varepsilon) = \{\Gamma_1, \cdots, \Gamma_q\}$, $q \in \mathbb{N}$ to be the finite set of all

connected embedded minimal hypersurfaces in the interior of N_ε with area bounded by $(\omega_p(\mathcal{C}(N))+1)$ and index bounded by p. We choose $\eta > 0$ small and for each $l \in \{1, \cdots, q\}$, pick $p_l \in \Gamma_l \backslash (\bigcup_{k\neq l} \Gamma_k)$ s.t. $B_{h_\varepsilon}(p_l, \eta) \cap (\bigcup_{k\neq l} \Gamma_k) = \emptyset$ and nonnegative functions $f_l \in C_c^\infty(B_{h_\varepsilon}(p_l, \eta)), f_l|_{\Gamma_l} \not\equiv 0$, s.t., $(\nabla_{h_\varepsilon} f_l)(x) \in T_x\Gamma_l$ for every $x \in \Gamma_l$. Then for the metric

$$\hat{h}_\varepsilon(t_1, t_2, \cdots, t_q) = \exp(2\sum_{l=1}^{q} t_l f_l) h_\varepsilon\,, \tag{A.2.13}$$

Γ_l is still minimal for each l.

It is possible to take $(t_1^{(i)}, \cdots, t_q^{(i)}) \subset (0,1)^q$ to be a sequence tending to 0 and for the metric $h_\varepsilon^i := \hat{h}_\varepsilon(t_1^{(i)}, \cdots, t_q^{(i)})$,

$$\mathrm{Area}_{h_\varepsilon^i}(\Gamma_1), \cdots, \mathrm{Area}_{h_\varepsilon^i}(\Gamma_q)\,, \tag{A.2.14}$$

are linearly independent over $\mathbb{Q}$. Sharp's compactness theorem shows that for i large enough, any connected embedded minimal hypersurface in $(N_\varepsilon, h_\varepsilon^i)$ with area bounded by $(\omega_p(\mathcal{C}(N), h)+1)$ and index bounded by p is one of $\{\Gamma_1, \cdots, \Gamma_q\}$ and thus non-degenerate, closed, contained inside $N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon])$.

Proposition 8.7 in [55] still implies the existence of a homotopy class Π_i of p-sweepouts with $\omega_p(N_\varepsilon, h_\varepsilon^i) = \mathbf{L}(\Pi_i)$, and one can observe (See details in Appendix A.A) that the proof for X. Zhou's Multiplicity One Theorem [86] works in our setting and induces a minimizing sequence $\{\Phi_j^{(i)}\}_j$ in Π_i with critical set $\mathbf{C}(\{\Phi_j^{(i)}\}_j)$ containing a multiplicity one, two-sided, embedded h_ε^i-minimal hypersurfaces $\{\Gamma_j^{(i)}\}_{j=1}^l$ with $\sum_{j=1}^l \mathrm{index}(\Gamma_j^{(i)}) \leq p$. Furthermore, $\{\Gamma_j^{(i)}\}_{j=1}^l$ is a subset of $\{\Gamma_1, \cdots, \Gamma_q\}$ mentioned above. The $\mathbb{Q}$-independency of

$$\{\mathrm{Area}_{h_\varepsilon^i}(\Gamma_1), \cdots, \mathrm{Area}_{h_\varepsilon^i}(\Gamma_q)\}\,, \tag{A.2.15}$$

implies that such a realization is unique.

Since all minimal hypersurfaces with bounded area $(\omega_p(\mathcal{C}(N), h) + 1)$ are far away from the boundary ∂N_ε and closed, Theorem 8.2 in [55] also works in our setting and leads to

$$\sum_{j=1}^{l} \text{index}(\Gamma_j^{(i)}) = p\,. \tag{A.2.16}$$

Taking the limit $i \to \infty$, Sharp's compactness theorem indicates the existence of a two-sided minimal hypersurface with multiplicity one, index p and area $\omega_p(N_\varepsilon, h_\varepsilon)$ (possibly with multiple components) inside $N_\varepsilon \backslash \gamma_\varepsilon(\partial N_\varepsilon \times [0, \delta_\varepsilon])$.

Step 3 As mentioned in the end of Step 1, the minimal hypersurface from Step 2 is in fact a g-minimal hypersurface in the interior of (N, g) with a distance to the boundary at least $\mathfrak{d}_2$. Take the limit $\varepsilon \to 0$, Sharp's compactness theorem again leads to the desired union of connected minimal hypersurfaces. □

A.3 Proof of Main Theorem

Firstly, we construct $(N, \partial N)$ by cutting out along two-sided stable minimal hypersurfaces. Indeed, suppose that $\{\Sigma_i\}_{i=1}^I$ is the union of all connected two-sided stable minimal hypersurfaces in M, and since g is bumpy, I is either a natural number or the ω ordinal. Thus, we may assume that $\text{Area}(\Sigma_i) \leq \text{Area}(\Sigma_{i+1})$ for any $i < I$, if $I \geq 2$.

If $I \neq 0$, $(N, \partial N)$ will be constructed inductively.

- Let N_1 be the metric completion of a connected component of $M \backslash \Sigma_1$.
- Suppose that $N_j (j \geq 1)$ is constructed. If there exists at least one hypersurface in $\{\Sigma_i\}_{i=1}^I$ in the interior of N_j, we may take the first such minimal hypersurface, denoted by Σ'_{j+1}, and let N_{j+1} be the metric completion of a

connected component of $N_j \backslash \Sigma'_{j+1}$. Otherwise, we will terminate the induction with $(N, \partial N) := (N_j, \partial N_j)$.

Thus, there will be three cases. If $I = 0$, we have M itself, or if $I \neq 0$, we obtain either $(N, \partial N)$ or a sequence of connected manifolds with boundary $\{(N_j, \partial N_j)\}$.

Case 1 $I = 0$.

We know that M has Frankel property for connected two-sided minimal hypersurfaces, i.e., any two connected two-sided minimal hypersurfaces should intersect with each other. Indeed, suppose this is not true and we have two connected two-sided minimal hypersurfaces Γ_1 and Γ_2, which must be unstable by assumption. By a standard area-minimization argument (See, for example, the proof of Lemma 11 in [76]), there exists a two-sided stable minimal hypersurface in M giving a contradiction to the assumption that $I = 0$.

Combining Frankel property and Theorem 8.3 in [55], we have a sequence of connected, smooth, closed, embedded, multiplicity one, two-sided, minimal hypersurfaces $\{\Gamma_p\}_{p=1}^{\infty}$ such that

$$\omega_p(M, g) = \mathrm{Area}(\Gamma_p), \quad \mathrm{index}(\Gamma_p) = p. \tag{A.3.1}$$

Case 2 $(N, \partial N) = (N_j, \partial N_j)$ for some j.

By Lemma 11 in [76], we know that $(N, \partial N)$ has Frankel property for two-sided minimal hypersurfaces in the interior. Suppose that $\partial N = \bigcup_{i=1}^{m} \Sigma_i$ and Σ_1 is the one with the largest area. Combining Frankel property and Theorem A.2.4, we have a sequence of connected, smooth, closed, embedded, two-sided minimal hypersurfaces $\{\Gamma_p\}_{p=1}^{\infty}$ such that

$$\mathrm{Area}(\Gamma_p) = \omega_p(\mathcal{C}(N)), \quad \mathrm{index}(\Gamma_p) = p. \tag{A.3.2}$$

Case 3 There exists an infinite sequence $\{(N_j, \partial N_j)\}$.

For each $(N_j, \partial N_j)$, suppose that $\partial N_j = \bigcup_{i=1}^{m_j} \Sigma_i^j$ where Σ_i^j is a connected stable minimal hypersurface and Σ_1^j is the one with the largest area. Since the metric g is bumpy, we have that

$$\lim_{j\to\infty} \text{Area}(\Sigma_1^j) = \infty\,. \tag{A.3.3}$$

Thus, for any fixed $p \in \mathbb{N}^+$, there exists a $j_p \in \mathbb{N}$ such that $\text{Area}(\Sigma_1^{j_p}) \geq p$ and by construction, we know that any two-sided stable minimal hypersurface in the interior of N_{j_p} has area no less than $\text{Area}(\Sigma_1^{j_p})$. In fact, this is true for any two-sided minimal hypersurface inside N_{j_p}.

Lemma A.3.1. *If Γ is a connected two-sided minimal hypersurface in $N_{j_p}\backslash\partial N_{j_p}$, then $\text{Area}(\Gamma) \geq \text{Area}(\Sigma_1^{j_p})$.*

Proof. We only need to deal with the case where Γ is unstable. In this case, Γ has an expanding neighborhood nearby inside N_{j_p}. Suppose that the conclusion does not hold, and then the two-sided unstable minimal hypersurface Γ has $\text{Area}(\Gamma) < \text{Area}(\Sigma_1^{j_p})$.

If Γ is non-separated in N_{j_p}, we can minimize the area in the cohomology class $[\Gamma] \in H_n(M;\mathbb{Z}_2)$ inside N_{j_p}. By maximum principle, we obtain a two-sided stable minimal hypersurface with at least one component Γ' in the interior of N_{j_p} with

$$\text{Area}(\Gamma') < \text{Area}(\Gamma) < \text{Area}(\Sigma_1^{j_p})\,, \tag{A.3.4}$$

giving a contradiction.

On the other hand, if Γ separates N_{j_p}, let $\tilde{N}$ be the metric completion of the connected component of $N_{j_p}\backslash\Gamma$ such that $\Sigma_1^{j_p} \subset \partial\tilde{N}$. Since Γ is unstable, there exists an expanding neighborhood of Γ in $\tilde{N}$ and then minimizing the homology class $[\Gamma]$ gives us a stable minimal hypersurface, whose connected component by maximum principle again is either contained in $\partial\tilde{N}\backslash\Gamma$ or in the interior of $\tilde{N}$. Note that the area of each component is strictly less than $\text{Area}(\Sigma_1^{j_p})$, so none of them could be inside

the interior of N_{j_p} and thus the interior of $\tilde{N}$. As a consequence, by the fact that $\partial\tilde{N}$ is the boundary of $\tilde{N}$, the minimal hypersurface obtained from the minimizing procedure in $[\Gamma]$ could only be exactly $\partial\tilde{N}\backslash\Gamma$, which also gives a contradiction from the following inequality:

$$\text{Area}(\partial\tilde{N}\backslash\Gamma) < \text{Area}(\Gamma) < \text{Area}(\Sigma_1^{j_p}) \leq \text{Area}(\partial\tilde{N}\backslash\Gamma)\,. \tag{A.3.5}$$

In summary, any two-sided minimal hypersurface in N_{j_p} has area no less than $\text{Area}(\Sigma_1^{j_p})$. □

Now, Let's define

$$\mathcal{N}(p) := \#\{\Sigma \subset M : \text{ a connected closed two-sided stable minimal hypersurface with } \text{Area}(\Sigma) \leq (p+1)\cdot\text{Area}(\Sigma_1^{j_p})\} < \infty\,. \tag{A.3.6}$$

The finiteness follows from the bumpiness of the metric (See Lemma 2.1 in [16]).

Following the idea in the proof of Lemma 2.2 in [16], we have the following counting estimate.

Lemma A.3.2. *Suppose that $\{\Gamma_k\}_{k=1}^l \subset N_{j_p}\backslash\partial N_{j_p}$ is a set of disjoint, connected, closed, two-sided, minimal hypersurfaces with area no greater than $(p+1)\cdot\text{Area}(\Sigma_1^{j_p})$, and then we have*

$$l \leq 2\mathcal{N}(p) + 1\,. \tag{A.3.7}$$

Proof. Consider $\tilde{N} = N_{j_p}\backslash\bigcup_{k=1}^l \Gamma_k$ which can be written as a disjoint union of $\{\tilde{N}_t\}_{t=1}^Q$, where $(\tilde{N}_t, \partial\tilde{N}_t)$ is a connected submanifold with boundary. Note that $Q \leq l+1$, since each cut would produce at most one more connected component.

Let $C_t = \#\{\text{connected components of } \partial\tilde{N}_t\}$ and it is apparent that

$$\sum_{t=1}^{Q} C_t \geq 2l\,. \tag{A.3.8}$$

Since each two-sided stable minimal hypersurface in M will be counted at most twice, the claim below implies

$$2\mathcal{N}(p) \geq \sum_{t=1}^{Q}(C_t - 1) = \sum_{t=1}^{Q} C_t - Q \geq l - 1\,, \tag{A.3.9}$$

where the last inequality follows from $Q \leq l + 1$.

Claim 8. *For each $\tilde{N}_t$, there are at least $(C_t - 1)$ two-sided stable minimal hypersurfaces in the metric completion of $\tilde{N}_t$ with area no greater than $(p+1)\cdot \mathrm{Area}(\Sigma_1^{j_p})$.*

Indeed, the proof is similar to that of Claim 2.3 in [16], but for completeness, we will present it here.

Proof of Claim. Let's prove it inductively on the number n_t of unstable components of $\partial\tilde{N}_t$.

If $n_t = 0$ or 1, then the claim holds trivially.

Let $k \in \mathbb{N}^+$, and suppose that the claim holds for any $n_t \leq k$. Now let's consider the case $n_t = k + 1$, and denote the components of $\partial\tilde{N}_t$ as $\Gamma_1, \cdots, \Gamma_{n_t}$ which are unstable and $\Gamma'_{n_t+1}, \cdots, \Gamma'_{C_t}$ which are stable. Note that $H_n(\tilde{N}_t; \mathbb{Z}_2) \ni [\Gamma_1] \neq 0$ since $C_t \geq n_t \geq 2$.

For the unstable component Γ_1, we can find $\Gamma'_1 \in [\Gamma_1] \neq 0$ minimizing area in the homology class, each of whose components by maximum principle is either in the interior of $\tilde{N}_t$ or one stable component of $\partial\tilde{N}_t$. We may denote the number of the components of Γ'_1 in the interior by m_t and the number of those on the boundary by l_t. Moreover, there exists a closed set $\Omega_1 \subset \tilde{N}_t$ with $\partial(\tilde{N}_t \backslash \Omega_1) = \Gamma'_1 \cup \Gamma_1$. Note that the number of connected components of Ω_1 is no greater than m_t, and the total number of connected components of $\partial\Omega_1$ is exactly $C_t - 1 - l_t + m_t$.

By induction, the claim holds for each component of Ω_1, so the total number of

connected stable minimal hypersurfaces with the area upper bound in Ω_1 is at least

$$(C_t - 1 - l_t + m_t) - m_t = C_t - 1 - l_t \,, \tag{A.3.10}$$

and thus, the number of those inside the metric completion of $\tilde{N}_t$ with the area upper bound is at least

$$(C_t - 1 - l_t) + l_t = C_t - 1 \,. \tag{A.3.11}$$

□

□

By Theorem A.2.4, for any $P \in \mathbb{N}$ large enough, there exists a union of smooth, connected, closed, embedded, two-sided minimal hypersurfaces $\Gamma_1^P, \cdots, \Gamma_{l_P}^P$ contained in $N_{j_p} \backslash \partial N_{j_p}$ such that

$$\begin{aligned} \omega_P(\mathcal{C}(N)) &= \sum_{k=1}^{l_P} \mathrm{Area}(\Gamma_k^P) \,, \\ P &= \sum_{k=1}^{l_P} \mathrm{index}(\Gamma_k^P) \,. \end{aligned} \tag{A.3.12}$$

By Lemma A.3.1, we know that $\mathrm{Area}(\Gamma_k^P) \geq \mathrm{Area}(\Sigma_1^{j_p}) \geq p$ for any k. Moreover, by Lemma A.3.2, we also know that the number of minimal hypersurfaces in $\{\Gamma_k^P\}$ above with area no greater than $(p+1) \cdot \mathrm{Area}(\Sigma_1^{j_p})$ is no greater than $2\mathcal{N}(p) + 1$, and hence, we can obtain an upper bound on l_P, i.e.,

$$l_P \leq 2\mathcal{N}(p) + 1 + \frac{\omega_P}{(p+1) \cdot \mathrm{Area}(\Sigma_1^{j_p})} \leq 2\mathcal{N}(p) + 1 + \frac{P}{p+1} + \frac{\tilde{C} P^{\frac{1}{n+1}}}{p+1} \,. \tag{A.3.13}$$

By the pigeonhole principle, we know that there exists a $\Gamma^P_{k'}$ with

$$\begin{aligned}\operatorname{index}(\Gamma^P_{k'}) &\geq \frac{P}{l_P} \\ &\geq \frac{P}{2\mathcal{N}(p)+1+\frac{P}{p+1}+\frac{\tilde{C}P^{\frac{1}{n+1}}}{p+1}} \\ &= \frac{p+1}{\frac{2(p+1)\mathcal{N}(p)+p+1}{P}+1+\tilde{C}P^{-\frac{n}{n+1}}} \\ &\geq p\,,\end{aligned} \tag{A.3.14}$$

when P is large enough.

In this case, we now can define $\Gamma_p := \Gamma^P_{k'}$ chosen above.

In summary, in each case, we obtain a sequence of connected two-sided minimal hypersurfaces with

$$\lim_{p\to\infty}\operatorname{Area}(\Gamma_p) = +\infty\,, \quad \lim_{p\to\infty}\operatorname{index}(\Gamma_p) = +\infty\,. \tag{A.3.15}$$

A.A X. Zhou's Multiplicity One Theorem

The strategy of X. Zhou's proof lies upon the approximation of minimal hypersurfaces by a sequence of min-max prescribed mean curvature (PMC) hypersurfaces. Roughly speaking, in a bumpy closed Riemannian manifold, for each p-width, he constructs a sequence of PMC hypersurfaces converging to a p-width minimal hypersurface Σ. Analysis on such a convergence implies that if the multiplicity of the convergence is greater than one, then either one could get a non-trivial Jacobi field, or a positive function φ satisfying the PDE,

$$L_\Sigma\varphi = \mathfrak{h}\,, \tag{A.A.1}$$

where L_Σ is the Jacobi operator and $\mathfrak{h}$ is the prescribed function. The first case can be excluded by the bumpiness while the second case is also excluded by choosing an appropriate $\mathfrak{h}$ such that the PDE has no positive solution. Therefore, one is able to show that the p-width minimal hypersurface should has multiplicity one. Note that the construction of min-max PMC hypersurfaces is based on the work by X. Zhou and J. Zhu [88].

However, the results above are all in the setting of a closed Riemmanian manifold, and here we need to deal with $(N_\varepsilon, \partial N_\varepsilon)$ which is a Riemannian manifold with boundary. Therefore, one would expect to resolve this issue by directly adapting their methods in the free boundary min-max setting but it seems that the analysis on the touching set near the boundary, if possible, would require tedious work on this adaption. In fact, in our setting, the multiplicity one theorem could be proved without such a general theory.

Firstly, for $\varepsilon > 0$ small enough, we may view $(N_\varepsilon, \partial N_\varepsilon)$ as an subset of some closed smooth Riemannian manifold $\tilde{N}_\varepsilon$ and then choose $\mathfrak{h}$ defined in $\tilde{N}_\varepsilon$ such that all minimal hypersurfaces with area bounded by $(\omega_p(\mathcal{C}(N), h) + 1)$ and Morse index bounded by p contained in the interior of N_ε (they are finitely many) satisfy the sign-changing property as in Lemma 4.2 in [86]. Meanwhile, $\mathfrak{h} \in \mathcal{S}(\tilde{N}_\varepsilon)$.

Second, define a cut-off function η supported in the interior of N_ε such that

$$\eta(x) \equiv 1\,, \text{ if } \operatorname{dist}_{h_\varepsilon}(x, \partial N_\varepsilon) \geq 1\,, \tag{A.A.2}$$

and define $\tilde{\mathfrak{h}} = \mathfrak{h} \cdot \eta$ on N_ε.

Now, on $(N_\varepsilon, \partial N_\varepsilon)$, given any associated (X, Z)-homotopy class Π of Φ_0 with

$$\omega_p(\mathcal{C}(N), h) + 1 \geq \mathbf{L}(\Pi) > \max_{x \in Z} \mathbf{M}(\partial \Phi_0(x)), \tag{A.A.3}$$

we would follow the proof of Theorem 3.6 in [86] with prescribed mean curvature

$\delta \cdot \tilde{\mathfrak{h}}$ where the number $\delta > 0$ is small enough. Note that for a pulled-tight min-max sequence $S = \{\Phi_l\} \in \Pi$, any varifold $V \in \mathbf{C}(S)$ is $\mathcal{A}^{\delta\tilde{\mathfrak{h}}}$-stationary.

Since $\mathfrak{h} \equiv 0$ near ∂N_ε, the regularity for almost-minimizing free boundary min-max minimal hypersurfaces in Li-Zhou [41] implies that there exists a $V_\delta \in \mathcal{C}(S)$, $\mathrm{spt}(V_\delta) \cap \{x : \mathrm{dist}_{h_\varepsilon}(x, \partial N_\varepsilon) < 1/2\}$ is either an empty set or a smooth hypersurface. Moreover, in the interior of $\{x : \mathrm{dist}_{h_\varepsilon}(x, \partial N_\varepsilon) > 1\}$, such a varifold V_δ is $\delta\mathfrak{h}$-almost minimizing in small annuli, since $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$ coincide there. Hence, by the maximum principle by B. White [80] and monotonicity formulae, if we fix ε small enough and take δ even smaller, V_δ should be contained in the interior of $\{x : \mathrm{dist}_{h_\varepsilon}(x, \partial N_\varepsilon) > 1\}$, since otherwise, the area of V_δ will be greater than $(\omega_p(\mathcal{C}(N), h) + 1)$. Thus, the proof of Theorem 3.6 in [86] implies that V_δ is a smooth, closed, multiplicity one, properly almost embedded hypersurface with Morse index bounded above by p as boundary of a Cacciopoli set.

Finally, we can follow Theorem 4.1 in [86], taking $\delta \to 0$, and obtain an embedded closed multiplicity one minimal hypersurface with area $\omega_p(N_\varepsilon, h_\varepsilon)$ and Morse index bounded by p in the interior of $(N_\varepsilon, \partial N_\varepsilon)$.

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