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Algebraic aspects of Iwasawa theory in the Gan--Gross--Prasad setting

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ALGEBRAIC ASPECTS OF IWASAWA THEORY IN THE
GAN–GROSS–PRASAD SETTING

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Abstract

This thesis sets up a framework to understand Iwasawa theory for the product of two unitary groups, generalizing phenomena observed in studying elliptic curves twisted by imaginary quadratic Hecke characters. The first part proves a formal theorem relating the different Iwasawa main conjectures arising in this picture. The second part interpolates the Selmer classes attached to the diagonal cycle appearing in the arithmetic Gan–Gross–Prasad conjecture. The final part is highly speculative and discusses the need for entirely new methods in the subject.

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Chapter 0

Introduction

Iwasawa theory has played a seminal role in the study of the arithmetic of elliptic curves, in particular the Birch–Swinnerton-Dyer conjecture. In the rank 1 case, a key object of study is the Heegner point. They are special points which can be constructed on elliptic curves depending on the choice of an auxiliary imaginary quadratic field $\mathcal{K}$. One important property is that if one considers such points over the anticyclotomic extensions of $\mathcal{K}$, there are simple relations between the resulting points. Kolyvagin [Kol88] used this to prove some of the first results towards the BSD conjecture for elliptic curves without CM, using a method now known as the method of Euler systems.

An Euler system consists of a collection of classes in Galois cohomology of various extensions of a base field satisfying norm relations. They can be used to bound the sizes of Selmer groups and deduce cases of the Bloch–Kato conjecture [BK90, FPR94], a vast generalization of the BSD conjecture. This part of the argument can be axiomatized, see for example [Rub00], [MR04], and the forthcoming work of Jetchev–Nekovář–Skinner. The question remains of their constructions. There have been roughly two types of constructions. The first is based on constructing special units, for example cyclotomic units, elliptic units, or Siegel units, cf. among others [Rub88, Kat04, LLZ14, KLZ17]. Unfortunately, we seem to have exhausted special elements of this type. The second method is based on constructing special cycles in algebraic varieties. This contains Kolyvagin’s work, and it is the focus of this thesis.

There is a wealth of examples that can be obtained based on cycles, see for example [LSZ22, DR14, BSV22, FJ21]. Recently, it has been realized that the theory of spherical varieties is a good organizing framework for these examples, and in fact there is an axiomatic treatment in [Loe21]. To study Iwasawa theory, one would also need a source of p -adic L -functions from interpolations,

which typically requires good integral representations for special values of L -functions. One natural setting in which we have both type of information is the Gan–Gross–Prasad conjecture for unitary groups, first formulated in [GGP12]. It directly encompasses Kolyvagin’s original setting of elliptic curves and Heegner points. The starting point of this thesis was an attempt to understand Iwasawa theory in this setting.

0.1 Motivation

Before explaining our results, we need to look closely at the phenomena in the lowest rank $n = 1$ which inspired them. This is the setting of an elliptic curve E twisted by an anticyclotomic character χ . Suppose the elliptic curve has Heegner points, then a lot is known about the Iwasawa theory and the Bloch–Kato conjecture. There are in fact two regions with fundamentally different behaviour:

Incoherent region If χ has finite order, then the Kummer image of the Heegner point defines an element in the Selmer group for $E \times \chi$. Its triviality depends on $L'(E \times \chi, 1)$ by the Gross–Zagier formula [GZ86]. If it is non-trivial, then Kolyvagin showed that the Selmer group has rank 1 [Kol88]. Moreover, we know it is non-trivial for most χ [CV07].

Coherent region If χ has infinite order, then the L -value $L(E \times \chi, 1)$ is generically non-zero [Bur20], so we expect the Selmer group to be trivial for most χ .

The key observation that began many progresses in the area is that on the Iwasawa theory level (which packages characters together into a family), the two regions are very closely related. We have to mention the foundational work of Bertolini–Darmon–Prasanna [BDP13]: they constructed a p -adic L -function $\mathcal{L}_p^{\text{BDP}}(\chi)$ interpolating $L(E \times \chi, 1)$ when χ has infinite order, but more importantly, they proved that when they take the limit $\chi \rightarrow \mathbf{1}$ p -adically, the value $\mathcal{L}_p^{\text{BDP}}(\mathbf{1})$ can be computed using the Heegner point. From the geometric side, as explained in [CH18], the norm-compatibility of the Heegner points allows us to form a big Heegner point interpolating over all finite order χ . It becomes *non-geometric* when specialized to an infinite order χ , so it can be used to bound Selmer groups in the coherent region too.

On the algebraic side, the difference between the coherent and incoherent region is the Selmer condition imposed. One can formulate two types of Iwasawa main conjectures: (a) Greenberg type identifying a torsion Selmer group with the p -adic L -function $\mathcal{L}_p^{\text{BDP}}$ [Gre94]; and (b) Perrin-Riou type which studies the index of Heegner points in certain compact Selmer groups, relating it to the torsion part of a rank-1 Selmer group of Greenberg type [PR87]. They are in fact equivalent given

the formula for $\mathcal{L}_p^{\text{BDP}}(1)$ in terms of Heegner points, and this fact has been used by [BCK21] to prove both conjectures at once.

In a different line of development, Howard [How07] interpolated the Heegner points as E varies in a Hida family. This allows one to access some information about higher weight modular forms without using a Kuga–Sato variety. It was shown in [Cas20] that the two resulting classes are the same. We will also generalize this part of Howard’s work.

0.2 Main results and organization of the thesis

We will now describe the structure of this thesis. At the same time, we will describe our main results and how they are generalizations of what we mentioned in the motivation.

The first chapter sets the ground work for p -adic interpolation by defining what we mean by a p -adic family of Galois representations. Its main contribution to the thesis is the construction of a multi-variable Coleman map (Theorem 1.4.1), to be used later. Its proof is a mild generalization of the ones in [LZ14] and [Och03], though their results are not sufficient for our purpose.

The second chapter is the first main part of the thesis. In it, we show in a large degree of generality how rank one results in Iwasawa theory can also imply rank zero results for a different Selmer condition (Theorem 2.3.2). The proof is an entirely formal computation using duality theorems in Galois cohomology. This abstracts the equivalence between the Greenberg main conjecture and the Perrin-Riou main conjecture. The equivalence is well-known to experts, though we find it useful to give an axiomatic treatments which clarifies the result. As an illustration, we give two applications to the elliptic curves setting which simplify previously existing proofs in the literature.

The third chapter is purely expository, reviewing the Gan–Gross–Prasad conjectures, Galois representations attached to unitary groups, and abelian type Shimura varieties of unitary groups. This can be skipped and referred to when needed.

The fourth chapter is another main part of the thesis. The main result is the construction of a big diagonal cycle class in Galois cohomology when we vary an automorphic form in a Hida family of maximum dimension. Along the way, we will also develop Hida theory using Emerton’s completed cohomology and ordinary parts functor. This by itself is a reformulation of known results and proofs, see for example [Hid95, KT17], and our formulation is based on the one which appears in [ACC⁺18], which is ultimately more convenient for us. Unfortunately, due to the lack of algebraic Künneth projectors, we have to make a highly non-canonical choice in the construction of the big diagonal cycle, which could make any attempt to prove an explicit reciprocity law or non-vanishing

troublesome.

Also in the fourth chapter, we initiate the study of the local behaviour of our class in cohomology. The results are not very refined. In particular, we rely on the theory of pseudo-representation to define the local condition at primes above p . A generalization of Ohta's work [Oht95] on the Eichler–Shimura isomorphism in family is required to remove this reliance on non-canonical constructions.

Finally, in the fifth chapter, we outline a largely conjectural framework to study Iwasawa theory in the Gan–Gross–Prasad setting. It describes the Selmer conditions using combinatorial data arising from the archimedean place and outlines how the results proven in this thesis can be applied in this picture. We describe a picture of wall crossing where the same cycle is expected to yield different p -adic L -functions, but also conversely, different cycles are expected give the same p -adic L -function.

0.3 Related works

We will now mention some related works in different lines of developments from that one we chose to pursue.

We have chosen the unitary setting. There is also an orthogonal version, more commonly known as the Gross–Prasad conjecture. The root systems here are more complicated, and in particular, we lose the direct base change comparison with GL_n that allowed W. Zhang's relative trace formula approach to eventually prove the global unitary conjectures. However, the p -adic theory is similar. In particular, it contains the triple product case (viewing $\mathrm{GL}_2 \times \mathrm{GL}_2$ as a form of SO_4). The wall-crossing phenomenon here is the observation that the triple product diagonal cycle describes p -adic L -functions in each of the three unbalanced regions [BSV22]. The corresponding framework in the case $\mathrm{SO}_4 \times \mathrm{SO}_5$ is also conjecturally described in [LZ21b].

There are also other spherical varieties with integral representations. One setting which has received attention recently is the doubling embedding $\mathrm{U}(n) \times \mathrm{U}(n) \hookrightarrow \mathrm{U}(n, n)$. The period for this embedding (called Friedberg–Jacquet period in literature) is expected to be related to the central value of the L -function for $\mathrm{U}(n, n)$ (its proof has been announced by J. Xiao and W. Zhang). In this setting, there is the same phenomenon that switching to a nearby signature gives rise to algebraic cycles, which are expected to be related to a p -adic L -function interpolating the Friedberg–Jacquet period. The construction of the Euler system and the p -adic L -function has been completed by A. Graham and S. W. A. Shah [Gra21, GS21].

Another aspect of Iwasawa theory that is very different from our approach here is based on level raising and lowering congruences. It can be used to prove rank 0 results without appealing to an Euler

system in the sense we have described, for example in the fundamental work of Bertolini–Darmon on the BSD conjecture without Heegner points [BD05]. Further generalizations led to impressive results such as W. Zhang’s Heegner indivisibility result [Zha14c] and the work [LTX⁺22] proving cases of the Bloch–Kato conjecture in the setting considered in this thesis. Howard’s bipartite Euler system [How06] is an attempt to combine both worlds and use Euler system techniques to prove *equalities* in main conjectures. It would be interesting to see how far it can be pushed in combination with our framework. In this direction, we are aware of an ongoing project led by L. Xiao to construct bipartite Euler systems in this setting.

Finally, there is the analytic side that we will not study in detail in this thesis, but is however very important for any application. We consider this to have two aspects

1. A p -adic part relating the special cycle constructed to a p -adic L -function in the interpolation sense. Result of this type are usually called explicit reciprocity laws. This is already present in the works of Coates–Wiles on elliptic curves [CW77]. In the case of a cyclotomic deformation, Perrin-Riou placed the results in a general framework [PR94, PR95]. Much of modern progress in the cycles case seems to depend on variations on syntomic cohomology, and we refer to [Bes16] for a survey, as well as [LZ21a] for a recent breakthrough. We hope the methods in the latter work can be applied in our case.
2. An archimedean part relating the special cycle to the *derivative* of a classical L -function. This would generalize the Gross–Zagier formula. Beyond the works of S. Zhang and his collaborators culminating in [YZZ13], very little appears to be known at the moment. The arithmetic Gan–Gross–Prasad conjecture that we will briefly describe is one such conjectural generalization.

There are further issues with non-degeneracy of the various realization maps, for which even less is known. This makes it difficult to get the full Bloch–Kato conjecture in the rank 1 case, even assuming we have the two points above.

0.4 Notations and Conventions

In this entire thesis, let $\mathcal{K}$ be an imaginary quadratic field, and let p be an odd prime which splits in $\mathcal{K}$. Fix embeddings $\mathcal{K} \rightarrow \mathbf{C}$ and $\mathcal{K} \rightarrow \bar{\mathbf{Q}}_p$, as well as an isomorphism $\bar{\mathbf{Q}}_p \simeq \mathbf{C}$ which is the identity on $\mathcal{K}$. This induces a distinguished place $\mathfrak{p}$ of $\mathcal{K}$ above p . We say a Hecke character of $\mathcal{K}$ has infinity type (a, b) if $z \in \mathcal{K}_\infty$ is sent to $z^a \bar{z}^b$. Let $\mathcal{K}^{\text{ac}}$ be the unique anticyclotomic $\mathbf{Z}_p$ -extension of $\mathcal{K}$, and fix an identification $\text{Gal}(\mathcal{K}^{\text{ac}}/\mathcal{K}) \simeq \mathbf{Z}_p$.

Let $v \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ be a place of $\mathcal{K}$ above p and j be an embedding $\mathcal{K}_v \rightarrow \bar{\mathbf{Q}}_p$. Given a Galois representation $\rho : \text{Gal}_{\mathcal{K}_v} \rightarrow \text{GL}_n(\bar{\mathbf{Q}}_p)$, we will define its Hodge–Tate weights with respect to j to be the non-trivial degrees of the n -dimensional $\bar{\mathbf{Q}}_p$ -vector space

$$\mathbf{D}_{\text{HT},j}(\rho_{\pi,v}) = (\rho \otimes_{j,\mathcal{K}_v} \mathbf{B}_{\text{HT},v})^{G_{\mathcal{K}_v}}$$

where $\mathbf{B}_{\text{HT},v} = \bigoplus_{k \in \mathbf{Z}} \widehat{\mathcal{K}_v}(k)$ is the Hodge–Tate period ring. There are two choices of j , exactly one of which is compatible with the fixed embedding $\mathcal{K} \rightarrow \bar{\mathbf{Q}}_p$. We write the Hodge–Tate weights of ρ as a tuple $(c_1, \dots, c_n; c'_1, \dots, c'_n)$, with the first n entries corresponding to the compatible embedding. If ρ is a global Galois representation, the resulting tuples are the same whether we restrict to $\mathfrak{p}$ or $\bar{\mathfrak{p}}$. Under this convention, the cyclotomic character $\text{Gal}_{\mathcal{K}} \rightarrow \mathbf{Q}_p^\times$ has weights $(-1, -1)$, and an automorphic character of $\text{U}(1)$ whose archimedean component is $z \mapsto z^n$ has an associated Galois representation with weight $(-n, n)$.

Let $\mathbb{A}$ denote the ring of adeles of $\mathbf{Q}$ and $\mathbb{A}_{\mathcal{K}}$ denote the ring of adeles of $\mathcal{K}$. We will systematically use the following convention: a superscript consisting of places indicates omitting those places, and a subscript consisting of places indicates only considering those places. For example, $\mathbb{A}_{\mathcal{K},f}$ is the ring of finite adeles of $\mathcal{K}$, and $\mathbb{A}_f^p = \mathbb{A}^{p\infty}$ is the ring of finite adeles of $\mathbf{Q}$ omitting p .

All of our Galois cohomology groups are defined using continuous cochains, as is standard in the subject. In particular, $\mathbf{R}\Gamma_{\text{cont}}$ denotes the bar complex of continuous cochains in an appropriate derived category, so its cohomology is the usual Galois cohomology.

Class field theory is always normalized geometrically.

Remark 0.4.1. Much of this thesis should generalize to the case of a general CM extension F/F^+ such that p is unramified in F^+ , and every place of F^+ above p splits in F . We have chosen the setting of $\mathcal{K}/\mathbf{Q}$ for simplicity of notations.

Chapter 1

p -adic family of Galois representations

In this section, we set up some notations and definitions for Iwasawa theory. In particular, we construct a Coleman map in our setting. Fix E , a finite extension of $\mathbf{Q}_p$. Let $\mathcal{O}$ be its ring of integers. They will serve as coefficients for the representations we consider.

1.1 Weight space

Let $\mathbf{T} = \mathbf{G}_m^r$ be the r -dimensional split torus with its standard cocharacters $\epsilon_1, \dots, \epsilon_r$. Let $\underline{\Lambda} = \mathcal{O}[[\mathbf{T}(\mathbf{Z}_p)]]$ be its completed group ring. This is a semi-local regular Noetherian ring. More precisely, we have the decomposition $\mathbf{T}(\mathbf{Z}_p) = (1 + p\mathbf{Z}_p)^r \times \Delta$, where $\Delta = \mu_{p-1}^r$. This induces an isomorphism $\underline{\Lambda} \simeq \Lambda_r[\Delta]$, where $\Lambda_r \simeq \mathcal{O}[[T_1, \dots, T_r]]$ via the map $T_i \mapsto [\epsilon_i(1 + p)] - 1$. Given a character $\psi : \Delta \rightarrow \mathcal{O}^\times$, we can pair with it to get a surjection $\mathrm{pr}_\psi : \underline{\Lambda} \rightarrow \Lambda_r$. Denote Λ_r with this $\underline{\Lambda}$ -algebra structure by Λ^ψ .

Let $\underline{\mathcal{W}}$ be the rigid analytic space attached to $\mathrm{Spf} \underline{\Lambda}$, then it is a disjoint union of open unit discs labelled by characters of Δ . Let $\mathcal{W}_\psi$ denote the component corresponding to ψ . We can view $\underline{\Lambda}$ as the bounded functions in the space of rigid functions $\mathcal{O}(\underline{\mathcal{W}})$. It can also be viewed as the subset of measures in the distribution space $\mathcal{H}(\underline{\mathcal{W}})$.

Given a cocharacter $\epsilon : \mathbf{G}_m \rightarrow \mathbf{T}$, we can define the character $\chi_\epsilon : \mathrm{Gal}_{\mathcal{K}} \twoheadrightarrow \mathrm{Gal}(\mathcal{K}^{\mathrm{ac}}/\mathcal{K}) \rightarrow \mathbf{Z}_p \rightarrow \Lambda_r^\times$, where the final step sends $z \in \mathbf{Z}_p$ to $[\epsilon(1 + pz)]$. We say a specialization $x : \Lambda \rightarrow \bar{\mathbf{Q}}_p$ is classical if $x \circ \chi_\epsilon$ is Hodge–Tate at p for all ϵ . In this case, we define the ϵ -weight $w_\epsilon(x) \in \mathbf{Z}$ to be negative

of the Hodge–Tate weight of $x \circ \chi_\epsilon$ at $\mathfrak{p}$. The subset of classical points of $\underline{\mathcal{W}}$ is denoted by $\underline{\mathcal{W}}^{\text{cl}}$.

1.2 p -adic family

Let $\mathbb{I}$ be a ring with a finite inclusion $\underline{\Lambda} \rightarrow \mathbb{I}$. Let $\underline{\mathcal{E}}$ denote the rigid space attached to $\text{Spf } \mathbb{I}$, so we have a finite morphism $\underline{\mathcal{E}} \rightarrow \underline{\mathcal{W}}$. We say a point of $\underline{\mathcal{E}}$ is classical if its image in $\underline{\mathcal{W}}$ is classical. In the intended applications, $\underline{\mathcal{E}}$ occurs as an irreducible component of the ordinary eigenvariety.

Definition 1.2.1. A p -adic family of Galois representations is a finitely generated $\mathbb{I}$ -module $\mathbf{T}$ with an action of $\text{Gal}_{\mathcal{K}}$ such that

1. The $\text{Gal}_{\mathcal{K}}$ -action is admissible in the sense of [Nek06, Section 3.2].
2. As a $\underline{\Lambda}$ -module, $\mathbf{T}$ is projective.

The first condition is a version of continuity. In the cases we will consider, it holds for a finite rank $\mathbb{I}$ -module if the action is continuous with respect to the topology induced from each of the maximal ideal of $\underline{\Lambda}$. The notion is also stable under Pontryagin duality and extensions, so we will usually not need to explicitly check it in practice.

If χ is a character valued in $\mathbb{I}^\times$, then we write $\mathbb{I}(\chi)$ to mean $\mathbb{I}$ with the Galois group acting through χ . This is admissible if and only if χ is continuous. In that case it defines a p -adic family of characters. In particular, for any $\alpha \in \mathbb{I}^\times$, we can form the unramified character α on $\text{Gal}_{\mathbf{Q}_p}$ sending a geometric Frobenius to α . Moreover, using the inclusion $\underline{\Lambda} \rightarrow \mathbb{I}$, the universal characters χ_ϵ are also characters on $\mathbb{I}$.

Fix $s \in \text{Gal}_{\mathbf{Q}}$ to have a non-trivial image in $\text{Gal}(\mathcal{K}/\mathbf{Q})$, then the map $g \mapsto sgs^{-1}$ defines an outer automorphism of $\text{Gal}_{\mathcal{K}}$. The transfer of $\mathbf{T}$ along this will be denoted by $\mathbf{T}^c$. Its isomorphism class is independent of the choice of s .

1.3 Commutative algebra

From now on, fix a tame character ψ and drop it from notations, so $\Lambda \simeq \mathcal{O}[[T_1, \dots, T_r]]$ carries a $\mathbb{T}(\mathbf{Z}_p)$ -action via ψ , and $\mathbb{I}$ is a finite local extension. In this section, we recall some results from commutative algebra on Λ .

The notation $(-)^{\vee}$ will always mean taking the (not derived) dual with respect to a dualizing module over the implied base ring. This is usually applied with $\mathbf{Z}_p$ or Λ , depending on the context. Both of them are Gorenstein, so they are their own dualizing module.

Definition 1.3.1. A p -adic family $\mathbf{T}$ is conjugate self-dual if there exists a Gal_K -invariant bilinear pairing $\mathbf{T} \otimes_{\Lambda} \mathbf{T}^c \rightarrow \Lambda(1)$ which induces an isomorphism $\mathbf{T}^c \simeq \mathbf{T}^{\vee}(1)$.

Now suppose $\mathbf{T}$ is conjugate self-dual. There are two types of duality pairing we want to consider for $\mathbf{T}$. For the ring $R = \Lambda$, the representations $\mathbf{T}$ and $\mathbf{T}^c$ satisfy condition 5.2.3(B) of [Nek06], so we get a good theory of duality for their Galois cohomology complexes viewed as Λ -modules. Let $(-)^*$ denote the Pontryagin dual, then we have $\mathbf{T}^* \simeq \mathbf{T}^{\vee} \otimes_{\Lambda} \Lambda^*$ as Λ -modules. The pair $\mathbf{T}$ and $\mathbf{T}^c \otimes_{\Lambda} \Lambda^*$ therefore satisfies condition 5.2.3(A) of [Nek06].

Finally, we define notations surrounding the characteristic ideal. Let P be a height one prime ideal of Λ . It is associated to a valuation v_P on Λ which can be computed as $v_P(x) = \text{length}_{\Lambda_P}(\Lambda_P/(x))$. The characteristic ideal of a torsion Λ -module X is defined to be

$$\text{char}_{\Lambda}(X) := \{x \in \Lambda \mid v_P(x) \geq \text{length}_{\Lambda_P}(X_P) \text{ for all } P \text{ of height } 1\}$$

It is clear that the characteristic ideal is multiplicative in exact sequences. A torsion module is pseudo-null if its characteristic ideal is the unit ideal. They form a Serre subcategory of the category of all Λ -modules. We use the notation $\mathbf{psMod}_{\Lambda}$ to denote the quotient abelian category. Though we will not use it, we remark that all torsion Λ -modules are isomorphic to a direct sum of the form $\bigoplus_i \Lambda/(f_i)$ in $\mathbf{psMod}_{\Lambda}$.

1.4 Coleman map

We want to generalize the classical Coleman logarithm $H^1(\mathbf{Q}_p, \Lambda(1)) \simeq \Lambda$ to our setting. Our result will follow from a slight modification of the constructions in [Och03]. We first recall some notations. Let $\mathbf{Q}_p^{\text{ur}}$ denote the maximal unramified extension of $\mathbf{Q}_p$ and $\Gamma_{\text{ur}} = \text{Gal}(\mathbf{Q}_p^{\text{ur}}/\mathbf{Q}_p)$. There is a canonical isomorphism $\hat{\mathbf{Z}} \rightarrow \Gamma_{\text{ur}}$ sending 1 to the geometric Frobenius σ .

Fix a surjection $\chi_{\text{ac}} : \text{Gal}_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_p^{\times}$ with Hodge–Tate weight -1 . Let $\epsilon : \mathbf{G}_m \rightarrow \mathbf{T}$ be a cocharacter, so we have a universal character $\chi_{\epsilon} : g \mapsto [\chi_{\text{ac}}(g)]_{\psi} \in \Lambda^{\times}$. This supersedes the notation used in the first section. Let $\alpha \in \mathbb{I}^{\times}$, which gives an unramified character α . Finally, let χ_{alg} be a Hodge–Tate character of $\text{Gal}_{\mathbf{Q}_p}$ valued in $\mathcal{O}^{\times}$. Let $\chi = \alpha \chi_{\epsilon} \chi_{\text{alg}} : \text{Gal}_{\mathbf{Q}_p} \rightarrow \mathbb{I}^{\times}$.

Theorem 1.4.1. *If $\psi \circ \epsilon|_{\mu_{p-1}}$ is non-trivial, then for any finitely generated p -torsion-free $\mathbb{I}$ -module M , we have an isomorphism of $\mathbb{I}$ -modules*

$$\text{Col} : H^1(\mathbf{Q}_p, M(\chi)) \simeq (M(\alpha) \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\text{ur}})^{\Gamma_{\text{ur}}}$$

Moreover, the right hand side is non-canonically isomorphic to M .

Proof. The group Γ_{ur} has p -cohomological dimension 1, so we have the following exact sequence

$$0 \rightarrow H^1(\Gamma_{\text{ur}}, M(\chi)^{\text{Gal}_{\mathbf{Q}_p^{\text{ur}}}}) \rightarrow H^1(\mathbf{Q}_p, M(\chi)) \rightarrow H^1(\mathbf{Q}_p^{\text{ur}}, M(\chi))^{\Gamma_{\text{ur}}} \rightarrow 0$$

By assumption, ϵ is non-trivial and M is p -torsion-free, so the $\text{Gal}_{\mathbf{Q}_p^{\text{ur}}}$ -invariant is trivial. Moreover, over $\mathbf{Q}_p^{\text{ur}}$, we can factor out the unramified character α . Therefore,

$$H^1(\mathbf{Q}_p, M(\chi)) \simeq (H^1(\mathbf{Q}_p^{\text{ur}}, \Lambda(\chi_{\text{alg}}\chi_{\epsilon})) \otimes_{\Lambda} M(\alpha))^{\Gamma_{\text{ur}}}$$

We now decouple the Galois action on Λ . The composite $\psi_1 := \psi \circ \epsilon|_{\mu_{p-1}}$ is a character $\mu_{p-1} \rightarrow \mathbf{Z}_p^{\times}$. Let $\Lambda_1^{\psi_1}$ be the component of $\mathbf{Z}_p[[\mathbf{Z}_p^{\times}]]$ defined by ψ_1 . It has a canonical Galois action by the universal character induced from χ_{ac} . Let $\Lambda_{r-1} \simeq \mathbf{Z}_p[[(1 + p\mathbf{Z}_p)^{r-1}]]$ with trivial Galois action. We will construct an Galois-equivariant isomorphism

$$\Lambda(\chi_{\epsilon}) \simeq \Lambda_1^{\psi_1} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{r-1}$$

As rings, both sides are completed group rings of $(1 + p\mathbf{Z}_p)^r$. Suppose first that there exists another map $\epsilon' : \mathbf{G}_m^{r-1} \rightarrow \mathbf{T}$ such that the product (ϵ, ϵ') is an isogeny of $\mathbf{G}_m^r$ with degree prime to p . This guarantees the induced map on $(1 + p\mathbf{Z}_p)^r$ is an isomorphism. It is then easy to check that this map also preserves the Galois actions. If no such ϵ' exists, then we need to consider the right hand side as the completed group ring of $(1 + p^a\mathbf{Z}_p)^r$ for some $a > 1$, but similar arguments work. This will not appear in the intended applications, so we will not go into details.

Since we are working over $\mathbf{Q}_p^{\text{ur}}$, we may replace χ_{ac} by χ_{cyc} at the cost of modifying α . It follows that $H^1(\mathbf{Q}_p^{\text{ur}}, \Lambda(\chi_{\text{alg}}\chi_{\epsilon})) \simeq H^1(\mathbf{Q}_p^{\text{ur}}, \Lambda_1^{\psi_1}(\chi_{\text{alg}})) \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{r-1}$. The group appearing on the right hand side is the ψ_1 -component of $H^1(\mathbf{Q}_p^{\text{ur}}, \underline{\Lambda}_1(\chi_{\text{alg}}))$ with its natural action through χ_{cyc} . Therefore, the Coleman map gives us the following short exact sequence

$$0 \rightarrow \mathbf{Z}_p \rightarrow H^1(\mathbf{Q}_p^{\text{ur}}, \underline{\Lambda}_1) \xrightarrow{\text{Log}} \underline{\Lambda}_1 \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\text{ur}} \rightarrow 0$$

making the following diagram commute for any classical point η

$$\begin{array}{ccc} H^1(\mathbf{Q}_p^{\text{ur}}, \underline{\Lambda}_1) & \xrightarrow{\text{Log}} & \underline{\Lambda}_1 \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p^{\text{ur}} \\ \downarrow \text{sp}_\eta & & \downarrow \text{sp}_\eta \\ H^1(\mathbf{Q}_p^{\text{ur}}, F_\eta(\eta)) & \xrightarrow{\log_\eta} & D_{\text{dR}}^{\text{ur}}(F_\eta(\eta)) \end{array}$$

We need to define $\log_\eta$. Write $\eta = \phi \chi_{\text{cyc}}^w$, where ϕ has finite order and $w \geq 1$ is the weight of η . Let p^s be the conductor of ϕ , then

$$\log_\eta(z) := \left(\frac{\sigma^{-1}}{p^{w-1}} \right)^s \left(1 - \frac{p^{w-1}\phi(p)}{\sigma^{-1}} \right) \left(1 - \frac{\sigma^{-1}\phi(p)}{p^w} \right) \frac{(-1)^{w-1}}{(w-1)!} \log_{\text{BK}}(z)$$

This form of the statement is taken from [Och03, Proposition 5.10]. The σ here represents the geometric Frobenius acting on $\mathbf{Q}_p^{\text{ur}}$, which explains the presence of inverses.

There is a $\text{Gal}_{\mathbf{Q}_p^{\text{ur}}}$ -equivariant isomorphism $\underline{\Lambda}_1(\chi_{\text{alg}}) \rightarrow \underline{\Lambda}_1$. In fact, for any $n \in \mathbf{Z}$ and $\xi \in \mu_{p^\infty}(\mathcal{O}^\times)$, consider the isomorphism $\underline{\Lambda}_1 \rightarrow \underline{\Lambda}_1$ induced from $\gamma \mapsto \xi \gamma^n$. A short calculation shows that the usual Galois action on the target gets transferred to usual action twisted by an algebraic character, and moreover all algebraic characters show up this way. Therefore, we have an isomorphism $H^1(\mathbf{Q}_p^{\text{ur}}, \underline{\Lambda}_1(\chi_{\text{alg}})) \simeq H^1(\mathbf{Q}_p^{\text{ur}}, \underline{\Lambda}_1)$. It satisfies the same commutative diagram as above, except every occurrence of η in the bottom row needs to be replaced by $\chi_{\text{alg}}\eta$.

By assumption, $\psi_1 \neq 1$, so we get an isomorphism $H^1(\mathbf{Q}_p^{\text{ur}}, \Lambda_1^{\psi_1}(\chi_{\text{alg}})) \simeq \Lambda_1^{\psi_1} \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\text{ur}}$. Going back through the chain of identifications gives us

$$H^1(\mathbf{Q}_p, M(\chi)) \simeq (M(\alpha) \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\text{ur}})^{\Gamma_{\text{ur}}}$$

Finally, the proof of [Och03, Lemma 3.3] gives an isomorphism of abelian groups between the right hand side and M . It is clear from the construction that this preserves the $\mathbb{L}$ -module structure. $\square$

Chapter 2

Selmer groups and main conjectures

This is the first main part of the thesis. In it, we will present a purely formal computation in Galois cohomology. Combined with the forthcoming work of Jetchev–Nekovář–Skinner, this shows how a rank-1 Euler system formally implies rank-0 results as well. Several concrete instances of this result have appeared in the literature.

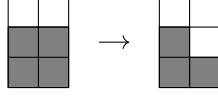
2.1 Ordinary family

Let $\mathbf{T}$ be a conjugate self-dual family of representations. We impose the following condition which is satisfied if $\mathbf{T}$ comes from an ordinary family.

There exists two $\mathrm{Gal}_{\mathcal{K}_p}$ -stable Λ -projective submodules $\mathbf{T}^{\mathbb{L}} \subseteq \mathbf{T}^{\mathbb{T}} \subseteq \mathbf{T}$ such that $\mathbf{T}^{\mathbb{T}}/\mathbf{T}^{\mathbb{L}}$ has $\mathbb{L}$ -rank 1 with Galois action given by a character of the form $\alpha\chi_{\epsilon}\chi_{\mathrm{alg}}$ for some non-trivial cocharacter ϵ and Hodge–Tate character χ_{alg} . (ord)

Using the isomorphism $\mathbf{T}^c \simeq \mathbf{T}^{\vee}(1)$, we can dualize the two submodules to obtain a sequence of $\mathrm{Gal}_{\mathcal{K}_p}$ -stable submodules $\mathbf{T}^{c,\mathbb{T}} \subseteq \mathbf{T}^{c,\mathbb{L}} \subseteq \mathbf{T}^c$, so for example, $\mathbf{T}^{c,\mathbb{T}}$ is the transfer of the annihilator of $\mathbf{T}^{\mathbb{T}}$ in $\mathbf{T}^{\vee}$. Equivalently, they are $\mathrm{Gal}_{\mathcal{K}_{\bar{p}}}$ -stable submodules of $\mathbf{T}$. The action of $\mathrm{Gal}_{\mathcal{K}_{\bar{p}}}$ on $\mathbf{T}^{c,\mathbb{L}}/\mathbf{T}^{c,\mathbb{T}}$ is given by $\alpha^{-1}\chi_{-\epsilon}\chi_{\mathrm{cyc}}\chi_{\mathrm{alg}}^{-1}$.

Remark 2.1.1. The notations used here is supposed to represent the following picture.



We think of $\mathbf{T}$ as the whole box, $\mathbf{T}^\top$ as the filled in subspace in the first diagram, and $\mathbf{T}^\perp$ as the one for the second diagram, removing a single $\mathbb{L}$ -rank-1 piece. The notations on the dual space are chosen so that the Selmer conditions $(\perp, \lrcorner)$ and $(\top, \rceil)$ we define later are self-dual.

The Coleman map gives the following result on the local cohomology of $\mathbf{T}^\top/\mathbf{T}^\perp$.

Corollary 2.1.2. *There is an isomorphism $\mathbf{R}\Gamma_{\text{cont}}(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}^\top/\mathbf{T}^\perp) \simeq (\mathbf{T}^\top/\mathbf{T}^\perp)[1]$ in $D^b(\mathbf{psMod}_\Lambda)$.*

Proof. Given Theorem 1.4.1, the claim follows from the statement that $H^i(\mathcal{K}_{\mathfrak{p}}, M)$ is pseudo-null if $i = 0, 2$, where M is an $\mathbb{L}$ -module with $\text{Gal}_{\mathcal{K}_{\mathfrak{p}}}$ acting through a character of the type specified. The case of $i = 0$ holds since $\epsilon \neq 1$ and $\mathbf{T}$ is Λ -projective. By the same reason, it holds for $M^\vee(1)$. We prove the $i = 2$ part for M and $M^\vee(1)$ at the same time.

By local duality in the form of [Nek06, Theorem 5.2.6], we have an isomorphism $\mathbf{R}\Gamma_{\text{cont}}(\mathcal{K}_{\mathfrak{p}}, M^\vee(1)) \simeq \mathbf{R}\text{Hom}(\mathbf{R}\Gamma_{\text{cont}}(\mathcal{K}_{\mathfrak{p}}, M), \Lambda)[2]$. After applying the known values for degrees 0 and 1, the spectral sequence computing the right hand side degenerates to

$$\text{Ext}_\Lambda^i(H^2(\mathcal{K}_{\mathfrak{p}}, M), \Lambda) = \begin{cases} H^2(\mathcal{K}_{\mathfrak{p}}, M^\vee(1)) & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

and similarly with the roles of M and $M^\vee(1)$ exchanged. When localized at a height one prime, the left hand side vanishes for $i \geq 2$ since Λ is regular. This proves the required conclusion. $\square$

2.2 Selmer groups and complexes

In this section, we recall various general facts about Selmer groups and Selmer complexes, in particular setting up notations for later use. Fix a finite set S of finite places of $\mathcal{K}$ such that $\mathbf{T}$ is unramified away from S and ∞ . Suppose also S contains $\mathfrak{p}, \bar{\mathfrak{p}}$.

Definition 2.2.1. Let $\mathcal{F}$ be a Selmer structure for $\mathbf{T}$, namely a choice of a subspace $\mathcal{F}_v \subseteq H^1(\mathcal{K}_v, \mathbf{T})$ for each $v \in S$. The compact Selmer group is

$$H_{\mathcal{F}}^1(\mathcal{K}, \mathbf{T}) := \ker \left(H^1(\mathcal{K}, \mathbf{T}) \rightarrow \bigoplus_v H^1(\mathcal{K}_v, \mathbf{T})/\mathcal{F}_v \right)$$

Let $\mathcal{F}'_v$ be the image of $\mathcal{F}_v$ under the map $H^1(\mathcal{K}_v, \mathbf{T}) \rightarrow H^1(\mathcal{K}_v, \mathbf{T} \otimes_{\Lambda} \Lambda^*)$. The Greenberg Selmer group is

$$\mathrm{Sel}_{\mathcal{F}}(\mathcal{K}, \mathbf{T}) := \ker \left(H^1(\mathcal{K}, \mathbf{T} \otimes_{\Lambda} \Lambda^*) \rightarrow \bigoplus_v H^1(\mathcal{K}_v, \mathbf{T} \otimes_{\Lambda} \Lambda^*) / \mathcal{F}'_v \right)$$

Its Pontryagin dual will be denoted by $X_{\mathcal{F}}(\mathcal{K}, \mathbf{T})$.

We will use $\spadesuit \in \{\sqcup, \sqcap\}$ to denote the (strict) Greenberg-type Selmer condition

$$\mathcal{F}_{\spadesuit}^{\mathfrak{p}} = \ker(H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}) \rightarrow H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}/\mathbf{T}^{\spadesuit}))$$

at $\mathfrak{p}$. The same symbols $\{\sqcap, \sqcup\}$ will also be used to denote the dual Selmer conditions at $\bar{\mathfrak{p}}$. Unless otherwise stated, the Selmer condition at a place away from p is the unramified subspace $H^1(\mathbf{F}_v, \mathbf{T}^{I_v})$, viewed as a subspace of $H^1(\mathcal{K}_v, \mathbf{T})$ by inflation. Therefore, we will use variations of the notation $H_{\clubsuit, \clubsuit}^1$ to denote $\spadesuit$ -condition at $\mathfrak{p}$, $\clubsuit$ -condition at $\bar{\mathfrak{p}}$, and the unramified condition everywhere else.

Definition 2.2.2. A local condition Δ_v is a map of complexes $U_v^+ \rightarrow C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_v}, \mathbf{T})$. Given a collection $\Delta = (\Delta_v)_{v \in S}$, the associated Selmer complex is

$$\widetilde{\mathbf{R}\Gamma}_{\Delta}(\mathcal{K}, \mathbf{T}) := \mathrm{Cone} \left(C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}, S}, \mathbf{T}) \oplus \bigoplus_{v \in S} U_v^+ \xrightarrow{\mathrm{res}_v - \Delta_v} \bigoplus_{v \in S} C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_v}, \mathbf{T}) \right) [-1]$$

Its cohomology groups will be denoted by $\widetilde{H}_{\Delta}^{\bullet}(\mathcal{K}, \mathbf{T})$.

Similarly as before, for $\spadesuit \in \{\sqcup, \sqcap\}$, we can define the local condition

$$\Delta_{\spadesuit}^{\mathfrak{p}} : C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_{\mathfrak{p}}}, \mathbf{T}^{\spadesuit}) \rightarrow C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_{\mathfrak{p}}}, \mathbf{T})$$

at $\mathfrak{p}$, and for $\clubsuit \in \{\sqcap, \sqcup\}$ at $\bar{\mathfrak{p}}$, we can define the condition $\Delta_{\clubsuit}^{\bar{\mathfrak{p}}}$ at $\bar{\mathfrak{p}}$. Away from p , we use the unramified local condition $C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_v}/I_v, \mathbf{T}^{I_v}) \rightarrow C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_v}, \mathbf{T})$.

By construction, the subspaces $\mathbf{T}^{\sqcup}$ and $\mathbf{T}^{c, \sqcup}$ are mutual annihilators. It follows from [Nek06, Proposition 6.7.6] that $\Delta_{\sqcup}^{\mathfrak{p}}$ and $\Delta_{\sqcup}^{\bar{\mathfrak{p}}}$ are orthogonal complements at the complex level. The same discussion holds with all $\sqcup$ replaced by $\sqcap$.

One version of global duality can be stated cleanly in this language. We have a Pontryagin duality pairing $\mathbf{T} \otimes_{\Lambda} (\mathbf{T}^c \otimes_{\Lambda} \Lambda^*) \rightarrow \Lambda^*(1)$. Subsection 7.8.4.3 of [Nek06] gives an isomorphism

$$\widetilde{\mathbf{R}\Gamma}_{\spadesuit, \clubsuit}(\mathcal{K}, \mathbf{T}) \simeq \mathrm{Hom}_{\mathrm{cont}}(\widetilde{\mathbf{R}\Gamma}_{\clubsuit, \spadesuit}(\mathcal{K}, \mathbf{T} \otimes_{\Lambda} \Lambda^*), \mathbf{Q}_p/\mathbf{Z}_p)[-3] \quad (*)$$

However, we will also need a version of duality arising from the pairing $\mathbf{T} \otimes_{\Lambda} \mathbf{T}^c \rightarrow \Lambda(1)$. This runs

into subtleties at ramified places away from p , since the unramified local conditions are no longer in perfect pairing, and we encounter error terms which contribute to Tamagawa numbers. More precisely, [Nek06, Subsection 7.8.4.4] gives an exact triangle

$$\widetilde{\mathbf{R}\Gamma}_{\clubsuit, \clubsuit}(\mathcal{K}, T) \rightarrow \mathbf{R}\mathrm{Hom}_{\Lambda}(\widetilde{\mathbf{R}\Gamma}_{\clubsuit, \spadesuit}(\mathcal{K}, T), \Lambda)[-3] \rightarrow \bigoplus_{v \in S - \{\mathfrak{p}, \bar{\mathfrak{p}}\}} \mathrm{Err}_v \rightarrow [1] \quad (\vee)$$

The error term above is the definition in [Nek06, Section 6.2.3] applied to the unramified local conditions. The necessary properties for our purposes are stated in the following proposition.

Proposition 2.2.3. *Let P be a height one prime ideal of Λ , then*

$$\mathrm{length}_{\Lambda_P}(H^i(\mathrm{Err}_v)_P) = \begin{cases} \mathrm{Tam}_v(T, P) & i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

where the Tamagawa number is defined by

$$\mathrm{Tam}_v(T, P) = \mathrm{length}_{\Lambda_P}(H^1(I_v, T_P)^{\mathrm{Frob}_v=1}_{\mathrm{tors}})$$

Proof. This follows from [Nek06, Subsection 7.6.10.7] since Λ is regular. The notation $\mathrm{Err}_v^{\mathrm{ur}}(\mathcal{D}, T)$ used there is exactly our Err_v . $\square$

In particular, the error terms are 0 when localized at a minimal prime, so it does not enter into rank calculations. In particular, the next proposition, usually credited to Greenberg or Wiles, follows easily from global duality and the Euler–Poincaré characteristic formula. We include the proof for completeness.

Proposition 2.2.4. *Let R be a Noetherian regular local domain. Let V be a projective R -module with an admissible action of $\mathrm{Gal}_{\mathcal{K}}$, and let $V^{\vee}(1) = \mathrm{Hom}_R(V, R(1))$. Suppose $V^{\mathrm{Gal}_{\mathcal{K}}} = (V^{\vee}(1))^{\mathrm{Gal}_{\mathcal{K}}} = 0$.*

For each $v \in S$, choose a $\mathrm{Gal}_{\mathcal{K}_v}$ -stable subspace $V_v \subseteq V$ and define the local condition Δ_v to be the Greenberg condition corresponding to it. Let $\Delta^{\vee}$ be the local condition for $V^{\vee}(1)$ which is the orthogonal complement to Δ , then

$$\mathrm{rank}_R \widetilde{H}_{\Delta}^1(\mathrm{Gal}_{\mathcal{K}, S}, V) - \mathrm{rank}_R \widetilde{H}_{\Delta^{\vee}}^1(\mathrm{Gal}_{\mathcal{K}, S}, V^{\vee}(1)) = -\mathrm{rank}_R V + \sum_{v \in S} [\mathcal{K}_v : \mathbf{Q}_v] \mathrm{rank}_R V_v$$

Proof. For each $v \in S$, we have $U_v^+ = C_{\mathrm{cont}}^{\bullet}(\mathrm{Gal}_{\mathcal{K}_v}, V_v)$. Let $\mathbf{str}$ denote the strict local condition where $U_v^+ = 0$ for each $v \in S$, then $\widetilde{\mathbf{R}\Gamma}_{\mathbf{str}}$ computes the compactly supported Galois cohomology in

the sense of [Nek06, Definition 5.3.1.1], and we have the following exact triangle

$$\widetilde{\mathbf{R}\Gamma}_{\text{str}} \rightarrow \widetilde{\mathbf{R}\Gamma}_{\Delta} \rightarrow \bigoplus_{v \in S} U_v^+ \rightarrow [1]$$

If $\chi(-)$ denotes the Euler–Poincaré characteristics, then we get

$$\chi(\widetilde{\mathbf{R}\Gamma}_{\Delta}) = \chi(\widetilde{\mathbf{R}\Gamma}_{\text{str}}) + \sum_{v \in S} \chi(U_v^+) = \text{rank}_R V - \sum_{v \in S} [\mathcal{K}_v : \mathbf{Q}_v] \text{rank}_R V_v$$

where we have applied Theorems 5.3.6 and 5.2.11 of [Nek06] to each of the two terms. Moreover, equation (v) and the remark after it shows that $\text{rank}_R \tilde{H}_{\Delta}^i = \text{rank}_R \tilde{H}_{\Delta^{\vee}}^{3-i}$. In particular,

$$\begin{aligned} \text{rank}_R \tilde{H}_{\Delta}^1 - \text{rank}_R \tilde{H}_{\Delta^{\vee}}^1 &= \text{rank}_R \tilde{H}_{\Delta}^1 - \text{rank}_R \tilde{H}_{\Delta}^2 \\ &= -\chi(\widetilde{\mathbf{R}\Gamma}_{\Delta}) + \text{rank}_R \tilde{H}_{\Delta}^0 \end{aligned}$$

Finally, observe that $\tilde{H}_{\Delta}^0(\text{Gal}_{\mathcal{K},S}, V) \subseteq V^{\text{Gal}_{\mathcal{K}}} = 0$, so the two error terms are 0. $\square$

In what follows, we will typically drop the $(\mathcal{K}, \mathbf{T})$ from the notation when no confusion arises this way. We now study some basic properties of Selmer groups and Selmer complexes. In our applications, we will typically have the following mild hypothesis

$$\mathbf{T}^{\text{Gal}_{\mathcal{K}}} = 0, \text{ and } \mathbf{T}_{\text{Gal}_{\mathcal{K}}} \text{ is pseudo-null} \quad (\text{no-pole})$$

Both statements will usually hold because the Galois action is sufficiently non-trivial. Assuming this, it is immediately seen that $\tilde{H}_{\clubsuit, \clubsuit}^0 = 0$, and $(*)$ then implies $\tilde{H}_{\clubsuit, \clubsuit}^3$ is pseudo-null. They are also related to torsions in $\tilde{H}^1$, as shown in the next proposition.

Proposition 2.2.5. *If (no-pole) holds, then the torsion subgroup of $\tilde{H}_{\clubsuit, \clubsuit}^1(\mathcal{K}, \mathbf{T})$ is pseudo-null.*

Proof. In equation (v), let A^1 be the first cohomology of the middle term, then a spectral sequence argument plus the fact that $\text{Hom}_{\Lambda}(-, \Lambda)$ is always torsion-free shows that $(A^1)_{\text{tors}} = \text{Ext}_{\Lambda}^1(\tilde{H}_{\clubsuit, \clubsuit}^3(\mathcal{K}, \mathbf{T}), \Lambda)$. By the discussion before this proposition, the first term inside Ext_{Λ}^1 is pseudo-null, so it has codimension at least 2. Recall from homological algebra that $\text{Ext}_{\Lambda}^i(M, \Lambda) = 0$ if $i < \text{codim}(M)$ [Nek06, Theorem 9.1.3(iii)]. Therefore, $(A^1)_{\text{tors}} = 0$. Finally, the long exact sequence attached to (v) has the form

$$A^0 \rightarrow \text{Err}^0 \rightarrow \tilde{H}_{\clubsuit, \clubsuit}^1(\mathcal{K}, \mathbf{T}) \rightarrow A^1$$

By Proposition 2.2.3, Err^0 is pseudo-null, and the desired result follows. $\square$

For each v , let $U_v^- = \text{Cone}(-\Delta_v)$, then by definition, we have an exact triangle

$$\widetilde{\mathbf{R}\Gamma}_{\clubsuit, \clubsuit} \rightarrow \mathbf{R}\Gamma(\text{Gal}_{\mathcal{K}, S}, \mathbf{T}) \rightarrow \bigoplus_{v \in S} U_v^- \rightarrow [1]$$

If Δ_v is the unramified local condition, then U_v^- is concentrated in degrees $[1, 2]$. If Δ_v is the Greenberg local condition defined by $\mathbf{T}^?$, then $U_v^- = C_{\text{cont}}^\bullet(\text{Gal}_{\mathcal{K}_v}, \mathbf{T}/\mathbf{T}^?)$, so it is concentrated in degrees $[0, 2]$. Taking cohomology gives the following exact sequences: for $\widetilde{H}^0$

$$0 \rightarrow \widetilde{H}_{\clubsuit, \clubsuit}^0 \rightarrow \mathbf{T}^{\text{Gal}_{\mathcal{K}}} \rightarrow (\mathbf{T}/\mathbf{T}^\spadesuit)^{\text{Gal}_{\mathcal{K}_p}} \oplus (\mathbf{T}/\mathbf{T}^\clubsuit)^{\text{Gal}_{\mathcal{K}_{\bar{p}}}}$$

and for $\widetilde{H}^1$

$$(\mathbf{T}/\mathbf{T}^\spadesuit)^{\text{Gal}_{\mathcal{K}_p}} \oplus (\mathbf{T}/\mathbf{T}^\clubsuit)^{\text{Gal}_{\mathcal{K}_{\bar{p}}}} \rightarrow \widetilde{H}_{\clubsuit, \clubsuit}^1 \rightarrow H_{\clubsuit, \clubsuit}^1 \rightarrow 0$$

To study $\widetilde{H}^2$, we use global duality in the form of equation $(*)$ to obtain the following exact sequence

$$0 \rightarrow X_{\clubsuit, \clubsuit} \rightarrow \widetilde{H}_{\clubsuit, \clubsuit}^2 \rightarrow H^2(\mathcal{K}_p, \mathbf{T}^\spadesuit) \oplus H^2(\mathcal{K}_{\bar{p}}, \mathbf{T}^\clubsuit)$$

The error terms with local invariants and co-invariants are related to trivial zeroes, cf. the computation in [Nek06, Section 0.10]. From an algebraic point of view, this is an error term which shows up in the control theorem. We will often impose the following hypothesis to rule them out at the family level.

$$(\mathbf{T}/\mathbf{T}^\spadesuit)^{\text{Gal}_{\mathcal{K}_p}} = 0, \spadesuit \in \{\lrcorner, \intercal\} \quad (\text{no-triv-zero})$$

Assuming it and (no-pole), we have $\widetilde{H}_{\clubsuit, \clubsuit}^1 = H_{\clubsuit, \clubsuit}^1$, and $X_{\clubsuit, \clubsuit} = \widetilde{H}_{\clubsuit, \clubsuit}^2$ up to a pseudo-null quotient. In the later applications in Chapter 5, the module $\mathbf{T}$ will come with a bi-filtration whose graded pieces are projective Λ -modules with non-trivial Galois action, which implies (no-triv-zero).

2.3 Equivalence of main conjectures

In classical Iwasawa theory, we want to study the compact $\mathbb{L}$ -modules $X_{\intercal, \intercal}$ and $X_{\lrcorner, \lrcorner}$ since they interpolate Bloch–Kato Selmer groups if $\mathbf{T}^\lrcorner$ and $\mathbf{T}^\intercal$ are chosen appropriately. They are singled out in parts because their local conditions are self-dual. We might expect the central values of the associated L -functions to not vanish generically in the family we are considering, in which case a

module like X would be torsion, and we have an Iwasawa main conjecture equating its characteristic ideal to a p -adic L -function, cf. [Gre94].

In the case which inspired the current axiomatization, there is a further feature, namely the L -functions are self-dual, and the sign of the function equation is constant in family. In the sign -1 case, the central values vanish automatically, and we instead expect X to have rank 1. This is the case Perrin-Riou considered in [PR87], leading to her Heegner point main conjecture.

In our setting, $X_{\mathbb{L},\mathbb{L}}$ and $X_{\mathbb{T},\mathbb{T}}$ will correspond to families with opposite signs. Our main result of this section is that a good global class can be used to relate the Iwasawa main conjecture and the Perrin-Riou main conjecture. To make this precise, we need to introduce the regulator map.

Definition 2.3.1. The regulator map is the following composite

$$\text{reg} : \tilde{H}_{\mathbb{T},\mathbb{T}}^1 \rightarrow H^1(\mathcal{K}_{\mathbb{P}}, \mathbf{T}^{\mathbb{T}}) \rightarrow H^1(\mathcal{K}_{\mathbb{P}}, \mathbf{T}^{\mathbb{T}}/\mathbf{T}^{\mathbb{L}}) \xrightarrow{\text{Col}} \mathbf{M}$$

where $\mathbf{M}$ is abstractly isomorphic to $\mathbf{T}^{\mathbb{T}}/\mathbf{T}^{\mathbb{L}}$, so by (ord), it is a rank-1 $\mathbb{L}$ -module which is projective over Λ .

Theorem 2.3.2. *Let $\mathbf{T}$ be an ordinary conjugate self-dual family of Galois representations. Suppose there exists an element $\mathbf{z} \in \tilde{H}_{\mathbb{T},\mathbb{T}}^1$ such that*

1. $\tilde{H}_{\mathbb{T},\mathbb{T}}^1/\mathbb{L} \cdot \mathbf{z}$ is a torsion $\mathbb{L}$ -module.
2. $\text{reg}(\mathbf{z})$ is non-torsion.

then $\text{rank}_{\mathbb{L}} \tilde{H}_{\mathbb{T},\mathbb{T}}^2 = 1$, $\text{rank}_{\mathbb{L}} \tilde{H}_{\mathbb{L},\mathbb{L}}^2 = 0$. Moreover, if (no-pole) holds, then the following equivalence holds

$$\text{char}_{\Lambda} \tilde{H}_{\mathbb{L},\mathbb{L}}^2 = \text{char}_{\Lambda}(\text{reg}(\mathbf{z}))^2 \iff \text{char}_{\Lambda}(\tilde{H}_{\mathbb{T},\mathbb{T}}^2)_{\text{tors}} = \text{char}_{\Lambda} \left(\frac{\tilde{H}_{\mathbb{T},\mathbb{T}}^1}{\mathbb{L} \cdot \mathbf{z}} \right)^2$$

In fact, the equalities can be replaced by either divisibility in the above equivalence.

In particular, if in addition (no-triv-zero) holds, then we can replace $\tilde{H}^1$ by H^1 and $\tilde{H}^2$ by X in the above statements to get an equivalence between two types of Iwasawa main conjectures.

Remark 2.3.3. Before proving the theorem, we will comment on how the hypotheses can be verified in practice. If $\mathbf{z}$ can be extended to an Euler system, then the forthcoming work of Jetchev–Nekovář–Skinner shows that condition (1) follows from (2) under some further technical hypotheses. We expect that any construction of a big Selmer class in $\tilde{H}^1$ could be extended to an Euler system by modifying the tame level appropriately. Certainly for the diagonal cycles we construct, the spherical

variety technique which first appeared in [LSZ22] and systematically developed in [Loe21] should apply. Progresses towards this have been made in Boumasmoud's thesis [Bou19].

Part (2) appears to be considerably deeper. For the theorem to be useful in practice, we typically want $\text{reg}(\mathbf{z})$ to be a p -adic L -function in the interpolation sense. Such results is known as an explicit reciprocity law, and one can prove the non-torsion property by showing the interpolated classical L -values are non-zero sufficiently often. In the cyclotomic case, this is typically done by analytic methods, though at the time of writing, there are already difficulties for central values of degree 4 L -functions. In the anticyclotomic case that we consider, these kinds of results have been obtained by considering the cycles directly using deep results related to homogeneous dynamics. The explicit reciprocity law then implies results about generic non-vanishing of classical L -values which appear to be out of reach with analytic methods. We found Vatsal's survey [Vat06] particularly helpful regarding this case.

Proof. For $\spadesuit \in \{\llcorner, \lrcorner\}$, the Selmer complex $\widetilde{\mathbf{R}\Gamma}_{\spadesuit, \spadesuit}$ is self-dual, so we have $\text{rank}_{\mathbb{I}} \widetilde{H}_{\spadesuit, \spadesuit}^1 = \text{rank}_{\mathbb{I}} \widetilde{H}_{\spadesuit, \spadesuit}^2$. Our hypothesis therefore implies $\text{rank}_{\mathbb{I}} \widetilde{H}_{\lrcorner, \lrcorner}^2 = 1$. We have the exact triangle

$$\widetilde{\mathbf{R}\Gamma}_{\llcorner, \lrcorner} \rightarrow \widetilde{\mathbf{R}\Gamma}_{\lrcorner, \lrcorner} \rightarrow U_{\mathfrak{p}} \rightarrow [1]$$

where $U_{\mathfrak{p}} := C_{\text{cont}}^{\bullet}(\text{Gal}_{\mathcal{K}_{\mathfrak{p}}}, \mathbf{T}^{\lrcorner}/\mathbf{T}^{\llcorner})$. By Corollary 2.1.2, $U_{\mathfrak{p}} \simeq \mathcal{M}[1]$ in $D^b(\mathbf{psMod}_{\Lambda})$. The associated long exact sequence contains the following piece

$$(\mathbf{T}^{\lrcorner}/\mathbf{T}^{\llcorner})^{\text{Gal}_{\mathcal{K}_{\mathfrak{p}}}} \rightarrow \widetilde{H}_{\llcorner, \lrcorner}^1 \rightarrow \widetilde{H}_{\lrcorner, \lrcorner}^1 \rightarrow H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}^{\lrcorner}/\mathbf{T}^{\llcorner})$$

By the assumption on the Galois action, the first term is trivial, and the Coleman map gives an isomorphism of the final term with a rank one $\mathbb{I}$ -module. Therefore, $\text{rank}_{\mathbb{I}} \widetilde{H}_{\lrcorner, \lrcorner}^1 \leq \text{rank}_{\mathbb{I}} \widetilde{H}_{\llcorner, \lrcorner}^1 + 1$. Moreover, assumption (2) implies the final morphism is surjective when tensored with $\text{Frac}(\mathbb{I})$, so equality holds and $\text{rank}_{\mathbb{I}} \widetilde{H}_{\llcorner, \lrcorner}^1 = 0$.

On the other hand, we can apply Proposition 2.2.4 to get

$$\text{rank}_{\Lambda} \widetilde{H}_{\lrcorner, \llcorner}^1 - \text{rank}_{\Lambda} \widetilde{H}_{\llcorner, \lrcorner}^1 = \text{rank}_{\Lambda} \mathbf{T}^{\lrcorner} + \text{rank}_{\Lambda} \mathbf{T}^{c, \llcorner} - \text{rank}_{\Lambda} \mathbf{T}$$

We can further convert all rank_{Λ} to $\text{rank}_{\mathbb{I}}$ since $\Lambda \rightarrow \mathbb{I}$ is finite. By assumption, $\mathbf{T}^{\llcorner}$ and $\mathbf{T}^{c, \llcorner}$ are mutual annihilators, so $\text{rank}_{\mathbb{I}} \mathbf{T}^{\llcorner} + \text{rank}_{\mathbb{I}} \mathbf{T}^{c, \llcorner} = \text{rank}_{\mathbb{I}} \mathbf{T}$. Again by ordinarity, $\text{rank}_{\mathbb{I}} \mathbf{T}^{\lrcorner} - \text{rank}_{\mathbb{I}} \mathbf{T}^{\llcorner} =$

1. Therefore,

$$\text{rank}_{\mathbb{I}} \tilde{H}_{\gamma, \mathbb{L}}^1 - \text{rank}_{\mathbb{I}} \tilde{H}_{\mathbb{L}, \gamma}^1 = 1$$

It follows that $\text{rank}_{\mathbb{I}} \tilde{H}_{\gamma, \mathbb{L}}^1 = 1$. By applying the argument in the previous paragraph to the exact triangle $\widetilde{\mathbf{R}\Gamma}_{\mathbb{L}, \mathbb{L}} \rightarrow \widetilde{\mathbf{R}\Gamma}_{\gamma, \mathbb{L}} \rightarrow U_{\mathfrak{p}} \rightarrow [1]$, it remains to show that $\tilde{H}_{\gamma, \mathbb{L}}^1 \rightarrow H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}^{\gamma}/\mathbf{T}^{\mathbb{L}})$ has non-torsion image. This again follows from assumption (2) since the regulator map factors through $\tilde{H}_{\gamma, \mathbb{L}}^1$.

We now verify the equivalence in the second part of the statement. First observe that by the rank calculation above and Proposition 2.2.5, we must have $\tilde{H}_{\mathbb{L}, \gamma}^1$ is pseudo-null. Let $C = \text{coker}(\text{reg})$. It is torsion since $\mathbf{M}$ has $\mathbb{I}$ -rank 1 and $\text{Im}(\text{reg})$ contains a non-torsion element by part (2) of the hypothesis. The long exact sequence attached to the exact triangle $\widetilde{\mathbf{R}\Gamma}_{\mathbb{L}, \gamma} \rightarrow \widetilde{\mathbf{R}\Gamma}_{\gamma, \gamma} \rightarrow U_{\mathfrak{p}} \rightarrow [1]$ gives

$$\tilde{H}_{\mathbb{L}, \gamma}^1 \rightarrow \tilde{H}_{\gamma, \gamma}^1 \rightarrow \mathbf{M} \rightarrow \tilde{H}_{\mathbb{L}, \gamma}^2 \rightarrow \tilde{H}_{\gamma, ad}^2 \rightarrow H^2(U_{\mathfrak{p}})$$

The two end terms are pseudo-null. In the middle, the map $\tilde{H}_{\gamma, \gamma}^1 \rightarrow \mathbf{M}$ is exactly the regulator map by definition. Therefore, we can split it into the following two short exact sequences:

$$\begin{aligned} (\text{pseudo-null}) \rightarrow \tilde{H}_{\gamma, \gamma}^1 \rightarrow \mathbf{M} \rightarrow C \rightarrow 0 \\ 0 \rightarrow C \rightarrow \tilde{H}_{\mathbb{L}, \gamma}^2 \rightarrow \tilde{H}_{\gamma, \gamma}^2 \rightarrow (\text{pseudo-null}) \end{aligned}$$

In the first sequence, after quotienting out by $\mathbb{I} \cdot \mathbf{z}$, we get the relation

$$\text{char}_{\Lambda}(C) \text{char}_{\Lambda} \frac{\tilde{H}_{\gamma, \gamma}^1}{\mathbb{I} \cdot \mathbf{z}} = \text{char}_{\Lambda}(\text{reg}(\mathbf{z}))$$

In the second sequence, since C is torsion, we can take the torsion subgroup without affecting the pseudo-null cokernel, so

$$\text{char}_{\Lambda}(\tilde{H}_{\mathbb{L}, \gamma}^2)_{\text{tors}} = \text{char}_{\Lambda} C \text{char}_{\Lambda}(\tilde{H}_{\gamma, \gamma}^2)_{\text{tors}}$$

Again by Proposition 2.2.5, the module C is also the cokernel of localization from $\tilde{H}_{\gamma, \mathbb{L}}^1$, so by the same argument as before, we have a third exact sequence

$$0 \rightarrow C \rightarrow \tilde{H}_{\mathbb{L}, \mathbb{L}}^2 \rightarrow \tilde{H}_{\gamma, \mathbb{L}}^2 \rightarrow (\text{pseudo-null})$$

This gives another equality of characteristic ideals. Combining it with the previous two, we see that

it remains to show that $\text{char}_\Lambda (\tilde{H}_{\ulcorner, \urcorner}^2)_{\text{tors}} = \text{char}_\Lambda \tilde{H}_{\urcorner, \ulcorner}^2$.

This is now a consequence of global duality. We work one prime at a time. Let P be a height one prime in Λ , and let $R = \Lambda_P$. It is a discrete valuation ring, and we need to prove $(\tilde{H}_{\ulcorner, \urcorner}^2)_{\text{tors}}$ and $\tilde{H}_{\urcorner, \ulcorner}^2$ have the same lengths after localizing at P . Since localization is exact, equation (V) still holds with Λ replaced by R everywhere. We apply it to get the following exact triangle

$$\widetilde{\mathbf{R}\Gamma}_{\ulcorner, \urcorner} \rightarrow \mathbf{R}\text{Hom}_R(\widetilde{\mathbf{R}\Gamma}_{\urcorner, \ulcorner}, R)[3] \rightarrow \text{Err} \rightarrow [1]$$

Let $A^\bullet$ denote the cohomology of the middle term. It can be computed by the spectral sequence

$$\text{Ext}_R^i(\tilde{H}_{\urcorner, \ulcorner}^j, R) \Rightarrow A^{i-j+3}$$

Since R is a DVR, the Ext-groups vanish for $i \geq 2$, and $\text{Ext}^1(-, R)$ computes the torsion module. Moreover, we have computed that $\text{rank}_R \tilde{H}_{\urcorner, \ulcorner}^j = 1$ if $j = 1$ and is 0 otherwise. The assumption (no-pole) implies $\tilde{H}_{\urcorner, \ulcorner}^0 = 0$, and by global duality (*) also implies $\tilde{H}_{\urcorner, \ulcorner}^3 = 0$. Finally, Proposition 2.2.5 implies that $\tilde{H}_{\urcorner, \ulcorner}^1$ is torsion-free, so it is abstractly isomorphic to R . The spectral sequence therefore degenerates on page 2 and gives

$$A^i = \begin{cases} R \oplus \tilde{H}_{\urcorner, \ulcorner}^2 & i = 2 \\ 0 & i \neq 2 \end{cases}$$

Plugging this into the long exact sequence and splitting it gives the following two short exact sequences

$$\begin{aligned} 0 \rightarrow \text{Err}^1 \rightarrow \tilde{H}_{\ulcorner, \urcorner}^2 \rightarrow N \rightarrow 0 \\ 0 \rightarrow N \rightarrow R \oplus \tilde{H}_{\urcorner, \ulcorner}^2 \rightarrow \text{Err}^2 \rightarrow 0 \end{aligned}$$

By Proposition 2.2.3, the two error terms are torsion and have the same length, so $N \simeq R \oplus ((\tilde{H}_{\ulcorner, \urcorner}^2)_{\text{tors}}/\text{Err}^1)$. But from the second sequence, $N \simeq R \oplus \ker(\tilde{H}_{\urcorner, \ulcorner}^2 \rightarrow \text{Err}^2)$. Taking the length of N_{tors} gives the desired equality. $\square$

Remark 2.3.4. 1. We can interchange $\ulcorner$ with $\urcorner$ throughout the entire theorem. This requires localizing at $\bar{\mathfrak{p}}$ instead of $\mathfrak{p}$, and the proof goes through with this change.

2. If we know that $\mathbb{I}$ is Gorenstein and $\mathbf{T}$ is projective over $\mathbb{I}$, then we can do the above computations over $\mathbb{I}$ instead of Λ . This has the effect of taking all characteristic ideals over $\mathbb{I}$, which

gives a refinement of the current statements.

3. We could replace the base ring Λ by any of its localizations without changing any argument, producing results over the localization. This is useful when the constructed lattice $\mathbf{T}$ has defects. For example, in Section 4.2, we are only able to produce a perfect duality between $\mathbf{T}$ and $\mathbf{T}^\vee(1)$ after inverting p . It should be possible in this case to carefully work through the arguments and identify exact error terms, though we do not pursue that here.
4. We do not expect an implication of the form $\text{rank}_\Lambda \tilde{H}_{\mathbb{L},\mathbb{L}}^2 = 0 \implies \text{rank}_\Lambda \tilde{H}_{\mathbb{L},\mathbb{L}}^2 = 1$ in this generality. Running through the argument would give a local class in $\tilde{H}_{\mathbb{L}}^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T})$, but it is difficult to show that this comes from a global class.

2.4 Applications

In this subsection, we present two applications of the above formalism. The first will recover one main theorem of [CH18] without the rank 0 Kolyvagin system argument. The second allows one to prove the rank 0 result in Kim Tuan Do's thesis directly without appealing to [KLZ17]. This section also hints at the wall crossing phenomena we will study in the more general Gan–Gross–Prasad setting.

2.4.1 Set-up

Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a modular form which is ordinary at p . It has an attached p -adic Galois representation $T_f : \text{Gal}_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is a finite extension of $\mathbf{Z}_p$. We assume for simplicity that T_f is residually irreducible. Let D_p be a decomposition group of $\text{Gal}_{\mathbf{Q}}$ at p , then T_f has the following D_p -stable filtration

$$0 \rightarrow T_f^- \rightarrow T_f \rightarrow T_f^+ \rightarrow 0$$

where D_p acts on T_f^- as $\alpha_p \chi_{\text{cyc}}$ and on T_f^+ as β_p , for some units $\alpha_p, \beta_p \in \mathcal{O}^\times$, identified as the unramified character sending Frobenius to that value. Both T_f^- and T_f^+ are 1-dimensional, and there is a perfect pairing on T_f exchanging T_f^- and T_f^+ .

In both applications, we will study the Bloch–Kato conjecture for anticyclotomic twists of f . We now set up the relevant notations. Fix an anticyclotomic character χ of our imaginary quadratic field $\mathcal{K}$. For simplicity, we assume the numbers N , $\text{disc}(\mathcal{K})$, $\text{cond}(\chi)$, p are pairwise coprime. Let $\mathcal{K}^{\text{ac}}$ be the anticyclotomic $\mathbf{Z}_p$ -extension of $\mathcal{K}$, $\Gamma^{\text{ac}} = \text{Gal}(\mathcal{K}^{\text{ac}}/\mathcal{K}) \simeq \mathbf{Z}_p$, $\Lambda^{\text{ac}} = \mathcal{O}[[\Gamma^{\text{ac}}]]$, and $\Psi : \text{Gal}_{\mathcal{K}} \rightarrow \Lambda^{\text{ac}}$

be the canonical character. Our p -adic family of Galois representation will be

$$\mathbf{T} = T_f \otimes_{\mathcal{O}} \Lambda^{\text{ac}}(\chi\Psi)$$

with $\text{Gal}_{\mathcal{K}}$ acting diagonally. This certainly satisfies (no-pole) by residual irreducibility. For the Selmer condition defined by either $\mathbf{T}$ or $T_f^- \otimes_{\mathcal{O}} \Lambda^{\text{ac}}(\chi\Psi)$, the condition (no-triv-zero) is easily checked. Moreover, the pairing on T_f induces a perfect pairing on $\mathbf{T}$.

2.4.2 Heegner points

The theory in this setting was very clearly set up in [BCK21], and this was the model on which we built our abstract framework. Suppose the generalized Heegner condition holds, namely the number of primes $\ell|N$ which is inert in $\mathcal{K}$ is even. Let $\mathbf{T}^{\mathbb{L}} = 0$ and $\mathbf{T}^{\top} = T_f^- \otimes_{\mathbf{Z}_p} \Lambda^{\text{ac}}(\chi\Psi)$. This satisfies (ord) since χ is unramified at p . The dual spaces are $\mathbf{T}^{c,\top} = \mathbf{T}^{\top}$ and $\mathbf{T}^{c,\mathbb{L}} = \mathbf{T}$.

The special class here is the Howard's big Heegner point [How04].

$$\mathbf{z}_c \in \varprojlim_n H^1(\mathcal{K}[cp^n], T_f^-) = H^1(\mathcal{K}[c], T_f^- \otimes \Lambda^{\text{ac}}(\Psi))$$

where we have used the notation $\mathcal{K}[c]$ to denote the ring class field of $\mathcal{K}$ with conductor c , and the equality is by a standard application of Shapiro's lemma. If $c = \text{cond}(\chi)$, then we can further project to the χ -isotypic part to produce an element $\mathbf{z}^{\chi} \in H^1(\mathcal{K}, \mathbf{T}^{\top})$. Since the dual condition is also defined by the same space, we have $\mathbf{z}^{\chi} \in H_{\top, \top}^1$.

To verify the hypothesis (2) of Theorem 2.3.2, we need to compute $\text{reg}(\mathbf{z}^{\chi})$, where $\mathbf{z}^{\chi} \in \varprojlim_n H^1(\mathcal{K}[p^n]_{\mathfrak{p}}, T_f^-)$. This is the explicit reciprocity law proven in [BCK21, Theorem 4.4]. A check is required to show that our Coleman map agrees with theirs, which ultimately came from [LZ14]. This can be done using the interpolation formulae, but in fact the constructions are the same: the family is 1-dimensional, so the decoupling step in our construction is unnecessary. It now follows from [Bur17] that the localization at $\mathfrak{p}$ is non-torsion.

Hypothesis (1) as well as one divisibility of the Perrin-Riou main conjecture are both proven in [How04]. It now follows from our theorem that

$$\text{char}_{\Lambda^{\text{ac}}} X_{\mathbb{L}, \mathbb{L}} | (\mathcal{L}_p^{\text{BDP}})^2$$

In classical terms, the Selmer group $X_{\mathbb{L}, \mathbb{L}}$ is the Greenberg Selmer group with the strict and relaxed

conditions at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively. The other side (with the square) is the p -adic L -function first constructed by Bertolini–Darmon–Prasanna [BDP13] interpolating the algebraic parts of the central values $L(\frac{1}{2}, f \times \chi')$ where the weights of χ' are greater than those of f . It follows by a standard application of the control theorem that $L(\frac{1}{2}, f \times \chi') \neq 0$ implies $\text{Sel}_{\text{Gr}}(V_f \otimes \chi') = 0$ under the previous assumptions on f .

Remark 2.4.1. By applying complex conjugation to our argument, we can prove a similar rank-0 results for the relaxed-strict Selmer group, corresponding to the case where the weight of χ' is less than 0. In this case, the result is just the conjugate of the one for positive weight. However, we will see that in general, one special cycle can control several fundamentally different rank 0 regions.

2.4.3 No Heegner points

We now consider the opposite of the above setting, so the number of prime $\ell|N$ which is inert in $\mathcal{K}$ is odd. In this case, we know that $L(\frac{1}{2}, f \times \chi) \neq 0$ for a generic finite order χ by the main result of [Vat03]. They can be interpolated into a p -adic L -function $\mathcal{L}_p$, and we have an Iwasawa main conjecture of the form

$$X_{\text{ord}}(E/\mathcal{K}) | (\mathcal{L}_p)$$

which implies a generic rank 0 result. This conjecture was established by Bertolini–Darmon [BD05] using congruence methods under mild technical hypotheses.

In his thesis [Do22], K. T. Do constructed an Euler system for the representation $V_f \otimes \chi'$, where χ' has large weight. In short, he specialized the Euler system constructed by Bertolini–Seveso–Venerucci [BSV22] in the balanced triple product setting to the case where two of the forms have CM by $\mathcal{K}$. As was further shown in Do’s thesis, the explicit reciprocity law proven in *loc. cit.* computes the big logarithm of his class as a p -adic L -function with certain interpolation properties coming from an easy factorization of the unbalanced triple product p -adic L -function.

In our formalism, let $\mathbf{T}^{\text{L}} = T_f^- \otimes \Lambda^{\text{ac}}(\chi\Psi)$ and $\mathbf{T}^{\text{r}} = \mathbf{T}$, so we start with information about the Greenberg Selmer group and want to derive a rank-0 result for the usual Selmer group. The above construction gives rise to a class z_{Do} satisfying the following properties

$$z_{\text{Do}} \in H_{\text{r}, \text{r}}^1, \quad \text{reg}(z_{\text{Do}}) = \mathcal{L}_p^?$$

where we once again need to observe that the regulator map defined in [BSV22] is the same as in our definition since they are both based on the constructions in [Och03]. The interpolation property as

well as the result of Vatsal cited earlier imply that $\mathcal{L}_p^?$ is non-torsion, so Jetchev–Nekovář–Skinner’s Euler system argument implies one divisibility of the Perrin-Riou main conjecture. Therefore, by Theorem 2.3.2,

$$\mathrm{char}_{\Lambda^{\mathrm{ac}}} X_{\mathrm{ord}} |(\mathcal{L}_p^?)^2$$

The usual control theorem argument implies a rank 0 result. This was deduced without appealing to cyclotomic constructions such as the Rankin–Eisenstein elements of [KLZ17]. While this is not known at the moment, the generalization of the results in the triple product setting to the totally real field case appears to be within reach, so this argument could lead to rank-0 results in those cases as well, where the cyclotomic constructions are not known.

Chapter 3

Backgrounds on the Gan–Gross–Prasad conjectures

This is a purely expository section recalling the automorphic and geometric backgrounds needed later in the thesis. It also sets up some notations and conventions, though they are all standard in the subject. A significant portion of the material is an elaboration on the fundamental work [GGP12].

3.1 Basic set-up

3.1.1 Fields and characters

For each place v of $\mathbf{Q}$, fix an embedding $\mathcal{K} \hookrightarrow \bar{\mathbf{Q}}_v$, which we suppose is compatible with the previously chosen embeddings for $v = p, \infty$. Let $\omega_{\mathcal{K}/\mathbf{Q}} : \mathbf{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbf{C}^\times$ denote the quadratic character, so $\omega_{\mathcal{K}/\mathbf{Q}}(\ell) = -1$ if ℓ is inert.

Let μ be a Hecke character of $\mathcal{K}$. It is conjugate self-dual if $\mu(\bar{z}) = \mu(z)^{-1}$. In this case, $\mu|_{\mathbb{A}^\times}$ is either 1 or $\omega_{\mathcal{K}/\mathbf{Q}}$, and we call μ conjugate orthogonal, resp. conjugate symplectic. In the conjugate-orthogonal case, μ is algebraic, so it has an attached p -adic representation by global class field theory, which we usually denote by the same letter. This construction accounts for all anti-cyclotomic characters of $\mathcal{K}$. A conjugate symplectic character is not algebraic since it has fractional archimedean weights.

3.1.2 Groups

Let E/F be a quadratic extension of either local or global fields. Let V be a Hermitian space defined with respect to it, then we can define its unitary similitude group and unitary group

$$\begin{aligned}\mathrm{GU}(V) &= \{(g, \lambda) \in \mathrm{GL}(V) \times F^\times \mid \langle gv, gw \rangle = \lambda \langle v, w \rangle\} \\ \mathrm{U}(V) &= \{g \in \mathrm{GL}(V) \mid (g, 1) \in \mathrm{GU}(V)\}\end{aligned}$$

We will typically use letters G, H to denote unitary groups and $\tilde{G}, \tilde{H}$ to denote the corresponding unitary similitude groups.

Let $n = \dim_E V$ and define the $n \times n$ matrix Φ by $\Phi_{ij} = (-1)^{i+1} \delta_{i, n+1-i}$. The L -group of the above groups have been computed in [Mor10, Section 2.3]

$${}^L\mathrm{GU}(V) = (\mathrm{GL}_n(\mathbf{C}) \times \mathbf{C}^\times) \rtimes W_F, \quad {}^L\mathrm{U}(V) = \mathrm{GL}_n(\mathbf{C}) \rtimes W_F$$

where the action of an element $c \in W_F$ which is non-trivial in $\mathrm{Gal}(E/F)$ is given respectively by

$$c(g, \lambda)c^{-1} = (\Phi g^{-t} \Phi^{-1}, \lambda \det(g)), \quad cgc^{-1} = \Phi g^{-t} \Phi^{-1}$$

The projection to the first factor ${}^L\mathrm{GU}(V) \rightarrow {}^L\mathrm{U}(V)$ is the dual of the inclusion $\mathrm{U}(V) \rightarrow \mathrm{GU}(V)$. We observe that this isomorphism depends on the choice of a root datum.

The L -parameters for $\mathrm{U}(V)$ are studied in detail in [GGP12, Section 8]. There is a one-to-one correspondence between the L -parameters and conjugate self-dual representations $W_E \rightarrow \mathrm{GL}_n(\mathbf{C})$ with sign $(-1)^{n+1}$, with one direction given by restriction. Moreover, the condition that ϕ is a discrete parameter is equivalent to $\phi|_{W_E}$ decomposes without multiplicity. Let r be the number of summands with sign $(-1)^{n+1}$, then the component group for ϕ is $(\mathbf{Z}/2\mathbf{Z})^r$.

3.1.3 Classification

In this section we do some explicit computations of the Galois cohomology of unitary groups. Suppose F is a global field. If a place v of F splits in E , then locally at v , we have $\mathrm{U}(V) \simeq \mathrm{GL}_n$ and $\mathrm{GU}(V) \simeq \mathrm{GL}_n \times \mathrm{GL}_1$. Both isomorphisms are canonical given a choice of a place of E above v . The same is true if $F_v = \mathbf{C}$, where the isomorphism depends on the choice of the embedding $F \hookrightarrow \mathbf{C}$.

Now suppose v does not split, then we also use v to denote the unique place of E above v . Let V_v be a Hermitian space of dimension n , defined relative to E_v/F_v , and let $G = \mathrm{U}(V_v)$. The

cohomology set $H^1(F_v, G)$ corresponds exactly to the set of Hermitian spaces of dimension n up to isomorphism since $U(V_v)$ is by definition the automorphisms of such a space. The pointed element is the space used to define G . Observe that even if G and G' are abstractly isomorphic, the Galois action on their $\overline{F_v}$ -points remember the underlying space. We also have a Kottwitz map first defined in [Kot86]. It is a priori a map of pointed sets

$$H^1(F_v, G) \rightarrow \pi_0(Z(\hat{G})^{W(F_v)})^\vee = \{1, \det\} \simeq \{\pm 1\}$$

Note that if v were split, then both sides would be trivial. We now describe this map in the non-archimedean and archimedean settings.

In the non-archimedean case, there are two isomorphism classes of spaces, which we label by $V_v^\pm$. Define V_v^+ to be the split space whose Hermitian form is given by Λ_n , the matrix with 1s on the anti-diagonal. If n is odd, then V_v^- can be given by $a\Lambda_n$, where a represents the non-trivial element in $F_v^\times / \mathbf{N}_{E_v/F_v} E_v^\times$. Both $V_v^\pm$ are split, and they are distinguished by the discriminant. If n is even, then V_v^- is the unique non-split Hermitian space of dimension n . The Kottwitz map is a bijection (as expected), and it is uniquely specified by sending the pointed element to 1. The base group in later applications will be have Hermitian form Λ_n , so the Kottwitz invariant of $V_v^\pm$ is $(-1)^{\pm 1}$.

In the archimedean case, $F_v = \mathbf{R}$, and there are $n + 1$ elements in the set $H^1(\mathbf{R}, G)$. We will compute the Kottwitz map explicitly. The Galois group is $\{1, c\}$, and we can represent a cocycle in $H^1(\mathbf{R}, -)$ by its value at complex conjugation. Suppose V_∞ is defined by the matrix J , then by definition,

$$H^1(\mathbf{R}, G) = \{A \in \mathrm{GL}_n(\mathbf{C}) \mid AJ\bar{A}^{-t}J = 1\} / \langle A \sim \bar{B}^t A B J \rangle$$

We can make the substitution $A' = AJ$, then this becomes more simply

$$H^1(\mathbf{R}, G) = \{A' \in \mathrm{GL}_n(\mathbf{C}) \mid A' \overline{A'}^{-t} = 1\} / \langle A' \sim \bar{B}^t A' B \rangle$$

Of course this is just twisting to the case $J = I_n$. A set of representatives is now $\{I_{a,b}\}$, where $a + b = n$, and $I_{a,b}$ is the diagonal matrix with a 1s and b 0s. The class represented by a matrix A' is exactly its signature. The base point is $A = I_n$ or $A' = J$. Let (p, q) be the signature of J .

All of the representative elements exist in the maximal torus $T = U(1)^n$, so the map $H^1(\mathbf{R}, T) \rightarrow H^1(\mathbf{R}, G)$ is surjective. To compute the Kottwitz map $H^1(\mathbf{R}, G) \rightarrow \mathbf{Z}/2\mathbf{Z}$, we follow the original definition in [Kot86] and first compute the Kottwitz map for T . This comes from the local duality

pairing

$$H^1(\mathbf{R}, T) \times H^1(\mathbf{R}, X^*(T)) \rightarrow H^2(\mathbf{R}, \mathbf{C}^\times) \simeq \mathbf{Z}/2\mathbf{Z}$$

Explicitly, if the two cocycles are represented by $t \in T(\mathbf{C})$ and $\chi \in X^*(T)$, then the value is the sign of $\chi(t)$. On the other side, $\pi_0(Z(\hat{T})^{W(\mathbf{R})}) = \{\text{diag}(\pm 1, \dots, \pm 1)\}$. The pairing between $H^1(\mathbf{R}, T)$ and $\pi_0(Z(\hat{T})^{W(\mathbf{R})})$ is the natural one under these identifications. Therefore, $I_{a,b}$, i.e. a Hermitian space of signature (a, b) , is sent to $(-1)^{b-q}$. This is compatible with the fact that the kernel is equal to the image of $H^1(\mathbf{R}, G_{\text{sc}})$, where in this case $G_{\text{sc}} = \text{SU}(p, q)$. Indeed, for the representative $I_{a,b}$ to come from G_{sc} , we exactly need $b - q$ to be even for the determinant condition.

Now we can globalize and consider the exact sequence of pointed sets

$$H^1(F, \text{U}(V)) \rightarrow \bigoplus_v H^1(F_v, \text{U}(V_v)) \rightarrow \pi_0(Z(\widehat{\text{U}(V)})^{W_F})^\vee \simeq \mathbf{Z}/2\mathbf{Z}$$

and a collection of local spaces is coherent if and only if the sum of their Kottwitz map is 0 [Kot86, Proposition 2.6]. This is just a rephrasing of the usual condition using signatures.

3.2 Representations of real groups

To describe the archimedean part of the local Gan–Gross–Prasad conjecture, we need to describe explicitly the discrete series of unitary groups. We also take this opportunity to introduce more notations and choices. This section is very technical, and its upshot is stated more clearly in the next section when we state the local conjecture. Appendix A of [Ato20] presents the same material with more detail, including an explicit verification of the endoscopic character identities.

3.2.1 Root data

In the rest of this section, the group $\text{U}(p, q)$ will denote the real unitary group defined using the Hermitian form $I_{p,q}$ (as opposed to an abstract group). The diagonal torus T is compact. Its characters will be denoted by $t_1, \dots, t_n$, so $z \in T$ is equal to $\text{diag}(t_1(z), \dots, t_n(z))$. Its cocharacters will be denoted by $e_1, \dots, e_n$. The roots of $\text{U}(p, q)_{/\mathbf{C}}$ are $\{t_i - t_j \mid 1 \leq i \neq j \leq n\}$. Let w be in the complex Weyl group. Up to multiplication by T , it is a permutation matrix, and we label it by a permutation σ so that $t_i(wxw^{-1}) = t_{\sigma(i)}(x)$ holds. Let $\Omega_{\mathbf{R}}^{(p)}$ be the real Weyl group, then in this labelling, it corresponds to σ fixing the subset $\{1, \dots, p\}$.

We now attach a binary word to $\Omega_{\mathbf{R}}^{(p)}$ -orbits on Weyl chambers. In fact, we may label the orbits by

subsets of $\{1, \dots, n\}$ of size p as follows: suppose $i_1 < \dots < i_p$ is the given subset and $j_1 < \dots < j_q$ is its complement, define a permutation σ sending i_α to α and j_β to $p + \beta$, then the attached orbit contains the Weyl chamber with positive roots $t_{\sigma(i)} - t_{\sigma(j)}$, $i < j$. Alternatively, the character in this chamber is

$$\lambda = \sum_{\alpha=1}^p a_{i_\alpha} t_\alpha + \sum_{\beta=1}^q a_{j_\beta} t_{p+\beta}$$

Let $\mathcal{C}$ be a Weyl chamber whose orbit is associated to the set S of size p , we define its binary word $\epsilon_{\mathcal{C}}$ to have 0 at indices in S and 1 elsewhere.

3.2.2 Pure inner forms

Following Vogan (as described in [GGP12, Section 9]), the pure inner forms of a group are the forms indexed by the set $H^1(F, G)$, and we should be considering their representations together. In our case, we can form the disjoint union

$$\mathbf{G} = \bigsqcup_{p=0}^n \mathrm{U}(p, n-p)$$

This is the disjoint union of *two* K -groups in the sense of Arthur [Art99], but we can still follow his prescription to define certain combinatorial data attached to $\mathbf{G}$ appearing in the endoscopic theory. Let $m = \frac{n}{2}$ if n is even and $\frac{n+1}{2}$ if n is odd. Let $m' = n - m$, then the group $\mathrm{U}(m, m')$ is quasi-split, and we will take it to be the base point.

Fix the twisting between the inner forms appearing in $\mathbf{G}$ to be the identity on complex points, then a representative for the twisting cocycle from $\mathrm{U}(p, n-p)$ to $\mathrm{U}(m, m')$ is $x^{(p)} = I_{p, n-p} I_{m, m'}$. Let $T = \mathrm{U}(1)^n$ be the compact diagonal maximal torus in each of $\mathrm{U}(p, n-p)$. This is preserved by all the twisting maps. Let $\mathfrak{E}(T) = H^1(\mathbf{R}, T) = \{\pm 1\}^n$. Given a group $\mathrm{U}(p, q)$ appearing in the disjoint union, we can define

$$\mathfrak{D}'_p(T) := \ker (H^1(\mathbf{R}, T) \rightarrow H^1(\mathbf{R}, \mathrm{U}(p, q))) \subseteq \mathfrak{E}(T)$$

It has size $\binom{n}{p}$. They do not partition $\mathfrak{E}(T)$. Instead we need to translate them by $x^{(p)}$

$$\mathfrak{E}(T) = \bigsqcup_{p=0}^n \mathfrak{D}_p(T), \quad \mathfrak{D}_p(T) = x^{(p)} \mathfrak{D}'_p(T)$$

Indeed, $\mathfrak{D}'_p(T)$ consists of diagonal (± 1) -matrices A such that $AI_{p, q}$ has the same signature as $I_{p, q}$,

so $\mathfrak{D}_p(T)$ consists of diagonal (± 1) -matrices which disagree with $I_{m,m'}$ at exactly q entries. In particular, the base point 1 is in $\mathfrak{D}_m(T)$.

Let $\{\pm 1\}$ act on $\mathrm{GL}_n(\mathbf{C})$ by the automorphism $g \mapsto \Phi_n^{-1} g^{-t} \Phi_n$, where $(\Phi_n)_{ij} = \delta_{i,n+1-j}(-1)^{i+1}$. Form the group $\mathrm{GL}_n(\mathbf{C}) \rtimes \{\pm 1\}$. This is isomorphic to the Galois form of the L -group of $\mathbf{G}$. The isomorphism depends on an identification of $X_*(T)$ with $X^*(T^\vee)$, where $T^\vee$ is the diagonal maximal torus of $\mathrm{GL}_n(\mathbf{C})$. Let $\mathcal{C}_*$ be a Weyl chamber in $\mathrm{U}(m, m')$ such that each simple root in $\mathcal{C}_*$ is non-compact. We label $\mathcal{C}_*$ by the permutation σ_* such that $\sum_i a_i t_i \in \mathcal{C}_*$ if and only if $a_{\sigma_*(1)} > \cdots > a_{\sigma_*(n)}$. We can then identify the coroot $e_{\sigma_*(i)} \in X_*(T)$ with the root $z_i \in X^*(T^\vee)$. This depends on $\mathcal{C}_*$, not just on the orbit by the real Weyl group, but the eventual labelling of L -packets will only depend on the orbit.

3.2.3 Discrete series

Let $\underline{a} = (a_1 > \cdots > a_n)$ be a regular infinitesimal character, so each $a_i \in \mathbf{Z} + \frac{n-1}{2}$. The discrete series with this infinitesimal character all form a single L -packet with parameter

$$\varphi(z) = \mathrm{diag}((z/\bar{z})^{a_1}, \dots, (z/\bar{z})^{a_n}), \quad \varphi(j) = \Phi_n$$

where Φ_n is the matrix with alternating 1 and -1 on the off-diagonal as before. In the form of the local Langlands correspondence proposed by Vogan, the L -packets with this parameter for all groups appearing in $\mathbf{G}$ can be combined into a single Vogan L -packet whose elements are indexed by characters of the component group for φ , cf. [GGP12, Section 9]. We describe it explicitly, unravelling the general definition of spectral endoscopic transfer factors by Shelstad [She08].

The component group A_φ is $\{\mathrm{diag}(\pm 1, \dots, \pm 1)\}$, which is easily seen by either direct computation or appealing to the results in [GGP12, Section 6]. A character of the component group can be represented as a binary word of length n . By Tate–Nakayama duality and the identification $X^*(T^\vee) \simeq X_*(T)$ from the previous section depending on a choice of σ_* , we can attach a binary word to each element of $H^1(\mathbf{R}, T)$. More explicitly, a diagonal matrix $\mathrm{diag}(t_1, \dots, t_n)$ is mapped to the word $\epsilon_1 \cdots \epsilon_n$, where $\epsilon_i = 0$ if $t_{\sigma_*(i)} > 0$ and 1 otherwise.

On each $\mathrm{U}(p, q)$, the discrete series is parametrized by $\Omega_{\mathbf{R}}^{(p)}$ -orbits on $X^*(T) + \rho_G$. With the fixed infinitesimal character $\underline{a}$ as before, this is equivalent to $\Omega_{\mathbf{R}}^{(p)}$ -orbits on Weyl chambers. Given a chamber $\mathcal{C}$, we now determine an element of $H^1(\mathbf{R}, T)$ attached to it. Viewing the word as a character of the component group in the way we have described above, this is the labelling of the L -packet from [GGP12, Section 9]. Let $\mathcal{C}_*$ be the Weyl chamber used in the previous section with

non-compact simple roots. Let w be a permutation matrix such that $\text{Ad}(w)$ sends the roots defining $\mathcal{C}$ to the roots defining $\mathcal{C}_*$. We attach to $\mathcal{C}$ the cocycle represented by the element

$$t = wI_{p,q}w^{-1}I_{m,m'}$$

Changing $\mathcal{C}$ to a different element of the $\Omega_{\mathbf{R}}^{(p)}$ -orbit corresponds to changing w to ww' for some $w' \in \Omega_{\mathbf{R}}^{(p)}$. This does not change t . Moreover, it is easy to check that t defines an element in $\mathfrak{D}_p(T)$. The endoscopic code attached to $\mathcal{C}$ is then the word attached to t using the recipe of the previous paragraph. One can then verify that this word is independent of the choice of $\mathcal{C}_*$ in its $\Omega_{\mathbf{R}}^{(m)}$ -orbit.

If n is odd, then the requirement that $\mathcal{C}_*$ contains only non-compact simple roots uniquely determines the $\Omega_{\mathbf{R}}^{(m)}$ -orbit, so the identification above is canonical. In this case, the binary code is simply $\epsilon_{\mathcal{C}} + 0101 \cdots 0$. However, there are two isomorphic representations in the Vogan L -packet, corresponding to the two isomorphic unitary groups $\text{U}(m, m-1)$ and $\text{U}(m-1, m)$ in the K -group. We made the choice to use $\text{U}(m, m-1)$ as the basic quasi-split group.

If n is even, then we have two different $\Omega_{\mathbf{R}}^{(m)}$ -orbits of $\mathcal{C}_*$, corresponding to the subsets $\{1, 3, \dots, n-1\}$ and $\{2, 4, \dots, n\}$. Given a choice of a Whittaker datum, exactly one of the two chambers correspond to the generic discrete series, and the binary code attached to that chamber should be the word which is entirely 0. We discuss this choice more precisely, following [GGP12, Section 12]. Let V be a split Hermitian space defined with respect to a quadratic extension k/k_0 . Fix a polarization $V = X \oplus X^\vee$, i.e. X and $X^\vee$ are both maximal isotropic subspaces, and the pairing induces a perfect duality between X and $X^\vee$. Given a basis $\{x_1, \dots, x_m\}$ for X , it corresponds to a dual basis $\{x_1^\vee, \dots, x_m^\vee\}$ for $X^\vee$. Let B be the Borel subgroup of $G = \text{U}(V)$ fixing the flag corresponding to this basis. Define characters $n_1, \dots, n_m$ of N

$$n_i(g) = \langle gx_{i+1}, x_i^\vee \rangle, \quad 1 \leq i \leq m-1, \quad n_m(g) = \langle gx_n^\vee, x_n^\vee \rangle$$

Given a character $\psi : k \rightarrow \mathbf{C}^\times$, we can form the character $\lambda : N \rightarrow \mathbf{C}^\times$, $g \mapsto \psi(\sum_{i=1}^m n_i(g))$. The Levi subgroup is a torus, and an element $t = (t_1, \dots, t_m)$ acts by

$$t\lambda(g) = \psi\left(\sum_{i=1}^{m-1} t_i t_{i+1}^{-1} n_i(g) + t_m \bar{t}_m n_m(g)\right)$$

One can check that $\text{Tr}_{k/k_0} n_m(g) = 0$ for all $g \in N$. It follows that by choosing an appropriate t , we can assume ψ is trivial on k_0 , and the T -orbits on λ can be identified with the $\mathbf{N}_{k/k_0} k^\times$ -orbits

on characters $\psi : k/k_0 \rightarrow \mathbf{C}^\times$. This identification does not depend on the polarization or the basis chosen, so it is canonical as soon as we realize G as a concrete unitary group. If k_0 is a local field, then this orbit has two elements.

Specializing to the case $k/k_0 = \mathbf{C}/\mathbf{R}$. The two orbits are represented by the characters

$$\psi_{\mathbf{C}} : z \mapsto \exp(2\pi(\bar{z} - z)), \quad \overline{\psi_{\mathbf{C}}} : z \mapsto \exp(2\pi(z - \bar{z}))$$

Using the above construction, the two orbits of Whittaker data can be represented by $(B, \psi_{\mathbf{C}})$ and $(B, \overline{\psi_{\mathbf{C}}})$. The generic representation with respect to $(B, \overline{\psi_{\mathbf{C}}})$ to the set $\{1, 3, \dots, n-1\}$, so the binary code attached to $\mathcal{C}$ is $\epsilon_{\mathcal{C}} + 01 \dots 01$. This fact can be found in [Ato20, Proposition A.3], though it can also be formally deduced a posteriori from H. He's proof of the local Gan–Gross–Prasad conjecture [He17] in our case.

3.3 Local conjecture

Let $W \subseteq V$ be two Hermitian spaces with dimensions $n-1$ and n respectively, defined using an extension of global fields E/F . Let $G = \mathrm{U}(W) \times \mathrm{U}(V)$. Let v be a place of F and fix a local additive character $\psi_v : E_v \rightarrow \mathbf{C}^\times$ which is trivial on F_v . With this choice of ψ_v , if ϕ is any conjugate self-dual representation of W_{E_v} , then Tate's local root number $\varepsilon_L(\frac{1}{2}, \phi, \psi_v)$ takes value in $\{\pm 1\}$ by a simple argument given in [GGP12, Section 5]. We will shorten the notation to $\varepsilon(\frac{1}{2}, \phi)$.

Let $\phi_v = \phi_{n,v} \otimes \phi_{n-1,v}^\vee$ be a tempered local Langlands parameter for G at v . For non-archimedean v , there is also a partition into Vogan L -packet, specified by endoscopic character identities. It depends on the same types of choices in the archimedean case, namely a Hermitian space in the n odd case and a character in the n even case. The existence of such a partition and its internal labelling is expected to be established in [KMSW14] and its sequels.

We can now state the local Gan–Gross–Prasad conjecture, which deals with the problem of whether a representation of $\mathrm{U}(W)$ appears in the restriction of a representation of $\mathrm{U}(V)$. Up to the appearance of a contragredient, this is equivalent to asking about restrictions along the diagonal embedding $\mathrm{U}(W) \hookrightarrow \mathrm{U}(W) \times \mathrm{U}(V)$, which will lead to cleaner statements. In the p -adic case, the conjecture is proven by Beuzart-Plessis [BP16, BP15] following the local trace formula approach pioneered by Waldspurger. They depend on the conjectural character identities alluded to above. In the archimedean case, it can be proven relatively easily using theta lifting. This was done by H. He in the discrete series case [He17] and by H. Xue in the general tempered case [Xue21].

Theorem 3.3.1. *For $i = n - 1, n$, there exists a unique $\pi_{i,v}$ in the Vogan L -packet of $\phi_{i,v}$ such that*

- *Relevancy:* if W_v° and V_v° are the Hermitian spaces underlying the representations $\pi_{n-1,v}, \pi_{n,v}$ respectively, then $V_v^\circ/W_v^\circ \simeq V_v/W_v$.
- *Restriction:* $\dim_{\mathbf{C}} \text{Hom}_{\text{U}(W_v^\circ)}(\mathbf{C}, \pi_{n-1,v} \otimes \pi_{n,v}) \neq 0$, and in fact it is equal to 1.

Moreover, define the following character of the component group

$$\chi_n : A_{\phi_n} \rightarrow \{\pm 1\}, \quad a \mapsto \varepsilon(\phi_{n,v}^{a=-1} \otimes \phi_{n-1,v}, \psi_v)$$

and similarly define $\chi_{n-1} : A_{\phi_{n-1}} \rightarrow \{\pm 1\}$. Let δ be the discriminant of the odd Hermitian space, and let $\psi'_v(x) = \psi_v(-2\delta x)$. If we label the Vogan L -packet for the even space using ψ'_v (as opposed to ψ_v), then $\pi_{i,v}$ has label χ_i for $i = n - 1, n$.

Now suppose $v = \infty$. We describe the distinguished representation explicitly using the labels introduced in the previous section. Choose $\psi_{\mathbf{C}}(z) = e^{2\pi(\bar{z}-z)}$, and suppose the parameter ϕ corresponds to discrete series with infinitesimal character

$$\phi_{n,\mathbf{R}} \leftrightarrow (a_1 > a_2 > \cdots > a_n), \quad \phi_{n-1,\mathbf{R}} \leftrightarrow (b_1 > b_2 > \cdots > b_{n-1})$$

We consider the pairs $(\phi_{n,\mathbf{R}}, \phi_{n-1,\mathbf{R}}^\vee)$. Since

$$\text{Hom}_{\text{U}(W_\infty^\circ)}(\mathbf{C}, \pi_{n-1,\mathbf{R}}^\vee \otimes \pi_{n,\mathbf{R}}) = \text{Hom}_{\text{U}(W_\infty^\circ)}(\pi_{n-1,\mathbf{R}}, \pi_{n,\mathbf{R}})$$

this is taking the restrictions problem point of view, which we find to be clearer. Let $e_i \in A_{\phi_n}$ be the identity matrix except -1 at entry (i, i) , then $\phi_n^{e_i=-1} = \mathbf{C}((z/\bar{z})^{a_i})$, so

$$\chi_n(e_i) = \varepsilon(\mathbf{C}((z/\bar{z})^{a_i} \otimes \phi_{n-1,\mathbf{R}}^\vee, \psi_{\mathbf{C}}) = \prod_{j=1}^{n-1} \varepsilon(\mathbf{C}((z/\bar{z})^{a_i-b_j}), \psi_{\mathbf{C}}) = (-1)^{\#\{j \mid a_i > b_j\}}$$

where the final equality follows from [Tat79, (3.2.5)]. Similarly, we have

$$\chi_{n-1}(f_j) = (-1)^{\#\{i \mid a_i > b_{n-j}\}}$$

where the appearance of $n - j$ instead of j is due to the contragredient on ϕ_{n-1} . If $\psi_{\mathbf{C}}$ is replaced by $\overline{\psi_{\mathbf{C}}}$, then the character corresponding to the even unitary group changes sign, which is the same operation as changing the Whittaker data to the opposite one. Therefore, the distinguished

representation does not depend on the choice of $\psi_{\mathbf{G}}$.

To specify the representation, we pick as the starting pair

$$\mathbf{U}(m-1, m-1) \subseteq \mathbf{U}(m, m-1) \text{ or } \mathbf{U}(m, m-1) \subseteq \mathbf{U}(m, m)$$

depending on the parity of n . This distinction is a side-effect of our choice of the odd unitary space. The discriminant of the odd Hermitian space is 1, so the labelling of the endoscopic packet for the even group uses the character $\overline{\psi_{\mathbf{G}}}$. By the description of the generic representation, we need to add 0101... to the above characters to recover the code for the Weyl chamber. The following easy combinatorial lemma checks that the resulting inner forms are relevant.

Lemma 3.3.2. *Let ϵ_n and ϵ_{n-1} be the words representing the Weyl chambers obtained from the above procedure. Let p_n respectively p_{n-1} be the number of 1s in them. If n is even, then $p_n - p_{n-1} = 1$, otherwise $p_n - p_{n-1} = 0$.*

Proof. We first show that if $a_i < b_j$ is a part of the weight interlacing condition, then $(\epsilon_n)_i = (\epsilon_{n-1})_j$. We will check that this holds for the classical branching law in the next example, and it's easy to see that changing the order of two consecutive weights preserve the property, so this is proven. It follows that if we change $a_i < b_j$ to $b_j < a_i$, then only $(\epsilon_n)_i$ and $(\epsilon_{n-1})_j$ gets flipped, so $p_n - p_{n-1}$ does not change. Again by the next example, $p_n - p_{n-1}$ takes the prescribed value in the case of the classical branching law, so we are done. $\square$

The upshot is that the distinguished archimedean representations are determined in a precise combinatorial recipe from the interlacing of weights. We give two examples to clarify this.

Example 3.3.3. This is compatible with the classical branching law for compact groups. Recall that if π_n is the representation of the compact Lie group $\mathbf{U}(n)$ with highest weight $(w_1 \geq w_2 \geq \dots \geq w_n)$, then $\pi_n|_{\mathbf{U}(n-1)}$ is a multiplicity-free direct sum of representations with highest weights $(v_1 \geq v_2 \geq \dots \geq v_{n-1})$ satisfying the interlacing condition

$$w_1 \geq v_1 \geq w_2 \geq v_2 \geq \dots \geq v_{n-1} \geq w_n$$

The infinitesimal character $\underline{a}$ is equal to $\underline{w} + \rho_{\mathbf{U}(n)}$, where $\rho_{\mathbf{U}(n)} = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2})$ is half of the sum of positive roots. Similarly for $\underline{b}$. Therefore, the interlacing condition in terms of infinitesimal characters is

$$a_1 > b_1 > a_2 > \dots > a_{n-1} > b_{n-1} > a_n$$

The distinguished characters correspond to the binary words

$$\chi_n : \cdots 1010, \quad \chi_{n-1} : \cdots 0101$$

If n is even, then the codes for the Weyl chamber for both groups are entirely 1, corresponding to the inner forms $U(0, n-1) \subseteq U(0, n)$. If n is odd, then they are all 0, corresponding to the inner forms $U(n-1, 0) \subseteq U(n, 0)$.

Example 3.3.4. This is also compatible with Blattner's formula in the case $U(n, 0) \subseteq U(n, 1)$ [HS75]. We consider the case n is odd for simplicity, otherwise the relevant signature is $U(0, n) \subseteq U(1, n)$, but the discussion is identical if we had chosen a different odd Hermitian space as our base space.

We use the notations in *loc. cit.* to state the formula. Suppose the Harish-Chandra parameter is

$$\lambda = (a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_{n+1}, a_k)$$

The positive roots associated with this system are

$$R_c^+ = \{t_i - t_j \mid 1 \leq i < j \leq n\}, \quad R_n^+ = \{t_1 - t_{n+1}, \cdots, t_{k-1} - t_{n+1}, -t_k + t_{n+1}, \cdots, -t_n + t_{n+1}\}$$

Let $(c_1, \cdots, c_{n+1})$ be a character whose components sum to 0, then $Q(c_1, \cdots, c_{n+1})$ is 1 if and only if $c_1, \cdots, c_{k-1} \geq 0$ and $c_k, \cdots, c_n \leq 0$. Since $(b_1 > \cdots > b_n)$ is the infinitesimal character on $U(n)$, we have $\mu + \rho_c = (b_1, \cdots, b_n, b_{n+1})$, where b_{n+1} is chosen so that $\sum_{i=1}^{n+1} b_i = \sum_{i=1}^{n+1} a_i$ and correspond to the character on $U(1)$. Moreover, we can compute that

$$\lambda + \rho_n = \left(a_1 + \frac{1}{2}, \cdots, a_{k-1} + \frac{1}{2}, a_{k+1} - \frac{1}{2}, \cdots, a_{n+1} - \frac{1}{2}, a_k + \frac{n-2k+2}{2} \right)$$

If the weight interlacing condition is

$$b_1 > a_1 > \cdots > b_{k-1} > a_{k-1} > a_k > a_{k+1} > b_k \cdots > a_{n+1} > b_n$$

then it's easy to check that the multiplicity given by Blattner's formula is 1. Moreover, in a weight interlacing condition with $b_i > b_{i+1}$ and no $a_{i'}$ in between, the transposition $(i \ i+1)$ does not change the value of Q , but it changes the sign, so the total sum is 0. Therefore, we have described the exact weight interlacing condition distinguished by $U(n, 0) \subseteq U(n, 1)$.

The distinguished representation predicted by the local GGP conjecture satisfy $\chi_n = 01 \cdots 010$ and χ_{n+1} differs from $10 \cdots 1010$ at the k -th bit. It is easy to see that they correspond exactly to the weight interlacing condition computed above.

3.4 Global conjecture

Let V, W, G be as in the previous section. Fix a global additive character $\psi : E \backslash \mathbb{A}_E \rightarrow \mathbf{C}^\times$ which is trivial on $\mathbb{A}_F$. Let $\phi = \phi_n \otimes \phi_{n-1}$ be a tempered global Langlands parameter for G . For each place v of F , the local conjecture (with additive character ψ_v) gives a relevant pair of Hermitian spaces $W_v^\circ \subseteq V_v^\circ$ and a representation π_v° on $G_v^\circ := \mathrm{U}(V_v^\circ) \times \mathrm{U}(W_v^\circ)$. We can therefore form their products

$$\pi^\circ := \bigotimes'_v \pi_v^\circ, \quad \mathbb{G}^\circ = \prod'_v G_v^\circ$$

Recall from the classification of Hermitian spaces that the collection of space (V_v°) comes from a global Hermitian space V if the sum of their Kottwitz invariants is 0. By the local computation, this is equivalent to $\varepsilon(\frac{1}{2}, \phi) = 1$. In this case, there is a global group G° defined over F such that $G^\circ(\mathbb{A}_F) = \mathbb{G}^\circ$.

Let χ_v be the endoscopic character attached to π_v° , then their product is a global character

$$\chi : A_\phi \rightarrow \prod_v A_{\phi_v} \rightarrow \{\pm 1\}$$

The Arthur multiplicity formula predicts that the multiplicity of π° in the discrete spectrum is 1 if $\chi = 1$ and 0 otherwise [GGP12, Section 25]. In the stable case, the global component group is $A_\phi = (\mathbf{Z}/2\mathbf{Z})^2$, with one factor from each group (for the purpose of this thesis, we may take this to be the definition of stable). Therefore, $\chi = 1$ is still equivalent to $\varepsilon(\frac{1}{2}, \phi) = 1$.

The global Gan–Gross–Prasad conjecture deals with the central L -value of π° , or rather the parameter ϕ . In the case $\varepsilon(\frac{1}{2}, \phi) = 1$, there is no trivial reason to expect the central L -value to be 0, and the conjecture relates it to a period integral. Once again, this is now known thanks to W. Zhang’s relative trace formula approach, cf. [Zha14b, Zha14a, BPLZZ21, BPCZ20], though we have omitted many intermediate works

Theorem 3.4.1. *For $\varphi \in \pi^\circ$, we have*

$$L\left(\frac{1}{2}, \pi^\circ\right) = C(\varphi) \left| \int_{[\mathrm{U}(W^\circ)]} \varphi(h) dh \right|^2$$

where $C(\varphi)$ is a product of elementary constants, abelian L -functions, and local zeta integrals. Moreover, φ can be chosen so that $C(\varphi) \neq 0$.

Remark 3.4.2. The global statement is complementary to the local statement, which explains why its proof does not depend on the yet incomplete endoscopic classification for non-quasi-split unitary groups.

We refer to this case as the *coherent case*, following terminologies from the Kudla program introduced by S. Zhang. In the incoherent case, i.e. $\varepsilon(\frac{1}{2}, \phi) = -1$, the central L -value vanishes for trivial reason, and the group $\mathbb{G}^\circ$ is no longer the adelic points of an algebraic group over F . Experiences in low ranks with the Gross–Zagier formula and its generalizations in [YZZ13] suggest that we should relate the central derivative of the L -function to the height of an algebraic cycle.

Very little is precisely conjectured in this case. We are only able to find conjectures stated in the minimal weight case for a totally real field, namely

$$\phi_{n-1,v} \leftrightarrow \left(-\frac{n-2}{2}, -\frac{n-4}{2}, \dots, \frac{n-2}{2}\right), \quad \phi_{n,v} \leftrightarrow \left(-\frac{n-1}{2}, -\frac{n-3}{2}, \dots, \frac{n-1}{2}\right)$$

for all archimedean v . Classical branching law already tells us the distinguished signature at all $v|\infty$ is $(0, n-1), (0, n)$. If we change it to $(1, n-2), (1, n-1)$ at exactly one archimedean place, then the collection becomes coherent. Denote these new Hermitian spaces by W' and V' , then the diagonal embedding of unitary groups $U(W') \hookrightarrow U(W') \times U(V')$ induces an embedding of Shimura varieties of dimensions $(n-2)$ and $(2n-3)$ respectively. The image is a cycle of codimension $(n-1)$, which is a precursor to the diagonal cycle. The following imprecise conjectural conjecture can be found in [GGP12, Section 27] and [Zha12].

Conjecture 3.4.3 (Arithmetic Gan–Gross–Prasad conjecture). *$L'(\frac{1}{2}, \phi) \neq 0$ if and only if the conjectural Beilinson–Bloch height of the diagonal cycle is non-zero.*

In Subsection 3.6.3, we will perform the constructions more carefully, having introduced the necessary data for Shimura varieties. In the next part of the thesis, we will p -adically interpolate their étale realizations in Hida families.

3.5 Galois representations

3.5.1 General properties

Let $G = \mathrm{U}(V)$ be a unitary group of rank n . Let $\pi = \bigotimes_v \pi_v$ be a (unitary) cuspidal automorphic representation of G , with π_∞ a discrete series with infinitesimal character $(a_1 > \cdots > a_n)$. Following [Ski12, Theorem B], there exists a p -adic Galois representation $\rho_\pi : \mathrm{Gal}_K \rightarrow \mathrm{GL}_n(\bar{\mathbf{Q}}_p)$ associated to π , in the sense that at all unramified and archimedean places,

$$L_v(\rho_\pi, s) = L_v\left(s - \frac{n-1}{2}, \pi\right)$$

Moreover, *loc. cit.* shows that ρ_π has weight $n-1$ and satisfies the relation $\rho_{\pi^\vee} \simeq \rho_\pi^c \simeq \rho_\pi^\vee(1-n)$. At the places above p , the representation is de Rham. It has Hodge–Tate weights $(\frac{n-1}{2} - a_i, \frac{n-1}{2} + a_i)$, following the conventions defined in the introduction. The expected archimedean L -factor given by this set of weights is given by

$$\prod_{i=1}^n \Gamma_{\mathbf{C}}\left(s + \frac{n-1}{2} + |a_i|\right)$$

which agrees with the one computed using the archimedean L -parameter.

Applying the previous discussion to a subspace $W \subseteq V$ of dimension $n-1$, we can form the tensor product $\rho := \rho_{\pi_n} \otimes \rho_{\pi_{n-1}}^\vee(1) = \rho_{\pi_n \otimes \pi_{n-1}^\vee}(n-1)$. It is arithmetically conjugate self-dual, i.e. $\rho^\vee(1) \simeq \rho^c$. In particular, its L -function has centre 0. Its Hodge–Tate weights are

$$\pm(b_j - a_i) - \frac{1}{2}$$

for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, where the sign is positive for the chosen embedding and negative for the conjugate one. The L -factor at infinity is

$$L_\infty(\rho, s) = \prod_{i=1}^n \prod_{j=1}^{n-1} \Gamma_{\mathbf{C}}\left(s + |b_j - a_i| + \frac{1}{2}\right)$$

In particular, the central value $s = 0$ is always critical, and the other critical values are those s such that $|s| \leq |b_j - a_i| + \frac{1}{2}$ for all i, j .

3.5.2 Ordinary representations

Now we assume that $\pi = \pi_n$ is unramified at p . Since p is split, we have a decomposition $V \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq V_0 \oplus \bar{V}_0$, where the first factor corresponds to $\mathfrak{p}$ and the second factor corresponds to $\bar{\mathfrak{p}}$.

Using this decomposition, we get $G \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq \mathrm{GL}(V_0)$. Viewing π_p as a representation of $\mathrm{GL}(V_0)$ using this isomorphism, let $p^{-a_1}\alpha_1, \dots, p^{-a_n}\alpha_n \in \bar{\mathbf{Q}}_p \simeq \mathbf{C}$ be its Satake parameters. Under this hypothesis, the representation ρ_π is crystalline at $\mathfrak{p}$, and its Frobenius eigenvalues are equal to $p^{\frac{n-1}{2}-a_i}\alpha_i$, $1 \leq i \leq n$. This was again shown in [Ski12, Theorem B(d)].

We further assume that π is ordinary in the sense of [SU14, Section 6.4.1], which is equivalent to

$$v_p(\alpha_i) = 0 \text{ for all } i \quad (\text{ord})$$

Equivalently, the Hodge polygon and the Newton polygon coincide for $D_{\mathrm{cris}}(\rho_\pi)$. It follows from the discussion of *loc. cit.* that the local Galois representation $\rho_\pi|_{\mathrm{Gal}_{K_{\mathfrak{p}}}}$ is upper triangular. More precisely, there exists a $\mathrm{Gal}_{K_{\mathfrak{p}}}$ -stable increasing filtration $0 = \mathrm{Fil}_0 \rho_\pi \subseteq \dots \subseteq \mathrm{Fil}_n \rho_\pi = \rho_\pi$ such that for $i = 1, \dots, n$, the graded piece $\mathrm{gr}_i \rho_\pi$ is 1-dimensional with $\mathrm{Gal}_{K_{\mathfrak{p}}}$ acting as $\alpha_i \chi_{\mathrm{cyc}}^{a_i - \frac{n-1}{2}}$, where we also use α_i to denote the unramified character such that $\alpha_i(\mathrm{Frob}_{\mathfrak{p}}) = \alpha_i$.

Choose a basis $\{v_1, \dots, v_n\}$ for ρ_π so that $\mathrm{Fil}_i \rho_\pi = \langle v_1, \dots, v_i \rangle$. Let $\{v_1^\vee, \dots, v_n^\vee\}$ be its dual basis for $\rho_\pi^\vee$. This induces a *decreasing* $\mathrm{Gal}_{K_{\mathfrak{p}}}$ -stable filtration $\rho_\pi^\vee = \mathrm{Fil}^0 \rho_\pi^\vee \supseteq \dots \supseteq \mathrm{Fil}^n \rho_\pi^\vee = 0$, with $\mathrm{Fil}^i \rho_\pi^\vee = \langle v_{i+1}^\vee, \dots, v_n^\vee \rangle$, with the action on the graded piece $\mathrm{Fil}^{i-1} \rho_\pi^\vee / \mathrm{Fil}^i \rho_\pi^\vee$ given by $\alpha_i^{-1} \chi_{\mathrm{cyc}}^{-a_i + \frac{n-1}{2}}$. We can transport this to a $\mathrm{Gal}_{K_{\mathfrak{p}}}$ -stable filtration on ρ_π using $\rho_\pi^\vee \simeq \rho_\pi^c(n-1)$. This whole package of data will be interpolated in the next section. The result is an instance of the axiomatic set-up of the previous section.

3.6 Shimura varieties

In this section, we recall various known facts about Shimura varieties attached to unitary groups and their cohomology. Fix a Hermitian space V and let $G = \mathrm{U}(V)$. Moreover, choose the Hermitian form on V so that at $\mathbf{R}$, it is given by $I_{p,q}$. Consider the Shimura datum

$$h : \mathbb{S} \rightarrow G_{\mathbf{R}}, \quad z \mapsto \mathrm{diag}((z/\bar{z})I_p, I_q)$$

Given a neat level structure $K \subseteq G(\mathbb{A}_f)$, this defines a Shimura variety which we denote by $\Sigma(V, K)$. They will be smooth algebraic varieties over $\mathcal{K}$, though in certain special cases, the reflex field is actually $\mathbf{Q}$. The associated symmetric space is $X = \mathrm{U}(p, q)/(\mathrm{U}(p) \times \mathrm{U}(q))$, which is connected of complex dimension pq , so

$$\Sigma(V, K)(\mathbf{C}) = G(\mathbf{Q}) \backslash (X \times G(\mathbb{A}_f)/K)$$

Let T be the torus of norm one elements of $\text{Res}_{\mathcal{K}/\mathbf{Q}} \mathbf{G}_m$, then there is a natural map $\det : G \rightarrow T$. This induces an isomorphism of components

$$\pi_0(\Sigma(V, K)) \simeq T(\mathbf{Q}) \backslash T(\mathbb{A}_f) / \det(K)$$

by [Mil05, Theorem 5.17]. The right hand side can be further embedded into a quotient of $\mathbb{A}_{\mathcal{K}, f}^\times$, and this isomorphism is $\text{Gal}_{\mathcal{K}}$ -equivariant.

If $K' \subseteq K$ are two neat levels, then there is a finite étale morphism $\Sigma(V, K') \rightarrow \Sigma(V, K)$ also defined over $\mathcal{K}$. Let $\Sigma(V)$ denote this projective system of varieties.

3.6.1 Betti cohomology

Given an algebraic representation V_ρ of G over $\mathbf{C}$, we can attach a local system on $\Sigma(V)$ defined to be the local sections of the vector bundle

$$G(\mathbf{Q}) \backslash (G(\mathbb{A}_f) \times X \times V_\rho) \rightarrow \Sigma(V)$$

Denote it by $\mathbb{L}_\rho$. Since $G_{\mathbf{C}} \simeq \text{GL}_n$, the algebraic representations are represented by their infinitesimal characters $(a_1 > \dots > a_n)$. They are all defined over $\mathcal{K}$. It follows that $\mathbb{L}_\rho$ can be defined over $\mathcal{K}$. Moreover, there is a p -adic étale version which we denote by the same notation. Its construction in the algebraic setting is explained in Section III.6 of [Mil90].

The L^2 -cohomology of $\Sigma(V)$ with coefficient in $\mathbb{L}_\rho$ can be computed using automorphic forms [BC83]. More precisely, for any d , we have

$$H_{L^2}^d(\Sigma(V)(\mathbf{C}), \mathbb{L}_\rho) = \bigoplus_{\pi: \pi_\infty \in \text{Coh}_\rho} m(\pi) H^d(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\rho) \otimes \pi_f$$

where $m(\pi)$ is the multiplicity of π in the discrete spectrum. The calculations of [VZ84] shows that if $d = pq$ is the middle degree, then the $(\mathfrak{g}, K_\infty)$ -cohomology occurring in the formula vanishes unless π_∞ is in the discrete series L -packet with infinitesimal character $\rho^\vee$, in which case it has dimension 1.

We are interested in the dimension of the multiplicity space. This can be computed using the still conjectural Arthur's multiplicity formula, as formulated in [Art89, Conjecture 8.1]. For this, suppose π_f is stable, in the sense that there exists a cusp form of $(\text{GL}_n)_{/\mathcal{K}}$ which is almost everywhere locally the base change of π_f . Equivalently, the global component group is equal to $\{\pm 1\}$. Arthur's

multiplicity formula gives

$$m(\pi_f \otimes \pi_\infty) = \begin{cases} 1 & \text{if } \langle \pi_\infty, -1 \rangle = \langle \pi_f, -1 \rangle \\ 0 & \text{if } \langle \pi_\infty, -1 \rangle = -\langle \pi_f, -1 \rangle \end{cases}$$

where the pairings are defined in Section 3.2 in the archimedean case and stated to exist in Section 3.3 in the p -adic case. Using the description there, we have the formula

$$\langle \pi_\infty, -1 \rangle = (-1)^{q + \lfloor \frac{n}{2} \rfloor}$$

independent of the Weyl chamber attached to π_∞ . Therefore, if π_f is the finite part of some automorphic representation in the discrete spectrum with a discrete series archimedean component, then $\pi_f \otimes \pi_\infty$ also occurs with multiplicity one for any π_∞ in the same L -packet. In particular,

$$\dim_{\mathbf{C}} H_{L^2}^{pq}(\Sigma(V)(\mathbf{C}), \mathbb{L}_\rho)[\pi_f] = \binom{n}{p}$$

Since Zucker's conjecture is proven [Loo88, SS90], we can replace the L^2 -cohomology by the intersection cohomology on the Bailey–Borel compactification $\Sigma^*(V)$. This defines a Galois representation $IH^{pq}[\pi_f]$ of $\text{Gal}_{\mathcal{K}}$, which is expected to be $(\wedge^p \rho_\pi)(r_p)$, where $r_p = \frac{1}{2}p(p-1)$. This equality was proven away from a finite set of bad primes in [Mor10].

In our eventual construction, we will instead be using compactly supported cohomology instead of intersection cohomology, mostly for convenience. The forms we study will be tempered cuspidal, so they do not contribute to the boundary cohomology. It follows that the π_f -isotypic components will be the same.

3.6.2 Coherent cohomology

A representation W_κ of $K_\infty = \text{U}(p) \times \text{U}(q)$ gives an automorphic vector bundle

$$\mathcal{E}_\kappa = G(\mathbf{Q}) \backslash (G(\mathbb{A}_f) \times (G(\mathbf{R}) \times W_\kappa) / K_\infty)$$

over $\Sigma(V)$. The action of $k \in K_\infty$ is by $(g, v)k = (gk, k^{-1}v)$. We index them by the highest weight of W_κ (with respect to the usual choice of positive roots), written as $(\kappa_1 \geq \dots \geq \kappa_p; \kappa_{p+1} \geq \dots \geq \kappa_n)$, so in particular they are all integers. Moreover, Harris [Har90] defined extensions $\mathcal{E}_\kappa^{\text{can}}$ and $\mathcal{E}_\kappa^{\text{sub}}$ over appropriate toroidal compactifications $\overline{\Sigma}(V)$. This picture also holds for the canonical model over

$\mathcal{K}$, and the detailed constructions can be found in [Mil90].

The choice of the Shimura datum defines a decomposition of the Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$, corresponding to the subspaces with weights 0, 1, -1 respectively. The following formula was proven for sufficiently regular weights by [Har90] and later extended to arbitrary weights by J. Su [Su19].

$$H^d(\overline{\Sigma}(V), \mathcal{E}_{\kappa}^{\text{can}}) = H^d(\text{Hom}_{K_{\infty}}(\wedge^{\bullet} \mathfrak{p}^-, \mathcal{A}(G) \otimes W_{\kappa}))$$

Both sides carry natural actions of $G(\mathbb{A}_f)$, and the identification is equivariant. Let π be a cuspidal automorphic representation on G such that π_{∞} is in the discrete series with infinitesimal character $(a_1 > \cdots > a_n)$. Take π_f -isotypic component of both sides, then the right hand side reduces to $H^d(\mathfrak{p}, K_{\infty}, \pi_{\infty} \otimes W_{\kappa})$ since π occurs with multiplicity 1, as computed in the previous subsection.

Let $\mathcal{C}$ be a Weyl chamber, and let $\{i_1, \dots, i_p\}$ denote the subset corresponding to its $\Omega_{\mathbf{R}}^{(p)}$ -orbit as described in Section 3.2. Suppose π_{∞} is the discrete series associated to this orbit. Let $\lambda_{\mathcal{C}}$ be the K_{∞} -dominant character in this chamber with the same infinitesimal character. The dual of π_{∞} is also a discrete series, and it has parameter $-\lambda_{\mathcal{C}}$. Let ρ_G denote the half sum of positive roots of G (with respect to the positive roots of K_{∞} chosen earlier plus $\mathfrak{p}^+$). Applying [Har90, Theorem 4.6.2], we get that $H^{\bullet}(\mathfrak{p}, K_{\infty}, \pi_{\infty} \otimes W_{\kappa}) = 0$ unless $-\lambda_{\mathcal{C}} = \kappa + \rho_G$, in which case it is 1-dimensional and concentrated in degree

$$d_{\mathcal{C}} := \#\{(a, b) \mid 1 \leq a \leq p, 1 \leq b \leq q, i_a < j_b\}$$

In particular, if $\lambda_{\mathcal{C}}$ gives a holomorphic discrete series, then $d_{\mathcal{C}} = 0$, so

$$\lambda_{\text{Hol}} = -(a_{q+1}t_1 + \cdots + a_nt_p + a_1t_{p+1} + \cdots + a_qt_n)$$

The condition on κ is then the usual one $\kappa_{p+1} \geq \cdots \geq \kappa_n > \kappa_1 \geq \cdots \geq \kappa_n$, where $\kappa_n - \kappa_1 \geq n$.

Given a finite part π_f which is stable, the computation of multiplicity in the previous subsection can be applied here as well. Let κ be a weight such that $\kappa + \rho_G$ is regular, then it determines a Weyl chamber $\mathcal{C}$. This discussion gives the dimension formula

$$\dim_{\mathcal{K}} H^{d_{\mathcal{C}}}(\overline{\Sigma}(G), \mathcal{E}_{\kappa}^{\text{can}})[\pi_f] = 1$$

The other degrees vanish. This gets more complicated if $\kappa + \rho_G$ is not regular or if π_f is not stable. In this thesis, we will only apply the computations here in the final section to predict the limitations

of certain approaches towards constructing p -adic L -functions.

3.6.3 Special cycle

We now briefly review the construction of the diagonal cycle and its étale realization outlined in [RSZ20], mainly as a motivation to the p -adic interpolation performed in the next part. They use a different Shimura datum which is of PEL-type, but since we are not working integrally, this is not a serious issue for our purpose.

Let $W \subseteq V$ be an embedding of Hermitian spaces such that the archimedean signatures are $(1, n-2)$ and $(1, n-1)$ respectively, then we get an embedding of Shimura varieties $\Sigma(W) \hookrightarrow \Sigma(W) \times \Sigma(V)$ with codimension $n-1$. This defines a cycle in a Chow group

$$[\Sigma(W)] \in \mathrm{CH}^{n-1}(\Sigma(W) \times \Sigma(V))$$

Conjecturally, there is a canonical projection to the null homological part, but without knowing this, *loc. cit.* applied a non-canonical sign projector defined in [MS19], which is an element in the Hecke algebra that kills all cohomology classes in even dimensions. This produces a non-canonical homologically trivial cycle

$$[\widetilde{\Sigma(W)}] \in \mathrm{CH}^{n-1}(\Sigma(W) \times \Sigma(V))_0$$

The p -adic étale Abel–Jacobi map (cf. [Nek00], where it is denoted by $cl_{1,X}$) can take this and produce a class in Galois cohomology, and we can further decompose the Galois module using Kummer decomposition

$$\begin{aligned} z &\in H_g^1(\mathcal{K}, H_{\text{ét}}^{2n-3}((\Sigma(W) \times \Sigma(V))_{/\mathcal{K}}, \mathbf{Q}_p(n-1))) \\ &\longrightarrow H_g^1(\mathcal{K}, H_{\text{ét}}^{n-1}(\Sigma(W)_{/\mathcal{K}}, \mathbf{Q}_p) \otimes H_{\text{ét}}^{n-2}(\Sigma(V)_{/\mathcal{K}}, \mathbf{Q}_p) \otimes \mathbf{Q}_p(n-1)) \end{aligned}$$

Now each of the cohomology groups is the étale cohomology in the middle dimension, so by the computation in Subsection 3.6.1, it contains Galois representations. In fact, if $\pi = \pi_n \otimes \pi_{n-1}^\vee$ is a stable representation of $G = \mathrm{U}(W) \times \mathrm{U}(V)$ which has minimal weight at infinity (corresponding to trivial coefficients), then we can project z further to obtain a class

$$z_{\mathrm{GGP}} \in H_g^1(\mathcal{K}, \rho_{\pi_n} \otimes \rho_{\pi_{n-1}^\vee}(n-1)) = H_g^1(\mathcal{K}, \rho)$$

In all of the above discussions, the subscript g denotes the “geometric” Selmer condition defined in

[FPR94, Section 3.1]. This is part of the formalism of the étale Abel–Jacobi map, though it is in fact expected to land in the more restrictive Bloch–Kato Selmer group $H_f^1(\mathcal{K}, \rho)$ [Nek00, Conjecture 2.1.2]. We will interpolate certain variants of this class in the next chapter.

Chapter 4

Interpolation of the diagonal cycle

In this section, which forms the second and main part of the thesis, we will construct a p -adic interpolation of the diagonal cycle arising from the arithmetic Gan–Gross–Prasad conjecture. In the next section, we will present a framework showing that this cycle is only a small piece of the very large puzzle of Iwasawa theory in this setting, and much of the other pieces are not within reach of currently known methods.

4.1 Completed cohomology

In this section, we define some completed cohomology groups which mildly generalize Emerton’s constructions in [Eme06]. Since it does not introduce additional difficulties, the setting here will be slightly more general than what is required for this thesis.

Let F be an arbitrary field of characteristic 0. Let I be a countable directed set and $X = (X_i)_{i \in I}$ be an inverse system of smooth algebraic varieties defined over F with finite étale transition maps. Let $? \in \{\text{ét}, c\}$ denote étale and compactly supported cohomology respectively. Given $\mathcal{F}$ a p -adic local system on X_0 , we can define the following spaces

$$H_{?, \text{Iw}}^\bullet(X, \mathcal{F}) := \varprojlim_n \varprojlim_i H_{?}^\bullet(X_i, \pi_i^*(\mathcal{F}/p^n \mathcal{F})), \quad \tilde{H}_{?}^\bullet(X, \mathcal{F}) := \varprojlim_n \varinjlim_i H_{?}^\bullet(X_i, \pi_i^*(\mathcal{F}/p^n \mathcal{F}))$$

and refer to them as the (absolute) Iwasawa cohomology and completed cohomology groups. In the definition of $H_{?, \text{Iw}}^\bullet$, the inverse limit in i is taken with respect to the trace map. The transition maps in $\tilde{H}_{?}^\bullet$ are the natural ones, so it differs from Emerton’s definition only in that we are working over F instead of $\bar{F}$. The notation conflicts with the notation for Selmer complexes, but this should not

cause any confusion. We will now define several classical constructions in this setting.

4.1.1 Duality pairing I

In this subsection, suppose F is algebraically closed. The comparison isomorphism between étale and singular cohomology allows us to replace the cohomology groups by their topological variants. Let $d = \dim X$, i.e. the common dimension of all X_i viewed as algebraic varieties, then topological Poincaré duality for smooth orientable manifolds gives an isomorphism

$$H^\alpha(X_i(\mathbf{C}), \mathcal{F}) \simeq H_{2d-\alpha}^{\text{BM}}(X_i(\mathbf{C}), \mathcal{F})$$

where $H_\bullet^{\text{BM}}$ is the Borel–Moore homology. Therefore, the Iwasawa cohomology we have defined is exactly Emerton’s completed homology $\tilde{H}_\alpha(X, \mathcal{F})$ in this geometric case, as defined in [CE12, Section 1.1]. The universal coefficient theorem then produces a short exact sequence

$$0 \rightarrow \text{Hom}(\tilde{H}_c^{2d-\alpha+1}(X, \mathbf{Z}_p)[p^\infty], \mathbf{Q}_p/\mathbf{Z}_p) \rightarrow H_{\text{Iw}}^\alpha(X, \mathbf{Z}_p) \xrightarrow{\text{P}} \text{Hom}(\tilde{H}_c^{2d-\alpha}(X, \mathbf{Z}_p), \mathbf{Z}_p) \rightarrow 0$$

which is stated as Theorem 1.1(3) without proof in [CE12]. Suppose $\alpha = d$, then the first term is 0 by the Calegari–Emerton conjecture, proven in our case by Hansen–Johansson [HJ20]. Therefore, we obtain a perfect $\mathbf{Z}_p$ -pairing between $H_{\text{Iw}}^d(X, \mathbf{Z}_p)$ and $\tilde{H}_c^d(X, \mathbf{Z}_p)$.

Now suppose all terms X_i are defined over a common subfield $\mathcal{K} \subseteq F$, then all terms of the final short exact sequence carries an action of $\text{Gal}(F/\mathcal{K})$. It is a priori not clear if the maps are Galois-equivariant. The map is equivariant if $\mathbf{Z}_p$ is replaced by the Tate twist $\mathbf{Z}_p(d)$, i.e. the map P from the above short exact sequence defines a Galois-equivariant map

$$\text{P} : H_{\text{Iw}}^\alpha(X, \mathbf{Z}_p) \rightarrow \text{Hom}(\tilde{H}_c^{2d-\alpha}(X, \mathbf{Z}_p), \mathbf{Z}_p(d))$$

Indeed, observe that this map is compatible with the Poincaré duality map in étale cohomology at each finite level, which are Galois equivariant.

4.1.2 Duality pairing II

Again suppose F is algebraically closed, so for each $? \in \{\text{ét}, c\}$, we can compute $H_{?, \text{Iw}}^i(X, \mathbf{Z}_p)$ using a finite simplicial complex $S_\bullet^?(X)$. Fix an element $0 \in I$ and let G_0 be the Galois group of X over X_0 , i.e. the inverse limit of the group of Deck transformations of X_i over X_0 as i ranges over I . The

simplicial complex then carries a natural action of the completed group ring $\mathbf{Z}_p[[G_0]]$.

Poincaré duality between H^α and $H_c^{2d-\alpha}$ at each level is defined on the complex level, and it respects the action of G . Therefore, there is a duality at the complex level

$$\mathbf{R}\mathrm{Hom}_{\mathbf{Z}_p[[G_0]]}(S_{\text{ét}}^\bullet(X), \mathbf{Z}_p[[G_0]])[2d] = S_c^\bullet(X) \quad \text{in } D^b(\mathbf{Z}_p[[G_0]])$$

Let $E^\alpha := \mathrm{Ext}_{\mathbf{Z}_p[[G]]}^\alpha(-, \mathbf{Z}_p[[G_0]])$, then we obtain a spectral sequence

$$E^\alpha(H_{\mathrm{Iw}}^\beta(X, \mathbf{Z}_p)) \implies H_{c, \mathrm{Iw}}^{\alpha-\beta+2d}(X, \mathbf{Z}_p)$$

This is stated without proof in Section 1.3 of [CE12], and the above is an elaboration of the proof sketched in Section 3 of [CE11]. Observe that the G_0 -structure on the left hand side is naturally on the right, so we need to move it to the left by using $g \mapsto g^{-1}$. For the Galois action, the same comment as in the previous section applies, including needing a Tate twist.

For our later applications, there is usually a bigger non-compact group G acting on the spaces, and we would like to extend the $\mathbf{Z}_p[[G_0]]$ -action on the terms on the left hand side to a g -action so that the spectral sequence is G_0 -equivariant. This can be done by using the ring $\mathbf{Z}_p[[G_0]] \otimes_{\mathbf{Z}_p[G_0]} \mathbf{Z}_p[G]$ as the base ring and observe that the spaces do not change.

4.1.3 Künneth decomposition

If $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ are two projective systems of the type considered before, then we can form their product $(X_i \times Y_j)_{(i,j) \in I \times J}$, and there is a natural map

$$\bigoplus_{\alpha_1 + \alpha_2 = \alpha} H_c^{\alpha_1}(X_i, \mathbf{Z}_p) \otimes H_c^{\alpha_2}(Y_j, \mathbf{Z}_p) \rightarrow H_c^\alpha(X_i \times Y_j, \mathbf{Z}_p)$$

Its construction is formal and can be found in [Mil80, Chapter IV Theorem 8.5]. Taking direct limit is an exact operation, so we obtain a map

$$\mathbf{K} : \bigoplus_{\alpha_1 + \alpha_2 = \alpha} \tilde{H}_c^{\alpha_1}(X, \mathbf{Z}_p) \otimes \tilde{H}_c^{\alpha_2}(Y, \mathbf{Z}_p) \rightarrow \tilde{H}_c^\alpha(X \times Y, \mathbf{Z}_p)$$

Its cokernel can also be identified in terms of certain Tor-groups, but we will not need that here.

4.1.4 Cycle class map

Suppose $\iota : (X_i)_{i \in I} \rightarrow (Y_j)_{j \in J}$ is a closed immersion of projective systems of smooth schemes with codimension c . More precisely, this means for any $i \in I$, there exists $j(i) \in J$ such that X_i is a closed subscheme of $Y_{j(i)}$, and that all these closed immersions are compatible with transition maps in a natural way. The classical cycle class map is constructed based on purity, so its compatibility with trace maps is formal, cf. [Mil80, Section IV.9]. Therefore, we can define

$$\iota_! : H_{\text{Iw}}^i(X, \mathbf{Z}_p) \rightarrow H_{\text{Iw}}^{i+2c}(Y, \mathbf{Z}_p(c))$$

simply as the inverse limit of the usual cycle class maps. This requires the technical hypothesis that the set $\{j(i) \mid i \in I\}$ is cofinal in J .

4.1.5 Abel–Jacobi map

If Y is a variety over F , then we have the Hochschild–Serre spectral sequence

$$E_2^{\alpha, \beta} = H^\alpha(F, H_{\text{cont}}^\beta(Y_{/\bar{F}}, \mathcal{F})) \implies H_{\text{cont}}^{\alpha+\beta}(Y, \mathcal{F})$$

where $H_{\text{cont}}^\bullet$ is the continuous étale cohomology defined in [Jan88]. Recall that in our setting, it is defined as

$$H_{\text{cont}}^\bullet(Y, \mathcal{F}) := R^\bullet \left(\varprojlim_n \Gamma_{\text{ét}}(Y, -) \right) (\mathcal{F}/p^n \mathcal{F})_{n \geq 0}$$

The spectral sequence computing the right hand side degenerates to the short exact sequence

$$0 \rightarrow \varprojlim_n H^{\alpha-1}(Y, \mathcal{F}/p^n \mathcal{F}) \rightarrow H_{\text{cont}}^\alpha(Y, \mathcal{F}) \rightarrow \varprojlim_n H^\alpha(Y, \mathcal{F}/p^n \mathcal{F}) \rightarrow 0$$

In particular, $H_{\text{cont}}^\bullet$ coincides with the usual definition of étale cohomology in two cases:

1. $\mathcal{F}$ is a torsion sheaf
2. $F = \bar{F}$

But they do not necessarily agree for example when $F = \mathbf{Q}$, which does not have good finiteness properties for its Galois cohomology.

The Hochschild–Serre spectral sequence does not degenerate in the generality that we consider, but $E_\infty^{0, \alpha}$ and $E_\infty^{1, \alpha-1}$ are both submodules of the corresponding terms in E_2 , so we can still define

the geometric cycle class map and Abel–Jacobi map

$$\begin{aligned} \mathrm{cl}_Y^\alpha &: H_{\mathrm{cont}}^\alpha(Y, \mathcal{F}) \rightarrow H_{\mathrm{\acute{e}t}}^\alpha(Y_{/\bar{F}}, \mathcal{F})^{\mathrm{Gal}_F} \\ \mathrm{AJ}_Y^\alpha &: H_{\mathrm{cont}}^\alpha(Y, \mathcal{F})_0 := \ker(\mathrm{cl}_Y^\alpha) \rightarrow H^1(F, H_{\mathrm{\acute{e}t}}^{\alpha-1}(Y_{/\bar{F}}, \mathcal{F})) \end{aligned}$$

Our goal is to extend this to Iwasawa cohomology. In each of the above two maps, replace Y by X_i and $\mathcal{F}$ by $\mathcal{F}/p^n \mathcal{F}$. The resulting maps are compatible as i and n vary, since the spectral sequence was deduced as a composition of derived pushforward maps. Therefore, by taking inverse limits, we get

$$\begin{aligned} \mathrm{cl}_X^\alpha &: H_{\mathrm{Iw}}^\alpha(X_i, \mathcal{F}) \rightarrow H_{\mathrm{Iw}}^\alpha(X_{/\bar{F}}, \mathcal{F})^{\mathrm{Gal}_F} \\ \mathrm{AJ}_X^\alpha &: H_{\mathrm{Iw}}^\alpha(X_i, \mathcal{F})_0 \rightarrow \varprojlim_{i,n} H^1(F, H_{\mathrm{\acute{e}t}}^{\alpha-1}((X_i)_{/\bar{F}}, \mathcal{F}/p^n \mathcal{F})) \end{aligned}$$

where we have repeatedly use the fact that $\varprojlim$ is left exact to pass various kernels inside the limit.

Recall again that our Galois cohomology is continuous cochain cohomology, so the short exact sequence relating $H_{\mathrm{cont}}^\bullet$ and $H_{\mathrm{\acute{e}t}}^\bullet$ becomes

$$0 \rightarrow \varprojlim_{i,n}^1 H^0(F, V_{i,n}^{\alpha-1}) \rightarrow H^1(F, H_{\mathrm{Iw}}^{\alpha-1}(X_{/\bar{F}}, \mathcal{F})) \rightarrow \varprojlim_{i,n} H^1(F, V_{i,n}^{\alpha-1}) \rightarrow 0$$

where $V_{i,n}^{\alpha-1} = H_{\mathrm{\acute{e}t}}^{\alpha-1}((X_i)_{/\bar{F}}, \mathcal{F}/p^n \mathcal{F})$ are finite abelian groups by the usual finiteness properties of étale cohomology. It follows that the inverse system $H^0(F, V_{i,n}^{\alpha-1})$ in (i, n) satisfies the Mittag-Leffler condition. Therefore, the kernel term in the short exact sequence is 0, and we recover an interpolated Abel–Jacobi map

$$\mathrm{AJ}_X^\alpha : H_{\mathrm{Iw}}^\alpha(X, \mathcal{F})_0 \rightarrow H^1(F, H_{\mathrm{Iw}}^{\alpha-1}(X_{/\bar{F}}, \mathcal{F}))$$

where $H_{\mathrm{Iw}}^\alpha(X, \mathcal{F})_0 = \ker(H_{\mathrm{Iw}}^\alpha(X, \mathcal{F}) \rightarrow H_{\mathrm{Iw}}^\alpha(X_{/\bar{F}}, \mathcal{F}))$.

4.2 Families in completed cohomology

We now specialize the general constructions in the previous section to the case of interest. Let $\mathbf{G}$ be a reductive group such that one can attach an abelian type Shimura variety of dimension d to $\mathbf{G}$. Moreover, suppose $\mathbf{G}$ splits over a number field F , so all of its reflex fields are contained in F . Fix a tame level $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$, then the neat open compact subgroups of $\mathbf{G}(\mathbb{Q}_p)$ form a directed set. This gives rise to an inverse system $\Sigma(\mathbf{G}, K^p) := (\Sigma(\mathbf{G}, K_p K^p))_{K_p}$ of the type considered in the previous

section defined over F . We are interested in the following groups

$$H_{\text{Iw}}^\bullet(K^p) := H_{\text{Iw}}^\bullet(\Sigma(\mathbf{G}, K^p)_{/\bar{F}}, \mathbf{Z}_p), \quad \tilde{H}_c^\bullet(K^p) := \tilde{H}_c^\bullet(\Sigma(\mathbf{G}, K^p)_{/\bar{F}}, \mathbf{Z}_p)$$

especially in the middle degree d . They carry commuting action of Gal_F , the p -adic Lie group action $\mathbf{G}(\mathbf{Q}_p)$, and the tame Hecke algebra $\mathcal{H}^p = C_c^\infty(\mathbf{G}(\mathbb{A}_f^p)/K^p)$, which is the convolution algebra of all smooth compactly supported K^p -bi-invariant functions on $\mathbf{G}(\mathbb{A}_f^p)$. Let Σ be the finite set of places away from p such that K^p is not a maximal compact subgroup, then we have the factorization

$$\mathcal{H}^p = \mathcal{H}^\Sigma \otimes \mathcal{H}^{\text{sph}}, \quad \mathcal{H}^{\text{sph}} := C_c^\infty(\mathbf{G}(\mathbb{A}_f^{p, \Sigma})/K^{p, \Sigma})$$

The spherical part $\mathcal{H}^{\text{sph}}$ is abelian. Emerton's approach to constructing eigenvarieties [Eme06] essentially consist of applying his locally analytic Jacquet functor to $\tilde{H}^\bullet$ to construct the spectral variety, and then use the action of $\mathcal{H}^{\text{sph}}$ to refine it to an eigenvariety. To get a good integral theory, we will use Emerton's ordinary part functor instead, which we will briefly recall.

In cases like ours where the group has Shimura varieties, Galois representations attached to cohomological automorphic forms are also expected to appear in the usual étale cohomology. Therefore, one expects $\tilde{H}^\bullet$ to contain families of such Galois representations. We will make this precise by constructing an ordinary p -adic family in the sense defined in Section 1 using $(\tilde{H}_c^\bullet)^\vee$.

4.2.1 Ordinary part

We first recall the definition of the category of p -adically continuous admissible representations over $\mathbf{Z}_p$ and its key properties, as defined in [Eme10a]. We will simply abbreviate this to admissible since this is the only notion needed here.

Definition 4.2.1. Let H be a p -adic Lie group. Let M be a $\mathbf{Z}_p$ -module with a continuous H -action (with respect to the p -adic topology on M). It is *admissible* if the following conditions are satisfied

1. M is p -adically complete and separated.
2. $M[p^\infty]$ has bounded exponent.
3. M/pM is a smooth admissible representation of H .

We will let $\text{Rep}_H^{\text{adm}}(\mathbf{Z}_p)$ denote the category of admissible representations as defined. On the dual side, let $\text{Rep}_H^{\text{aug, fg}}(\mathbf{Z}_p)$ denote the category of $\mathbf{Z}_p[H]$ -modules which can be completed to a

finitely generated $\mathbf{Z}_p[[H']]$ -modules for some (and hence any) open compact $H' \subseteq H$. A superscript fl denotes the subcategory of p -torsion-free objects.

Theorem 4.2.2 (Proposition 2.4.10 of [Eme10a]). *The category $\text{Rep}_H^{\text{adm}}(\mathbf{Z}_p)$ is abelian. Taking $\text{Hom}(-, \mathbf{Z}_p)$ is an $\mathbf{Z}_p$ -linear anti-equivalence of category between $\text{Rep}_H^{\text{adm}}(\mathbf{Z}_p)^{\text{fl}}$ and $\text{Rep}_H^{\text{aug, fg}}(\mathbf{Z}_p)^{\text{fl}}$.*

Theorem 4.2.3. *The $\mathbf{Z}_p$ -representation of $\mathbf{G}(\mathbf{Q}_p)$ on $\tilde{H}_c^\bullet(K^p)$ is admissible.*

Proof. This result is stated without proof in [CE12]. As stated in [Eme06], the cohomology can be computed by an explicit finite simplicial complex at each level. The direct limit of each term is easily shown to be admissible, so the result follows since the category is abelian. $\square$

We are now ready to define Emerton's ordinary part functor $\text{Ord}_{\mathbf{B}}$. For simplicity of notation, we will assume there is a given isomorphism $\mathbf{G}(\mathbf{Q}_p) \simeq \text{GL}_n(\mathbf{Q}_p)$. Let $\mathbf{B}$ denote the standard Borel subgroup of upper triangular matrices, $\mathbf{T} \simeq \mathbf{G}_m^n$ be its maximal torus, and $\mathbf{N}$ be its unipotent radical. We will typically use the usual letter to denote $\mathbf{Q}_p$ -points of the group defined by the corresponding bold letter. The group T contains the following two subsets

$$T^+ = \{(t_1, \dots, t_n) \in T \mid v_p(t_1) \geq \dots \geq v_p(t_n)\}, \quad T^{++} = \{(t_1, \dots, t_n) \in T \mid v_p(t_1) > \dots > v_p(t_n)\}$$

where v_p is the p -adic valuation on $\mathbf{Q}_p$. They have the property that T^+ preserves $\mathbf{N}(\mathbf{Z}_p)$ and T^{++} contracts $\mathbf{N}(\mathbf{Z}_p)$ under the conjugation action. Form the monoid algebras $\mathbf{Z}_p[T]$ and $\mathbf{Z}_p[T^+]$, then given a smooth admissible G -representation V , define

$$\text{Ord}_{\mathbf{B}}(V) = \text{Hom}_{\mathbf{Z}_p[T^+]}(\mathbf{Z}_p[T], V^{\mathbf{N}(\mathbf{Z}_p)})_{T\text{-finite}}$$

The action of $t \in \mathbf{Z}_p[T^+]$ on $V^{\mathbf{N}(\mathbf{Z}_p)}$ is defined by summing over the cosets of $\mathbf{N}(\mathbf{Z}_p)/t\mathbf{N}(\mathbf{Z}_p)t^{-1}$. Now, if M is admissible, then define

$$\text{Ord}_{\mathbf{B}}(M) = \varprojlim_n \text{Ord}_{\mathbf{B}}(M/p^n M)$$

The resulting module carries a natural T action coming from the first variable. The following theorem summarizes the properties of this functor we will require. The proofs are all found in the paper [Eme10a].

Theorem 4.2.4. 1. $\text{Ord}_{\mathbf{B}}$ is a left exact functor $\text{Rep}_G^{\text{adm}}(\mathbf{Z}_p) \rightarrow \text{Rep}_T^{\text{adm}}(\mathbf{Z}_p)$.

2. There is a closed embedding $\text{ev} : \text{Ord}_{\mathbf{B}}(M) \hookrightarrow M^{\mathbf{N}(\mathbf{Z}_p)}$ given by evaluation at 1.

3. Let U be an admissible representation of T , then there is an adjunction

$$\mathrm{Hom}_G(\mathrm{Ind}_{\overline{\mathbf{B}}}^{\mathbf{G}} U, M) = \mathrm{Hom}_T(U, \mathrm{Ord}_{\mathbf{B}}(M))$$

where $\overline{\mathbf{B}}$ is the opposite Borel of lower triangular matrices, and the parabolic induction consists of continuous functions.

Using the anti-equivalence of category stated above, we can define the following functor

$$R(M) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Ord}_{\mathbf{B}}(M), \mathbf{Z}_p)$$

This is a finitely generated p -torsion-free $\mathbf{Z}_p[[\mathbf{T}(\mathbf{Z}_p)]]$ -module with the additional action of $\mathbf{Z}^n$ coming from the valuation part of T . Observe that this ring is the Iwasawa algebra $\underline{\Lambda}_n$ defined in Section 1. Applying the $\mathbf{Z}_p$ -dual also kills the torsion of $\mathrm{Ord}_{\mathbf{B}}(M)$. However, item (2) of the theorem shows that the p^∞ -torsion of $\mathrm{Ord}_{\mathbf{B}}(M)$ can be controlled by the torsion of M , so this will not be a serious issue later.

Finally we want to study the relation of $\mathrm{Ord}_{\mathbf{B}}$ with other related methods of taking the ordinary part. Take any $t \in T^{++}$, then we have the following result.

Lemma 4.2.5 (Lemma 3.2.1 of [Eme10b]). *Let X be a locally T^+ -finite $\mathbf{Z}_p[T^+]$ -module, then the natural maps in the following diagram are all isomorphisms*

$$\mathrm{Hom}_{\mathbf{Z}_p[T^+]}(\mathbf{Z}_p[T], X)_{T\text{-finite}} \rightarrow \mathrm{Hom}_{\mathbf{Z}_p[t]}(\mathbf{Z}_p[t^{\pm 1}], X)_{t\text{-finite}} \rightarrow X \otimes_{\mathbf{Z}_p[t]} \mathbf{Z}_p[t^{\pm 1}]$$

This can be applied to finite level cohomology groups to show that $\mathrm{Ord}_{\mathbf{B}}$ amounts to simply inverting the action of t . In [ACC⁺18], the authors chose to use this latter definition to define the ordinary part. This has the advantage of better homological algebra properties, since the conjectured right derived functor constructed in [Eme10b] is in fact the derived functor with this definition (since localization is exact). However, it is not known if the result is still admissible.

4.2.2 Hida theory for a single group

We will now apply the functor R defined above to completed cohomology. Keeping the group $\mathbf{G}$ as before, in particular fixing an isomorphism $\mathbf{G}(\mathbf{Q})_p \simeq \mathrm{GL}_n(\mathbf{Q}_p)$ and a tame level K^p . Define the module

$$\tilde{T} := R(\tilde{H}_c^d(\Sigma(\mathbf{G}, K^p)_{/\bar{F}}, \mathbf{Z}_p))$$

which we observed is a finitely generated $\underline{\Lambda}_n$ -module. This still carries commuting actions of Gal_F and $\mathcal{H}^p$. Let $\mathbb{T}^\Sigma$ be the closure of the $\mathbf{Z}_p$ -algebra generated by the image of the commutative algebra $\mathcal{H}^{\text{sp}}_c$ in the endomorphisms of all $H_c^i(\Sigma(\mathbf{G}, K_p K^p), \mathbb{L}/p^n)$ as K_p varies over wild levels, $\mathbb{L}$ varies over automorphic local systems, and $n \geq 0$. This is a semilocal $\mathbf{Z}_p$ -algebra which acts on every cohomology groups we have used. Moreover, we can define $\mathfrak{h}$ to be the image of $\mathbb{T}^\Sigma$ and $\underline{\Lambda}_n$ in $\text{End}_{\underline{\Lambda}_n}(\tilde{\mathcal{T}})$, so by admissibility, $\mathfrak{h}$ is a finite $\underline{\Lambda}_n$ -algebra. By definition, $\tilde{\mathcal{T}}$ is an $\mathfrak{h}$ -module. We will think of the rigid analytic space attached to $\mathfrak{h}$ as the ordinary eigenvariety. The upshot of this section is the following theorem.

Theorem 4.2.6. *If $\mathfrak{m}$ is a “good” maximal ideal of $\mathbb{T}^\Sigma$, then the pair $(\tilde{\mathcal{T}}_n, \underline{\mathbb{L}}_n) := (\tilde{\mathcal{T}}_{\mathfrak{m}}, \mathfrak{h}_{\mathfrak{m}})$ is a conjugate self-dual family of p -adic representations interpolating compactly supported cohomology.*

Before proving the theorem, it is necessary to define “good”. The following definition is simply the exact condition to make the theorem work. After the proof, we will discuss what kind of actually verifiable conditions imply “good”.

Definition 4.2.7. A maximal ideal $\mathfrak{m}$ is “good” in the sense above if $H_{?, \text{Iw}}^\bullet(K^p)_{\mathfrak{m}}$ is concentrated in the middle degree d , and in that degree, it is projective over $\mathbf{Z}_p[[K_p]]$ for any open compact subgroup $K_p \subseteq \mathbf{G}(\mathbf{Q}_p)$.

Remark 4.2.8. The definition of $\tilde{\mathcal{T}}_{\mathfrak{m}}$ still makes sense if $\mathfrak{m}$ is not “good”, so the method of the next section can be used to define a big diagonal form valued in it. However, the local behaviour is more subtle, and we do not expect perfect control and projectivity. There will be a map $\tilde{\mathcal{T}} \rightarrow \tilde{\mathcal{T}}^\vee$, but we are unable to even prove that it is non-trivial.

In fact, parts of the above vanishing hypothesis holds automatically. These kind of results were previously conjectured by Calegari–Emerton in [CE12]. Based on Scholze’s perfectoid Shimura variety and the methods of Caraiani–Scholze [CS17], it has been proven in many cases by Hansen–Johansson [HJ20]. We state their result as it applies to our situation

Theorem 4.2.9 (Theorem 1.4 of *loc. cit.*). *If Leopoldt’s conjecture holds for $Z(\mathbf{G})$, then*

1. *For $i > d$, all groups $\tilde{H}_?^i(K^p)$, $H_{?, \text{Iw}}^i(K^p)$ vanish.*
2. *For $i < d$, the groups $H_{?, \text{Iw}}^i(K^p)$ are torsion over $\mathbf{Z}_p[[K_p]]$ for K_p as before.*
3. *The natural map $H_{c, \text{Iw}}^d(K^p) \simeq H_{\text{Iw}}^d(K^p)$ is an isomorphism, and both groups are p -torsion-free.*

Remark 4.2.10. Since Leopoldt’s conjecture holds trivially for an imaginary quadratic field, the hypothesis is satisfied for our eventual application.

We are now ready to prove theorem 4.2.6. As was known before, the key part is the control theorem, namely the big module over the weight space we constructed should specialize to cohomology at finite levels at classical points. Classically, this is split into two parts: horizontal control varies the weight of the local system, and vertical control varies the level. The derived versions of the two theorems in our setting were proven in Section 5.2 of [ACC⁺18], which amounts to pushing derived functors around. Before stating the results, we set up some notations for Iwahori-type wild levels. Given integers $c \geq b \geq 0$, let

$$K_{(b,c)} = \{A \in \mathrm{GL}_n(\mathbf{Z}_p) \mid A \pmod{p^b} \in \mathbf{N}(\mathbf{Z}/p^b\mathbf{Z}), A \pmod{p^c} \in \mathbf{B}(\mathbf{Z}/p^b\mathbf{Z})\}$$

So in particular $K_{(0,1)} = \ker(\mathrm{GL}_n(\mathbf{Z}_p) \rightarrow \mathbf{B}(\mathbf{Z}/p\mathbf{Z}))$ is the classical Iwahori level. Further define $T_b = \ker(\mathbf{T}(\mathbf{Z}_p) \rightarrow \mathbf{T}(\mathbf{Z}/p^b\mathbf{Z}))$, an open compact subgroup of $\mathbf{T}(\mathbf{Z}_p)$.

Proposition 4.2.11. *Let W be an algebraic representation of $\mathbf{G}$ with highest weight λ . Let $\mathbb{L}_\lambda$ be the associated p -adic local system. Define $\mathbf{R}\mathrm{Ord}_{\mathbf{B}} = \mathbf{R}\Gamma(\mathbf{N}(\mathbf{Z}_p), -) \otimes_{\mathbf{Z}_p[t]} \mathbf{Z}_p[t^{\pm 1}]$ for any $t \in T^{++}$, then*

$$\mathbf{R}\Gamma(T_b, \mathcal{O}(\lambda^\vee) \otimes \mathbf{R}\mathrm{Ord}_{\mathbf{B}}(\widetilde{\mathbf{R}\Gamma}_c(\Sigma(\mathbf{G}, K^p)_{/\bar{F}}, \mathbf{Z}_p))) = \mathbf{R}\Gamma_c(\Sigma(\mathbf{G}, K_{(b,c)}K^p), \mathbb{L}_\lambda)^{\mathrm{ord}}$$

This $\mathbf{T}^\Sigma$ -equivariant. The classical U_p -operators on the right hand side correspond to the T -action on the left hand side.

Proof of Theorem 4.2.6. Throughout this proof, all modules are localized at $\mathfrak{m}$, and we will drop it from notation. Using the duality between Iwasawa and completed cohomology, the “good” hypothesis implies the vanishing of $\tilde{H}_c^\bullet$. Moreover, the anti-equivalence of category between admissible G -representations and augmented $\mathbf{Z}_p[[K_p]]$ -module shows that $\tilde{H}_c^d(K^p)$ is an injective object. Therefore, the above spectral sequence degenerates to

$$\mathbf{R}\Gamma(T_b, \mathcal{O}(\lambda^\vee) \otimes \mathrm{Ord}_{\mathbf{B}}\tilde{H}_c^d(K^p)) = \mathbf{R}\Gamma_c(\Sigma(\mathbf{G}, K_{(b,c)}K^p), \mathbb{L}_\lambda)^{\mathrm{ord}}$$

However, by the short exact sequence [CE12, (1.1)], the right hand side is also concentrated in degree d since we are assuming $\tilde{H}_c^d(K^p)$ is p -torsion-free. Specializing to that degree and taking the dual, we get a perfect control theorem

$$\mathrm{Ord}_{\mathbf{B}}\tilde{H}_c^d(K^p) \otimes_{\underline{\Lambda}_n} \mathbf{Z}_p[(\mathbf{Z}/p^b\mathbf{Z})^n](\lambda) = H^d(\Sigma(\mathbf{G}, K_{(b,c)}), \mathbb{L}_\lambda)^{\mathrm{ord}}$$

where we have also applied Poincaré duality on the right hand side and replaced $\lambda^\vee$ with λ .

Finally, we want to invoke the derived Poincaré duality pairing constructed in Subsection 4.1.2. Observe that by the discussions in Subsection 4.1.1,

$$\tilde{\mathbf{T}} = \mathrm{Ord}_{\mathbf{B}}(H_{\mathrm{Iw}}^d(\Sigma(\mathbf{G}, K^p), \mathbf{Z}_p)^\vee)^\vee$$

There is a derived duality relating the inside term to its compactly supported version, and we want to take it outside. From the spectral sequence, we get a map

$$H_{\mathrm{Iw}}^d(K^p) \rightarrow \mathrm{Hom}_{\mathbf{Z}_p[[K_p]]}(H_{c, \mathrm{Iw}}^d(K^p), \mathbf{Z}_p[[K_p]])$$

We claim that applying the functor $M \mapsto (\mathrm{Ord}_{\mathbf{B}} M^\vee)^\vee$ to this gives a map

$$\tilde{\mathbf{T}} \rightarrow \mathrm{Hom}_{\underline{\Lambda}_n}(\tilde{\mathbf{T}}_c, \underline{\Lambda}_n)$$

where $\tilde{\mathbf{T}}_c$ is defined in the same way as $\tilde{\mathbf{T}}$ except with compactly supported cohomology. Indeed, let $M = H_{c, \mathrm{Iw}}^d(K^p)$ and $V = M^\vee$, then we can compute

$$\begin{aligned} (\mathrm{Ord}_{\mathbf{B}} \mathrm{Hom}_{\mathbf{Z}_p[[K_p]]}(M, \mathbf{Z}_p[[K_p]]))^\vee &\simeq (\mathrm{Ord}_{\mathbf{B}} \mathrm{Hom}_G(\mathcal{C}^0(K_p), V))^\vee \\ &\rightarrow \mathrm{Hom}_T(\mathcal{C}^0(T \cap K_p), \mathrm{Ord}_{\mathbf{B}} V)^\vee \\ &\simeq \mathrm{Hom}_{\underline{\Lambda}_n}((\mathrm{Ord}_{\mathbf{B}} V)^\vee, \underline{\Lambda}_n) \end{aligned}$$

where the middle arrow follows from the adjunction between parabolic induction and ordinary part, and we are assuming K_p has an Iwasawa decomposition. By Theorem 4.2.9, we may replace $\tilde{\mathbf{T}}_c$ with $\tilde{\mathbf{T}}$ to obtain a self-pairing. The perfect control theorem implies by density and projectivity that the resulting map is a perfect pairing. It also tells us how to insert Galois action: we need to decorate $\tilde{\mathbf{T}}$ with a complex conjugation and add a Tate twist by d to $\underline{\Lambda}_n$. $\square$

We now comment on the condition of “good”. In his ICM address, Emerton made a conjecture that $\mathfrak{m}$ being non-Eisenstein implies $\mathfrak{m}$ is “good” in our sense [Eme14, Conjecture 3.3]. He did not give a precise definition of non-Eisenstein since the Galois representations he required was not known to exist. In our case, his conjectural definition of non-Eisenstein should correspond to residually irreducible. Before making the conjecture, he provided some plausibility arguments and observed that Arthur’s conjecture would imply his conjecture after inverting p . We are unable to follow his argument, since CAP forms can be supercuspidal at a finite number of places, which a priori may

include p .

We can however give some promising answers. In light of the results of Hansen–Johansson, we only need to impose $H_{\text{Iw}}^i(K^p)_{\mathfrak{m}} = 0$ for $i < d$. The corresponding version for the compactly supported version follows by duality. Recall the definition

$$H_{c,\text{Iw}}^i(K^p) = \varprojlim_{K_p} H_c^i(\Sigma(\mathbf{G}, K_p K^p), \mathbf{Z}_p)$$

so the result certain follows if we know the vanishing of the individual cohomology groups. For $\mathbf{G} = \text{GU}(n, n)$ and F is a totally real field not equal to $\mathbf{Q}$, this is the main result of [CS21], where the conditions imposed on $\mathfrak{m}$ are (1) $\tilde{\mathbf{T}}_{\mathfrak{m}}$ is residually of length at most 2; and (2) The same Galois representation residually distinguished at an unramified prime $\ell \neq p$. Their methods have been considerably simplified by Koshikawa [Kos21], and we have been informed by L. Hamann that the results for a general unitary similitude group should be within reach and is being worked on by M. Santos. The other parts of this thesis is insensitive to the difference between $\text{U}(V)$ and $\text{GU}(V)$, so we could eventually apply her result to the PEL-type Shimura variety instead.

4.2.3 Product of two unitary groups

We now specialize the above construction to the Gan–Gross–Prasad setting. Given $\pi = \pi_n \otimes \pi_{n+1}$ an automorphic form of the type we have been considering. Further suppose it is ordinary at p and residually irreducible, and that it decomposes generically at an $\ell \neq p$. For $i = n, n+1$, we have constructed a p -adic family $\tilde{\mathbf{T}}_i$ lying over a big ordinary Hecke algebra $\mathbb{I}_i$ interpolating ρ_{π_i} . They are both projective over the respective weight spaces. Now define

$$\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_n \hat{\otimes} \tilde{\mathbf{T}}_{n+1}(n)$$

This is a projective module over $\underline{\Lambda}_n \hat{\otimes} \underline{\Lambda}_{n+1}$, which is the weight space with $2n+1$ variables. The twist is needed to make $\tilde{\mathbf{T}}$ self-dual, so it really interpolates Galois representations of weight -1 . These are the ρ_{π} defined in Section 3.5. The tame Hecke operators away from Σ and the U_p -operators are absorbed into $\mathbb{I} = \mathbb{I}_n \hat{\otimes} \mathbb{I}_{n+1}$. This still leaves us with the action of the ramified Hecke algebra $\mathcal{H}^{\Sigma}$. Correspondingly, its specialization at those places contains a multiplicity space of the form $\pi_v^{K_v}$. We would like to use the local multiplicity one statements to kill this space.

As usual, let $\mathbf{H} = \text{U}(W)$ and $\mathbf{G} = \text{U}(W) \times \text{U}(V)$ be unitary groups. Suppose we are starting with an automorphic representation π on $\mathbf{G}$ which is distinguished for $\mathbf{H}$. By choosing the tame

level and maximal ideal $\mathfrak{m}$ from the previous section with respect to π , we may ensure ρ_π is a part of the family. Recall the following statement of local multiplicity one: given a local L -packet Π_{φ_v} for G , we have

$$\sum_{\pi_v \in \Pi_{\varphi_v}} \dim_{\mathbf{C}} \pi_v^{H(\mathbf{Q}_v)} = 1$$

If we know the level of π_v , then we can replace $H(\mathbf{Q}_v)$ by an appropriate Hecke algebra in the above expression without changing the dimensions. Consider the invariants

$$\mathbf{T} = \tilde{\mathbf{T}}^{\prod_{v \in \Sigma} C_c^\infty(H(\mathbf{Q}_v)/K_{n,v})}$$

Since the ramified tame Hecke algebra commutes with all other actions considered, Theorem 4.2.6 goes through without change to give

$$\mathbf{T}_x = (H_c^{n-1}(\Sigma(\mathbf{H}, K_{p,n}K_n^p), \mathbf{Z}_p) \otimes H_c^n(\Sigma(\mathbf{G}, K_{p,n+1}K_{n+1}^p), \mathbf{Z}_p))[\lambda_x]^{\text{ord}, \prod_{v \in \Sigma} C_c^\infty(\mathbf{H}(\mathcal{K}_v)/K_{n,v})}$$

where we have restricted ourselves to the constant coefficient sheaves since the general case is similar, and we ultimately will only need the results in this case by projectivity. By strong multiplicity one (applied to the base change to $\mathcal{K}$), exactly one automorphic form π contributes to this cohomology, so

$$\dim \mathbf{T}_x = n(n+1) \prod_{v \in \Sigma} \dim \left((\pi_v^\vee)^{K_{n,v}} \otimes \pi_{n+1,v}^{K_{n+1,v}} \right)^{C_c^\infty(\mathbf{H}(\mathbf{Q}_v)/K_{n,v})}$$

By the above coarse multiplicity one theorem, the multiplicity part of the right hand side is either 0 or 1. Moreover, if all $v \in \Sigma$ are split in $\mathcal{K}$, then we would be done since there are no L -packet structures.

Let $\underline{\mathcal{E}}^1$ be the support of $\mathbf{T}$ in $\mathbb{I}$, then we have seen that $\mathbf{T}$ has rank $n(n+1)$ over the classical points of $\underline{\mathcal{E}}^1$. Since we started with an automorphic representation π which is distinguished for the representations, $\underline{\mathcal{E}}^1$ is non-empty. A rigidity of automorphic type theorem would imply $\underline{\mathcal{E}}^1$ is open in the analytic topology. This is unfortunately not available to us due to the lack of a local Langlands correspondence in family. Instead, following the idea of [Dis22, Lemma 3.3.4], we may apply averaging to show that $\mathbf{T}$ is a direct summand of $\tilde{\mathbf{T}}$, and hence also projective. This requires possibly inverting p , so we would need to work with $\underline{\mathbb{A}}_n[\frac{1}{p}]$ in the future. In summary, we have proven

Proposition 4.2.12. *Let π be a stable cuspidal automorphic representation of $\text{U}(W) \times \text{U}(V)$ which is unramified and ordinary at p and cohomological at infinity. There exists a p -adic family of conjugate*

self-dual Galois representations of rank $n(n+1)$ associated to the Hida family passing through π if

1. All ramified primes of π split in $\mathcal{K}$; or
2. We instead work with a Hida family over $\underline{\Delta}[\frac{1}{p}]$.

This is not satisfactory, but in the thesis, killing multiplicity space is only needed in a context where we also need the filtration, and we would need to invert p already in those situations.

4.3 Big diagonal cycle

Having done the preparations in the previous section, we will now define the big diagonal cycle. This result can be viewed as a direct generalization of big Heegner points, first constructed in [How07] and recently in full generality in [Dis22]. The two constructions have the same underlying ideas, but Disegni phrased it in terms of Hida's ordinary completed homology. Our approach here makes fuller use of tools in representation theory developed by Emerton, and in particular, we do not take the ordinary part until the final step. This should allow one to also construct interpolated classes in finite slope families.

We again put ourselves in the Gan–Gross–Prasad setting. Let $W \subseteq V$ be a pair of Hermitian spaces as before, with signatures $(1, n-1)$ and $(1, n)$. Let K_n^p and K_{n+1}^p be fixed tame levels for $U(W)$ and $U(V)$ respectively, chosen so that we have an closed embedding of projective systems $\Sigma(W, K_n^p) \hookrightarrow \Sigma(V, K_{n+1}^p)$. By taking its graph, we get another closed embedding $\iota : \Sigma(W, K_n^p) \hookrightarrow \Sigma(W, K_n^p) \times \Sigma(V, K_{n+1}^p)$ with codimension n . Starting with the cycle class map we defined earlier and composing with the restriction to the geometric map, we obtain

$$\begin{aligned} H_{\text{Iw}}^0(\Sigma(W, K_n^p), \mathbf{Z}_p) &\xrightarrow{\iota_!} H_{\text{Iw}}^{2n}(\Sigma(W, K_n^p) \times \Sigma(V, K_{n+1}^p), \mathbf{Z}_p(n)) \\ &\xrightarrow{\text{cl}^{2n}} H_{\text{Iw}}^{2n}(\Sigma(W, K_n^p)_{/\bar{\mathcal{K}}} \times \Sigma(V, K_{n+1}^p)_{/\bar{\mathcal{K}}}, \mathbf{Z}_p(n))^{\text{Gal}_{\mathcal{K}}} \end{aligned}$$

The Iwasawa cohomology group degree is equal to $H_{2n-2}^{\text{BM}}(-, \mathbf{Z}_p(n))$, which has degree 1 below the middle degree, so it has codimension at least 1 viewed as a $\mathbf{Z}_p[[K_{p,n} \times K_{p,n+1}]]$ -module by the Calegari–Emerton conjecture. In particular, it is torsion. Let h be a non-zero element of its annihilator, then $h \cdot \iota_!$ lands in the null homologous part.

Using this trivialization, we can now follow the classical construction and consider the following

composite

$$\begin{aligned}
H_{\text{Iw}}^0(\Sigma(W, K_n^p), \mathbf{Z}_p) &\xrightarrow{\iota_!} H_{\text{Iw}}^{2n}(\Sigma(W, K_n^p) \times \Sigma(V, K_{n+1}^p), \mathbf{Z}_p(n)) \\
&\xrightarrow{\times h} H_{\text{Iw}}^{2n}(\Sigma(W, K_n^p) \times \Sigma(V, K_{n+1}^p), \mathbf{Z}_p(n))_0 \\
&\xrightarrow{\text{AJ}^{2d}} H^1(\mathcal{K}, H_{\text{Iw}}^{2n-1}(\Sigma(W, K_n^p)_{/\bar{\mathcal{K}}} \times \Sigma(V, K_{n+1}^p)_{/\bar{\mathcal{K}}}, \mathbf{Z}_p)(n)) \\
&\xrightarrow{\text{P}} H^1(\mathcal{K}, \tilde{H}_c^{2n-1}(\Sigma(W, K_n^p)_{/\bar{\mathcal{K}}} \times \Sigma(V, K_{n+1}^p)_{/\bar{\mathcal{K}}}, \mathbf{Z}_p)^\vee \otimes \mathbf{Z}_p(n)) \\
&\xrightarrow{\text{K}^\vee} H^1(\mathcal{K}, \tilde{H}_c^{n-1}(\Sigma(W, K_n^p)_{/\bar{\mathcal{K}}}, \mathbf{Z}_p)^\vee \hat{\otimes} \tilde{H}_c^n(\Sigma(V, K_{n+1}^p)_{/\bar{\mathcal{K}}}, \mathbf{Z}_p)^\vee \otimes \mathbf{Z}_p(n)) \\
&\xrightarrow{\text{ev}^\vee} H^1(\mathcal{K}, (\tilde{\mathbf{T}}_n \hat{\boxtimes} \tilde{\mathbf{T}}_{n+1}) \otimes \mathbf{Z}_p(n)) \\
&\longrightarrow H^1(\mathcal{K}, \mathbf{T})[p^{-1}]
\end{aligned}$$

We recall the definition of each map: $\iota_!$, $\times h$ have been discussed, AJ^{2d} is the interpolated Abel–Jacobi map, P is the Poincaré duality map from Subsection 4.1.1, $\text{K}^\vee$ is the dual to the Künneth decomposition from Subsection 4.1.3, $\text{ev}^\vee$ is the dual of the canonical lift, and the final unnamed arrow is localizing at a maximal ideal followed by taking invariants as in the previous section.

The source of the composite is canonically identified with

$$\left(\varprojlim_{K_p} \mathbf{Z}_p[\pi_0(\Sigma(W, K_n^p K_p))] \right)^{\text{Gal}_{\mathcal{K}}} = \mathbf{Z}_p[[T(\mathbf{Q}) \backslash T(\mathbb{A}_f) / \det(K_n^p)]]^{\text{Gal}_{\mathcal{K}}}$$

where $T = \det U(W)$ is isomorphic to $U(1)$, and the action of $\text{Gal}_{\mathcal{K}}$ is via class field theory. Since T is isotropic, the integration measure is a non-trivial element in the space, and in fact it is the only one up to $\mathbf{Z}_p$ -multiple. Its image in $H^1(\mathcal{K}, \mathbf{T})$ will be denoted by $\mathbf{z}_{\text{GPP}}$.

Remark 4.3.1. The analogous big diagonal cycle in the triple product setting (a special case of the orthogonal Gross–Prasad conjectures) was constructed in [BSV22]. They make systematic use of Steven’s theory of p -adic distributions. This can be viewed as using only Iwahoric level structures instead of all possible level structures. We also find the duality between Iwasawa cohomology and completed cohomology more natural than the duality between two modules of analytic functions.

Of course, multiplying by an element h is a highly non-canonical operation. However, we believe this is substantially the same as choosing a sign projector, as was done in [RSZ20] when they needed to trivialize the cycle for the height pairing. We hope that a better understanding of the absolute Iwasawa cohomology in the degree we consider could lead to a more canonical choice.

4.4 Filtration and local properties

One defect of the theory is that we are unable to directly construct a filtration on $\tilde{\mathbf{T}}_n$. This type of results usually fall under the name of Eichler–Shimura isomorphism, and we believe it is within reach in the ordinary setting. Instead, we have to use the usual workaround of comparing our construction with an abstract one obtained from the theory of pseudo-representations. The result is that the study of the Selmer condition has to be done over an unspecified dense affinoid subset of the weight space. This does not affect the eventual application of Theorem 2.3.2 since we can localize Λ throughout the proof.

Theorem 4.4.1. *Let $(\mathbf{T}_n, \mathbb{I}_n)$ be the p -adic family of Galois representations and the big ordinary Hecke algebra for a single unitary group of rank n defined in Section 4.2. Suppose further that $\mathbf{T}_n$ is residually absolutely irreducible. There exists an affinoid open subset $\underline{\mathcal{A}} \subseteq \underline{\mathcal{W}}$ over which it comes with a Gal_{K_p} -stable filtration $\mathrm{Fil}_i \mathbf{T}'$ with the non-zero graded pieces*

$$\mathrm{gr}_i \mathbf{T}' = \mathrm{Fil}_i \mathbf{T}' / \mathrm{Fil}_{i-1} \mathbf{T}' \simeq \mathbb{I}_n|_{\underline{\mathcal{A}}}(\Psi_i)$$

for $i = 1, \dots, n$, where Ψ_i is a character of Gal_{K_p} valued in $(\mathbb{I}_n|_{\underline{\mathcal{A}}})^\times$.

Proof. Using the theory of pseudo-representations, one can construct a representation $\mathbf{T}'$ over $\mathrm{Frac}(\mathbb{I})$ with the required filtration which specializes to the classical Galois representations considered in Section 3.5. This was done in [SU14, Section 7.2] in the case V is split, but the argument works in general. Inverting the denominators which occur gives us a dense Zariski open subset of $\underline{\mathcal{W}}$. Finally, Carayol’s generalization of the classical Brauer–Nesbitt argument [Car94] gives an abstract isomorphism between $\mathbf{T}'$ and $\mathbf{T}$. $\square$

We need to recall more notations to specify the character Ψ_i . As in the set-up for weight space, let χ_i be the composition of the universal anticyclotomic character with the i -th standard coroot for the diagonal maximal torus. Let ξ_i be the unramified character sending a Frobenius at $\mathfrak{p}$ to the (normalized) Hecke operator defining the i -th Satake parameter. Finally, let ω be the character attached to the central character of the family, then $\Psi_i = \xi_i \chi_i \omega^c$, as one can check by comparing with the description in Section 3.5. The key point is that this satisfies the condition of Theorem 1.4.1. Moreover, due to the presence of χ_i , the Gal_{K_p} -invariants are trivial, and the coinvariants are pseudo-null. It follows that the conditions (no-pole) and (no-triv-zero) holds for $\mathbf{T}$ restricted to $\underline{\mathcal{A}}$.

Finally, by applying the theorem to both $\mathbf{T}_n$ and $\mathbf{T}_{n+1}$, we can obtain a bi-filtration on $\mathbf{T}$ after restricting to $\underline{\mathcal{A}}$, the product of the two good affinoid subsets. To simplify notations, we drop the restriction to $\underline{\mathcal{A}}$ from notations. In particular, we can define a subspace

$$\mathbf{T}_{\text{diag}} := \bigcup_{i=1}^n ((\text{Fil}^i \mathbf{T}_n)^\vee \otimes \text{Fil}_i \mathbf{T}_{n+1})$$

This is $\text{Gal}_{K_{\mathfrak{p}}}$ -stable, so it can be used to define Selmer conditions at $\mathfrak{p}$, and its dual defines a Selmer condition at $\bar{\mathfrak{p}}$. The reason for the subscript **diag** will be explained in the next chapter.

We want to study the behaviour of $\mathbf{z}_{\text{GGP}}$ locally. This is done by specialization at a dense set of smooth characters, using the following easy lemma from commutative algebra [How07, 2.1.7]: if M is a finitely generated module over a normal ring R , and $a \in M$ is mapped to zero in $M/\mathfrak{p}$ for a Zariski-dense subset of height one primes of R , then a is torsion.

Proposition 4.4.2. *On an open dense subset of $\mathcal{W}$, the following holds*

1. *For ℓ a place of $\mathcal{K}$ not above p such that the tame Hecke algebra is spherical, $\text{res}_\ell(\mathbf{z}_{\text{GGP}})$ is unramified.*
2. *$\text{res}_{\mathfrak{p}}(\mathbf{z}_{\text{GGP}}) \in \ker(H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}) \rightarrow H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}/\mathbf{T}_{\text{diag}}))$, and similarly for $\bar{\mathfrak{p}}$.*

Proof. For ℓ as in the first part, the Shimura variety has a smooth integral model over $\mathcal{O}_\ell$, so all the representations we study are unramified there, and we could have developed the theory using Galois cohomology unramified at ℓ . This observation proves (1).

For the second claim, since the coefficient system is constant, the archimedean types of π_n and π_{n+1} are both the minimal type. It follows by the interpolation property that $\mathbf{T}_{\text{diag}}$ specialized at such a point is a Panchishkin subspace, i.e. it contains exactly the non-positive Hodge–Tate weights. By [Nek00, Theorem 3.1], $(\mathbf{z}_{\text{GGP}})_x \in H_g^1(\mathcal{K}, \mathbf{T}_x)$ automatically. After restricting weight space to the open subsets of those x whose weights are all distinct, there is no difference between H_f^1 and H_g^1 . Indeed, as explained in *loc. cit.*, the difference in their dimensions can be computed from the $\varphi = 1$ subspace for φ the crystalline Frobenius, which vanishes by our ordinarity and weight assumptions. Moreover the Panchishkin condition is equivalent to the H_f^1 -condition, so $(\mathbf{z}_{\text{GGP}})_x$ becomes 0 in $H^1(\mathcal{K}_{\mathfrak{p}}, (\mathbf{T}/\mathbf{T}_{\text{diag}})_x)$.

By the discussion before the theorem, we have shown that the image of $\mathbf{z}_{\text{GGP}}$ in $H^1(\mathcal{K}_{\mathfrak{p}}, \mathbf{T}/\mathbf{T}_{\text{diag}})$ is torsion. It remains to show that this latter group is torsion-free. This follows from stratifying the space using the filtration and applying Coleman map to each piece. $\square$

Unfortunately, the behaviour of the Shimura at the ramified primes is a difficult question. In particular, the weight monodromy conjecture for its toroidal compactification appears to be not known in full generality, so we are unable to give information about the cycle at the ramified places away from p .

Chapter 5

Framework and further speculations

5.1 Algebraic side

We will now go back to the motivation of this project, which is to understand the Bloch–Kato conjecture for the Galois representations arising in the Gan–Gross–Prasad setting. More precisely, let $\pi = \pi_n^\vee \otimes \pi_{n+1}$ be a cohomological stable cuspidal representation of $G = \mathrm{U}(W) \times \mathrm{U}(V)$, then we have seen in Section 3.5 that we can attach a p -adic Galois representation

$$\rho : \mathrm{Gal}_{\mathcal{K}} \rightarrow \mathrm{GL}_{n(n+1)}(E)$$

for some E finite extension of $\mathbf{Q}_p$. It is conjugate self-dual, so the Bloch–Kato conjecture states

$$\mathrm{ord}_{s=\frac{1}{2}} L(s, \pi) \stackrel{?}{=} \dim_E H_f^1(\mathcal{K}, \rho)$$

Here, the subscript f is the Bloch–Kato local condition. It means unramified away from p , but the conditions at the two primes above p are more subtle. However, it is easy to describe if ρ satisfies the Panchishkin condition: if ρ contains a $\mathrm{Gal}_{\mathcal{K}_p}$ -stable subspace ρ^- such that the Hodge–Tate weights of ρ^- are non-positive and the Hodge–Tate weights of ρ/ρ^- are positive. Since ρ is pure of weight -1 , we know that

$$H_f^1(\mathcal{K}_p, \rho) = \ker(H^1(\mathcal{K}_p, \rho) \rightarrow H^1(\mathcal{K}_p, \rho/\rho^-))$$

by [Nek07, Proposition 3.3.2(2)]. The Hodge–Tate weights of ρ are directly related to the relative sizes of the infinitesimal characters for π_n and π_{n+1} . We describe it using a “weight interlacing string”.

Definition 5.1.1. Given two discrete series representations $\pi_{n,\infty}$ and $\pi_{n+1,\infty}$ with infinitesimal characters $(b_1 > \cdots > b_n)$ and $(a_1 > \cdots > a_{n+1})$ respectively. Define their weight interlacing string as follows: first write the numbers $a_1, \dots, a_{n+1}, b_1, \dots, b_n$ in descending orders, then forget the subscript to obtain a string of length $2n + 1$ with exactly $(n + 1)$ *As* and n *Bs*.

Recall that $a_i \in \mathbf{Z} + \frac{n}{2}$, but $b_j \in \mathbf{Z} + \frac{n-1}{2}$, so they cannot be equal. Now if π is ordinary (and unramified) at p , then all Selmer conditions are Panchishkin, and they are naturally indexed by weight interlacing strings. This is very explicit. As described in Section 3.5, choose a basis $\{w_1, \dots, w_n\}$ for ρ_n which is compatible with the filtration. Similarly, choose a basis $\{v_1, \dots, v_{n+1}\}$ for ρ_{n+1} . Then a basis for ρ is $\{v_i \otimes w_j^\vee\}$. Given a weight interlacing string $\square$, we can define a subspace

$$\rho_\square := \langle v_i \otimes w_j^\vee \mid a_i - b_j > 0 \rangle \subseteq \rho$$

By compatibility with filtration, this is $\text{Gal}_{\mathcal{K}_p}$ -stable and therefore defines a Selmer condition. At the place $\bar{p}$, the dual space is

$$\rho_\square^c := \langle v_i^\vee \otimes w_j \mid a_i - b_j < 0 \rangle \subseteq \rho^c$$

These two Selmer conditions together with the unramified condition defines a Selmer group $H_\square^1(\mathcal{K}, \rho)$. If $\square$ comes from the weight interlacing at the archimedean place of π , then the previous discussion shows that $H_\square^1(\mathcal{K}, \rho) = H_f^1(\mathcal{K}, \rho)$. Pictorially, we view one such subspace as a tableau, for example

a_4			
a_3			
a_2			
a_1			
	b_3	b_2	b_1

In this diagram, each square represents a basis element, and the filled in area represents $\rho_\square$, or equivalently $a_i > b_j$. Since the sequences $\underline{a}$ and $\underline{b}$ are decreasing, the shaded part is lower-left closed. The terminology of tableaux is taken from [Har13, Section 1.4], where the structure is considered to determine the Deligne period for π .

Example 5.1.2. The case $n = 1$ is the case of a modular form f twisted by an anticyclotomic Hecke character χ . In this case, there are 3 weight interlacing strings: ABA , AAB , BAA . The filtration

on ρ_2 is the usual ordinary filtration on the Tate module. Therefore, in more classical notations

$$H_{ABA}^1 = H_{\text{ord}}^1, \quad H_{AAB}^1 = H_{\text{rel, str}}^1, \quad H_{BAA}^1 = H_{\text{str, rel}}^1$$

At the archimedean place, if f has weight k , then its infinitesimal character is $(\frac{k-1}{2}, -\frac{k-1}{2})$. There is a subtlety in identifying modular forms with automorphic forms on $\text{U}(1, 1)$, but this is fine if f has trivial character, which implies $k \in 2\mathbf{Z}$. The infinitesimal character of χ is (ℓ) , with $\ell \in \mathbf{Z}$. If $|\ell| < \frac{k-1}{2}$, then we are in the first setting. If $|\ell| > \frac{k-1}{2}$, then we are in one of the last two cases depending on the sign of ℓ . The two applications we gave in Section 2.4 are instances where we study the relations between the BAA region and the ABA region.

We now consider what happens when we interpolate. In Section 4.2, we constructed a p -adic family of Galois representations $\mathbf{T}$ lying over a weight space with $2n + 1$ variables. It is of the expected rank $n(n + 1)$ when viewed as a module over the Hecke algebra $\mathbb{I}$. The weight interlacing strings can be used to partition $\mathcal{W}^{\text{cl}}$ into disjoint regions $\mathcal{W}^{\square}$, each of which is Zariski dense in the entire weight space. Over an affinoid open subsets, it carries a bi-filtration of the type we described above, so weight interlacing still corresponds to Selmer conditions. We will proceed as if it is defined over all of $\mathcal{W}$ since this section is mainly speculative. One might then expect an Iwasawa main conjecture of the form

$$\tilde{H}_{\square}^2(\mathcal{K}, \mathbf{T}) \text{ is torsion, and } \text{char}_{\underline{\Lambda}} \tilde{H}_{\square}^2(\mathcal{K}, \mathbf{T}) = (\mathcal{L}_p^{\square})$$

where $\tilde{H}_{\square}^2(\mathcal{K}, \mathbf{T})$ is the module interpolating Bloch–Kato Selmer groups over $\mathcal{W}^{\square}$, defined as in Section 2.2. This is indeed the expectation for a cyclotomic deformation, where we expect generic non-vanishing of L -values in the family. However, we are studying anticyclotomic deformation. The crucial difference that all of the representations in our family are conjugate self-dual, so its global ε -factor (always at the center) is in $\{\pm 1\}$. In the -1 , case the central L -value is 0 for trivial reasons. The next lemma from [Nek07] shows that the finite ε -factor does not change in a family.

Lemma 5.1.3. *If $\mathbf{T}$ is a p -adic family of Galois representations satisfying the Panchishkin condition, then there exists $\varepsilon_{\mathbf{T}} \in \{\pm 1\}$ such that for all classical points $x \in \mathcal{W}^{\text{cl}}$, the specialization $\mathbf{T}_x : \text{Gal}_{\mathcal{K}} \rightarrow \text{GL}_n(\bar{\mathbf{Q}}_p)$ has finite ε -factor equal to $\varepsilon_{\mathbf{T}}$.*

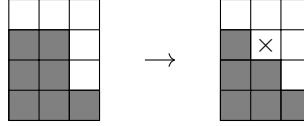
By a computation similar to the one made in Section 3.3, the archimedean sign counts the number of positive Hodge–Tate weights (for the fixed additive character $\psi_{\mathbf{C}}$), so it is a function of the weight

interlacing string. We denote it by $\varepsilon(\square)$. Therefore, in our family, there is a systematic vanishing of L -values. Therefore, we need two types of Iwasawa main conjectures,

Coherent case If $\varepsilon_{\mathbf{T}}\varepsilon(\square) = 1$, then $\tilde{H}_{\square}^2(\mathcal{K}, \mathbf{T})$ is torsion, and its characteristic ideal is a p -adic L -function.

Incoherent case If $\varepsilon_{\mathbf{T}}\varepsilon(\triangle) = -1$, then $\tilde{H}_{\triangle}^2(\mathcal{K}, \mathbf{T})$ has rank 1 over $\mathbb{I}$, and the characteristic ideal of its torsion is equal to the index of a special element in $\tilde{H}_{\triangle}^1(\mathcal{K}, \mathbf{T})$.

Our main result Theorem 2.3.2 was axiomatized to relate these two conjectures. In fact, given $\square$ containing a substring AB , let $\triangle$ be obtained by replacing it with BA . We say the words $\square$ and $\triangle$ are nearby. On the tableau, this corresponds to the operation of deleting an extremal square, for example



The quotient $\mathbf{T}_{\square}/\mathbf{T}_{\triangle}$ is then the rank 1 space corresponding to the square deleted, so we can apply the theorem. Further observe that $\varepsilon(\triangle)\varepsilon(\square) = -1$, so exactly one of them is incoherent. Using the terminology in Remark 2.3.3, we obtain the following expectation

Expectation 5.1.4. Given a family $\mathbf{T}$, let $\triangle$ be an incoherent word for $\mathbf{T}$, i.e. $\varepsilon_{\mathbf{T}}\varepsilon(\triangle) = -1$. Suppose we have a special cycle $\mathbf{z} \in \tilde{H}_{\triangle}^1(\mathcal{K}, \mathbf{T})$ in the sense that the $\mathbb{I}$ -module it generates has torsion cokernel, then

$$\text{char}_{\underline{\Lambda}} \tilde{H}_{\triangle}^2(\mathcal{K}, \mathbf{T})_{\text{tors}} \supseteq \text{char}_{\underline{\Lambda}} \left(\frac{\tilde{H}_{\triangle}^1(\mathcal{K}, \mathbf{T})}{\underline{\Lambda} \cdot \mathbf{z}} \right)^2$$

and for any nearby word $\square$, if the explicit reciprocity law holds, then

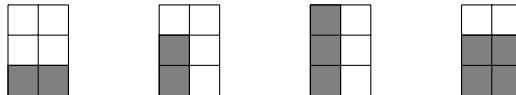
$$\text{char}_{\underline{\Lambda}} \tilde{H}_{\square}^2(\mathcal{K}, \mathbf{T}) \supseteq (\mathcal{L}_p^{\square})$$

Moreover, equality in any of the statements implies equality in all other statements.

To make this expectation a reality, we need three ingredients

1. A special cycle $\mathbf{z}_{\triangle}$.
2. A p -adic L -function for $\square$.
3. An explicit reciprocity law relating them.

We discuss (1) here and leave (2) and (3) to the next section. The diagonal cycle we have constructed lies in the Selmer condition defined by the word $\mathbf{diag} = AB \cdots ABA$ which represents perfect interlacing. In picture, it is the second tableau in the previous diagram, so it should be related to p -adic L -functions in the $2n$ tableaux nearby to it. For $n = 2$, they are



Beyond it, we are not aware of any systematic construction of cycles, or even Selmer classes. Even in this case, if $\varepsilon_{\mathbf{T}}$ has the wrong sign, then $\mathbf{diag}$ would be coherent, and the construction of Selmer classes for any word is not known to us systematically.

Example 5.1.5. In the $n = 1$ setting we have discussed before, there are only three diagrams, so the diagonal word controls everything. The condition that $\mathbf{diag}$ is incoherent is exactly the sign condition guaranteed by the Heegner condition and its generalization. This is the case we understand well, and the first application in Section 2.4 completes the expected picture.

In the opposite setting when $\mathbf{diag}$ is coherent, we saw the issue that there is no natural geometric construction of cycles in either of the incoherent regions. K. T. Do's work supplied the Selmer class necessary to push through the above expectations in the same way, and one might hope to generalize his line of work to higher rank cases.

5.2 Analytic side

From algebraic considerations, we are already expecting a p -adic L -function attached to each weight interlacing string. This can also be seen on the analytic side, though there are more mysterious phenomena here.

The global Gan–Gross–Prasad formula expresses the central L -value in terms of a period integral, which can potentially be interpolated to construct a p -adic L -function. The archimedean data, i.e. the signatures of W , V and the exact nature of the discrete series π_{∞}° , should determine the transcendental period. Harris has made computations in [Har13] conjecturally comparing them to Deligne periods. Since the underlying geometry is so different, we are also not expecting a uniform p -adic L -function for all archimedean data. In fact, we propose the following imprecise conjecture.

Conjecture 5.2.1. *For each of the $\binom{2n+1}{n}$ weight interlacing strings $\square$, there exists a p -adic L -function $\mathcal{L}_p^{\square}$ interpolating $L(\frac{1}{2}, \pi)$ in that region.*

There are currently on-going works in progress by Harris and collaborators to construct some of the p -adic L -functions from higher Hida theory, see for example [Har21]. However, there is a fundamental limitation to this approach. Recall that we want to interpolate $\int_{[H]} \varphi_n(h) \varphi_{n+1}(h) dh$. Each φ_i lies in a discrete series, and computations done in section 3.6 shows that they contribute to exactly one degree d_i in the coherent cohomology of the Shimura variety. The integral is interpreted as a cup product pairing $\langle \Delta_* \varphi_n, \varphi_{n+1} \rangle$, so the minimal compatibility requirement simplifies to

$$d_n + d_{n+1} = \dim_{\mathbf{C}} \Sigma(H)$$

We now compute what this means in terms of weight interlacing. For simplicity, suppose n is odd, the other case being entirely analogous. Let $(b_1 > \cdots > b_n)$ and $(a_1 > \cdots > a_{n+1})$ be the sequence of numbers representing their infinitesimal characters. Suppose the distinguished signatures are (p, q) and $(p, q+1)$, then we need $d_n + d_{n+1} = pq$. Let ϵ_n and ϵ_{n+1} be the codes for the distinguished Weyl chambers. Recall from Section 3.3 that

$$(\epsilon_n)_j = (-1)^{j + \#\{i | a_i > b_j\}}, \quad (\epsilon_{n+1})_i = (-1)^{i+1 + \#\{j | a_i > b_j\}}$$

By the computations made in Subsection 3.6.2, d_{n+1} is the number of occurrences of 0 before 1 in ϵ_{n+1} , and the same holds not for d_n , but instead for $pq - d_n$ due to the presence of the contragredient. For example, in [Har21], π_n is anti-holomorphic and π_{n+1} is holomorphic, so both $\pi_n^\vee$ and π_{n+1} are holomorphic (as expected, since a holomorphic discrete series should restrict to another holomorphic discrete series). The chamber labels are

$$\epsilon_n = \underbrace{1 \cdots 1}_q \underbrace{0 \cdots 0}_p, \quad \epsilon_{n+1} = \underbrace{1 \cdots 1}_{q+1} \underbrace{0 \cdots 0}_p$$

The corresponding tableau consists of removing a block of p contiguous squares from **diag** from the end.

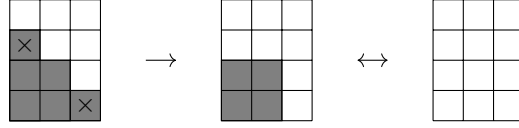
$$p = 2 \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \blacksquare & & \\ \hline \blacksquare & \blacksquare & \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \blacksquare & & \\ \hline \blacksquare & \times & \\ \hline \blacksquare & \blacksquare & \times \\ \hline \end{array}$$

In general, the dimension equality reduces to stating that the number of 0s before 1s in ϵ_n and ϵ_{n+1} are equal. One way of achieving this is to remove an arbitrary set of squares from the diagonal. The resulting strings satisfy $\epsilon_{n+1} = 1 \oplus \epsilon_n$. By taking complex conjugation of the result, we can also add a contiguous block.

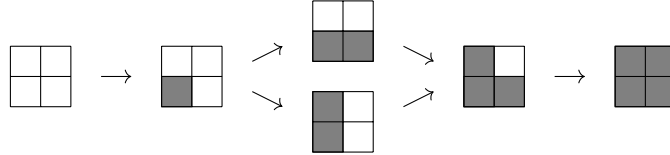
This does not account for all weight interlacing strings satisfying the numerology. The point is as follows: we have defined a function

$$\mathbf{dist} : \{\text{weight interlacing strings}\} \rightarrow \{\text{pairs of chambers}\}$$

It turns out that both sides have the same size, but the map is not a bijection. Indeed, an operation of the form $AABB \rightarrow BBAA$ does not change the image. On a tableau, this corresponds to removing an extremal 2×2 square, for example



We suspect that this operation alone explains the fibres of $\mathbf{dist}$. If two words have the same image, then the archimedean data describing their period integrals are identical. This is a genuinely new phenomenon beyond the GL_2 -case. If $\square$ and $\square'$ are in the same fibre, then we expect an equality $\mathcal{L}_p^\square = \mathcal{L}_p^{\square'}$. On the algebraic side, in the process of deleting the 2×2 square, we see the following 6 regions:



The two diagrams at the end are coherent. Since their p -adic L -functions are equal, we expect a relation between the special classes for the diagrams in the second and fourth column. The simplest case is the relation between the diagonal cycle and the conjectural one attached to weight interlacing string $BABAA$. This new string distinguishes the archimedean data $U(2,0) \hookrightarrow U(2,1)$, with the form on $U(2,1)$ in the holomorphic discrete series.

Finally, the $2n$ words nearby to the diagonal cycles correspond to the p -adic L -functions constructible from cup products in coherent cohomology using only unitary groups of signature $(1, n)$. We do not have a satisfactory explanation for this except to remark that the constructions we have seen of p -adic L -functions are less sensitive to the finite ε -factor, in the sense that for example, the construction of [Har21] only requires the coherency condition to choose non-trivial local sections.

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