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Derived Structures in the Duality of Automorphic Periods

Chen, Eric Y.

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Derived Structures in the Duality of Automorphic Periods

Eric Y. Chen

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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by the Department of
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Abstract

Built on top of the global Langlands correspondence – a matching of automorphic forms on a reductive group G and Galois representations into its Langlands dual group $\check{G}$ – is the question of functoriality, i.e., the construction and detection of natural transfers of automorphic forms between varying G, predicted by maps between varying $\check{G}$. Traditionally, this involved understanding natural numerical invariants attached to both sides of the correspondence: computing automorphic periods on the one hand, and understanding analytic properties of various Langlands L-functions on the other. Recently, Ben-Zvi–Sakellaridis–Venkatesh proposed a duality framework that extends Langlands duality to these numerical invariants, in which automorphic periods and L-functions are extracted from Hamiltonian actions of G and $\check{G}$, respectively. We expand on this perspective by computing new automorphic periods and by recasting old computations in the context of this emergent duality. In the process, we encounter and introduce the notion of nonabelian L-functions as the Galois-side counterpart to our new automorphic computations, and we suggest that they may be understood as reflecting the presence of singularities or derived structures, on the Hamiltonian $\check{G}$-action.

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In writing this thesis, I can't help but think that the material that happened to be collected here, in one place by me, owes itself entirely to my interactions with others, mathematical or otherwise, and that my own self played as small or as large a part as any other person in this acknowledgement. I can't even claim that I was the one who has suffered the most in the completion of this thesis. In the days leading up to the deadline, it was surely my patient and generous adviser Akshay Venkatesh who has suffered the most, along with our departmental hero Jill LeClair.

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I'd better stop now, and finish my final edits.

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Chapter 1

Introduction

1.1 Hamiltonian actions, automorphic Periods, and L-Functions

The theory of integral representations of L-functions provides a direct link between automorphic objects and Galois-theoretic objects, and the search for arithmetic interpretations of automorphic periods has proven indispensable for understanding finer structures in the Langlands correspondence. These structures have emerged under the umbrella term of "relative Langlands", and they encode, for instance, the relationship between automorphic branching laws, functoriality, and analytic behavior of L-functions.

For example, consider the following automorphic period investigated by Friedberg–Jacquet [FJ93]. Let $G = \mathrm{GL}_{2n}$ over a global field F, and let π be a cuspidal automorphic representation of G with trivial central character. Consider the subgroup

$$H = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \cong \mathrm{GL}_n \times \mathrm{GL}_n \subset G$$

The main theorem of *loc. cit.* provides a link between the three questions raised above.

Theorem 1.1.1 (Friedberg–Jacquet)**.** Consider the automorphic period $P : \pi \to \mathbf{C}$ defined by the integral

$$P(f) := \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} f(g)\, dg$$

Then we have the following:

1. If the (partial) exterior square L-function of π does not have a pole at $s = 1$, then $P(f) = 0$

2. Suppose the (partial) exterior square L-function of π has a pole at $s = 1$, then there is a vector $f \in \pi$

so that

$$P(f) = L(1/2, \pi)$$

where the equality is understood to be valid away from a finite set of ramified places.

To paraphrase, Friedberg–Jacquet show that the part of the cuspidal spectrum of G distinguished by H, that is to say, the part which admits an H-invariant period, is exactly the part whose L-parameters factor through the symplectic group $\mathrm{Sp}_{2n} \subset \check{G}$.

Although theorems of the above form are ubiquitous in the literature (see [JS88] [GJ01][GJ] for a few examples), a unifying theory has only recently been suggested by the work of Ben-Zvi–Sakellaridis–Venkatesh [BZSV]. Through their physically inspired point of view (see Section 1.3 for a primer on the physical perspective), automorphic periods for G and L-functions for L-parameters valued in $\check{G}$ are encoded by certain Hamiltonian G-spaces M and $\check{G}$-spaces $\check{M}$, respectively.

We explain the procedure here briefly; for details on precise definitions see Section 2. Given a Hamiltonian G-space M, one can in principle find a representation of $G_{\mathbb{A}}$ which geometrically quantizes $M_{\mathbb{A}}$, and by averaging over rational points one obtains automorphic functions which transform as the geometric quantization of $M_{\mathbb{A}}$, and we may choose a distinguished such automorphic function Θ_M. The **automorphic M-period** of an automorphic form f on G is then defined as the L^2-pairing

$$P_M(f) := \langle f, \Theta_M \rangle_{L^2([G])}$$

On the other hand, suppose we are given a *polarized* Hamiltonian $\check{G}$-space $\check{M}$, i.e., $\check{M} = T^*\check{X}$ is a cotangent bundle, and the Hamiltonian action is induced from a $\check{G}$-action on $\check{X}$. Then the **spectral $\check{M}$-period** of an L-parameter φ is defined as the following sum of L-functions:

$$L_{\check{X}}(\varphi) = \sum_{x \in \mathrm{Fix}(\varphi, \check{M})} L(\varphi, T_x\check{X})$$

whenever the sum makes sense. Note that each summand is an evaluation of some ordinary L-function attached to an Artin representation.

In this language, the statement of Friedberg–Jacquet's theorem can be reformulated as follows: consider the G-space $M = T^*(H\backslash G)$, and the $\check{G}$-space $\check{M} = T^*(\mathbf{A}^{2n} \times_{\mathrm{Sp}_{2n}} \check{G})$. Then for a cuspidal automorphic representation π and a suitably chosen vector $f \in \pi$ with L-parameter φ_f, we have

$$P_X(f) = L_{\check{X}}(\varphi_f)$$

So far the translation into Hamiltonian actions is nothing but a reformulation, but one essentially new ingredient proposed by Ben-Zvi–Sakellaridis–Venkatesh is that the structure of Langlands duality should extend to a more refined duality between the spaces M and $\check{M}$ themselves. More precisely, if $M = T^*X$ is a Hamiltonian G-space and $\check{M} = T^*\check{X}$ is a Hamiltonian $\check{G}$-space, and we have a theorem of the form

$$P_X(f) = L_{\check{X}}(\varphi_f)$$

then we should expect a "dual" theorem

$$P_{\check{X}}(f) \stackrel{?}{=} L_X(\varphi_f)$$

where now f is an automorphic form on $\check{G}$ with L-parameter φ_f valued in G. In other words, integral representations of L-functions should come in Langlands dual pairs. From the perspective of number theory this is a very practical method to generate new integral representations of L-functions from old ones: if one understands how to recast an existing integral representation in the above form, there is automatically a conjectural "dual" integral representation, where the automorphic and Galois sides are switched.

1.2 Towards singular and derived periods

The generality of Hamiltonian spaces that Ben-Zvi–Sakellaridis–Venkatesh work in includes (at least) cotangent bundles of smooth affine spherical varieties. However, the translation from an existing integral representation of an L-function to the language of Hamiltonian actions is not always straightforward even in the simplest examples, and we will see that singular and nonaffine G-spaces are immediately necessary to understand these examples. We propose that in these cases, there are often singularities and derived structures on the Hamiltonian spaces that must be taken into account in order for the duality to be manifest.

For instance, consider the following result of Ginzburg [Gin91] representing the adjoint L-function of SL_3:

Theorem 1.2.1 (Ginzburg)**.** Let $G = \mathrm{SL}_3$, let $E^*_{\mathrm{G}_2}(g,s)$ be a (suitably normalized) degenerate Eisenstein series on the simple group G_2, and we consider G as a subgroup of G_2 via the long root parabolic. For a cuspidal automorphic representation π of G, there is a vector $f \in \pi$ with L-parameter φ_f such that

$$\int_{\mathrm{SL}_3(F)\backslash\mathrm{SL}_3(\mathbb{A})} f(g)E^*_{\mathrm{G}_2}(g,s)\,dg = \frac{L(\varphi_f, \mathrm{Ad}, 3s-1)}{\zeta(6s-2)\zeta(9s-3)}$$

We note crucially that the Eisenstein series $E^*_{\mathrm{G}_2}(g,s)$ is only *partially completed*; we have opted to

distribute some of the factors of the *completed* Eisenstein series to the right hand side of the equation, for reasons that will soon become apparent.

The naïve guess, following the recipe outlined in the previous Section, would be to consider the pair of spaces

$$X = P\backslash \mathrm{G}_2 \text{ and } \check{X} = \mathrm{Ad}_{\mathrm{PGL}_3}$$

where $P \subset \mathrm{G}_2$ is the parabolic with respect to which the Eisenstein series E_{G_2} was induced. However, this proves to unsatisfactory for several reasons. On the automorphic side, the normalization of the Eisenstein series needs to be taken into account, and thus X needs to be modified by a certain affine cone to capture the relevant normalization factors. This process already forces us to replace X by an affine but singular variety, or a dense open orbit which is smooth but possibly nonaffine.

On the spectral side, the problem is more serious. The generic stabilizer of $\mathrm{Ad}_{\mathrm{PGL}_3}$ in $\check{G}$ is a maximal torus; as such, it would give rise to a nonunique automorphic period which is not known to unfold to any local model, let alone be related to L-functions. Furthermore, we see abelian L-factors $\zeta(6s-2)\zeta(9s-3)$ on the right hand side which cannot be explained by the spectral $\check{X}$-period.

In fact, these phenomena abound in the integral representations considered by Ginzburg and his collaborators [GR94b] [Gin95] [BG92]. Their philosophy is that one should be able to find an automorphic period computing the L-function for many prehomogeneous vector spaces V, i.e., $\check{G}$-representations with a dense open orbit, via generalizations of the Rankin–Selberg method. These prehomogeneous vector spaces arising from graded Lie algebras were studied by Kac [Kac80], who also computed the degrees of generators in the ring of $\check{G}$-invariant polynomials $\mathbf{C}[V]^{\check{G}}$. Through experience with computations, Ginzburg noticed a striking numerical pattern: the integral representations he finds for $L(V,s)$ often naturally come with abelian L-factors in the denominator

$$\frac{L(V,s)}{\zeta(d_1 s)\zeta(d_2 s)\cdots\cdots\zeta(d_k s)}$$

such that $d_1,\ldots,d_k$ are exactly the invariant degrees computed by Kac.

We interpret these phenomena as follows. Suppose $f_1,\ldots,f_k \in \mathbf{C}[V]^{\check{G}}$ are the homogeneous generators of the invariant ring, with degrees $d_1,\ldots,d_k$. Then the correct $\check{X}$ to consider is not V, but the vanishing locus of $f_1,\ldots,f_k$

$$\check{X} = Z(f_1,\ldots,f_k)$$

which one may term the *nilpotent cone* of V. In the case where $\check{X}$ is a complete intersection singularity this is an especially reasonable notion since its spectral period coincides with the natural definition of an L-function of the tangent complex at the singular fixed point; for more on this, see Proposition 2.6.1. In

the case of Ginzburg's adjoint L-function discussed above, our interpretation suggests that $\check{X}$ should be the usual nilpotent cone of $\check{G}$. Theorem 3.7.2 applied to this case expresses the underlying BZSV duality for Ginzburg's integral representation:

Theorem 1.2.2 (C.-Venkatesh, Special case of Theorem 3.7.2)**.** Let X be the affine cone over the G_2-Grassmannian, thought of as a $\mathrm{SL}_3 \times \mathbf{G}_m$-space, with the extra $\mathbf{G}_m$ scaling the affine cone. Let $\check{X}$ be the nilpotent cone of $\mathrm{PGL}_3 \times \mathbf{G}_m$, thought of as a PGL_3-space, with the extra $\mathbf{G}_m$ scaling the nilpotent cone. Then X and $\check{X}$ are weakly BZSV dual.

For the definition of weakly BZSV dual Hamiltonian actions, see Section 2.5.

On the other hand, all known integral representations of L-functions that unfold to unique models rely on the Casselman–Shalika formula, which expresses the local Whittaker function of a spherical representation in terms of Weyl characters of the dual group. In principle, Sakellaridis' generalizations of the Casselman–Shalika formula [Sak12] [Sak18] should play a similar role for automorphic periods that unfold to more exotic unique models. For instance, the following formula would be essential in understanding integrals that unfold to the Shalika model.

Theorem 1.2.3 (Sakellaridis, Theorem 2.1 [Sak18])**.** Let $G = \mathrm{GL}_{2n}$ over a nonarchimedian local field F with uniformizer ϖ and residue field $\mathbf{F}_q$, and let π be an unramified principle series representation of $G(F)$ with Satake parameter Sat_π. Let $\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}$ be (a choice of) the set of positive short roots of $\mathrm{Sp}_{2n} \subset \check{G}$.

1. If Sat_π is not symplectic, then π has no Shalika model.

2. Suppose Sat_π is symplectic. For a suitable normalization, the spherical Shalika function of π is given by

$$\Omega(\varpi^{(\lambda_1,\dots,\lambda_n,1^n)}) = \begin{cases} \sum_{w \in W_{\mathrm{Sp}}} (-1)^w w \left[e^{\rho_{\mathrm{Sp}}+\lambda} \prod_{\alpha \in \Phi_{\mathrm{Sp}}^{+,\mathrm{s}}} (1 - q^{-1} e^{-\alpha})(\mathrm{Sat}_\pi) \right] \\ \qquad\qquad\qquad\qquad \text{if } \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

where W_{Sp} is the Weyl group of Sp_{2n}, and ρ_{Sp} is the half sum of positive roots of Sp_{2n}.

If we formally set $q^{-1} = 0$, then Sakellaridis' formula reduces to the Weyl numerator for the symplectic group valued at λ; thus, we may understand these more general spherical functions as a q^{-1}-deformation of the Casselman–Shalika formula. We propose that these extra terms reflect the presence of derived structures on the spectral side of the dual pair in question. For instance, applying our calculations in the $n = 3$ case we speculate on a BZSV-type duality underlying Ginzburg–Rallis' integral representation of the exterior cube L-function on GL_6 [GR00]:

Theorem 1.2.4 (C. $n = 3$ case of Theorem 4.6.2)**.** Let $G = \mathrm{GL}_6/\mu_3$, and $\check{G}$ its Langlands dual group. Let $X \subset \wedge^3 \mathbf{A}^6$ be the vanishing locus of the unique G-semiinvariant of degree 4. Let $\check{X} = \mathrm{Ind}_{\mathrm{GSp}_6}^{\check{G}} \overline{V \backslash \mathrm{GSp}_6}$, where $V \subset \mathrm{GSp}_6$ is a maximal unipotent subgroup. Let $\check{X}^- := \mathrm{Ind}_{\mathrm{GSp}_6}^{\check{G}} \overline{U[M,M] \backslash \mathrm{GSp}_6}$ where $P = MU$ is the Siegel parabolic.

- Let f be an unramified cusp form on G with L-parameter φ_f. If φ_f is not symplectic, then $P_X(f) = 0$. If φ_f is symplectic, then up to a global factor of the discriminant,

$$P_X(f) = (*)\, L_{\check{X}}(\varphi_f) \cdot \prod_v \mathrm{gtr}(\mathrm{Fr}_v \mid \wedge^\bullet \mathfrak{U})$$

  where $\mathfrak{U}$ is a certain finite dimensional vector space (see the discussion following Proposition 4.6.1 for the definition of $\mathfrak{U}$ and the definition of "graded trace" gtr of F_v).

- Let f be an unramified cusp form on $\check{G}$ with L-parameter φ_f. Then up to a global factor of the discriminant,

$$P_{\check{X}^-}(f) = (*)\, L_X(\varphi_f)$$

Evidently, there is asymmetry between X and its BZSV dual that remains to be explained in this statement. We propose a possible interpretation in Remark 4.6.3; in short, we explain that the dual "space" to X ought to be $\check{X}^-$, along with a sheaf of algebras obtained from pushforward under a derived enhancement of $\check{X}$. A similar structure is present in the BZSV duality underlying Ginzburg–Rallis' integral representation of the exterior square L-function on GSp_6 (see Theorem 5.5.3). However, we emphasize that this is far from the last word on this issue; we merely propose this as a first step towards understanding the underlying duality in these examples.

In summary, we see that in order to understand, in the duality framework, the generalized Rankin–Selberg method developed by Ginzburg and his collaborators, and the Casselman–Shalika-type formulae of Sakellaridis, one is forced already to consider automorphic and spectral periods of singular and nonaffine spaces. Our suggestion can be summarized by the following two rough slogans:

Proposal 1. Normalization factors of Eisenstein series in the Rankin–Selberg method reflect the fact that the automorphic side should be replaced by a suitable affine cone, and the spectral side should be replaced by a nilpotent cone.

Proposal 2. Automorphic periods unfolding to more general unique models reflect the fact that the spectral side should be replaced by certain derived vector bundles.

The goal of this thesis is to substantiate these proposals by automorphic period calculations.

1.3 Some physical perspectives

To finish the Introduction, we include a short exposition of the more general picture of BZSV duality and give some motivation from the perspective of geoemtric Langlands and mathematical physics. This will not be essential in understanding the body of this thesis, but nonetheless it gives a heuristic explanation for the numerical coincidences that we present, and makes it clear that our calculations are but one layer in a systematic *relative Langlands duality* proposed by BZSV which has manifestations in various settings in which one may study a Langlands-type correspondence.

The perspective that the geometric Langlands correspondence for a Riemann surface Σ should be viewed as a duality between topological quantum field theories (TQFTs) can be traced back at least to Kapustin–Witten's foundational paper [KW07]. In their work, for a pair of Langlands dual reductive groups G and $\check{G}$, automorphic forms on G are interpreted as certain A-branes on $T^*\mathrm{Bun}_G(\Sigma)$ and Galois representations are interpreted as certain B-branes on the moduli space of local systems $\mathrm{Loc}_{\check{G}}(\Sigma)$. In this language, the categorical Langlands conjecture can be understood as homological mirror symmetry between $T^*\mathrm{Bun}_G(\Sigma)$ and $\mathrm{Loc}_{\check{G}}(\Sigma)$.

Let's now consider Σ an algebraic curve over a finite field $\mathbf{F}$, with field of functions F, and let $\overline{\Sigma} = \Sigma \times_{\mathbf{F}} \overline{\mathbf{F}}$ be its base change to the algebraic closure. The Langlands correspondence for Σ can be viewed as an equivalence of two 4-dimensional TQFTs: the A-model A_G and the B-model $B_{\check{G}}$. The (unramified) equivalence contains at least the following structures:

- (Global Langlands). Let Σ be our algebraic curve over $\mathbf{F}$, which we view as a 3-manifold. We expect an identification of vector spaces

$$A_G(\Sigma) := \mathrm{Fun}(\mathrm{Bun}_G(\Sigma)) \cong \mathcal{O}(\mathrm{Loc}_{\check{G}}(\Sigma)) =: B_{\check{G}}(\Sigma)$$

  between certain functions on $\mathrm{Bun}_G(\Sigma)$, or automorphic forms, and the ring of functions on the moduli space of $\check{G}$-local systems.

- (Local Langlands). Let $O = \mathbf{F}[[t]]$, $K = \mathrm{Frac}\, O$, and $D^\circ = \mathrm{Spec}\, K$ be the algebraic punctured disk over $\mathbf{F}$, which we view as a 2-manifold. We expect an equivalence of categories

$$A_G(D^\circ) := D_{G(O)}(\mathrm{Gr}_G) \cong \mathrm{Coh}_{\check{G}}(\check{\mathfrak{g}}^*[2]) =: B_{\check{G}}(D^\circ)$$

  between $G(O)$-equivariant constructible sheaves on Gr_G and $\check{G}$-equivariant coherent sheaves on $\check{\mathfrak{g}}^*$ (with a grading shift). This is the derived geometric Satake of Bezrukavnikov-Finkelberg [BF08].

Let X be a smooth affine G-variety, and $\Phi \in \mathcal{S}(X(\mathbb{A}))$ be the Schwartz function $\Phi = \otimes_v \mathbf{1}_{\mathcal{O}_v}$. The data of the polarized Hamiltonian G-space $M = T^*X$ defines naturally the following structures on A_G:

- (Global relative Langlands). An automorphic form

$$\Theta_X(g) := \sum_{\gamma \in X(F)} \Phi(\gamma g) \in A_G(\Sigma)$$

- (Local relative Langlands). Consider the "variety of triples" R_X of Bezrukavnikov-Finkelberg-Nakajima [Nak16][BFN18]:

$$R_X := \left\{ (E, \eta, s) \ : \ \begin{array}{c} (E, \eta) \in \mathrm{Gr}_G = G(K)/G(O) \\ s \in X_E(K) \text{ such that } s\eta \in X_E(O) \end{array} \right\}$$

  where X_E is the associated X-bundle to the G-bundle E. Then there is a natural map $\pi_X : R_X \to \mathrm{Gr}_G$ by forgetting the section s, and we have an object

$$(\pi_X)_* \, \omega_{R_X} \in A_G(D^\circ)$$

We can summarize these structures by saying that M defines a boundary field theory for A_G, i.e., the choice of a vector in the vector space $A_G(\Sigma)$ and the choice of an object in the category $A_G(D^\circ)$.

Similarly, if $\check{M} = T^*\check{X}$ is a polarized Hamiltonian $\check{G}$-variety with $\check{X}$ smooth and affine, we can attach a boundary field theory to $B_{\check{G}}$:

- (Global relative Langlands). A function on $\mathrm{Loc}_{\check{G}}(\Sigma)$ defined as follows: for a local system $\rho : \pi_1(\Sigma) \to \check{G}$,

$$L(\check{X}, \rho) := \sum_{x \in \mathrm{Fix}(\rho, \check{X})} L(0, \rho, \widehat{\mathcal{O}}^*_{\check{X}, x})$$

  where $\widehat{\mathcal{O}}^*_{\check{X}, x}$ is the ($\mathbf{C}$-linear dual) of the completed ring of functions at $x \in \check{X}$. We think of this expression as the L-function of $\check{X}$.

- (Local relative Langlands). Let $\mu : \check{M} \to \check{\mathfrak{g}}^*$ be the moment map defining the Hamiltonian action. Then we have the object

$$\mu_* \mathcal{O}^*_{\check{M}} \in B_{\check{G}}(D^\circ)$$

At the level of global Langlands, we see that if X and $\check{X}$ defined equivalent boundary field theories under the identification $A_G(\Sigma) = B_{\check{G}}(\Sigma)$, we can extract the following numerical consequence: for every

(cuspidal) normalized Hecke eigenform $f \in A_G(\Sigma)$ with L-parameter ρ_f, an equality

$$\langle f, \Theta_X \rangle_{\mathrm{Bun}_G(\Sigma)} = L(\check{X}, \rho_f)$$

We can call the left hand side the automorphic X-period of f, and the right hand side the spectral $\check{X}$-period of ρ_f. Many results in the theory of integral representations of L-functions can be rephrased in this way, among them are Tate's integral, Hecke's integral for the standard L-function of a modular form, the Rankin-Selberg integral, and the Gan–Gross–Prasad integral. Similarly, at the level of local Langlands, the equivalence of boundary field theories implies an equivalence of functors out of $A_G(D^\circ) \cong B_{\check{G}}(D^\circ)$:

$$\mathrm{Hom}_{A_G}(\pi_* \omega_{R_M}, \cdot\,) = \mathrm{Hom}_{B_G}(\mu_* \mathcal{O}_{\check{M}}, \cdot\,)$$

which we can view as a categorifying a computation of local periods.

One essentially new ingredient proposed by Ben-Zvi–Sakellaridis–Venkatesh is that the structure of duality between A_G and $B_{\check{G}}$ should extend to a more refined duality between the boundary field theories associated to M and $\check{M}$. More precisely, if $M = T^*X$ and $\check{M} = T^*\check{X}$ define equivalent boundary conditions for A_G and $B_{\check{G}}$ respectively, then they should define equivalent boundary conditions for B_G and $A_{\check{G}}$, respectively, as well. Roughly speaking, the numerical consequence one may deduce from this presupposition is that integral representations of L-functions should come in pairs:

$$\langle f, \Theta_X \rangle_{\mathrm{Bun}_G(\Sigma)} = L(\check{X}, \rho_f) \overset{?}{\iff} \langle f, \Theta_{\check{X}} \rangle_{\mathrm{Bun}_{\check{G}}(\Sigma)} = L(X, \rho_f)$$

This provides a heuristic explanation for the existence of the duality structure which seems to exist in the theory of integral representations of L-functions explained in the Introduction.

1.4 Organization of this thesis

In Chapter 2 we give an overview of the parts of BZSV relative Langlands duality that are most relevant to us, namely, their conjectures at the level of numerical invariants. In particular we state the key definitions that are used throughout this thesis. Following this, we explain general principles with which we venture beyond spherical automorphic periods, and record some recurring computational methods.

In Chapter 3 we study a class of new automorphic periods that unfold abstractly to the Whittaker model, and establish BZSV duality in this setting. In particular, Ginzburg's integral representation of the adjoint L-function of GL_3 [Gin91] and Garrett's triple product L-function [Gar87] are examples of this general

calculation, although some modifications and interpretations are required. Namely, one needs to pass to certain affine cones and nilpotent cones of the Hamiltonian actions. The results in this section are joint work with Akshay Venkatesh.

Retaining general lessons from the previous section, in Chapter 4 we examine another class of new automorphic periods that unfold abstractly to the Shalika model, and propose a BZSV duality in this setting by derived enhancements. In particular, Ginzburg–Rallis' integral representation of the exterior cube L-function of GL_6 was the motivation for studying this class of periods, given the expected modifications from the Whittaker case. In Chapter 5 we examine an unfolding to the Ginzburg–Jiang model. Ginzburg–Rallis' integral representation for the exterior square L-function of GSp_6 [GR94a] was the motivation for studying this period. Again we apply the philosophies from the Whittaker and Shalika examples, this time there is one more step: to apply Sakellaridis' general Casselman–Shalika formula for the spherical Ginzburg–Jiang function.

1.5 Glossary of basic notation and assumptions

We collect the basic notations and assumptions that are valid throughout the entire thesis.

- Following typical convention, we write $\mathbf{Z}$ for the integers, $\mathbf{Q}$ for the rational numbers, $\mathbf{R}$ for the real numbers, and $\mathbf{C}$ for the complex numbers. We write $\mathbf{G}_m$ and $\mathbf{G}_a$ for the multiplicative and additive groups, respectively, considered as group schemes over $\mathbf{Z}$. We write $\mathbf{A}^n$ for the affine space of dimension n, considered as a $\mathbf{Z}$-scheme.

- We fix a global function field F over $\mathbf{F} = \mathbf{F}_q$. For every place v of F, we write F_v for the local field at v, $\mathcal{O}_v$ for the ring of integers in F_v, and ϖ_v for a uniformizer of F_v, and q_v is the size of the residue field $\mathcal{O}_v/(\varpi_v)$.

- If G is a split reductive group defined over $\mathbf{Z}$, we let $\check{G}$ be the split reductive group defined over $\mathbf{Z}$, whose root datum is dual to that of G. We emphasize that $\check{G}$ is viewed as an algebraic group; in particular, we can view it as a complex algebraic group $\check{G}_{\mathbf{C}}$ when regarded as a target of L-parameters, but we need its structure as an algebraic group over $\mathbf{Z}$ when it is placed on the automorphic side.

- We write $\mathbb{A} = \mathbb{A}_F$ for the ring of adeles of F, and $\mathcal{O} \subset \mathbb{A}$ is the subring of everywhere integral elements.

- For a reductive group G over F, we write $[G] = G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$ for the adelic double quotient of G.

- We will consider automorphic forms as functions on $[G]$; in particular, we only consider *everywhere unramified* automorphic forms. The central character of an automorphic form f will be denoted χ_f.

- All reductive groups G we work with will be pinned, either evidently by taking diagonal matrices as tori, upper triangular matrices as the Borel, etc, or otherwise will be explicitly specified. We write A_G for this maximal torus. We write $X^*(\,\cdot\,)$ (resp. $X_*(\,\cdot\,)$) for the lattice of characters (resp. lattice of cocharacters), and $X^*_{\mathrm{dom}}(\,\cdot\,)$ (resp. $X_*^{\mathrm{dom}}(\,\cdot\,)$) for the submonoid of dominant characters (resp. submonoid of dominant cocharacters).

- Let G be a split reductive group defined over $\mathbf{Z}$. For a place v, we often write $G_v = G(F_v)$, $K_v \subset G_v$ for the maximal compact subgroup $G(\mathcal{O}_v)$ when the context is clear.

- Let H be an algebraic group over F, which may or may not be unimodular. The measure on $H(\mathbb{A})$ we assume will always be the left-invariant Haar measure, normalized so that $\mathrm{vol}(H(\mathcal{O})) = 1$, *unless otherwise specified.* This does happen: occasionally we will use a *ψ-self-dual* measure on certain unipotent subgroups, in which case we will decorate the measure as $d^\psi u$. The measure on H_v is also the left-invariant Haar measure, normalized so that $\mathrm{vol}(H(\mathcal{O}_v)) = 1$, unless otherwise specified.

- The group actions we consider will always be right actions, which we assume to be defined over some localization of $\mathbf{Z}$, i.e., a split reductive group G over $\mathbf{Z}[1/M]$ acting on a variety X defined over $\mathbf{Z}[1/M]$. In particular, assuming that q not dividing M, it makes sense to consider the $\mathcal{O}_v$-points of X for every place v of F.

- In particular, if we consider linear representations on finite dimensional vector spaces, we will follow the conventions of the previous point: representations are right actions on row vectors.

- Let X be variety with G-action as above. Then we define the Schwartz space

$$\mathcal{S}(X_{\mathbb{A}}) := C_c^\infty(X_{\mathbb{A}}, \mathbf{C})$$

of locally constant compactly supported $\mathbf{C}$-valued functions on $X_{\mathbb{A}}$. This space admits an right action of $G_{\mathbb{A}}$ by the following formula: for $g \in G_{\mathbb{A}}$ and $\Phi \in \mathcal{S}(X_{\mathbb{A}})$,

$$(\Phi * g)(x) := \Phi(xg)$$

This action will be further unitarily normalized in Section 2.2.

- We write Fr_v for the geometric Frobenius at a place v of F. Suppose $V = \bigoplus_{k \gg -\infty} V_k$ is a graded $\mathbf{C}$-vector space where each V_k is finite dimensional, and Fr_v acts on V by a graded linear operator. Then we define the graded trace of Fr_v on V as the following expression whenever it converges:

$$\mathrm{gtr}(\mathrm{Fr}_v | V) := \sum_{k \gg -\infty} \mathrm{tr}\big(\mathrm{Fr}_v q_v^{-k/2} \,|\, V_k\big)$$

- Suppose Φ is some system of positive simple roots of a root system. For a subset $J \subset \Phi$ and a weight λ we will often use the notation

$$\langle J, \lambda \rangle := \sum_{\check{\alpha} \in J} \langle \lambda \check{\alpha}, \lambda \rangle$$

and thus confound J as the subset with J as the sum of its elements.

Chapter 2

Relative Langlands Duality after Ben-Zvi–Sakellaridis–Venkatesh

In this section, we introduce fundamental players in relative Langlands duality, in the sense of BZSV: a numerical invariant of automorphic forms on G given a G-space X, and a numerical invariant of L-parameters into $\check{G}$ given a $\check{G}$-space $\check{X}$. These objects, termed the *automorphic X-period* and the *spectral $\check{X}$-period*, will be defined in Section 2.2 and Section 2.3, respectively. In Section 2.4, we describe how we have chosen to normalize our automorphic forms, and explain how certain local spherical functions will be normalized at some place where the Whittaker functional ψ ramifies, which is inevitable. In Section 2.5 we make the fundamental definition of *weak duality*, as an approximation of BZSV duality. Briefly, the difference is that we will only endeavour to verify that the Hamiltonian actions we consider are BZSV dual when "restricted to the cuspidal spectrum", although we also allow some further flexibility which seems to be essential for the eventual generalization of BZSV duality to a larger genre of spaces. Finally, in Section 2.6 and Section 2.7 we record some relevant computational philosophies that will be used implicitly throughout all subsequent chapters.

2.1 Setup

Let Σ be a curve of genus g over $\mathbf{F} = \mathbf{F}_q$ and with function field F. Let $K^{1/2} = K_\Sigma^{1/2}$ be a choice of spin structure, and we fix a rational section ν of $K^{1/2}$ which determines also a rational 1-form $\omega = \nu^{\otimes 2}$. Note

that ω determines a global additive character

$$\psi : F\backslash \mathbb{A}_F \longrightarrow \mathbf{F}^\times$$

$$(f_v) \longmapsto \sum_v \mathrm{Res}_v(f\omega)$$

which we will think of as $\mathbf{C}^\times$-valued by composing with a character of $\mathbf{F}^\times$ that we fix forever. The divisor $\partial := \mathrm{div}(\omega) = \sum_v n_v[v]$ has even coefficients because we have a square root of it, and we write $\partial^{1/2} := \mathrm{div}(\nu) = \sum_v (n_v/2)[v]$, which can be thought of as an element $\partial^{1/2} = (\varpi_v^{n_v/2})$ in $\mathbb{A}_F^\times$. The local additive character ψ_v on F_v thus has conductor $\partial_v^{-1}\mathcal{O}_v = \varpi_v^{-n_v}$. We always write $n_v = 2m_v$.

Let G be a split reductive group over F. One basic paradigm shift proposed by Ben-Zvi–Sakellaridis–Venkatesh [BZSV] (henceforth referred to as BZSV) is that the data of a graded Hamiltonian G-variety M gives rise to numerical invariants of both automorphic forms on G, and L-parameters valued in G. More precisely, one has an automorphic functional

$$P_M : \left\{\text{automorphic forms on } G\right\} \longrightarrow \mathbf{C}$$

and a numerical invariant of L-parameters

$$L_M : \left\{\text{L-parameter valued in } G\right\} \longrightarrow \mathbf{C}$$

attached to M, such that P_M is a certain automorphic period, and L_M behaves as an L-function. In this Section, we explain these definitions and BZSV's duality conjecture at the level of these numerical invariants, give useful computational tricks along the way, and outline some general philosophy for going beyond the BZSV setting.

To start, we make the following definitions.

Definition 2.1.1. A graded Hamiltonian G-space M is a symplectic variety M equipped with a G-action and a G-equivariant moment map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

along with an extra commuting $\mathbf{G}_m$-action on M. We will refer to this extra $\mathbf{G}_m$ as $\mathbf{G}_{\mathrm{gr}}$.

Definition 2.1.2. A graded Hamiltonain G-space M is polarized if $M \cong T^*X$ is isomorphic to the cotangent bundle of a G-variety X, and the action on M is the induced one from X.

In the following, we will mostly work in the polarized setting, and use the symbol "X" to refer to a

chosen polarization, so a "G-variety X" really refers to the "Hamiltonian G-space $M = T^*X$" for the BZSV setup. We will use the symbol "M" when necessary to emphasize that we are considering a Hamiltonian space without chosen (or does not have) a polarization with respect to the G-action.

Finally, given a (possibly singular) G-variety X, we distinguish a volume G-eigenform on the smooth locus of X, with associated eigencharacter

$$\eta = \eta_X : G \longrightarrow \mathbf{G}_m$$

which defines, by duality, a cocharacter

$$\eta = \eta_X : \mathbf{G}_m \longrightarrow \check{G}$$

which play an essential role in unitarily normalizing the automorphic period and normalizing the functional equation of the L-functions.

2.2 Automorphic period

Suppose X admits a $G \times \mathbf{G}_{\mathrm{gr}}$-action (where, again, we are considering the underlying Hamiltonian G-space $M = T^*X$), and admits an eigenmeasure $\eta = \eta_X$. We consider the following unitarily-normalized action on $\mathcal{S}(X(\mathbb{A}_F))$: for a Schwartz function $\Phi \in \mathcal{S}(X(\mathbb{A}))$,

$$((g,\lambda) * \Phi)(x) := |\eta(g,\lambda)|^{1/2}\, \Phi(x(g,\lambda))$$

The normalized θ-series of X on $G(\mathbb{A}_F)$ is defined by automorphic averaging

$$\begin{aligned}\theta_X^{\mathrm{norm}}(g) &:= q^{\frac{g-1}{2}(\dim(X)-\dim(G))} \sum_{x \in X(F)} ((g,\partial^{1/2}) * \Phi)(x) \\ &= q^{\frac{g-1}{2}(\dim(X)-\dim(G))} |\eta(g,\partial^{1/2})|^{1/2} \sum_{x \in X(F)} \Phi(x(g,\partial^{1/2}))\end{aligned}$$

Note that the power of q here is $\dim(\mathrm{Bun}_H)/2$, where H is the generic stabilizer on X. From this we can define the **(normalized) automorphic X-period** for an unramified automorphic form f on G by L^2-pairing with the normalized θ-series

$$P_X^{\mathrm{norm}}(f) := \langle f, \theta_X^{\mathrm{norm}} \rangle_{L^2([G])} = \int_{[G]} \theta_X^{\mathrm{norm}}(g) f(g)\, dg \tag{2.1}$$

where the measure on $\mathrm{Bun}_G(\mathbf{F}) = [G]$ is the groupoid measure. In summary, the automorphic X-period measures the contribution of $\mathcal{S}(X(\mathbb{A}))$ to the automorphic spectrum of G.

The following lemma will handle covergence of $P_X(f)$ in all cases that we encounter here.

Lemma 2.2.1. Suppose G (split, reductive) acts linearly on $X \subset \mathbf{A}^n$ and contins the scaling $\mathbf{G}_m$. Let f be a cusp form on G. Then the integral defining $P_X(f)$ is absolutely and uniformly convergent if the real part of the central character of f is sufficiently large.

Proof. In the context of this proof, $\mathbf{G}_m$ will always mean the central $\mathbf{G}_m \subset G$. Being cuspidal, f has compact support modulo center, i.e. there exists a compact subset $\Omega \subset G_{\mathbb{A}}$ such that f is supported inside $\Omega \cdot \mathbf{G}_m(\mathbb{A})$. Since there are only a finite number of $G_{\mathfrak{o}}$-orbits on $\Omega \cdot G_{\mathfrak{o}}$ this integral amounts to a finite sum of integrals over the adelic points of $\mathbf{G}_m$, each of which is bounded by an integral of the form

$$\int_{\lambda \in \mathbb{A}^\times} \theta_X(\lambda g_0)|\lambda|^s d\lambda.$$

where s arises from the central character of f. This is bounded, in absolute value, by a suitable multiple of

$$\int_{\mathbb{A}^\times} \Phi(x_0\lambda)|\lambda|^s d\lambda$$

where Φ is a Schwartz function on the adelic points of the ambient affine space, and x_0 any basepoint. In turn, we can majorize Φ by a coordinate-wise product $\varphi_1(x_1)\dots\varphi_n(x_n)$ of Schwartz functions, from which we see that the above integral is bounded by a similar integral in the case $n = 1$, where it is an itnegral of Iwasawa-Tate type. □

Remark 2.2.2. In the case when M is a nonpolarized (or nonpolarizable) Hamiltonian G-space, one follows a similar philosophy: through geometric quantization, one should obtain a representation of G which is realized concretely as "functions on a Lagrangian of M", and apply the analogous formulae as above to measure the extent to which this representation occurs in the automorphic spectrum. We will not encounter any nonpolarizable symplectic spaces in this thesis, but we mention that theta correspondences come from these types of Hamiltonian actions.

2.3 Spectral period

On the L-function side, we first record the definition from [BZSV] which works well when considering smooth spaces with discrete fix points. For T a $\check{G} \times \mathbf{G}_{\mathrm{gr}}$-representation and $\varphi : \mathrm{Gal}_F \to \check{G}$ an L-parameter,

we view $T = \bigoplus_k T_k$ as a graded Gal_F-representation, with grading coming from $\mathbf{G}_{\mathrm{gr}}$. Then we write

$$L(s,T) := \prod_k L(s+k/2, T_k)$$

for the L-function "sheared" by the $\mathbf{G}_{\mathrm{gr}}$-grading. Furthermore, we consider the normalization

$$L^{\mathrm{norm}}(s,T) := \epsilon(s,T)^{-1/2} L(s,T)$$

Under this normalization, the functional equation (along with the fact that $\epsilon(s,T)\epsilon(1-s,T^\vee) = 1$) is equivalent to the normalized L-function being invariant under $(T,s) \leftrightarrow (T^\vee, 1-s)$:

$$L(s,T) = \epsilon(s,T)L(1-s,T^\vee) \iff L^{\mathrm{norm}}(T,s) = L^{\mathrm{norm}}(T^\vee, 1-s)$$

Combining the two types of twists, we have

$$L^{\mathrm{norm}}(s,T) = \prod_k \epsilon^{-1/2}(s+k/2, T_k) L(s+k/2, T_k)$$

Let's highlight the role of the choice of spin structure on the spectral side. The ϵ-factor is by definition

$$\epsilon(s,T) := \det(T)(\partial) q^{(1/2-s)(2g-2)\dim(T)}$$

so our choice of spin structure affords a preferred choice of square root

$$\epsilon(s,T)^{1/2} = \det(T)(\partial^{1/2}) q^{(1/2-s)(g-1)\dim(T)}$$

For a smooth $\check{G}$-variety $\check{X}$ and a $\check{G}$-valued parameter φ with finitely many fix points on $\check{X}$, we define the **(normalized) spectral $\check{X}$-period** for an L-parameter $\varphi : \mathrm{Gal}_F \to \check{G}$ by

$$L_{\check{X}}(\varphi) := \sum_{x \in \mathrm{Fix}(\varphi, \check{X})} L^{\mathrm{norm}}(0, T_x)$$

Now we extend the above definition to include a much general class of spaces, which makes sense (at least) for locally ringed spaces with $\check{G}$-action, which we term **nonabelian L-functions**. We will propose a normalization which is the most naïve extension of the above definitions. However, we emphasize that one sees in our computations that the normalized automorphic and normalized spectral periods will *not* match exactly in singular cases, so evidently there remains something to be understood about how to properly

normalize these nonabelian L-functions.

Now suppose $\varphi : \mathrm{Gal}_F \to \check{G}$ is an L-parameter and $x \in \check{X}$ is fixed by $\varphi \times \mathbf{G}_m$. We define the *local L-function* at a place v to be:

$$L^{(v)}_{\check{X},x}(\varphi) := \text{graded trace of Frobenius at } v \text{ on the completed local ring } \mathcal{O}_x \text{ at } x,$$

which we understand to be the limit of the corresponding traces evaluated on $\mathcal{O}/\mathfrak{m}_x^N$ as $N \to \infty$, or, what is the same, the sum of the traces taken over $\mathfrak{m}_x^N/\mathfrak{m}_x^{N+1}$. Here, both the $\mathbf{G}_m$ action and the Frobenius action is via pullback, $\lambda \cdot f(x) = f(x\lambda)$.

Lemma 2.3.1. Suppose that $\rho : \check{G} \hookrightarrow \mathrm{GL}_n$ contains the central $\mathbf{G}_m$, and that $\check{X} = \mathbf{A}^n$. Then $L^{(v)}_{\check{X},0}(\varphi)$ coincides with the the local factor $L(\frac{1}{2}, \varphi_v, \rho)$.

Proof. Here the $\mathfrak{m}_x^N/\mathfrak{m}_X^{N+1}$ corresponds to $\mathrm{Sym}^n \mathrm{std}$ as a GL_n-representation. Therefore, the graded trace is given by

$$\sum_n q^{-n/2} \mathrm{trace}(\mathrm{Fr}_v | \mathrm{Sym}^k \mathbf{A}^n),$$

which becomes $\det(1 - q^{-1/2}\mathrm{Fr}_v)^{-1}$ as required. □

Finally, we define the **(normalized) nonlinear L-function** or **normalized spectral period** by the formula

$$L_{\check{X}}(\varphi) := (\check{\eta} \circ \varphi)(\partial^{-1/2}) \Delta^{\frac{\dim X - \gamma}{4}} \sum_{x \in \mathrm{Fix}(\varphi, X)} \prod_v L^{(v)}_{\check{X},x}(\varphi) \tag{2.2}$$

It is understood to be defined only when the fixed locus is finite. By means of the same reasoning as in Lemma 2.3.1, we see that it recovers the previous definition in the smooth case, and that the product in (2.2) is absolutely convergent.

2.4 Normalization of automorphic forms

2.4.1 Whittaker normalization.

We will use Whittaker normalization of our automorphic forms.

Thinking of $f \in \pi$ as a $\mathbf{C}$-valued function, we prefer to use a **Whittaker normalization**: let $W(f)$ denote the Whittaker period of f with probability normalization, i.e. with reference to adelic uniformization:

$$W_f = \int_{n \in U(F)\backslash U(\mathbb{A})} \psi(u) f(u) du. \tag{2.3}$$

where the measure on $U(\mathbb{A})$ is normalized to give mass one to the quotient. In the case of GL_2, for example, we then have the following Fourier expansion for a cuspidal function f:

$$f(g) = \sum_{t \in T_F} W_f(tg),$$

where t is the torus of diagonal matrices with $(2,2)$ entry equal to 1.

Of particular importance to us is the value W_f^0 of W at the point $a_0 = e^{2\rho^\vee}(\partial^{-1/2})$.

$$W_f^0 := W_f(a_0), \tag{2.4}$$

For example in the case of GL_2 we have $a_0 = \prod a_{0,v}$ where

$$a_0 = \prod_v \begin{pmatrix} \varpi_v^{-m_v} & 0 \\ 0 & \varpi_v^{m_v} \end{pmatrix} \tag{2.5}$$

For later use we observe then that for a in the adelic torus we have

$$W_f(a) = W_f^0 \prod_v W_v^{\mathrm{un}}(a_v a_{0,v}^{-1}) \tag{2.6}$$

where W_v^{un} is the standard normalization of the unramified Whittaker function; that is to say, W_v^{un} evaluated at a_v corresponding to a co-character $\chi \in X_*(T)$ is zero unless χ is dominant, in which it equals $q^{-\langle \rho, \chi \rangle}$ times the trace of the irreducible representation of $\check{G}$ with highest weight $\chi \in X^*(\check{T})$ evaluated at the Satake parameter.

$$W_v^{\mathrm{un}}(e^{\chi}(\varpi_v) a_{0,v}^{-1}) = W_v^{\mathrm{un}}(e^{\chi + 2m_v\check{\rho}}(\varpi_v)) = q^{-\langle \chi + 2m_v\check{\rho}\rangle, \rho} s_{\chi + 2m_v\check{\rho}}. \tag{2.7}$$

The normalized Whittaker period is given by

$$P_{\mathrm{Whitt}}(f) = q^{\beta_{\mathrm{Whitt}}/2} W_f^0, \beta_{\mathrm{Whitt}} = (g-1)(\dim(U) - \langle 2\rho, 2\rho^\vee \rangle). \tag{2.8}$$

For example:

$$\beta_{\mathrm{Whitt}}^{\mathrm{SL}_2} = -(g-1), \beta_{\mathrm{Whitt}}^{\mathrm{SL}_3} = (g-1) \times (3-8) = -5(g-1).$$

We say that an automorphic form is **Whittaker normalized** if $P_{\mathrm{Whitt}}^{\mathrm{norm}}(f) = 1$.

2.4.2 Shalika normalization

Mimicking the Whittaker normalization, we standardize a normalization of automorphic forms on $G = \mathrm{GL}_{2n}$ which we call Shalika normalized. Conceptually, this is the easiest normalization one should adopt when comparing with any automorphic period that unfolds to the Shalika model.

First we set the constants

$$\beta_{\mathrm{Sha}} = (g-1)\big(\dim(U) - \langle 2\rho_X, 2\check{\rho}_X\rangle\big)$$

where $U \cong \mathrm{Mat}_n$ the unipotent part of the Shalika subgroup

$$\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad m \in \mathrm{GL}_n \text{ and } X \in \mathrm{Mat}_n$$

with $\check{\rho}_X$ the distinguished cocharacter into G

$$\check{\rho}_X(t) = \mathrm{diag}(t^{1/2}\mathrm{Id}_n, t^{-1/2}\mathrm{Id}_n)$$

We also fix the cocharacter into G

$$\mathbf{j}(t) = \mathrm{diag}(t\mathrm{Id}_n, \mathrm{Id}_n)$$

For a cusp form f on GL_{2n}, we define its **(global) Shalika function** as

$$\Omega_f(g) := \int_{[U][\mathrm{GL}_n]} f(umg)\Psi(u)\delta_P^{-1}(m)\, d^{\psi}u\, dm$$

where d^{ψ} is the measure giving mass one to $[U]$, and $\mathrm{GL}_n(\mathcal{O})$ has mass one. Consider the following normalization scheme:

Definition 2.4.1. Let $f \in \pi$ be a cuspidal automorphic form on GL_{2n}. We say that f is Shalika normalized if

$$\Omega_f(\mathbf{j}(\partial^{-1})) = q^{-\beta_{\mathrm{Sha}}/2}\left[\operatorname*{Res}_{s=1} L(\wedge^2, \pi, s)\right] \tag{2.9}$$

By the work of Jacquet–Shalika (see Proposition 1 of [JS88]), the integral defining Ω_f is nonvanishing if and only if the exterior square L-function has a pole at $s = 1$, so we may simply interpret the above definition as vacuously true for automorphic forms with no Shalika model. Up to a translation by the center (i.e., noting

that the cocharacters $\mathbf{j}$ and $\check{\rho}_X$ are equivalent mod center), we see that (2.9) is analogous to the Whittaker normalization.

Suppose f is a cusp form with nonvanishing Shalika function Ω_f. As with Whittaker normalization, if f is Shalika normalized, then by local multiplicity one of the Shalika model, the value of Ω_f at a given cocharacter is computed by $\Omega_f(\mathbf{j}(\partial^{-1}))$ and local functionals evaluated at a shift determined by the ramification of ψ. Precisely, let f be a Shalika normalized cusp form. Then for a torus element $a = (a_1, \dots, a_n, \mathrm{Id}_n)$,

$$\Omega_f(a) = q^{-\beta_{\mathrm{Sha}}/2} \left[\operatorname*{Res}_{s=1} L(\wedge^2, \pi, s) \right] \prod_v \Omega^{\mathrm{ur}}_{f,v}(a_v \mathbf{j}(\partial_v)) \tag{2.10}$$

where $\Omega^{\mathrm{ur}}_{f,v}$ is the unique $\mathrm{GL}_n(F_v) \times (U_v, \psi_v^{\mathrm{ur}})$-equivariant distribution on π_v whose value at the identity is 1. We will often write the local factors as

$$\Omega_{f,v}(\,\cdot\,) := \Omega^{\mathrm{ur}}_{f,v}(\,\cdot\,\mathbf{j}(\partial_v)) \tag{2.11}$$

and term it the **(normalized) local Shalika function of** f.

2.4.3 Ginzburg–Jiang normalization

Mimicking the Shalika normalization, we standardize a normalization of automorphic forms on $G = \mathrm{GSp}_6$ which we call Ginzburg–Jiang normalized. Conceptually, this is the easiest normalization one should adopt when comparing with any automorphic period that unfolds to the Ginzburg–Jiang model.

First we set the constants

$$\beta_{\mathrm{GJ}} = (g-1)\big(11 - \langle 2\rho_X, 2\check{\rho}_X \rangle\big)$$

with $\check{\rho}_X$ the distinguished cocharacter into G

$$\check{\rho}_X(t) = \mathrm{diag}(t^{1/2}\mathrm{Id}_2, \mathrm{Id}_2, t^{-1/2}\mathrm{Id}_2)$$

The 11 is secretly the dimension of the Ginzburg–Jiang subgroup

$$S = \begin{bmatrix} 1 & X & Y \\ 0 & 1 & X^* \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \; m \in \mathrm{GL}_2,\; X \in \mathrm{Mat}_2,\; Y = \begin{bmatrix} x & y \\ y + \det(X) & z \end{bmatrix}$$

We let U be the unipotent part of the Ginzburg–Jiang subgroup, whose elements we parametrize by $u(X, Y)$,

and we consider the following character of U

$$\Psi(u(X,Y)) := \psi(\mathrm{tr}(X))$$

We also fix the cocharacter into G

$$\mathbf{j}(t) = \mathrm{diag}(t\mathrm{Id}_n, \mathrm{Id}_n)$$

For a cusp form f on GSp_6, we define its **(global) Ginzburg–Jiang function** as

$$\Omega_f(g) := \int_{[U][\mathrm{GL}_2]} f(umg)\Psi(u)\delta_S^{-1}(m)\, d^\psi u\, dm$$

where d^ψ is the measure giving mass one to $[U]$, and $\mathrm{GL}_2(\mathcal{O})$ has mass one. Consider the following normalization scheme:

Definition 2.4.2. Let $f \in \pi$ be a cuspidal automorphic form on GL_{2n}. We say that f is Ginzburg–Jiang normalized if

$$\Omega_f(\mathbf{j}(\partial^{-1})) = q^{-\beta_{\mathrm{GJ}}/2}\left[\operatorname*{Res}_{s=1} L(\mathrm{Spin}, \pi, s)\right] \tag{2.12}$$

By the work of Bump–Ginzburg (see Section 2 of [BG92]), the integral defining Ω_f is nonvanishing if and only if the spin L-function has a pole at $s = 1$, so we may simply interpret the above definition as vacuously true for automorphic forms with no Shalika model. Up to a translation by the center (i.e., noting that the cocharacters $\mathbf{j}$ and $\check{\rho}_X$ are equivalent mod center), we see that (2.12) is analogous to the Whittaker normalization.

Suppose f is a cusp form with nonvanishing Ginzburg–Jiang function Ω_f. As with Whittaker normalization, if f is Ginzburg–Jiang normalized, then by local multiplicity one of the Ginzburg–Jiang model (see Theorem 1 of [Vo97]), the value of Ω_f at a given cocharacter is computed by $\Omega_f(\mathbf{j}(\partial^{-1}))$ and local functionals evaluated at a shift determined by the ramification of ψ. Precisely, let f be a Ginzburg–Jiang normalized cusp form. Then for a torus element $a = (a_1^{-1}, a_2^{-1}, 1, 1, a_1, a_2)$,

$$\Omega_f(a) = q^{-\beta_{\mathrm{GJ}}/2}\left[\operatorname*{Res}_{s=1} L(\mathrm{spin}, \pi, s)\right] \prod_v \Omega^{\mathrm{ur}}_{f,v}(a_v \mathbf{j}(\partial_v)) \tag{2.13}$$

where $\Omega^{\mathrm{ur}}_{f,v}$ is the unique $\mathrm{GL}_2(F_v) \times (U_v, \psi_v^{\mathrm{ur}})$-equivariant distribution on π_v whose value at the identity is

1. We will often write the local factors as

$$\Omega_{f,v}(\,\cdot\,) := \Omega^{\mathrm{ur}}_{f,v}(\,\cdot\,\mathbf{j}(\partial_v)) \tag{2.14}$$

and term it the **(normalized) local Ginzburg–Jiang function of** f.

2.5 Weak duality

Let G_1, G_2 be a Langlands dual pair of reductive groups, and let $M_1 = T^*X_1$ be a polarized graded Hamiltonian G_1-space, and $M_2 = T^*X_2$ be a polarized graded Hamiltonian G_1-space.

Definition 2.5.1. Let G_1, G_2 be a pair of Langlands dual reductive groups. A pair of Hamiltonian actions (G_1, X_1) and (G_2, X_2) are **weakly BZSV dual** if the following holds: for any choice of $\mathbf{F}, \Sigma, \psi$ as above, for f_i any Whittaker normalized, everywhere unramified cuspidal automorphic form on G_i with Langlands parameter φ_i valued in $\check{G}_i$, we have equalities

$$P_{X_1}(f_1) = (*)\, L_{X_2}(\varphi_1)$$

$$P_{X_2}(f_2) = (*)\, L_{X_1}(\varphi_2)$$

where $(*)$ is a rational power of the discriminant $\Delta_F = q^{2g-2}$, independent of the data $\mathbf{F}, \Sigma, \psi, f_i$ above.

The equalities are understood to hold whenever the fixed locus of φ_i on the X_i are discrete. In particular, if this fixed locus is empty, the period must vanish.

Remark 2.5.2. In fact, the anomalous half integral powers of discriminant do not appear in many examples of weak BZSV duality between smooth Hamiltonian actions. However, they do appear in our calculations for singular spaces, and their role in the duality framework warrants further study.

As our terminology suggests, we do not claim X_1, X_2 to be dual if the above numerical equivalences are satisfied, but merely that they are "weakly" dual. We make this modest definition essentially because we are only testing them on the cuspidal part of the automorphic spectrum. In our experience, establishing weak duality between a pair of spaces essentially specifies the dual pair up to birational equivalence, as should be familiar from automorphic period calculations: there is often a "main" open dense orbit which contributes to the period of a cuspidal automorphic form, and several "negligible" orbits which do not contribute unless the automorphic form was a suitable Eisenstein series. This may seem like a needlessly picky point, especially if one is only interested in the cuspidal spectrum; however, recall that when interpreted spectrally, changing

the birational type of a space has the potential of altering the spectral period drastically. Most notably, by removing special fix points from a space one can force the spectral period to vanish on a larger class of L-parameters.

2.6 Beyond sphericity

As mentioned in the Introduction, one can view the present thesis as taking a few first steps towards the relative Langlands program in a nonspherical setting. To explain this, we quickly review the role of sphericity in the study of automorphic periods, and point out motivation as well as new phenomenon in the directions we study.

For the most part, integral representations of L-functions for a given reductive group G are obtained by the following three steps:

1. Let H be another reductive group such that $G \subset H$, and let $P \subset H$ be some parabolic subgroup. One considers an Eisenstein series $E_P(h) = \sum_{\gamma \in P_F \backslash H_F} \Phi(\gamma g)$ on H induced from P, and for a cuspidal automorphic form f on G one unfolds the integral

$$\int_{[G]} f(g) E_P(g)\, dg = \sum_{\gamma_i \in G_F \backslash H_F / P_F} \int_{[\mathrm{Stab}_G(\gamma_i)]} f(s)\, ds \overset{(*)}{=} \int_{[\mathrm{Stab}_G(\gamma_0)]} f(s)\, ds$$

   where in the step $(*)$ one argues that the stabilizers of γ_i, except for a distinguished γ_0, all contain some unipotent radical of G, and thus the $[\mathrm{Stab}_G(\gamma_i)]$-integrals all vanish except for the $[\mathrm{Stab}_G(\gamma_0)$-term, by cuspidality of f.

2. Next, one argues by Fourier expansion and further unfolding, until one has a formula of the form

$$\int_{[\mathrm{Stab}_G(\gamma_0)]} f(s)\, ds = \int_{\omega_{\mathbb{A}}} W_f(a)\, da$$

   where $W_f(g) = \int_{[N]} f(ng)\psi(n)\, dg$ is the Whittaker function of f (provided that f is generic), and $\omega_{\mathbb{A}}$ is some adelic region. By local multiplicity one of the Whittaker model, one can factorize this latter integral as

$$\prod_v \int_{\omega_v} W_v(a_v)\, da_v$$

   where W_v is a suitably normalized local Whittaker function.

3. For almost all places v, W_v is spherical, so one can apply the formula of Casselman and Shalika [**?**] to

convert evaluations of the spherical Whittaker function into Weyl characters of the dual group:

$$\int_{\omega_v} W_f(a_v)\, da_v = \sum_{\chi} W_f(\varpi^\chi) = \sum_{\chi} \delta^{1/2}(\varpi^\chi) s_\chi(\mathrm{Sat}_v)$$

Here, χ is summed over some region of dominant cocharacters of G (or equivalently, dominant characters of $\check{G}$), s_χ is the Weyl character of $\check{G}$ with highest weight χ, and Sat_v is the Satake parameter of f at v.

4. Finally, one identifies the previous expression with some products and quotients of local L-factors to obtain an integral representation of an L-function.

Note that at least for cuspidal f, one may think of the integral $\int_{[G]} f(g)E_P(g)\, dg$ as the $[\mathrm{Stab}_G(\gamma_0)]$-period, or in newly introduced terminology, the automorphic $(\mathrm{Stab}_G(\gamma_0)\backslash G)$-period.

Recall that a subgroup $S \subset G$ is spherical if the variety $S\backslash G$ is a spherical G-variety, i.e., that there is a dense open Borel orbit in $S\backslash G$. At first approximation, requiring that S be spherical in G renders the above strategy feasible because it implies that the local intertwining spaces $\mathrm{Hom}_{S_v}(\pi_v, \mathbf{C})$ have small dimension, and hence that the $[S]$-period has a chance of Euler-factorizing on representation-theoretic grounds.

In this thesis, we will advocate for the necessity and effectiveness of removing aspects of the sphericity assumption. More specifically, we emphasize two emergent types of spaces: singular cones and derived vector bundles. The presence of singular cones is relatively better understood, whereas the presence of derived vector bundles is more mysterious and it remains to be seen what would ultimately be the correct definitions.

The apparent importance of considering singular cones can be traced back to Ginzburg's observation [GR94a] that integral representations of L-functions often come with extra zeta factors. Concretely, let π be an automorphic representation of G, and let V be an algebraic representation of its dual group $\check{G}$; then one should not look for an integral representation of the L-function $L(\varphi, V, s)$, but rather the expression

$$\frac{L(\pi, V, s)}{\zeta(d_1 s)\cdots\zeta(d_k s)} \tag{2.15}$$

where $d_1, \ldots, d_k$ are degrees of homogeneous invariants in $\mathbf{C}[V]^{\check{G}}$. Let's write $f_1, \ldots, f_k \in \mathbf{C}[V]^{\check{G}}$ for the corresponding homogeneous invariants of degrees $d_1, \ldots, d_k$; then one defines the nilpotent cone of V as the vanishing locus of these invariants:

$$Z(f_1, \ldots, f_k) \subset V$$

In some nice situations, notably when $Z(f_1, \ldots, f_k)$ is locally complete intersection at the origin, one may in-

terpret the expression (2.15) as the spectral period of the nilpotent cone of V, as the following straightforward calculation demonstrates.

Proposition 2.6.1. Suppose V is a linear $G \times \mathbf{G}_{\mathrm{gr}}$-representation, $X \subset V$ is an affine $G \times \mathbf{G}_{\mathrm{gr}}$-subvariety, and $0 \in X \subset V$ is the unique φ-fixed point for an everywhere unramified L-parameter $\varphi : \mathrm{Gal}_F \to G$. Suppose $f_1, \dots, f_r$ are transverse equations defining X near 0, which are $\mathbf{G}_{\mathrm{gr}}$-homogeneous with degrees $d_1, \dots, d_r$ and G-semiinvariant with characters $\chi_1, \dots, \chi_r$ as before. Suppose $V = V[k_1] \oplus \cdots \oplus V[k_s]$ is its $\mathbf{G}_{\mathrm{gr}}$-weight space decomposition. Then the spectral period of φ with respect to X is

$$L_X^{\mathrm{norm}}(\varphi) = \frac{\prod_{i=1}^r \chi_i(\partial^{1/2})}{\det_V(\partial^{1/2})} q^{\frac{g-1}{2}(\sum_{j=1}^s k_j - \sum_{i=1}^r d_i + s - r)} \prod_v \mathrm{gtr}(\mathrm{Fr}_v \,|\, \mathcal{O}_X) = \frac{L(V, \varphi, 1/2)}{\prod_{i=1}^r \zeta(d_i/2)}$$

Proof. Factoring the left hand side over primes, we see that the Proposition can be split into the two following independent facts:

$$(1) \quad \frac{L_v(0, V)}{\prod_{i=1}^r L_v(d_i/2, \chi_i)} = \mathrm{gtr}(\mathrm{Fr}_v \,|\, \mathcal{O}_X^*)$$

$$(2) \quad \frac{\epsilon^{-1/2}(0, V)}{\prod_{i=1}^r \epsilon^{-1/2}(d_i/2, \chi_i)} = \frac{\prod_{i=1}^r \chi_i(\partial^{1/2})}{\det_V(\partial^{1/2})} q^{\frac{g-1}{2}(\sum_{j=1}^s k_j - \sum_{i=1}^r d_i + s - r)}$$

Indeed, (1) is the unnormalized version of the Proposition, and (2) is the leftover normalization factor, which can be verified directly from definition.

Let's prove (1). The isomorphism

$$\mathrm{Sym}^\bullet(V)[k] \cong \bigoplus_{n_1 k_1 + \cdots + n_s k_s = k} \mathrm{Sym}^{n_1}(V[k_1]) \otimes \cdots \mathrm{Sym}^{n_s}(V[k_s])$$

implies the following identity for the sheared L-function

$$\begin{aligned} L_v(0, V) &= \prod_{i=1}^r L_v(k_i/2, V[k_i]) \\ &= \sum_{k \in \mathbf{Z}} q^{-k/2} \sum_{n_1 k_1 + \cdots + n_s k_s = k} \prod_{i=1}^r \mathrm{tr}(\mathrm{Fr}_v | \mathrm{Sym}^{n_i}(V^*[k_i]) \\ &= \sum_{k \in \mathbf{Z}} \mathrm{tr}(\mathrm{Fr}_v q^{-k/2} \,|\, \mathrm{Sym}^\bullet(V^*)[k]) \end{aligned}$$

With this we see that by extracting the equality of $q^{-n/2}$'s coefficients from both sides of (1), we get the assertion

$$(1') \quad \sum_{d_I + k = n} (-1)^{|I|} \mathrm{tr}(\mathrm{Fr}_v | \mathrm{Sym}^\bullet(V^*)[k]) \chi_I(\mathrm{Fr}_v) = \mathrm{tr}(\mathrm{Fr}_v | \mathcal{O}_X[n])$$

where $I \subset \{1, \dots, r\}$ is an indexing subset, and $d_I = \sum_{i \in I} i$, $\chi_I = \prod_{i \in I} \chi_i$. But (1') is exactly the

inclusion-exclusion principle applied to the quotient $\mathrm{Sym}^\bullet(V)[k] \to \mathcal{O}_X^*[k]$; explicitly, we have the Koszul resolution

$$\cdots \to \bigoplus_{i,j} f_i f_j \mathrm{Sym}^\bullet(V^*)[n-d_i-d_j] \to \bigoplus_i f_i \mathrm{Sym}^\bullet(V^*)[n-d_i] \to \mathrm{Sym}^\bullet(V^*)[n] \to \mathcal{O}_X[n] \to 0$$

Taking alternating traces we recover (1'). □

The theme of taking nilpotent cones is present in most calculations in this thesis, but for the simplest examples see Section 3.

Remark 2.6.2. If X is *not* LCI at the origin, we cannot expect L_X to have the beloved and expected features of usual L-functions: functional equation and analytic continuation. Estermann's phenomenon tells us that when considering an arbitrary Euler product

$$L(s) := \prod_v \frac{Q_v(q_v^{-s})}{R_v(q_v^{-s})}$$

for $Q_v, R_v \in \mathbf{C}[q_v^{-s}]$ monic polynomials, if Q_v/R_v cannot be written as a ratio of cyclotomic polynomials in q_v^{-s}, then $L(s)$ has a natural boundary at $\mathrm{Re}(s) > 0$. We will point out later that this is the case for the nonabelian L-functions that fall out of automorphic computations.

The still very nascent idea of endowing spaces with derived structure stems from generalizations of Step (2) in the above outline. Namely, one might unfold to unique models besides the Whittaker model. Sakellaridis has proven vast generalizations of the Casselman–Shalika formula [Sak12], expressing local spherical functions in unique models in terms of the Satake parameter. In this generality, Sakellaridis' formula looks like a deformation of the Casselman–Shalika formula: the evaluation at a dominant (suitably defined in this setting) cocharacter χ has a main term given by the Weyl character of the dual group of the spherical variety with highest weight χ. The other terms appear in the form of an exterior algebra, which we interpret as derived structures on the BZSV dual space. As the role of "derived spaces" is still very murky at the moment, we will be largely agnostic of the exact category or framework within which we place our derived spaces. In fact, for the definition of the automorphic period only the underlying classical scheme will be relevant (since only its F-rational points are relevant), and for the spectral period we will simply regard these derived spaces as a a topological space equipped with global functions that form a dg-algebra. This is enough to discuss Galois fix points and Frobenius traces, hence enough to define spectral periods. For examples of this nature, see Section 4 and Section 5.

2.7 Residues and distinction

Automorphic periods are often linked to questions about Langlands functoriality in the following prototypical statement: let π be an automorphic representation of G, and let P be a certain automorphic period, then

$$P \text{ vanishes on } \pi \text{ unless the } L\text{-parameter of } \pi \text{ factors through a specific subgroup of } \check{G}.$$

In other words, automorphic periods often distinguish automorphic forms whose L-parameters are special in some sense.

Although at first glance mysterious, this is just a reformulation of the fact that the zeta function has a pole at $s = 1$. In more detail, such a distinction statement often has an underlying integral representation of the form

$$P_s(f) = L(V, \pi, s)$$

where π is an automorphic representation of G, $f \in \pi$ is an automorphic form, s is a complex parameter (coming from an Eisenstein series, for example), and V is an algebraic representation of the dual group $\check{G}$. Suppose P is obtained by taking the residue at $s = 1$ of P_s; then we see that

$$P \text{ vanishes on } \pi \text{ unless the } L(V, \pi, s) \text{ has a pole at } s = 1$$

and asking for $L(V, \pi, s)$ to have a pole at $s = 1$ is asking for the zeta function to appear as factor in this L-function, in other words, asking for the L-parameter to have an invariant line in the representation V. In agregate, we expect a prototypical theorem of the following form:

> P vanishes on π unless the L-parameter of π factors through a specific subgroup $H \subset \check{G}$, which is the stabilizer group of an invariant line in some $\check{G}$-representation V. In the case when the L-parameter of π does factor through H, the period P is related to the L-value $L(V \ominus \mathbf{1}, \pi, 1)$.

The spectral side of this situation can be understood using the BZSV formulation in a straightforward manner: if H is a reductive subgroup of $\check{G}$ and Y is an H-variety, then we can define the **induction** of Y from H to $\check{G}$ by

$$\mathrm{Ind}_H^{\check{G}} Y := H \backslash \big(Y \times \check{G} \big)$$

where H acts by

$$h \cdot (y, g) = (yh^{-1}, hg)$$

To elucidate the $\mathbf{G}_{\mathrm{gr}}$ grading on this space, we note that $\mathrm{Ind}_H^{\check{G}} Y$ can alternatively be viewed as follows: take the $\mathbf{G}_{\mathrm{gr}}$-action on $T^*Y \times T^*G$ by the $\mathbf{G}_{\mathrm{gr}}$-action on the first factor, producted with scaling the cotangent fiber

of T^*G with weight 2. Then this $\mathbf{G}_{\mathrm{gr}}$-action commutes with the H-action, so we can consider the symplectic reduction of $T^*Y \times T^*G$ by H, which gives $T^*(\mathrm{Ind}_H^{\check{G}} Y)$, on which there is an inherited $\mathbf{G}_{\mathrm{gr}}$-action.

Lemma 2.7.1. The spectral period of $X = \mathrm{Ind}_H^{\check{G}} Y$ distinguishes L-parameters with image in $\check{G}$ conjugate to H, and for such a parameter φ,

$$L_X(\varphi) = \sum_{c \in C_\varphi} L(\check{\mathfrak{g}}/\mathfrak{h}, \varphi^c, 1) L_Y(\varphi)$$

where c is summed over equivalence classes in

$$c \in C_\varphi := \big\{ g \in G : \mathrm{Im}(\mathrm{Ad}_g(\varphi)) \subset H \big\}/H$$

and φ^c is the c-conjugate of φ.

Proof. Let $(y, g) \in X$ be an arbitrary point; then an element $g_1 \in G$ fixes (y, g) if and only if there is some element $h \in H$ such that

$$yh = y \text{ and } h^{-1} g g_1 = g$$

i.e., g_1 is conjugate to some $h \in \mathrm{Stab}_H(y)$. In particular, if the image of an L-parameter φ is not conjugate to H, then it will not have any fix points on X, rendering $L_X(\varphi) = 0$.

Now suppose the image of φ can be conjugated into H; we will assume without loss of generality that the image of φ is contained in H to begin with. The above calculation shows that the fix points of φ are exactly $\mathrm{Fix}(\varphi, Y) \times C_\varphi$.

The completed local ring at such a fix point (y, c) is exactly

$$\widehat{\mathcal{O}}_{X,(y,c)} \cong \widehat{\mathcal{O}}_{Y,y} \mathbin{\widehat{\otimes}} \widehat{\mathcal{O}}_{\mathrm{Id},H\backslash G}$$

where the action is given by φ^c on $\widehat{\mathcal{O}}_{\mathrm{Id},H\backslash G}$. So we can calculate

$$\begin{aligned} L_X(\varphi) &= \sum_{(c,y) \in \mathrm{Fix}(\varphi, X)} \mathrm{gtr}\big(\varphi(\mathrm{Fr}_v) | \widehat{\mathcal{O}}^*_{X,(c,y)}\big) \\ &= \sum_{c \in C_\varphi} \sum_{y \in \mathrm{Fix}(\varphi, Y)} \mathrm{gtr}(\varphi^c(\mathrm{Fr}_v) \,|\, \mathrm{Sym}^\bullet(\mathfrak{h}\backslash\check{\mathfrak{g}})^*) \mathrm{gtr}(\varphi(\mathrm{Fr}_v) \,|\, \widehat{\mathcal{O}}^*_{Y,y}) = L(\check{\mathfrak{g}}/\mathfrak{h}, \varphi, 1) L_Y(\varphi) \\ &= \sum_{c \in C_\varphi} L(\check{\mathfrak{g}}/\mathfrak{h}, \varphi^c, 1) L_Y(\varphi) \end{aligned}$$

where we evaluated the L-function of $\check{\mathfrak{g}}/\mathfrak{h}$ at 1 since the grading on $\mathrm{Sym}^k(\check{\mathfrak{g}}/\mathfrak{h})^*$ is given grading $2k$ by

definition. □

Chapter 3

Unfolding to the Whittaker Model

The material of this Chapter is joint work with Akshay Venkatesh, and partially copied from a paper in preparation [CV]. In this section, we consider automorphic periods that unfold "in one step" to the Whittaker model. Among these are our original motivations: Ginzburg's integral representation of the adjoint L-function for SL_3 [Gin91] and Garret's integral representation of the triple product L-function for SL_2 [Gar87]. The strategy of this section provides the general outline for other examples that we study:

- We consider a certain singular $\check{G}$-variety $\check{X}$ and compute its spectral period by an invariant theory calculation; this is accomplished by Proposition 3.3.1.
- The automorphic period of the analogous singular G-variety X is designed to unfold to the Whittaker model, and after Euler factorizing, we will integrate local Whittaker functions over a region dictated by the integral points of X. The computation of integral points is accomplished by Proposition 3.5.1 and Proposition 3.5.2.
- Finally, our main result in this section is Theorem 3.7.2: under a root-theoretic condition (3.22) on the Langlands dual pair of reductive groups, we observe that the automorphic X-period and the spectral $\check{X}$-period agree on cusp forms. Thus the pair $(X, \check{X})$ are in weak BZSV duality.

3.1 Group theoretical setup

Let G be a split reductive group over F, with one dimensional center $Z(G)^0 \cong \mathbf{G}_m$. Let $B = UA \subset G$ be a Borel subgroup with unipotent radical U, maximal split torus A, and let $\Psi : U \to \mathbf{G}_a$ be a Whittaker character. Let $U^0 = \mathrm{Ker}(\Psi) \subset U$, then the normalizer of U^0 in B is generated by U^0 and the torus

$\mathrm{Im}(\check{\rho}) \subset A$, where $\check{\rho}$ is the half sum of positive coroots. Note that

$$\langle \check{\rho}, \alpha \rangle = 1 \text{ for all simple roots } \alpha$$

Let $\theta : G/[G,G] \to \mathbf{G}_m$ be an isomorphism, and let $\check{\theta} : \mathbf{G}_m \dashrightarrow Z(G)$ be such that $\langle \theta, \check{\theta} \rangle = 1$ (where we allow ourselves to pass to a finite cover to define this cocharacter).

For a concrete version of a setup like this, let μ_k be the kernel of the morphism $[G,G] \times \mathbf{G}_m \to G$; then θ corresponds to the character $t \mapsto t^k$ on the $\mathbf{G}_m$ factor, whereas $\check{\theta}$ corresponds to the projection of $t \mapsto (\mathrm{Id}, t^{1/k})$. The dual group to G is then a k-fold cover of $[G,G]^\vee \times \mathbf{G}_m$ and the dual to θ projects to $t \mapsto t^k$, whereas the dual to $\check{\theta}$ is given by $z \mapsto z^{1/k}$.

We will suppose that

$$\check{\rho}' := \check{\theta} + \check{\rho} \in X_*(A)$$

which, a priori, lives inside $X_*(A) \otimes \mathbf{Q}$, in fact belongs to the cocharacter lattice. We write $\mathbf{i} = e^{\check{\rho}'}$ for this embedding of $\mathbf{G}_m$ inside G and T for the image of $\mathbf{i}$.

Numerology. We use the notation

$$a = \langle \rho, \check{\rho} \rangle.$$

and we will take

$$\gamma = \frac{1}{a-1}$$

We further require that the cocharacter of $\check{G}$ dual to θ is given by γ times $\check{\theta}_{\check{G}}$, and vice versa; that is to say, under the duality isomorphisms

$$X^*(A) \simeq X_*(\check{A}) \text{ and } X^*(\check{A}) \simeq X_*(A)$$

we have

$$\theta_G \leftrightarrow \gamma \check{\theta}_{\check{G}}, \check{\theta}_G \leftrightarrow \gamma^{-1} \theta_{\check{G}}. \tag{3.1}$$

Forms. Note that at a smooth point in X, the top exterior power of the tangent space as a representation of A is given by the weight

$$-(-2\rho + \alpha_0) \text{ for some (and any) simple root } \alpha_0$$

where the extra minus sign reflects the fact that we are taking the right adjoint action instead of the usual one.

In particular, the weight with which $\mathbf{G}_m \xrightarrow{\mathbf{i}} T$ scales a volume eigenform η on X is

$$\langle \check{\rho}', 2\rho - \alpha_0 \rangle = 2\langle \rho, \check{\rho} \rangle - 1 = 2a - 1, a = \langle \check{\rho}, \rho \rangle. \tag{3.2}$$

and correspondingly the eigenform character of $G \times \mathbf{G}_m$ is given by

$$\eta(g, \lambda) = \theta(g)^{2a-1} \lambda^{2a-1}.$$

Consider the (nonunimodular) group

$$S = \left\{ u\, \mathbf{i}(\lambda) \in U^0 \times T \right\} \subset G$$

We write $X = \overline{S\backslash G}$, the affine closure of $S\backslash G$, viewed as a conical G-variety where the $\mathbf{G}_{\mathrm{gr}}$ scaling action is given by $\gamma\check{\theta}$.

3.2 Examples

3.2.1 The adjoint representation of PSL_3, after Ginzburg

Here we take $G = \mathrm{PSL}_3 \times \mathbf{G}_m$; the isomorphism θ is the projection to $\mathbf{G}_m$, and $\check{\theta}$ the inclusion of $\mathbf{G}_m$; the embedding $\mathbf{i}$ sends λ to $\mathrm{diag}(\lambda, 1, \lambda^{-1})$. Let $\mathfrak{pgl}_3 = \mathfrak{sl}_3$ be its Lie algebra, understood as trace-free 3×3 matrices, and take

$$\gamma_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{pgl}_3. \tag{3.3}$$

Then the rule $(g, z) \mapsto z \cdot (\mathrm{Ad}(g)^{-1} \cdot \gamma_0)$ identifies $S\backslash G = g^{-1}\gamma_0 g$ an open subset of the nilpotent cone inside $\mathfrak{pgl}_3$, and X with that nilpotent cone. The action of both $\mathbf{G}_m \subset G$ and $\mathbf{G}_{\mathrm{gr}}$ is by scaling. The scaling factor of the eigenform under both $\mathbf{G}_m \subset G$ and $\mathbf{G}_{\mathrm{gr}}$ is given by $\eta(g, \lambda) = \theta(g)^3 \lambda^3$, since $2\langle \rho, \check{\rho} \rangle - 1 = 3$. In particular, $\dim X - \gamma = 3$.

3.2.2 The triple product example, after Garrett

The relevant pair of Langlands dual groups for Garrett's triple product integral is

$$G = \frac{\left\{ (g_1, g_2, g_3, z) \in \mathrm{GL}_2^3 \times \mathbf{G}_m : z^3 = \det(g_1 g_2 g_3) \right\}}{(\lambda, \mu, \nu, 1) : \lambda\mu\nu = 1}$$

$$\check{G} = \frac{\{(g_1, g_2, g_3) \in \mathrm{GL}_2^3 : \det(g_1) = \det(g_2) = \det(g_3)\}}{\Delta\mu_3}$$

There are concrete models for the spaces $X = \overline{S\backslash G}$ and $\check{X} = \overline{\check{S}\backslash\check{G}}$, which we describe now.

Consider the G-representation on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ defined by

$$(v_1 \otimes v_2 \otimes v_3)(g_1, g_2, g_3, z) = z^{-1}\big(v_1 g_1 \otimes v_2 g_2 \otimes v_3 g_3\big)$$

which we think of as a right $G \times \mathbf{G}_{\mathrm{gr}}$-space acting on row vectors, with $\mathbf{G}_{\mathrm{gr}}$ acting by scaling; equivalently via the following morphism $\mathbf{G}_m \to G$:

$$t \longmapsto \left(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}^{\times 3}, t^2 \right) \subset G \tag{3.4}$$

We now set

$$X = \text{vanishing locus of the unique quartic semiinvariant,}$$

which we will describe explicitly now. Thinking of $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ as a cube of numbers with side length 2, we take a pair of opposing faces which gives us two 2×2 matrices A and B. Then we consider the binary quadratic form

$$Q(x, y) := \det(Ax + By)$$

whose discriminant $f = \mathrm{disc}(Q)$ is the quartic semiinvariant defining X.

Let $\check{X}$ be the affine cone of the Plücker embedding of $\mathrm{LGr}(3, 6)$, the Grassmannian of Lagrangian 3-planes in $\mathbf{C}^6$, into $\wedge^3\mathbf{C}^6$. In other words, every nonzero point of $\check{X}$ can be represented by a (L, ν), where $L \subset \mathbf{C}^6$ is a Lagrangian and $\nu \in \wedge^3 L$ is a volume form. For $v_1 \wedge v_2 \wedge v_3 \in \check{X} \subset \wedge^3\mathbf{C}^6$

$$(v_1 \wedge v_2 \wedge v_3)(g_1, g_2, g_3) = (v_1 g_1 \wedge v_2 g_2 \wedge v_3 g_3)$$

We consider $\check{X}$ as a right $\check{G} \times \mathbf{G}_{\mathrm{gr}}$-space, with $\mathbf{G}_{\mathrm{gr}}$ acting by scaling, or equivaelently through the following (virtual, i.e. defined after passing to a cover) cocharacter:

$$\mathbf{i} : t \longmapsto \begin{bmatrix} t^{1/3} & 0 \\ 0 & t^{1/3} \end{bmatrix}^{\times 3} \subset \check{G} \tag{3.5}$$

The volume form on $\check{X}$ has $\check{G} \times \mathbf{G}_{\mathrm{gr}}$-eigenvalue

$$\eta_{\check{X}}(g_1, g_2, g_3; \lambda) = \left(\prod \det(g_i) \right)^2 \lambda^4. \tag{3.6}$$

Here we take $\mathbf{G}_{\mathrm{gr}}$ to act through $\check{\theta}^2$. Since $2\langle \rho, \check{\rho}' \rangle - 1 = 2(3/2) - 1 = 2$ the scaling character of the eigenform is

$$\eta(g, \lambda) = \theta(g)^2 \lambda^4.$$

Again, $\dim X - \gamma = 3$.

On the dual side to this example we again let $\mathbf{G}_{\mathrm{gr}}$ act by $\check{\theta}^2$, again $2\langle \rho, \check{\rho}' \rangle - 1 = 2(3/2) - 1 = 2$ and the eigenform character for $G \times \mathbf{G}_{\mathrm{gr}}$ scales a volume form is thereby

$$\eta(g, \lambda) = \theta(g)^2 \lambda^4$$

3.3 Geometry of $U^0\backslash G$ and $TU^0\backslash G$

Let $\Psi : U \to \mathbf{G}_a$ be the unique homomorphism restricting to the identity on each pinned root subgroup, and let U^0 be the kernel of U; we will denote the induced map $U \to \mathbf{G}_a$ also by $u \mapsto \bar{u}$. We put $S = TU^0$. As an example, we can take $G = \mathrm{SL}_3 \times \mathbf{G}_m$, and θ the projection to the second coordinate; then

$$S = \{ \begin{bmatrix} x & a & b \\ 0 & 1 & a \\ 0 & 0 & 1/x \end{bmatrix} \times x \}.$$

More examples are given in the following section.

In any case, our main concern in this section is the geometry of the space

$$X = \overline{S\backslash G}, S = TU^0$$

be the affine closure of $S\backslash G$. More formally, this connotes the scheme over $\mathbf{Z}$ whose ring of functions consists of left TU^0-invariant functiosn on $\mathbf{Z}[G]$, i.e. elements of $\mathbf{Z}[G]$ which are left TU^0-invariant considered as regular functions on $G_{\mathbb{C}}$. This is a flat affine scheme over $\mathbf{Z}$, on which G (as a group scheme over $\mathbf{Z}$) acts. We will be interested in X over various fields k, which we shall denote by X_k if there danger of confusion.

Our first goal is to compute the ring of functions on X; this is accomplished by the following proposition:

Proposition 3.3.1. Let k be a field of characteristic zero. Let $k[X]$ be the ring of regular functiosn on X_k. Then, as G-representation,

$$\mathcal{O}_X \cong \bigoplus_{\lambda} V_\lambda^*$$

where V_λ is the representation of highest weight λ and the sum is taken over all λ that satisfy the inequalities and λ must satisfy the inequalities

$$0 \leq \langle \lambda, \check{\rho}' \rangle \leq \min_{\check{\alpha} \text{ simple}} \langle \lambda, \check{\alpha} \rangle. \tag{3.7}$$

We say that a weight λ is **relevant** if it satisfies these inequalities. A relevant weight is automatically dominant. We will analyze the cone of relevant weights more carefully in §3.4.

Proof. (of Proposition 3.3.1) We have

$$k[X] \simeq \bigoplus_{\lambda} V_\lambda^S \otimes V_\lambda^*. \tag{3.8}$$

where we associate to $v_\lambda \otimes v_\lambda^*$ the function $\langle v_\lambda, g v_\lambda^* \rangle$. On the other hand, by Borel-Weil, the rule sending $v \in V$ to the function $\langle gv, v^* \rangle$ for v^* a highest weight vector of the dual representation $V_{-w\lambda}$ gives

$$V_\lambda \simeq \{ f \in k[G] : f(bg) = w\lambda(b) f(g) \}. \tag{3.9}$$

where G acts by right translation on the right hand side; here w denotes the long Weyl element for G. This induces an injection

$$\mathrm{I} : V_\lambda^{U^0} \to \text{polynomials on } \mathbf{G}_a$$

sending a function f to its restriction to $f(wu)$, which descends from U to $U^0 \backslash U \simeq \mathbf{G}_a$. Under this map, the action of U/U^0 is by translation. We claim that the image of I precisely consists of polynomials P that have degree at most $\langle \lambda, \check{\alpha} \rangle$ for all simple roots α. Any polynomial P on $\mathbf{G}_a$ determines a function f_P on the big cell BwB satisfying (3.9), and we must determine when f_P extends to G; for this it is sufficient to check that it extends over all codimension one B-orbits, which reduces us to the SL_2-case. In the SL_2-case, this amounts to checking that the polynomial P has degree at most $\langle \lambda, \check{\alpha} \rangle$: the pole of P at infinity is exactly the degree of P, and it needs to be less than $\langle \lambda, \check{\alpha} \rangle$ to be a section of the equivariant line bundle on G/B defined by (3.9).

In order that a vector $v \in V^{U^0}$ be fixed by T, we must have $f(wut) = f(wu)$ or equivalently $f(t^w w t^{-1} u t) = f(t)$. Now, writing $t_\lambda = x^{\check{\rho}}$, $t^{-1} u t$ effects a dilation by x^{-1} on $U/U^0 \simeq \mathbf{G}_a$, whereas the term in front cor-

responds to

$$x^{\langle w\lambda, w\check{\rho}'\rangle} = x^{\langle \lambda, \check{\rho}'\rangle}$$

In other words, the polynomial $P = x^i$ corresponds to a T-invariant function precisely when $\langle \check{\rho}', \lambda\rangle - i = 0$ for some i between 0 and $\min\langle \lambda, \alpha^\vee\rangle$. This is precisely the relevancy condition. In fact, we have proved that the function $\tilde{P}_\lambda$ on G that satisfies $f(bg) = w\lambda(b)f(g)$ and

$$\tilde{P}_\lambda(wu) = \bar{u}^{\langle \check{\rho}', \lambda\rangle}. \tag{3.10}$$

corresponds under (3.9) to the unique line in V_λ^S. □

3.4 The cone of relevant weights

The set defined by (3.4) defines a convex cone inside $X^*(A) \otimes \mathbf{R}$, that is to say, a closed convex set invariant by scaling under $\mathbf{R}_{>0}$. We will describe this more explicitly, with the following notation: We write $\varpi_\alpha \in X^*(A) \otimes \mathbf{R}$ for the dual basis to the simple coroots $\check{\alpha}$. and

$$\check{\rho} = \sum_{\check{\alpha}} n_{\check{\alpha}}\check{\alpha}, \rho = \sum_\alpha n_\alpha \alpha,$$

$$a = \langle \rho, \check{\rho}\rangle = \sum n_\alpha = \sum \check{n}_\alpha.$$

where the α-sums range over simple roots or simple coroots. At the last stage we used that $\langle \rho, \check{\alpha}\rangle = \langle \check{\rho}, \alpha\rangle = 1$ for α simple.

Note that each element $\varpi_\alpha - n_{\check{\alpha}}\theta$ is orthogonal to $\check{\rho}'$ and $\langle \varpi_\alpha - n_{\check{\alpha}}\theta, \check{\alpha}\rangle = 1$, so it belongs to the relevant cone. Writing $a = \langle \check{\rho}, \rho\rangle$, we see that the element $\rho - (a-1)\theta$ pairs to 1 with each simple coroot and with $\check{\rho}'$, so it also belongs to the relevant cone.

Lemma 3.4.1. The vectors named above, that is,

$$\{\varpi_\alpha - n_{\check{\alpha}}\theta : \alpha \text{ simple root}\} \cup \{\rho - (a-1)\theta\} \tag{3.11}$$

span the extreme rays of the (real) cone of relevant weights, that it to say, the relevant cone consists of all linear combinations

$$\sum_{\alpha \text{ simple}} x_\alpha (\varpi_\alpha - n_\alpha\theta) + y(\rho - (a-1)\theta) \tag{3.12}$$

for nonnegative reals x_α, y.

Proof. First of all, note that $\{\varpi_\alpha - n_{\check\alpha}\theta : \alpha \text{ simple root}\} \cup \{\rho - (a-1)\theta\}$ forms a basis of $X^*(A) \otimes \mathbf{R}$; indeed, the set $\{\varpi_\alpha - n_{\check\alpha}\theta : \alpha \text{ simple root}\}$ is evidently linearly independent, and as noted previously we have

$$\langle \varpi_\alpha - n_{\check\alpha}, \check\rho' \rangle = 0 \text{ while } \langle \rho - (a-1)\theta, \check\rho' \rangle = 1$$

so the whole set is linearly independent of size equal to $\dim(X^*(A) \otimes \mathbf{R})$. Thus, for any $\lambda \in X^*(A)$ we can write

$$\sum_{\alpha \text{ simple}} x_\alpha \left(\varpi_\alpha - n_{\check\alpha}\theta\right) + y(\rho - (a-1)\theta)$$

for some uniquely determined real numbers x_α and y. Suppose now λ is a relevant dominant weight; our goal is to show that the above numbers x_α and y are nonnegative. We can directly compute

$$y = \langle \lambda, \check\rho' \rangle \geq 0$$

by (3.4). Next, we set

$$\lambda^* := \lambda - y(\rho - (a-1)\theta) = \sum_{\alpha \text{ simple}} x_\alpha(\varpi_\alpha - n_{\check\alpha}\theta)$$

and compute

$$x_\alpha = \langle \lambda^*, \check\alpha \rangle = \langle \lambda, \check\alpha \rangle - y\langle \rho - (a-1)\theta, \check\alpha \rangle = \langle \lambda, \check\alpha \rangle - y \geq 0$$

again by (3.4). □

3.5 Integral points on X

Our next task is to understand integral points on X in terms of the Iwasawa decomposition. We then use the following criterion for integrality, *valid in all sufficiently large residue characteristics*, which will amount to checking integrality for all coefficients of a one-variable polynomial. This residual characteristic assumption is important only for the automorphic period calculation.

Proposition 3.5.1. Let F_v be a nonarchimedean field. There exists q_0 such that, if $q \geq q_0$, the following holds:

> Let $g \in B(F_v)$; then $Sg \in X(F_v)$ defines a point in $X(\mathcal{O}_v)$ if and only if for each relevant weight λ the rule
>
> $$U \ni u \mapsto \tilde P_\lambda(wug^{-1}) \tag{3.13}$$
>
> *a priori* in $F_v[U]$, lies in fact within $\mathcal{O}_v[U]$. Here $\tilde P_\lambda$ is as (3.10).

Proof. By definition $x := Sg$ is integral if and only if $f(x) \in \mathcal{O}_v$ whenever $f \in \mathcal{O}_v[X]$. Now the ring of regular functions $\mathbf{Z}[X]$ is given – by definition – by $\mathbf{Z}$-span of

$$\mathbf{Z}[G] \cap \left(\bigoplus_{\lambda \in \Lambda} V_\lambda^S \otimes V_\lambda^* \right) \to \mathbf{Q}[X].$$

where the sum is taken over relevant weights Λ, and the map was described after (3.8).

Since the ring $\mathbf{Z}[X]$ is generated, it can be generated by a finite subset $\Lambda_0 \subset \Lambda$, which we can harmlessly enlarge to contain one element that lies along each extremal ray. Inverting a finite collection of primes as necessary – replacing, thus, $\mathbf{Z}$ by $\mathbf{Z}' := \mathbf{Z}[\frac{1}{M}]$ – we may further assume that the integral structure structure on the left coincides with that arising from (3.8). Let us write

$$\mathsf{V}_\lambda = \mathbf{Z}'\text{-lattice in } V_\lambda \text{ from (3.9)}$$

and V_λ^* for the dual lattice in V_λ^*. Thus, we may suppose that

$$\bigoplus_{\lambda \in \Lambda_0} \mathsf{V}_\lambda^S \otimes \mathsf{V}_\lambda^*$$

generates $\mathbf{Z}'[X]$. In what follows we shall suppose that the residue characteristic of $\mathcal{O}_v$ is invertible on $\mathbf{Z}'$; this determines the integer q_0 of the statement.

Therefore, in order to show that $Sg \in X(F_v)$ is integral, it is sufficient to show that $f_\lambda(x) \in \mathcal{O}$ whenever f_λ lies in the image of $\mathsf{V}_\lambda^S \otimes \mathsf{V}_\lambda^*$, and $\lambda \in \Lambda_0$. By definition of the integral structure on V_λ^*, this is equivalent to asking that

$$g^{-1}\mathsf{V}_\lambda^S \subset \mathsf{V}_\lambda \otimes \mathcal{O}_v. \tag{3.14}$$

We proved in (3.10) that V_λ^S corresponds, under (3.9), to the F_v-linear span of the single function given on BwU by the rule $wu \mapsto \bar{u}^{\check{\rho}', \lambda}$; one readily sees, then, that this function generates the integral structure V_λ^S.

The condition stated in the proposition is then clearly necessary for (3.14). For sufficiency, assume that (3.13) holds; we will essentially apply "Hartog's theorem" to see that $\tilde{P}_\lambda$ extends: Consider G as a smooth scheme over $\mathcal{O}_v$. The function

$$(b, u) \mapsto w\lambda(b) \cdot \tilde{P}(wug^{-1})$$

defines a function on $\mathcal{B} \times \mathcal{U}$, where we use script to denote the smooth group schemes over $\mathcal{O}_v$. The morphism $(b, u) \mapsto bwu$ is an open immersion $\mathcal{B} \times \mathcal{U} \to \mathcal{G}$, and therefore the aboe function defines a function (a section of the structure sheaf) on its image, which we shall call the open cell in $\mathcal{G}$. This section of the structure sheaf

extends, moreover, to the union of the open cell with the generic fiber of $\mathcal{G}$.

In other words, we are given a scheme $\mathcal{G}$, a subset $U \subset \mathcal{G}$ whose codimension is ≥ 2, and a section of $\mathcal{O}$ over V. The algebraic form of Hartog's theorem asserts that the map $\Gamma(\mathcal{G}, \mathcal{O}) \to \Gamma(V, \mathcal{O})$ is bijective. (In detail, we choose a point x in the complement of V and let R be the local ring. The function on V corresponds to an element of the fraction field of R which belongs to every localization at a height one prime ideal, which therefore belongs to R, i.e. extends over x.)

□

We can now prove the main result of this section, a classification of integral points:

Proposition 3.5.2. Let F_v be a local field with residue characteristic q_v with $q_v \geq q_0$ in the notation of Proposition 3.5.1. Let $\chi \in X_*(A)$. Write $a = \langle \rho, \check{\rho} \rangle$. Then

$$Su_t \varpi^\chi \in X(F_v)$$

is integral, where u_t is an arbitrary preimage of t under $U \mapsto \mathbf{G}_a$, if and only if:

(a) $\chi \in X_*(A)$ has nonegative pairing with each of $-\varpi_\alpha + n_{\check{\alpha}}\theta$ and $(a-1)\theta + \alpha - \rho$, where α varies over simple roots.

(b) The valuation of t is bounded below by $\langle \chi, \rho - (a-1)\theta \rangle$.

For later use, we note that, as long as $a > 1$, which excludes SL_2 and SL_2^2, we can summarize these conditions thus:

$$\langle \chi, \theta \rangle \geq \langle \chi, \frac{\rho - \alpha}{a-1} \rangle \text{ and } \langle \chi, n_{\check{\alpha}}^{-1} \varpi_\alpha \rangle, \tag{3.15}$$

$$v(t) \geq \langle \chi, \rho - (a-1)\theta \rangle. \tag{3.16}$$

Proof. Applying the criterion of Proposition 3.5.1 to $g = u_t \varpi^\chi$, we want to check if

$$u \mapsto \tilde{P}_\lambda(w\varpi^{-\chi}[\mathrm{Ad}(\varpi^\chi)u]u_{-t}) = \varpi^{-\langle \lambda, \chi \rangle} \tilde{P}([\mathrm{Ad}(\varpi^\chi)u]\, u_{-t})$$

lies in $\mathcal{O}_v[U]$. All that matters in the last expression is the projection of $[\mathrm{Ad}(\varpi^\chi)u]\, u_{-t}$ to $U^0 \backslash U$. Let x_α, for α a simple root, be the various coordinates of U in the simple root spaces; then this projection is given, with reference to $U^0 \backslash U \overset{\Psi}{\simeq} \mathbf{G}_a$, by

$$= \sum \varpi^{\langle \chi, \alpha \rangle} x_\alpha - t.$$

We are therefore asking that the following expresion in $F_v[x_\alpha]$ lies inside $\mathcal{O}_v[x_\alpha]$:

$$\varpi^{-\langle\lambda,\chi\rangle}\left[\sum \varpi^{\langle\chi,\alpha\rangle}x_\alpha - t\right]^d \tag{3.17}$$

where $d = \langle\lambda, \check{\rho}'\rangle$.

The binomial formula shows that the polynomial in x_α, t that appears above is integral if and only if the extremal coefficinets involving x_α^d and t^d are integral.

These correspond to the integrality, for each α, of

$$\varpi^{-\langle\lambda,\chi\rangle}\varpi^{d\langle\alpha,\chi\rangle}, \varpi^{-\langle\lambda,\chi\rangle}t^d$$

or formulated in terms of valuations, with the substitution $d = \langle\lambda, \check{\rho}'\rangle$.

$$-\langle\lambda,\chi\rangle + \langle\lambda,\check{\rho}'\rangle\langle\alpha,\chi\rangle \geq 0 \text{ and } -\langle\lambda,\chi\rangle + \langle\lambda,\check{\rho}'\rangle v(t) \geq 0. \tag{3.18}$$

We want these linear inequalities to be valid for all characters λ that lie in the relevant set (3.4).

Now, a linear inequality of the type (3.18) is valid on all integral points in a homogeneous cone if and only if it is valid at all real points in the cone, for positive rescalings of integral points are dense; and, further, that is the case if and only if it is valid for each of a set of generators for extremal rays. Consequently, it suffices to impose the conditions (3.18) above solely at the set of extremal vectors prescribed in (3.12).

For $\lambda = \varpi_\alpha - n_{\check{\alpha}}\theta$, so that λ is orthogonal to $\check{\rho}'$, the inequalities coincide and become $\langle -\varpi_\alpha + n_{\check{\alpha}}\theta, \chi\rangle \geq 0$. Moreover, these extreme weights impose no constraint on t. The remaining weight is $\lambda = \rho - (a-1)\theta$ where, we recall, $a = \langle\rho, \check{\rho}\rangle$, which can be described equivalently as $a = \sum n_\alpha = \sum \check{n}_{\check{\alpha}}$. The first inequality places the following constraint on χ: $\langle -\lambda + \alpha, \chi\rangle \geq 0$ and the second inequality imposes a condition on t, namely,

$$v(t) \geq \langle\rho - (a-1)\theta, \chi\rangle.$$

3.6 The automorphic period

Let f be an automorphic form on G, which we Whittaker normalize so that $W_f^0 = 1$, and let $\Phi = \otimes\Phi_v \in \mathcal{S}(X_{\mathbb{A}})$ be a factorizable Schwartz function; in our applications, we always take $\Phi_v(x) = \mathbf{1}_{X(\mathcal{O}_v)}(xe^{\gamma\check{\theta}}(\partial_v^{1/2}))$.

Lemma 3.6.1. Let f be a Whittaker normalized, everywhere unramified cusp form on G. Then the automor-

phic X-period of f Euler factorizes as

$$P_X(f) = q^{(g-1)\beth} \prod_v \int_{U_v^0 \backslash G_v} \Phi_v(g) W_v(g) \eta(g)^{1/2} \, dg,$$

where $\beth = (2a-1)(1-\gamma/2)$.

In the Ginzburg case $a = 2, \gamma = 1, \beth = 3/2$ and the Garrett case $a = 3/2, \gamma = 2, \beth = 0$.

Proof. The automorphic period attached to X is given by

$$\begin{aligned} P_X(f) &= q^{\frac{-\beta_{\mathrm{Whitt}}}{2}} q^{\frac{-\dim(S)}{2}(g-1)} |\eta^{1/2}(\partial^{1/2})| \int_{[G]} f(g) \sum_{x \in S_F \backslash G_F} \Phi(xg) \eta^{1/2}(g) \, dg \\ &= q^{\frac{-\beta_{\mathrm{Whitt}}}{2}} q^{\frac{-\dim(S)}{2}(g-1)} |\eta^{1/2}(\partial^{1/2})| \int_{S_F \backslash G_{\mathbb{A}}} \Phi(g) f(g) \, dg \\ &= q^{\frac{-\beta_{\mathrm{Whitt}}}{2}} q^{\frac{-\dim(S)}{2}(g-1)} |\eta^{1/2}(\partial^{1/2})| \int_{S_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Phi(g) \int_{[S]} f(sg) \eta^{1/2}(sg) \, ds \, dg \end{aligned}$$

where ds here is the left Haar measure on S. Writing $S = U^0 \times \mathbf{G}_m$ in the evident way, and δ the modular character of $\mathbf{G}_m$ acting on U^0, the inner integral becomes

$$\begin{aligned} &\int_{[S]} f(sg) \eta^{1/2}(sg) \, ds \\ &= \int_{[U^0][\mathbf{G}_m]} f(u\mathbf{i}(\lambda)g) \eta(\lambda g)^{1/2} \, \delta^{-1}(\mathbf{i}(\lambda)) \, d\lambda \, du \\ &= q^{(u-1)(g-1)} \int_{[U^0][\mathbf{G}_m]} f(u\mathbf{i}(\lambda)g) \eta(\lambda g)^{1/2} \, \delta^{-1}(\mathbf{i}(\lambda)) \, d\lambda \, d^\psi u \\ &= q^{(u-1)(g-1)} \sum_{\alpha \in F^\times} \int_{[U][\mathbf{G}_m]} f(u\mathbf{i}(\lambda)g) \eta(\lambda g)^{1/2} \delta^{-1}(\mathbf{i}(\lambda)) \psi_\alpha(u) \, d\lambda \, d^\psi u \end{aligned}$$

where in the second to last step we switched to the Fourier self-dual measure in the u-variable, so and we write $u = \dim(U)$ and $|\psi|$ for the number of ψ-generic roots in U^0, or equivalently the number of simple roots of G. The identity

$$\psi(\mathrm{Ad}_{\mathbf{i}(\alpha)}(u)) = \psi_\alpha(u)$$

allows us to combine the sum over $\alpha \in F^\times$ with the adelic integral over $\mathbf{G}_m$ to get

$$\begin{aligned} &= q^{(u-1)(g-1)} \int_{[U]} \int_{\mathbb{A}^\times} f(u\mathbf{i}(\lambda)g) \eta(\lambda g)^{1/2} \delta^{-1}(\mathbf{i}(\lambda)) \psi(u) \, d\lambda \, d^\psi u \\ &= q^{(u-1)(g-1)} \int_{\mathbb{A}^\times} W_f(\mathbf{i}(\lambda)g) \eta(\lambda g)^{1/2} \delta^{-1}(\mathbf{i}(\lambda)) \, d\lambda \end{aligned}$$

Returning to the original period P_X and recognizing the factorizability of the inner adelic S-integral, we

have

$$P_X(f) = Q \prod_v \int_{S_v \backslash G_v} \Phi_v(g) \int_{F_v^\times} W_v(\mathbf{i}(\lambda)g)\eta(\lambda g)^{1/2}\delta_B^{-1}(\mathbf{i}(\lambda))\, d\lambda\, dg$$
$$= Q \prod_v \int_{U_v^0 \backslash G_v} \Phi_v(g) W_v(g)\eta(g)^{1/2}\, dg$$

where in the last step we folded the inner $F_v^\times$ integral back into the outer $S_v\backslash G_v$ integral, and Q is the global constant

$$Q = q^{\frac{-\beta_{\text{Whitt}}}{2}} q^{\frac{-\dim(S)}{2}(g-1)} |\eta^{1/2}(\partial^{1/2})| q^{(u-1)(g-1)} = q^{\beth(g-1)}$$

where, using (2.8) and $\dim(S) = \dim(U)$,

$$\beth = \big(2\langle \rho, \check{\rho}\rangle - 1 - \frac{\langle \eta, \gamma\check{\theta}\rangle}{2}\big) = (2a-1)(1-\gamma/2) \tag{3.19}$$

which equals $3/2$ in the Ginzburg case $a = 2, \gamma = 1$ and 0 in the Garrett case $a = 3/2, \gamma = 2$. □

Let P_v be the local factor $P_v := \int_{U_v^0\backslash G_v} \Phi_v(g)W_v(g)\eta(g)^{1/2}\, dg$ appearing in the statement of the Lemma. Here, the measure that is used is the integrally normalized measure both on U_v^0 and G_v. By Iwasawa decomposition, we can write

$$U_v^0 \backslash G_v / G(\mathcal{O}_v) \cong F_v \times A_v$$

via the Whittaker character $\psi : U^0\backslash U \cong \mathbf{G}_a$. Using this parametrization to compute the local period, we have

$$P_v = \int_{A_v} (\eta^{1/2}\delta^{-1})(a) \Big[\int_{F_v} \Phi_v(ua)\psi(u)\, du\Big] W_v(a)\, da$$
$$= \int_{A_v} (\eta^{1/2}\delta^{-1})(a) \Big[\int_{F_v} \mathbf{1}_{X(\mathcal{O}_v)}(ua\varpi^{\gamma m_v\check{\theta}})\psi(u)\, du\Big] W_v(a)\, da$$
$$= \sum_{\chi \in X_*(A)} (\eta^{1/2}\delta^{-1})(\varpi^\chi) W_v(\varpi^\chi) \Big[\int_{F_v} \mathbf{1}_{X(\mathcal{O}_v)}(u\varpi^{\chi+\gamma m_v\check{\theta}})\psi(u)\, du\Big]$$

Let us now compute the inner integral over F_v. According to the integrality inequalities (3.15) and (3.16), the following conditions on $u\varpi^{\chi+\gamma m_v\check{\theta}}$ guarantee the nonvanishing of the Schwartz function $\mathbf{1}_{X(\mathcal{O}_v)}$:

$$\langle \chi, \theta\rangle \geq \text{ all of } \langle \chi, \frac{\rho-\alpha}{a-1}\rangle - \gamma m_v \text{ and } \langle \chi, n_{\check{\alpha}}^{-1}\varpi_\alpha\rangle - \gamma m_v \text{ for } \alpha \text{ simple roots}$$

$$v(u) \geq \langle \chi, \rho - (a-1)\theta \rangle - m_v$$

Since the conductor of ψ is ϖ^{-2m_v}, the F_v-integral vanishes unless

$$\langle \chi, \rho - (a-1)\theta \rangle - m_v \geq -2m_v \iff \frac{\langle \chi, \rho \rangle - m_v}{a-1} \geq \langle \chi, \theta \rangle$$

When the integral over ψ does not vanish, it contributes the volume of its range of integration, so we have

$$\begin{aligned} P_v &= \sum_{\chi} (\eta^{1/2}\delta^{-1})(\varpi^\chi) W_v(\varpi^\chi)\, q^{-\langle \chi, \rho-(a-1)\theta\rangle + m_v} \\ &= q^{-2m\langle \rho, \check{\rho}\rangle} \sum_{\chi} \eta^{1/2}(\varpi^\chi) s_{\chi + 2m\check{\rho}}(\mathrm{Sat}_v)\, q^{(a-1)\left(\langle \chi, \theta\rangle + \gamma m_v\right)} \end{aligned}$$

where s_χ is the Weyl character with highest weight χ if χ is dominant, and 0 otherwise; here we used (2.6) and in particular (2.7) at the last step: the factor $\delta^{-1}(\varpi^\chi)$ cancels with two factors of $q^{-\langle \chi, \rho\rangle}$.

Let us make the change of variables

$$\chi_{\text{new}} = \chi_{\text{old}} + 2m\check{\rho}$$

then the above expression becomes

$$P_v = q^{-2m\langle \rho, \check{\rho}\rangle}\, q^{m\langle \eta, \check{\rho}\rangle} \sum_{\chi} \eta^{1/2}(\varpi^\chi) s_\chi(\mathrm{Sat}_v)\, q^{(a-1)\left(\langle \chi, \theta\rangle + \gamma m_v\right)}$$

where now χ is now ranging over dominant coweights of A subject to the bounds

$$\frac{\langle \chi, \rho\rangle}{a-1} \geq \langle \chi, \theta\rangle + (2+\gamma)m_v \geq \frac{\langle \chi, \rho - \alpha\rangle}{a-1} \text{ and } \langle \chi, n_{\check{\alpha}}^{-1}\varpi_\alpha\rangle \tag{3.20}$$

We make another substitution on χ, this time only affecting the center

$$\chi_{\text{new}} = \chi_{\text{old}} + (2+\gamma)m_v\check{\theta}$$

then we have the following bounds on $\chi \in X_*(A)_{\mathrm{dom}}$

$$\frac{\langle \chi, \rho\rangle}{a-1} \geq \langle \chi, \theta\rangle \geq \frac{\langle \chi, \rho - \alpha\rangle}{a-1} \text{ and } \langle \chi, n_{\check{\alpha}}^{-1}\varpi_\alpha\rangle \tag{3.21}$$

and the local period becomes

$$P_v = q^{-2m_v\langle\rho,\check{\rho}\rangle}\, q^{m\langle\eta,\check{\rho}\rangle} \sum_{\chi:(3.21)} \eta^{1/2}(\varpi^{\chi-(2+\gamma)m_v\check{\theta}}) s_{\chi-(2+\gamma)m_v\check{\theta}}(\mathrm{Sat}_v)\, q^{(a-1)\left(\langle\chi,\theta\rangle-2m_v\right)}$$
$$= q^{-2m_v a} q^{-2(a-1)m_v} q^{\left(\frac{(2+\gamma)\langle\eta,\check{\rho}\rangle}{2}\right)m_v} \chi_f \circ \check{\theta}(\partial_v^{-1-\gamma/2}) \sum_{\chi:(3.21)} q^{\langle\chi,(a-1)\theta-\eta/2\rangle}\, s_\chi(\mathrm{Sat}_v)$$

Further substituting $\eta = (2a-1)\theta$ the exponent of q outside becomes

$$2m - 4ma + (2+\gamma)(2a-1)m/2 = (2a-1)m(-2+1+\gamma/2) = (2a-1)m(-1+\gamma/2) = -m\rrbracket,$$

where we used (3.19), and factor inside the summation becomes

$$q^{-m\rrbracket}\chi_f \circ \check{\theta}(\partial_v^{-1-\gamma/2}) \sum_{\chi:(3.21)} q^{\langle\chi,-\theta/2\rangle}\, s_\chi(\mathrm{Sat}_v).$$

We note, moreover, that:

- $\langle\chi,\theta\rangle$ is precisely the character by which $\mathbf{G}_m$ acts, through the morphism $\mathbf{G}_m \to \check{G}$ dual to θ, on the irreducible representation of $\check{G}$ with highest weight θ;
- The eigenmeasure character on the dual side equals $(2a-1)\theta_{\check{G}}$, which corresponds duially via (3.1) to $\gamma(2a-1)\check{\theta}_G = \frac{2a-1}{a-1}\check{\theta}_G$. On the other hand, $2+\gamma = 2+\frac{1}{a-1} = \frac{2a-1}{a-1}$.

Returning to the global period and combining all the local constants, we arrive at the main result on the automorphic side.

Theorem 3.6.2. Assume that the residual characteristic is large enough so that Proposition 3.5.1 holds. Let $X = \overline{U^0\backslash G}$ be viewed as a conical G-variety with $\mathbf{G}_{\mathrm{gr}}$-acting through $\gamma\check{\theta}_G$. Then the automorphic X-period of a Whittaker normalized unramified cusp form f on G is given by an Euler product

$$P_X(f) = \chi_f(\check{\theta}(\partial^{-1-\gamma/2})) \prod_v \sum_{\chi:(3.21)} q^{-\langle\chi,\theta/2\rangle}\, s_\chi(\mathrm{Sat}_v)$$

3.7 Comparison with spectral period

Having computed the automorphic $X = \overline{U^0\backslash G}$-period, we compare it with the spectral period of

$$\check{X} := \overline{\check{U}^0\backslash\check{G}}$$

Throughout this section, we will assume a simplification of the bounds in (3.21), namely

> The points $n_{\check{\alpha}}^{-1}\varpi_\alpha$ lie in the convex cone spanned by $\frac{\rho-\alpha}{a-1}$, as α ranges through simple roots. In particular, (3.21) reduces to the simpler form

$$\frac{\langle \chi, \rho\rangle}{a-1} \geq \langle \chi, \theta_1\rangle \geq \frac{\langle \chi, \rho-\alpha\rangle}{a-1} \tag{3.22}$$

We verify this simplifying assumption in the examples most pertinent to us.

Lemma 3.7.1. If G is either of the following groups or their Langlands duals

- $\mathrm{PSL}_3 \times \mathbf{G}_m$ (as in Ginzburg's adjoint example), or
- $$\frac{\{(g_1,g_2,g_3) \in \mathrm{GL}_2^3 : \det(g_1)=\det(g_2)=\det(g_3)\}}{\Delta\mu_3}$$
  (as in Garrett's triple product example),

then Assumption (3.22) is satisfied.

Proof. Suppose G is one of the groups in the Ginzburg adjoint example. Let α_1, α_2 be the two simple roots of G so that $\rho = \alpha_1 + \alpha_2$, we see that $a = 2$ and $n_{\check{\alpha}} = 1$ for all simple $\check{\alpha}$. So assumption (3.22) is the assumption that the fundamental weights lie in the convex cone spanned by α_1, α_2.

Now suppose G is a group in the Garrett triple product example. Let $\alpha_1, \alpha_2, \alpha_3$ be the three simple roots of G, so that $\rho = (1/2)(\alpha_1+\alpha_2+\alpha_3)$. We see that $a = 3/2$ and $n_{\check{\alpha}} = 1/2$, so assumption (3.22) is the assumption that fundamental weights lie in the convex cone spanned by

$$\beta_k := \alpha_i + \alpha_j - \alpha_k \text{ for } i,j,k \in \{1,2,3\} \text{ distinct}$$

since $\varpi_i = \alpha_i/2$, and so

$$\varpi_i = \frac{1}{4}\Big[\beta_j + \beta_k\Big] \text{ for } i,j,k \in \{1,2,3\} \text{ distinct}$$

□

By definition, the normalized spectral period of a $\check{G}$-valued L-parameter φ is

$$L_{\check{X}}(\varphi) = q^{(g-1)\frac{\dim(G/U)-(2a-1)\gamma}{2}}(\check{\eta}\circ\varphi)(\partial^{-1/2})\prod_v \mathrm{gtr}\big(\mathrm{Fr}_v \,|\, \widehat{\mathcal{O}}^*_{\check{X},0}\big)$$

The role of the character $\check{\eta} \circ \varphi$ on the spectral side is as follows: the composition

$$\Gamma_F \xrightarrow{\varphi} \check{G} \xrightarrow{\check{\eta}} \mathbf{G}_m$$

factors through $\Gamma_F^{\mathrm{ab}} \cong [\mathbf{G}_m]$, so we may evaluate it on the idele class $\partial^{-1/2}$. The role of this quantity on the automorphic side, i.e., taking $\check{\eta}$ now to mean a cocharacter into G, we are considering the composition

$$\mathbf{G}_m \xrightarrow{\check{\eta}} G \xrightarrow{\omega} \mathbf{G}_m$$

where ω is the central character of the automorphic form whose L-parameter is φ. Then we have an equation

$$(\check{\eta} \circ \varphi)(\partial^{-1/2}) = (\omega \circ \check{\eta})(\partial^{-1/2})$$

where the latter quantity can be calculated using (3.1) and (3.2) as

$$(\omega \circ \check{\theta}_G^{(2a-1)\gamma})(\partial^{-1/2}) = (\omega \circ \check{\theta}_G)(\partial^{-1-\gamma/2})$$

which agrees exactly with the central character factor appearing in Theorem 3.6.2.

According to Proposition 3.3.1, we can calculate the graded $\check{G}$-representation on $\mathcal{O}^*_{\check{X}}$ as

$$\bigoplus_{\chi \text{ relevant}} V_\chi$$

where, writing $\check{\rho}_2' = \check{\rho}_2 + \check{\theta}_2$, equation (3.4) reads as

$$0 \leq \langle \chi, \check{\rho} + \check{\theta} \rangle \leq \min_{\check{\alpha} \text{ simple}} \langle \chi, \check{\alpha} \rangle$$

Subtracting $\langle \chi, \check{\rho} \rangle$ from and taking the negative of these inequalities, we arrive at

$$\langle \chi, \check{\rho}_2 \rangle \geq \langle \chi, -\check{\theta}_2 \rangle \geq \max_{\alpha \text{ simple}} \langle \check{\rho}_2 - \check{\alpha}, \chi \rangle \tag{3.23}$$

- In the Ginzburg adjoint cases for $G = \mathrm{SL}_3 \times \mathbf{G}_m$ or $\mathrm{PGL}_3 \times \mathbf{G}_m$, since $a - 1 = 1$ and $\check{\theta}_2 = \theta_1$ under the isomorphism $X_*(A_2) = X^*(A_1)$, we see that (3.21) and (3.23) are exactly equivalent.
- In the Garrett triple product cases, since $a - 1 = 1/2$, and $\check{\theta}_2 = \theta_1/2$ under the isomorphism $X_*(A_2) \cong X^*(A_1)$, we see that (3.21) and (3.23) are exactly equivalent.

Next, note that when χ satisfies (3.23) so that V_χ occurs with multiplicity one in $\mathcal{O}^*_{\check{X}}$, the $\mathbf{G}_{\mathrm{gr}}$-grading on V_χ is $\langle \chi, \gamma\check{\theta}_{\check{G}}\rangle$.

- In the Ginzburg adjoint cases,

$$\langle \chi, \theta_1/2\rangle = \langle \chi, \check{\theta}_2/2\rangle = \big(\mathbf{G}_{\mathrm{gr}}\text{weight on } V_\chi\big)/2$$

  because $\mathbf{G}_{\mathrm{gr}}$ acts by $\check{\theta}_2$.

- In the Garrett triple product cases,

$$\langle \chi, \theta_1/2\rangle = \langle \chi, \check{\theta}_2\rangle = \big(\mathbf{G}_{\mathrm{gr}}\text{weight on } V_\chi\big)/2$$

  because $\mathbf{G}_{\mathrm{gr}}$ acts by $2\check{\theta}_2$.

In summary, we have the following agreement of automorphic and spectral periods:

Theorem 3.7.2 (C.-Venkatesh)**.** Let $G, \check{G}$ be a pair of Langlands dual groups satisfying Condition (3.22), and assume that the residual characteristic is large enough so that Proposition 3.5.1 holds. Consider the G-space $X = \overline{TU^0\backslash G}$ and the $\check{G}$-space $\check{X} = \overline{\check{T}\check{U}^0\backslash\check{G}}$; then X and $\check{X}$ are weakly BZSV dual: for all Whittaker normalized cusp forms f on G with L-parameter φ_f, we have

$$P_X(f) = q^{(g-1)\frac{(2a-1)\gamma - \dim(G/U)}{2}} L_{\check{X}}(\varphi_f)$$

and for all Whittaker normalized cusp forms f on $\check{G}$ with L-parameter φ_f, we have

$$P_{\check{X}}(f) = q^{(g-1)\frac{(2a-1)\gamma - \dim(\check{G}/\check{U})}{2}} L_X(\varphi_f)$$

Remark 3.7.3. In most cases, the spectral period $L_{\check{X}}$ that we see in Theorem 3.6.2 will not be (finite sum, products, quotients, of) Langlands L-functions. For instance, we take X to be the nilpotent cone of PGL_3, i.e., Ginzburg's adjoint example, in which case the graded SL_3-representation on $\mathcal{O}_{\check{X}}$ has $V_{(a,b)}$ occurring with multiplicity $\min(a,b)+1$, and the $\mathbf{G}_{\mathrm{gr}}$-grading comes in a string

$$\max(a,b), \max(a,b)+1, \ldots, a+b$$

Therefore, the dimension of weight k can be calculated by Weyl's dimension formulas as

$$\dim(\mathcal{O}_{\check{X}}[k]) = \sum_{a,b \leq k} \frac{(a+1)(b+1)(a+b+2)}{2}$$

which yields the generating function

$$\sum_{k \geq 0} \dim(\mathcal{O}_{\check{X}}[k]) x^k = \frac{(1+x)(1+8x+x^2)}{(1-x)^6}$$

In particular, the quadratic factor in the numerator tells us that $\check{X}$ is not $\mathbf{G}_{\mathrm{gr}}$-equivariantly LCI in a $\check{G}$-representation, and Estermann's phenomenon also tells us that $L_{\check{X}}$ has a natural boundary viewed as a function of the $\mathbf{G}_{\mathrm{gr}}$-character (i.e., the L-function's evaluation point). However, the Euler product defining $L_{\check{X}}$ converges in some half plane of central characters where the functional $\langle \cdot, \check{\theta} \rangle$ has large real part.

Chapter 4

Unfolding to the Shalika Model

In this section, we consider automorphic periods that unfold "in one step" to the Shalika model; this is a locally unique model which is known to distinguish functorial lifts from odd orthogonal groups by work of Jacquet–Shalika [JS88]. Our original was Ginzburg–Rallis' integral representation for the exterior cube L-function of GL_6, although we consider the analogous automorphic and spectral periods on $G = \mathrm{GL}_{2n}$.

- We consider a certain G-variety $X \subset \wedge^n \mathbf{A}^{2n}$ and compute its spectral period by an invariant theory calculation; this is accomplished in Section 4.2.
- The automorphic period of the same G-variety X is designed to unfold to the Shalika model, and after Euler factorizing, we will integrate local Shalika functions over a region dictated by the integral points of X. The computation of integral points is accomplished by Section 4.3 and Proposition 3.5.2.
- Finally, our main result in this section is Theorem 4.6.2: we interpret, with weak BZSV duality as motivation, the automorphic X-period as the spectral period as a derived enhancement of a space $\check{X}$. The automorphic computation in reverse is difficult for large n, but for $n = 3$ reduces essentially to the aforementioned work of Ginzburg–Rallis.

4.1 Setup

Consider the reductive group over F

$$G = \frac{\mathrm{GL}_{2n} \times \mathbf{G}_m}{z\mathrm{Id}_{2n} \times z^{-n}}$$

with 1-dimensional center, and we fix a cocharacter

$$\check{\theta}(t) = \mathrm{diag}(t\mathrm{Id}_n, t^{-1}\mathrm{Id}_n)$$

Let P be the (n,n)-block Siegel parabolic, whose Levi decomposition we write as $P = UM$. Identifying U with the additive group of $(n \times n)$-matrices, we consider the additive character

$$\Psi : \begin{bmatrix} 1_n & X \\ 0 & 1_n \end{bmatrix} \longmapsto \psi(\mathrm{tr}(X))$$

and we let

$$U^0 := \mathrm{Ker}(\Psi) = \left\{ \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} : X \in \mathrm{Mat}_n,\ \mathrm{tr}(X) = 0 \right\}$$

let $M^0 \subset M$ be the following diagonal subgroup isomorphic to GL_n

$$M^0 := \left\{ \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} : m \in \mathrm{GL}_n \right\}$$

Let $V = \mathbf{A}^{2n}$ be the standard representation of GL_{2n} (which we view as a right action on row vectors), with basis

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$$

then the induced representation on $\wedge^n \mathbf{A}^{2n}$ factors through G, and we take the following distinguished vector in $\wedge^n \mathbf{A}^{2n}$

$$\gamma_0 = e_1 f_2 \cdots f_n + f_1 e_2 f_3 \cdots f_n + \cdots + f_1 \cdots f_{n-1} e_n$$

Lemma 4.1.1. The stabilizer of γ_0 in G is

$$S = U^0 M^0 \begin{bmatrix} \lambda \mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n \end{bmatrix} \times \lambda^{-1} \mathrm{det}^{-1}(m) \subset G$$

Proof. We first show that $U^0 M^0 \subset \mathrm{Stab}_G(\gamma_0)$. For $u \in U^0$,

$$\gamma_0 \cdot u = \gamma_0 + \Psi(u) f_1 \cdots f_n = \gamma_0$$

To check that M^0 stabilizes γ_0, it suffices to show that γ_0 is stabilized by the diagonal torus, the upper triangular unipotent subgroup, and the permutation matrices in M^0. We write a diagonal torus element in M^0 as

$$(t_{\hat{1}}^{-1}, \ldots, t_{\hat{n}}^{-1}, t_1, \ldots, t_n)$$

where $t_i \in \mathbf{G}_m$ and $t_{\hat{i}} = \prod_{j \neq i} t_j$. So the monomial in γ_0 with an e_i is scaled by

$$t_{\hat{i}}^{-1} \cdot \prod_{j \neq i} t_j = 1$$

and γ_0 is invariant under this torus. For invariance under unipotent elements, we consider the generating elements

$$u = \begin{bmatrix} 1_n + E_{ij} & 0 \\ 0 & 1_n + E_{ij} \end{bmatrix}$$

for $i < j$ where E_{ij} is the $(n \times n)$ matrix with a 1 at the (i,j)th entry and 0's everywhere else. Note that the monomials in γ_0 for which e_i and e_j do not appear are stabilized by u, and these two distinguished monomials are sent to themselves plus negatives of each other. Finally, for invariance under symmetric matrices, we consider transpositions $\sigma = (i,\, i+1)$ for which the element in M^0 looks like

$$\begin{bmatrix} -\sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

Again, all monomials in γ_0 for which e_i, e_{i+1} do not appear, are evidently invariant, and these two distinguished monomials are exchanged under σ, so γ_0 is invariant. □

We take the G-variety

$$X = \overline{\gamma_0 \cdot G} \subset \wedge^n \mathbf{A}^{2n}$$

We distinguish a cocharacter into G

$$\mathbf{j} : t \longmapsto \mathrm{diag}(t\mathrm{Id}_n, \mathrm{Id}_n) \subset G$$

In other words, we can write

$$S = U^0 M^0 \mathbf{j}(\mathbf{G}_m)$$

$$u \times m \times \lambda \longmapsto um \begin{bmatrix} \lambda 1_n & 0 \\ 0 & 1_n \end{bmatrix} \times \lambda^{-1} \det^{-1}(m) \subset G$$

Note that $\check{\theta}$ normalizes S so it acts on X by inverse scaling on the left, which gives X the structure of a conical variety.

Remark 4.1.2. Ginzburg and Rallis constructed an integral on $\check{G}$ representing the exterior square L-function

(Theorem 4, [GR00]), whose right hand side is

$$\frac{L(\pi \otimes \chi, \wedge^3, 2s-1/2)}{L(\chi^2, 4s)L(\chi^4, 8s-2)}$$

The choice of our G-space X is informed by applying our general principle of taking nilpotent cones.

4.2 Spectral period

In this section, we calculate the ring of functions on X with the following assumption:

$$\text{the action map } \overline{S\backslash G} \longrightarrow \overline{\gamma_0 \cdot G} \text{ by } g \mapsto \gamma_0 g \text{ is an isomorphism of affine varieties.} \tag{4.1}$$

This can be verified directly for $n = 2, 3$ (see, for the $n = 3$ case, the orbit classifications of Cohen-Helminck [CH88]); for larger n we do not check this, but Proposition 4.3.1 provides some evidence for its possible validity. To be precise, we use this assumption because we calculate the ring of functions on $\overline{S\backslash G}$ (for the spectral period calculation), yet we use $\overline{\gamma_0 \cdot G}$ to calculate integrality conditions (for the automorphic period calculation).

With the above assumption, we write

$$X = \overline{S\backslash G}$$

and identify functions on X with left S-invariant functions on G. By Peter–Weyl, we have

$$\mathcal{O}_X = \bigoplus_{\lambda \in X^*_{\mathrm{dom}}(A)} V_\lambda^{U^0 M^0 \mathbf{j}(\mathbf{G}_m)} \boxtimes V_\lambda^*$$

as a G-representation. To compute these multiplicities $V_\lambda^{U^0 M^0}$ we realize the representation V_λ concretely via Borel–Weil theory. Let $P \subset G$ be the (n,n)-block upper triangular parabolic with Levi decomposition $P = UM$. For a dominant weight λ of A_M, we let W_λ be the highest weight representation of M with highest weight λ, which we induce to P by letting U act trivially. Consider the explicit Borel–Weil model of the highest weight representation V_λ (where we implicitly view λ also as a weight of A_G):

$$V_\lambda = \left\{ f : G \to W_\lambda^* : f(pg) = p \cdot f(g) \right\}$$

Viewing this space as sections of an equivariant bundle, restriction to the cell $PwU \subset P\backslash G$

$$I : V_\lambda \ni f \longmapsto \big[u \mapsto f(wu)\big] \in \mathbf{C}[U, W_\lambda^*]$$

is an injective homomorphism of M-representations since the big cell is dense in the flag variety. In particular, the U^0-invariant functions further descend to a function on $\mathbf{C}[U/U^0, W_\lambda^*] \cong \mathbf{C}[\mathbf{G}_a, W_\lambda^*] \cong \mathbf{C}[t] \otimes W_\lambda^*$ where the identification comes from the character $\Psi : U/U^0 \cong \mathbf{G}_a$. A polynomial in t extends to a section of the vector bundle if and only if its degree is at most $\langle \lambda, \check{\alpha} \rangle$, for all $\check{\alpha}$ simple coroots contained in the parabolic P.

Now $M^0 \cong \mathrm{GL}_n = \mathrm{diag}(m, m) \times \det^{-1}(m)$ leaves the coordinate t invariant, while $\mathbf{j}(\mathbf{G}_m) = \mathrm{diag}(\lambda \mathrm{Id}_n, \mathrm{Id}_n)$ scales t by weight 1. So, if we view $W_\lambda^* = U_{\lambda^{(1)}} \boxtimes U_{\lambda^{(2)}}$ as the box tensor product of two irreducible representations U_{μ_1} and U_{μ_2} of GL_n, we find a S-invariant vector in $t^k \otimes W_\lambda^*$ if and only if $U_{\mu_1} \otimes \det^{k-1} \cong U_{\mu_2}^*$. In terms of the original weight λ on G, we find a S-invariant vector $t^k \otimes \mathrm{ev}$ in V_λ (where ev is the natural pairing $U_{\mu_1} \otimes \det^{k-1} \otimes U_{\mu_2}^* \to \det^{k-1}$) if and only if λ is of the form

$$\lambda = (\mu_1, \dots, \mu_n, -\mu_n + (-1-k), \dots, -\mu_1 + (-1-k)) \in X^*_{\mathrm{dom}}(G)$$

We call such dominant weights of G weights of **symplectic type**.

On such a representation, $\mathbf{G}_{\mathrm{gr}}$ acts by the cocharacter $\check{\theta}$, which gives a V_λ of the above form the grading

$$\langle \lambda, \check{\theta} \rangle = 2\mu_1 + \cdots + \mu_n + 2n(k+1)$$

Summarizing these calculations, we arrive at the following

Proposition 4.2.1. The spectral X-period of an irreducible G-valued parameter with Satake parameter Sat_v at a place v is given by

$$L_X(\varphi) = q^{(g-1)\frac{\dim(G/S) - \langle \eta, \check{\theta} \rangle}{2}} (\check{\eta} \circ \varphi)(\partial^{-1/2}) \prod_v \sum_{\mu_1 \geq \cdots \geq \mu_n \geq 0; k \geq 0} q^{-(|\mu| + n(k+1))} s_{(\mu, \mu^* + (-1-k))}(\mathrm{Sat}_v)$$

where $|\mu| = \mu_1 + \cdots + \mu_n$ and $\mu^* = -w_0 \mu$ is the dual highest weight to μ.

4.3 Integrality

In evaluating the automorphic X-period, in particular once we unfold it to an Euler product of local integrals, we will need a parametrization of integral points in $X(\mathcal{O}_v)$, which we calculate in this section. We start with Iwasawa decomposition $G_v = P_v K_v = U_v M_v K_v$ to find double coset representatives for $S_v \backslash G_v / K_v$ as

follows. Using the Whittaker character to identify

$$U_v^0 \backslash U_v \xrightarrow{\sim} F_v$$

$$u(t) = \mathrm{Id}_{2n} + tE_{1,n+1} \leftrightarrow t$$

we are reduced to finding representatives for $M_v^0 \backslash M_v / K_v$. Left translating by S_v, we can move any element $\mathrm{diag}(g_1, g_2) \in M_v$ to the form

$$\begin{bmatrix} g & 0 \\ 0 & \mathrm{Id}_n \end{bmatrix}$$

Finally, left multiplication by $S_v \cap K_v$ and right multiplication by $P_v \cap K_v$, we are allowed to take the g-entry in the double coset space $\mathrm{GL}_n(\mathcal{O}_v) \backslash \mathrm{GL}_n(F_v) / \mathrm{GL}_n(\mathcal{O}_v)$, which is identified with $X_*^{\mathrm{dom}}(\mathrm{GL}_n)$ by Cartan decomposition. The action of $\mathbf{j}(F_v) \subset S_v$ reduces us to considering χ up to diagonal translation. In conclusion,

$$\begin{aligned} &S_v \backslash M_v / K_v \\ &= \left\{ u(t)\varpi^\chi := \begin{bmatrix} \mathrm{Id}_n & \mathrm{diag}(t, 0, \ldots, 0) \\ 0 & \mathrm{Id}_n \end{bmatrix} \begin{bmatrix} \varpi^\chi & 0 \\ 0 & \mathrm{Id}_n \end{bmatrix} : \chi \in X_*^{\mathrm{dom}}(\mathrm{PGL}_n) \right\} \end{aligned}$$

We will often write $\chi = (\chi_1, \ldots, \chi_n)$ as an n-tuple of integers by identifying it with the coweights of the diagonal torus in GL_n, and up to diagonal translation we may assume that $\chi_n = 0$ so that χ is of the form $(\chi_1 \geq \chi_2 \geq \cdots \geq \chi_{n-1} \geq \chi_n = 0)$.

The action of such a coset representative $u_t \varpi^\chi$ brings γ_0 to the following form:

$$\gamma_0 \cdot u(t)\varpi^\chi = t f_{1\ldots,n} + \sum_{i=1}^{n} \varpi^{\chi_i} f_{1,\cdots,i-1} e_i f_{i+1,\cdots,n}$$

In summary, we have the following conditions on $u(t)\varpi^\chi$ for the nonvanishing of the Schwartz function $\Phi_v(\cdot) = \mathbf{1}_{X(\mathcal{O}_v)}(\cdot\, \varpi^{m_v})$:

$$v(t) + m_v \geq 0 \text{ and } \chi_i + m_v \geq 0 \tag{4.2}$$

These conditions ensure that $\gamma_0 \cdot v(t)\varpi^\chi \varpi^{m_v}$ defines an integral point in the G-orbit closure $\overline{\gamma_0 \cdot G}$.

Recall again that we are under Assumption (4.1). In any case, we have the following piece of evidence for the above expectation, which is good enough for us to proceed:

Proposition 4.3.1. Let $X_1 = \overline{\gamma_0 \cdot G}$ and $X_2 = \overline{S \backslash G}$. Then under the map $\phi : X_2 \to X_1$ obtained by acting

on the base point γ_0 we have

$$u(x)\varpi^\chi \in X_2(\mathcal{O}_v) \text{ if and only if } \phi(u(x)\varpi^\chi)) \in X_1(\mathcal{O}_v)$$

Proof. We have already identified the set of integral points in X_1 above, and it remains to compute the condition for a point $u(x)\varpi^\chi \in G(F_v)$ to define an integral point in $X_2(\mathcal{O}_v)$.

To this end, we employ analogous methods of Proposition 3.5.1 and Proposition 3.5.2. We have already computed the ring of S-invariant functions on G preceding the statement of Proposition 4.2.1, so it remains to examine on which points $u(x)\varpi^{\chi,r}$ do these functions take integral values.

Recall that in the Borel–Weil realization of the representation V_λ for $\lambda = (\mu, \mu^* + (-1-k))$ of symplectic type, we have an S-invariant vector of the form

$$P_k := t^k \otimes \mathrm{ev} \in V_\lambda^S = \mathcal{O}_v[t] \otimes W_\lambda$$

where ev is the following vector: since λ is of symplectic type so $W_\lambda|_{\mathrm{GL}_n \times \mathrm{GL}_n} = U_\mu \boxtimes U_{\mu^*+(-1-k)}$, there is an M^0-equivariant evaluation $\mathrm{ev} : U_\mu \otimes U_{\mu^*+(-1-k)} \to \det^{-1-k}$. The value of P_k on $u(x)\varpi^\chi$ can be calculated explicitly as

$$P_k(u(x)\varpi^\chi) = \left[\sum_{i=1}^{n} \varpi^{\chi_i} - x\right]^k$$

Taking valuations of this expression, and dividing by k (which we may take to be positive) we see that integrality of $P_k(u(x)\varpi^\chi)$ translates equivalently to

$$\chi_i \geq 0 \text{ for all } i \text{ and } v(x) \geq 0$$

which are the same conditions as (4.2). □

4.4 Automorphic period

Let f be a cuspidal automorphic form on G with central character χ_f, which we assume to be Shalika normalized (for the definition, see (2.9)). We write Q for the following normalization global power of discriminant

$$Q = q^{-\beta_{\mathrm{Sha}}/2} q^{-\dim(S)(g-1)/2} \eta^{1/2}(\partial^{1/2})$$

where we recall that β_{Sha} is the following constant

$$\beta_{\mathrm{Sha}} = (g-1)\big(\dim(\mathrm{Mat}_n) - \langle 2\rho_X, 2\check{\rho}_X\rangle\big)$$

with $\check{\rho}_X$ the distinguished cocharacter

$$\check{\rho}_X(t) = \mathrm{diag}(t^{1/2}\mathrm{Id}_n, t^{-1/2}\mathrm{Id}_n)$$

Later this same cocharacter will also be referred to as ρ_{Sp}, since we can view it as the half sum of positive roots of the symplectic subgroup of the Langlands dual group of G. We set

$$\check{\theta}_X = 2\check{\rho}_X$$

Then the automorphic period attached to X can be unfolded as follows:

$$\begin{aligned} P_X(f) &= Q\int_{[G]} f(g)\sum_{\gamma\in X_F}\Phi(\gamma g)\eta^{1/2}(g)\,dg \\ &= Q\int_{S_{\mathbb{A}}\backslash G_{\mathbb{A}}}\Phi(g)\int_{[S]} f(sg)\eta^{1/2}(sg)\,ds\,dg \end{aligned}$$

Focusing on the inner $[S]$-integral and applying Fourier expansion on $U^0\subset U$, we have

$$\begin{aligned} &\int_{[S]} f(sg)\eta^{1/2}(sg)\,ds \\ &= \int_{[\mathbf{j}(\mathbf{G}_m)M^0U^0]} f(um\mathbf{j}(a)g)\eta^{1/2}(m\mathbf{j}(a)g)\delta_S^{-1}(\mathbf{j}(a)m)\,du\,d(m,a) \\ &= q^{|\psi|(g-1)}\sum_{\alpha\in F^\times}\int_{[\mathbf{j}(\mathbf{G}_m)M^0U]} f(um\mathbf{j}(a)g)\eta^{1/2}(m\mathbf{j}(a)g)\delta_S^{-1}(\mathbf{j}(a)m)\psi(\alpha\mathrm{tr}(u))\,d^\psi u\,d(m,a) \\ &= q^{|\psi|(g-1)}\int_{\mathbb{A}^\times}\int_{[M^0U]} f(um\mathbf{j}(a)g)\eta^{1/2}(m\mathbf{j}(a)g)\delta_S^{-1}(\mathbf{j}(a)m)\Psi(u)\,d^\psi u\,d(m,a) \\ &= q^{|\psi|(g-1)}\int_{\mathbb{A}^\times}\Omega_f(\mathbf{j}(a)g)\delta_S^{-1}(\mathbf{j}(a))\,da \end{aligned}$$

where Ω_f is the $(M^0,\eta^{1/2})\times(U,\Psi)$-equivariant Shalika function on G given by

$$g\mapsto\Omega_f(g) = \int_{[U][\mathrm{GL}_n]} f(umg)\Psi(u)\delta_P^{-1}(m)\,d^\psi u\,dm$$

Returning to the global period, we can fold $\mathbf{j}(\mathbb{A}^\times)$ back into the outer integral to obtain

$$P_X(f) = q^{\beth(g-1)} \int_{M^0_{\mathbb{A}} U^0_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Phi(g) \Omega_f(g) \eta^{1/2}(g)\, dg$$

where $\beth$ is the following global constant

$$\beth = 2\langle \rho_X, \check{\rho}_X \rangle - \frac{\dim(S)}{2} - \frac{\dim(U)}{2} - \frac{\langle \eta, \check{\theta}_X \rangle}{2} + |\psi|$$

By local uniqueness of the Shalika model, we see that the global period factorizes as

$$P_X(f) = q^{\beth(g-1)} \left[\operatorname*{Res}_{s=1} L(\pi, \wedge^2, 1) \right] \prod_v P_v$$

where we define the local periods as

$$P_v = \int_{M^0_v U^0_v \backslash G_v} \Phi(g) \Omega_v(g) \eta^{1/2}(g)\, dg$$

where $\Omega_v = \Omega_{f,v}$ is the normalized local Shalika functions of f, normalized as in (2.10) and (2.11).

We begin to evaluate it in terms of the Satake parameter of f at v. Using the local integrality computations of the previous section to evaluate the Schwartz function Φ on the double coset representatives, the local period can be now written as a sum over cocharacters that satisfy those bounds: using the fact that the conductor of ψ is $-2m_v$,

$$\begin{aligned} P_v &= \sum_{\chi : (4.2)} \left[\int_{v(t) \geq -m_v} \psi(t)\, dt \right] \Omega_f(\varpi^\chi) \eta^{1/2}(\varpi^\chi) \\ &= \sum_{\chi : (4.2)} q^{m_v}\, \Omega_f(\varpi^\chi) \eta^{1/2}(\varpi^\chi) \end{aligned}$$

4.5 Casselman–Shalika–Sakellaridis Formula

In order to evaluate the local period in terms of Satake parameters, one needs a formula for the Shalika function $\Omega(\varpi^\chi)$ analogous to the Casselman–Shalika formula.

Sakellaridis provided exactly such a formula in [Sak18], and more generally for unramified spherical functions on general spherical varieties in [Sak12]. We record his main result here, appropriately normalized for our application.

Theorem 4.5.1 ([Sak18] Theorem 2.1)**.** Let Ω^{ur} be the normalized unramified (in the sense of (2.11)) spherical Shalika function of an unramified principal series π_v of GL_{2n} with Satake parameter Sat_v. Then Ω^{ur} is uniquely determined by its values on the elements

$$\varpi^{\chi} = \mathrm{diag}(\varpi^{\chi_1,\dots,\chi_n,1\dots,1}) \in \mathrm{GL}_{2n}(F_v)$$

and these evaluations are given by the following explicit formula: $\Omega(\varpi^{\chi})$ vanishes unless $\chi_1 \geq \cdots \geq \chi_n \geq 0$, and in this case,

$$\Omega^{\mathrm{ur}}(\varpi^{\chi}) = \delta_{\mathrm{Sp}}^{1/2}(\varpi^{\chi}) \frac{\mathcal{A}\big(e^{\check{\rho}_{\mathrm{Sp}}+\chi} \prod_{\alpha\in\Phi^+_{\mathrm{Sp}}} (1-q^{-1}e^{-\check{\alpha}})\big)}{\mathcal{A}\big(e^{\check{\rho}_{\mathrm{Sp}}} \prod_{\alpha\in\Phi^+_{\mathrm{Sp}}} (1-q^{-1}e^{-\check{\alpha}})\big)}(\mathrm{Sat}_v)$$

where $\check{\rho}_{\mathrm{Sp}}$ is the half sum of positive coroots of $\mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$, $\Phi^{+,\mathrm{s}}_{\mathrm{Sp}}$ is the set of positive simple short roots of Sp_{2n}, and $\mathcal{A}$ is the Weyl alternator for the symplectic group

$$\mathcal{A}(x) = \sum_{w\in W_{\mathrm{Sp}}} (-1)^w\, w(x)$$

Note that if we formally set $q^{-1} = 0$, then the formula becomes

$$\Omega^{\mathrm{ur}}(\varpi^{\chi})|_{q^{-1}=0} = \delta_{\mathrm{Sp}}^{1/2}(\varpi^{\chi}) \frac{\mathcal{A}\big(e^{\check{\rho}_{\mathrm{Sp}}+\chi}\big)}{\mathcal{A}\big(e^{\check{\rho}_{\mathrm{Sp}}}\big)}(\mathrm{Sat}_v) = \delta_{\mathrm{Sp}}^{1/2}(\varpi^{\chi}) s_{\chi}(\mathrm{Sat}_v)$$

which, since χ may be viewed as a dominant Sp_{2n}-valued cocharacter, can be interpreted as a q^{-1}-deformation of the usual Casselman–Shalika formula for the symplectic group.

We will rewrite the denominator in the following way as to make the comparison with the Weyl character formula more direct (see, for example, the proof of Proposition 2 of [WW00]):

$$\mathcal{A}\left(e^{\check{\rho}_{\mathrm{Sp}}} \prod_{\alpha\in\Phi^+_{\mathrm{Sp}}} (1-q^{-1}e^{-\check{\alpha}})\right) = \mathcal{A}(e^{\check{\rho}_{\mathrm{Sp}}})\omega(q)$$

where

$$\omega(q) = \prod_i \frac{1-q^{d_i}}{1-q}$$

and the d_i's are the exponents of Sp_{2n}.

Expanding the numerator of Sakellaridis' formula, we have

$$\mathcal{A}\left(e^{\check{\rho}_{\mathrm{Sp}}+\chi}\prod_{\alpha\in\Phi^+_{\mathrm{Sp}}}(1-q^{-1}e^{-\check{\alpha}})\right)=\sum_{J\subset\Phi^+_{\mathrm{Sp}}}(-q)^{-|J|}\mathcal{A}(e^{\check{\rho}_{\mathrm{Sp}}+\chi-J})$$

where we sum over all subsets $J\subset\Phi^+_{\mathrm{Sp}}$. So Sakellaridis' formula can be directly expressed in terms of Weyl characters of the dual group as

$$\Omega^{\mathrm{ur}}(\varpi^\chi)=\frac{\delta_{\mathrm{Sp}}^{1/2}(\varpi^\chi)}{\omega(q)}\sum_{J\subset\Phi^+_{\mathrm{Sp}}}(-q)^{-|J|}s_{\chi-J}(\mathrm{Sat}_v)\tag{4.3}$$

where the Weyl character s_χ is formally applied at a possibly nondominant χ by the usual quotient of Weyl antisymmetric expressions

$$s_\chi=\frac{\mathcal{A}(e^{\check{\rho}_{\mathrm{Sp}}+\chi})}{\mathcal{A}(e^{\check{\rho}_{\mathrm{Sp}}})}$$

Note that equation (4.3) is in spirit much closer to the Casselman–Shalika-type formula of Sakellaridis (Theorem 7.2.1 of [Sak12]), although the combinatorial assumptions (namely Statement 7.1.5 of *loc. cit*) appear to be tedious to verify.

Returning to the local period calculation and keeping in mind (2.11), we make the shift

$$\chi_{\mathrm{new}}=\chi_{\mathrm{old}}+2m_v$$

in the bounds (4.2) to obtain

$$\chi-m_v\geq 0\tag{4.4}$$

and using the fact that Ω has central character χ_f, we can evaluate the local period as

$$\begin{aligned}
P_v &= q^{m_v} \sum_{\chi:(4.2)} \eta^{1/2}(\varpi^\chi)\Omega_f(\varpi^\chi) \\
&= q^{m_v} \sum_{\chi:(4.2)} \eta^{1/2}(\varpi^\chi)\Omega_f^{\mathrm{ur}}(\varpi^{\chi+2m_v}) \\
&= q^{m_v}\eta(\varpi^{-m_v}) \sum_{\chi:(4.4)} \eta^{1/2}(\varpi^\chi)\Omega_f^{\mathrm{ur}}(\varpi^\chi) \\
&= q^{m_v}\eta(\varpi^{-m_v}) \sum_{\chi:(4.4)} (\eta\delta_{\mathrm{Sp}})^{1/2}(\varpi^\chi) \sum_{J\subset\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}} (-q)^{-|J|} s_{\chi-J}(\mathrm{Sat}_v) \\
&= q^{m_v}\eta(\varpi^{-m_v})\delta_{\mathrm{Sp}}^{1/2}(\varpi^{m_v})\chi_f(\partial_v^{-1/2}) \sum_{\chi\in X^*_{\mathrm{dom}}(\mathrm{GSp})} (\eta\delta_{\mathrm{Sp}})^{1/2}(\varpi^\chi) \sum_{J\subset\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}} (-q)^{-|J|} s_{\chi-J}(\mathrm{Sat}_v) \\
&= q^{m_v} q^{m\langle\eta,\check\rho_{\mathrm{Sp}}\rangle} q^{-m_v\langle\rho_{\mathrm{Sp}},2\check\rho_{\mathrm{Sp}}\rangle}\chi_f(\partial_v^{-1/2}) \sum_{\chi\in X^*_{\mathrm{dom}}(\mathrm{GSp})} (\eta\delta_{\mathrm{Sp}})^{1/2}(\varpi^\chi) \sum_{J\subset\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}} (-q)^{-|J|} s_{\chi-J}(\mathrm{Sat}_v)
\end{aligned}$$

Collecting global constants in the Euler product, we have the following explicit expression for the global automorphic period attached to X:

Theorem 4.5.2. Consider $G = \mathrm{GL}_{2n}/\mu_n$ for $n \geq 1$. Let $X = \overline{S\backslash G}$ and f a Shalika normalized unramified cusp form on G with L-parameter φ_f, which we assume to be symplectic, and whose Satake parameter at a place v is Sat_v. Then

$$P_X(f) = (*)\,\chi_f(\partial^{-1/2})L(\wedge_0^2,\varphi_f,1)\prod_v \sum_\chi q_v^{-\langle\chi,\rho_{\mathrm{Sp}}+\eta/2\rangle} \sum_{J\subset\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}} (-q_v)^{-|J|} s_{\chi-J}(\mathrm{Sat}_v)$$

where the sum ranges through $\chi \in X^*_{\mathrm{dom}}(\mathrm{GSp}_{2n})$ that pair positively with the similitude character, and $(*)$ is the following explicit power of the discriminant

$$(*) = q^{\frac{g-1}{2}\left(2|\psi|+2-\dim(S)-\dim(U)\right)}$$

Remark 4.5.3. In the $n = 1$ case, we interpret $\Phi_{\mathrm{Sp}_2}^{+,\mathrm{s}} = \emptyset$, and we recover Hecke's period.

Note that the Weyl character $s_\chi(\mathrm{Sat}_v)$ occurs in the Euler factor at v exactly $2^{|\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}|}$ times, and each time with a weight labeled by

$$(-q_v)^{-|J|}q_v^{-\langle\chi+J,\rho_{\mathrm{Sp}}\rangle} = (-q_v)^{-|J|}q_v^{-\langle J,\check\theta+\eta\rangle/2}q_v^{-\langle\chi,\check\theta+\eta\rangle/2} \tag{4.5}$$

4.6 Duality

Consider the Langlands dual group to G

$$\check{G} = \left\{(g; z) \in \mathrm{GL}_{2n} \times \mathbf{G}_m : z^n = \det(g)\right\}$$

For the rest of this Section, we fix the following symplectic form

$$e_{1,2n} + e_{2,2n-1} + \cdots + e_{n,n+1}$$

in $\mathbf{A}^{2n}$, which induces an embedding of $\mathrm{GSp}_{2n} \subset \mathrm{GL}_{2n}$, where the diagonal torus of GSp_{2n} is parametrized by

$$\mathrm{diag}(a_1 b, \ldots, a_n b, a_n^{-1}, \ldots, a_1^{-1})$$

This further induces an inclusion of $H = \mathrm{GSp}_{2n}$ as a subgroup in $\check{G}$

$$h \longmapsto (h, \nu(h))$$

where ν is the similitude character of H. Let

$$\theta_{\check{G}} : \check{G}/[\check{G}, \check{G}] \longrightarrow \mathbf{G}_m$$

$$(g, z) \longmapsto z$$

be our distinguished character, and

$$\check{\theta}_{\check{G}}(t) = (t^{1/2}\mathrm{Id}_{2n}, t)$$

be its dual rational cocharacter which defines the $\mathbf{G}_{\mathrm{gr}}$-action. Note that $\theta_{\check{G}}|_H = \nu$.

Let $N \subset G$ be the upper triangular unipotent subgroup, and let $V = H \cap N$ be the corresponding maximal upper triangular unipotent subgroup in H. Consider the $\check{G}$-space

$$\check{X} = \mathrm{Ind}_H^{\check{G}} \overline{V \backslash H}$$

Note that $\dim(\check{X}) = 3n^2$. We have not calculated the automorphic $\check{X}$-period, although the spectral $\check{X}$-period can be readily calculated.

Proposition 4.6.1. The spectral period of $\check{X}$ distinguishes symplectic similitude parameters. For such a

parameter φ, we have

$$L_{\check{X}}(\varphi) = q^{\frac{g-1}{2}\left(\dim(\check{X})-\langle\check{\theta}_{\check{G}},\eta_{\check{X}}\rangle\right)} L(\wedge_0^2, \varphi, 1)(\check{\eta}\circ\varphi)(\partial^{-1/2}) \prod_v \sum_{\chi\in X^*_{\mathrm{dom}}(H):\langle\chi,\nu\rangle\geq 0} q^{-\langle\check{\theta}_{\check{G}},\chi\rangle/2}\,\mathrm{tr}(\mathrm{Fr}_v \mid V_\chi)$$

Proof. The L-factor appearing in front comes from the tangent space $\check{\mathfrak{g}}/\mathfrak{h} \cong \wedge_0^2$ as an H-representation, applying Lemma 2.7.1 . The sum in the Euler factor comes from the Peter–Weyl computation

$$\mathcal{O}_{\overline{V\backslash H}} = \bigoplus_\chi V_\chi^V \boxtimes V_\chi^*$$

There is only one invariant vector in V_χ^V, namely the line with highest weight. Furthermore, requiring that the resulting matrix coefficients in $V_\chi^V \boxtimes V_\chi^*$ extend to the cone point in $\overline{V\backslash H}$ requires that χ pairs nonnegatively with the similitude character ν. □

Comparing this with Theorem 4.5.2 we see that $P_X(f)$ and $L_{\check{X}}(\varphi_f)$ are almost in agreement, but the local Euler factors of P_X involve a shift over subsets of $\Phi^{+,\mathrm{s}}_{\mathrm{Sp}}$. In particular, in light of Equation (4.5), we could read the local factor as

$$\left[\sum_{J\subset\Phi^{+,\mathrm{s}}_{\mathrm{Sp}}} (-q_v)^{-|J|} q_v^{-\langle J,\check{\theta}+\eta\rangle/2}\right] \mathrm{gtr}(\mathrm{Fr}_v \mid \mathcal{O}^*_{\overline{V\backslash H}}) \tag{4.6}$$

where, for a subset $J \subset \Phi^{+,\mathrm{s}}_{\mathrm{Sp}}$ and weight λ we write

$$\langle J, \lambda\rangle := \sum_{\check{\alpha}\in J} \langle\check{\alpha}, \lambda\rangle$$

We prefer at this moment to be agnostic of the precise definition of derived spaces, and we will state our preliminary weak duality statement as follows in its raw numerical form, leaving interpretations to the remark which follows its statement:

Theorem 4.6.2. Let X (resp. $\check{X}$) be the G-space (resp. $\check{G}$-space) as above, Then

- (For all n). Let f be a Shalika normalized, unramified cusp form on G with L-parameter φ_f in $\check{G}$.

$$P_X(f) = \begin{cases} (*_1)\, L_{\check{X}}(\varphi_f) \prod_v \left[\sum_{J\subset\Phi^{+,\mathrm{s}}_{\mathrm{Sp}}} (-q_v)^{-|J|} q_v^{-\langle J,\check{\theta}+\eta\rangle/2}\right] & \text{if } \varphi_f \text{ is symplectic} \\ 0 \text{ otherwise} \end{cases}$$

where $(*)$ is the following explicit power of the discriminant:

$$(*_1) = q^{\frac{g-1}{2}\left[-\dim(\check{X})+\langle\check{\theta}_{\check{G}},\eta_{\check{X}}\rangle+2|\psi|+2-\dim(S)-\dim(U)\right]}$$

- (For $n = 3$). Let $\check{X}^- = \mathrm{Ind}_{\check{H}}^{\check{G}} \overline{U[M,M]\backslash H}$, where $P = MU$ is the Siegel parabolic in H, so that $\overline{U[M,M]\backslash H}$ is the affine cone over the Lagrangian–Grassmannian in the Plücker embedding into $\wedge^3\mathbf{A}^6$. Let f be a Whittaker normalized cusp form on $\check{G}$ with L-parameter φ_f in G. Up to a rational power of the discriminant $(*_2)$,

$$P_{\check{X}^-}(f) = (*_2)\, L_X(\varphi_f)$$

Proof. The first bullet point is the automorphic period computation we carried out in this section. The second bullet point follows from the integral representation of the exterior cube L-function of Ginbzurg–Rallis [GR00] (see Proposition 1.1 for the global integral representation and Proposition 2.1 for the local unramified computation). We will explain this computation while nailing down the constant $(*_2)$ in a formal writeup. □

Remark 4.6.3. We remark on some speculations regarding Theorem 4.5.2 as a type of assertion about weak duality, despite its apparent asymmetry as stated. Again, the pseudo-definitions in this remark is far from the final verdict on the correct interpretation in the end.

We would like to interpret the bracketed expression appearing in the above evaluation of P_X more conceptually as follows: consider the derived vector space

$$\mathfrak{U} := \bigoplus_{\check{\alpha}\in\Phi_{\mathrm{Sp}}^{+,\mathrm{s}}} \mathbf{C}_{\check{\alpha}}[1]$$

where $\mathbf{C}_{\check{\alpha}}$ is just a 1-dimensional complex vector space labeled by J. We can consider the derived vector bundle

$$\check{\mathfrak{X}} := \mathrm{Ind}_{\check{H}}^{\check{G}} \overline{V\backslash\big(H\times\mathfrak{U}\big)}$$

whose underlying classical scheme is $\check{X}^-$. We make a computation of the spectral $\check{\mathfrak{X}}$-period adopting the two following conventions:

1. The ring of functions on a vector space T in degree -1 is the exterior algebra on its dual; in other words,

$$\mathcal{O}_{T[1]} = \bigoplus_{k=0}^{\dim(V)} \wedge^k T^*[k]$$

2. The Frobenius trace on a vector space $T[k]$ placed in degree $-k$ is given by

$$\mathrm{tr}(\mathrm{Fr}_v|T[k]) := (-q_v)^{-k}\mathrm{tr}(\mathrm{Fr}_v|T)$$

Now let φ be an H-valued L-parameter which fixes the cone point in $\check{X}$. Formally calculating the spectral $\check{\mathfrak{X}}$-period of an H-valued L-parameter φ we have the following local factor:

$$\mathrm{gtr}\big(F_v|\mathcal{O}^*_{\check{\mathfrak{X}}}\big) = \mathrm{gtr}(\mathrm{Fr}_v|\mathcal{O}^*_{\mathfrak{U}})\,\mathrm{gtr}(\mathrm{Fr}_v|\mathcal{O}^*_{\overline{V\backslash H}}) = \Big[\sum_{k\geq 0}(-q_v)^{-k}\mathrm{gtr}(\mathrm{Fr}_v|\wedge^k\mathfrak{U}^*)\Big]\mathrm{gtr}(\mathrm{Fr}_v|\mathcal{O}^*_{\overline{V\backslash H}}) \tag{4.7}$$

But the L-parameter acts trivially on the derived vectorial direction, and the graded trace of F_v on $\wedge^k\mathfrak{U}^*[-1]$ is exactly

$$\mathrm{gtr}(\mathrm{Fr}_v\mid \wedge^k\mathfrak{U}^*[-1]) = \sum_{J\subset\Phi^{+,\mathrm{s}}_{\mathrm{Sp}},\,|J|=k} q^{-\langle J,\check{\theta}+\eta\rangle/2}$$

So we see that expression (4.6) agrees with the spectral $\check{\mathfrak{X}}$-period calculated as in (4.7).

Next we explain our interpretation of of $\check{X}^-$. We contend that $\check{X}^-$ is the correct underlying geometric space, but the correct sheaf of rings to take is $\pi_*\mathcal{O}_{\check{\mathfrak{X}}}$ where $\pi : \check{\mathfrak{X}} \to \check{X}^-$ is the natural map. In other words, we consider the object weakly dual to X as the pair

$$(\check{X}^-, \pi_*\mathcal{O}_{\check{\mathfrak{X}}})$$

along with its $\check{G}$-action.

Chapter 5

Unfolding to the Ginzburg–Jiang Model

In this section, we consider automorphic periods on $G = \mathrm{GSp}_6$ that unfold "in one step" to the Ginzburg–Jiang model; this is a locally unique model which is known to distinguish functorial lifts from $\mathrm{G}_2 \subset \mathrm{GSpin}_7$ by work of Ginzburg–Jiang [GJ01]. Our original motivation was Ginzburg–Rallis' integral representation for the exterior square L-function of GSpin_7 [GR94a].

- We consider a certain G-variety $X \subset \wedge^2 \mathbf{A}^6 \ominus \mathbf{1}$ and compute its spectral period by an invariant theory calculation; this is accomplished in Section 5.2.
- The automorphic period of the same G-variety X is designed to unfold to the Ginzburg–Jiang model, and after Euler factorizing, we will integrate local Shalika functions over a region dictated by the integral points of X. The computation of integral points is accomplished by Section 5.3.
- Finally, our main result in this section is simply the explicit formula for the automorphic X-period, given by Theorem 5.5.3. The shape of the result is very similar to a mix of the preceding Shalika and Whittaker cases, and we expect that a similar interpretation in terms of derived structures on the spectral side can be made, starting with Ginzburg–Rallis' integral representation for the exterior square L-function of GSpin_7 in [GR94a].

5.1 Setup

We start with the following pair of Langlands dual reductive groups

$$G = \mathrm{GSp}_6 \text{ and } \check{G} = \mathrm{GSpin}_7$$

Let ω be the symplectic form scaled by G along the similitude character ν. Consider the G-space

$$X = V(f_2, f_3) \subset \wedge^2 \mathbf{A}^6 \ominus \mathbf{C}\omega$$

the vanishing locus of an invariant quadratic and an invariant cubic. These invariant forms are

$$f_2(\alpha) = \alpha \wedge \alpha \wedge \omega$$

$$f_3(\alpha) = \alpha \wedge \alpha \wedge \alpha$$

Remark 5.1.1. Ginzburg and Rallis constructed an integral on $\check{G}$ representing the exterior square L-function (Theorem 4, [GR94a]), whose right hand side is

$$\frac{L(\pi \otimes \chi, \wedge_0^2, 3s-1)}{L(\chi, 3s)L(\chi^2, 6s-2)L(\chi^3, 9s-3)}$$

where $\wedge_0^2 = \wedge^2 \ominus \nu$ is the 14-dimensional irreducible representation of G. The choice of our G-space X is informed by applying our general principle of taking nilpotent cones.

To fix bases for further calculations, we choose

$$\omega = e_{15} + e_{26} + e_{34}$$

and we realize $\wedge_0^2 \mathbf{C}^6$ as the orthogonal complement to ω in $\wedge^2 \mathbf{C}^6$, with e_{ij}'s being an orthonormal basis.

Since $\dim X = 12$ and $\dim G = 22$, the open dense orbit in X has a 10-dimensional stabilizer. To describe this stabilizer, we consider the subgroup

$$\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4 = \mathrm{GSp}(\langle e_3, e_4 \rangle) \boxtimes \mathrm{GSp}(\langle e_1, e_2, e_5, e_6 \rangle) \subset \mathrm{GSp}_6$$

where $\boxtimes$ means the subgroup of the direct product satisfying the following condition: for $g_1 \in \mathrm{GL}_2$ and $g_2 \in \mathrm{GSp}_4$, we have

$$\det(g_1) = \nu(g_2)$$

where ν is the similitude character. Within this $\boxtimes$-product there is a $\mathrm{GL}_2 \times \mathrm{GL}_2$ sitting as the Levi of a Siegel

parabolic in the GSp_4-factor. In terms of matrices, we have

$$\begin{bmatrix} A & 0_2 & 0_2 \\ 0_2 & B & 0_2 \\ 0_2 & 0_2 & {}^tA^{-1}\det(B) \end{bmatrix} \cong \mathrm{GL}_2 \times \mathrm{GL}_2 \subset \mathrm{GL}_2 \boxtimes \mathrm{GSp}_4 \subset G$$

The unipotent part of this $\mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$-Levi can be explicitly described as follows: they are matrices of the form

$$U = U(X,Y) := \begin{bmatrix} 1_2 & X & Y \\ & 1_2 & X^* \\ & & 1_2 \end{bmatrix}$$

where, if $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$X^* = \begin{bmatrix} b & d \\ -a & -c \end{bmatrix}$$

and Y is of the form

$$Y = \begin{bmatrix} x & y \\ y + \det(X) & z \end{bmatrix}$$

Lemma 5.1.2. Consider $\gamma_0 = e_{35} + e_{46} \in X$. Then the orbit of γ_0 is dense, and the stabilizer of γ_0 is the 10-dimensional subgroup

$$S = \mathrm{diag}(\lambda^2, \lambda, \lambda, 1, \lambda^{-1}, 1) \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1 & X & Y \\ 0 & 1 & X^* \\ 0 & 0 & 1 \end{bmatrix} \subset \mathrm{GSp}_6$$

for $A \in \mathrm{SL}_2$, $\mathrm{tr}(X) = 0$, and Y of the form

$$Y = \begin{bmatrix} x & y \\ y + \det(X) & z \end{bmatrix}$$

Proof. To check invariance under the SL_2, it suffices to check for

$$A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then e_{35} gets sent to

$$e_{35} \longmapsto e_{35},\ e_{35} + xe_{45},\ e_{46}$$

respectively, and e_{46} gets sent to

$$e_{46} \longmapsto e_{46},\ e_{46} - xe_{45},\ e_{35}$$

so we see that γ_0 is invariant under the SL_2.

To check invariance under the unipotent factor, it suffices to check the action of the lower right 4×4-block since γ_0 only involves the indices $3, 4, 5, 6$. The condition that $\mathrm{tr}(X) = 0$ becomes the condition that X^* has the following form:

$$X^* = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Then e_{35} gets sent to

$$e_{35} \longmapsto (e_3 + ae_5 + be_6) \wedge e_5 = e_{35} - be_{56}$$

and e_{46} gets sent to

$$e_{46} \longmapsto (e_4 + be_5 + ce_6) \wedge e_6 = e_{46} + be_{56}$$

so we see that γ_0 is again invariant. □

Alternatively, we can combine the $\lambda \in \mathrm{GL}_1$ and the $A \in \mathrm{SL}_2$ parts of H to write

$$S = \begin{bmatrix} A\det(A) & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1 & X & Y \\ 0 & 1 & X^* \\ 0 & 0 & 1 \end{bmatrix} =: M^0U^0$$

where A now ranges through GL_2.

We distinguish the following cocharacter $\check{\rho}_X$ into G given by

$$\check{\rho}_X = \mathrm{diag}(t^{-1/2}\mathrm{Id}_2, \mathrm{Id}_2, t^{1/2}\mathrm{Id}_2)$$

and set $\check{\theta}_X := 2\check{\rho}_X$, through which $\mathbf{G}_{\mathrm{gr}}$ acts on X, via the usual formula

$$Sx \cdot \lambda := S\check{\theta}_X(\lambda)^{-1}x$$

5.2 Spectral period

To calculate the spectral period of X, we first observe that the natural map

$$\alpha : \overline{S\backslash G} \longrightarrow X = \overline{\gamma_0 \cdot G}$$

induced by the action $g \mapsto \gamma_0 \cdot g$ is an isomorphism. Indeed, since all smaller G-orbits in X have codimension ≥ 2, by the algebraic version of Hartog's theorem, α is a morphism between affine varieties inducing isomorphism on global functions, hence an isomorphism of varieties.

By Peter-Weyl, we have

$$\mathcal{O}_X = \bigoplus_{\lambda \in X^*_{\mathrm{dom}}(A)} V_\lambda^S \boxtimes V_\lambda^*$$

as a G-representation, and we need to compute the multiplicities V_λ^S. For this, we appeal to Borel–Weil theory once more.

Let $P = UM \subset G$ be the parabolic subgroup stabilizing the isotropic 2-plane $\langle e_1, e_2$, so that it has the "same shape" as S. For a dominant weight λ of A_M, we let W_λ be the highest weight representation of M with highest weight λ, which we induce to P by letting U act trivially. Consider the explicit Borel–Weil model of the highest weight representation V_λ (where we implicitly view λ also as a weight of A_G):

$$V_\lambda = \left\{ f : G \to W_\lambda^* : f(pg) = p \cdot f(g) \text{ for } p \in P \right\}$$

Viewing this space as sections of an equivariant bundle, restriction to the cell $PwU \subset P\backslash G$

$$I : V_\lambda \ni f \longmapsto [u \mapsto f(wu)] \in \mathbf{C}[U, W_\lambda^*]$$

is an injective homomorphism of M-representations since the big cell is dense in the flag variety $P\backslash G$. In particular, the U^0-invariant functions further descend to a function on $\mathbf{C}[U/U^0, W_\lambda^*] \cong \mathbf{C}[t] \otimes W_\lambda^*$, where the identification comes from the character $\Psi : U/U^0 \cong \mathbf{G}_a$.

Now M^0 leaves the coordinate t invariant, while $\mathbf{j}(\mathbf{G}_m)$ scales Ψ by weight 1 and hence t by weight -1. Viewing W_λ^* as a representation of $M \cong \mathrm{GL}_2 \boxtimes \mathrm{GSp}_4$ (where the box tensor means, as before, that the determinant of the GL_2-factor agrees with the similitude of the GSp_4-factor), we can write $W_\lambda^* = U_\mu \boxtimes T_\nu$ as the box tensor product of two irreducible representations. Then the vector

$$t^k \otimes \mathrm{ev} \in \mathbf{C}[t] \otimes U_\mu \boxtimes T_\nu$$

gives a nonzero S-invariant if and only if λ is of the form

$$\lambda = (\mu - (1+k), \mu^*)$$

where $\mu^* = -w_0\mu$ is the dual highest weight to μ. We call such dominant weights of G weights of G_2 **type**.

Summarizing these calculations, we arrive at the following

Proposition 5.2.1. The spectral X-period of an irreducible G-valued parameter φ is

$$L_X(\varphi) = q^{(g-1)\frac{\dim(G/S)-\langle\eta,\check{\theta}\rangle}{2}}(\check{\eta}\circ\varphi)(\partial^{-1/2})\prod_v \sum_{\lambda \text{ of } \mathrm{G}_2 \text{ type}} q^{-\langle\lambda,\check{\theta}\rangle/2}\, s_\lambda(\mathrm{Sat}_v)$$

5.3 Integrality

In evaluating the automorphic X-period, in particular once we unfold it to an Euler product of local integrals, we will need a parametrization of integral points in $X(\mathcal{O}_v)$, which we calculate in this section. We start with Iwasawa decomposition $G_v = P_vK_v = U_vM_vK_v$ where P is the parabolic subgroup stabilizing the isotropic 2-plane $\langle e_1, e_2\rangle$; this is the parabolic of the "same shape" as S:

$$P = \begin{bmatrix} A & 0_2 & 0_2 \\ 0_2 & B & 0_2 \\ 0_2 & 0_2 & {}^tA^{-1}\det(B) \end{bmatrix} \begin{bmatrix} 1_2 & X & Y \\ 0_2 & 1_2 & X^* \\ 0_2 & 0_2 & 1_2 \end{bmatrix}$$

for $A, B \in \mathrm{GL}_2$, $X \in \mathrm{Mat}_2$, and Y of the form $Y = \begin{bmatrix} x & y \\ y+\det(X) & z \end{bmatrix}$.

First, we use the additive character to identify

$$\Psi : U_v^0\backslash U_v \xrightarrow{\sim} F_v$$

$$U(X,Y) \longmapsto \psi(\mathrm{tr}(X))$$

$$u(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t^* \\ 0 & 0 & 1 \end{bmatrix} \leftrightarrow t$$

where we abuse notation to write $t = \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix}$ for $t \in F_v$, so that $t^* = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$.

Now for the reductive part of P, let $g \in M$ be arbitrary. We will split the GL_2 factors occuring in g into $\mathrm{GL}_1 \times \mathrm{SL}_2$ as

$$g = \begin{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} A & 0 & 0 \\ 0 & \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} B & 0 \\ 0 & 0 & \begin{bmatrix} \lambda^{-1}\mu & 0 \\ 0 & \mu \end{bmatrix} {}^tA^{-1} \end{bmatrix}$$

for $A, B \in \mathrm{SL}_2$ and $\lambda, \mu \in \mathrm{GL}_1$. Left multiplication by

$$\begin{bmatrix} B^{-1} & & \\ & B^{-1} & \\ & & {}^tB \end{bmatrix} \mathrm{diag}(\mu^{-2}, \mu^{-1}, \mu^{-1}, 1, \mu, 1)$$

puts our matrix into the form

$$g = \begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^tA^{-1} \end{bmatrix}$$

where the matrix A is now ranging over GL_2. Next, note that the action of right multiplication by

$$\begin{bmatrix} \det(k_2)k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & {}^tk_1^{-1} \end{bmatrix} \in P \cap K_v$$

and left multiplication by

$$\begin{bmatrix} \det(k_2)^{-1}k_2^{-1} & 0 & 0 \\ 0 & k_2^{-1} & 0 \\ 0 & 0 & {}^tk_2 \end{bmatrix} \in S_v \cap K_v$$

has the effect of

$$\begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^tA^{-1} \end{bmatrix} \longmapsto \begin{bmatrix} k_2^{-1}Ak_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^t(k_2^{-1}Ak_1)^{-1} \end{bmatrix}$$

So by Cartan decomposition in A, we may take A to be of the form $\mathrm{diag}(\varpi^{(a,b)})$ for $a \geq b$. In conclusion,

we may assume that our matrix g is of the form

$$g = u(t)\varpi^{(a,b)} := \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t^* \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varpi^{(-a,-b)} & & \\ & 1 & \\ & & \varpi^{(a,b)} \end{bmatrix}$$

for $a \geq b$. These form a complete set of coset representatives for $S_v \backslash G_v / K_v$.

This matrix sends γ_0 to

$$\gamma_0 g = \varpi^a e_{35} + \varpi^{a+b} t e_{65} + \varpi^b e_{46}$$

For this to be an $\mathcal{O}_v$-point, we require

$$a, b \geq 0 \text{ and } v(t) + a + b \geq 0$$

Summarizing this computation, we have the following conditions on $u(t)\varpi^{(a,b)}$ for the nonvanishing of the Schwartz function $\Phi_v(\,\cdot\,) = \mathbf{1}_{X(\mathcal{O}_v)}(\,\cdot\,\varpi^{m_v})$:

$$a + m_v,\ b + m_v,\ v(t) + a + b + m_v \geq 0 \tag{5.1}$$

5.4 Automorphic period

Let f be a cuspidal automorphic form on G with central character χ_f. We write Q for the following normalization global constant

$$Q = q^{-\beta_{\mathrm{GJ}}} q^{-\dim(S)(g-1)/2} \eta^{1/2}(\partial^{1/2})$$

where β_{GJ} is the following constant

$$\beta_{\mathrm{GJ}} = (g-1)(\dim(S) - \langle 2\rho_X, 2\check{\rho}_X \rangle)$$

and $\check{\rho}_X$ is the distinguished cocharacter

$$\check{\rho}_X(t) = \mathrm{diag}(t^{-1/2}, t^{1/2}, \mathrm{Id}_2, t^{1/2}.t^{-1/2})$$

Later this same cocharacter will also be referred to as ρ_{G_2}, since we can view it as a character of the G_2-subgroup inside $\check{G}$. We set

$$\check{\theta}_X := 2\check{\rho}_X$$

Then the automorphic period attached to X can be unfolded as follows:

$$\begin{aligned} P_X(f) &= Q \int_{[G]} f(g) \sum_{\gamma \in X_F} \Phi(\gamma g) \eta^{1/2}(g)\, dg \\ &= Q \int_{S_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Phi(g) \int_{[S]} f(sg)\, \eta^{1/2}(sg)\, ds\, dg \end{aligned}$$

Focusing on the inner $[S]$-integral and applying Fourier expansion on $U^0 \subset U$, we have

$$\begin{aligned} &\int_{[S]} f(sg) \eta^{1/2}(sg)\, ds \\ &= \int_{[\mathbf{j}(\mathbf{G}_m) M^0 U^0]} f(um\mathbf{j}(a)g) \eta^{1/2}(m\mathbf{j}(a)g) \delta_S^{-1}(\mathbf{j}(a)m)\, du\, d(m,a) \\ &= q^{2(g-1)} \sum_{\alpha \in F^\times} \int_{[\mathbf{j}(\mathbf{G}_m) M^0 U^0]} f(um\mathbf{j}(a)g) \eta^{1/2}(m\mathbf{j}(a)g) \delta_S^{-1}(\mathbf{j}(a)m) \Psi_\alpha(u)\, d^\psi u\, d(m,a) \end{aligned}$$

At this point, we consider the following change of variables on the $u = u(X,Y)$-integral:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto X_\alpha = \begin{bmatrix} \alpha a & \alpha^2 b \\ c & \alpha d \end{bmatrix}$$

$$Y = \begin{bmatrix} x & y \\ y + \det(X) & z \end{bmatrix} \longmapsto Y_\alpha = \begin{bmatrix} \alpha^3 x & \alpha^2 y \\ \alpha^2(y + \det(X)) & \alpha z \end{bmatrix}$$

The point of this change of variables being that

$$\mathrm{Ad}_{\mathbf{j}(\alpha)}(u(X,Y)) = u(X_\alpha, Y_\alpha)$$

and also $\Psi(u(X_\alpha, Y_\alpha)) = \Psi_\alpha(u(X,Y))$. So we can combine the sum over $\alpha \in F^\times$ with the $[\mathbf{j}(\mathbf{G}_m)]$-integration to get

$$\begin{aligned} &= q^{2(g-1)} \int_{\mathbb{A}^\times} \int_{[M^0 U]} f(um\mathbf{j}(a)g) \eta^{1/2}(m\mathbf{j}(a)g) \delta_S^{-1}(\mathbf{j}(a)m) \Psi_\alpha(u)\, d^\psi u\, d(m,a) \\ &= q^{2(g-1)} \int_{\mathbb{A}^\times} \Omega_f(\mathbf{j}(a)g) \delta_S^{-1}(\mathbf{j}(a))\, da \end{aligned}$$

where Ω_f is the $M^0 \times (U, \Psi)$-equivariant **Ginzburg–Jiang function** on G given by

$$g \mapsto \Omega_f(g) = \int_{[M^0 U]} f(um) \eta^{1/2}(g) \delta_S^{-1}(m) \Psi(u)\, d^\psi u\, dm$$

Returning to the global period, we can fold $\mathbf{j}(\mathbb{A}^\times)$ back into the outer integral to obtain

$$P_X(f) = q^{\beth(g-1)} \int_{M^0_{\mathbb{A}} U^0_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Phi(g) \Omega_f(g) \eta^{1/2}(g)\, dg$$

where $\beth$ is the following global constant

$$\beth = 2\langle \rho_X, \check{\rho}_X \rangle - \dim(S) - \frac{\langle \eta, \check{\theta}_X \rangle}{2} + 2$$

S.C. Vo's multiplicity one result (Theorem 1, [Vo97]) implies the Euler factorization of the Ginzburg-Jiang model, and the Ginzburg-Jiang function Ω_f can be expressed in terms of local spherical functions. We normalize the spherical functions as before: its evaluation at the most antidominant nonvanishing point gives the local L-factor of $L_v(1, \mathrm{Std}_{\mathrm{G}_2})$, where $\mathrm{Std}_{\mathrm{G}_2}$ is the standard 7-dimensional representation of G_2. We will be more preicse about this normalization when calculating the local integrals. The local L-factor here comes from the distinction properties of the Ginzburg–Jiang model. Globally, the Ginzburg–Jiang function is proportional to the residue of the spin L-function, which detects functorial lifts from G_2 (see Theorem 1, [BG92]).

In conclusion, our global integral can be Euler factorized as

$$P_X(f) = q^{\beth(g-1)} \prod_v P_v$$

where we define the local periods as

$$P_v = \int_{M^0_v U^0_v \backslash G} \Phi(g) \Omega_v(g) \eta^{1/2}(g)\, dg$$

Using the local integrality computations of the previous section to evaluate the Schwartz function Φ on double coset representatives, the local period can be now written as a sum over cocharacters that satisfy those bound in (5.1). Furthermore, using the fact that the conductor of ψ is $-2m_v$, the integral over dt of ψ will vanish unless

$$-a - b - m_v \geq -2m_v \iff m_v \geq a + b$$

In conclusion, we have the following conditions on (a, b) for its correpsonding double coset representative to contribute to the local integral:

$$m_v \geq a + b \text{ and } a + m_v, b + m_v \geq 0 \tag{5.2}$$

$$P_v = \sum_{(a,b):(5.2)} \left[\int_{v(t) \geq -a-b-m_v} \psi(t)\, dt \right] \Omega_v(\varpi^{(a,b)}) \eta^{1/2}(\varpi^{(a,b)})$$
$$= q^{m_v} \sum_{(a,b):(5.2)} q^{a+b} \Omega_v(\varpi^{(a,b)}) \eta^{1/2}(\varpi^{(a,b)})$$

where the integral of ψ, when nonvanishing, simply contributes the volume of its domain of integration.

5.5 Casselman–Shalika–Sakellaridis Formula

In order to evaluate the local periods in terms of Satake parameters, one needs a formula for the Ginzburg–Jiang function analogous to the Casselman–Shalika formula, which is again, afforded by Sakellaridis' general formula for spherical functions. Although not directly addressed by [Sak18], in Section 9 of *loc. cit* an outline is given for how to adapt the strategy more generally.

The spherical variety in question is $Z = M^0 U \backslash_{\Psi} G$, where Ψ is the additive character on U given by $\Psi(u(X,Y)) = \psi(\mathrm{tr}(X))$. We start with the parabolic subgroup $P = UL$ of stabilizing the isotropic 2-plane $\langle e_1, e_2 \rangle$, where $L \cong \mathrm{GL}_2 \times \mathrm{GL}_2$ is the Levi subgroup, realized concretely as

$$(A,B) \leftrightarrow \begin{bmatrix} A & 0_2 & 0_2 \\ 0_2 & B & 0_2 \\ 0 & 0 & {}^tA^{-1}\det(B) \end{bmatrix}$$

as before. Then the stabilizer of our character Ψ on U is exactly ${}^{\Delta}\mathrm{GL}_2$. Consider the inducing spherical L-variety

$$Y = {}^{\Delta}\mathrm{GL}_2 \backslash L \cong \mathrm{GL}_2$$

where L acts by

$$y \cdot (g_1, g_2) = g_2^{-1} y g_1$$

In other words, Y is just the group case spherical variety for GL_2. Then the spherical G-variety we are considering is Ψ-twisted parabolic induction

$$Z = \mathrm{Ind}_P^G (Y, \Psi)$$

The following Proposition follows from an explicit calculation of this induction process. A review of the relevant theory of spherical varieties can be found in Section 2 of [SV17].

Proposition 5.5.1. The dual group $\check{G}_Z$ of the spherical variety Z is isomorphic to G_2.

Proof. We look at the following parametrization of a diagonal 4-dimensional Cartan subgroup of GSp_6:

$$\mathrm{diag}(h_1, h_2, h_3, h_3^{-1}, h_1^{-1}, h_2^{-1}) \cdot \mathrm{diag}(h_4, \ldots, h_4)$$

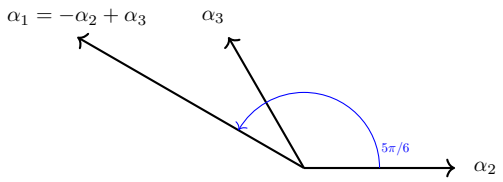
To compute the dual group, we start with the inducing variety Y, which contributes the spherical root of type G

$$\alpha_1 := -h_1^\vee + h_2^\vee + 2h_3^\vee$$

We also have two ψ-generic roots (of type (U, ψ))

$$\alpha_2 := h_1^\vee - h_3^\vee \quad \alpha_3 := h_2^\vee + h_3^\vee$$

Then we see that $\alpha_1 = -\alpha_2 + \alpha_3$, and the geometry of these three roots forms the following picture:



which evidently generates a root system of type G_2. Since $\check{G}_2 \cong \mathrm{G}_2$, we have $\check{G}_Z \cong \mathrm{G}_2$. □

In particular, $\check{G}_Z$ is equipped with a canonical maximal torus $\check{A}_Z$, which is dual to the quotient torus

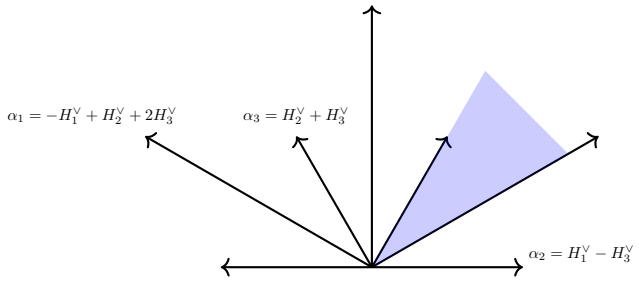
$$\mathrm{GSp}_6 \supset A \twoheadrightarrow A_Z = \mathrm{diag}(h_1 h_3^{-1}, h_2 h_3)$$

The cocharacters $\varpi^{(r,s)}$ of A_Z defined by

$$\mathbf{G}_m \longrightarrow A \longrightarrow A_Z$$

$$x \longmapsto \mathrm{diag}(x^{-r}, x^{-s}, 1, 1, x^r, x^s) \longmapsto (x^r, x^s)$$

for $r \geq s \geq 0$ are most important to us. This constraint translates to a constraint on the weight space of $\check{A}_Z$, which, juxtaposed with the root system of $\check{G}_Z$, is shown below. Note that α_2, α_3 are the weights of $\check{A}_Z$ dual to the coweights we indexed by r, s respectively.



The blue region gives the constraint of $r \geq s \geq 0$, and we will refer to it as the dominant Weyl chamber in the following.

Finally we are ready for an explicit formula of the spherical Ginzburg-Jiang function.

Theorem 5.5.2 (Sakellaridis)**.** Let λ be an antidominant coweight of G in the above sense, and let Ω be the spherical Ginzburg-Jiang function associated to an unramified principal series representation of G_v with Satake parameter Sat_v, normalized so that its value at the identity is 1. Then $\Omega = 0$ unless Sat_v is conjugate to $\mathrm{G}_2 \subset \check{G}$, in which case

$$\Omega(\varpi^\lambda) = \frac{\delta_{\mathrm{G}_2}^{1/2}(\varpi^\lambda) \sum_{w \in W_{\mathrm{G}_2}} (-1)^w \, e^{w(\rho+\lambda)} \prod_{\alpha \in \Phi_{\mathrm{G}_2}^{+,\mathrm{long}}} \left(1 - q^{-1} e^{-w\alpha}\right)}{\sum_{w \in W_{\mathrm{G}_2}} (-1)^w \, e^{w\rho} \prod_{\alpha \in \Phi_{\mathrm{G}_2}^{+,\mathrm{long}}} \left(1 - q^{-1} e^{-w\alpha}\right)} (\mathrm{Sat}_v)$$

Proof. For details we refer to the Appendix. The short story is that the calculations done in [Sak18] works for any "Shalike-type" situation, where the significance of the short positive symplectic roots are that they are the positive spherical roots of type G. □

The denominator factors as the Weyl denominator times a simple factor (see, for example, the proof of Proposition 2 of [WW00]) as

$$\sum_{w \in W_{\mathrm{G}_2}} (-1)^w \, e^{w\rho} \prod_{\alpha \in \Phi_{\mathrm{G}_2}^{+,\mathrm{long}}} \left(1 - q^{-1} e^{-w\alpha}\right) = \omega(q) \sum_{w \in W_{\mathrm{G}_2}} (-1)^w \, e^{w\rho}$$

where $\omega(q) = 1 + q^{-1}$. This allows us to write the spherical function directly in terms of Weyl characters of G_2 as

$$\Omega(\varpi^{(a,b)}) = \frac{\delta_{\mathrm{G}_2}^{1/2}(\varpi^{(a,b)})}{\omega(q)} \sum_{J \subset \Phi_{\mathrm{G}_2}^{+,\mathrm{long}}} (-q)^{-|J|} s_{(a,b)-J}(\mathrm{Sat}_v)$$

Returning to the local period calculation, we make the shift

$$(a,b)_{\mathrm{new}} = (a,b)_{\mathrm{old}} + (2m_v, 2m_v)$$

in the bounds (5.2) to obtain

$$a, b \geq m_v \text{ and } 5m_v \geq a + b \tag{5.3}$$

and using the fact that Ω has central character χ_f, we can evaluate the local period as

$$\begin{aligned}
P_v &= q^{m_v} \sum_{(a,b):(5.2)} q^{a+b} \eta^{1/2}(\varpi^{(a,b)}) \Omega_v(\varpi^{(a,b)}) \\
&= q^{m_v} q^{m\langle \eta, \check{\theta} \rangle} \sum_{(a,b):(5.3)} \eta^{1/2}(\varpi^{(a,b)}) \Omega_v^{\mathrm{ur}}(\varpi^{(a,b)}) \\
&= q^{m_v} q^{m\langle \eta, \check{\theta} \rangle} \sum_{(a,b):(5.3)} (\delta_{\mathrm{G}_2}\eta)^{1/2}(\varpi^{(a,b)}) \sum_{J \subset \Phi_{\mathrm{G}_2}^{+,\mathrm{long}}} (-q)^{-|J|} s_{(a,b)-J}(\mathrm{Sat}_v) \\
&= q^{m_v} q^{m\langle \eta, \check{\theta} \rangle/2} q^{-2\langle \rho_{\mathrm{G}_2}, \check{\rho}_{\mathrm{G}_2} \rangle} \chi_f(\partial^{-1/2}) \sum_{(a,b):(5.3)} (\delta_{\mathrm{G}_2}\eta)^{1/2}(\varpi^{(a,b)}) \sum_{J \subset \Phi_{\mathrm{G}_2\rangle}^{+,\mathrm{long}}} (-q)^{-|J|} s_{(a,b)-J}(\mathrm{Sat}_v)
\end{aligned}$$

where we used the fact that $\check{\theta}_X = 2\check{\rho}_X = \rho_{\mathrm{G}_2}$.

Collecting global constants in the Euler product, we have the following explicit expression for the global automorphic period attached to X:

Theorem 5.5.3. Let $G = \mathrm{GSp}_6$. Let $X = \overline{S \backslash G}$ and f a Ginzburg–Jiang normalized unramified cusp form on G with L-parameter φ_f, which we assume to be conjugate to G_2 in $\check{G} = \mathrm{GSpin}_7$, and whose Satake parameter at a place v is Sat_v. Then

$$P_X(f) = (*)\, \chi_f(\partial^{-1/2}) L(\varphi_f, \check{\mathfrak{g}}/\mathfrak{g}_2, 1) \prod_v \sum_{\chi:(5.3)} q^{-\langle \rho_{\mathrm{G}_2} + \eta/2, \chi \rangle} \sum_{J \subset \Phi_{\mathrm{G}_2}^{+,\mathrm{long}}} (-q)^{-|J|} s_{\chi - J}(\mathrm{Sat}_v)$$

where $(*)$ is the following explicit power of the discriminant

$$q^{\frac{g-1}{2}(2|\psi| + 2 - \dim(S) - \dim(U))}$$

We note that the number of times the Weyl character s_χ occurs in an Euler factor is $2^{|\Phi_{\mathrm{G}_2}^{+,\mathrm{long}}|}$, and it occurs with weights

$$(-q_v)^{-|J|} q_v^{-\langle \chi + J, \rho_{\mathrm{G}_2} + \eta/2 \rangle} = (-q_v)^{-|J|} q_v^{-\langle J, \check{\theta} + \eta/2 \rangle} q_v^{-\langle \chi, \check{\theta} + \eta/2 \rangle}$$

5.6 Duality

Let $N \subset \check{G} = \mathrm{GSpin}_7$ be the upper triangular unipotent subgroup, and let $V = H \cap N$ be the corresponding maximal unipotent subgroup of $H = \mathrm{G}_2$. Consider the $\check{G}$-space

$$\check{X} = \mathrm{Ind}_H^{\check{G}} \overline{V \backslash H}$$

Proposition 5.6.1. The spectral period of $\check{X}$ distinguishes G_2-parameters. For such a parameter φ, we have

$$L_{\check{X}}(\varphi) = (*)\, L(\check{\mathfrak{g}}/\mathfrak{g}_2, \varphi, 1))(\check{\eta} \circ \varphi)(\partial^{-1/2}) \prod_v \sum_{\chi \in X^*_{\mathrm{dom}}(\mathrm{G}_2)} q^{-\langle \check{\theta}+\eta, \chi \rangle/2} \mathrm{tr}(\mathrm{Fr}_v | V_\chi)$$

where $(*)$ is the following power of the discriminant

$$q^{\frac{g-1}{2}\left(\dim(\check{X}) - \langle \check{\theta}, \eta_{\check{X}} \rangle\right)}$$

Proof. The L-factor appearing in front comes from the tangent space $\check{\mathfrak{g}}/\mathfrak{h} \cong \wedge_0^2$ as an H-representation. The sum in the Euler factor comes from the Peter–Weyl computation

$$\mathcal{O}_{\overline{V \backslash H}} = \bigoplus_\chi V_\chi^V \boxtimes V_\chi^*$$

There is only one invariant vector in V_χ^V, namely the line with highest weight. □

Comparing this with Theorem 5.5.3 we see that $P_X(f)$ and $L_{\check{X}}(\varphi_f)$ are almost in agreement, but the local factors of P_X involve a shift over subsets of $\Phi^{+,\mathrm{long}}_{\mathrm{G}_2}$. In particular, we could read the local factor as

$$\left[\sum_{J \subset \Phi^{+,\mathrm{long}}_{\mathrm{G}_2}} (-q_v)^{-|J|} q_v^{-\langle J, \check{\theta}+\eta \rangle/2} \right] \mathrm{gtr}(\mathrm{Fr}_v \mid \mathcal{O}^*_{\overline{V \backslash H}})$$

where, as in the Shalika case, we define for a weight λ the pairing

$$\langle J, \lambda \rangle := \sum_{\check{\alpha} \in J} \langle \check{\alpha}, \lambda \rangle$$

So we are in a situation similar to the unfolding to the Shalika model, and we have the following preliminary step towards a weak duality statement.

Theorem 5.6.2. Let X (resp. $\check{X}$) be the G-space (resp. $\check{G}$-space) as above, Then

- Let f be a Ginzburg–Jiang normalized, unramified cusp form on G with L-parameter φ_f in $\check{G}$.

$$P_X(f) = \begin{cases} (*)\, L_{\check{X}}(\varphi_f) \prod_v \left[\sum_{J \subset \Phi^{+,\mathrm{long}}_{\mathrm{G}_2}} (-q_v)^{-|J|} q_v^{-\langle J, \check{\theta}+\eta \rangle/2} \right] & \text{if } \varphi_f \text{ is in } \mathrm{G}_2 \\ 0 \text{ otherwise} \end{cases}$$

  where $(*)$ is the following explicit power of the discriminant

$$(*_1) = q^{\frac{g-1}{2}\left[-\dim(\check{X}) - \langle \check{\theta}, \eta_{\check{X}} \rangle + 2|\psi| + 2 - \dim(S) - \dim(U)\right]}$$

- Let $\check{X}^- = \mathrm{Ind}_H^{\check{G}} \overline{U[M,M]\backslash H}$, where $P = MU$ is the parabolic in H whose Levi contains the short root, so that $\overline{U[M,M]\backslash H}$ is the affine cone over the G_2-Grassmannian $P\backslash H$. Let f be a cusp form on $\check{G}$ with L-parameter φ_f in G. Up to a half integral power of the discriminant

$$P_{\check{X}^-}(f) = (*_2)\, L_X(\varphi_f)$$

Proof. The first bullet point is the summary of our automorphic and spectral calculations. The second bullet point follows from the computation in Section 4 of [GR94a]. We will explain this computation while nailing down the constant $(*)$ in a formal writeup. □

Remark 5.6.3. A similar interpretation from the point of weak duality can be made, along the lines of Remark 4.6.3. Namely, we consider the derived vector space

$$\mathfrak{U} := \bigoplus_{\check{\alpha} \in \Phi^{+,\mathrm{long}}_{\mathrm{G}_2}} \mathbf{C}_{\check{\alpha}}$$

and form the derived vector bundle

$$\check{\mathfrak{X}} := \mathrm{Ind}_H^{\check{G}} \overline{V\backslash H \times \mathfrak{H}}$$

We consider the map $\pi : \check{\mathfrak{X}} \to \check{X}^-$ and take the pair $(\check{X}^-, \pi_* \mathcal{O}_{\check{\mathfrak{X}}})$ along with its $\check{G}$-action as our candidate weak dual to X.

Bibliography

[BF08] R.V. Bezrukavnikov and M.V. Finkel'berg. Equivariant satake category and kostant–whittaker reduction. *Moscow Mathematical Journal*, 8(1):39–72, 2008.

[BFN18] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, ii. *Adv. Theor. Math. Phys.*, 2018.

[BG92] Daniel Bump and David Ginzburg. Spin L-functions on symplectic groups. *International Mathematics Research Notices*, 1992.

[BZSV] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh. *Dualities in the relative Langlands program*.

[CH88] Arjeh Cohen and Aloysius Helminck. Trilinear alternating forms on a vector space of dimension 7. *Communications in Algebra*, 1988.

[CV] Eric Y. Chen and Akshay Venkatesh. Examples of duality for singular automorphic periods. in preparation.

[FJ93] Solomon Friedberg and Hervé Jacquet. Linear periods. *Journal für die reine und angewandte Mathematik*, 1993.

[Gar87] Paul Garrett. Decomposition of eisenstein series: Ranking triple product. *Annals of Mathematics*, 1987.

[Gin91] David Ginzburg. A Rankin-Selberg integral for the adjoint representation of GL_3. *Inventiones mathematicae*, 1991.

[Gin95] David Ginzburg. On spin L-functions for orthogonal groups. *Duke Mathematical Journal*, 1995.

[GJ] Roger Godement and Hervé Jacquet. *Zeta functions of simple algebras*. Springer Berlin, Heidelberg.

[GJ01] David Ginzburg and Dihua Jiang. Periods and liftings: From G2 to C3. *Israel Journal of Mathematics*, 2001.

[GR94a] D. Ginzburg and S. Rallis. A tower of Rankin–Selberg integrals. *International Mathematics Research Notices*, (5), 1994.

[GR94b] David Ginzburg and Stephen Rallis. A tower of Rankin–Selberg integrals. *International Mathematics Research Notices*, 1994.

[GR00] David Ginzburg and Stephen Rallis. The exterior cube L-function for GL_6. *Compositio Mathematica*, 2000.

[JS88] Hervé Jacquet and Joseph Shalika. Exterior square L-functions. *Automorphic forms, Shimura varieties, and L-functions, Vol. II*, 1988.

[Kac80] Victor Kac. Some remarks on nilpotent orbits. *Journal of algebra*, 64, 1980.

[KW07] Anton Kapustin and Edward Witten. Electric-magnetic duailty and the geometric Langlands program. *Communications in number theory and physics*, 1(1):1–236, 2007.

[Nak16] Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, i. *Adv. Theor. Math. Phys.*, 2016.

[Sak12] Yiannis Sakellaridis. Spherical functions on spherical varieties. *American Journal of Mathematics*, 135(5):1291–1381, 2012.

[Sak18] Yiannis Sakellaridis. A Casselman-Shalika formula for the Shalika model of GL_n. *Canadian Journal of Mathematics*, 2018.

[SV17] Yiannis Sakellaridis and Akshay Venkatesh. *Periods and harmonic analysis on spherical varieties*. Société mathématique de France, 2017.

[Vo97] San Cao Vo. The spin L-function on the symplectic group GSp(6). *Israel Journal of Mathematics*, 1997.

[WW00] N.R. Wallach and J. Willenbring. On some q-analogs of a theorem of Kostant–Rallis. *Canad. J. Math.*, 52, 2000.