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Smoothing conic Kahler metrics and the conical Kahler-Ricci flow

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SMOOTHING CONIC KÄHLER METRICS AND
THE CONICAL KÄHLER-RICCI FLOW

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Abstract

In this thesis, we first give two different smoothing methods for conic Kähler metrics. One is constructing an approximating sequence of smooth Kähler metrics with the same lower Ricci curvature bound with the original conic Kähler metric with some lower Ricci curvature bound based on Tian's approximation method for the conic Kähler-Einstein metric. Another one is constructing a smooth approximating sequence of Kähler metrics with uniformly bisectional curvature upper bound, based on C. Li and Y. Rubinstein's bisectional curvature upper bound estimate for a standard conic metric. Then we will use approximation method to construct solutions to the conical Kähler-Ricci flow which preserves the conic structure. After C^0 and C^2 -estimates for potential functions, we finally obtain a $C^{2,\alpha}$ -estimate based on Tian's method in his PKU thesis.

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To my parents

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Chapter 1

Introduction

1.1 Main theme: conic Kähler metrics and conical Kähler-Ricci flow

Kähler geometry is a special kind of complex geometry which has a Riemannian structure compatible with the complex structure. One basic problem in Kähler geometry is to find canonical metrics on Kähler manifolds. Canonical means that it depends on the complex structure and is unique up to biholomorphic automorphisms. One typical canonical metric is Kähler-Einstein metric, i.e. $Ric(\omega) = \lambda\omega$, where the sign of λ depends on the sign of the first Chern class $c_1(M)$. When $c_1(M)$ is negative or zero, the existence problem was solved by Aubin [1] and Yau [50]. The most difficult case is when $c_1(M)$ is positive. Recently, this folklore Yau-Tian-Donaldson conjecture that the existence of positive Kähler-Einstein metrics is equivalent to K-stability was confirmed in [44] [6] [7] [8].

In the solution of this conjecture, conic Kähler metrics play a critical role. As the generalization of smooth Kähler metric, the conic Kähler metric itself also attracts a lot of research interests, for instance, see [2] [3] [4] [13] [15] [20] [21] [26] or a nice survey by Y. Rubinstein [31]. Similarly, the existence and regularity of canonical

conic metrics is also the central problem in this field. The continuity path in cone angles bridges conic metrics and smooth Kähler metrics, which is very important in the study of this folklore conjecture, and is believed to play roles in future study of Kähler geometry.

On the other hand, Ricci flow has become a powerful tool in the study of finding canonical metrics in differential geometry since R. Hamilton's breakthrough work [18]. Tian–Zhang [47], Song–Tian [35] [36] [37] developed an analytic approach to minimal model program by using Kähler-Ricci flow. Their idea can be also extended to conical Kähler-Ricci flow—a flow which preserves the conic singularities. This flow may be helpful in future study of pairs and minimal model program.

1.2 Content of the thesis

In Chapter 2, we will first introduce the basic settings of conic Kähler metrics, then make use of Tian's smooth approximation of the conic Kähler-Einstein metric [44] to prove that similar approximation exists for conic Kähler metrics with lower Ricci curvature bound:

Theorem 1.2.1. *(Shen 14' [32]) For Kähler manifold (M, D) where D is a simple normal crossing divisor, assume that we have a smooth background Kähler metric ω_0 and a conic Kähler metric $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ with cone angle $2\pi\beta_i$ ($0 < \beta_i < 1, 1 \leq i \leq m$) along each irreducible component D_i of D and φ is a smooth real function on $M \setminus D$. If the conic Kähler metric ω has a lower Ricci curvature bound μ , then for any $\delta > 0$, there exists a smooth Kähler metric ω_δ with the same lower Ricci curvature bound μ which converges to ω in the Gromov-Hausdorff topology on M and in the smooth topology outside D as δ tends to 0.*

In Chapter 3, based on C. Li and Y. Rubinstein's bisectional curvature estimate in the appendix of [21], we construct a sequence of smooth potential functions which

approximate the potential function of the conic metric. We can prove that the corresponding smooth Kähler metrics approximate the conic Kähler metric with uniformly upper bisectional curvature bound. Moreover, actually we can prove this approximation can be generalized to some special cases of simple normal crossing divisors:

Theorem 1.2.2. *(Shen 15') For Kähler manifold (M, D) where D is a simple normal crossing divisor satisfying either all cone angles $\beta_i \leq \frac{1}{2}$ or D is composed by some irreducible divisors free of triple singularity, let $\omega = \omega_0 + \sum_{i=1}^m \sqrt{-1} \partial \bar{\partial} ||S_i||_i^{2\beta_i}$, where S_i is the holomorphic section of each irreducible divisor D_i , and β_i is the cone angle along D_i . Then there exists a sequence of smooth Kähler metrics ω_ϵ which converges to ω outside D as ϵ tends to 0, such that the bisectional curvature of ω_ϵ is uniformly bounded from above on M independent of ϵ .*

In Chapter 4, following the approximation method of Liu–Zhang [28] and Guanancia–Paun [15] we will extend the maximal time existence of the solution to Kähler-Ricci flow to the conical Kähler-Ricci flow, which preserves the conic singularities:

Theorem 1.2.3. *(Shen 14'[33]) The conical Kähler-Ricci flow can exist until the Kähler class of the evolving metric degenerates. Moreover, during the maximal time interval, the solution is $C^{2,\alpha}$ bounded.*

In this chapter, we establish the C^0 and C^2 -estimates for the solution. In the end of this chapter, we will sketch a new proof for this theorem based on our bisectional curvature estimate in Chapter 2.

In Chapter 5, based on Tian's method in his master's thesis [38], we will give a parabolic version of his estimate to establish the $C^{2,\alpha}$ bound for conical Kähler-Ricci flow and complete the proof of the main theorem in Chapter 4.

Chapter 2

Smoothing conic metric I: lower Ricci curvature bound

2.1 Introduction

Let (M, g) be a complex manifold of complex dimension n with a Hermitian metric g . We say M is a Kähler manifold if the associated $(1, 1)$ -form ω of the metric g is a closed form, i.e, $d\omega = 0$. We can call ω a Kähler metric on M . Now we can introduce the conic Kähler metric according to Tian:

Definition 2.1.1. (*Tian 96'[41]*) *We call a Kähler current ω a conic Kähler metric along a smooth divisor D with cone angle $2\pi\beta$ ($\beta \in (0, 1)$), if it is a smooth Kähler metric outside the divisor D , and in local coordinates $(z_1, \dots, z_n)$ near a point $p \in D$, if $D = (z_1 = 0)$ locally, then ω is asymptotic to the model metric*

$$\omega_\beta = \sqrt{-1} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right).$$

Moreover, in the thesis, instead of a smooth divisor, we can consider the divisor D as a simple normal crossing divisor, i.e, there are m irreducible divisors $D_1, \dots, D_m$

intersecting transversally at some point $p \in M$, and in the local coordinate neighborhood U we can write $D_i = z_i = 0$. Now we say the current ω is a conic metric along each D_i with cone angles $2\pi\beta_i$, if it is asymptotic to the model conic metric

$$\omega_\beta = \sqrt{-1} \left(\sum_{i=1}^m \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{i=m+1}^n dz_i \wedge d\bar{z}_i \right).$$

In recent breakthrough progress on Kähler-Einstein metrics on Fano manifolds (See [44] [6] [7] [8]), conic Kähler metrics supplies an important continuity path. The main tool is an extension of Cheeger-Colding-Tian theory [5] to conic Kähler-Einstein metrics. This extension allows one to establish partial C^0 -estimate which is known for long to be crucial in proving the existence of Kähler-Einstein metrics. To extend Cheeger-Colding-Tian theory from smooth case to conic case, in [44], Tian proved a sharp approximation theorem: any conic Kähler-Einstein metric can be approximated by smooth Kählermetrics with the same lower Ricci curvature bound in the Cheeger-Gromov sense.

The main idea for proving this sharp approximation came from [43], which gives a method of proving the equivalence of C^0 -estimate and the properness of the Lagrangian of corresponding complex Monge-Ampere equation. First, we can define so called twisted Ding energy $F_\omega(\varphi)$ or twisted Mabuchi energy $\nu_\omega(\varphi)$ as [26], which are Lagrangians of corresponding complex Monge-Ampere equation, for the conic Kähler-Einstein metric. Then we can prove these two energies are both proper with respect to the functional $J_\omega(\varphi)$. After that, we perturb this singular complex Monge-Ampere equation, and prove that corresponding energies are also proper after such perturbation. Then, we make use of C^0 -estimate in [44] to get a new C^0 -estimate for perturbed complex Monge-Ampere equation. Finally, according to compactness theorem, we can prove that perturbed Kähler metrics converge to the original conic

Kähler-Einstein metric in Cheeger–Gromov sense, and converge smoothly in C^∞ sense outside the divisor.

Now a more general problem is to understand the structures of Kähler manifolds with lower Ricci curvature bound. A natural question is whether we can also approximate arbitrary conic Kähler metric by smooth Kähler metrics with the same lower Ricci curvature bound. We observe that the method above in [44] can apply if we can get suitable complex Monge–Ampere equations and define suitable energies for them. Moreover, instead of pluri-anti-canonical divisors in the original proof, we can generalize our result to simple normal crossing divisors.

We say the conic Kähler metric ω has a lower Ricci curvature bound μ if there exists a nonnegative $(1, 1)$ –form Ω such that (we may assume that $\Omega \neq 0$ otherwise we come back to conic Kähler-Einstein case.)

$$Ric(\omega) = \mu\omega + \sum_{i=1}^m 2\pi(1 - \beta_i)[D_i] + \Omega. \quad (2.1)$$

This equation is in the sense of currents on M and in classic sense outside singular part D . Consider these equations and apply Tian’s method [44] for conic Kähler-Einstein metrics, we can prove Theorem 1.2.1 as below:

Theorem 2.1.2. *For Kähler manifold (M, D) where D is a simple normal crossing divisor, assume that we have a smooth Kähler metric ω_0 and a conic Kähler metrics $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ with cone angle $2\pi\beta_i$ ($0 < \beta_i < 1, 1 \leq i \leq m$) along each irreducible component D_i of D and φ is a smooth real function on $M \setminus D$. If the conic metric ω has a lower Ricci curvature bound μ , then for any $\delta > 0$, there exists a smooth Kähler metric ω_δ with the same lower Ricci curvature bound μ which converges to ω in the Gromov-Hausdorff topology on M and in the smooth topology outside D as δ tends to 0.*

This theorem generalized Tian's approximation theorem for conic Kähler-Einstein metrics to arbitrary conic metrics with a lower Ricci curvature bound. Note that here we do not assume the sign of μ . Due to Aubin and Yau, the cases $\mu < 0$ and $\mu = 0$ are somewhat easier to solve. The difficulty will be when $\mu > 0$, i.e. Fano case. In the following section, we set up the complex Monge-Ampere equations and perturb it, and derive C^0 -estimate for nonpositive μ . And we will deal with the case $\mu > 0$ in the remaining parts of this chapter.

2.2 Basic set up and the case $\mu \leq 0$

First, as we have the equation (2.1), we can have such an expression for Ricci curvature of the background metric ω_0 :

$$Ric(\omega_0) = \mu\omega_0 + \Omega_0, \quad (2.2)$$

where Ω_0 is a $(1, 1)$ -form. Comparing equations (2.1) and (2.2), we have

$$\sqrt{-1}\partial\bar{\partial}\log\frac{\omega^n}{\omega_0^n} = -\mu\varphi + \Omega_0 - \Omega - \sum_{i=1}^m(1-\beta_i)(R(\|\cdot\|_i) + \sqrt{-1}\partial\bar{\partial}\log\|S_i\|_i^2),$$

where $\omega = \omega_0 + \varphi$ is the conic Kähler metric. As each D_i is an irreducible positive divisor, we set S_i as its defining holomorphic section, with $(\|\cdot\|_i)$ as the Hermitian product on the associated line bundle $[D_i]$, and the curvature of this bundle is defined as $R(\|\cdot\|_i) := -\sqrt{-1}\partial\bar{\partial}\log\|\cdot\|_i^2$. Then we get the equation above just from Poincaré-Lelong equation

$$2\pi[D] = \sqrt{-1}\partial\bar{\partial}\log|S|^2 = \sqrt{-1}\partial\bar{\partial}\log\|S\|^2 + R(\|\cdot\|).$$

Note that the left hand side of (2.2)(2.1) lie both in the cohomology class $c_1(M)$, we deduce that

$$\Omega_0 - \Omega - \sum_{i=1}^m (1 - \beta_i) (R(\|\cdot\|_i) = \sqrt{-1} \partial \bar{\partial} h_0. \quad (2.3)$$

where h_0 is a smooth function on M and we note that $\frac{1}{2\pi} R(\|\cdot\|_i)$ represents $c_1(D_i)$.

Then we get our complex Monge-Ampere equation:

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{h_0 - \mu \varphi - \sum_{i=1}^m (1 - \beta_i) \log \|S_i\|_i^2 + c} \omega_0^n, \quad (2.4)$$

where the constant c is chosen so that

$$\int_M (e^{h_0 - \sum_{i=1}^m (1 - \beta_i) \log \|S_i\|_i^2 + c} - 1) \omega_0^n = 0.$$

The reason that why c can be chosen is that near the singular set, the function $e^{-\sum_{i=1}^m (1 - \beta_i) \log \|S_i\|_i^2}$ behaviors as $|z_1|^{2(\beta_1 - 1)} \dots |z_m|^{2(\beta_m - 1)}$, which is locally integrable.

As [44], we can choose such an approximation equation:

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{h_\delta - \mu \varphi} \omega_0^n, \quad (2.5)$$

where

$$h_\delta = h_0 - \sum_{i=1}^m (1 - \beta_i) \log(\delta + \|S_i\|_i^2) + c_\delta$$

and the constant c_δ is chosen such that

$$\int_M (e^{h_0 - \sum_{i=1}^m (1 - \beta_i) \log(\delta + \|S_i\|_i^2) + c_\delta} - 1) \omega_0^n = 0.$$

Here c_δ is uniformly bounded. If we have a solution φ_δ for (2.5), then we get a smooth

Kähler metric $\omega_\delta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\delta$ with Ricci curvature as below:

$$\begin{aligned}
Ric(\omega_\delta) &= Ric(\omega_0) + \mu\sqrt{-1}\partial\bar{\partial}\varphi_\delta - \sqrt{-1}\partial\bar{\partial}h_\delta \\
&= \mu\omega_0 + \Omega_0 + \mu\sqrt{-1}\partial\bar{\partial}\varphi_\delta - \sqrt{-1}\partial\bar{\partial}h_0 + \sum_{i=1}^m (1 - \beta_i)\sqrt{-1}\partial\bar{\partial}\log(\delta + \|S_i\|_i^2) \\
&= \mu\omega_\delta + \Omega + \sum_{i=1}^m (1 - \beta_i)(R(\|\cdot\|_i) + \frac{\|S_i\|_i^2}{\delta + \|S_i\|_i^2}\sqrt{-1}\partial\bar{\partial}\log\|S_i\|_i^2 \\
&\quad + \sqrt{-1}\frac{\delta DS_i \wedge \overline{DS_i}}{(\delta + \|S_i\|_i^2)^2}) \\
&= \mu\omega_\delta + \Omega + \sum_{i=1}^m (1 - \beta_i)(\frac{\delta}{\delta + \|S_i\|_i^2}R(\|\cdot\|_i) + \sqrt{-1}\frac{\delta DS_i \wedge \overline{DS_i}}{(\delta + \|S_i\|_i^2)^2}),
\end{aligned}$$

note that $\|S_i\|_i^2\sqrt{-1}\partial\bar{\partial}\log\|S_i\|_i^2 = \|S_i\|_i^2 \cdot 2\pi[D_i] = 0$. We can see that if we have a solution φ_δ for small $\delta > 0$, the Ricci curvature of ω_δ is always greater than μ .

By the computation above, we have a corollary which asserts the openness of the solvable set for the continuity path below:

Lemma 2.2.1. *Consider the continuity path of the equation (2.5)*

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\delta - t\varphi}\omega_0^n, \quad (2.6)$$

and set the interval I_δ as its solvable interval, then $0 \in I_\delta$ and this interval is open.

Proof. $0 \in I_\delta$ follows from Calabi-Yau theorem. By the computation above and [44], it's easy to have $\lambda_1(-\Delta_t)$ is strictly larger than t . Then the openness of I_δ follows. $\square$

So now, to solve the equation (2.5), we need to set up C^0 -estimate for φ_δ . We first consider the cases that $\mu = 0$ and $\mu < 0$. Actually by Calabi-Yau Theorem and Aubin's work (see [50] [1]), we can get C^0 -estimates for these cases. The main difficulty lies in the case $\mu > 0$, which we will deal with in the following sections.

2.3 Twisted functionals for complex Monge-Ampere equations, bounded from below

Following [12] [43] [26], we can still define corresponding functionals for our complex Monge-Ampere equation (2.4). First, we define generalized energy functionals as below, by Aubin:

Definition 2.3.1.

$$(1) J_{\omega_0}(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_0^i \wedge \omega_\varphi^{n-i-1},$$

$$(2) I_{\omega_0}(\varphi) = \frac{1}{V} \int_M \varphi (\omega_0^n - \omega_\varphi^n),$$

where $V = \int_M \omega_0^n$, $\omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$.

Note that these functionals are well defined even in conic case (See [31]). It's easy to check that

$$0 \leq \frac{n+1}{n} J_{\omega_0}(\varphi) \leq I_{\omega_0}(\varphi) \leq (n+1) J_{\omega_0}(\varphi).$$

Next let's define two functionals which are both Lagrangians of the equation (2.4).

For simplicity here we set

$$H_0 = h_0 - \sum_{i=1}^m (1 - \beta_i) \log \|S_i\|_i^2 + c,$$

and we can choose a family φ_t connecting 0 and φ

Definition 2.3.2. (1) We define twisted Ding functional as

$$F_{\omega_0, \mu}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^n - \frac{1}{\mu} \log \left(\frac{1}{V} \int_M e^{H_0 - \mu \varphi} \omega_0^n \right), \quad (2.7)$$

(2) we define twisted Mabuchi functional as

$$\begin{aligned}
\nu_{\omega_0, \mu}(\varphi) &= -\frac{n}{V} \int_0^1 \int_M \dot{\varphi} (\text{Ric}(\omega_\varphi) - \mu \omega_\varphi - \sum_{i=1}^m 2\pi(1 - \beta_i)[D_i] - \Omega \wedge \omega_\varphi^{n-1}) dt \\
&= \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega_0^n} \omega_\varphi^n + \frac{1}{V} \int_M H_0(\omega_0^n - \omega_\varphi^n) - \mu(I_{\omega_0}(\varphi) - J_{\omega_0}(\varphi)) \\
&= \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega_0^n} \omega_\varphi^n + \frac{1}{V} \int_M H_0(\omega_0^n - \omega_\varphi^n) + \mu(F_{\omega_0}^0(\varphi) + \frac{1}{V} \int_M \varphi \omega_\varphi^n),
\end{aligned}$$

where

$$F_{\omega_0}^0(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^n,$$

is defined by Mabuchi [23].

These definitions are similar to smooth case [43] and conic Kähler-Einstein case [26]. We can check that they are well defined for conic case. From [43][26], we know that to get C^0 -estimate for φ_δ , we need to prove the corresponding twisted Ding functional is proper with respect to the generalized energy $J_{\omega_0}(\varphi)$. Now let's recall the definition of properness:

Definition 2.3.3. Suppose the twisted Ding functional $F_{\omega, \mu}(\phi)$ (twisted Mabuchi functional $\nu_{\omega, \mu}(\phi)$) is bounded from below, i.e. $F_{\omega, \mu}(\phi) \geq -c_\omega$ ($\nu_{\omega, \mu}(\varphi) \geq -c_\omega$), we say it is proper on $P_c(M, \omega)$, if there exists an increasing function $f : [-c_\omega, \infty) \rightarrow \mathbb{R}$, and $\lim_{t \rightarrow \infty} f(t) = \infty$, such that for any $\phi \in P_c(M, \omega)$, we have

$$F_{\omega, \mu}(\phi) \geq f(J_\omega(\phi)) \quad (\nu_{\omega, \mu}(\phi) \geq f(J_\omega(\phi))),$$

where $\phi \in P_c(M, \omega)$ is a smooth function on $M \setminus D$ such that $\omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi$ is a conic metric with the prescribed angles along each component of D .

There are a lot of properties for these functionals, which are parallel to [43] [25] [26]. First we just put two basic facts here and the proofs are in [43] [26]:

Proposition 2.3.4. (1) Given a path $\{\phi_t\}$ in $P_c(M, \omega)$, we have

$$\begin{aligned}\frac{d}{dt}J_\omega(\phi_t) &= -\frac{1}{V} \int_M \dot{\phi}_t(\omega_\phi^n - \omega^n), \\ \frac{d}{dt}F_\omega^0(\phi_t) &= -\frac{1}{V} \int_M \dot{\phi}_t \omega_\phi^n,\end{aligned}$$

(2) $F_{\omega, \mu}(\phi)$, $F_\omega^0(\phi)$ and $\nu_{\omega, \mu}(\phi)$ satisfy the cocycle condition:

$$\begin{aligned}F_{\omega, \mu}(\phi) + F_{\omega_\phi, \mu}(\psi - \phi) &= F_{\omega, \mu}(\psi), \\ F_\omega^0(\phi) + F_{\omega_\phi}^0(\psi - \phi) &= F_\omega^0(\psi), \\ \nu_{\omega, \mu}(\phi) + \nu_{\omega_\phi, \mu}(\psi - \phi) &= \nu_{\omega, \mu}(\psi),\end{aligned}$$

We note that in (2), the last two follow directly from differentiation. For $F_{\omega_\phi, \mu}$, we need to choose corresponding function h_ϕ parallel to h_0 in the equation (2.3). Whatever ω_ϕ is smooth or conic along D , we can write $Ric(\omega_\phi) = \mu\omega_\phi + \Omega_\phi$ or $Ric(\omega_\phi) = \mu\omega_\phi + \sum_{i=1}^k 2\pi(1 - \beta_i)[D_i] + \Omega_\phi$, where Ω_ϕ is not necessarily nonnegative. Then all the arguments in smooth case will apply.

From (1) we have a useful corollary, by Ding [11]:

Corollary 2.3.5. For $0 < t < 1$, we have

$$J_\omega(t\phi) \leq t^{\frac{n+1}{n}} J_\omega(\phi).$$

Proof. Consider the path $\{t\phi\}_{0 \leq t \leq 1}$, then we have

$$\frac{d}{dt}J_\omega(t\phi) = -\frac{1}{V} \int_M \phi(\omega_{t\phi}^n - \omega^n) = \frac{I_\omega(t\phi)}{t} \geq \frac{n+1}{n} \frac{J_\omega(t\phi)}{t},$$

integrate this inequality then the corollary follows. $\square$

Now we discuss some relations among these functionals and their behaviors under

different background metrics. First we have a lemma on the generalized energy J_ω , see [26] and [30] for its proof:

Lemma 2.3.6. *suppose $\omega_2 = \omega_1 + \sqrt{-1}\partial\bar{\partial}\varphi$, then for any $\phi \in P_c(M, \omega_1) \cap P_c(M, \omega_2)$, we have*

$$|J_{\omega_1}(\phi) - J_{\omega_2}(\phi)| \leq C(\omega_1, \omega_2).$$

From this lemma and the cocycle property of $F_{\omega, \mu}(\phi), \nu_{\omega, \mu}(\phi)$, we observe that the properties of bounded from below and properness are independent of the choice of metrics in the same Kähler class.

Next we want to know the relation between $F_{\omega, \mu}(\phi), \nu_{\omega, \mu}(\phi)$. We want to prove that these two properties of the two functionals are actually equivalent. Actually these are similar to the proofs by Berman [2] and Li-Sun [26], and we'd like to use the proof in [25] and [31]:

Lemma 2.3.7. *(1) there exists a constant $C > 0$ such that*

$$\nu_{\omega, \mu}(\phi) \geq \mu F_{\omega, \mu}(\phi) - C,$$

(2) suppose ψ solves $\omega_\psi^n = e^{H_0 - \mu\phi}$ by Calabi-Yau theorem, then we have

$$\mu F_{\omega, \mu}(\phi) + \frac{1}{V} \int_M H_0 \omega^n \geq \nu_{\omega, \mu}(\psi),$$

In particular, by (1)(2) we know that $F_{\omega, \mu}$ is bounded from below is equivalent to that $\nu_{\omega, \mu}$ is bounded from below,

(3) in case that $\nu_{\omega, \mu}(\phi) \geq C_1 J_\omega(\phi) - C_2$ where $C_1, C_2 > 0$, there exist constants $c, C' > 0$ such that

$$F_{\omega, \mu}(\phi) \geq c\nu_{\omega, \mu}(\phi) - C'.$$

Proof. (1) We modify the expression of twisted Mabuchi functional in the definition

as below:

$$\begin{aligned}
\nu_{\omega,\mu}(\phi) &= \frac{1}{V} \int_M \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n + \frac{1}{V} \int_M H_0(\omega^n - \omega_\phi^n) + \mu(F_\omega^0(\phi) + \frac{1}{V} \int_M \phi \omega_\phi^n) \\
&= \mu F_{\omega,\mu}(\phi) + \frac{1}{V} \int_M H_0 \omega^n + \frac{1}{V} \int_M \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n - \frac{1}{V} \int_M (H_0 - \mu \phi) \omega_\phi^n \\
&\quad + \log \left(\frac{1}{V} \int_M e^{H_0 - \mu \phi} \omega^n \right) \\
&= \mu F_{\omega,\mu}(\phi) + \frac{1}{V} \int_M H_0 \omega^n + \log \left(\frac{1}{V} \int_M e^{H_0 - \mu \phi - \log \frac{\omega_\phi^n}{\omega^n}} \omega_\phi^n \right) \\
&\quad - \frac{1}{V} \int_M (H_0 - \mu \phi - \log \frac{\omega_\phi^n}{\omega^n}) \omega_\phi^n,
\end{aligned}$$

then (1) follows from the concavity of logarithm.

(2) Still make use of the definition and cocycle property, we have

$$\begin{aligned}
\nu_{\omega,\mu}(\psi) &= \frac{1}{V} \int_M \log \frac{\omega_\psi^n}{\omega^n} \omega_\psi^n + \frac{1}{V} \int_M H_0(\omega^n - \omega_\psi^n) + \mu(F_\omega^0(\psi) + \frac{1}{V} \int_M \psi \omega_\psi^n) \\
&= \frac{1}{V} \int_M (H_0 - \mu \phi) \omega_\psi^n + \frac{1}{V} \int_M H_0(\omega^n - \omega_\psi^n) + \mu(F_\omega^0(\psi) + \frac{1}{V} \int_M \psi \omega_\psi^n) \\
&= \frac{1}{V} \int_M H_0 \omega^n + \mu \left(F_\omega^0(\phi) - F_{\omega_\psi}^0(\phi - \psi) + \frac{1}{V} \int_M (\psi - \phi) \omega_\psi^n \right) \\
&= \frac{1}{V} \int_M H_0 \omega^n + \mu \left(F_{\omega,\mu}(\phi) + \log \left(\frac{1}{V} \int_M e^{H_0 - \mu \phi} \omega^n \right) - J_{\omega_\psi}(\phi - \psi) \right),
\end{aligned}$$

then (2) follows from that $e^{H_0 - \mu \phi} \omega^n = \omega_\psi^n$ and $J_{\omega_\psi}(\phi - \psi) \geq 0$.

(3) From the assumption, we have a small $\delta > 0$ such that $\nu_{\omega,\mu+\delta}(\phi) = \nu_{\omega,\mu}(\phi) - \delta(I - J)_\omega(\phi)$ is bounded from below, so is $F_{\omega,\mu+\delta}(\phi)$ by (2). Then we can compute that

$$\begin{aligned}
F_{\omega,\mu}(\phi) &= F_\omega^0(\phi) - \frac{\mu + \delta}{\mu} \frac{1}{\nu + \delta} \log \left(\frac{1}{V} \int_M e^{H_0 - (\mu + \delta) \frac{\mu}{\mu + \delta} \phi} \omega^n \right) \\
&= F_\omega^0(\phi) + \frac{\mu + \delta}{\mu} (F_{\omega,\mu+\delta}(\frac{\mu}{\mu + \delta} \phi) - F_\omega^0(\frac{\mu}{\mu + \delta} \phi)) \\
&\geq J_\omega(\phi) - \frac{\mu + \delta}{\mu} J_\omega(\frac{\mu}{\mu + \delta} \phi) - C' \\
&\geq (1 - (\frac{\mu}{\mu + \delta})^{\frac{1}{n}}) J_\omega(\phi) - C',
\end{aligned}$$

where the last inequality follows from Corollary 2.3.5. $\square$

To prove properness of the functionals in case of the existence of the conic metric $\omega = \omega_\varphi$, we need to verify that they are bounded from above:

Theorem 2.3.8. *If the singular Monge-Ampere equation (2.4) has a solution φ , i.e there exists a conic Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfying the equation (2.2), then φ attains the minimum of the functional $F_{\omega_0, \mu}$ on the space $P_c(M, \omega_0)$. In particular $F_{\omega_0, \mu}$ is bounded from above.*

Proof. A parallel result is proved in [26], but we'd like to extend Ding-Tian's proof [12] [43] to our conic case. Let's consider the continuity path of the complex Monge-Ampere equation:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{H_0 - t\varphi_t}\omega_0^n. \quad (2.8)$$

Now we know that when $t = \mu$ this equation is solvable. Actually by [3], we know that it is also solvable when $t = 0$. Now when $0 < t < \mu$, by implicit function theorem, we need to consider whether the linearized operator of the (2.8), $\Delta_t + t$, is invertible. We know that in smooth case, by Bochner's formula, as $Ric(\omega_t) > t\omega_t$, it is invertible and we can prove the openness of solvable set for t . However, in conic case, [21] gives a parallel result. By their argument, we have Δ_t as the Friedrich extension of the Laplacian associated to $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$ and $\lambda_1(-\Delta_t) > t$, so the openness is true. We can set $\{\varphi_t\}$ as a continuous family of solutions of (2.8), then we can do computations as [43] in a weak sense.

First take the derivative of (2.8) with respect to t , we have

$$\Delta_t \dot{\varphi}_t = -\varphi_t - t\dot{\varphi}_t,$$

where Δ_t is in a weak sense as [21]. As for all t , we have $\int_M e^{H_0 - t\varphi_t}\omega_0^n = V$, take the

derivative with respect to t we get

$$\int_M (\varphi_t + t\dot{\varphi}_t) e^{H_0 - t\varphi_t} \omega_0^n = 0.$$

Now make use of the formulas in the beginning of this section, we have

$$\begin{aligned} \frac{d}{dt}(I_{\omega_0}(\varphi_t) - J_{\omega_0}(\varphi_t)) &= \frac{1}{V} \int_M \dot{\varphi}_t (\omega_0^n - \omega_t^n) - \frac{1}{V} \int_M \varphi_t \Delta_t \dot{\varphi}_t \omega_t^n - \frac{1}{V} \int_M \dot{\varphi}_t (\omega_0^n - \omega_t^n) \\ &= \frac{1}{V} \int_M \varphi_t (\varphi_t + t\dot{\varphi}_t) \omega_t^n \\ &= -\frac{d}{dt} \left(\frac{1}{V} \int_M \varphi_t e^{H_0 - t\varphi_t} \omega_0^n \right) + \frac{1}{V} \int_M \dot{\varphi}_t e^{H_0 - t\varphi_t} \omega_0^n \\ &= -\frac{d}{dt} \left(\frac{1}{V} \int_M \varphi_t \omega_t^n \right) - \frac{1}{tV} \int_M \varphi_t \omega_t^n. \end{aligned}$$

From this, we have

$$\frac{d}{dt} (t(I_{\omega_0}(\varphi_t) - J_{\omega_0}(\varphi_t))) - (I_{\omega_0}(\varphi_t) - J_{\omega_0}(\varphi_t)) = -\frac{d}{dt} \left(\frac{1}{V} \int_M \varphi_t \omega_t^n \right), \quad (2.9)$$

Integrating this from 0 to t , we have

$$t(I_{\omega_0}(\varphi_t) - J_{\omega_0}(\varphi_t)) - \int_0^t (I_{\omega_0}(\varphi_s) - J_{\omega_0}(\varphi_s)) ds = -\frac{t}{V} \int_M \varphi_t \omega_t^n.$$

By the definition, it's just

$$-\int_0^t (I_{\omega_0}(\varphi_s) - J_{\omega_0}(\varphi_s)) ds = t(J_{\omega_0}(\varphi_t) - \frac{1}{V} \int_M \varphi_t \omega_0^n) = tF_{\omega_0}^0(\varphi_t). \quad (2.10)$$

As we have $\int_M e^{H_0 - \mu\varphi} \omega_0^n = V$, we derive that $F_{\omega_0, \mu}(\varphi) \leq 0$.

Now let's consider $\phi \in P_c(M, \omega_0)$. If $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ is smooth, then we have

$$Ric(\omega_\phi) = \mu\omega_\phi + \Omega_\phi,$$

where Ω_ϕ is not necessarily nonnegative. Compare it with (2.2), we have

$$(\omega_\phi + \sqrt{-1}\partial\bar{\partial}(\varphi - \phi))^n = e^{h_\phi - \mu(\varphi - \phi) - \sum_{i=1}^m (1 - \beta_i) \log \|S_i\|_i^2 + c_\phi} \omega_\phi^n,$$

where we take $\sqrt{-1}\partial\bar{\partial}h_\phi = \Omega_\phi - \Omega - \sum_{i=1}^m (1 - \beta_i)(R(\|\cdot\|_i))$. Then all the arguments are parallel and we have $F_{\omega_\phi, \mu}(\varphi - \phi) \leq 0$. Now let's consider the case when ω_ϕ is conic along D. Here we have the equation

$$Ric(\omega_\phi) = \mu\omega_\phi + \sum_{i=1}^k 2\pi(1 - \beta_i)[D_i] + \Omega_\phi.$$

Compare it with (2.2), we have

$$(\omega_\phi + \sqrt{-1}\partial\bar{\partial}(\varphi - \phi))^n = e^{h_\phi - \mu(\varphi - \phi) + c_\phi} \omega_\phi^n,$$

where we have $\sqrt{-1}\partial\bar{\partial}h_\phi = \Omega_\phi - \Omega$. In this case, all the arguments are similar to smooth case and we get the same conclusion. Now by cocycle condition, we have

$$F_{\omega_0, \mu}(\phi) = F_{\omega_0, \mu}(\varphi) - F_{\omega_\phi, \mu}(\varphi - \phi) \geq F_{\omega_0, \mu}(\varphi).$$

This theorem is proved. □

2.4 Log α -invariant, properness of twisted energies

We want to prove the properness of the twisted Ding energy. First we want to introduce log α -invariant, then let's see how to use this invariant to prove the properness of twisted Mabuchi energy in case that μ is small. Then make use of concavity of energies to prove the properness of energies in general case.

Recall that α -invariant in smooth case was introduced by Tian [39] in 1980s. In [2] this invariant is generalized to conic case. We introduce so-called log α -invariant here,

following [26]:

Definition 2.4.1. Fix a smooth volume form vol , for any Kähler class $[\omega]$, we define $\log \alpha$ -invariant as below:

$$\alpha(\omega, D) = \sup\{\alpha > 0 : \exists C_\alpha < \infty \quad s.t. \quad \frac{1}{V} \int_M e^{\alpha(\sup \phi - \phi)} \frac{vol}{\prod_{i=1}^m |S_i|^{2(1-\beta_i)}} \leq C_\alpha$$

$$for \text{ any } \phi \in P_c(M, \omega)\}.$$

Berman [2] has an estimate for the positive lower bound of $\log \alpha$ -invariant in conic case, i.e, there exists a positive number α_0 such that $\alpha(\omega, D) \geq \alpha_0 > 0$. Using this estimate, we can prove that the twisted Mabuchi energy is proper when μ is small enough:

Theorem 2.4.2. Suppose $\alpha(\omega, D) \geq \alpha_0 > \frac{n}{n+1}\mu > 0$, then we have

$$\nu_{\omega_0, \mu}(\phi) \geq \epsilon J_{\omega_0}(\phi) - C,$$

where ϵ, C are constants depending on α_0, μ .

Proof. Similar to [21] [26] [43], make use of logarithm property, for $\frac{n}{n+1}\mu < \alpha < \alpha_0$, we have

$$\begin{aligned} \log C_\alpha &\geq \log \left(\frac{1}{V} \int_M e^{\alpha(\sup \phi - \phi)} \frac{e^{H_0} \omega_0^n}{\prod_{i=1}^m |S_i|^{2(1-\beta_i)}} \right) \\ &\geq \log \left(\frac{1}{V} \int_M e^{\alpha(\sup \phi - \phi) - \log \frac{\prod_{i=1}^m |S_i|^{2(1-\beta_i)} \omega_\phi^n}{\omega_0^n} + H_0} \omega_\phi^n \right) \\ &\geq \frac{1}{V} \int_M \left(H_0 - \frac{\prod_{i=1}^m |S_i|^{2(1-\beta_i)} \omega_\phi^n}{\omega_0^n} \right) \omega_\phi^n + \frac{\alpha}{V} \int_M (\sup \phi - \phi) \omega_\phi^n \\ &\geq \frac{1}{V} \int_M \left(H_0 - \frac{\prod_{i=1}^m |S_i|^{2(1-\beta_i)} \omega_\phi^n}{\omega_0^n} \right) \omega_\phi^n + \alpha I_{\omega_0}(\phi). \end{aligned}$$

By the definition of twisted Mabuchi energy, we have

$$\begin{aligned}
\nu_{\omega_0, \mu}(\phi) &= \frac{1}{V} \int_M \log \frac{\omega_\phi^n}{\omega_0^n} \omega_\phi^n + \frac{1}{V} \int_M H_0(\omega_0^n - \omega_\phi^n) - \mu(I_{\omega_0}(\phi) - J_{\omega_0}(\phi)) \\
&\geq \log C_\alpha + \frac{1}{V} \int_M H_0 \omega_0^n + \alpha I_{\omega_0}(\phi) - \mu(I_{\omega_0}(\phi) - J_{\omega_0}(\phi)) \\
&\geq (\alpha - \frac{n}{n+1} \mu) I_{\omega_0}(\phi) - C \\
&\geq (\frac{n+1}{n} \alpha - \mu) J_{\omega_0}(\phi) - C.
\end{aligned}$$

The proof is finished. $\square$

As the equivalence of the properness of twisted Ding energy and Mabuchi energy, we have such a corollary very easily:

Corollary 2.4.3. *When $\alpha(\omega, D) \geq \alpha_0 > \frac{n}{n+1} \mu > 0$, we have*

$$F_{\omega_0, \mu}(\phi) \geq \epsilon J_{\omega_0}(\phi) - C,$$

where ϵ, C are constants depending on α_0, μ .

Until now we only get the properness when μ is small enough. For general case, we need to apply continuity method and the concavity property of the energy which is shown below to increase μ . Here is a lemma which allows us to increase μ and see also in [26]:

Lemma 2.4.4. *Suppose $0 < \mu_0 < \mu_1$, write $\mu = (1-t)\mu_0 + t\mu_1$ where $0 \leq t \leq 1$, we have*

$$\mu F_{\omega_0, \mu}(\phi) \geq (1-t)\mu_0 F_{\omega_0, \mu_0}(\phi) + t\mu_1 F_{\omega_0, \mu_1}(\phi).$$

Proof. It follows from the concavity of the function $f(h) = -\log(\frac{1}{V} \int_M e^{-h} \omega_0^n)$. $\square$

Now we can prove our main theorem in this section and similar results also appear in [26] [44]:

Theorem 2.4.5. *For $t \in (0, \mu]$ and any $\phi \in P_c(M, \omega_0)$ there exist constants ϵ, C_ϵ such that*

$$F_{\omega_0, t}(\phi) \geq \epsilon J_{\omega_0}(\phi) - C_\epsilon. \quad (2.11)$$

Proof. We apply the continuity path similar to [21], i.e, the equation (2.8). In our case, we may assume that $\Omega \neq 0$. Then by [21], we have that $\lambda_1(-\Delta_t) > t$ for all $t \in (0, \mu]$, which allows us to prove the openness at $t = \mu$. So now when $\bar{\mu} = \mu + \delta$ where δ is very small, we have a solution $\bar{\varphi}$ for (2.8) where μ is replaced by $\bar{\mu}$. By theorem 2.3.8, $F_{\omega_0, \bar{\mu}}(\phi)$ is bounded from below. As we have the corollary above which asserts that when $t > 0$ is very small $F_{\omega_0, t}(\phi)$ is proper, by the lemma above, we know that for all $t \in (0, \mu]$ the twisted Ding energy is proper, i.e.

$$F_{\omega_0, t}(\phi) \geq \epsilon J_{\omega_0}(\phi) - C_\epsilon. \quad (2.12)$$

We finish the proof. □

2.5 C^0 -estimate for approximating solution: the case $\mu > 0$

Recall that in Section 2 we set up the approximating complex Monge-Ampere equation (2.5), which is expected to give us a smooth approximation of conic Kähler metric $\omega = \omega_0 + \varphi$. We also prove C^0 -estimate for φ_δ when $\mu \leq 0$. In this section, we want to make use of the properness of corresponding energy functionals to prove C^0 -estimate when $\mu > 0$. Then the first step is to prove the properness of the new approximating twisted Ding energy, which can be deduced from the last section:

Lemma 2.5.1. *We introduce the new approximating twisted Ding energy as below:*

$$F_{\delta,t}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^n - \frac{1}{t} \log \left(\frac{1}{V} \int_M e^{h_\delta - t\varphi} \omega_0^n \right), \quad (2.13)$$

which is the Lagrangian of the approximating complex Monge-Ampere equation (2.6) in the continuity path. Then we have

$$F_{\delta,t}(\phi) \geq \epsilon J_{\omega_0}(\phi) - C(\epsilon, \delta, t)$$

.

Proof. In the end of last section, we proved that

$$F_{\omega_0,t}(\phi) \geq \epsilon J_{\omega_0}(\phi) - C_\epsilon.$$

Note that

$$\begin{aligned} h_\delta &= h_0 - \sum_{i=1}^m (1 - \beta_i) \log(\delta + \|S_i\|_i^2) + c_\delta \\ &\leq h_0 - \sum_{i=1}^m (1 - \beta_i) \log \|S_i\|_i^2 + c_\delta = H_0 - c + c_\delta, \end{aligned}$$

We have

$$F_{\delta,t}(\phi) \geq F_{\omega_0,t}(\phi) + \frac{c - c_\delta}{t},$$

and the lemma follows very easily. $\square$

Now we will follow [43] to finish C^0 -estimate for φ_δ . Similar to the equation (2.10), we have

$$- \int_0^t (I_{\omega_0}(\varphi_{\delta,s}) - J_{\omega_0}(\varphi_{\delta,s})) ds = t(J_{\omega_0}(\varphi_{\delta,t}) - \frac{1}{V} \int_M \varphi_{\delta,t} \omega_0^n) = tF_{\omega_0}^0(\varphi_{\delta,t}),$$

where $\varphi_{\delta,t}$ solves the equation (2.6). By this equation, we can estimate $F_{\delta,\mu}(\varphi_{\delta,t})$ that

$$\begin{aligned}
F_{\delta,\mu}(\varphi_{\delta,t}) &= F_{\omega_0}^0(\varphi_{\delta,t}) - \log \left(\frac{1}{V} \int_M e^{h_\delta - \mu \varphi_{\delta,t}} \omega_0^n \right) \\
&\leq -\log \left(\frac{1}{V} \int_M e^{h_\delta - t \varphi_{\delta,t} - (\mu-t) \varphi_{\delta,t}} \omega_0^n \right) \\
&= -\log \left(\frac{1}{V} \int_M e^{-(\mu-t) \varphi_{\delta,t}} \omega_{\delta,t}^n \right) \\
&\leq \frac{\mu-t}{\mu} \frac{1}{V} \int_M \varphi_{\delta,t} \omega_{\delta,t}^n.
\end{aligned}$$

To finish the estimate, we need a useful lemma as below:

Lemma 2.5.2. $\|\varphi_{\delta,t}\|_{C^0} \leq C(1 + J_{\omega_0}(\varphi_{\delta,t}))$.

Proof. First we note that $Ric(\omega_{\delta,t}) > t$, and the volume preserved. Then we have uniform Sobolev and Poincaré constants when t doesn't tend to 0. We observe that $n + \Delta_0 \varphi_{\delta,t} > 0$, then we get

$$0 \leq \sup \varphi_{\delta,t} \leq \frac{1}{V} \int_M \varphi_{\delta,t} \omega_0^n + C$$

by Green's formula. On the other hand, we have $n - \Delta_{\delta,t} \varphi_{\delta,t} > 0$, by Moser's iteration, we have

$$-\inf \varphi_{\delta,t} \leq -\frac{C}{V} \int_M \varphi_{\delta,t} \omega_{\delta,t}^n + C.$$

By normalization condition, $\varphi_{\delta,t}$ changes sign, we have

$$\|\varphi_{\delta,t}\|_{C^0} \leq \sup \varphi_{\delta,t} - \inf \varphi_{\delta,t} \leq C(1 + I_{\omega_0}(\varphi_{\delta,t})) \leq C(1 + J_{\omega_0}(\varphi_{\delta,t})).$$

□

In the proof we have

$$0 \leq -\inf \varphi_{\delta,t} \leq -\frac{C}{V} \int_M \varphi_{\delta,t} \omega_{\delta,t}^n + C,$$

then we have

$$\frac{1}{V} \int_M \varphi_{\delta,t} \omega_{\delta,t}^n \leq C,$$

which gives $F_{\delta,\mu}(\varphi_{\delta,t}) \leq C$. Combine the two lemmas above, we conclude the C^0 -estimate for φ_δ and get the following theorem similar to Tian [42] [44]:

Theorem 2.5.3. *For each $\delta > 0$, the approximating complex Monge-Ampere equation (2.5) has a unique smooth solution φ_δ , which gives us a smooth Kähler metric $\omega_\delta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\delta$ such that $\text{Ric}(\omega_\delta) \geq \mu\omega_\delta$.*

2.6 Convergence when δ tends to 0

In the last section we proved C^0 -estimate for φ_δ . We also note that in the approximating complex Monge-Ampere equation (2.5), the constant c_δ is uniformly bounded. Then the constant $C(\epsilon, \delta, t)$ in Lemma 5.1 is uniform with respect to δ . According to this observation, we conclude that our C^0 -estimate for φ_δ is uniform with respect to δ , i.e, $\sup |\varphi_\delta| \leq C_0$. Based on this, first we can give C^2 -estimate for φ_δ by Chern-Lu Inequality (see [21]):

Lemma 2.6.1.

$$C_1\omega_0 \leq \omega_\delta \leq \frac{C_2\omega_0}{\prod_{i=1}^m (\delta + \|S_i\|^2)^{(1-\beta_i)}}. \quad (2.14)$$

Proof. First we have $\sup |\varphi_\delta| \leq C_0$ and $\text{Ric}(\omega_\delta) \geq \mu\omega_\delta$. Take Δ as the Laplacian for ω_δ and normal coordinate around a point p for ω_δ , i.e, $g_{i\bar{j}}(p) = \delta_{ij}$, $dg_{i\bar{j}}(p) = 0$. We may also take $g_{0i\bar{j}}(p) = g_{0i\bar{i}}\delta_{ij}$, i.e, diagonal for ω_0 , then we can compute that

$$\begin{aligned} \Delta \text{tr}_{\omega_\delta} \omega_0 &= g^{i\bar{i}}(g^{k\bar{l}}g_{0k\bar{l}})_{i\bar{i}} = g^{i\bar{i}}(g^{k\bar{k}})_{i\bar{i}}g_{0k\bar{k}} + g^{i\bar{i}}g^{k\bar{k}}(g_{0k\bar{k}})_{i\bar{i}} \\ &= g^{i\bar{i}}R_{i\bar{i}}{}^{k\bar{k}}(g)g_{0k\bar{k}} - g^{i\bar{i}}g^{k\bar{k}}R_{i\bar{i}k\bar{k}}(g_0) + g^{i\bar{i}}g^{k\bar{k}}g^{l\bar{l}}(g_{0k\bar{l}})_i(g_{0l\bar{k}})_{\bar{i}} \\ &= R^{k\bar{k}}g_{0k\bar{k}} - g^{i\bar{i}}g^{k\bar{k}}R_{i\bar{i}k\bar{k}}(g_0) + g_0^{i\bar{i}}g^{k\bar{k}}g^{l\bar{l}}(g_{0k\bar{l}})_i(g_{0l\bar{k}})_{\bar{i}} \\ &\geq -g^{i\bar{i}}g^{k\bar{k}}R_{i\bar{i}k\bar{k}}(g_0) + g_0^{i\bar{i}}g^{k\bar{k}}g^{l\bar{l}}(g_{0k\bar{l}})_i(g_{0l\bar{k}})_{\bar{i}}, \end{aligned}$$

and the last inequality follows from $Ric(\omega_\delta) \geq \mu\omega_\delta$. Now we have

$$\begin{aligned}\Delta \log tr_{\omega_\delta} \omega_0 &= \frac{\Delta tr_{\omega_\delta} \omega_0}{tr_{\omega_\delta} \omega_0} - \frac{|\nabla tr_{\omega_\delta} \omega_0|^2}{|tr_{\omega_\delta} \omega_0|^2} \\ &\geq \frac{(tr_{\omega_\delta} \omega_0) g_0^{i\bar{i}} g^{k\bar{k}} g^{l\bar{l}} (g_{0k\bar{l}})_i (g_{0l\bar{k}})_{\bar{i}} - g^{i\bar{i}} g^{k\bar{k}} g^{l\bar{l}} (g_{0k\bar{k}})_i (g_{0l\bar{l}})_{\bar{i}}}{|tr_{\omega_\delta} \omega_0|^2} \\ &\quad - \frac{g^{i\bar{i}} g^{k\bar{k}} R_{i\bar{i}k\bar{k}}(g_0)}{tr_{\omega_\delta} \omega_0} \geq -a tr_{\omega_\delta} \omega_0,\end{aligned}$$

where the bisectional curvature of ω_0 is less than a and the last inequality follows from $g_0^{i\bar{i}} tr_{\omega_\delta} \omega_0 \geq g^{i\bar{i}}$. As we have $\sup |\varphi_\delta| \leq C_0$, we take $u = \log tr_{\omega_\delta} \omega_0 - (a+1)\varphi_\delta$, then we will have

$$\Delta u \geq tr_{\omega_\delta} \omega_0 - n(a+1) = e^{u+n(a+1)} - n(a+1).$$

By maximal principle $u \leq C(a)$, and we then get $tr_{\omega_\delta} \omega_0 \leq C'$, which will give us that $C_1 \omega_0 \leq \omega_\delta$. For the other side, make use of the complex Monge-Ampere equation (2.5) and the inequality we get, we can easily deduce that

$$\omega_\delta \leq \frac{C_2 \omega_0}{\prod_{i=1}^m (\delta + \|S_i\|^2)^{(1-\beta_i)}}.$$

□

From this lemma, by C^3 -estimate in [50] (or see [43]) and regularity theory we can prove that for any $l > 2$ and compact set $K \in M \setminus D$, there exists a uniform constant $C(l, K)$ independent of δ such that:

$$\|\varphi_\delta\|_{C^l(K)} \leq C(l, K). \quad (2.15)$$

As we have got all the estimates we need, we can prove the main theorem below, following [44]:

Theorem 2.6.2. *As δ tends to 0, the smooth Kähler metric ω_δ converge to the conic Kähler metric ω in the Gromov-Hausdorff topology on M and in the smooth topology outside the divisor D .*

Proof. In this proof, we first consider the case that D is an irreducible divisor. As we have high order estimates (2.14) (2.15) outside the divisor D , it suffices to prove ω_δ converges to ω in the Gromov-Hausdorff topology. Note that for all ω_δ we have $Ric(\omega_\delta) \geq \mu, Vol(M, \omega_\delta) = V$, to apply compactness theorem of Cheeger-Gromov (e.g. see Chapter 10 in [29]), we only need to bound the diameter for all ω_δ . In case that $\mu > 0$ we can get it directly by Meyer's theorem. However, as we have the estimate (2.14), it's easy to control the length of arbitrary geodesics outside the divisor. And in the neighborhood of some irreducible divisor, say D , we make use of local coordinates and set $r = |z_1|$, where $\{z_1 = 0\}$ locally defines the divisor D . Now we know that $\|S\|$ here is almost r near the divisor and we consider the length of a short geodesic γ transverse to D that:

$$L(\gamma, \omega_\delta) \approx C \int_0^{r_0} \frac{dr}{(\delta + r^2)^{\frac{1-\beta}{2}}} \leq C \int_0^{r_0} \frac{dr}{r^{1-\beta}} \leq \frac{Cr_0^\beta}{\beta}.$$

Along the geodesics almost tangential to D we almost have $dz_1 = 0$ so in all cases the diameter with respect to ω_δ is uniformly bounded. Now by compactness theorem, without loss of generality, (M, ω_δ) converge to a length space $(\bar{M}, \bar{d})$ in Gromov-Hausdorff topology. To prove the theorem we need to prove that $(\bar{M}, \bar{d})$ coincides with (M, ω) . As we have high order estimate (2.15) outside the divisor D , there exists an open set U in $\bar{M}$ which is equivalent to $M \setminus D$, and the equivalence $i : M \setminus D \rightarrow U$ induces an isometry between $(M \setminus D, \omega|_{M \setminus D})$ and $(U, \bar{d})$. Now we note that $M \setminus D$ is geodesically convex with respect to ω , i.e. any two points $p, q \in M \setminus D$, there exists a minimal geodesic $\gamma \subset M \setminus D$ joining them. Actually we only need to consider the case when p, q are in the small neighborhood of $o \in D$. In this case we know that the

metric ω is almost the standard conic metric around a point $o \in D$, which behaviors like

$$\omega_{o,c} = \sqrt{-1} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right).$$

Now we assume that $|z_1(p)| = |z_1(q)| = \epsilon$, $|z_i(p)|, |z_i(q)| \approx \epsilon$ where $\epsilon > 0$ is small enough and $2 \geq i \geq n$. First we choose the segment connecting p and q across the point $o \in D$. By the estimate above we know that

$$d(p, o) + d(o, q) \approx \frac{2\epsilon^\beta}{\beta}.$$

On the other hand we choose a segment γ' whose projection on z_1 coordinate is almost a geodesic in the cone with angle β , by standard computation we know that

$$L(\gamma') \approx C\epsilon + 2 \sin \frac{\pi\beta}{2} \frac{\epsilon^\beta}{\beta}.$$

As ϵ is small and $\beta < 1$, we conclude that the geodesic connecting p and q doesn't cross the point $o \in D$. In general case we only need to choose p', q' as the case above to replace p, q and connect p, p' and q, q' respectively then all the argument follows. As $M \setminus D$ is geodesically convex, by the C^2 -estimate in (2.14), we can estimate as above to show that for each point $o \in D$, a radical short line connecting o and a point outside the divisor is always rectifiable and absolutely continuous with respect to local coordinates, thus we can see that M is the metric completion of $M \setminus D$. Moreover, the equivalence i extends to a Lipschitz map from (M, ω) onto $(\bar{M}, \bar{d})$ (we still denote this map as i) and the Lipschitz constant is 1. What remains to do is to prove i is an isometry between (M, ω) and $(\bar{M}, \bar{d})$. As $(\bar{M}, \bar{d})$ is a metric completion of $M \setminus D$, we only need to prove that for $p, q \in M \setminus D$,

$$d_\omega(p, q) = \bar{d}(i(p), i(q)).$$

First we observe that $\bar{D} = i(D)$ is the Gromov-Hausdorff limit of D under the convergence of (M, ω_δ) to $(\bar{M}, \bar{d})$, whose Hausdorff measure is 0, by the C^2 -estimate in (2.14). Now we only need to prove that for any $\bar{p}, \bar{q} \in \bar{M} \setminus \bar{D}$ there exists a minimizing geodesic $\gamma \subset \bar{M} \setminus \bar{D}$ joining $\bar{p}, \bar{q}$. If not, we will have

$$\bar{d}(\bar{p}, \bar{q}) < d_\omega(p, q),$$

where $\bar{p} = i(p), \bar{q} = i(q)$. Then there exists a small $r > 0$ such that:

- (1) $B_r(\bar{p}, \bar{d}) \cap \bar{D} = \emptyset, B_r(\bar{q}, \bar{d}) \cap \bar{D} = \emptyset$, where $B_r(\cdot, \bar{d})$ is a geodesic ball in $(\bar{M}, \bar{d})$;
 - (2) $\bar{d}(\bar{x}, \bar{y}) < d_\omega(x, y)$, where $\bar{x} = i(x) \in B_r(\bar{p}, \bar{d})$ and $\bar{y} = i(y) \in B_r(\bar{q}, \bar{d})$. From these two we know that any minimizing geodesic γ connecting $\bar{x}$ and $\bar{y}$ intersects with $\bar{D}$.
- As $r > 0$ is small, and i is an isometry outside the divisor D , we have

$$B_r(\bar{p}, \bar{d}) = i(B_r(p, \omega)), \quad B_r(\bar{q}, \bar{d}) = i(B_r(q, \omega)).$$

Choosing a small tubular neighborhood T of D in M whose closure is disjoint from both $B_r(p, \omega)$ and $B_r(q, \omega)$. When the radius of such tubular is small enough we can make $Vol(\partial T)$ arbitrary small. Now we can choose $p_\delta, q_\delta \in M$ and neighborhood T_δ of D with respect to ω_δ such that as $\delta \rightarrow 0$, $p_\delta, q_\delta, T_\delta$ converge to $\bar{p}, \bar{q}, i(T)$ in Gromov-Hausdorff topology. By volume convergence theorem of Colding,

$$\lim_{\delta \rightarrow 0+} Vol(\partial T_\delta, \omega_\delta) = Vol(\partial T, \omega),$$

so $Vol(\partial T_\delta, \omega_\delta)$ can be also arbitrary small as $\delta \rightarrow 0$. Also by convergence, when δ is small enough, $B_r(p_\delta, \omega_\delta), B_r(q_\delta, \omega_\delta)$ and T_δ are mutually disjoint. By (2), any minimizing geodesic γ_δ connecting any $w \in B_r(p_\delta, \omega_\delta)$ and $z \in B_r(q_\delta, \omega_\delta)$ intersects with T_δ . Now we need an estimate due to Gromov:

Lemma 2.6.3. $c(\mu)r^{2n} \leq Vol(B_r(q_\delta, \omega_\delta), \omega_\delta) \leq C(L, \mu, n, r)Vol(\partial T_\delta, \omega_\delta)$, where $L =$

$\bar{d}(\bar{p}, \bar{q})$.

Proof. The first inequality follows from the Ricci lower bound and Gromov's relative volume comparison theorem directly. For the second inequality, by Chap.9 in [29], we set $\lambda(t, \theta)$ as the volume density function where t is the distance from p_δ . We also set $\lambda_k(t, \theta)$ as the standard volume density function of the space form with constant curvature $k = \frac{\mu}{n-1}$. By the argument in [29] we know that the map

$$t \rightarrow \frac{\lambda(t, \theta)}{\lambda_k(t, \theta)}$$

is nonincreasing in t . In our case, we consider the geodesics from p_δ to $z \in B_r(q_\delta, \omega_\delta)$. According to the construction, we have $r < d(p_\delta, z_T) < d(p_\delta, z)$, $L - r < d(p_\delta, z) < L + r$, where z_T is the intersection point of the geodesics from p_δ to z and ∂T_δ , and $L \approx d(p_\delta, q_\delta)$. Along ∂T_δ , we have

$$\frac{\lambda(z_T)}{\lambda_k(z_T)} \geq \frac{\lambda(z)}{\lambda_k(z)}.$$

Let $S \in S^{2n-1}$ and $C(S)$ denote the part which all the geodesics from p_δ to $z \in B_r(q_\delta, \omega_\delta)$ lie in and corresponding geodesic cone, and set $t(\theta)$ as the distance from p_δ to each point of ∂T_δ , then we have

$$\begin{aligned} Vol(\partial T_\delta) &\geq \int_{\partial T_\delta \cap C(S)} \lambda(t, \theta) = \int_S t^{2n-1}(\theta) \lambda(t, \theta) d\theta \\ &\geq \int_S \lambda(L') \frac{\lambda_k(t(\theta))}{\lambda_k(L')} t^{2n-1}(\theta) d\theta \geq C \int_S \lambda(L') L'^{2n-1} d\theta, \end{aligned}$$

where $L - r < L' < L + r$. Now we take the integral of this inequality, we will have that

$$Vol(\partial T_\delta) \geq \frac{2C}{r} \int_{L-r}^{L+r} \int_S \lambda(L') L'^{2n-1} d\theta dt \geq C(L, \mu, n, r) Vol(B_r(q_\delta, \omega_\delta), \omega_\delta).$$

Then the lemma follows. $\square$

As we have known that $Vol(\partial T_\delta, \omega_\delta)$ can be also arbitrary small as δ tends to 0, the lemma above leads to a contradiction. Then i can extend to a isometry from (M, ω) onto $(\bar{M}, \bar{d})$, and the theorem follows when D is irreducible. In case that D has a simple normal crossing, we only need to see that near the crossing point, we can project the geodesic to the direction corresponding to each irreducible divisor, then by the argument above in each direction, the geodesic will not cross the divisor. Hence in general case the theorem still follows. $\square$

Chapter 3

Smoothing conic metric II: Upper bound of bisectional curvature

3.1 Introduction

Conic Kähler metric is very useful in the study of Kähler geometry. However, as it is singular near the divisor, in applications, it is useful to consider smoothing metrics. In last chapter, we can see that to apply Cheeger-Colding-Tian theory to study the compactness of conical Kähler-Einstein metrics, we need to approximate them by smooth Kähler metrics with the same lower Ricci curvature bound. To study different problems we may need different approximating methods. In [4] [15], Campana-Guenancia-Paun and Guenancia-Paun developed a smoothing method of conic metrics to study existence problems of conical Kähler-Einstein metrics. As their approximating metrics have only lower bisectional curvature bounds depending on approximating coefficient ϵ , they need to construct some auxiliary functions to get control in the Aubin-Yau Inequality and obtain the Laplacian estimate. In [3] and [21], they proved that for the case of irreducible divisor with cone angle $\beta \leq \frac{1}{2}$ the bisectional curvature is uniformly bounded away from the divisor. In the appendix of

[21], C. Li and Y. Rubinstein proved that there exists an upper bisectional curvature bound in case of an irreducible divisor without cone angle condition, and consequently the Laplacian estimate in [21] follows naturally from this estimate by use of Chern-Lu Inequality. Moreover, Y. Rubinstein proved that this upper bisectional curvature bound exists for some special cases of simple normal crossing divisors. In his survey [31], he gave a simple proof, due to J. Sturm.

But it is still not easy to do analysis on conic metric directly although we have such upper bound estimate. For convenience of analysis, in this chapter, we will prove Theorem 1.2.2, i.e, construct smoothing sequence of the conic metric $\omega = \omega_0 + \sum_{i=1}^m \sqrt{-1} \partial \bar{\partial} ||S_i||_i^{2\beta_i}$ with uniformly upper bisectinal curvature bound in some special cases of simple normal crossing divisors. This will not only give us smoothing metrics, but also the uniformly upper bisectional curvature bound globally, even on divisors! By computation, we note that the principle part of the metric tensors are $\frac{1}{||S_i||_i^{2(1-\beta_i)}}, i = 1, \dots, m$. One natural smoothing construction is to construct the metric tensors with the principle part $\frac{1}{(||S_i||_i^2 + \epsilon^2)^{(1-\beta_i)}}, i = 1, \dots, m$. This is just what Campana-Guenancia-Paun and Guenancia-Paun did in their works. Unfortunately, this smoothing sequence does not have uniformly upper or lower bisectional curvature bound near the divisors. To overcome this difficulty, it is necessary to make cut-off for the potential functions near the divisors.

As in the appendix of [21], we denote $\hat{g}, g$ as the Kähler metrics associated to ω_0, ω . Here we only deal with the most complicated case that the point we consider is near the intersection of all divisors. For other cases the estimates will be similar but easier. First we extend Lemma A.2 in [21] which came from [46] originally to choose appropriate local holomorphic frames and coordinate system near the intersection (actually in private communication Y. Rubinstein also gave such extension).

Lemma 3.1.1. *There exists $\epsilon_0 > 0$ such that if $0 \leq \text{dist}_{\hat{g}}(p, D_r) \leq \epsilon_0$ for all $r = 1, \dots, m$, then we can choose local holomorphic frames e_r of each holomorphic line*

bundle $[D_r]$ and local holomorphic coordinates $(z_1, \dots, z_n)$ valid in a neighborhood of p , such that (i) $S_r = z_r e_r$, and $a_r := \|e_r\|_r^2$ satisfies $a_r(p) = 1$, $da_r(p) = 0$, $\frac{\partial^2 a_r(p)}{\partial z_i \partial \bar{z}_j} = 0$, and (ii) $\hat{g}_{i\bar{j}}(p) = 0$ for $i \leq m, j > m$, and $\hat{g}_{i\bar{j},k}(p) = \frac{\partial}{\partial z_k} \omega_0(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})|_p = 0$, whenever $j > m$.

Proof. Actually this proof can be modified from the proof of Lemma A.2 in [21]. Fix any point q on the intersection of $D_1, \dots, D_m$, and choose local holomorphic frames $e'_1, \dots, e'_m$ and holomorphic coordinates $(w_1, \dots, w_n)$ in $B_{\hat{g}}(q, \epsilon(q))$ for some sufficient small $\epsilon(q)$. Let $S_r = f'_r e'_r$ with f'_r holomorphic functions and $\|e'_r\|_r^2 = c_r$. Let $e_r = F_r e'_r$ for some nonvanishing holomorphic functions F_r to be determined later. Then we have $a_r = \|F_r e'_r\|_r^2 = |F_r|^2 c_r$. Now fix any point $p \in B_{\hat{g}}(q, \epsilon(q))$. In order for a_r to satisfy condition (i) with respect to coordinates $(w_1, \dots, w_n)$ at point p , we choose F_r such that $F_r(p) = c_r(p)^{-\frac{1}{2}}$ and

$$\begin{aligned} \partial_{w_i} F_r(p) &= -c_r^{-1} F_r(p) \partial_{w_i} c_r(p) = -c_r^{-\frac{3}{2}} \partial_{w_i} c_r(p) \\ \partial_{w_i} \partial_{w_j} F_r(p) &= -c_r^{-1} (F_r \partial_{w_i} \partial_{w_j} c_r + \partial_{w_i} c_r \partial_{w_j} F_r + \partial_{w_i} F_r \partial_{w_j} c_r)(p) \\ &= -c_r^{-\frac{3}{2}} \partial_{w_i} \partial_{w_j} c_r(p) + 2c_r^{-\frac{5}{2}} \partial_{w_i} c_r \partial_{w_j} c_r(p). \end{aligned}$$

Since $c_r = \|e'_r\|_r^2$ is nonzero as $\epsilon(q)$ is small, which implies $|w - w(p)|$ is small, we can assume $F_r \neq 0$ in $B_{\hat{g}}(q, \epsilon(q))$. Now $S_r = f'_r e'_r = f_r e_r$ with $f_r = f'_r F_r^{-1}$ holomorphic functions. As $D_r = \{z_r = 0\}$ are smooth divisors, we can assume that $\partial_{w_r} f_r(q) \neq 0$, but $\partial_{w_s} f_r(q) = 0$ for $s \neq r$ among $1, \dots, m$, and choose small $\epsilon(q)$, we can assume that $\partial_{w_r} f_r \neq 0$ but $\partial_{w_s} f_r(q)$ sufficiently small for $s \neq r$ among $1, \dots, m$, in $B_{\hat{g}}(q, \epsilon(q))$. Then by the inverse function theorem,

$$z_1 = f_1(w_1, \dots, w_n), \dots, z_m = f_m(w_1, \dots, w_n), z_{m+1} = w_{m+1}, \dots, z_n = w_n$$

are holomorphic coordinates in $B_{\hat{g}}(q, \epsilon(q)/2)$ and now $S_r = f_r(w) e_r = z_r e_r$. By the

chain rule, (i) holds locally in the new coordinates. By covering argument (i) holds globally.

For (ii), we denote by $w^1, \dots, w^n$ the coordinates obtained in (i). First we can make a coordinate change such that at the point p , $\frac{\partial}{\partial z_j}_{j>m}$ perpendicular to $\frac{\partial}{\partial z_j}_{j\leq m}$. Let

$$\tilde{z}^k = w^k - w^k(p) + \frac{1}{2}b_{st}^k(w^s - w^s(p))(w^t - w^t(p)),$$

with $b_{st}^k = b_{ts}^k$, define a new coordinate system. Then we have that

$$\begin{aligned} \omega_0\left(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^j}\right) &= \omega_0\left(\frac{\partial}{\partial \tilde{z}^i}, \frac{\partial}{\partial \bar{\tilde{z}}^j}\right) + \hat{g}_{t\bar{j}}b_{is}^t(w^s - w^s(p)) + \overline{\hat{g}_{i\bar{t}}b_{sj}^t(w^s - w^s(p))} \\ &\quad + O\left(\sum_{r=1}^n |w^r - w^r(p)|^2\right), \\ d_{i\bar{j}k} &:= \frac{\partial}{\partial w^k} \omega_0\left(\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^j}\right)|_p = \frac{\partial}{\partial \tilde{z}^k} \omega_0\left(\frac{\partial}{\partial \tilde{z}^i}, \frac{\partial}{\partial \bar{\tilde{z}}^j}\right) + \hat{g}_{t\bar{j}}(p)b_{ik}^t =: e_{i\bar{j}k} + \hat{g}_{t\bar{j}}(p)b_{ik}^t. \end{aligned}$$

Let $\hat{g}'_{r\bar{s}} := \hat{g}_{r\bar{s}}$, for $r, s > m$, and denote the inverse of the $(n-m) \times (n-m)$ matrix $[\hat{g}'_{r\bar{s}}]$ by $[\hat{g}'^{r\bar{s}}]$. Let $b_{ik}^r = 0$ for $r = 1, \dots, m$. Then for each $j > m$, the equations can be rewritten as $d_{i\bar{j}k} - \sum_{t>m} \hat{g}_{t\bar{j}}(p)b_{ik}^t = e_{i\bar{j}k}$. Hence if we define $b_{ik}^t = \sum_{j>m} \hat{g}'^{t\bar{j}} d_{i\bar{j}k}$ for each $t > m$, we will have that $e_{i\bar{j}k} = 0$ for each $j > m$. Finally, we set $z^r = \tilde{z}^r + w^r(p)$ for each $r = 1, \dots, n$, then these coordinates satisfy conditions (i) and (ii). $\square$

In the following sections we will compute the curvature expressions and obtain desired estimates under such coordinates.

3.2 Smoothing potential function and bisectional curvature

First, we have known that the conic metric $\omega = \omega_0 + \sum_{r=1}^m \sqrt{-1} \partial \bar{\partial} \|S_r\|_r^{2\beta_r}$ has the principle part $\sum_{r=1}^m \beta_r^2 \frac{dz_r \wedge d\bar{z}_r}{\|S_r\|_r^{2(1-\beta_r)}}$. In this sense we hope to construct a positive smooth nondecreasing function $F_{r,\epsilon}(t)$ with the property that $F_{r,\epsilon}(t) \rightarrow t$ as ϵ tends to 0, such that the smoothing metric has the principle part $\sum_{r=1}^m \beta_r^2 \frac{dz_r \wedge d\bar{z}_r}{F_{r,\epsilon}(\|S_r\|_r^2)^{1-\beta_r}}$. Assume that the corresponding potential function is $\sum_{r=1}^m \psi_{r,\epsilon}(\|S_r\|_r^2)$, by simple computation we have that $(\psi'_{r,\epsilon}(t)t)' = \frac{1}{F_{r,\epsilon}(t)^{1-\beta_r}}$, then we obtain that

$$\psi_{r,\epsilon}(t) = \beta_r^2 \int_0^t \frac{1}{u} \int_0^u F_{r,\epsilon}(v)^{\beta_r-1} dv du, \quad (3.1)$$

and we can denote $H_{r,\epsilon}(t) := \int_0^t F_{r,\epsilon}(v)^{\beta_r-1} dv$. We can write the smoothing metric as below:

$$\omega_\epsilon = \omega_0 + \sum_{r=1}^m \sqrt{-1} \partial \bar{\partial} \psi_{r,\epsilon}(\|S_r\|_r^2). \quad (3.2)$$

Using the coordinates we constructed in Lemma 3.1.1, we can compute the metric tensors and their derivatives as below (for simplicity we use g , F_r , H_r , ψ_r to denote the metric associated to ω_ϵ and $F_{r,\epsilon}$, $H_{r,\epsilon}$, $\psi_{r,\epsilon}$):

$$\begin{aligned} g_{i\bar{j}} &= \hat{g}_{i\bar{j}} + \sum_{r=1}^m \frac{\beta_r^2}{F_r(\|S_r\|_r^2)^{1-\beta_r}} \frac{(\|S_r\|_r^2)_i (\|S_r\|_r^2)_{\bar{j}}}{\|S_r\|_r^2} + \sum_{r=1}^m \beta_r^2 H_r(\|S_r\|_r^2) (\log \|S_r\|_r^2)_{i\bar{j}} \\ &= \hat{g}_{i\bar{j}} + \sum_{r=1}^m \left(\frac{\beta_r^2}{F_r(\|S_r\|_r^2)^{1-\beta_r}} \frac{(\|S_r\|_r^2)_i (\|S_r\|_r^2)_{\bar{j}}}{\|S_r\|_r^2} - \beta_r^2 H_r(\|S_r\|_r^2) \Theta_{r,i\bar{j}} \right), \end{aligned}$$

where the last equality comes from Poincare-Lelong equation and the fact that $H_r(0) = 0$, and $\Theta_r = -\sqrt{-1} \partial \bar{\partial} \log a_r$ represents the curvature form of the line bundle $[D_r]$.

Now for the first order derivatives, we have

$$\begin{aligned}
g_{i\bar{j},k} &= \hat{g}_{i\bar{j},k} + \sum_{r=1}^m \frac{\beta_r^2(\beta_r - 1)F'_r(\|S_r\|_r^2)}{F_r(\|S_r\|_r^2)^{2-\beta_r}} \frac{(\|S_r\|_r^2)_i(\|S_r\|_r^2)_{\bar{j}}(\|S_r\|_r^2)_k}{\|S_r\|_r^2} \\
&+ \sum_{r=1}^m \frac{\beta_r^2}{F_r(\|S_r\|_r^2)^{1-\beta_r}} \frac{(\|S_r\|_r^2)_{ik}(\|S_r\|_r^2)_{\bar{j}} + (\|S_r\|_r^2)_i(\|S_r\|_r^2)_{\bar{j}k}}{\|S_r\|_r^2} \\
&- \sum_{r=1}^m \frac{\beta_r^2}{F_r(\|S_r\|_r^2)^{1-\beta_r}} \left(\frac{(\|S_r\|_r^2)_i(\|S_r\|_r^2)_{\bar{j}}(\|S_r\|_r^2)_k}{\|S_r\|_r^4} + \Theta_{r,i\bar{j}}(\|S_r\|_r^2)_k \right) \\
&- \sum_{r=1}^m \beta_r^2 H_r(\|S_r\|_r^2) \Theta_{r,i\bar{j},k},
\end{aligned}$$

Note that $\|S_r\|_r^2 = a_r|z_r|^2$, considering (i) of Lemma 3.1.1, we can obtain the second order derivatives at p:

$$\begin{aligned}
g_{i\bar{j},k\bar{l}}(p) &= \hat{g}_{i\bar{j},k\bar{l}} + \sum_{r=1}^m \frac{\beta_r^2(\beta_r - 1)(\beta_r - 2)F'_r(|z_r|^2)^2}{F_r(|z_r|^2)^{3-\beta_r}} |z_r|^2 \delta_{ri} \delta_{rj} \delta_{rk} \delta_{rl} \\
&+ \sum_{r=1}^m \frac{\beta_r^2(\beta_r - 1)(F''_r(|z_r|^2)|z_r|^2 + F'_r(|z_r|^2))}{F_r(|z_r|^2)^{2-\beta_r}} \delta_{ri} \delta_{rj} \delta_{rk} \delta_{rl} \\
&+ \sum_{r=1}^m \frac{\beta_r^2((\beta_r - 1)F'_r(|z_r|^2)|z_r|^2 + F(|z_r|^2))}{F_r(|z_r|^2)^{2-\beta_r}} (a_{r,i\bar{j}} \delta_{rk} \delta_{rl} + a_{r,i\bar{l}} \delta_{rk} \delta_{rj} + a_{r,k\bar{j}} \delta_{ri} \delta_{rl} + a_{r,k\bar{l}} \delta_{ri} \delta_{rj}) \\
&+ \sum_{r=1}^m \frac{\beta_r^2}{F_r(|z_r|^2)^{1-\beta_r}} ((a_{r,ik\bar{l}} \delta_{rj} + a_{r,i\bar{j}k} \delta_{rl}) z_r + (a_{r,i\bar{j}\bar{l}} \delta_{rk} + a_{r,\bar{j}k\bar{l}} \delta_{ri}) \bar{z}_r) \\
&+ \sum_{r=1}^m \frac{\beta_r^2}{F_r(|z_r|^2)^{1-\beta_r}} (a_{r,i\bar{j}} a_{r,k\bar{l}} + a_{r,i\bar{l}} a_{r,k\bar{j}}) |z_r|^2 - \sum_{r=1}^m \beta_r^2 H_r(|z_r|^2) \Theta_{r,i\bar{j},k\bar{l}}.
\end{aligned}$$

Meanwhile we can also obtain the metric tensors and the first order derivatives at point p:

$$\begin{aligned}
g_{i\bar{j}}(p) &= \hat{g}_{i\bar{j}} + \sum_{r=1}^m \left(\frac{\beta_r^2 \delta_{ri} \delta_{rj}}{F_r(|z_r|^2)^{1-\beta_r}} - \beta_r^2 H_r(|z_r|^2) \Theta_{r,i\bar{j}} \right) \\
g_{i\bar{j},k}(p) &= \hat{g}_{i\bar{j},k} + \sum_{r=1}^m \frac{\beta_r^2(\beta_r - 1)F'_r(|z_r|^2)}{F_r(|z_r|^2)^{2-\beta_r}} \bar{z}_r \delta_{ri} \delta_{rj} \delta_{rk} \\
&+ \sum_{r=1}^m \frac{\beta_r^2 \bar{z}_r}{F_r(|z_r|^2)^{1-\beta_r}} (a_{r,i\bar{j}} \delta_{rk} + a_{r,k\bar{j}} \delta_{ri}) + \sum_{r=1}^m \beta_r^2 H_r(|z_r|^2) a_{r,i\bar{j}k}.
\end{aligned}$$

For $1 \leq r, s \leq m, r \neq s$ and $m+1 \leq t, t' \leq n$, we can easily have

$$g^{r\bar{t}}(p) = O(F_r^{1-\beta_r}), \quad g^{t\bar{t}'} = O(1). \quad (3.3)$$

We need the following lemma to give a more precise estimate for $g^{r\bar{s}}(p) (1 \leq r, s \leq m)$:

Lemma 3.2.1.

$$g^{r\bar{r}}(p) = \frac{\beta_r^{-2} F_r^{1-\beta_r}}{1 + c_r(p) F_r^{1-\beta_r}} + O(F_r^{2(1-\beta_r)} \sum_{s=1}^m (F_s^{1-\beta_s} + F_s^{\beta_s})), \quad (3.4)$$

$$g^{r\bar{s}}(p) = \beta_r^{-2} \beta_s^{-2} F_r^{1-\beta_r} F_s^{1-\beta_s} ((-1)^{r+s} \hat{g}_{s\bar{r}} + o(1)) \quad (r \neq s), \quad (3.5)$$

where $c_r(p) := \beta_r^{-2} \frac{\det(\hat{g}_{i\bar{j}})_{i,j=r,m+1,\dots,n}}{\det(\hat{g}_{i\bar{j}})_{i,j=m+1,\dots,n}}(p)$ and $0 < C_1 < c_r(p) < C_2$ for all $p \in M$ and $r = 1, \dots, m$.

Proof. Actually we only need to prove the result for $r = 1, s = 2$. For simplicity we denote $a_{i\bar{j}} := \hat{g}_{i\bar{j}}(p) - \sum_{r=1}^m \beta_r^2 H_r(|S_r|^2) \Theta_{r,i\bar{j}}$, $b_r = \frac{\beta_r^2}{F_r^{1-\beta_r}}$, then we know that $g_{i\bar{j}} = a_{i\bar{j}} + b_i \delta_{ij}$ where $i \leq m$. By determinant rule we know that $g^{1\bar{1}}(p) = \frac{G_{1\bar{1}}}{\det g}$ where $G_{1\bar{1}}$ represents the cofactor of $g_{1\bar{1}}$. Let us compute $G_{1\bar{1}}$ and $\det g$ respectively. We can denote $A_{r_1, r_2, \dots, r_k, R}$ as the $(k+n-m)$ -th minor $\det(a_{i_p \bar{i}_q})_{r_1, \dots, r_k, m+1, \dots, n}$, where $1 \leq r_1 < \dots < r_k \leq m$. Now make use of determinant rules, we can have such

decomposition of $\det g$:

$$\begin{aligned}
\det g &= \begin{vmatrix} b_1 & 0 & \cdots & 0 \\ a_{2\bar{1}} & a_{2\bar{2}} + b_2 & \cdots & a_{2\bar{n}} \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} + \begin{vmatrix} a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1\bar{n}} \\ a_{2\bar{1}} & a_{2\bar{2}} + b_2 & \cdots & a_{2\bar{n}} \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} \\
&= \begin{vmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} + \begin{vmatrix} b_1 & 0 & \cdots & 0 \\ a_{2\bar{1}} & a_{2\bar{2}} & \cdots & a_{2\bar{n}} \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} \\
&+ \begin{vmatrix} a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1\bar{n}} \\ 0 & b_2 & \cdots & 0 \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} + \begin{vmatrix} a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1\bar{n}} \\ a_{2\bar{1}} & a_{2\bar{2}} & \cdots & a_{2\bar{n}} \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} = \cdots + \cdots
\end{aligned}$$

$$= \begin{vmatrix} b_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & & \cdots & b_m & & 0 \\ a_{m+1,\bar{1}} & & \cdots & & & a_{m+1,\bar{n}} \\ & & \cdots & & & \\ a_{n\bar{1}} & & \cdots & & & a_{n\bar{n}} \end{vmatrix}$$

$$\begin{aligned}
& + \left(\begin{vmatrix} b_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & b_{m-1} & & 0 & \\ a_{m,\bar{1}} & \cdots & & & a_{m,\bar{n}} & \\ & \cdots & & & & \\ a_{n\bar{1}} & \cdots & & & a_{n\bar{n}} & \end{vmatrix} + \cdots + \begin{vmatrix} a_{1\bar{1}} & a_{1\bar{2}} & \cdots & 0 & \cdots & a_{1\bar{n}} \\ 0 & b_2 & \cdots & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & b_m & & 0 & \\ a_{m+1,\bar{1}} & \cdots & & & a_{m+1,\bar{n}} & \\ & \cdots & & & & \\ a_{n\bar{1}} & \cdots & & & a_{n\bar{n}} & \end{vmatrix} \right) \\
& + \cdots + \begin{vmatrix} a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1\bar{n}} \\ & & \cdots & \\ a_{m\bar{1}} & & \cdots & a_{m\bar{n}} \\ & & \cdots & \\ a_{n\bar{1}} & & \cdots & a_{n\bar{n}} \end{vmatrix} \\
& = b_1 b_2 \cdots b_m A_R + b_1 b_2 \cdots b_m \sum_{r=1}^m \frac{A_{r,R}}{b_r} + b_1 b_2 \cdots b_m \sum_{1 \leq r < s}^m \frac{A_{r,s,R}}{b_r b_s} + \cdots + \det(a_{i\bar{j}}).
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
G_{1\bar{1}} &= b_2 b_3 \cdots b_m \left(A_R + \sum_{r=2}^m \frac{A_{r,R}}{b_r} + \sum_{2 \leq r < s}^m \frac{A_{r,s,R}}{b_r b_s} \right) + \cdots + \det(a_{i\bar{j}})_{2 \leq i, j \leq m}, \\
G_{1\bar{2}} &= -a_{2\bar{1}} b_3 \cdots b_m A_R + \cdots .
\end{aligned}$$

Now we can have such estimate:

$$\frac{G_{1\bar{1}}}{\det g} = b_1^{-1} \frac{1 + \sum_{r=2}^m \frac{A_{r,R}}{A_R b_r} + \sum_{2 \leq r < s}^m \frac{A_{r,s,R}}{A_R b_r b_s} + \cdots}{1 + \sum_{r=1}^m \frac{A_{r,R}}{A_R b_r} + \sum_{1 \leq r < s}^m \frac{A_{r,s,R}}{A_R b_r b_s} + \cdots} = \frac{b_1^{-1}}{B},$$

where

$$\begin{aligned}
B &= (1 + \sum_{r=1}^m \frac{A_{r,R}}{A_R b_r} + \sum_{1 \leq r < s}^m \frac{A_{r,s,R}}{A_R b_r b_s} + \dots) [1 - (\sum_{r=2}^m \frac{A_{r,R}}{A_R b_r} + \sum_{2 \leq r < s}^m \frac{A_{r,s,R}}{A_R b_r b_s} + \dots) \\
&\quad + (\sum_{r=2}^m \frac{A_{r,R}}{A_R b_r} + \sum_{2 \leq r < s}^m \frac{A_{r,s,R}}{A_R b_r b_s} + \dots)^2 + \dots] \\
&= 1 + \frac{A_{1,R}}{A_R b_1} + \frac{1}{A_R b_1} \sum_{r=2}^m \frac{1}{b_r} (A_{1,r,R} - \frac{A_{1,R} A_{r,R}}{A_R}) + O(\frac{1}{b_1} \sum_{r=2}^m \frac{1}{b_r^2}).
\end{aligned}$$

Note that $a_{i\bar{j}} := \hat{g}_{i\bar{j}}(p) - \sum_{s=1}^m \beta_s^2 H_s(\|S_s\|_s^2) \Theta_{s,i\bar{j}} = \hat{g}_{i\bar{j}}(p) + O(\sum_{s=1}^m F_s^{\beta_s})$, (3.4) follows.

(3.5) follows from the computation of $G_{1\bar{2}}$ more easily. $\square$

Take two unit vectors $\eta = \eta^i \frac{\partial}{\partial z_i}, \nu = \nu^i \frac{\partial}{\partial z_i} \in T_p^{1,0} M$, so that $g(\eta, \eta)|_p = g(\nu, \nu)|_p = 1$. Then from the expression of $g_{i\bar{j}}$ we have

$$\eta^r, \nu^r = O(F_r^{\frac{1-\beta_r}{2}}), \eta^t, \nu^t = O(1) \text{ for } r = 1, \dots, m, t = m+1, \dots, n. \quad (3.6)$$

By the definition of bisectonal curvature, we set

$$Bisec_\omega(\eta, \nu) = R(\eta, \bar{\eta}, \nu, \bar{\nu}) = R_{i\bar{j}k\bar{l}} \eta^i \bar{\eta}^j \nu^k \bar{\nu}^l = \sum_{i,j,k,l} (\Lambda_{i\bar{j}k\bar{l}} + \Pi_{i\bar{j}k\bar{l}}),$$

with $\Lambda_{i\bar{j}k\bar{l}} := -g_{i\bar{j},k\bar{l}} \eta^i \bar{\eta}^j \nu^k \bar{\nu}^l$, and $\Pi_{i\bar{j}k\bar{l}} := g^{p\bar{q}} g_{i\bar{q},k} g_{p\bar{j},\bar{l}} \eta^i \bar{\eta}^j \nu^k \bar{\nu}^l$ (no summations on i, j, k, l). By (3.3)-(3.6) and note that $|z_r|^2 = O(F_r(|z_r|^2))$ we have $|\Lambda_{i\bar{j}k\bar{l}}| \leq C$ except for the terms $\sum_{r=1}^m \Lambda_{r\bar{r}r\bar{r}}$, hence

$$\begin{aligned}
\sum_{i,j,k,l} \Lambda_{i\bar{j}k\bar{l}}(p) &= O(1) - \sum_{r=1}^m \left[\frac{\beta_r^2 (\beta_r - 1) (\beta_r - 2) F'_r(|z_r|^2)^2 |z_r|^2}{F_r(|z_r|^2)^{3-\beta_r}} \right. \\
&\quad \left. + \frac{\beta_r^2 (\beta_r - 1) (F''_r(|z_r|^2) |z_r|^2 + F'_r(|z_r|^2))}{F_r(|z_r|^2)^{2-\beta_r}} \right] |\eta^r|^2 |\nu^r|^2. \quad (3.7)
\end{aligned}$$

For the other part in the bisectonal curvature formula, we have such a lemma:

Lemma 3.2.2. *In case that no three irreducible divisors intersect or all angles $\beta_i \leq \frac{1}{2}$, there exists a uniform constant $C > 0$ such that for every $p \in M$,*

$$\sum_{i,j,k,l} \Pi_{i\bar{j}k\bar{l}}(p) \leq C + \sum_{r=1}^m \frac{\beta_r^2(\beta_r - 1)^2 F_r'(|z_r|^2)^2 |z_r|^2}{F_r(|z_r|^2)^{3-\beta_r}} |\eta^r|^2 |\nu^r|^2. \quad (3.8)$$

Proof. By Brendle's computation in [3], we can easily bound all the terms if $\beta_i \leq \frac{1}{2}$ for all i . Now we consider the general case. As lemma A.3 in [21], we define a bilinear Hermitian form of two tensors $a = [a_{i\bar{q}k}], b = [b_{j\bar{p}l}] \in (\mathbb{C})^3$ satisfying $a_{i\bar{q}k} = a_{k\bar{q}i}, b_{j\bar{p}l} = b_{l\bar{p}j}$ by setting

$$\langle [a_{i\bar{q}k}], [b_{j\bar{p}l}] \rangle := \sum_{i,j,k,l,p,q} g^{p\bar{q}} (\eta^i g_{i\bar{q},k} \nu^k) (\overline{\eta^j g_{j\bar{p},l} \nu^l}).$$

Obviously it is a nonnegative bilinear form. We denote by $\|\cdot\|$ the associated norm.

Then $\sum_{i,j,k,l} \Pi_{i\bar{j}k\bar{l}} = \|[a_{i\bar{j}k}]\|^2$. We write

$$g_{i\bar{j},k} = A_{i\bar{j}k} + B_{i\bar{j}k} + D_{i\bar{j}k} + E_{i\bar{j}k}$$

with $A_{i\bar{j}k} := \hat{g}_{i\bar{j},k}$, $B_{i\bar{j}k} := \sum_{r=1}^m \beta_r^2 H_r(|z_r|^2) a_{r,i\bar{j}} \delta_{rk}$, $D_{i\bar{j}k} := \sum_{r=1}^m \frac{\beta_r^2 \bar{z}_r}{F_r(|z_r|^2)^{1-\beta_r}} (a_{r,i\bar{j}} \delta_{rk} + a_{r,k\bar{j}} \delta_{ri})$, $E_{i\bar{j}k} := \sum_{r=1}^m \frac{\beta_r^2 (\beta_r - 1) F_r'(|z_r|^2)}{F_r(|z_r|^2)^{2-\beta_r}} \bar{z}_r \delta_{ri} \delta_{rj} \delta_{rk}$. Denote $A := [A_{i\bar{q}k}]$ and similarly B, D, E . Using (3.3) and note that $|z_r|^2 = O(F_r(|z_r|^2))$ and $H_r(|z_r|^2) = O(F_r(|z_r|^2)^{\beta_r})$, we can bound $\|A + B + D\|^2$ easily. For the crossing terms of A, B, D and E, we have that

$$\begin{aligned} 2\operatorname{Re}\langle A, E \rangle &= 2\operatorname{Re}\left(\sum_{i,j,k,r} g^{r\bar{j}} \hat{g}_{i\bar{j},k} \overline{E_{r\bar{r}r}}\right) \\ &\leq C \sum_{i,j,k} |\hat{g}_{i\bar{j},k}|^2 + \delta \sum_r F_r^{1-\beta_r} \|E_{r\bar{r}r}\|^2 \leq C + \delta \sum_r F_r^{1-\beta_r} \|E_{r\bar{r}r}\|^2, \end{aligned}$$

where δ can be chosen small enough. By the same argument, we also have that

$$2\operatorname{Re}\langle B + D, E \rangle \leq C + \delta \sum_r F_r^{1-\beta_r} \|E_{r\bar{r}r}\|^2.$$

Now let us consider $\|E\|^2$: In case that at most two divisors intersect transversely, we can take $m = 2$. As there exists a uniform constant $c_0 < 1$ such that $|\hat{g}_{1\bar{2}}|^2(p) \leq c_0 \hat{g}_{1\bar{1}}(p) \hat{g}_{2\bar{2}}(p)$, By (3.4) and (3.5), we have that

$$\begin{aligned} \|E\|^2(p) &= \sum_{r=1}^2 g^{r\bar{r}} |E_{r\bar{r}r}|^2 + \sum_{r \neq s}^2 g^{r\bar{s}} E_{r\bar{r}r} \overline{E_{s\bar{s}s}} \\ &\leq \sum_{r=1}^2 \frac{\beta_r^{-2} F_r^{1-\beta_r}}{1 + c_r(p) F_r^{1-\beta_r}} |E_{r\bar{r}r}|^2 + \sum_{r=1}^2 c_0 \hat{g}_{r\bar{r}}(p) \beta_r^{-4} F_r^{2(1-\beta_r)} |E_{r\bar{r}r}|^2 \\ &= \sum_{r=1}^2 \beta_r^{-2} F_r^{1-\beta_r} \left(\frac{1 + c_0 \hat{g}_{r\bar{r}}(p) \beta_r^{-2} F_r^{1-\beta_r}}{1 + c_r(p) F_r^{1-\beta_r}} + o(1) \right) |E_{r\bar{r}r}|^2. \end{aligned}$$

As in our coordinate system $c_r(p) = \frac{A_{r,R}}{A_R} = \hat{g}_{r\bar{r}}$, by choosing $\delta > 0$ such that $2\delta + c_0 < 1$, we obtain the lemma. $\square$

Remark 3.2.3. *When more than three divisors intersect transversely, i.e. $m > 2$, we do not have that*

$$[\hat{g}_{r\bar{s}}] < 2\operatorname{diag}[\hat{g}_{1\bar{1}}, \dots, \hat{g}_{m\bar{m}}].$$

Then we cannot control the crossing terms in $\|E\|^2$ by its diagonal terms so well as $m = 2$.

The Theorem 1.2.2 will follow from the lemma below. Moreover, if Lemma 3.2.2 is true in general case, then Theorem 1.2.2 is also true in general case:

Lemma 3.2.4. *There exist positive functions $F_{r,\epsilon}(t)$ such that $F_{r,\epsilon}(t) \rightarrow t$ as ϵ tends to 0 and*

$$\frac{(F_{r,\epsilon} F_{r,\epsilon}'' - F_{r,\epsilon}'^2)t + F_{r,\epsilon} F_{r,\epsilon}'}{F_{r,\epsilon}^{1+\beta_r}} \leq C_r, \quad (3.9)$$

where C_r is independent of ϵ .

We will construct such functions and prove this lemma in the next section.

3.3 Proof of Lemma 3.2.4

To satisfy that $F_{r,\epsilon}(t) \rightarrow t$ as ϵ tends to 0, Let us assume the form of the function $F_{r,\epsilon}(t) := \epsilon + t\chi_{r,\epsilon}(t)$ where $\chi_{r,\epsilon}(t)$ is a bump function depending only on ϵ and β_r . Considering the inequality (3.9), dropping the index r , we have that

$$\frac{\epsilon(\chi_\epsilon + 3t\chi'_\epsilon + t^2\chi''_\epsilon) + t^2(\chi_\epsilon\chi'_\epsilon + t\chi_\epsilon\chi''_\epsilon - t\chi'^2_\epsilon)}{(\epsilon + t\chi_\epsilon)^{1+\beta}} \leq C. \quad (3.10)$$

Let us begin to construct χ_ϵ . First, consider a function $\chi(t)$ defined on $[0, +\infty)$ satisfying that

$$\chi'(t) = c_k(t - N)^k(N + 1 - t)^k$$

on $[N, N + 1]$ and $\chi''(t) = 0$ outside $[N, N + 1]$. Here k, N are positive integers to be determined later and c_k satisfies that $\chi(N + 1) = 1$. By computation, we have that

$$\chi''(t) = kc_k(2N + 1 - 2t)(t - N)^{k-1}(N + 1 - t)^{k-1}$$

on $[N, N + 1]$ and $\chi''(t) = 0$ outside $[N, N + 1]$, and we also have

$$\chi(t) = \begin{cases} 0 & 0 \leq t \leq N \\ c_k \left(\frac{(t-N)^{k+1}(N+1-t)^k}{k+1} + \sum_{i=1}^k \frac{k(k-1)\cdots(k-i+1)(t-N)^{k+i+1}(N+1-t)^{k-i}}{(k+1)(k+2)\cdots(k+i+1)} \right) & N \leq t \leq N + 1 \\ 1 & N + 1 \leq t < \infty \end{cases} \quad (3.11)$$

Although this function is not smooth, we can make some smoothing modification near N and $N+1$ without changing lower order terms. Then we can make this function smooth and still denote by $\chi(t)$. As the modification does not change lower order terms, we can still do estimates for the function defined above. Now by direct

computation, on $[N, N + 1]$ we have that

$$\begin{aligned} \chi\chi'' - \chi'^2 &= -c_k^2(t - N)^{2k}(N + 1 - t)^{k-1} \left[\frac{(N + 1 - t)^{k+1}}{k + 1} + \frac{2k(t - N)(N + 1 - t)^k}{(k + 1)(k + 2)} \right. \\ &\quad \left. + \sum_{i=1}^{k-1} \frac{2(i + 1)k^2(k - 1) \cdots (k - i + 1)}{(k + 1) \cdots (k + i + 2)} + k \frac{k!(t - N)^{k+1}}{(k + 1) \cdots (2k + 1)} \right]. \end{aligned}$$

For each k , if we choose sufficiently large N , on $[0, +\infty)$ we will have that

$$\chi\chi' + t(\chi\chi'' - \chi'^2) \leq 0. \quad (3.12)$$

Now given an exponential $\delta < 1$, if we take $\chi_\epsilon(t) := \chi(\frac{t}{\epsilon^\delta})$, by (3.12), we will have that

$$\chi_\epsilon\chi'_\epsilon + t(\chi_\epsilon\chi''_\epsilon - \chi'^2_\epsilon) \leq 0.$$

So now we only need to find δ depending on N, k, β such that the

$$\frac{\epsilon(\chi_\epsilon + 3t\chi'_\epsilon + t^2\chi''_\epsilon)}{(\epsilon + t\chi_\epsilon)^{1+\beta}}$$

is bounded independent of ϵ . First, if we choose $\delta \leq \frac{1}{1+\beta}$, then this function is bounded when $t > (N + 1)\epsilon^\delta$. The main difficulty is the case when $t - N\epsilon^\delta$ is small. By the expansion of χ, χ', χ'' , we find that when $t - N$ is small, $\chi' = O(\chi^{\frac{k}{k+1}})$, $\chi'' = O(\chi^{\frac{k-1}{k+1}})$. By scaling, we know that we only need to bound

$$\frac{\epsilon\chi_\epsilon^{\frac{k-1}{k+1}}}{(\epsilon + t\chi_\epsilon)^{1+\beta}}$$

when $t - N\epsilon^\delta$ is small. Now as we have that

$$\frac{\epsilon\chi_\epsilon^{\frac{k-1}{k+1}}}{(\epsilon + t\chi_\epsilon)^{1+\beta}} \leq \min\left\{ \frac{\epsilon\chi_\epsilon^{\frac{k-1}{k+1}}}{\epsilon^{1+\beta}}, \frac{\epsilon\chi_\epsilon^{\frac{k-1}{k+1}}}{(t\chi_\epsilon)^{1+\beta}} \right\},$$

to make $\chi_\epsilon^{\frac{k-1}{k+1}} \leq \epsilon^\beta$, by the expansion of χ , we have that $\frac{t}{\epsilon^\delta} - N \leq C\epsilon^{\frac{\beta}{k-1}}$. Actually we have bounded this function on this short interval near $N\epsilon^\delta$. Now when $\frac{t}{\epsilon^\delta} - N > C\epsilon^{\frac{\beta}{k-1}}$, we have that

$$\frac{\epsilon \chi_\epsilon^{\frac{k-1}{k+1}}}{(t\chi_\epsilon)^{1+\beta}} \leq \frac{C\epsilon^{1-\delta(1+\beta)}}{\chi_\epsilon^{\beta+\frac{2}{k+1}}} \leq C\epsilon^{1-\delta(1+\beta)-\beta\frac{k-1}{k+1}(\beta+\frac{2}{k+1})}.$$

Now choose a large k such that $1 - \beta\frac{k-1}{k+1}(\beta + \frac{2}{k+1}) > 0$, then we can choose $\delta > 0$ such that

$$1 - \delta(1 + \beta) - \beta\frac{k-1}{k+1}(\beta + \frac{2}{k+1}) \geq 0,$$

then we will have that

$$\frac{\epsilon(\chi_\epsilon + 3t\chi'_\epsilon + t^2\chi''_\epsilon)}{(\epsilon + t\chi_\epsilon)^{1+\beta}} \leq C. \quad (3.13)$$

Combine this inequality and (3.12), we obtain (3.10), which completes the proof of Lemma 3.2.4.

Chapter 4

Unnormalized conical Kähler-Ricci flow

4.1 Introduction

Kähler-Ricci flow has become a powerful tool in the study of Kähler geometry for many years. In [47], Tian-Zhang adapted Tsuji's idea [48] to consider Kähler-Ricci flow in the level of cohomology class on projective manifolds. They proved that Kähler-Ricci flow can exist in maximal time interval, on which the cohomology class of the Kähler form $\omega(t)$ stays nondegenerate. Base on this maximal time existence, they obtained existence results and limit behavior analysis in case that the canonical line bundle is big or big and nef. Later, in the series of papers [35] [36] [37], Song-Tian extended this idea to construct an analytic method of minimal model program in algebraic geometry.

Recently, conic Kähler metric has become an attracting topic for geometers, especially conical Kähler-Einstein metrics. Similarly, to study conic metrics, conical Kähler-Ricci flow has been thought as a useful tool, and there have been some works

on it, e.g., [24] [9] [10] [28] [49]. First, we recall that a Kähler current ω is a conical Kähler metric with angle $2\pi\beta, 0 < \beta < 1$ along the divisor D , if ω is smooth away from D and asymptotically equivalent along D to the model conic metric

$$\sqrt{-1} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right),$$

where $(z_1, z_2, \dots, z_n)$ are local holomorphic coordinates and $D = \{z_1 = 0\}$ locally. We call ω a conic Kähler-Einstein metric with angle $2\pi\beta (0 < \beta < 1)$ along the divisor D if it is a conic Kähler metric and satisfies the equation

$$Ric(\omega) = \mu\omega + 2\pi(1 - \beta)[D]$$

in the sense of currents, and a smooth Kähler-Einstein metric outside the divisor D . In [9] [10] [28], they set $\mu = \beta$ and studied such normalized conical Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -Ric(\omega) + \beta\omega + 2\pi(1 - \beta)[D]$$

starting with a conical Kähler metric with angle $2\pi\beta$ along the anticanonical divisor D . In [28], Liu-Zhang used smooth approximation of conic metric, which was set up by Campana-Guenancia-Păun in [4], [15] to obtain a long time solution for the normalized normalized conical Kähler-Ricci flow. Moreover, they proved that when $\beta \in (0, \frac{1}{2}]$, the flow converges to a conical Kähler-Einstein metric if it exists.

Now we can consider whether the conical Kähler-Ricci flow exists on general Kähler manifolds starting with arbitrary conic Kähler metric, and how long it will last, moreover, what will happen when the flow is extinct. Actually, we can extend the work of Tian-Zhang [47] and Song-Tian [37] to conic case. And in the study of these problems, we can generalize the irreducible divisor D to be a simple normal

crossing divisor, i.e, $D = \sum_{i=1}^l (1 - \beta_i) D_i$ ($l \leq n$), where each D_i is irreducible and $\beta_i \in (0, 1)$. Locally near a point $p \in \bigcap_i D_i$, $D_i = \{z_i = 0\}$. Then a conic metric ω along each D_i with cone angle $2\pi\beta_i$ is asymptotically equivalent to the model metric

$$\omega_\beta = \sqrt{-1} \left(\sum_{i=1}^l \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{i=l+1}^n dz_i \wedge d\bar{z}_i \right).$$

And the unnormalized conical Kähler-Ricci flow is

$$\begin{cases} \frac{\partial}{\partial t} \omega = -Ric(\omega) + 2\pi \sum_{i=1}^l (1 - \beta_i) [D_i] \\ \omega(0) = \omega^* = \omega_0 + \sum_{i=1}^l k \sqrt{-1} \partial \bar{\partial} ||S_i||_i^{2\beta_i} \end{cases} \quad (4.1)$$

where the initial metric is conic Kähler metric ω^* with conic angles $2\pi\beta_i$ along D_i . Suppose S_i is a holomorphic section of the bundle associated to the divisor D_i then this conic Kähler metric can be written as $\omega^* = \omega_0 + \sum_{i=1}^l k \sqrt{-1} \partial \bar{\partial} ||S_i||_i^{2\beta_i}$, where ω_0 is a smooth Kähler metric on M , k is a small positive constant and $|| \cdot ||_i^2$ is a Hermitian metric associated to $[D_i]$. Similar to Tian-Zhang and Song-Tian, we can consider this flow equation (4.1) in cohomology level and obtain maximal time existence theorem 1.2.3:

Theorem 4.1.1. *Let*

$$T_0 := \sup \{ t | [\omega_0] - t(c_1(M) - \sum_{i=1}^l (1 - \beta_i) [D_i]) > 0 \},$$

then starting with a conic Kähler metric ω^* defined above, the unnormalized conical Kähler-Ricci flow (4.1) has a unique solution on $[0, T_0)$, which is smooth outside the divisors D_i and $C^{2,\alpha}$ along the divisors D_i .

we can prove this theorem by Tian-Zhang's method [47]. However, as the conic metric is not smooth, we need to apply Liu-Zhang's approach, making use of the

approximation developed in [15], to obtain a family of solutions for the approximating equation, then we prove this family of solutions converge to the solution of the original flow equation. Finally, we prove the solution is smooth outside the divisors D_i and $C^{2,\alpha}$ along the divisor in suitable sense.

We recall that in [47] [35] [36] [37], Kähler-Ricci flow is used to study minimal model program. More precisely, they analyzed the behavior of Kähler-Ricci flow near the singular time or infinite time on different types of projective manifolds. As we mentioned before that conic Kähler metric has becoming an interesting topic, we hope that unnormalized conical Kähler-Ricci flow can play some similar role in the study of conic Kähler metrics on projective manifolds. We note that conical Kähler-Ricci flow always preserves conical singularities so it's natural to expect that we can have a good picture of the long time behavior besides conical singularities. Similarly, if we assume that so-called twisted canonical line bundle $K_M + \sum_{i=1}^l (1 - \beta)[D_i]$ is big, we have such a theorem, as [47],

Theorem 4.1.2. *If the twisted canonical line bundle is big on the projective manifold M , starting with a conic Kähler metric ω^* , as t tends to $T_0 < \infty$ in 4.1.1, the solution converges to a Kähler current which is a smooth Kähler metric outside the divisor D and the stable base locus set of $K_M + \sum_{i=1}^l (1 - \beta)[D_i]$, say E , and $C^{2,\alpha}$ with conic angle $2\pi\beta_i$ along $D_i \setminus E$. Moreover, on smooth part, the flow solution converges to the limiting metric in the local C^∞ -sense and $C^{2,\alpha}$ along the divisor $D \setminus E$.*

Furthermore, if we assume the twisted canonical line bundle $K_M + \sum_{i=1}^l (1 - \beta)[D_i]$ is big and nef, by Theorem 4.1.1, we can see that $T_0 = \infty$. To get a more clear picture of the limiting metric, we can normalized such conical Kähler-Ricci flow equation (4.1) to

$$\frac{\partial}{\partial t} \omega = -\omega - Ric(\omega) + 2\pi \sum_{i=1}^l (1 - \beta)[D_i]. \quad (4.2)$$

We note that the solutions to these two flow equations are different only by scaling. Then similarly we have such a theorem

Theorem 4.1.3. *All conditions follow 4.1.2, additionally, if the twisted canonical line bundle $K_M + \sum_{i=1}^l (1 - \beta)[D_i]$ is big and nef, then starting from a conic Kähler metric ω^* , the normalized conical Kähler-Ricci flow (4.2) converges to a current as t tends to infinity such that modulo the stable base locus E , the limiting current is a conical Kähler-Einstein metric which is $C^{2,\alpha}$ with conic angle $2\pi\beta_i$ along $D_i \setminus E$, and independent of the choice of the initial conic metric.*

Later we will introduce the definitions of bigness and numerical effectiveness. For the proof, we will make use of Kodaira's lemma and set up A priori estimates outside the base locus. Note that we still need Liu-Zhang's approach to overcome the conic singularities.

For simplicity, in this chapter, we only study the conic metrics along only one irreducible divisor D with cone angle $2\pi\beta$, and in the end we will briefly discuss how to generalize the proofs to simple normal crossing case. We will also sketch another smoothing method for the conical ka-Ricci flow based on Chapter 3.

4.2 Maximal time existence

4.2.1 approximation, and C^0 -estimate

Now we restrict the situation to $l = 1$, i.e, there is only one irreducible divisor D . Then the unnormalized conical Kähler-Ricci flow is

$$\begin{cases} \frac{\partial}{\partial t}\omega = -Ric(\omega) + 2\pi(1 - \beta)[D] \\ \omega(0) = \omega^* = \omega_0 + k\sqrt{-1}\partial\bar{\partial}||S||^{2\beta} \end{cases} \quad (4.3)$$

As [47] [37], first we want to transform the conical Kähler-Ricci flow equation (4.3) to Monge-Ampere flow equation. We set $T_\delta = T_0 - \delta$, where $T_0 := \sup\{t | [\omega_0] - t(c_1(M) - (1 - \beta)[D]) > 0\}$, as we defined before. We know that when $t \in [0, T_\epsilon]$, the cohomology class $[\omega_0] - t(c_1(M) - (1 - \beta)[D])$ is positive. Then when $t \in [0, T_\delta]$, there exist a hermitian metric $\|\cdot\|$ on the holomorphic line bundle associated to the divisor D and a volume form Ω on M , such that

$$\omega_t = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(\|\cdot\|)) > 0 \quad (4.4)$$

where $R(\|\cdot\|) := -\sqrt{-1}\partial\bar{\partial}\log\|\cdot\|^2$ represents the curvature of the bundle $[D]$. Now we can write

$$\bar{\omega}_t = \omega_t + k\sqrt{-1}\partial\bar{\partial}\|S\|^{2\beta},$$

where k is so small that $\bar{\omega}_t$ is positive as $t \in [0, T_\delta]$. Then if we set $\omega = \bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$, we can write the conical Kähler-Ricci flow (4.3) as following:

$$\frac{\partial}{\partial t}(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi) = \sqrt{-1}\partial\bar{\partial}\log(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n + (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log|S|^2.$$

Plug the equation (4.4) into it, we obtain a Monge-Ampere flow equation:

$$\frac{\partial}{\partial t}\varphi = \log \frac{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} + \log \|S\|^{2(1-\beta)} \quad (4.5)$$

To approximate the solution to this equation, we first use the smoothing metric in [4] that

$$\omega_{t,\epsilon} = \omega_t + k\sqrt{-1}\partial\bar{\partial}\chi(\epsilon^2 + \|S\|^2), \quad (4.6)$$

where

$$\chi(\epsilon^2 + t) = \beta \int_0^t \frac{(\epsilon^2 + r)^\beta - \epsilon^{2\beta}}{r} dr.$$

Before approximating the flow equation, we want to introduce some useful properties of this function $\chi(\epsilon^2 + t)$. First, we can find that for each $\epsilon > 0$, $\omega_{t,\epsilon}$ is a smooth Kähler metric, and as ϵ tends to 0, $\omega_{t,\epsilon}$ converges to a conic metric $\overline{\omega}_t$ in the sense of currents globally on M and in C_{loc}^∞ sense outside D . Moreover, for each $\epsilon > 0$, this function is smooth, and there exist constants $C > 0$ and $\gamma > 0$ independent of ϵ such that as t is finite $0 \leq \chi(\epsilon^2 + t) \leq C$, and $\omega_{0,\epsilon} \geq \gamma\omega_0$. As for $t \in [0, T_\delta]$ ω_t is comparable to ω_0 we can still use a constant γ such that $\omega_{t,\epsilon} \geq \gamma\omega_0$.

Now we can introduce our approximation equation of (4.5) as following

$$\begin{cases} \frac{\partial}{\partial t}\varphi_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n}{\Omega} + \log(\|S\|^2 + \epsilon^2)^{1-\beta} \\ \varphi_\epsilon(\cdot, 0) = 0 \end{cases} \quad (4.7)$$

Equivalently, this equation can be written as the form of generalized Kähler-Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t}\omega_{\varphi_\epsilon} = -Ric(\omega_{\varphi_\epsilon}) + (1 - \beta)\sqrt{-1}\partial\bar{\partial} \log \frac{\|S\|^2 + \epsilon^2}{\|\cdot\|^2} \\ \omega_{\varphi_\epsilon}(\cdot, 0) = \omega_{0,\epsilon} \end{cases} \quad (4.8)$$

Note that the elliptic version of such approximation equations was set up in [44]. The main steps here are also similar to Tian's approximation. We need to set up C^0 -estimate, C^2 -estimate and high order derivative estimate step by step and finally complete approximation argument. In the rest of this subsection we will complete C^0 -estimate and leave other steps to the next subsection.

Now let's consider the equation (4.7). We can rewrite this equation as following

$$\frac{\partial}{\partial t}\varphi_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}.$$

To get an upper bound for φ_ϵ , we can make use of maximal principle to get that

$$\frac{\partial}{\partial t} \sup \varphi_\epsilon \leq \sup \log \frac{\omega_{t,\epsilon}^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}.$$

Recall the definition of the function $\chi(\epsilon^2 + t)$, we know from [4],

$$\sqrt{-1} \partial \bar{\partial} \chi(\epsilon^2 + \|S\|^2) = \frac{\beta^2 \sqrt{-1} DS \wedge \bar{D} S}{(\epsilon^2 + \|S\|^2)^{1-\beta}} - \beta((\epsilon^2 + \|S\|^2)^\beta - \epsilon^{2\beta}) R(\|\cdot\|).$$

From this computation, we observe that near the divisor D ,

$$\frac{\omega_{t,\epsilon}^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} \approx C,$$

which is independent of ϵ . So we can get a uniform upper bound that $\varphi_\epsilon \leq C_1$. Do the same argument for the lower bound, make use of maximal principle again, we can give a uniform C^0 -estimate that

$$|\varphi_\epsilon| \leq C_1.$$

Next we hope to bound the time derivative for φ_ϵ . We can apply the technique developed in [37]. For simplicity, we write $\rho = -Ric(\Omega) + (1 - \beta)R(\|\cdot\|) = \dot{\omega}_{t,\epsilon}$. Take the derivative of (4.7), we have

$$\frac{\partial}{\partial t} \dot{\varphi}_\epsilon = \Delta \dot{\varphi}_\epsilon + tr_\omega \rho, \tag{4.9}$$

where $\omega = \omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon$, and Δ is the Laplacian w.r.t ω . First, we compute that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)(t\dot{\varphi}_\epsilon - \varphi_\epsilon - nt) &= t tr_\omega \rho + n - tr_\omega \omega_{t,\epsilon} - n \\ &= -tr_\omega \omega_{0,\epsilon} < 0. \end{aligned}$$

From this we have $\dot{\varphi}_\epsilon \leq n + \frac{C_1}{t}$. Combined with the equation (4.7) at time $t=0$, and

local existence of parabolic equation, we obtain a uniform upper bound for $\dot{\varphi}_\epsilon$. Now we try to derive the lower bound for $\dot{\varphi}_\epsilon$,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\varphi}_\epsilon + A\varphi_\epsilon - n \log t) &= \text{tr}_\omega(\rho + A\omega_{t,\epsilon}) + A \log \frac{\omega^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - An - \frac{n}{t} \\
&\geq C\left(\frac{\omega_{0,\epsilon}^n}{\omega^n}\right)^{\frac{1}{n}} + A \log \frac{\omega^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - An - \frac{n}{t} \\
&\geq C_1\left(\frac{\omega_{0,\epsilon}^n}{\omega^n}\right)^{\frac{1}{n}} - \frac{C_2}{t}.
\end{aligned}$$

Let's explain these two inequalities. For the first, note that we can choose A sufficiently large such that $\rho + A\omega_{t,\epsilon} \geq \omega_{0,\epsilon}$ in the time interval, then this inequality follows from Schwarz inequality. For the second, note that in the proof of C^0 -estimate, we have got that $\frac{\omega_{0,\epsilon}^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} \approx C$, then this inequality follows from the behavior of logarithmic functions. Then by maximal principle, at the minimal point of the function $\dot{\varphi}_\epsilon + A\varphi_\epsilon - n \log t$, we have that $\omega^n \geq C_4 t^n \omega_{0,\epsilon}^n$. Make use of this inequality and the relation of $\omega_{0,\epsilon}$ and Ω again, we get that

$$\begin{aligned}
\dot{\varphi}_\epsilon + A\varphi_\epsilon - n \log t &= \log \frac{\omega^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} + A\varphi_\epsilon - n \log t \\
&\geq \log \frac{C_4 t^n \omega_{0,\epsilon}^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} + A\varphi_\epsilon - n \log t \geq C_5.
\end{aligned}$$

As φ_ϵ is uniformly bounded, we get a uniform lower bound for $\dot{\varphi}_\epsilon$.

4.2.2 C^2 -estimate, and high order estimate outside the divisor

Now we continue our proof. We want to set up a uniform C^2 -estimate and high order estimate for φ_ϵ . First, like [37] [4] [28], we prove a Laplacian estimate, which is essentially an application of generalized Aubin-Yau inequality:

Theorem 4.2.1. *Let φ_ϵ solve the equation (4.7). As we have*

$$|\varphi_\epsilon| \leq C, |\dot{\varphi}_\epsilon| \leq C,$$

on $[0, T_\delta]$, then on this time interval, there exists a uniform constant A which is independent of ϵ , such that

$$A^{-1}\omega_{0,\epsilon} \leq \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon \leq A\omega_{0,\epsilon}.$$

Proof. First we take an holomorphic orthonormal coordinates $(z_1, \dots, z_n)$ at a point p for the metric $\omega_{0,\epsilon}$, say, $g_{0i\bar{j}} = \delta_{ij}$, such that $g_{i\bar{j}}(\omega_{t,\epsilon}) = \lambda_i \delta_{ij}$, $\varphi_{\epsilon i\bar{j}} = \varphi_{\epsilon i\bar{i}} \delta_{ij}$. Write $g_{i\bar{j}}$ as the metric of $\omega = \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon$ and assume that under such coordinates, $\frac{\partial g_{i\bar{j}}}{\partial z_k} = 0$. From approximation flow equations (4.7) (4.8), we compute that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \log tr_{\omega_{0,\epsilon}} \omega &= \frac{(\frac{\partial}{\partial t} - \Delta) tr_{\omega_{0,\epsilon}} \omega}{tr_{\omega_{0,\epsilon}} \omega} + \frac{|\nabla tr_{\omega_{0,\epsilon}} \omega|^2}{|tr_{\omega_{0,\epsilon}} \omega|^2} \\ &= \frac{1}{tr_{\omega_{0,\epsilon}} \omega} \left\{ g_0^{i\bar{i}} (-R_{i\bar{i}} + (1-\beta) \partial_i \partial_{\bar{i}} \log \frac{\|S\|^2 + \epsilon^2}{\|\cdot\|^2}) \right. \\ &\quad \left. - g_{i\bar{i}} g^{k\bar{k}} (-R_{k\bar{k}l}^i g_0^{l\bar{l}} + R_{i\bar{i}k\bar{k}}(\omega_{0,\epsilon}) + \sum_j \frac{\partial g_{0i\bar{j}}}{\partial \bar{z}_k} \frac{\partial g_{0j\bar{i}}}{\partial z_k}) \right\} + \frac{|\nabla tr_{\omega_{0,\epsilon}} \omega|^2}{|tr_{\omega_{0,\epsilon}} \omega|^2} \\ &= \frac{1}{tr_{\omega_{0,\epsilon}} \omega} \left\{ (1-\beta) \Delta_{\omega_{0,\epsilon}} \log \frac{\|S\|^2 + \epsilon^2}{\|\cdot\|^2} - g_{i\bar{i}} g^{k\bar{k}} R_{i\bar{i}k\bar{k}}(\omega_{0,\epsilon}) \right. \\ &\quad \left. - g_{i\bar{i}} g^{k\bar{k}} \sum_j \frac{\partial g_{0i\bar{j}}}{\partial \bar{z}_k} \frac{\partial g_{0j\bar{i}}}{\partial z_k} \right\} + \frac{1}{|tr_{\omega_{0,\epsilon}} \omega|^2} \sum_{i,j} g^{k\bar{k}} g_{i\bar{i}} g_{j\bar{j}} \frac{\partial g_{0i\bar{i}}}{\partial z_k} \frac{\partial g_{0j\bar{j}}}{\partial \bar{z}_k} \\ &\leq \frac{1}{tr_{\omega_{0,\epsilon}} \omega} \left\{ (1-\beta) \Delta_{\omega_{0,\epsilon}} \log \frac{\|S\|^2 + \epsilon^2}{\|\cdot\|^2} - R(\omega_{0,\epsilon}) \right. \\ &\quad \left. - \sum_{i \leq j} R_{i\bar{i}j\bar{j}}(\omega_{0,\epsilon}) \left(\frac{\lambda_i + \varphi_{\epsilon i\bar{i}}}{\lambda_j + \varphi_{\epsilon j\bar{j}}} + \frac{\lambda_j + \varphi_{\epsilon j\bar{j}}}{\lambda_i + \varphi_{\epsilon i\bar{i}}} - 2 \right) \right\}. \end{aligned}$$

As [28], we choose a function $\chi_\rho(\|S\|^2 + \epsilon^2) = \rho \int_0^{\|S\|^2 + \epsilon^2} \frac{(\epsilon^2 + r)^\rho - \epsilon^{2\rho}}{r} dr$. By the computation

in [4] [15] we have that

$$R_{i\bar{i}j\bar{j}}(\omega_{0,\epsilon}) \geq -C_1 - \frac{C_2}{(\|S\|^2 + \epsilon^2)^{1-\beta}}.$$

Meanwhile, we have

$$\sqrt{-1}\partial\bar{\partial}\chi_\rho(\epsilon^2 + \|S\|^2) = \frac{\rho^2\sqrt{-1}DS \wedge \bar{D}S}{(\epsilon^2 + \|S\|^2)^{1-\rho}} - \rho((\epsilon^2 + \|S\|^2)^\rho - \epsilon^{2\rho})R(\|\cdot\|),$$

so if we choose suitable constants ρ, C we can obtain that

$$R_{i\bar{i}j\bar{j}}(\omega_{0,\epsilon}) \geq -C - C'\chi_{\rho i\bar{i}},$$

then we have such estimate

$$\begin{aligned} & \frac{1}{tr_{\omega_{0,\epsilon}}\omega} \left\{ - \sum_{i \leq j} R_{i\bar{i}j\bar{j}}(\omega_{0,\epsilon}) \left(\frac{\lambda_i + \varphi_{\epsilon i\bar{i}}}{\lambda_j + \varphi_{\epsilon j\bar{j}}} + \frac{\lambda_j + \varphi_{\epsilon j\bar{j}}}{\lambda_i + \varphi_{\epsilon i\bar{i}}} - 2 \right) \right\} \\ & \leq \frac{1}{\sum_i (\lambda_i + \varphi_{\epsilon i\bar{i}})} \sum_{i < j} \left\{ \frac{\lambda_i + \varphi_{\epsilon i\bar{i}}}{\lambda_j + \varphi_{\epsilon j\bar{j}}} (C + C'\chi_{\rho j\bar{j}}) + \frac{\lambda_j + \varphi_{\epsilon j\bar{j}}}{\lambda_i + \varphi_{\epsilon i\bar{i}}} (C + C'\chi_{\rho i\bar{i}}) \right\} \\ & \leq \sum_i \frac{C + C'\chi_{\rho i\bar{i}}}{\lambda_i + \varphi_{\epsilon i\bar{i}}} = C tr_{\omega}\omega_{0,\epsilon} + C'\Delta\chi_\rho. \end{aligned}$$

Now we want to estimate the other term. First We have that

$$\begin{aligned} & (1 - \beta)\Delta_{\omega_{0,\epsilon}} \log \frac{\|S\|^2 + \epsilon^2}{\|\cdot\|^2} - R(\omega_{0,\epsilon}) \\ & = \Delta_{\omega_{0,\epsilon}} \log \frac{(\|S\|^2 + \epsilon^2)^{1-\beta}\omega_{0,\epsilon}^n}{\Omega} - tr_{\omega_{0,\epsilon}} Ric(\Omega) + (1 - \beta)tr_{\omega_{0,\epsilon}} R(\|\cdot\|). \end{aligned}$$

Now recall that $\omega_{0,\epsilon} \geq \gamma\omega_0$, and $Ric(\Omega)$, $R(\|\cdot\|)$ are uniformly bounded independent of ϵ , moreover by [4] [15],

$$\sqrt{-1}\partial\bar{\partial} \log \frac{(\|S\|^2 + \epsilon^2)^{1-\beta}\omega_{0,\epsilon}^n}{\Omega} \leq C\omega_{0,\epsilon} + C\sqrt{-1}\partial\bar{\partial}\chi_\rho,$$

we can bound this term. Considering that $tr_{\omega_{0,\epsilon}}\omega \cdot tr_{\omega}\omega_{0,\epsilon} \geq n$, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)(\log tr_{\omega_{0,\epsilon}}\omega + C'\chi_\rho - B\varphi_\epsilon) \\ & \leq Ctr_{\omega}\omega_{0,\epsilon} + Ctr_{\omega_{0,\epsilon}}\omega - B\dot{\varphi}_\epsilon + Bn - Btr_{\omega}\omega_{t,\epsilon}. \end{aligned}$$

As $\omega_{0,\epsilon}$ and $\omega_{t,\epsilon}$ are equivalent, and $\dot{\varphi}_\epsilon$ are uniformly bounded, choose a suitable constant B , we obtain that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log tr_{\omega_{0,\epsilon}}\omega + C'\chi_\rho - B\varphi_\epsilon) \leq C - tr_{\omega}\omega_{0,\epsilon}.$$

By the maximal principle, at the maximal point p of $\log tr_{\omega_{0,\epsilon}}\omega + C'\chi_\rho - B\varphi_\epsilon$ we have $tr_{\omega}\omega_{0,\epsilon}(p) \leq C$. By the approximation flow equation (4.7), we know that

$$\frac{\omega^n}{\omega_{0,\epsilon}^n} = e^{\dot{\varphi}_\epsilon} \frac{\Omega}{(\|S\|^2 + \epsilon^2)^{1-\beta} \omega_{0,\epsilon}^n}$$

is uniformly bounded from above and away from 0, then we obtain that at that point p ,

$$tr_{\omega_{0,\epsilon}}\omega(p) \leq \frac{\omega^n}{\omega_{0,\epsilon}^n} (tr_{\omega}\omega_{0,\epsilon})(p)^{n-1} \leq C.$$

As at that point p , $\log tr_{\omega_{0,\epsilon}}\omega + C'\chi_\rho - B\varphi_\epsilon$ attains the maximal, we conclude that on the whole manifold M , $tr_{\omega_{0,\epsilon}}\omega \leq C$. Now make use of the inequality above again we have that

$$tr_{\omega}\omega_{0,\epsilon} \leq \frac{\omega^n}{\omega_{0,\epsilon}^n} (tr_{\omega_{0,\epsilon}}\omega)^{n-1} \leq C,$$

which gives us a uniform constant $A > 0$ such that

$$A^{-1}\omega_{0,\epsilon} \leq \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon \leq A\omega_{0,\epsilon}.$$

□

Now let's consider high order estimates. We know that until now we don't have a uniform C^3 -estimate for conic Kähler-Einstein metrics. However, to get C^∞ -convergence away from the divisor, we only need to have local high order estimates away from the divisor. Note that from the two forms of approximation flow equations (4.7) (4.8), compare with [28], we find that our generalized Kähler-Ricci flow has the form that

$$\frac{\partial}{\partial t}\omega = -Ric(\omega) + \theta, \quad (4.10)$$

where $\theta = (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log \frac{\|S\|^2 + \epsilon^2}{\|\cdot\|^2}$ is uniformly bounded away from the divisor.

And we can also define that

$$S = |\nabla_0 g|_\omega^2 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} \nabla_{0i} g_{k\bar{q}} \bar{\nabla}_{0j} g_{p\bar{l}},$$

where g is the metric of $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(\|\cdot\|)) + \sqrt{-1}\partial\bar{\partial}\varphi$, and ∇_0 denotes the covariant derivatives with respect to ω_0 . Then here we can apply Proposition 2.2 in [28] directly:

Theorem 4.2.2. *Let $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi$ solve the generalized Kähler-Ricci flow (4.10) and satisfy*

$$A^{-1}\omega_0 \leq \omega \leq A\omega_0 \quad \text{on} \quad B_r(p) \times [0, T].$$

Then there exist constant C', C'' such that

$$S \leq \frac{C'}{r^2}, \quad |Rm(\omega)|^2 \leq \frac{C''}{r^4}$$

on $B_{\frac{r}{2}}(p) \times [0, T]$. Here C' depends on $\omega_0, T, A, \|\varphi(\cdot, 0)\|_{C^3(B_r(p))}, \|\theta\|_{C^1(B_r(p))}$ and C'' depends on $\omega_0, T, A, \|\varphi(\cdot, 0)\|_{C^4(B_r(p))}, \|\theta\|_{C^2(B_r(p))}$.

Locally outside the divisor, metrics $\omega_0, \omega_{0,\epsilon}, \omega_{t,\epsilon}$ are equivalent, and derivatives of these metrics are also uniformly bounded on the time interval, then we have C^3 -local estimates for φ_ϵ under the metric ω_0 . By this theorem, using bootstrap methods we

can also get local uniform high order estimates outside the divisor D .

4.2.3 uniqueness and convergence

In this subsection we want to examine how the approximation solutions converge and what the limit looks like. First, in the sections above we proved that on time interval $[0, T_\delta]$ we have a priori estimates for the solution of approximation flow equation (4.7). Now we hope to solve this equation on $[0, T)$. What remains to do is to prove the solution is independent of the choice of δ . We can prove this as [47]. We recall that for each $\delta > 0$, we can find a volume form Ω such that $\omega_t = \omega_0 - t(Ric(\Omega) - (1 - \beta)R(\|\cdot\|)) > 0$ on $[0, T_\delta]$. Now we suppose that for another $\delta' > 0$, we have Ω' such that $\omega'_t = \omega_0 - t(Ric(\Omega') - (1 - \beta)R(\|\cdot\|)) > 0$. Now we assume that $\Omega' = e^f \Omega$ where f is a smooth function on M and φ'_ϵ solves the equation

$$\frac{\partial}{\partial t} \varphi'_\epsilon = \log \frac{(\omega'_t + \sqrt{-1} \partial \bar{\partial} \varphi'_\epsilon)^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega'}, \quad \varphi'_\epsilon(\cdot, 0) = 0.$$

Now put $\overline{\varphi}_\epsilon = \varphi'_\epsilon + tf$, as we have $Ric(\Omega') = Ric(e^f \Omega) = Ric(\Omega) - \sqrt{-1} \partial \bar{\partial} f$, we compute that

$$\begin{aligned} \frac{\partial}{\partial t} \overline{\varphi}_\epsilon &= \frac{\partial}{\partial t} \varphi'_\epsilon + f = \log \frac{(\omega'_t + \sqrt{-1} \partial \bar{\partial} \varphi'_\epsilon)^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega'} + f \\ &= \log \frac{(\omega_t + t\sqrt{-1} \partial \bar{\partial} f + \sqrt{-1} \partial \bar{\partial} \varphi'_\epsilon)^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{e^f \Omega} + f \\ &= \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \overline{\varphi}_\epsilon)^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}, \end{aligned}$$

and $\overline{\varphi}_\epsilon(\cdot, 0) = 0$. From this we find that $\overline{\varphi}_\epsilon$ just solves the equation (4.7). By the uniqueness of the solution for (4.7) we know $\overline{\varphi}_\epsilon = \varphi_\epsilon$, which means that for arbitrary δ, δ' , their corresponding solutions are the same essentially. So we can glue these solutions together to get a solution for (4.7) on the time interval $[0, T)$.

Finally, we want to prove that as ϵ tends to 0, the solution φ_ϵ to (4.7) converges to a solution of unnormalized conical Kähler-Ricci flow in suitable sense. By the argument above, for simplicity we can prove this on some closed time interval $[0, T_\delta]$. We can prove this as [28]. Recall that we have uniform bound for $\varphi_\epsilon, \dot{\varphi}_\epsilon$ and Laplacian estimate $A^{-1}\omega_{0,\epsilon} \leq \omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon \leq A\omega_{0,\epsilon}$. For any compact set $K \subset\subset M \setminus D$ we have a number N which depends on A and K such that $N^{-1}\omega_0 \leq \omega \leq N\omega_0$, and on this set $K \log(\|S\|^2 + \epsilon^2)$ converges to $\log\|S\|^2$, $\omega_{0,\epsilon}$ converges to ω^* in C_{loc}^∞ sense. Let K approximate $M \setminus D$ and ϵ_i tend to 0, on time interval $[0, T_\delta]$, by diagonal rule we have a sequence which is denoted by $\varphi_{\epsilon_i}(t)$, and converges to a function $\varphi(t)$, which is smooth away from D , in C_{loc}^∞ sense outside D . By 4.2.1 we know that $\omega_{\varphi(t)} = \bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ are conic Kähler metrics with cone angle $2\pi\beta$ along the divisor D on the whole time interval.

Now let's show that the limit potential $\varphi(t)$ we defined above satisfies the conical Kähler-Ricci flow equation (4.5) globally on $M \times [0, T_\delta]$ in the sense of currents. For any $(n-1, n-1)$ -form η , by the argument above, as ϵ_i tends to 0, we have

$$\begin{aligned} \int_M \sqrt{-1}\partial\bar{\partial}\frac{\partial}{\partial t}\varphi_{\epsilon_i} \wedge \eta &= \int_M \log \frac{(\omega_{\epsilon_i,t} + \sqrt{-1}\partial\bar{\partial}\varphi_{\epsilon_i})^n (\|S\|^2 + \epsilon_i^2)^{1-\beta}}{\Omega} \wedge \sqrt{-1}\partial\bar{\partial}\eta \\ &\rightarrow \int_M \log \frac{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n \|S\|^{2(1-\beta)}}{\Omega} \wedge \sqrt{-1}\partial\bar{\partial}\eta \\ &= \int_M \sqrt{-1}\partial\bar{\partial} \log \frac{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n \|S\|^{2(1-\beta)}}{\Omega} \wedge \eta. \end{aligned}$$

On the other hand, let $K \subset\subset M \setminus D$ and when K approximates $M \setminus D$, we have $\int_{M \setminus D} \sqrt{-1}\partial\bar{\partial}\eta$ tends to 0. As we have $\dot{\varphi}_{\epsilon_i}, \dot{\varphi}$ are uniformly bounded independent of ϵ , we see that

$$|\int_M (\dot{\varphi}_{\epsilon_i} - \dot{\varphi}) \wedge \sqrt{-1}\partial\bar{\partial}\eta| \leq |\int_K (\dot{\varphi}_{\epsilon_i} - \dot{\varphi}) \wedge \sqrt{-1}\partial\bar{\partial}\eta| + |\int_{M \setminus K} (\dot{\varphi}_{\epsilon_i} - \dot{\varphi}) \wedge \sqrt{-1}\partial\bar{\partial}\eta|.$$

The first integral tends to 0 as φ_{ϵ_i} converges to φ in the sense of $C^\infty(K)$, and the second one tends to as $\dot{\varphi}_{\epsilon_i}, \dot{\varphi}$ are uniformly bounded and K approximates $M \setminus D$. From above we can see that $\varphi(t)$ satisfies the conical Kähler-Ricci flow equation (4.5) globally on $M \times [0, T)$ in the sense of currents.

To prove the uniqueness we argue as Wang [49], which comes from Jeffery's argument [20]. First we need to verify that $\varphi(t)$ is Hölder continuous with respect to ω_0 . As [28] write $\phi = \varphi + k||S||^{2\beta}$, in any closed time interval, say $[0, T_\delta]$, we have uniform bound for $\phi, \dot{\phi}$. Then we rewrite the equation of conical Kähler-Ricci flow (4.5) as

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{\dot{\phi}} \frac{\Omega}{||S||^{2(1-\beta)}}$$

away from D . As $\beta \in (0, 1)$ we find a constant η such that $2(1-\beta)(1+\eta) < 2$ such that

$$\int_M e^{(1+\eta)(\dot{\phi} - \log ||S||^{2(1-\beta)})} \Omega \leq C \int_M \frac{\Omega}{||S||^{2(1-\beta)(1+\eta)}} \leq C.$$

By Kolodziej's L^p estimate [22] we know that $\varphi(t)$ is Hölder continuous with respect to ω_0 on $[0, T_\delta]$, actually, on $[0, T)$. Now suppose there are two solutions $\varphi_1(t), \varphi_2(t)$ which satisfy all the properties above and solve the conical Kähler-Ricci flow on the maximal time interval, then we set $\tilde{\varphi}_1 = \varphi_1(t) + a||S||^{2p}$ and write $v = \tilde{\varphi}_1 - \varphi_2$. Compare corresponding conical Kähler-Ricci flow equations, we have that

$$\frac{\partial}{\partial t} v = \frac{\partial}{\partial t} (\varphi_1 - \varphi_2) = \log \frac{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_1)^n}{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_2)^n} = \underline{\Delta} v - a \underline{\Delta} ||S||^{2p}, \quad (4.11)$$

where

$$\underline{\Delta} := \int_0^1 (\bar{g}_t + \sqrt{-1}\partial\bar{\partial}((1-s)\varphi_1 + s\varphi_2))^{i\bar{j}} \partial_i \bar{\partial}_{\bar{j}} ds.$$

Outside the divisor D we can compute that

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}||S||^{2p} &= p||S||^{2p}\sqrt{-1}\partial\bar{\partial}||S||^2 + \sqrt{-1}p^2||S||^{2p-2}\partial||S||^2 \wedge \bar{\partial}||S||^2 \\ &\geq -p||S||^{2p}R(||\cdot||).\end{aligned}$$

By the estimates above, we know that the metric $\bar{g}_t + \sqrt{-1}\partial\bar{\partial}((1-s)\varphi_1 + s\varphi_2)$ is equivalent to the initial conic metric ω^* and C^α outside the divisor D . Then from (4.11) we obtain that

$$\frac{\partial}{\partial t}v \leq \underline{\Delta}v + aC,$$

by maximal principle, we have

$$v(t) \leq \sup v(0) + aCt = a(Ct + ||S||^{2p}).$$

Let a tend to 0, we have $\varphi_1 \leq \varphi_2$, and we can prove $\varphi_2 \leq \varphi_1$ by the same argument. Finally we obtain the uniqueness of the limit solution.

4.2.4 $C^{2,\alpha}$ -estimate on the divisor D

To complete the proof of Theorem 4.1.1, we need to give a $C^{2,\alpha}$ -estimate for $\varphi(t)$, as this estimate allows us to apply inverse function theorem to obtain the existence of the solution to conical Kähler-Ricci flow (4.5). In this part, we will only introduce the definition of these norms under conic setting and leave the proof to the last chapter. Let's describe the basic construction. First we can fix a unit polydisk $\mathbb{D}^n \subset \mathbb{C}$ with the origin as the center, and the divisor $D = \{z_1 = 0\}$. Then the standard conic metric attached with $(\mathbb{C}, D)$ is that

$$\omega_\beta := \sqrt{-1} \left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^n dz_i \wedge d\bar{z}_i \right),$$

which defines a Riemannian metric g_β and induces a distance d_β . For simplicity, define $\bar{d}(p_1, p_2) := d_\beta(x_1, x_2) + |t_1 - t_2|^{\frac{1}{2}}$ as the spacetime distance, where $p_i = (x_i, t_i)$. Now for a locally integrable function on $\mathbb{D}^n \times [0, T)$ and $\alpha \in (0, 1)$, we define the Hölder norm as [15]:

$$[f]_\alpha := \sup_{\mathbb{D}^n \times [0, T)} |f| + \sup_{p_1 \neq p_2} \frac{|f(p_1) - f(p_2)|}{\bar{d}(p_1, p_2)^\alpha}.$$

We say the function f is C^α if the norm $[f]_\alpha < +\infty$. Consider the vector fields $\xi_1 = |z_1|^{1-\beta} \frac{\partial}{\partial z_1}, \xi_k = \frac{\partial}{\partial z_k}$, for $k = 2, \dots, n$, then we say a $(1, 0)$ -form τ is C^α if $\tau(\xi_i)$ is C^α for any $i = 1, \dots, n$, and say a $(1, 1)$ -form σ is C^α if $\sigma(\xi_i, \xi_j)$ is C^α for any $i, j = 1, \dots, n$. And we say f is $C^{2,\alpha}$ if $f, \partial f, \sqrt{-1}\partial\bar{\partial}f, \bar{\partial}f$ are all C^α .

Consider the equation (4.5), as ω_t is always smooth on the maximal interval, when r is sufficient small we can write $\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t) = \sqrt{-1}\partial\bar{\partial}u$ locally, then we only need to show that u is $C^{2,\alpha}$. By computation before we know that $Ric(\omega_{beta}) = 2\pi(1 - \beta)[D] - \sqrt{-1}\partial\bar{\partial}F$ for some smooth function f , so the equation for u can be written as

$$\frac{\partial}{\partial t}u = \log \frac{(\sqrt{-1}\partial\bar{\partial}u)^n}{\omega_\beta^n} + F.$$

Take the covariant derivatives with respect to ω_β , we have

$$\frac{\partial}{\partial t}u_{k\bar{l}} = u^{i\bar{j}}u_{k\bar{l}i\bar{j}} - u^{i\bar{q}}u^{p\bar{j}}u_{i\bar{j}k}u_{p\bar{q}\bar{l}} + F_{k\bar{l}}.$$

Unfortunately, classic Evans-Krylov-Safanov estimate for nonlinear PDE can't be used in conic setting. In conic Kähler-Einstein case, as the Ricci curvature of target metric has lower bound, by Cheeger-Colding-Tian theory [5], the metric near the conic singularities is close to a standard flat conic metric, which indicates that perturbed Schauder estimate can be applied. Alternately, Tian extended his master thesis to conic case and gave a proof of $C^{2,\alpha}$ -estimate in case of conic Kähler-Einstein metrics. In Chapter 5, we will follow Tian's approach to prove this theorem:

Theorem 4.2.3. *u is $C^{2,\alpha}$ -bounded on the divisor D .*

This theorem completes the proof for the first main theorem 4.1.1.

4.3 Limit behavior near the singular time when the twisted canonical bundle is big

In the last section we discussed how long the unnormalized conical Kähler-Ricci flow would last. Now we want to understand what will happen when the flow runs to singular time. During the remaining part of this article we assume that M is an n -dimensional projective manifold. In [47] [37], to study minimal model program, they mainly considered the case that the canonical line bundle K_M is big, or big and nef (numerical effective). First let's recall their definitions:

Definition 4.3.1. *Suppose L is a holomorphic line bundle on a projective manifold M , then we say L is big if $[c_1(L)]^n = \int_M c_1(L)^n > 0$. And we say L is nef if for any algebraic curve C on M , $c_1(L)(C) = \int_C c_1(L) \geq 0$.*

To study the limit behavior near the singular time, they mainly made use of so-called Kodaira's lemma, which was proved by Kawamata and used in degenerate Monge-Ampere equations by Tsuji [48] first:

Lemma 4.3.2. *Let L be a line bundle on a projective manifold M . If L is big, then there exists an effective line bundle E and two positive numbers a, b such that $L - \delta E$ is positive for any $\delta \in (a, b)$. Moreover, if L is big and nef, then the conclusion above is true for $\delta \in (0, b)$.*

Now we can begin the proof of theorem 4.1.2. As the last section, we can do similar constructions and consider the approximation flow equation (4.7) again that

$$\frac{\partial}{\partial t} \varphi_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon)^n}{\Omega} + \log(\|S\|^2 + \epsilon^2)^{1-\beta}.$$

Although $\omega_{t,\epsilon}$ may not be a Kähler metric near the singular time T , as it's controlled from above, we can still make use of maximal principle to get a uniformly upper bound for φ_ϵ . To get a lower bound, we need to apply Kodaira's lemma to get a family of regular background metrics. Recall that we define $\omega_t = \omega_0 - t(\text{Ric}(\Omega) - (1-\beta)R(\|\cdot\|))$, and we assume that the twisted canonical bundle $K_M + (1-\beta)[D]$ is big, then we can easily see that the current $[\omega_t]$ is big on the time interval $[0, T_0]$. By Kodaira's lemma, we have a $\delta \in (a, b)$ and an effective divisor E with a Hermitian metric $\|\cdot\|'$ such that $\omega_t + \delta\sqrt{-1}\partial\bar{\partial}\log\|\cdot\|'^2 > 0$. If we define S' as a local holomorphic section for E , we will have

$$\omega_t + \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|^2 > 0.$$

Set $\omega'_{t,\epsilon} = \omega_{t,\epsilon} + \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|^2$, we can rewrite the equation (4.7) as

$$\frac{\partial}{\partial t}(\varphi_\epsilon - \delta\log\|S'\|^2) = \log \frac{(\omega'_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}(\varphi_\epsilon - \delta\log\|S'\|^2))^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}. \quad (4.12)$$

Note that in this equation, $\omega'_{t,\epsilon}$ is a smooth metric, by the argument in the last section, we know that

$$\log \frac{\omega'^n_{t,\epsilon} (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}$$

is bounded from below by a constant depending on δ . By maximal principle, we know that $\varphi_\epsilon - \delta\log\|S'\|^2$ is bounded from below by C_δ . Now we get a C^0 -estimate for φ_ϵ that

$$C_\delta + \delta\log\|S'\|^2 \leq \varphi_\epsilon \leq C.$$

Now let's begin to estimate $\dot{\varphi}_\epsilon$. For simplicity, we denote $\varphi_{\epsilon,\delta} = \varphi_\epsilon - \delta\log\|S'\|^2$, and $\omega = \omega'_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_{\epsilon,\delta}$. Then the equation (4.12) can be written as

$$\frac{\partial}{\partial t}\varphi_{\epsilon,\delta} = \log \frac{\omega^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}.$$

Take the derivative of this equation, we obtain that

$$\frac{\partial}{\partial t} \dot{\varphi}_{\epsilon, \delta} = \Delta \dot{\varphi}_{\epsilon, \delta} + \text{tr}_\omega \rho,$$

which is the same with (4.9). Note that $\sqrt{-1} \partial \bar{\partial} \log \|S'\|^2 = -R(\|\cdot\|')$ outside the base divisor E , and $C_\delta \leq \varphi_{\epsilon, \delta} \leq C - \delta \log \|S'\|^2$, we can compute as [37],

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log \|S'\|^2) \\ &= \text{tr}_\omega \rho - A^2 \dot{\varphi}_{\epsilon, \delta} + A^2 (n - \text{tr}_\omega \omega'_{t, \epsilon}) + A \text{tr}_\omega R(\|\cdot\|') \\ &= \text{tr}_\omega (\rho - A^2 \omega'_{t, \epsilon} + AR(\|\cdot\|')) - A^2 (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log \|S'\|^2) \\ & \quad + (nA^2 - A^4 \varphi_{\epsilon, \delta} + A^3 \log \|S'\|^2). \end{aligned}$$

Choose suitable large constant A such that the first term is negative, make use of C^0 -estimate for $\varphi_{\epsilon, \delta}$ and $\log \|S'\|^2$ is bounded from above, we have that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log \|S'\|^2) &\leq -A^2 (\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log \|S'\|^2) \\ &\quad + C - C' \log \|S'\|^2. \end{aligned}$$

Now make use of maximal principle, note that the maximal of $\dot{\varphi}_{\epsilon, \delta} - A^2 \varphi_{\epsilon, \delta} + A \log \|S'\|^2$ can only be obtained outside E , we can finally get that

$$\dot{\varphi}_{\epsilon, \delta} \leq C - C' \log \|S'\|^2,$$

where the constants C, C' may depend on δ . Similarly, to get a lower bound, we can

compute that

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\varphi}_{\epsilon,\delta} + A^2\varphi_{\epsilon,\delta} - A\log \|S'\|^2) \\
&= tr_{\omega}\rho + A^2\log \frac{\omega^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - nA^2 + A^2tr_{\omega}\omega'_{t,\epsilon} - Atr_{\omega}R(\|\cdot\|') \\
&= tr_{\omega}\left(\rho + \frac{A^2}{2}\omega'_{t,\epsilon} - AR(\|\cdot\|')\right) - nA^2 \\
&\quad + A^2\left(\log \frac{\omega^n}{\omega'^n_{t,\epsilon}} + \frac{tr_{\omega}\omega'_{t,\epsilon}}{2} + \log \frac{\omega'^n_{t,\epsilon}(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}\right).
\end{aligned}$$

Choose A large enough such that the first term becomes positive, then make use of Schwarz inequality and the property of logarithmic functions, we obtain that

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\varphi}_{\epsilon,\delta} + A^2\varphi_{\epsilon,\delta} - A\log \|S'\|^2) \\
&\geq -nA^2 + A^2\left(\log \frac{\omega^n}{\omega'^n_{t,\epsilon}} + \frac{1}{2}\left(\frac{\omega'^n_{t,\epsilon}}{\omega^n}\right)^{\frac{1}{n}} + \log \frac{\omega'^n_{t,\epsilon}(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}\right) \\
&\geq A^2\left(-\log \frac{\omega^n}{\omega'^n_{t,\epsilon}} + \log \frac{\omega'^n_{t,\epsilon}(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega}\right) - C_1 \\
&= -A^2(\dot{\varphi}_{\epsilon,\delta} + A^2\varphi_{\epsilon,\delta} - A\log \|S'\|^2) + 2A^2\log \frac{\omega'^n_{t,\epsilon}(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} \\
&\quad + A^4\varphi_{\epsilon,\delta} - A^3\log \|S'\|^2 - C_1 \\
&\geq -A^2(\dot{\varphi}_{\epsilon,\delta} + A^2\varphi_{\epsilon,\delta} - A\log \|S'\|^2) - C_2,
\end{aligned}$$

By maximal principle and argue as above, we can get a lower bound for $\dot{\varphi}_{\epsilon,\delta}$ that

$$\dot{\varphi}_{\epsilon,\delta} \leq -C + C'\log \|S'\|^2.$$

Now we can summarize these estimates and obtain such a theorem:

Theorem 4.3.3. *Suppose the twisted canonical bundle $K_M + (1 - \beta)[D]$ is big on a projective manifold M , and E is an effective divisor such that $[\omega_t] - \delta[E] = [\omega_0] - t[c_1(M) - (1 - \beta)c_1(D)] - \delta[E] > 0$ on the time interval $[0, T_0]$ for a positive $\delta \in (a, b)$.*

Let φ_ϵ solves the approximation flow equation (4.7), then we have that

$$C_\delta + \delta \log ||S'||'^2 \leq \varphi_\epsilon \leq C, \quad |\dot{\varphi}_\epsilon| \leq C - C' \log ||S'||'^2.$$

Now we can derive a Laplacian estimate as before, which is also an application of generalized Schwarz Lemma.

Theorem 4.3.4. *On the time interval $[0, T_0]$ there exist constants $C, \alpha > 0$ such that $tr_{\omega_{0,\epsilon}} \omega \leq C ||S'||'^{-2\alpha}$.*

Proof. As theorem 4.2.1, we first compute that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\log tr_{\omega_{0,\epsilon}} \omega + C' \chi_\rho) \leq C tr_\omega \omega_{0,\epsilon}.$$

As in the proof of theorem 4.3.3, we define

$$H = \log tr_{\omega_{0,\epsilon}} \omega + C' \chi_\rho - A^2 \varphi_{\epsilon,\delta} + A \log ||S'||'^2,$$

then for sufficiently large $A > 0$, we have that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)H \\ & \leq tr_\omega (C \omega_{0,\epsilon} - A^2 \omega'_{t,\epsilon} + AR(|| \cdot ||')) - A^2 \dot{\varphi}_{\epsilon,\delta} + nA^2 \\ & \leq -tr_\omega \omega_{0,\epsilon} + C - C' \log ||S'||'^2. \end{aligned}$$

By maximal principle, we know that the maximal of H can only be obtained outside the divisor E , and when this maximal is obtained, we have

$$tr_\omega \omega_{0,\epsilon} \leq C - C' \log ||S'||'^2,$$

which indicates that at the maximal of H,

$$tr_{\omega_{0,\epsilon}} \omega \leq \frac{\omega^n}{\omega_{0,\epsilon}^n} (tr_{\omega} \omega_{0,\epsilon})^{n-1} = e^{\dot{\varphi}_\epsilon} \frac{\Omega}{(\|S\|^2 + \epsilon^2)^{1-\beta} \omega_{0,\epsilon}^n} \leq C \|S'\|'^{-2\alpha'}.$$

Then the theorem follows. $\square$

As in the last section, we can give local high order estimate for φ_ϵ outside the divisors D and E.

Proposition 4.3.5. *For any $K \subset\subset M \setminus (D \cup E)$ and $k > 0$, there exists $C_{k,K}$ such that*

$$\|\varphi_\epsilon\|_{C^k(K \times [0, T_0])} \leq C_{k,K}.$$

By this proposition, we can argue as the last section. We put a sequence of compact set to approximate the regular part $M \setminus (D \cup E)$. On each compact set K, as we have uniform estimates for high order derivatives, we can choose a sequence ϵ_i such that φ_{ϵ_i} converges on $K \times [0, T_0]$ in C_{loc}^∞ sense. By diagonal argument, we can take a sequence, say, φ_{ϵ_i} converges to a function φ on $(M \setminus (D \cup E)) \times [0, T_0]$ in the C_{loc}^∞ sense, which is smooth on $M \setminus (D \cup E)$. By theorem 4.3.4 we know that

$$\omega_\varphi = \omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi$$

is a conic Kähler metric with angle $2\pi\beta$ along the part of divisor D outside the divisor E on $[0, T_0]$. Now as above, we want to prove that φ_{ϵ_i} converges to φ globally in the sense of currents. Other arguments are the same except for the part near the divisor E. Note that when we consider the integral

$$\int_M \log \frac{(\bar{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n \|S\|^{2(1-\beta)}}{\Omega} \wedge \sqrt{-1} \partial \bar{\partial} \eta,$$

near the divisor E, by theorem 4.3.4, the integral function grows as $\log \|S\|'$, so this

integral exists. On the other hand, by theorem 4.3.4, as $\dot{\varphi}, \dot{\varphi}_{\epsilon_i}$ grows like $\log ||S||'$, then it's also true that

$$|\int_M (\dot{\varphi}_{\epsilon_i} - \dot{\varphi}) \wedge \sqrt{-1} \partial \bar{\partial} \eta|$$

tends to 0. So we know that the limit $\varphi(t)$ outside the divisor E satisfies conical Kähler-Ricci flow equation on $[0, T_0]$ in the sense of currents, and as t tends to T_0 , $\omega_\varphi = \bar{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ converges smoothly to a Kähler metric on $M \setminus (D \cup E)$, and conic along $D \setminus E$ with cone angle $2\pi\beta$. Now we want to claim that the limit is unique, i.e. for any sequence $\{\epsilon_i\}$ which tends to 0, the sequence φ_{ϵ_i} will converge to the same limit in our sense. Actually this follows from the uniqueness of the solution on the time interval $[0, T_0)$. Then consider that in any compact set in $M \setminus (D \cup E)$, as $\dot{\varphi}_{\epsilon_i}$ is uniformly bounded, we can conclude that the limit of any sequence φ_{ϵ_i} converges to the same limit, say, $\varphi(t)$ at T_0 , which completes our claim. Finally, note that the base locus E may not be unique, however, we can consider the intersection of these E 's and call it the stable base locus, and still denote it as E . By the argument above, we can obtain a unique limit current $\omega_\varphi = \omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi$ outside E at time T_0 , and this current is a smooth Kähler metric outside $D \cup E$, and conic along $D \setminus E$ with cone angle $2\pi\beta$. Finally, $C^{2,\alpha}$ -estimate in the proof of theorem 4.1.1 still holds here. Now we complete the proof of theorem 4.1.2.

4.4 Limit behavior in case of big and nef twisted canonical bundle

In this section we want to understand the limit behavior when the twisted canonical bundle is big and nef. By theorem 4.1.1 we know that this conical Kähler-Ricci flow exists forever. In this case we find that $\omega_t = \omega_0 - t(\text{Ric}(\Omega) - (1 - \beta)R(|| \cdot ||))$ will blow up at infinite time. To capture the limit behavior more precisely, we can consider

normalized conical Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\omega - Ric(\omega) + 2\pi(1 - \beta)[D]. \quad (4.13)$$

Note that if we define $\tilde{\omega}(t) := (1 + t)\omega(\log(1 + t))$, then we can compute that

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{\omega}(t) &= \omega(\log(1 + t)) + (-Ric(\omega(\log(1 + t)) - \omega(\log(1 + t)) + 2\pi(1 - \beta)[D])) \\ &= -Ric(\tilde{\omega}(t)) + 2\pi(1 - \beta)[D], \end{aligned}$$

which means that the solution of unnormalized conical Kähler-Ricci flow is actually the same with normalized conical Kähler-Ricci flow only modulo a scaling. As we did in the proof of theorem 4.1.1, we can consider this equation in the level of cohomology class. Then we can write

$$\omega_t = e^{-t}\omega_0 + (1 - e^{-t})(-Ric(\Omega) + (1 - \beta)R(|| \cdot ||)),$$

and set $\bar{\omega}_t = \omega_t + k\sqrt{-1}\partial\bar{\partial}||S||^{2\beta}$ and $\omega = \bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$. Now we can write the equation (4.13) in scalar form:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}\varphi = \log \frac{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} + \log ||S||^{2(1-\beta)} - \varphi - k||S||^{2\beta} \\ \varphi(\cdot, 0) = 0 \end{array} \right. \quad (4.14)$$

As before, we can consider an approximation flow equation of (4.14)

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}\varphi_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n (||S||^2 + \epsilon^2)^{1-\beta}}{\Omega} - \varphi_\epsilon - k\chi(\epsilon^2 + ||S||^2) \\ \varphi_\epsilon(\cdot, 0) = 0 \end{array} \right. \quad (4.15)$$

where

$$\omega_{t,\epsilon} := e^{-t}\omega_0 + (1 - e^{-t})(-Ric(\Omega) + (1 - \beta)R(\|\cdot\|)) + k\sqrt{-1}\partial\bar{\partial}\chi(\epsilon^2 + \|S\|^2)$$

as before.

First consider C^0 -estimate for φ_ϵ . By maximal principle, as

$$\log \frac{\omega_{t,\epsilon}^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - k\chi(\epsilon^2 + \|S\|^2)$$

is bounded from above, it's easy to see that $\varphi_\epsilon \leq C$. On the other hand, we recall lemma 4.3.2 as the twisted canonical bundle $K_M + (1 - \beta)[D]$ is big and nef, there exists an effective divisor E such that $\omega_{t,\epsilon} + \delta\sqrt{-1}\partial\bar{\partial}\log \|S'\|^2 > 0$ for all $t \in [0, \infty)$ and any $\delta \in (0, a)$, where S' is a local defining section of E . Then the equation (4.15) can be written as

$$\begin{aligned} \frac{\partial}{\partial t}(\varphi_\epsilon - \delta\sqrt{-1}\partial\bar{\partial}\log \|S'\|^2) &= \log \frac{(\omega'_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}(\varphi_\epsilon - \delta\sqrt{-1}\partial\bar{\partial}\log \|S'\|^2))^n}{\Omega \|S'\|^{2\delta}} \\ &\quad + \log(\|S\|^2 + \epsilon^2)^{1-\beta} - (\varphi_\epsilon - \delta\sqrt{-1}\partial\bar{\partial}\log \|S'\|^2) - k\chi(\epsilon^2 + \|S\|^2), \end{aligned}$$

where $\omega'_{t,\epsilon} = \omega_{t,\epsilon} + \delta\sqrt{-1}\partial\bar{\partial}\log \|S'\|^2$. By maximal principle again, as

$$\log \frac{\omega_{t,\epsilon}'^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega \|S'\|^{2\delta}} - k\chi(\epsilon^2 + \|S\|^2)$$

is uniformly bounded from below, we have that $\varphi_\epsilon \geq -C_\delta + \delta\sqrt{-1}\partial\bar{\partial}\log \|S'\|^2$. Now let's consider the estimate of $\dot{\varphi}_\epsilon$. Denote $\rho = -Ric(\Omega) + (1 - \beta)R(\|\cdot\|)$ again and

recall in [47], we have that

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta\right)(e^t \dot{\varphi}_\epsilon - \dot{\varphi}_\epsilon - \varphi_\epsilon - nt) \\
&= -tr_\omega(\omega_0 - \rho) + e^{-t} tr_\omega(\omega_0 - \rho) + \dot{\varphi}_\epsilon - \dot{\varphi}_\epsilon + n - tr_\omega(e^{-t}\omega_0 + (1 - e^{-t})\rho \\
&\quad + k\sqrt{-1}\partial\bar{\partial}\chi) - n = -tr_\omega(\omega_0 + k\sqrt{-1}\partial\bar{\partial}\chi) = -tr_\omega\omega_{0,\epsilon} < 0,
\end{aligned}$$

then by maximal principle, for $t > t_0 > 0$, we have $\dot{\varphi}_\epsilon \leq Cte^{-t}$. On the other hand, consider

$$H = \varphi_\epsilon + (1 - e^{t-T})\dot{\varphi}_\epsilon - \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|^2,$$

for $t < T$ and compute that

$$\left(\frac{\partial}{\partial t} - \Delta\right)H = -n + tr_\omega(\rho + e^{-T}(\omega_0 - \rho) + k\sqrt{-1}\partial\bar{\partial}\chi) = -n + tr_\omega\omega'_{T,\epsilon}.$$

We know that for $t \rightarrow \infty$, H tends to positive infinity near E , so the minimal point of H is attained away from E , where $-n + tr_\omega\omega'_{T,\epsilon} \leq 0$, by maximal principle. Then we have $\omega \geq \frac{\omega'_{T,\epsilon}}{n}$. At the minimal point of H , we have that

$$\begin{aligned}
H &= \varphi_\epsilon + (1 - e^{t-T})\left(\log \frac{\omega^n(\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - \varphi_\epsilon - k\chi\right) - \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|^2 \\
&\geq e^{t-T}\varphi_\epsilon + (1 - e^{t-T})\left(\log \frac{\omega'^n_{T,\epsilon}(\|S\|^2 + \epsilon^2)^{1-\beta}}{n^n\Omega} - k\chi\right) - \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|^2 \\
&\geq -C_\delta,
\end{aligned}$$

then we have that

$$(1 - e^{t-T})\dot{\varphi}_\epsilon \geq -\varphi_\epsilon + \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|^2 - C_\delta \geq -C_\delta.$$

Now let T tend to infinity, we have that $\dot{\varphi}_\epsilon \geq -C_\delta$. When we begin to study Laplacian estimate for φ_ϵ , we note that the evolution equation for $\log tr_{\omega_{0,\epsilon}}\omega$ is different from

before only by a constant. So do the same computation we can derive that

$$tr_{\omega_0, \epsilon} \omega \leq \frac{C}{\|S'\|^{2\alpha}}$$

again. By the argument in [28] again we can derive local high order estimate for φ_ϵ outside the base locus E and the divisor D that

$$\|\varphi_\epsilon\|_{C^k([0, \infty) \times K)} \leq C(k, K)$$

for any $K \subset\subset M \setminus (D \cup E)$.

Similar to last two sections, we can choose a sequence of compact sets to approximate $M \setminus (D \cup E)$, and a sequence φ_{ϵ_i} , by diagonal method, such that φ_{ϵ_i} converges to a function $\varphi(t)$ on $[0, \infty) \times M \setminus (D \cup E)$ in C_{loc}^∞ sense, which is smooth on $M \setminus (D \cup E)$. By the same argument as the last section we know that this convergence is also in the sense of currents globally. Similarly, as time tends to infinity, $\omega_\varphi = \bar{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ is smooth on $M \setminus (D \cup E)$ where E is the intersection of all base divisors, i.e. stable base locus of the twisted canonical bundle, and conic along $D \setminus E$ with cone angle $2\pi\beta$. As the flow exists forever, we hope to analyze the limit behavior of $\varphi(t)$ and ω_φ . As $\dot{\varphi}_\epsilon \leq Cte^{-t}k$ we can see that near the infinite time, φ_ϵ is nonincreasing. Note that we also have $\varphi_\epsilon \geq -C_\delta + \delta\sqrt{-1}\partial\bar{\partial} \log \|S'\|^2$, which indicates that φ_ϵ converges uniformly on arbitrary compact set $K \subset\subset M \setminus (D \cup E)$. And we just see that φ_ϵ is nonincreasing near the infinite time, so we can conclude that $\dot{\varphi}_\epsilon$ converges to 0 in $M \setminus (D \cup E)$. And we find that convergence properties here are independent of ϵ , we can see that $\varphi(t)$ converges uniformly on each compact set $K \subset\subset M \setminus (D \cup E)$, and $\dot{\varphi}$ converges to 0 in $M \setminus (D \cup E)$. Then recall the flow equation (4.14) that the right hand side of it tends to 0. Now set $\bar{\omega}_\infty = \bar{\omega}(\infty) = -Ric(\Omega) + (1 - \beta)R(\|\cdot\|) + k\sqrt{-1}\partial\bar{\partial}\|S\|^{2\beta}$, and φ_∞ as the limit of $\varphi(t)$, write $\omega_\infty = \bar{\omega}_\infty + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$. By (4.14), we conclude

that

$$\begin{aligned} Ric(\omega_\infty) &= Ric(\Omega) - \sqrt{-1}\partial\bar{\partial}\varphi_\infty - k\sqrt{-1}\partial\bar{\partial}\|S\|^{2\beta} + \log\|S\|^{2(1-\beta)} \\ &= -\omega_\infty + R(\|\cdot\|) + \sqrt{-1}\partial\bar{\partial}\log\|S\|^{2(1-\beta)} = -\omega_\infty + 2\pi(1-\beta)[D], \end{aligned}$$

which means that outside the stable base locus, the current ω_φ tends to a conical Kähler-Einstein metric, with cone angle $2\pi\beta$ along $D \setminus E$ as t tends to infinity. Moreover, $C^{2,\alpha}$ -estimate still holds. Finally, as [47], we can prove that the limit current is independent of the choice of the initial conic metric. First, by the same argument as in the proof of theorem 4.1.1, we can see that the solution is independent of the choice of volume form Ω . Now fix Ω , and see whether different initial conic Kähler metrics matter. Recall that for any $\delta \in (0, a)$, we have the estimates

$$-C_\delta + \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|'^2 \leq \varphi_\epsilon \leq C, \quad -C_\delta + \delta\sqrt{-1}\partial\bar{\partial}\log\|S'\|'^2 \leq \dot{\varphi}_\epsilon \leq C,$$

which indicates that $\varphi_\epsilon + \dot{\varphi}_\epsilon \leq C$. And we also know that $\dot{\varphi}_\epsilon$ tends to 0 as t tends to infinity. Then we obtain that

$$\begin{aligned} \int_M \frac{e^{\varphi_{\epsilon,\infty} + k\chi}}{(\|S\|^2 + \epsilon^2)^{1-\beta}} \Omega &= \lim_{t \rightarrow \infty} \int_M \frac{e^{\varphi_\epsilon + \dot{\varphi}_\epsilon + k\chi}}{(\|S\|^2 + \epsilon^2)^{1-\beta}} \Omega \\ &= \lim_{t \rightarrow \infty} \int_M (\omega_t + \sqrt{-1}\partial\bar{\partial}(\varphi_\epsilon + k\chi))^n = \lim_{t \rightarrow \infty} \int_M \omega_t^n. \end{aligned}$$

Now suppose we have two initial conic metrics $\omega^* = \omega_0 + k\sqrt{-1}\partial\bar{\partial}\|S\|^{2\beta}$ and $\omega'^* = \omega'_0 + k'\sqrt{-1}\partial\bar{\partial}\|S\|^{2\beta}$. Without loss of generality, we can suppose that $\omega_0 < \omega'_0$ and $k < k'$, otherwise we can replace ω'^* by $\omega^* + \omega'^*$, then we can write the following flow

equations:

$$\begin{aligned}\frac{\partial}{\partial t}(\varphi_\epsilon + k\chi) &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}(\varphi_\epsilon + k\chi))^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - (\varphi_\epsilon + k\chi) \\ \frac{\partial}{\partial t}(\varphi'_\epsilon + k'\chi) &= \log \frac{(\omega'_t + \sqrt{-1}\partial\bar{\partial}(\varphi'_\epsilon + k'\chi))^n (\|S\|^2 + \epsilon^2)^{1-\beta}}{\Omega} - (\varphi'_\epsilon + k'\chi), \\ \varphi_\epsilon(\cdot, 0) &= \varphi'_\epsilon(\cdot, 0) = 0,\end{aligned}$$

where $\omega'_t = e^{-t}\omega'_0 + (1 - e^{-t})(-Ric(\Omega) + (1 - \beta)R(\|\cdot\|)) = \omega_t + e^{-t}(\omega'_0 - \omega_0)$. Take the difference, and denote $\varphi_{\epsilon,k} = \varphi_\epsilon + k\chi$ and $\varphi'_{\epsilon,k'} = \varphi'_\epsilon + k'\chi$, we have that

$$\begin{aligned}\frac{\partial}{\partial t}(\varphi_{\epsilon,k} - \varphi'_{\epsilon,k'}) &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_{\epsilon,k})^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_{\epsilon,k} + e^{-t}(\omega'_0 - \omega_0) - \sqrt{-1}\partial\bar{\partial}(\varphi_{\epsilon,k} - \varphi'_{\epsilon,k'}))^n} \\ &\quad - (\varphi_{\epsilon,k} - \varphi'_{\epsilon,k'}),\end{aligned}$$

and $(\varphi_{\epsilon,k} - \varphi'_{\epsilon,k'})(\cdot, 0) = (k - k')\chi \leq 0$. By maximal principle, we have $\varphi_{\epsilon,k} - \varphi'_{\epsilon,k'} \leq 0$ at any time, so at infinity, we have that $\varphi_{\epsilon,\infty} + k\chi \leq \varphi'_{\epsilon,\infty} + k'\chi$. However, as $\int_M \frac{e^{\varphi_{\epsilon,\infty} + k\chi}}{(\|S\|^2 + \epsilon^2)^{1-\beta}} \Omega = \lim_{t \rightarrow \infty} \int_M \omega_t^n = \int_M \frac{e^{\varphi'_{\epsilon,\infty} + k'\chi}}{(\|S\|^2 + \epsilon^2)^{1-\beta}} \Omega$, we can obtain that $\varphi_{\epsilon,\infty} + k\chi = \varphi'_{\epsilon,\infty} + k'\chi$. Take the limit as ϵ tends to 0, we complete the proof for the uniqueness of the limit conical Kähler-Einstein current, which also completes the proof for theorem 4.1.3.

4.5 Further remarks

Until now we proved three main theorems under the assumption of one irreducible divisor. Now we discuss how to generalize our proofs to the case of simple normal crossing divisors briefly. By [15], we can add up all the approximation metrics to all divisors, and all the properties are preserved. In Laplacian estimate, we note that in [15], they proved that the approximation metric has a lower bound for bisectional curvature, which allows us to obtain the Laplacian estimate. The proof of $C^{2,\alpha}$ -estimate

is essentially the same, see Chapter 4.

Now let us sketch approach of setting approximating conical Kähler-Ricci flow based on Chapter 3. For simplicity we only consider the case of one irreducible divisor. Actually we can do similar approximating setting whenever the conditions of Theorem 1.2.2 are satisfied. For the conic Monge-Ampere type equation (4.5), we consider another approximating flow equation:

$$\begin{cases} \frac{\partial}{\partial t} \varphi_\epsilon = \log \frac{(\omega_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon)^n}{\Omega} + \log F_\epsilon(\|S\|^2)^{1-\beta} \\ \varphi_\epsilon(\cdot, 0) = 0 \end{cases} \quad (4.16)$$

where

$$\omega_{t,\epsilon} := \omega_t + k\sqrt{-1} \partial \bar{\partial} \psi_\epsilon(\|S\|^2),$$

and $F_\epsilon, \psi_\epsilon$ are defined as in Chapter 3. By the same computation we can obtain C^0 -estimate for φ_ϵ . For Laplacian estimate, instead of Aubin-Yau type inequality, we make use of Chern-Lu type inequality

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_\omega \omega_{0,\epsilon} \leq \frac{g^{i\bar{i}} g^{k\bar{k}} R_{i\bar{i}k\bar{k}}(\omega_{0,\epsilon}) - (1-\beta) \sum_i g^{i\bar{i}} g^{i\bar{i}} \left(\log \frac{F_\epsilon(\|S\|^2)}{\|\cdot\|^2}\right)_{i\bar{i}}}{\operatorname{tr}_\omega \omega_{0,\epsilon}}.$$

By Theorem 1.2.2 we know that $R_{i\bar{i}k\bar{k}}(\omega_{0,\epsilon}) \leq C$ uniformly. Then we can derive a Laplacian estimate more directly.

Note that in [37], unnormalized Kähler-Ricci flow can smooth initial Kähler metrics with rough and degenerate data. So naturally, we hope unnormalized Kähler-Ricci flow can play similar roles in the study of conic Kähler metrics and we will discuss this phenomenon in the future.

Chapter 5

The $C^{2,\alpha}$ -estimate for conical Kähler-Ricci flow

Regularity of complex Monge-Ampere equations in conic setting is an important issue in the study of conic Kähler metrics. As the C^0 -estimate and C^2 -estimate is not enough to obtain the solution to existence problems, we need to get at least $C^{2,\alpha}$ - estimate to guarantee that the inverse function theorem can be applied. For conic metrics, in smooth part, it is easy to apply Calabi's C^3 -estimate as smooth case [50]. For the conic singularities, when the cone angle $\beta < \frac{1}{2}$, even Calabi's C^3 -estimate still holds [3]. However, when $\beta \geq \frac{1}{2}$, the C^3 -estimate seems not to hold any more so we can expect only a $C^{2,\alpha}$ -estimate. In conic setting, we are still not sure that whether classic Evans-Krylov-Safanov theory (see [16]) holds, as we are not clear whether the subtle argument of choosing squares and measure estimates can be extended to conic setting. Actually there have been some works on extension of this classic theory to conic setting [9] [10] [15] [21] [49]. When the metric is sufficiently close to the standard flat conic metric, perturbation method of the Schauder estimate can be used to derive a $C^{2,\alpha}$ -estimate [7]. However we don't know the metric involved can be made sufficiently close to a standard flat conic metric before we establish an

estimate than the C^2 -estimate. In [45], Tian adapted the method of his master thesis [38] to conic case and proved the $C^{2,\alpha}$ -estimate for conical complex Monge-Ampere equations without the closeness assumption. The main idea of Tian's approach is to compare the second derivative to harmonic forms whose regularity was studied before.

In this paper, we extend Tian's idea in [45] to derive a $C^{2,\alpha}$ -estimate for conical Kähler-Ricci flow, which completes the proof of the existence theorem in last chapter. Conical Kähler-Ricci flow is another interesting topic in the study of conic Kähler metrics and there have been some works on it [9] [10] [28] [24] [33] [49]. To establish the existence of conical Kähler-Ricci flow we need a parabolic version of $C^{2,\alpha}$ -estimate. In deriving our estimate, similar to what Tian did, we need to use regularity properties of heat equations.

Now let's recall the construction of unnormalized Kähler-Ricci flow in [33]. Suppose the divisor $D = \sum_{i=1}^l (1 - \beta_i) D_i$ ($l \leq n$) is a simply normal crossing divisor on a n dimensional compact Kähler manifold (M, ω_0) , where each D_i is an irreducible divisor and ω_0 is a smooth background Kähler metric. Now consider unnormalized Kähler-Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} \omega = -Ric(\omega) + 2\pi \sum_{i=1}^l (1 - \beta_i) [D_i] \\ \omega(0) = \omega^* = \omega_0 + \sum_{i=1}^l k \sqrt{-1} \partial \bar{\partial} ||S_i||^{2\beta_i} \end{cases} \quad (5.1)$$

where S_i is a defining holomorphic section of $[D_i]$, and we know that when k is a small positive number ω^* is a conic Kähler metric with cone angles $2\pi\beta_i$ along each D_i . In last chapter we apply the methods of [47] [37] to consider this equation in the cohomology level, i.e, write $\bar{\omega}_t = \omega^* - t(Ric(\Omega) - \sum_{i=1}^l (1 - \beta_i) R(|| \cdot ||_i))$, and $\omega = \bar{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi$, where Ω is a smooth volume form and $R(|| \cdot ||_i)$ is corresponding curvature form of the line bundle $[D_i]$. Then the equation (5.1) can be transformed

to a Monge-Ampere type equation:

$$\begin{cases} \frac{\partial}{\partial t} \varphi = \log \frac{(\bar{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} + \sum_{i=1}^l \log \|S_i\|^{2(1-\beta_i)} \\ \varphi(\cdot, 0) = 0 \end{cases} \quad (5.2)$$

Recall that in last chapter, we apply the approximation techniques of [4] [15] [28] and estimates in [47] [37] to obtain C^0 -estimate and Laplacian estimate for φ , i.e, for some uniform constant A ,

$$A^{-1} \bar{\omega}_0 \leq \omega = \bar{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi \leq A \bar{\omega}_0 \quad (5.3)$$

on the time interval $[0, T]$ where

$$T < T_0 := \sup\{t > 0 \mid [\omega_0] - t(c_1(M) - \sum_{i=1}^l (1 - \beta_i)[D_i]) > 0\}.$$

What is remaining in last chapter is a Hölder estimate for $\sqrt{-1} \partial \bar{\partial} \varphi$, i.e, the $C^{2,\alpha}$ -estimate. We will follow the approach in [45] to get the following main theorem in this chapter by comparing the flow solution with the solutions to heat equations:

Theorem 5.0.1. *For any $\alpha \in (0, \bar{\beta}^{-1} - 1, 1)$, where $\bar{\beta} := \max\{\beta_1, \dots, \beta_l\}$, there exist constants $r_0 \in (0, 1)$ and $C_\alpha > 0$ such that for any $(x_0, t_0) \in M \times (0, T_0)$ and $0 < r < r_0$,*

$$\iint_{P_r(x_0, t_0)} |\nabla^3 \varphi|^2 \bar{\omega}_0^n dt \leq C_\alpha r^{2n+2\alpha}, \quad (5.4)$$

where $P_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0]$, and $B_r(x_0)$ is the geodesic ball with center x_0 and radius r , and ∇ is the covariant derivative, the norm is taken with respect to $\bar{\omega}_0$.

Provided this theorem, by standard analysis arguments, we have the following corollary, which gives the $C^{2,\alpha}$ -estimate:

Corollary 5.0.2. *For any $\alpha \in (0, \min\{\bar{\beta}^{-1} - 1, 1\})$, $\sqrt{-1} \partial \bar{\partial} \varphi$ is C^α -bounded with*

respect to $\bar{\omega}_0$.

The note will be organized as below. First, begin with the Monge-Ampere type equation (4.5), we derive a second order parabolic equation for complex Hessian of φ . Then we review some facts on heat equation, which can be modified from [19]. Finally, we follow [45], to give a parabolic version of comparison argument.

5.1 Preliminaries

In this section we'll obtain a parabolic equation for complex Hessian, and make some necessary preparation for the estimate. First, we note that the Theorem 5.0.1 can be proved locally. As Calabi's 3rd derivative estimate can be used outside the divisor, we only need to consider that $x_0 \in D$. Now choose a local coordinate system $(U; z_1, \dots, z_n)$ around $x_0 = (0, \dots, 0)$ and $D_i \cap U = \{z_i = 0\}$. Now U can be identified with an open set in $\mathbb{C}$. By Laplacian estimate (5.3) we have

$$C^{-1}\omega_\beta \leq \omega \leq C\omega_\beta, \quad (5.5)$$

where ω_β is the standard conic flat metric on $\mathbb{C}$:

$$\omega_\beta = \sqrt{-1} \left(\sum_{i=1}^l \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{i=l+1}^n dz_i \wedge d\bar{z}_i \right).$$

By $\partial\bar{\partial}$ -lemma (see [43]), we can write $\omega = \sqrt{-1}\partial\bar{\partial}u$ for some spacetime neighborhood $U \times (t', t_0]$. Now consider the Monge-Ampere type equation (4.5), as $\sqrt{-1}\partial\bar{\partial}u = \omega =$

$\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\bar{\varphi}$, we have that

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\frac{\partial}{\partial t}u &= -Ric(\Omega) + \sum_{i=1}^l(1-\beta_i)R(\|\cdot\|_i) + \sqrt{-1}\partial\bar{\partial}\log \frac{(\bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}\bar{\varphi})^n \prod_{i=1}^l \|S_i\|^{2(1-\beta_i)}}{\Omega} \\ &= \sqrt{-1}\partial\bar{\partial}(\log \frac{(\sqrt{-1}\partial\bar{\partial}u)^n}{\omega_\beta^n} + F),\end{aligned}$$

Here F is defined as

$$\sqrt{-1}\partial\bar{\partial}F = \sqrt{-1}\partial\bar{\partial}\log \omega_\beta^n + \sum_{i=1}^l(1-\beta_i)(R(\|\cdot\|_i + \sqrt{-1}\partial\bar{\partial}\log \|S_i\|^2),$$

so F is a smooth function. Then we obtain a scalar equation:

$$\frac{\partial}{\partial t}u = \log \frac{(\sqrt{-1}\partial\bar{\partial}u)^n}{\omega_\beta^n} + F. \quad (5.6)$$

As $\bar{\omega}_t$ is a conic Kähler metric which has the same singular part with the initial conic metric, it suffices to prove that $\sqrt{-1}\partial\bar{\partial}u$ is C^α -bounded. Now take the covariant derivatives of the equation (5.6) twice with respect to ω_β , we obtain a parabolic equation for $\sqrt{-1}\partial\bar{\partial}u$:

$$\frac{\partial}{\partial t}u_{k\bar{l}} = u^{i\bar{j}}u_{k\bar{l}i\bar{j}} - u^{i\bar{q}}u^{p\bar{j}}u_{i\bar{j}k}u_{p\bar{q}\bar{l}} + F_{k\bar{l}}. \quad (5.7)$$

Obviously, with respect to ω_β , $u_{k\bar{l}}$ is uniformly bounded. Unfortunately we can't use Krylov-Safanov estimate directly to get a Hölder estimate. We'll make use of the properties of heat equation to derive the estimate. For this purpose, we need to do some preparation.

As [45], we can define $\mathbf{B}_\beta(r)$ to be an angle domain in $\mathbb{C}^l \times \mathbb{C}^{n-l}$ consisting of all $(w_1, \dots, w_l, w')$ satisfying $\sum_{i=1}^l |w_i|^2 + |w'|^2 \leq r^2$, where $w_i = \rho_i e^{\sqrt{-1}\theta_i}$ for $\rho_i \in [0, r]$ and $\theta_i \in [0, 2\pi\beta_i]$, $1 \leq i \leq l$. Now we have such a Sobolev type inequality, which can be

proved in standard way([17] [27]) by Sobolev inequality in [45]:

Lemma 5.1.1. *There is a constant C_β which depends on $1 - \beta_i, i = 1, \dots, l$ such that for any smooth function h on $\mathbf{B}_\beta \times (-1, 0]$ with boundary conditions:*

$$h(w_1, \dots, \rho_i e^{\sqrt{-1}2\pi\beta_i}, \dots, w_l, w') = e^{\sqrt{-1}2\pi(1-\beta_i)} h(w_1, \dots, \rho_i, \dots, w_l, w') \quad (5.8)$$

for each i , then we have

$$\left(\iint_{\mathbf{B}_\beta \times [-1, 0]} |h|^{2(1+\frac{1}{n})} \omega_\beta^n dt \right)^{\frac{n}{n+1}} \leq C_\beta \left(\sup_t \int_{\mathbf{B}_\beta} |h|^2 \omega_\beta^n + \iint_{\mathbf{B}_\beta \times [-1, 0]} |\nabla h|^2 \omega_\beta^n dt \right), \quad (5.9)$$

or for $s \in (\frac{n}{n+1}, 1)$,

$$\iint_{\mathbf{B}_\beta \times [-1, 0]} |h|^2 \omega_\beta^n dt \leq C_{\beta, s} \left(\sup_t \int_{\mathbf{B}_\beta} |h|^{2s} \omega_\beta^n + \iint_{\mathbf{B}_\beta \times [-1, 0]} |\nabla h|^{2s} \omega_\beta^n dt \right)^{\frac{1}{s}}. \quad (5.10)$$

Now we prove a lemma that outside the divisor D , the scalar curvature is uniformly bounded from below, which is useful in Moser's iteration with respect to evolving metrics and weak Harnack inequality below:

Lemma 5.1.2. *Suppose the evolving conic metrics $\omega(t)$ satisfy the conical Kähler-Ricci flow (4.3), then outside the divisor D , the scalar curvature $R \geq -\frac{C}{t}$, which only depends on initial metric.*

Proof. We know that the conclusion is true in case of smooth Ricci flow. In conic case, the conclusion at the initial time always holds. Outside the divisor D we have the evolution equation for scalar curvature [18]

$$\frac{\partial}{\partial t} R = \Delta R + |Ric|^2,$$

and now we can write this equation as following:

$$(\frac{\partial}{\partial t} - \Delta)(R + \frac{C}{t} - c \sum_{i=1}^l \log \|S_i\|^2) = |Ric|^2 + \frac{C}{t^2} - c \sum_{i=1}^l \text{tr}_\omega(R(\|\cdot\|_i) + 2\pi[D_i]),$$

where c is a positive constant. It is easy to see that the space minimum will be attained outside the divisor, so by maximal principle [18], as $R(\|\cdot\|_i)$ is uniformly bounded and $\omega \geq C\omega_0$ where ω_0 is a smooth background metric, we have that

$$\min(R + \frac{C}{t} - c \sum_{i=1}^l \log \|S_i\|^2)(t) \geq \min(R + \frac{C}{t} - c \sum_{i=1}^l \log \|S_i\|^2)(0) - cC't.$$

Now let c tend to 0, we obtain our result. $\square$

Now make use of above two lemmas, we can prove an inequality similar to mean value inequality for subharmonic functions:

Proposition 5.1.3. *Suppose $v \geq 0$ satisfy that $(\frac{\partial}{\partial t} - \Delta_\beta)v \leq C_0$, or $(\frac{\partial}{\partial t} - \Delta_\omega)v \leq C_0$, where ω satisfies conical Kähler-Ricci flow (4.3) on $P(1) = B(1) \times (-1, 0]$, we have that*

$$\sup_{P(1/2)} v \leq C \left(\iint_{R(1)} v \omega_\beta^n dt + C_0 \right),$$

or replace ω_β by ω .

Proof. Before the L^1 -control, we first prove a L^2 -control that for any $\theta \in (1/2, 1)$, by Moser's iteration as [17]:

$$\sup_{P(\theta)} v \leq C \left(\frac{1}{(1-\theta)^{n+1}} \left(\iint_{P(1)} v^2 \omega_\beta^n dt \right)^{1/2} + C_0 \right). \quad (5.11)$$

Without loss of generality, we assume that $C_0 = 0$, otherwise we take $v' = v - C_0 t$ to replace v . We prove the first case. First we choose $\eta = v \xi^2 \chi_{[-r^2, -\sigma]}(t)$ where ξ is a cut-off function with compact support in $P(1)$, and χ is a characteristic function. Define $P^\sigma(r) = B(r) \times [-r^2, -\sigma]$ and take η as a test function for the first inequality,

we have

$$\frac{1}{2} \iint_{P^\sigma(r)} \left(\frac{\partial}{\partial t} \xi^2 v^2 + \xi^2 |\nabla v|^2 \right) \omega_\beta^n dt \leq C \iint_{P^\sigma(r)} (|\nabla \xi|^2 + |\xi \dot{\xi}|) v^2 \omega_\beta^n dt. \quad (5.12)$$

For $\theta \leq r' < r \leq 1$, choose $\sigma \leq r'^2$ such that

$$\int_{B(r')} v^2 \omega_\beta^n|_{t=-\sigma} \geq \frac{1}{2} \sup_{t \in [-r'^2, 0]} \int_{B(r')} v^2 \omega_\beta^n. \quad (5.13)$$

Choose $\xi(x, t) = \xi_1(x) \xi_2(t)$, where ξ_1 is 1 on $B(r')$, and 0 outside $B(r)$ with $|\nabla \xi_1| \leq \frac{C}{r-r'}$, and ξ_2 is 1 on $[-r'^2, 0]$, and 0 on $[-1, -r^2]$ with $|\dot{\xi}_2| \leq \frac{C}{(r-r')^2}$. By the equation (5.12), we obtain that

$$\begin{aligned} \sup_{t \in [-r'^2, 0]} \int_{B(r')} v^2 \omega_\beta^n &\leq 2 \int_{B(r')} v^2 \omega_\beta^n|_{t=-\sigma} \leq 2 \iint_{P^\sigma(r)} \frac{\partial}{\partial t} \xi^2 v^2 \omega_\beta^n dt \\ &\leq \frac{C}{(r-r')^2} \iint_{P^\sigma(r)} v^2 \omega_\beta^n dt \leq \frac{C}{(r-r')^2} \iint_{P(r)} v^2 \omega_\beta^n dt. \end{aligned}$$

Choose test function $\eta = v \xi^2 \chi_{[-r^2, 0]}(t)$ then the integral domain of (5.12) will be $P(r)$ and we obtain that

$$\iint_{P(r')} |\nabla v|^2 \omega_\beta^n dt \leq \frac{C}{(r-r')^2} \iint_{P(r)} v^2 \omega_\beta^n dt.$$

Now by Sobolev type inequality (5.9), we obtain that

$$\left(\iint_{P(r')} v^{2(1+\frac{1}{n})} \omega_\beta^n dt \right)^{\frac{n}{n+1}} \leq \frac{C}{(r-r')^2} \iint_{P(r)} v^2 \omega_\beta^n dt.$$

By computation, $(\frac{\partial}{\partial t} - \Delta_\beta) v^p \leq 0$ for $p \geq 1$. So similar argument is true that

$$\left(\iint_{P(r')} v^{2p(1+\frac{1}{n})} \omega_\beta^n dt \right)^{\frac{n}{n+1}} \leq \frac{C}{(r-r')^2} \iint_{P(r)} v^{2p} \omega_\beta^n dt.$$

Then by Moser's iteration, we obtain the inequality (5.11). Now by (5.11) and

Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
\|v\|_{L^\infty(P(r'))} &\leq C \left(\frac{1}{(r-r')^{n+1}} \|v\|_{L^2(P(r))} + C_0 \right) \\
&\leq C \left(\frac{1}{(r-r')^{n+1}} \|v\|_{L^\infty(P(r))}^{1/2} \|v\|_{L^1(P(r))}^{1/2} + C_0 \right) \\
&\leq \frac{\|v\|_{L^\infty(P(r))}}{2} + C \left(\frac{1}{(r-r')^{2n+2}} \|v\|_{L^1(P(r))} + C_0 \right),
\end{aligned}$$

then by an iteration lemma (Lemma 4.3 on Page 75 of [19]), we get that

$$\|v\|_{L^\infty(P(\theta))} \leq C \left(\frac{1}{(1-\theta)^{2n+2}} \|v\|_{L^1(P(r))} + C_0 \right)$$

which completes the proof of the first case. For the proof of the second case, note that in the derivation of similar inequality to (5.12), there is an extra term $-R\xi^2 v^2$ in the right hand side integral which comes from the derivative of ω^n . However by Lemma 5.1.2 this term is bounded from above by $C\xi^2 v^2$, then all the argument for the first case follows. $\square$

From this proposition, we can easily get following corollaries:

Corollary 5.1.4. *For the function h which solves heat equation $(\frac{\partial}{\partial t} - \Delta)h = 0$ and satisfies the same boundary condition (5.8) of Lemma 5.1.1 on $B_\beta \times (-1, 0]$, and any $r < 1$, we have that*

$$\iint_{B_\beta(r) \times (-r^2, 0]} |\nabla h|^2 dw \wedge d\bar{w} dt \leq Cr^{2n-2+2\bar{\beta}^{-1}} \iint_{B_\beta \times (-1, 0]} |\nabla h|^2 dw \wedge d\bar{w} dt. \quad (5.14)$$

Moreover, for any $\sigma < r < 1$, and function v which solves $(\frac{\partial}{\partial t} - \Delta_\beta)v = 0$ on $P(1)$, we have that

$$\iint_{P(\sigma)} |\nabla v|^2 \omega_\beta^n dt \leq C \left(\frac{\sigma}{r} \right)^{2n-2+2\bar{\beta}^{-1}} \iint_{P(r)} |\nabla v|^2 \omega_\beta^n dt. \quad (5.15)$$

Proof. Let's see what the boundary condition (5.8) means. We choose coordinate change $z_i = (\beta_i w_i)^{\frac{1}{\beta_i}}$ for $i = 1 \dots, l$ and $z_i = w_i$ for other i , which maps the angle domain B_β to the ball with standard conic metric. Then we find that

$$\begin{aligned}
& \frac{\partial h(w(z), t)}{\partial z_i}(\dots, z_i e^{\sqrt{-1}2\pi\beta_i}, \dots) = \frac{\partial h(w, t)}{\partial w_i} \frac{\partial w_i}{\partial z_i}(\dots, z_i e^{\sqrt{-1}2\pi\beta_i}, \dots) \\
& = e^{\sqrt{-1}2\pi(1-\beta_i)} \frac{\partial h(w, t)}{\partial w_i}(\dots, z_i, \dots) (\beta_i w_i (z_i e^{\sqrt{-1}2\pi\beta_i}))^{1-\beta_i^{-1}} \\
& = e^{\sqrt{-1}2\pi(1-\beta_i)} \frac{\partial h(w, t)}{\partial w_i} \frac{\partial w_i}{\partial z_i}(\dots, z_i, \dots) e^{\sqrt{-1}2\pi(\beta_i-1)} \\
& = \frac{\partial h(w, t)}{\partial w_i} \frac{\partial w_i}{\partial z_i}(\dots, z_i, \dots) = \frac{\partial h(w(z), t)}{\partial z_i}(\dots, z_i, \dots),
\end{aligned}$$

which allows us to consider h as a function on the ball with standard conic metric. Note that $\omega_\beta = \sqrt{-1} \sum dw_i \wedge d\bar{w}_i$, So (5.15) follows from (5.14) and we only need to prove (5.14).

Note that the functions $\zeta_i = w_i^{\frac{k}{\beta_i}-1}$ satisfy boundary condition (5.8). By Fourier expansion of one function satisfying the same boundary condition we can find that functions $\zeta_i = w_i^{\frac{k}{\beta_i}-1}$ constitute a basis of such functions satisfying this boundary condition (We want to thank Y. Rubinstein for showing the details). As $k \geq 1$, for all i , choose the smallest exponential $2(\min(\beta_i^{-1}) - 2) = 2(\bar{\beta}^{-1} - 2)$ and integrate on $P(r)$, we obtain our result. $\square$

On the other hand, we still need a weak Harnack inequality as Theorem 4.15 in [19]. Here we need to consider a new spacetime neighborhood $P'(r) = B(r) \times (-r^2, r^2)$ to replace $P(r)$. Then we will have

Proposition 5.1.5. *Suppose $\omega = \sqrt{-1}\partial\bar{\partial}u$ solves the conical Kähler-Ricci flow (4.3) on $M \times [-T, T](T > 1)$, and a nonnegative function $v \in H^1(P'(r))$ satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta'\right)v \geq -k, \tag{5.16}$$

where Δ' denotes the Laplacian with respect to ω . Then we will have

$$\iint_{P'(1)} v \omega_\beta^n dt \leq C \left(\inf_{P'(\frac{1}{2})} v + k \right). \quad (5.17)$$

Proof. The proof is similar to the elliptic version of weak Harnack inequality in Theorem 4.15 of [19] so we just sketch the proof. We set $\bar{v} = v + k$ and $w = \bar{v}^{-1}$. Choose a cut-off function $\xi(x, t) = \xi_1(x)\xi_2(t)$, where ξ_1 is 1 on $B(\frac{1}{2})$, and 0 near $\partial B(1)$ with $|\nabla \xi_1| \leq 4$, and ξ_2 is 1 on $[-\frac{1}{4}, \frac{1}{4}]$, and 0 near $t = \pm 1$ with $|\dot{\xi}_2| \leq 16$. Take $\bar{v}^{-2}\xi$ as the test function of (5.16) we obtain that

$$\iint_{P'(1)} \xi \left(\Delta' - \frac{\partial}{\partial t} \right) w \omega^n dt \leq \iint_{P'(1)} \xi \frac{C}{\bar{v}^2} \omega^n dt.$$

By Proposition 5.1.3 we obtain that

$$\sup_{P'(\frac{1}{2})} \bar{v}^{-p} \leq C \iint_{P'(1)} \bar{v}^{-p} \omega^n dt \leq C \iint_{P'(1)} \bar{v}^{-p} \omega_\beta^n dt,$$

for any $p > 0$, which indicates that

$$\inf_{P'(\frac{1}{2})} v + k \geq C \left(\iint_{P'(1)} \bar{v}^p \omega_\beta^n dt \iint_{P'(1)} \bar{v}^{-p} \omega_\beta^n dt \right)^{-1/p} \left(\iint_{P'(1)} v \omega_\beta^n dt \right)^{1/p}.$$

Now we need to prove

$$\iint_{P'(1)} \bar{v}^p \omega_\beta^n dt \iint_{P'(1)} \bar{v}^{-p} \omega_\beta^n dt \leq C$$

for some $p = p_0 > 0$. It suffices to prove that

$$\iint_{P'(1)} e^{p_0|h|} \omega_\beta^n dt \leq C,$$

where $h = \log \bar{v} - \frac{1}{|P'(1)|_{\omega_\beta}} \iint_{P'(1)} \bar{v} \omega_\beta^n dt$. As $e^{p_0|h|} = 1 + |p_0 h| + \frac{|p_0 h|^2}{2!} + \dots + \frac{|p_0 h|^l}{l!} + \dots$

it suffices to bound $\iint_{P'(1)} |h|^l \omega_\beta^n dt$ for any positive integer l .

Denote $\det(u_{k\bar{l}})u^{i\bar{j}}$ by $U^{i\bar{j}}$ and by [45], we have $U_i^{i\bar{j}} = 0$ (See also Lemma 5.2.1 in the next section). Now take $\bar{v}^{-1}\xi^2$ as the test function in (5.16), we have that

$$\begin{aligned}
& - \iint_{P'(1)} \frac{k}{\bar{v}} \xi^2 \omega_\beta^n dt \leq \iint_{P'(1)} \bar{v}^{-1} \xi^2 \left(\frac{\partial}{\partial t} - \Delta' \right) \bar{v} \omega_\beta^n dt \\
& = \iint_{P'(1)} \xi^2 \left(\frac{\partial}{\partial t} h - \bar{v}^{-1} U^{i\bar{j}} \bar{v}_{i\bar{j}} \det(u_{k\bar{l}}^{-1}) \right) \omega_\beta^n dt \\
& = \iint_{P'(1)} \left(-2h\xi\dot{\xi} + 2\xi\xi_{\bar{j}} \bar{v}^{-1} U^{i\bar{j}} \bar{v}_i \det(u_{k\bar{l}}^{-1}) + \xi^2 (-\bar{v}^{-2} U^{i\bar{j}} \bar{v}_i \bar{v}_{\bar{j}} \det(u_{k\bar{l}}^{-1}) + U^{i\bar{j}} h_i \det(u_{k\bar{l}}^{-1})_{\bar{j}}) \right) \omega_\beta^n dt \\
& = \iint_{P'(1)} \left(-2h\xi\dot{\xi} + 2\xi\xi_{\bar{j}} h_i u^{i\bar{j}} - \xi^2 |\nabla' h|^2 - h U^{i\bar{j}} (2\xi\xi_i \det(u_{k\bar{l}}^{-1})_{\bar{j}} + \xi^2 \det(u_{k\bar{l}}^{-1})_{i\bar{j}}) \right) \omega_\beta^n dt
\end{aligned}$$

For the last term, we have that

$$\begin{aligned}
-U^{i\bar{j}} \det(u_{k\bar{l}}^{-1})_{i\bar{j}} & = U^{i\bar{j}} \left(\frac{(\log \det(u_{k\bar{l}}))_i}{\det(u_{k\bar{l}})} \right)_{\bar{j}} \\
& = u^{i\bar{j}} ((\log \det(u_{k\bar{l}}))_{i\bar{j}} - \det(u_{k\bar{l}})^{-2} \det(u_{k\bar{l}})_i \det(u_{k\bar{l}})_{\bar{j}}) \\
& = -R(\omega) - |\nabla' \det(u_{k\bar{l}})|^2 \leq C - |\nabla' \det(u_{k\bar{l}})|^2,
\end{aligned}$$

where the last inequality follows from Lemma 5.1.2. Make use of this and Cauchy-Schwarz inequality, we have

$$\iint_{P'(1)} \xi^2 |\nabla' h|^2 \omega_\beta^n dt \leq C \iint_{P'(1)} (|\nabla' \xi|^2 + k \bar{v}^{-1} \xi^2 - 2h\xi\dot{\xi} + C(\xi^2 + |\nabla \xi|^2) h) \omega_\beta^n dt, \quad (5.18)$$

where ∇' denotes the gradient with respect to ω , which also have corresponding Sobolev inequality and Poincare inequality as it is equivalent to ω_β . Note that in the new spacetime domain $P'(1)$, the first term in the Sobolev inequality (5.9) can be canceled and by Poincare inequality we have that

$$\iint_{P'(1)} |h|^2 \omega_\beta^n dt \leq C.$$

For $l > 2$, take $\bar{v}^{-1}\xi^2h^{2l}$ as the test function in (5.16) and the similar estimate above, we have that

$$\begin{aligned} \iint_{P'(1)} \xi^2 h^{2l} |\nabla' h|^2 \omega_\beta^n dt &\leq \iint_{P'(1)} (2l \xi^2 h^{2l-1} |\nabla' h|^2 + k \bar{v}^{-1} \xi^2 h^{2l} \\ &\quad + 2\xi \nabla' \xi \cdot h^{2l} \nabla' h + \frac{1}{2l+1} (C(|\nabla \xi|^2 + \xi^2) h^{2l+1} - 2\xi \dot{\xi} h^{2l+1})) \omega_\beta^n dt. \end{aligned}$$

Apply Cauchy-Schwarz inequality and Young's inequality, we obtain that

$$\iint_{P'(1)} |\nabla'(\xi h^{l+1})|^2 \omega_\beta^n dt \leq C \iint_{P'(1)} \left((2l)^{2l+2} \xi^2 |\nabla' h|^2 + (2l)^\alpha (|\nabla' \xi|^2 + |\dot{\xi}| + C) |h|^{2(l+1)} \right) \omega_\beta^n dt,$$

where $\alpha > 0$. Then follow [19] to use Moser's iteration, we can prove that there exists $p_0 > 0$ such that $\iint_{P'(1)} e^{p_0|h|} \omega_\beta^n dt \leq C$.

Finally, we need to prove that the inequality (5.17) is true for $p = 1$. We only need to take $\xi^2 \bar{v}^{-\alpha}$ as the test function to (5.16) where $\alpha \in (0, 1)$, then we have

$$\frac{2\alpha}{(1-\alpha)^2} \iint_{P'(1)} \xi^2 |\nabla' \bar{v}^{\frac{1-\alpha}{2}}|^2 \omega_\beta^n dt \leq \iint_{P'(1)} \left(\frac{C(|\nabla \xi|^2 + \xi^2) - 2\xi \dot{\xi}}{1-\alpha} + \xi^2 + \frac{2}{\alpha} |\nabla' \xi|^2 \right) \bar{v}^{1-\alpha} \omega_\beta^n dt.$$

As [19], the proposition follows from Sobolev inequality and finite steps of iterations. □

We end this section by a lemma which indicates that Corollary 5.0.2 follows from Theorem 5.0.1:

Lemma 5.1.6. *Suppose $u \in H_{loc}^1(P(1))$ satisfies*

$$\iint_{P(r)} |\nabla u|^2 \omega_\beta^n dt \leq C r^{2n+2\alpha},$$

then $u \in C^\alpha$ at the origin.

Proof. We only need to note that by smallest square theorem and Poincare Inequality,

$$\begin{aligned} \iint_{P(r)} \left| u - \frac{1}{|P(r)|} \iint_{P(r)} u \omega_\beta^n dt \right|^2 \omega_\beta^n dt &\leq \int_{-r^2}^0 \int_{B(r) \times \{t\}} \left| u - \frac{1}{|B(r)|} \int_{B(r) \times \{t\}} u \omega_\beta^n \right|^2 \omega_\beta^n dt \\ &\leq C r^2 \iint_{P(r)} |\nabla u|^2 \omega_\beta^n dt \leq C r^{2n+2+2\alpha}, \end{aligned}$$

then this lemma follows from the same argument of Theorem 3.1 on Page 48 of [19]. $\square$

5.2 Proof of main Theorem 5.0.1

We begin the proof of main theorem 5.0.1 along Tian's second approach in [45]. We denote $\det(u_{k\bar{l}})u^{i\bar{j}}$ by $U^{i\bar{j}}$ and recall two lemmas in [45]:

Lemma 5.2.1. *For each j , $U_i^{i\bar{j}} = 0$.*

Proof.

$$U_i^{i\bar{j}} = (\det(u_{k\bar{l}})u^{i\bar{j}})_i = \det(u_{k\bar{l}})(u^{k\bar{l}}u_{k\bar{l}i}u^{i\bar{j}} - u^{i\bar{l}}u^{k\bar{j}}u_{k\bar{l}i}) = 0,$$

as we can switch j and l . $\square$

Lemma 5.2.2. *For any positive definite Hermitian matrix $(\lambda_{i\bar{j}})$, we have*

$$|\det(\lambda_{p\bar{q}})\lambda^{i\bar{j}}u_{i\bar{j}} - \det(u_{p\bar{q}}) - (n-1)\det(\lambda_{p\bar{q}})| \leq C \sum_{i,j} |u_{i\bar{j}} - \lambda_{i\bar{j}}|^2, \quad (5.19)$$

where $\lambda^{i\bar{j}}$ denotes the inverse of $\Lambda = (\lambda_{i\bar{j}})$ and C is a constant depending only on the norms of Λ and $u_{i\bar{j}}$.

Proof. Consider the function f of matrix $u_{i\bar{j}}$ that $f(u_{i\bar{j}}) = \det(\lambda_{p\bar{q}})\lambda^{i\bar{j}}u_{i\bar{j}} - \det(u_{p\bar{q}}) - (n-1)\det(\lambda_{p\bar{q}})$ at the point $\Lambda = (\lambda_{i\bar{j}})$ that $f(\lambda_{i\bar{j}}) = 0$ and

$$\frac{\partial f}{\partial u_{i\bar{j}}} \Big|_{u_{i\bar{j}}=\lambda_{i\bar{j}}} = \det(\lambda_{p\bar{q}})\lambda^{i\bar{j}} - \det(u_{p\bar{q}})u^{i\bar{j}} \Big|_{u_{i\bar{j}}=\lambda_{i\bar{j}}} = 0.$$

Then this lemma follows from Taylor expansion of f at point $u_{i\bar{j}} = \lambda_{i\bar{j}}$. $\square$

From now on we will modify Tian's arguments in [45] to parabolic case. First, we have

Lemma 5.2.3. *There exists some $q > 2$ which may depend on $\beta, \|u_{i\bar{j}}\|_{L^\infty}$ and $\|F_{i\bar{j}}\|_{L^\infty}$ such that for any $P(2r) \subset U \times (-1, 0]$ around $(y, 0)$, we have*

$$\left(\frac{1}{r^{2n+2}} \iint_{P(r)} (1 + |\nabla \omega|)^q \omega_\beta^n dt \right)^{\frac{2}{q}} \leq \frac{C}{r^{2n+2}} \iint_{P(2r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt, \quad (5.20)$$

where C denotes a uniform constant.

Proof. First assume that $y = x_0$ and move t_0 to be 0. Define $\lambda_{i\bar{j}} := r^{-(2n+2)} \iint_{P(r)} u_{i\bar{j}} \omega_\beta^n dt$ for $i, j = 1, \dots, n$. By using unitary transformation if necessary, we assume that $\lambda_{i\bar{j}} = 0$ if $i \neq j$ and $i, j > l$. By Laplacian estimate (5.5), we have $A^{-1}\mathbf{I} \leq \Lambda = (\lambda_{i\bar{j}}) \leq A\mathbf{I}$. Choose a cutoff function $\eta = \eta_1(x)\eta_2(t)$ on $P(r)$ such that $\eta_1(x) = 1$ on $B(y, \frac{3r}{4})$, $\eta_1(x) = 0$ near $\partial B(y, 1)$, and $\eta_2(t) = 1$ on $[-\frac{9r^2}{16}, 0]$, $\eta_2(t) = 0$ near $t = -r^2$, with $|\nabla \eta_1|^2, |\nabla^2 \eta_1|, |\dot{\eta}_2| \leq \frac{C}{r^2}$. Make use of the equation (5.7) and Lemma 5.2.1, denote

$\lambda_k = \lambda_{k\bar{k}}$ and $\lambda = \lambda_1 \cdots \lambda_n$, we have

$$\begin{aligned}
& \iint_{P(r)} U^{i\bar{j}} \left(\sum_{k=1}^n \frac{\lambda}{\lambda_k} u_{k\bar{k}} - \det(u_{p\bar{q}}) - (n-1)\lambda \right) \eta_{i\bar{j}} \omega_\beta^n dt \\
&= \iint_{P(r)} U^{i\bar{j}} \left(\left(\sum_{k=1}^n \frac{\lambda}{\lambda_k} u_{k\bar{k}i\bar{j}} \right) \eta - \det(u_{p\bar{q}}) \eta_{i\bar{j}} \right) \omega_\beta^n dt \\
&\geq \iint_{P(r)} \sum_{k=1}^n \frac{\lambda}{\lambda_k} \det(u_{p\bar{q}}) (u^{i\bar{q}} u^{p\bar{j}} u_{i\bar{j}k} u_{p\bar{q}\bar{k}} + \frac{\partial}{\partial t} u_{k\bar{k}} - f_{k\bar{k}}) \eta \omega_\beta^n dt - Cr^{2n} \\
&\geq c \iint_{P(r)} \eta |\nabla \omega|^2 \omega_\beta^n dt + \int_{B(r) \times \{0\}} \sum_{k=1}^n \frac{\lambda}{\lambda_k} \det(u_{p\bar{q}}) u_{k\bar{k}} \eta \omega_\beta^n \\
&\quad - \iint_{P(r)} \sum_{k=1}^n \frac{\lambda}{\lambda_k} u_{k\bar{k}} \det(u_{p\bar{q}}) (\dot{\eta} + \eta \frac{\partial}{\partial t} (\log \det(u_{p\bar{q}}))) \omega_\beta^n dt - Cr^{2n} \\
&\geq c \iint_{P(r)} \eta |\nabla \omega|^2 \omega_\beta^n dt + \iint_{P(r)} \sum_{k=1}^n \frac{\lambda}{\lambda_k} u_{k\bar{k}} \det(u_{p\bar{q}}) (R - \dot{\eta}) \omega_\beta^n dt - Cr^{2n} \\
&\geq c \iint_{P(r)} \eta |\nabla \omega|^2 \omega_\beta^n dt - Cr^{2n},
\end{aligned}$$

where the last inequality uses the Lemma 5.1.2 that the scalar curvature stays uniformly bounded from below. By Lemma 5.2.2, we have

$$\iint_{P(\frac{3r}{4})} |\nabla \omega|^2 \omega_\beta^n dt \leq C \left(r^{2n} + r^{-2} \iint_{P(r)} \sum_{i,j} |u_{i\bar{j}} - \lambda_{i\bar{j}}|^2 \omega_\beta^n dt \right).$$

By Sobolev inequality (5.10) in Lemma 5.1.1,

$$\iint_{P(\frac{3r}{4})} (1 + |\nabla \omega|)^2 \omega_\beta^n dt \leq Cr^{(2n+2)(1-\frac{1}{s})} \left(\iint_{P(r)} (1 + |\nabla \omega|)^{2s} \omega_\beta^n dt \right)^{\frac{1}{s}}, \quad (5.21)$$

where $s \in (\frac{n}{n+1}, 1)$. This inequality holds for parabolic neighborhoods with center point on or not on the divisor. Then by a covering argument that for any parabolic neighborhood $P((y, t), 2r) \subset U \times (-1, 0]$,

$$\frac{1}{r^{2n+2}} \iint_{P(r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt \leq C \left(\frac{1}{r^{2n+2}} \iint_{P(2r)} (1 + |\nabla \omega|)^{2s} \omega_\beta^n dt \right)^{\frac{1}{s}}.$$

Then (5.20) follows from Gehring's inverse Hölder inequality [14]. $\square$

Similar to [45], we can define a $(1, 1)$ -form v solving a heat equation:

$$\frac{\partial}{\partial t} v = \sum_{i=1}^n \frac{1}{\lambda_i} v_{i\bar{i}} \text{ on } P(r) \quad v = \omega \text{ on } B(r) \times \{-r^2\} \bigcup \partial B(r) \times (-r^2, 0]. \quad (5.22)$$

Then we have such a lemma:

Lemma 5.2.4. *For any $P((y, t), 4r) \subset U \times (-1, 0]$, and $\sigma < r$, we have*

$$\begin{aligned} & \iint_{P(\sigma)} |\nabla \omega|^2 \omega_\beta^n dt - cr^{2n+2} \\ & \leq c \left[\left(\frac{\sigma}{r} \right)^{2n-2+2\bar{\beta}^{-1}} + \left(\frac{1}{r^{2n}} \iint_{P(\sigma)} (1 + |\nabla \omega|^2) \omega_\beta^n dt \right)^{\frac{q-2}{q}} \right] \iint_{P(r)} |\nabla \omega|^2 \omega_\beta^n dt. \end{aligned} \quad (5.23)$$

Proof. Take $\hat{\omega} = \omega - v$ as the test function for the Hessian equation (5.7), considering (5.22), integrate by parts, we have

$$\begin{aligned} & - \iint_{P(r)} \hat{\omega}_{l\bar{k}} (u^{i\bar{q}} u^{p\bar{j}} u_{i\bar{j}k} u_{p\bar{q}l} - F_{k\bar{l}}) \omega_\beta^n dt \\ & = \iint_{P(r)} \hat{\omega}_{l\bar{k}} \left(\frac{\partial}{\partial t} u_{k\bar{l}} - u^{i\bar{j}} u_{k\bar{l}i\bar{j}} \right) \omega_\beta^n dt \\ & = \iint_{P(r)} \hat{\omega}_{l\bar{k}} \left(\frac{\partial}{\partial t} \hat{\omega}_{k\bar{l}} + \sum_{i=1}^n \frac{1}{\lambda_i} v_{i\bar{i}k\bar{l}} - u^{i\bar{j}} u_{k\bar{l}i\bar{j}} \right) \omega_\beta^n dt \\ & = \iint_{P(r)} \left(\frac{1}{2} \frac{\partial}{\partial t} |\hat{\omega}|^2 + \hat{\omega}_{l\bar{k}} \left(- \sum_{i=1}^n \frac{1}{\lambda_i} \hat{\omega}_{i\bar{i}k\bar{l}} + \left(\frac{1}{\lambda_i} \delta_{ij} - u^{i\bar{j}} \right) u_{k\bar{l}i\bar{j}} \right) \right) \omega_\beta^n dt \\ & = \int_{B(r) \times \{0\}} \frac{|\hat{\omega}|^2}{2} \omega_\beta^n + \iint_{P(r)} \sum_{i=1}^n \frac{1}{\lambda_i} |\hat{\omega}_{k\bar{l}i}|^2 \omega_\beta^n dt \\ & \quad - \iint_{P(r)} \left(\left(\frac{1}{\lambda_i} \delta_{ij} - u^{i\bar{j}} \right) \hat{\omega}_{l\bar{k}j} u_{k\bar{l}i} + \hat{\omega}_{l\bar{k}} u^{i\bar{q}} u^{p\bar{j}} u_{k\bar{l}i} u_{p\bar{q}j} \right) \omega_\beta^n dt, \end{aligned}$$

then by Cauchy-Schwarz inequality, and note that $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, we

obtain that

$$\iint_{P(r)} |\nabla \hat{\omega}|^2 \omega_\beta^n dt \leq c \left(r^{2n+2} + \iint_{P(r)} (|\hat{\omega}| + |u_{i\bar{j}} - \lambda_i \delta_{ij}|^2) |\nabla \omega|^2 \omega_\beta^n dt \right). \quad (5.24)$$

By Lemma 5.2.3 and Poincare inequality, we have

$$\begin{aligned} & \iint_{P(r)} |u_{i\bar{j}} - \lambda_i \delta_{ij}|^2 |\nabla \omega|^2 \omega_\beta^n dt \\ & \leq r^{2n+2} \left(r^{-2n-2} \iint_{P(r)} |\nabla \omega|^q \omega_\beta^n dt \right)^{\frac{2}{q}} \left(r^{-2n-2} \iint_{P(r)} |u_{i\bar{j}} - \lambda_i \delta_{ij}|^{\frac{2q}{q-2}} \omega_\beta^n dt \right)^{\frac{q-2}{q}} \\ & \leq C \left(r^{-2n-2} \iint_{P(r)} |u_{i\bar{j}} - \lambda_i \delta_{ij}|^2 \omega_\beta^n dt \right)^{\frac{q-2}{q}} \iint_{P(r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt \\ & \leq C \left(r^{-2n} \iint_{P(r)} |\nabla \omega|^2 \omega_\beta^n dt \right)^{\frac{q-2}{q}} \iint_{P(r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt. \end{aligned} \quad (5.25)$$

For the other part on the right hand side of (5.24), we may assume $\frac{q}{q-2} > 2$, otherwise the following second inequality follows from Hölder inequality:

$$\begin{aligned} & \iint_{P(r)} |\hat{\omega}| |\nabla \omega|^2 \omega_\beta^n dt \\ & \leq r^{2n+2} \left(r^{-2n-2} \iint_{P(r)} |\nabla \omega|^q \omega_\beta^n dt \right)^{\frac{2}{q}} \left(r^{-2n-2} \iint_{P(r)} |\hat{\omega}|^{\frac{q}{q-2}} \omega_\beta^n dt \right)^{\frac{q-2}{q}} \\ & \leq C \left(r^{-2n-2} \iint_{P(r)} |\hat{\omega}|^2 \omega_\beta^n dt \right)^{\frac{q-2}{q}} \iint_{P(r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt \\ & \leq C \left(r^{-2n} \iint_{P(r)} |\nabla \hat{\omega}|^2 \omega_\beta^n dt \right)^{\frac{q-2}{q}} \iint_{P(r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt \\ & \leq C \left(r^{-2n} \iint_{P(r)} (1 + |\nabla \omega|^2) \omega_\beta^n dt \right)^{\frac{q-2}{q}} \iint_{P(r)} (1 + |\nabla \omega|)^2 \omega_\beta^n dt, \end{aligned} \quad (5.26)$$

where the last inequality follows from (5.24). Now apply Corollary 5.1.4 to v , we have

that

$$\begin{aligned}
\iint_{P(\sigma)} |\nabla \omega|^2 \omega_\beta^n dt &\leq \iint_{P(\sigma)} (|\nabla \hat{\omega}|^2 + |\nabla v|^2) \omega_\beta^n dt \\
&\leq c \left(\frac{\sigma}{r}\right)^{2n-2+2\bar{\beta}-1} \iint_{P(r)} |\nabla v|^2 \omega_\beta^n dt + \iint_{P(r)} |\nabla \hat{\omega}|^2 \omega_\beta^n dt \\
&\leq c \left(\frac{\sigma}{r}\right)^{2n-2+2\bar{\beta}-1} \iint_{P(r)} (|\nabla \omega|^2 + |\nabla \hat{\omega}|^2) \omega_\beta^n dt + \iint_{P(r)} |\nabla \hat{\omega}|^2 \omega_\beta^n dt \\
&\leq c \left(\frac{\sigma}{r}\right)^{2n-2+2\bar{\beta}-1} \iint_{P(r)} |\nabla \omega|^2 \omega_\beta^n dt + C \iint_{P(r)} |\nabla \hat{\omega}|^2 \omega_\beta^n dt,
\end{aligned} \tag{5.27}$$

Now combine (5.24) to (5.27), (5.23) follows. $\square$

To conclude the main theorem, similar to [45], we still need a lemma which controls the right hand side terms in (5.23):

Lemma 5.2.5. *For any $\epsilon_0 > 0$, there is an l depending only on $\epsilon_0, \|\Delta u\|_{L^\infty}, \|\Delta F\|_{L^\infty}$ satisfying: For any $\tilde{r} > 0$ with $P((y, t), \tilde{r}) \subset U \times (-1, 0]$, there exists $r \in [2^{-l}\tilde{r}, \tilde{r}]$ such that*

$$r^{-2n} \iint_{P(r)} (1 + |\nabla \omega|^2) \omega_\beta^n dt \leq \epsilon_0. \tag{5.28}$$

Proof. Actually we only need to consider the integral of $|\nabla \omega|^2$ and use the domain $P'(r)$ to replace the smaller $P(r)$. Now by the Hessian equation (5.7) again, we have

$$\left(\Delta' - \frac{\partial}{\partial t}\right) \Delta u = u^{i\bar{q}} u^{p\bar{j}} g^{k\bar{l}} u_{i\bar{j}k} u_{p\bar{q}l} - \Delta F, \tag{5.29}$$

where Δ' denotes the Laplacian of ω , and g denotes the metric of ω_β .

Now denote $M_r = \sup_{P'(r)} \Delta u$, and choose a cut-off function $\eta = \eta_1(x) \eta_2(t)$ on $P(r)$ such that $\eta_1(x) = 1$ on $B(y, \frac{r}{2})$, $\eta_1(x) = 0$ near $\partial B(y, 1)$, and $\eta_2(t) = 1$ on $[-\frac{r^2}{4}, 0]$, $\eta_2(t) = 0$ near $t = -r^2$, with $|\nabla \eta_1|^2, |\nabla^2 \eta_1|, |\dot{\eta}_2| \leq \frac{C}{r^2}$. Make use of the

estimate in (5.18) of Proposition 5.1.5, we have

$$\begin{aligned}
\iint_{P'(r)} \eta^2 |\nabla \omega|^2 \omega_\beta^n dt &\leq Cr^{2n+2} + C \iint_{P'(r)} \eta^2 \left(\frac{\partial}{\partial t} - \Delta' \right) (M_r - \Delta u) \omega_\beta^n dt \\
&\leq Cr^{2n+2} + C \iint_{P'(r)} (-2\eta \dot{\eta} - \Delta' \eta^2 + C |\nabla \eta|^2 - R(\omega) \eta) (M_r - \Delta u) \omega_\beta^n dt \\
&\leq Cr^{2n+2} + \frac{C}{r^2} \iint_{P'(r)} (M_r - \Delta u) \omega_\beta^n dt.
\end{aligned}$$

Make use of weak Harnack inequality (5.17) in Proposition 5.1.5, we obtain that

$$\begin{aligned}
\left(\frac{r}{2}\right)^{-2n} \iint_{P'(\frac{r}{2})} |\nabla \omega|^2 \omega_\beta^n dt &\leq C \left(r^{-2n-2} \iint_{P'(r)} (M_r - \Delta u) \omega_\beta^n dt + r^2 \right) \\
&\leq C \left(\inf_{P'(\frac{r}{2})} (M_r - \Delta u) + r^2 \right) \leq C (M_r - M_{\frac{r}{2}} + r^2).
\end{aligned}$$

Hence, if (5.28) doesn't hold for $r = \frac{\tilde{r}}{2}, \frac{\tilde{r}}{2^2}, \dots, \frac{\tilde{r}}{2^k}$, then

$$k\epsilon_0 \leq C(M_{\tilde{r}} - M_{\frac{\tilde{r}}{2^k}} + 2\tilde{r}^2).$$

This is impossible for sufficiently large k . Then the lemma follows. $\square$

Finally we can complete the proof of Theorem 5.0.1 by an iteration argument as [45]. Suppose $\sigma = \lambda r$ where $0 < \lambda < 1$. Then suppose r satisfies Lemma 5.2.5, and denote $H(r) = \iint_{P(r)} |\nabla \omega|^2 \omega_\beta^n dt$, by Lemma 5.2.4 we obtain that

$$\sigma^{-2n} H(\sigma) + \sigma^{2(\bar{\beta}^{-1}-1)} \leq c(\lambda^{2(\bar{\beta}^{-1}-1)} + \frac{\epsilon_0}{\lambda^{2n}}) r^{-2n} H(r) + (c \frac{r^{4-2\bar{\beta}^{-1}}}{\lambda^{2n}} + \lambda^{2\bar{\beta}^{-1}-2}) r^{2(\bar{\beta}^{-1}-1)}.$$

Now for $\alpha \in (0, \bar{\beta}^{-1} - 1, 1\}$, choose $\lambda \in (0, 1)$, such that

$$\log 2(1+c) \leq -2(\bar{\beta}^{-1} - 1 - \alpha)\lambda.$$

Then we choose $\theta = 2(1+c)\lambda^{2(\bar{\beta}^{-1}-1)} < 1$. After that we choose $\epsilon_0 \leq \frac{\theta\lambda^{2n}}{2}$ and r_0 such

that

$$c \frac{r^{4-2\bar{\beta}^{-1}}}{\lambda^{2n}} + \lambda^{2\bar{\beta}^{-1}-2} \leq \theta,$$

where we need to note that we can assume $\bar{\beta} \geq \frac{1}{2}$ otherwise the high order estimate follows from Calabi's 3rd estimate directly ([3]). Then we can obtain that

$$\sigma^{-2n} H(\sigma) + \sigma^{2(\bar{\beta}^{-1}-1)} \leq \theta(r^{-2n} H(r) + r^{2(\bar{\beta}^{-1}-1)}).$$

Now by iteration method, for $r < r_0$ and any positive integer k we have that

$$\begin{aligned} (\lambda^k r)^{-2n} H(\lambda^k r) + (\lambda^k r)^{2(\bar{\beta}^{-1}-1)} &\leq \theta^k (r^{-2n} H(r) + r^{2(\bar{\beta}^{-1}-1)}) \\ &= (2(1+c)\lambda^{2(\bar{\beta}^{-1}-1-\alpha)})^k (\lambda^k r)^{2\alpha} r^{-2\alpha} (r^{-2n} H(r) + r^{2(\bar{\beta}^{-1}-1)}). \end{aligned}$$

By the choice of λ above, $2(1+c)\lambda^{2(\bar{\beta}^{-1}-1-\alpha)} \leq 1$, then the main theorem 5.0.1 follows.

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