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Counting simple knots via arithmetic invariants

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Counting simple knots via arithmetic invariants

Alison Beth Miller

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

Knot theory and arithmetic invariant theory are two fields of mathematics that rely on algebraic invariants. We investigate the connections between the two, and give a framework for addressing asymptotic counting questions relating to knots and knot invariants.

We study invariants of simple $(2q-1)$-knots when q is odd; these include the Alexander module and Blanchfield pairing. In the case that $q = 1$, simple 1-knots are exactly knots as classically defined. In the high-dimensional cases of $q \geq 3$, the theory is different, and simple knots are exactly classified by these algebraic invariants.

These invariants connect to arithmetic invariant theory by way of the theory of Seifert matrices, which are related to the $\mathbb{Z}$-orbits of the adjoint representation $\mathrm{Sym}^2(2g)$ of the algebraic group Sp_{2g}. We classify the orbits of this representation over general fields and over $\mathbb{Z}$. These techniques are modeled after those of Bhargava, Gross, and Wood in arithmetic invariant theory, but also have much in common with methods used by Trotter and others in the topological context. We explain how our results fit into this topological context of the Alexander module, Blanchfield pairing, and related invariants.

In the final section, we look at how this connection can be used to asymptotically count simple knots and Seifert hypersurfaces ordered by the size of their Alexander polynomial. For knots of genus 1, the theory of binary quadratic forms yields an explicit count for Seifert surfaces. We also conjecture heuristics for the asymptotic number of genus 1 knots. These heuristics imply that most such knots have Alexander polynomial of the form $pt^2 + (1-2p)t + p$ where p is a positive prime number. Using sieve methods, we obtain an upper bound for the asymptotic count of such knots that agrees with our heuristics.

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To Ross Miller (1954-2013)

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Chapter 1

Introduction

In this thesis, we will explore a connection between two separate branches of mathematics: arithmetic invariant theory and knot theory. This connection comes about by way of knot invariants that have natural arithmetic structure. We will consider the question of counting these arithmetic objects, and the consequences for counting high-dimensional simple knots.

Arithmetic invariant theory, a term coined by Bhargava and Gross in [3], deals with the questions of the following form. Suppose G is an algebraic matrix group defined over a ring R, and V is a finite-dimensional representation of G. What are the orbits of $G(R)$ acting on $V(R)$? What sorts of invariants classify them? This is a well-studied problem when R is an algebraically closed field, in which case the invariants are always polynomial functions on V; however, if R is non-algebraically closed field, or a general ring, there is additional arithmetic complexity.

The goal of knot theory is different: to understand the topologically distinct ways that one manifold can be embedded into another. Classical knots are embedded circles in S^3, but knots can also be defined in higher dimensions: an n-knot is an embedding of S^n in S^{n+2}. Knot theory also concerns itself with invariants; there are

many ways to construct invariants of knots. However, it is in general much harder to completely classify knots by their invariants.

Some knot invariants, in particular those related to the Alexander module and to the Seifert pairing, have natural arithmetic structure; they can be thought of as ideal classes of rings, or as $\mathbb{Z}$-orbits of a representation of Sp_{2g}. These structures were analyzed by a number of knot theorists, including Milnor [25] and Trotter[30]. The arguments they made were of a similar nature to ones made in work by Bhargava and Gross [3] and of Wood [32]. In Chapter 3, we will reinterpret their arguments in the context of arithmetic invariant theory.

The bijections given by arithmetic invariant theory have made it possible to count a wide range of objects, including points on elliptic curves, finite dimensional $\mathbb{Z}$-algebras, and ideal classes of orders in number fields. In this thesis, using the connection between arithmetic invariant theory and algebraic knot invariants, we will extend the range of these techniques to count high-dimensional knots as well.

1.1 Historical background

Historically, the first researchers to look at algebraic-number-theoretic invariants of knots were Fox and Smythe [12], studying classical knots in S^3. Their construction implicitly used the Alexander module via a presentation matrix. Hillman [13] later recast their work in terms of module theory and simultaneously generalized the results to links in S^3. Many of the results from that paper are recapitulated in [14]. The generalization to links meant that Hillman worked with Laurent polynomial rings in more variables, which we shall not be concerned with.

At the same time, Kearton, Levine, and others were developing the general higher-dimensional theory of n-knots, that is, knotted copies of S^n in S^{n+2}.

(Some parts of the theory of higher-dimensional knots also have similar algebraic structures: see [21] and [14] for more details.)

Meanwhile, the Alexander module had been studied as an object in its own right. One thing that was realized early on was that the Alexander module had additional structure; in fact it was endowed with a canonical pairing, known as the Blanchfield pairing (introduced by Blanchfield [7]). Knot theorists studied properties of the Alexander module, and also worked on the question of classifying the possible modules that could be realized as Alexander modules of some knot. Most of the work done involved higher-dimensional generalizations.

When $n = 2q-1$ is odd, one productive way to understand the Alexander module of an n-knot K is to choose an embedded hypersurface V in S^{n+2} with boundary equal to K. The homology group $H_q(V, \mathbb{Z})$ then carries a natural non-symmetric $\mathbb{Z}$-valued pairing, known as the "Seifert pairing" such that the skew-symmetric part of the Seifert pairing is the intersection pairing on S. The Seifert pairing is not an invariant of the knot K, but depends upon the hypersurface V. For one thing, the rank of the homology group $H_1(V, \mathbb{Z})$ depends upon the choice of V, but even if one fixes the rank the isomorphism class of Seifert pairing can still depend on the choice of V. In order to construct a true invariant one must impose a weaker equivalence relation called "S-equivalence" on matrices, which has the property that any two Seifert matrices for the same knot are S-equivalent.

The relevance of Seifert pairings for us is that knowing a Seifert matrix for a knot is sufficient information to compute the Alexander module along with its Blanchfield pairing. In fact (as shown, e.g., by Trotter [29]), two knots have isomorphic Alexander modules and Blanchfield pairings if and only if they have Seifert matrices that are S-equivalent. Therefore, one approach to finding number-theoretic invariants of knots is to look for invariants of the Seifert pairing (here considered as a pairing on a free $\mathbb{Z}$-

module, not as an S-equivalence class) and then to determine which of them become knot invariants.

Another feature of the Seifert pairing is that it is easy to construct matrices with desired Seifert pairing. For this reason, the Seifert matrix approach was used to study the realization problem for Alexander modules. In particular, it was used by Trotter [30] who built on previous work of Levine [20], to give an algebraic criterion for which $\mathbb{Z}[t, t^{-1}]$-modules are middle-dimensional Alexander modules.

Although in general these invariants are not complete invariants, there is one important class of knots for which they are. These are the *simple n-knots*, those whose complement has the homotopy type of a circle up until the middle dimension. Simple n-knots are fully classified by their Alexander module and Blanchfield pairing. They were introduced by Levine [19], and classified in multiple equivalent ways by Levine [19], Kearton [17], and Farber [10].

Using this classification, Bayer and Michel [1] proved that there are only finitely many equivalence classes of simple knots with a fixed squarefree Alexander polynomial. They also proved a similar theorem for simple Seifert hypersurfaces.

1.2 Outline

This thesis has two main goals. The first is to revisit the knot invariants described above from the point of view of arithmetic invariant theory. The second is to explore the new question of finding asymptotics for the number of simple $(2q-1)$-knots with Alexander polynomials of bounded height. By the result of Bayer and Michel [1], we know this number is finite: an answer to this question thus gives an average-case quantitative version of their results.

Chapter 2 covers background material from high-dimensional knot theory. In Chapter 3 we consider the question of the $\mathrm{Sp}_{2g}(R)$ orbits on $2g \times 2g$ Seifert matrices

in the cases where R is a field or $R = \mathbb{Z}$. This is an arithmetic question that arises naturally from the topological question of the classification of Seifert hypersurfaces, and which has been studied extensively by topologists for that reason. We study it from the point of view of arithmetic invariant theory, and explain the connection to previous work done by topologists on the question.

Chapter 4 explains the connection between the arithmetic invariants from Chapter 2 and knot invariants related to the Alexander polynomial.

Chapter 5 deals with the question of counting simple knots and simple Seifert hypersurfaces. We prove asymptotics for counting simple Seifert hypersurfaces of genus 1, and give heuristics, based on standard conjectures in number theory, for the asymptotics of the number of simple knots of genus 1. In particular, we conjecture that most simple knots of genus 1 have Alexander polynomials of the form $pt^2+(1-2p)t+p$. We use sieve theory to prove an upper bound for the number of knots with Alexander polynomial of that form.

Chapter 2

Background on n-knots

2.1 Basic concepts and definitions in the theory of n-knots

We are interested in counting n-knots by way of invariants with interesting arithmetic structure. Informally, an n-knot is a knotted copy of S^n in S^{n+2}. There are multiple ways of making this rigorous; the following two definitions are both commonly used:

Definition.

(i) An *n-knot* K is a topologically embedded copy of S^n in S^{n+2} that is locally flat (locally homeomorphic to $\mathbb{R}^n \subset \mathbb{R}^{n+2}$). Equivalence is given by ambient isotopy.

(ii) An *n-knot* is a smoothly embedded submanifold K of S^{n+2} which is homeomorphic to S^n (but not necessarily diffeomorphic; K might be an exotic sphere). Equivalence is induced by orientation-preserving diffeomorphism of the S^{n+2}.

In both cases, we will consider both S^n and S^{n+2} to be oriented, so that reversing the orientation of either or both may give a different knot.

Although it is far from obvious, classification results we use will give the same answer regardless of which of the two definitions above is used. I will talk about

knots and equivalence with the understanding that all statements hold using either formulation.

We will be interested in the following class of n-knots, which has a nice classification.

Definition. The knot K is called *simple* if $\pi_i(S^{n+2}-K) = \pi_i(S^1)$ for all $i \leq (n-1)/2$.

Simple knots are interesting not only because of their classification, but also because they show up "in nature": for instance, knots coming from singularities of algebraic hypersurfaces are simple [24].

The following theorem yields an equivalent condition for being simple. First we recall the definition of a Seifert hypersurface (also called a Seifert manifold), which generalizes that of a Seifert surface from classical knot theory.[1]

Definition. A *Seifert hypersurface* for an n-knot K is a compact oriented $(n+1)$-manifold V with boundary, embedded in S^{n+2} in such a way that $\partial V = K$.

(Again there is a choice of definitions, and again it ultimately doesn't matter.)

One reason why Seifert hypersurfaces are a useful tool in knot theory is that they are guaranteed to exist for any knot. We will reference Farber [10] for the statement of the following general proposition about the existence of Seifert hypersurfaces, which was proved by Levine [19] in the case of $n \geq 4$ and by others in the low-dimensional cases.

Theorem 2.1.1 (Farber [10], Theorem 0.5)**.** *For $n \neq 2$, an n-knot K bounds an r-connected Seifert hypersurface if and only if $\pi_i(S^{n+2} - K) = \pi_i(S^1)$ for all $i \leq r$.*

Comparing this theorem with our definition of a simple knot motivates the following definition:

[1] Earlier drafts of this paper used the term "Seifert manifold", but I have changed the terminology because it is more precise and the term "Seifert manifold" is used for other things inside topology.

Definition. A *simple Seifert hypersurface* in S^{n+2} is an $(n+1)$-manifold V with boundary embedded in S^{n+2} satisfying the following topological conditions (which are intrinsic to V and do not depend on the embedding):

- V is $\frac{n-1}{2}$-connected
- ∂V is homeomorphic to S^n.

The case $r = (n-1)/2$ of Theorem 2.1.1 may now be restated as follows.

Corollary 2.1.2. *If V is a simple Seifert hypersurface in S^{n+2}, ∂V is a simple $(n-1)$-knot, and V is a Seifert hypersurface for ∂V. Conversely, any simple n-knot K has at least one simple Seifert hypersurface; in fact, infinitely many.*

We can define equivalence of Seifert hypersurfaces analogously to equivalence of knots.

Remark. The reason that our definitions put conditions on the homotopy groups up to dimension $(n-1)/2$ and no higher is because this is the strongest such condition on the homotopy type that we may impose without forcing triviality of the knot. For any larger value of r, calculations with the Hurewicz theorem and Poincare duality show that any r-connected Seifert hypersurface V is contractible. For $n \geq 5$, the h-cobordism theorem then implies that V is diffeomorphic to a ball, and that $K = \partial V$ is the unknot. (In a bit more generality, this implication has also been shown to be true for all n except possibly 2 and 4 [10]. In the case for $n = 4$, K is unknotted if and only if K is diffeomorphic to S^4. In the case $n = 2$ I believe this implication is now known as a consquence of the 3-d Poincare conjecture.)

One may consider the homology of a simple Seifert hypersurface V. All homology groups except H_q are forced to be trivial, but the rank of H_q gives us an important numerical invariant.

Definition. If V is a simple Seifert hypersurface in S^{2q+1}, the *genus* of V is defined as $\frac{1}{2}\operatorname{rk} H_q(V, \mathbb{Z}) = \frac{1}{2}\operatorname{rk} H_q(V, \mathbb{Q})$.

This definition always yields a whole number, because $H_q(V, \mathbb{Z})$ has a nondegenerate skew-symmetric pairing (the intersection pairing).

We can also define the genus of a knot. This is more complicated, because the homology of the knot complement does not give us any information. Indeed, the complement $S^{2q+1} - K$ of a simple knot always has the same homology as S^1.

However, because $H_1(S^{2q+1} - K) \cong \mathbb{Z}$, there is a unique covering space $(S^{2q+1} - K)^{\text{cyc}}$ of $S^{2q+1} - K$ with infinite cyclic covering group.

We can then define the genus of a knot intrinsically.

Definition. Let K be a simple knot in S^{2q+1} with $q \geq 3$. We define the *genus* of K as $\frac{1}{2} \operatorname{rk} H_q((S^{2q+1} - K)^{\text{cyc}}, \mathbb{Q})$.

Theorem 2.1.3. *Let K be a simple knot in S^{2q+1}, with $q \geq 3$. Then* $\text{genus}(K)$ *is always finite, an integer, and equal to* $\min(\text{genus}(V))$ *where V ranges over all simple Seifert hypersurfaces such that $\partial V = K$.*

Proof. This follows from the existence of minimal Seifert hypersurfaces, as proved in [11]. □

Remark. In the theory of classical knots in S^3, the genus is typically defined as the minimum genus of a Seifert surface. We choose the equivalent definition in terms of the infinite cyclic cover here because it is simpler to state.

The homology group $H_q((S^{2q+1} - K)^{\text{cyc}})$ will also be important to us as an object in its own right, because it is an Alexander module for K. We will refer to it as "the Alexander module of K", because the Alexander modules coming from the ith homology when $i \neq q$ turn out to be all trivial.

We now go into detail regarding this definition.

Let t be a generator of the group of deck transformations of $(S^{2q+1} - K)^{\text{cyc}}$: using the orientations on K and on S^{2q+1}, we can make a canonical choice of t.

The group of deck transformations acts on $H_q((S^{2q+1} - K)^{\text{cyc}}, \mathbb{Z})$, and thus makes $H_q((S^{2q+1} - K)^{\text{cyc}}, \mathbb{Z})$ into a module over the group ring $\mathbb{Z}[t, t^{-1}]$.

Definition. The *Alexander module* Alex_K of a knot K is the $\mathbb{Z}[t, t^{-1}]$-module $H_q((S^{2q+1} - K)^{\text{cyc}}, \mathbb{Z})$.

The Alexander module also possesses an important duality pairing, known as the Blanchfield pairing. We will not get into the details of its construction here, but we will state its existence.

Theorem 2.1.4 (Blanchfield, [7])**.** *There is a canonical perfect pairing, the* Blanchfield pairing

$$\text{Bl} : \text{Alex}_K \times \text{Alex}_K \to \mathbb{Q}(t)/(\mathbb{Z}[t, t^{-1}]),$$

which is Hermitian with respect to the involution of $\mathbb{Z}[t, t^{-1}]$ *interchanging* t *with* t^{-1}*.*

2.2 The classification of simple Seifert hypersurfaces by Sp_{2g}-orbits

The remainder of this chapter will be dedicated to the classifications of simple Seifert hypersurfaces and of simple knots by algebraic data. We will follow the standard expository path due to Kearton, Levine, Trotter, and Farber, which starts with the classification of Seifert hypersurfaces and then passes to knots.

As before, we will study simple knots of genus g inside S^{2q+1}, where g and q are fixed, and q is at least 3. The results below will depend on the choice of g, but will be independent of q.

We first consider simple Seifert hypersurfaces inside S^{2q+1}. If V is a simple Seifert hypersurface, the homology group $H_q(V, \mathbb{Z})$ posseses a natural $\mathbb{Z}$-valued skew-

symmetric perfect pairing, namely the intersection pairing. We denote it by

$$\alpha, \beta \mapsto \alpha \cap \beta \in \mathbb{Z}.$$

Additionally, $H_q(V, \mathbb{Z})$ posseses a pairing which depends on the embedding of V inside S^{2q+1}. For γ and γ' in $H_q(V, \mathbb{Z})$, we define the *Seifert pairing* $p(\gamma, \gamma')$ of γ with γ' as

$$p(\gamma, \gamma') = \mathrm{lk}(\gamma, (\gamma')^+),$$

where $(\gamma')^+$ is the cycle in $H_q(S^{2q+1} - V, \mathbb{Z})$ produced by pushing γ' off the hypersurface V in the positive normal direction (defined in terms of the orientation on V and S^{2q+1}) and $\mathrm{lk} : H_q(V, \mathbb{Z}) \times H_q(S^{2q+1} - V, \mathbb{Z}) \to \mathbb{Z}$ is the linking pairing given by Alexander duality.

We can recover the intersection pairing from the Seifert pairing p by the formula

$$\gamma \cap \gamma' = p(\gamma, \gamma') - p(\gamma', \gamma).$$

In some situations, instead of using p, it will be more useful for us to use the symmetrized pairing

$$q(\gamma, \gamma') = p(\gamma, \gamma') + p(\gamma', \gamma).$$

If we know the intersection pairing on $H_q(V, \mathbb{Z})$, we can recover either pairing p or q for the other.

Note that q is constrained to be congruent to the intersection pairing mod 2.

It will sometimes be useful to us to pick a basis for $H_q(V, \mathbb{Z})$, thereby identifying $H_q(V, \mathbb{Z})$ with $\mathbb{Z}^{2g}$.

Definition. The *Seifert matrix* P for V with respect to a $\mathbb{Z}$-basis B of $H_q(V, \mathbb{Z})$ is the matrix of the asymmetric pairing p on $H_q(V, \mathbb{Z})$ with respect to the basis B.

Because the intersection pairing is a perfect pairing, any Seifert matrix P must satisfy $\det(P - P^T) = 1$. We will see later that this condition is sufficient for a matrix P to be a Seifert matrix of some Seifert hypersurface.

Another consequence of the intersection pairing being a perfect pairing is that we can, if we wish, choose a basis of $H_q(V, \mathbb{Z})$ in which the intersection pairing is identified with the standard symplectic pairing $\langle v, w \rangle = v^T J w$ on $\mathbb{Z}^{2g}$, where $J = J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

In this setting, the parity condition on the quadratic form q and on its associated matrix Q motivates the following definition:

Definition. We say that a symmetric bilinear form m on $\mathbb{Z}^{2g}$ is *Seifert-parity* if

$$m(x, y) \equiv \langle x, y \rangle \pmod 2 \text{ for all } x, y \in \mathbb{Z}^{2g}.$$

We say that a symmetric $2g \times 2g$ matrix Q is *Seifert-parity* if Q is the matrix of a Seifert-parity symmetric bilinear form, or equivalently,

$$Q \equiv J \pmod 2.$$

(Note that the Seifert-parity condition is a condition on the parity on the symmetric matrix Q, not on the Seifert matrix itself!)

First we introduce some algebraic preliminaries. Let $\mathrm{Sp}_{2g}(\mathbb{Z})$ be the group of integer matrices preserving the standard symplectic form $\langle v, w \rangle = v^T J w$ on $\mathbb{Z}^{2g}$, where $J = J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

In this way we can go from a simple Seifert hypersurface V to a $2g \times 2g$ Seifert-parity symmetric matrix Q.

The crucial topological fact is that one can also go in the other direction. The following correspondence is due to Levine[19].

Theorem 2.2.1. *If Q is a Seifert-parity symmetric $2g \times 2g$ matrix, there is a unique pair $V, (\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ with the properties that*

(a) V is a simple Seifert hypersurface in S^{2q+1}

(b) $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ constitute a basis for $H_q(V, \mathbb{Z})$

(c) $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0$, and $\alpha_i \cap \beta_j = -\beta_i \cap \alpha_j = \delta_{ij}$.

(d) the symmetrized pairing $q(\alpha, \beta) = p(\alpha, \beta) + p(\beta, \alpha)$ has matrix Q with respect to the basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$.

Very crude sketch. This fact is proved as part of the proof of Lemma 3 in [19]. We provide a sketch of that proof here to show the basic details. We note that the last two conditions listed may equivalently be stated as: the Seifert pairing p on $H_q(V)$ has pairing matrix $\frac{J+Q}{2}$ with respect to the basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$. Define $P = \frac{J+Q}{2}$.

Start with a copy V_0 of D^{2q} inside S^{2q+1}. We create V by attaching $2g$ different handles to V_0 along its boundary. Each handle is homeomorphic to $D^q \times D^q$, and is glued to V_0 at the boundary by an attaching map $S^q \times D^q \to \partial V_0$. The resulting manifold is $(q-1)$-connected, and the handle decomposition yields an explicit basis for $H^q(V; \mathbb{Z})$; each handle contributes one generator. By appropriately embedding the handles in S^{2q+1}, we can ensure that the Seifert pairing on $H^q(V; \mathbb{Z})$ has matrix P. This gives us items (2) and (3); the one thing left to check is that ∂V is in fact homeomorphic to S^{2q-1}. It is a straightforward calculation (using the fact that $\det J = 1$) to check that the boundary ∂V has the same homology as S^{2q-1}. As well, ∂V is simply-connected by construction (it has a cell decomposition using no 1-cells), so ∂V is homeomorphic to S^{2q-1} by the (high-dimensional) Poincare conjecture.

To show uniqueness, one uses the fact from handlebody theory that any simple Seifert hypersurface V has a handle decomposition of the type described above. The only obstructions to deforming one such handle decomposition into another come from the linking numbers of the handles, so V is unique up to isotopy. □

Remark. In the classical theory of Seifert surfaces in S^3, the construction above will still produce Seifert manifolds with desired pairing. However, this construction is very far from unique.

Theorem 2.2.1 can be restated as saying that there is a bijection between the set of simple Seifert hypersurfaces with homology basis and the set of Seifert-parity $2g \times 2g$ symmetric matrices. This can be stated more concisely using the following definition.

Definition. A *symplectic basis* for $H_q(V, \mathbb{Z})$ is a basis $B = (\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ for $H_q(V, \mathbb{Z})$ such that $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0$ and $\alpha_i \cap \beta_j = -\beta_i \cap \alpha_j = \delta_{ij}$.

Corollary 2.2.2. *There is a bijection*

$$\left\{ \begin{array}{l} \text{simple Seifert hypersurfaces} \\ V \text{ of genus } 2g \text{ equipped with} \\ \text{a symplectic basis } B \text{ for} \\ H_q(V, \mathbb{Z}) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{Seifert-parity } 2g \times 2g \text{ sym-} \\ \text{metric matrices} \end{array} \right\}$$

given by sending the pair (V, B) to the matrix of the symmetric pairing $q(\gamma, \gamma')$ with respect to the basis B.

Furthermore, this induces a bijection

$$\left\{ \begin{array}{l} \text{simple Seifert hypersurfaces} \\ V \text{ of genus } 2g \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \mathrm{Sp}_{2g}(\mathbb{Z})\text{-equivalence classes} \\ \text{of Seifert-parity } 2g \times 2g \\ \text{symmetric matrices} \end{array} \right\}$$

where the Sp_{2g} action is given by $(A, Q) \mapsto A^T Q A$.

We can also state a version of this corollary using the Seifert matrix P of the asymmetric pairing instead of Q.

Corollary 2.2.3. *There is a bijection*

$$\left\{ \begin{array}{l} \text{simple Seifert hypersurfaces} \\ V \text{ of genus } 2g \text{ equipped with} \\ \text{a symplectic basis } B \text{ for} \\ H_q(V, \mathbb{Z}) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 2g \times 2g \text{ matrices } P \text{ with} \\ P - P^T = J \end{array} \right\}$$

given by sending the pair (V, B) to the matrix of the Seifert pairing $p(\gamma, \gamma')$ with respect to the basis B.

Furthermore, this induces a bijection

$$\left\{ \begin{array}{l} \text{simple Seifert hypersurfaces} \\ V \text{ of genus } 2g \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \mathrm{Sp}_{2g}\text{-orbits on } 2g \times 2g \text{ ma-} \\ \text{trices } P \text{ with } P - P^T = J \end{array} \right\}$$

where the Sp_{2g} action is given by $(A, J) \mapsto A^T J A$.

Also, if we drop the requirement that the basis be symplectic, we obtain a bijection

$$\left\{ \begin{array}{l} \text{simple Seifert hypersurfaces} \\ V \text{ of genus } 2g \text{ equipped with} \\ \text{a (not necessarily symplec-} \\ \text{tic) basis for } H_q(V, \mathbb{Z}) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 2g \times 2g \text{ matrices } P \text{ such that} \\ \det(P - P^T) = 1 \end{array} \right\}$$

given by sending a Seifert hypersurface V with basis (γ_i) to the matrix P of the Seifert pairing $p(\gamma, \gamma')$ with respect to the basis (γ_i).

2.3 The classification of simple knots

We will now summarize the classification theorems for simple knots given by Levine, Trotter, Kearton, and Farber, without going into much detail. As before, let $q \geq 3$ be a fixed odd integer, and g be a fixed positive integer.

Theorem 2.3.1 (Classification of simple knots)**.** *[19, 20, 30, 17, 10] The following are in bijection with each other:*

(i) simple $2q-1$ knots of genus $\leq g$

(ii) S-equivalence classes of $2g \times 2g$ Seifert matrices P [19]

(iii) Alexander modules of genus $\leq g$ equipped with Blanchfield pairing. [20, 30]

(iv) R-equivalence classes of $\mathbb{Z}[z]$-modules with isometric structures [10]

Remark. In (ii), S-equivalence is an equivalence relation strictly weaker than Sp_{2g}-equivalence, in that it allows operations that alter the size of the matrix as well as changes of basis.

Regarding (iii), Levine [20] gave explicit conditions for when a $\mathbb{Z}[t, t^{-1}]$-module is the Alexander module of a simple knot. We will give equivalent conditions when we discuss conjugate-self-balanced $\mathcal{O}_\Delta$-modules in the next chapter.

We mention (iv) for the sake of completeness, but will not go into any detail, except to mention it is related to the conjugate-self-balanced R_f-modules discussed in the next chapter.

Chapter 3

The arithmetic of Sp_{2g}-orbits on $2g \times 2g$ symmetric matrices

Motivated by the classification of simple Seifert hypersurfaces in terms of Sp_{2g}-orbits, we now study Sp_{2g}-orbits on $2g \times 2g$-symmetric matrices from the point of view of arithmetic invariant theory. We will use general methods from arithmetic invariant theory to put these orbits in correspondence with natural arithmetic objects. We will start by analyzing this in the simplest case, of algebraically closed fields, and then moving on to general fields.

This classification was already understood by topologists. We will however present here a proof modeled on that given in [3], using Galois cohomology.

3.1 The orbits over an algebraically closed field and algebraic invariants

Let k be a field of characteristic different from 2. Since 2 is a unit in K, the Seifert-parity condition for $2g \times 2g$-symmetric matrices is vacuous over k. We wish to understand all Sp_{2g} orbits on $2g \times 2g$ symmetric matrices over k.

We will rewrite our representation $\mathrm{Sym}^2(k^{2g})$ in a manner that is more convenient for us. Note that the standard representation k^{2g} of Sp_{2g} is self-dual (via the the symplectic pairing), so we also have $\mathrm{Sym}^2(k^{2g}) \cong \mathrm{Sym}^2((k^{2g})^*) \cong (\mathrm{Sym}^2(k^{2g}))^*$ (since we are in characteristic not 2.) Write $V = \mathrm{Sym}^2(k^{2g})^*$

Now $V = (\mathrm{Sym}^2(k^{2g}))^*$ is the space of symmetric bilinear forms on the standard representation of Sp_{2g}. We will show that V is isomorphic to the adjoint representation of Sp_{2g}, that is, the representation of Sp_{2g} acting by conjugation on its Lie algebra.

Let W be a $2g$-dimensional vector space with a unimodular skew-symmetric pairing $\langle , \rangle$. By definition, the automorphism group of W is equal to $\mathrm{Sp}_{2g}(k)$. Its Lie algebra $\mathfrak{sp}_{2g}(k)$ consists of the skew-self-adjoint endomorphisms T of W; that is, those endomorphisms T satisfying $\langle Tw_1, w_2\rangle + \langle w_1, Tw_2\rangle = 0$ for all $w_1, w_2 \in W$.

We now proceed to identify V as well with the representation of skew-self-adjoint endomorphisms of T of W.

By definition, the space of symmetric bilinear forms on W is isomorphic to V. An element $q \in V$ correspends to a symmetric bilinear form $q(x, y)$ on W, which in turn gives rise to an endomorphism T_q of W determined by the property that $\langle x, T_q(y)\rangle = q(x, y)$. The fact that q is symmetric means that

$$\langle x, T_q(y)\rangle = q(x, y) = q(y, x) = \langle T_q(y), x\rangle = -\langle T_q(x), y\rangle.$$

Therefore T_q is skew-self-adjoint with respect to the form $\langle , \rangle$, hence lies in the Lie algebra $\mathrm{Sp}_{W,\langle,\rangle} = \mathrm{Sp}_{2g}$. By the same reasoning in reverse, any skew-self-adjoint operator T on V is equal to of T_q for some quadratic form $q(x, y)$ (because $\langle , \rangle$ is unimodular).

It is easily checked that this correspondence is Sp_{2g}-equivariant, and hence that V is isomorphic to the adjoint representation $\mathfrak{sp}_{2g}$ as a representation of Sp_{2g}.

It is a fact of Lie theory [8][Ch. 8, §8.3, §13.3 (V)] that the invariant ring of the adjoint representation is generated by the coefficients of the characteristic polynomial

of T_q. If J and Q are explicit matrices for the bilinear forms $\langle x, y\rangle$ and $q(x, y)$, respectively, then the operator T_q has matrix $J^{-1}Q$. Explicitly, the characteristic polynomial of T_q equals

$$e_q(x) = \text{char}(T_q) = \det(xI - (J^{-1}Q)) = \det(xJ - Q) = x^{2g} + c_2x^{2g-2} + c_4x^{2g-4} + \cdots + c_{2g}.$$

where the coefficients $c_2, c_4 \ldots, c_{2g}$ of the polynomial $e_q(x)$ are all polynomial functions in the entries of Q. Note that $e_q(x)$ has only even degree coefficients of x, that is to say, it is an even polynomial in x (or equivalently, a polynomial in x^2). Define $e_q(x) := \text{char}(T_q)$, so that the ring of algebraic invariants of V is generated by the coefficients $c_2, c_4, \ldots, c_{2n}$ of $e(x)$.

These algebraic invariants allow us to classify the Sp_{2g}-orbits over an algebraically closed field.

Theorem 3.1.1. *Let k be an algebraically closed field. For any monic squarefree even polynomial $e(x)$ of degree $2g$ over k, there is a unique Sp_{2g}-orbit $[q]$ with characteristic polynomial equal to $e(x)$.*

Theorem 3.1.2. *The map $[q] \mapsto e_q$ gives a bijection between the Sp_{2g}-orbits on W with squarefree characteristic polynomial and the space of all monic squarefree even polynomials $e(x) \in k[x]$.*

Remark. If $e(x)$ is not squarefree, there will be multiple distinct Sp_{2g}-orbits on W with characteristic polynomial $e(x)$. We will not go into the details of how to classify those, because we will not need them in this paper.

3.2 The orbits over a general field

Even when k is not algebraically closed, we may still define the invariant characteristic polynomial $e_q(x)$.

The proof that the map $[q] \mapsto e_q$ is a bijection (for squarefree characteistic polynomials) no longer holds, but we can still prove that it is a surjection in general.

Proposition 3.2.1. *Let k be an algebraically closed field. Then there is a way of associating to any squarefree even polynomial $e(x)$ of degree $2g$ over k a "distinguished" Sp_{2g}-orbit $[q]$ with characteristic polynomial equal to $e(x)$.*

Proof. We will construct a "distinguished orbit" by mimicking the construction of [3] for the case of odd special orthogonal groups.

Let $e(x)$ be an arbitirary monic even polynomial of degree $2g$, and let $E = k[x]/e(x)$. Then E is a $2g$-dimensional k-algebra. It has basis $1, \beta, \beta^2, \ldots, \beta^{2n-1}$, where β is the image of x in E. Because $e(x)$ is an even polynomial in x, E has an involution that switches β and $-\beta$; call that involution $\alpha \mapsto \overline{\alpha}$. Let E_1 be the fixed subalgebra of that involution.

Let $W = E$, and let $T : W \to W$ be the map given by multiplication by β. The map T has our desired characteristic polynomial $e(x)$. We wish to make $T = T_q$, but first we need to pick a skew-symmetric pairing $\langle , \rangle$. We define such a pairing by letting $\langle \lambda, \mu \rangle$ be the coefficient of β^{2g-1} in $\lambda\overline{\mu}$. This is clearly skew-symmetric, and its determinant can be computed to be 1, so it is nondegenerate. We now let q be the symmetric bilinear form corresponding to T (with respect to $\langle , \rangle$), that is, $q(\lambda, \mu) = \langle \lambda, T\mu \rangle$, so $T_q = T$. Then π maps q to $\mathrm{char}(T_q) = \mathrm{char}(T) = e$. Since f was arbitrary, this means that the map π is surjective, that is, any monic even polynomial can be attained as $\mathrm{char}(T_q)$ for some $q \in (\mathrm{Sym}^2(k^{2g}))^*$. □

Remark. We call this k-orbit "distinguished" because it mimics the construction of the distinguished k-orbit of Bhargava and Gross. However, we do not currently have a criterion for "distinguishing the distinguished orbit," that is telling if a specific orbit is equal to the distingished orbit. It would be nice to have such a criterion, and to compare it with the distinguished orbit given by the Kostant section [28].

Fix an even polynomial $e(x)$. We now know that there is at least one k-orbit on V with characteristic polynomial equal to $e(x)$. We will now use methods of Galois cohomology to analyze the set of all such k-orbits.

In this context, it will be convenient for us to think of V as being the adjoint representation of Sp_{2g}.

Because there is only one orbit over the separable closure $\overline{k}$, we may use principles of Galois cohomology (see [27]) to conclude that the orbits with characteristic polynomial $e(x)$ are in one to one correspondence with elements of the kernel of the map of (nonabelian) Galois cohomology sets

$$H^1(k, G_T) \to H^1(k, \mathrm{Sp}_{2g}),$$

where G_T is the stabilizer of T in Sp_W, that is, G_T is the algebraic group given by $\{g \in \mathrm{End}(W) : gTg^{-1} = T\}$. However, $H^1(k, \mathrm{Sp}_{2g})$ is trivial, so we can in fact say that the orbits are in one-to-one correspondence with elements of $H^1(k, G_T)$.

We now compute G_T. As in [3][Section 5], any element that commutes with T must lie in the subalgebra generated by T, hence takes the form of multiplication by some element $\alpha \in E$. On the other hand, multiplication by α preserves $\langle , \rangle$ if and only if $\alpha\overline{\alpha} = 1$.

It then follows from principles of Galois cohomology that these orbits are in one-to-one correspondence with elements of the group $H^1(k, \mathrm{Res}_{E_1/k} U_1(E/E_1))$ (where $U_1(E/E_1)$ is the 1-dimensional unitary group of the quadratic extension E/E_1). Using the short exact sequence

$$1 \to \mathrm{Res}_{E_1/k} U_1(E/E_1) \to \mathrm{Res}_{E/k} \mathbb{G}_m \to \mathrm{Res}_{E_1/k} \mathbb{G}_m \to 1,$$

we conclude that $H^1(k, G_T) \cong E_1^*/N(E)^*$.

We can summarize our results in the following proposition:

Proposition 3.2.2. *The set of orbits V/Sp_{2g} of $\operatorname{Sp}_{2g}(k)$ acting on $V = \operatorname{Sym}^2(k^{2g}) \cong \operatorname{Sym}^2(k^{2g})^*$ has the following properties:*

There is a surjective map π from V/Sp_{2g} to the space of monic even polynomials $e(x)$ of degree $2g$ in x (that is, ones that can be expressed as a polynomial of degree g in x^2).

For a fixed squarefree polynomial $e(x)$, the set $\pi^{-1}(e)$ of orbits that map to the polynomial e is in one-to-one correspondence with the group E_1^*/NE^*.

(Here, as above, $E = k[x]/e(x)$, E_1 is the subalgebra of E fixed under the involution sending x to $-x$, and N is the norm map of algebras.)

This gives us a classification of the $\operatorname{Sp}_{2g}(k)$-orbits on $\operatorname{Sym}^2(k^{2g})$ with separable characteristic polynomial, or equivalently of the orbits of $\operatorname{Sp}_{2g}(k)$ acting on $2g \times 2g$ symmetric matrices Q.

In future sections it will often be more convenient to work with the asymmetric matrix $P = \frac{Q+J}{2}$ instead. We now restate the results above in terms of this matrix P.

Let $\mathcal{A}$ be the space of all matrices $P \in M_{2g}(k)$ such that $P - P^T = J$. Then $\mathcal{A}$ is an affine space over $V = \operatorname{Sym}^2(k^{2g})$, and it has an Sp_{2g} action compatible with its affine space structure. The map $Q \mapsto \frac{Q+J}{2}$ is a bijection between V and $\mathcal{A}$.

We can define an Sp_{2g}-invariant polynomial of an element $P \in \mathcal{A}$ by

$$f_P(y) = \operatorname{char}(J^{-1}P) = y^{2g} + d_1 y^{2g-1} + d_2 y^{2g-2} + \cdots + d_{2g} y^0.$$

This polynomial $f_P(y)$ has the symmetry property that $f_P(y) = f_P(1-y)$. The space of such polynomials f_P is a g-dimensional affine subspace of the space of polynomials of degree g. (Note that the coefficients of f_P are not algebraically independent, because they are linearly dependent.)

If $P = \frac{Q+J}{2}$, the polynomials $e_Q(x)$ and $f_P(y)$ are related by

$$e_Q(x) = 2^{2g} f_P\left(\frac{x-1}{2}\right) \quad \text{and} \tag{3.1}$$

$$f_P(y) = 2^{-2g} e_Q(2y-1). \tag{3.2}$$

We now state, as a corollary of Proposition 3.2.2, a classification result for Sp_{2g}-orbits on $\mathcal{A}$.

Corollary 3.2.3. *The set of orbits $\mathcal{A}/\mathrm{Sp}_{2g}$ of $\mathrm{Sp}_{2g}(k)$ acting on the affine space*

$$\mathcal{A} = \{P \in M_{2g}(k) \mid P - P^t = J\}$$

has the following properties:

Thre is a surjective map π from $\mathcal{A}/\mathrm{Sp}_{2g}$ to the space of monic polynomials $f(y)$ of degree $2g$ in y satisfying $f(y) = f(1-y)$.

For a fixed squarefree polynomial $f(y)$, the set $\pi^{-1}(y)$ of orbits that map to the polynomial f is in one-to-one correspondence with the group

$$F_1^*/NF^*,$$

where $F = k[y]/f(y)$, F_1 is the subalgebra of F fixed under the involution sending $y \mapsto 1-y$, and N is the norm map of algebras.

Proof. Change variables to $Q = 2P - J$, and apply Proposition 3.2.2. □

Remark. Another way of thinking of the proof above is the following. We are interested in classifying orbits of GL_W acting on the set of triples (a, q, T) such that $a(x, y)$ $(= \langle x, y\rangle)$ is a skew-symmetric bilinear pairing on W, $q(x, y)$ is a symmetric bilinear

pairing, and T satisfies

$$q(x,y) = a(x,Ty) = -a(Tx,y) = -q(Tx,T^{-1}y).$$

Given any two among (a,q,T), the third is uniquely determined.

Hence to study the orbits, one can fix any one of (a,q,T) and look at the orbit of the stabilizer of that one on either of the other two.

If we fix a (which has the advantage that there is only a single $\mathrm{GL}(W)$-orbit of a's with nonzero determinant), we get the action of the symplectic group $\mathrm{Sp}(a)$ on either Sym^2 of the standard representation, or on the Lie algebra $\mathfrak{sp}(a)$, depending upon whether we choose q or T. This is the question we have studied above.

If we fix q (in which case classifying the $\mathrm{GL}(W)$-orbits on the space of quadratic forms is a non-trivial problem; see Milnor and Husemoller [26]), we get the action of the orthogonal group $O(q)$ on either $\bigwedge^2$ of the standard representation, or on the Lie algebra $\mathfrak{o}(q)$, depending upon whether we choose q or T. This was studied in [3].

However, if, as above, we first fix T, things look more different. First, the possible orbits of T over k^{sep} are given by the characteristic polynomials of T, along with extra data if T is not semisimple. Once this is fixed, the stabilizer of any given such T is $\mathrm{Cent}(T) \subset \mathrm{GL}(W)$, which equals $k[T]^*$ if T is semisimple.

At this point, it does not make much difference whether we study q-orbits or a-orbits. In the above we have focused on the a-orbits. We are now looking at skew-symmetric pairings $\langle , \rangle$ on W with the additional condition that $\langle Tx, y\rangle = -\langle x, Ty\rangle$. This is a fairly restrictive condition that is conducive to further analysis.

Remark. One might ask what happens in characteristic 2. In this case, looking at orbits of symmetric matrices Q with $Q = J \pmod 2$ is the wrong question; in characteristic 2, a matrix which is congruent mod 2 to J is just equal to J, so the orbit question on matrices Q becomes trivial. Rather, we should go back to our Seifert

matrix P with $P - P^t \equiv J \pmod 2$, and so $P - P^t = J$. This matrix P is no longer uniquely determined by Q. We will study this question in the next section in the full generality of Dedekind domains, and the results we get will specialize to characteristic 2.

3.3 The orbits over $\mathbb{Z}$

We now move to the case of most interest in knot theory, namely the $\mathbb{Z}$-orbits. In this case the classification of orbits is less simple, but the orbits can still be put into correspondence with arithmetic objects, namely "conjugate-self-balanced" modules or ideal classes over certain $\mathbb{Z}$-algebras.

We will define these objects before establishing the correspondence. We first define a $\mathbb{Z}$-algebra R_f that will serve an analogous function to the k-algebra F of the previous section. Let $f \in \mathbb{Z}[y]$ be a polynomial such that $f(y) = f(1-y)$. Define $R_f = \mathbb{Z}[\gamma] \cong \mathbb{Z}[y]/f(y)$. The monogenic $\mathbb{Z}$-algebra R_f possesses an involution $a \mapsto \overline{a}$ such that $\overline{\gamma} = 1 - \gamma$.

We now define a conjugate self-balanced R_f-module.

Definition. A *conjugate-self-balanced R_f-module* is an R_f-module M equipped with a bilinear pairing $\phi : M \otimes \overline{M} \to R_f$ such that

(a) M is a free $\mathbb{Z}$-module of rank equal to the degree of f

(b) the characteristic polynomial of γ acting on M is exactly f

(c) for $m \in M$ and $n \in \overline{M}$, $\phi(m,n) = \phi(\overline{n}, \overline{m})$

(d) ϕ induces an isomorphism $M \cong \mathrm{Hom}_{R_f}(\overline{M}, R_f)$.

Remark. This definition is consistent with the definition of a balanced pair of modules defined by Wood[32] in the following sense. We homogenize f to make a binary $2g$-ic

form F. Then the ring R_F defined by Wood associated to the binary $2g$-ic form F is identical to our ring R_f, and the ideal I_F is principal: $I_F = (i_f)$ for $i_f = f'(\gamma)$. If M is a conjugate-self-balanced R_f-module with pairing ϕ, then the pair $(M, \overline{M})$ equipped with the map $i_f \cdot \phi : M \otimes \overline{M} \to I_F$ is a balanced pair in the sense of Wood.

(Note that the identification of these two defintions is not quite canonical, in the sense that we had to choose a generator i_f for I_F; the choice $i_f = -f'(\gamma)$ would have been equally good.)

Remark. Note the similarities of this definition to the "Levine conditons" introduced in [20] to classify Alexander modules of knots, and later used by Trotter in [30] to give another proof of the classificaton via Seifert matrices.

A large family of conjugate-self-balanced R_f modules are isomorphic as R_f modules to ideals of R_f. This will lead us to define a notion of "conjugate-self-balanced ideal class".

We will first need to define the absolute norm of a fractional ideal of R_f.

Definition. For a fractional ideal I of R_f, let $|I|$ denote the absolute ideal norm of I. This is a function from the set of ideals of I to R_f uniquely determined by the following two properties.

- $|I|$ is equal to the index $[R_f : I]$ when I is an ideal of R
- $|\kappa I| = |(\kappa)|I$ for any $\kappa \in R_f$ and any fractional ideal I of R_f.

Definition. A conjugate-self-balanced ideal class of R_f is an equivalence class of pairs (I, α) where I is a fractional ideal of R_f and $\alpha \in R_f \otimes \mathbb{Q}$, such that $I\overline{I} \subset \alpha$ and $|I|^2 = |(\alpha)|$, modulo the equivalence $(I, \alpha) \sim (\kappa I, \kappa\overline{\kappa}\alpha)$ where κ is any invertible element of $R_f \otimes \mathbb{Q}$.

If $[I, \alpha]$ is a conjugate-self-balanced ideal class of R_f, then I is also a conjugate-self-balanced R_f module with self-balancing map given by $a \otimes b \mapsto ab/\alpha$. The proof

that this is in fact a self-balancing map is the same as the proof of Theorem 5.5 in [32].

If f is squarefree the converse holds; every conjugate-self-balanced R_f-modules comes from a unique conjugate-self-balanced ideal class of R_f. Again, this follows from the proof of Theorem 5.5 in [32].

We now show that Sp_{2g}-orbits exactly parametrize conjugate-self-balanced modules. The ideas behind this proof are similar to those found in Trotter [30].

In this case, we will use the bijection we already saw in knot theory, which puts Seifert-parity $2g \times 2g$-symmetric matrices Q in one-to-one correspondence with $2g \times 2g$ matrices P such that $P - P^t = J$.

Theorem 3.3.1. *Let $f \in \mathbb{Z}[y]$ be a monic polynomial of degree $2g$ with $f(y) = f(1-y)$.*

The set of Sp_{2g}-equivalence classes of matrices P with $P - P^t = J$ and given characteristic polynomial $f(y) = \det(yJ - P)$ is in one-to-one correspondence with the set of classes of conjugate-self-balanced R_f-modules.

Before proving this, we define a $\mathbb{Z}$-linear map $c_{2g-1} : R_f \to \mathbb{Z}$ motivated by definitions of Wood [32] and of Trotter [30, 29] that will be useful here.

Definition. We define $c_{2g-1} : R \to \mathbb{Z}$ to be the linear functional that maps an element $\sum_{i=0}^{2g-1} c_i \gamma^i \in R$ to the coefficent c_{2g-1} of γ^{2g-1}.

Note that for any $x \in R_f$, $c_{2g-1}(\overline{x}) = -c_{2g-1}(x)$.

We now establish some useful properties of c_{2g-1}. The lemma below is a special case of Proposition 2.5 in [32].

Lemma 3.3.2. *(a) The $\mathbb{Z}$-bilinear pairing $r_1, r_2 \mapsto c_{2g-1}(r_1 r_2)$ on R_f is a perfect pairing.*

(b) If N is any R_f-module, then the map $c_{2g-1} : R_f \to \mathbb{Z}$ induces an isomorphism

$$(c_{2g-1})_* : \mathrm{Hom}_{R_f}(N, R_f) \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}).$$

given by $\psi \mapsto c_{2g-1} \circ \psi$.

Proof. For (i), we observe that the pairing matrix with respect to the basis given by powers of y has 0s above the antidiagonal and 1s on the antidiagonal.

For (ii), we apply (i) to get an isomorphism

$$R_f \cong \mathrm{Hom}_{\mathbb{Z}}(R_f, \mathbb{Z}).$$

Applying the functor $\mathrm{Hom}(N, \cdot)$ gives an isomorphism

$$\mathrm{Hom}_{R_f}(N, R_f) \cong \mathrm{Hom}_{R_f}(N, \mathrm{Hom}_{\mathbb{Z}}(R_f, \mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}),$$

which is exactly $(c_{2g-1})_*$. □

Proof. We will construct explicit maps going in both directions. The ideas here are the same as those used in [32].

We first construct a map $M \mapsto P_M$ from conjugate-self-balanced R_f-modules to Sp_{2g}-orbits.

Given a conjugate-self-balanced module M, equipped with an isomorphism $\phi : M \otimes \overline{M} \to \mathbb{R}_f$, define a $\mathbb{Z}$-valued pairing on M by

$$\langle m_1, m_2 \rangle_M = c_{2g-1}(\phi(m_1 \otimes \overline{m_2})).$$

This pairing is easily checked to be skew-symmetric. The fact that it is a perfect pairing over $\mathbb{Z}$ follows by composing the given isomorphism $M \cong \mathrm{Hom}_{R_f}(\overline{M}, R_f)$

induced by ϕ with the isomorphism $(c_{2g-1})_* : \mathrm{Hom}_{R_f}(\overline{M}, R_f) \cong \mathrm{Hom}_{\mathbb{Z}}(\overline{M}, \mathbb{Z})$ from Lemma 3.3.2.

Choose a basis $\mathcal{B}$ of M with respect to which $\langle , \rangle_M$ has matrix equal to J. Then let P_M be the matrix of the asymmetric $\mathbb{Z}$-valued pairing

$$m_1, m_2 \mapsto \phi(\gamma m_1 \otimes \overline{m_2})$$

on M with respect to $\mathcal{B}$. It is then easily checked that $P - P^T = J$ and that $J^{-1}P$ has the desired characteristic polynomial.

Now we construct a map $P \mapsto M_P$ going in the other direction, from Sp_{2g}-orbits to conjugate-self-balanced R_f-modules.

To define M_P as an $R_f = \mathbb{Z}[\gamma]$ module, it suffices to specify the structure of M_P as a $\mathbb{Z}$-module and the action of γ on M_P. Let M_P equal the $\mathbb{Z}$-module $\mathbb{Z}^{2g}$ where γ acts as multiplication by $T_P := J^{-1}P$. By the Cayley-Hamilton theorem, $f(T_P) = 0$, so M_P is a well-defined $\mathbb{Z}[\gamma]$-module.

We now run our previous construction in reverse. Let $\langle , \rangle$ be the standard (with matrix J) symplectic $\mathbb{Z}$-valued pairing on $\mathbb{Z}^{2g} \cong M_P$. Applying Lemma 3.3.2 again, in the reverse direction, we conclude that there is a unique R_f-linear map $\phi_P : M_P \otimes \overline{M_P} \to R_f$ such that

$$\langle m_1, m_2 \rangle = c_{2g-1}(\phi(m_1 \otimes \overline{m_2}))$$

for all $m_1, m_2 \in M_P$, and that ϕ_P makes M_P into a conjugate-self-balanced R_f-module. □

The following proposition summarizes all of our results:

Proposition 3.3.3. *Let f be a squarefree monic polynomial of degree $2g$ satisfying $f(y) = f(1-y)$. Then the following objects are in bijection with each other:*

- *Conjugate-self-balanced ideal classes of the ring* $R_f = \mathbb{Z}[y]/f(y)$
- *Conjugate-self-balanced modules over* $R_f = \mathbb{Z}[y]/f(y)$
- Sp_{2g}*-equivalence classes of Seifert-parity* $2g \times 2g$ *symmetric matrices* Q *such that the characteristic polynomial of* $J^{-1}Q$ *is equal to* $2^{2g}f((x-1)/2)$
- Sp_{2g}*-equivalence classes of asymmetric pairing matrices* P *with* $P - P^T = J$ *and such that the characteristic polynomial of* $J^{-1}P$ *is equal to* $f(y)$
- *Simple Seifert hypersurfaces* V *such that, for one or any symplectic basis* B *of* $H^q(V)$*, the matrix* P *of the asymmetric Seifert pairing* $p(\gamma, \gamma') = \mathrm{lk}(\gamma, (\gamma')^+)$ *with respect to* B *satisfies the property that the characteristic polynomial of* $(J^{-1}P)$ *is equal to* $f(y)$
- *Simple Seifert hypersurfaces* V *such that the Alexander polynomial* $\Delta(t)$ *of the knot* ∂V *is equal to* $(1-t)^{2g}f(1/(1-t))$.

Chapter 4

Connections to the Alexander module and related knot invariants

So far, we have only discussed invariants of Seifert hypersurfaces. In this section we will discuss how they relate to known invariants of knots.

Let V be a Seifert hypersurface in S^{2q+1}, and let P be a Seifert matrix for V with respect to a symplectic basis of $H_q(V)$.

The matrix P is unique up to Sp_{2g}-equivalence. Hence we can apply the invariants constructed in the previous section. We define an invariant polynomial

$$f_V(y) = \det(yI - J^{-1}P). \tag{4.1}$$

We also define an invariant conjugate-self-balanced module $[M_V, \phi_V]$ over the ring $R_{f_V} = \mathbb{Z}[y]/f_V(y)$. When the polynomial $f_V(y)$ is squarefree, we can also define the corresponding conjugate-self-balanced ideal class $[I_V, \alpha_V]$.

In this chapter, we will show:

- after change of variables, f_V becomes an Alexander polynomial for the boundary knot $K = \partial V$.

- after localization, the conjugate-self-balanced module M_V becomes identified with the Alexander module of K.

4.1 The Alexander polynomial

We now describe the relationship between the polynomial f_V and the Alexander polynomial of K coming from q-dimensional homology. As a result, we will see that f_V depends only on the boundary knot $K = \partial V$ and on the genus g of V.

We first review the standard definition of the Alexander polynomial in terms of the Alexander module.

Recall that we have defined $(S^{2q+1} - K)^{\text{cyc}}$ to be the unique infinite cyclic cover of $S^{2q+1} - K$. Let t be the "positively oriented" generator of the group of deck transformations of $(S^{2q+1} - K)^{\text{cyc}}$ (where this orientation is defined in terms of the orientations on S^{2q+1} and on K). Then t has an induced action on all homology groups of $(S^{2q+1} - K)^{\text{cyc}}$. We can extend this action to give a $\mathbb{Z}[t, t^{-1}]$-module structure on those homology groups. These modules are referred to as the Alexander modules of K. In the setting of simple knots, the only nontrivial Alexander module is the one coming from qth homology, which we will refer to as *the Alexander module*.

Definition. We define the *Alexander module* Alex_K of K as the $\mathbb{Z}[t, t^{-1}]$-module $H_q((S^{2q+1} - K)^{\text{cyc}}, \mathbb{Z})$.

We may define the Alexander polynomial for this Alexander module.

Definition. The *Alexander polynomial* Δ_K is the generator of the first Fitting ideal of the $\mathbb{Z}[t, t^{-1}]$-module Alex_K, which is well-defined up to multiplication by units in $\mathbb{Z}[t, t^{-1}]$.

We have a choice about how to normalize it. The standard choice is the unique normalization which makes $\Delta_K(t)$ is a polynomial in t with nonzero constant term satisfying $\Delta_K(1) = 1$, which is always possible.

We note the following facts from high-dimensional knot theory that are true about the Alexander polynomial using the standard normalization.

- The Alexander polynomial $\Delta_K(t)$ has degree equal to $2\,\mathrm{genus}(K)$.
- The Alexander polynomial satisfies the identity $\Delta_K(t) = t^{2\,\mathrm{genus}(K)}\Delta_K(t^{-1})$.
- $\Delta_K(1) = 1$.
- The Alexander polynomial annihilates the Alexander module.

Remark. The last two facts are still true for classical knots. The first becomes false (using the standard definition of knot genus as the minimal genus of a Seifert surface). The second fact is false as stated but becomes true after replacing $2\,\mathrm{genus}(K)$ with the degree of $\Delta_K(t)$.

If V is a Seifert hypersurface for K (not necessarily of minimal genus), the Alexander polynomial of $\Delta_K(t)$ can be obtained from the polynomial $f_V(t)$ by a change of variables. For this purpose, it is better to normalize the Alexander polynomial differently, as we will do below.

Note that $\Delta_{K,V}(t) = t^{\mathrm{genus}(V)-\mathrm{genus}(K)}\Delta_K(t)$. The functional equation now becomes $\Delta_{K,V}(t) = t^{2\,\mathrm{genus}(V)}\Delta_{K,V}(t^{-1})$, and the property $\Delta_{K,V}(1) = 1$ is still satisfied. Note that $\Delta_{K,V}(t)$ depends only on K and the genus of V, not on the choice of Seifert hypersurface V for K. When V is a minimal genus Seifert hypersurface, we have $\mathrm{genus}(V) = \mathrm{genus}(K)$, hence $\Delta_{K,V}(t) = \Delta_K(t)$.

We now state the theorem showing that $f_V(t)$ is an Alexander polynomial.

Theorem 4.1.1. *The polynomials $\Delta_{K,V}(t)$ and $f_V(y)$ are related in the following manner:*

$$f_V(y) = y^{2\,\mathrm{genus}(V)}\Delta_{K,V}(1 - 1/y) \tag{4.2}$$

and

$$\Delta_{K,V}(t) = (1-t)^{2\,\mathrm{genus}(V)} f_V(1/(1-t)). \tag{4.3}$$

Proof. This follows from the standard formula $\Delta_{K,V}(t) = \det(Pt - P^T)$ and algebraic manipulations. □

4.2 Conjugate-self-balanced $\mathcal{O}_\Delta$-modules

We now wish to compare the Alexander module Alex_K of a knot K with the invariant conjugate-self-balanced module M_V constructed in Section 3.3.

Again, let V be a simple Seifert hypersurface with boundary knot K. Let $f(y) = f_V(y)$ be the invariant polynomial for V, and let $\Delta(t) = \Delta_K(t)$ be the Alexander polynomial of K. For convenience we will restrict our attention to the case when the matrix P is nonsingular, which is equivalent to the condition that $\mathrm{genus}\, K = \mathrm{genus}\, V$. This implies that $\Delta_{K,V}$ has nonzero constant term and is equal to $\Delta = \Delta_K$.

To do this, we must first change our point of view on the Alexander module. As it stands, the Alexander module is a module over the ring $\mathbb{Z}[t, t^{-1}]$, while M_V is a module over $R_f = \mathbb{Z}[y]/f_V(y)$. However, the Alexander module is annihilated by the Alexander polynomial $\Delta(t)$, and thus can also be viewed as a module over the quotient ring, which we now give a name.

Definition. Let

$$\mathcal{O}_\Delta = \mathbb{Z}[t, t^{-1}]/\Delta(t).$$

Let θ denote the image of t in $\mathcal{O}_\Delta$, so that $\mathcal{O}_\Delta = \mathbb{Z}[\theta, \theta^{-1}]$.

The ring $\mathcal{O}_\Delta$ has a natural involution interchanging θ with θ^{-1}. We denote it by $x \mapsto \overline{x}$.

We would ultimately like to say that the the Alexander module Alex_K is a conjugate-self-balanced module for $\mathcal{O}_\Delta$, but to do this we must first define what this means.

We define it in a manner similar to Definition 3.3, but with the first condition modified because $\mathcal{O}_\Delta$ need not be a finitely-generated $\mathbb{Z}$-module.

Definition. A *conjugate-self-balanced $\mathcal{O}_\Delta$-module* is an $\mathcal{O}_\Delta$-module M equipped with a bilinear pairing $\phi : M \otimes \overline{M} \to \mathcal{O}_\Delta$ such that

(a) M is a finitely generated $\mathcal{O}_\Delta$-module.

(b) $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free $\mathbb{Q}$-module of rank equal to the degree of f.

(c) The characteristic polynomial of θ acting on $M \otimes_{\mathbb{Z}} \mathbb{Q}$ by multiplication is exactly Δ.

(d) For $m \in M$ and $n \in \overline{M}$, $\phi(m, n) = \phi(n, m)$.

(e) ϕ induces an isomorphism $M \cong \text{Hom}_{\mathcal{O}_\Delta}(\overline{M}, \mathcal{O}_\Delta)$.

In this context we can also define the related concept of a conjugate-self-balanced ideal class. First we define a $\mathbb{Z}$-valued absolute norm on the set of fractional ideals of $\mathcal{O}_\Delta$ exactly as in Definition 3.3. We can then carry over verbatim the definition of a conjugate-self-balanced ideal class from 3.3.

Definition. A *conjugate-self-balanced ideal class* of $\mathcal{O}_\Delta$ is an equivalence class of pairs (I, α) where I is a fractional ideal of $\mathcal{O}_\Delta$ and $\alpha \in \mathcal{O}_\Delta \otimes_{\mathbb{Z}} \mathbb{Q}$, such that $I\overline{I} \subset \alpha$ and $|I|^2 = |(\alpha)|$, modulo the equivalence $(I, \alpha) \sim (\kappa I, \kappa\overline{\kappa}\alpha)$ for any invertible element κ of $\mathcal{O}_\Delta \otimes_{\mathbb{Z}} \mathbb{Q}$.

***TODO:*add results about ideals**

4.3 The Alexander module as a conjugate-self-balanced module

We will use the Blanchfield pairing, introduced in Theorem 2.1.4, to endow the Alexander module with a conjugate-self-balancing map.

Theorem 4.3.1. *If K is a simple knot with Alexander polynomial $\Delta(t)$, the Alexander module* Alex_K *is a conjugate-self-balanced $\mathcal{O}_\Delta$-module with balancing map given by*

$$\phi_K(m,n) = \Delta(t) \cdot \text{Bl}(m,\overline{n}) \in \mathbb{Z}[t,t^{-1}]/\Delta(t) \cong \mathcal{O}_\Delta.$$

Furthermore, the map sending a knot K to the conjugate-self-balanced module (Alex_K, ϕ_K) *induces a bijection:*

$$\left\{ \begin{array}{l} \text{simple knots with Alexander polynomial } \Delta(t) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{CSB modules } (M,\phi) \text{ over the ring } \mathcal{O}_\Delta = \\ \mathbb{Z}[t,t^{-1}]/\Delta(t) \end{array} \right\}.$$

Proof. Theorem 4.4 of Levine [20] implies that (Alex_K, ϕ_K) is a conjugate-self-balanced module, and Theorem 12.1 of the same paper shows that any conjugate-self-balanced module can be realized in this way. The classification of simple knots by Alexander module and Blanchfield pairing then tells us that the map $K \mapsto (\text{Alex}_K, \phi_K)$ is a bijection. □

4.3.1 Comparing conjugate-self-balanced modules

We now have two different conjugate-self-balanced modules, M_V and Alex_K. We will show that they are related via change of base ring.

We first define a homomorphism between the base rings R_f and $\mathcal{O}_\Delta$. Recall that $R_f = \mathbb{Z}[\gamma] = \mathbb{Z}[y]/f(y)$, and $\mathcal{O}_\Delta = \mathbb{Z}[\theta,\theta^{-1}] = \mathbb{Z}[t,t^{-1}]/\Delta(t)$. Note that the element

$1-\theta$ is a unit of $\mathcal{O}_\Delta$, because $\Delta(1)=1$, and that $\frac{1}{1-\theta}$ is a root of the polynomial f by (4.3). Hence the following homomorphism is well-defined:

Definition. Let $\iota : R_f \to \mathcal{O}_\Delta$ be the homomorphism from R_f to $\mathcal{O}_\Delta$ determined by $f(\gamma) = \frac{1}{1-\theta}$.

Under our assumption that $\operatorname{genus} K = \operatorname{genus} V$, it is easily checked that ι is an injection. Therefore ι identifies R_f with the subring of $\mathcal{O}_\Delta$ generated by $\frac{1}{1-\theta}$.

In the other direction, with the same assumption, one can also identify $\mathcal{O}_\Delta$ with the localization $R_f[(\gamma\overline{\gamma})^{-1}] = R_f[\gamma^{-1}, (1-\gamma)^{-1}]$. Here the element θ is identified with $1-\gamma^{-1}$, and θ^{-1} is identified with $1-(1-\gamma)^{-1}$. Under our assumption that $\operatorname{genus}(K) = \operatorname{genus}(V)$, this induces an identification of the ring $\mathcal{O}_\Delta \otimes \mathbb{Q} = \mathbb{Q}[\theta, \theta^{-1}] = \mathbb{Q}[\theta]$ with $R_f \otimes \mathbb{Q} = \mathbb{Q}[\gamma]$.

We now give the map from $\mathrm{CSB}(R_f)$ to $\mathrm{CSB}(\mathcal{O}_\Delta)$.

Proposition 4.3.2. *The map ι induces a map ι_* from the set $\mathrm{CSB}(R_f)$ of conjugate-self-balanced R_f-modules to the set $\mathrm{CSB}(\mathcal{O}_\Delta)$ of conjugate-self-balanced $\mathcal{O}_\Delta$-modules, given by*

$$(M, V) \mapsto (M \otimes_{R_f} \mathcal{O}_\Delta, \phi \otimes_{R_f} \mathcal{O}_\Delta).$$

Proof. All conditions except the last are straightforward to check.

For the last one, we are given an isomorphism

$$M \cong \operatorname{Hom}_{R_f}(\overline{M}, R_f).$$

Tensoring both sides with $\mathcal{O}_\Delta$ gives

$$M \otimes_{R_f} \mathcal{O}_\Delta \cong \operatorname{Hom}_{R_f}(\overline{M}, R_f) \otimes_{R_f} \mathcal{O}_\Delta.$$

Now, because $\mathcal{O}_\Delta$ is a flat R_f-module (it is a localization), it follows that (see exercise 1.2.8 in [22])

$$\mathrm{Hom}_{R_f}(\overline{M}, R_f) \otimes_{R_f} \mathcal{O}_\Delta \cong \mathrm{Hom}_{\mathcal{O}_\Delta}(\overline{M} \otimes_{R_f} \mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

Composing the two isomorphisms, we see that $M \otimes_{R_f} \mathcal{O}_\Delta$ also satisfies the final condition for a conjugate-self-balanced module. □

We now show that ι_* maps the conjugate-self-balanced module (M_V, ϕ_V) constructed from the Seifert pairing on V to the conjugate-self-balanced module $(\mathrm{Alex}_K, \phi_K)$ coming from the Alexander module of K.

Theorem 4.3.3. *We have*

$$\iota_*(M_V, \phi_V) = (\mathrm{Alex}_K, \phi_K).$$

Proof. Let P be a Seifert matrix for V. We will use explicit presentations, due to Kearton, of the Alexander module and Blanchfield pairing of K in terms of P.

Kearton [17][§1] gave the following presentation of the Alexander module

$$\mathcal{O}^{2g} \to \mathcal{O}^{2g} \to \mathrm{Alex}_K \to 0.$$

where the first map is multiplication by $\theta P - P^t$. We will show that

$$\iota_*(M_V) = M_V \otimes_{R_f} \mathcal{O}_\Delta$$

fits into the same exact sequence.

First, note that, because R_f is monogenic, there is a short exact sequence of R_f-modules.

$$R_f \otimes_{\mathbb{Z}} R_f \to R_f \otimes_{\mathbb{Z}} R_f \to R_f \to 0$$

where the first map is multiplication by $1 \otimes \gamma - \gamma \otimes 1$ and the second map is induced by the multiplication map $R_f \times R_f \to R_f$.

By right exactness of the tensor product, we may tensor this sequence over R_f with M_V. This gives:

$$R_f \otimes_{\mathbb{Z}} M_V \to R_f \otimes_{\mathbb{Z}} M_V \to M_V \to 0.$$

Recall that we constructed M_V by setting $M_{\text{Arith}} = \mathbb{Z}^{2g}$ as a $\mathbb{Z}$-module and having γ act as left multiplication by $J^{-1}P$. Using this identification of M_V with $\mathbb{Z}^{2g}$, we obtain a short exact sequence

$$R_f^{2g} \to R_f^{2g} \to M_V \to 0,$$

where now the matrix of the first map is given by $J^{-1}P - \gamma = J^{-1} \cdot (P - \gamma J)$, as desired.

Now, tensoring over R_f with $\mathcal{O}_\Delta$, we get a short exact sequence

$$\mathcal{O}_\Delta^{2g} \to \mathcal{O}_\Delta^{2g} \to M_V \to 0.$$

where the first map is left multiplication by $J^{-1} \cdot (P - \iota(\gamma)J)$.

But

$$P - \iota(\gamma)J = P - \frac{1}{1-\theta}J = (1-\theta)^{-1}((1-\theta)P - (P - P^T)) = -(1-\theta)^{-1}(\theta P - P^T).$$

Because J is invertible and $-(1-\theta)^{-1}$ is a unit of $\mathcal{O}_\Delta$, this means that there is another short exact sequence

$$\mathcal{O}_\Delta^{2g} \to \mathcal{O}_\Delta^{2g} \to M_V \to 0.$$

where the first map is left multiplication by $\theta P - P^t$. This shows the first part.

To compare the conjugate-self-balancing maps, we use the presentation of the Blanchfield pairing due to Kearton in [17, §8]. He shows that the Blanchfield pairing $\mathrm{Alex} \times \mathrm{Alex} \to \mathbb{Q}(t, t^{-1})/\mathbb{Z}[t, t^{-1}]$ is induced, in the presentation above, by the pairing on $\mathcal{O}_\Delta^{2g}$ with matrix $(t-1)(tP - P^t)^{-1}$.

By the first half, we have $\mathrm{Alex}_K \cong M_V \otimes_{R_f} \mathcal{O}_\Delta$, and so we can identify M_V with a submodule of $\mathcal{O}_\Delta$. Then it suffices to show that the restriction of ϕ_K to a bilinear pairing $M_V \times \overline{M_V} \to \mathcal{O}_\Delta$ agrees with the pairing ϕ_V.

Recall from the proof of Theorem 3.3.1 that the pairing ϕ_V is uniquely characterized by the property that

$$\langle m_1, m_2 \rangle = c_{2g-1}(\phi_V(m_1 \otimes \overline{m_2}))$$

for all $m_1, m_2 \in M_V$.

It suffices to check that the restriction of ϕ_K has the same property.

Using Kearton's presentation matrix for the Blanchfield pairing, it is a simple computation to show that the matrix of the restriction of $\phi_K(m_1 \otimes \overline{m_2})$ to M_V, with respect to the standard basis, is given by $(P - \gamma(J))^{\mathrm{ad}}$ (where ad refers to the adjoint matrix from Cramer's rule), and therefore the restricted pairing takes values in R_f.

We now apply c_{2g_1}. The only terms with a γ^{2g-1} in the expansion of $(P - \gamma J)^{\mathrm{ad}}$ come from the $(-\gamma J)^{\mathrm{ad}} = -\gamma^{2g-1} J^{\mathrm{ad}}$, and so $c_{2g-1}((P - \gamma J)^{\mathrm{ad}}) = -J^{\mathrm{ad}} = J$.

Hence the pairing $c_{2g-1}(\phi_V(m_1 \otimes \overline{m_2}))$ also has matrix J with respect to the standard basis of $M \cong \mathbb{Z}^{2g}$, as desired.

□

4.3.2 Conjugate-self-balanced ideal classes of Dedekind rings

Before we move on to counting, we will note some ways in which the theory of conjugate-self-balanced ideal classes is particularly nice when the base ring is a Dedekind domain.

Recall that a Dedekind domain is a Noetherian integral domain of Krull dimension 1 which is integrally closed in its field of fractions. The rings R_f and $\mathcal{O}_\Delta$ are always Noetherian and of Krull dimension 1, so they will be Dedekind domains precisely when f (respectively Δ) is an irreducible polynomial and when R_f (respectively $\mathcal{O}_\Delta$) is integrally closed.

Let A be a general Dedekind domain. Because every ideal of A is invertible, the set of conjugate-self-balanced ideal classes of A forms a group with the natural multiplication induced by multiplication of ideals: $[I, \alpha] \cdot [I', \alpha'] = [II', \alpha\alpha']$. This group, which we will denote CSB_A, has a natural homomorphism to the ideal class group $\mathrm{Cl}(R)$ (see [18] for definition of the ideal class group of a Dedekind ring). This homomorphism is given by forgetting α, and its image is exactly the kernel of the norm map from $\mathrm{Cl}(R)$ to $\mathrm{Cl}(R_1)$. On the other hand, the kernel of this homomorphism is the image of the norm homomorphism on the unit groups $R^\times \to R_1^\times$. We can summarize this by saying that the group of conjugate-self-balanced ideal classes fits into a short exact sequence

$$0 \to R_1^\times / N(R^\times) \to \mathrm{CSB}_R \to \ker(N : \mathrm{Cl}(R) \to \mathrm{Cl}(R_1)) \to 0.$$

Chapter 5

Counting

5.1 Finiteness theorems for Seifert hypersurfaces and simple knots

We would now like to put some notion of "size" on a knot in such a way that there are only finitely many simple knots of bounded size. The following finiteness theorems will be useful to us here.

Theorem 5.1.1 (Bayer and Michel,[1], Propositions 6 and 8)**.**

- *If $\Delta \in \mathbb{Z}[t]$ is a squarefree polynomial, there are only finitely many simple knots K with $\Delta_K = \Delta$.*
- *For any squarefree $f \in \mathbb{Z}[y]$, there are only finitely many simple Seifert hypersurfaces V such that $f_V = f$.*

Therefore, we can and will measure the size of K by the size of the Alexander polynomial of K. For knots K of fixed genus g, we can measure this size by the size of the coefficients of $\Delta_K(t)$, or of $f_V(t)$ (where V is any genus g Seifert hypersurface). There is still the question of how to "weight" the different coefficients with respect to each other.

One way is to treat all coefficients equally; this leads to the standard definition of the height of a polynomial.

Definition. The *height* $\mathrm{ht}(P)$ of a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0 \in \mathbb{Z}[x]$ is equal to

$$\max_{0 \le i \le n} |a_i|.$$

One advantage of using height is that, for fixed g, the asymptotics for $\mathrm{ht}(f_V(y))$ are identical to those for $\mathrm{ht}(\Delta_{K,V}(t))$.

Note that $\mathrm{ht}(P) < X$ if and only if all coefficients of P have absolute value at most X.

For specific constrained families of polynomials, there may be other reasonable ways to define size. For instance, for monic polynomials, we may defined a "weighted height" function modeled after the definition in [4]

Definition. The *weighted height* $\mathrm{ht}_w(P)$ of a monic polynomial

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0 \in \mathbb{Z}[x]$$

is equal to

$$\max_{0 \le i \le n-1} |a_i|^{n(n-1)/(n-i)}.$$

This definition is nice because it behaves nicely with respect to rescaling the variable, that is, replacing $P(x)$ with $P(ax)$. However, such transformations do not have an obvious interpretation in knot theory.

Since there are only finitely many integer polynomials with bounded height, we obtain the following finiteness statement as a corollary of Theorem 5.1.1:

Corollary 5.1.2. • *For any $q \ge 3$ and $g \ge 0$, and for any X, there are only finitely many equivalence classes of simple Seifert hypersurfaces V in S^{2q+1} of genus g such that the Alexander polynomial $\Delta_K(t)$ is squarefree of height $\le X$.*

- *For any $q \geq 3$ and $g \geq 0$, and for any X, there are only finitely many simple $2q-1$-knots of genus g such that the Alexander polynomial $\Delta_K(t)$ is squarefree of height $\leq X$.*

5.2 Strategies

We will now use the algebraic machinery developed in the previous chapter to tackle the following asymptotic counting question:

Question. How many simple $2q-1$-knots of genus g have squarefree Alexander polynomial of height $\leq X$?

We will also ask the analogous question for Seifert hypersurfaces:

Question. How many simple Seifert hypersurfaces of genus g in S^{2q+1} have squarefree Alexander polynomial of height $\leq X$?

Both of these (originally topological) questions can be converted into purely arithmetic questions by means of Theorem 2.2.1. We now discuss these arithmetic questions.

We have seen that Seifert hypersurfaces are in bijection with the Sp_{2g} orbits on Seifert-parity symmetric matrices. These orbits can be counted using general methods from the geometry of numbers for counting orbits of an algebraic group over $\mathbb{Z}$ acting on a lattice in a vector space.

The most naïve of these is the following: find a fundamental domain $\mathcal{R}$ for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{Sym}^2(\mathbb{R}^{2g})$, and let $\mathcal{R}_X$ be the subregion of R consting of points with height $\leq X$. Then the orbits with Alexander polynomial of height $\leq X$ are in bijection with lattice points in $\mathcal{R}_X$. We can get an approximate count of the latter using the volume of $\mathcal{R}_X$.

This does not quite work as stated, because $\mathrm{Sp}_{2g}(\mathbb{Z})$ does not act properly discontinuously on all of $\mathrm{Sym}^2(\mathbb{R}^{2g})$. First, we need to omit the points where the Alexander

polynomial has repeated roots. This splits $\mathrm{Sym}^2(\mathbb{R}^{2g})$ into several connected components. On some of these components, $\mathrm{Sp}_{2g}(\mathbb{Z})$ does act properly discontinuously and we can count orbits by counting lattice points in a fundamental domain. On the others, there is no fundamental domain, and the counting problem is substantially harder.

For example, in the case of genus 1, there are three connected components. The ones coming from definite quadratic forms have fundamental domains, and their counting question is very well-understood. The indefinite component is less well understood, though heuristics suggest it contains a small number of orbits compared to the others.

There are two difficulties with this approach. The first is that error terms become more complicated to estimate as g gets large. A more specific issue is that the region $\mathcal{R}$ is unbounded, with a cusp going off to infinity. In order to get a finite error term, one must "chop off" the region at some point along the cusp, and deal with the cusp separately.

In the case $g = 1$, the region $\mathcal{R}$ is 3-dimensional, and it is possible to do this estimation explicitly. This was done, in the equivalent context of positive definite binary quadratic forms, by a number of people, including Mertens [23]. We will use these same methods to prove Theorem 5.3.2, which counts positive definite binary quadratic forms with a congruence condition on the discriminant, which is an essential ingredient to our sieving argument.

For larger g, this approach becomes more complicated. [5, 6, 4, 31] have recently developed more streamlined methods for asymptotic orbit-counting. They count orbits by averaging over a continuously varying family of fundamental domains. The author is currently working on applying these methods to count genus g Seifert hypersurfaces for $g \geq 1$.

5.3 Counting genus 1 Seifert hypersurfaces

5.3.1 2×2 Seifert-parity symmetric matrices, binary quadratic forms, and ideals in quadratic rings

In the simplest example where V has genus 1, the polynomial f_V will be of the form $x^2 - x + m$, for an arbitrary integer m. If $m \neq 0$, the boundary knot K will also have genus 1, and the Alexander polynomial Δ_K of the boundary knot K will be $mt^2 + (1-2m)t + m$. We henceforward assume that m is nonzero.

The corresponding ring $R_f = \mathbb{Z}[y]/(y^2-y+m) = \mathbb{Z}[\frac{1+\sqrt{1-4m}}{2}]$ is a quadratic ring of $\mathbb{Z}$ with discriminant $1-4m$. If $1-4m$ is not square, R_f is an order in the quadratic field $\mathbb{Q}[\sqrt{1-4m}]$, and if $1-4m$ is squarefree, R_f is the entire ring of integers of $\mathbb{Q}[\sqrt{1-4m}]$.

The ring $\mathcal{O}_\Delta$ also has a nice form: it is equal to $\mathbb{Z}[\frac{1}{m}, \sqrt{1-4m}]$, and can be viewed as a quadratic extension of the ring $\mathbb{Z}[\frac{1}{m}]$ with discriminant $1-4m$.

In this setting, all the "size functions" we have considered are the same up to linear change of variables. The height of f_v is $|m|$ and the height of Δ_K is $|2m-1|$.

Now we consider conjugate-self-balanced ideal classes in the rings R_f and $\mathcal{O}_\Delta$.

We now show that any ideal class of these rings can be equipped with a (not necessarily unique) balancing map.

Lemma 5.3.1. *(i) If I is an ideal of R_f, and $a = |I| = [R_f : I]$ is the absolute norm of the ideal I, then $[I, a]$ and $[I, -a]$ are (possibly equal) conjugate-self-balanced ideal classes of R_f.*

(ii) The same is true of $\mathcal{O}_\Delta$.

(iii) The set of conjugate-self-balanced ideal classes of $R_f = \mathbb{Z}[\frac{1+\sqrt{1-4m}}{2}]$ can be identified with the set of oriented ideal classes of R_f, as defined in [2]. These are the ideal classes I of R_f equipped with an isomorphism of $\mathbb{Z}$-modules $\bigwedge^2 I \to \mathbb{Z}$.

(iv) The number of simple Seifert hypersurfaces of genus 1 *with Alexander polynomial* $mt^2 + (1-2m)t + m$ *is equal to the narrow class number of* R_f.

Proof. Part (i) amounts to saying that $|(a)| = a^2$ and $I\overline{I} \subset (a)$. These are both standard facts from the theory of imaginary quadratic orders. Part (ii) is analogous.

For (iii), recall the map $c_1 = c_{2g-1}$ from Definition 3.3. We have explicitly

$$c_1(x + y\gamma) = y.$$

By the proof of Theorem 3.3.1, the map $I \times I \to \mathbb{Z}$ given by $i_1, i_2 \mapsto c_1(i_1\overline{i_2}/a)$ is a perfect skew-symmetric pairing, and so induces an orientation on I. The other direction of the proof of Theorem 3.3.1 shows that any orientation on I arises in this way.

Finally, (iv) follows from the definition of the narrow class number as the number of oriented ideal classes of I. □

Fortunately for us, the asymptotics for the average behavior of the narrow class number of quadratic rings are well-understood, as we shall see next.

5.3.2 Counting Seifert forms of negative discriminant

Because the Seifert form of a Seifert hypersurface uniquely determines the isotopy type of its boundary knot, one can obtain an upper bound for the number of genus 1 knots with Alexander polynomial $mt^2 + (1-2m)t + m$ by counting the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of Seifert-parity binary quadratic forms with that Alexander polynomial. It is easier to do this counting if we average over all m within a certain range.

It turns out that one gets two different types of behavior depending upon whether the form is definite (signature 2 or -2), or indefinite (signature 0).

The problem of counting binary quadratic forms of bounded discriminant is a classical one in number theory and the additional condition of Seifert parity is very minor: in fact, it is just a parity condition on the discriminant. In the definite case, one can obtain an asymptotic result via the geometry of numbers as done in [23]. The indefinite case is much harder; the same methods used in the definite case obtain an upper bound of the same size, but standard heuristics suggest that the answer is much lower. Our geometry of numbers argument here will be essentially the same as that used in [23].

Ultimately we will prove a more general result that will be useful in the next section:

Theorem 5.3.2. *Let X be a real number, and let s be a positive integer with $s \leq X$. The number of Seifert-parity definite ($m > 0$) binary quadratic forms with Alexander polynomial $mt^2 + (1-2t)x + m$ where $|m| < X$ and $s \mid m$ is asymptotically equal to*

$$c_1 \rho(s) X^{3/2} + O(\rho(s) s X \max(s, \log X)). \tag{5.1}$$

Here $\rho(s) = \prod_{p|s} \frac{p+1}{p^2}$. The value of the constant c_1 can be made explicit, as can the constant implicit in the big-O notation for the error term, but we will not compute them here.

The number of Seifert-parity indefinite ($m \leq 0$) binary quadratic forms with Alexander polynomial $mt^2 + (1-2m)t + m$ where $|m| < X$ is bounded by a constant times $X^{3/2}$.

Remark. The bound for the indefinite case is extremely weak. The heuristics of Hooley [15] suggest that the actual asymptotic should be $X(\log X)^2$.

It is also possible to get an upper bound of the same form as (5.1) for the number of indefinite forms with $s \mid m$, but this is not useful to us.

Before we prove Theorem 5.3.2, we first change the normalization of our binary quadratic form to produce an integer-valued quadratic form $ax^2 + bxy + cy^2$.

Lemma 5.3.3. *There is a bijection between Seifert-parity binary quadratic forms with Alexander polynomial $mt^2+(1-2m)t+m$ and binary quadratic forms $ax^2+bxy+cy^2$ with discriminant $D = b^2 - 4ac = 1 - 4m$.*

Proof. Indeed, any Seifert-parity binary quadratic form has matrix of the form

$$\begin{pmatrix} 2p & 2q+1 \\ 2q+1 & 2r \end{pmatrix}$$

and so can be written, as a binary quadratic form, as $2(px^2+(2q+1)xy+ry^2)$. Hence, after dividing out the 2, we get the corresponding binary quadratic form. Conversely, any binary quadratic form $ax^2 + bxy + cy^2$ with discriminant congruent to 1 mod 4 must have b odd, and hence lies in the image of this map. □

Proof. We are now ready to prove Theorem 5.3.2. We do the definite case first. We will count positive definite forms only, as it changes the answer only by a factor of 2.

By the lemma above, it suffices to get an asymptotic for the number of SL_2-equivalence classes of binary quadratic forms $ax^2 + bxy + cy^2$ with discriminant D with $-X < D < 0$ and $D \equiv 1 \pmod{4s}$. Note that this condition is also equivalent to: b is odd and $D \equiv 1 \pmod{s}$.

By standard fundamental domain arguments, any SL_2-orbit of positive definite binary quadratic forms has a unique representative (a, b, c) with $0 \leq a \leq c$ and $-a < b \leq a$. We call such a triple (a, b, c) "reduced". Again, the Seifert-parity condition means that b is still odd.

Thus to count the total number of SL_2-orbits of positive definite binary quadratic forms with discriminants in the range $[0, -X]$ and $\equiv 1 \pmod{4s}$, we need only count

the number of lattice points (a, b, c) in the region

$$\mathcal{R}_X = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq z, -x < y \leq x, 0 < y^2 - 4xz \leq X\}$$

such that $b^2 - 4ac \equiv 1 \pmod{4s}$.

Our strategy will be to use Davenport's theorem on counting integer lattice points inside regions.

Theorem 5.3.4 (Davenport [9])**.** *Let R be a bounded semi-algebraic region in $\mathbb{R}^n$, defined by k polynomial inequalities of degree at most ℓ. Then the number of points $(a, b, c) \in \mathbb{Z}^n \cap R$ can be asymptotically expressed as*

$$\mathrm{vol}(R) + \epsilon(R)$$

with the error term $\epsilon(R)$ bounded in size by $\epsilon(R) < \kappa \max(\overline{R}, 1)$ where $\overline{R}$ runs over all projections of R onto subspaces of $\mathbb{R}^n$ spanned by coordinate axes, and $\kappa = \kappa(n, m, k, \ell)$ is some explicit constant depending only on n, m, k, and ℓ.

We will be unable to apply Davenport's theorem to $\mathcal{R}_X$, because some of its shadows have infinite volume. Therefore we will use a truncated region, $\mathcal{R}_{X,t} = \{(x, y, z) \in \mathcal{R} \mid z < t\}$.

We now state some facts about the size of the various projections onto coordinate axes and planes. Let P_x, P_y, P_z denote the projection maps onto the x, y, z axes respectively, and likewise define $P_{x,y}, P_{x,z}, P_{y,z}$ as the projections onto the coordinate planes.

Lemma 5.3.5. *We have the following bounds:*

$$\mathrm{vol}(R_{X,t}) = c_0 X^{3/2} + O(t^{-3} X^3), \tag{5.2}$$

$$\text{area}(P_{x,y}(R_{X,t})) \qquad = O(X) \tag{5.3}$$

$$\text{area}(P_{x,z}(R_{X,t})) \quad = O(X \log(tX^{-1/2}))) \tag{5.4}$$

$$\text{area}(P_{y,z}(R_{X,t})) \quad = O(X \log(tX^{-1/2})), \tag{5.5}$$

and

$$\text{length}(P_x(R_{X,t})) \quad = O(X^{1/2}) \tag{5.6}$$

$$\text{length}(P_y(R_{X,t})) \quad = O(X^{1/2}) \tag{5.7}$$

$$\text{length}(P_z(R_{X,t})) \qquad = t. \tag{5.8}$$

Proof. This is a calculation. □

We first prove (5.1) for the case when $s = 1$. Note that if (a, b, c) is a lattice point in R, then c can be at most $\frac{1}{4}(X + 1) \leq X$. Therefore all lattice points in R_X also belong to the truncated region $R_{X,X}$, and we may count the latter instead. To do this, we estimate the number of lattice points in the region $R_{X,t}$ for general t and then set $t = X$.

Note that we are not counting all lattice points, only the ones with b odd. However, we can apply an affine transformation to the second coordinate, to convert our question into one about counting lattice points.

We then apply Davenport's theorem, and set $t = X$

$$\#(R_{X,t} \cap \mathbb{Z}^3) = c_1 X^{3/2} + O(\max(X \log(tX^{-1/2}), t)) = c_1 X^{3/2} + O(X \log X). \tag{5.9}$$

Now we do the same analysis, but with s a general positive integer. The following lemma will allow us to replicate the argument above.

Lemma 5.3.6. *Let s be a squarefree integer. The set of triples of integers $(a, b, c) \in \mathbb{Z}^3$ such that $b^2 - 4ac \equiv 1 \pmod 4 s$ is a union of $\rho(s)s^3$ translates of the lattice $s\mathbb{Z} \times (2s)\mathbb{Z} \times s\mathbb{Z}$. (Where as before $\rho(s) = \prod_{p|s} \frac{p+1}{p^2}$).*

Proof. By the Chinese Remainder Theorem we may reduce to the case when s is prime.

First we consider the case when s is an odd prime. We count the number of solutions to $b^2 - 4ac = 1$ in $(\mathbb{Z}/p\mathbb{Z})^3$. If $b \not\equiv \pm 1 \pmod p$, there are $p - 1$ possibilites for the pair (a, c), while if $b \equiv \pm 1 \pmod p$, then there are $2p - 1$ possibilities. This gives a total of $p^2 + p = \rho(p)p^3$ as desired.

The power of 2 case may be done by hand: to get $b^2 - 4ac$ equal to 1 mod 8, we need b to be odd, and at least one of a and c to be even. The set of such is a union of 6 translates of the lattice $2\mathbb{Z} \times 4\mathbb{Z} \times 2\mathbb{Z}$. □

Let $L_1, L_2, \cdots, L_{\rho(s)s^3}$ be the translates of $s\mathbb{Z}^3$ in the lemma above. We bound the number of points in each L_i individually. Doing a homothety by a factor of s^{-1}, and applying Davenport's lemma again, we get

$$\begin{aligned}\#(R_{X,t} \cap L_i) =& c_1 s^{-3} X^{3/2} \\ &+ O(s^{-2} X \max(\log(tX^{-1/2})), 1) + O(s^{-1}X^{1/2}) + O(s^{-1}t) + O(1).\end{aligned} \tag{5.10}$$

Setting $X = t$ and combining terms, we obtain

$$\#(R_X \cap L_i) = c_1 s^{-3} X^{3/2} + O(s^{-2} X \log(X)) + O(s^{-1}X). \tag{5.11}$$

Now, we sum over all L_i to get a total of

$$\rho(s)(c_1 X^{3/2} + O(sX \log(X)) + O(s^2 X)), \tag{5.12}$$

as desired.

The upper bound for indefinite forms can be proved in a similar manner, by noting that any indefinite form is SL_2-equvalent to at least one such with (a, b, c) with $0 \leq |a| \leq |c|$ and $-|a| < b \leq |a|$.

Alternatively, it can be proved by showing a result using regulators. □

5.3.3 Counting Seifert forms of positive discriminant

We now consider counting the Seifert forms with positive description, namely, the indefinite quadratic forms. The crude bound given above is far from tight. We expect the following heuristic:

Heuristic 1. *The total number of indefinite Seifert-parity quadratic forms whose Alexander polynomial is* $mt^2 + (1-2m)t + m$ *with* $m \in [1, X]$ *is asymptotic to*

$$c_3 X (\log X)^2.$$

The justification for this is that, as we have seen, indefinite Seifert-parity quadratic forms are the same as indefinite binary quadratic forms with odd discrimiant. Hooley [15] gives an asymptotic proportional to $X(\log X)^2$ for all indefinite binary quadratic forms, and

Comparing with the bound for definite quadratic forms, we see that, according to the heuristic, most genus 1 Seifert pairings have symmetric part which is definite.

5.4 Counting genus 1 simple knots

5.4.1 The map ι_*

We now move to the question of conuning equivalence classes of genus 1 simple knots, which we have put in bijection with conjugate-self-balanced ideal classes of $\mathcal{O}_\Delta$.

It is harder to count conjugate-self-balanced ideal classes of $\mathcal{O}_\Delta$, because they are not in one-to-one correspondence with $\mathbb{Z}$-orbits of an arithmetic group. However, we do have the surjection

$$\iota_* : \mathrm{CSB}(R_f) \twoheadrightarrow \mathrm{CSB}(\mathcal{O}_\Delta).$$

This map is a homomorphism of monoids, and when $1 - 4m$ is squarefree it is also a group homomorphism. We wish to know what we can say about its fibers.

Because we are working with conjugate-self-balanced ideal classes of both R_f and $\mathcal{O}_\Delta$, we will use subscripts to clarify: we will write $[I, \alpha]_{R_f}$ to specify that we mean an element of $\mathrm{CSB}(R_f)$.

In that case, we have the following criterion:

Lemma 5.4.1. *Suppose that I and J are ideals of R_f such that $I\mathcal{O}_\Delta = J\mathcal{O}_\Delta$. Then $I = \mathfrak{a}J$ for some fractional ideal $\mathfrak{a}$ of R_f of the form*

$$\mathfrak{a} = \prod_{\mathfrak{p}|m} \mathfrak{p}^{e_\mathfrak{p}} \tag{5.13}$$

where the product is over all ideals $\mathfrak{p}$ of R_f dividing m.

Proof. The condition $I = \mathfrak{a}J$ holds if and only if it holds locally at all primes of R_f, that is, if and only if $I(R_f)_\mathfrak{q} = \mathfrak{a}J(R_f)_\mathfrak{q}$ for all primes $\mathfrak{q}$ of R_f.

First consider the case when $\mathfrak{q}$ divides m. Then $\mathfrak{q}$ is invertible and the localization $(R_f)_\mathfrak{q}$ is a DVR. So we can pick $e_\mathfrak{q}$ to satisfy the condition at $\mathfrak{q}$.

For each $\mathfrak{q}$ that does not divide m, the ring $\mathcal{O}_\Delta = R_f[1/m]$ is contained in $(R_f)_\mathfrak{q}$, and so the condition $I(R_f)_\mathfrak{q} = J(R_f)_\mathfrak{q}$ is automatically satisfied. □

Because $\mathcal{O}_\Delta = R_f[1/m]$ is obtained from R_f by inverting the element m, it will be useful for us to know the factorization of the ideal mR_f in R_f. Fortunately for us, mR_f has an explicit factorization as a product of distinct prime ideals of R_f, given

by

$$mR_f = \prod_{p|m} (p, \gamma) \cdot (p, 1 - \gamma). \tag{5.14}$$

where p ranges over all prime factors of m in $\mathbb{Z}$.

The following observation is easily checked, and will be useful later on: if $\mathfrak{p}$ is a prime factor of mR_f, then $\mathfrak{p}\overline{\mathfrak{p}}$ is a principal ideal generated by some prime factor (p) of m. As a consequence, $\mathfrak{p}$ is an invertible ideal of R_f.

We may then refer to the primes appearing in (5.14) as "the prime factors of mR_f".

We now use these prime factors to define a special subgroup of K of $\mathrm{CSB}(R_f)$ which we will see we can view as the "kernel" of the monoid homomorphism ι.

Definition. Let K be the subgroup of the monoid $\mathrm{CSB}(R_f)$ generated by

$$[\mathfrak{p}(\overline{\mathfrak{p}})^{-1}, 1]_{R_f},$$

where $\mathfrak{p}$ ranges over the set of prime factors of mR_f.

Because $\mathfrak{p}\overline{\mathfrak{p}}$ is a principal ideal for any prime factor $\mathfrak{p}$ of R_f, this definition can also be restated as follows: K is the subgroup of $\mathrm{CSB}(R_f)$ generated by $[\mathfrak{p}^2, |\mathfrak{p}|^2] = [\mathfrak{p}, |\mathfrak{p}|]^2$ where $\mathfrak{p}$ ranges over the prime factors of mR_f.

Theorem 5.4.2. *A conjugate-self-balanced ideal class $[I, \alpha]_{R_f} \in \mathrm{CSB}(R_f)$ maps to the identity under ι_* if and only if $[I, \alpha]_{R_f} \in K$.*

More generally, the ideals $[I, \alpha]_{R_f}, [J, \beta]_{R_f} \in \mathrm{CSB}(R_f)$ map to the same element of $\mathrm{CSB}(\mathcal{O}_\Delta)$ if and only if $[I, \alpha]_{R_f} = k[J, \beta]_{R_f}$ for some $k \in K$.

Proof. We will prove the second part, which is a strictly stronger statement (because $\mathrm{CSB}(R_f)$ is a monoid and not a group).

For the "only if" direction, it suffices to show that generators of K are sent to the identity by ι_*. This is easily checked.

For the other direction, suppose that

$$\iota([I,\alpha]_{R_f}) = \iota([J,\beta]_{R_f}).$$

It follows from the definitions that $I\mathcal{O}_\Delta = cJ\mathcal{O}_\Delta$ and $\alpha = c\bar{c}\beta$ for some $c \in \operatorname{Frac}\mathcal{O}_\Delta = \operatorname{Frac} R_f$. Let $I' = c^{-1}I$, so that

$$[I,\alpha]_{R_f} = [c^{-1}I, cc^{-1}\alpha] = [I',\beta]_{R_f}.$$

Hence $I'\mathcal{O}_\Delta = J\mathcal{O}_\Delta$, and Lemma 5.4.1 applies, so we can write $I' = \mathfrak{a}J$, where $\mathfrak{a}$ is a product of (possibly negative) powers of primes dividing m.

By comparing $I'\overline{I'}$ with $J\overline{J}$, and localizing at each prime dividing m, we can show that $\mathfrak{a}$ is a product of ideals of the form $\mathfrak{p}^{-1}\overline{\mathfrak{p}}$, where $\mathfrak{p} \mid m$, as desired. □

This theorem has the following useful consequences. Recall that the 2-rank of an abelian group G (with the action written multiplicatively) is the rank of the $\mathbb{F}_2$-vector space G/G^2.

Corollary 5.4.3. *The 2-rank of* $\mathrm{NCl}(\mathcal{O}_\Delta)$ *is the same as the 2-rank of the narrow class group* $\mathrm{NCl}(R_f)$.

Remark. This 2-rank can be computed explicitly using genus theory.

Corollary 5.4.4. *Suppose that $m > 0$. Then the map $\iota_* : \mathrm{NCl}(R_f) \to \mathrm{NCl}(\mathcal{O}_\Delta)$ is a bijection if and only if m is prime.*

Proof. **Proof of ⇒:** Suppose that m is not prime, and let p be a prime factor of m.

Then the ideal (p,γ) is a prime factor of m with norm p, but cannot be principal because $\mathbb{R}_f = \mathbb{Z}[\frac{1-4m}{2}]$ does not contain any elements of norm strictly less than m.

Proof of ⇐: In this case $m = p$ is prime, and $\gamma(1-\gamma) = p$. Hence pR_f factors as the product of the two principal ideals $(p,\gamma) = (\gamma)$ and $(p,1-\gamma) = (1-\gamma)$. □

In this case, we will wish to divide up into cases, depending upon whether or not m is prime. As seen above, if m is a prime, the map ι_* is a bijection, whereas in the case where m is composite, it cannot be one-to-one. Therefore we will get a larger answer in the case when m is prime. We quantify this below.

We can use the bounds in Theorem 5.3.2, to show

Theorem 5.4.5. *The total number of distinct Alexander modules (with pairing) having Alexander polynomial equal to $pt^2 + (1-2p)t + p$ for p running over all primes in the range $[1, X]$ is (unconditionally) bounded above by $O(X^{3/2}/\log X)$.*

Proof. We will prove this by proving the following statement, which will imply the desired result. For any real number z, let $P(z) = \prod_{p \text{ prime} < z} p$. Let α be a real constant, whose value will be chosen later, but will satisfy $\alpha < 1$.

Then it will suffice to show:

$$\sum_{\substack{X \leq m \leq 2X \\ (m, P(X^\alpha))=1}} \#((a,b,c) \in \mathbb{Z}^3 : b^2 - 4ac = 1 - 4m) = O(X^{3/2}/\log X). \tag{5.15}$$

5.4.2 A sieving argument

We follow the approach of Rosser's sieve [16], modifying the terminology to suit our approach. We name the quantity that we wish to bound:

$$S(X, z) := \sum_{\substack{X \leq m \leq 2X \\ (m, P(z))=1}} \#\{(a,b,c) \in \mathbb{Z}^3 : b^2 - 4ac = 1 - 4m\}$$

for $z = X^\alpha$.

To apply the sieve, we need estimates on the following quantities for all squarefree d:

$$S_d(X, z) := \sum_{\substack{m \in [X, 2X] \\ d | m}} \#\{(a,b,c) \in \mathbb{Z}^3 : b^2 - 4ac = 1 - 4m\}. \tag{5.16}$$

Applying Theorem 5.3.2 for two different values of X, and subtracting, we obtain he following estimate for $S_d(X, z)$

$$S_d(X, z) = c_4\rho(d)X^{3/2} + R_d(X, z), \tag{5.17}$$

where the error term $R_d(X, Z)$ is bounded by

$$|R_d(X, z)| \leq c_5 X\rho(d)(\max(d, \log X)). \tag{5.18}$$

(Here c_4 and c_5 are explicit constants.)

We now have the inputs to the sieve, and need to calculate the "sieving density," also known as the "dimension". The following inequality is analogous to (1.3) in [16]: for all $z > w \geq 2$ we have

$$\prod_{w\leq p<z} (1 - \rho(p))^{-1} \leq \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + \frac{K}{\log w}\right). \tag{5.19}$$

where $\kappa = 1$ and K is a sufficiently large constant. This is true by comparing to the product $\prod_{w<p<z}(1 - 1/p)$ and applying Mertens' formula for the asymptotic growth of the latter.

Therefore we may apply (the first half of) Theorem 1.4 of [16] with $y = z$ (so that $s = 1$) to obtain

$$S(X, z) < X^{3/2} \prod_{p<z} (1 - \rho(p)) \left(F(1) + e^{\sqrt{K}} Q(1)(\log z)^{-1/3}\right) + \sum_{d<z \text{ squarefree}} |R_d(X, z)| \tag{5.20}$$

where $F(s)$, $Q(s)$ are specific functions defined in [16]; we will not need any properties of them, just that $F(1)$ and $Q(1)$ are constants.

Using our previous result that $\prod p < z(1 - \rho(p)) \sim 1/\log(z)$, we see that the first term is $O(X^{3/2}/\log z)$.

Applying (5.18) to the second term gives

$$\sum_{d<z \text{ squarefree}} |R_d(X,z)| \leq c_5 X \sum_{d<z \text{ squarefree}} \rho(d) d(\max(d, \log X)). \tag{5.21}$$

We estimate $\rho(d)$. Since d is squarefree, we can write d as a product of distinct primes: $d = \prod_{i=1}^{n_d} p_i$. Then

$$\begin{aligned} \rho(d) &= \prod_{i=1}^{n_d} \frac{p_i + 1}{p_i^2} \\ &= \frac{1}{d} \prod_{i=1}^{n_d} \left(1 + \frac{1}{p_i}\right) \\ &= \frac{1}{d} \sum_{d' | d} \frac{1}{d'} \\ &\leq \frac{1}{d} \sum_{1 \leq d' \leq z} \frac{1}{d'} \\ &\leq \frac{\log z + 1}{d}. \end{aligned} \tag{5.22}$$

Plugging (5.22) into the sum in (5.21), we obtain

$$\begin{aligned} \sum_{d<z \text{ squarefree}} |R_d(X,z)| &\leq c_5 X (\log z + 1) \sum_{d<z} \max(d, \log x) \\ &\leq c_5 (\log z + 1) \sum_{d<z} (d + \log X) \\ &\leq c_5 X (\log z + 1)(z^2 + z \log X) \\ &\leq c_5 X (\alpha \log X + 1)(X^{2\alpha} + X^{\alpha} \log X). \end{aligned} \tag{5.23}$$

In the last line we have set $z = X^{\alpha}$.

For fixed α and X varying, we get the following asymptotics for our remainder term

$$\sum_{d<z \text{ squarefree}} |R_d(X,z)| = O(X^{1+2\alpha} \log X). \tag{5.24}$$

This will be $o(X^{3/2}/\log X)$ for any $\alpha < 1$, and so our error term is bounded. $\square$

5.4.3 Heuristics

Heuristically, we expect that this theorem is sharp. That is, we expect

Heuristic 2. *The total number of distinct Alexander modules (with pairing) having Alexander polynomial equal to $pt^2 + (1-2p)t + p$ for p running over all primes in the range $[1, X]$ is asymptotic to $O(X^{3/2}/\log X)$.*

The error bound in Theorem 5.3.2 is too large to be used in the sieve to obtain a lower bound. Although one can get better estimates by using the geometry of numbers more carefully, this technique does not appear to give a sufficient improvement.

For all other discriminants, we expect there to be substantially fewer Alexander modules.

Heuristic 3. *The total number of distinct Alexander modules (with pairing) having Alexander polynomial equal to $mt^2 + (1-2m)t + m$ for m running over all integers in the range $[-X, X]$ that are not positive primes is (heuristically) bounded above by $O(X(\log X)^2)$.*

Combining the two heuristics above suggests the following, weaker, heuristic

Heuristic 4. *Most Alexander modules (with pairing) have Alexander polynomial of the form $pt^2 + (1-2p)t + p$ for $p > 0$ prime. Equivalently, most simple knots have Alexander polynomal of the same form.*

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