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Abelianization of Stable Envelopes in Symplectic Resolutions

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Abelianization of Stable Envelopes in Symplectic Resolutions

Daniel Shenfeld

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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by the Department of
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Abstract

Stable envelopes, introduced by Maulik and Okounkov, form a basis for the equivariant cohomology of symplectic resolutions. We study the case of Nakajima quiver varieties, where the resolution is a hyperkähler quotient. We relate the stable basis to that of the associated quotient by a maximal torus, and obtain a formula for the transition between the stable basis and the fixed point basis, using the root system and combinatorial data from the torus quotient.

We compute the transition matrix explicitly in the case of cotangent bundles to partial flag varieties, and show that for Grassmannians, it is given by rational Schur polynomials. This recovers in a geometric way the diagonalization of the Hamiltonian in the XXX spin chain. As a second application, we study the case of the Hilbert scheme of points on the plane, and obtain a novel formula expressing Schur polynomials in terms of Jack polynomials.

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The work presented here stems from the author's attempt to understand Andrei's paper with Davesh Maulik, "Quantum groups and quantum cohomology". I would like to thank Davesh for his interest in this work and for many conversations and explanations, which often supplied an intriguing point of view complementary to Andrei's.

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New York, 2013

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Chapter 1

Introduction

Let X be a symplectic resolution, namely a smooth complex algebraic variety equipped with a holomorphic symplectic form ω, which is a resolution of singularities. We will be interested in the situation where X is obtained as a hyperkähler quotient by a reductive group G. This includes most cases of interest, in particular Nakajima quiver varieties. We will treat in detail two particular examples:

1. Cotangent bundles to type A partial flag varieties;

2. The Hilbert scheme of points on $\mathbb{C}^2$.

In [22], Maulik and Okounkov introduce *stable envelopes*, which form a basis of lagrangian classes in $H^*_A(X)$, where A is a torus whose action scales ω by a non-trivial character. In that paper, the stable basis is used as a tool for the computation of the operators of quantum multiplication by a divisor in quiver varieties, and as a basic ingredient in establishing important cases of the Nekrasov-Shatashvili correspondence [28, 27]: a deep connection between the algebra of quantum multiplication and the commuting quantum integrals of motion in an associated quantum integrable system.

In this work we will not be using the "quantum" part of the theory except as a motivation. In fact, we believe that the stable basis is of great interest in itself, being closely related to many key objects in representation theory and in mathematical

physics. For example, in case (1) above it is essentially given by characteristic cycles of Verma modules (see 2.2.3). In case (2), identifying the cohomology ring with the ring of symmetric polynomials, the stable basis corresponds to Schur polynomials (6.1.2).

The main focus of our work is computing the transition matrix between the stable basis and the fixed-point basis. It turns out that for this end it is very fruitful to study the *abelian quotient*, i.e. the quotient of the same space by the action of a maximal torus $T \leq G$. Hyperkähler quotients by a torus are called *hypertoric varieties*; much like toric varieties, their geometry can be described by combinatorial methods, in this case by using hyperplane arrangements. Smooth hypertoric varieties are symplectic resolution, and using their combinatorial description, we show in (3.3.5) how to explicitly express the stable basis in terms of certain tautological classes.

The next natural step is "abelianization", namely relating the stable basis in X to that of the abelian quotient. We describe this relation in our main theorem (4.3.1), and use it to obtain a formula (4.3.3) giving the restriction of the stable basis to fixed components. In particular, for any Nakajima quiver variety we have a constructive way of expressing the stable basis in the fixed point basis. We study in detail the two cases mentioned above and obtain explicit formulas. In the first case, and specifically for cotangent bundles to Grassmannians, we recover in a geometric way known results related to diagonalization of the Hamiltonian in the XXX spin chain, as well as the restriction of Schubert classes to fixed points (5.3). For Hilb, the fixed point basis is given by suitably normalized Jack polynomials, and in (6.3.6) we obtain a novel formula expressing Schur polynomials in terms of Jack polynomials.

Relating the structures of abelian and non-abelian quotients has been the topic of several studies. In the hyperkähler setting, Hausel and Proudfoot [15] have studied the relation between their cohomology rings, relying on previous work by Martin [21] in the symplectic case, and Ellingsrud-Strømme [7] in the GIT setting; see also [1]

for conjectural relations between Gromov-Witten invariants in the Kähler case. We expect that the results presented here will prove useful in understanding the relation between the quantum cohomology of the abelian and non-abelian hyperkähler quotients, and complete descriptions for the quantum rings. This is the focus of ongoing work of the author with Michael McBreen; a first step, giving a full presentation of the quantum cohomology ring of hypertoric varieties, appears in [23].

From the point of view of the Nekrasov-Shatashvili theory, a notion of K-theoretic stable basis would be immensely useful. A definition for what a stable K-class may be has been proposed to us by Maulik and Okounkov, with the caveat that it is not clear how to prove the existence of the stable sheaf for a general symplectic resolution. Here once again we expect the abelianization procedure to prove useful: in the abelian case the stable sheaf is nothing but the structure sheaf of the stable envelope, and an analogous abelianization construction gives the stable sheaf and allows us to compute its restriction to fixed points. While the author has some preliminary results in this direction, they are not presented here.

The structure of this thesis is as follows. In chapter 2 we review a few facts about symplectic resolutions and define the stable basis. For context, we briefly review some of its applications in the Nekrasov-Shatashvili theory as presented in [22]. In chapter 3 we introduce hypertoric varieties and give some background on their geometry. We then show how to express the stable basis in terms of tautological classes and in the fixed point basis.

In chapter 4 we study the abelianization procedure for stable bases. We start by studying the geometric relation between abelian and non-abelian hyperkähler quotients, and use it to derive a formula for the restriction of the stable basis to fixed components. In chapters 5 and 6 we apply our formula to the two examples above. In particular, we obtain a new formula expanding Schur polynomials in the Jack basis.

Chapter 2

Stable Envelopes

2.1 Symplectic resolutions

2.1.1 Basic setting

We briefly review a few basic results on symplectic resolutions. A more detailed treatment can be found in [5] and the references within. A symplectic resolution X is a holomorphic symplectic variety, such that the canonical map

$$\pi : X \to X_0 = Spec(H^0(X, \mathcal{O}_X))$$

is projective and birational. We assume that X admits an action by a reductive group $\mathbf{G} = G \times \mathbb{C}^*$, satisfying the following conditions:

1. The G action is Hamiltonian;

2. The $\mathbb{C}^*$ action scales the symplectic form by a nontrivial character $\hbar$;

3. The fixed point locus X^g is proper for some $g \in \mathbf{G}$.

Deformation

The deformations of (X, ω) are classified by the period map, namely the image of the symplectic form ω in $H^2(X, \mathbb{C})$:

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow \phi \\ [\omega] & \longrightarrow & H^2(X, \mathbb{C}) \end{array} \tag{2.1}$$

The fibers of ϕ are symplectic resolutions, and the generic fiber is affine. Primitive curve classes $\beta \in H_2(X, \mathbb{Z})$ are only effective along hyperplanes in the base where the pairing $\omega(\beta)$ vanises; we call these *root hyperplanes*, and their union the *discriminant locus*. When X is a hyperkähler quotient, $\tilde{X}$ is obtained by varying the level of the moment map, as explained in (4.1.1).

2.1.2 Nakajima quiver varieties

Arguably the most important family of symplectic resolutions, and the one we will be primarily interested in, is that of Nakajima quiver varieties. We give a brief description to fix notation; a more thorough introduction can be found in e.g. [10, 22, 25].

Let Q be a finite quiver with vertex set I and edges E. Given dimension vectors $\mathbf{v}_i, \mathbf{w}_i$ indexed by I, define complex vector spaces V_i, W_i of corresponding dimensions, and let $G = \prod_i \mathrm{GL}(V_i)$. Let θ be a character of G and $\xi \in Z(\mathfrak{g}^*)$ a fixed point of the coadjoint action. Let

$$\mathbb{V} = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i \in I} \mathrm{Hom}(W_i, V_i).$$

Identifying $T^* \mathrm{Hom}(V_i, V_j) = \mathrm{Hom}(V_i, V_j) \oplus \mathrm{Hom}(V_j, V_i)$, we write a point in $T^*\mathbb{V}$ by

data $(X_{ij}, Y_{ij}, \mathbf{i}_i, \mathbf{j}_i)$ where

$$X_{ij} \in \mathrm{Hom}(V_i, V_j) \qquad Y_{ij} \in \mathrm{Hom}(V_j, V_i)$$
$$\mathbf{i}_i \in \mathrm{Hom}(W_i, V_i) \qquad \mathbf{j}_i \in \mathrm{Hom}(V_i, W_i).$$

Identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form, the G action and the moment map $\mu : T^*\mathbb{V} \to \mathfrak{g}^*$ are given by

$$\begin{aligned} g(X, Y, \mathbf{i}, \mathbf{j}) &= (gXg^{-1}, gYg^{-1}, g\mathbf{i}, \mathbf{j}g^{-1}) \\ \mu(X, Y, \mathbf{i}, \mathbf{j}) &= \sum_{i \in I} [X, Y] + \mathbf{ij} \end{aligned} \tag{2.2}$$

We now define the Nakajima variety $\mathcal{M}_{\theta,\xi}$ to be the hyperkähler quotient

$$\mathcal{M}_{\theta,\xi} = T^*\mathbb{V} /\!\!/_\theta G = \mu^{-1}(\xi)^{\theta-ss}/G$$

where θ-ss denotes the θ-semistable points and we take the GIT quotient. For generic values of θ, $\mathcal{M}_{\theta,\xi}$ is smooth, hence a symplectic resolution. When $\xi = 0$, the $\mathbb{C}^*$ action scaling the fibers of $T^*\mathbb{V}$ descends to the quotient and our basic assumptions are satisfied. The following well-known examples are treated in detail in chapters 5, 6.

Example 2.1.1. *Let Q be a type A quiver with vertices $1..r$ and edges between adjacent vertices. Let*

$$k_1 < k_2 < ... < k_r < n$$

and put $\mathbf{v} = (k_1, ..., k_r), \mathbf{w} = (0, ..., 0, n)$. Choosing the character $\theta = \prod \det$, $\mathcal{M}_{\theta,0}$ is the cotangent bundle to the variety of partial flags of dimensions $k_1, ..., k_r$ in $\mathbb{C}^n$.

Example 2.1.2. *Let Q be the Jordan quiver with one vertex and one loop from the vertex to itself, and put $\mathbf{v} = n, \mathbf{w} = 1$. Choosing again $\theta = \det$, $\mathcal{M}_{\theta,0}$ is the Hilbert scheme of n points on $\mathbb{C}^2$.*

Hypertoric varieties and abelianization

Another family of symplectic resolutions which will be important to us is that of *hypertoric varieties*, introduced in more detail in the next chapter. These are the hyperkähler quotients $T^*V/\!\!/T$ where V is a complex vector space and T is a torus. Smooth hypertoric varieties are symplectic resolutions.

For a Nakajima quiver variety $\mathcal{M} = T^*\mathbb{V}/\!\!/G$, we can construct the hypertoric variety $T^*\mathbb{V}/\!\!/T$, where T is a maximal torus of G. We call this the *abelianization* of $\mathcal{M}$. The relation between $\mathcal{M}$ and its abelianization is of primary interest to us.

2.2 The stable basis

Let X be a smooth algebraic variety admitting an action by an algebraic torus A. Restriction to the fixed locus X^A gives a map in equivariant cohomology:

$$res : H_A^*(X) \to H_A^*(X^A).$$

The idea of the stable basis is to construct, for a symplectic resolution, a canonical map in the other direction. We assume:

1. X^A is proper;
2. X is a formal A variety (see e.g. [11]).

These assumptions will hold in all cases of interest. We denote by $A_0 \leq A$ the stabilizer of ω_X in A, and by $\mathfrak{a}, \mathfrak{a}_0$ the Lie algebras of A, A_0 respectively. In general, we denote by $F_A(X)$ the set of fixed components of the A action on X. The reference for this section is [22], chapter 3. We first establish a few preliminaries.

2.2.1 Basic constructions

Chamber decomposition

The cocharacters $\sigma : \mathbb{C}^* \to A_0$ form a full rank integral lattice in $\mathfrak{a}_0$. Put $\mathfrak{a}_0^{\mathbb{R}} = \mathrm{Cochar}(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$. The normal bundle to any component of X^{A_0} splits as a sum of A_0-equivariant line bundles, whose weights define hyperplanes in $\mathfrak{a}_0^{\mathbb{R}}$. Their union over all fixed loci divides $\mathfrak{a}_0^{\mathbb{R}}$ into open *chambers* $\mathfrak{C}_i$. If a cocharacter σ lands in a chamber, then $X^\sigma = X^{A_0}$.

Stable leaves

Fix a chamber $\mathfrak{C}$ and a cocharacter $\sigma : \mathbb{C}^* \to A_0$ landing in it, and let $Z \in F_{A_0}(X)$. The *leaf* of Z is defined as its attracting locus

$$m_X(Z) = \{x \in X : \lim_{t \to 0} \sigma(t) \cdot x \in Z\}.$$

This is a vector bundle over Z. We sometimes use the notation $m(X, Z)$, or suppress X when no confusion may arise. We denote the closure of $m_X(Z)$ by $\overline{m}_X(Z)$. There is a partial order on $F_{A_0}(X)$, defined as the transitive closure of the relation

$$\overline{m}(Z) \cap Z' \neq \emptyset \Rightarrow Z \leq Z'.$$

If X is affine, this ordering is trivial: $m_X(Z)$ is already closed. Define the *slope* of Z by

$$slope(Z) = \bigcup_{Z' \geq Z} m(Z').$$

The slope is closed in X ([22], 3.2.7). We say that a class $\gamma \in H_A^*(X)$ is supported on $slope(Z)$ if $\gamma|_{X \setminus slope(Z)} = 0$ in $H_A^*(X \setminus slope(Z))$; equivalently, as a Borel-Moore class, γ is pushed forward from $slope(Z) \hookrightarrow X$. Here and in what follows we use the restriction symbol to denote the natural restriction maps in equivariant cohomology.

Polarization

For $Z \in F_{A_0}(X)$, we have an A-invariant decomposition of the normal bundle $N_X Z$,

$$N_X Z = N_+ \oplus N_-$$

into the A_0-weights on which σ is positive or negative respectively. The symplectic form yields an identification

$$N_+ = N_-^\vee \otimes \hbar$$

so that the A_0 equivariant Euler class $e(N_X Z)$ is a perfect square. A choice of ε_Z for each Z such that $\varepsilon_Z^2 = e(N_X Z)$ is called a *polarization* of the symplectic resolution. Note that

$$\overline{leaf(Z)}|_Z = \pm\varepsilon_Z.$$

While the choice of a polarization is arbitrary, certain choices are geometrically natural and save on signs. If X is a cotangent bundle, we can choose the polarization defined by the zero section. If X is a hyperkähler quotient $T^*V/\!\!/G$, it admits a polarization induced by the natural polarization on T^*V.

Degree in A_0

Since A_0 acts trivially on X^{A_0}, we have

$$H_A^*(X^{A_0}) = H_\hbar^*(X^{A_0}) \bigotimes_{\mathbb{C}[\hbar]} \mathbb{C}[\mathfrak{a}],$$

where $H_\hbar^*$ stands for A/A_0 equivariant cohomology. Here and in what follows, all cohomologies are taken with complex coefficients unless otherwise indicated. Though this is immaterial to us, we note that for symplectic resolutions, all odd cohomologies

vanish. Any splitting

$$\mathbb{C}[\mathfrak{a}] \cong \mathbb{C}[\mathfrak{a}/\mathfrak{a}_0] \otimes \mathbb{C}[\mathfrak{a}_\circ]$$

leads to the same filtration on $H_A^*(X^{A_0})$, defining the A_0-degree.

2.2.2 Definition of the stable basis

For $Z \in F_{A_0}(X)$, the idea of the stable envelope is to construct a class supported on the slope, and lowering the equivariant A_0 degree. More formally,

Theorem 2.2.1 ([22],3.3.4)**.** *Let X be a symplectic resolution, and fix a chamber $\mathfrak{C}$ and a polarization ε. There exists a unique map of $H_A^*(pt)$ modules*

$$stab : H_A^*(X^{A_0}) \to H_A^*(X)$$

depending on $\mathfrak{C}, \varepsilon$, called the stable map *satisfying the following conditions for any $Z \in F_{A_0}(X), \gamma \in H_\hbar^*(Z)$:*

(support)	$stab(\gamma)$ *is supported on slope*(Z);
(normalization)	$stab(\gamma)\|_Z = \pm e_Z \cap \gamma$, *with sign according to* ε;
(degree)	$\deg_{A_0} stab(\gamma)\|_{Z'} < \frac{1}{2} \cdot \operatorname{codim}_X Z'$, *for all* $Z' > Z$.

In particular, $res \circ stab : H_A^*(X^{A_0}) \to H_A^*(X^{A_0})$ is given by an upper-triangular matrix in the ordered fixed point basis, and we therefore obtain a basis for cohomology. We also use the terms stable envelope and stable basis, with the former typically indicating the Borel-Moore cycle. The degree condition follows from the following stronger condition:

(divisibility)	$stab(\gamma)\|_{Z'}$ is divisible by $\hbar$ for all $Z' > Z$.

The construction of the stable basis is done by successive corrections: for $Z, Z' \in F_A(X)$ with $Z \cap \overline{m}(Z')$ nonempty, one shows that an appropriate integral multiple of $m(Z')$ can be added to $m(Z)$ in a way that the degree condition is satisfied. The construction continues by induction for the next fixed components. In fact, the stable basis is given by a Lagrangian correspondence:

Proposition 2.2.2 ([22], 3.5.1)**.** *There exists an A-invariant Lagrangian cycle $\mathcal{L}_{\mathfrak{C}}$ on $X \times X^{A_0}$, proper over X, such that stab is the induced correspondence $H_A^*(X^{A_0}) \to H_A^*(X)$.*

2.2.3 Stable envelopes via deformation

The good deformation properties of symplectic resolutions provide a useful geometric characterization of the stable basis. The reference for this subsection is [22], 3.7.

Choosing a line in $B \subset H^2(X)$, generic in the sense that it does not intersect the discriminant locus away from the origin, we obtain from the diagram 2.1 a family over $\mathbb{C}$ whose fibers away from 0 are affine, and so that the A action extends to $\tilde{X}$. Put $\tilde{X}^\circ = \tilde{X} \setminus X$. For $Z \in F_{A_0}(X)$, let $\tilde{Z} \in F_{A_0}(\tilde{X})$ be the component satisfying $\tilde{Z} \cap X = Z$, and finally let $\tilde{Z}^\circ = \tilde{Z} \cap \tilde{X}^\circ$.

Since the fibers away from zero are affine, $m(\tilde{X}^0, \tilde{Z}^0)$ is closed in $\tilde{X}^0$. Specializing this cycle to the central fiber, we obtain an algebraic cycle in X; as a Borel-Moore cycle, this is precisely $stab_X(Z)$. Indeed, the divisibility condition follows at once since we are specializing a cycle that restricts to 0 on all fixed components other than $\tilde{Z}_0$. This construction does not depend on the choice of the deformation. As a correspondence, the stable map is thus given by specializing the leaf of the diagonal

$$\Delta^\circ \subset \tilde{X}^\circ \underset{B^\circ}{\times} (\tilde{X}^\circ)^{A_0}. \tag{2.3}$$

2.2.4 Stable envelopes in cotangent bundles

Suppose that $X = T^*M$ for some projective variety M on which A_0 acts with isolated fixed points. Let $A = A_0 \times \mathbb{C}^*$, with the induced symplectic action by A_0 and $\mathbb{C}^*$ acting by scaling the fibers with weight $\hbar$.

Choose a chamber in A_0 and let $z \in M^{A_0}$ be a fixed point and $\mathbb{C}_{m(M,z)}$ be the structure sheaf over its leaf. Then as a Borel-Moore cycle,

$$stab_X(z) = CC(\iota_!(\mathbb{C}_{m(M,Z)})),$$

where CC denotes the characteristic cycle (see e.g. [9] for the definition), and ι is the inclusion in M. To see this, recall that the characteristic cycle is supported on the closures of conormals bundles to the orbits in $\overline{m}(M, Z)$ and the multiplicity of $T^*_{m(M,Z)}$ itself is 1, hence the support and normalization conditions. For the degree condition, we note that each successive correction can be described as a specialization from a family of cycles in $X^0 = X \setminus M$ [9], and the previous argument holds.

Example 2.2.3. *Let G be a reductive group, P a parabolic subgroup, $X = T^*G/P$. Let σ be a cocharater of a maximal torus. Then stable leaves are conormal bundles to Schubert cells, and the stable envelope is the characteristic cycles of a Verma module.*

We remark that in general, the transition from the conormal (leaf) basis to the stable basis is not given by Kazhdan-Lusztig numbers as one might expect, because the characteristic cycle of the irreducible intermediate extension is not necessarily irreducible [2]. This is true, however, for Grassmannians [6].

2.3 Stable envelopes and the Nekrasov-Shatashvili Correspondence

The stable basis was introduced in [22] by Maulik and Okounkov to study the Nekrasov-Shatashvili correspondence and the operators of quantum multiplication for Nakajima quiver varieties. While we believe that the stable basis is of independent interest, and this correspondence is not directly related to the work presented here, we give a brief summary of a few of the ideas presented there in order to illustrate the role of the stable basis in the general theory.

2.3.1 Stable envelopes and Steinberg Correspondences

Let $X = \mathcal{M}(\mathbf{w}) = \sqcup_{\mathbf{v}} \mathcal{M}(\mathbf{v}, \mathbf{w})$, and $X_0 = \mathcal{M}_0(\mathbf{w})$ be its affinization. The *Steinberg variety* of X is defined as

$$Z = X \underset{X_0}{\times} X.$$

By a theorem of Kaledin, the irreducible components of Z are all half-dimensional. They act as lagrangian correspondences on $H^*_{\mathbf{G}}(X)$; we call them *Steinberg correspondences*.

For a divisor u on X, a very general argument presented in [5] shows that the operator of quantum multiplication by u is given by

$$u * — = u \cap — + \hbar \sum_{\beta} (u, \beta) f(q^{\beta}) Z_{\beta}(—),$$

where $\beta \in H_2(X)$ varies over effective primitive curve classes in X, f is a power series and Z_β is a Steinberg correspondence. Computing the operators of quantum multiplication therefore boils down to understanding the action of Steinberg correspondences. The stable basis enables us to localize this action in the following sense. Let $\mathbf{w} = \sum_{i=1}^{n} \mathbf{w}_i$ be any decomposition. Let A_0 be the torus acting with weight

z_i on the corresponding subspace $W_i \leq W$. Every component of X^{A_0} is a symplectic resolution, in fact a product of Nakajima varieties with framing vectors $\mathbf{w}_i$. Let $\iota : Z \to X$ be the inclusion of such a component, ε a polarization of Z. Then we have

Proposition 2.3.1 ([22], 3.4.2)**.** *For any A_0-invariant Lagrangian class $[L]$ on X, there exists a unique Lagrangian cycle $\mathrm{Res}_Z L$ supported on $L \cap Z$ called the residue of L, such that*

$$\iota^*[L] = \varepsilon[\mathrm{Res}_Z L] + ...$$

where dots stand for terms of lower A_0 degree.

Let Θ be a Steinberg correspondence, and Θ^A its residue on the Steinberg variety of X^{A_0}. Then we have the following commutative diagram ([22], 4.6.1):

$$\begin{array}{ccc} H^*(X^{A_0}) & \xrightarrow{stab_{\mathfrak{C}}} & H^*X \\ {\scriptstyle \Theta^A}\downarrow & & \downarrow{\scriptstyle \Theta} \\ H^*(X^{A_0}) & \xrightarrow{stab_{\mathfrak{C}}} & H^*X \end{array} \tag{2.4}$$

This is a key result in the identification of the operators of quantum multiplication by a divisor given in [22], 10.2.1.

2.3.2 Stable envelopes and Yangians

On the quantum-integrable side, the authors of [22] construct a certain geometric Yangian, acting on $H^*_{\mathbf{G}}(\mathcal{M}(\mathbf{w}))$, and identify the algebra of quantum multiplication with the commutative Baxter subalgebras generated by transfer matrices. Naturally, the R-matrix plays a central role in this theory, and its construction rests on the definition of the stable basis.

For a symplectic resolution X and a chamber $\mathfrak{C}$, the map

$$stab_{\mathfrak{C}} : H^*_A(X^{A_0}) \to H^*_A(X)$$

becomes an isomorphism after inverting $e(N_-)$, and we can define the R-matrix

$$R_{\mathfrak{C}',\mathfrak{C}} = stab_{\mathfrak{C}'}^{-1} \circ stab_{\mathfrak{C}'} \in \mathrm{End}(H_A^*(X^{A_0})) \otimes \mathbb{Q}(\mathfrak{a}).$$

The Yang-Baxter equation is an easy consequence of this definition. The reader will note that our interest in restricting *stab* to fixed points amounts to "computing half the R-matrix". Theorem (2.4) shows that the Steinberg algebra intertwines R matrices.

For any decomposition $\mathbf{w} = \sum \mathbf{w}_i$ as above we have a decomposition

$$H_A^*(\mathcal{M}(\mathbf{w})^{A_0}) = \bigotimes H_A^*(\mathcal{M}(\mathbf{w}_i)).$$

and we obtain a collection of R matrices

$$R_{ij}(a_i - a_j) \in \mathrm{End}\left(H_A^*(\mathcal{M}(\mathbf{w}_i)) \otimes H_A^*(\mathcal{M}(\mathbf{w}_j))\right) \otimes \mathbb{Q}(\mathfrak{a})$$

satifying the Yang-Baxter equation. In this setting, a general construction identifies the Yangian as a subalgebra of $\mathrm{End}\left(H_A^*(\mathcal{M}(\mathbf{w}))\right)$ generated by the $u \to \infty$ matrix coefficients of R-matrices applied to an auxiliary space ([22], 5.2). One can also identify Baxter subalgebras in the Yangian, also called Bethe subalgebras, generated by the commuting transfer matrices. These are the subalgebras which Maulik-Okounkov theory identifies with the quantum algebra of $\mathcal{M}(\mathbf{w})$. A key ingredient of this identification are certain *shift operators*, which on the one hand are identified as quantum operators, and on the other hand their conjugation by *stab* recovers the action of the R matrix, again showing the central role played by the stable basis.

Chapter 3

The Stable Basis in Hypertoric Varieties

In this chapter we study the stable basis for abelian hyperkähler quotients. After reviewing the definitions of hypertoric varieties and their associated hyperplane arrangements, we show how to read off stable leaves and stable envelopes from the arrangement. We then express the stable basis in terms of tautological classes defined below. As a result, we obtain expressions for the stable basis in terms of the fixed-point basis in a purely combinatorial way.

3.1 Hypertoric Varieties

3.1.1 Definitions

We review here the definition and properties of hypertoric varieties which we will need in the sequel. A fuller treatment may be found in e.g. [29] and the references therein. Consider the torus $T^n = (\mathbb{C}^*)^n$ acting symplectically on $T^*\mathbb{C}^n$. Setting $\mathfrak{t}^n = \mathrm{Lie}(T^n)$,

the moment map $\mu_n : T^*\mathbb{C}^n \to (\mathfrak{t}^n)^*$ is given by

$$\mu_n(z, w) = (z_1 w_1, ..., z_n w_n).$$

Let $T^k \leq T^n$ be an algebraic subtorus, $T^d = T^n/T^k$ $(d = n-k)$, and let $\mathfrak{t}^k, \mathfrak{t}^d$ be their respective Lie algebras. We have the exact sequence

$$0 \to \mathfrak{t}^k \xrightarrow{\iota} \mathfrak{t}^n \xrightarrow{a} \mathfrak{t}^d \to 0$$

and dualizing,

$$0 \to (\mathfrak{t}^d)^* \xrightarrow{a^*} (\mathfrak{t}^n)^* \xrightarrow{\iota^*} (\mathfrak{t}^k)^* \to 0.$$

We will often identify elements of $\mathfrak{t}^k$ with their images in $\mathfrak{t}^n$. Taking $\mu_k = \iota^* \circ \mu_n$ we obtain a moment map for the T^k action on $T^*\mathbb{C}^n$. Fix a character θ of T^k and a level $\xi \in (\mathfrak{t}^k)^*$. We define the associated *hypertoric variety* by

$$\mathfrak{M}_{\theta,\xi} = \mu_k^{-1}(\xi) /\!/_\theta T^k,$$

where we take the GIT quotient with respect to the linearization determined by θ.

The induced T^d action on $\mathfrak{M}_{\theta,\xi}$ preserves the holomorphic symplectic form. There is a further action of $\mathbb{C}^*$ dilating the fibers of $T^*\mathbb{C}^n$, which scales the symplectic form by $\hbar$. This also preserves $\mu_k^{-1}(0)$, and visibly descends to an action of $\mathbb{C}^*$ on $\mathfrak{M}_{\theta,0}$ commuting with the T^d action. In the notation of (2.1.1), $G = T^d$ and $\mathbf{G} = T^d \times \mathbb{C}^*$.

3.1.2 Hyperplane arrangements

The geometry of hypertoric varieties can be described by means of a hyperplane arrangement. The Lie algebras $\mathfrak{t}^k, \mathfrak{t}^n$ and $\mathfrak{t}^d$ inherit integral structures from the associated tori. Let $(\mathfrak{t}^k)^*_{\mathbb{R}} = (\mathfrak{t}^k)^*_{\mathbb{Z}} \otimes \mathbb{R}$, and define $(\mathfrak{t}^n)^*_{\mathbb{R}}$ and $(\mathfrak{t}^d)^*_{\mathbb{R}}$ analogously.

Choose a lift $\hat{\theta}$ of θ to $(\mathfrak{t}^n)^*$, with coordinates $\hat{\theta}_i$. Write e_i for the standard

generators of $(\mathfrak{t}^n)_{\mathbb{Z}}$ and a_i for their images in $(\mathfrak{t}^d)_{\mathbb{Z}}$. Define hyperplanes $H_1, ..., H_n$ in $(\mathfrak{t}^d)^*_{\mathbb{R}}$ by

$$H_i = \left\{ x \in (\mathfrak{t}^d)^*_{\mathbb{R}} : a_i \cdot x + \hat{\theta}_i = 0 \right\}. \tag{3.1}$$

These are the intersections of $(\mathfrak{t}^d)^*_{\mathbb{R}} + \hat{\theta}$ with the coordinate hyperplanes of $(\mathfrak{t}^n)^*_{\mathbb{R}}$. We call the collection of oriented affine hyperplanes $\mathcal{A} = \{H_i\}_{i=1}^n$ the *hyperplane arrangement* associated to the hypertoric variety $\mathfrak{M}_{\theta,0}$. Each hyperplane H_i divides $(\mathfrak{t}^d)^*$ into a positive and a negative half space:

$$x \in \begin{cases} \mathbb{H}_i^+ & \text{if } a_i \cdot x + \hat{\theta}_i \geq 0. \\ \mathbb{H}_i^- & \text{if } a_i \cdot x + \hat{\theta}_i \leq 0. \end{cases}$$

The arrangement $\mathcal{A}$ is called

- *Simple* if every subset of m hyperplanes with nonempty intersection intersects in codimension m;
- *Unimodular* if any collection of d independent vectors in $\{a_1, ..., a_n\}$ spans $\mathfrak{t}^d$ over $\mathbb{Z}$;
- *Smooth* if it is simple and unimodular.

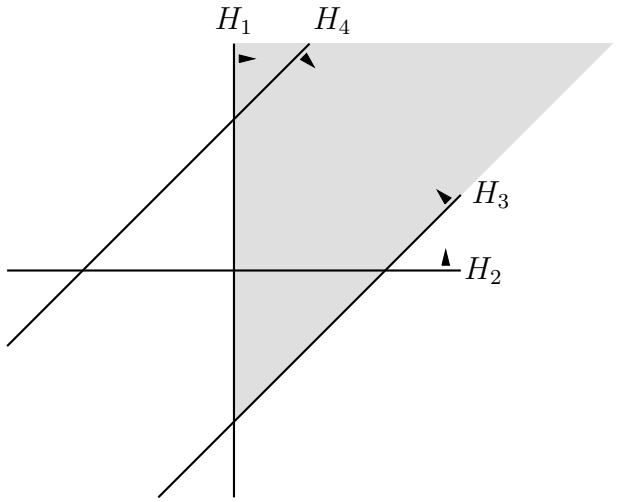
The associated hypertoric variety is smooth if and only if the arrangement is smooth. The affinization map is the canonical GIT map $\mathfrak{M}_{\theta,0} \to \mathfrak{M}_{0,0}$, and it is birational. In particular, smooth hypertoric varieties are symplectic resolutions. *We assume from now on that $\mathfrak{M}_{\theta,0}$ is smooth.* To reduce clutter, in the sequel we fix θ and write $\mathfrak{M}$ for $\mathfrak{M}_{\theta,0}$.

Example 3.1.1. *The hypertoric variety $T^*\mathbb{P}^n$ is obtained as the quotient of $T^*\mathbb{C}^{n+1}$ by the action of the diagonal torus. The corresponding hyperplane arrangement is composed of $n+1$ hyperplanes bounding a simplex in $\mathbb{R}^n$. Dividing instead by the*

complementary torus $\{(\zeta_1, ..., \zeta_{n+1}) \in (\mathbb{C}^*)^{n+1} | \prod \zeta_i = 1\}$ *we obtain the* $\tilde{\mathcal{A}}_n$ *surface, a crepant resolution of a type* A_n *singularity. The corresponding arrangement is simply* $n+1$ *points on a line.*

Example 3.1.2. *The following example is related to the abelianization of* Hilb_2*, the Hilbert scheme of two points on* $\mathbb{C}^2$*; the latter is isomorphic to* $T^*\mathbb{P}^1 \times \mathbb{C}^2$*, but the quotient structure is different than the description in the previous example.*

Let $n = 4, k = d = 2$*, with* $\mathfrak{t}^k$ *spanned by* $\{(1, -1, 1, 0), (-1, 1, 0, 1)\}$*. Let* $\theta = (1, 1), \hat{\theta} = (0, 0, 1, 1)$*. We choose the basis* $b_{12} = (1, 0, -1, 1), b_{21} = (0, 1, 1, -1)$ *for* $(\mathfrak{t}^d)^*$ *(this notation is compatible with the treatment in chapter 6). The corresponding arrangement is given by the coordinate axes and the lines* $b_{21} = b_{12} \pm 1$*. In the figure, arrows point towards the positive half planes. The shaded region is a cone in the arrangement, see (3.2.2) below.*



3.1.3 Cohomology

Consider the character of T^n given by $diag(\zeta_1, ..., \zeta_n) \mapsto \zeta_i$. Restricting to T^k, we obtain an induced $T^d \times \mathbb{C}^*$-equivariant line bundle on $\mathfrak{M}$, with equivariant Euler class u_i, corresponding to the divisor $z_i = 0$. Recall that $\hbar$ is the weight of the symplectic form under the $\mathbb{C}^*$ action; the divisor $w_i = 0$ thus corresponds to the class $\hbar - u_i$.

Notation 3.1.3. *For $S \subseteq \{1..n\}$, let $H_S = \cap_{i \in S} H_i$.*

Definition 3.1.4. *A* circuit *$S \subseteq \mathcal{A}$ is a minimal subset satisfying $H_S = \emptyset$. The orientation defines a splitting $S = S^+ \sqcup S^-$, where $i \in S^\pm$ if $H_{S\setminus\{i\}}$ lies in $\mathbb{H}_i^\pm$ respectively. Alternatively, S corresponds to a minimal relation in $\mathfrak{t}^d$:*

$$\sum_{i \in S^+} a_i \quad - \sum_{i \in S^-} a_i = 0.$$

Theorem 3.1.5 ([14])**.** *$H^\bullet_{T^d \times \mathbb{C}^*}(\mathfrak{M})$ is generated by $u_1, ..., u_n, \hbar$, subject to the relations*

$$\prod_{i \in S^+} u_i \prod_{i \in S^-} (\hbar - u_i) \tag{3.2}$$

for all circuits S. $H^\bullet_\hbar(\mathfrak{M})$ is obtained by imposing in addition all the linear relations

$$\sum c_i u_i \in \ker \iota^*. \tag{3.3}$$

Notation 3.1.6. *We will frequently encounter products such as those above, and it will be convenient to introduce the following formal "star notation":*

$$x^{*(s)} = \begin{cases} x & \text{if } s > 0 \\ \hbar - x & \text{if } s < 0 \\ 1 & \text{if } s = 0 \end{cases}$$

For example, given a circuit S, define the sign vector $(s_1, ..., s_n)$ by $s_i = \pm 1$ if $i \in S^\pm$, $s_i = 0$ if $i \notin S$. Then (3.2) is succinctly expressed as $\prod u_i^{*(s_i)}$.

3.2 The stable basis from the arrangement

3.2.1 Arrangement constructions

Toric lagrangians

While we do not follow any particular reference in this section, much of the material is contained in [3]. To avoid trivialities, it will be convenient to assume the following irreducibility condition:

$$a_i \notin \mathfrak{t}^k \text{ for any } i.$$

If $a_i \in \mathfrak{t}^k$, then $\mathfrak{M}$ is the product of $T^*\mathbb{C}$ with a smaller hypertoric variety. The arguments below are easily modified to accommodate this case.

Definition 3.2.1. *Given $U \subseteq \{1..n\}$, define the corresponding* chamber *in the arrangement as the intersection*

$$\left(\bigcap_{i\in U} \mathbb{H}_i^+\right) \cap \left(\bigcap_{i\notin U} \mathbb{H}_i^-\right)$$

If the chamber is non-empty, there is a corresponding toric lagrangian subvariety of $\mathfrak{M}$ defined as follows. Let

$$\mathcal{E}_U^+ = \{(z,w) \in T^*\mathfrak{t}^n \big| z_i = 0 \quad \forall i \in U\} \tag{3.4}$$

$$\mathcal{E}_U^- = \{(z,w) \in T^*\mathfrak{t}^n \big| w_i = 0 \quad \forall i \in U\} \tag{3.5}$$

and set

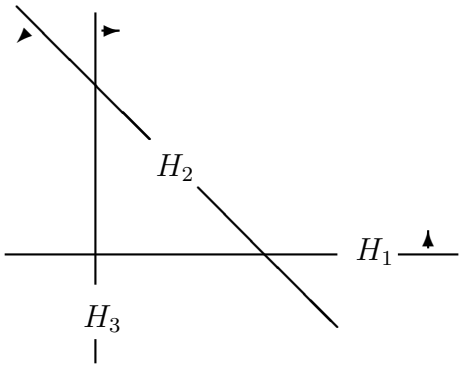
$$\mathfrak{L}_U = (\mu^{-1}(\lambda) \cap (\mathcal{E}_U^+ \cap \mathcal{E}_{\mathcal{A}\setminus U}^-)/\!/_\theta T^k.$$

The image of $\mathfrak{L}_U$ under the moment map $\mathfrak{M} \to (\mathfrak{t}^d)^*$ is precisely the corresponding chamber in the arrangement [29]. Note that the smoothness conditions on the arrangement imply that $\mathfrak{L}_U$ is a smooth toric variety. It is clear that all T^d invariant

lagrangians are obtained in this way. We define the *core* $\mathcal{C} \subset \mathfrak{M}$ as the union of all $\mathfrak{L}_U \subset \mathfrak{M}$. Note that this is the image of $\mu_n^{-1}(0) = \{(z_i, w_i) : z_i = 0 \text{ or } w_i = 0 \quad \forall i\}$. The core is a T^d-eqivariant deformation retraction of $\mathfrak{M}$ [14, 29]. Using the restriction formulas below, one easily checks:

$$[\mathcal{C}] = \hbar^d[\mathfrak{M}]. \tag{3.6}$$

Example 3.2.2. *The figure illustrates the hyperplane arrangement for* $T^*\mathbb{P}^2$. $\mathfrak{L}_{123}$ *is the zero section, and its image under the moment map is the central simplex, while* $\mathfrak{L}_1$ *is the fiber over* $[1:0:0] \in \mathbb{P}^2$*, corresponding to the cone above the topmost vertex.*



Hypertoric subvarieties

Given $Q = \{i_1, ..., i_r\} \subset \{1..n\}$, there is a hypertoric subvariety $\mathfrak{M}_Q \subset \mathfrak{M}$ whose hyperplane arrangement is given by the intersection of the ambient arrangement with H_Q. It is defined by

$$\mathfrak{M}_Q = (\mu^{-1}(\lambda) \cap \mathcal{E}_Q^+ \cap \mathcal{E}_Q^-)/\!/_\theta T^k. \tag{3.7}$$

Let $T^Q \leq T^d$ be the subtorus such whose Lie algebra $\mathfrak{t}^Q$ is generated by $\{a_{i_1}, ..., a_{i_r}\}$. Note that T^Q fixes $\mathfrak{M}_Q$, and that $\dim T^Q = r$ by the smoothness condition on the arrangement. We denote by $\{\alpha_{i_1}, ..., \alpha_{i_r}\}$ the dual basis on $(\mathfrak{t}^Q)^*$.

More generally, let $A \leq T^d$ be any subtorus. The connected components of $\mathfrak{M}^A$

are the subvarieties $\mathfrak{M}_Q$ above for minimal subsets Q such that $A \leq T^Q$. In particular, if $A = T^d$, $\mathfrak{M}^A$ is discrete, and the fixed points correspond to the intersection of d hyperplanes in the arrangement.

3.2.2 Stable leaves and stable envelopes

Stable leaves

Consider a subvariety $\mathfrak{M}_Q$ as above and a chamber in the arrangement H_Q defined by $U^0 \subseteq \mathcal{A} \setminus Q$, and corresponding to a Lagrangian $\mathfrak{L}^Q_{U^0} \subseteq \mathfrak{M}_Q$. Let $\sigma \leq T^Q$ be a generic cocharacter, and let

$$U^+ = \{j \in Q, (\alpha_j, \sigma) > 0\}.$$

Then $U^0 \cup U^+$ defines a chamber in the ambient arrangement $\mathcal{A}$, and it is clear that σ attains its minimum on this chamber precisely on the face defined by Q. It is easy to see from the definitions that the corresponding Lagrangian $\mathfrak{L}_{U^0 \cup U^+}$ is the stable leaf of $\mathfrak{L}^Q_{U^0}$. Since any T^d invariant cycle is obtained in this way, we see that

Proposition 3.2.3. *The closure of the stable leaf of any irreducible T^d invariant cycle in $\mathfrak{M}$ is smooth.*

This property makes the stable basis particularly simple in the hypertoric case: expressed in the stable leaf basis, all nonzero coefficients must be ± 1.

Stable envelopes

With the same notation, define the *cone* of $\mathfrak{L}^Q_{U^0}$ as the union of all chambers in the intersection

$$(\cap_{j \in U^+} \mathbb{H}_j^+) \cap (\cap_{j \in Q \setminus U^+} \mathbb{H}_j^-), \tag{3.8}$$

see (3.1.2) for an example. The corresponding lagrangian in $\mathfrak{M}$ is a union of leaves, which is ordered by the minimal value attained by σ on each chamber. This order agrees with the partial order on stable leaves (cf. [3], 2.6). It is easy to see by induction that the lagrangian corresponding to the cone is contained in the slope of $\mathfrak{M}_Q$; in fact, up to sign this is precisely its stable envelope, as we will see below.

Example 3.2.4. *Let $X = T^*\mathbb{P}^n$ with the action by $(\mathbb{C}^*)^n$. The fixed points are indexed by $\{0, ..., n\}$, where the fixed point i corresponds to the intersection $H_{\mathcal{A}\setminus\{i\}}$ in the arrangement. Choosing σ to induce the natural order on the fixed points, we have an inclusion*

$$pt \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq ... \subseteq \mathbb{P}^n$$

and the closure of the stable leaves are the conormal bundles $N^*\mathbb{P}^i$. It is clear from the arrangement (see e.g. 3.2.2 for $n = 2$) that the cone of the chamber corresponding to the leaf m_i is the union of m_i, m_{i-1}, so as a cycle

$$stab_i = N^*\mathbb{P}^i + N^*\mathbb{P}^{i-1}.$$

Remark 3.2.5. *In [4], the authors introduce an analogue of the BGG category $\mathcal{O}$ for hypertoric varieties. The irreducible modules in this category are supported on the lagrangians corresponding to chambers of the arrangement, and the standard modules are supported on the cones, each chamber with multiplicity 1. This is another example of the correspondence between stable envelopes and standard modules in a suitable highest weight category (cf. 2.2.3).*

3.3 The stable basis in terms of tautological classes

3.3.1 Restriction of tautological classes to fixed points

Let $p_V \in \mathfrak{M}^{T^d \times \mathbb{C}^*}$ be a fixed point corresponding to a d-tuple of hyperplanes indexed by $V \subset \mathcal{A}$ and intersecting at a point H_V. The action of $T^d \times \mathbb{C}^*$ on $T_p\mathfrak{M}$ is described as follows [13]. Since we assume $\mathfrak{M}$ is smooth, the normals $\{a_j\}_{j \in V}$ form a basis for $\mathfrak{t}^d$. Let α_j be the dual basis. Define $\mathcal{A}_V^{\pm} \subset \mathcal{A}$ by

$$\mathcal{A}_V^{\pm} = \{H_i : H_V \notin H_i, H_V \in \mathbb{H}_i^{\pm}\}.$$

It is easy to check from the moment map conditions that [13, 14]:

$$\begin{cases} z_i = w_i = 0 & \text{if} \quad i \in V \\ z_i \neq 0, w_i = 0 & \text{if} \quad i \in \mathcal{A}_V^+ \\ z_i = 0, w_i \neq 0 & \text{if} \quad i \in \mathcal{A}_V^- \end{cases} \tag{3.9}$$

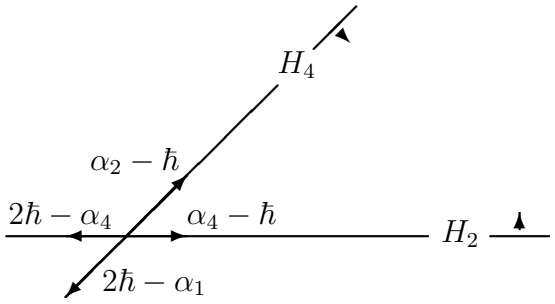
The tangent space at p_V splits into a sum of $T^d \times \mathbb{C}^*$-stable lines corresponding to the $2d$ edges extending from H_V. Given an edge e, there is a single $H_i \in V$ not containing e. The associated weight is

$$\begin{cases} \alpha_i - \left(\alpha_i, \sum_{j \in \mathcal{A}_V^-} a_j\right) \hbar & \text{if } e \text{ extends into } \mathbb{H}_i^+; \\ \hbar - \left(\alpha_i - \left(\alpha_i, \sum_{j \in \mathcal{A}_V^-} a_j\right)\hbar\right) & \text{if } e \text{ extends into } \mathbb{H}_i^-. \end{cases}$$

More generally, the normal weights to a hypertoric subvariety $\mathfrak{M}_Q$ are given by the same formula restricted to $\mathfrak{t}^Q$ weights. It is an easy exercise to check that this is well defined.

Example 3.3.1. *Consider the arrangement in (3.1.2), and let $V = \{2, 4\}$. In the notation of (3.1.2), we have $\alpha_2 = b_{12} + b_{21}, \alpha_4 = b_{12}$, and $\mathcal{A}_V^- = \{1\}$. The associated*

weights are indicated in the figure; note that the weights of opposite edges always add to $\hbar$.



The restriction of the tautological class u_i to p_V is given by:

Proposition 3.3.2 ([13],3.5)**.**

$$u_i|_{p_V} = \begin{cases} 0 & \text{if } i \in \mathcal{A}_V^+ \\ \hbar & \text{if } i \in \mathcal{A}_V^- \\ \alpha_i - \left(\alpha_i, \sum_{j \in \mathcal{A}_V^-} a_j\right) \hbar & \text{if } i \in V. \end{cases} \tag{3.10}$$

Observation 3.3.3. *Note that if $\mathfrak{M}_Q$ is a hypertoric subvariety, and the hyperplane H_i is transverse to H_Q, then from the definitions, $u_i|_{\mathfrak{M}_Q} = u_i \in H_\hbar^*(\mathfrak{M}_Q)$. We will not distinguish in the notation between u_i as a tautological class in $\mathfrak{M}$ and $\mathfrak{M}_Q$.*

Remark 3.3.4. *We will need the following observation. Let $L \subseteq \mathfrak{M}_Q$ be a $\mathbb{C}_\hbar^*$-invariant (but not necessarily T^d-invariant) cycle, such that the $L \cap \mathfrak{M}_Q^{\mathbb{C}^*}$ is discrete, and let $x \in L^{\mathbb{C}^*}$. Then $u_i|_x \in H^*_{T^Q \times \mathbb{C}^*}(L)$ is given by the same formula (3.10) after restricting to T^Q weights. The same proof as in [13], which explicitly computes the weights along the fiber over T^d fixed points, works for this case as well.*

3.3.2 The stable basis in terms of tautological classes

Given $Q \subset \mathcal{A}$, consider the hypertoric subvariety $\mathfrak{M}_Q \subset \mathfrak{M}$, and let $\mathcal{C}_Q \subset \mathfrak{M}_Q$ be its core. Let $T_Q \subset T^d$ be generated by a_i for $i \in Q$; recall that T_Q fixes $\mathfrak{M}_Q$. Choose a chamber $\mathfrak{C} \subset \mathfrak{t}_Q = \mathrm{Lie}(T_Q)$, and put $Q^+_{\mathfrak{C}} = \{i : (\alpha_i, v) > 0 \text{ for } v \in \mathfrak{C}\}$. Then by (3.3.2)

$$\Big(\prod_{i \in Q^+_{\mathfrak{C}}} u_i \prod_{i \in Q \setminus Q^+_{\mathfrak{C}}} \hbar - u_i\Big)\Big|_{\mathfrak{M}_Q} \tag{3.11}$$

is a polarization of the normal bundle of $\mathfrak{M}_Q$ with respect to the torus $T_Q \times \mathbb{C}^*$. Denote it $\epsilon_{\mathfrak{C}}$.

Theorem 3.3.5. *Let $\gamma \in H^*_\hbar(\mathfrak{M}_Q)$ be given by a polynomial p_γ in the tautological classes of $\mathfrak{M}_Q$. Then for the chamber $\mathfrak{C}$ and polarization $\epsilon_{\mathfrak{C}}$,*

$$stab_{\mathfrak{M}}(\gamma) = p_\gamma \cdot \prod_{i \in Q^+_{\mathfrak{C}}} u_i \prod_{i \in Q \setminus Q^+_{\mathfrak{C}}} \hbar - u_i. \tag{3.12}$$

Proof. First note that this expression is well defined: if $p_\gamma = 0$ in $H^*_\hbar(\mathfrak{M}_Q)$, then either it contains a circuit in $\mathfrak{M}_Q$, in which case (3.12) contains a circuit in $\mathfrak{M}$; or p_γ contains a linear relation as in (3.3), but these vanish in the restriction from T^d to T^Q weights.

From the definition of the stable basis and (3.3.3), it is enough to prove the case $\gamma = 1$. Moreover, since the core $\mathcal{C}^Q$ is a deformation retraction of $\mathfrak{M}^Q$ and $\mathcal{C}^Q \subseteq \mathcal{C}$, we may work on the core and assume $\gamma = [\mathcal{C}^Q] = \hbar^{d-|Q|}$ (3.6). Since T^Q-equivariant tautological classes lift naturally to T^d equivariant classes, we can work in T^d equivariant cohomology and restrict to T^Q equivariant parameters. Then from the description of the slope in (3.2.2) and the restriction rules (3.10), we see that (3.12) is supported on the slope. The divisibility condition in (2.2.2) again follows from (3.10), while the normalization condition is clear.

□

Chapter 4

Abelianiation of Stable Envelopes

In this chapter we prove our main theorem (4.3.1), relating stable envelopes in abelian and non-abelian hyperkähler quotients. We also obtain the restriction formula (4.3.3) of stable envelopes to fixed points. Though many of the results hold in greater generality, it will be convenient to assume that our quotients are quiver varieties.

4.1 Abelianization of hyperkähler quotients

4.1.1 Hyperkähler quotients from the GIT point of view

We adopt here the GIT point of view for hyperkähler quotients. The interplay between the various abelian and non-abelian quotients plays an important role in what follows; we review the definitions and notation for the reader's convenience.

Let V be a complex vector space, $X = T^*V$ with the canonical symplectic form ω. Let G be a complex reductive group acting on V, and $T \leq G$ a maximal torus. The G action on V induces a canonical hamiltonian action on X; let $\mu_G : X \to \mathfrak{g}^*$ be the moment map. Composing with the projection $pr : \mathfrak{g}^* \to \mathfrak{t}^*$, we obtain a moment map $\mu_T = pr \circ \mu_G$ for the T action on X. Let $\xi \in Z(\mathfrak{g}^*)$, i.e. ξ is a fixed point of the

coadjoint action. We may also think of ξ as an element of $\mathfrak{t}^*$. We denote

$$X^{\xi}_{\mathrm{G}} = \mu_G^{-1}(\xi) \qquad\qquad X^{\xi}_{\mathrm{T}} = \mu_T^{-1}(\xi).$$

There are closed inclusions

$$X^{\xi}_{\mathrm{G}} \subseteq X^{\xi}_{\mathrm{T}} \subseteq X.$$

We use similar notation for other subgroups of G. When $\xi = 0$ we put $X_{\mathrm{G}} = X^0_{\mathrm{G}}$, and similarly for T.

Fix a generic character θ of G once and for all. For a variety Y on which G acts, we call a point $y \in Y$ G-semistable if it is (θ, G)-semistable in the GIT sense, and denote the G-semistable locus of Y by $Y^{\mathrm{G}-ss}$. Viewing θ as a character of T, we define T-semistability similarly; there are open inclusions

$$Y^{\mathrm{G}-ss} \subseteq Y^{\mathrm{T}-ss} \subseteq Y.$$

We denote the GIT quotient of the G, resp. T-semistable loci by $/\!\!/_{\mathrm{G}}, /\!\!/_{\mathrm{T}}$, e.g.:

$$Y/\!\!/_{\mathrm{G}}G = Y^{\mathrm{G}-ss}/G \qquad\qquad Y/\!\!/_{\mathrm{G}}T = Y^{\mathrm{G}-ss}/T.$$

Letting now $Y = X^{\xi}_{\mathrm{G}}$ or $Y = X^{\xi}_{\mathrm{T}}$, we obtain the hyperkähler quotient by G and the corresponding abelian quotient:

$$X^{\xi}/\!\!/G := X^{\xi}_{\mathrm{G}}/\!\!/_{\mathrm{G}}G \qquad\qquad X^{\xi}/\!\!/T := X^{\xi}_{\mathrm{T}}/\!\!/_{\mathrm{T}}T,$$

as well as the *intermediate quotients* $X^{\xi}_{\mathrm{G}}/\!\!/_{\mathrm{G}}T, X^{\xi}_{\mathrm{G}}/\!\!/_{\mathrm{T}}T$. These play a central role in what follows and we denote

$$X^{\xi}/\!\!/^{G}_{T} := X^{\xi}_{\mathrm{G}}/\!\!/_{\mathrm{G}}T.$$

When $\xi = 0$, we again suppress it from the notation.

Remark 4.1.1. *For generic values of ξ, which are regular values for μ_G, μ_T on* all *of X, the quotients $X^\xi/\!/G, X^\xi/\!/T, X^\xi/\!/\frac{G}{T}$ are smooth and affine. In this setting, the family $X^{t\xi}/\!/G, t \in \mathbb{C}$ is a deformation of $X/\!/G$ over $\mathbb{C}$ as in (2.2.3), and similarly for the various other quotients introduced here. All the constructions presented below for $\xi = 0$ extend equivariantly to the families obtained by varying ξ.*

Remark 4.1.2. *These constructions and all other results presented in this section extend with obvious modifications when T is replaced by any maximal rank reductive group. These will be important for us later, but for ease of presentation we only use the T quotient here.*

Example 4.1.3. *Let $X/\!/G = T^*Gr(k,n)$ be the cotangent bundle to the Grassmannian, obtained as the Nakajima quiver variety of type A_1. Then $X/\!/T \cong T^*(\mathbb{P}^{n-1})^k$ (see 5.1.2). More generally, replacing T with a reductive group $L \cong \prod \mathrm{GL}(k_i)$, $\sum k_i = n$, we have $X/\!/L = \prod T^*Gr(k_i, n)$.*

4.1.2 Smoothness

We are interested in the case where $X/\!/G$ is a symplectic resolution, in particular, it is smooth. We therefore assume:

1. 0 is a regular value for the μ_G, μ_T actions restricted to the semistable loci;
2. θ is generic for both the G and T actions.

The latter condition means that all semistable points are actually stable, and that the G, T actions on the respective stable loci is locally free.

Proposition 4.1.4. *If $X/\!/G$ is a Nakajima quiver variety satisfying the above conditions, then $X/\!/T$ is smooth (in particular, a symplectic resolution).*

Proof. We need to show that T stabilizers are trivial on T-stable points which are not G-stable. Let x be such a point; then the stabilizer G_x is connected ([10], 1.3.2). Since G is of type A, $G_x \cap T$ is connected. Being finite, it is trivial. □

4.1.3 The Martin Diagram

The various quotients we have introduced are related by the following diagram due to Martin (cf. [21, 15, 7]):

$$\begin{array}{ccccc} X/\!/\!/_T^G & \xrightarrow{j} & X_G/\!/\!/_T T & \xrightarrow{i} & X/\!/\!/T \\ \Big\downarrow{\scriptstyle \pi} & & & & \\ X/\!/\!/G & & & & \end{array} \tag{4.1}$$

Here, i is a closed inclusion and j is an open embedding; we put $\iota = i \circ j$. π is a flat bundle with affine fiber G/T. We call its fibers the *vertical* fibers. The symplectic forms ω_G, ω_T of $X/\!/\!/G, X/\!/\!/T$ are *compatible* in the sense that

$$\iota^* \omega_T = \pi^* \omega_G.$$

In particular, the vertical fibers are isotropic; in other words, ω_T matches the vertical directions with the normal bundle to $X/\!/\!/_T^G$ in $X/\!/\!/T$.

Note that since the Weyl group $W = W(G,T)$ normalizes T, it acts on all the abelian quotients.

Remark 4.1.5. *Let $x \in X/\!/\!/_T^G$. Then locally at x,*

$$X/\!/\!/T \cong X/\!/\!/G \times T^*G/T.$$

This follows from a general fact about the local structure of symplectic quotients ([12], §41). We will only be using this description to highlight the geometry, and therefore favor this abuse of notation in the interest of clarity.

Remark 4.1.6. *Hyperkähler quotients may also be described as symplectic quotients by a* compact *Lie group. The quotients $X/\!/\!/G, X/\!/\!/T$ remain the same when described*

as the quotients by maximal compact subgroups of G, T respectively. This is not true for intermediate quotients: the quotient denoted by $\mu_G^{-1}(0)/T$ in [15] corresponds to what we denote below by $\mu_G^{-1}(0)/B$. Their inclusion $\mu_G^{-1}(0)/T \hookrightarrow X/\!\!/T$ is not algebraic, as it requires the C^∞ isomorphism $G/T \cong T^*G/B$.

4.1.4 Cohomology

Line bundles

Let Δ be the root system of G. For a character α of T and the corresponding 1-dimensional representation $\mathbb{C}_\alpha$, there is an induced line bundle over $X/\!\!/T$,

$$\mathcal{L}_\alpha = X_{\mathrm{T}} \underset{\mathrm{T}}{\times} \mathbb{C}_\alpha.$$

We denote its Euler class by e_α.

Following Martin [21], we observe that restricting μ_G to X_{T} defines a T-equivariant map to $\ker(pr : \mathfrak{g}^* \to \mathfrak{t}^*)$, which descends to an equivariant section s of the bundle $\oplus_{\alpha\in\Delta}\mathcal{L}_\alpha$. The assumption that 0 is a regular value of μ_G implies that this section is generic. The zeroset of s is $X_{\mathrm{G}}/\!\!/_{\mathrm{T}}T$, therefore the Euler class of its normal bundle in $X/\!\!/T$ is

$$Eu(X_{\mathrm{G}}/\!\!/_{\mathrm{T}}T) = Eu\left(\bigoplus_{\alpha\in\Delta}\mathcal{L}_\alpha\right) = \prod_{\alpha\in\Delta} e_\alpha. \tag{4.2}$$

Note again that these constructions extend to the family obtained by varying ξ.

Torus action

Assume now that $X = T^*V$ admits an algebraic action by an additional complex torus A, satisfying the following conditions:

1. The A action and the G action commute.

2. A scales the symplectic form by a non-trivial character $\hbar$.

3. The moment map μ_G is A-equivariant, where A acts on $\mathfrak{g}^*$ by scaling by $\hbar$.

These assumptions hold, for example, for the framing action in Nakajima varieties. Note that the first assumption implies that the semistable loci are A-stable, and therefore the A action descends to the quotients. The third assumption implies that the bundles $\mathcal{L}_\alpha$ are A-equivariant. We denote by $A_0 \leq A$ the stabilizer of ω_G, and by $H^*_\hbar$ equivariant cohomology with respect to A/A_0.

Abelianization

By an equivariant version of a standard theorem of Borel, π^* is an isomorphism onto the W-invariant part of $H^*_A(X/\!/^G_T)$. Then we have:

Theorem 4.1.7 ([15]). *The map*

$$\Theta^G_T = (\pi^*)^{-1} \circ \iota^* : H^*_A(X/\!/T)^W \to H^*_A(X/\!/G).$$

is surjective and its kernel is the annihilator of

$$e = \prod_{\alpha \in \Delta} e_\alpha(\hbar - e_\alpha) \in H^*_A(X/\!/T)^W.$$

In particular, classes supported on the G-unstable locus kill e.

Lifts [21, 7]

There are Kirwan maps:

$$\kappa_G : H^*_{A\times G}X \to H^*_A(X/\!/G) \qquad\qquad \kappa_T : H^*_{A\times T}X \to H^*_A(X/\!/T)$$

which we assume to be surjective; this holds in the quiver case after inverting A weights [15]. We also have the standard restriction map

$$r^G_T : H^*_{A\times G}(X) \to H^*_{A\times T}(X)^W.$$

We call a class $\dot{\gamma} \in H^*_A(X//T)$ a *lift* of $\gamma \in H^*_A(X//G)$ if there exists $\bar{\gamma}$ such that

$$\gamma = \kappa_G(\bar{\gamma}) \qquad\qquad \dot{\gamma} = \kappa_T \circ r^G_T(\bar{\gamma})$$

We can choose $\dot{\gamma}$ to be W invariant. In this case $\Theta^G_T(\dot{\gamma}) = \gamma$, i.e.

$$\iota^*(\dot{\gamma}) = \pi^*(\gamma). \tag{4.3}$$

Thinking of γ as a cycle, in the local description (4.1.5) we have $\dot{\gamma} \cong \gamma \times T^*G/T$.

4.2 Fixed components

4.2.1 Fixed components and abelianization

Lifts of fixed components

Let $\dot{Z} \in F_{A_0}(X//T), x \in \dot{Z}$. The fiber over x in X^{T-ss} is an A_0-stable T-orbit on which T acts freely. Since the A_0 and T actions commute, we obtain a map $\Phi_{\dot{Z}} : A_0 \to T$ defined by

$$ay = \Phi_{\dot{Z}}(a)y \tag{4.4}$$

for any point y in this orbit.By continuity, $\Phi_{\dot{Z}}$ is independent of x. We put

$$\alpha(\dot{Z}) := \Phi^*_{\dot{Z}}(\alpha),$$

where $\Phi^*_{\dot{Z}} : \mathfrak{t}^* \to \mathfrak{a}^*$ is the induced map. Since A acts on $\mathfrak{g}^*$ with weight $\hbar$, we see that

$$e_\alpha|_{\dot{Z}} = \hbar + \alpha(\dot{Z}) \in H^*_A(pt). \tag{4.5}$$

Now let $Z \in F_{A_0}(X/\!/G), x \in Z$, and consider the A_0 action on the fiber in X^{G-ss} over x. Indetifying

$$\pi^{-1}(x) \cong G/T$$

and for $y \in G/T$, we obtain maps $\Phi_{Z,y} : A_0 \to G$ using the same definition 4.4, and the dependence on y is G-equivariant. It is clear that $y \in (X/\!/T)^{A_0}$ iff the image of $\Phi_{Z,y}(A_0)$ lands in T, and that all the W-conjugates of y are also fixed. In this case,

Observation 4.2.1. *The vertical directions at y on which A_0 acts with weight 0 are identified with the roots contained in the centralizer G_Z of A_0.*

Let W_Z, Δ_Z be the Weyl group and root system of G_Z respectively. These are well defined up to conjugation, and we conclude:

Proposition 4.2.2.

$$(X/\!/T)^{A_0} \cap \pi^{-1}(x) \cong G_Z/T \times W/W_Z \tag{4.6}$$

We call any component $\dot{Z} \in F_{A_0}(X/\!/T)$ intersecting $\pi^{-1}(x)$ a *lift* of Z; the terminology is justified by the lemma below. We see that Z has exactly $|W/W_Z|$ lifts.

Abelianization of fixed components

Suppose $X/\!/G$ is a quiver variety, and recall that we had $X = T^*V$. Choose a lift of Z and let $V_Z \leq V$ be the subspace which the A_0 action coincides with the $\Phi_Z(A_0)$ action. It is shown in ([22], 2.3) that Z is itself a quiver variety, obtained as

$$Z = T^*V_Z/\!/G_Z. \tag{4.7}$$

Lemma 4.2.3. *Let $\dot{Z}$ be a lift of Z. Then*

$$\dot{Z} \cong T^*V_Z /\!/ T.$$

Proof. By choosing a conjugate of Φ_Z so that A_0 lands in T, it is clear that the T orbits in T^*V_Z are A_0-stable. In (4.7), the moment map and the stability conditions for G_Z are obtained by restriction from G. Since $T \leq G_Z$, they agree with the ones in $X/\!/T$. □

In particular, the lifts of any $\gamma \in H^*_\hbar(Z)$ are defined as in (4.1.4). We refer the reader to (6.3.1) for interesting examples in the abelianization of $\mathrm{Hilb}_n\mathbb{C}^2$.

4.2.2 Extended Martin Diagram

Fix a Borel subgroup $B \leq G$ containing T. Let P be a parabolic subgroup containing B, $L \ltimes U$ its Lévi decomposition. Recall the notation

$$\mu_P = (\mathfrak{g}^* \to \mathfrak{p}^*) \circ \mu, \quad X_{\mathrm{P}} = \mu_P^{-1}(0).$$

We first note

Proposition 4.2.4. *The geometric quotient of X_{G} by P exists.*

Proof. The proof given in ([7], 2.5) for Borel subgroups also holds for parabolics. □

We have the following extended version of the diagram (4.1):

$$\begin{array}{ccccccc} & & & & X_{\mathrm{G}}/\!/_{\mathrm{L}}L & \xrightarrow{i_G} & X_{\mathrm{P}}/\!/_{\mathrm{L}}L \xrightarrow{i_P} X/\!/L \\ & & & \overset{j_G}{\nearrow} & & \overset{j_P}{\nearrow} & \overset{j_L}{\nearrow} \\ X_{\mathrm{G}}/\!/_{\mathrm{G}}L & \xrightarrow{\iota_G} & X_{\mathrm{P}}/\!/_{\mathrm{G}}L & \xrightarrow{\iota_P} & X_{\mathrm{L}}/\!/_{\mathrm{G}}L & & \\ \downarrow{\scriptstyle \pi_P} & & & & & & \\ X_{\mathrm{G}}/\!/_{\mathrm{G}}P & & & & & & \\ \downarrow{\scriptstyle \pi_G} & & & & & & \\ X/\!/G & & & & & & \end{array} \tag{4.8}$$

Here, the maps j are open inclusions of the G-semistable in the L-semistable loci, and the maps i, ι are closed inclusions. Note that ι_G is a regular embedding, whereas i_G need not be. π_P is an affine bundle with fiber $U = P/L$, and π_G is proper with fiber G/P. We abbreviate $X/\!/P = X_{\mathrm{G}}/\!/_{\mathrm{G}}P, X/\!/_L^G = X_{\mathrm{G}}/\!/_{\mathrm{G}}L$.

From now on we consider the case where $L = G_Z$ for some $Z \in F_{A_0}(X/\!/G)$ and P is the unique parabolic containing B whose Lévi factor is L. Let j_G^A, π_G^A etc. be the restrictions of the maps in (4.8) to the A_0-fixed locus. Then using (4.7,4.2.1), we note that:

Observation 4.2.5.

1. *The maps j^A, i^A, ι^A (with any subscript), and π_P^A are isomorphisms;*

2. *The fibers of π_G^A are discrete.*

4.2.3 Dominant lifts

Let $\Delta = \Delta^+ \sqcup \Delta^-$ be the decomposition of Δ into positive and negative roots with respect to B. Fix a generic cocharacter σ of A_0, Z as above. The following is a key definition:

Definition 4.2.6. *We say that a lift $\dot{Z} \in F_{A_0}(X/\!\!/L)$ of Z is* dominant *if σ pairs positively with all the roots $\Delta^+ \setminus \Delta_Z^+$ under the induced map $\Phi_{\dot{Z}} : A_0 \to T$.*

Clearly, Z has precisely one dominant lift; its other lifts are obtained as its conjugates under W/W_Z. The following simple lemma characterizes the dominant lift, and is the geometric crux of the argument for proving the abelianization theorem. We retain the notation of the previous section; recall that m denotes the stable leaf (2.2.1).

Proposition 4.2.7. *The following are equivalent:*

1. *$\dot{Z}$ is the dominant lift of Z;*
2. *$\pi_P^{-1}(\pi_P^A(\dot{Z})) \subseteq m(X/\!\!/_L^G, \dot{Z})$;*
3. *$m(X/\!\!/P, \pi_P^A(\dot{Z}))$ is transverse to the fiber $\pi_G^{-1}(Z)$.*
4. *$m(X/\!\!/L, \dot{Z}) \subseteq X_{\mathrm{P}}/\!\!/_{\mathrm{L}} L$;*

Proof. The equivalence of the first three claims follows directly from the definition of a dominant lift by examining the weights of the A_0 action on the vertical directions. To see (4), note that it is enough to check the inclusion locally on the tangent space, since $X_{\mathrm{P}}/\!\!/_{\mathrm{L}} L$ is a closed A_0-stable subvariety. Similarly to 4.2, the Euler class of the normal bundle to $X_{\mathrm{P}}/\!\!/_{\mathrm{L}} L$ in $X/\!\!/L$ is

$$Eu(X_{\mathrm{P}}/\!\!/_{\mathrm{L}} L) = \prod_{\alpha \in \Delta^- \setminus \Delta_Z^-} e_\alpha. \tag{4.9}$$

The weights of the A_0 action on the repelling vertical directions are given by $-\alpha(\dot{Z})$ for $\alpha \in \Delta^+ \setminus \Delta_Z^+$. The symplectic form must match them with the attractive normal weights $h + \alpha(\dot{Z})$, which are precisely those contained in $X_{\mathrm{P}}/\!\!/_{\mathrm{L}} L$. The claim follows. □

4.3 Abelianization theorem

4.3.1 Main Theorem

Our goal in this section is to relate stable envelopes on $X/\!/G$ and $X/\!/T$. We will see that taking stable envelopes does not commute with abelianization; instead, we have to restrict to dominant lifts. We use σ and the standard polarization induced from $X = T^*V$ to define stable envelopes with respect to the A action. We retain the notation of the previous section, and denote stable envelopes in $X/\!/G$ by $stab_G$, and similarly for T, L.

Theorem 4.3.1. *Let $Z \in F_{A_0}(X/\!/G)$, $\dot{Z}, \ddot{Z}$ its dominant lifts to $X/\!/L, X/\!/T$ respectively. Let $\gamma \in H^*_\hbar(Z)$, $\dot{\gamma}, \ddot{\gamma}$ its lifts to $\dot{Z}, \ddot{Z}$. Let*

$$\Theta : H^*_A(X/\!/T)^{W_L} \to H^*_A(X/\!/L) \qquad \Theta^A : H^*_\hbar(\ddot{Z})^{W_L} \to H^*_\hbar(\dot{Z})$$

be as in (4.1.7), so that $\Theta^A(\ddot{\gamma}) = \dot{\gamma}$. Then

1. *$stab_T(\ddot{\gamma})$ is W_L-invariant and*

$$\Theta\left(stab_T(\ddot{\gamma})\right) = stab_L(\dot{\gamma}). \tag{4.10}$$

2. *There exist classes $s^{hor}_P(\dot{\gamma}) \in H^*_A(X_P/\!/_G L)$, $s^{ver}_P(\dot{\gamma}) \in H^*_A(X/\!/P)$ such that*

$$j^*_L(stab_L(\dot{\gamma})) = \iota_{P*}(s^{hor}_P(\dot{\gamma})) \tag{4.11}$$

$$\iota^*_G(s^{hor}_P(\dot{\gamma})) = \pi^*_P(s^{ver}_P(\dot{\gamma})) \tag{4.12}$$

$$\pi_{G*}(s^{ver}_P(\dot{\gamma})) = stab_G(\gamma) \tag{4.13}$$

Remark 4.3.2. *For an equivalent formulation of (2), construct the quotient $X_{\mathrm{P}}/\!/_{\mathrm{G}}P$. Completing the diagram (4.8), we obtain correspondences*

$$X_{\mathrm{P}}/\!/_{\mathrm{G}}P \to X/\!/L \qquad\qquad X_{\mathrm{P}}/\!/_{\mathrm{G}}P \to X/\!/G.$$

Then the theorem asserts that there exists a class in $H^*_A(X_{\mathrm{P}}/\!/_{\mathrm{G}}P)$ mapping to $stab_L(\dot{\gamma})$, $stab_G(\gamma)$ respectively.

Proof. We prove both parts of the theorem together, first for $\gamma = [Z]$. Recall from (2.3) that $stab_G(Z)$ is obtained by specializing the stable leaves on a family of affine deformations. Since all the relevant operations commute with specialization ([8], 10.1), it is enough to prove the theorem on the affine fibers. Note in particular that since there are no unstable points in the affine fiber, when specializing the open inclusion j, any corrections for leaves of fixed components supported in the G-unstable locus cannot involve leaves of G-stable components.

Fix a generic $\xi \in \mathfrak{g}^*$ as in (4.1.1), and let Z^ξ be the fixed component specializing to Z, and similarly for $\dot{Z}, \ddot{Z}$. We denote the specialization to the central fiber by $\lim_{\xi\to 0}$. Since $X^\xi/\!/G, X^\xi/\!/L, X^\xi/\!/T$ are all affine and $X^\xi/\!/P$ fibers over $X^\xi/\!/G$, the following are algebraic cycles (cf. 2.2.1):

$$m_G = m(X^\xi/\!/G, Z) \qquad m_L = m(X^\xi/\!/L, \dot{Z}) \qquad m_T = m(X^\xi/\!/T, \ddot{Z})$$
$$m_{\substack{G\\L}} = m(X^\xi/\!/_{\substack{G\\L}}, \dot{Z}) \qquad m_P^{ver} = m(X^\xi/\!/P, \pi_P^A(\dot{Z}))$$

Part (1) says that in the diagram (4.1) with $G = L$, $\pi^* m_L = \iota^* m_T$ which is obvious since $\pi^{-1}(\dot{Z})$ is fixed by A_0. For part (2) we show:

1. $m_L \subseteq X^\xi_{\mathrm{P}}/\!/_{\mathrm{L}}L$
2. $m_{\substack{G\\L}} = \pi_P^*(m_P^{ver})$
3. $m_G = (\pi_G)_*(m_P^{ver})$

This essentially follows from (4.2.7). The first claim is its part (4). For the second claim, since both sides are closed, irreducible, A_0-invariant cycles, it is enough to check the claim locally near $\dot{Z}$, where it follows from part (2). The last claim follows similarly from part (3). By the first claim, there is a class m_P^{hor} pushing forward to m_L, and clearly $m_{\substack{G\\L}} = \iota_G^* m_P^{hor}$. The theorem follows readily by specializing to the central fiber, where we take

$$s_P^{ver} = \lim_{\xi \to 0} m_P^{ver} \qquad\qquad s_P^{hor} = \lim_{\xi \to 0} m_P^{hor}. \tag{4.14}$$

To show the theorem for general $\gamma \in H_\hbar^*(Z)$, recall from (2.2.2) that the map $stab_L$ is given by a correspondence in $X/\!/L \times (X/\!/L)^{A_0}$, proper over $X/\!/L$. It is clear that restricting to $X/\!\!/L \times \dot{Z}$ computes $stab_L$ on $\dot{Z}$; we still call this correspondence $stab_L$, and similarly for G, T. Our argument above shows that the theorem holds for these correspondences, e.g.

$$(\Theta \otimes \Theta^A)(stab_T) = stab_L$$
$$(\pi_G \times \pi_G^A)_*(s_P^{ver}) = stab_G$$

We show the first part of the theorem (4.10). Denote projections to $X/\!\!/T$, $X/\!\!/L$, $X/\!/\frac{L}{T}$ by p_T, p_L, p_T^L, projections to $\dot{Z}, \ddot{Z}, \pi^{-1}(\dot{Z})$ by p_Z, and restrictions of maps in the diagram (4.1) to $\dot{Z}, \ddot{Z}$ by the subscript Z. Then using the property (4.3) of lifts, and since ι is a composition of an open inclusion with a regular embedding and π is flat,

$$\pi^* stab_L(\dot{\gamma}) = \pi^* p_{L*}(stab_L \cap p_Z^* \dot{\gamma}) = p_{T*}^L((\pi \times \pi_Z)^*(stab_L \cap p_Z^* \dot{\gamma})) =$$
$$p_{T*}^L((\iota \times \iota_Z)^* stab_T \cap p_Z^* \pi_Z^* \dot{\gamma}) = p_{T*}^L((\iota \times \iota_Z)^* stab_T \cap p_Z^* \iota_Z^* \ddot{\gamma}) =$$
$$p_{T*}^L (\iota \times \iota_Z)^* (stab_T \cap p_Z^* \ddot{\gamma}) = \iota^* p_{T*}(stab_T \cap p_Z^* \ddot{\gamma}) = \iota^* stab_T(\ddot{\gamma}).$$

This implies that $stab_T$ is W_L-invariant since π^* lands in the restriction of the W_L

invariant part. Note that the necessary properness condition for p_T^L hold ([8], 1.7).

For the second part the computation is similar, in fact simpler. We show, for example, (4.13), using analogous notation:

$$\pi_{G*}(s_P^{ver}(\dot{\gamma})) = \pi_{G*}p_{P*}(s_P^{ver} \cap p_Z^*\dot{\gamma}) = p_{G*}(\pi_G \times \pi_G^A)_*(s_P^{ver} \cap p_Z^*\dot{\gamma}) \stackrel{1}{=}$$
$$p_{G*}\left((\pi_G \times \pi_G^A)_*(s_P^{ver} \cap (\pi_G \times \pi_G^A)^* p_Z^*\gamma)\right) \stackrel{2}{=} p_{G*}\left((\pi_G \times \pi_G^A)_*(s_P^{ver}) \cap p_Z^*\gamma\right) =$$
$$p_{G*}(stab_G \cap p_Z^*\gamma) = stab_G(\gamma).$$

Here 1 follows from the fact that π_G^A is an isomorphism when restricted to any lift (4.2.5), and 2 is the projection formula. This concludes the proof of the theorem.

□

4.3.2 Restriction formula

We obtain a formula for the restriction of the stable basis to fixed points as follows.

Corollary 4.3.3. *With the notation of (4.3.1) and $Z' \in F_{A_0}(X/\!/G)$, $\ddot{Z}'$ any lift to $X/\!/T$,*

$$stab_G(\gamma)|_{Z'} = \sum_{w \in W/W_Z} \frac{stab_T(\ddot{\gamma})|_{w \cdot \ddot{Z}'}}{\prod_{\alpha \in \Delta^- \setminus \Delta_Z^-} \alpha(w \cdot \ddot{Z}') \cdot (\hbar + \alpha(w \cdot \ddot{Z}'))} \tag{4.15}$$

The sum is to be understood as restricted to the support of $stab_T(\ddot{\gamma})$, i.e. terms where the numerator vanishes are discarded even if there are 0 weights in the denominator.

Proof. By part (1) of the theorem and (4.3),

$$stab_T(\ddot{\gamma})|_{w \cdot \ddot{Z}'} = stab_L(\dot{\gamma})|_{w \cdot \dot{Z}'}$$

By (4.11), $Eu(X_{\mathrm{P}}/\!/_{\mathrm{G}}L)$ divides $\iota_G^*(s_P^{hor})$ in $H_A^*(X/\!/_L^G)$. Therefore, using (4.5,4.9), and

noting that $\alpha(\dot{Z}') = \alpha(\ddot{Z}')$, we have:

$$\iota_G^*(s_P^{hor})|_{w\cdot\dot{Z}'} = \frac{stab_T|_{w\cdot\ddot{Z}'}}{\prod\limits_{\alpha\in\Delta^-\setminus\Delta_Z^-}(\hbar+\alpha(w\cdot\ddot{Z}'))}.$$

Since π_P is an affine bundle and by (4.12), the restriction $s_P^{ver}|_{\pi_P^A(w\cdot\dot{Z}')}$ is given by the same formula. Finally, to compute $stab_G = \pi_{G*}(s_P^{ver})$, we may use the localization formula, since the fixed points along the fiber are isolated, and we have assumed formality. We obtain:

$$stab_G|_{Z'} = \sum_{w\in W/W_Z} \frac{s_P^{ver}|_{\pi_P^A(w\cdot\dot{Z}')}}{\prod\limits_{\alpha\in\Delta^-\setminus\Delta_Z^-}\alpha(w\cdot\ddot{Z}')}$$

the denominator being the product of weights along the fiber. The claim follows. □

Remark 4.3.4. *To use this formula in practice for quiver varieties, one may proceed as follows. If γ is given as a polynomial in Chern classes of tautological bundles, its lift can be computed explicitly by replacing each Chern class with an elementary symmetric polynomial in the Chern roots, which are tautological classes on $X/\!\!/T$. Then we express $stab_T(\ddot{\gamma})$ in tautological classes using (3.3.5).*

Next, it may be more convenient to compute the restriction of $stab_L(\dot{\gamma})$ rather than $stab_T(\ddot{\gamma})$. If the $\hbar$ fixed points along the fiber of π are isolated, then we may apply (3.3.4) at any of these points to obtain the result as an A weight. We will use this method in chapter 6.

Chapter 5

Abelianization for Type A Partial Flag Varieties

Fix $k_1 < ... < k_r < k_{r+1} = n$, and let $\mathcal{F} = \mathcal{F}(k_1, ..., k_r, n)$ be the variety of partial flags

$$\mathbb{C}^{k_1} \subseteq \mathbb{C}^{k_2} \subseteq ... \subseteq \mathbb{C}^{k_r} \subseteq \mathbb{C}^n.$$

The cotangent bundle $X_G = T^*\mathcal{F}$ is the Nakajima variety of a type A_r quiver with the dimension vector $(k_i)_{i=1}^r$ and framing vector $(0, ..., 0, n)$. Recall that the stable envelope can be identified with the characteristic cycle of a Verma module (2.2.3). In this chapter we apply the abelianization formula to compute the restriction of the stable envelope to fixed points. When $r = 1$, i.e. $X_G = T^*Gr(k_1, n)$, the corresponding quantum integrable system is the XXX spin chain, and we recover in a geometric way an explicit diagonalization of the Hamiltonian at $q = 0$. We also recover the fixed-point restriction of equivariant Schubert classes.

Notation 5.0.5. *For the rest of this paper, we simplify the notation for hyperkähler quotients, and put* $X_G = X//G, X_T = X//T$.

5.1 Partial flag varieties and their abelianization

5.1.1 Hyperkähler construction

Let $G = \prod_{i=1}^{r} \mathrm{GL}_{k_i}(\mathbb{C})$. The cotangent bundle $X_G = T^*\mathcal{F}$ is obtained as the hyperkähler quotient

$$T^* \bigoplus_{l=1}^{r} \mathrm{Hom}(\mathbb{C}^{k_l}, \mathbb{C}^{k_{l+1}}) /\!/ G,$$

where GL_{k_l} acts on the l-th factor on the left, and on the $l-1$-th factor on the right. For $A_l \in \mathrm{Hom}(\mathbb{C}^{k_l}, \mathbb{C}^{k_{l+1}}), B_l \in (\mathbb{C}^{k_{l+1}}, \mathbb{C}^{k_l}), \mathbf{i} \in \mathrm{Hom}(\mathbb{C}^{k_r}, \mathbb{C}^n), \mathbf{j} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{k_r})$, the moment map is given by (2.2)

$$\mu(A, B, \mathbf{i}, \mathbf{j}) = \sum_{l=1}^{r-1} [A_l, B_l] + \mathbf{ij}$$

and we use the product of the determinants as the stability parameter. For the abelianization X_T we use the product of diagonal tori $T = \prod_{i=1}^{r} T^{k_i}$, and the Weyl group is $W = \prod S_{k_i}$. It is easy to check the smoothness conditions (4.1.2) for X_T.

5.1.2 Description of the abelian quotient

The Grassmannian case

In the simplest case $r = 1$, i.e. $X_G = T^*Gr(k, n)$, the construction is particularly simple, and $X_T = (T^*\mathbb{P}^{n-1})^k$. Explicitly, for a map $S : \mathbb{C}^k \to \mathbb{C}^n$ and Q in the fiber over S, the semistable points are those with $\mathrm{rk}(S) = k$, and the moment map for the $G = \mathrm{GL}(k)$ action is given by $\mu_G(S, Q) = Q \cdot S$. Thinking of S as an element of Gr and identifying the fiber over it with $\mathrm{Hom}(\mathbb{C}^n/S, S)$, we obtain $X/\!/G = T^*Gr$.

Consider now the abelian quotient. The T-semistable points are the pairs (S, Q) as above where no row of S vanishes; $\mu_T^{-1}(0)$ consists of pairs (S, Q) where the diagonal of $Q \cdot S$ vanishes. Dividing by the T action we obtain the claim.

The general case

For $r > 1$, we can write the zero section $\mathcal{F} \subseteq X_G$ as the Kähler quotient

$$\bigoplus_{i=1}^{r} \mathrm{Hom}(\mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}}) /\!/ G.$$

Corresponding to the description of $\mathcal{F}$ as a tower of Grassmannians, its abelianization $\mathcal{F}_T$ is a toric variety which can be described as a tower of products of projective bundles [1]:

$$\begin{array}{ccc} (\mathbb{P}^{k_{i+1}-1})^{k_i} \longrightarrow \mathcal{F}_T^i & = & \mathbb{P}(V_{i+1}) \times_{\mathcal{F}_T^{i+1}} \ldots \times_{\mathcal{F}_T^{i+1}} \mathbb{P}(V_{i+1}) \\ \downarrow & & \\ \mathcal{F}_T^{i+1} & & \end{array} \tag{5.1}$$

where $\mathcal{F}_T^i$ is the abelianization of $\mathcal{F}(k_i, ..., k_r, n)$, and

$$V_{i+1} = \bigoplus_{j=1}^{k_{i+1}} \mathcal{O}_{\mathcal{F}_T^{i+1}}(0, ..., 0, \underset{j}{-1}, 0, ..., 0)$$

is the vector bundle on $\mathcal{F}_T^{i+1}$ corresponding to the tautological bundle on $\mathcal{F}^{i+1}$.

The hyperkähler abelianization X_T contains $T^*\mathcal{F}_T$ as a dense open subset, and we have further $X_T^{G-ss} \subseteq T^*\mathcal{F}_T$. X_T is obtained by gluing together flopped versions of $T^*\mathcal{F}_T$, but we do not give a full description here.

5.1.3 Torus fixed points

Take $A_0 = (\mathbb{C}^*)^n$, acting on $\mathcal{F}$ through its natural action on $\mathbb{C}^n$ and on X_G symplectically. Let $\mathbb{C}^*$ act on X_G by scaling the fibers with weight $\hbar$, and put $A = A_0 \times \mathbb{C}^*$. We choose the chamber corresponding to a cocharacter of A_0 acting with distinct weights $\lambda_1 > \lambda_2 > ... > \lambda_n$. The stable leaves are conormal bundles to Schubert cells.

Fixed points in X_G correspond to flags where each subspace is spanned by coor-

dinate axes. The number of fixed points is

$$\binom{n}{k_1, k_2 - k_1, ..., n - k_r} = \binom{k_2}{k_1}\binom{k_3}{k_2}...\binom{n}{k_r}.$$

We use two ways to index a fixed point P, corresponding to the RHS and LHS above:

1. The 'quiver' notation by subsets $\mathfrak{p}^l \subseteq \{1..k_{l+1}\}, l = 1..r$ of order k_{l+1}. We put $\mathfrak{p}^{r+1} = \{1..n\}$.

2. The 'flag' notation by subsets

$$p^1 \subset p^2 \subset ... \subset p^r \subset p^{r+1} = \{1..n\}$$

   of order $k_1, k_2, ..., k_r, n$.

These are related by

$$p^l_i = p^{l+1}_{\mathfrak{p}^l_i}. \tag{5.2}$$

As one might expect, and we show below, fixed point lifts in X_T are also discrete, indexed by a choice of ordering on each p^l. The l-th factor of W acts on $\mathfrak{p}^l$ by permuting the factors and on $\mathfrak{p}^{l-1}$ through its action on $\{1..k_l\}$, and by its action on p^l alone in the flag notation. Choosing the Borel of upper triangular matrices in all factors of G, the dominant lift is given by taking the ascending order on each subset.

5.2 Abelianization of the stable basis

5.2.1 The stable basis in the abelian quotient

Let $\mathfrak{t}^N = \bigoplus_{l=1}^r \mathrm{Hom}(\mathbb{C}^{k_l}, \mathbb{C}^{k_{l+1}}), \mathfrak{t}^K = Lie(T)$ (corresponding to the tori denoted $\mathfrak{t}^n, \mathfrak{t}^k$ in chapter 3), $\mathfrak{t}^d = \mathfrak{t}^N/\mathfrak{t}^K$. Recall the notation (z^l_{ij}, w^l_{ij}) for the coordinates of $T^*\mathfrak{t}^N$. In accordance with our previous notation, we also denote by $e^l_{ij}$the standard generators

on $\mathfrak{t}^N$, ϵ^l_{ij} the dual basis, u^l_{ij} the associated tautological classes on X_T.

The fixed point P indexed in the quiver notation corresponds to the intersection of all hyperplanes except $u^l_{i,\mathfrak{p}^l_i}$. Let $a^l_{ij}, j \neq \mathfrak{p}^l_i$ be the normals to the hyperplanes defining P; these span $\mathfrak{t}^d$. Denote by α^l_{ij} be the dual basis. The coordinate $\mathfrak{t}^l_i$ of $\mathfrak{t}^K$ embeds in $\mathfrak{t}^N$ as $-\sum_j e^{l-1}_{ji} + \sum_j e^l_{ij}$. Identifying $(\mathfrak{t}^d)^*$ with its image in $(\mathfrak{t}^N)^*$ as usual, we have

$$\alpha^l_{ij} = \epsilon^l_{ij} - \epsilon^l_{i,\mathfrak{p}^l_i} + \epsilon^{l+1}_{j,\mathfrak{p}^{l+1}_j}. \tag{5.3}$$

Considering the dominant lift $\dot{P}$ of P and using the flag notation, the weight of a^l_{ij} under the A_0 action is

$$\lambda_{p^{l+1}_j} - \lambda_{p^l_i}$$

as can be readily seen by induction. In particular, if the λ-s are distinct there are no zero weights and the lifts are discrete as claimed. It is also easy to see that this weight is negative iff $j < \mathfrak{p}^l_i$ (now in the quiver notation!). Recall the star notation from (3.4); the polarization induced from the quotient structure matches the polarization (3.11) and we conclude that the stable envelope is given by

$$stab_T(\dot{P}) = \prod_l \prod_{i,j} (u^l_{ij})^{*(j-p^l_i)}. \tag{5.4}$$

5.2.2 Abelianization

Roots

The T orbit corresponding to the fixed point P is spanned by the coordinates z^l_{i,s^l_i}. Induction again shows that the identification of the A action as a subtorus of T is given by

$$\Phi_P(\lambda_1, ..., \lambda_n) \mapsto \left((\lambda_{p^1_1}, ..., \lambda_{p^1_{k_1}}), ..., (\lambda_{p^r_1}, ..., \lambda_{p^r_{k_r}}) \right)$$

The root (ij) of the l-th factor in G is therefore given by

$$\lambda_{p_i^l} - \lambda_{p_j^l}, \tag{5.5}$$

proving also that the ascending order on S gives the dominant lift as claimed.

Fixed point restriction

We now compute the restrictions of tautological classes to fixed points, using (3.3.2). Since there are no w coordinates in the T orbit corresponding to P, it follows from (3.9) that $\hbar$ does not appear in the restriction. Let $\sigma = (\sigma_1, ..., \sigma_r) \in W$ and let $\sigma\dot{P}$ be the corresponding lift. Then restricting from T^d to A-equivariant cohomology, we obtain

$$u_{ij}^l|_{\sigma\dot{P}} = \lambda(p^{l+1}_{\sigma_j^{l+1}}) - \lambda(p^l_{\sigma_i^l}). \tag{5.6}$$

Note that the restriction vanishes when u_{ij}^l is not one of the defining hyperplanes, as required.

Abelianization formula

Applying (5.2–5.6) to our abelianization formula (4.3.3), we obtain:

Theorem 5.2.1. *Let $P = (p^l), Q = (q^l)$ be A fixed points in $T^*\mathcal{F}$. Then*

$$stab(T^*\mathcal{F}, P)|_Q = \sum_{\sigma^1..\sigma^r} \prod_{l=1}^{r} \frac{\prod\limits_{\substack{i=1..k_l \\ j=1..k_{l+1}}} \left(\lambda(q^{l+1}_{\sigma_j^{l+1}}) - \lambda(q^l_{\sigma_i^l})\right)^{*\left(\lambda(p_j^{l+1}) - \lambda(p_i^l)\right)}}{\prod\limits_{1\le j<i\le k_l} \left(\lambda(q^l_{\sigma_i^l}) - \lambda(q^l_{\sigma_j^l})\right) \cdot \left(\hbar + \lambda(q^l_{\sigma_i^l}) - \lambda(q^l_{\sigma_j^l})\right)}$$

were the sum is over $\sigma^i \in S_{k_i}$, and $\sigma^{r+1} = id$.

Remark 5.2.2. *In the recent paper [31], the authors obtain analogous formulas for the stable basis; while they to not provide an explicit restriction formula, their proof*

involves a restriction argument. Their results originate from the quantum-integrable side of the theory, by means of certain weight functions*, which play the role of our* $stab_T$*. They proceed to obtain the formulas for* $stab_G$ *by explicit computation. Our derivation provides a geometric counterpart to theirs, and an interesting example of the interplay between the two sides of the Nekrasov-Shatashvili correspondence.*

5.3 The Grassmannian case

One of the most basic examples of the Nekrasov-Shatashvili correspondence is between the inhomogeneous XXX spin chain and $X = T^*Gr$. Under this identification, the stable basis corresponds to the coordinate (spin) basis, and the fixed-point basis corresponds to the Bethe eigenbasis at $q = 0$ (this follows from [22], though an equivalent statement was known to Nekrasov and Shatashvili). Our restriction formula therefore gives an explicit diagonalization of the Hamiltonian at $q = 0$. While this result is not new, we believe that our geometric derivation is of interest. We state the result for this case separately:

Proposition 5.3.1. *Let* $X_G = T^*Gr(k,n)$*,* $P = (p_1, ..., p_k), Q = (q_1, ..., q_k)$ *two fixed points. Then*

$$stab(T^*Gr, P)|_Q = \sum_{\sigma \in S_k} \frac{\prod_{i=1}^{k} \left(\prod_{j=1}^{p_i - 1} (\hbar - \lambda_j + \lambda_{q_{\sigma(i)}}) \cdot \prod_{j=p_i+1}^{n} (\lambda_j - \lambda_{q_{\sigma(i)}}) \right)}{\prod_{1 \le j < i \le k} (\lambda_{q_{\sigma(i)}} - \lambda_{q_{\sigma(j)}}) \cdot (\hbar + \lambda_{q_{\sigma(i)}} - \lambda_{q_{\sigma(j)}})}$$

These are the *rational Schur polynomials*, introduced by Rains [30] as part of a broader family of rational interpolation polynomials.

Remark 5.3.2. *We can use this result to recover the fixed point restriction of equivariant Schubert classes in the Grassmannian [16, 18], given by factorial Schur polynomials ([20], 1.3.20), by taking the limit* $\hbar \to \infty$ *as follows. Let* $\Omega(p)$ *be the Schubert*

class in $Gr(k,n)$ corresponding to p. Then up to sign,

$$\Omega(p)|_q = \lim_{\hbar\to\infty} \frac{stab(T^*Gr, p)|_q}{\hbar^{\dim \Omega(p)}}$$

Since this restriction might be computed by taking semismall resolutions, we are using here the fact that for Grassmannians, the transition between the conormal basis and the stable basis is given by Kazhdan-Lusztig numbers ([6], see also 2.2.3).

Chapter 6

Abelianization for $\mathrm{Hilb}_n\mathbb{C}^2$ and Symmetric Polynomials

In this chapter we apply our abelianization formula to Hilb_n, the Hilbert scheme of n points on $\mathbb{C}^2$. Under the identification of the cohomology ring of Hilb with symmetric functions, the fixed point basis and the stable basis are mapped to Jack polynomials and Schur polynomials respectively. Thus, we obtain a formula (6.3.6) expressing Schur polynomials in the Jack basis. To the author's knowledge, this is the first formula of its kind.

6.1 Stable basis of Hilb_n

6.1.1 Hyperkähler quotient

We begin by recording a few basic facts about Hilb_n; more detail can be found in e.g. [26]. Recall that this is the Nakajima variety corresponding to the quiver with a single vertex and one edge loop. It is expressed as the hyperkähler quotient

$$\mathrm{Hilb}_n = (X, Y, \mathbf{i}, \mathbf{j})/\!/\mathrm{GL}(n), \tag{6.1}$$

where $X, Y \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, $\mathbf{i} \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n)$, $\mathbf{j} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$. The GL action and the moment map are given by

$$g(X, Y, \mathbf{i}, \mathbf{j}) = (gXg^{-1}, gYg^{-1}, g\mathbf{i}, \mathbf{j}g^{-1})$$
$$\mu : (X, Y, \mathbf{i}, \mathbf{j}) \mapsto [X, Y] + \mathbf{ij}$$

Choosing the character $\theta = \det$ of $\mathrm{GL}(n)$ as the stability parameter, the stability condition reads: $\mathbf{i}$ is a cyclic vector for the action of X, Y, and $\mathbf{j} = 0$. In particular, on the stable locus the moment map restricts to $[X, Y]$. The isomorphism with Hilb_n is made explicit by sending a codimension n ideal $\mathcal{I}$ to the action of x, y on $\mathbb{C}[x, y]/\mathcal{I}$ and the inclusion $\mathbf{i} : 1 \to \mathbb{C}[x, y]/\mathcal{I}$.

Torus action

Let $A \cong (\mathbb{C}^*)^2$ act on $\mathbb{C}^2$ with weights t_1, t_2, and the induced action on Hilb_n. The weight of the symplectic form under this action is

$$\hbar = t_1 + t_2$$

and its stabilizer is the subtorus $A_0 \cong \mathbb{C}^*$ acting with opposite weights. The fixed point locus is discrete, consisting of monomial ideals indexed by partitions of n, where the partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ corresponds to the ideal

$$\mathcal{I}_\lambda = \langle x^{\lambda_1}, x^{\lambda_2} y, ..., x^{\lambda_k} y^{k-1}, y^k \rangle.$$

6.1.2 Stable basis

We now explain the identification of the fixed point and stable bases with Jack and Schur polynomials. These results in fact predate the definition of the stable basis;

see, e.g. [32, 24, 19]. For notational consistency, we follow the treatment in [22], 17.2. Identifying the Nakajima operator α_{-k} acting on the y-axis with multiplication by the power-sum polynomial p_k, we obtain the standard isomorphism

$$H_A^*(\mathrm{Hilb}) \cong \Lambda \otimes H_A^*(pt) \tag{6.2}$$

where Λ is the ring of symmetric polynomials and

$$\mathrm{Hilb} = \bigsqcup_n \mathrm{Hilb}_n.$$

In [22], the standard inner-product on cohomology is twisted by a sign corresponding to what is called in *loc. cit.* the "canonical theta characteristic". Transporting this to cohomology, we obtain the Jack inner product

$$\langle p_k, p_l \rangle = \delta_{kl} k(-t_1/t_2)$$

with parameter $-t_1/t_2$, denoted α in [20]. By definition, Gram-Schmidt orthogonalization of this inner product with respect to the basis $\{m_\lambda\}$ of monomial symmetric functions gives the Jack polynomials. On the other hand, the orthogonality relations on the fixed point basis imply that up to suitable normalization, these are the same. We define

$$\mathbf{J}_\lambda = t_2^{|\lambda|} \cdot J_\lambda \tag{6.3}$$

where J_λ is the integral Jack polynomial defined in [20]. This normalization implies

$$\mathbf{J}_\lambda = \prod_{\square \in \lambda} (t_2(l(\square)+1) - t_1 a(\square)) m_\lambda + ... \tag{6.4}$$

where

$$l(\square) = \lambda'_j - i \qquad\qquad a(\square) = \lambda_i - j$$

are the leg length and arm length of a box $\square = (i, j)$ in λ. Then we have

Proposition 6.1.1 ([32, 24, 19])**.** *Under the identification (6.2),* $\mathbf{J}_\lambda$ *is the class of the fixed point* $[\mathcal{I}_\lambda]$*.*

We now consider the stable basis, under the chamber

$$\mathfrak{C} = \{t_1 < 0\}$$

of A_0, and the polarization defined by the Euler class of N_- at $\mathcal{I}_\lambda$, where we identify the product in (6.4) with the Euler class of N_+. The induced order on partitions is the dominance order, and we have

Proposition 6.1.2 ([32])**.** *Under the identification (6.2), the Schur polynomial* s_λ *is the class of the stable envelope of* $[\mathcal{I}_\lambda]$*.*

The proof is immediate, since Schur polynomials are triangular with respect to Jack polynomials, and proportional to them modulo $\hbar$; (6.4) fixes the normalization.

6.2 The Hypertoric Hilbert Scheme

We define the hypertoric Hilbert scheme to be the abelianization of Hilb, i.e. the quotient

$$\mathrm{THilb} = (X, Y, \mathbf{i}, \mathbf{j})//T^n,$$

where $T^n \leq \mathrm{GL}_n$ is the diagonal torus. This is a rather interesting, if little studied, hypertoric variety, and we digress briefly to mention a few basic details about it. We

do not have a moduli theoretic interpretation of THilb, and it would be an interesting problem to find one.

6.2.1 Hyperplane arrangement

In our hypertoric notation, we write

$$\mathrm{THilb} = T^*\mathbb{C}^{n^2+n}/\!/T^n.$$

Writing as usual (z, w) for the coordinates of $T^*\mathbb{C}^{n^2+n}$, the z coordinates correspond to $X, \mathbf{i}$, and the w coordinates correspond to $Y^t, \mathbf{j}$.

The torus $\mathfrak{t}^n = diag(\kappa_1, ..., \kappa_n)$, corresponding to $\mathfrak{t}^k$ in the notation of chapter 3, acts with weight $\kappa_i - \kappa_j$ on the coordinate $z_{ij} = X_{ij}$, and weight κ_i on the coordinate $z_i^{\mathrm{fr}} = \mathbf{i}_i$. We denote the tautological classes associated with z_{ij} by u_{ij}, and those associated with z_i^{fr} by u_i^{fr}. The torus A acts as a subtorus of T^{n^2+n}, with the weight t_1 on the coordinates z_{ij} and 0 on the coordinates z_i^{fr}.

We denote the coordinates of $\mathfrak{t}^{n^2+n}$ by $e_{ij}, e_i^{\mathrm{fr}}$; the dual basis by ϵ; and the image of the coordinates e in the quotient torus $T^d = T^{n^2+n}/T^n$ by b (here we depart from the notation of chapter 3 to reserve the letter a for partition arm length). Once a basis for T^d or its subtorus has been chosen, the dual coordinates are denoted by β.

The stability parameter is $\sum \kappa_i$, and lifts to the character $\sum_i \epsilon_i^{\mathrm{fr}}$ of $\mathfrak{t}^{n^2+n}$. Choosing the images of the e_{ij} coordinates as a basis for T^d we have

$$\beta_{ij} = \epsilon_{ij} - \epsilon_i^{\mathrm{fr}} + \epsilon_j^{\mathrm{fr}}$$

Under this notation, we obtain the $n^2 + n$ hyperplanes

$$b_{ij} = 0 \qquad\qquad \sum_j b_{ij} - b_{ji} + 1 = 0$$

Example 6.2.1. *Let $n = 2$. Splitting away the diagonal factor corresponding to the hyperplanes b_{ii}, we obtain the arrangement described in example (3.1.2). The Weyl group action is given by the reflection along the $b_{12} = b_{21}$ diagonal.*

Circuits

An easy calculation shows that there are $2^n - 1$ circuits in the arrangement: for every non-empty subset $S \subseteq \{1..n\}$, the following is a minimal relation:

$$\sum_{i \in S} b_i^{\mathrm{fr}} + \sum_{\substack{i \in S \\ j \notin S}} b_{ij} - \sum_{\substack{i \notin S \\ j \in S}} b_{ij}. \tag{6.5}$$

6.2.2 Geometry of THilb

Cohomology

Using (3.1.5, 6.5) we conclude:

Proposition 6.2.2. *$H^*_{T^d \times \mathbb{C}^*}(\mathrm{THilb})$ is generated by $u_{ij}, u_i^{\mathrm{fr}}, \hbar$ subject to the relations:*

$$\prod_{\substack{i \in S \\ j \notin S}} u_{ij} \prod_{\substack{i \notin S \\ j \in S}} (\hbar - u_{ij}) \prod_{i \in S} u_i^{\mathrm{fr}} \qquad\qquad \forall S \subseteq \{1..n\}, S \neq \emptyset$$

Stability

The stability conditions for a point in $\mu_T^{-1}(0)$ are explained in [17]. Briefly, for each circuit one obtains a condition on the vanishing of some of the coordinates z, w. We do not give the full details here, and instead state directly the application for THilb: $(z, w) \in \mu_T^{-1}(0)$ is stable iff for every nonempty $S \subseteq \{1..n\}$, at least one of the following coordinates does not vanish:

$$z_{ij} \quad (i \in S, j \notin S); \qquad w_{ij} \quad (i \notin S, j \in S); \qquad z_i^{\mathrm{fr}} \quad (i \in S).$$

Equivalently,

Proposition 6.2.3. $(X, Y, \mathbf{i}, \mathbf{j}) \in \mu_T^{-1}(0)$ *is stable iff any subspace containing* $\mathbf{i}$ *and stable under* X, Y *is not contained in any coordinate hyperplane.*

6.3 Abelianization

6.3.1 Fixed point loci in THilb

Let $\mathcal{I}_\lambda \in \mathrm{Hilb}_n^A$, and let $V_\lambda = \mathbb{C}[x,y]/\mathcal{I}_\lambda$. The $W = S_n$ action permutes the monomial basis for V_λ; this is equivalent to a Young tableau of shape λ. It will be convenient to write the T^n weights in the appropriate box, as in the example below. We use *zero based* coordinates for boxes; we call the top left box $\mathbf{o} = (0,0)$ the *origin* of λ. For the box $i = (i_y, i_x)$ let

$$p(i) = p_\lambda(i) = i_y - i_x;$$

writing partitions in the "Russian" convention, rotated by 45°, this is the projection to the horizontal axis. We will see that for the Borel of *lower* triangular matrices (since we chose the chamber $t_1 < 0$), the dominant lift corresponds to any ordering compatible with p. We choose such an ordering, increasing along the diagonals, as our reference.

Example 6.3.1. *We will use the partition* $\lambda = (3,3)$ *as a running example; e.g., we write the* T *action on* V_λ *as*

κ_4	κ_2	κ_1
κ_6	κ_5	κ_3

In this example the coordinates of the box labeled 1 *are* $(0,2)$.

Consider the lift of λ to data $X, Y, \mathbf{i}$ as in (6.1) such that

$$\begin{aligned} X_{ij} = z_{ij} \neq 0 \qquad &\text{iff} \quad i_y = j_y, \quad j_x = i_x - 1 \\ Y_{ij} = w_{ji} \neq 0 \qquad &\text{iff} \quad i_x = j_x, \quad j_y = i_y + 1 \end{aligned} \tag{6.6}$$

Since the A-weights of $\mathbf{i} = z^{\text{fr}}$ are 0, one checks that under the associated embedding $\Phi_\lambda : A \to T$, the box i must have weight $i_x t_1 + i_y t_2$. The root (i, j) on A_0 evaluates to $(p(j) - p(i))\, t_1$, so that the roots in the centralizer G_λ of A_0 are those associated with pairs of boxes along a diagonal. We denote the corresponding Weyl group W_λ. From this argument it is also clear that the lift we have picked is indeed the dominant lift when $t_1 < 0$ (see figure below). Finally, A embeds into T with distinct weights, so no roots vanish on it; therefore

Observation 6.3.2. *In the diagram (4.1), the $\hbar$ fixed points along the fiber of π are discrete, and therefore (3.3.4) applies.*

Example 6.3.3. *For $\lambda = (3, 3)$, Φ_λ is given by*

0	t_1	$2t_1$
t_2	t_1+t_2	$2t_1+t_2$

Explicitly, restricting to A_0 with $t_1 = -1$:

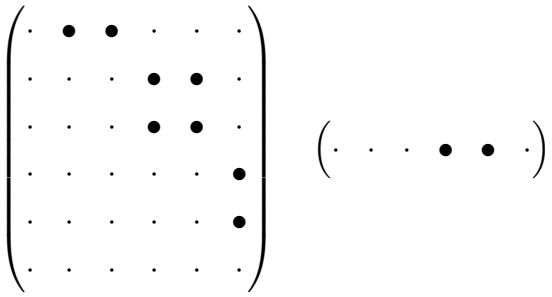
0	-1	-2
1	0	-1

G_λ contains the positive roots $(3, 2)$ and $(5, 4)$.

Let $\tilde{\mathcal{I}}_\lambda$ be a lift of $\mathcal{I}_\lambda$ to THilb. By the abelianization lemma (4.2.3), this is the hypertoric variety obtained from THilb by restricting to the hyperplanes where the A_0 action is compatible with $\Phi_\lambda(A_0)$. These are the coordinates e_{ij} where $p(j) = p(i) + 1$,

and the coordinates e_i^{fr} with $p(i) = 0$. In the language of [22], the point $\mathcal{I}_\lambda$ is written as a type A quiver variety with vertices indexed by the weights of A_0, framed at 0, with edges between consecutive weights, and dimension vector given by the weight multiplicity. The lift is the abelianization of this quiver.

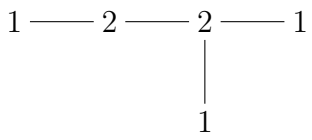
Example 6.3.4. *For $\lambda = (3,3)$ the dominant lift is obtained by restricting the arrangement of* THilb *to the coordinates indicated in the figure.*



Note that we have 10 *hyperplanes, and dividing by T^6,*

$$\dim \tilde{\mathcal{I}}_\lambda = 2 \cdot (10 - 6) = 2 \dim(G_\lambda / T)$$

as expected. Alternatively, the lift corresponds to the abelianization of the Nakajima quiver



where the indicated dimensions correspond to the multiplicities of the weights -2, -1, 0, 1.

6.3.2 Fixed point restriction and the stable basis of THilb

Fixed point restriction

Following (4.3.4) we study the A-equivariant restriction of tautological classes to the lift (6.6), thought of as an A-fixed point in $\tilde{\mathcal{I}}_\lambda$. The normals to the defining hyperplanes for $\tilde{\mathcal{I}}_\lambda$ form a basis for its stabilizing torus, call it T^Q. These are the b_{ij} where $p(j) \neq p(i)+1$, and b_i^{fr} with $p(i) \neq 0$. The dual basis is given as follows. Choose a path $j = j_0, j_1, ..., j_m = i$ in the diagram λ such that the boxes j_k, j_{k+1} are adjacent. Let

$$\epsilon^*_{j_k,j_{k+1}} = \begin{cases} \epsilon_{j_k,j_{k+1}} & \text{if} \quad p(j_{k+1}) = p(j_k)+1 \\ -\epsilon_{j_{k+1},j_k} & \text{if} \quad p(j_{k+1}) = p(j_k)-1 \end{cases}$$

Then we have

$$\beta_{ij} = \epsilon_{ij} + \epsilon^*_{j,j_1} + \epsilon^*_{j_1,j_2} + ... + \epsilon^*_{j_{m-1},j_m}.$$

This is easy to check keeping in mind that b_i^{fr} is the image in $\mathfrak{t}^d$ of $\sum_j e_{ji} - e_{ij}$. One also sees in a similar manner that as a basis for $(\mathfrak{t}^Q)^*$ this is independent of the chosen path, i.e. that any closed path in λ kills $\mathfrak{t}^Q$. Similarly, let j be any box such that $p(j) = 0$. For convenience we choose $j = \mathbf{o}$, the origin. Choose a path $i = i_0, i_1, ..., i_m = \mathbf{o}$ as above. Then

$$\beta_i^{\mathrm{fr}} = \epsilon^*_{i,i_1} + \epsilon^*_{i_1,i_2} + ... + \epsilon^*_{i_{m-1},\mathbf{o}}$$

Applying the restriction rules (3.10) and using (3.9), we see that we also get a contribution of $-\hbar$ from every vertical step in the chosen paths.

Introduce the following notation for the relative arms and legs of two boxes i, j in λ:

$$a_{ij}^{\lambda} = j_x - i_x \qquad\qquad l_{ij}^{\lambda} = j_y - i_y$$

Restricting to A weights sends any ϵ_{ij} to t_1, and since $\hbar = t_1 + t_2$, we obtain:

$$\begin{aligned} u_{ij}|_{\tilde{\mathcal{I}}_\lambda} &= (1 + a_{ij}^{\lambda})t_1 + l_{ij}^{\lambda} t_2 = h_{\lambda}^{*}(i,j) \\ v_i|_{\tilde{\mathcal{I}}_\lambda} &= a_{i,\mathbf{o}}^{\lambda} t_1 + l_{i,\mathbf{o}}^{\lambda} t_2 \end{aligned} \tag{6.7}$$

These are the restrictions to dominant lifts; the other lifts are obtained by applying the W action. For $\sigma \in W$ we write $a_{ij}^{\sigma\lambda} = a_{\sigma_i,\sigma_j}^{\lambda}$ etc.

Stable basis of THilb

Taking $t_2 = -t_1$ and applying the results of the previous section for the defining hyperplanes of $\tilde{\mathcal{I}}_\lambda$ we obtain their A_0 weights. Since we only need their signs, we take $t_1 = -1$. These are given by

$$\begin{aligned} 1 + p(i) - p(j) \quad &\text{on} \quad b_{ij} \\ p(i) \quad &\text{on} \quad b_i^{\mathrm{fr}}. \end{aligned}$$

Note how these vanish precisely on the defining hyperplanes.

Example 6.3.5. *For $\lambda = (3,3)$, these weights are*

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -2 \\ 2 & 1 & 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 & 0 & -1 \\ 3 & 2 & 2 & 1 & 1 & 0 \\ 3 & 2 & 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 2 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Recall the star notation (3.4). We conclude from (3.3.5) that in the chamber $t_1 < 0$,

$$stab(\mathrm{THilb}, \lambda) = \prod_{1 \le i,j \le n} u_{ij}^{*(1+p(i)-p(j))} \prod_{1 \le i \le n} (u_i^{\mathrm{fr}})^{*(p(i))} \tag{6.8}$$

Roots

The last ingredient needed for our abelianization formula is the roots. From the discussion in (6.3.1), the root (i,j) is given by

$$a_{ij}t_1 + l_{ij}t_2.$$

6.3.3 Schur polynomials and Jack polynomials

Recall that in Hilb, the stable basis corresponds to Schur polynomials and the fixed-point basis corresponds to Jack polynomials. Let

$$N_\lambda = \prod_{\square \in \lambda} (t_2(l(\square)+1) - t_1 a(\square)) \cdot (t_2 l(\square) - t_1(a(\square)+1))$$

be the product of normal weights to Hilb_n at λ. Since $\mathbf{J}_\lambda$ are the classes of fixed points in $H_A^*(\mathrm{Hilb})$, the restriction formula will give us the coefficients of $\mathbf{J}_\lambda/N_\lambda$ in the expansion of Schur polynomials. Up to a constant factor, this is known as the

"C" normalization of Jack functions.

Plugging in the results of the previous section, our restriction formula (4.3.3) gives:

Theorem 6.3.6. *Let*

$$s_\lambda = \sum c_{\lambda,\nu} \mathbf{J}_\nu / N_\nu.$$

Then

$$c_{\lambda,\nu} = \sum_{\sigma \in W/W_\lambda} \frac{\prod_{i,j=1}^{n} \left((1+a_{ij}^{\sigma\nu})t_1 + l_{ij}^{\sigma\nu} t_2\right)^{*(1+a_{ij}^{\lambda} - l_{ij}^{\lambda})} \prod_{i=1}^{n} \left(a_{i,\mathbf{o}}^{\sigma\nu} t_1 + l_{i,\mathbf{o}}^{\sigma\nu} t_2\right)^{*(-a_{i,\mathbf{o}}^{\lambda} + l_{i,\mathbf{o}}^{\lambda})}}{\prod_{\substack{a_{ij}^{\lambda} \neq l_{ij}^{\lambda} \\ j<i}} \left(a_{ij}^{\sigma\nu} t_1 + l_{ij}^{\sigma\nu} t_2\right) \left((1+a_{ij}^{\sigma\nu})t_1 + (1+l_{ij}^{\sigma\nu})t_2\right)}$$

We remind the reader that terms whose numerator is 0 are discarded even if the denominator vanishes.

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