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A Birch and Swinnerton-Dyer formula for high-weight modular forms

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A BIRCH AND SWINNERTON-DYER FORMULA
FOR HIGH-WEIGHT MODULAR FORMS

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OF PRINCETON UNIVERSITY
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Abstract

The Birch and Swinnerton-Dyer conjecture – which is one of the seven million-dollar Clay Mathematics Institute Millennium Prize Problems – and its generalizations to modular forms, motives, etc. are a significant focus of current number theory research.

A 2017 article of Jetchev, Skinner and Wan proved a Birch and Swinnerton-Dyer formula at a prime p for certain rational elliptic curves of rank 1. I generalize and adapt that article’s arguments to prove an analogous formula for certain modular forms. For newforms f of even weight higher than 2 with Galois representation V_f containing a Galois-stable lattice T_f , let $W_f = V_f/T_f$ and let K be an imaginary quadratic field in which the prime p splits as $v_0\bar{v}_0$. My main result is that under some conditions, the p -index of the size of the Shafarevich-Tate group $\text{III}(K, W_f)$ is twice the p -index of a logarithm of the Abel-Jacobi map of a Heegner cycle defined by Bertolini, Darmon and Prasanna.

Significant original adaptations I make to the 2017 arguments are (1) a generalized version of a previous calculation of the size of the cokernel of the localization-modulo-torsion map $H_f^1(K, T_f) \rightarrow H_f^1(K_{v_0}, T_f)/\text{tor}$, and (2) a comparison of different Heegner cycles.

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Part A

Introduction, setup and background

Chapter 1

Introduction and outline

1.1 Introduction

A major theme in modern number-theoretic research is that analytic objects (like L -functions) can yield information about algebraic or geometric objects (like Galois characters and groups of rational points on elliptic curves). In this introductory section, I briefly summarize some of the history of this exploration; the reference [Be14] gives details associated with some of the material covered here. In the next section, I outline the contribution of my thesis to the subject.

Some early examples of L -functions were the Riemann zeta function $\zeta(s)$ (investigated for real $s > 1$ by Euler [Ne99, p. 431]) and the L -functions of Dirichlet, used to prove his theorem that there are infinitely many primes in any arithmetic progression of integers with common difference coprime to all the terms [Ne99, pp. 469-470]. These L -functions were generalized by Hecke to L -functions of so-called “*größencharaktere*” [Ne99, pp. 470, 480-483]; Tate’s thesis [Ta50] reformulated Hecke’s results adelically.

The genesis of p -adic L -functions (see [CoSu06, chapter 1] for details) came with Kummer’s criterion for testing whether primes are regular, and its refinement by Herbrand, of which the converse would later be shown by Ribet. Calculations in the

proofs of these results suggested that the Riemann zeta function had a p -adic version: for negative odd integers n_1 and n_2 , if $n_1 \equiv n_2 \not\equiv 1 \pmod{p-1}$, then $\zeta(n_1)$ and $\zeta(n_2)$ are in $\mathbb{Q} \cap \mathbb{Z}_p \subseteq \mathbb{Q}_p$ and $\zeta(n_1) \equiv \zeta(n_2) \pmod{p}$ [CoSu06, Theorem 1.1.3]. Kubota and Leopoldt constructed such a p -adic L -function, and Iwasawa showed that it could be seen as an element of (the ring of fractions of) an Iwasawa algebra, as part of his development of Iwasawa theory.

The main conjecture for cyclotomic fields relates the Kubota-Leopoldt p -adic L -function to the characteristic ideal of a module over the Iwasawa algebra. It is now a theorem, having received three different proofs: one by Mazur and Wiles [MaWi84] via modular curve arithmetic, one by Wiles [Wi90] adapting Ribet's proof of half of the Herbrand-Ribet theorem, and one by Rubin [Ru90] via Euler system arguments.

We now consider the case of elliptic curves (a reference for this material is Silverman [Si10, pp. 451-453]). Calculations (described in [Sw10]) of the numbers of points on rational elliptic curves over $\mathbb{F}_\ell$ for primes ℓ led to the Birch and Swinnerton-Dyer (BSD) conjecture:

Conjecture 1.1.1 (BSD). *Suppose an elliptic curve $E/\mathbb{Q}$ is given. Let the analytic rank of E be the order of the zero of $L(E, s)$ at $s = 1$. Then the analytic rank of E equals the (algebraic) rank of the finitely generated abelian group $E(\mathbb{Q})$, and*

$$\frac{1}{R(E/\mathbb{Q}) \cdot \Omega_E} \cdot \lim_{s \rightarrow 1} \frac{L(E, s)}{(s-1)^{\text{rank } E(\mathbb{Q})}} = \frac{\#\text{III}(E/\mathbb{Q}) \cdot \prod_{\ell \mid \infty} c_\ell(E/\mathbb{Q})}{(\#E(\mathbb{Q})_{\text{tor}})^2}$$

where $R(E/\mathbb{Q})$ is the regulator of E over $\mathbb{Q}$ (using the height pairing that is twice the pairing in [Si10]), the period Ω_E is given by $\Omega_E = \int_{E(\mathbb{R})} |\omega|$ for the differential form ω on E as in [Si10], $\text{III}(E/\mathbb{Q})$ is the Shafarevich-Tate group of E over $\mathbb{Q}$, and for finite rational primes ℓ , the Tamagawa number $c_\ell(E/\mathbb{Q})$ is the index in $E(\mathbb{Q}_\ell)$ of the subgroup of points with nonsingular reduction modulo ℓ .

So far, the main progress has been for analytic and algebraic rank 0 and rank 1

cases of the BSD conjecture. I recall the following results.

- (a) Rank 0: For CM elliptic curves $E/\mathbb{Q}$, Coates and Wiles proved that if $L(E, 1) \neq 0$, then $E(\mathbb{Q})$ is finite; that is, analytic rank 0 implies algebraic rank 0.
- (b) Rank 1: For (modular) elliptic curves $E/\mathbb{Q}$ with analytic rank at least 1, the Gross-Zagier formula [GrZa86] writes $L'(E, 1)$ in terms of the height pairing of a Heegner point, and implies that $L'(E, 1) \neq 0$ if and only if the Heegner point is nontorsion; so analytic rank 1 implies algebraic rank at least 1.
- (c) Rank 1: In the late 1980s and early 1990s, Kolyvagin discovered the technique of Euler systems and proved that for general (modular) elliptic curves $E/\mathbb{Q}$, if the analytic rank is at most 1, then it is equal to the algebraic rank and the Shafarevich-Tate group of E is finite.
- (d) Rank 0: The work of Kato [Ka04, Theorem 18.4] uses Euler systems to prove, among other things, that if the “rank” of the p -adic L-function of a general elliptic curve $E/\mathbb{Q}$ is 0, then $E(\mathbb{Q})$ is finite; that is, p -adic analytic rank 0 implies algebraic rank 0.

Although (b) and (c) were proved for modular elliptic curves at first, the modularity assumption can be removed because of the Modularity Theorem – proved by Wiles [Wi95], Taylor and Wiles [TaWi95] and the later papers [Di96, CoDiTa99, Br01] of Breuil, Conrad, Diamond and Taylor – that any elliptic curve $E/\mathbb{Q}$ is modular.

Using some of the advances described above, as well as the main conjecture for GL_2 proved by Skinner and Urban [SkUr14], a paper of Bhargava, Skinner and Zhang [BhSkZh14] has proved the BSD conjecture for more than 66% of elliptic curves over $\mathbb{Q}$ arranged by height.

The BSD conjecture has a version for modular forms [Ka04, p. 280]:

Conjecture 1.1.2 (BSD for modular forms). *Suppose a newform $f \in S_k(\Gamma_1(N), \mathbb{C})$ of weight $k \geq 2$ with Fourier coefficient $a(1, f) = 1$ is given. Let the analytic rank*

of f be the order of the zero of $L(f, s)$ at $s = k/2$, and let the algebraic rank of f be the corank of the Selmer group of a Galois-stable lattice in the standard Galois representation associated to f . Then the analytic and algebraic ranks of f are equal.

These BSD conjectures are thought to be part of a larger framework: certain values of L-functions of motives are potentially described by the Bloch-Kato conjectures [BlKa90] and the Beilinson conjectures [Be85, De79].

1.2 Main result and outline of proof

In the recent paper [JeSkWa17] of Jetchev, Skinner and Wan, the following result towards the BSD conjecture was proved; in essence, it is the part of the BSD formula at a prime p .

Theorem 1.2.1. *[JeSkWa17, Theorem 1.2.1] Assume that*

- (a) *the elliptic curve $E/\mathbb{Q}$ is semistable,*
- (b) *the rational prime p is odd and does not divide the conductor of E ,*
- (c) *the Galois representation $E[p]$ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{F}_p$ is irreducible,*
- (d) *E has analytic rank 1 (so, by the Gross-Zagier formula and the results of Kolyvagin, E also has algebraic rank 1 and $\text{III}(E/\mathbb{Q})$ has finite cardinality), and*
- (e) *if E has supersingular reduction at p , then $a_p(E) = 0$.*

Then

$$\text{ind}_p \left(\frac{L'(E, 1)}{R(E/\mathbb{Q})\Omega_E} \right) = \text{ind}_p \left(\# \text{III}(E/\mathbb{Q}) \cdot \prod_{\ell} c_{\ell}(E/\mathbb{Q}) \right).$$

The broad structure of the proof went as follows. For suitable auxiliary imaginary quadratic fields K' and K'' , the following results were obtained. I write $\text{ind}_p x$ for the p -index of x , that is, the number of factors of p dividing x ; for example, $\text{ind}_p(p^n) = n$ for $n \in \mathbb{Z}$.

- (a) A theorem obtained from Brooks, linked to work of Bertolini, Darmon and Prasanna [JeSkWa17, Proposition 5.1.7]: For a certain Heegner point $z_{K'} \in E(K')$, a certain differential form ω_E on E , and a certain L-function L_{BDP} of Bertolini, Darmon and Prasanna, we have

$$2\text{ind}_p \log_{\omega_E}(z_{K'}) + 2\text{ind}_p((1 - a_p(E) + p)/p) = \text{ind}_p L_{BDP}(1).$$

- (b) Interpolating and comparing L -functions [JeSkWa17, Corollary 5.3.2]: For a certain L-function L_{Wan} of Wan, we have $\text{ind}_p L_{BDP}(1) = \text{ind}_p L_{Wan}(1)$.
- (c) Iwasawa theory [JeSkWa17, Proposition 6.2.1], relying on a result of Wan that is half of an Iwasawa main conjecture: For a certain cohomology-related quantity $C(E[p^\infty])$, we have

$$\text{ind}_p L_{Wan}(1) \leq \text{ind}_p(C(E[p^\infty]) \# H_{ac}^1(K', E[p^\infty])).$$

- (d) Galois cohomology [JeSkWa17, (3.5d)]: We have

$$\begin{aligned} & \text{ind}_p(C(E[p^\infty]) \# H_{ac}^1(K', E[p^\infty])) \\ &= \text{ind}_p(\#\text{III}(E/K')) + 2\text{ind}_p \log_{\omega_E}(z_{K'}) + 2\text{ind}_p((1 - a_p(E) + p)/p) \\ & \quad - 2\text{ind}_p(E(K') : \mathbb{Z}z_{K'}) + (p\text{-indices of Tamagawa factors}). \end{aligned}$$

Points (a) to (d) yield

$$2\text{ind}_p(E(K') : \mathbb{Z}z_{K'}) - (p\text{-indices of Tamagawa factors}) \leq \text{ind}_p(\#\text{III}(E/K')[p^\infty]).$$

- (e) Euler systems [JeSkWa17, Theorem 4.4.1], relying on a result of Nekovář: We

have

$$\mathrm{ind}_p(\#\mathrm{III}(E/K'')[p^\infty]) \leq 2\mathrm{ind}_p(E(K'') : \mathbb{Z}z_{K''}).$$

Applying Gross-Zagier formulas for the Heegner points $z_{K'}$ and $z_{K''}$, re-writing the Shafarevich-Tate groups $\mathrm{III}(E/K')$ and $\mathrm{III}(E/K'')$ in terms of $\mathrm{III}(E/\mathbb{Q})$ and III of quadratic twists of E , and applying a previously known rank 0 case of the BSD conjecture produced Theorem 1.2.1.

In this thesis, I replace the elliptic curve E with a modular form f of weight larger than 2. Adapting the arguments used in [JeSkWa17], I show the following theorem. For a modular form $f = \sum_{n=1}^{\infty} a(n, f)q^n$, let the $\mathbb{Z}$ -algebra generated by the Fourier coefficients $a(n, f)$ of f be $O(f)$, and let the field generated by the coefficients $a(n, f)$ be $\mathbb{Q}(f)$ with ring of integers $O_{\mathbb{Q}(f)}$. Definitions are given in section 6.2 for the Heegner cycle Δ_{Ma17} , section 6.3 for $\mathrm{ind}_p(c, B)$, section 6.4 for Galois representations, and section 6.7 for Abel-Jacobi maps AJ with various superscripts and subscripts.

Theorem 1.2.2. *Assume the following.*

- *A prime p is fixed.*
- *$L/\mathbb{Q}_p$ is an unramified finite-degree field extension, with ring of integers O_L ; a uniformizer ϖ_L of L is fixed. (There should be no confusion between this L and the L of L -functions; the context will determine which is which.)*
- *$f \in S_k(\Gamma_0(N))$ is a non-CM newform of conductor N with $a(1, f) = 1$ such that $N \geq 5$ is an odd integer, $k > 2$ is an even integer, $k/2$ is not congruent to 0 or 1 modulo $p-1$, and $\mathbb{Q}(f) \subseteq L$. The O_L -module T_f is an O_L -lattice in the Galois representation V_f of f , with $V_f = T_f \otimes_{O_L} L$, and I define $W_f = T_f \otimes_{O_L} (L/O_L)$.*
- *The imaginary quadratic field extension $K/\mathbb{Q}$ has discriminant $D_K \equiv 1 \pmod{4}$ and the Hilbert class field K_1 of K is contained in L . Let $h_K = [K_1 : K]$ be the class number of K .*

- $T_f/\varpi_L T_f$ is an irreducible G_K -representation of dimension at least 2.
- The following Heegner hypothesis holds: each prime factor of N splits or ramifies in K , at least one prime factor of N ramifies in K , and every prime $q \mid N$ ramifying in K is such that $q^2 \nmid N$. This means that there is an ideal $\mathfrak{C}$ of O_K for which the standard map $\mathbb{Z} \hookrightarrow O_K$ induces an isomorphism $\mathbb{Z}/N\mathbb{Z} \cong O_K/\mathfrak{C}$; fix such an ideal $\mathfrak{C}$.
- The prime p splits in K as $p = v_0 \bar{v}_0$.
- For some representatives $\mathfrak{a}$ of the class group of K , the norms $N(\mathfrak{a})$ are p -adic units when viewed as elements of $\widehat{\mathbb{Q}}_p$ via a fixed isomorphism $\widehat{\mathbb{Q}}_p \cong \mathbb{C}$.
- We have $p \geq k/2$, the Fourier coefficient $a(p, f)$ is a p -adic unit, the prime p does not divide $(k-2)! \cdot 6N\phi(N)D_K h_K \cdot (O_{\mathbb{Q}(f)} : O(f))$, and if $g \in GL_2(O_{\mathbb{Q}(f)} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ and $\det g$ is a $(k-1)$ th power in $\mathbb{Z}_p^\times$, then g is in the image of the representation $\rho_{f,p} : G_{\mathbb{Q}} \rightarrow GL_2(O_{\mathbb{Q}(f)} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ corresponding to T_p . (This excludes only finitely many primes p for any given f and K .)
- The L -function-related constant $C_{W_{an}}$ from subsection 5.2.3 is a p -adic unit.
- The cohomology class $C_0 = \text{cor}_{K_1/K} A J_{K_1}(\Delta_{Ma17}) \in H_f^1(K, T_f)$ has a non-torsion image in $H_f^1(K_{v_0}, T_f)$ under localization.
- The prime p is coprime to the product of the two fractions

$$\frac{\#H^0(K_{\bar{v}_0}, W_f) \prod_{v \in S'} (\#H_{ur}^1(K_v, W_f))}{\#H^0(K_{v_0}, W_f)}$$

and

$$\frac{(O_L : O_L \log_{\omega}(\text{loc}_{v_0} C_0))^2 (H_f^1(K_{v_0}, T_f) : \exp_{\omega}(pO_L))^2}{p^{2[L:\mathbb{Q}_p]-k} (H_f^1(K, T_f) : O_L C_0)^2}.$$

Here, S' is the set of finite places v of K such that $v \nmid p$, V_f is ramified at v , and v is above a rational prime that splits in K ; the differential form ω and its associated maps $\log_\omega$ and $\exp_\omega$ are described in Theorem 9.3.1 and section 6.9.

- For each place $v \mid p$ of K , the O_L -module $H_f^1(K_v, W_f)$ has corank 1, and the localization map $H_f^1(K, W_f) \rightarrow H_f^1(K_v, W_f)$ restricts to a map

$$(\mathrm{im} \ AJ_K) \otimes_{O_L} (L/O_L) \rightarrow (\mathrm{im} \ AJ_{K_v}) \otimes_{O_L} (L/O_L)$$

of which the kernel is torsion.

- We have $\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} \ AJ_{K_1}) = \mathrm{ind}_p(AJ_{K_1}(\Delta_{Ma17}), \mathrm{im} \ AJ_{K_1})$.
- Let $x_1 \in X_1(N)$ be a point whose projection to $X_0(N)$ corresponds to the isogeny $\mathbb{C}/O_K \rightarrow \mathbb{C}/\mathfrak{C}^{-1}$ via CM theory; then when the $h_K \cdot \phi(N)$ elements of $\mathrm{Gal}(K_1/K) \times (\Gamma_0(N)/\Gamma_1(N))$ act on the point of $X_1(N)$ associated to E_{x_1} , the corresponding $h_K \cdot \phi(N)$ images of that point are distinct.
- The map

$$\log_{\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1}, L}^{1, E_{x_1}, \mathbb{Z}_p} : H_f^1(L, J_{BeDaPr}^{\mathbb{Z}_p}) \rightarrow L$$

described in (6.4) has image in O_L .

Then we have

$$2\mathrm{ind}_p AJ_L^{1, E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) = \mathrm{ind}_p \# \mathrm{III}(K, W_f).$$

Like the intermediate results of Jetchev, Skinner and Wan mentioned above, my result links a Shafarevich-Tate group to a function of a Heegner cycle; the Abel-Jacobi map replaces the logarithm.

First, in chapter 2 I go through basic definitions and details of the setup. In chapters 3 to 6 I review some background on class field theory, modular forms, L-functions,

algebraic geometry and cohomology. Finally, chapters 7 to 12 prove Theorem 1.2.2. My argument is of the form “ $x_1 \leq x_2 \leq \cdots \leq x_6 \leq x_1$, so all x_i are equal”, and I allocate one chapter to each of the circular chain’s six inequalities except one. (Some of the inequalities are known from the start to be equalities.) In detail, the six parts are as follows; note the similarity with the proof of Jetchev, Skinner and Wan [JeSkWa17]. The notation will be explained in later chapters of the thesis; here, I simply note what is linked to what in each part, and I give the relevant formulas.

First, a formula of Bertolini, Darmon and Prasanna [BeDaPr13] links the logarithm of the Abel-Jacobi map of a Heegner cycle to a p -adic L -function:

$$2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1}) = k + \mathrm{ind}_p P_1(L_{BDP}^{-,N_p}).$$

Second, p -adic L -functions are interpolated and compared:

$$\mathrm{ind}_p P_1(L_{BDP}^{-,N_p}) = \mathrm{ind}_p P_1(L_{Wan}^{-,N_p}).$$

Third, half of an Iwasawa main conjecture links a p -adic L -function to Galois cohomology:

$$\mathrm{ind}_p P_1(L_{Wan}^{-,N_p}) \leq \mathrm{ind}_p (C(W_f) \# H_{ac}^1(K, W_f)).$$

Fourth, Galois cohomology is linked to the Shafarevich-Tate group of f :

$$\#H_{ac}^1(K, W_f) = \#\mathrm{III}(K, W_f) \cdot (\#\delta_{v_0})^2.$$

Fifth, an Euler-system-related result links Sha to the Abel-Jacobi image of a Heegner cycle:

$$\mathrm{ind}_p(\#\mathrm{III}(K, W_f)) \leq 2\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}).$$

Sixth, Heegner cycles are compared, and we return to the starting point:

$$2\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}) \leq 2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13})(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1}).$$

Combining these six results yields an inequality

$$\begin{aligned} & 2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) \\ & \leq 2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) + \mathrm{ind}_p(\text{a } p\text{-adic unit}) \end{aligned}$$

so that there is equality in all six parts – the last two parts in particular. This yields

$$2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) = \mathrm{ind}_p \# \mathrm{III}(K, W_f)$$

which is Theorem 1.2.2.

Chapter 2

Setup and notation

2.1 General notation

For $n \in \mathbb{Z}_{>0}$, S_n is the symmetric group of permutations of $\{1, 2, \dots, n\}$, and the elements of S_n are bijections $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$.

Let G_{tor} be the torsion subgroup of an abelian group G ; I sometimes write $G/\text{tor} := G/G_{\text{tor}}$ for brevity. For a $\mathbb{Z}_p$ -module M , let M_{div} be the maximal p -divisible subgroup of M .

For a function F whose domain is a quotient group G/H or $H \backslash G$ (for a normal subgroup H of a group G), I sometimes abuse notation by confusing F with the function obtained from F by precomposing with the projection $G \twoheadrightarrow G/H$ or $G \twoheadrightarrow H \backslash G$. For $g \in G$, I sometimes write $F(g)$ for the value of F at the coset gH or Hg . If the quotient group G/H has finite cardinality – that is, if the normal subgroup H of G has finite index in G – then that index is denoted $(G : H)$.

For fields F_1 and F_2 with $F_1 \supseteq F_2$, I write $[F_1 : F_2]$ for the F_2 -vector-space dimension of F_1 .

2.2 Number fields and characters

In this section, let F be a number field. The ring of integers of F is written O_F . For a finite place v of F , the v -adic completion F_v has ring of integers $O_{F,v}$ (explicitly, $O_{F,v} := \{x \in F_v : |x|_v \leq 1\}$) and, in a standard way, I abuse notation by denoting the maximal ideal of $O_{F,v}$, as well as that ideal's intersection with O_F , as v . The adèles, finite adèles, ideles and finite ideles over F are written $\mathbb{A}_F$ (as in Neukirch [Ne99, section VI.1]), $\mathbb{A}_{F,f}$, $\mathbb{A}_F^\times$, $\mathbb{A}_{F,f}^\times$ respectively. An element of any one of these four spaces is written $z = (z_v)_v$, with $z_v \in F_v$ the component of the adèle z at the (finite or infinite) place v of F . We have the embedding $\mathbb{A}_{F,f}^\times \hookrightarrow \mathbb{A}_F^\times$ which appends 1s at the infinite places of F . We have the embeddings $F_v^\times \hookrightarrow \mathbb{A}_{F,f}^\times$ for finite places v , and $F_v^\times \hookrightarrow \mathbb{A}_F^\times$ for all places v , where each embedding appends 1s at places other than v . For an adèle or finite adèle $z = (z_v)_v$ and a product M of powers of finite primes of F , write $z_M = (z_v)_{v|M} \in \prod_{v|M} F_v =: F_M$. For $x \in F^\times$, write $x_M \in \mathbb{A}_{F,f}^\times$ for the finite idele with component x at the places $v \mid M$ and component 1 at all other places. For $x \in F^\times$, let $x_f \in \mathbb{A}_{F,f}^\times$ be the finite idele with component x at all finite places v of F . For a finite idele $z \in \mathbb{A}_{F,f}^\times$ with $z_v \in v^{a(v)} O_{F,v}^\times$ and $a(v) \in \mathbb{Z}$ for each finite prime v of F , define the fractional ideal $zO_F = \prod_{v \nmid \infty} v^{a(v)}$ of F .

Consider a character (that is, a continuous group homomorphism) $\eta : F_v^\times \rightarrow \mathbb{C}^\times$, where v is a finite place of F . Consider all the nonzero ideals C of $O_{F,v}$ such that

$$\text{If } x \in O_{F,v}^\times \text{ satisfies } x \equiv 1 \pmod{C}, \text{ then } \eta(x) = 1. \quad (2.1)$$

Among those ideals, the largest is called the conductor $C(\eta)$ of η . (By a no-small-subgroups argument, a nonzero ideal C satisfying (2.1) always exists: the subgroups $1 + v^n O_{F,v}$ of $O_{F,v}^\times$ (where $1 \leq n \in \mathbb{Z}$) form a basis of open neighborhoods of 1 in $F_v^\times$, so the inverse image under η of the open neighborhood $U = \{z \in \mathbb{C} : |z - 1| < 1/2\}$ of 1 in $\mathbb{C}$ contains some subgroup $1 + v^n O_{F,v}$, whose image under η is a subgroup of $\mathbb{C}^\times$

contained in U , which must be $\{1\}$.) In particular, if η is the trivial character whose image is $\{1\}$, then $C(\eta) = O_{F,v}$. The character η is called unitary when its image is contained in $\{z \in \mathbb{C}^\times : |z| = 1\}$.

For a Hecke character $\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$, let χ_v be the restriction of $\chi : \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ to $F_v^\times$. Consider all the nonzero ideals C of O_F such that for each finite place v of F , we have $C(\chi_v) \supseteq CO_{F,v}$; among those ideals, the largest is called the conductor $C(\chi)$ of χ . (This exists by another no-small-subgroups argument, using the basis of neighborhoods of 1 in $\mathbb{A}_{F,f}^\times$ consisting of products $\prod_{v \nmid \infty} U_v$ where $U_v = O_{F,v}^\times$ for all but finitely many v , and where the other U_v are of the form $U_v = 1 + v^{a(v)}O_{F,v}$ for $1 \leq a(v) \in \mathbb{Z}$.) For a fractional ideal $\mathfrak{a} = \prod_{v \nmid \infty} v^{a(v)}$ of F with each $a(v) \in \mathbb{Z}$, if $\mathfrak{a}$ is coprime to $C(\chi)$, then I write $\chi(\mathfrak{a})$ for the value of χ at an idele $z \in \mathbb{A}_{F,f}^\times$ with $zO_F = \mathfrak{a}$ and $z_{C(\chi)} = 1_{C(\chi)}$. In particular, for $a \in F^\times$ coprime to $C(\chi)$, we have $\chi(aO_F) = \chi(z)$ for an idele $z = (z_v)_v \in \mathbb{A}_{F,f}^\times$ such that $z_{C(\chi)} = 1_{C(\chi)}$ and, for $v \nmid C(\chi)\infty$, $z_v = a$.

For a number field F , as in [BeDaPr13], let $N : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ be the Hecke character of conductor O_F such that for fractional ideals $\mathfrak{a}$ of F , the number $N(\mathfrak{a})$ is the positive element of $\mathbb{Q}$ that generates the fractional ideal $N_{F/\mathbb{Q}}\mathfrak{a}$ of $\mathbb{Q}$ as a $\mathbb{Z}$ -module. In particular, for an integral ideal $\mathfrak{a}$ of F , we have $N(\mathfrak{a}) = (O_F : \mathfrak{a})$, and for a nonzero principal fractional ideal aO_K of an imaginary quadratic field K , we have $N(aO_K) = a\bar{a}$.

2.3 Galois groups

I follow the notation of [SkUr14, subsection 2.1.2]. The Galois group of a number field F is denoted $G_F = \text{Gal}(\bar{F}/F)$. For a set S of finitely many finite places of F , let F^S/F be the maximum extension of F that is unramified at every finite place outside S , and define $G_{F,S} = \text{Gal}(F^S/F)$. For a finite place v of F , a decomposition group

and inertia group are respectively written as $G_{F,v}$ and $I_{F,v}$, or simply as G_v and I_v if F is understood.

2.4 The prime p and p -adic numbers

Fix a prime $p \geq 5$. Let $\widehat{\overline{\mathbb{Q}}}_p$ be the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and fix a field isomorphism $\widehat{\overline{\mathbb{Q}}}_p \cong \mathbb{C}$. Write $\text{ind}_p : \widehat{\overline{\mathbb{Q}}}_p^\times \rightarrow \mathbb{Q}_{>0}$ for the multiplicative p -adic valuation with $\text{ind}_p(p^n) = n$ for $n \in \mathbb{Z}$.

Let $L/\mathbb{Q}_p$ be an unramified finite-degree field extension with a fixed uniformizer ϖ_L and ring of integers O_L (that is, O_L is the set of $x \in L$ such that $x = 0$ or $\text{ind}_p(x) \geq 0$). Let $\widehat{O}_L^{ur}$ be the ring of integers of the completion $\widehat{L^{ur}}$ of the maximal unramified extension L^{ur} of L .

2.5 The modular form f

Let $f = \sum_{n=1}^{\infty} a(n, f)q^n \in S_k^{\text{new}}(\Gamma_0(N))$ be a non-CM newform with $a(1, f) = 1$, where $k > 2$ is an even integer, $N \geq 5$ is odd and $p \nmid (k-2)! \cdot N\phi(N)$. Let $\mathbb{Q}(f) = \mathbb{Q}(a(n, f) : n \in \mathbb{Z}_{>0})$ be the number field generated over $\mathbb{Q}$ by all the $a(n, f)$, and let $O(f) = \mathbb{Z}[a(n, f) : n \in \mathbb{Z}_{>0}]$ be the ring generated as a $\mathbb{Z}$ -algebra by all the $a(n, f)$. Assume that $p \nmid (O_{\mathbb{Q}(f)} : O(f))$. Fix a uniformizer $\varpi_{\mathbb{Q}(f)}$ for the completion of $\mathbb{Q}(f)$ at the prime above the rational prime p . Assume $\mathbb{Q}(f) \subseteq L$ (which implies that the rational prime p does not ramify in $\mathbb{Q}(f)$).

2.6 The imaginary quadratic field K

Let $D_K < -3$ be a square-free integer such that $D_K \equiv 1 \pmod{4}$. Let $K = \mathbb{Q}(\sqrt{D_K})$ be the imaginary quadratic number field with discriminant D_K ; note that $O_K^\times = \{\pm 1\}$. Let the Hilbert class field, class group and class number of K be respectively K_1 , Cl_K

and $h_K = [K_1 : K] = \#Cl_K$. Assume $K_1 \subseteq L$.

Assume that $p \nmid D_K h_K$ and that p splits in K as $p = v_0 \overline{v_0}$. Define the set $N_p = \{v_0, \overline{v_0}\}$. The valuation corresponding to v_0 yields an embedding $K \hookrightarrow \overline{\mathbb{Q}_p} \subseteq \widehat{\overline{\mathbb{Q}_p}}$; composing that map with the isomorphism $\widehat{\overline{\mathbb{Q}_p}} \cong \mathbb{C}$ results in an embedding $K \hookrightarrow \mathbb{C}$.

Following Hida and Tilouine [HiTi93, p. 192], for a Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$, define the Hecke character $\chi^* : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ so that $\chi^*(z) = |z|_{\mathbb{A}, K} \cdot \chi(z^c)^{-1}$.

2.7 The Heegner hypothesis

Assume the Heegner hypothesis for f and K consisting of the following three assumptions.

- (a) Each rational prime factor of N splits or ramifies in K ,
- (b) At least one rational prime factor of N ramifies in K , and
- (c) If a rational prime $q \mid N$ is such that q ramifies in K , then $q^2 \nmid N$.

Under these assumptions, there is an ideal $\mathfrak{C} \subseteq O_K$ such that the inclusion $\mathbb{Z} \hookrightarrow O_K$ induces an isomorphism $\mathbb{Z}/N\mathbb{Z} \cong O_K/\mathfrak{C}$. We can and do take $\mathfrak{C}$ so that $N\mathbb{Z} = \mathfrak{C}\mathfrak{C}^c$.

Chapter 3

Class field theory

In this chapter, I review standard material on class field theory and Hecke characters.

I use and adapt notation from [Hi88], [Hi91] and [SkUr14].

3.1 Class field theory and Galois extensions

For $M \in \mathbb{Z}_{>0}$, write

$$Z(M) = \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_{\mathbb{Q},f}^\times / U_{\mathbb{Q}}(Mp^\infty) \cong \widehat{\mathbb{Z}}^\times / U_{\mathbb{Q}}(Mp^\infty)$$

where

$$U_{\mathbb{Q}}(Mp^\infty) = \{z \in \widehat{\mathbb{Z}}^\times : z_p = 1, z_\ell \in 1 + M\mathbb{Z}_\ell \text{ for finite } \ell \neq p\}.$$

Note that $Z(M)$ is the ray class group modulo Mp^∞ over $\mathbb{Q}$.

For $p \nmid M$, we identify $\mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times \cong Z(M)$ by sending $(y_p, y_M \bmod M\mathbb{Z})$ to the class of a finite idele $z \in \mathbb{A}_{\mathbb{Q},f}^\times$ with $z_p = y_p$, $z_M \in \prod_{\ell|M} \mathbb{Z}_\ell^\times$, $z_M \equiv y_M \bmod M$ (interpreting via the Chinese remainder theorem), and $z_\ell = 1$ for finite $\ell \nmid Mp$.

For all $M \in \mathbb{Z}_{>0}$, we have the cyclotomic character $\epsilon : Z(M) \rightarrow \mathbb{Z}_p^\times$ such that if $z \in r\widehat{\mathbb{Z}}^\times \subseteq \mathbb{A}_{\mathbb{Q},f}^\times$ for $r = |z|_{\mathbb{A},\mathbb{Q}}^{-1} \in \mathbb{Q}_{>0}^\times$, then $\epsilon(z) = z_p/r \in \mathbb{Q}_p^\times$. Geometrically normalized Artin reciprocity identifies ϵ with the Galois character describing the

Galois action on roots of unity with order a power of p [SkUr14, section 2.2.3]. For a p -adic Galois representation U and an integer n , let $U(\epsilon^n) := U \otimes \epsilon^n$ be U twisted by the n th power of the cyclotomic character ϵ . (This notation is analogous to that of [SkUr14, Proposition 3.4]. The representation $U(\epsilon^n)$ is often written $U(n)$ in the literature, but since the cyclotomic character appears quite often in this paper, I keep the notation uniform by using ϵ throughout. This also makes clear which choice of normalization for the cyclotomic character I am using: ϵ , not Hida's character $\mathcal{N}$ which is my ϵ^{-1} [Hi91, p. 324].)

There is the classical Teichmüller character $\omega : \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times$ such that for $y \in \mathbb{Z}_p^\times$, we have $\omega(y)^p = \omega(y) \equiv y \pmod{p\mathbb{Z}_p}$. The Teichmüller character's image is the set of roots of $x^{p-1} - 1$ in $\mathbb{Z}_p^\times$. Define a Teichmüller character $\omega : Z(M) \rightarrow \mathbb{Z}_p^\times$, with the same image as the previous character ω , so that

- (a) If the embedding $\mathbb{Z}_p^\times \hookrightarrow \mathbb{A}_{\mathbb{Q},f}^\times$ sends $y \in \mathbb{Z}_p^\times$ to $y_p \in \mathbb{A}_{\mathbb{Q},f}^\times$ in the equivalence class $[y_p] \in Z(M)$, then $\omega([y_p]) = \omega(y)$ (so $\epsilon([y_p]) \equiv \omega(y) \pmod{p\mathbb{Z}_p}$); and
- (b) Each element $(1, y_M \pmod{M\mathbb{Z}}) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$ corresponds to an element of $Z(M)$ in the kernel of $\omega : Z(M) \rightarrow \mathbb{Z}_p^\times$.

(Under geometrically normalized reciprocity, this ω corresponds to the inverse of the character denoted ω in both [SkUr14, section 2.2.4] and [Wa18, Theorems 1.1-1.2].)

Recall that we have fixed an imaginary quadratic field K in which the rational prime $p \geq 5$ splits as $p = v_0 \bar{v}_0$. The embeddings $\mathbb{Z}_p \hookrightarrow O_{K,v_0}$ and $\mathbb{Z}_p \hookrightarrow O_{K,\bar{v}_0}$ are isomorphisms. Let the Teichmüller characters $\omega_{v_0} : O_{K,v_0}^\times \rightarrow \mathbb{Z}_p^\times$ and $\omega_{\bar{v}_0} : O_{K,\bar{v}_0}^\times \rightarrow \mathbb{Z}_p^\times$ send the embeddings of $y \in \mathbb{Z}_p$ in respectively O_{K,v_0} and $O_{K,\bar{v}_0}$ to $\omega(y)$.

For a nonzero ideal $\mathfrak{a}$ of O_K , write

$$Z_K(\mathfrak{a}) = K^\times \backslash \mathbb{A}_{K,f}^\times / U_K(\mathfrak{a}p^\infty) \cong K^\times \backslash \mathbb{A}_K^\times / (U_K(\mathfrak{a}p^\infty) K_\infty^\times)$$

where

$$U_K(\mathfrak{a}p^\infty) = \{z \in \widehat{O}_K^\times : z_{v_0} = z_{\bar{v}_0} = 1, z_v \in 1 + \mathfrak{a}O_{K,v} \text{ for finite } v \nmid p\}.$$

Note that $Z_K(\mathfrak{a})$ is the ray class group modulo $\mathfrak{a}p^\infty$ over K . (The group $Z_K(\mathfrak{C})$ is written $G_\infty(\mathfrak{C})$ in [HiTi93]; here the notation is aligned with $Z(M)$ as in [Hi91].)

If $\mathfrak{a}$ is relatively prime to $p = v_0\bar{v}_0$ and $(O_K : \mathfrak{a}) > 1$ is odd, then we have a group isomorphism

$$O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times Cl_K \times ((O_K/\mathfrak{a})^\times / \{\pm 1\}) \cong Z_K(\mathfrak{a}) \quad (3.1)$$

similar to $\mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times \cong Z(M)$: send each element of the left hand side to the class of an element $z \in \mathbb{A}_{K,f}^\times$ with the same v_0 and $\bar{v}_0$ components, the same $K_{\mathfrak{a}}$ component modulo $\mathfrak{a}$, and the same image in $Cl_K \cong (\text{image of } K^\times) \backslash (\oplus_{v \nmid \infty} v^\mathbb{Z})$. (Proof: Because of the direct product

$$\widehat{O}_K^\times \cong ((\Pi_{v \nmid p\mathfrak{a}\infty} O_{K,v}^\times) \times (\Pi_{v|\mathfrak{a}}(1 + \mathfrak{a}O_{K,v}))) \times (O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times (O_K/\mathfrak{a})^\times)$$

we obtain

$$\begin{aligned} & K^\times \backslash \mathbb{A}_K^\times / (U_K(\mathfrak{a}p^\infty) K_\infty^\times) \\ & \cong K^\times \backslash ((\oplus_{v \nmid \infty} v^\mathbb{Z}) \times \widehat{O}_K^\times \times K_\infty^\times) / ((\Pi_{v \nmid p\mathfrak{a}\infty} O_{K,v}^\times) \times (\Pi_{v|\mathfrak{a}}(1 + \mathfrak{a}O_{K,v}))) \times K_\infty^\times \\ & \cong K^\times \backslash ((\oplus_{v \nmid \infty} v^\mathbb{Z}) \times (O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times (O_K/\mathfrak{a})^\times)) \\ & \cong Cl_K \times (O_K^\times \backslash (O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times (O_K/\mathfrak{a})^\times)) \\ & \cong Cl_K \times O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times ((O_K/\mathfrak{a})^\times / \{\pm 1\}) \end{aligned}$$

since $O_K^\times = \{\pm 1\}$.)

Given a multiplicative injection $(\mathbb{Z}/M\mathbb{Z})^\times \hookrightarrow (O_K/\mathfrak{a})^\times$, there is an embedding

$$\frac{Z(M)}{\{\pm 1\}} \cong \mathbb{Z}_p^\times \times \frac{(\mathbb{Z}/M\mathbb{Z})^\times}{\{\pm 1\}} \hookrightarrow (O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times Cl_K) \times \frac{(O_K/\mathfrak{a})^\times}{\{\pm 1\}} \cong Z_K(\mathfrak{a})$$

that is the direct product of the map

$$\begin{aligned} \mathbb{Z}_p^\times &\rightarrow O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times Cl_K \\ y &\mapsto (y, y, 1) \end{aligned}$$

and the map $(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\} \rightarrow (O_K/\mathfrak{a})^\times/\{\pm 1\}$ induced from the given injection. See also [HiTi93, p. 239].

I use and adapt notation from [SkUr14, section 2.1]. Let K^{ab}/K be the maximum abelian extension of fields over K ; then $(K^\times \backslash \mathbb{A}_K^\times)^\wedge \cong \text{Gal}(K^{ab}/K)$, where the left hand side is the profinite completion of $K^\times \backslash \mathbb{A}_K^\times$. Let this reciprocity isomorphism identify a uniformizer of the maximal ideal of K_v (viewed as an idele modulo $K^\times$) with a geometric Frobenius of the decomposition group at v , for each finite place v of K . From now on in this section, I take subquotients of that geometrically normalized isomorphism to obtain all isomorphisms marked $\cong$; this identifies certain subquotients of $(K^\times \backslash \mathbb{A}_K^\times)^\wedge$ with certain Galois groups $\text{Gal}(L'/L)$, $K^{ab} \supseteq L' \supseteq L \supseteq K$.

The group $Z_K(\mathfrak{C})$, viewed as the quotient $(K^\times \backslash \mathbb{A}_K^\times)^\wedge / (U_K(\mathfrak{C}p^\infty)K_\infty^\times)^\wedge$, may be identified with the Galois group of the corresponding extension of K contained in K^{ab} . Inside that extension, let $K_\pm/K$ be the $\mathbb{Z}_p^2$ -extension such that $K_\pm$ is the fixed field of $Z_K(\mathfrak{C})_{\text{tor}}$; then $Z_K(\mathfrak{C})/\text{tor} \cong \text{Gal}(K_\pm/K) =: \Gamma_K$ using the standard isomorphism from Galois theory. The map (3.1) with $\mathfrak{a} = \mathfrak{C}$ restricts to an isomorphism from

$$\omega_{v_0}[O_{K,v_0}^\times] \times \omega_{\bar{v}_0}[O_{K,\bar{v}_0}^\times] \times Cl_K \times ((O_K/\mathfrak{C})^\times/\{\pm 1\}) \quad (3.2)$$

to $Z_K(\mathfrak{C})_{\text{tor}}$. This means that $K_\pm/K$ is the largest $\mathbb{Z}_p^2$ -extension of K inside the extension corresponding to $Z_K(\mathfrak{C})$. (There is no p -part in $(O_K/\mathfrak{C})^\times/\{\pm 1\}$ or Cl_K

since $p \nmid \phi(N)h_K$.)

The composition of group maps

$$(1 + v_0 O_{K,v_0}) \times (1 + \bar{v}_0 O_{K,\bar{v}_0}) \hookrightarrow O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \hookrightarrow \mathbb{A}_K^\times \twoheadrightarrow Z_K(\mathfrak{C}) \twoheadrightarrow Z_K(\mathfrak{C})/\text{tor} \cong \Gamma_K$$

is an isomorphism

$$i : (1 + v_0 O_{K,v_0}) \times (1 + \bar{v}_0 O_{K,\bar{v}_0}) \xrightarrow{\cong} \Gamma_K$$

which I use to identify these two groups. I sometimes abuse notation by writing $i(y_{v_0}, y_{\bar{v}_0})$ for the class of that element in quotients of Γ_K , i.e., Galois groups of extensions of K inside $K_\pm$.

Combine the following: (3.1), the internal direct products $O_{K,v_0}^\times \cong (1 + v_0 O_{K,v_0}) \times \omega_{v_0}[O_{K,v_0}^\times]$ and $O_{K,\bar{v}_0}^\times \cong (1 + \bar{v}_0 O_{K,\bar{v}_0}) \times \omega_{\bar{v}_0}[O_{K,\bar{v}_0}^\times]$, the map i , the identification of (3.2) with $Z_K(\mathfrak{C})_{\text{tor}}$, and the Galois-theory map $Z_K(\mathfrak{C})/\text{tor} \cong \Gamma_K$. We obtain

$$\begin{aligned} Z_K(\mathfrak{C}) &\cong O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times \times Cl_K \times ((O_K/\mathfrak{C})^\times / \{\pm 1\}) \\ &\cong ((1 + v_0 O_{K,v_0}) \times \omega_{v_0}[O_{K,v_0}^\times]) \times ((1 + \bar{v}_0 O_{K,\bar{v}_0}) \times \omega_{\bar{v}_0}[O_{K,\bar{v}_0}^\times]) \times Cl_K \times \frac{(O_K/\mathfrak{C})^\times}{\{\pm 1\}} \\ &\cong ((1 + v_0 O_{K,v_0}) \times (1 + \bar{v}_0 O_{K,\bar{v}_0})) \times \left(\omega_{v_0}[O_{K,v_0}^\times] \times \omega_{\bar{v}_0}[O_{K,\bar{v}_0}^\times] \times Cl_K \times \frac{(O_K/\mathfrak{C})^\times}{\{\pm 1\}} \right) \\ &\cong \Gamma_K \times Z_K(\mathfrak{C})_{\text{tor}} \\ &\cong (Z_K(\mathfrak{C})/\text{tor}) \times Z_K(\mathfrak{C})_{\text{tor}} \end{aligned}$$

which yields a group isomorphism $Z_K(\mathfrak{C}) \cong (Z_K(\mathfrak{C})/\text{tor}) \times Z_K(\mathfrak{C})_{\text{tor}}$. That isomorphism sends an element $\gamma \in Z_K(\mathfrak{C})_{\text{tor}}$ to $(1, \gamma)$, and it sends the class $\gamma \in Z_K(\mathfrak{C})$ of $(y_{v_0}, \bar{y}_{v_0}) \in (1 + v_0 O_{K,v_0}) \times (1 + \bar{v}_0 O_{K,\bar{v}_0})$ (seen as an element of $\mathbb{A}_K^\times$) to $([\gamma], 1)$, where the coset $[\gamma] = \gamma + Z_K(\mathfrak{C})_{\text{tor}} \in Z_K(\mathfrak{C})/\text{tor}$ corresponds to $i(y_{v_0}, \bar{y}_{v_0})$ under the Galois-theory map $Z_K(\mathfrak{C})/\text{tor} \cong \Gamma_K$.

In $K_{\pm}/K$, the maximum extension unramified at $\bar{v}_0$ (respectively, v_0) is the extension K_{v_0}/K (respectively, $K_{\bar{v}_0}/K$) such that K_{v_0} is the fixed field of $i[1 + \bar{v}_0 O_{K, \bar{v}_0}]$ (respectively, $K_{\bar{v}_0}$ is the fixed field of $i[1 + v_0 O_{K, v_0}]$) in $K_{\pm}$. We have

$$1 + v_0 O_{K, v_0} \cong \frac{(1 + v_0 O_{K, v_0}) \times (1 + \bar{v}_0 O_{K, \bar{v}_0})}{1 + \bar{v}_0 O_{K, \bar{v}_0}} \stackrel{i}{\cong} \frac{\text{Gal}(K_{\pm}/K)}{\text{Gal}(K_{\pm}/K_{v_0})} \cong \text{Gal}(K_{v_0}/K)$$

in which the first map is given by $y_{v_0} \mapsto (\text{class of } (y_{v_0}, 1))$, and

$$1 + \bar{v}_0 O_{K, \bar{v}_0} \cong \frac{(1 + v_0 O_{K, v_0}) \times (1 + \bar{v}_0 O_{K, \bar{v}_0})}{1 + v_0 O_{K, v_0}} \stackrel{i}{\cong} \frac{\text{Gal}(K_{\pm}/K)}{\text{Gal}(K_{\pm}/K_{\bar{v}_0})} \cong \text{Gal}(K_{\bar{v}_0}/K)$$

in which the first map is given by $y_{\bar{v}_0} \mapsto (\text{class of } (1, y_{\bar{v}_0}))$. So we obtain an isomorphism $1 + v_0 O_{K, v_0} \cong \text{Gal}(K_{v_0}/K)$ sending y_{v_0} to the class of $i(y_{v_0}, 1)$, and an isomorphism $1 + \bar{v}_0 O_{K, \bar{v}_0} \cong \text{Gal}(K_{\bar{v}_0}/K)$ sending $y_{\bar{v}_0}$ to the class of $i(1, y_{\bar{v}_0})$. The projection $pr_{v_0} : \Gamma_K \rightarrow \text{Gal}(K_{v_0}/K)$ sends $g = i(y_{v_0}, y_{\bar{v}_0}) \in \Gamma_K$ to the class of g in $\text{Gal}(K_{v_0}/K) \cong \Gamma_K / \text{Gal}(K_{\pm}/K_{v_0})$, which is the class of $i(y_{v_0}, 1)$.

The field extension K_+/K (respectively, K_-/K) is the cyclotomic (respectively, anticyclotomic) extension such that K_+ is the fixed field of $\{i(y, y^{-1}) : y \in 1 + p\mathbb{Z}_p\}$ (respectively, K_- is the fixed field of $\{i(y, y^{+1}) : y \in 1 + p\mathbb{Z}_p\}$) in $K_{\pm}$. Use the embeddings $\mathbb{Z}_p \hookrightarrow O_{K, v_0}$ and $\mathbb{Z}_p \hookrightarrow O_{K, \bar{v}_0}$ to identify the groups

$$(1 + p\mathbb{Z}_p)^2 \cong (1 + v_0 O_{K, v_0}) \times (1 + \bar{v}_0 O_{K, \bar{v}_0}).$$

We have

$$\{(y, y^{+1}) : y \in 1 + p\mathbb{Z}_p\} \cong \frac{(1 + p\mathbb{Z}_p)^2}{\{(y, y^{-1}) : y \in 1 + p\mathbb{Z}_p\}} \stackrel{i}{\cong} \frac{\text{Gal}(K_{\pm}/K)}{\text{Gal}(K_{\pm}/K_+)} \cong \text{Gal}(K_+/K)$$

in which the first map is given by $(y, y) \mapsto (\text{class of } (y, y))$, and

$$\{(y, y^{-1}) : y \in 1 + p\mathbb{Z}_p\} \cong \frac{(1 + p\mathbb{Z}_p)^2}{\{(y, y^{+1}) : y \in 1 + p\mathbb{Z}_p\}} \stackrel{i}{\cong} \frac{\text{Gal}(K_{\pm}/K)}{\text{Gal}(K_{\pm}/K_-)} \cong \text{Gal}(K_-/K)$$

in which the first map is given by $(y, y^{-1}) \mapsto (\text{class of } (y, y^{-1}))$. In this way, we may view the Galois group $\Gamma_K^+ := \text{Gal}(K_+/K)$ (respectively, $\Gamma_K^- := \text{Gal}(K_-/K)$) as a copy of the group $\{(y, y^{+1}) : y \in 1 + p\mathbb{Z}_p\}$ (respectively, $\{(y, y^{-1}) : y \in 1 + p\mathbb{Z}_p\}$). The group Γ_K^+ (respectively, Γ_K^-) is topologically generated by the element $\gamma_+ = i((1 + p)^{1/2}, (1 + p)^{1/2})$ (respectively, $\gamma_- = i((1 + p)^{1/2}, (1 + p)^{-1/2})$), or more precisely, by the class of that element in the quotient of $\text{Gal}(K_{\pm}/K)$ by $\text{Gal}(K_{\pm}/K_+)$ (respectively, by $\text{Gal}(K_{\pm}/K_-)$). The projection $pr_{ac} : \Gamma_K \twoheadrightarrow \Gamma_K^-$ sends $g = i(y_{v_0}, y_{\bar{v}_0}) \in \Gamma_K$ to the class of g in $\Gamma_K^- \cong \Gamma_K / \text{Gal}(K_{\pm}/K_-)$, which is the class of $(gg^{-c})^{1/2} = i(y_{v_0}^{1/2} y_{\bar{v}_0}^{-1/2}, y_{v_0}^{-1/2} y_{\bar{v}_0}^{1/2})$.

The squaring maps $sq : Z_K(\mathfrak{C}) \rightarrow Z_K(\mathfrak{C})$, $sq : \Gamma_K \rightarrow \Gamma_K$ and $sq : \text{Gal}(K_{v_0}/K) \rightarrow \text{Gal}(K_{v_0}/K)$ send each element g to $g^2 = g \circ g$.

Let O be a discrete valuation ring O with $\overline{\mathbb{Q}_p} \supseteq O \supseteq \mathbb{Z}_p$. The projection pr_{ac} and the two maps sq are extended O -linearly and continuously to yield the O -algebra maps $pr_{ac} : O[[\Gamma_K]] \twoheadrightarrow O[[\Gamma_K^-]]$, $sq : O[[Z_K(\mathfrak{C})]] \rightarrow O[[Z_K(\mathfrak{C})]]$ and $sq : O[[\Gamma_K]] \rightarrow O[[\Gamma_K]]$.

Let $c : z \mapsto \bar{z}$ be the conjugation map on $\mathbb{C}$ (or on any subfield of $\mathbb{C}$ stable under conjugation). The group $\text{Gal}(K/\mathbb{Q}) = \{1, c\}$ acts on Γ_K via conjugation: $c \in \text{Gal}(K/\mathbb{Q})$ sends $g \in \Gamma_K$ to $cgc^{-1} \in \Gamma_K$. The element c acts on Γ_K^+ and Γ_K^- as 1 and -1 respectively.

3.2 Complex and p -adic Hecke characters

Suppose a character $\tilde{\chi} : K^{\times} \backslash \mathbb{A}_{K,f}^{\times} \rightarrow \overline{\mathbb{Q}_p}^{\times}$ is such that for some integers t and u , there is a neighborhood U of 1 in $O_{K,v_0}^{\times} \times O_{K,\bar{v}_0}^{\times}$ such that for all $(z_{v_0}, z_{\bar{v}_0}) \in U$, we have

$$\tilde{\chi}(\dots, 1, 1, z_{v_0}, z_{\bar{v}_0}, 1, 1, \dots) = z_{v_0}^t z_{\bar{v}_0}^u. \quad (3.3)$$

Then $\tilde{\chi}$ is called the p -adic avatar of the so-called algebraic (complex) Hecke character $\chi = \tilde{\chi}^{alg} : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$, where for $z \in \mathbb{A}_K^\times$, we have

$$\chi(z) = (z_\infty^t \bar{z}_\infty^u) \cdot (z_{v_0}^{-t} \bar{z}_{\bar{v}_0}^{-u}) \tilde{\chi}(z_f) \quad (3.4)$$

(identifying $(z_{v_0}^{-t} \bar{z}_{\bar{v}_0}^{-u}) \tilde{\chi}(z_f) \in \overline{\mathbb{Q}}_p^\times$ with a number in $\mathbb{C}^\times$ via the isomorphism $\widehat{\overline{\mathbb{Q}}_p} \cong \mathbb{C}$). Note that if $(z_{v_0}, z_{\bar{v}_0}) \in U$ then, letting $z_f^{(p)} \in \mathbb{A}_{K,f}^\times$ have the same component as z_f at each place $v \nmid p$ and the component 1 at the places v_0 and $\bar{v}_0$ over p , we have

$$\chi(z) = (z_\infty^{-t} \bar{z}_\infty^{-u}) \cdot \tilde{\chi}(z_f^{(p)}).$$

Conversely, given a Hecke character $\chi : K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow \mathbb{C}^\times$ such that $\chi(z_\infty, 1, 1, \dots) = z_\infty^t \bar{z}_\infty^u$ identically, the p -adic avatar $\tilde{\chi} : K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ is defined by (3.4), and the corresponding Galois character $\sigma_\chi : G_K \rightarrow \overline{\mathbb{Q}}_p^\times$ sends the geometric Frobenius at any $v \nmid p$ to χ_v of a uniformizer at v (I use notation from [SkUr14, subsection 2.2.1]).

Recall the identification

$$Z_K(\mathfrak{C}) \cong \Gamma_K \times Z_K(\mathfrak{C})_{\text{tor}} \cong (Z_K(\mathfrak{C})/\text{tor}) \times Z_K(\mathfrak{C})_{\text{tor}}.$$

Characters $P : \Gamma_K \rightarrow \overline{\mathbb{Q}}_p^\times$ and $\psi : Z_K(\mathfrak{C})_{\text{tor}} \rightarrow \overline{\mathbb{Q}}_p^\times$, respectively, can be precomposed with the projections $Z_K(\mathfrak{C}) \twoheadrightarrow Z_K(\mathfrak{C})/Z_K(\mathfrak{C})_{\text{tor}} \cong \Gamma_K$ and $Z_K(\mathfrak{C}) \twoheadrightarrow Z_K(\mathfrak{C})_{\text{tor}}$ to yield characters P and ψ from $Z_K(\mathfrak{C})$ to $\overline{\mathbb{Q}}_p^\times$, whose product $P\psi : Z_K(\mathfrak{C}) \rightarrow \overline{\mathbb{Q}}_p^\times$ is such that the element of $Z_K(\mathfrak{C})$ identified with $(\sigma, \zeta) \in \Gamma_K \times Z_K(\mathfrak{C})_{\text{tor}}$ is sent by $P\psi$ to $P(\sigma)\psi(\zeta)$. This gives a character $P\psi : K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ (taking the previous character $P\psi$ and precomposing with $K^\times \backslash \mathbb{A}_{K,f}^\times \twoheadrightarrow Z_K(\mathfrak{C})$) which, if it is of the form (3.3) in some neighborhood of 1 in $O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times$ for some integers t and u , is the p -adic avatar of an algebraic Hecke character $(P\psi)^{alg} : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$.

A continuous character $P_{ac} : \Gamma_K^- \rightarrow \overline{\mathbb{Q}}_p^\times$ gives a character $P = P_{ac} \circ pr_{ac} : \Gamma_K \rightarrow \overline{\mathbb{Q}}_p^\times$.

For a discrete valuation ring O with $\overline{\mathbb{Q}}_p \supseteq O \supseteq \mathbb{Z}_p$, a character $\psi : Z_K(\mathfrak{C})_{\text{tor}} \rightarrow O^\times$ yields (again using the identification $Z_K(\mathfrak{C}) \cong \Gamma_K \times Z_K(\mathfrak{C})_{\text{tor}}$)

- (a) a continuous O -algebra map $\psi_\pm : O[[Z_K(\mathfrak{C})]] \rightarrow O[[\Gamma_K]]$ such that restrictions are given by $\psi_\pm|_{Z_K(\mathfrak{C})_{\text{tor}}} = \psi$ and $\psi_\pm|_{\Gamma_K} = \text{identity}$, and
- (b) a continuous O -algebra map $\psi_{ac} = pr_{ac} \circ \psi_\pm : O[[Z_K(\mathfrak{C})]] \twoheadrightarrow O[[\Gamma_K^-]]$ such that restrictions are given by $\psi_{ac}|_{Z_K(\mathfrak{C})_{\text{tor}}} = \psi$ and $\psi_{ac}|_{\Gamma_K} = pr_{ac}$.

Chapter 4

Modular forms

In this chapter, I review standard facts related to modular forms in their various incarnations.

4.1 Modular forms, adelically and p -adically

This section introduces adelic and p -adic modular forms, proceeding as in papers of Hida [Hi88, Hi91] and using and adapting notation from those articles.

Let $F : GL_2(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$ be a function. For $u \in GL_2(\mathbb{A}_{\mathbb{Q}})$ with $u_{\infty} \in \mathbb{R}^{\times} SO_2(\mathbb{R})$ and $k, w \in \mathbb{Z}$, define

$$F|_{k,w}u : GL_2(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C} : g \mapsto (\det u_{\infty})^w j(u_{\infty}, i)^{-k} F(gu^{-1})$$

with $j(y, i) = ci + d$ for $y \in GL_2^+(\mathbb{R})$ of the form $y = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. This gives an action of $\mathbb{R}^{\times} SO_2(\mathbb{R}) \times GL_2(\mathbb{A}_{\mathbb{Q},f})$ on functions $GL_2(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$.

For $k, w \in \mathbb{Z}$ and an open compact subgroup S of $GL_2(\mathbb{A}_{\mathbb{Q},f})$ (so $S \subseteq GL_2(\widehat{\mathbb{Z}}) = \prod_{\ell \neq \infty} GL_2(\mathbb{Z}_{\ell})$), $S_{k,w}(S, \mathbb{C})$ is the space of $F : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$ satisfying:

- (a) Invariance: $F|_{k,w}u = F$ for $u \in S \cdot \mathbb{R}^{\times} SO_2(\mathbb{R})$.

- (b) Holomorphicity: $\partial F_1 / \partial \bar{\tau} = 0$, where for F satisfying invariance, the function $F_1 : \mathfrak{h} \rightarrow \mathbb{C}$ given by

$$F_1(\tau) = (\det t_\infty)^{-w} j(t_\infty, i)^k F(t_\infty) \text{ for } \tau = t_\infty i \in \mathfrak{h} \text{ and } t_\infty \in GL_2^+(\mathbb{R})$$

is the classical modular form corresponding to F ; see also [CoKiMu04, pp. 13-15].

- (c) Cuspidality: For all $g \in GL_2(\mathbb{A}_\mathbb{Q})$ and any Haar measure da on $\mathbb{Q} \backslash \mathbb{A}_\mathbb{Q}$,

$$\int_{\mathbb{Q} \backslash \mathbb{A}_\mathbb{Q}} F \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g \right) da = 0.$$

- (d) Growth condition: For some $C \in \mathbb{R}_{>0}$, for all $\tau \in \mathfrak{h}$, $|\text{im}(\tau)^{k/2} F_1(\tau)| \leq C$.

For $k, w, w' \in \mathbb{Z}$, the map $A : S_{k,w}(S, \mathbb{C}) \rightarrow S_{k,w'}(S, \mathbb{C}) : F \mapsto |\det|_{\mathbb{A}, \mathbb{Q}}^{w'-w} F$ is an isomorphism; $(A(F))|_{k,w'} u = A(F|_{k,w} u)$ for $u \in S \cdot \mathbb{R}^\times SO_2(\mathbb{R})$; $(A(F))_1 = F_1$. If k is even and $w' = k/2$, then $A(F) \in S_{k,k/2}(S, \mathbb{C})$ is called the unitarization of F and is written F^u .

Examples of S include, for $M \in \mathbb{Z}_{>0}$ (see [Hi88, p. 308], [Hi91, p. 322]),

$$\begin{aligned} U(M) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbb{Z}}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{M\widehat{\mathbb{Z}}} \right\} \\ U_1(M) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbb{Z}}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{M\widehat{\mathbb{Z}}} \right\} \\ V_1(M) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbb{Z}}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{M\widehat{\mathbb{Z}}} \right\} \\ U_0(M) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbb{Z}}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{M\widehat{\mathbb{Z}}} \right\} \end{aligned}$$

For $U(M) \subseteq S \subseteq U_0(M)$ and a cusp form $F \in S_{k,w}(S, \mathbb{C})$, the associated classical form F_1 has a Fourier expansion $F_1(\tau) = \sum_{n=1}^{\infty} a(n, F_1) q^n$, where $q = \exp(2\pi i \tau)$. Write $a(n, F) = a(n, F_1)$ for a Fourier coefficient.

The conjugate F_1^c of F_1 is a cusp form with Fourier series $F_1^c(\tau) = \overline{F_1(-\bar{\tau})} = \sum_{n=1}^{\infty} \overline{a(n, F_1)} q^n$, where the line denotes complex conjugation. If $F \in S_{k,w}(S, \mathbb{C})$ has F_1 as its classical modular form, then there is a cusp form

$$F^c \in S_{k,w} \left(\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot S \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{C} \right)$$

with associated classical modular form $(F^c)_1 = F_1^c$.

Similarly to [Sh71, chapter 3], the Hecke algebra for $U(M)$ with coefficients in $\mathbb{Z}$ consists of $\mathbb{Z}$ -linear combinations of elements $[U(M)gU(M)]$ with g in

$$\Delta(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathbb{Z}}) \cap GL_2(\mathbb{A}_{\mathbb{Q},f}) : a_M \in \mathbb{Z}_M^\times, c \in M\widehat{\mathbb{Z}} \right\}$$

(see [Hi89, p. 143]). Such an element $[U(M)gU(M)]$ acts on $F \in S_{k,w}(U(M), \mathbb{C})$ by

$$F|_{k,w}[U(M)gU(M)] = \sum_j F|_{k,w}((\det g_j)^{-1} g_j)$$

for a disjoint union $U(M)gU(M) = \sqcup_j U(M)g_j$, $g_j \in GL_2(\mathbb{A}_{\mathbb{Q},f})$. (To obtain this from the definition of [Hi88, pp. 305-306], apply the involution of [Hi88, p. 307].)

For $r \in \mathbb{Z}_{>0}$, let $T(r) = \sum [U(M)gU(M)]$, with the sum over the finitely many cosets $U(M)gU(M)$ contained in $\{g \in \Delta(M) : (\det g)\mathbb{Z} = r\mathbb{Z}\}$ (adapting [Hi88, pp. 309, 311]); this is the classical Hecke operator (see also [Ge75, proof of Lemma 3.7] and [Hi91, section 2]). We have $T(\ell) = [U(M)\text{diag}(1, \ell)U(M)]$. If $F \in S_{k,w}(U(M), \mathbb{C})$,

then for primes $\ell \nmid M\infty$,

$$(F|_{k,w}T(\ell))(g) = F \left(g \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \right)_\ell + \sum_{b=0}^{\ell-1} F \left(g \begin{pmatrix} \ell & b \\ 0 & 1 \end{pmatrix} \right)_\ell$$

(see [Ge75, proof of Lemma 3.7]), and for $\ell \mid M$,

$$(F|_{k,w}T(\ell^m))(g) = \sum_{b=0}^{\ell^m-1} F \left(g \begin{pmatrix} \ell^m & b \\ 0 & 1 \end{pmatrix} \right)_\ell.$$

This means that if $F \in S_{k,w}(U(M), \mathbb{C}) \cap S_{k,w}(U(M'), \mathbb{C})$ for two different M, M' either both divisible by ℓ or both not divisible by ℓ , then $F|_{k,w}T(\ell)$ is the same whether $T(\ell)$ is viewed

- (a) as a sum of elements $[U(M)gU(M)]$ acting on $S_{k,w}(U(M), \mathbb{C})$, or
- (b) as a sum of elements $[U(M')gU(M')]$ acting on $S_{k,w}(U(M'), \mathbb{C})$.

See also [Hi91, section 2].

For $z \in \mathbb{A}_{\mathbb{Q},f}^\times$ with $z_M \in 1 + M\mathbb{Z}_M$ and $z\mathbb{Z} = \mathfrak{a}$ a fractional ideal of $\mathbb{Z}$ coprime to M , write $\langle z \rangle = \langle \mathfrak{a}^{-1} \rangle = T(\mathfrak{a}^{-1}, \mathfrak{a}^{-1}) = [U(M)\text{diag}(z^{-1}, z^{-1})U(M)]$, so that $F|_{k,w}\langle z \rangle = F|_{k,w}z$ for $F \in S_{k,w}(U(M), \mathbb{C})$. If $\mathfrak{a}$ is known, then so is $T(\mathfrak{a}^{-1}, \mathfrak{a}^{-1})$. In particular, $(F|_{k,w}\langle \ell\mathbb{Z} \rangle)(g) = F(g\ell_\ell)$ for primes $\ell \nmid M\infty$ [HiTi93, p. 238].

From now on, let O be any ring with $O_L \subseteq O \subseteq \overline{\mathbb{Q}}_p \subseteq \widehat{\overline{\mathbb{Q}}_p} \cong \mathbb{C}$, and let $U(M) \subseteq S \subseteq U_0(M)$; I follow [Hi91, sections 2-3]. Write $S(p^\alpha) = S \cap U(p^\alpha)$ and

$$\begin{aligned} S_{k,w}(S, O) &= \{F \in S_{k,w}(S, \mathbb{C}) : \text{all } a(n, F) \in O\} \\ S_{k,w}(S(p^\infty), O) &= \varinjlim_n S_{k,w}(S(p^n), O) \\ \overline{S}_{k,w}(S(p^\infty), O) &= (\text{completion of } S_{k,w}(S(p^\infty), O) \text{ under the norm } |\cdot| \text{ such that} \\ &\quad |F| = \sup_{n=0}^\infty |a(n, F)|_p \text{ for } F \in S_{k,w}(S(p^\infty, O))). \end{aligned}$$

The Hecke algebra $h_{k,w}(S, O)$ is the O -subalgebra of $\mathbb{C}$ -endomorphisms of the space $S_{k,w}(S, \mathbb{C}) \subseteq S_{k,w}(U(M), \mathbb{C})$ generated by the operators $T(\ell^m)$ (ranging over all rational primes ℓ) and the $\langle z \rangle$ operators. The Hecke algebra $h_{k,w}(S(p^\infty), O) = \varprojlim_n h_{k,w}(S(p^n), O)$ (with standard restriction maps used for the limit and with n ranging over the positive integers) acts on $\overline{S}_{k,w}(S(p^\infty), O)$. Let $h_{k,w}^{ord}(S(p^\infty), O)$ and $\overline{S}_{k,w}^{ord}(S(p^\infty), O)$ be the nearly ordinary parts of $h_{k,w}(S(p^\infty), O)$ and $\overline{S}_{k,w}(S(p^\infty), O)$ respectively, and let e be the ordinary projector, as described by Hida in [Hi88, pp. 312-313] and (writing “ord” instead of “n,ord”) [Hi91, p. 335].

Write $h_{k,w}(M, O) = h_{k,w}(V_1(M)(p^\infty), O)$, $h_{k,w}^{ord}(M, O) = h_{k,w}^{ord}(V_1(M)(p^\infty), O)$. Considering the maps

$$\begin{aligned} \{z \in \mathbb{A}_{\mathbb{Q},f}^\times : z_{Mp} = 1\} &\rightarrow \frac{\left(\begin{array}{c} \text{Fractional ideals of} \\ \mathbb{Z} \text{ coprime to } Mp \end{array} \right)}{\langle r\mathbb{Z} : r \in 1 + Mp^n\mathbb{Z} \rangle} &\rightarrow h_{k,w}(V_1(M)(p^n), O) \\ z &\mapsto [z\mathbb{Z}] = [\mathfrak{a}] &\mapsto \langle \mathfrak{a}^{-1} \rangle = \langle z \rangle \end{aligned}$$

and taking the projective limit over n , we get a continuous multiplicative map

$$\begin{aligned} Z(M) = \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_{\mathbb{Q},f}^\times / U_{\mathbb{Q}}(Mp^\infty) &\rightarrow h_{k,w}(M, O) \\ z &\mapsto \langle z \rangle \end{aligned}$$

(see [Hi91, p. 334] and [Hi88, sections 2-3]). In this way, an element $z \in Z(M)$ is viewed as corresponding to the element $\langle z \rangle$ of the Hecke algebra $h_{k,w}(M, O)$.

For $a \in \mathbb{Z}_p^\times$ corresponding to the adèle $a_p \in \mathbb{A}_{\mathbb{Q},f}^\times$, the projective limit of the Hecke operators

$$\mathbf{T}(a_p) := a^{w-1} [U(Mp^n) \text{diag}(a_p, 1) U(Mp^n)] \in h_{k,w}(V_1(M)(p^n), O)$$

over all positive-integer values of n , is a Hecke operator $\mathbf{T}(a_p) \in h_{k,w}(M, O)$, as

described in [Hi91, pp. 330-332].

The perfect pairing

$$\begin{aligned} \bar{S}_{k,w}(V_1(M)(p^\infty), O) \times h_{k,w}(V_1(M)(p^\infty), O) &\rightarrow O \\ (F, H) &\mapsto a(1, F|H) \end{aligned}$$

yields isomorphisms between one of its arguments and $\text{Hom}_O(\cdot, O)$ of the other ([Hi91, Theorem 3.1]; see also [Hi88, Theorem 5.3]). Applying $\otimes_O \bar{\mathbb{Q}}_p$ yields a perfect pairing over $\bar{\mathbb{Q}}_p$ given by the same formula with each O replaced by $\bar{\mathbb{Q}}_p$.

4.2 Hida families and parameterizations

This section's purpose is to introduce Hida families of modular forms. I follow Hida [Hi91, pp. 335-337].

Let O be as before. We have $O[[\Gamma_K]] \cong O[[t_{\mathbb{Q}}, t_{\bar{v}_0}]]$ with $\gamma_+ = i((1+p)^{1/2}, (1+p)^{1/2}) \leftrightarrow 1+t_{\mathbb{Q}}$ and $\gamma_{\bar{v}_0} = i(1, 1+p) \leftrightarrow 1+t_{\bar{v}_0}$. Let the O -algebra $I \supseteq O[[\Gamma_K]]$ be such that I is contained in the integral closure of $O[[\Gamma_K]]$ in a finite-degree field extension of the quotient field of $O[[\Gamma_K]]$.

Let $\lambda : h_{k,w}(M, O) \rightarrow I$ be an O -algebra map such that for $\sigma \in 1 + p\mathbb{Z}_p$ corresponding to $z = [\sigma_p^{-1}] \in Z(M)$, the map λ sends $\langle z \rangle$ to $i(\sigma, \sigma) \in \Gamma_K \subseteq O[[\Gamma_K]]$, and if $\sigma \in 1 + p\mathbb{Z}_p \cong 1 + \bar{v}_0 O_{K, \bar{v}_0}^\times$, then λ sends $\mathbf{T}(\sigma_p^{-1})$ to $i(1, \sigma) \in \Gamma_K \subseteq O[[\Gamma_K]]$.

Let $P : I \rightarrow \bar{\mathbb{Q}}_p$ be an O -algebra map so that for some finite-order multiplicative characters $\varepsilon_P : 1 + p\mathbb{Z}_p \rightarrow \bar{\mathbb{Q}}_p^\times$ and $\varepsilon'_P : 1 + p\mathbb{Z}_p \rightarrow \bar{\mathbb{Q}}_p^\times$,

$$(a) \quad P(i(\sigma, \sigma)) = \sigma^{k-2w} \varepsilon_P(\sigma) \text{ for } \sigma \in 1 + p\mathbb{Z}_p, \text{ and}$$

$$(b) \quad P(i(1, \sigma)) = \sigma^{1-w} \varepsilon'_P(\sigma) \text{ for } \sigma \in 1 + p\mathbb{Z}_p.$$

Call such P arithmetic, following Hida [Hi91, pp. 316, 335-337] as well as Skinner and Urban [SkUr14, section 3.3.8]. Note that restricting P to Γ_K and then precomposing

that restricted map with the projections $K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow Z_K(\mathfrak{C}) \rightarrow \Gamma_K$ yields a p -adic Hecke character $\tilde{\chi} : K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ such that (3.3) holds for $(t, u) = (k-1-w, 1-w)$ and $U = (\ker \varepsilon_P \cap \ker \varepsilon'_P)^2 \subseteq O_{K,v_0}^\times \times O_{K,\bar{v}_0}^\times$.

The O -algebra map $P \circ \lambda : h_{k,w}(M, O) \rightarrow \overline{\mathbb{Q}}_p$ yields the $\overline{\mathbb{Q}}_p$ -algebra map $\lambda(P)$ as the composite

$$h_{k,w}(M, \overline{\mathbb{Q}}_p) \xrightarrow{e} h_{k,w}^{ord}(M, \overline{\mathbb{Q}}_p) \hookrightarrow h_{k,w}(M, \overline{\mathbb{Q}}_p) \xrightarrow{(P \circ \lambda) \otimes \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p.$$

Define the characters $\psi_P : Z(M) \rightarrow \overline{\mathbb{Q}}_p^\times$ and $\psi'_P : \mathbb{Z}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ by

$$\begin{aligned} \psi_P(\zeta[\sigma_p^{-1}]) &= \epsilon(\zeta)^{k-2w} \cdot \lambda(P)(\langle \zeta \rangle) \cdot \varepsilon_P(\sigma) \\ \psi'_P(\zeta'\sigma) &= (\zeta')^{w-1} \cdot \lambda(P)(\mathbf{T}((\zeta')_p^{-1})) \cdot \varepsilon'_P(\sigma) \end{aligned}$$

for $\zeta \in Z(M)_{tor}$, $\zeta' \in (\mathbb{Z}_p^\times)_{tor}$ and $\sigma \in 1 + p\mathbb{Z}_p$.

The characters ψ_P and ψ'_P have finite order: for some $\alpha \in \mathbb{Z}_{>0}$, ψ_P and ψ'_P factor through $\psi_P : (\mathbb{Z}/Mp^\alpha\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ and $\psi'_P : (\mathbb{Z}/p^\alpha\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ respectively.

Pre-composing ψ_P with $\mathbb{A}_{\mathbb{Q},f}^\times \rightarrow Z(M)$ yields a character $\psi_P : \mathbb{A}_{\mathbb{Q},f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$, which restricts to its p -adic part $\psi_{P,p} : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ (with $\mathbb{Q}_p^\times$ embedded in $\mathbb{A}_{\mathbb{Q},f}^\times$ in the usual way). Extend ψ'_P multiplicatively to a character $\psi'_{P,p} : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ by letting $\psi'_{P,p}(p) = 1$.

Define the space $\overline{S}_{k,w}^{ord}(Mp^\alpha, \psi'_P, \psi_P, \overline{\mathbb{Q}}_p)$ to be

$$\left\{ F \in \overline{S}_{k,w}^{ord}(U(Mp^\alpha), \overline{\mathbb{Q}}_p) : \begin{aligned} F|_{k,w}\langle z \rangle &= \epsilon(z)^{2w-k} \psi_P(z) F \text{ for } z \in Z(M), \\ F|_{k,w}\mathbf{T}(a_p^{-1}) &= a^{1-w} \psi'_P(a) F \text{ for } a \in \mathbb{Z}_p^\times \end{aligned} \right\}.$$

Then via Hecke algebra duality, $\lambda(P)$ corresponds to an eigenform $F(\lambda, P) \in \overline{S}_{k,w}^{ord}(Mp^\alpha, \psi'_P, \psi_P, \overline{\mathbb{Q}}_p)$ so that $a(1, F(\lambda, P)) = 1$ and, for each element H of the Hecke algebra,

$$F(\lambda, P)|H = \lambda(P)(H) \cdot F(\lambda, P).$$

The map λ is a cuspidal Hida family; it can be viewed as corresponding to the collection of ordinary normalized eigenforms $F(\lambda, P)$ ranging over the arithmetic points $P : I \rightarrow \overline{\mathbb{Q}}_p$.

4.3 Theta series

In this section, I describe classical theta series, and then I fit them into a Hida family. See [Hi81, p. 257], [HiTi93, pp. 234-238] and [Ko11, sections 5.1-5.2].

Let $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ be a Hecke character with conductor $C(\chi) \subseteq O_K$ so that there is a positive integer n , together with a finite-order character $\psi : (O_K/C(\chi))^\times \rightarrow \mathbb{C}^\times$, such that for all $a \in O_K$ coprime to $C(\chi)$, we have $\chi(aO_K) = a^n \psi(a)^{-1}$. (The p -adic avatar $\tilde{\chi}$ of χ satisfies (3.3) for $(t, u) = (-n, 0)$ and some U . For $a \in O_K$ coprime to $C(\chi)$ we have $\chi(a_\infty) = a^{-n}$ and $\chi(a_{C(\chi)}) = \psi(a)$.) Then the theta series of χ is

$$\theta_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) q^{N_{K/\mathbb{Q}} \mathfrak{a}}$$

with L-series

$$L(s, \theta_\chi) = L(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) (N_{K/\mathbb{Q}} \mathfrak{a})^{-s}$$

where the sums range over nonzero integral ideals $\mathfrak{a}$ of O_K coprime to $C(\chi)$. If $\varphi_K : (\mathbb{Z}/|D_K|\mathbb{Z})^\times \rightarrow \{\pm 1\}$ is the Legendre-symbol character of $K/\mathbb{Q}$ such that $\varphi_K(\ell) = \left(\frac{D_K}{\ell}\right)$ for odd rational primes ℓ , then for the character $\varepsilon : (\mathbb{Z}/|D_K|(N_{K/\mathbb{Q}}C(\chi))\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ with $\varepsilon(m) = \varphi_K(m)\psi^{-1}(m)$ for $m \in \mathbb{Z}_{>0}$, we have $\theta_\chi \in S_{n+1}(|D_K|(N_{K/\mathbb{Q}}C(\chi)), \varepsilon)$.

The modular form θ_χ will now be fit into a Hida family.

Let the character $P_{-n,0} : \Gamma_K \cong Z_K(\mathfrak{C})/Z_K(\mathfrak{C})_{\text{tor}} \rightarrow \overline{\mathbb{Q}}_p^\times$ be the character such that $P_{-n,0}(i(y_{v_0}, y_{\bar{v}_0})) = y_{v_0}^{-n}$ for $y_{v_0} \in 1+p\mathbb{Z}_p \cong 1+v_0O_{K,v_0}$ and $y_{\bar{v}_0} \in 1+p\mathbb{Z}_p \cong 1+\bar{v}_0O_{K,\bar{v}_0}$.

Suppose we have a character $\psi : Z_K(\mathfrak{C})_{\text{tor}} \rightarrow \overline{\mathbb{Q}}_p^\times$. If ψ is interpreted as a finite-order character $\mathbb{A}_{K,f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ whose restriction to $\widehat{O}_K^\times$ corresponds to the Dirichlet

character $(O_K/C(\psi))^\times \rightarrow \overline{\mathbb{Q}}_p^\times$, then we have $C(\psi) \mid \mathfrak{C}v_0\bar{v}_0$, because ψ factors through

$$\begin{aligned} & \omega_{v_0}[O_{K,v_0}^\times] \times \omega_{\bar{v}_0}[O_{K,\bar{v}_0}^\times] \times Cl_K \times (O_K/\mathfrak{C})^\times / \{\pm 1\} \\ & \cong (O_K/v_0)^\times \times (O_K/\bar{v}_0)^\times \times Cl_K \times (O_K/\mathfrak{C})^\times / \{\pm 1\} \end{aligned}$$

and $C(\psi)$ is determined by the restriction of ψ to $\widehat{O}_K^\times$.

For each ideal $\mathfrak{a}$ of O_K coprime to $C(\psi)p$, let

$$[\mathfrak{a}] \in Z_K(\mathfrak{C}) \cong K^\times \backslash \mathbb{A}_{K,f}^\times / U_K(\mathfrak{C}p^\infty)$$

be the class of a finite idele $z \in \mathbb{A}_{K,f}^\times$ for which $zO_K = \mathfrak{a}$ and $z_{C(\psi)} = 1_{C(\psi)}$. For an ideal $\mathfrak{a}$ of O_K not coprime to $C(\psi)p$, let $[\mathfrak{a}] = 0 \in O[[Z_K(\mathfrak{C})]]$.

In this paragraph, assume that $C(\omega_{v_0}^{-n}\psi) = \mathfrak{C}v_0\bar{v}_0$. Then the character $P_{-n,0}\omega_{v_0}^{-n}\psi : Z_K(\mathfrak{C}) \rightarrow \overline{\mathbb{Q}}_p^\times$ yields a character $\tilde{\chi} = P_{-n,0}\omega_{v_0}^{-n}\psi : K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$, which is the p -adic avatar of a Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ such that for $a \in O_K$ coprime to $\mathfrak{C}v_0\bar{v}_0$, we have

$$\chi(aO_K) = \tilde{\chi}(aO_K) = \tilde{\chi}(a\mathfrak{C}v_0\bar{v}_0)^{-1} = a^n\psi^{-1}(a) \quad (4.1)$$

(viewing $\chi(aO_K)$ and $\tilde{\chi}(aO_K)$ as the values of χ and $\tilde{\chi}$ respectively at the finite idele $a_f/a_{\mathfrak{C}v_0\bar{v}_0}$). Furthermore, as in [HiTi93, pp. 234-238], consider the O -algebra map $\lambda_1 : h_{2n+1,1}(pN|D_K|, O) \rightarrow O[[Z_K(\mathfrak{C})]]$ that sends $T(\ell)$ to $\sum_{\mathfrak{a}: N_{K/\mathbb{Q}}\mathfrak{a}=\ell} [\mathfrak{a}]$. Compose λ_1 with $(\omega_{v_0}^{-n}\psi)_\pm : O[[Z_K(\mathfrak{C})]] \rightarrow O[[\Gamma_K]]$ to obtain the Hida family $\lambda : h_{2n+1,1}(N|D_K|p, O) \rightarrow O[[\Gamma_K]]$ that sends $T(\ell)$ to $\sum_{\mathfrak{a}: N_{K/\mathbb{Q}}\mathfrak{a}=\ell} (\omega_{v_0}^{-n}\psi)_\pm[\mathfrak{a}]$. Then

$$P_{-n,0}(\lambda(T(\ell))) = \sum_{\mathfrak{a}: N_{K/\mathbb{Q}}\mathfrak{a}=\ell} \chi(\mathfrak{a}) \quad (4.2)$$

for ℓ coprime to $\mathfrak{C}p$, and $F(\lambda, P_{-n,0}) = \theta_\chi$. Thus the Hida family λ interpolates θ_χ .

In this paragraph, assume that $C(\omega_{v_0}^{-n}\psi) = \mathfrak{C}v_0$. As before, $P_{-n,0}\omega_{v_0}^{-n}\psi : Z_K(\mathfrak{C}) \rightarrow \overline{\mathbb{Q}}_p^\times$ yields a character $\tilde{\chi} = P_{-n,0}\omega_{v_0}^{-n}\psi : K^\times \backslash \mathbb{A}_{K,f}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$, which is the p -adic avatar

of a Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ such that for $a \in O_K$ coprime to $\mathfrak{C}$, we have $\chi(aO_K) = a^n \psi^{-1}(a)$: if a is also coprime to v_0 , then (4.1) holds (viewing $\chi(aO_K)$ and $\tilde{\chi}(aO_K)$ as being evaluated at the finite idele $a_f/a_{\mathfrak{C}v_0}$), and if a is not coprime to v_0 , then we have

$$\chi(aO_K) = a^n \tilde{\chi}(aO_K) = a^n \tilde{\chi}(a_{\mathfrak{C}})^{-1} = a^n \psi^{-1}(a)$$

(evaluating $\chi(aO_K)$ and $\tilde{\chi}(aO_K)$ at $a_f/a_{\mathfrak{C}}$). As before, compose λ_1 with the map $\psi_{\pm} : O[[Z_K(\mathfrak{C})]] \rightarrow O[[\Gamma_K]]$ to obtain the Hida family $\lambda : h_{2n+1,1}(N|D_K|, O) \rightarrow O[[\Gamma_K]]$. Again, we have (4.2) for ℓ coprime to $\mathfrak{C}p$. On the complex upper half plane, define the function $U\theta_{\chi}$ so that $U\theta_{\chi}(s) = \theta_{\chi}(s) - \chi(\bar{v}_0)\theta_{\chi}(ps)$; then $U\theta_{\chi}$ has q -expansion

$$U\theta_{\chi} = \sum_{n=1}^{\infty} \begin{Bmatrix} a(n, \theta_{\chi}) & p \nmid n \\ 0 & p \mid n \end{Bmatrix} q^n$$

and $U\theta_{\chi} = F(\lambda, P_{-n,0})$. This p -stabilized Hida family appears in p -adic L -function interpolation formulas later in the thesis.

4.4 Local characters and representations

I introduce notation related to local characters and their associated local representations, adapting the notation in [Hi91], [HiTi93] and [Wa18]. The following Gauss sums and characters η and η' appear in p -adic L -function interpolation formulas later.

For a (not necessarily unitary) character $\eta : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$, define

$$g_p(\eta) = \sum_{u \in (\mathbb{Z}_p/C(\eta))^\times} \eta^{-1}(u) \exp\left(-2\pi i \frac{u}{C(\eta)}\right)$$

as a local Gauss sum, where the ideal $C(\eta)$ of $\mathbb{Z}_p$ is identified with a generator of the form p^t , $t \in \mathbb{Z}$. (The Gauss sum $g_p(\eta)$ is written $G(\eta^{-1})$ in [Hi91]. Following a usual convention, I interpret the expression $\exp(-2\pi i x)$ with $x \in \mathbb{Q}_p$ to mean $e^{-2\pi i ap^{-n}}$

where $x = ap^{-n} + b$ for some $a \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}_p$.) Similarly, for a (not necessarily unitary) character $\eta : K_{v_0}^\times \rightarrow \mathbb{C}^\times$, define

$$g_{v_0}(\eta) = \frac{\eta(C(\eta))}{N_{K/\mathbb{Q}}(C(\eta))} \sum_{u \in (O_{K,v_0}/C(\eta))^\times} \eta^{-1}(u) \exp \left(-2\pi i \operatorname{Tr}_{K_{v_0}/\mathbb{Q}_p} \left(\frac{u}{2\delta C(\eta)} \right) \right)$$

as a local Gauss sum, where the ideal $C(\eta)$ of O_{K,v_0} is identified with a generator of the form p^t , $t \in \mathbb{Z}$, and where the number $\delta \in (\mathbb{R}_{>0}i) \cap K$ is chosen so that the fractional ideal $\{\operatorname{Tr}_{K/\mathbb{Q}}(x\bar{y}/(2\delta)) : x, y \in O_K\}$ of $\mathbb{Q}$ is coprime to pN . (For a Hecke character $\lambda : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ with $\lambda_{v_0} = \eta^{-1}$, the Gauss sum $g_{v_0}(\eta)$ is written $W_p(\lambda)$ in [HiTi93, p. 192], taking the same δ as in that source.)

For (not necessarily unitary) characters $\eta : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ and $\eta' : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$, the representation $\pi(\eta, \eta')$ (principal series) or $\sigma(\eta, \eta')$ (special with $\eta' = \eta| \cdot |_p^{-1}$) is the infinite-dimensional irreducible subquotient of the representation induced (from the upper-triangular Borel to GL_2) by the character

$$c : \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto \eta(a)\eta'(d)|a/d|_p^{1/2}.$$

That is, $\pi(\eta, \eta')$ or $\sigma(\eta, \eta')$ is the infinite-dimensional irreducible subquotient of the representation of locally constant functions $F : GL_2(\mathbb{Q}_p) \rightarrow \mathbb{C}$ such that $F(bg) = c(b)F(g)$ for $g \in GL_2(\mathbb{Q}_p)$ and b in the Borel subgroup of $GL_2(\mathbb{Q}_p)$. The group $GL_2(\mathbb{Q}_p)$ acts on these representations by right translation.

4.5 Global automorphic representations

I introduce automorphic representations and associated concepts, continuing to use and adapt notation from Hida [Hi91].

Let k be even and consider $F \in S_{k,k/2}(S, \mathbb{C})$ with $U(M) \subseteq S \subseteq U_0(M)$. Let $\pi(F)$

be the automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ generated by the right translations of F . For the classical modular form $f = F_1$ corresponding to F , write $\pi(f) = \pi(F)$.

If f is an eigenform for all but finitely many Hecke operators $T(\ell)$, then $\pi(F)$ is irreducible and there is an isomorphism $\pi(F) \cong \widehat{\otimes}_v \pi_v(F)$ over all (finite and infinite) places of $\mathbb{Q}$, where $\widehat{\otimes}$ is the completed tensor product and $\pi_v(F)$ is a certain representation of $GL_2(\mathbb{Q}_v)$ (see [MaPa07, p. 333] and [JaLa70]).

Let P , λ and $F(\lambda, P)$ be as in section 4.2. Write $\pi(P) = \pi(F(\lambda, P))$ and $\pi_v(P) = \pi_v(F(\lambda, P))$. Let the characters $\eta_{P,p} : \mathbb{Q}_p^{\times} \rightarrow \mathbb{C}^{\times}$ and $\eta'_{P,p} : \mathbb{Q}_p^{\times} \rightarrow \mathbb{C}^{\times}$ be such that $\pi_p(P)$ is equivalent to either $\pi(\eta_{P,p}, \eta'_{P,p})$ (principal series) or $\sigma(\eta_{P,p}, \eta'_{P,p})$ (special with $\eta'_{P,p} = \eta_{P,p}|\cdot|_p^{-1}$).

4.6 The W factors

I introduce W factors relating to the Atkin-Lehner involution here. References for this section are [DiSh10, exercises 1.5.4, 5.5.1 and section 5.10] and [Hi91, pp. 344-345] (where formula (4.10b) should have no minus sign).

In this section, let M be arbitrary (a multiple of p or not). An Atkin-Lehner-involution-like map w_M for adelic cusp forms $F \in S_k(U(M), \mathbb{C})$ is given by

$$F \mapsto F|_k \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}_f$$

which corresponds via $F \mapsto F_1$ to the classical map w_M defined by

$$F_1 \mapsto F_1|_k \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}$$

for $F_1 \in S_k(\Gamma_1(M))$. For primitive $F \in S_{k,w}(U(M), \mathbb{C})$ of conductor M , Hida [Hi91, pp. 344-345] describes

- (a) a number $W(F) \in \mathbb{C}$ such that $|W(F)| = 1$ and $M^{k/2-1}w_M F = W(F)F^c$, and
- (b) a local factor decomposition $W(F) = W'(F)W_p(F)$, where $W'(F)$ is thought of as the product of W factors away from p .

The Atkin-Lehner involution is $i^k M^{k/2-1}w_M$; this yields $W(F^c)W(F) = (-1)^k$.

To compare notation with Bertolini, Darmon & Prasanna [BeDaPr13, pp. 1088-1089]: If that source's f is my F_1 , then that source's w_f is my $i^k W(F) = i^k / W(F^c)$.

4.7 Petersson inner product

Let f_1 and f_2 be classical modular forms, both in $S_{k'}(\Gamma_0(M), \chi)$ for some Dirichlet character χ . Their Petersson inner product is defined to be

$$\langle f_1, f_2 \rangle_{\Gamma_0(M)} = \int_{\Gamma_0(M) \backslash \mathfrak{h}} f_1(\tau) \overline{f_2(\tau)} y^{k'} \frac{dx dy}{y^2}$$

where $\tau = x + iy \in \mathbb{C}$. For adelic modular forms F and G both in $S_{k',w}(U(M), \mathbb{C})$, one can write (if the following is well defined)

$$\langle F, G \rangle_{U_0(M)} = \langle F_1, G_1 \rangle_{\Gamma_0(M)} = \int_{X_0(M)} F(g) \overline{G(g)} |\det g|_{\mathbb{A}, \mathbb{Q}}^{k'-2w} d\mu_M$$

for a certain measure μ_M on the space

$$X_0(M) := (\mathbb{A}_{\mathbb{Q}}^{\times} GL_2(\mathbb{Q})) \backslash GL_2(\mathbb{A}_{\mathbb{Q}}) / (U_0(M) \times (\mathbb{R}_{>0}^{\times} SO_2(\mathbb{R})))$$

for which we have $X_0(M) \cong \Gamma_0(M) \backslash \mathfrak{h}$ as usual [Hi91, pp. 338, 342-343].

Define $D(s, F_1, G_1) = \sum_{n=1}^{\infty} a(n, F_1) a(n, G_1) / n^s$. Using the Rankin-Selberg method (for the details, see [Si17, p. 35] for example), one can obtain the equation

$$\langle F_1, G_1 \rangle_{\Gamma_0(M)} = \text{vol}(\Gamma_0(M) \backslash \mathfrak{h}) \cdot \frac{\Gamma(k')}{(4\pi)^{k'}} \text{res}_{s=k'} D(s, F_1, G_1^c). \quad (4.3)$$

Chapter 5

L -functions

5.1 The Rankin-Selberg L -function

This section introduces the Rankin-Selberg L -function, following Hida and Tilouine [HiTi93, pp. 243-244].

Consider classical cusp forms g and h that are eigenforms for the Hecke operators away from their levels. Let the adelic forms $G \in \overline{S}_{k_G, w_G}^{ord}(M, 1, \psi_G, \mathbb{C})$ and $H \in \overline{S}_{k_H, w_H}^{ord}(N, 1, \psi_H, \mathbb{C})$ be such that $g = G_1 \in S_{k_G}(\Gamma_1(M), \psi_G)$ and $h = H_1 \in S_{k_H}(\Gamma_1(N), \psi_H)$ are Hecke eigenforms, where ψ_G and ψ_H are finite-order characters. The automorphic Rankin-Selberg L -function of g and h , or equivalently of G and H , can be expressed as a product of Euler factors using adelic representations [HiTi93, pp. 243-244]. Let $\pi(G) \cong \widehat{\otimes}_v \pi_v(G)$ and, if $\pi_v(G)$ is principal series or special, $\pi_v(G) = \pi(\eta_{G,\ell}, \eta'_{G,\ell})$ or $\sigma(\eta_{G,\ell}, \eta'_{G,\ell})$ respectively; use similar notation for H . For finite primes ℓ , define the set $S_\ell(G)$ to be $\{\ell^{w_G - (k_G+1)/2} \eta_{G,\ell}(\ell), \ell^{w_G - (k_G+1)/2} \eta'_{G,\ell}(\ell)\}$ if $\pi_\ell(G)$ is spherical, $\{\ell^{w_G - (k_G+1)/2} \eta_{G,\ell}(\ell)\}$ if $\pi_\ell(G)$ is minimal but not spherical and not supercuspidal, and $\emptyset$ if $\pi_\ell(G)$ is supercuspidal; similarly for H . Then

$$L(s, \pi(g) \times \pi(h)) = L(s, \pi(G) \times \pi(H)) = \prod_{\ell \nmid \infty} L_\ell(\ell^{-s})^{-1}$$

where for each finite rational prime ℓ , we have

$$L_\ell(X) = \prod_{\alpha_\ell(G) \in S_\ell(G)} \prod_{\alpha_\ell(H) \in S_\ell(H)} (1 - \alpha_\ell(G)\alpha_\ell(H)X).$$

(The factor $L_\ell(X)$ is written $\mathfrak{D}_q(X)$ in Hida and Tilouine [HiTi93].)

One can express $L(s, \pi(\theta_{\chi^{-1}}) \times \pi(f))$ in terms of the classical Rankin-Selberg L-function $L(s, \theta_{\chi^{-1}} \times f)$. For $f \in S_k(\Gamma_0(N))$ and a Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$, the ℓ -Euler factor of $L(s, \theta_{\chi^{-1}} \times f)$ for ℓ coprime to the conductors of χ and f is

$$\prod_{v|\ell} \prod_{\gamma \in \{\alpha_\ell, \beta_\ell\}} (1 - \chi^{-1}(v)\gamma\ell^{-s})^{-1} = \prod_{v|\ell} (1 - \chi^{-1}(v)a(\ell, f)\ell^{-s} + \chi^{-2}(v)\ell^{k-1-2s})^{-1}$$

where α_ℓ and β_ℓ are such that $\alpha_\ell + \beta_\ell = a(\ell, f)$ and $\alpha_\ell\beta_\ell = \ell^{k-1}$, and we have

$$L(s, \theta_{\chi^{-1}} \times f) = L(s + \tfrac{1}{2}, \pi(\theta_{\chi^{-1}}) \times \pi(f))$$

(see Bertolini, Darmon and Prasanna [BeDaPr13, pp. 1088-1089]).

I write $L^p(s, \theta_{\chi^{-1}} \times f)$ for the L-function $L(s, \theta_{\chi^{-1}} \times f)$ without the Euler factor at p (that is, without the Euler factor at each place over p).

The Rankin-Selberg L-functions $L(s, \theta_{\chi^{-1}} \times f)$ and $L^p(s, \theta_{\chi^{-1}} \times f)$ have meromorphic continuations to $\mathbb{C}$; see for example [CoKiMu04, pp. 222-224].

5.2 p -adic L -functions

This section gives interpolation formulas for different p -adic L -functions. Let k be a fixed even positive integer. Throughout this section, the character $P_{ac} : \Gamma_K^- \rightarrow \overline{\mathbb{Q}}_p^\times$ and the anticyclotomic character $\psi : Z_K(\mathfrak{C})_{\text{tor}} \rightarrow \overline{\mathbb{Q}}_p^\times$ are allowed to vary so that, for the characters $P = P_{ac} \circ pr_{ac} : \Gamma_K \rightarrow \overline{\mathbb{Q}}_p^\times$ and $P\psi : Z_K(\mathfrak{C}) \rightarrow \overline{\mathbb{Q}}_p^\times$, there is a positive integer $n \geq k/2$, depending on P_{ac} , such that $(P\psi)^{\text{alg}}(z_\infty) = z_\infty^n \bar{z}_\infty^{-n}$. Define the

variable $j = (2n - k)/2$, which is always a non-negative integer. Each interpolation formula evaluates $P_{ac} : \widehat{O}_L^{ur}[[\Gamma_K^-]] \rightarrow \overline{\mathbb{Q}}_p$ at a p -adic L -function.

5.2.1 The Katz p -adic L -function of Hida and Tilouine

A 1993 paper of Hida and Tilouine constructs the Katz p -adic L -function $L_{Katz}^-(K) \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$ as follows. There is an element $\mu \in \widehat{O}_L^{ur}[[Z_K(\mathfrak{C})]]$ satisfying an interpolation formula [HiTi93, Theorem II on p. 193]; define $L_{Katz}^-(K) = \psi_{ac}(sq(\widetilde{|\cdot|_{\mathbb{A},K} \cdot \mu}))$, where $\widetilde{|\cdot|_{\mathbb{A},K} \cdot \mu}$ is interpreted as a measure¹ in $\widehat{O}_L^{ur}[[Z_K(\mathfrak{C})]]$. Let the Hecke character $\xi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ with infinity type $z_\infty \mapsto z_\infty^{k+j} \bar{z}_\infty^{-j}$ be such that $\xi \xi^{-c}$ has p -adic avatar $(P\psi) \circ sq$ and infinity type $z_\infty \mapsto z_\infty^{2n} \bar{z}_\infty^{-2n}$. Then the interpolation formula for $L_{Katz}^-(K)$ is

$$P_{ac}(L_{Katz}^-(K)) = P_{ac}\psi_{ac}(sq(\widetilde{|\cdot|_{\mathbb{A},K} \cdot \mu})) = (P\psi)(sq(\widetilde{|\cdot|_{\mathbb{A},K} \cdot \mu})) = F_{93}C_{1,93}C_{2,93}^n \quad (5.1)$$

where the function F_{93} and the constants $C_{1,93}$ and $C_{2,93}$ are given by

$$F_{93} = L(1, \xi \xi^{-c})_{g_{v_0}}((\xi^{-1} \xi^c)_{v_0}) \Gamma(2n+1) \cdot (1 - p^{-1} \xi \xi^{-c}(\bar{v}_0))(1 - \xi \xi^{-c}(\bar{v}_0)) \prod_{v|\mathfrak{C}} (1 - (Nv)^{-1} \xi \xi^{-c}(v))$$

$$C_{1,93} = \frac{2\mathrm{im}(\delta)}{\pi}$$

$$C_{2,93} = \left(\frac{\Omega_p}{\Omega_\infty}\right)^4 \left(\frac{\pi}{\mathrm{im}(\delta)}\right)^2$$

for certain periods $\Omega_p \in \widehat{O}_L^{ur}$, $\Omega_\infty \in \mathbb{C}$ defined in [HiTi93], using the same δ as in section 4.4. (To compare notation with [HiTi93]: that source's m_0 , d and λ are, in

¹Recall that the function $\tilde{\chi} : Z_K(\mathfrak{C}) \rightarrow \overline{\mathbb{Q}}_p^\times$ is the p -adic avatar of a complex-valued Hecke character χ . A function can be multiplied by a measure to give a measure (if defined); in the case of the real line, a continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$ and the standard measure dx on $\mathbb{R}$ yield the measure Fdx , of which the integral over $\mathbb{R}$ is $\int Fdx$ (if defined).

my notation, 2 , $2n - 1$ and $|\xi\xi^{-c}| \cdot |\mathbb{A}_K|$ respectively. See also [Wa18, Definition 7.9], which gives the main terms in the interpolation formula, written in a different way.)

5.2.2 The L -function of Hida

For Hida families $\lambda : h_{2n+1}(N|D_K|, \widehat{O}_L^{ur}) \rightarrow \widehat{O}_L^{ur}[[\Gamma_K]]$ and $\lambda' : h_k(N, \widehat{O}_L^{ur}) \rightarrow \widehat{O}_L^{ur}[[\Gamma_K]]$, a 1991 paper of Hida gives $\mathcal{D} \in \widehat{O}_L^{ur}[[\Gamma_K]] \widehat{\otimes}_{\widehat{O}_L^{ur}} \widehat{O}_L^{ur}[[\Gamma_K]]$ so that for arithmetic $\widehat{O}_L^{ur}$ -algebra maps $P_1 : \widehat{O}_L^{ur}[[\Gamma_K]] \rightarrow \overline{\mathbb{Q}}_p$ and $Q : \widehat{O}_L^{ur}[[\Gamma_K]] \rightarrow \overline{\mathbb{Q}}_p$, $(P_1 \otimes Q)(\mathcal{D})$ is given by an interpolation formula [Hi91, p. 317].

Let Q be fixed so that $F(\lambda', Q) = f^c$, $\psi'_Q = 1$, $k(Q) = k$ and $w(Q) = 1 - v$ for a certain integer v . Write $D_Q \in \widehat{O}_L^{ur}[[\Gamma_K]]$ for $((sq \circ pr_{v_0}) \otimes Q)(\mathcal{D}) \in \widehat{O}_L^{ur}[[\Gamma_K]] \widehat{\otimes}_{\widehat{O}_L^{ur}} \widehat{O}_L^{ur} \cong \widehat{O}_L^{ur}[[\Gamma_K]]$, and write $D_Q^- = pr_{ac}(D_Q) \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$.

Let $P_1 = P \circ sq \circ pr_{v_0}$. Assume $\psi'_{P_1} = 1$ (this is actually an assumption about the Hida family λ). Assume that each of $p^{\gamma(p)} = C(\eta_{Q,p})$, $p^{\gamma'(p)} = C(\eta'_{Q,p})$ and $p^{\delta(p)} = C(\eta'_{P_1,p})$ has at least one factor of p . Assume $F(\lambda, P_1)$ is p -ordinary.

Then Hida's interpolation formula gives

$$P_{ac}(D_Q^-) = P_{ac}(pr_{ac}(D_Q)) = P(D_Q) = F_{91} C_{1,91} C_{2,91}^n \quad (5.2)$$

with

$$F_{91} = \frac{1}{W'(F(\lambda, P_1))} \cdot \frac{g_p(\eta_{Q,p}) g_p(\eta'_{Q,p})}{g_p(\eta'_{P_1,p})} \cdot \frac{\Gamma(n + \frac{1}{2}k) \Gamma(n - \frac{1}{2}k + 1) p^{\delta(p)}}{\eta_{P_1,p}(p^{\gamma(p)+\gamma'(p)}) \eta'_{P_1,p} \eta_{P_1,p}^{-1}(p^{\delta(p)})} \cdot \frac{L^p(0, F(\lambda, P_1) \times f)}{\langle F(\lambda, P_1), F(\lambda, P_1) \rangle_{\Gamma_0(N|D_K|)}}$$

$$C_{1,91} = \eta_{Q,p}(p^{\gamma(p)+\gamma'(p)}) \cdot \psi_Q \psi'_Q(-1) \frac{\sqrt{N|D_K|}}{2\pi} W'(f^c)$$

$$C_{2,91} = \frac{1}{16\pi^2 |D_K|}.$$

(To compare with [Hi91]: Hida's P and Q are my P_1 and Q respectively. In Hida, we

have $M = K$, $F = \mathbb{Q}$, $d = 1_f$, $k(P) = \text{my } k(P_1) = 2n + 1$, $w(P) = \text{my } w(P_1) = 1$, $k(Q) = k$, $w(Q) = 1 - j$, $n(P) = \text{my } n(P_1) = 2n - 1$, $v(P) = \text{my } v(P_1) = 0$, $n(Q) = k - 2$ and $v(Q) = j$, where $k + 2j = 2n$. Hida's N and J are, in my notation, $|D_K|N = |D_K|N_{K/\mathbb{Q}}\mathfrak{C}$ and N respectively. Hida's ξ_p and ξ'_p are my $\eta_{Q,p}$ and $\eta'_{Q,p}$ respectively; see also [Ge75, pp. 65-66].)

5.2.3 The L -function of Wan

For this subsection, let $I = \widehat{O}_L^{ur}[[\Gamma_K]]$. Fix an irreducible component of $I \widehat{\otimes}_{O_L} \widehat{O}_L^{ur}$, and let that component's associated ring have normalization $\widehat{I}^{ur}$ (see [Wa18, p. 2]).

For a set S of finitely many places of K including v_0 and $\bar{v}_0$, Wan [Wa18, section 7.4] defines two related p -adic L -functions in $\widehat{I}^{ur}[[\Gamma_K]]$ which I call $\widetilde{L}_{Wan}^S$ and L_{Wan}^S ; Wan calls them $\mathcal{L}_{\mathbf{f}, \mathcal{K}, \xi}^{S, \text{Hida}}$ and $\mathcal{L}_{\mathbf{f}, \mathcal{K}, \xi}^S$ respectively. For the Hida family $Q : I \rightarrow \overline{\mathbb{Q}}_p$ corresponding to f , write $\widetilde{L}_{Wan}^S(f) = Q(\widetilde{L}_{Wan}^S)$ and $L_{Wan}^S(f) = Q(L_{Wan}^S)$; let their images under pr_{ac} be $\widetilde{L}_{Wan}^{-,S}(f)$ and $L_{Wan}^{-,S}(f)$ respectively.

We have $\widetilde{L}_{Wan}^{-,N_p}(f) = D_Q^- L_{Katz}^-(K) \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$, and $L_{Wan}^{-,N_p}(f) = C_{Wan} \widetilde{L}_{Wan}^{-,N_p}(f)$ for a constant $C_{Wan} \in O_{\overline{\mathbb{Q}}_p}$ which Wan calls $C_{\mathbf{f}, K, \xi}$. Starting with $\widetilde{L}_{Wan}^{N_p}$ or $L_{Wan}^{N_p}$ and omitting the Euler factors at primes over $S \setminus N_p$ yields $\widetilde{L}_{Wan}^S$ or L_{Wan}^S respectively.

5.2.4 The BDP L -function

A 2013 paper of Bertolini, Darmon and Prasanna gives a p -adic L -function $L_{BDP}^{-,N_p}(f)$ as follows. (That paper's interpolation formula [BeDaPr13, (5.2.3)] defines a number $L_p(f, \chi)$, which is what I call $P_{ac}(L_{BDP}^{-,N_p}(f))$, using a term $L_{alg}(f, \chi^{-1}, 0)$ which is defined earlier in the same article [BeDaPr13, Theorems 5.5 and 4.6, and (5.1.11)].)

Let the Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$ be so that $P\psi$ is the p -adic avatar of $\chi^{-1}N^{k/2}$ (so $(\chi^{-1}N^{k/2})N^n$ has infinity type $z_\infty \mapsto z_\infty^0 \bar{z}_\infty^{-2n}$). Let $\mathfrak{b}$ be an ideal of O_K coprime to Np , and let b_N be a number in O_K , such that $\mathfrak{b}\mathfrak{C} = b_N O_K$. Choose the periods Ω_p and Ω_∞ in [BeDaPr13, (5.2.2) and (5.1.16)] to be the same as those

in the previous formula of Hida and Tilouine, following the argument of [JeSkWa17, section 5.2]. (In fact, if the two Ω_p values are allowed to have a p -adic unit as their quotient – and if the two Ω_∞ values have the corresponding complex number as their quotient – then the only change to my argument would be to multiply the number $\tilde{C}_2$ in chapter 7 by the appropriate p -adic unit.) Then for $n \geq k/2$, the element $L_{BDP}^{-,N_p}(f) \in \widehat{O}_L^{ur}[[\Gamma_K^-]]$ satisfies

$$P_{ac}(L_{BDP}^{-,N_p}(f)) = F_{13}C_{1,13}C_{2,13}^n \quad (5.3)$$

with

$$F_{13} = \Gamma\left(n + \frac{1}{2}k\right) \Gamma\left(n - \frac{1}{2}k + 1\right) (P\psi)^{alg}(\mathfrak{b}) \psi_{(O_K/\mathfrak{C})^\times}(N_{K/\mathbb{Q}}\mathfrak{b}) \\ \cdot (1 - \chi^{-1}(\bar{v}_0)a(p, f) + \chi^{-2}(\bar{v}_0)p^{k-1})^2 \cdot L(0, \theta_{\chi^{-1}} \times f)$$

$$C_{1,13} = \frac{2\sqrt{|D_K|}}{4\pi(-1)^{k/2}} W(F(\lambda', Q))$$

$$C_{2,13} = \frac{\Omega_p^4 \pi^2}{\Omega_\infty^4} \cdot \frac{b_N^2 |\mathfrak{b}|_{\mathbb{A},K}}{-N}$$

where $\psi_{(O_K/\mathfrak{C})^\times}$ is obtained by precomposing ψ with the projection from $Z_K(\mathfrak{C})_{\text{tor}}$ to its $(O_K/\mathfrak{C})^\times / \{\pm 1\}$ part, using the decomposition (3.2).

(To compare with [BeDaPr13]: That source's $k + 2j$, with $j \in \mathbb{Z}_{\geq 0}$, is my $2n$; the source's $\mathfrak{N}$, c and χ are, in my notation, $\mathfrak{C}$, 1 and χ respectively. Since f has nebentypus 1, the set called by the two names $S(f)$ and S_f in [BeDaPr13, pp. 1094–1095] is empty.)

Note that [BeDaPr13, Proposition 5.10] gives this p -adic L -function as merely a continuous function sending certain Hecke characters χ to $P_{ac}(L_{BDP}^{-,N_p}(f))$. In fact, the p -adic L -function yields a measure; this is mentioned in [Bu15] and, for f of weight 2, [JeSkWa17]. Indeed, chapter 7's equations will show that $L_{BDP}^{-,N_p}(f)$ is just Wan's

p -adic L -function multiplied by a certain unit.

As in [JeSkWa17, section 5.1], for a set S of finitely many places of K such that $S \supseteq N_p$, define

$$L_{BDP}^{-,S}(f) = L_{BDP}^{-,N_p}(f) \cdot \prod_{v \in S \setminus N_p} L_v(f)$$

where $L_v(f)$ is the Euler factor at v .

Chapter 6

Algebraic geometry and cohomology

6.1 Kuga-Sato varieties and projections

I refer to [Ne92], [Ma17], [BeDaPr13] as references. As in chapter 1, I assume that $N \geq 5$ and that the prime p is coprime to $2N\phi(N) \cdot (k-2)!$.

As usual, let $Y(N)$, $Y_1(N)$, $Y_0(N)$ be the open modular curves over $\mathbb{Q}$, and let $X(N)$, $X_1(N)$, $X_0(N)$ be the complete modular curves over $\mathbb{Q}$. (See [DiSh10] or other textbooks on modular forms.)

Each of the curves $X(N)$ and $X_1(N)$ has a standard proper smooth model over $\text{Spec } \mathbb{Z}[1/N]$, hence also over $\text{Spec } \mathbb{Z}_p$ since $p \nmid N$. Define

$$X(N)_{\mathbb{F}_p} = (\text{proper smooth model of } X(N)) \times_{\mathbb{Z}_p} \mathbb{F}_p;$$

then there is a mod p reduction map $\text{red}_p : X(N)(\mathbb{C}_p) \rightarrow X(N)_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$. Similarly, there is a mod p reduction map $\text{red}_p : X_1(N)(\mathbb{C}_p) \rightarrow X_1(N)_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$; see [BeDaPr13, section 1.3].

Let $j : Y(N) \hookrightarrow X(N)$ be the standard inclusion map. Let $\pi : \mathcal{E}_Y(\Gamma(N)) \rightarrow Y(N)$

be the universal elliptic curve. Let $\mathcal{E}_X(\Gamma(N)) \rightarrow X(N)$ be the universal generalized elliptic curve. The $(k-2)$ th power of $\mathcal{E}_X(\Gamma(N))$ over $X(N)$ has as its standard desingularization the Kuga-Sato variety $\tilde{\mathcal{E}}^{k-2}(\Gamma(N))$. (Nekovář [Ne92] writes $Y(N)$, $X(N)$, $\mathcal{E}_Y(\Gamma(N))$, $\mathcal{E}_X(\Gamma(N))$ as M_N , $\overline{M}_N$, X_N , $\overline{X}_N$ respectively.)

Define the corresponding concepts where $\Gamma_1(N)$ replaces $\Gamma(N)$: let $j_1 : Y_1(N) \hookrightarrow X_1(N)$ be the inclusion, let $\pi_1 : \mathcal{E}_Y(\Gamma_1(N)) \rightarrow Y(N)$ be the universal elliptic curve, let $\mathcal{E}_X(\Gamma_1(N)) \rightarrow X_1(N)$ be the universal generalized elliptic curve, and let its r th power have the Kuga-Sato variety $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ as the standard desingularization. See [BeDaPr13].

Fixing the “forget a $(\mathbb{Z}/N\mathbb{Z})$ -basis vector” map $X(N) \rightarrow X_1(N)$, we get a map $\mathcal{E}_X(\Gamma(N)) \rightarrow \mathcal{E}_X(\Gamma_1(N))$, which yields maps $P_r : \tilde{\mathcal{E}}^r(\Gamma(N)) \rightarrow \tilde{\mathcal{E}}^r(\Gamma_1(N))$.

Define the projection operators

$$\pi_B = \left| \frac{\Gamma_0(N)}{\Gamma(N)} \right|^{-1} \sum_{\sigma \in \Gamma_0(N)/\Gamma(N)} \sigma \in \mathbb{Q} \left[\frac{\Gamma_0(N)}{\Gamma(N)} \right]$$

and

$$\pi_{B,1} = \left| \frac{\Gamma_0(N)}{\Gamma_1(N)} \right|^{-1} \sum_{\sigma \in \Gamma_0(N)/\Gamma_1(N)} \sigma \in \mathbb{Q} \left[\frac{\Gamma_0(N)}{\Gamma_1(N)} \right].$$

Since p is coprime to each of $|\Gamma_0(N)/\Gamma(N)| = N\phi(N)$ and $|\Gamma_0(N)/\Gamma_1(N)| = N$, we have $\pi_B \in \mathbb{Z}_p[\Gamma_0(N)/\Gamma(N)]$ and $\pi_{B,1} \in \mathbb{Z}_p[\Gamma_0(N)/\Gamma_1(N)]$.

In the following definition, the underlying set of a semidirect product of groups is interpreted to be the direct product of the underlying sets of the groups in the usual way (see [Mi10, Chapter 3, “Semidirect products” section, p. 44]). For $t \in \{0, 1, 2\}$ and $r \in \mathbb{Z}_{>0}$, I define the group $G(t, r) = ((\mathbb{Z}/N\mathbb{Z})^t \rtimes \{\pm 1\})^r \rtimes S_r$ so that, as described below, $G(0, r)$ acts on the r th power A^r of any elliptic curve A (see [BeDaPr13, p. 1052]), $G(1, r)$ acts on $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ (see [BeDaPr13, pp. 1056-1057]) and $G(2, r)$ acts on $\tilde{\mathcal{E}}^r(\Gamma(N))$ (see [Sc90, section 1.1] and [Ne92, section 2]).

(a) Action of $(\mathbb{Z}/N\mathbb{Z})^t$, for $t = 1$ or $t = 2$: Consider a fiber of $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ at a point

of $Y_1(N)$, or a fiber of $\tilde{\mathcal{E}}^r(\Gamma(N))$ at a point of $Y(N)$. Such a fiber is an r th power of some elliptic curve E . Each of the r subgroups $(\mathbb{Z}/N\mathbb{Z})^t$ in the definition of $G(t, r)$ translates a different one of the fiber's r components E by sections of order dividing N for $\Gamma_1(N)$ or $\Gamma(N)$ respectively.

- (b) Action of $\{\pm 1\}$, for $t \in \{0, 1, 2\}$: Consider A^r , or a fiber E^r of $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ at a point of $Y_1(N)$, or a fiber E^r of $\tilde{\mathcal{E}}^r(\Gamma(N))$ at a point of $Y(N)$. Each of the r subgroups $\{\pm 1\}$ in the definition of $G(t, r)$ multiplies a different one of the r components of A^r or E^r by ± 1 .
- (c) The subgroup S_r permutes those components: an element $s \in S_r$, which is a bijection $s : \{1, 2, \dots, r\} \rightarrow \{1, 2, \dots, r\}$, takes a point $(x_1, \dots, x_r)$ in A^r or E^r to $(x_{s^{-1}(1)}, x_{s^{-1}(2)}, \dots, x_{s^{-1}(r)})$.
- (d) A general element $\sigma \in G(t, r)$ acts on A^r , or on points of $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ over $Y_1(N)$, or on points of $\tilde{\mathcal{E}}^r(\Gamma(N))$ over $Y(N)$, as follows. Suppose $\sigma = ((z_i, \varepsilon_i)_{i=1}^r, s)$ where each z_i is in $(\mathbb{Z}/N\mathbb{Z})^t$, each ε_i is in $\{\pm 1\}$ and s is in S_r . Then the action of σ first applies s (permuting components), then applies the elements $\varepsilon_i \in \{\pm 1\}$ (multiplying the i th component by ± 1), then applies the elements $z_i \in (\mathbb{Z}/N\mathbb{Z})^t$ (translating the i th component by a section of order dividing N).
- (e) The actions over $Y_1(N)$ and $Y(N)$ extend continuously to actions over $X_1(N)$ and $X(N)$ (viewing $Y_1(N)$, $Y(N)$, $X_1(N)$ and $X(N)$ as topological spaces).

For $t \in \{0, 1, 2\}$, let $c_t : G(t, r) \rightarrow \{\pm 1\}$ be the group map that is 1 on each $(\mathbb{Z}/N\mathbb{Z})^t$ factor, the identity (that is, -1 is sent to -1) on each $\{\pm 1\}$ factor, and the sign map on S_r . Then define

$$\pi_{t,r} = \frac{1}{\#G(t, r)} \sum_{\sigma \in G(t, r)} c_t(\sigma) \cdot \sigma \in \mathbb{Q}[G(t, r)].$$

If $p > r$ then, since p is coprime to $\#G(t, r) = (2N^t)^r \cdot r!$, we have $\pi_{t,r} \in \mathbb{Z}_p[G(t, r)]$. In particular, I assume that p is coprime to $(k-2)!$, so $\pi_{2,k-2} \in \mathbb{Z}_p[G(2, k-2)]$; this example will be used later. (In [BeDaPr13], the projections $\pi_{1,r}$ and $\pi_{0,r}$ are written as ϵ_W and ϵ_A respectively. In [Ne92, Ma17], the projection $\pi_{2,k-2}$ is written as Π_ϵ .)

For a field $F \supseteq \mathbb{Q}$, there is a standard isomorphism of $\pi_{1,r}H_{dR}^*(\tilde{\mathcal{E}}^r(\Gamma_1(N))/F)$ with the parabolic cohomology group as given in [BeDaPr13, section 2.1].

Take a field $F \supseteq K_1$. For an elliptic curve A defined over F , where A has complex multiplication by O_K , one may choose F -vector space generators ω_A and η_A of $H_{dR}^{1,0}(A/F)$ and $H_{dR}^{0,1}(A/F)$ respectively; then $\pi_{0,r}H_{dR}^*(A^r/F) = \text{Sym}^r H_{dR}^1(A/F)$ is generated as an F -vector space by the $r+1$ elements

$$\omega_A^j \eta_A^{r-j} = \binom{r}{j}^{-1} \sum_{S \subseteq \{1,2,\dots,r\}} \left(\left(\bigwedge_{s \in S} pr_s^* \omega_A \right) \wedge \left(\bigwedge_{s \in \{1,2,\dots,r\}-S} pr_s^* \eta_A \right) \right)$$

with $j \in \{0, 1, \dots, r\}$; see [BeDaPr13, section 1.4]. Also, we have an isomorphism [BeDaPr13, Proposition 2.5]

$$S_{r+2}(\Gamma_1(N), F) \otimes \text{Sym}^r H_{dR}^1(A/F) \cong \text{Fil}^{r+1} \pi_{1,r} \pi_{0,r} H_{dR}^{2r+1}(\tilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r/F)$$

sending $f \otimes \eta$ to $\omega_f \wedge \eta$ for the differential form ω_f corresponding to f in the standard way. (For details of the definition of ω_f , see [BeDaPr13, section 1.1], in particular [BeDaPr13, equations (1.1.1), (1.1.5) and (1.1.12)].)

6.2 Chow groups and Heegner cycles

This section gives definitions related to Chow groups and defines Heegner cycles that will be used later in the paper.

For an algebraic variety U defined over a field F , the Chow group $CH^a(U/F)$ is the group of finite sums $\sum_i c_i U_i$ where all $c_i \in \mathbb{Z}$ and where the U_i are classes, under

rational equivalence, of codimension- a cycles in U defined over F . The subgroup $CH_0^a(U/F)$ is the group of such sums where the U_i can be represented by codimension- a cycles defined over F that are homologically equivalent to zero up to torsion (see [Ha77, p. 426] and [Ne00, section 1]).

See [Ma17, section 4.1]. For $\mathfrak{C}$ the ideal from the Heegner hypothesis on f and K , the isogeny $\mathbb{C}/O_K \rightarrow \mathbb{C}/\mathfrak{C}^{-1}$ gives $x_0 \in X_0(N)(K)$ by CM theory. Choose an $x \in X(N)$ that is sent to x_0 under the standard map $X(N) \rightarrow X_0(N)$; the fiber E_x for $\mathcal{E}_X(\Gamma(N)) \rightarrow X(N)$ at x (which is also the fiber E_{x_1} for $\mathcal{E}_X(\Gamma_1(N)) \rightarrow X_1(N)$ at the image x_1 of x in $X_1(N)$) is an elliptic curve with complex multiplication by O_K , so the variety $\text{Graph}(\sqrt{D_K})$ exists in E_x^2 . We have an embedding $i_x : E_x^{k-2} \hookrightarrow \tilde{\mathcal{E}}^{k-2}(\Gamma(N))$. Define the Heegner cycle

$$\Delta_{Ma17} = \pi_B \pi_{2,k-2}(i_x)_*(\text{Graph}(\sqrt{D_K})^{k/2-1}) \in CH^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

and let

$$Z_{Ma17} = N_{K_1/K} \Delta_{Ma17} \in N_{K_1/K}(CH^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$$

be the image of Δ_{Ma17} under the norm map $N_{K_1/K} = \sum_{g \in \text{Gal}(K_1/K)} g$.

See [Ca13, section 3]. Similarly, with an embedding $i_{x_1} : E_{x_1}^{k-2} \hookrightarrow \tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$, we define

$$\Delta_{Ca13} = \pi_{B,1} \pi_{1,k-2}(i_{x_1})_*(\text{Graph}(\sqrt{D_K})^{k/2-1}) \in CH^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

as a Heegner cycle. Also, for an ideal $\mathfrak{a}$ of O_K , define the modified Heegner cycle

$$\Delta_{Ca13,\mathfrak{a}} = \pi_{B,1} \pi_{1,k-2}(\text{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1}) \in CH^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/K_1) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

For an isogeny $\varphi : A \rightarrow A'$ between elliptic curves A and A' , where A' has $\Gamma_1(N)$ structure, consider

$$\text{Graph}(\varphi)^r \subseteq (A')^r \times A^r \subseteq \tilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r$$

(embedding in the fiber in $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ at the point linked to A') and the corresponding Heegner cycle $\Delta_\varphi = \pi_{1,r}\pi_{0,r}(\text{Graph}(\varphi)^r)$ in $\tilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r$ (where $\pi_{1,r}$ and $\pi_{0,r}$ act on $\tilde{\mathcal{E}}^r(\Gamma_1(N))$ and A^r respectively). For each nonzero integral ideal $\mathfrak{a}$ of O_K and elliptic curve A , we have a “modulo $\mathfrak{a}$ -torsion” isogeny $\varphi(A, \mathfrak{a}) : A \rightarrow A/A[\mathfrak{a}]$ (see [BeDaPr13, formula 1.4.7]).

Choose representatives $\mathfrak{a}$ of the class group of K so that the numbers $N(\mathfrak{a})$ seen as elements of $\widehat{\mathbb{Q}}_p$ are p -adic units. Then, taking a sum over the classes $[\mathfrak{a}]$ of the class group of K , define

$$Z_{BeDaPr13} = \frac{1}{(k/2 - 1)!} \sum_{[\mathfrak{a}]} \frac{1}{N^{k/2-1}(\mathfrak{a})} \cdot \Delta_{\varphi(E_{x_1}, \mathfrak{a})} \in CH(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

6.3 Cohomology

I briefly recall the definitions of continuous cohomology and describe restriction and corestriction, following [NeScWi13, pp. 12-13, 45-48, 51, 61, 67]. Let G be a topological group and let U be a G -module.

The $\mathbb{Z}$ -module $X^n(G, U)$ of continuous functions from G^{n+1} to U is a G -module: an element $g \in G$ acts on an element $F \in X^n(G, U)$ to yield $gF \in X^n(G, U)$ so that

$$(gF)(g_0, g_1, \dots, g_n) = g \cdot F(g^{-1}g_0, g^{-1}g_1, \dots, g^{-1}g_n).$$

There is the resolution of G -modules

$$0 \rightarrow U \xrightarrow{\partial} X^0(G, U) \xrightarrow{\partial} X^1(G, U) \xrightarrow{\partial} \dots \rightarrow X^n(G, U) \xrightarrow{\partial} X^{n+1}(G, U) \rightarrow \dots$$

where $\partial : U \rightarrow X^0(G, U)$ sends $u \in U$ to the constant map $G \rightarrow U$ with image $\{u\}$, and for an element $F \in X^n(G, U)$, the function $\partial(F)$ sends $(g_0, \dots, g_{n+1}) \in G^{n+2}$ to

$$F(g_1, g_2, \dots, g_{n+1}) + \left(\sum_{m=1}^n (-1)^m F(g_1, g_2, \dots, g_{m-1}, g_{m+1}, g_{m+2}, \dots, g_{n+1}) \right) \\ + (-1)^{n+1} F(g_0, g_1, \dots, g_n).$$

The cohomology groups are

$$H^0(G, U) = \ker (X^0(G, U)^G \xrightarrow{\partial} X^1(G, U)^G) \cong U^G$$

and, for integers $n \geq 1$,

$$H^n(G, U) = \frac{\ker(X^n(G, U)^G \xrightarrow{\partial} X^{n+1}(G, U)^G)}{\operatorname{im}(X^{n-1}(G, U)^G \xrightarrow{\partial} X^n(G, U)^G)}.$$

For a closed normal subgroup H of G , the cohomology group $H^n(H, U)$ is a G -module: an element $g \in G$ acts on $H^n(H, U)$ by the conjugation map $g_* : H^n(H, U) \rightarrow H^n(H, U)$ induced by the map $g_* : X^n(H, U) \rightarrow X^n(H, U)$ sending $F \in X^n(H, U)$ to $g_*F \in X^n(H, U)$ so that for $(h_0, h_1, \dots, h_n) \in H^{n+1}$, we have

$$(g_*F)(h_0, h_1, \dots, h_n) = g \cdot F(g^{-1}h_0g, g^{-1}h_1g, \dots, g^{-1}h_ng).$$

In fact, for $h \in H$ and $F \in X^n(H, U)$ we have $h_*F = F$, so G/H acts on $H^n(H, U)$.

For a closed subgroup H of G , there are restriction maps $\operatorname{res}_{G/H} : H^n(G, U) \rightarrow H^n(H, U)$ induced by the maps $X^n(G, U)^G \rightarrow X^n(H, U)^H$ that send $F \in X^n(G, U)^G$

(note that F is a function from G^{n+1} to U) to its restriction on $H^{n+1} \subseteq G^{n+1}$; if H is also normal in G , then $\text{res}_{G/H}$ has image inside $H^n(H, U)^{G/H}$.

Suppose H is a finite-index closed subgroup of G , and let $S \subseteq G$ be a set of representatives of the cosets in $H \backslash G$, that is, for each $g \in G$ there is exactly one representative $r(g) \in S$ such that $Hg = Hr(g)$; then there are corestriction maps $\text{cor}_{G/H} : H^n(H, U) \rightarrow H^n(G, U)$ induced by the maps $\text{cor} : X^n(H, U)^H \rightarrow X^n(G, U)^G$ that send $F \in X^n(H, U)^H$ to $\text{cor}(F) \in X^n(G, U)^G$ so that for $(g_0, g_1, \dots, g_n) \in G^{n+1}$, we have

$$(\text{cor}(F))(g_0, g_1, \dots, g_n) = \sum_{s \in S} s^{-1} \cdot F(sg_0r(sg_0)^{-1}, sg_1r(sg_1)^{-1}, \dots, sg_nr(sg_n)^{-1}).$$

For H a finite-index closed subgroup of G , the restriction and corestriction maps satisfy the equation $\text{cor}_{G/H} \circ \text{res}_{G/H} = (G : H)$, and if H is also normal in G , then the equation $\text{res}_{G/H} \circ \text{cor}_{G/H} = \sum_{s \in S} s_*$ holds, in which the right hand side is the norm map $N_{G/H} : H^n(H, U) \rightarrow H^n(H, U)^{G/H}$.

If G acts on finitely generated free R -modules U_1 and U_2 , where R is a commutative ring with 1, then G acts on the R -module $\text{Hom}_R(U_2, U_1)$ (an element $g \in G$ acts on $F \in \text{Hom}_R(U_2, U_1)$ to yield $gF \in \text{Hom}_R(U_2, U_1)$ so that $(gF)(u_2) = g \cdot F(g^{-1}u_2)$ for $u_2 \in U_2$), and $H^1(G, \text{Hom}_R(U_2, U_1))$ is isomorphic to the group $\text{Ext}^1(U_2, U_1)$ whose elements are the (classes of) extensions $U_1 \hookrightarrow E \twoheadrightarrow U_2$ in the category of continuous R -module representations of G . An isomorphism is given by sending the class of an extension $U_1 \xrightarrow{f_1} E \xrightarrow{f_2} U_2$, for which some map $s : U_2 \rightarrow E$ satisfies $f_2 \circ s = 1_{U_2}$, to the class of the function in $X^1(G, \text{Hom}_R(U_2, U_1))$ sending $(g_0, g_0g) \in G^2$ to the map g_0F , where F is defined by

$$F : u_2 \mapsto f_1^{-1}(g \cdot s(g^{-1}u_2) - s(u_2)).$$

(In [Wa97, Proposition 4], the special case $U_1 = U_2$ is proved, and a generalization of

that proof yields the result for all U_1 and U_2 .)

For the representation V_f to be defined in section 6.4, this isomorphism takes the submodule $H_f^1(K_{v_0}, V_f)$ to the group $\text{Ext}_{\text{cris}}^1(\mathbb{Q}_p, V_f)$ of crystalline extensions $V_f \hookrightarrow E \twoheadrightarrow \mathbb{Q}_p$ of $G_{K_{v_0}}$ -modules over $\mathbb{Q}_p$ [Ne00, section 3.4].

I recall the following well-known results.

Lemma 6.3.1. (*[Ta76, Proposition 2.3], [Ru00, Proposition B.2.4]*) *Let $L/\mathbb{Q}_p$ be a finite-degree extension of fields. Let O_L be the ring of integers of L . Let G be a compact topological group acting linearly and continuously on an O_L -module $T \cong O_L^r$ for some $r \in \mathbb{Z}_{\geq 0}$. Define $V = T \otimes_{O_L} L$ and $W = T \otimes_{O_L} (L/O_L)$ as in [Ta76, Ru00]. Referring to the long exact sequence in cohomology coming from the short exact $T \hookrightarrow V \twoheadrightarrow W$: for each $n \in \mathbb{Z}_{\geq 0}$, the torsion subgroup of $H^n(G, T)$ is the image of the connecting map $H^{n-1}(G, W) \rightarrow H^n(G, T)$ and an isomorphism is given by $H^n(G, T) \otimes_{O_L} L \rightarrow H^n(G, V)$.*

Lemma 6.3.2. (*[Ru00, Proposition B.2.7]*) *Assume that the topological group G is among:*

- *The Galois group of a finite or infinite Galois extension of a number field unramified away from finitely many places.*
- *The absolute Galois group of a local field.*
- *The absolute inertia group of a local field whose residue field has characteristic $\neq p$.*

Assume G acts linearly and continuously over the $\mathbb{Z}_p$ -module U . Then there are the implications:

- *U has finite size $\Rightarrow$ so does $H^n(G, U)$ for each $n \in \mathbb{Z}_{\geq 0}$.*
- *U is finitely generated over $\mathbb{Z}_p \Rightarrow$ so is $H^n(G, U)$ for each $n \in \mathbb{Z}_{\geq 0}$.*

- U is finitely generated over $\mathbb{Q}_p \Rightarrow$ so is $H^n(G, U)$ for each $n \in \mathbb{Z}_{\geq 0}$.

Let $L/\mathbb{Q}_p$ be an unramified finite-degree extension of fields. Let G and a finitely-generated O_L -module U be as in Lemma 6.3.2. Let B be an O_L -submodule of $H^n(G, U)$. For an element $c \in B$ that is not in B_{tor} , define $\text{ind}_p(c, B)$ to be the maximum of the set

$$\{M \in \mathbb{Z} : M \geq 0 \text{ and there is } c' \in B \text{ such that } c - p^M c' \in B_{\text{tor}}\}.$$

Intuitively, just as the p -index ind_p of a positive integer is the number of factors of p in the prime factorization of that integer, so $\text{ind}_p(c, B)$ can be viewed as the number of factors of p in the class c thought of as an element of B/B_{tor} . (Although it does not make much difference since $L/\mathbb{Q}_p$ is unramified, I am considering the index not with respect to the uniformizer ϖ_L of L , but instead with respect to p . This version of the index is adapted to an Euler-system-related result of Masoero which I shall use later.)

6.4 Galois representations

Recall the projectors $\pi_B \in \mathbb{Z}_p[\Gamma_0(N)/\Gamma(N)]$, $\pi_{B,1} \in \mathbb{Z}_p[\Gamma_0(N)/\Gamma_1(N)]$ and $\pi_{2,k-2} \in \mathbb{Z}_p[G(2, k-2)]$ from section 6.1.

The Galois representation T_p linked to f can be defined as follows [Ma17, Ne92]: For the p -adic sheaf $\mathcal{F} = \varprojlim_n \mathcal{F}_n$ over $Y(N)$ with the sheaves

$$\mathcal{F}_n = \text{Sym}^{k-2}(R^1 \pi_* (\mathbb{Z}/p^n)_{\mathcal{E}_Y(\Gamma(N))})$$

over $Y(N)$, define the Galois representations

$$\begin{aligned} J_p &= \pi_B H_{et}^1(X(N) \otimes \overline{\mathbb{Q}}, j_* \mathcal{F})(\epsilon) \\ T_p &= \{x \in J_p : I_f x = 0\} \end{aligned}$$

where I_f is the kernel of the $O(f)$ -algebra map from the Hecke algebra with coefficients in $\mathbb{Z}$ to $O_{\mathbb{Q}(f)}$ sending $T(\ell)$ to $a(\ell, f)$. As mentioned in [Ne92, p. 102], because f is a newform, a map $R : J_p \rightarrow T_p$ exists such that R respects Hecke operators, R is $G_{\mathbb{Q}}$ -equivariant and for some non-negative integer c , the restriction of R to T_p is multiplication by p^c . Note that by [Ne92, Proposition 2.1] (which comes from [Sc90, Theorem 1.2.1]) and [Ne92, Lemma 2.2], we have that $H_{et}^1(X(N) \otimes \overline{\mathbb{Q}}, j_* \mathcal{F})$ is torsion free (this is a nontrivial result) and there are isomorphisms

$$\begin{aligned} \pi_{2,k-2} H_{et}^*(\tilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}/p^n)(\epsilon^{k/2-1}) &\cong H_{et}^1(X(N) \otimes \overline{\mathbb{Q}}, j_* \mathcal{F}_n) \\ &\cong H_{et}^1(X(N) \otimes \overline{\mathbb{Q}}, j_* \mathcal{F})/p^n \end{aligned}$$

so that identifying π_B with a projection on $\pi_{2,k-2} H_{et}^*(\tilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$ yields

$$J_p \cong \pi_B \pi_{2,k-2} H_{et}^*(\tilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)(\epsilon^{k/2}).$$

Using the standard map $O_{\mathbb{Q}(f)} \otimes \mathbb{Z}_p \twoheadrightarrow O_{\mathbb{Q}(f), \varpi_{\mathbb{Q}(f)}} \hookrightarrow O_L$ for the first tensor product below, define

$$\begin{aligned} T_f &= T_p \otimes_{O_{\mathbb{Q}(f)} \otimes \mathbb{Z}_p} O_L \\ V_f &= T_f \otimes_{O_L} L \\ W_f &= T_f \otimes_{O_L} (L/O_L). \end{aligned}$$

The usual short exact sequence $T_f \hookrightarrow V_f \twoheadrightarrow W_f$ and maps $p^{-n} : T_f \rightarrow W_f[p^n]$ for $n \in \mathbb{Z}_{>0}$ exist, as in Rubin [Ru00, sections 1.1-1.2].

Let V_f^a (respectively, V_f^g) be the Deligne/Scholl representations over L , pure¹ of weight $1-k$ (respectively, $k-1$), with $\det(xI - F) = x^2 - a_\ell x + \ell^{k-1}$ the characteristic polynomial of arithmetic (respectively, geometric) Frobenius F at $\ell \nmid Np$ [DiIm95, section 12.5]. Then:

- $L(V_f^a, s) = L(f, s)$ at all but finitely many primes.
- $V_f^g = \text{Hom}_L(V_f^a, L)$.
- $V_f^g(\epsilon^{k/2})$ is self-dual by a Poincare duality map $V_f^g(\epsilon^{k/2}) \times V_f^g(\epsilon^{k/2}) \rightarrow L(\epsilon)$ [NePl00, section 1.3], so $V_f^g(\epsilon^{k/2}) \cong V_f^a(\epsilon^{1-(k/2)})$.
- $V_f \cong V_f^g(\epsilon^{k/2}) \cong V_f^a(\epsilon^{1-(k/2)})$ is pure of weight -1 and $\text{Hom}_L(V_f, L(\epsilon)) \cong V_f$.

Let $T_f^g := T_f(\epsilon^{-k/2})$, so that $T_f^g \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V_f^g$ as Galois representations.

For the p -adic sheaf $\mathcal{F}^1 = \text{Sym}^{k-2}(R^1\pi_*(\mathbb{Z}_p)_{\mathcal{E}_Y(\Gamma_1(N))})$ over $Y_1(N)$, we similarly have an isomorphism (see [Sc96, section 2.8])

$$\pi_{1,k-2}H_{et}^*(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p) \cong H_{et}^1(X_1(N) \otimes \overline{\mathbb{Q}}, j_{1,*}\mathcal{F}^1)$$

and I define

$$J_p^1 = \pi_{B,1}H_{et}^1(X_1(N) \otimes \overline{\mathbb{Q}}, j_{1,*}\mathcal{F}^1)(\epsilon) \cong \pi_{B,1}\pi_{1,k-2}H_{et}^*(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)(\epsilon)$$

after identifying $\pi_{B,1}$ with a projector on $\pi_{1,k-2}H_{et}^*(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$. Note that $\mathcal{F}^1 = \pi_{Y(N) \rightarrow Y_1(N),*}\mathcal{F}$ (where $\pi_{Y(N) \rightarrow Y_1(N),*}$ has the obvious meaning). In fact, we have

$$J_p^1 = \pi_{B,1}H_{et}^1(X_1(N) \otimes \overline{\mathbb{Q}}, j_{1,*}\mathcal{F}^1)(\epsilon) \cong \pi_B H_{et}^1(X(N) \otimes \overline{\mathbb{Q}}, j_*\mathcal{F})(\epsilon) = J_p$$

(see [Ne95, section II.2.5] for the analogous result with $X_0(N)$ instead of $X_1(N)$).

¹“Pure of weight w ” means that the eigenvalues of geometric Frobenius at v have absolute value $(Nv)^{w/2}$.

6.5 Selmer group conditions

I define the Bloch-Kato Selmer groups following [Ma17, section 2.2]. (Note the small difference in principle between this and [Ne92], which relaxes all conditions for places over N .) For a number field F , letting I_v be the inertia group in G_{L_v} for a place $v \nmid \infty$ of F , define

$$H_f^1(F_v, V_f) = \begin{cases} \ker(H^1(F_v, V_f) \rightarrow H^1(I_v, V_f)) & v \nmid p \\ \ker(H^1(F_v, V_f) \rightarrow H^1(F_v, V_f \otimes_{\mathbb{Q}_p} B_{cris})) & v \mid p \end{cases}$$

and let $H_f^1(F_v, T_f)$ (respectively, $H_f^1(F_v, W_f[p^n])$) be the inverse image (respectively, image) of $H_f^1(F_v, V_f)$ under the standard map $H^1(F_v, T_f) \rightarrow H^1(F_v, V_f)$ (respectively, $p^{-n} : H^1(F_v, T_f) \rightarrow H^1(F_v, W_f[p^n])$). Define the global Bloch-Kato Selmer groups

$$\begin{aligned} H_f^1(F, T_f) &= \{c \in H^1(F, T_f) : \forall \text{ places } v \text{ of } F : c_v \in H_f^1(F_v, T_f)\} \\ H_f^1(F, V_f) &= \{c \in H^1(F, V_f) : \forall \text{ places } v \text{ of } F : c_v \in H_f^1(F_v, V_f)\} \\ H_f^1(F, W_f) &= \{c \in H^1(F, W_f) : \forall \text{ places } v \text{ of } F : c_v \in H_f^1(F_v, W_f)\}. \end{aligned}$$

The Shafarevich-Tate group is

$$\text{III}_f(F, W_f) := H_f^1(F, W_f) / H_f^1(F, W_f)_{\text{div}} \tag{6.1}$$

which has finite cardinality since the O_L -module $H_f^1(F, W_f)$ has finite corank.

As in [JeSkWa17, section 2.3.4], I define the anticyclotomic Selmer groups

$$H_{ac}^1(K_v, V_f) = \begin{cases} \ker(H^1(K_v, V_f) \rightarrow H^1(I_v, V_f)) & \text{split } v \nmid p\infty \\ H^1(K_v, V_f) & v = \bar{v}_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_{ac}^1(K, V_f) = \{c \in H^1(K, V_f) : \forall \text{ places } v \text{ of } K : c_v \in H_{ac}^1(K_v, V_f)\}.$$

In the same way as for the Bloch-Kato Selmer groups, define the local cohomology groups $H_{ac}^1(K_v, T_f)$ and $H_{ac}^1(K_v, W_f)$ by taking preimages and images of $H_{ac}^1(K_v, V_f)$, and define the cohomology groups $H_{ac}^1(K, T_f)$ and $H_{ac}^1(K, W_f)$ as the groups of global elements localizing to elements of $H_{ac}^1(K_v, T_f)$ and $H_{ac}^1(K_v, W_f)$ respectively at all v .

6.6 Etale cohomology and the cycle class map

For a smooth algebraic variety U with $\dim U \leq p/2$, there is a cycle class map

$$cl_n : CH^*(U) \rightarrow \sum_{r=0}^{\dim U} H_{et}^{2r}(U, (\mathbb{Z}/p^n\mathbb{Z})(\epsilon^r))$$

for each $n \in \mathbb{Z}_{>0}$, and the projective limit of the maps cl_n is a p -adic cycle class map

$$cl : CH^*(U) \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \sum_{r=0}^{\dim X} H_{et}^{2r}(U, \mathbb{Z}_p(\epsilon^r))$$

which yields

$$cl : CH^*(U) \otimes_{\mathbb{Z}} \mathbb{Q}_p \rightarrow \sum_{r=0}^{\dim X} H_{et}^{2r}(U, \mathbb{Q}_p(\epsilon^r))$$

after tensoring with $\mathbb{Q}_p$; see [Mi80, section VI.9], [Mi13, section 23].

6.7 The p -adic Abel-Jacobi map

As in Nekovář [Ne00, section 1], for any smooth proper variety U defined over a field F and any non-negative integer n , there is a p -adic Abel-Jacobi map

$$AJ_F^U : CH_0^n(U/F) \rightarrow H^1(F, H^{2n-1}(\overline{U}_{et}, \mathbb{Z}_p))(\epsilon^n)$$

coming from the cycle class map and Hochschild-Serre spectral sequence. The Abel-Jacobi map is Galois equivariant (see [Ne92, Proposition 4.2] and [Ma17, section 3.2]) and commutes with pushforwards and pullbacks of correspondences ([Ne92, proof of Proposition 4.2]; see [Ki83, section 2] for the complex algebraic geometry version of this result).

I consider the p -adic Abel-Jacobi map in the following three different settings.

As described by Nekovář [Ne92, chapters 2-4] and Masoero [Ma17, sections 3.1-3.3], for any field F containing $\mathbb{Q}$, there is a p -adic Abel-Jacobi map (extending the map $\Phi_{p,L}$ in [Ma17] by $\mathbb{Z}_p$ -linearity)

$$\Phi : CH_0^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N))/F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow H^1(F, H_{et}^{k-1}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2}))).$$

Composing Φ with the map that is $H^1(F, \cdot)$ of the composite

$$H_{et}^{k-1}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2})) \xrightarrow{\pi_B \pi_{2,k-2}} J_p \xrightarrow{R} T_p$$

and then applying $\otimes O_L$ or $\otimes L$ yields compatible Abel-Jacobi maps

$$\begin{aligned} AJ_F : CH_0^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N))/F) \otimes O_L &\rightarrow H_f^1(F, T_f) \\ AJ_F : CH_0^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N))/F) \otimes L &\rightarrow H_f^1(F, V_f). \end{aligned}$$

See Castella [Ca13]. Similarly,

$$\Phi_{Ca13} : CH_0^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow H^1(L, H_{et}^{k-1}(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2})))$$

is a p -adic Abel-Jacobi map which, together with the composition of maps

$$H_{et}^{k-1}(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \otimes \overline{F}, \mathbb{Z}_p(\epsilon^{k/2})) \xrightarrow{\pi_B, 1 \pi_{1,k-2}} J_p^1 \cong J_p \xrightarrow{R} T_p$$

and the application of $\otimes O_L$, yields an Abel-Jacobi map

$$AJ_F^1 : CH_0^{k/2}(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))/F) \otimes O_L \rightarrow H^1(F, T_f).$$

See Bertolini, Darmon and Prasanna [BeDaPr13, sections 3.1-3.4], taking that paper's r to be $k - 2$. Let F be a field containing K_1 , let A and A' be elliptic curves over $\mathbb{Q}$, let A' have $\Gamma_1(N)$ structure, and let $\varphi : A \rightarrow A'$ be an isogeny for which the Heegner cycle Δ_φ is defined over F . (Note that $\tilde{\mathcal{E}}^r(\Gamma_1(N)) \times A^r$ is defined over $\mathbb{Q}$ [BeDaPr13, p. 1056].) Define

$$\begin{aligned} J_{BeDaPr} &= \pi_{1,k-2}\pi_{0,k-2}H_{et}^{2k-3}(\overline{\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2}}, \mathbb{Q}_p(\epsilon^{k-1})) \\ J_{BeDaPr}^{\mathbb{Z}_p} &= \pi_{1,k-2}\pi_{0,k-2}H_{et}^{2k-3}(\overline{\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2}}, \mathbb{Z}_p(\epsilon^{k-1})) \\ J_{BeDaPr}^{dR} &= \pi_{1,k-2}\pi_{0,k-2}H_{dR}^{2k-3}((\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2})/F)(\epsilon^{k-2}). \end{aligned}$$

There are compatible p -adic Abel-Jacobi maps

$$\begin{aligned} AJ_F^{1,A} : CH_0^{k-1}((\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2})/F) \otimes_{\mathbb{Z}} \mathbb{Z}_p &\rightarrow H_f^1(F, J_{BeDaPr}^{\mathbb{Z}_p}) \\ AJ_F^{1,A} : CH_0^{k-1}((\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times A^{k-2})/F) \otimes_{\mathbb{Z}} \mathbb{Q}_p &\rightarrow H_f^1(F, J_{BeDaPr}). \end{aligned}$$

6.8 Shafarevich-Tate-like groups

Following Masoero [Ma17, section 3.3], I define a variant of the Shafarevich-Tate group which will be used in Chapter 9.

For each positive $n \in \mathbb{Z}$, the map $p^{-n} : T_f \rightarrow W_f[p^n]$ and the inclusion $W_f[p^n] \hookrightarrow W_f[p^{n+1}]$ yield maps on cohomology

$$\begin{aligned} p^{-n} : H^1(K, T_f) &\rightarrow H^1(K, W_f[p^n]) \\ 1 : H^1(K, W_f[p^n]) &\rightarrow H^1(K, W_f[p^{n+1}]). \end{aligned}$$

These maps combine to form a commutative diagram

$$\begin{array}{ccc}
H^1(K, T_f) & \xrightarrow{p^{-1}} & H^1(K, W_f[p]) \\
\downarrow p & & \downarrow 1 \\
\vdots & & \vdots \\
\downarrow p & & \downarrow 1 \\
H^1(K, T_f) & \xrightarrow{p^{-n}} & H^1(K, W_f[p^n]) \\
\downarrow p & & \downarrow 1 \\
H^1(K, T_f) & \xrightarrow{p^{-(n+1)}} & H^1(K, W_f[p^{n+1}]) \\
\downarrow p & & \downarrow 1 \\
\vdots & & \vdots
\end{array}$$

from which the direct limit of the maps $p^{-n} : H^1(K, T_f) \rightarrow H^1(K, W_f[p^n])$ is a map

$$H^1(K, T_f) \otimes_{O_L} (L/O_L) \rightarrow H^1(K, W_f). \quad (6.2)$$

Define $\text{III}_{p^n}(K, W_f)$ to be the quotient of $H_f^1(K, W_f[p^n])$ by the image under the map $p^{-n} : H^1(K, T_f) \rightarrow H^1(K, W_f[p^n])$ of $(\text{im } AJ_K)/p^n(\text{im } AJ_K)$. Similarly, define the Shafarevich-Tate-like group $\text{III}(K, W_f)$ to be the quotient of $H_f^1(K, W_f)$ by the image under the map (6.2) of $(\text{im } AJ_K) \otimes_{O_L} (L/O_L)$. Then we have a commutative diagram of short exact sequences

$$\begin{array}{ccccc}
(\text{im } AJ_K)/p(\text{im } AJ_K) & \xhookrightarrow{p^{-1}} & H_f^1(K, W_f[p]) & \twoheadrightarrow & \text{III}_p(K, W_f) \\
\downarrow p & & \downarrow 1 & & \downarrow 1 \\
\vdots & & \vdots & & \vdots \\
\downarrow p & & \downarrow 1 & & \downarrow 1 \\
(\text{im } AJ_K)/p^n(\text{im } AJ_K) & \xhookrightarrow{p^{-n}} & H_f^1(K, W_f[p^n]) & \twoheadrightarrow & \text{III}_{p^n}(K, W_f) \\
\downarrow p & & \downarrow 1 & & \downarrow 1 \\
(\text{im } AJ_K)/p^{n+1}(\text{im } AJ_K) & \xhookrightarrow{p^{-(n+1)}} & H_f^1(K, W_f[p^{n+1}]) & \twoheadrightarrow & \text{III}_{p^{n+1}}(K, W_f) \\
\downarrow p & & \downarrow 1 & & \downarrow 1 \\
\vdots & & \vdots & & \vdots
\end{array}$$

in which each term has finite size, and these sequences' direct limit is

$$(\mathrm{im} AJ_K) \otimes_{O_L} (L/O_L) \hookrightarrow H_f^1(K, W_f) \twoheadrightarrow \mathrm{III}(K, W_f)$$

which yields a surjection $\mathrm{III}(K, W_f) \twoheadrightarrow \mathrm{III}_f(K, W_f)$ by the definition of $\mathrm{III}_f(K, W_f)$ in (6.1), since $(\mathrm{im} AJ_K) \otimes_{O_L} (L/O_L)$ is divisible. For general modular forms f and imaginary quadratic fields K , $\mathrm{III}(K, W_f)$ has not been proved to have finite cardinality (to my knowledge), although $\mathrm{III}_f(K, W_f)$ does have finite cardinality.

6.9 Logarithms on local cohomology

In this section, I define logarithm maps on local cohomology groups.

Recall that $H^1(K_{v_0}, V_f) \cong \mathrm{Ext}^1(\mathbb{Q}_p, V_f)$ via an isomorphism which takes the subgroup $H_f^1(K_{v_0}, V_f)$ to $\mathrm{Ext}_{\mathrm{cris}}^1(\mathbb{Q}_p, V_f)$.

Define

$$\tilde{V} := \pi_B \pi_{2,k-2} H_{dR}^{k-1}(\tilde{\mathcal{E}}^{k-2}(\Gamma(N))/L)(\epsilon^{k/2})$$

and let $\mathrm{Ext}_{ffm}^1(L, \tilde{V})$ consist of the classes of extensions $\tilde{V} \hookrightarrow E \twoheadrightarrow L$ of filtered Frobenius modules [BeDaPr13, sections 3.2-3.3]. As in [Ne93, Proposition 1.21 and Corollary 1.22] and [BeDaPr13, sections 3.2-3.3], since the extension $L/\mathbb{Q}_p$ is unramified, an etale-versus-de-Rham-cohomology comparison theorem of Faltings [Fa89, Theorem 5.6] yields a map

$$\mathrm{comp} : \mathrm{Ext}_{\mathrm{cris}}^1(\mathbb{Q}_p, J_p \otimes L) \xrightarrow{\cong} \mathrm{Ext}_{ffm}^1(L, \tilde{V}) \cong \tilde{V}/\mathrm{Fil}^0 \tilde{V}.$$

From the inclusion $T_p \hookrightarrow J_p$, we obtain a map

$$H_f^1(K_{v_0}, \mathrm{Hom}(\mathbb{Q}_p, V_f)) \cong H_f^1(K_{v_0}, \mathrm{Hom}(\mathbb{Q}_p, T_p \otimes L)) \rightarrow H_f^1(K_{v_0}, \mathrm{Hom}(\mathbb{Q}_p, J_p \otimes L))$$

which, because of the isomorphism between $H_f^1(K_{v_0}, \cdot)$ and $\text{Ext}_{cris}^1(\mathbb{Q}_p, \cdot)$, is identified with a map

$$J' : \text{Ext}_{cris}^1(\mathbb{Q}_p, V_f) \cong \text{Ext}_{cris}^1(\mathbb{Q}_p, T_p \otimes L) \rightarrow \text{Ext}_{cris}^1(\mathbb{Q}_p, J_p \otimes L)$$

of which the image is sent by comp to a module which I'll call $\tilde{U} \subseteq \tilde{V}/\text{Fil}^0 \tilde{V}$.

Define $\tilde{\tilde{V}}$ to be the annihilator of $\text{Fil}^0 \tilde{V}$ with respect to the Poincare duality map $\tilde{V} \times \tilde{V} \rightarrow L$ (see also [NePl00, section 1.3.4]); then that duality gives an isomorphism $J : \tilde{V}/\text{Fil}^0 \tilde{V} \cong \text{Hom}_L(\tilde{\tilde{V}}, L)$.

I define the logarithm as the composite

$$\log : J \circ \text{comp} \circ J' : H_f^1(K_{v_0}, V_f) \cong \text{Ext}_{cris}^1(\mathbb{Q}_p, V_f) \rightarrow \text{Hom}_L(\tilde{\tilde{V}}, L)$$

and, for a differential form $\eta \in \tilde{\tilde{V}}$, I define $\log_\eta : H_f^1(K_{v_0}, V_f) \rightarrow L$ as the map sending $C \in H_f^1(K_{v_0}, V_f)$ to $(\log C)(\eta)$; here I adapt notation of [JeSkWa17, section 3.5].

Those maps $\log$ and $\log_\eta$ were adapted from the following logarithms of Bertolini, Darmon and Prasanna [BeDaPr13, sections 3.1-3.4]. Let F be a field containing K_1 ; let A be an elliptic curve over $\mathbb{Q}$. There is an isomorphism

$$\log_F^{1,A} : H_f^1(F, J_{BeDaPr}) \xrightarrow{\cong} \text{Hom}_F(\text{Fil}^1 J_{BeDaPr}^{dR}, F)$$

given in [BeDaPr13, p. 1070] as the composition of three vertical maps in that source's diagram: the isomorphism $H_f^1(F, \cdot) \cong \text{Ext}_{cris}^1(\mathbb{Q}_p, \cdot)$, then a comparison isomorphism, then a Poincare duality map. Precomposing $\log_F^{1,A}$ with the canonical map $H_f^1(F, J_{BeDaPr}^{\mathbb{Z}_p}) \rightarrow H_f^1(F, J_{BeDaPr})$ yields a map

$$\log_F^{1,A,\mathbb{Z}_p} : H_f^1(F, J_{BeDaPr}^{\mathbb{Z}_p}) \rightarrow \text{Hom}_F(\text{Fil}^1 J_{BeDaPr}^{dR}, F).$$

From $\log_F^{1,A}$ and $\log_F^{1,A,\mathbb{Z}_p}$, for each $\eta \in \text{Fil}^1 J_{BeDaPr}^{dR}$ we obtain compatible maps

$$\log_{\eta,F}^{1,A} : H_f^1(F, J_{BeDaPr}) \rightarrow F \quad (6.3)$$

$$\log_{\eta,F}^{1,A,\mathbb{Z}_p} : H_f^1(F, J_{BeDaPr}^{\mathbb{Z}_p}) \rightarrow F \quad (6.4)$$

sending a cohomology class C to $(\log_F^{1,A} C)(\eta)$ and $(\log_F^{1,A,\mathbb{Z}_p} C)(\eta)$ respectively. We may take $\eta = \omega_f \wedge \omega_A^{k/2-1} \eta_A^{k/2-1}$ for the differential forms ω_f , ω_A and η_A in [BeDaPr13].

Part B

Argument

Chapter 7

Comparing p -adic L -functions

The line of argument of [JeSkWa17] is followed and adapted. (Note: The places $v, \bar{v}$ in [JeSkWa17, section 5] are our $\bar{v}_0, v_0$ respectively, and that source's infinity type $z_\infty^a \bar{z}_\infty^b$ is our infinity type $z_\infty^b \bar{z}_\infty^a$.) Throughout this chapter, the arithmetic $\widehat{O}_L^{ur}$ -algebra map $P_{ac} : \widehat{O}_L^{ur}[[\Gamma_K^-]] \rightarrow \widehat{O}_L^{ur}$ is as in section 5.2 (see also section 3.2) and the character χ is as in subsection 5.2.4.

7.1 Comparison and missing factor

By section 5.2's interpolation formulas (5.1), (5.2) and (5.3), using that section's functions F_* and constants $C_{1,*}$ and $C_{2,*}$, we have

$$\begin{aligned} P_{ac}(L_{Katz}^-(K)) &= F_{93} C_{1,93} C_{2,93}^n \\ P_{ac}(D_Q^-) &= F_{91} C_{1,91} C_{2,91}^n \\ P_{ac}(L_{BDP}^{-,N_p}(f)) &= F_{13} C_{1,13} C_{2,13}^n. \end{aligned}$$

From this, we obtain

$$P_{ac}(\widetilde{L}_{Wan}^{-,N_p}(f)) = P_{ac}(D_Q^- L_{Katz}^-(K)) = C(f, P_{ac}) P_{ac}(L_{BDP}^{-,N_p}(f))$$

writing $C(f, P_{ac}) = \tilde{F} \tilde{C}_1 \tilde{C}_2^n$ where we define

$$\tilde{F} := \frac{F_{91}F_{93}}{F_{13}}, \quad \tilde{C}_1 := \frac{C_{1,91}C_{1,93}}{C_{1,13}}, \quad \tilde{C}_2 := \frac{C_{2,91}C_{2,93}}{C_{2,13}}.$$

As in the argument of [JeSkWa17, section 5.2], $\tilde{F}$ is a constant times the n th power of a constant. (We have $U\theta_{\chi^{-1}} = F(\lambda, P_1)$ and

$$\frac{L(1, \xi\xi^{-c})\Gamma(2n+1)}{\langle U\theta_{\chi^{-1}}, U\theta_{\chi^{-1}} \rangle_{\Gamma_0(N|D_K|)}} = (\text{constant})(16\pi^2)^n$$

by equation (4.3) and the fact that

$$\text{res}_{s=2n+1} D(s, U\theta_{\chi^{-1}}, U\theta_{\chi^{-1}}^c) = (\text{constant})L(1, \xi\xi^{-c})$$

holds.) So we can write $C(f, P_{ac}) = C_1 C_2^n$ for constants C_1, C_2 .

7.2 Interpolation

Lemma 7.2.1. *There is a constant $C_1 \in \widehat{O}_L^{ur}[1/p]^\times$ and a p -adic unit $u \in \widehat{O}_L^{ur}[[\Gamma_K^-]]^\times$ such that for all P_{ac} with $\phi(pN) = (p-1)\phi(N) \mid n$, we have $P_{ac}(C_1 u) = C(f, P_{ac})$.*

Proof. Let C_1 be as in section 7.1, and let u be such that $P_{ac}(u) = C_2^n$ identically; this is possible since $\psi^{\phi(pN)} = 1$ and the infinity type exponents of $P\psi$ are $\pm n$. $\square$

The maps P_{ac} as in Lemma 7.2.1 are dense in $\text{Spec } \widehat{O}_L^{ur}[[\Gamma_K^-]]$, so $\tilde{L}_{Wan}^{-,N_p}(f) = D_Q^- L_{Katz}^-(K) = C_1 u L_{BDP}^{-,N_p}(f)$. So we have shown the following theorem (proved in [JeSkWa17, Corollary 5.3.2] for f of weight 2; we have carried out their argument for f of higher weight).

Theorem 7.2.2. *In $\widehat{O}_L^{ur}[[\Gamma_K^-]] \otimes_{O_L} L = \widehat{O}_L^{ur}[[\Gamma_K^-]][1/p]$, we have*

$$(L_{BDP}^{-,N_p}(f)) = (\tilde{L}_{Wan}^{-,N_p}(f)) \supseteq (L_{Wan}^{-,N_p}(f)).$$

Chapter 8

From Wan's L -function to cohomology: Iwasawa theory

This chapter proves one direction of an Iwasawa main conjecture and, as a consequence, links Wan's L -function to the cohomology of the $(\mathbb{Q}_p/\mathbb{Z}_p)$ -representation W_f of the modular form f .

8.1 Notation

I briefly recall previous material and introduce some new notation for this chapter. I use and adapt notation from [SkUr14, Wa18, Hi91].

As discussed in section 4.2, let $\lambda : h_{k,w}(N, O_L) \rightarrow I$ be a cuspidal Hida family with coefficients in the integrally closed domain I , where I is a finite-rank $\mathbb{Z}_p[[t]]$ -module, and let $Q : I \rightarrow \overline{\mathbb{Q}}_p$ be a continuous $\mathbb{Z}_p$ -algebra map corresponding via λ to the classical cusp form f of even weight k . Let $L/\mathbb{Q}_p$ be an unramified finite-degree field extension with ring of integers $O_L = Q[I]$ and uniformizer ϖ_L . Let T_λ and T_f^g be the Galois representations coming from λ and f . (T_λ is described in [SkUr14, subsection 3.3.10], where it is denoted by $\rho_{\mathbf{f}}$; Hecke duality identifies that source's $\mathbf{f}$ with my λ .) Assume that the residual representation $\overline{T_f^g}$ is irreducible. We have $T_\lambda \cong I^2$

and $T_f^g \cong O_L^2$, and for a sufficiently large extension $L/\mathbb{Q}_p$, we have $T_\lambda \otimes_I O_L \cong T_f^g$. Let $\widehat{O}_L^{ur}$ be the completion of the maximum unramified extension of O_L . Choose an irreducible component of $I \widehat{\otimes}_{O_L} \widehat{O}_L^{ur}$, and let the normalization of that component's associated ring be $\widehat{I}^{ur}$ (see [Wa18, p. 2]).

Let $\Psi_K : G_K \twoheadrightarrow \Gamma_K \subseteq O_L[[\Gamma_K]]^\times$, $\Psi_- : G_K \twoheadrightarrow \Gamma_K^- \subseteq O_L[[\Gamma_K^-]]^\times$ be the standard projections. Write $(\cdot)^* = \text{Hom}_{O_L}(\cdot, L/O_L)$ for the Pontryagin dual. The module $O_L[[\Gamma_K]]$ acts on $O_L[[\Gamma_K]]^*$ so that $(xF)(y) = F(yx)$ for $x, y \in \Gamma_K$ and $F \in O_L[[\Gamma_K]]^*$. Define the modules

$$\begin{aligned} T_{\lambda,K,\xi} &= T_\lambda \sigma_{\xi^{-c}}(\epsilon^{2-(\kappa/2)}) \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]](\Psi_K^{-c}) \\ T_{f,K,\xi} &= T_f^g \sigma_{\xi^{-c}}(\epsilon^{2-(\kappa/2)}) \otimes_{O_L[[\Gamma_K]]} O_L[[\Gamma_K]](\Psi_K^{-c}). \end{aligned}$$

For any finite set S of finite places of K , such that S includes $v_0, \bar{v}_0$ and all places at which V_f ramifies, define the modules

$$\begin{aligned} \text{Sel}_{\lambda,K,\xi}^S &= \{c \in H^1(K, T_{\lambda,K,\xi} \otimes I[[\Gamma_K]]^*) : c \text{ unramified at } \bar{v}_0 \text{ and outside } S\} \\ \text{Sel}_{f,K,\xi}^S &= \{c \in H^1(K, T_{f,K,\xi} \otimes O_L[[\Gamma_K]]^*) : c \text{ unramified at } \bar{v}_0 \text{ and outside } S\} \\ X_{\lambda,K,\xi}^S &= (\text{Sel}_{\lambda,K,\xi}^S)^* \\ X_{f,K,\xi}^S &= (\text{Sel}_{f,K,\xi}^S)^* \\ \widehat{X}_{\lambda,K,\xi}^S &= X_{\lambda,K,\xi}^S \otimes_{I[[\Gamma_K]]} \widehat{I}^{ur}[[\Gamma_K]] \\ \widehat{X}_{f,K,\xi}^S &= X_{f,K,\xi}^S \otimes_{O_L[[\Gamma_K]]} \widehat{O}_L^{ur}[[\Gamma_K]]. \end{aligned}$$

8.2 Main conjecture for Hida families

Wan proved the following main conjecture (see [Wa18, p. 85]; the result is in the final proof of that source's Theorem 1.2):

Theorem 8.2.1 (Main conjecture). *Assume some nebentypus-1 weight-2 specialization f_0 of a Hida family λ satisfies:*

- f_0 is the ordinary stabilization of a newform of level divisible by some odd prime

q not split in K .

- The Galois representation $T_{f_0}^g$ has irreducible residual representation $\overline{T_{f_0}^g}|_{G_K}$, and $\overline{T_{f_0}^g}$ is ramified at q .

Suppose the Hecke character $\xi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}$ is of infinity type $z_\infty \mapsto z_\infty^u \overline{z_\infty}^{-u}$ for some positive integer u divisible by $p-1$, and is such that the p -adic avatar of $\xi| \cdot |_K^u (\omega_{v_0} \cdot \omega_{\overline{v}_0})$ factors through Γ_K .

Let S be a set of finitely many places of K , including all places dividing pND_K .

Then, letting $P_1, \dots, P_t$ be the height 1 primes in $\widehat{I}^{ur}[[\Gamma_K]]$ dividing L_{Wan}^S that are pullbacks of height 1 primes in $\widehat{I}^{ur}$, we have

$$L_{Wan}^S \widehat{I}^{ur}[[\Gamma_K]]_{p, P_1, \dots, P_t} \supseteq \text{Fitt}_{\widehat{I}^{ur}[[\Gamma_K]]_{p, P_1, \dots, P_t}} \widehat{X}_{\lambda, K, \xi}^S$$

in which the notation $\widehat{I}^{ur}[[\Gamma_K]]_{p, P_1, \dots, P_t}$ indicates localization with respect to the primes P_i and p as in [Wa18, p. 85].

8.3 From Hida families to modular forms

We follow the argument in the proof of [Wa18, Theorem 1.2], spelling out some of the steps. In this section, S is a set of finitely many places of K including all places over pND_K .

Recall $L_{Wan}^S(f) = Q(L_{Wan}^S) \in \widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L$ for the arithmetic point Q corresponding to f via the Hida family λ . Applying Q to Theorem 8.2.1's result, and noting $\text{Fitt}_{R/I}(M/IM) = (\text{Fitt}_R M)(R/I)$ for an R -module M and ideal I in a noetherian ring R (for which see [Ei95, Corollary 20.5], [SkUr14, section 3.1.5] and [Wa18, section 2.2]), I obtain

$$L_{Wan}^S(f)(\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L) \supseteq \text{Fitt}_{\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L} \left(\frac{\widehat{X}_{\lambda, K, \xi}^S \otimes_{O_L} L}{(\ker Q)(\widehat{X}_{\lambda, K, \xi}^S \otimes_{O_L} L)} \right)$$

and therefore

$$L_{Wan}^S(f)(\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L) \supseteq \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L} \left(\frac{\widehat{X}_{\lambda,K,\xi}^S \otimes_{O_L} L}{(\ker Q)(\widehat{X}_{\lambda,K,\xi}^S \otimes_{O_L} L)} \right)$$

because the characteristic ideal is the minimum principal ideal containing the Fitting ideal (see, for instance, the last sentence in the proof of [JeSkWa17, Corollary 3.4.2]).

Now Wan [Wa18, Proposition 2.4] proved an $O_L[[\Gamma_K]]$ -module version of the following result for f of weight 2; that argument still holds for higher weight and is outlined here.

Theorem 8.3.1. *There is an $\widehat{O}_L^{ur}[[\Gamma_K]]$ -module exact sequence*

$$M \rightarrow \widehat{X}_{\lambda,K,\xi}^S / (\ker Q) \widehat{X}_{\lambda,K,\xi}^S \rightarrow \widehat{X}_{f,K,\xi}^S \rightarrow 0$$

where $M \otimes_{O_L} L$ has annihilator of codimension ≥ 2 in $\text{Spec } \widehat{O}_L^{ur}[[\Gamma_K]] \otimes L$, i.e., is pseudo-null. In $\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L$, this implies

$$\text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L} \left(\frac{\widehat{X}_{\lambda,K,\xi}^S \otimes_{O_L} L}{(\ker Q)(\widehat{X}_{\lambda,K,\xi}^S \otimes_{O_L} L)} \right) = \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L} (\widehat{X}_{f,K,\xi}^S \otimes_{O_L} L)$$

and therefore

$$L_{Wan}^S(f)(\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L) \supseteq \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]] \otimes_{O_L} L} (\widehat{X}_{f,K,\xi}^S \otimes_{O_L} L).$$

Proof: (Outline). I first work with $O_L[[\Gamma_K]]$ -modules and then pass to $\widehat{O}_L^{ur}[[\Gamma_K]]$ -modules. Apply $T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} (\cdot)^*$ to the short exact sequence $(\ker Q)[[\Gamma_K]] \hookrightarrow I[[\Gamma_K]] \xrightarrow{Q} O_L[[\Gamma_K]]$ to get the short exact

$$\begin{aligned} T_{f,K,\xi} \otimes_{O_L[[\Gamma_K]]} O_L[[\Gamma_K]]^* &\xrightarrow{Q^*} T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]]^* \\ &\xrightarrow{res} T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} ((\ker Q)[[\Gamma_K]])^* \end{aligned} \quad (8.1)$$

(noting that $T_{\lambda,K,\xi} \otimes_I O_L \cong T_{f,K,\xi}$) for which $I_{\bar{v}_0}$ -cohomology gives

$$\begin{aligned} H^0(I_{\bar{v}_0}, T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]]^*) &\xrightarrow{s} H^0(I_{\bar{v}_0}, T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} ((\ker Q)[[\Gamma_K]])^*) \\ &\rightarrow H^1(I_{\bar{v}_0}, T_{f,K,\xi} \otimes_{O_L[[\Gamma_K]]} O_L[[\Gamma_K]]^*) \xrightarrow{Q^*} H^1(I_{\bar{v}_0}, T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]]^*) \end{aligned}$$

so we have an exact sequence

$$\begin{aligned} 0 \rightarrow (\text{coker } s)^{G_{\bar{v}_0}} &\rightarrow H^1(I_{\bar{v}_0}, T_{f,K,\xi} \otimes_{O_L[[\Gamma_K]]} O_L[[\Gamma_K]]^*)^{G_{\bar{v}_0}} \\ &\xrightarrow{Q^*} H^1(I_{\bar{v}_0}, T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]]^*)^{G_{\bar{v}_0}}. \end{aligned}$$

By the argument of [SkUr14, Proposition 3.7], we have

$$H^1(G_{K,S}, T_{f,K,\xi} \otimes_{O_L[[\Gamma_K]]} O_L[[\Gamma_K]]^*) \xrightarrow{Q^*} H^1(G_{K,S}, T_{\lambda,K,\xi} \otimes_{I[[\Gamma_K]]} I[[\Gamma_K]]^*)[\ker Q]$$

and considering the part of the above module mapped to 0 by the restriction map in the last two terms of (8.1), we get an exact sequence

$$0 \rightarrow \text{Sel}_{f,K,\xi}^S \rightarrow \text{Sel}_{\lambda,K,\xi}^S[\ker Q] \rightarrow (\text{coker } s)^{G_{\bar{v}_0}}[\ker Q]$$

Taking Pontryagin duals, applying $\otimes_{O_L[[\Gamma_K]]} \widehat{O}_L^{ur}[[\Gamma_K]]$ and setting

$$M = \frac{((\text{coker } s)^{G_{\bar{v}_0}})^*}{\ker Q} \otimes_{O_L[[\Gamma_K]]} \widehat{O}_L^{ur}[[\Gamma_K]]$$

gives the required exact sequence; more detailed reasoning as in Wan [Wa18] proves that $M \otimes_{O_L} L$ is pseudo-null. $\square$

8.4 From Greenberg to anticyclotomic: characteristic ideals

The following arguments are adapted from [JeSkWa17, section 3.4] (that source's Σ is my $S \setminus N_p$). In this section, the set S is as before.

The actions of G_K on $M = T_{f,K,\xi} \otimes_{O_L} O_L[[\Gamma_K^-]]^*$ and $\mathcal{M} = T_{f,K,\xi} \otimes_{O_L} O_L[[\Gamma_K]]^*$ are given by (G_K -action on $T_{f,K,\xi}$) $\otimes \Psi_-^{-1}$ and (G_K -action on $T_{f,K,\xi}$) $\otimes \Psi_K^{-1}$ respectively.

Recall that $p = v_0 \bar{v}_0$ in K , we define $N_p = \{v_0, \bar{v}_0\}$, and K^S/K is the maximum extension of K unramified away from S . For $\widetilde{M} \in \{M, \mathcal{M}\}$ and $\bullet \in \{\text{ac}, \text{Gr}\}$:

- I identify $H^1(K^S/K, \widetilde{M})$ with the space of classes in $H^1(K, \widetilde{M})$ unramified at all primes outside S (see [Ru00, Lemma 1.5.3]).
- Let $H_{\bullet}^1(K, \widetilde{M}) \subseteq H^1(K^S/K, \widetilde{M})$ be the space of classes in $H^1(K, \widetilde{M})$ satisfying the following conditions:
 - If $\bullet = \text{ac}$ (“ac” is for “anticyclotomic”): no condition at v_0 , unramified at finite primes outside N_p splitting in K , 0 at all other primes.
 - If $\bullet = \text{Gr}$ (“Gr” is for “Greenberg”): no condition at v_0 , unramified at all other primes.

Let $H_{\bullet, S \setminus N_p}^1(K, \widetilde{M}) \subseteq H^1(K^S/K, \widetilde{M})$ be the space of classes in $H^1(K, \widetilde{M})$ satisfying the above conditions at primes outside $S \setminus N_p$ (but not necessarily at primes in $S \setminus N_p$: the conditions are relaxed at these primes).

- Define

$$\begin{aligned} X_{\bullet, S \setminus N_p}(\widetilde{M}) &= (H_{\bullet, S \setminus N_p}^1(K, \widetilde{M}))^* \\ \widehat{X}_{\bullet, S \setminus N_p}(M) &= X_{\bullet, S \setminus N_p}(M) \otimes_{O_L[[\Gamma_K^-]]} \widehat{O}_L^{ur}[[\Gamma_K^-]] \\ \widehat{X}_{\bullet, S \setminus N_p}(\mathcal{M}) &= X_{\bullet, S \setminus N_p}(\mathcal{M}) \otimes_{O_L[[\Gamma_K]]} \widehat{O}_L^{ur}[[\Gamma_K]] \end{aligned}$$

Note that $Sel_{f,K,\xi}^S = H_{Gr, S \setminus N_p}^1(K, \mathcal{M})$ and $X_{f,K,\xi}^S = X_{Gr, S \setminus N_p}(\mathcal{M})$.

Now $M = \mathcal{M}[\gamma_+ - 1] = T_{f,K,\xi} \otimes_{O_L} (\text{morphisms } O_L[[\Gamma_K]] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \text{ factoring through } O_L[[\Gamma_K^-]])$.

An isomorphism $H^1(K^S/K, M) \cong H^1(K^S/K, \mathcal{M})[\gamma_+ - 1]$ is given by the cohomology map coming from $M \hookrightarrow \mathcal{M}$. (Proof: In the long exact cohomology sequence of $M \hookrightarrow \mathcal{M} \xrightarrow{\gamma_+ - 1} \mathcal{M}$, we have $(\gamma_+ - 1)\mathcal{M}^{K^S/K} = \mathcal{M}^{K^S/K}$, so the H^0 -to- H^1 connecting map is 0: see the proof in the Appendix inspired by [Sk14, section 2.7].)

Inside these two groups, $H_{ac,S \setminus N_p}^1(K, M) \subseteq H_{Gr,S \setminus N_p}^1(K, \mathcal{M})[\gamma_+ - 1]$ since ac is a stricter condition than Gr. So the restriction map $X_{Gr,S \setminus N_p}(\mathcal{M}) \rightarrow X_{ac,S \setminus N_p}(M)$ has $(\gamma_+ - 1)X_{Gr,S \setminus N_p}(\mathcal{M})$ in its kernel: if $x \in H_{ac,S \setminus N_p}^1(K, M) \subseteq H_{Gr,S \setminus N_p}^1(K, \mathcal{M})[\gamma_+ - 1]$ and $F = (\gamma_+ - 1)G$, $G \in X_{Gr,S \setminus N_p}(\mathcal{M})$ then $F(x) = G((\gamma_+ - 1)x) = G(0) = 0$. Therefore, there is a restriction map $X_{Gr,S \setminus N_p}(\mathcal{M})/(\gamma_+ - 1)X_{Gr,S \setminus N_p}(\mathcal{M}) \rightarrow X_{ac,S \setminus N_p}(M)$.

I note the following result.

Theorem 8.4.1. [JeSkWa17, Lemma 3.4.1]

- The inclusion map $H_{ac,S \setminus N_p}^1(K, M) \rightarrow H_{Gr,S \setminus N_p}^1(K, \mathcal{M})[\gamma_+ - 1]$ has finite cokernel.
- The restriction map $X_{Gr,S \setminus N_p}(\mathcal{M})/(\gamma_+ - 1) \rightarrow X_{ac,S \setminus N_p}(M)$ has finite kernel.

Adapting the proof of [JeSkWa17, Corollary 3.4.2], I show

Theorem 8.4.2.

$$\frac{(\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]] + \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]]}\widehat{X}_{Gr,S \setminus N_p}(\mathcal{M})}{(\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]]} \supseteq \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}\widehat{X}_{ac,S \setminus N_p}(M).$$

Proof. The $O_L[[\Gamma_K^-]]$ -module map $X_{Gr,S \setminus N_p}(\mathcal{M})/(\gamma_+ - 1) \rightarrow X_{ac,S \setminus N_p}(M)$ has finite-order kernel. So in $O_L[[\Gamma_K^-]]$, writing $(\gamma_+ - 1) := (\gamma_+ - 1)O_L[[\Gamma_K]]$, we have

$$\frac{(\gamma_+ - 1) + \text{Fitt}(X_{Gr,S \setminus N_p}(\mathcal{M}))}{(\gamma_+ - 1)} \supseteq (\gamma_+ - 1, \varpi_L)^r \text{Fitt}(X_{ac,S \setminus N_p}(M))$$

for some $r \in \mathbb{Z}_{\geq 0}$. Apply $\cdot \otimes_{O_L[[\Gamma_K^-]]} \widehat{O}_L^{ur}[[\Gamma_K^-]]$; in $\widehat{O}_L^{ur}[[\Gamma_K^-]]$, writing $(\gamma_+ - 1) := (\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]]$, we get

$$\begin{aligned} \frac{(\gamma_+ - 1) + \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]]} \widehat{X}_{Gr, S \setminus N_p}(\mathcal{M})}{(\gamma_+ - 1)} &\supseteq \frac{(\gamma_+ - 1) + \text{Fitt}_{\widehat{O}_L^{ur}[[\Gamma_K]]}(\widehat{X}_{Gr, S \setminus N_p}(\mathcal{M}))}{(\gamma_+ - 1)} \\ &\supseteq (\gamma_+ - 1, \varpi_L)^r \text{Fitt}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}(\widehat{X}_{ac, S \setminus N_p}(M)) \end{aligned}$$

where the leftmost ideal is principal and therefore contains $\text{Fitt}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}(\widehat{X}_{ac, S \setminus N_p}(M))$ as well as that ideal's minimum principal superideal $\text{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]} \widehat{X}_{ac, S \setminus N_p}(M)$. $\square$

Theorem 8.4.2 implies

$$\frac{(\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]] + \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K]]} \widehat{X}_{f, K, \xi}^S}{(\gamma_+ - 1)\widehat{O}_L^{ur}[[\Gamma_K]]} \supseteq \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]} \widehat{X}_{ac, S \setminus N_p}(M).$$

8.5 Half of an Iwasawa main conjecture

This section and the next follow and adapt arguments of Jetchev, Skinner and Wan [JeSkWa17, chapter 6].

From Theorems 8.3.1 and 8.4.2, we have the following over $\widehat{O}_L^{ur}[[\Gamma_K^-]][1/p]$.

Theorem 8.5.1. *For S including all places of K dividing pND_K , we have*

$$L_{Wan}^{-, S}(f) \cdot \widehat{O}_L^{ur}[[\Gamma_K^-]][1/p] \supseteq \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]][1/p]}(\widehat{X}_{ac, S \setminus N_p}(M)).$$

In this section, I briefly outline how to remove both the p -power denominators and the assumption that S includes the places of K over ND_K to obtain half of an Iwasawa main conjecture:

Theorem 8.5.2. *Assume C_{Wan} is a p -adic unit. For any set S of finitely many*

places of K containing N_p (possibly $S = N_p$), we have

$$L_{Wan}^{-,S}(f) \cdot \widehat{O}_L^{ur}[[\Gamma_K^-]] \supseteq \text{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}(\widehat{X}_{ac, S \setminus N_p}(M)).$$

The key result is a theorem of Hsieh ([Hs14, Theorem B]; see also the Remark on the previous page in that source). Recall the isomorphism $\widehat{O}_L^{ur}[[\Gamma_K^-]] \cong \widehat{O}_L^{ur}[[t]]$ sending γ^- to $1 + t$.

Theorem 8.5.3 (Hsieh's $\mu = 0$). *As in the Weierstrass preparation theorem, factor the p -adic L -function $L_{BDP}^{-,N_p}(f)$ as $L_{BDP}^{-,N_p}(f) = p^\mu R(t)U(t)$, where $\mu \in \mathbb{Q}$, $U(t) \in \widehat{O}_L^{ur}[[t]]^\times$ and the monic distinguished polynomial $R(t) \in \widehat{O}_L^{ur}[t]$ is chosen so that $\deg R$ is minimized. Then $\mu = 0$.*

I now assume that C_{Wan} is a p -adic unit. Under that assumption, the reasoning of [JeSkWa17, Theorem 6.1.6] goes through and proves Theorem 8.5.2; I outline the argument here.

Proof outline for Theorem 8.5.2. First, I remove from Theorem 8.5.1 the assumption that S contain all primes of K dividing ND_K ; then I apply the $\mu = 0$ result.

By a result of Jetchev, Skinner and Wan [JeSkWa17, Proposition 3.3.3], for a set S of finitely many places of K , the direct product of localization maps

$$H^1(K^S/K, M) \rightarrow \prod_{v \in S} \frac{H^1(K_v, M)}{H_{ac}^1(K_v, M)}$$

is surjective. So, for sets S_1 and S_2 each including finitely many places of K , if

$N_p \subseteq S_1 \subseteq S_2$ then

$$\begin{aligned}
& \text{char}_{O_L[[\Gamma_K^-]]} X_{ac}^{S_2}(M) \\
&= \text{char}_{O_L[[\Gamma_K^-]]} X_{ac}^{S_1}(M) \cdot \text{char}_{O_L[[\Gamma_K^-]]} \prod_{v \in S_2 - S_1} \text{Hom}_{O_L} \left(\frac{H^1(K_v, M)}{H_{ac}^1(K_v, M)}, L/O_L \right) \\
&= \text{char}_{O_L[[\Gamma_K^-]]} X_{ac}^{S_1}(M) \cdot \prod_{v \in S_2 - S_1} L_v(f). \tag{8.2}
\end{aligned}$$

Letting S_2 contain all primes of K over pND_K , taking $S = S_2$ in Theorem 8.5.1 and applying (8.2) yields the conclusion of Theorem 8.5.1 for S_1 , which need not contain all the places of K over ND_K .

Since C_{Wan} is a p -adic unit, Theorem 7.2.2 becomes

$$(L_{BDP}^{-, N_p}(f)) = (L_{Wan}^{-, N_p}(f)).$$

Because of Hsieh's $\mu = 0$ result, Theorem 8.5.1's conclusion now implies Theorem 8.5.2's conclusion for $S = N_p$, hence for $S \supseteq N_p$. $\square$

8.6 Consequences

Under the identification $\widehat{O}_L^{ur}[[\Gamma_K^-]] \cong \widehat{O}_L^{ur}[[t]]$ with $\gamma^- \mapsto 1 + t$, the continuous $\widehat{O}_L^{ur}$ -algebra map $P_1 : \widehat{O}_L^{ur}[[\Gamma_K^-]] \rightarrow \overline{\mathbb{Q}}_p$ that sends each element of Γ_K^- to 1 is equivalent to substituting $t = 0$ in an element of $\widehat{O}_L^{ur}[[t]]$.

Let $F_{ac}(t) \in \widehat{O}_L^{ur}[[t]]$ be a generator for $\text{char}_{\widehat{O}_L^{ur}[[\Gamma_K^-]]}(\widehat{X}_{ac, \emptyset}(M))$. Applying P_1 to Theorem 8.5.2 with $S = N_p$, we get $P_1(L_{Wan}^{-, N_p}(f)) \mid F_{ac}(0)$ in $\widehat{O}_L^{ur}$. But by the Greenberg control theorem of Jetchev, Skinner and Wan [JeSkWa17, Theorem 3.3.1], we have

$$\text{ind}_p F_{ac}(0) = \text{ind}_p C(W_f) \# H_{ac}^1(K, W_f)$$

where

$$C(W) := \#H^0(K_{v_0}, W) \cdot \#H^0(K_{\bar{v}_0}, W) \cdot \prod_{v \in S'} \#H_{ur}^1(K_v, W)$$

in which S' is the set of finite places v of K such that $v \nmid p$, V_f is ramified at v , and v is above a rational prime that splits in K . (In [JeSkWa17], my $C(W)$ is written $C^\emptyset(W)$.) So:

Theorem 8.6.1. *We have*

$$\mathrm{ind}_p P_1(L_{Wan}^{-, N_p}(f)) \leq \mathrm{ind}_p (C(W_f) \# H_{ac}^1(K, W_f)).$$

8.7 Appendix

In this appendix, I write $T = T_{f,K,\xi}$; recall from section 8.4 that $\mathcal{M} = T \otimes_{O_L} O_L[[\Gamma_K]]^*$.

I shall prove that $\mathcal{M}^{K^S/K} = (\gamma_+ - 1)\mathcal{M}^{K^S/K}$. It is enough to prove that given $t \in T$ and a continuous $\mathbb{Z}_p$ -map $F : O_L[[\Gamma_K]] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$, there is a continuous $\mathbb{Z}_p$ -map $F' : O_L[[\Gamma_K]] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ where for $h \in O_L[[\Gamma_K]]$ we have $F(h) = F'((\gamma_+ - 1)h)$ and, if $t \otimes F \in \mathcal{M}^{K^S/K}$, then $t \otimes F' \in \mathcal{M}^{K^S/K}$. (This will then imply that for any sum of pure tensors $\sum_i t_i \otimes F_i \in \mathcal{M}^{K^S/K}$, since each pure tensor $t_i \otimes F_i$ is in $(\gamma_+ - 1)\mathcal{M}^{K^S/K}$, their sum is also in $(\gamma_+ - 1)\mathcal{M}^{K^S/K}$.)

With $O_L[[\Gamma_K]] = \varprojlim_n O_L[\Gamma_n^+ \oplus \Gamma_n^-]$, [Ru00, Proposition B.1.1(ii)] gives

$$\varinjlim_n \mathrm{Hom}_{\mathbb{Z}_p}(O_L[\Gamma_n^+ \oplus \Gamma_n^-], \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p, cont}(O_L[[\Gamma_K]], \mathbb{Q}_p/\mathbb{Z}_p) = O_L[[\Gamma_K]]^*$$

Say that under this isomorphism, F corresponds to $[F_m]$ for some $\mathbb{Z}_p$ -map $F_m : O_L[\Gamma_m^+ \oplus \Gamma_m^-] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$, so that for $h = (h_n)_n \in O_L[[\Gamma_K]] = \varprojlim_n O_L[\Gamma_n^+ \oplus \Gamma_n^-]$ we have $F(h) = F_m(h_m)$.

Let $\{\alpha_a\}$ be a $\mathbb{Z}_p$ -basis for O_L . For $j \in \{0, 1, \dots, p^m - 1\}$ and α_a , let

$$F_m \left(\alpha_a \gamma_{m,-}^j \sum_{i=0}^{p^m-1} \gamma_{m,+}^i \right) = p^{-k_{aj}} x_{aj}$$

where $k_{aj} \in \mathbb{Z}_{\geq 0}$, $x_{aj} \in \mathbb{Z}_p/p^{k_{aj}}\mathbb{Z}_p$ and $x_{aj} \notin p\mathbb{Z}_p/p^{k_{aj}}\mathbb{Z}_p$. Let $r = m + \max_{a,j} k_{aj}$ and define the $\mathbb{Z}_p$ -map $F_r : O_L[\Gamma_r^+ \oplus \Gamma_r^-] \twoheadrightarrow O_L[\Gamma_m^+ \oplus \Gamma_m^-] \xrightarrow{F_m} \mathbb{Q}_p/\mathbb{Z}_p$; then for $j \in \{0, 1, \dots, p^r - 1\}$ and α_a , we have

$$F_t \left(\alpha_a \gamma_{r,-}^j \sum_{i=0}^{p^r-1} \gamma_{r,+}^i \right) = p^{r-m} F_m \left(\alpha_a \gamma_{r,-}^j \sum_{i=0}^{p^m-1} \gamma_{r,+}^i \right) = 0$$

since $r - m \geq \max k_{aj}$. The map F corresponds to $[F_r]$.

For $h \in O_L[\Gamma_r^+ \oplus \Gamma_r^-]$, $h = \sum_a \sum_{i=0}^{p^r-1} \sum_{j=0}^{p^r-1} a_{aij} \alpha_a \gamma_{r,+}^i \gamma_{r,-}^j$ with $a_{aij} \in \mathbb{Z}_p$, letting $a_{a,-1,j} = a_{a,p^r-1,j}$, we have

$$(\gamma_{r,+} - 1)h = \sum_a \sum_{i=0}^{p^r-1} \sum_{j=0}^{p^r-1} (a_{a,i-1,j} - a_{aij}) \alpha_a \gamma_{r,+}^i \gamma_{r,-}^j$$

which is 0 iff $(a_{a,i-1,j} = a_{aij} \text{ for all } (a, i, j))$ iff

$$h = \sum_a \sum_{j=0}^{p^r-1} \left(a_{aj} \alpha_a \gamma_{r,-}^j \sum_{i=0}^{p^r-1} \gamma_{r,+}^i \right)$$

for some $a_{aj} \in \mathbb{Z}_p$, in which case $F_r(h) = 0$.

So there is a $\mathbb{Z}_p$ -map $F'_r : O_L[\Gamma_r^+ \oplus \Gamma_r^-] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ where for $h \in O_L[\Gamma_r]$, we have $F'_r((\gamma_{r,+} - 1)h) = F_r(h)$. This determines F'_r on only the kernel of $\gamma_{r,+} \mapsto 1 : O_L[\Gamma_r^+ \oplus \Gamma_r^-] \rightarrow O_L[\Gamma_r^-]$, so we may also choose F'_r so that it sends $O_L[\Gamma_r^-]$ to 0. The class of F'_r corresponds to $F' \in \text{Hom}_{\mathbb{Z}_p, \text{cont}}(O_L[[\Gamma_K]], \mathbb{Q}_p/\mathbb{Z}_p)$ with $F'((\gamma_+ - 1)h) = F(h)$ for $h \in O_L[[\Gamma_K]]$, that is, $(\gamma_+ - 1) \cdot F' = F$. Also, F' sends $O_L[[\Gamma_K^-]]$ to 0.

So $O_L[[\Gamma_K]]^* \xrightarrow{\gamma_+ - 1} O_L[[\Gamma_K]]^*$ is surjective. Its kernel Z consists of $F : O_L[[\Gamma_K]] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ factoring through $\Gamma_K^+ \mapsto 1 : O_L[[\Gamma_K^+ \oplus \Gamma_K^-]] \rightarrow O_L[[\Gamma_K^-]]$. We have $Z \cong$

$\text{Hom}_{\mathbb{Z}_p}(O_L[[\Gamma_K^-]], \mathbb{Q}_p/\mathbb{Z}_p)$. Since $T \cong O_L^2$ is flat as an O_L -module, we have a short exact G_K -module sequence

$$0 \rightarrow T \otimes_{O_L} Z \rightarrow T \otimes_{O_L} O_L[[\Gamma_K]]^* \xrightarrow{1 \otimes (\gamma_+^{-1})} T \otimes_{O_L} O_L[[\Gamma_K]]^* \rightarrow 0.$$

Choose $t \in T$; assume that for all $g \in G_K$ we have $g(t \otimes F) = \rho(g) \cdot t \otimes \Psi_K^{-1}(g) \cdot F = t \otimes F \in \mathcal{M}$, that is, $\rho(g) \cdot t \otimes (\gamma_+ - 1) \Psi_K^{-1}(g) \cdot F' = t \otimes (\gamma_+ - 1) \cdot F' \in \mathcal{M}$. Then by the last paragraph, we have $x = \rho(g) \cdot t \otimes \Psi_K^{-1}(g) \cdot F' - t \otimes F' \in T \otimes_{O_L} Z$. But we chose F' to send $O_L[[\Gamma_K^-]]$ to 0, so, interpreting $T \otimes_{O_L} O_L[[\Gamma_K]]^*$ and $T \otimes_{O_L} Z$ as the $\mathbb{Z}_p$ -modules $\text{Hom}_{\mathbb{Z}_p, \text{cont}}(O_L[[\Gamma_K]], (\mathbb{Q}_p/\mathbb{Z}_p)^2)$ and $\text{Hom}_{\mathbb{Z}_p}(O_L[[\Gamma_K^-]], (\mathbb{Q}_p/\mathbb{Z}_p)^2)$ respectively via $e_1 \otimes F_1 + e_2 \otimes F_2 \mapsto (h \mapsto (F_1(h), F_2(h)))$, x must send $O_L[[\Gamma_K^-]]$ to $(0, 0)$; hence $x = 0$. So $t \otimes F \in \mathcal{M}^{K^S/K} \Rightarrow t \otimes F' \in \mathcal{M}^{K^S/K}$. This completes the proof.

Chapter 9

From cohomology to III

In this chapter, I adapt an argument of Jetchev, Skinner and Wan [JeSkWa17, section 3.5] to prove an identity $\#H_{ac}^1(K, W_f) = \#\text{III}(K, W_f) \cdot (\#\delta_{v_0})^2$ and to describe $\#\delta_{v_0}$.

9.1 A result from the literature

I recall a result of Jetchev, Skinner and Wan.

Proposition 9.1.1. *[JeSkWa17, Proposition 3.2.1] Let $L/\mathbb{Q}_p$ be a finite-degree field extension, with O_L the ring of integers of L , m_L its maximal ideal, and k_L its residue field. Let the L -vector space V of dimension 2 have an L -linear continuous action by $G_{\mathbb{Q}}$. Let V contain an O_L -module T of O_L -rank 2, so that T is stable under the action by $G_{\mathbb{Q}}$. Define $W = V/T$.*

Assume the following eight statements about (T, V, W) .

(U) The G_K -representation V is unramified outside finitely many places of K .

(P) The G_K -representation V is pure of weight -1 .

(D) As G_K -representations, the dual $\text{Hom}_L(V, L(1))$ is isomorphic to V^c , which is defined as V with a new G_K -action so that $g \in G_K$ sends $v \in V$ to $cgc^{-1}(v)$, for the complex conjugation map $c \in G_{\mathbb{Q}}$.

- (L) As a representation of each of the local Galois groups $G_{K_{v_0}}$ and $G_{K_{\bar{v}_0}}$, V is semistable.
- (H) Every Hodge-Tate weight of V is either 0 or not divisible by $p - 1$.
- (I) The semisimplification of $T/m_L T$ is irreducible as a G_K -representation over k_L .
- (C) The O_L -modules $H_f^1(K, W)_{\text{div}}$, $H_f^1(K_{v_0}, W)$ and $H_f^1(K_{\bar{v}_0}, W)$ each have corank 1, that is, are isomorphic to L/O_L .
- (S) Both of the localization maps $H_f^1(K, W)_{\text{div}} \rightarrow H_f^1(K_{v_0}, W)$ and $H_f^1(K, W)_{\text{div}} \rightarrow H_f^1(K_{\bar{v}_0}, W)$ are surjections.

Define δ_{v_0} to be the cokernel of the localization map

$$H_f^1(K, T) \rightarrow H_f^1(K_{v_0}, T)/H_f^1(K_{v_0}, T)_{\text{tor}}.$$

Then

$$\#H_{ac}^1(K, W) = \#\text{III}_f(K, W) \cdot (\#\delta_{v_0})^2.$$

(Note that in this result, $L/\mathbb{Q}_p$ need not be unramified.)

9.2 Results and arguments

Theorem 9.2.1. *Suppose the following hold.*

- (a) *Congruence:* $k/2$ is not congruent to 0 or 1 modulo $p - 1$.
- (b) *Residual irreducibility:* $T_f/\varpi_L T_f$ is an irreducible G_K -representation of dimension at least 2.
- (c) *Rank 1:* The O_L -module $\text{im } AJ_K$ has rank 1.
- (d) *Finiteness of Sha:* $\text{III}(K, W_f)[p^\infty]$ has finite cardinality as a set.

(e) *Localization:* For each place $v \mid p$ of K , the localization map $H_f^1(K, W_f) \rightarrow H_f^1(K_v, W_f)$ restricts to a map

$$(\mathrm{im} \ AJ_K) \otimes_{O_L} (L/O_L) \rightarrow (\mathrm{im} \ AJ_{K_v}) \otimes_{O_L} (L/O_L)$$

of which the kernel is torsion.

(f) *Local corank 1:* For each place $v \mid p$ of K , the O_L -module $H_f^1(K_v, W_f)$ has corank 1.

Define δ_{v_0} to be the cokernel of the localization map

$$\mathrm{loc}_{v_0}/\mathrm{tor} : H_f^1(K, T_f) \rightarrow H_f^1(K_{v_0}, T_f)/H_f^1(K_{v_0}, T_f)_{\mathrm{tor}}.$$

Then

$$\#H_{ac}^1(K, W_f) = \#\mathrm{III}(K, W_f) \cdot (\#\delta_{v_0})^2 \tag{9.1}$$

and $H_f^1(K, T_f) \cong O_L$.

Proof. I show that for $(T, V, W) = (T_f, V_f, W_f)$, the hypotheses of Proposition 9.1.1 and the equation $\mathrm{III}(K, W_f) = \mathrm{III}_f(K, W_f)$ are true. (For the definitions of T_f , V_f , W_f and the two III groups, see sections 6.4, 6.5 and 6.7.) By Proposition 9.1.1, this is enough to prove (9.1); I then finally prove $H_f^1(K, T_f) \cong O_L$.

As mentioned in [JeSkWa17, section 3.5], the hypotheses (U) and (P) hold by a result of Saito, and the hypotheses (D) (because $V_f^c \cong V_f \cong \mathrm{Hom}_L(V_f, L(\epsilon))$), (L) (because $p \nmid N$) and (H) (by assumption (a), because V_f has Hodge-Tate weights $-k/2$ and $(k/2) - 1$) also hold.

By assumption (b), the G_K -representation $T_f/\varpi_L T_f$ is irreducible, so it is equal to its semisimplification and hypothesis (I) holds.

In the short exact sequence

$$(\operatorname{im} AJ_K) \otimes_{O_L} (L/O_L) \hookrightarrow H_f^1(K, W_f) \twoheadrightarrow \operatorname{III}(K, W_f)$$

of O_L -modules, the first term has corank 1 because $\operatorname{im} AJ_K$ has rank 1 (assumption (c)), and the third term has corank 0 by assumption (d), so $H_f^1(K, W_f)$ and $H_f^1(K, W_f)_{\operatorname{div}}$ have corank 1. Hence, by assumption (f), (C) holds.

Let $v \mid p$ be a place of K . The O_L -module $\operatorname{im} AJ_K$ has rank 1, so it is the direct sum of O_L and some torsion O_L -module (possibly 0). The O_L -module $(\operatorname{im} AJ_K) \otimes_{O_L} (L/O_L)$ has corank 1, and it is isomorphic as an O_L -module to L/O_L . By assumption (e), $(\operatorname{im} AJ_K) \otimes_{O_L} (L/O_L)$ is sent by the localization map to $(\operatorname{im} AJ_{K_v}) \otimes_{O_L} (L/O_L)$ with torsion kernel, so $(\operatorname{im} AJ_{K_v}) \otimes_{O_L} (L/O_L)$ has corank at least 1. But $(\operatorname{im} AJ_{K_v}) \otimes_{O_L} (L/O_L)$ is an O_L -submodule of $H_f^1(K_v, W_f)$, which has corank 1 by assumption (f). So as O_L -modules, $(\operatorname{im} AJ_{K_v}) \otimes_{O_L} (L/O_L) = H_f^1(K_v, W_f) \cong L/O_L$, and each $c' \in H_f^1(K_v, W_f)$ is the image of some $c \in (\operatorname{im} AJ_K) \otimes_{O_L} (L/O_L) \subseteq H_f^1(K, W_f)_{\operatorname{div}}$. This implies (S).

There are quotient maps

$$\begin{array}{ccc} H_f^1(K, W_f) & & \\ \downarrow & & \\ \operatorname{III}(K, W_f) & = & H_f^1(K, W_f) / ((\operatorname{im} AJ_K) \otimes_{O_L} (L/O_L)) \\ \downarrow & & \\ \operatorname{III}_f(K, W_f) & = & H_f^1(K, W_f) / H_f^1(K, W_f)_{\operatorname{div}}. \end{array}$$

Since $(\operatorname{im} AJ_K) \otimes_{O_L} (L/O_L)$ is divisible, it is the maximal p -divisible subgroup of $H_f^1(K, W_f)$ (because $H_f^1(K, W_f)$ has corank 1), so $\operatorname{III}(K, W_f) = \operatorname{III}_f(K, W_f)$.

For a uniformizer $\lambda_L \in m_L$ of L , taking the long exact G_K -cohomology of the

short exact sequence

$$T_f \xrightarrow{\lambda_L} T_f \rightarrow T_f/\lambda_L T_f$$

implies that the sequence

$$(T_f/\lambda_L T_f)^{G_K} \rightarrow H^1(K, T_f) \xrightarrow{\lambda_L} H^1(K, T_f)$$

is exact. The left term is 0: it is an O_L -submodule of the G_K -module $T_f/m_L T_f$, and $T_f/m_L T_f$ is irreducible and not 1-dimensional (assumption (b)). Therefore $\lambda_L : H^1(K, T_f) \rightarrow H^1(K, T_f)$ is injective and $H^1(K, T_f)$ is torsion free. Now $H_f^1(K, T_f)$ is finitely generated as an O_L -module by Lemma 6.3.2, and

$$\text{rank}_{O_L} H_f^1(K, T_f) = \dim_L H_f^1(K, V_f) = \text{corank}_{O_L} H_f^1(K, W_f) = 1 \quad (9.2)$$

so $H_f^1(K, T_f) \cong O_L$. (Proof of (9.2): By Lemma 6.3.1, we have $H^1(G_K, T_f) \otimes_{O_L} L \cong H^1(G_K, V_f)$. There is no divisible part in the cokernel of $H^1(K, V_f) \rightarrow H^1(K, W_f)$, since that cokernel is the image of the connecting map $H^1(K, W_f) \rightarrow H^2(K, T_f)$, which is the torsion subgroup of $H^2(K, T_f)$ by Lemma 6.3.1, and $H^2(K, T_f)$ is a finitely generated O_L -module by Lemma 6.3.2.) $\square$

9.3 Finding $\#\delta_{v_0}$

I now adapt the argument of [JeSkWa17, section 3.5] to find a formula for $\#\delta_{v_0}$, which can be substituted into (9.1). Recall that δ_{v_0} is the cokernel of the localization-modulo-torsion map $\text{loc}_{v_0}/\text{tor}$.

Define $C_0 = \text{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma17}) \in H_f^1(K, T_f)$. Assume that the image of C_0 in $H_f^1(K_{v_0}, T_f)$ is not torsion, and that

- (i) $H_f^1(K, T) \cong O_L$ as O_L -modules.

(ii) $H_f^1(K_{v_0}, T)/\text{tor} \cong O_L$ is a torsion-free rank-1 O_L -module.

(If the hypotheses of Proposition 9.2.1 hold, then both (i) and (ii) above are true.)
Then $\text{loc}_{v_0}/\text{tor} : O_L \cong H_f^1(K, T_f) \rightarrow H_f^1(K_{v_0}, T_f)/\text{tor} \cong O_L$ is injective, because of the assumption that $(\text{loc}_{v_0}/\text{tor})(C_0) \neq 0$.

Recall the map $\log_\omega$ for differential forms $\omega \in \tilde{V}$ as described in section 6.9.

Theorem 9.3.1. *Choose the differential form $\omega \in \tilde{V}$ so that $\log_\omega$ restricts to an isomorphism $\log_\omega : H_f^1(K_{v_0}, T_f)/\text{tor} \xrightarrow{\cong} O_L$ with inverse $\exp_\omega$. (Here, $H_f^1(K_{v_0}, T_f)/\text{tor}$ is interpreted as a subgroup of $H_f^1(K_{v_0}, V_f) \cong (H_f^1(K_{v_0}, T_f)/\text{tor}) \otimes_{O_L} L$.) Then*

$$\#\delta_{v_0} = \frac{(O_L : O_L \log_\omega(\text{loc}_{v_0} C_0)) \cdot (H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L))}{p^{[L:\mathbb{Q}_p]} \cdot \#H^0(K_{v_0}, W_f) \cdot (H_f^1(K, T_f) : O_L C_0)}. \quad (9.3)$$

Proof. I argue as in [JeSkWa17], replacing that source's A_f , $\#A_f[\mathfrak{p}^\infty](\mathbb{F}_p)$, ω_f and P with $H_f^1(\cdot, T_f)$, $(H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L))$, ω and C_0 respectively.

The rank-1 O_L -modules $H_f^1(K_{v_0}, T_f)/\text{tor} \supseteq (\text{loc}_{v_0}/\text{tor})H_f^1(K, T_f) \supseteq O_L \text{loc}_{v_0} C_0$ are of finite index in one another, so we have

$$\begin{aligned} \#\delta_v &= (H_f^1(K_{v_0}, T_f)/\text{tor} : (\text{loc}_{v_0}/\text{tor})H_f^1(K, T_f)) \\ &= (H_f^1(K_{v_0}, T_f)/\text{tor} : O_L \text{loc}_{v_0} C_0) / ((\text{loc}_{v_0}/\text{tor})H_f^1(K, T_f) : O_L \text{loc}_{v_0} C_0) \\ &= (H_f^1(K_{v_0}, T_f)/\text{tor} : O_L \text{loc}_{v_0} C_0) / (H_f^1(K, T_f) : O_L C_0) \end{aligned} \quad (9.4)$$

because $\text{loc}_{v_0}/\text{tor}$ is injective.

Since $\log_\omega : H_f^1(K_{v_0}, T_f)/\text{tor} \rightarrow O_L$ is an isomorphism, the numerator in the last fraction in (9.4) is

$$\begin{aligned} &(H_f^1(K_{v_0}, T_f)/\text{tor} : O_L \text{loc}_{v_0} C_0) \\ &= (\log_\omega(H_f^1(K_{v_0}, T_f)/\text{tor}) : \log_\omega(O_L \text{loc}_{v_0} C_0)) \\ &= (O_L : O_L \log_\omega(\text{loc}_{v_0} C_0)) / (O_L : \log_\omega(H_f^1(K_{v_0}, T_f)/\text{tor})). \end{aligned} \quad (9.5)$$

Finally, the denominator in the last fraction in (9.5) is

$$\begin{aligned}
& (O_L : \log_\omega(H_f^1(K_{v_0}, T_f)/\text{tor})) \\
&= (O_L : pO_L) / (\log_\omega(H_f^1(K_{v_0}, T_f)/\text{tor}) : pO_L) \\
&= (O_L : pO_L) / (H_f^1(K_{v_0}, T_f)/\text{tor} : \exp_\omega(pO_L)) \\
&= (O_L : pO_L) \cdot \#H_f^1(K_{v_0}, T_f)_{\text{tor}} / (H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L)) \\
&= (O_L : pO_L) \cdot \#H^0(K_{v_0}, W_f) / (H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L)). \tag{9.6}
\end{aligned}$$

For the last equation, note that $H_f^1(K_{v_0}, T_f)_{\text{tor}} = H^1(K_{v_0}, T_f)_{\text{tor}}$ which, by Lemma 6.3.1, is isomorphic to the image of the connecting map $H^0(K_{v_0}, W_f) \rightarrow H^1(K_{v_0}, T_f)$; but this image is isomorphic to $H^0(K_{v_0}, W_f)$ since $V_f^{G_{K_{v_0}}} = 0$.

Noting that $(O_L : pO_L) = p^{[L:\mathbb{Q}_p]}$ (because p does not ramify in $L/\mathbb{Q}_p$) and combining (9.4), (9.5) and (9.6) yields (9.3). $\square$

Chapter 10

From III to Heegner cycles: an Euler system result

10.1 Masoero's theorem

In this section, I link the order of III to the Heegner cycles Δ_{Ma17} and Z_{Ma17} defined in section 6.2 by describing and slightly adapting a result of Masoero [Ma17], which that paper proved by adapting arguments of Kolyvagin [Ko90, Ko91] (as re-organized by McCallum [Mc91]) about Euler systems and Shafarevich-Tate groups.

Theorem 10.1.1. (*[Ma17, Theorem 7.3 and next sentence, Corollary 7.11]; see also [Ne92, Theorem 13.1]). Assume the following:*

- (a) *The integer $k \geq 4$ is even. The integer N satisfies $N \geq 3$.*
- (b) *The eigenform $f \in S_k^{new}(\Gamma_0(N))$ is not CM. We have $a(1, f) = 1$.*
- (c) *The imaginary quadratic number field $K = \mathbb{Q}(\sqrt{D_K})$ has negative discriminant $D_K \equiv 1 \pmod{4}$.*
- (d) *Every rational prime factor of N splits or ramifies in K .*

(e) The prime p does not ramify in $\mathbb{Q}(f)$.

(f) The prime p does not divide $(k-2)! \cdot 6N\varphi(N)D_K h_K \cdot (O_{\mathbb{Q}(f)} : O(f))$.

(g) If $g \in GL_2(O_{\mathbb{Q}(f)} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ and $\det g$ is a $(k-1)$ th power in $\mathbb{Z}_p^\times$, then g is in the image of the representation $\rho_{f,p} : G_{\mathbb{Q}} \rightarrow GL_2(O_{\mathbb{Q}(f)} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ corresponding to T_p .

(h) The cohomology class $C_0 = \text{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma17}) \in H^1(K, T_f)$ is not torsion.

Then

$$(\text{im } AJ_K) \otimes \mathbb{Q} = L \cdot C_0 \subseteq H_f^1(K, V_f),$$

the group $\text{III}(K, W_f)$ has finite cardinality, and

$$\text{ind}_p \# \text{III}(K, W_f) \leq 2 \text{ind}_p(AJ_{K_1}(Z_{Ma17}), \text{im } AJ_{K_1}).$$

Remarks 10.1.2. (a) Although Masoero's paper assumes that every prime dividing N splits in K , the paper's argument goes through under the more general assumption (d) above, replacing “splits” with “splits or ramifies”. The only place where Masoero uses the splitting assumption is to deduce the existence of an ideal $\mathfrak{C}$ of O_K for which $\mathbb{Z}/N\mathbb{Z} \cong O_K/\mathfrak{C}$ [Ma17, section 4.1]; such a $\mathfrak{C}$ still exists if each prime over N may split or ramify.

(b) Conditions (e)-(g) exclude only finitely many p ; see [Ma17, section 4.2]. Masoero assumes $p \nmid h_K$ to define Kolyvagin classes. (This forces a certain restriction map to be an isomorphism; see the argument between Remark 4.2 and Proposition 4.3 in [Ma17].) To adapt Masoero's reasoning to the case $p \mid h_K$, one might need to use universal Euler system arguments along the lines of Rubin [Ru00, sections 4.2-4.4]; see in particular [Ru00, Remark 4.4.3].

(c) The definition of $\text{ind}_p(c, B)$ is that of section 6.3. I emphasize that this index is not with respect to the uniformizer ϖ_L of L , but instead with respect to p .

Chapter 11

Comparing Heegner cycles: Abel-Jacobi maps

Recall the groups $G(t, r)$ and the projections π_B and $\pi_{B,1}$ from section 6.1, as well as the cycles and varieties from section 6.2.

11.1 The chapter's main result

In this chapter, I prove the following result.

Theorem 11.1.1. *Assume $p \nmid 2D_K N\phi(N)$. Assume that*

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}) = \mathrm{ind}_p(AJ_{K_1}(\Delta_{Ma17}), \mathrm{im} AJ_{K_1}) \quad (11.1)$$

and that when the $h_K \cdot \phi(N)$ elements of $\mathrm{Gal}(K_1/K) \times (\Gamma_0(N)/\Gamma_1(N))$ act on the point of $X_1(N)$ associated to E_{x_1} , the corresponding $h_K \cdot \phi(N)$ images of that point are distinct. Let $f \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be a newform of even weight $k > 2$ with $k/2 \leq p$. Assume that the map (6.4) with $\eta = \omega_f \wedge \omega_A^{k/2-1} \eta_A^{k/2-1}$, $A = E_{x_1}$ and $F = L$ has

image in O_L . Then

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}) \leq \mathrm{ind}_p AJ_L^{1, E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right). \quad (11.2)$$

Remarks 11.1.2. (a) Recall that the p -index $\mathrm{ind}_p : \widehat{\mathbb{Q}}_p^\times \rightarrow \mathbb{Q}_{>0}$ satisfies $\mathrm{ind}_p(p^n) = n$ for $n \in \mathbb{Z}$.

(b) We always have

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}) = \mathrm{ind}_p(\mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma17}), \mathrm{im} AJ_K) \quad (11.3)$$

because

$$AJ_{K_1}(Z_{Ma17}) = \mathrm{res}_{K_1/K}(\mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma17}))$$

and, noting that $p \nmid h_K$,

$$\begin{aligned} \mathrm{cor}_{K_1/K} AJ_{K_1}(Z_{Ma17}) &= \mathrm{cor}_{K_1/K} \mathrm{res}_{K_1/K}(\mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma17})) \\ &= h_K \mathrm{cor}_{K_1/K} AJ_{K_1}(\Delta_{Ma17}). \end{aligned}$$

(To see why these identities imply (11.3), the reader can use versions of the next section's lemmas with pushforwards and pullbacks replaced by the maps $\mathrm{res}_{K_1/K}$ and $\mathrm{cor}_{K_1/K}$.) Equation (11.1) says that the $\mathrm{cor}_{K_1/K}$ on the right hand side of (11.3) can be removed without changing the p -indices, so (11.1) is stronger than (11.3).

11.2 Correspondences, Galois actions and p -indices

The following two lemmas are key tools in my argument.

Lemma 11.2.1. *Let U_1 and U_2 be varieties both defined over K_1 , with associated Abel-Jacobi maps $AJ_{K_1}^{U_1}$ and $AJ_{K_1}^{U_2}$. Let Δ_1 and Δ_2 be cycles, defined over K_1 , in U_1 and*

U_2 respectively. Let P be a correspondence from U_1 to U_2 , with induced pushforward and pullback maps P_* and P^* between the Chow groups of U_1 and U_2 . Write

$$M_1 := \text{ind}_p(AJ_{K_1}^{U_1}(\Delta_1), \text{im } AJ_{K_1}^{U_1})$$

if this number is well defined, and write

$$M_2 := \text{ind}_p(AJ_{K_1}^{U_2}(\Delta_2), \text{im } AJ_{K_1}^{U_2})$$

if this number is well defined.

- (a) Assume that $AJ_{K_1}^{U_2}(\Delta_2)$ is not torsion and that for some $\alpha \in \mathbb{Z}_p^\times$, we have $P_*\Delta_1 = \alpha\Delta_2$. Then $AJ_{K_1}^{U_1}(\Delta_1)$ is also not torsion, M_1 and M_2 are both well defined, and $M_1 \leq M_2$.
- (b) Assume that $AJ_{K_1}^{U_1}(\Delta_1)$ is not torsion and that for some $\alpha \in \mathbb{Z}_p^\times$, we have $\alpha\Delta_1 = P^*\Delta_2$. Then $AJ_{K_1}^{U_2}(\Delta_2)$ is also not torsion, M_1 and M_2 are both well defined, and $M_1 \geq M_2$.

Proof. (a) Assume for a contradiction that $AJ_{K_1}^{U_1}(\Delta_1)$ is torsion. Then so is the cycle

$$P_*AJ_{K_1}^{U_1}(\Delta_1) = AJ_{K_1}^{U_2}(P_*\Delta_1) = \alpha AJ_{K_1}^{U_2}(\Delta_2),$$

so $AJ_{K_1}^{U_2}(\Delta_2)$ is also torsion, contrary to assumption. So $AJ_{K_1}^{U_1}(\Delta_1)$ is not torsion.

Therefore, there are non-torsion classes $C_1 \in \text{im } AJ_{K_1}^{U_1}$ and $C_2 \in \text{im } AJ_{K_1}^{U_2}$ for

which $AJ_{K_1}^{U_1}(\Delta_1) = p^{M_1}C_1$ and $AJ_{K_1}^{U_2}(\Delta_2) = p^{M_2}C_2$, and we have

$$\begin{aligned}
M_2 &= \text{ind}_p(\alpha AJ_{K_1}^{U_2}(\Delta_2), \text{im } AJ_{K_1}^{U_2}) && (\text{since } \alpha \in \mathbb{Z}_p^\times) \\
&= \text{ind}_p(AJ_{K_1}^{U_2}(P_*\Delta_1), \text{im } AJ_{K_1}^{U_2}) && (\text{by assumption}) \\
&= \text{ind}_p(P_*AJ_{K_1}^{U_1}(\Delta_1), \text{im } AJ_{K_1}^{U_2}) && (\text{Abel-Jacobi maps commute} \\
&&& \text{with correspondences}) \\
&= \text{ind}_p(p^{M_1}P_*(C_1), \text{im } AJ_{K_1}^{U_2}) \\
&= M_1 + \text{ind}_p(P_*(C_1), \text{im } AJ_{K_1}^{U_2}) \geq M_1.
\end{aligned}$$

(b) In the argument for part (a), replace P_* with P^* and swap M_1 with M_2 , U_1 with U_2 , Δ_1 with Δ_2 and C_1 with C_2 . We obtain $M_1 \geq M_2$. $\square$

Lemma 11.2.2. *Let U be a variety defined over K , with associated Abel-Jacobi map $AJ_{K_1}^U$. Let Δ be a cycle defined over K_1 in U . Suppose that*

$$\text{ind}_p(AJ_{K_1}^U(\Delta), \text{im } AJ_{K_1}^U)$$

is well defined. Then for any $\sigma \in \text{Gal}(K_1/K)$, we have

$$\text{ind}_p(AJ_{K_1}^U(\Delta), \text{im } AJ_{K_1}^U) = \text{ind}_p(AJ_{K_1}^U(\sigma\Delta), \text{im } AJ_{K_1}^U).$$

Proof. If a non-torsion class $C_1 \in \text{im } AJ_{K_1}^U$ satisfies $AJ_{K_1}^U(\Delta) = p^{M_1}C_1$ for some $M_1 \in \mathbb{Z}_{\geq 0}$, then applying $\sigma \in \text{Gal}(K_1/K)$ and noting that Abel-Jacobi maps are Galois equivariant yields $AJ_{K_1}^U(\sigma\Delta) = p^{M_1}(\sigma C_1)$, and σC_1 is non-torsion since C_1 is non-torsion.

Conversely, if a non-torsion class $C_2 \in \text{im } AJ_{K_1}^U$ satisfies $AJ_{K_1}^U(\sigma\Delta) = p^{M_2}C_2$ for some $M_2 \in \mathbb{Z}_{\geq 0}$, then applying σ^{-1} similarly yields $AJ_{K_1}^U(\Delta) = p^{M_2}(\sigma^{-1}C_2)$, and $\sigma^{-1}C_2$ is non-torsion.

The desired result follows immediately. $\square$

For the rest of this chapter, the hypotheses of Theorem 11.1.1 are assumed.

11.3 From Masoero to Castella

First, in this section, I link the p -index of Masoero's Heegner cycle Δ_{Ma17} to the p -index of Castella's Heegner cycle Δ_{Ca13} .

The “forget the second $(\mathbb{Z}/N\mathbb{Z})$ -basis vector” map $\mathcal{E}(\Gamma(N)) \rightarrow \mathcal{E}(\Gamma_1(N))$ gives maps $P_r : \tilde{\mathcal{E}}^r(\Gamma(N)) \rightarrow \tilde{\mathcal{E}}^r(\Gamma_1(N))$. Passing to Chow groups, we obtain pushforward maps $P_{r,*}$ and pullback maps P_r^* .

To help perform the calculations below, I define a $\mathbb{Q}$ -linear map¹

$$P_{r,*} : \mathbb{Q}[G(2, r) \times (\Gamma_0(N)/\Gamma(N))] \rightarrow \mathbb{Q}[G(1, r) \times (\Gamma_0(N)/\Gamma_1(N))]$$

so that the element

$$(((z_{11}, z_{12}, \varepsilon_1), \dots, (z_{r1}, z_{r2}, \varepsilon_r)), s) \in ((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \{\pm 1\}^r \rtimes S_r = G(2, r)$$

(where each z_{ij} is in $\mathbb{Z}/N\mathbb{Z}$, each ε_i is in $\{\pm 1\}$ and $s \in S_r$) is sent by $P_{r,*}$ to

$$(((z_{11}, \varepsilon_1), \dots, (z_{r1}, \varepsilon_r)), s) \in ((\mathbb{Z}/N\mathbb{Z}) \rtimes \{\pm 1\}^r \rtimes S_r = G(1, r),$$

and an element $b \in \Gamma_0(N)/\Gamma(N)$ is sent by $P_{r,*}$ to the image of b under the quotient map $\Gamma_0(N)/\Gamma(N) \rightarrow \Gamma_0(N)/\Gamma_1(N)$. Then, for any $\sigma \in \mathbb{Q}[G(2, r) \times (\Gamma_0(N)/\Gamma(N))]$, any cycle Z of $\tilde{\mathcal{E}}^r(\Gamma(N))$ and any cycle Z_1 of $\tilde{\mathcal{E}}^r(\Gamma_1(N))$, we have

$$P_{r,*}(\sigma \cdot Z) = P_{r,*}(\sigma) \cdot P_{r,*}(Z) \tag{11.4}$$

¹My reason for using the same notation $P_{r,*}$ for different maps is explained by (11.4) and (11.5).

and

$$P_r^*(P_{r,*}(\sigma) \cdot Z_1) = \sigma \cdot P_r^*(Z_1). \quad (11.5)$$

Equations (11.4) and (11.5) are easily shown by first considering the cases $\sigma \in G(2, r)$ and $\sigma \in \Gamma_0(N)/\Gamma(N)$, then extending by $\mathbb{Q}$ -linearity.

The maps $P_{k-2,*}$ and P_{k-2}^* act on the cycles Δ_{Ma17} and Δ_{Ca13} in the following simple way.

Proposition 11.3.1. *We have*

$$P_{k-2,*}(\Delta_{Ma17}) = \Delta_{Ca13} \quad (11.6)$$

and

$$P_{k-2}^*(\Delta_{Ca13}) = N \cdot \Delta_{Ma17}. \quad (11.7)$$

Proof. For the pushforward, we have

$$P_{k-2,*}(i_x)_*((\text{Graph}(\sqrt{D_K}))^{k/2-1}) = (i_{x_1})_*((\text{Graph}(\sqrt{D_K}))^{k/2-1}),$$

so by (11.4), for each $\sigma \in G(2, k-2)$ and each $b \in \Gamma_0(N)/\Gamma(N)$, we have

$$\begin{aligned} P_{k-2,*}(b\sigma(i_x)_*((\text{Graph}(\sqrt{D_K}))^{k/2-1})) \\ = P_{k-2,*}(b)P_{k-2,*}(\sigma)(i_{x_1})_*((\text{Graph}(\sqrt{D_K}))^{k/2-1}). \end{aligned} \quad (11.8)$$

Multiplying (11.8) by

$$\frac{1}{|\Gamma_0(N)/\Gamma(N)|} \cdot \frac{c_2(\sigma)}{|G(2, k-2)|}$$

(recall that the expression $c_2(\sigma)$ was defined in section 6.1) and then summing over all $\sigma \in G(2, k-2)$ and all $b \in \Gamma_0(N)/\Gamma(N)$ yields (11.6).

For the pullback, we have

$$P_{k-2}^*(i_{x_1})_*((\text{Graph}(\sqrt{D_K}))^{k/2-1}) = \sum_{\tilde{x}} (i_{\tilde{x}})_*((\text{Graph}(\sqrt{D_K}))^{k/2-1})$$

where $\tilde{x}$ runs over the N points in the inverse image of x_1 under the map $X(N) \rightarrow X_1(N)$. Arguing as before, using (11.5) instead of (11.4), yields (11.7). $\square$

The Abel-Jacobi map commutes with correspondences, so applying the Abel-Jacobi map to (11.6) and (11.7) yields

$$P_{k-2,*}AJ_{K_1}(\Delta_{Ma17}) = AJ_{K_1}^1(\Delta_{Ca13}) \quad (11.9)$$

and

$$P_{k-2}^*AJ_{K_1}^1(\Delta_{Ca13}) = N \cdot AJ_{K_1}(\Delta_{Ma17}). \quad (11.10)$$

Since $p \nmid N$ and $AJ_{K_1}(\Delta_{Ma17})$ is not torsion, applying Lemma 11.2.1 yields the following result.

Proposition 11.3.2. *We have*

$$\text{ind}_p(AJ_{K_1}(\Delta_{Ma17}), \text{im } AJ_{K_1}) = \text{ind}_p(AJ_{K_1}^1(\Delta_{Ca13}), \text{im } AJ_{K_1}^1). \quad (11.11)$$

11.4 Galois action on Castella's Heegner cycle

By the theory of complex multiplication, there is a bijection between elements $\sigma \in \text{Gal}(K_1/K)$ and ideal classes $[\mathfrak{a}]$ of O_K so that for an element σ and its corresponding class $[\mathfrak{a}]$, the elliptic curve σE_{x_1} corresponds to the same point of $X_1(N)$ as $E_{x_1}/E_{x_1}[\mathfrak{a}]$. It is easily checked that for σ corresponding to $[\mathfrak{a}]$ in this way, we have $\sigma \Delta_{Ca13} = \Delta_{Ca13, \mathfrak{a}}$. Applying Lemma 11.2.2, we obtain:

Proposition 11.4.1. *For a nonzero ideal $\mathfrak{a}$ of O_K , we have*

$$\mathrm{ind}_p(AJ_{K_1}^1(\Delta_{Ca13}), \mathrm{im} AJ_{K_1}^1) = \mathrm{ind}_p(AJ_{K_1}^1(\Delta_{Ca13, \mathfrak{a}}), \mathrm{im} AJ_{K_1}^1).$$

11.5 From Castella to BDP

As in [Ca13, proof of Lemma 3.4], define Π_{k-2} to be the image of

$$\begin{aligned} \tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k/2-1} &\hookrightarrow \tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times (\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}) \\ (w, a) &\mapsto (w, (w, (a, \sqrt{D_K} \cdot a))) \end{aligned}$$

and view Π_{k-2} as a correspondence from $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}$ to $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$.

Proposition 11.5.1. *We have*

$$\Delta_{Ca13, \mathfrak{a}} = \pi_{B,1} \Pi_{k-2} \Delta_{\varphi(E_{x_1}, \mathfrak{a})}. \quad (11.12)$$

Proof. For $\sigma_1 = ((z_i, \varepsilon_{1i})_{i=1}^{k-2}, s_1) \in G(1, k-2)$ and $\sigma_0 = ((\varepsilon_{0i})_{i=1}^{k-2}, s_0) \in G(0, k-2)$, we have

$$\Pi_{k-2} \sigma_1 \sigma_0 \mathrm{Graph}(\varphi(E_{x_1}, \mathfrak{a}))^{k-2} = \sigma \mathrm{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1} \quad (11.13)$$

where

$$\sigma = ((z_i, \varepsilon_{1i} \varepsilon_{0, s_0 \circ s_1^{-1}(i)})_{i=1}^{k-2}, s_1 \circ s_0^{-1}) \in G(1, k-2).$$

Multiplying (11.13) by

$$\frac{c_1(\sigma_1)}{|G(1, k-2)|} \cdot \frac{c_0(\sigma_0)}{|G(0, k-2)|}$$

and then summing over all $(\sigma_1, \sigma_0) \in G(1, k-2) \times G(0, k-2)$, we obtain

$$\Pi_{k-2} \Delta_{\varphi(E_{x_1}, \mathfrak{a})} = \pi_{1, k-2}(\mathrm{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1}) \quad (11.14)$$

and applying $\pi_{B,1}$ yields (11.12). $\square$

Define $Q_{\mathfrak{a}}$ to be the correspondence from $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$ to itself that sends a cycle to its intersection with the fiber at $E_{x_1}/E_{x_1}[\mathfrak{a}]$ in $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$.

Proposition 11.5.2. *We have*

$$\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})} = \phi(N) \cdot Q_{\mathfrak{a}}\Delta_{Ca13, \mathfrak{a}}. \quad (11.15)$$

Proof. By definition, Δ_{Ca13} is

$$\sum_{b \in \Gamma_0(N)/\Gamma_1(N)} \frac{1}{\phi(N)} \sum_{\sigma \in G(1, k-2)} \frac{c_1(\sigma)}{|G(1, k-2)|} b\sigma(\text{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1}). \quad (11.16)$$

An assumption in Theorem 11.1.1 is that the $h_K \cdot \phi(N)$ images of the point in $X_1(N)$ corresponding to E_{x_1} under the $h_K \cdot \phi(N)$ elements of $\text{Gal}(K_1/K) \times (\Gamma_0(N)/\Gamma_1(N))$ are all different. Because of this, applying $Q_{\mathfrak{a}}$ to Δ_{Ca13} eliminates the terms in (11.16) involving a nontrivial $b \in \Gamma_0(N)/\Gamma_1(N)$ and preserves the terms in (11.16) with $b = 1$. Therefore $\phi(N) \cdot Q_{\mathfrak{a}}\Delta_{Ca13, \mathfrak{a}}$ is equal to

$$\sum_{\sigma \in G(1, k-2)} \frac{c_1(\sigma)}{|G(1, k-2)|} \sigma(\text{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1}) = \pi_{1, k-2}(\text{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1})$$

which is $\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})}$ by (11.14), so (11.15) is proved. $\square$

Let $\sigma \in \text{Gal}(K_1/K)$ correspond to the ideal class $[\mathfrak{a}]$ as before. Define $R_{\mathfrak{a}}$ to be the correspondence from $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$ to $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}$ given by the variety in

$$(\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N)) \times E_{x_1}^{k-2}) \times \tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$$

whose points are precisely the points of the form

$$\begin{aligned} &(((w_1 \bmod E_{x_1}[\mathfrak{a}], \dots, w_{k-2} \bmod E_{x_1}[\mathfrak{a}])_{E_{x_1}/E_{x_1}[\mathfrak{a}]}, (w_1, \dots, w_{k-2})), \\ &(w_1 \bmod E_{x_1}[\mathfrak{a}], \dots, w_{k/2-1} \bmod E_{x_1}[\mathfrak{a}], x_1, \dots, x_{k/2-1})_{E_{x_1}/E_{x_1}[\mathfrak{a}]} \end{aligned}$$

where the w_i are points in E_{x_1} and the x_i are points in $E_{x_1}/E_{x_1}[\mathfrak{a}]$. The subvariety $\text{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1}$ of $\tilde{\mathcal{E}}^{k-2}(\Gamma_1(N))$ is interpreted as the variety of points of the form

$$(w_1, \dots, w_{k/2-1}, \sqrt{D_K}w_1, \dots, \sqrt{D_K}w_{k/2-1}).$$

Proposition 11.5.3. *We have*

$$\Delta_{\varphi(E_{x_1}, \mathfrak{a})} = \pi_{1,k-2}\pi_{0,k-2}R_{\mathfrak{a},*}\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})}. \quad (11.17)$$

Proof. By (11.14), we have

$$\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})} = \pi_{1,k-2}(\text{Graph}(\sqrt{D_K})_{E_{x_1}/E_{x_1}[\mathfrak{a}]}^{k/2-1})$$

and applying $\pi_{1,k-2}\pi_{0,k-2}R_{\mathfrak{a},*}$ yields (11.17). $\square$

Combining Lemma 11.2.1 with Propositions 11.5.1, 11.5.2 and 11.5.3 in that order yields

$$\begin{aligned} \text{ind}_p(AJ_{K_1}^{1,E_{x_1}}(\Delta_{\varphi(E_{x_1}, \mathfrak{a})}), \text{im } AJ_{K_1}^{1,E_{x_1}}) &\leq \text{ind}_p(AJ_{K_1}^1(\Delta_{Ca13, \mathfrak{a}}), \text{im } AJ_{K_1}^1) \\ &\leq \text{ind}_p(AJ_{K_1}^1(\Pi_{k-2}\Delta_{\varphi(E_{x_1}, \mathfrak{a})}), \text{im } AJ_{K_1}^1) \\ &\leq \text{ind}_p(AJ_{K_1}^{1,E_{x_1}}(\Delta_{\varphi(E_{x_1}, \mathfrak{a})}), \text{im } AJ_{K_1}^{1,E_{x_1}}), \end{aligned}$$

which means that all of the p -indices are equal. The additional fact that

$$\text{ind}_p(AJ_{K_1}^{1,E_{x_1}}(\Delta_{\varphi(E_{x_1}, \mathfrak{a})}), \text{im } AJ_{K_1}^{1,E_{x_1}}) \leq \text{ind}_p(AJ_L^{1,E_{x_1}}(\Delta_{\varphi(E_{x_1}, \mathfrak{a})}), \text{im } AJ_L^{1,E_{x_1}})$$

now implies:

Proposition 11.5.4. *We have*

$$\mathrm{ind}_p(AJ_{K_1}^1(\Delta_{Ca13,\mathfrak{a}}), \mathrm{im} AJ_{K_1}^1) \leq \mathrm{ind}_p(AJ_L^{1,E_{x_1}}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im} AJ_L^{1,E_{x_1}}).$$

11.6 Conclusion

By (11.1) and Propositions 11.3.2, 11.4.1 and 11.5.4, we have

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}) \leq \mathrm{ind}_p(AJ_L^{1,E_{x_1}}(\Delta_{\varphi(E_{x_1},\mathfrak{a})}), \mathrm{im} AJ_L^{1,E_{x_1}})$$

for each nonzero ideal $\mathfrak{a}$ of O_K . Now recall the cycle

$$Z_{BeDaPr13} = \frac{1}{(k/2 - 1)!} \sum_{[\mathfrak{a}]} \frac{1}{N^{k/2-1}(\mathfrak{a})} \Delta_{\varphi(E_{x_1},\mathfrak{a})}$$

and note that the numbers $(k/2 - 1)!$ and $N(\mathfrak{a})$ are coprime to p . Since $Z_{BeDaPr13}$ is a $\mathbb{Z}_p$ -linear combination of cycles whose Abel-Jacobi images' p -indices are at least the p -index of $AJ_{K_1}(Z_{Ma17})$, it follows that

$$\mathrm{ind}_p(AJ_{K_1}(Z_{Ma17}), \mathrm{im} AJ_{K_1}) \leq \mathrm{ind}_p(AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}), \mathrm{im} AJ_L^{1,E_{x_1}}). \quad (11.18)$$

The map (6.4) with $\eta = \omega_f \wedge \omega_A^{k/2-1} \eta_A^{k/2-1}$, $A = E_{x_1}$ and $F = L$ has image in O_L by assumption, so

$$(\mathrm{RHS} \text{ of } (11.18)) \leq \mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right). \quad (11.19)$$

Equations (11.18) and (11.19) imply (11.2), so Theorem 11.1.1 is proved.

Chapter 12

Final argument

I now prove Theorem 1.2.2. Assume all the hypotheses of this theorem.

By a result of Bertolini, Darmon and Prasanna, [BeDaPr13, Theorem 5.13] (with that source's $\chi = \mathbb{N}^{k/2}$, $j = k/2 - 1$, $r = k - 2$, $c = 1$, $\varepsilon_f = 1$), noting the correspondence between chapter 8's P_1 and subsection 5.2.4's P_{ac} corresponding to $\chi = \mathbb{N}^{k/2}$, we have

$$P_1(L_{BDP}^{-,N_p}(f)) = (1 - p^{-k/2}a(p, f) + p^{-1})^2 \cdot \left(\frac{1}{(k/2 - 1)!} \sum_{[\mathfrak{a}]} \frac{1}{\mathbb{N}^{k/2-1}(\mathfrak{a})} AJ_L^{1,E_{x_1}}(\Delta_{\varphi_{\mathfrak{a}}}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) \right)^2.$$

Since $a(p, f)$ is a p -adic unit and $k/2 \geq 2$, this implies

$$2\mathrm{ind}_p AJ_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) = k + \mathrm{ind}_p P_1(L_{BDP}^{-,N_p}(f)). \quad (12.1)$$

By Theorem 7.2.2 (comparing L -functions), we have

$$\mathrm{ind}_p P_1(L_{BDP}^{-,N_p}(f)) \leq \mathrm{ind}_p P_1(L_{Wan}^{-,N_p}(f)). \quad (12.2)$$

By Theorem 8.6.1 (from Iwasawa theory), we have

$$\begin{aligned} \text{ind}_p P_1(L_{W_{an}}^{-, N_p}(f)) &\leq \text{ind}_p(\#H^0(K_{v_0}, W_f)) + \text{ind}_p(\#H^0(K_{\bar{v}_0}, W_f)) \\ &\quad + \text{ind}_p(\#H_{ac}^1(K, W_f)) + \sum_{v \in S'} \text{ind}_p(\#H_{ur}^1(K_v, W_f)). \end{aligned} \quad (12.3)$$

By Theorem 10.1.1, the O_L -module $\text{im } AJ_K$ has rank 1 and the group $\text{III}(K, W_f)$ has finite cardinality, so all the hypotheses of Theorems 9.2.1 and 9.3.1 (which use Galois cohomology and p -adic Hodge theory) hold. By those theorems, we have

$$\begin{aligned} &\text{ind}_p(\#H_{ac}^1(K, W_f)) \\ &= \text{ind}_p(\#\text{III}(K, W_f)) - 2[L : \mathbb{Q}_p] + 2\text{ind}_p(O_L : O_L \log_\omega(\text{loc}_{v_0} C_0)) \\ &\quad + 2\text{ind}_p(H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L)) - 2\text{ind}_p(\#H^0(K_{v_0}, W_f)) \\ &\quad - 2\text{ind}_p(H_f^1(K, T_f) : O_L C_0). \end{aligned} \quad (12.4)$$

Again by Theorem 10.1.1 (from Euler systems), we have

$$\text{ind}_p(\#\text{III}(K, W_f)) \leq 2\text{ind}_p(AJ_{K_1}(Z_{Ma17}), \text{im } AJ_{K_1}). \quad (12.5)$$

By Theorem 11.1.1 (comparing Heegner cycles), we have

$$2\text{ind}_p(AJ_{K_1}(Z_{Ma17}), \text{im } AJ_{K_1}) \leq 2\text{ind}_p AJ_L^{1, E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right). \quad (12.6)$$

Each of (12.1) to (12.6) is either an equation or an inequality in the $\leq$ direction.

Combining these six statements in that order yields

$$\begin{aligned}
& 2\text{ind}_p A J_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) \\
& \leq 2\text{ind}_p A J_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) - \text{ind}_p(\#H^0(K_{v_0}, W_f)) \\
& \quad + \text{ind}_p(\#H^0(K_{\bar{v}_0}, W_f)) + \sum_{v \in S'} \text{ind}_p(\#H_{ur}^1(K_v, W_f)) \\
& \quad + 2\text{ind}_p(O_L : O_L \log_\omega(\text{loc}_{v_0} C_0)) + 2\text{ind}_p(H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L)) \\
& \quad + k - 2[L : \mathbb{Q}_p] - 2\text{ind}_p(H_f^1(K, T_f) : O_L C_0). \tag{12.7}
\end{aligned}$$

By assumption, the prime p is coprime to the product of

$$\frac{\#H^0(K_{\bar{v}_0}, W_f) \prod_{v \in S'} (\#H_{ur}^1(K_v, W_f))}{\#H^0(K_{v_0}, W_f)}$$

and

$$\frac{(O_L : O_L \log_\omega(\text{loc}_{v_0} C_0))^2 (H_f^1(K_{v_0}, T_f) : \exp_\omega(pO_L))^2}{p^{2[L:\mathbb{Q}_p]-k} (H_f^1(K, T_f) : O_L C_0)^2},$$

so equality occurs in (12.7). Therefore equality occurs in each of (12.1) to (12.6). In particular, equality occurs in (12.5) and (12.6), which implies

$$2\text{ind}_p A J_L^{1,E_{x_1}}(Z_{BeDaPr13}) \left(\omega_f \wedge \omega_{E_{x_1}}^{k/2-1} \eta_{E_{x_1}}^{k/2-1} \right) = \text{ind}_p \# \text{III}(K, W_f)$$

and the main Theorem 1.2.2 is proved.

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