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Koszul dual bordered algebras and the wrapped Fukaya category

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KOSZUL DUAL BORDERED ALGEBRAS AND THE WRAPPED FUKAYA CATEGORY

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Abstract

By slicing the Heegaard diagram for a given 3-manifold in a particular way, it is possible to construct $\mathcal{A}_\infty$ -bimodules, the tensor product of which retrieves the Heegaard Floer homology of the original 3-manifold. The first step in this is to construct algebras corresponding to the individual slices. In this dissertation, we construct Koszul dual $\mathcal{A}_\infty$ -algebras $\mathcal{A}$ and $\mathcal{B}$, called star algebras, for a particular star-shaped class of slice, called star diagrams. Using $\mathcal{A}_\infty$ -bimodules over $\mathcal{A}$ and $\mathcal{B}$, we then verify the Koszul duality relation, Theorem 1.0.2.

The second chapter establishes an isomorphism between endomorphism algebras from the wrapped Fukaya category of the star diagrams, and the star algebras constructed in the first chapter. By viewing the star algebras as $\mathcal{A}_\infty$ -deformations of underlying associative algebras and making several calculations with Hochschild cohomology, we verify that they are unique with a given set of generators and basic $\mathcal{A}_\infty$ -relations. We then make model calculations in order to establish that the endomorphism algebras have these generators and basic operations, so that the desired isomorphism follows.

Acknowledgements

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I am grateful for many friends at Forbes College and in the Princeton music department who supported me as I worked on this thesis, especially Maria and Rich Garlock at Forbes, and Ruth Ochs in the music department. I am also very grateful to the staff of the Department of Mathematics, especially Jill LeClair. I would also like to thank my friends Nancy Kim and Elinor Detmer, for interludes of conversation that helped to maintain sanity. I especially thank Rodrigo Córdova Rosado for his constant friendship (and occasional snacks) during the many evenings we worked together at Forbes College.

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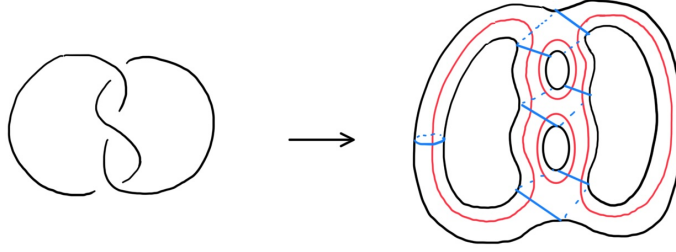
Chapter 1

Introduction

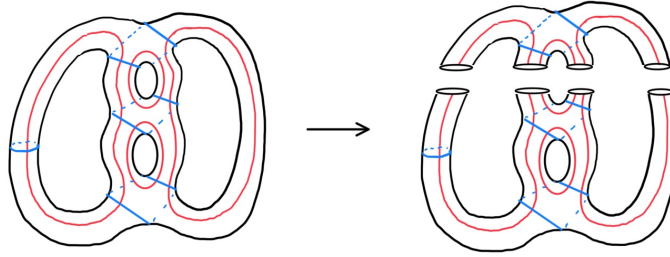
Heegaard Floer homology is a three-manifold invariant which can be used to retrieve classical data about the underlying manifold (see e.g. [23] and [24]). It is constructed using a Heegaard diagram for the given three-manifold, as well as pseudo-holomorphic disks associated to this Heegaard diagram. One advantage of the Heegaard Floer package over gauge theory is that it is often more easily computable, while also retrieving much of the same data.

The aim in bordered Heegaard Floer constructions, as in e.g. [15], [16], [17], is to retrieve this data by associating algebraic objects to subsections of the original Heegaard diagram rather than the whole. More specifically, we first cut up the Heegaard diagram into two or more pieces and associate an algebraic object to each piece. Then we recombine these objects (usually by taking a tensor product) to get back the original Heegaard Floer homology. Since the algebraic objects corresponding to the cut-up pieces of the Heegaard diagram can be more easily computable than the whole Heegaard Floer complex, this often speeds up computations for the whole three-manifold.

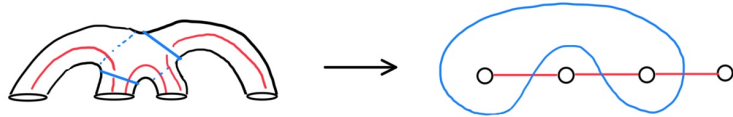
Knot Floer homology (see [33], [26]) is a knot invariant closely related to Heegaard Floer homology, which also admits a bordered construction. Bordered knot Floer homology allows us to retrieve the knot Floer homology corresponding to a given knot K from modules and bimodules associated to different parts of a braid presentation of K – see e.g. [27], [28]. More specifically, we start with a knot diagram for K and take a tubular neighborhood, labelled so as to make it a Heegaard diagram. For instance, in the case of the the left-handed trefoil, this looks like



Notice that the blue β -circles encode the crossings in this diagram. Then we can slice the diagram horizontally:



and in this case, the top portion of the diagram becomes a sphere with four punctures, labelled with three red α -arcs and a β -circle, that is



The first step in the bordered knot Floer construction is to associate an $\mathcal{A}_\infty$ -algebra $\mathcal{A}$ to the horizontal slice. The β -circle does not have any bearing on $\mathcal{A}$, so we are really just associating an algebra to the planar graph



Figure 1.1: Linear graph from [26]

See the final section of [26] for this construction. The papers [27], [28] also contain further details of the construction and its applications. One way to further understand this algebra is to construct a Koszul dual algebra $\mathcal{B}$ for $\mathcal{A}$. In [29], the authors do just this, associating an algebra $\mathcal{B}$ to the blue portion of the following figure:

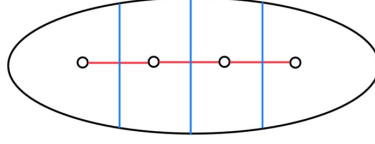


Figure 1.2: Planar picture from [29]

just as $\mathcal{A}$ is associated to the red portion. They then prove that $\mathcal{A}$ and $\mathcal{B}$ are Koszul dual.

It is also of interest to consider more complicated planar graphs, which may arise in horizontal slices corresponding to other tangles. In this thesis, we consider one such class of planar graph – namely star shaped graphs as in

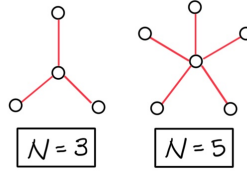


Figure 1.3: α -graphs corresponding to the algebras $\mathcal{A}$

where N denotes the number of 1-valent vertices, and is usually assumed greater than two.

In this thesis, we start with a graph as in Figure 1.3, and construct a corresponding $\mathcal{A}_\infty$ -algebra $\mathcal{A}$. We then verify that this is a bona-fide, weighted $\mathcal{A}_\infty$ -algebra – which in this case has a variety of more complicated non-zero operations. Then, in order to better understand these higher operations, we consider diagrams as in

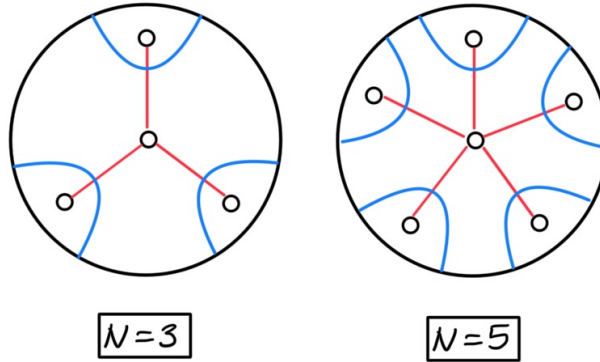


Figure 1.4: Full graphs motivating $\mathcal{A}$ and $\mathcal{B}$

and associate to the blue portion of the diagram a weighted $\mathcal{A}_\infty$ -algebra $\mathcal{B}$, verifying in the course of the construction that $\mathcal{B}$ satisfies $\mathcal{A}_\infty$ -relations and is therefore a bona-fide $\mathcal{A}_\infty$ -algebra.

The ultimate goal is to prove that the full, weighted algebras are Koszul dual in the sense of [14], that is:

Theorem 1.0.1. *With $\mathcal{A}$ and $\mathcal{B}$ constructed as in Sections 2.2 and 2.3, and DD and AA bimodules ${}^{\mathcal{A}}X^{\mathcal{B}}$ and ${}_{\mathcal{B}}Y_{\mathcal{A}}$ constructed in Sections 2.7 and 2.5, respectively, we have the following relations*

$${}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \simeq {}^{\mathcal{A}}\text{id}_{\mathcal{A}}, \quad (1.1)$$

and

$${}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \simeq {}^{\mathcal{B}}\text{id}_{\mathcal{B}}. \quad (1.2)$$

However, we do not currently have a formal notion of weighted DA-bimodules over weighted algebras, so instead, we prove the easier theorem:

Theorem 1.0.2. *With $\mathcal{A}_0$ and $\mathcal{B}_0$ the associative algebras over $\mathbb{F}_2$ constructed as in Sections 2.4, and with DD and AA bimodules ${}^{\mathcal{A}_0}X^{\mathcal{B}_0}$ and ${}_{\mathcal{B}_0}Y_{\mathcal{A}_0}$ constructed in Sections 2.7 and 2.5, respectively, we have the following relations*

$${}^{\mathcal{A}_0}X^{\mathcal{B}_0} \boxtimes {}_{\mathcal{B}_0}Y_{\mathcal{A}_0} \simeq {}^{\mathcal{A}_0}\text{id}_{\mathcal{A}_0}, \quad (1.3)$$

and

$${}_{\mathcal{B}_0}Y_{\mathcal{A}_0} \boxtimes {}^{\mathcal{A}_0}X^{\mathcal{B}_0} \simeq {}^{\mathcal{B}_0}\text{id}_{\mathcal{B}_0}. \quad (1.4)$$

(For formal definitions of AA and DD bimodules, see Sections 2.1.3 and 2.1.6; the id bimodule is defined in Section 2.1.4. The specific bimodules X and Y involved in this result are constructed in Sections 2.5 and 2.7. For further discussion of this definition of Koszul duality as it relates to the definition in [14], see Section 2.1.7.)

The Koszul duality result in Theorem 1.0.2 can be placed in a larger framework having to do with various folklore duality conjectures. It is expected that to any surface with a handle decomposition one can associate an appropriately weighted algebra. Then look at a dual decomposition of the surface. It is then expected that the algebra associated with this dual decomposition will be Koszul dual to one another. See e.g. [19] [29], [31], [36]. The duality result of Theorem 1.0.2 can be interpreted as a special case of this general principle.

The second chapter constructs an isomorphism connecting the combinatorially defined star algebras of Theorem 1.0.2, and endomorphism algebras associated to the wrapped Fukaya category (see e.g. [2], [12], [34]), of the star diagrams, Figure 1.5, associated to the construction of the star alge-

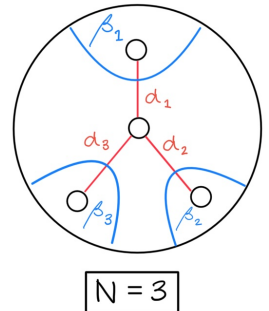


Figure 1.5

bras from the first chapter. The star diagrams from Figure 1.5 can also be viewed as symplectic manifolds which are conical near the boundary, in which the red α -arcs $\{\alpha_1, \dots, \alpha_N\}$ and the blue β -arcs $\{\beta_1, \dots, \beta_N\}$ can each be viewed as a set of Lagrangians. We can then construct endomorphism algebras having to do with the *wrapped Fukaya category* for this set-up. However, in order to use the existing literature about the wrapped Fukaya category, we have to consider two different manifolds:

$$M_\alpha = S^2 \setminus \{p_1, \dots, p_{N+1}\} \quad (1.5)$$

$$M_\beta = S^2 \setminus \{p_0, \dots, p_N\}, \quad (1.6)$$

where the points are indexed in the same way as the corresponding boundary circles in Figure 3.1. With this background, the main result of the second chapter is as follows.

Theorem 1.0.3. *Fix $N \in \mathbb{N}$. Let M_α, M_β be as in (1.5). Let $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$ denote the α - and β -arcs in M respectively, as labelled in Figure 1.5. Let $\mathcal{W}(M_\alpha), \mathcal{W}(M_\beta)$ denote the wrapped Fukaya categories of the respective manifolds, as defined in Section 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be the $\mathcal{A}_\infty$ -algebras for this star diagram, as constructed in Section 3.4. Then there exist $\mathcal{A}_\infty$ -isomorphisms*

$$\mathcal{A} \cong \text{End}_{\mathcal{W}(M_\alpha)}(\alpha_1 \oplus \dots \oplus \alpha_N), \quad (1.7)$$

and

$$\mathcal{B} \cong \text{End}_{\mathcal{W}(M_\beta)}(\beta_1 \oplus \dots \oplus \beta_N). \quad (1.8)$$

The Koszul duality result in Theorem 1.0.2 can be placed in a larger framework having to do with various folklore duality conjectures. It is expected that to any surface with a handle decomposition one can associate an appropriately weighted algebra. Then look at a dual decomposition of the surface. It is then expected that the algebra associated with this dual decomposition will be Koszul dual to one another. See e.g. [29], [31], or [36] for a sutured perspective. The duality result of Theorem 1.0.2 can be interpreted as a special case of this general principle.

From an algebraic standpoint, this thesis exhibits two new phenomena. First, it gives concretely computable examples of $\mathcal{A}_\infty$ -structures of various types – operationally bounded, non-operationally bounded, and weighted – all with a clearly discernible geometric foundation. At the same time, the Koszul duality relation in Theorem 1.0.2 says that the data of the more complicated, non-operationally bounded algebra $\mathcal{A}$ is still in some sense captured in far simpler algebra $\mathcal{B}$.

These more complicated $\mathcal{A}_\infty$ -structures arise when we allow disks in a bordered Heegaard diagram to cross punctures in the partial Heegaard diagram, and this thesis also gives explicitly computed examples of what happens on the algebraic structures when we do this. (For analogous computations in the bordered Heegaard Floer case, see e.g. [17].) In particular, these orbits are what force us to include higher multiplications and weighted operations on both algebras. To keep track of all of the algebra structures, we therefore have to develop new notation. See Sections 2.2 and 2.3 for more details.

In a similar vein, this thesis also gives an exposition of (and concrete examples for) the system of graphical calculus by which $\mathcal{A}_\infty$ -operations and relations can be represented using planar trees. While this is discussed in full in [18], a reader encountering this notation for the first time could also view Sections 2.1 and 2.6 as a very brief introduction to this way of managing the arithmetic of $\mathcal{A}_\infty$ -structures.

Chapter 2

Koszul duality for star-shaped partial Heegaard diagrams

The aim of this first chapter is to present the proof of Theorem 1.0.2. The chapter is structured as follows. In Section 2.1, we define the notions of $\mathcal{A}_\infty$ -algebras, maps, and bimodules used in the rest of the thesis. Section 2.6 contains the algebraic background needed for the DD -bimodule construction in Section 2.7, following [18]. In Sections 2.2 and 2.3, we construct the pair of dual $\mathcal{A}_\infty$ -algebras $\mathcal{A}$ and $\mathcal{B}$ for this class of diagram and verify that these satisfy the $\mathcal{A}_\infty$ -relations defined in Section 2.1. In Sections 2.5 and 2.7, we construct the bimodules needed to prove the duality relation. Finally in Section 2.8, we verify the duality result, Theorem 1.0.2.

2.1 Definitions

The goal of this section is to set up the machinery and graphical calculus that will be used to construct the algebraic objects in the following sections. It should be emphasized again that this section is meant rather to introduce the graphical calculus for $\mathcal{A}_\infty$ -algebras used in this thesis, than to serve as a comprehensive introduction to $\mathcal{A}_\infty$ -structures. For such an introduction, see for instance [7], [8].

While much of the exposition in this section follows [16], [18], [27], [28], etc., some of the notation and definitions are specific to this thesis. The reader less familiar with the $\mathcal{A}_\infty$ -structures discussed here should therefore concentrate mainly on the arithmetic of trees used to calculate $\mathcal{A}_\infty$ -operations and relations: it is this which will be used in the body of the thesis.

2.1.1 $\mathcal{A}_\infty$ -algebras, and operations as trees

In this subsection we will introduce both unweighted and weighted $\mathcal{A}_\infty$ -algebras. It is possible to view unweighted $\mathcal{A}_\infty$ -algebras as a special case of weighted $\mathcal{A}_\infty$ -algebras and jump immediately to the latter definition; however, we define unweighted $\mathcal{A}_\infty$ -algebras first, separately, because many earlier papers (e.g. [14], [27], [29]) only deal with unweighted $\mathcal{A}_\infty$ -algebras, whereas the weighted case only appears later in the literature.

We start with the definition of an unweighted $\mathcal{A}_\infty$ -algebra. Start with a ring R of characteristic two, usually $\mathbb{F}[V_0, \dots, V_{N+1}]$, where $\mathbb{F} = \mathbb{F}_2$. An *unweighted $\mathcal{A}_\infty$ -algebra* is a R -module $\mathcal{A}$ equipped with R -homomorphisms

$$\mu_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A} \quad (2.1)$$

for $n \in \mathbb{Z}_{\geq 0}$ and the tensor products are over R , where the maps μ_n are called *unweighted multiplications*, and are required to satisfy the relations:

$$\sum_{\substack{1 \leq r \leq n \\ 0 \leq j \leq (n-r-1)}} \mu_{n-r+1}(a_1, \dots, a_j, \mu_r(a_{j+1}, \dots, a_{j+r}), a_{j+r+1}, \dots, a_n) = 0 \quad (2.2)$$

Equation (2.2) is called the *$\mathcal{A}_\infty$ -relation for $\mathcal{A}$ with n inputs*. An R -module $\mathcal{A}$ equipped with maps as in (2.1) is an unweighted $\mathcal{A}_\infty$ -algebra if and only if it satisfies the relations in (2.2) for each choice of $a_1, \dots, a_n \in \mathcal{A}$ and $n \in \mathbb{Z}_{\geq 0}$.

The unweighted μ_0 is a map $R \rightarrow \mathcal{A}$ and is determined by its action on $1 \in R$, so it can be viewed as an element of $\mathcal{A}$. In more familiar language, μ_0 is curvature (if it is nonzero), μ_1 is the differential, and μ_2 is the usual multiplication.

We usually work with the case where μ_0 , the unweighted curvature, is zero. To illustrate some of the basic $\mathcal{A}_\infty$ -relations, we assume for the rest of the section that $\mu_0 = 0$. In this case, for small n , the $\mathcal{A}_\infty$ -relations are, in more familiar language

- $n = 1$: $\mu_1 \circ \mu_1 \equiv 0$ i.e. $d^2 = 0$;
- $n = 2$: $(\mu_1 \circ \mu_2)(a, b) = \mu_2(\mu_1(a), b) + \mu_2(a, \mu_1(b))$ is the Leibniz rule, i.e

$$d(ab) = (da)b + a(db)$$

A *corolla* is defined to be a planar tree with a single internal vertex, n input edges and one output edge, as in.

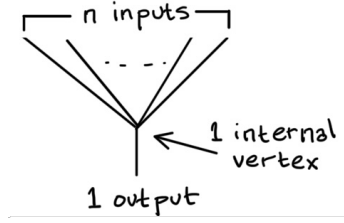
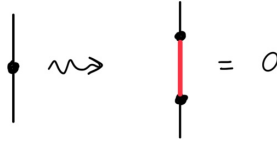


Figure 2.1: Corolla corresponding to μ_n

In this thesis, we will most commonly present $\mathcal{A}_\infty$ -operations and relations in terms of trees. Operations, in particular, are written in terms of corollas: the corolla from Figure 2.1 is defined to represent μ_n .

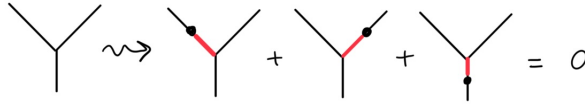
The next step is to represent the $\mathcal{A}_\infty$ -relations in terms of trees. First, define *pushing out an edge* in a planar tree to mean replacing an internal vertex of the original tree with an edge in the new tree. (This will be illustrated below.) The $\mathcal{A}_\infty$ -relation corresponding to a planar tree is defined to be as the sum of all the ways to push out a single internal edge in T . For instance, in the following cases:

- $d^2 = 0$:

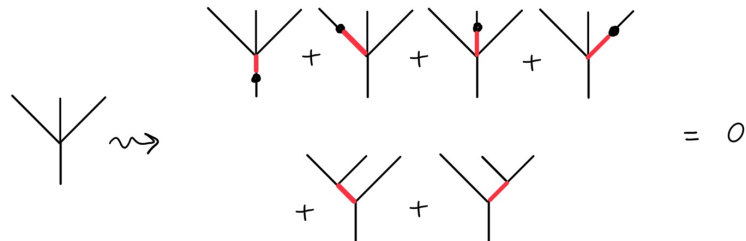


(The red edge represents the new, pushed out edge.)

- Leibniz rule:



- Homotopy associativity:



Traditional associativity would say that the last two terms sum to zero. Homotopy associativity implies that associativity holds on the level of homology.

Occasionally, as we did above, we will not label the input elements at the top of a tree and just write out the trees.

Next, we define a *weighted $\mathcal{A}_\infty$ -algebra*. The basic data for a weighted $\mathcal{A}_\infty$ algebra is an R -module $\mathcal{A}$, equipped with *operations*, i.e. maps

$$\mu_n^{\mathbf{w}} : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}. \quad (2.3)$$

where $\mathbf{w}$ is a $\mathbb{Z}_{\geq 0}$ -linear combination of basis vectors $\{\mathbf{e}_i\}_{i=0}^{N+1}$ of basis vectors, for a single fixed $N \in \mathbb{N}$.

Operations in a weighted $\mathcal{A}_\infty$ -algebra are required to satisfy weighted $\mathcal{A}_\infty$ -relations, which we now define. First, given weight vectors

$$\begin{aligned} \mathbf{w} &= \sum_{i=0}^{N+1} k_i \mathbf{e}_i \\ \mathbf{w}' &= \sum_{i=0}^{N+1} k'_i \mathbf{e}_i \end{aligned}$$

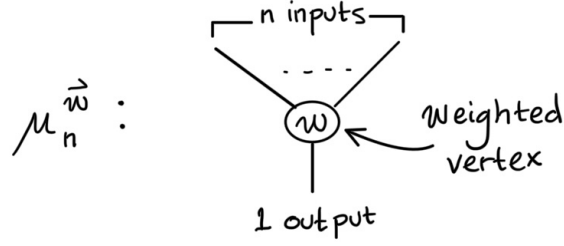
where $k_i, k'_i \in \mathbb{Z}_{\geq 0}$ for each i , we say that $\mathbf{w}' \leq \mathbf{w}$ if and only if $k'_i \leq k_i$ for each i . Now look at $a_1, \dots, a_n \in \mathcal{A}$ and a weight vector $\mathbf{w} \in V$. Define the *weight- $\mathbf{w}$ $\mathcal{A}_\infty$ relation for the n elements $a_1, \dots, a_n$* as

$$\sum_{\substack{0 \leq r \leq n \\ 0 \leq j \leq (n-r-1) \\ 0 \leq \mathbf{w}' \leq \mathbf{w}}} \mu_{n-r+1}^{\mathbf{w}-\mathbf{w}'}(a_1, \dots, a_j, \mu_r^{\mathbf{w}'}(a_{j+1}, \dots, a_{j+r}), a_{j+r+1}, \dots, a_n) = 0 \quad (2.4)$$

We also assume throughout that $\mu_0^0 \equiv 0$.

Now, an R -module $\mathcal{A}$ equipped with maps as in (2.3) is a *weighted $\mathcal{A}_\infty$ algebra* if and only if the maps from (2.3) satisfy (2.4) for each $\mathbf{w} \in V$ and $a_1, \dots, a_n \in \mathcal{A}$.

We can also represent (2.4) in terms of weighted trees; each $\mu_n^{\mathbf{w}}$ is written as the following tree:

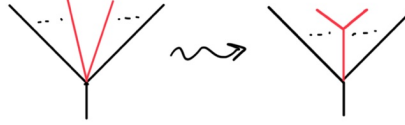


The weighted operations we will most commonly encounter will be μ_0 s – namely, $\mu_0^{e_i} = U_i$, for $0 \leq i \leq N + 1$ – which will correspond to picking up an orbit in the bimodule.

Next, we discuss how to express the $\mathcal{A}_\infty$ -relations in terms of planar trees. As in the unweighted case, the $\mathcal{A}_\infty$ -relation for a given (weighted) tree is the sum over the number of ways to push out an edge. It is clear that this corresponds to (2.4); that is, if we represent a given operation $\mu_n^{\mathbf{w}}(a_1, \dots, a_n)$ as a weighted tree T , then the sum over the ways to push out an edge in T is a representation, in terms of planar trees, of (2.4).

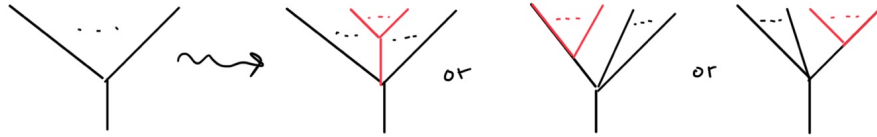
There are four types of terms that can appear in the $\mathcal{A}_\infty$ relation for a given tree: a pull, a split, a push, or a differential. These are defined as follows:

- Pull:



In a pull, we bring together two of the input elements.

- Split: (Where we are assuming the internal operation is a higher multiplication)



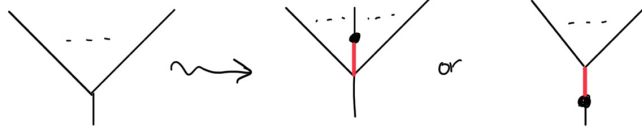
The distinction between splits and pulls is largely artificial; both correspond to compositions of $\mathcal{A}_\infty$ -operations. However, it will be useful, in the verification of the $\mathcal{A}_\infty$ -relations below, to have distinct notations for pulling together two input elements, and pulling together more than two input elements.

- Push:



Here we insert an edge which adds a new input edge, with input the element of $\mathcal{A}$ corresponding to $\mu_0^{e_i}$.

- Differential:



Here we insert an edge into either an input edge or an output edge of the graph. We distinguish between a differential and a more general pull because in the specific algebras we deal with in Sections 2.2 and 2.3, differentials will behave differently from other operations and will need to be treated separately in the verification of $\mathcal{A}_\infty$ -relations.

2.1.2 Homomorphisms of $\mathcal{A}_\infty$ -algebras

Start with $\mathcal{A}_\infty$ -algebras $\mathcal{A}$ and $\mathcal{B}$. A *homomorphism* $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a family of maps $\varphi_k : \mathcal{A}^{\otimes k} \rightarrow \mathcal{B}$, where $k \in \mathbb{Z}_{\geq 0}$, satisfying a structure relation expressed as follows. First, define total maps $F^\varphi : \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{B})$, for each weight $\mathbf{w}$ as above, as

$$F^\varphi(a_1 \otimes \cdots \otimes a_n) = \sum_{\substack{1 \leq i_1 < \cdots < i_\ell < n \\ \mathbf{w}_1 + \cdots + \mathbf{w}_\ell = \mathbf{w}}} \varphi_{i_1}^{\mathbf{w}_1}(a_1, \dots, a_{i_1}) \otimes \cdots \otimes \varphi_{i_\ell}^{\mathbf{w}_\ell}(a_{i_\ell+1}, \dots, a_n)$$

so for instance, in terms of trees and with two input elements, $F^{\varphi,0}$ would act as

$$F^\varphi(a_1, a_2) := \begin{array}{c} a_1 \quad a_2 \\ | \quad | \\ \varphi_1 \quad \varphi_1 \\ | \quad | \end{array} + \begin{array}{c} a_1 \quad a_2 \\ \quad \cup \\ \varphi_2 \\ | \end{array}$$

Next, write the *total differential* $\overline{D}^{\mathcal{A}} : \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{A})$ as

$$\overline{D}^{\mathcal{A}}(a_1 \otimes \cdots \otimes a_n) = \sum_{i_1 \leq i_2} a_1 \otimes \cdots \otimes \mu_{i_2-i_1+1}(a_{i_1}, \dots, a_{i_2}) \otimes \cdots \otimes a_n$$

and likewise for $\mathcal{B}$. The $\mathcal{A}_\infty$ -relations for the homomorphism φ state that

$$F^\varphi \circ \overline{D}^{\mathcal{A}} + \overline{D}^{\mathcal{B}} \circ F^\varphi = 0 \quad (2.5)$$

In terms of trees, this looks like

$$\begin{array}{c} | \\ \varphi_1 \\ | \\ \mu_1 \\ | \end{array} + \begin{array}{c} | \\ \mu_1 \\ | \\ \varphi_1 \\ | \end{array} = 0$$

for one input – that is, φ_1 is a chain map – and for two inputs and one output,

$$\begin{array}{c} \varphi_1 \quad \varphi_1 \\ \diagdown \quad \diagup \\ | \end{array} + \begin{array}{c} \diagdown \quad \diagup \\ \varphi_2 \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \varphi_2 \\ | \end{array} + \begin{array}{c} \diagdown \quad \diagup \\ \varphi_2 \\ | \end{array} + \begin{array}{c} \diagdown \quad \diagup \\ \varphi_1 \\ | \end{array} = 0$$

Figure 2.2: The $\mathcal{A}_\infty$ relation for two inputs

and so on. (All terms with more than one output will cancel in pairs automatically by the $\mathcal{A}_\infty$ -relations for lower numbers of inputs.)

The composition $\psi \circ \varphi$ of $\mathcal{A}_\infty$ -homomorphisms φ and ψ are defined by the rule that

$$F^{\psi \circ \varphi} = F^\psi \circ F^\varphi. \quad (2.6)$$

Additionally, when we equip our $\mathcal{A}_\infty$ -algebras with a Maslov grading, m (to be defined formally later) we require $\mathcal{A}_\infty$ -homomorphisms to satisfy the rule

$$m(\varphi_k(a_1, \dots, a_k)) = \sum_{i=1}^k m(a_i) + k - 1 \quad (2.7)$$

So, for instance, φ_1 preserves total grading.

2.1.3 AA bimodules

The basic data for any type of $\mathcal{A}_\infty$ -bimodule is a R -module X and two *unweighted* $\mathcal{A}_\infty$ -algebras, $\mathcal{A}$, and $\mathcal{B}$, over R . An *AA-bimodule* ${}_B Y_{\mathcal{A}}$ is an R -module Y equipped with maps

$$m_{j|\mathbf{x}|i} : \mathcal{B}^{\otimes j} \otimes Y \otimes \mathcal{A}^{\otimes i} \rightarrow Y, \quad (2.8)$$

This operation can be expressed in terms of trees as:

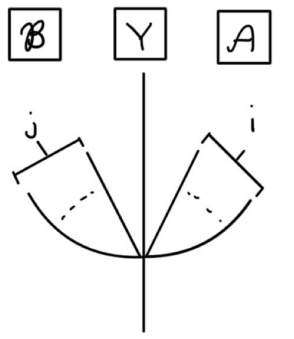


Figure 2.3: The corolla for $m_{j|\mathbf{x}|i}$

that is, with i inputs on the $\mathcal{A}$ -side, and j inputs on the $\mathcal{B}$ -side. As in the case of $\mathcal{A}_\infty$ algebras, we require that the operations of 2.8 satisfy $\mathcal{A}_\infty$ relations. That is, for a fixed $i, j \in \mathbb{N}$, $\mathbf{x} \in Y$, $a_1, \dots, a_i \in \mathcal{A}$, $b_1, \dots, b_j \in \mathcal{B}$, the corresponding $\mathcal{A}_\infty$ -relation is the requirement that:

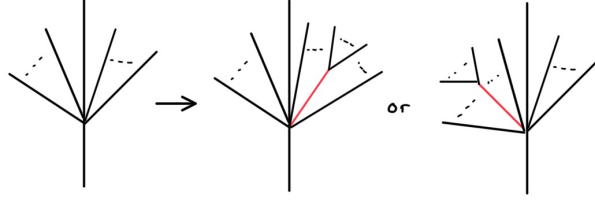
$$\begin{aligned} & \sum_{i' \leq i''} m_{j|\mathbf{x}|i-(i''-i')}(b_j, \dots, b_1, \mathbf{x}, a_1, \dots, \mu_{i''-i'+1}(a_{i'}, \dots, a_{i''}), \dots, a_i) \\ & + \sum_{j' \leq j''} m_{j-(j''-j')|\mathbf{x}|i}(b_j, \dots, \mu_{j''-j'+1}(b_{j'}, \dots, b_{j''}), \dots, b_1, \mathbf{x}, a_1, \dots, a_i) \\ & + \sum_{i' < i, j' < j} m_{(j-j')|\mathbf{y}|(i-i')}(b_j, \dots, b_{j'+1}, \mathbf{y}, a_{i'+1}, \dots, a_i) = 0, \quad (2.9) \end{aligned}$$

where in the third line, $\mathbf{y} = m_{j'|\mathbf{x}|i'}(b_{j'}, \dots, b_1, \mathbf{x}, a_1, \dots, a_{i'})$.

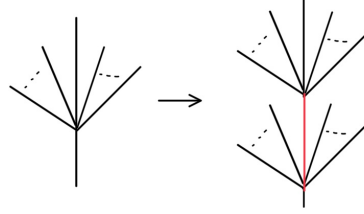
We can also write the $\mathcal{A}_\infty$ -relations in terms of trees; the $\mathcal{A}_\infty$ -relation is equivalent to the requirement that that all ways to add a single edge to this tree sum to zero.

In this unweighted case, there are three ways an edge can be added: namely, a pull, a split, or a differential. In the case of *AA*-bimodules, these look a little different than they did in the case of algebras, so we give examples here:

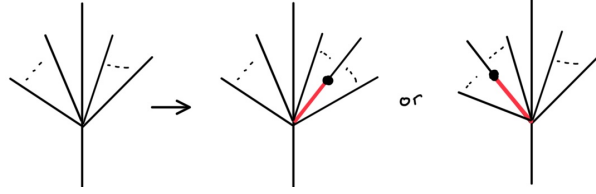
Pull Brings together a subset of the elements on either side, as in



Split Decomposes the original tree into a composition of two bimodule operations, as in:



Differential Takes a differential of one of the input elements on either side, as in:



2.1.4 DA bimodules

If $\mathcal{A}, \mathcal{B}$ are again *unweighted* $\mathcal{A}_\infty$ -algebras over a ring R , then a *DA bimodule* ${}^{\mathcal{B}}M_{\mathcal{A}}$ is a R -bimodule N with R -linear maps

$$\delta_{j+1}^1 : M \otimes \mathcal{A}^{\otimes j} \rightarrow \mathcal{B} \otimes M$$

Then we define $\delta^1 = \sum_{j=0}^{\infty} \delta_{j+1}^1$, and iterated maps (to be used in verifying the $\mathcal{A}_\infty$ relations)

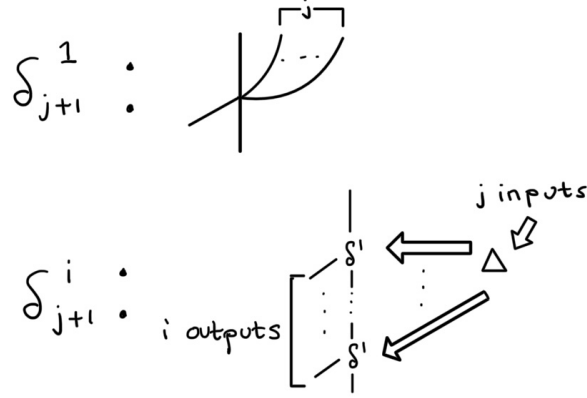
$$\delta^i : M \otimes \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{B}^{\otimes i} \otimes M$$

inductively according to the rule:

$$\begin{aligned} \delta^0 &= \mathbf{I}_M \\ \delta^{(i+1)} &= (\mathbf{I}_{\mathcal{B}^{\otimes i}} \otimes \delta^1) \circ (\delta^i \otimes \mathbf{I}_{\mathcal{T}^*(\mathcal{A})}) \otimes (\mathbf{I}_M \otimes \Delta) \end{aligned}$$

where $\Delta : \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{A}) \otimes \mathcal{T}^*(\mathcal{A})$ is the subdivision map, or more technically, comultiplication.

In terms of trees, this looks like



Then we define

$$\delta^M = \sum_{i=0}^{\infty} \delta^i : M \otimes \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{B}) \otimes M,$$

and the $\mathcal{A}_\infty$ relations which any DA bimodule is required to satisfy are

$$\delta^M \circ (\mathbf{I}_M \otimes \overline{D}^{\mathcal{A}}) + (\overline{D}^{\mathcal{B}} \otimes \mathbf{I}_M) \circ \delta^M = 0 \quad (2.10)$$

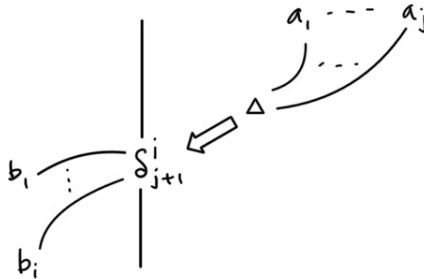
where again, the total differential

$$\overline{D}^{\mathcal{A}} : \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{T}^*(\mathcal{A})$$

is given by

$$\overline{D}^{\mathcal{A}}(a_1 \otimes \cdots \otimes a_n) = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n-i+1}} a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu_j(a_i, \dots, a_{i+j-1}) \otimes a_{i+j} \otimes \cdots \otimes a_n$$

and likewise for $\overline{D}^{\mathcal{B}}$. In terms of trees, (2.10) means that we start with a set of inputs $a_1, \dots, a_j$ and the picture:



then take the sum of all ways to pull together some subset of the $\mathcal{A}$ -inputs (i.e. add an edge on the $\mathcal{A}$ -side), the sum of all ways to pull together some subset of the $\mathcal{B}$ -outputs (i.e. add an edge after the δ -operation), and add them; the $\mathcal{A}_\infty$ -relations says this sum is zero.

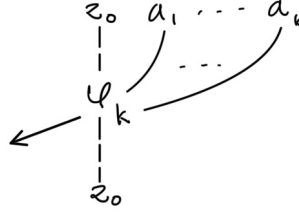
Later on, in Section 2.1.5, we will discuss the DA -bimodule determined by a given $\mathcal{A}_\infty$ homomorphism, and when a given DA -bimodule can be obtained in this way; but we leave that discussion for the section on duality, where it is needed.

2.1.5 DA bimodules from homomorphisms

For this section, we will work over the ground ring $R = \mathbb{F}_2$, and consider only unweighted $\mathcal{A}_\infty$ -algebras $\mathcal{A}$ and $\mathcal{B}$, and unweighted $\mathcal{A}_\infty$ -homomorphisms. Now, an $\mathcal{A}_\infty$ homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ defines a DA bimodule $Z = {}^{\mathcal{B}}[\varphi]_{\mathcal{A}}$ in the following way. Viewed as an R -module, Z is defined to be R itself – so it has a single generator over R . The basic DA bimodule operations $\delta_{j+1}^1 : Z \otimes \mathcal{A}^{\otimes j} \rightarrow \mathcal{B} \otimes Z$ are defined as

$$\delta_{j+1}^1(z_0 \otimes a_1 \otimes \cdots \otimes a_j) = \varphi_j(a_1, \dots, a_j) \otimes z_0,$$

where z_0 is the generator of Z over R . In terms of trees, this looks like



where the dashed line indicates that z_0 does not interact with φ – it just passes through trivially. That this is a bona fide DA bimodule follows from the $\mathcal{A}_\infty$ -relations for φ .

From [15] we have the following facts about this construction:

Proposition 2.1.1. (Lemma 2.2.50 in [15]) *Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be unweighted $\mathcal{A}_\infty$ -algebras with ground ring a field $\mathbb{F}$. If ${}^{\mathcal{B}_0}Z_{\mathcal{A}_0}$ is an unweighted DA bimodule over $\mathcal{A}_0$ and $\mathcal{B}_0$ such that the underlying $\mathbb{F}$ -module of Z is $\mathbb{F}$ itself, and $\delta_1^1 = 0$, then there exists an $\mathcal{A}_\infty$ homomorphism $\varphi : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ such that $Z = {}^{\mathcal{B}_0}[\varphi]_{\mathcal{A}_0}$.*

Proposition 2.1.2. (Analogous to Theorem 4 in [15]) *Let $\mathcal{A}_0$ be an unweighted $\mathcal{A}_\infty$ -algebra. Then the identity map $\text{id} : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ induces the identity DA bimodule ${}^{\mathcal{A}_0}\text{id}_{\mathcal{A}_0}$, that is,*

$${}^{\mathcal{A}_0}[\text{id}]_{\mathcal{A}_0} \simeq {}^{\mathcal{A}_0}\text{id}_{\mathcal{A}_0}$$

Remark 2.1.3. Actually, Theorem 4 of [15] says that the identity map on a surface decorated with a pointed matched circle induces the identity bimodule, rather than that the identity homomorphism

$\mathcal{A}_0 \rightarrow \mathcal{A}_0$ induces the identity. However, the steps of the proof (and the subsidiary lemmas) in [15] translate to this case without issue, so there is no problem.

2.1.6 DD bimodules

An *unweighted DD bimodule* ${}^{\mathcal{A}}X^{\mathcal{B}}$ over $\mathcal{A}$ and $\mathcal{B}$ is an R -module with a single basic operation:

$$\delta^1 : X \rightarrow \mathcal{A} \otimes \mathcal{B} \otimes X.$$

We apply δ^1 repeatedly to get

$$\delta^n = \underbrace{\delta^1 \circ \dots \circ \delta^1}_{n \text{ times}} : X \rightarrow (\mathcal{A} \otimes \mathcal{B})^{\otimes n} \otimes X.$$

Writing $\mathbf{I}$ for the identity on X , the $\mathcal{A}_{\infty}$ relation for X is

$$\sum_n (\mu_n \otimes \mathbf{I}) \circ \delta^n = 0 \tag{2.11}$$

where the sum is over all $n \geq 0$, and the μ_n denote the operations on $\mathcal{A} \otimes \mathcal{B}$, as defined below, in Section 2.6.2. That this sum is finite must be will generally follow from an argument using gradings; for our case, see Lemmadd3.

Remark 2.1.4. For readers familiar with the notion of type D modules, it is worth noting that a *DD bimodule* over $\mathcal{A}$ and $\mathcal{B}$ is actually just a *type-D module* over $\mathcal{A} \otimes \mathcal{B}$.

2.1.7 Koszul duality

This subsection adapts the relevant definition from Section 8 of [14] to the current setting. The aim here is to define what it means for two $\mathcal{A}_{\infty}$ algebras to be Koszul dual. We also point out the way this definition differs from the one in [14], and how it is related to the usual notion of Koszul duality, as in e.g. [6] or [32].

Start with $\mathcal{A}_{\infty}$ algebras $\mathcal{A}$ and $\mathcal{B}$ over a single ground ring R , as usual. Define a *Koszul dualizing bimodule* to be a DD bimodule ${}^{\mathcal{A}}X^{\mathcal{B}}$ which is quasi-invertible. We say that $\mathcal{A}$ and $\mathcal{B}$ are *Koszul dual* if they admit a Koszul dualizing bimodule X .

Here, we prove that the algebras $\mathcal{A}$ and $\mathcal{B}$ constructed in Sections 2.2 and 2.3 are dual by constructing a Koszul dualizing bimodule X in Section 2.7, and a quasi-inverse Y in Section 2.5. Theorem 1.0.2 simply verifies that Y is in fact a quasi-inverse for X .

Our definition for a Koszul dualizing bimodule is less stringent than the one found in Definition 8.5 of [14]. There, the authors consider the following set-up. An $\mathcal{A}_\infty$ algebra $\mathcal{A}$ is a *dg-algebra* if $\mu_n \equiv 0$ for $n \neq 1$ or 2 . An *augmentation map* on a dg-algebra $\mathcal{A}$ is a R -module homomorphism $\epsilon : \mathcal{A} \rightarrow R$; a dg-algebra admitting an augmentation map is said to be *augmented*. If $\mathcal{A}$ is an augmented dg-algebra, then let

$$\mathcal{A}_+ = \ker \epsilon.$$

An $\mathcal{A}_\infty$ bimodule (of any of type) is *rank n* over R if it is isomorphic to R^n as an $R - R$ -bimodule. With this set-up, the authors of [14] start with augmented dg-algebras $\mathcal{A}$ and $\mathcal{B}$, and define a *Koszul dualizing bimodule* for $\mathcal{A}$ and $\mathcal{B}$ to be a *DD* bimodule ${}^{\mathcal{A}}X^{\mathcal{B}}$ such that

- X is quasi-invertible;
- X is rank 1;
- the operation δ^1 on X has image in $\mathcal{A}_+ \otimes X \otimes \mathcal{B}_+$;

In our case, we drop the requirements that X be rank 1 and have specific image, since we are dealing with more complicated algebras than in [14]. In the first place, our algebras $\mathcal{A}$ and $\mathcal{B}$ will not be dg algebras, let alone augmented dg-algebras.

We are still able to obtain the quasi-isomorphism result in Theorem 1.0.2, however, because of the symmetries inherent in the star diagrams.

2.2 The α -bordered algebra $\mathcal{A}$

The object of this section is to define a weighted $\mathcal{A}_\infty$ algebra $\mathcal{A}$, for star diagrams of the form

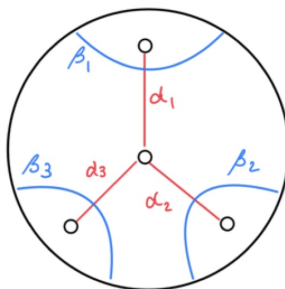


Figure 2.4: The star-diagram for $N = 3$, with α - / β -arcs labelled

That is, we are given, for $N \geq 3$, a diagram consisting of $N + 2$ boundary components (the black circles), N red spokes (the α -arcs) and N blue arcs going out to the outer boundary component (the

β -arcs). The algebra $\mathcal{A}$ is called “ α -bordered” because the generators, some of which are pictured below, in Figure 2.5, for $N = 3$, correspond to Reeb chords on the boundary circles, which are adjacent to α -arcs:

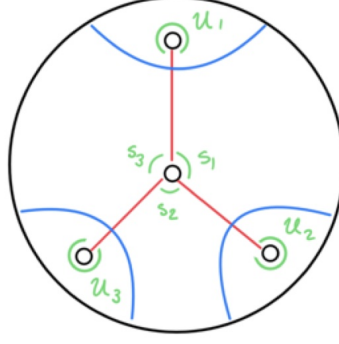


Figure 2.5: Some of the short chords which will correspond to basic algebra elements

There is a corresponding β -bordered algebra, $\mathcal{B}$, which will be defined in Section 2.3.

2.2.1 Algebra elements and basic operations

Fix $N \geq 3$ and the usual ground ring $R = \mathbb{F}_2[V_0, \dots, V_{N+1}]$. The weighted $\mathcal{A}_\infty$ algebra $\mathcal{A}$ is, discounting $\mathcal{A}_\infty$ operations, an algebra (in the usual sense) over the ring R with generators $\{I_i\}_{i=1}^N$, $\{U_i\}_{i=1}^N$, and $\{s_{i(i+1)}\}_{i=1}^N$, where the indices for the s_{ij} are counted mod N . Define

$$s_{ij} = s_{i(i+1)}s_{(i+1)(i+2)} \cdots s_{(j-1)j},$$

for each $0 < j - i < N$ (counted in $\mathbb{N}$, not mod N). Note in particular that we have

$$s_{ii} = s_{i(i+1)}s_{(i+1)(i+2)} \cdots s_{(i-1)i}$$

which is a non-zero element of $\mathcal{A}$. As shorthand, we will almost always write

$$s_i := s_{i(i+1)}$$

for each $1 \leq i \leq N$ for these generators. The geometric reason we still initially defined the s_i using the notation $s_{i(i+1)}$ will be made clear in the discussion surrounding Figure 2.6, below.

We define simple multiplication (i.e. μ_2) by the following rules:

$$\begin{aligned}
U_i U_j &= \begin{cases} 0 & i \neq j \\ U_i^2 & i = j \end{cases} \\
U_i s_{jk} &= s_{ij} U_k = 0 \quad \text{for all } i, j, k \\
s_{ij} s_{k\ell} &= \begin{cases} s_{i\ell} & j = k \\ 0 & \text{otherwise} \end{cases} \\
I_i^2 &= I_i \\
I_i I_j &= 0 \text{ if } i \neq j \\
I_i U_j &= U_j I_i = \begin{cases} U_i & i = j \\ 0 & \text{otherwise} \end{cases} \\
I_i s_j &= \begin{cases} s_i & i = j \\ 0 & \text{otherwise} \end{cases} \\
s_j I_i &= \begin{cases} s_{i-1} & j = i - 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

We summarize the last three lines of this by saying that U_i lives in the i th idempotent, and that s_i starts in the i th idempotent and ends in the $(i + 1)$ st idempotent. All other elements are products of the basic U_i and s_i . In particular, let

$$U_{N+1} = \sum_{i=1}^N s_{ii}$$

so that

$$\begin{aligned}
U_i U_{N+1} &= 0 \quad \text{for each } 1 \leq i \leq N \\
s_{ij} U_{N+1} &= s_{ij} s_{jj} \\
U_{N+1} s_{ij} &= s_{ii} s_{ij}
\end{aligned}$$

We define the *initial idempotent* of an algebra element $a \in A$ to be the (unique) I_i such that $I_i a \neq 0$,

and the *final idempotent* of a to be the (unique) I_j such that $aI_j \neq 0$. In this case, we will often say that an element a *starts* in the i th idempotent and *ends* in the j th idempotent.

It follows from the above that U_i ($1 \leq i \leq N$) starts and ends in the i th idempotent, and s_{ij} starts in the i th idempotent and ends in the j th idempotent. Likewise, U_{N+1} has a single summand that starts and ends in the i th idempotent, for each $1 \leq i \leq N$.

Throughout the following exposition, we say that two elements are *multipliable* if they have non-zero product, and *cannot be multiplied* or are *non-multipliable* if they have product zero. Hence, by the above multiplication rules two elements $a_1, a_2 \in \mathcal{A}$ are multipliable (i.e. $a_1 a_2 \neq 0$) *only if* the final idempotent of a_1 is the same as the initial idempotent of a_2 . This is, however, not always sufficient, as we see from the $U_i s_{ij} = 0$ (for $1 \leq i \leq N$).

Graphically, each U_i with $1 \leq i \leq N$ corresponds to the path all the way around the i -th boundary circle (which is at the end of a spoke), while s_{ij} with $0 < j - i < N$ is a short chord around the central boundary circle, for instance

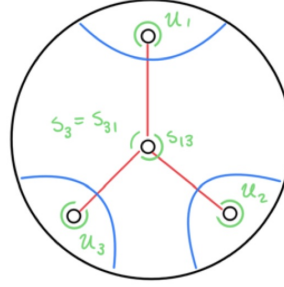


Figure 2.6: Some short chords corresponding to elements of $\mathcal{A}$

This gives a geometric motivation for the notion of initial and final idempotents, that is, each U_i starts and ends on the i th spoke / α -arc, while each s_{ij} connects the i th α -arc to the j -th α -arc. The notation $s_{i(i+1)}$ for the short-chords s_i around the central boundary circle keeps track of the initial and final idempotent, and is sometimes used when we want to emphasize where one of these chords starts / ends.

Figure 2.6 also gives a geometric reason why the s_{ij} and U_i (except U_{N+1}) cannot be multiplied. Multiplication corresponds geometrically to concatenation of adjacent chords, and even the nearest two U_i nearest to a given s_{jk} are each separated from it by an α -arc. In the pseudo-holomorphic picture of things, for U_i and s_{jk} to be multipliable would imply that this α -arc could be compressed to a point, which is not allowed. This is also the underlying reason why it makes sense that U_i and U_j cannot be multiplied.

Also, each s_{ii} corresponds the long chord around the central boundary circle beginning and ending at the central spoke, as in Figure 2.7.

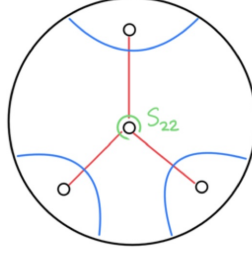


Figure 2.7: One of the N long chords

This then makes U_{N+1} in some sense “the” long chord around the central boundary circle.

2.2.2 Allowable graphs

We have now defined the multiplication on $\mathcal{A}$ (i.e. the unweighted μ_2). The object of this subsection is to define the other (weighted and unweighted) operations which will make $\mathcal{A}$ into a bona-fide weighted $\mathcal{A}_\infty$ algebra.

The key tool for this will be so-called *allowable graphs*, defined below. These graphs will be analogous to those in the construction of the weighted bordered torus algebra in [17]. After defining allowable graphs, we will define non-zero higher operations to be $\mathcal{A}$ to be in one-to-one correspondence with these allowable graphs.

We now define allowable graphs. First, a *labelled graph* is a properly embedded planar graph in the disk, with each sector around each vertex labelled with an element of $\mathcal{A}$. For fixed N , define the *basic allowable (unweighted) planar graphs* to be the set of $2N$ -valent graphs with:

- one internal vertex;
- one marked vertex on the boundary called the *root*;
- sectors labeled by a cyclic permutation of the elements $U_1, s_1, \dots, U_N, s_N$;

Each such graph corresponds to a corolla with $2N$ inputs and one output. For example, in the case $N = 3$, one such basic allowable graph is:

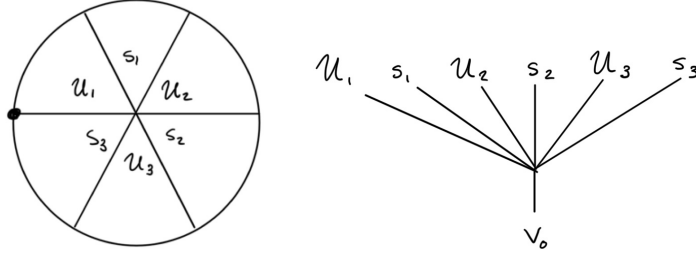


Figure 2.8: A basic allowable graph with corresponding corolla

To define which other graphs are allowable, we need to define *allowable concatenations of basic graphs*. Assign to each idempotent $I_1, \dots, I_N$ a color. Note that in the sequence $U_1, s_1, \dots, U_N, s_N$, the final idempotent of each algebra element in the sequence is the initial idempotent of the next algebra element in the sequence.

This means that the labelling of each basic allowable planar graph induces a coloring of its branches corresponding to the idempotent in which each branch lies. One example of this is illustrated in Figure 2.9: note that there are N colors, and each sector labelled with U_i bordered by edges colored with the i -th color, and each sector labelled with an s_i is bordered by edges colored with the i -th and $(i+1)$ -st colors, counted clockwise around the vertex.

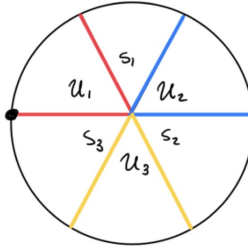


Figure 2.9: A basic allowable graph, with coherent coloring

Define an *allowable concatenation of two basic graphs* G_1 and G_2 as follows. G_1 and G_2 are both embedded in disks. To concatenate these, we are going to take a boundary connected sum of these two disks in the following way: look at the root vertex r of G_2 . There is a unique vertex v of G_1 such that if we take a connected sum of the two disks, gluing r to v , then

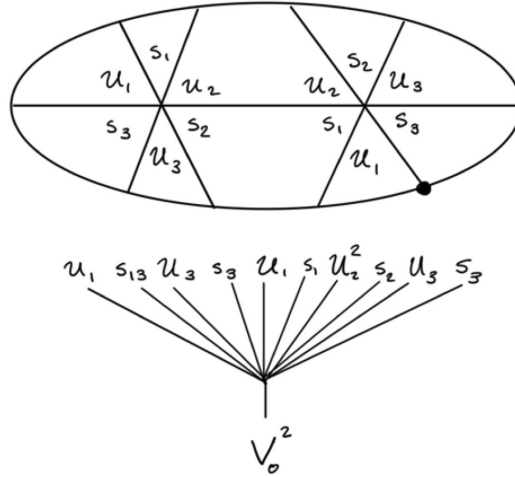
- (a) The edge adjacent to v in G_1 has the same color as the edge adjacent to r in G_2 ; in the connected sum, these two edges will merge into a single edge;
- (b) The two elements of $\mathcal{A}$ labelling sectors in the region above this edge are multipliable when read in order as we travel clockwise around the boundary;

- (c) The two elements of $\mathcal{A}$ labelling sectors in the region below this edge are multipliable when read in order as we travel clockwise around the boundary.

Gluing the two graphs in this way gives the unique allowable concatenation of G_1 and G_2 . In general, an *allowable concatenation of basic graphs* is obtained by the process described above, by induction on the number of internal vertices: starting with an allowable concatenation of basic planar graphs G_1 , with $(n - 1)$ internal vertices, and an basic graph G_2 , look at the edge adjacent to the root r of G_2 . There will be at least one vertex v of G_1 satisfying conditions (a)-(c) above. The connected planar graph obtained by taking a connected sum of the G_1 and G_2 , gluing a neighborhood of v to a neighborhood of r , is called an *allowable concatenation of G_1 and G_2* . It is clear that in most cases, when $n > 2$, there will be more than one such allowable concatenation.

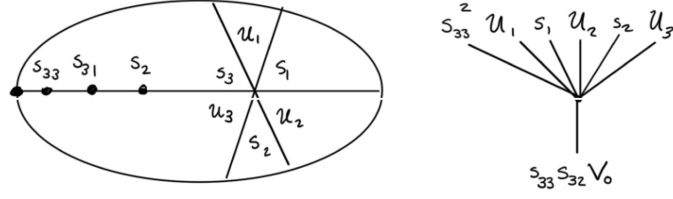
In general, define an *allowable unweighted planar graph* to be an allowable concatenation of some number of basic planar graphs.

In practice, we usually suppress the coloring of these graphs, and just label the regions as in

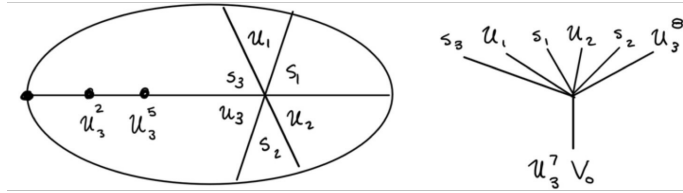


Later on, in Section 2.2.4, we will show that every such allowable unweighted planar graph corresponds to a unique corolla, and that every corolla satisfying certain conditions corresponds to such a graph.

Next, *extended unweighted planar graphs* are defined by starting with an allowable unweighted planar graph adding a sequence of labelled 2-valent vertices to the edge adjacent the root in the following way. We require that the labels are either all on the left of the root (looking into the disk from the direction of the root), in which case the operation is called *left-extended*, or on the right of the root, in which case the operation is called *right extended*. For instance, in the case $N = 3$, we could have



which is left-extended, or

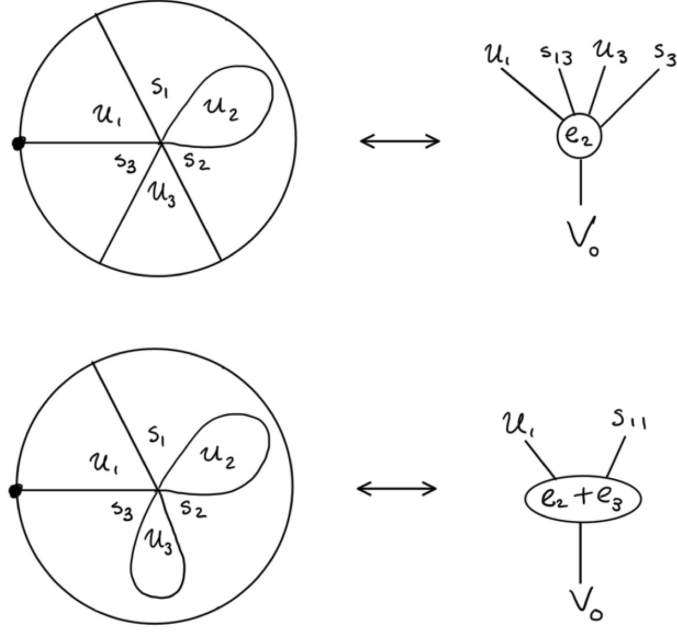


which is right-extended. We do not allow 2-valent vertices on any other edges.

Weighted operations likewise correspond to colored planar graphs, this time allowing certain cycles, namely, for each $1 \leq i \leq N + 1$,

$$\mu_{\circ}^{\bar{e}_i} : \begin{array}{c} \textcircled{e_i} \\ | \\ u_i \end{array}$$

(Recall that the weights live in space $\mathbb{Z}_{\geq 0}[V_0, \dots, V_{N+1}]$. Inclusion of weighted operations induces higher multiplications, for instance, in the case $N = 3$,

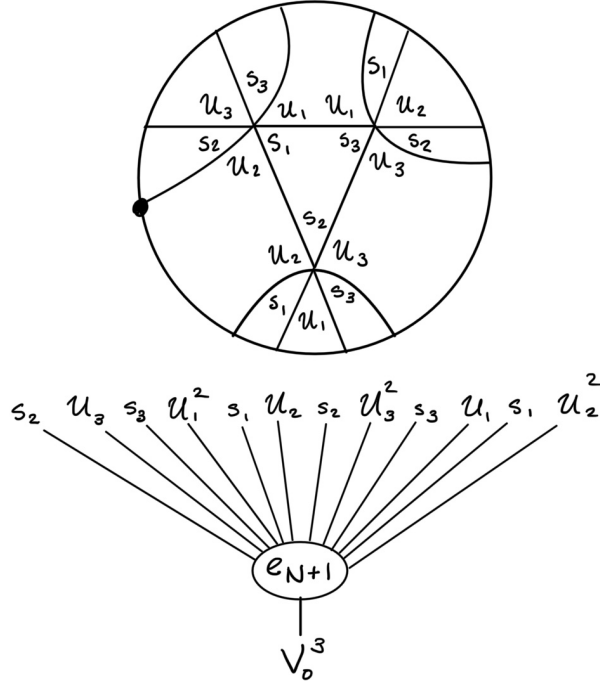


We call these cycles (for $1 \leq i \leq N$) *petal cycles* which induce *petal weight* in the corresponding trees. In general, these basic petal-weighted operations are of the form

$$\mu_{2N-2k}^{\mathbf{w}}(a_1, \dots, a_{2N-2k}) = V_0$$

where $\mathbf{w} = \sum_{j=1}^k \mathbf{e}_{i_j}$ for distinct i_j 's, and for suitable a_i as detailed above. (This description for weighted operations will be formalized and made more specific in Theorem 2.2.7.)

We also have weighted higher multiplications corresponding to *internal cycles*, for instance, in case $N = 3$,



and the other operations which correspond to moving the basepoint around to any of the other outer vertices. In general, these basic, internally weighted, operations are of the form

$$\mu_{N(2N-2)}^{\mathbf{e}_{N+1}}(a_1, \dots, a_{N(2N-2)}) = V_0^N$$

for suitable a_i , as detailed above. (Again, see Theorem 2.2.7 for a formal description of these weighted higher operations.)

2.2.3 Maslov and Alexander gradings for $\mathcal{A}$

In order to specify which trees define allowable graphs, we will need two notions of grading. The Maslov grading m is a map from the set of labelled trees to $\mathbb{Z}$. It is required to satisfy the following property:

$$m(\mu_n^{\mathbf{w}}(a_1, \dots, a_n)) = \sum_{i=1}^n m(a_i) + m(\mathbf{w}) + n - 2 \quad (2.12)$$

where $\mathbf{w}$ is the weight and $a_1, \dots, a_n \in \mathcal{A}$. We set

$$\begin{aligned}
m(U_i) &= m(s_{ij}) = 0 \text{ for each } i, j \\
m(\mathbf{e}_i) &= 2 \text{ for each } 1 \leq i < N + 1 \\
m(\mathbf{e}_0) &= -(2N - 2) \\
m(\mathbf{e}_{N+1}) &= 3 \\
m(V_0) &= 2N - 2 \\
m(V_i) &= -2 \text{ for each } 1 \leq i < N + 1 \\
m(V_{N+1}) &= -2
\end{aligned}$$

Notice that in each case, this fits with (2.12), and that the $m(\mathbf{e}_{N+1}) = 3$ is independent of choice of N .

The second grading is called the *Alexander grading*, and is defined as follows. Let X be S^1 minus $2N$ points, one corresponding to each endpoints of a β -arc. The 0-th homology of X is just $\mathbb{Z}^{2N}$. Write the obvious generators as $\bar{k}$ for $1 \leq k \leq 2N$. The Alexander grading is a map from the generators of $\mathcal{A}$ to $H_0(X; \mathbb{Z}) \cong \mathbb{Z}^{2N}$, such that

$$\begin{aligned}
A(U_i) &= \overline{2i - 1} \text{ for each } 1 \leq i \leq N \\
A(s_{ij}) &= \sum_{k=i}^{j-1} \overline{2k} \\
A(\mathbf{e}_i) &= \overline{2i - 1} \text{ for each } 1 \leq i \leq N \\
A(\mathbf{e}_{N+1}) &= \sum_{k=1}^N \overline{2k} \\
A(V_0) &= \sum_{k=1}^{2N} \bar{k}
\end{aligned}$$

extended to all products of these generators by stipulating that $A(ab) = A(a) + A(b)$.

The Alexander grading of a sequence can also be defined, as:

$$A(a_1, \dots, a_n) = \sum_{i=1}^n a_i.$$

The Alexander grading of a weighted sequence is

$$A(\mathbf{w}, a_1, \dots, a_n) = A(\mathbf{w}) + A(a_1, \dots, a_n).$$

We require that operations preserve the Alexander grading.

Graphically, all this looks like

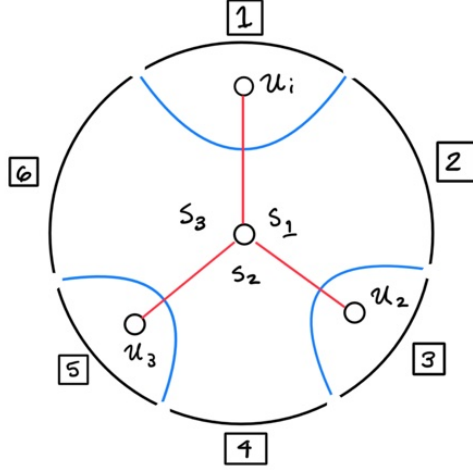


Figure 2.10: The Alexander gradings for basic elements of $\mathcal{A}$

where the boxed numbers are the Alexander grading corresponding to each component of the cut-out S^1 along the boundary, and we can see why each algebra element fits into the Alexander grading to which it is assigned. As usual, we are just looking at $N = 3$ as an example, but this also holds for other N .

2.2.4 $\mathcal{A}_\infty$ -relations

We are nearly ready to verify the $\mathcal{A}_\infty$ relations for $\mathcal{A}$. The next step is to give a condition for an unweighted corolla to correspond to an allowable unweighted planar graph (i.e. to an unweighted operation). Before we can state the condition, we need some further definitions. A sequence $(a_1, \dots, a_n)$ of elements of $\mathcal{A}$ is *left extended* if the initial element a_1 is not basic, but can be written as a product of more than one of the basic elements (U_i 's and s_i 's)¹; it is *right extended* if a_n can be written as a product of more than one of the basic elements of $\mathcal{A}$. The sequence is *centered* if it is neither left nor right extended. Note that none of these definitions require that the elements other than initial / final elements are basic.

The condition is as follows:

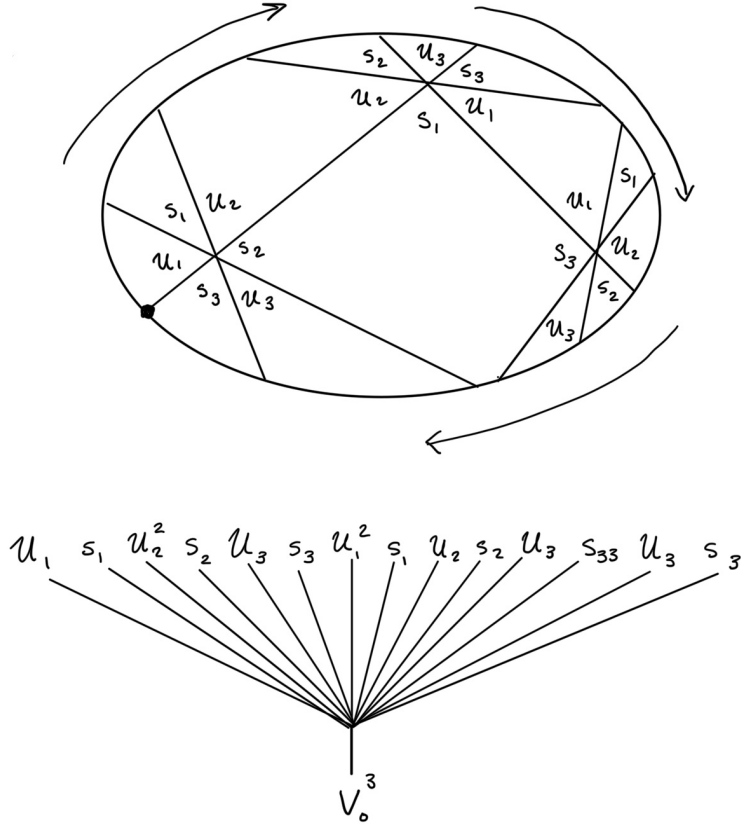
Theorem 2.2.1. (Unweighted higher multiplications) *Let T be an unweighted tree whose inputs are labelled with elements of $\mathcal{A}$, say $a_1, \dots, a_n$, counted from left to right, with $n > 2$. Then T*

¹While we usually view any s_{ij} as basic, even if $j > i + 1$ (in which case s_{ij} can be decomposed as a product of more than one s_i), we only want to look at single s_i 's for this case.

corresponds to an allowable unweighted planar graph if and only if the following conditions hold:

- (i) (The idempotents match up) *The initial idempotent of a_i is the final idempotent of a_{i-1} for each $1 < i \leq n$;*
- (ii) (The Maslov grading works out) *$n = j(2N - 2) + 2$ for some $j \in \mathbb{N}$;*
- (iii) (Extensions) *The sequence $(a_1, \dots, a_n)$ is either centered, or left or right extended, but not both at once;*
- (iv) (The Alexander grading works out) *The condition varies depending on whether the sequence $(a_1, \dots, a_n)$ is centered, left extended, or right extended.*
 - *Centered: $A(a_1, \dots, a_n) = j \cdot \sum_{k=1}^{2N} \bar{k}$;*
 - *Left extended: Write $a_1 = a'_1 \cdot a''_1$, where $a''_1 = U_i$ or s_i for some i . Then we require that $A(a_1, \dots, a_n) = A(a'_1) + j \cdot \sum_{k=1}^{2N} \bar{k}$ for the same j as in (ii);*
 - *Right extended: Write $a_n = a''_n \cdot a'_n$ where $a''_n = U_i$ or s_i for some i . Then we require that $A(a_1, \dots, a_n) = A(a'_n) + j \cdot \sum_{k=1}^{2N} \bar{k}$ (again, for the same j as in (ii);*
- (v) (Correct output) *The output is V_0^j if the $(a_1, \dots, a_n)$ is centered $a'_1 V_0^j$ if it is left extended, and $V_0^j a'_n$ if it is right extended;*

Proof. Any suitable graph yields a labelled tree by reading off the labels on the sectors of the graph counting clockwise from the root; for instance



which in this case, is an unweighted μ_{14} . Because of how we constructed our graphs, the resulting tree satisfies (i)-(v).

For the other direction, suppose a tree T satisfies conditions (i)-(v), and is centered, with the inputs labelled by $(a_1, \dots, a_n)$. Then we can construct a simply connected, $2N$ -valent graph for T inductively, in the following way. First of all, if $j = 1$, so $n = 2N$, then T determines some choice of root vertex in the graph of Figure 2.11:

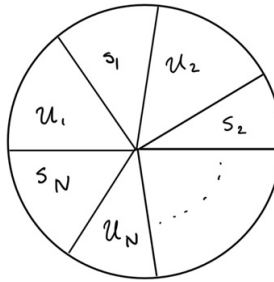


Figure 2.11: A basic allowable planar graph before choice of root

That is, it uniquely determines a suitable graph. Now suppose that for all $j' < j$, if T is a centered tree satisfying (i)-(v), we can define a unique, suitable corresponding graph, and look at a T with

$j(2N - 2) + 2$ inputs. Because

- T has $n = j(2N - 2) + 2$ inputs;
- These inputs satisfy (ii) (the condition on idempotents);
- The inputs satisfy

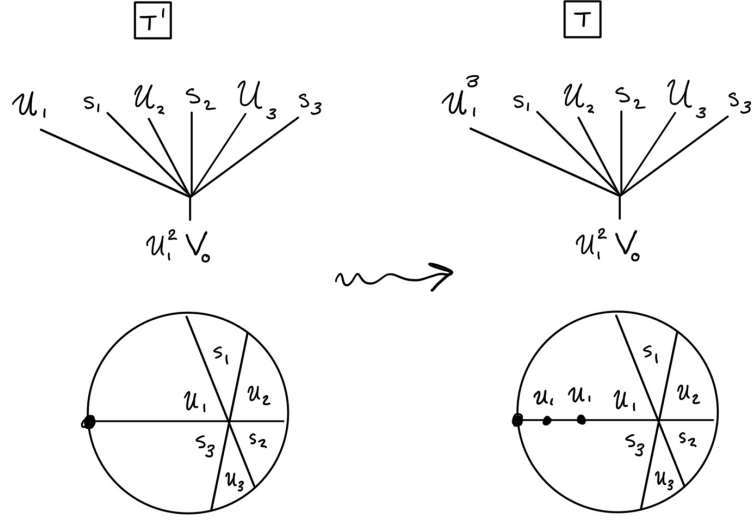
$$A(a_1, \dots, a_n) = j \cdot \sum_{k=1}^{2N} \bar{k}$$

it follows that at most $2j - 2$ of these elements are non-basic, and hence, we can find at least one subsequence of $2N - 2$ consecutive basic elements among the $(a_1, \dots, a_n)$; say $a_{i+1}, \dots, a_{i+2N-1}$. Look at a_i and a_{i+2N} , and write

$$\begin{aligned} a_i &= a'_i a''_i \\ a_{i+2N} &= a''_{i+2N} a'_{i+2N}, \end{aligned}$$

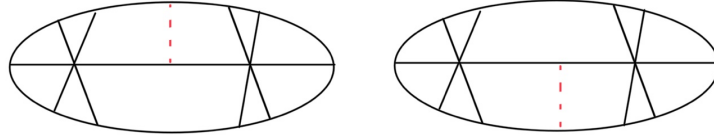
where a''_i and a''_{i+2N} are basic elements of $\mathcal{A}$. (Again, basic elements are $\{U_i\}_{i=1}^N$ or $\{s_i\}_{i=1}^N$.) It is now clear that the initial idempotent of a''_i is the same as the final idempotent of a''_{i+2N} ; so now we have a sequence $a''_i, a_{i+1}, \dots, a_{i+2N-1}, a''_{i+2N}$ of basic elements, which by the rules on idempotents, must be some cyclic permutation of $U_1, s_1, \dots, U_N, s_N$. Hence, it determines some choice of root vertex on the single-vertex graph in Figure 2.11. Remove the sequence $a''_i, a_{i+1}, \dots, a_{i+2N-1}, a''_{i+2N}$ from $a_1, \dots, a_n$, to get $a_1, \dots, a_{i-1}, a'_i, a'_{i+2N}, a_{i+2N+1}, \dots, a_n$. Look at the tree T' with these elements as inputs, and with output V_0^{j-1} . Then it is clear that this T' still satisfies (i)-(v), but the output power of V_0 is less than j ; hence the inductive hypothesis holds and we can get a unique, suitable rooted graph corresponding to T' . Now attach the root of the graph corresponding to $a''_i, a_{i+1}, \dots, a_{i+2N-1}, a''_{i+2N}$ to the edge between a'_i, a'_{i+2N} . This is clearly a graph that corresponds to T .

Now, consider a left-extended tree T satisfying (i)-(iii) and (v), and write $a_1 = a'_1 a''_1$, where a''_1 is a basic element. The output of T is $a'_1 V_0^j$ for j as in the statement. The tree T' with inputs $(a''_1, a_2, \dots, a_n)$ and output V_0^j is a centered tree satisfying (i)-(v) and affords a unique graph. Now decompose a'_1 into a product of basic elements, say $a_1 = p_1 \cdots p_k$, and mark the initial of the graph for T' with points labelled $p_1, \dots, p_k$; for instance:

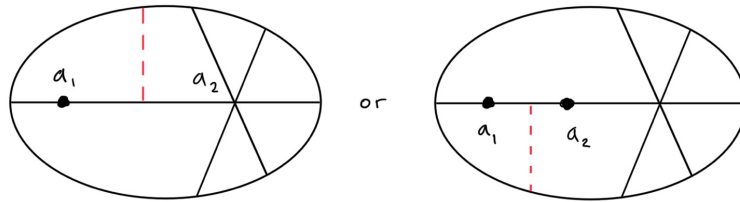


This gives the graph for T , which is clearly allowable. The situation for a right-extended tree is analogous. $\square$

Before we can verify the $\mathcal{A}_\infty$ -relations for $\mathcal{A}$, we need to define the notion of *augmented graphs*, which we will use in the proof. There are two types. The first is an allowable graph in which we choose one internal edge (that is, an edge not intersecting the boundary) and draw a dotted line from this edge to the boundary. Note that the direction of drawing the edge matters, so

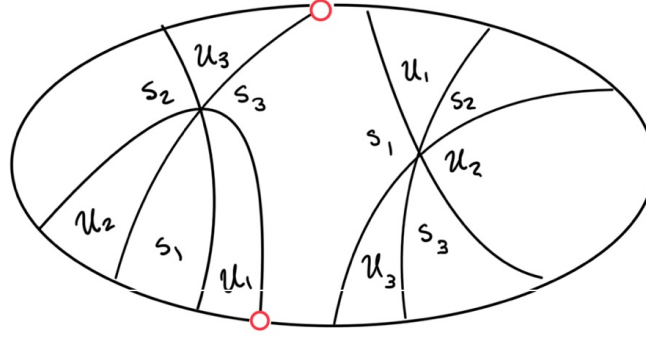


are distinct. We are also allowed to draw this dotted line on the on an edge adjacent to an “extended” vertex, for instance



Note that we can only draw this dotted edge on the side of the extension (so in the first case, only on the top, and in the second, only on the bottom). Note also that we cannot draw this dotted edge between the extended vertex and the root. We will prove that that these augmented graphs are in one to one correspondence with pulls in Theorem 2.2.7.

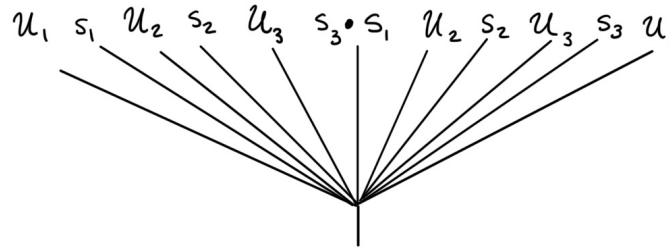
The other kind of augmented graph are two vertex, two component graphs of the form



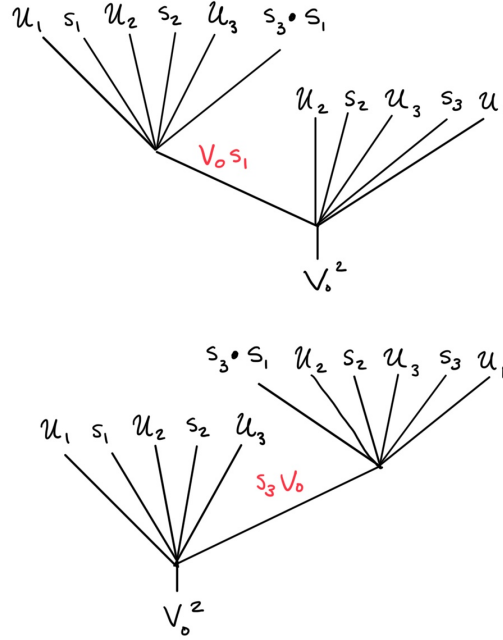
Here, we are again using $N = 3$ as an example. All augmented graphs of the second kind (for $N = 3$) can be obtained from this one by permuting the labels of the first component one step counterclockwise, and the labels of the second component one step clockwise. We only consider graphs with $N = 3$, and the argument when $N > 3$ is entirely analogous.

Lemma 2.2.2. *Let T be a tree satisfying conditions (i), (iii) and (iv) from Theorem 2.2.1, and with $n = 2 \cdot (2N - 2) + 3$, with no multipliable pairs. Then T corresponds to a unique augmented graph of the first kind, and hence, to a cancelling pair of splits as pictured above.*

Proof. The hollow roots in the graph above indicate the two possible ways to resolve this into allowable compositions; this graph above corresponds to the tree



and the two ways to resolve it are



Thus, the second kind of augmented graph are in one to one correspondence with this particular kind of split. A tree admits this particular kind of split if and only if satisfies (i), (iii), and (iv) from Theorem 2.2.1, has $2(2N) - 1$ inputs – that is, one extra from the required number from (ii) of Theorem 2.2.1 – and has no multipliable pairs. Essentially, the multipliable pair (which leads to the extra input) is the pair of elements on the two ends. But while those correspond to adjacent elements in the graph, they are not adjacent in the tree. $\square$

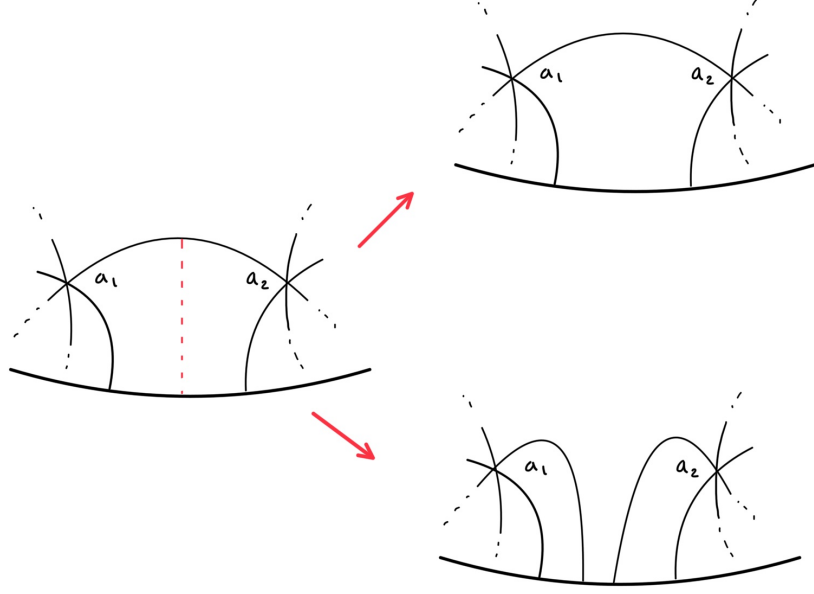
We are now ready to prove the $\mathcal{A}_\infty$ -relations for unweighted trees labelled with elements of $\mathcal{A}$.

Theorem 2.2.3. (Unweighted $\mathcal{A}_\infty$ -relations) *The $\mathcal{A}_\infty$ relations for unweighted trees labelled with elements of $\mathcal{A}$ are satisfied – that is, the sum over all ways to add an edge to a given tree T is always zero.*

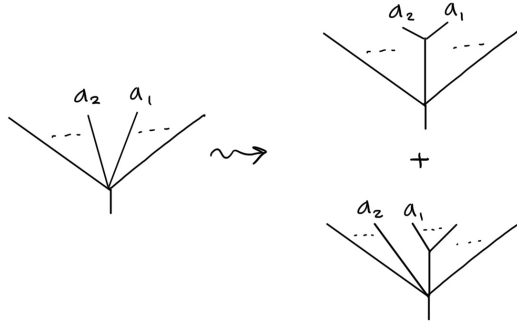
Proof. First of all we can restrict our attention to $\mathcal{A}_\infty$ relations corresponding to trees with at least three inputs. This is because $\mathcal{A}$ has $\mu_1 = 0$, so the $\mathcal{A}_\infty$ relations for any pair of elements $a, b \in \mathcal{A}$ (i.e. $\mu_1(ab) + \mu_1(a) \cdot b + a \cdot \mu_1(b) = 0$) are trivial.

To prove our claim for this case, we are going to show that non-vanishing $\mathcal{A}_\infty$ relations in $\mathcal{A}$ (i.e. trees T for which at least one of the trees corresponding to adding an edge to T is a non-zero operation) are in one-to-one correspondence with augmented graphs as described above.

Note first that any augmented graph of the first kind yields exactly two suitable graphs (i.e. operations, or compositions of operations) in the following ways:

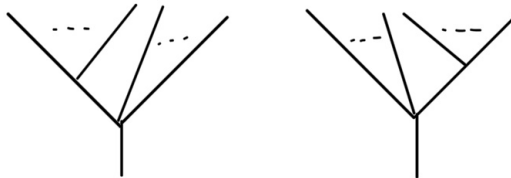


which corresponds to the $\mathcal{A}_\infty$ relation



The a_1, a_2 , are assumed to be multipliable, and as usual, we were using $N = 3$ as a shorthand for general N , specifically in depicting the generally $2N$ -valent vertices above. That the second graph above has two disjoint components (and hence, corresponds to a composition i.e. a split) follows from the fact that our graphs do not contain cycles, and therefore, cutting an edge gives two components. Also, we are assuming without loss of generality that the root is on the component adjacent to a_2 .

That there are non-trivial inputs on either side of the internal operation containing a_1 follows from the fact that the root is not on an edge bordering the sector containing a_2 . This implies, in particular, that the only way to obtain a composition of either of the forms



as the result of an $\mathcal{A}_\infty$ -relation is in the particular kind of split dealt with in Lemma 2.2.2, that is, from the second kind of augmented graph.

Next, we consider the case of augmented graphs of the first kind.

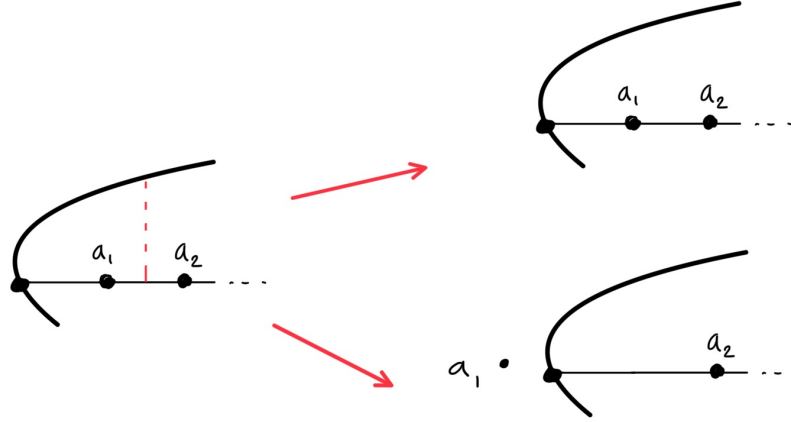
Lemma 2.2.4. *Each augmented graph of the first kind corresponds to a tree T satisfying conditions (i), (iii), and (iv) from Theorem 2.2.1, and in place of (ii), satisfies*

(ii') The number of inputs is $n = j(2N - 2) + 3$, where $j \in \mathbb{Z}_{\geq 1}$, and T contains a single multipliable pair;

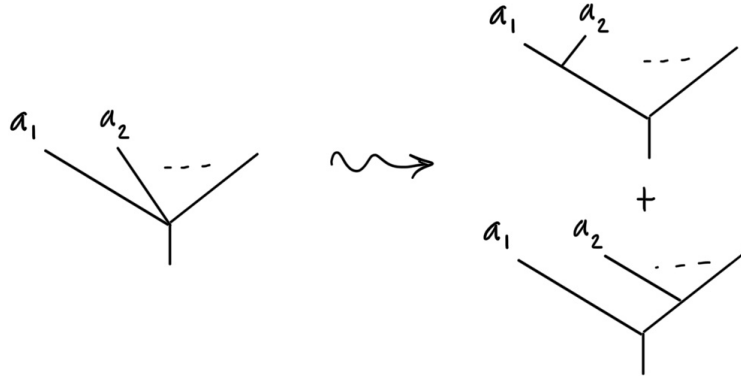
In this case, T admits exactly one pull. In addition, any tree T satisfying these conditions corresponds to a unique augmented graph of the first kind (and hence admits exactly one pull).

Remark 2.2.5. As usual, when we say T “admits” a split or pull, we mean that the tree T' obtained from T by this method of adding an internal edge represents a non-zero operation.

Proof of Lemma 2.2.4. We first show that each augmented graph of the first kind corresponds to a tree satisfying (i), (ii)', (iii), and (iv). If the dotted edge is drawn between two vertices on the initial edge (at least one of which must be extended), we get:



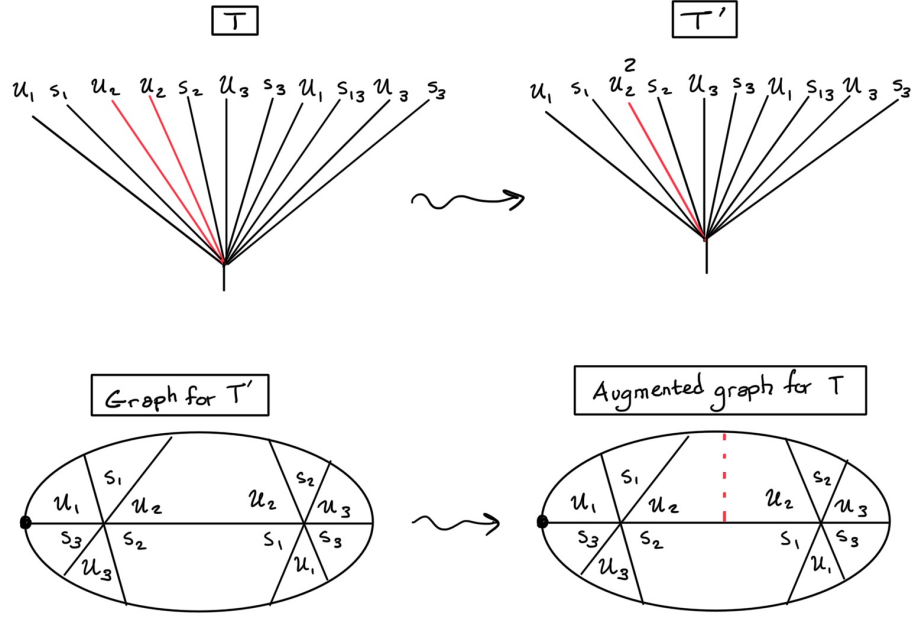
which, on the level of trees, corresponds to



Note that the a_2 vertex is not necessarily 2-valent; it can be the first of the $2N$ -valent vertices, even though the 2-valent case is pictured above. The case in which we cut on the initial edge for a right-extended tree is also analogous.

By the definition of augmented graphs (of the first kind) the graphs appearing on right hand sides of the pictures above were allowable. Hence, the corresponding trees on the right hand sides of the pictures above are operations.

It remains to show that any tree T satisfying (i), (ii)', (iii), and (iv) corresponds to a unique augmented graph of the first kind. This follows because a tree admits a pull if and only if it has a single multipliable pair. To get an augmented graph from an arbitrary tree satisfying the conditions from Lemma 2.2.4, we pull together the two multipliable elements in T to get bona fide operation T' , which has a well-defined, allowable graph. Then we draw a dotted line between the two elements which are multiplied in T' but not in T , for instance:



It is also clear that by reversing this process, we can get back from an augmented graph to a tree of the form from Lemma 2.2.4. $\square$

Lastly, we need to show that if T admits a split that is not of the particular kind dealt with in Lemma 2.2.2, then T is of the form from Lemma 2.2.4. Suppose T admits such a split. Then drawn as trees, the resulting compositions look like

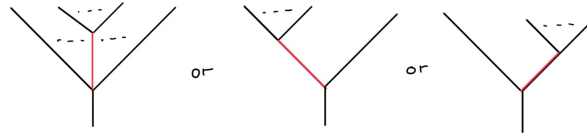
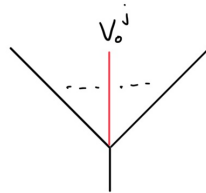


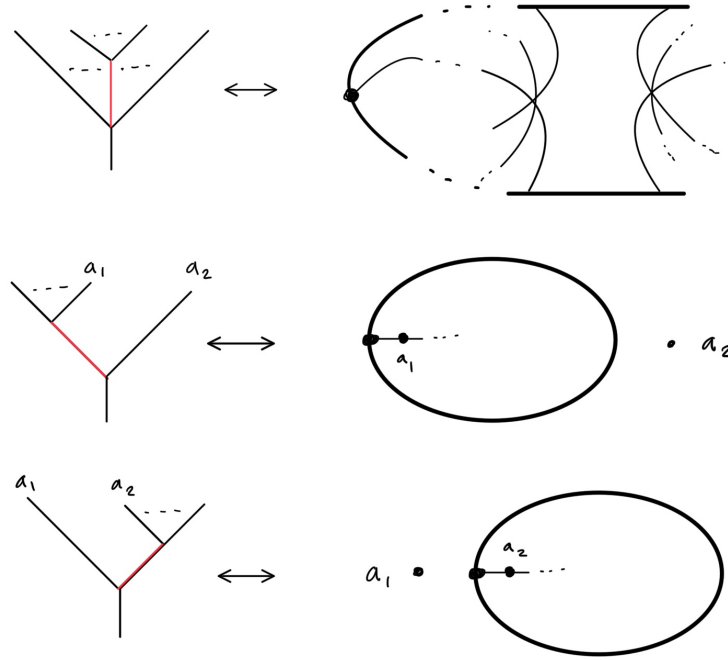
Figure 2.12

In the first case, inner operation must be either left or right extended. If it were not, then the tree obtained by performing a split would be of the form

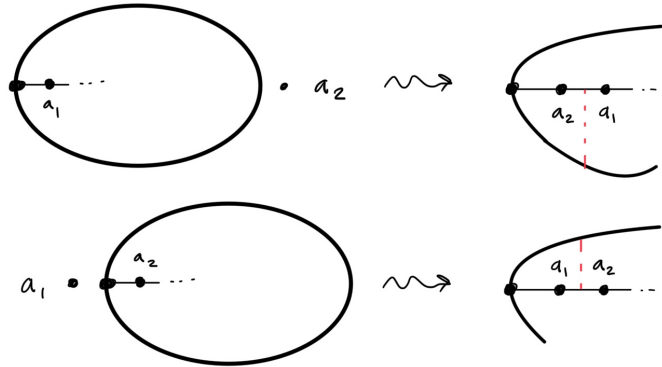


i.e. it would contain at least one multipliable pair. It would therefore not be an operation since by the condition on Maslov gradings, operations cannot have multipliable pairs. In the other cases, the inner operation cannot be centered; there is no problem with a μ_2 involving a raw power of V_0 .

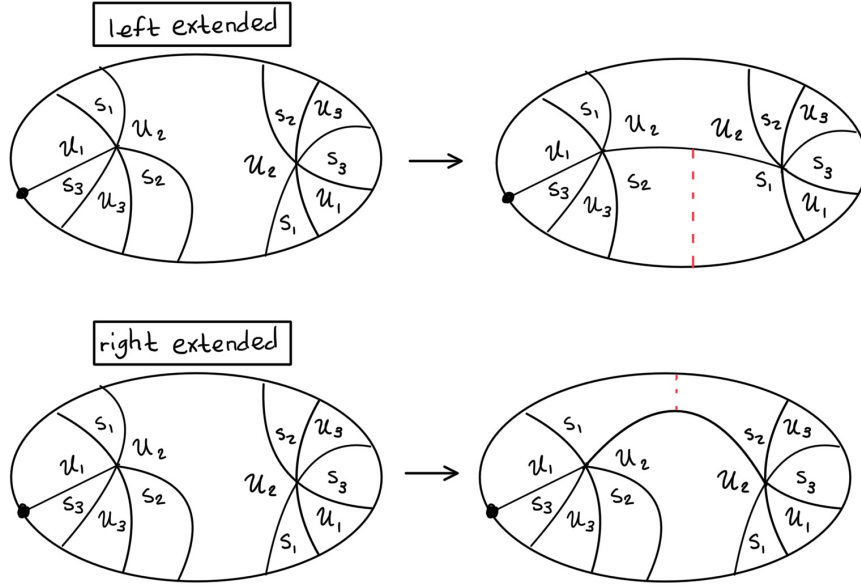
By an argument analogous to the one in the proof of Theorem 2.2.1, we can show that the trees pictured in Figure 2.12 each correspond to one of the following types of allowable graph:



Notice that each of these graphs has exactly two disjoint components. (We are counting the singleton elements on the left and right of the second and third, respectively, as formal “connected graphs”.) In the second and third cases, the two-component graphs pictured above clearly came from pushing out the dotted edge of an augmented graph, that is,



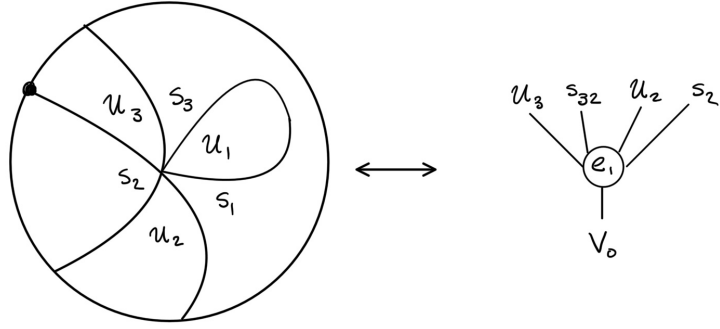
drawn above with corresponding (original, given) trees. Likewise, in the first case, because the internal operation of the composition was either left or right extended, we can use this to unambiguously determine an augmented graph that gives rise to our given two component graph by pushing out an edge. For instance



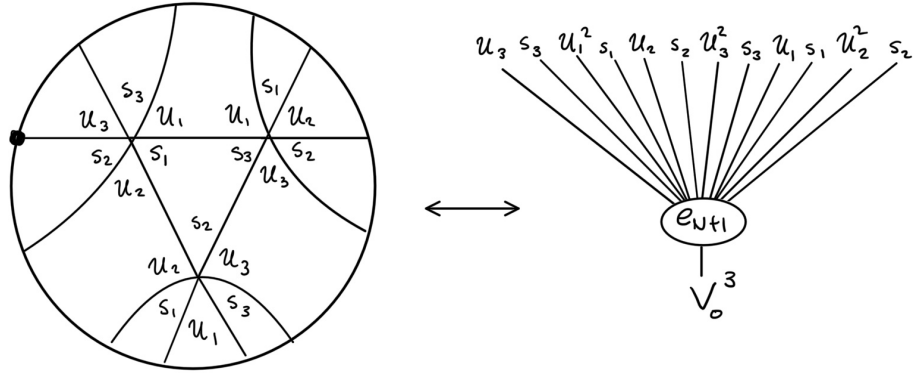
Hence, in each of the three cases, our split corresponds to a unique augmented graph of the first kind. By Lemma 2.2.4, these augmented graphs each give rise to a unique pull. This means that splits (of this kind) and pulls always cancel. The other kind of splits always cancel in pairs, as discussed before. Since these are the only possible ways to get non-zero terms in the $\mathcal{A}_\infty$ -relation for a given unweighted tree, we have proved our claim. $\square$

Remark 2.2.6. We did not really need to show that non-vanishing $\mathcal{A}_\infty$ -relations in the unweighted case were in one-to-one correspondence with augmented graphs. We could just have shown that a tree admits a pull or a split if and only if it satisfies the conditions of Lemma 2.2.4, and that in this case it admits exactly one of each – without mentioning augmented graphs at all. However, the augmented graphs will be important in the weighted case, which is why we include the subsidiary part of the argument above.

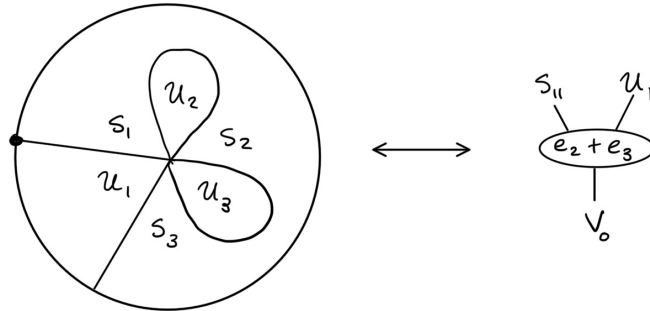
The next step is to run through the same process, allowing cycles. There are two kinds of cycles. The first are *petal cycles*, that is $\mathbf{e}_i$ -weight for $1 \leq i \leq N$, which, in the trees and graphs, correspond to e.g.



The second are *internal cycles*, i.e. $\mathbf{e}_{N+1}$ -weight, which shows up as e.g. (in the case $N = 3$ and with a particular choice of root)



Before we can consider the case with cycles, we need to modify our definition of left / right extensions. In the unweighted case, a tree was extended whenever the left- (resp. right-) most element was non-basic. However, in the weighted case, this is not sufficient. For instance the weighted sequence $\mathbf{w} = \mathbf{e}_2 + \mathbf{e}_3$, with $(a_1, a_2) = (s_{11}, U_1)$ gives tree and graph,



which is a centered graph (there are no 2-valent vertices), but has left-most element non-basic. Hence, simply saying that a weighted tree is left- (resp. right-) extended if the left- (resp. right-) most element is non basic is clearly not enough.

Recall that in the weighted case, the other way of identifying non-centered trees was that these had unbalanced Alexander grading, i.e

$$A(a_1, \dots, a_n) = j \cdot \sum_{k=1}^{2N} \bar{k} + \text{some non-zero term} \quad (2.13)$$

where j was assumed maximal in the sense that the non-zero term does not include another factor of $\sum_{k=1}^{2N} \bar{k}$. This is necessary and sufficient to identify left- (resp. right-) extended trees in the unweighted case. We will use an analogous condition to define left- and right-extended trees in the weighted case. Using the fact that $A(\mathbf{e}_i) = \overline{2i-1}$ for each $1 \leq i \leq N$, the analogous condition to (2.13), for the petal-weighted case, is

$$A(\mathbf{w}, a_1, \dots, a_n) = j \cdot \sum_{k=1}^{2N} \bar{k} + \text{some non-zero term} \quad (2.14)$$

where again, j is maximal in the sense that the non-zero term does not contain an extra factor of $\sum_{k=1}^{2N} \bar{k}$. We also need to define the notion of an *offset term*. Look at a weighted sequence $(\mathbf{w}, a_1, \dots, a_n)$ satisfying (2.14). If we can write $a_1 = \alpha a'_1$, with $\alpha, a'_1 \in \mathcal{A}$ and

$$A(\mathbf{w}, a'_1, a_2, \dots, a_n) = j \cdot \sum_{k=1}^{2N} \bar{k},$$

or if we can write $a_n = a'_n \alpha$, with $\alpha, a'_n \in \mathcal{A}$ and

$$A(\mathbf{w}, a_1, \dots, a_{n-1}, a'_n) = j \cdot \sum_{k=1}^{2N} \bar{k},$$

then we say that $(\mathbf{w}, a_1, \dots, a_n)$ has a *well-defined offset term*, namely α . Any sequence which has an offset term on both left and right is discarded. If a weighted sequence satisfies (2.14) and has a single well-defined offset term, then it is *left extended* if the offset is on the left, and *right-extended* if the offset is on the right. If, on the other hand,

$$A(\mathbf{w}, a_1, \dots, a_n) = j \cdot \sum_{k=1}^{2N} \bar{k}, \quad (2.15)$$

then this is a *centered* sequence. Again, there are sequences which are not centered, left, or right extended (i.e. those with off-sets on both right and left) but these do not correspond to suitable graphs, so we do not consider them.

We define the *total magnitude* of a weight

$$\mathbf{w} = \sum_{j=0}^{N+1} k_j \mathbf{e}_j$$

as $|\mathbf{w}| = k = \sum_{j=0}^{N+1} k_j$.

Finally, recall that operations are still *defined* to correspond to suitable (and now possibly non-simply-connected) graphs. For petal cycles, the condition for a tree to be an operation is as follows.

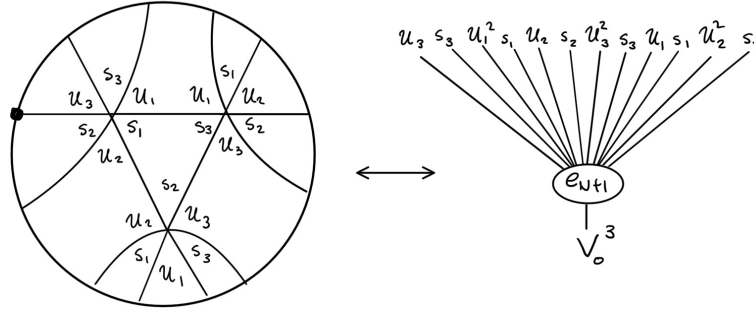
Theorem 2.2.7. (Weighted higher operations) *Let T be a tree with inputs $a_1, \dots, a_n \in \mathcal{A}$, with $n \geq 0$, and some non-zero weight $\mathbf{w}$ which is some sum of the $\{\mathbf{e}_i\}_{i=1}^{N+1}$ with total magnitude k . Then T represents an operation if and only if it satisfies the following conditions:*

- (i) (The idempotents match up) *The initial idempotent of a_i is the final idempotent of a_{i-1} for each $1 < i \leq n$;*
- (ii) (The Maslov grading works out) *$n = j(2N - 2) + 2 - 2k$ where $j \in \mathbb{N}$ and again, $|\mathbf{w}| = k$;*
- (iii) (Extensions) *The sequence $(\mathbf{w}, a_1, \dots, a_n)$ is either centered, left or right extended – in particular, this means that there cannot be an offset on both left and right ends of the sequence, or anywhere but at one of the ends;*
- (iv) (The Alexander grading works out) *If $(\mathbf{w}, a_1, \dots, a_n)$ is centered, then we require it satisfy (2.15); if it is left or right extended, we require that it satisfy (2.14);*
- (v) (Correct output) *The output is V_0^j if the $(a_1, \dots, a_n)$ is centered, and αV_0^j if it is extended, where α is the offset term, as usual;*

Proof. First we prove that any allowable graph, which by definition corresponds to a valid operation, corresponds to a tree satisfying the required conditions. As before, it is clear how to read off a tree from an allowable graph (this time allowing cycles) – just count inputs traveling counterclockwise along the boundary, as before. That the resulting tree satisfies (i) and (iii) is obvious. For (v), we just *define* the output to be V_0^j , where j is the number of vertices. For (ii), we count the number, j , of vertices, and note that each petal cycle drops the number of sectors (and hence, n) by 2 without changing j (from the unweighted case), and each internal cycle likewise drops the number of sectors by two, without changing j . Since each $\mathbf{e}_i$ adds 2 to the Maslov grading of a sequence, and the output is still V_0^j , the Maslov gradings work out. For (iv), note that each $2N$ -valent vertex contributes a $j \cdot \sum_{k=1}^{2N} \bar{k}$ to the Alexander grading of the tree, and the 2-valent vertices (if there are any) contribute the off-set term, so the Alexander grading is fine.

For the reverse, start with a tree T satisfying (i)-(v), with some non-zero weight. (If there is no weight, we are back to the case of Theorem 2.2.1, in which case we are done.) The issue is how to “distribute” the weight in the graph so as to make it an operation which gives back T when we read off the elements from each sector (as in the previous paragraph). For each petal weight, we can (by comparing the sequences of idempotents and Alexander gradings of the input elements) find a unique spot in the tree where a U_i is missing between a (multiplied) pair $s_{i-1} \cdot s_i$. Define T_1 by replacing this $s_{(i-1)(i+1)}$ with the sequence (s_{i-1}, U_i, s_i) , and removing the $\mathbf{e}_i$ from the weight. This removes all petal weights from the tree. Now we only have (possibly) internal weights. Notice also that T_1 also satisfies all the conditions (i)-(v) from the statement.

In order to verify that any tree satisfying (i)-(v) gives rise to an allowable graph, we now just need to show that any tree satisfying conditions (i)-(v) *with only $\mathbf{e}_{N+1}$ -weight*, has a (unique) corresponding graph. This is obvious in the base case, which for $N = 3$, looks like



up to choice of root.

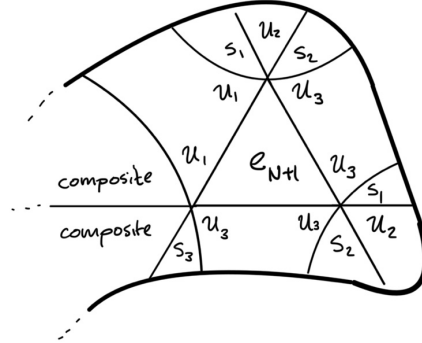
Now assume we have proven this for all $n' < n$, and T_1 is a tree satisfying (i)-(v), with only $\mathbf{e}_{N+1}$ weight and n inputs. Write $(k\mathbf{e}_{N+1}, a_1, \dots, a_n)$ for the weighted input sequence of T_1 . The condition (ii) on n forces there to be at least one *maximal sequence* of basic elements, which we will now define. The meaning of maximal will depend on k . Notice that in the unweighted case, n depended only on the number of vertices (j). In this case, every additional $\mathbf{e}_{N+1}$ drops the number of inputs by 2, because two of the sectors get folded together to make the canonical graph (pictured above in the case $N = 3$). This means that, depending on k , the maximal length sequence of basic elements will take one of the following forms

1. (The chunk corresponds to an unweighted piece of graph) A sequence of basic elements, specifically, a length $2N - 2$ subsequence $a_{i+1}, \dots, a_{i+2N-1}$ which was the middle $2N - 2$ elements of a cyclic permutation of $U_1, s_1, \dots, U_N, s_N$ (e.g, in the case $N = 3$, one such sequence would be s_2, U_3, s_3, U_1).

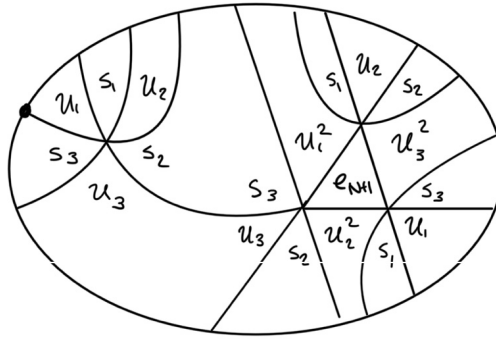
If we are dealing with this case again, the subsequence determines a labeled graph with a single $2N$ -valent vertex. We can excise the subsequence from $a_1, \dots, a_n$ as we did in the proof of Theorem 2.2.1 to get a shorter sequence $(ke_{N+1}, b_1, \dots, b_{n-(2N-2)})$ which still satisfies the conditions (i)-(v), and being of length $< n$, corresponds to a graph (with internal cycles). Attaching the smaller subgraph corresponding to the excised subsequence to the correct edge as we did there, we have a graph for T_1 , which is clearly unique.

2. (The chunk corresponds to a piece of the graph containing internal cycle) The other option is that the maximal sequence of basic elements (the extremal portion of the tree) is a subsequence, of length either $N(2N-2)-2$ or $N(2N-2)-3$, which consists of the middle terms of a cyclic permutation of the sequence from the outside of the basic graph for internal weight. In the case $N=3$, this basic graph is pictured above.

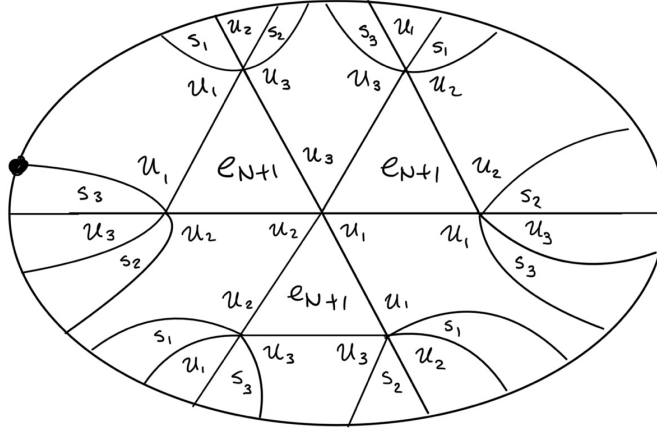
We can use this subsequence to label the outer sectors of the basic graph for internal weight, except for two or three (adjacent) ones, so for instance



which could appear if this internal cycle was attached to an unweighted basic chunk, e.g.



On the other hand, this maximal sequence could be much shorter, as in the case



Here, the maximal sequence is of length $7 = N(2N-2) - (2N-1)$ (so we lost all of the possible labels around the vertex attached to the rest of the graph).

This is what was meant by the ambiguity in the length of the subsequence. Viewed on the level of the graph we are going to construct, if this “extremal” chunk of the graph is attached to an unweighted bit, then we are looking at the first picture, and the subsequence is of length $N(2N-2) - 2$ – we only lose the two labels attached to the adjacent unweighted bit. On the other hand, if it is directly attached to one or more internally weighted chunks, the maximal sequence of unaffected elements (which look like a sequence from the basic internally weighted pictures) becomes shorter, down to $N(2N-2) - (2N-1)$. This is not an issue, however, since in each case, we can still retrieve enough information about this extremal bit to excise it and reduce to a shorter graph, which is what we want.

Now write the subsequence as $a_{i+1}, \dots, a_{i+m-1}$. The condition on the idempotents forces a_i and a_{i+m} to be the product of the two missing elements from the canonical internal weight chunk, multiplied, on the left and the right respectively, by some other elements a'_i and a'_{i+m} . There are two subcases to consider, corresponding to the two pictures above.

In the first subcase, the initial idempotent of a'_i and the final idempotent of a'_{i+m} match up (and $m = N(2N-2) + 1$) and we can consider the tree T_2 with weighted input sequence

$$((k-1)\mathbf{e}_{N+1}, a_1, \dots, a_{i-1}, a'_i, a'_{i+m}, a_{i+m+1}, \dots, a_n),$$

and output V_0^{j-N} . This will still satisfy (i)-(v), and since it satisfies the inductive hypothesis, it gives back a well-defined graph. Attaching the chunk from the first picture above to the correct edge of this graph, we get back a graph for T_1 .

In the second subcase, the idempotents of a'_i and a'_{i+m} do not match up (because, from the perspective of the graph we are trying to construct, there is an s_j between them), but they will be of the form

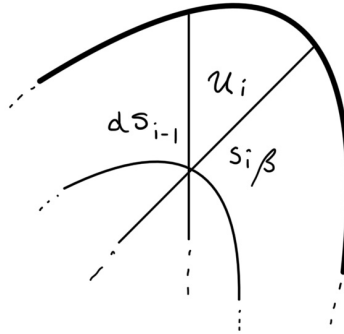
$$\begin{aligned} a'_i &= U_{j-1}^p \\ a'_{i+m} &= U_j^q, \end{aligned}$$

where $p, q \in \mathbb{N}$, and $j, (j-1)$ are counted mod N . In this case, we consider the tree T_2 with weighted input sequence

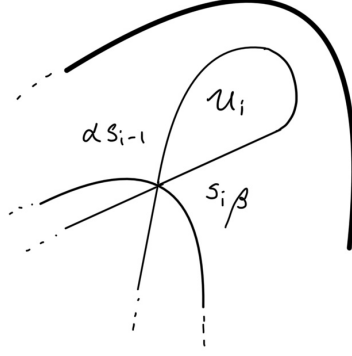
$$((k-1)\mathbf{e}_{N+1}, a_1, \dots, a_{i-1}, a'_i, s_j, a'_{i+m}, a_{i+m+1}, \dots, a_n).$$

and output $V_0^{j-(N-1)}$. This still satisfies (i)-(v) and is of length strictly less than n , so we have a graph for T_2 . Attaching the chunk from the second picture above to the correct edge of this graph, we get back a well-defined graph for T_1 .

This now gives us a way to associate a graph to any internally weighted tree T_1 satisfying (i)-(v). To conclude, we recall how T_1 was initially obtained from our original T (which included petal weights in addition to internal weights). In place of each petal weight $\mathbf{e}_i$ (and corresponding $\alpha s_{(i-1)(i+1)}\beta$) we inserted an $\alpha s_{i-1}, U_i, s_i\beta$ into the sequence in the proper place. (There could be other terms multiplying the $s_{(i-1)(i+1)}$, but this is the only relevant part.) This corresponds to a portion of the graph for T_1 that looks like



Replacing each such section (which originally corresponded to a petal weight) with



The input sequence corresponding to this new graph is the same as the one for T_1 , except with each subsequence $\alpha s_{i-1}, U_i, s_i\beta$ (which originally corresponded to a petal weight) replaced with $\alpha s_{(i-1)(i+1)}\beta$, with an extra $\mathbf{e}_i$ in the weight; in other words, it is precisely the original T . $\square$

Next, we discuss the $\mathcal{A}_\infty$ -relations. There are, in this case, three ways to add an edge to a tree T : a push, a pull, and a split. First, we have the following analogue of Lemma 2.2.4:

Lemma 2.2.8. *A weighted tree T admits a pull precisely when it satisfies conditions (i), (iii), and (iv) from Theorem 2.2.7, and in place of (ii), satisfies*

(ii') $n = j(2N - 2) + 3 - 2k$, where $j \in \mathbb{Z}_{\geq 1}$, n is the number of inputs, and k is the total magnitude of the weight of T ;

In this case, T admits exactly one pull, and T can be expressed uniquely as an augmented graph. In addition, each augmented graph corresponds to a tree satisfying these conditions.

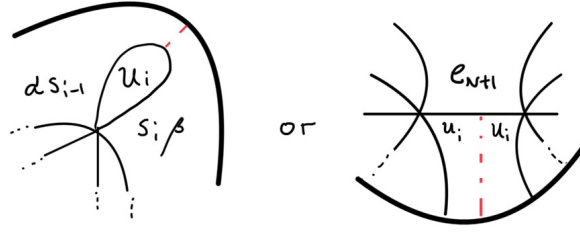
This condition is exactly the same as Lemma 2.2.4, except that (ii)' is modified to fit the weighted case. The proof of the part about pulls is obvious, and is omitted. The only thing that needs verification is the part about augmented graphs, but this follows exactly as in the proof of Lemma 2.2.4, and can also be omitted.

In order to verify the $\mathcal{A}_\infty$ relations, we just need to show that each split or push corresponds to an augmented graph, and that each augmented graph corresponds to a split or push, but not both. The second part is easier:

Lemma 2.2.9. *Every augmented graph gives rise to exactly one split or one push, but not both.*

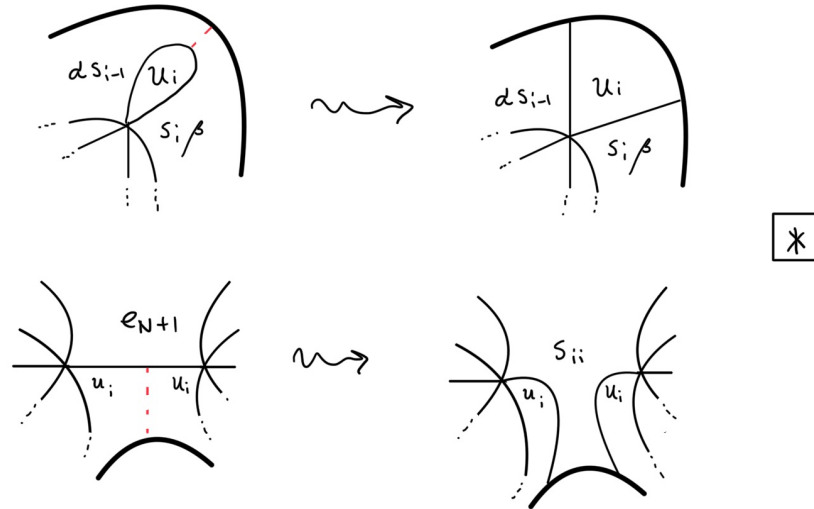
Proof. Choose an augmented graph G of the first kind. The dotted line in G be draw in one of five ways. First we have the three options from the unweighted case – the dotted line going from a true internal edge to the boundary, or from the initial edge to the boundary in the left extended case,

and from the final edge to the boundary in the right extended case. We also have the following two new options:

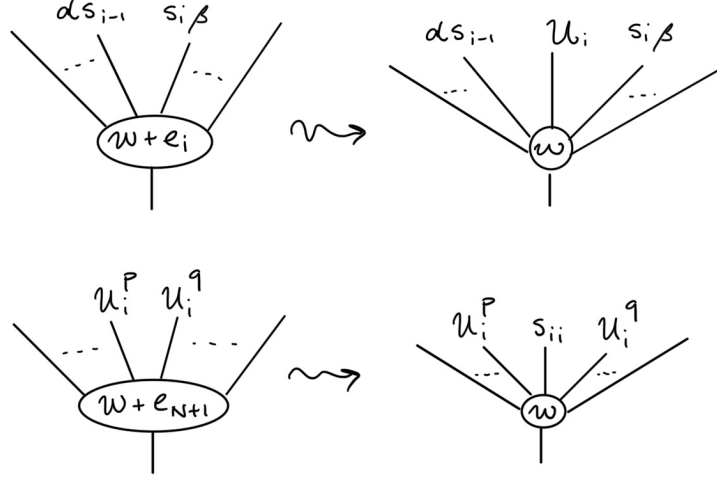


To get the pull from these diagrams (as we did in Lemma 2.2.8) we just erase the dotted line. For each augmented graph, there is a corresponding tree given by reading off the elements clockwise around the boundary from the root in the usual way, and counting the dotted line as an unmultiplied multipliable pair. For the graphs above, the multipliable pairs are $(\alpha s_{i-1}, s_i\beta)$ and (U_i, U_i) , respectively. As per Lemma 2.2.8, these trees satisfy the conditions (i), (iii), and (iv), and have one too many inputs, corresponding to the multipliable pairs pictured above.

For our present situation, we look back at the original augmented graph, and “open out” the edge abutting the dotted line, in the following way



In terms of trees, these are pushes, that is



By the remark at the end of the previous paragraph, these are bona fide operations. In the first case, this follows by a counting argument as in the proof of the $\mathcal{A}_\infty$ -relations for the unweighted case. For the other two, this is just because we have balanced out the Maslov grading by deleting a weight term and adding an input, without messing up any of the other conditions.

That the second kind of augmented graph always gives rise to the same specific kind of split follows exactly as in the unweighted case. $\square$

Now we need to verify the other direction, namely:

Lemma 2.2.10. *Every split or push gives rise to a unique augmented graph.*

Remark 2.2.11. Graphically, pushing a cycle out of an augmented graph looks like the two starred pictures from the proof of the previous lemma. The goal of the “push” part of the proof of Lemma 2.2.10 is to retrieve this picture from the tree for a push.

Proof. That a split gives rise to a unique augmented graph follows as in the proof of Theorem 2.2.3; the weight has no effect on the proof.

For the case of pushing out a basic element of weight (i.e. $\mathbf{e}_i$ for $1 \leq i \leq N + 1$) we need to remember what exactly it means for a tree T to admit a push. Writing the weighted input sequence for T as $(\mathbf{w}, a_1, \dots, a_n)$, T admits a push (of weight $\mathbf{e}_i$) if and only if we can find $1 \leq j \leq n$ so that the tree T' with the same output as T , and input sequence

$$(\mathbf{w} - \mathbf{e}_i, a_1, \dots, a_{j-1}, U_i, a_j, \dots, a_n) \quad (2.16)$$

satisfies (i)-(v) from Theorem 2.2.7 and is therefore an operation. Suppose first that $1 \leq i \leq N -$ that is, we are pushing out petal weight. Then in order for the sequence from (2.16) to determine

an operation, we need $a_{j-1} = \alpha s_{i-1}$ and $a_j = s_i \beta$, for $\alpha, \beta \in \mathcal{A}$, left-multipliable with s_{i-1} and right-multipliable with s_i , respectively. This means that the original weighted input sequence for T is:

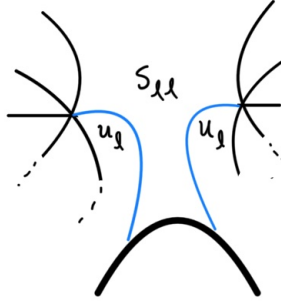
$$(\mathbf{w}, a_1, \dots, s_{i-1}, s_i, \dots, a_n). \quad (2.17)$$

It must still satisfy (i), (iii), (iv), and (v) from Theorem 2.2.7, and comparing the sequences from (2.16) and (2.17), we see that it must also satisfy (ii)' from Lemma 2.2.8. This means that T corresponds to a unique augmented graph.

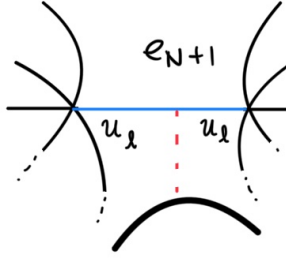
Now, consider the case where $i = N + 1$. Then since the idempotents in the sequence from (2.16) work out, and it contains no multipliable pairs, there must be some $1 \leq k \leq N$, as well as $p, q \in \mathbb{N}$, so that $a_{j-1} = U_\ell^p$, $a_j = U_\ell^q$. Remembering that U_{N+1} is actually a sum of one term in each idempotent, and in any given situation all the terms except the one in the correct idempotent will drop out, the weighted input sequence for T' looks like

$$(\mathbf{w} - \mathbf{e}_{N+1}, a_1, \dots, U_\ell^p, s_{\ell\ell}, U_\ell^q, \dots, a_n) \quad (2.18)$$

This determines a well defined graph, and the portion of this graph corresponding to the $U_\ell^p, s_{\ell\ell}, U_\ell^q$ segment looks like



(Here, the coloring on the two bottom edges is to facilitate the next part of the proof, and does not indicate anything else.) To get an augmented graph for T , zip the two colored edges together, and add a dotted edge out to the boundary, as



This clearly gives rise to a tree, say T'' . Zipping these edges together (and adding the dotted edge, to keep the two powers of U_ℓ separate) added an $\mathbf{e}_{N+1}$ -weight and deleting the $s_{\ell\ell}$, but did not change anything of the other inputs or weight from the bona fide suitable graph for T' . This means that these are the only ways T'' differs from T' , that is, $T'' = T$, and we have found an augmented graph for T , as desired. This is clearly unique, because the place where we are allowed to push out the $\mathbf{e}_{N+1}$ is well-defined, which completely determines T' , and hence the augmented graph. Thus, the proof is complete. $\square$

Now, recall that by definition, all terms of $\mathcal{A}_\infty$ -relation corresponding to a given tree T vanish identically unless T admits a push, a split, or a pull. Combining Theorem 2.2.3 with Lemmas 2.2.8, 2.2.9, and 2.2.10, we have now shown that in all cases when one term of the $\mathcal{A}_\infty$ -relation for some T is non-vanishing, there is a unique cancelling term, that is:

Theorem 2.2.12. *$\mathcal{A}$ satisfies the $\mathcal{A}_\infty$ -relations, and is a bona fide $\mathcal{A}_\infty$ -algebra.*

2.3 The β -bordered algebra $\mathcal{B}$

2.3.1 Algebra elements and basic operations

The algebra $\mathcal{B}$ (which will be Koszul dual to $\mathcal{A}$, as shown in Section 2.8) is constructed as follows. $\mathcal{B}$ is an $R = \mathbb{F}[V_0, \dots, V_{N+1}]$ -module with generators $\{I_i\}_{i=1}^N$, $\{\rho_i\}_{i=1}^N$, and $\{\sigma_i\}_{i=1}^N$, where the I_i are

called idempotents. Simple multiplication (i.e. μ_2) is defined by

$$\rho_i \rho_j = 0 \text{ for each } i, j$$

$$\sigma_i \sigma_j = 0 \text{ for each } i, j$$

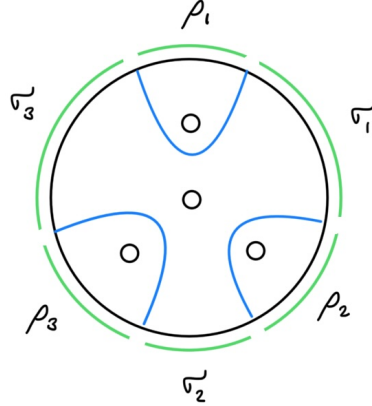
$$I_i^2 = I_i$$

$$I_i I_j = 0 \text{ for each } i \neq j$$

$$I_i \rho_i = \rho_i I_i = \rho_i$$

$$I_{i+1} \sigma_i = \sigma_i I_i = \sigma_i$$

Heuristically, we take the last two lines of this to mean that the ρ_i each live in the i -th idempotent (i.e. the initial and final idempotents are both i) and each σ_i has initial idempotent i and final idempotent $i + 1$. This makes geometric sense, since the ρ_i and σ_i represent Reeb chords on the star diagram:



For each i , ρ_i stays above the i -th spoke, while the σ_i starts roughly above the i th spoke and ends in the $(i + 1)$ st spoke.

We also assume the V_i (in the ground ring) are central elements which can be multiplied by anything.

Notice that according to the rules above, two basic elements can be multipliable² only if the initial idempotent of the first is the final idempotent of the second, with idempotents counted from right to left. So $\rho_i \sigma_i = 0$ for each i , but $\sigma_i \rho_i \neq 0$. Starting / ending in the same idempotent is not, however, enough to guarantee that two elements are multipliable. For instance, $\rho_i \rho_i = \sigma_{i+1} \sigma_i = 0$.

²As in the previous section, two algebra elements are said to be *multipliable* or *can be multiplied* if and only if they give non-zero product.

We also define

$$U_0 := \sum_{i=1}^N \rho_i \sigma_i \rho_{i+1} \sigma_{i+1} \cdots \rho_{i+N-1} \sigma_{i+N-1} \\ + \sum_{i=1}^N \sigma_i \rho_{i+1} \sigma_{i+1} \rho_{i+2} \cdots \sigma_{i+N-1} \rho_i$$

where, in the last definition, all indices are counted mod N . Geometrically, U_0 is the sum of all the long chords around the outer boundary circle.

We define $d\rho_i = V_i$ for each $1 \leq i \leq N$, and one weighted μ_0 , namely

$$\mu_{\bullet}^{\vec{e}_0} = \mathcal{U}_{\bullet} : \begin{array}{c} \textcircled{e_0} \\ | \\ \mathcal{U}_{\bullet} \end{array}$$

There are no other weighted operations. (Formally, weights live in the space $\{n \cdot \mathbf{e}_0 : n \in \mathbb{Z}_{\geq 0}\}$, but since $\mu_0^{\mathbf{e}_0}$, we do not usually worry about this.)

Since $dU_0 = 0$ by direct computation, all $\mathcal{A}_{\infty}$ relations for weighted operations are satisfied. This weighted μ_0 does not arise in any way in any of the $\mathcal{A}_{\infty}$ -relations for higher multiplication defined below, since all higher multiplications are unweighted. Therefore, we make no further comment about it in the subsection that follows, in which we prove the $\mathcal{A}_{\infty}$ relations for unweighted operations.

2.3.2 Higher operations and $\mathcal{A}_{\infty}$ -relations

We also define additional nonzero μ_k for $1 \leq k \leq N$, according to the following rules.

Let a sequence of algebra elements $(\tau_k, \dots, \tau_1)$ in $\mathcal{B}$ be *in an allowable sequence of idempotents* if the initial idempotent of τ_{i+1} equals the final idempotent of τ_i , for each i .

Define the *length of an algebra element* $\tau \in \mathcal{B}$ by decomposing it into a product of σ s and ρ s: this decomposition is unique, and the length of τ is defined to be the number of σ s that appears in this product. (We do not count the V_i 's when determining the length because they are elements of the ground ring.) We then define the *total length* of $(\tau_k, \dots, \tau_1)$ as the sum of the lengths of each of the individual elements,

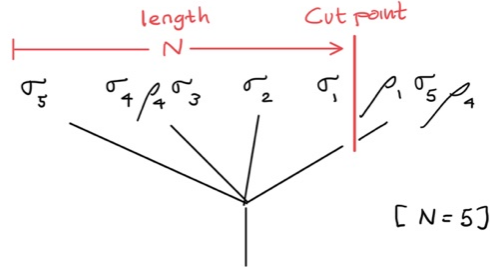
$$\ell(\tau_k, \dots, \tau_1) = \sum_{j=1}^k \ell(\tau_j).$$

We say that a sequence $(\tau_k, \dots, \tau_1)$ with $\ell(\tau_k, \dots, \tau_1) \geq N$ *stretches too far* if

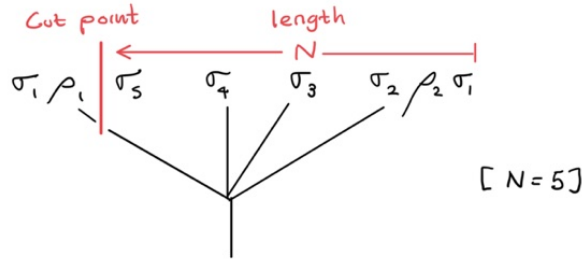
(S1) for some $j > 1$, the sequence $(\tau_k, \dots, \tau_j)$ is of total length $\geq N$, and

(S2) for some $j < k$, the sequence $(\tau_j, \dots, \tau_1)$ is of total length $\geq N$;

If either of these two fails, then we can define a notion called *the cut point* in the sequence $\tau_k, \dots, \tau_1$, which will be useful in proving the $\mathcal{A}_\infty$ relations. Suppose that (S1) does not hold. Then counting backwards from the final idempotent of τ_k , the point at which we have backed up exactly N idempotets will be somewhere in the decomposition (into ρ s and σ s) of τ_1 . (Otherwise, there would be a $j > 1$ such that $(\tau_k, \dots, \tau_j)$ was of length N , that is, $(\tau_k, \dots, \tau_1)$ would satisfy (S1).) We define this to be the *cut point*. For instance:



is the cut point in one case where (S1) fails. If (S2) fails, then counting forwards from the initial idempotent of τ_1 , the point at which we have gone exactly N steps is somewhere in the decomposition of τ_k . In this case we let this be the *cut point*. For instance



is the cut point for one cases where (S2) fails. If both (S1) and (S2) fail – that is, $(\tau_k, \dots, \tau_1)$ does not stretch too far no matter which direction we count from – we have ambiguity in the definition of the cut point. In this case, we just define it counting back from the final idempotent of τ_k . After we have defined the operations, we will show that this ambiguity does not matter.

We say that $(\tau_k, \dots, \tau_1)$ is *left flanked* if $\tau_k = \rho_{\lambda'} \tau'_k$, where λ' is the final idempotent of τ_k . The sequence is *right flanked* if $\tau_1 = \tau'_1 \rho_\lambda$, where λ is the initial idempotent of τ_1 . It is *doubly flanked* if it is both left and right flanked.

A sequence is *impermissibly flanked* if

(F1) $(\tau_k, \dots, \tau_1)$ is left-flanked, and satisfies (S2) but not (S1);

(F2) $(\tau_k, \dots, \tau_1)$ is right-flanked, and satisfies (S1) but not (S2);

(F3) $(\tau_k, \dots, \tau_1)$ is doubly flanked;

The difficulty in each of these cases, is that the cut point would have to be defined counting from the same end on which the sequence is flanked. As we will see later, impermissible flanking always forces $\mu_k(\tau_k, \dots, \tau_1) = 0$.

More generally, a sequence $(\tau_k, \dots, \tau_1)$ with $\ell(\tau_k, \dots, \tau_1) \geq N$ is said to be *allowable* only when

- (i) $k \leq N$;
- (ii) $(\tau_k, \dots, \tau_1)$ is in an allowable sequence of idempotents, meaning in particular, no jumps;
- (iii) The sequence does not stretch too far;
- (iv) The sequence is not impermissibly flanked;
- (v) Each τ_j ($1 < j < k$), is of the form

$$\sigma_{\lambda_j} \rho_{\lambda_j} \sigma_{\lambda_j-1} \cdots \rho_{\lambda_{j-1}+1} \sigma_{\lambda_{j-1}+1}$$

where we may possibly have $\lambda_j = \lambda_{j-1} + 1$, i.e. $\tau_j = \sigma_{\lambda_j}$, and we also require

$$\tau_1 = \begin{cases} \sigma_{\lambda_1} \rho_{\lambda_1} \sigma_{\lambda_1-1} \cdots \rho_{\lambda_0+1} \sigma_{\lambda_0} \\ \text{or} \\ \sigma_{\lambda_1} \rho_{\lambda_1} \sigma_{\lambda_1-1} \cdots \rho_{\lambda_0+1} \sigma_{\lambda_0} \rho_{\lambda_0} \end{cases}$$

and

$$\tau_k = \begin{cases} \sigma_{\lambda_k} \rho_{\lambda_k} \sigma_{\lambda_k-1} \cdots \rho_{\lambda_{k-1}+1} \sigma_{\lambda_{k-1}+1} \\ \text{or} \\ \rho_{\lambda_k+1} \sigma_{\lambda_k} \rho_{\lambda_k} \sigma_{\lambda_k-1} \cdots \rho_{\lambda_{k-1}+2} \sigma_{\lambda_{k-1}+1} \end{cases}$$

For instance, with $N = 3$, allowable sequences include

$$(\sigma_3, \sigma_2, \sigma_1), (\rho_2 \sigma_1, \sigma_3, \sigma_2) \\ (\sigma_2, \sigma_1 \rho_1 \sigma_3 \rho_3 \sigma_2), \text{ and } (\rho_1 \sigma_3 \rho_3 \sigma_2 \rho_2 \sigma_1),$$

but not

$$(\sigma_2 \rho_2 \sigma_1), (\rho_1 \sigma_3, \sigma_2, \sigma_1 \rho_1)$$

$$(\sigma_1 \rho_3, \sigma_3, \sigma_2), \text{ or } (\rho_1, \rho_1, \rho_1),$$

etc.

By contrast, a sequence $(\tau_k, \dots, \tau_1)$ with $\ell(\tau_k, \dots, \tau_1) < N$ is allowable if and only if $k = 2$.

When defining μ_k s, there are two general cases to consider.

Case 1: $\ell(\tau_k, \dots, \tau_1) < N$. Then we have only μ_2 s and μ_1 s. In this case, we have the following lemma:

Lemma 2.3.1. (a) $d(\sigma_j \rho_j \sigma_{j-1} \cdots \rho_{i+1} \sigma_i) = 0$ for all $i \leq j$ and where the product alternates between ρ_λ 's and σ_λ 's in decreasing order;

(b) $d(\sigma_j \rho_j \cdots \sigma_i \rho_i) = \sigma_j \rho_j \cdots \sigma_i V_i$ for each $i \leq j$, with alternating product as in (a);

(c) $d(\rho_i \sigma_{i-1} \cdots \rho_{j+1} \sigma_j) = V_i \sigma_{i-1} \cdots \rho_{j+1} \sigma_j$, for each $j < i$, product alternating as above;

(d) $d(\rho_j \sigma_{j-1} \cdots \rho_{i+1} \sigma_i \rho_i) = V_j \sigma_{j-1} \cdots \rho_{i+1} \sigma_i \rho_i + \rho_j \sigma_{j-1} \cdots \rho_{i+1} \sigma_i V_i$.

(e) $d(\tau V) = (d\tau)V$ where τ is a product of ρ 's and σ 's and V is some product of $\{V_i\}_{i=1}^N$.

Proof of Lemma 2.3.1. This is by induction on the number of terms in the product (which could be more than $j - i$ since we are counting mod N). $\square$

Lemma 2.3.1 suffices to verify the $\mathcal{A}_\infty$ relation for any tree with inputs $\tau_k, \dots, \tau_1$ such that $\ell(\tau_k, \dots, \tau_1) < N$. In the remaining cases, we will verify $\mathcal{A}_\infty$ -relations for trees labelled with a sequences of length $\geq N$.

Case 2: $\ell(\tau_k, \dots, \tau_1) \geq N$ and $k = 1$ – that is, we are dealing with a differential. This differential, $d\tau$, is defined to be non-zero precisely in the following cases:

Case 2.1 τ is unflanked;

Write $\tau = \sigma_j \rho_j \sigma_{j-1} \cdots \rho_{i+1} \sigma_i$. Then (cutting from the left and the right, respectively) we can write

$$\tau = \begin{cases} \underbrace{(\sigma_j \rho_j \cdots \rho_{j+2} \sigma_{j+1})}_{\text{length } N} \cdot \underbrace{\rho_{j+1} \sigma_{j+1} \cdots \rho_{i+1} \sigma_i}_{\text{length } \ell(\tau) - N} & \text{from the left} \\ \underbrace{\sigma_j \rho_j \cdots \sigma_i \rho_i}_{\text{length } \ell(\tau) - N} \cdot \underbrace{(\sigma_{i+N-1} \rho_{i+N-1} \cdots \rho_{i+1} \sigma_i)}_{\text{length } N} & \text{from the right} \end{cases} \quad (2.19)$$

Then we define

$$d\tau = \prod_{\substack{\lambda \neq (j+1) \\ N \text{ total Vs}}} V_\lambda \cdot \underbrace{(\rho_{j+1}\sigma_{j+1} \cdots \rho_{i+1}\sigma_i)}_{\text{length } \ell(\tau)-N} + \prod_{\substack{\lambda \neq i \\ N \text{ total Vs}}} V_\lambda \cdot \underbrace{(\sigma_j\rho_j \cdots \sigma_i\rho_i)}_{\text{length } \ell(\tau)-N} \quad (2.20)$$

Case 2.2 τ is left-flanked;

Write $\tau = \rho_{j+1}\sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i$. Then we cut from the right, and write

$$\tau = \underbrace{\rho_{j+1}\sigma_j\rho_j \cdots \sigma_i\rho_i}_{\text{length } \ell(\tau)-N} \cdot \underbrace{(\sigma_{i+N-1}\rho_{i+N-1} \cdots \rho_{i+1}\sigma_i)}_{\text{length } N}$$

and define

$$d\tau = V_{j+1} \cdot \underbrace{\sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i}_{\text{length } \ell(\tau)} + \prod_{\substack{\lambda \neq i \\ N \text{ total Vs}}} V_\lambda \cdot \underbrace{(\rho_{j+1}\sigma_j\rho_j \cdots \sigma_i\rho_i)}_{\text{length } \ell(\tau)-N} \quad (2.21)$$

Case 2.3 τ is right-flanked;

Write $\tau = \sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i\rho_i$. Then we cut from the left, and write

$$\tau = \underbrace{(\sigma_j\rho_j \cdots \rho_{j+2}\sigma_{j+1})}_{\text{length } N} \cdot \underbrace{\rho_{j+1}\sigma_{j+1} \cdots \rho_{i+1}\sigma_i\rho_i}_{\text{length } \ell(\tau)-N}$$

and define

$$d\tau = V_i \cdot \underbrace{\sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i}_{\text{length } \ell(\tau)} + \prod_{\substack{\lambda \neq (j+1) \\ N \text{ total Vs}}} V_\lambda \cdot \underbrace{(\rho_{j+1}\sigma_{j+1} \cdots \rho_{i+1}\sigma_i\rho_i)}_{\text{length } \ell(\tau)-N} \quad (2.22)$$

Case 2.4 τ is doubly flanked;

Write $\tau = \rho_{j+1}\sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i\rho_i$. In this case, we just define

$$d\tau = V_{j+1} \cdot \sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i\rho_i + V_i \cdot \rho_{j+1}\sigma_j\rho_j\sigma_{j-1} \cdots \rho_{i+1}\sigma_i. \quad (2.23)$$

Remark 2.3.2. Essentially, in both cases 2.2 and 2.3, the first term of the differential is calculated as in the $\text{length} < N$ case, and the second term is calculated analogous to the more general $\text{length} \geq N$ case, by at a point $\text{length } N$ from the unflanked side differentiating that $\text{length}-N$ piece, and leaving the rest fixed.

In Case 2.1, there is no analogous nonzero differential from the length $< N$ case, so both terms are by analogy to the length $\geq N$, $k > 1$ cases from below.

In Case 2.3, there is no analogous non-zero differential from the length $\geq N$, $k > 1$ cases, so both terms are calculated as they would be if $\ell(\tau) < N$.

Case 3: $\ell(\tau_k, \dots, \tau_1) \geq N$, $k > 1$. If $(\tau_k, \dots, \tau_1)$ is not allowable, then $\mu_k(\tau_k, \dots, \tau_1) = 0$. If it is allowable, then we define $\mu_k(\tau_k, \dots, \tau_1)$ in the following way.

Case 3.1: $(\tau_k, \dots, \tau_1)$ is unflanked and stretches too far from the left, but not the right;

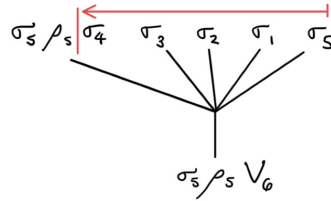
Then we can compute the cut-point counting up from the right, and write

$$(\tau_k, \dots, \tau_1) = (\tau'_k | \sigma_{i+N-1} B_{i+N-1} \cdots B_{i+1} \sigma_i) \quad (2.24)$$

where $|$ is the cut point, i is the initial idempotent of τ_1 , each $B_j = \rho_j$ or denotes a comma, but not both, and τ'_k is the leftover part of the decomposition of τ_k to the left of the cut-point. In this case, we define

$$\mu_k(\tau_k, \dots, \tau_1) = \tau'_k \cdot V_{N+1} \prod \{V_j : B_j = \rho_j\} \quad (2.25)$$

For instance, when $N = 5$, we could have



Case 3.2: $(\tau_k, \dots, \tau_1)$ is unflanked and stretches too far from the right, but not the left;

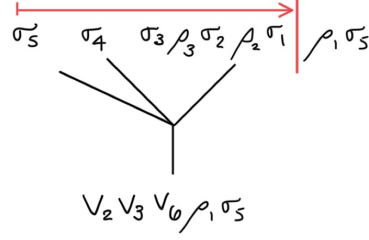
Then the cut-point is defined counting down from the final idempotent of τ_k . This means we can write

$$(\tau_k, \dots, \tau_1) = (\sigma_i B_i \sigma_{i-1} B_{i-1} \cdots B_{i-N+2} \sigma_{i-N+1} | \tau'_1), \quad (2.26)$$

where $i + 1$ is the final idempotent of τ_k , the $|$ denotes the cut point, each B_j is either a comma or ρ_j (but not both), and τ'_k is the leftover part of the decomposition of τ_1 which lies to the right of the cut point. We then define

$$\mu_k(\tau_k, \dots, \tau_1) = V_{N+1} \prod \{V_j : B_j = \rho_j\} \cdot \tau'_1. \quad (2.27)$$

For instance, with $N = 5$, we have



Case 3.3: $(\tau_k, \dots, \tau_1)$ is unflanked and does not stretch too far in either direction;

Then we can write $(\tau_k, \dots, \tau_1)$ as in (2.24) or as in (2.26). In fact, in this case the decompositions given by counting the cut from the left versus from the right differ in a very predictable way:

Lemma 2.3.3. *Suppose $k \geq 1$, $\ell(\tau_k, \dots, \tau_1) \geq N$, and the sequence is unflanked, as above. Then we can count the cut both from the left and from the right, to get*

$$(\tau_k, \dots, \tau_1) = \begin{cases} (\sigma_{j-1} B_{j-1} \sigma_{j-2} \cdots B_{j+1} \sigma_j | \tau'_1) & \text{(from the left)} \\ (\tau'_k | \sigma_{i+N-1} B'_{i+N-1} \sigma_{i+N-2} \cdots B'_{i+1} \sigma_i) & \text{(from the right)} \end{cases} \quad (2.28)$$

where $i < j \bmod N$ are the initial and final idempotents of the sequence, respectively.

Writing the sequence in this way, $B_\lambda = B'_\lambda$ for each λ which appears more than once, and in fact, writing

$$\Lambda = \sigma_{j-1} \rho_{j-1} \cdots \rho_{i+1} \sigma_i,$$

we have

$$\tau'_1 = \rho_j \Lambda, \quad \tau'_k = \Lambda \rho_i. \quad (2.29)$$

Remark 2.3.4. The statement of Lemma 2.3.3 is true even when $k = 1$ (i.e. we are dealing with a μ_1) not just in the current case. However, in that case, it is patently obvious, so we exclude that from the proof.

Proof. First, note that on the overlap of the σ/B parts of the two decompositions, the B_λ and B'_λ have to match, because these are just decompositions of the same sequence. Hence, if $(\tau_k, \dots, \tau_1)$ is of length N (i.e the “overlap” is the whole sequence) then τ'_1 and τ'_k are trivial, and the two ways of writing the sequence in (2.28) are actually the same. Next, suppose $\ell(\tau_k, \dots, \tau_1) > N$ (i.e. suppose we have B_λ other than those in the overlap of the two decompositions from (2.28)). Because the total sequence does not stretch too

far in either direction, it is either a μ_1 – in which case *all* B_λ s and B'_λ s are ρ_λ s – or it is of length $\leq 2N$. Hence, every idempotent appears at most twice in this sequence.

Suppose now that λ is *not* in the overlap, but that there is both a B_λ and a B'_λ . This means $\lambda < j \bmod N$ and $\lambda > i \bmod N$; we put in the exclusive bound because if e.g. $\lambda = j$, then there is a B'_j in the 2nd decomposition of (2.28), but no B_j the first, because that is one step past the final idempotent of τ_k , and likewise when $\lambda = i$. Because the sequence does not stretch too far in either direction, when λ appears on the right, it is part of the right-most term of the sequence (otherwise $(\tau_k, \dots, \tau_1)$ would satisfy (S1)), and when it appears on the left, it is part of the left-most term (otherwise $(\tau_k, \dots, \tau_1)$ would satisfy (S2)). In terms of idempotents, this means that:

$$\begin{aligned} i < \lambda \leq i' \bmod N; \\ j' \leq \lambda < j \bmod N, \end{aligned}$$

where i' denotes the final idempotent of τ_1 and j' denotes the initial idempotent of τ_k , respectively. Moreover, if $\lambda = i'$, then the sequence $(\tau_k, \dots, \tau_2)$ would be of length $\geq N$, i.e. $(\tau_k, \dots, \tau_1)$ would stretch too far from the left, and likewise, if $\lambda = j'$, then $(\tau_k, \dots, \tau_1)$ stretches too far from the right. Hence, we actually have

$$i < \lambda < i' \bmod N \text{ and } j' < \lambda < j \bmod N. \quad (2.30)$$

Now decompose τ_1, τ_k into products of ρ s and σ s. Because (2.30) says that λ appears in the interior of the decompositions of both τ_1 or τ_k , it must correspond to a ρ_λ both in τ_1 and in τ_k . Hence, $B_\lambda = \rho_\lambda$ in the top line of (2.28), where the λ is to the far left, and $B'_\lambda = \rho_\lambda$ in the bottom line, where the λ is to the far right.

We can now apply the decomposition from (2.28) to get

$$\begin{aligned} \tau'_1 &= B_j \sigma_{j-1} B_{j-1} \cdots B_{i+1} \sigma_i \\ &= \rho_j \sigma_{j-1} \rho_{j-1} \cdots \rho_{i+1} \sigma_i \\ \tau'_k &= \sigma_{j-1} B_{j-1} \cdots B_{i+1} \sigma_i B_i \\ &= \sigma_{j-1} \rho_{j-1} \cdots \rho_{i+1} \sigma_i \rho_i \end{aligned}$$

which is precisely (2.29). □

With this done, we define

$$\begin{aligned} \mu_k(\tau_k, \dots, \tau_1) &= \tau'_k V_{N+1} \prod \{V_\lambda : j < \lambda < i \bmod N, B_\lambda = \rho_\lambda\} \\ &\quad + \tau'_1 V_{N+1} \prod \{V_\lambda : j < \lambda < i \bmod N, B_\lambda = \rho_\lambda\} \end{aligned} \quad (2.31)$$

Using Lemma 2.3.3, this becomes

$$\begin{aligned} \mu_k(\tau_k, \dots, \tau_1) &= \Lambda \rho_i \cdot V_{N+1} \prod \{V_\lambda : j < \lambda < i \bmod N, B_\lambda = \rho_\lambda\} \\ &\quad + \rho_j \Lambda \cdot V_{N+1} \prod \{V_\lambda : j < \lambda < i \bmod N, B_\lambda = \rho_\lambda\} \end{aligned} \quad (2.32)$$

So for instance, with $N = 5$, we could have

$$\begin{array}{c} \sigma_5 \rho_5 \sigma_4 \rho_4 \sigma_3 \quad \sigma_2 \rho_2 \sigma_1 \rho_1 \sigma_5 \\ \swarrow \quad \searrow \\ \hline = \overbrace{\sigma_5 \rho_5 \sigma_4 \rho_4 \sigma_3 \quad \sigma_2 \rho_2 \sigma_1}^{\text{red arrow}} \rho_1 \sigma_5 \quad + \quad \sigma_5 \rho_5 \overbrace{\sigma_4 \rho_4 \sigma_3 \quad \sigma_2 \rho_2 \sigma_1}^{\text{red arrow}} \rho_1 \sigma_5 \\ \swarrow \quad \searrow \quad \quad \quad \swarrow \quad \searrow \\ \boxed{\begin{array}{c} \nu_2 \nu_4 \nu_5 \nu_6 \rho_1 \sigma_5 \quad + \quad \nu_1 \nu_2 \nu_4 \nu_6 \sigma_5 \rho_5 \end{array}} \end{array}$$

Case 3.4: $(\tau_k, \dots, \tau_1)$ is left-flanked but of length N ;

Here, we can technically cut from either direction, but the cut-point is trivial since the sequence is of length N . We can just write

$$(\tau_k, \dots, \tau_1) = (\rho_i \sigma_{i+N-1} B_{i+N-1} \sigma_{i+N-2} B_{i+N-2} \cdots B_{i+1} \sigma_i),$$

where again, each B_j is either a comma or a ρ_j but not both. Define

$$\mu_k(\tau_k, \dots, \tau_1) = \rho_i V_{N+1} \prod \{V_j : B_j = \rho_j\}. \quad (2.33)$$

so for instance, in the case $N = 5$, we could have

$$\begin{array}{c} \overleftarrow{\text{red arrow}} \\ \rho_5 \sigma_5 \quad \sigma_4 \quad \sigma_3 \rho_3 \sigma_2 \quad \sigma_1 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \rho_5 \nu_3 \nu_6 \end{array}$$

Case 3.5: $(\tau_k, \dots, \tau_1)$ is right flanked but of length N ;

As in the previous case, we write

$$(\tau_k, \dots, \tau_1) = (\sigma_{i+N-1} B_{i+N-1} \sigma_{i+N-2} B_{i+N-2} \cdots B_{i+1} \sigma_i \rho_i),$$

and define

$$\mu_k(\tau_k, \dots, \tau_1) = \rho_i V_{N+1} \prod \{V_j : B_j = \rho_j\} \quad (2.34)$$

Case 3.6: $(\tau_k, \dots, \tau_1)$ is left-flanked and of length $> N$;

Because $(\tau_k, \dots, \tau_1)$ is not inadmissibly flanked, is not right flanked and it must not stretch too far counting from the right; in this case we write

$$(\tau_k, \dots, \tau_1) = (\tau'_k | \sigma_{i+N-1} B_{i+N-1} \cdots B_{i+1} \sigma_i)$$

as in (2.24), but this time with the understanding that the τ'_k ends with a ρ_j (where j is the final idempotent of τ_k). We then define $\mu_k(\tau_k, \dots, \tau_1)$ using (2.25).

Case 3.7: $(\tau_k, \dots, \tau_1)$ is right-flanked and of length $> N$;

Again, because the sequence is not impermissibly flanked, it must not be flanked on the left, and does not stretch too far from that direction, so we can write $(\tau_k, \dots, \tau_1)$ as in (2.26), but with the understanding that in this case, τ'_1 begins with a ρ , and define

$$\mu_k(\tau_k, \dots, \tau_1) = V_{N+1} \prod \{V_j : B_j = \rho_j\} \cdot \tau'_1.$$

as in (2.27).

The point in Cases 3.4 through 3.7 is that except in the trivial case (when the sequence is of length N , so there is no real cut) we never want to define the cut counting down from the same end of the sequence where the sequence is flanked. Suppose the sequence $(\tau_k, \dots, \tau_1)$ is of length $> N$, and we did define the cut counting from the same direction in which the sequence was flanked. Then

(assuming the sequence did not stretch too far in that direction) we would get a cut that looked like

$$(\tau_k, \dots, \tau_1) = \begin{cases} (\rho_{i+1}\sigma_i B_i \cdots B_{i-N+2}\sigma_{i-N+1} | \tau'_1) & \text{if we had left flanking and counted from the left} \\ (\tau'_k | \sigma_{i+N-1} B_{i+N-1} \cdots B_{i+1}\sigma_i \rho_i) & \text{if we had right flanking and counted from the right} \end{cases}$$

where $\tau'_1 = \rho_{i+1}\omega_1$ and $\tau'_k = \omega_k \rho_i$, for some $\omega_1, \omega_k \in \mathcal{B}$. By analogy with the definitions above, in left flanked case we should have

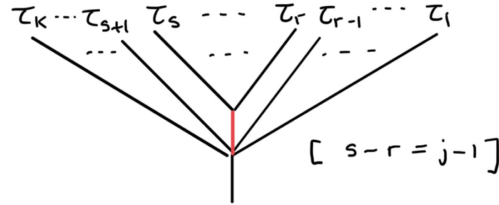
$$\begin{aligned} \mu_k(\tau_k, \dots, \tau_1) &= \rho_{i+1} \cdot V_{N+1} \prod \{V_j : B_j = \rho_j\} \cdot \tau'_1 \\ &= \rho_{i+1} \tau'_1 \cdot V_{N+1} \prod \{V_j : B_j = \rho_j\} \\ &\quad \text{(Because all the } V_\lambda \text{ are central)} \\ &= \rho_{i+1} \cdot \rho_{i+1}\omega_1 \cdot V_{N+1} \prod \{V_j : B_j = \rho_j\} \\ &= 0 \end{aligned}$$

because ρ_λ cannot be multiplied with one another. Likewise, in the right flanked case,

$$\mu_k(\tau_k, \dots, \tau_1) = \tau'_k \cdot V_{N+1} \prod \{V_j : B_j = \rho_j\} \cdot \rho_i = 0.$$

We avoid this by never counting the cut from the same direction a sequence is flanked. This is also why we rule out doubly flanked sequences – because in this case, there is no way to define the cut.

The next step is to verify the $\mathcal{A}_\infty$ relations. Again, the $\mathcal{A}_\infty$ relations, as stated in (2.2), also say that for any tree T labelled with elements of $\mathcal{B}$, the sum of all ways to add an edge to T is zero. Adding an edge corresponds either to a differential, or to pulling together j adjacent elements of the sequence $\tau_k, \dots, \tau_1$ with which the input edges of T are labeled, that is:



So the $\mathcal{A}_\infty$ relation for T counts the number of ways to pull together j elements and get a non-zero composition of algebra operations.

Proposition 2.3.5. $\mathcal{B}$ satisfies the $\mathcal{A}_\infty$ relations, and is therefore a bona fide $\mathcal{A}_\infty$ -algebra.

Proof. Look at a sequence $\tau := (\tau_k, \dots, \tau_1)$. We have two notions of length. The first is k , the actual number of elements in the sequence. The second is $\ell := \ell(\tau)$. If $\ell < N$, then the only operations we have are the usual μ_2 and the μ_1 , so the $\mathcal{A}_\infty$ relations for τ are trivial. For the remainder of the proof, we will deal with only $\ell \geq N$.

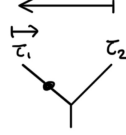
If $k = 1$, then we go through Cases 2.1-2.4, and it is clear that all double differentials vanish.

Suppose now that $k = 2$, $A = \ell(\tau_2)$ and $B = \ell(\tau_1)$. We are assuming with $\ell = A + B \geq N$, so there are the following cases to consider

Case II.1 $A, B < N$: This is just the Leibniz rule, as in the low-length case.

Case II.2 $A \geq N$, $B < N$, $\ell < 2N$; There are 16 cases, corresponding to whether τ_1, τ_2 are flanked, and whether each is left, right, or doubly flanked. The verification in each case follows directly from the rules for high-length products, above.

Case II.3 $A \geq N$, $B < N$, $\ell \geq 2N$; Same situation as Case II.2. The only difference is that since $\ell \geq 2N$, the relations may involve terms of the form



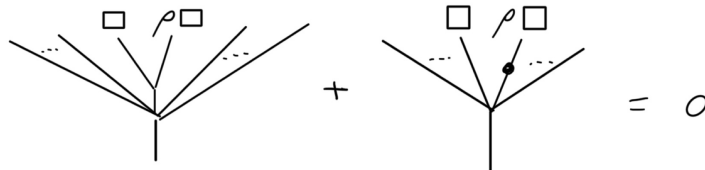
in the case where τ_1 is not right flanked and τ_2 is not left-flanked. Here, again, the arrow denotes the direction in which the cut point is computed if the operation is of length

Case II.4 $A < N$, $B \geq N$, $\ell < 2N$; Analogous to Case II.2;

Case II.5 $A < N$, $B \geq N$, $\ell \geq 2N$; Analogous to Case II.3

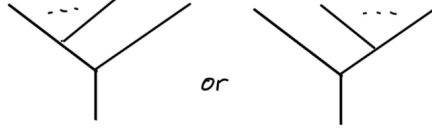
Case II.6 $A, B \geq N$ (so $\ell \geq 2N$ is implied); This is the straight Leibniz rule again, since τ stretches too far in both directions so high-length products are all trivial;

Now suppose $k > 2$. Then there are three cases to consider. The first option is that τ contains a multipliable pair (in the traditional sense). No high-length multiplication can involve this pair; hence, the $\mathcal{A}_\infty$ relation is trivial, unless this is the only multipliable pair, in which case it looks like



Here, we assumed without loss of generality that the right term of the multipliable pair was the one with an exposed ρ ; the other case is analogous.

The second option is that one of the terms in the $\mathcal{A}_\infty$ relation has the second operation a μ_2 , i.e. one of the terms looks like



We are going to verify the $\mathcal{A}_\infty$ relations for cases in which a term like the left picture appears, since the other situation is completely analogous. In what follows, we will the internal lengths in this diagram as

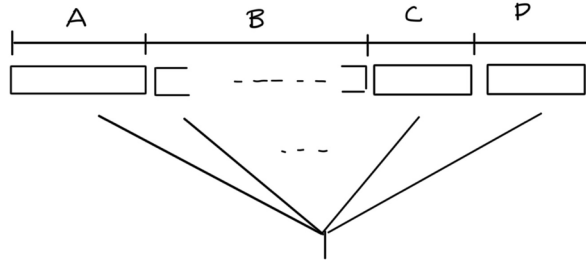
$$A := \ell(\tau_k)$$

$$B := \ell(\tau_{k-1}, \dots, \tau_3)$$

$$C := \ell(\tau_2)$$

$$D := \ell(\tau_1),$$

that is,



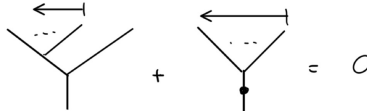
We subdivide the cases according to the cases 3.1-3.7 from the definition of higher multiplications:

Case 3.1 For this case the base assumptions are $A + B \geq N$, $B + C < N$ (so $A + B + C \geq N$ is implied).

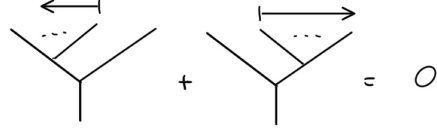
Within this, we need to subdivide both by the various possibilities for the internal lengths, and by whether τ_1 is unflanked, left / right flanked, or doubly flanked.

(3.1.1) τ_1 unflanked;

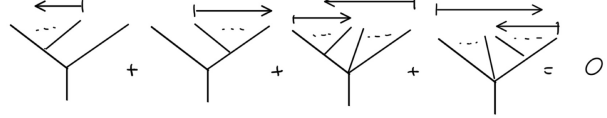
(3.1.1.1) $A, D < N$, $B + C + D < N$ ($\ell < 2N$ implied);



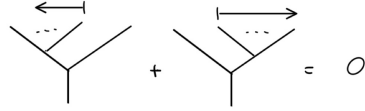
(3.1.1.2) $A, D < N, B + C + D \geq N, \ell < 2N$;



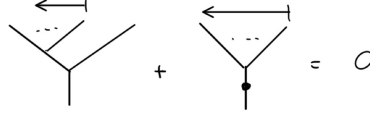
(3.1.1.3) $A, D < N, B + C + D \geq N, \ell \geq 2N$;



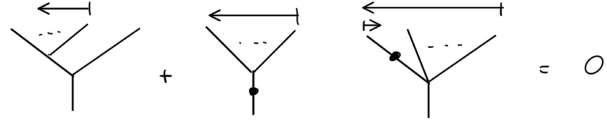
(3.1.1.4) $A < N, D \geq N$ ($B + C + D \geq N$ and $\ell \geq 2N$ implied);



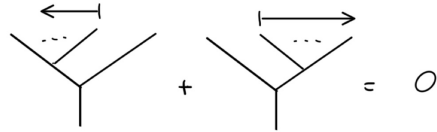
(3.1.1.5) $A \geq N, D < N, B + C + D < N, \ell < 2N$;



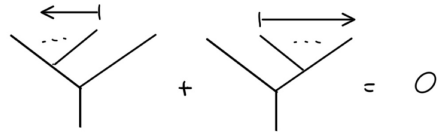
(3.1.1.6) $A \geq N, D < N, B + C + D < N, \ell \geq 2N$;



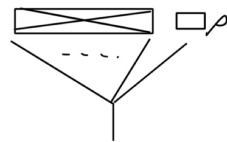
(3.1.1.7) $A \geq N, D < N, B + C + D \geq N$ ($\ell \geq 2N$ implied);



(3.1.1.8) $A, D \geq N$ ($B + C + D \geq N$ and $\ell \geq 2N$ implied);



(3.1.2) τ_1 right flanked, i.e. the tree looks like



(3.1.2.1) $A, D < N, B + C + D < N$ ($\ell < 2N$ implied);

$$+ \quad = \quad 0$$

(3.1.2.2) $A, D < N, B + C + D \geq N, \ell < 2N$; Same relation as (3.1.1.2);

(3.1.2.3) $A, D < N, B + C + D \geq N, \ell \geq 2N$; Same relation as (3.1.1.2) – the other two terms from (3.1.1.3) drop out because of flanking;

(3.1.2.4) $A < N, D \geq N (B + C + D \geq N \text{ and } \ell \geq 2N \text{ implied})$; Same relation as (3.1.1.2)

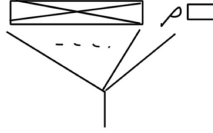
(3.1.2.5) $A \geq N, D < N, B + C + D < N, \ell < 2N$; Same as (3.1.2.1);

(3.1.2.6) $A \geq N, D < N, B + C + D < N, \ell \geq 2N$; Same as (3.1.2.1);

(3.1.2.7) $A \geq N, D < N, B + C + D \geq N (\ell \geq 2N \text{ implied})$; Same as (3.1.1.2);

(3.1.2.8) $A, D \geq N (B + C + D \geq N \text{ and } \ell \geq 2N \text{ implied})$; Same as (3.1.1.2);

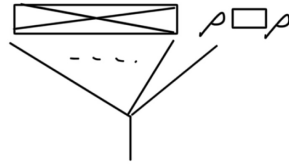
(3.1.3) τ_1 left flanked, i.e. the tree looks like



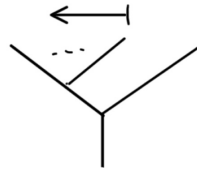
There are non-zero terms in the $\mathcal{A}_\infty$ relation if and only if $B + C + D < N$, in which case the two terms are

$$+ \quad = \quad 0$$

(3.1.4) τ_1 is doubly flanked, i.e. the tree looks like



In this case, if we evaluated the inner high-length product from the right, i.e.



Then we would get adjacent ρ s; hence, we would have to evaluate from the left. But this is impossible since $A + B \geq N$. Hence, the $\mathcal{A}_\infty$ relation for a string of this form does not have a term with μ_2 as the external operation, and will be dealt with in the last part of this proof.

Case 3.2 For this case the base assumptions are $A + B < N$, $B + C \geq N$ (so again, $A + B + C \geq N$ is implied). Also, the inner multiplication will be evaluated from the left, the two elements multiplied by the outer μ_2 will not be multipliable (in the usual sense). Hence, the string $\mu(\tau_k, \dots, \tau_2), \tau_1$ must be of length $\geq N$, which gives $\ell \geq 2N$. It follows that $C + D \geq N$, and also $D < N$ since the outer (high length) μ_2 will be evaluated from the right. Subdividing as above:

(3.2.1) τ_1 unflanked; The only thing we can control is whether $C < N$ or $\geq N$. But this does not actually affect the diagrams that appear in the $\mathcal{A}_\infty$ relation because even when $C \geq N$, the (one or two) term(s) of the differential do not give non-zero compositions. Hence in both cases, the $\mathcal{A}_\infty$ relation is

The diagram shows two tree structures separated by a plus sign, followed by an equals sign and the letter O. Each tree has a root node with two children. The left tree has a horizontal arrow pointing left above the root, and a horizontal arrow pointing right above the left child. The right tree has a horizontal arrow pointing right above the root, and a horizontal arrow pointing left above the right child. Both trees have three dots between the children of the root, indicating a sequence of nodes.

(3.2.2) τ_1 right flanked; In this case, since the inner multiplication is evaluated from the left, the resulting sequence will be doubly flanked. Since it is also of length $\geq N$, it does not contribute, and this case is trivial.

(3.2.3) τ_1 left flanked; Here, there are the same two subcases as in Case (3.2.1), but here both of those trees vanish for flanking reasons (because the outer multiplication is still high length).

(3.2.4) τ_1 doubly flanked; Again, both contributing terms vanish in all subcases.

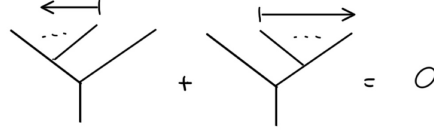
Case 3.3 For this case, the base assumptions are $A + B, B + C < N$ and $A + B + C \geq N$.

(3.3.1) τ_1 unflanked;

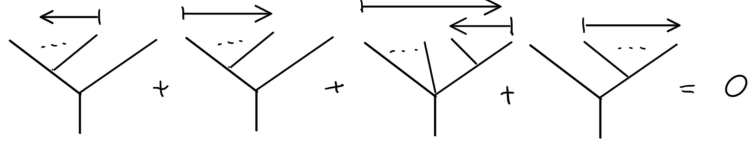
(3.3.1.1) $D < N$, $B + C + D < N$ ($\ell < 2N$ implied);

The diagram shows two tree structures separated by a plus sign, followed by an equals sign and the letter O. Each tree has a root node with two children. The left tree has a horizontal arrow pointing left above the root, and a horizontal arrow pointing right above the left child. The right tree has a horizontal arrow pointing right above the root, and a horizontal arrow pointing left above the right child. Both trees have three dots between the children of the root, indicating a sequence of nodes. The right tree has a small dot on the vertical line of its root node.

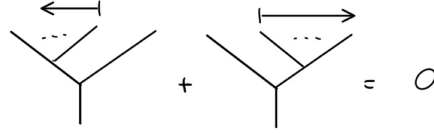
(3.3.1.2) $D < N$, $B + C + D \geq N$, $\ell < 2N$;



(3.3.1.3) $D < N$, $B + C + D \geq N$, $\ell \geq 2N$;

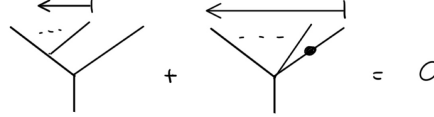


(3.3.1.4) $D \geq N$ ($B + C + D \geq N$ and $\ell \geq 2N$ implied);



(3.3.2) τ_1 right flanked;

(3.3.2.1) $D < N$, $B + C + D < N$ ($\ell < 2N$ implied);



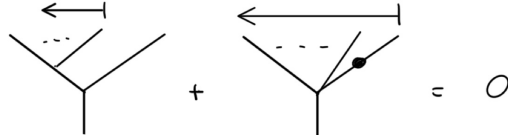
(3.3.2.2) $D < N$, $B + C + D \geq N$, $\ell < 2N$; Same as (3.1.1.2);

(3.3.2.3) $D < N$, $B + C + D \geq N$, $\ell \geq 2N$; Same as (3.1.1.2);

(3.3.2.4) $D \geq N$ ($B + C + D \geq N$ and $\ell \geq 2N$ implied); Same as (3.1.1.2);

(3.3.3) τ_1 left flanked;

(3.3.3.1) $B + C + D < N$;



(3.3.3.2) $B + C + D \geq N$; Same as (3.1.1.2)

(3.3.4) τ_1 doubly flanked; Because the overall string is right flanked, we can never evaluate the outer multiplication from the right. This means that in both the $B + C < N$ and $\geq N$ cases, the picture is precisely the one from Case (3.3.3.2), above.

Case 3.4 Here, since $(\tau_k, \dots, \tau_2)$ is left-flanked and length N , the situation is much simpler. The base assumption is $A + B + C = N$.

(3.4.1) τ_1 unflanked; There are three cases:

(3.4.1.1) $D < N$, $B + C + D < N$; Same as (3.1.1.1);

(3.4.1.2) $D < N$, $B + C + D \geq N$; Same as (3.1.1.2);

(3.4.1.3) $D \geq N$; Same as (3.1.1.2);

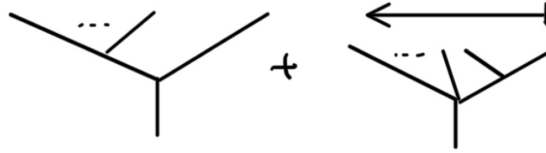
(3.4.2) τ_1 right flanked – same single case as in (3.4.1);

(3.4.3) τ_1 left flanked – vanishes;

(3.4.4) τ_1 doubly flanked – vanishes;

Case 3.5 Here, $(\tau_k, \dots, \tau_2)$ is right-flanked and the base assumption is again $A + B + C = N$.

(3.5.1) τ_1 unflanked; There are no cases, just



(3.5.2) τ_1 right flanked – same single case as in (3.5.1);

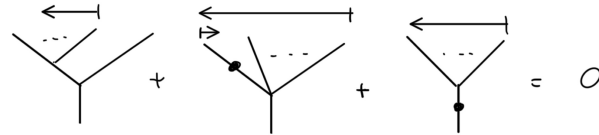
(3.5.3) τ_1 left flanked – all terms vanish because too much flanking;

(3.5.4) τ_1 doubly flanked – vanishes;

Case 3.6 Here, $(\tau_k, \dots, \tau_2)$ is left flanked and the base assumptions are $A + B + C \geq N$, $B + C < N$, or else there is no contributing term of the form we want.

(3.6.1) τ_1 unflanked; We have the same 8 subcases as in Case (3.1.1):

(3.6.1.1) $A, D < N$, $B + C + D < N$ ($\ell < 2N$ implied);



(3.6.1.2) $A, D < N$, $B + C + D \geq N$, $\ell < 2N$; Same as (3.1.1.2);

(3.6.1.3) $A, D < N$, $B + C + D \geq N$, $\ell \geq 2N$; Same as (3.1.1.2);

(3.6.1.4) $A < N$, $D \geq N$ ($B + C + D \geq N$ and $\ell \geq 2N$ implied); Same as (3.1.1.2);

(3.6.1.5) $A \geq N$, $D < N$, $B + C + D < N$, $\ell < 2N$; Same as (3.6.1.1);

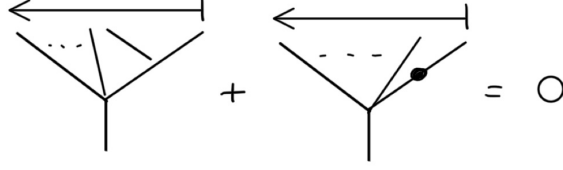
(3.6.1.6) $A \geq N$, $D < N$, $B + C + D < N$, $\ell \geq 2N$; Same as (3.6.1.1);

(3.6.1.7) $A \geq N$, $D < N$, $B + C + D \geq N$ ($\ell \geq 2N$ implied); Same as (3.1.1.2);

(3.6.1.8) $A, D \geq N$ ($B + C + D \geq N$ and $\ell \geq 2N$ implied); Same as (3.1.1.2);

(3.6.2) τ_1 right flanked: then there are only two cases:

(3.6.2.1) $B + C + D < N$;



(3.6.2.2) $B + C + D \geq N$; all terms vanish and the relation is trivial;

(3.6.3) τ_1 left flanked – No contributing term with μ_2 as the external operation;

(3.6.4) τ_1 doubly flanked – same;

Case 3.7 Here, $(\tau_k, \dots, \tau_2)$ is right flanked and the base assumptions are again $A+B+C \geq N$, $A+B < N$, or else there is no contributing term of the form we want.

(3.7.1) τ_1 unflanked; The only possible cases are

(3.7.1.1) $B + C + D < N$; Same as (3.3.3.1);

(3.7.1.2) $B + C + D \geq N$; Same as (3.1.1.2);

(3.7.2) τ_1 right flanked – no cases, just the same relation as in Case (3.1.1.2);

(3.7.3) τ_1 left flanked – all terms vanish because too much flanking;

(3.7.4) τ_1 doubly flanked – same as (3.7.3);

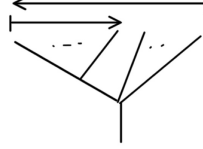
Now what is left is the case in which $k \geq 3$, τ contains no multipliable pairs, and no term of the $\mathcal{A}_\infty$ relation for τ has a μ_2 as the second operation of the composition. This means that for each non-vanishing term of the $\mathcal{A}_\infty$ relation, both operations in the composition (internal and external) must be high length. Therefore, if the first operation was $\mu_j(\tau_s, \dots, \tau_r)$, with $1 < r \leq s < k$ (i.e. was not based at either the far left or the far right of the tree) then we would end up with a multipliable pair in the resulting sequence $(\tau_k, \dots, \tau_{s+1}, \tau', \tau_{r-1}, \dots, \tau_1)$. This cannot happen, since the outer operation is at least a μ_3 .

Hence, we can now restrict ourselves to the case when $r = 1$ or $s = k$ or both. If $r = 1$, $s = k$, then this is just the statement that $d \circ \mu_j = 0$ for any high length operation μ_j . This follows easily from the definitions of the high length μ_j . Now suppose $r > 1$, $s = k$.

Because we are also that the composition we are looking at does not have a μ_2 as the outer operation, the picture is



that is, $s = k$, $r \geq 3$. In particular, this means that the given operation (if it is non-vanishing) must be evaluated in the directions



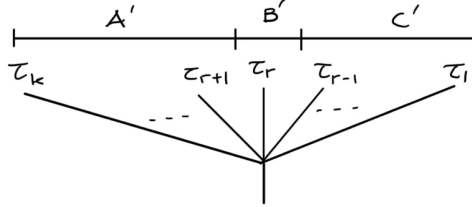
Writing the internal lengths as

$$A' := \ell(\tau_k, \dots, \tau_{r+1})$$

$$B' := \ell(\tau_r)$$

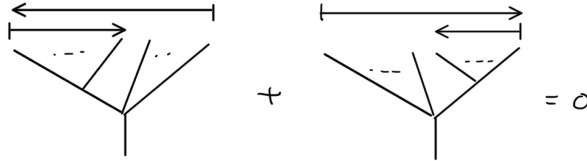
$$C' := \ell(\tau_{r-1}, \dots, \tau_1),$$

that is,



the composition we are looking at is non-vanishing if and only if $A' < N$, $A' + B' \geq N$, and $C' < N$.

This means that the $\mathcal{A}_\infty$ relation is precisely



The other case ($r = 1$, $s < k$) is entirely analogous, and will be omitted. Thus the proof is complete, and $\mathcal{B}$ is a bona fide $\mathcal{A}_\infty$ algebra. □

2.3.3 Maslov and Alexander gradings for $\mathcal{B}$

We define

$$\begin{aligned} m(\rho_i) &= m(\sigma_i) = -1 \text{ for each } i \\ m(V_i) &= -2 \text{ for each } 1 \leq i \leq N+1 \\ m(\mathbf{e}_{N+1}) &= 1 \\ m(U_0) &= -2N \end{aligned}$$

(This last one is technically not a definition, but a computation from the definition of U_0 and the rules for the ρ_i and σ_i .) We know from Section 2.2.3 that also

$$\begin{aligned} m(V_0) &= 2N - 2 \\ m(\mathbf{e}_i) &= 2 \text{ for each } 1 \leq i \leq N+1 \\ m(\mathbf{e}_0) &= -(2N - 2) \end{aligned}$$

Note that all of the new definitions still satisfy (2.12). That all the higher operations satisfy (2.12) follows from the conditions (on length, etc.) which we placed on the operations when we defined them.

For the Alexander gradings, we have

$$\begin{aligned} A(\rho_i) &= A(U_i) = \overline{2i-1} \text{ for each } 1 \leq i \leq N \\ A(\sigma_i) &= A(s_i) = \overline{2i} \text{ for each } 1 \leq i \leq N \\ A(\mathbf{e}_0) &= \sum_{k=1}^{2N} \overline{k} \\ A(V_i) &= A(U_i) = \overline{2i-1} \text{ for each } 1 \leq i \leq N \\ A(V_{N+1}) &= A(\mathbf{e}_{N+1}) = \sum_{k=1}^N \overline{2k} \end{aligned}$$

Again, it is clear that all operations on $\mathcal{B}$ preserve the Alexander grading. This therefore defines a suitable $\mathcal{A}_\infty$ -algebra with operationally bounded higher multiplications, satisfying all the properties we need it to.

2.4 Underlying Associative algebras

Because we do not currently have a formal notion of weighted DA -bimodules, the main Koszul duality result, Theorem 1.0.2, will include only the simplest, unweighted versions of our $\mathcal{A}_\infty$ -algebras, the associative algebras defined below.

Define $\mathcal{A}_0$ to be an associative R -algebra with the same basic generators as $\mathcal{A}$:

- $I_i, i = 1, \dots, N$
- $U_i, i = 1, \dots, N;$
- $s_i, i = 1, \dots, N$

The rules for multiplication (μ_2) are as in Section 2.2.1. $\mathcal{A}_0$ is equipped with the same Maslov and Alexander grading as $\mathcal{A}$, and the notions of *initial* and *final idempotents* are defined as in that section.

Likewise, define $\mathcal{B}_0$ to be an associative algebra with generators

- $I_j, j = 1, \dots, N;$
- $\rho_j, j = 1, \dots, N;$
- $\sigma_j, j = 1, \dots, N;$

Simple multiplication on $\mathcal{B}_0$ (i.e. μ_2), and Maslov and Alexander gradings for $\mathcal{B}_0$, are all defined as in Section 2.3.1.

2.5 The AA bimodule

The goal of this section is to define the *unweighted* AA bimodule ${}_{\mathcal{B}_0}Y_{\mathcal{A}_0}$ which, with the DD -bimodule ${}^{\mathcal{A}_0}X^{\mathcal{B}_0}$ defined in Section 2.7, will fit into the duality relation of Theorem 1.0.2.

Y is defined to be a $\mathbb{F}_2$ -module, generated by the intersections of α and β arcs in the usual diagram

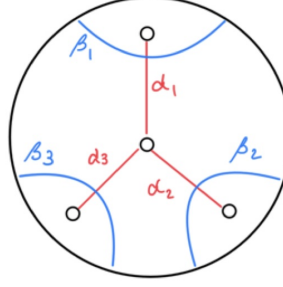


Figure 2.13: The full star diagram with α -/ β -arcs labelled

We notate these as $\mathbf{x} = \{i\}$ for $1 \leq i \leq N$, where $\mathbf{x} = \{i\}$ is in the i -th idempotent. We also view each V_i as an element of the ground ring. Note that in general, a generator of a bimodule has one left idempotent and one right idempotent. However, all generators of Y are supported in the same idempotent on both left and right.

Recall from Section 2.1.3 that $\mathcal{A}_\infty$ operations on an AA bimodule ${}_B Y_A$ are in one to one correspondence with corollas with a designated central leaf, and labelled to the right of this leaf with $\mathcal{A}$ -elements and on the left with $\mathcal{B}$ -elements. See Figure 2.3. This is the basic definition. In this section we are going to manually define the non-zero operations, and show that they can be written in terms of this definition.

We do that as follows. In order to define a valid AA bimodule that satisfies the $\mathcal{A}_\infty$ -relations, we will use the following steps:

1. Define the set of non-zero operations manually in a way specific to the case we are dealing with – *not* in terms of corollas, for this step;
2. Show that each operation, as manually defined in (1), can be expressed as a labelled corolla;
3. Exhibit a condition (Proposition 2.5.3) which a corolla T must satisfy in order to correspond with a non-zero operation, as defined in (1);
4. Show that the AA bimodule with generators as above, and non-zero operations – now written, properly, in terms of corollas – as in Proposition 2.5.3 satisfies the $\mathcal{A}_\infty$ relations.

For step (1), we define non-zero bimodule operations to be certain allowable types of planar graphs in the disk. A non-zero operation, often called just an *operation* is defined as follows. Start with a 4-valent planar graph embedded in $\mathcal{D}$ which admits a coloring of the following type: one involving in two colors – in our case, red and blue, with red bounding $\mathcal{A}$ -sectors and blue bounding $\mathcal{B}$ -sectors – such that each vertex looks like:



We require that each red sector is labelled with an $\mathcal{A}$ -algebra element, each blue sector is labelled with a $\mathcal{B}$ -algebra element, and each mixed sector is labelled with a number $1 \leq i \leq N$, chosen so that the idempotent along the incoming (i.e. left) red / blue strand matches the initial idempotent of the $\mathcal{A}$ - or $\mathcal{B}$ -label in the all-red or all-blue sector, and the idempotent along the outgoing (i.e. right) red / blue strand matches the final idempotent of the $\mathcal{A}$ - or $\mathcal{B}$ -label in this sector. With these conditions, the three allowable types of labelling are

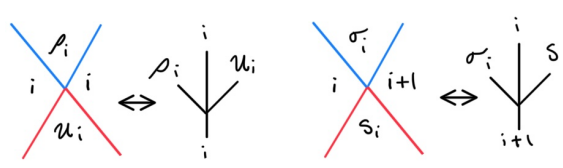
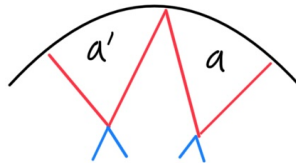


Figure 2.14: The basic allowable labelings for intersections

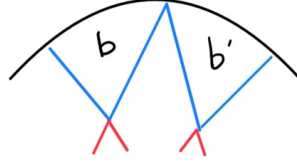
We define each basic graph in Figure 2.14 to correspond to a corolla as notated above. These are the only three types of allowable basic graphs. A 4-valent graph, colored and labeled as described in the previous paragraph, is an operation if and only if it satisfies the conditions below. Where possible, we specify the tree (or partial tree) corresponding to each conditions. Define an *allowable graph* as follows.

Op. 1 : The graph does not contain *composite* regions – i.e. regions that are bordered by a boundary edge, red edge, and blue edge, and more than one of at least one of these kinds. The difficulty with these regions is that the extra edges can be pushed out the boundary – that is, graphs containing composite regions do not correspond to operations.

This has a notable consequence, namely that there are no unmultiplied multipliable pairs; more specifically, if we have $a, a' \in \mathcal{A}$ labelling adjacent but disconnected sectors such as

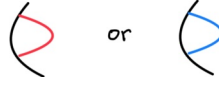


then $a \cdot a' = a' \cdot a = 0$, and if $b, b' \in \mathcal{B}$ labelling adjacent but disconnected sectors such as



then $b \cdot b' = b' \cdot b = 0$.

Op. 2 : One vertex of each edge must be internal, so we cannot have e.g.

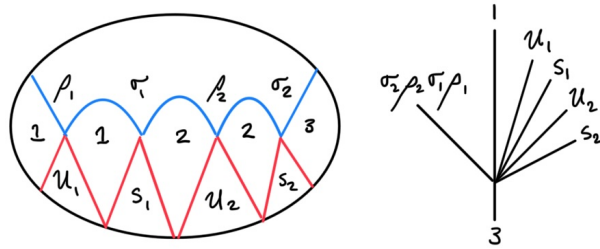


Op. 3 : The graph must be rooted in the following sense: at least one of the edges intersects the boundary, which by the preceding conditions implies that there must be at least one region of the form



We choose one such region as the designated region or *root* idempotent.

Op. 4 : The sequence of elements in a given region must be multipliable, read clockwise according to the orientation of the main disk; as in



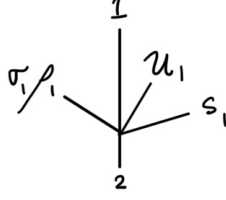
Before we talk about how to associate a tree to any allowable graph, we need to talk about Maslov and Alexander gradings. We make the following definition for weighted input sequences: for $a_1, \dots, a_n \in \mathcal{A}, \mathbf{x} \in Y, b_1, \dots, b_k \in \mathcal{B}$,

$$m(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n) = m(\mathbf{x}) + \sum_{i=1}^n m(a_i) + \sum_{i=1}^k m(b_i) \quad (2.35)$$

We also stipulate that $m(\mathbf{x}) = 0$ for each $\mathbf{x} \in Y$. We define the Alexander grading of an input sequence as

$$A(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n) = \sum_{i=1}^n A(a_i) + \sum_{i=1}^k A(b_i) \quad (2.36)$$

We also need to talk about the kind of modified matching between elements on the $\mathcal{A}$ -side and the $\mathcal{B}$ side that will be crucial to deciding which trees correspond to operations. Let $\eta_1, \dots, \eta_r, \tau_1, \dots, \tau_\ell$, be the basic inputs on the $\mathcal{A}$ and $\mathcal{B}$ -sides, respectively, so in the case of the tree,



we have

$$\eta_1 = U_1, \eta_2 = s_1$$

$$\tau_1 = \rho_1, \tau_2 = \sigma_1$$

An in sequence $(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n)$ *admits a matching* if the corresponding sequence of basic elements, which we now write as

$$(\eta_1, \dots, \eta_r, \tau_1, \dots, \tau_\ell)$$

satisfy the following conditions:

Mat. 1: $r = \ell$; write this single number as α ;

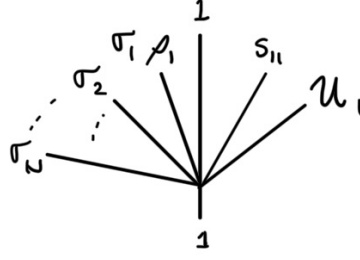
Mat. 2: The resulting matching

$$h : \{\eta_1, \dots, \eta_r\} \rightarrow \{\tau_1, \dots, \tau_r\} \quad (2.37)$$

given by $\eta_i \mapsto \tau_i$ for each i preserves Alexander grading.

An input sequence *admits a matching* if and only if we can construct h , as in (2.37), with the stipulated properties.

Remark 2.5.1. Notice that the condition “admits a matching” is stronger than the condition we will place on Alexander grading (namely (2.39), below). The matching matches elements with the same Alexander grading, so trees that admit a matching automatically satisfy (2.39). However, the second tree discussed at the end of the definition of matchings,



satisfies conditions (i) and (ii), as well as (2.39), but does not admit a matching (because we are not allowed to rearrange the τ_i and η_i). We include the explicit Alexander grading condition (2.39) in what follows for clarity rather than strictly for completeness.

We can also define the input sequence corresponding to an allowable graph, by reading the $\mathcal{A}$ -inputs counterclockwise along the boundary from the root sector, and reading the $\mathcal{B}$ -inputs clockwise along the boundary from the root sector, and adding in weight wherever it is found. We will see in Lemma 2.5.2 that every allowable graph admits a matching.

With this in mind, we now show that:

Lemma 2.5.2. *Any allowable graph G with $\mathcal{A}$ -inputs $a_1, \dots, a_n$, initial element $\mathbf{x} \in Y$, $\mathcal{B}$ -inputs $b_1, \dots, b_k$ satisfies*

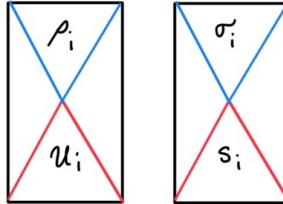
$$m(\mathbf{x}) + \sum_{i=1}^n m(a_i) + \sum_{i=1}^k m(b_i) + k + n - 1 = 0 \quad (2.38)$$

and

$$A(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n) = 0 \bmod 2 \quad (2.39)$$

and weighted input sequence $b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n$ admits a matching in the sense described above.

Proof of Lemma 2.5.2. To get a matching from G , just go vertex by vertex clockwise from the root and match inputs across vertices. For (2.38), we are going to use induction on the number of vertices in a given allowable graph. This is obvious for the basic graphs



by a straight calculation. Suppose now that every allowable k -vertex graph satisfies (2.38) and (2.39). Let G be an allowable graph with $(k + 1)$ vertices. Pick one of the vertices to excise, and assume first that this is a vertex of the form below:

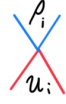
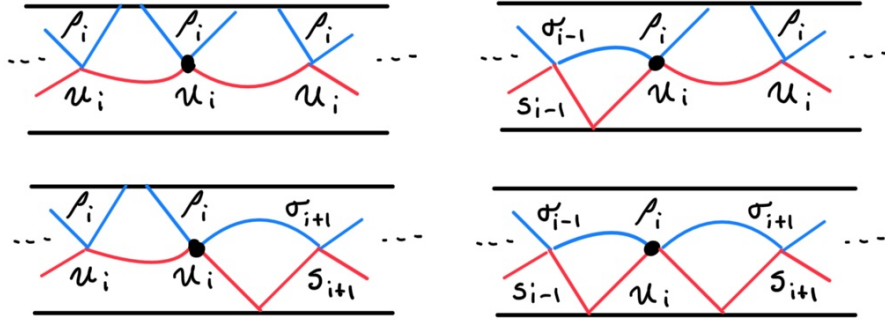


Figure 2.15

Then the part of the graph surrounding this vertex looks like one of



First, calculate the contribution from each one of these to the left hand side of (2.38) – e.g. for the top left graph,

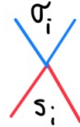
$$3m(\rho_i) + m(U_i^3) + 3 + 1 - 1 = 0$$

and then remove the vertex in question, and recalculate – so, for the top left diagram again,

$$2m(\rho_i) + m(U_i^2) + 2 + 1 - 1 = 0$$

to see that the total contribution of this part of the graph does not change. The resulting graph is also an allowable graph, and hence, we can apply the induction hypothesis.

If the vertex looked like a



then we can go through the options for adjacent vertices, and in each case, excise this vertex in such a way as to not change the sum on the left hand side of (2.38). This means that we can apply the induction hypothesis in this case, as well. $\square$

Notice that in the limited cases, such as some of the ones treated above, it is reasonably obvious how to associate a tree to a graph. In general, we do that as follows.

Proposition 2.5.3. (Identifying operations in Y) *Any allowable graph G , as defined above, corresponds to a tree with input sequence*

$$(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n) \quad (2.40)$$

and output $\mathbf{y} \in Y$ satisfying the following properties

(i) (Idempotents)

- *The initial idempotent of a_i (resp. b_i) is the final idempotent of a_{i-1} (resp. b_{i-1}) for each $1 < i \leq n$ (resp. $1 < i \leq k$);*
- *The initial idempotent of a_1 (resp. b_1) is the idempotent of $\mathbf{x}$;*
- *The final idempotent of a_n (resp. b_k) is the idempotent of $\mathbf{y}$;*

(ii) (Maslov grading) *The input sequence satisfies (2.38);*

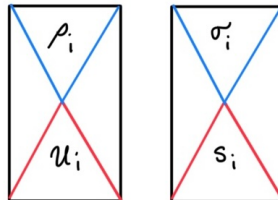
(iii) (Alexander grading) *The input sequence satisfies (2.39) and admits a matching;*

Under these conditions, the associated tree is unique with this property. Moreover, every tree the input sequence of which satisfies (i)-(iii) determines a unique allowable graph.

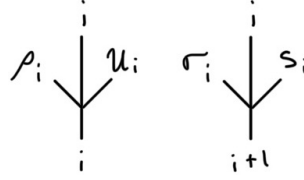
Proof of Proposition 2.5.3. Look at a given allowable graph G . We can read off the input sequence by looking at the labels of the all-red and all-blue sectors of each vertex. If a sector is part of a cycle, then it goes to weight, otherwise it goes to the appropriate side. The $\mathbf{x} \in Y$ comes from the idempotent of the root. That the resulting input sequence $(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n)$ satisfies (ii) and (iii) follows directly from Lemma 2.5.2, so we only need to verify that

- (a) We can choose an arrangement of $(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n)$ that satisfies (i), and
- (b) There is a single well-defined idempotent corresponding to an element $\mathbf{y} \in Y$, which is the final idempotent of both a_n and b_k , and hence the output element for the tree T corresponding to G .

The trick is that (a) comes from (iii) from the statement. This works by induction on the number of vertices in G . The base case is the graphs



These clearly satisfy (a) and (b) and correspond to the trees



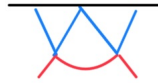
Now suppose we know that there is an arrangement of the inputs satisfying (i) and a well-defined final idempotent for all allowable graphs with ℓ or fewer vertices and input sequence $(b_k, \dots, b_1, \mathbf{x}, a_1, \dots, a_n)$ with $k \leq k_0, n \leq n_0$. Look at a graph with $\ell + 1$ vertices. Then as in the proof of Lemma 2.5.2, we can excise a single vertex (which we can assume without loss of generality is not the final vertex) without changing that this is an allowable graph, G' . By the induction hypothesis, we can get an (allowable) tree T' corresponding to G' . Then looking at the point in G' from which we excised our original vertex, we can find the corresponding point in T' , and add back the appropriate algebra elements (properly multiplied) to get an allowable tree T for G .

The second half of the proof deals with getting an allowable graph back from a given tree T , the inputs of which satisfy (i)-(iii). We use the given matching to get a graph G for T by induction on α , the number of inputs on either side, counting weights (with notation as in the definition of the matching).

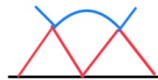
We now need to verify that

1. G satisfies Op. 1-4 (i.e. is a bona fide operation);
2. G gives back the original tree T when we read off the elements as we did in the first half of the proof;

We do (1) first. This graph is clearly rooted – we just choose the initial idempotent $\mathbf{x}$ as the root, which is well-defined because the graph was constructed inductively. The sequence of elements in a given region is multipliable by construction of G – as we walked down the pairs of $\mathcal{A}$ - and $\mathcal{B}$ -elements, we connected vertices as



when the adjacent $\mathcal{A}$ elements are multipliable and as



when the $\mathcal{B}$ elements are multipliable. This also gives us back the fact that all multipliable pairs are multiplied.

Now we can use the first part of the proof to get back an allowable tree T' from G , and T' will have the same sequence of basic elements (in the same order) as T . Also, there are no unmultiplied multipliable pairs of elements in T' . The only thing that could go wrong is that there are such pairs in T' (because everything else is the same by construction). But T' satisfies (2.38), and if T had one or more multipliable pairs (with everything else the same as T' , as it has) then it could not also satisfy (2.38). Since both T and T' satisfy (2.38), it follows that since everything else matches, they must be identical. $\square$

The last step of the section is to verify the $\mathcal{A}_\infty$ relations for Y – that is, show that given any tree T , the sum over all ways to add an edge to T is zero. Again, the four ways to add an edge to a tree are a pull or a split, as noted in Section 2.1.3. (In our case, since we are dealing only with $\mathcal{A}_0$ and $\mathcal{B}_0$, there are no non-zero differentials.)

First, we note that:

Lemma 2.5.4. *Start with a tree T and look at the left hand side of (2.38), namely*

$$\sum_{i=1}^n m(a_i) + \sum_{i=1}^k m(b_i) + k + n - 1 \quad (2.41)$$

corresponding to the inputs of T . If T gives rise to a non-zero operation (or a composition of non-zero operations, as for a split) by one of the four actions above, then this quantity is $+1$. In particular, a pull shifts the quantity of (2.41) by -1 .

Proof. For a pull, we recall that

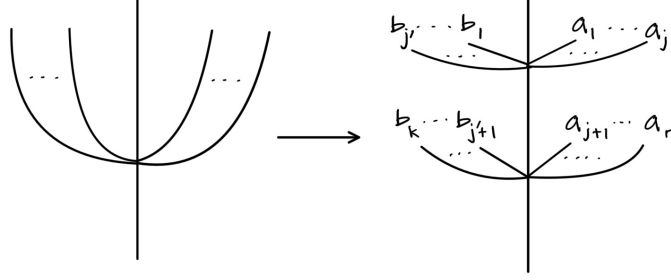
$$m(\mu_j(a_1, \dots, a_j)) = \sum m(a_j) + j - 2 \quad (2.42)$$

Let T be the initial tree and T' the one that results after pulling together $a_1, \dots, a_j$ (which is an adjacent string of elements on either the $\mathcal{A}$ -side or the $\mathcal{B}$ -side). Write C for then the expression of (2.41), for T , and D for the expression on the right hand side of (2.42). Then the expression of (2.41) is, for T' ,

$$C + D - j + 1 - \sum m(a_j) = C - 1.$$

If T' corresponds to a non-zero operation, then $C = +1$.

Now look at a split, where by splitting T , we obtain a composition of trees each corresponding to a *valid* bimodule operation. Suppose the operation on the top pulls together $a_1, \dots, a_j$ and $b_1, \dots, b_{j'}$ so:



Because T' corresponds to a valid operation, we have

$$\sum_{i=1}^j m(a_i) + \sum_{i=1}^{j'} m(b_i) + j + j' - 1 = 0 \quad (2.43)$$

Likewise,

$$\sum_{i=j+1}^n m(a_i) + \sum_{i=j'+1}^k m(b_i) + (n - j) + (k - j') - 1 = 0 \quad (2.44)$$

Summing the quantities from (2.43) and (2.44) tells us that the left hand side of (2.41) is $+1$. $\square$

As in the case of $\mathcal{A}$, we introduce the notion of *augmented graphs*, which will correspond to non-trivial relations. An augmented graph is a graph satisfying Op. 2 through 4, and contains exactly one composite region. We first show that every such graph corresponds to a pair of operations.

There is only one type of composite region that appears in this unweighted case, namely

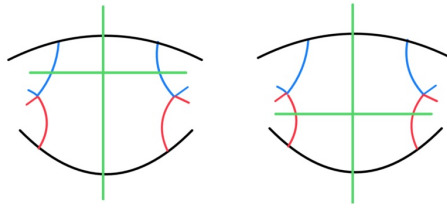
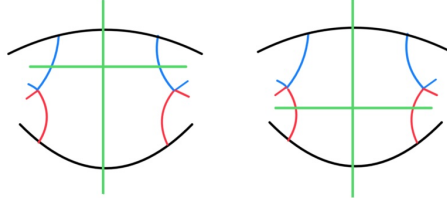


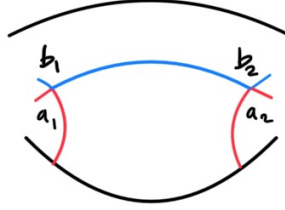
Figure 2.16

This region can be resolved in two ways so that the resulting graph satisfies Op. 1 - Op. 4; on the level of trees, these correspond to a pull and a split, respectively.



In the first case, we are assuming the pair of elements in the adjacent blue sectors are multipliable, and the pair in the adjacent red sectors is not; in the second, we are assuming the reverse. The two green lines represent the two actions possible for this graph. The red pair and the blue pair cannot both be multipliable, since the sequence of input elements for this graph admits a matching, and any multipliable pair of basic $\mathcal{A}$ -elements matches with a non-multipliable pair of $\mathcal{B}$ -elements, and vice versa.

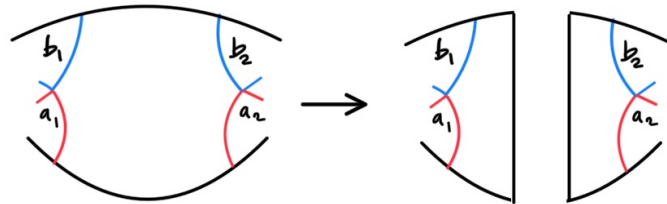
For now, look at the first picture, and replace it with the graph:



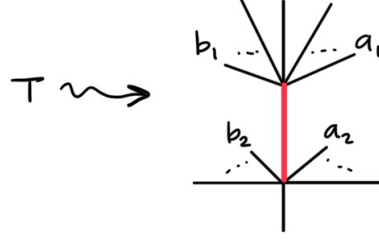
Since everything else about the original augmented graph was allowable, and we have now eliminated the objectionable composite region, this is an allowable graph, and corresponds to a nonzero operation T' .

To get back a tree T corresponding to the original graph G , find the pair of elements that were multiplied to get from G to G' , and unmultiply them on the level of trees. Since nothing else has changed (and nothing else changed to go from G to G'), this tree T corresponds to G .

But notice that because the elements b_1 and b_2 (which will show up on the $\mathcal{B}$ -side of T) are unmultiplied, we can also modify T by splitting:



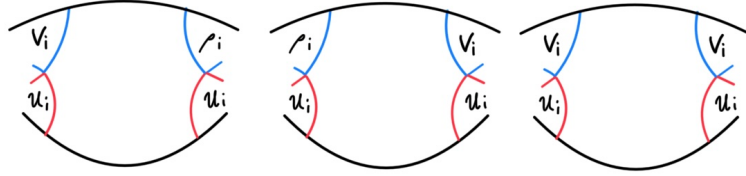
which corresponds to:



To conclude that both trees (top and bottom) correspond to non-zero operations, just note that the two graphs (left and right) of the split are each allowable, and hence correspond to non-zero operations. Since the trees corresponding to these sections of the graph are just T' and T'' , the conclusion follows.

The case where the two $\mathcal{B}$ -elements are multipliable, but not the pair of $\mathcal{A}$ -elements, is analogous.

Finally, we have to deal with the case where both pairs are multipliable. But the only labelled graphs of the type pictured above where both the red pair of elements and the blue pair of elements are multipliable are:



and we do not consider any of these three case, because in any of these cases, the full augmented graph G would correspond to a tree T with

$$m(\mathbf{x}) + \sum_{i=1}^n m(a_i) + \sum_{i=1}^k m(b_i) + k + n - 1 = 2$$

which, by Lemma 2.5.4, cannot yield a non-zero operation (or composition of non-zero operations) by any of the four actions.

Remark 2.5.5. Our graph G is never allowed to have “composite” regions. The issue is not that we cannot define a corresponding tree: we can, by the exact same methods used above when we associated a tree to an allowable graph in Lemma 2.5.3. The issue is that when we look at the the resulting tree T , and evaluate the expression from (2.41), we will get $+2$ (assuming that the rest of the graph G is allowable). By Lemma 2.5.4, this means that no way of “resolving” this region (by applying one of our four actions to the corresponding tree and considering the graph corresponding to the result) will yield an allowable graph. Hence, no graph G with such a “composite” region can correspond to a non-trivial relation.

The main lemma in the proof of the $\mathcal{A}_\infty$ -relations for Y is now as follows:

Lemma 2.5.6. *The non-trivial relations T (i.e. trees where adding some single edge gives a non-trivial operation) are in one-to-one correspondence with augmented graphs as described above.*

Proof. We have already shown that augmented graphs each correspond to a non-trivial relation. Now start with a non-trivial relation, some tree T . Any resulting tree T' (from adding an edge to T) is a non-trivial operation, and therefore satisfies the conditions of Proposition 2.5.3, and corresponds to an allowable graph G' . To get from T to T' , we performed one of the four allowable actions (pull, split, push, or differential). We can mark, on T and T' , where this operation happened, and the corresponding location on G' .

Going from T to T' corresponds to zipping up a single composite region. To see this, we examine the allowable graph G' for T' , and identify the point in the tree where T' differs from T . Now consider a graph G containing a composite region as in Figure 2.16 at this point (and having all other regions satisfying Op. 1 - Op. 3); this graph will correspond to T . Note that this G will have the same number of vertices (with the same labels) as G' , and T has the same sequence of basic inputs. This means that T corresponds to the augmented graph T' . $\square$

Proposition 2.5.7. *Y satisfies the $\mathcal{A}_\infty$ -relations (2.9) and is a valid AA-bimodule.*

Proof. The trick is that each augmented graph can be resolved into an allowable graph in precisely two ways. In the one-to-one correspondence between relations and augmented graphs from Lemma 2.5.6, this shows us that any tree T which has a non-trivial relation can be resolved into a non-zero operation in exactly two ways; that is, the $\mathcal{A}_\infty$ relations hold. $\square$

2.6 Diagonals and tensor products

The goal of this section is to define *weighted diagonals*, especially in the context of our algebras $\mathcal{A}$ and $\mathcal{B}$, and discuss the role they play in describing operations on $\mathcal{A}_\infty$ -tensor products, both of algebras (e.g. $\mathcal{A} \otimes \mathcal{B}$) and bimodules (e.g. $X \boxtimes Y$). This will be relevant to the verification of the $\mathcal{A}_\infty$ -relations for the DD -bimodule ${}^{\mathcal{A}}X^{\mathcal{B}}$ in Section 2.7, since this module is in fact just a type- D module over $\mathcal{A} \otimes \mathcal{B}$.

2.6.1 Diagonals

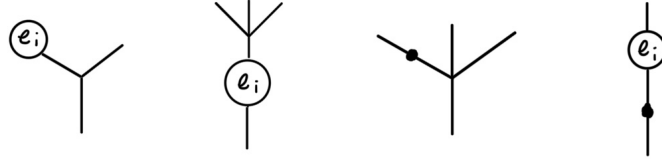
The main source for this is Section 6 of [18]. In this section, we summarize the definitions and results most relevant the constructions that follow, modified to fit this situation. The main definition of

this section is a *weighted diagonal*, given in (2.50). The other definitions that lead up to this point in the section are intended entirely to prepare the reader to understand the stipulations of this most important definition.

The basic set-up is as follows. Throughout, a *weight vector* $\mathbf{w}$ is defined to be a finite $\mathbb{Z}_{\geq 0}$ -linear combination of the vectors $\{\mathbf{e}_i\}_{i=0}^{N+1}$, viewed an element of the $\mathbb{Z}_{\geq 0}$ -linear space V generated by these basis vectors. A *weighted tree* T is a planar tree where each internal vertex v of T is labelled with a particular weight vector $\mathbf{w}(v) \in V$. The *total weight* $\mathbf{w}(T)$ of T is the sum of all the $\mathbf{w}(v)$, over all the internal vertices of T . Define $\text{wt}(T)$ to denote the magnitude of the total weight $\mathbf{w}(T)$ of T . Define the *formal dimension* $\dim T$ of a tree T with n inputs, total weight $\mathbf{w}(T)$, and $|v|$ internal vertices as

$$\dim T = n + 2\text{wt}(T) - |v| - 1 \quad (2.45)$$

In this section, we will consider only a subset of the total set of trees, namely *stably weighted trees*. To define *stably weighted*, we start by looking at a given vertex v in a tree T , with corresponding weight $\mathbf{w}(v)$. This vertex $(v, \mathbf{w}(v))$ is defined to be *stable* when v is 3-valent, $|\mathbf{w}(v)| > 0$, or both. The tree T is *stably weighted* if all its vertices are stable. So, for instance, considering

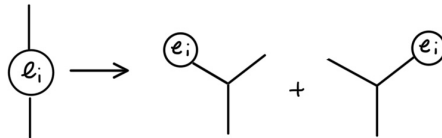


the left two trees are stably weighted, and the right two are not.

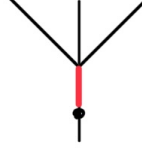
Define $X_k^{n, \mathbf{w}}$ to be the set of stably weighted trees with n inputs, total weight $\mathbf{w}$, and formal dimension k . Define a *differential* on $X_k^{n, \mathbf{w}}$ as follows. Let ∂T be sum of all the ways to replace a vertex of T with an edge in such a way that the result is still a stably weighted tree. For instance:



or



Notice that when we look at the ways to push out an edge from the 3-input corolla, above, we do not get



as a summand of the result, because this tree is not stably weighted (it has a 2-valent vertex). ∂ is clearly a differential, i.e. $\partial^2 = 0$, and drops dimension by 1 in each case.

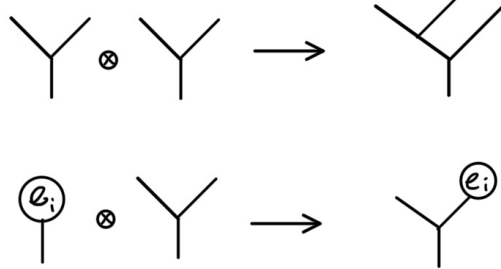
Common conventions are as follows: we usually want our trees to be in dimension 0, and we usually suppress the dimension and speak of $X_*^{n,\mathbf{w}} = \bigoplus_{k \in \mathbb{Z}} X_k^{n,\mathbf{w}}$ as a chain complex. The *input leaves* of a tree T are defined to be the edges adjacent to the 1-valent input vertices of T . The *output leaf* of T is the edge adjacent to the 1-valent output vertex of T .

We also use special symbols for the *corollas* – trees with one internal vertex, either unweighted or weighted, and any number of inputs. We let $\Psi_n^{\mathbf{w}}$ denote corolla with n inputs and weight $\mathbf{w}$ placed at its single vertex. We permit $n = 0$, and in particular, the popsicles

$$\Psi_{\circ}^{\vec{e}_i} : \quad \text{corolla with } e_i \text{ at the vertex}$$

as well as Ψ_1^0 , which will give back the differential of an element.

We also want to compose trees, e.g.



We do this explicitly with *stacking maps* $\phi_{i,j,n;\mathbf{v},\mathbf{w}}$. To define these, we need a little more notation, namely the *gluing map* $\circ_i$; for each $S \in X_*^{m,\mathbf{v}}$, $T \in X_*^{n,\mathbf{w}}$, $T \circ_i S$ is defined as the result when we attach the output leaf of S to the i -th input of T . Here, we are not considering the actual labelling of the inputs / outputs, but only the trees themselves. For instance, the two stacked trees above are:

$$\begin{aligned} \text{Y-tree} \circ_1 \text{Y-tree} &= \text{stacked Y-trees} \\ \text{Y-tree} \circ_2 \text{pop}(e_i) &= \text{Y-tree with pop}(e_i) \text{ on an output edge} \end{aligned}$$

With this notation, we further define the *stacking map*

$$\phi_{i,j,n;\mathbf{v},\mathbf{w}} : X_*^{(j-i+1),\mathbf{v}} \otimes X_*^{(n+i-j),\mathbf{w}} \rightarrow X_*^{n,\mathbf{v}+\mathbf{w}} \quad (2.46)$$

for $1 \leq i \leq n$ (so we can choose any branch to stack into) and $i-1 \leq j \leq n$, as

$$\phi_{i,j,n;\mathbf{v},\mathbf{w}}(S \otimes T) = T \circ_i S \quad (2.47)$$

So the examples given above can be written in these terms as

$$\begin{aligned} \phi_{312;00} \left(\Psi_2^o \otimes \Psi_2^o \right) &\simeq \text{Y-shape diagram} \\ \phi_{121;\vec{e}_i,0} \left(\bar{\Psi}_2^{\vec{e}_i} \otimes \Psi_2^o \right) &= \text{Y-shape diagram with a circle labeled } e_i \text{ on the right branch} \end{aligned}$$

When we define weighted diagonals, we will work over our usual ground ring $R = \mathbb{F}_2[V_0, \dots, V_{N+1}]$.

We define *weight for tensor products of trees* as

$$\begin{cases} \text{wt}_1(V_0^{a_0} \cdots V_{N+1}^{a_{N+1}} \cdot S \otimes T) = a_0 + \text{wt}(S) \\ \text{wt}_2(V_0^{a_0} \cdots V_{N+1}^{a_{N+1}} \cdot S \otimes T) = \sum_{i=1}^{N+1} a_i + \text{wt}(T) \end{cases} \quad (2.48)$$

where again $\text{wt}(S), \text{wt}(T)$ just denote the magnitudes of the total weights of S, T , respectively.

The last piece of the set-up is the *generalized set of trees* we will use in our definition of the diagonal. We start with $X_*^{n,\mathbf{w}}$ and add

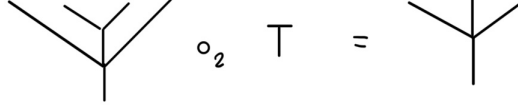
- (i) The *stump*, $\top$, with no vertices and no inputs. Making the (formal) stipulation that $\top$ actually has -1 internal vertices, (2.45) gives $\dim \top = 0$. We write $\tilde{\mathcal{T}}_{0,0}$ for this stump.
- (ii) The *shoot*, $\downarrow$, with one input and no internal vertices, so that $\dim \downarrow = 0$. We write $\tilde{\mathcal{T}}_{1,0}$ for this shoot.

We then define

$$\tilde{X}_*^{n,\mathbf{w}} = \begin{cases} X_*^{n,\mathbf{w}} & n > 0 \\ R(\tilde{\mathcal{T}}_{0,0}) & n = 0, \mathbf{w} = 0; \\ R(\tilde{\mathcal{T}}_{1,0}) & n = 1, \mathbf{w} = 0; \end{cases} \quad (2.49)$$

We extend the differential to the complex $\tilde{X}_*^{n,\mathbf{w}}$ by letting it be the same as usual when $n > 0$, and vanishing on $\tilde{X}_*^{0,0}, \tilde{X}_*^{1,0}$. To extend the gluing / composition map $\circ_i$, we make the following definitions:

- (i) For $\top$, and any suitable T , we set $T \circ_i \top = 0$ unless the vertex adjacent to the i -th input leaf of T is 3-valent – i.e. near this vertex, T looks like the two-pronged corolla Ψ_2^0 – in which case, $T \circ_i \top$ is the tree obtained from T by removing the i -th input and the adjacent leaf. For instance:



- (ii) Any composition with $\downarrow$ is the identity $\tilde{X}^{n,\mathbf{w}} \rightarrow \tilde{X}^{n,\mathbf{w}}$.

We are now ready to define diagonals. A *weighted diagonal* is a family of maps

$$\Gamma_*^{n,\mathbf{w}} : X_*^{n,\mathbf{w}} \rightarrow \bigoplus_{|\mathbf{w}-\mathbf{w}_1-\mathbf{w}_2| \geq 0} \tilde{X}^{n,\mathbf{w}_1} \otimes \tilde{X}^{n,\mathbf{w}_2} \otimes R \quad (2.50)$$

satisfying conditions **WD1.** through **WD4.**, where $n \in \mathbb{Z}_{\geq 0}$ and $\mathbf{w}$ ranges over all finite sums of the basic weight vectors $\{\mathbf{e}_i\}_{i=0}^{N+1}$. The rules we require Γ to satisfy are as follows:

WD1. (Preserves dimension) For $n, \mathbf{w}$, and $T \in X_*^{n,\mathbf{w}}$,

$$\dim \Gamma(T) = \dim T \quad (2.51)$$

with the convention that $\dim(S \otimes T) = \dim S + \dim T$ for any tensor product of trees;

WD2. (Preserves weight) For any tree $T \in X^{n,\mathbf{w}}$, we have

$$\text{wt}_1(\Gamma^{n,\mathbf{w}}(T)) = \text{wt}_2(\Gamma^{n,\mathbf{w}}(T)) = \text{wt}(T) = |\mathbf{w}|. \quad (2.52)$$

where the weights on each side are defined as in (2.48).

WD3. (Stacking) The requirement is

$$\Gamma^{n,\mathbf{v}+\mathbf{w}} \circ \phi_{i,j,n;\mathbf{v},\mathbf{w}} = \sum_{\substack{|\mathbf{v}-\mathbf{v}_1-\mathbf{v}_2| \geq 0 \\ |\mathbf{w}-\mathbf{w}_1-\mathbf{w}_2| \geq 0}} (\phi_{i,j,n;\mathbf{v}_1,\mathbf{w}_1} \otimes \phi_{i,j,n;\mathbf{v}_2,\mathbf{w}_2}) \circ (\Gamma^{(j-i+1),\mathbf{v}} \otimes \Gamma^{(n+i-j+1),\mathbf{w}}) \quad (2.53)$$

WD4. (Non-degeneracy / Base-case requirements)

- (a) (Base case for unweighted) By **WD2.**, $\Gamma^{2,0}$ is a map from $X^{2,0} \rightarrow X^{2,0} \otimes X^{2,0}$. But $X^{2,0}$ is canonically isomorphic to R , with generator the unique two pronged corolla. As already noted, we then require that $\Gamma^{2,0}$ is the canonical isomorphism:

$$\Gamma_{*}^{2,0} \left(\text{Y-shape} \right) = \text{Y-shape} \otimes \text{Y-shape}$$

- (b) (Base case for weighted) We require that $\Gamma^{0, \mathbf{e}_i}$ is the predetermined seed for $\mathbf{e}_i$,

$$\begin{aligned} \Gamma_{*}^{0, \vec{\mathbf{e}}_0} \left(\text{circle } e_0 \text{ on a stem} \right) &= \top \otimes \text{circle } e_0 \text{ on a stem} \\ \Gamma_{*}^{0, \vec{\mathbf{e}}_i} \left(\text{circle } e_i \text{ on a stem} \right) &= \text{circle } e_i \text{ on a stem} \otimes \top \quad (1 \leq i \leq N+1) \end{aligned}$$

- (c) (Ruling out bad cases) At most one of each pair of trees in the diagonal is a stump or a shoot, that is, the image of $\Gamma^{n, \mathbf{w}}$ is in

$$\bigoplus_{|\mathbf{w} - \mathbf{w}_1 - \mathbf{w}_2| \geq 0} (\tilde{X}_{*}^{n, \mathbf{w}_1} \otimes X_{*}^{n, \mathbf{w}_2} \otimes R + X_{*}^{n, \mathbf{w}_1} \otimes \tilde{X}_{*}^{n, \mathbf{w}_2} \otimes R) \quad (2.54)$$

In practice, diagonals satisfying **WD1.- WD4.** are constructed using the following recipe, rather than found in nature. Starting from base-cases, or *seeds*, which we choose manually at the beginning, a diagonal is built by induction on the number of inputs and total weight. The seed for the unweighted part of the diagonal (i.e. the $\Gamma_{*}^{n,0}$, which have only unweighted trees in their domains) is by definition always

$$\Gamma_{*}^{2,0} \left(\text{Y-shape} \right) = \text{Y-shape} \otimes \text{Y-shape}$$

From this comes the unweighted part of the diagonal, via an acyclic models type construction: for an example and further details of this part of the construction, see the discussion, surrounding (2.56), below.

To construct the weighted part of the diagonal, i.e. the maps $\Gamma^{n, \mathbf{w}}$ with $|\mathbf{w}| > 0$, we have to manually choose our seed trees. We choose a set of seed trees T_i , each of which is required to be a linear combination, with coefficients either 0 or 1, of the following elements:

$$V_i \cdot \Psi_0^{\mathbf{e}_i} \otimes \top, \quad V_0 \cdot \top \otimes \Psi_0^{\mathbf{e}_i} \quad (2.55)$$

where i ranges over $1 \dots, N + 1$. We then define $\Gamma^{0, \mathbf{e}_i}(\Psi_0^{\mathbf{e}_i}) = T_i$ for each $\mathbf{e}_i$, $0 \leq i \leq N + 1$. To construct the rest of the diagonal, we again apply an inductive acyclic models type construction; that is, choose $n \geq 0$ and a weight $\mathbf{w}$, and a tree $T \in X_*^{n, \mathbf{w}}$. Assume we have already defined $\Gamma_*^{m, \mathbf{v}}$ for the cases

- $m \leq n$, $|\mathbf{v}| < |\mathbf{w}|$ and
- $m < n$, $|\mathbf{v}| \leq |\mathbf{w}|$

This means that we know what $\Gamma_*^{*,*} \partial T$ is, and one can show that it is a cycle. Then, since the associahedron is acyclic, we can guarantee that $\Gamma_*^{*,*} \partial T = \partial T'$ for some

$$T' \in \bigoplus_{|\mathbf{w} - \mathbf{w}_1 - \mathbf{w}_2| \geq 0} \tilde{X}_*^{n, \mathbf{w}_1} \otimes \tilde{X}_*^{n, \mathbf{w}_2} \otimes R. \quad (2.56)$$

We then choose such a T' , and define $\Gamma^{n, \mathbf{w}} T = T'$.

In the remainder of this subsection, we give further details about the inductive step by which we construct the diagonal from a the base-case. We work with the unweighted case for this example.

The base case is given from **WD4**., so the next step is to define $\Gamma^{3,0} \Psi_3^0$, that is, when we ask

$$\Gamma_*^{3,0} \left(\text{Y-shape} \right) = ?$$

we have

$$\partial \left(\text{Y-shape} \right) = \text{Y-shape}_1 + \text{Y-shape}_2$$

The stacking rules give us

$$\begin{aligned} \text{Y-shape}_1 &= \phi_{312;00} \left(\text{Y-shape} \otimes \text{Y-shape} \right) \\ \text{Y-shape}_2 &= \phi_{323;00} \left(\text{Y-shape} \otimes \text{Y-shape} \right) \end{aligned}$$

so that

$$\begin{aligned}
\Gamma_{*}^{3,0} \text{ (Y-shape) } &= (\phi_{312;00} \otimes \phi_{312;00}) \circ (\Gamma_{*}^{2,0} \otimes \Gamma_{*}^{2,0})(Y \otimes Y) \\
&= (\phi_{312;00} \otimes \phi_{312;00})(Y \otimes Y \otimes Y \otimes Y) \\
&= \text{ (Y-shape) } \otimes \text{ (Y-shape) }
\end{aligned}$$

and likewise,

$$\Gamma_{*}^{3,0} \text{ (inverted Y-shape) } = \text{ (inverted Y-shape) } \otimes \text{ (inverted Y-shape) }$$

Incidentally, this holds (by induction) for all compositions of μ_2 s, so e.g.

$$\Gamma_{*}^{5,0} \left(\text{ (V-shape with 4 inputs) } \right) = \text{ (V-shape with 4 inputs) } \otimes \text{ (V-shape with 4 inputs) }$$

and so on, for any number of inputs, provided that all operations involved are μ_2 s. But, using this for the case $n = 3$, we get

$$\Gamma_{*}^{3,0} \partial \text{ (Y-shape) } = \text{ (Y-shape) } \otimes \text{ (Y-shape) } + \text{ (Y-shape) } \otimes \text{ (Y-shape) }$$

And we have

$$\begin{aligned}
\partial \left(\text{ (Y-shape) } \otimes \text{ (Y-shape) } \right) &= \boxed{\text{ (Y-shape) } \otimes \text{ (Y-shape) }} + \text{ (Y-shape) } \otimes \text{ (Y-shape) } \\
\partial \left(\text{ (Y-shape) } \otimes \text{ (inverted Y-shape) } \right) &= \boxed{\text{ (Y-shape) } \otimes \text{ (inverted Y-shape) }} + \text{ (Y-shape) } \otimes \text{ (inverted Y-shape) } \\
\partial \left(\text{ (inverted Y-shape) } \otimes \text{ (Y-shape) } \right) &= \text{ (inverted Y-shape) } \otimes \text{ (Y-shape) } + \boxed{\text{ (inverted Y-shape) } \otimes \text{ (Y-shape) }} \\
\partial \left(\text{ (inverted Y-shape) } \otimes \text{ (inverted Y-shape) } \right) &= \text{ (inverted Y-shape) } \otimes \text{ (inverted Y-shape) } + \boxed{\text{ (inverted Y-shape) } \otimes \text{ (inverted Y-shape) }}
\end{aligned}$$

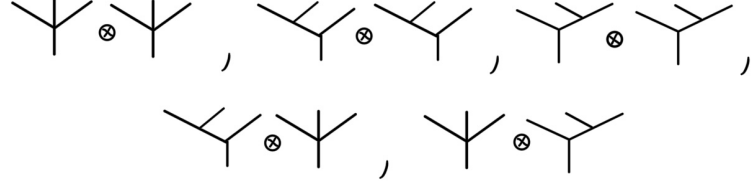
that is, we have two choices that give the correct differential, namely:

$$\left[\text{ (Y-shape) } \otimes \text{ (Y-shape) } + \text{ (Y-shape) } \otimes \text{ (inverted Y-shape) } \right] \text{ or } \left[\text{ (Y-shape) } \otimes \text{ (inverted Y-shape) } + \text{ (inverted Y-shape) } \otimes \text{ (Y-shape) } \right]$$

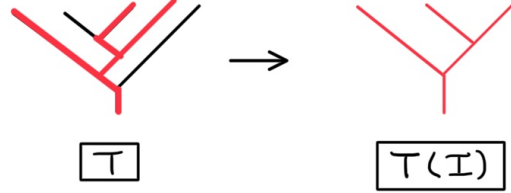
We choose:

$$\mathcal{R}_{*}^{3,0}(\text{tree}) = \text{tree} \otimes \text{tree} + \text{tree} \otimes \text{tree}$$

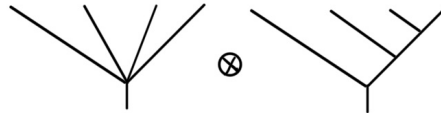
This is what is called the *right-moving* choice, and in general, we will always choose the right-moving options. For general trees, the notion of being *right-moving* is defined as follows. The basic right-moving pairs of trees are



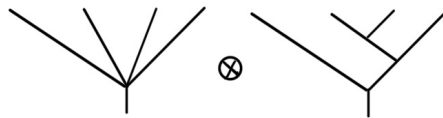
For more general trees, we need to use the notion of a *profile tree* corresponding to a subset of the inputs. For $T \in X_{*}^{n,\mathbf{w}}$ and $I := \{i_1 < \dots < i_k\}$ with $i_1 \geq 1$, $i_k \leq n$, the profile tree corresponding to I , written $T(I)$ is the tree that results when we delete the inputs whose indices are not in I , and then delete any 2-valent vertices that result. For instance, with $I = (1, 3, 4)$, and a particular choice of T , this looks like:



The profile tree for a product $S \otimes T$ of trees (both with n inputs) corresponding to the set I is written $(S \otimes T)(I)$, and is defined to be $S(I) \otimes T(I)$. A product of trees $S \otimes T$ is defined to be *right-moving* if every profile tree $(S \otimes T)(I)$ corresponding to a 3-element subset I is one of the basic right-moving pairs. So, for instance,



is right moving, while



is not. Note that “right-moving or not” does not depend on the weight, or its distribution in the tree. So when determining whether a tree is or is not right-moving, we can ignore weight. It is a result of Masuda-Thomas-Tonks-Vallette, [22], cited in [18], that we can find a right-moving diagonal in the unweighted case, which is all we will need to prove Theorem 1.0.2.

2.6.2 $\mathcal{A}_\infty$ -structures for tensor products of algebras

So far, all discussion of diagonals has been entirely on the level of trees, with no reference to the actual inputs (which in our case, will be algebra elements). The notion of a weighted diagonal allows us to define the $\mathcal{A}_\infty$ -structure for a tensor product of algebras $\mathcal{A} \otimes \mathcal{B}$, in the following way.

First, it is clear from the construction that a weighted diagonal Γ is a chain map on the complex of trees. Next, we label the inputs with algebra elements. The data for an operation on $\mathcal{A} \otimes \mathcal{B}$ is as follows

- $\Gamma_*^{n,\mathbf{w}}T$, for $T \in X_*^{n,\mathbf{w}}$, $n \in \mathbb{Z}_{\geq 0}$, and $\mathbf{w}$ ranging over finite linear combinations of the basic weight vectors;
- A collection $\{a_i \otimes b_i\}_{i=1}^n$ of pairs in $\mathcal{A} \otimes \mathcal{B}$.

The corresponding operation is now the sum over all products of trees which make up $\Gamma_*^{n,\mathbf{w}}T$, with the inputs of the left tree (of each pair) labelled with $\{a_i\}_{i=1}^n$, and the inputs of the right tree labelled with $\{b_i\}_{i=1}^n$. We use the usual maps

$$\begin{aligned}\mu : X_*^{n,\mathbf{w}} &\rightarrow \text{Mor}(\mathcal{A}^{\otimes n}, \mathcal{A}) \\ \nu : X_*^{n,\mathbf{w}} &\rightarrow \text{Mor}(\mathcal{B}^{\otimes n}, \mathcal{B})\end{aligned}$$

to go from trees to valid operations, on the $\mathcal{A}$ -side and the $\mathcal{B}$ -side respectively. We further stipulate that $\mu(\top) = V_0$, and $\nu(\top) = \sum_{i=1}^N V_i$. For instance,

or

$$\begin{aligned}\mu_2(u_i \otimes \rho_i, u_i \otimes \rho_i) &= (\mu \otimes \nu) \circ \Gamma_*^{2,0} \left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \end{array} \right) (u_i \otimes \rho_i, u_i \otimes \rho_i) \\ &= \begin{array}{c} u_i \quad u_i \quad \rho_i \quad \rho_i \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array} \otimes \begin{array}{c} \rho_i \quad \rho_i \\ \diagdown \quad \diagup \\ \text{---} \end{array} \\ &= 0\end{aligned}$$

That the operations on $\mathcal{A} \otimes \mathcal{B}$ determined in this way satisfy $\mathcal{A}_\infty$ relations follows directly from the fact that Γ is a chain map, and from the original $\mathcal{A}_\infty$ relations on $\mathcal{A}$ and $\mathcal{B}$.

2.6.3 Box tensor products of bimodules

The goal of this subsection is to define operations for the box tensor product of a DD - and an AA -bimodule, i.e. ${}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}}$, which will be a DA -bimodule. We will not need this until Section 2.8, but it is convenient to discuss it now. As an R -module, $X \boxtimes Y$ is just the usual tensor product $X \otimes Y$. In general, the operations on $X \boxtimes Y$ will be determined by a diagonal, analogous to the one for the product of two *unweighted* algebras. That the box tensor product, defined in this way, is a bona fide DA -bimodule, is shown in Section 7 of [18]. However, the actual process of computing the operations for the tensor product quickly becomes very complicated, since, as soon as an operation has more than one $\mathcal{A}$ -input, we need to consider what are called *primitive trees*. In general, an operation on $X \boxtimes Y$ will look something like Figure 2.17,

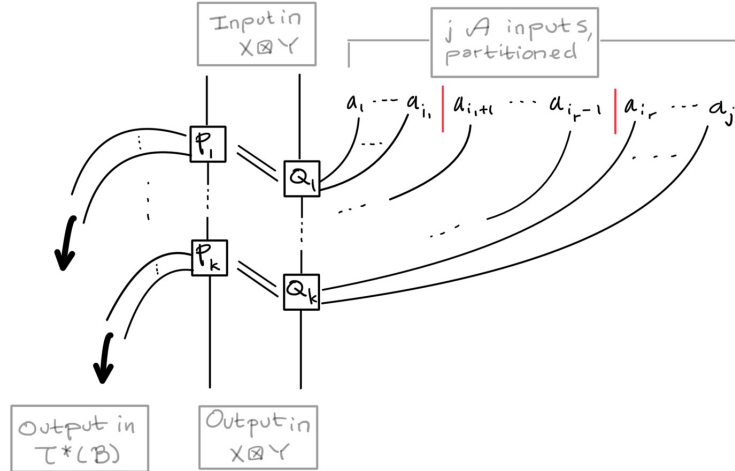


Figure 2.17: A generic operation for $X \boxtimes Y$

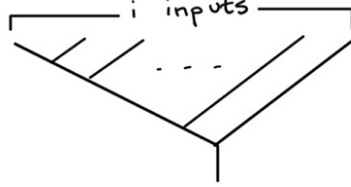
where the (P_i, Q_i) are so-called primitive pairs of trees. This will not be used any further, so we will not give an actual definition here (for a rigorous treatment, see [18]). In practice, we will use Propositions 2.1.1 and 2.1.2 to study box tensors of $\mathcal{A}_\infty$ -modules in terms of $\mathcal{A}_\infty$ -homomorphisms.

This will mean that we will only compute – or need – operations with zero or one $\mathcal{A}$ -input, so it suffices to discuss the *basic operations* on $X \boxtimes Y$, that is, those which can be computed directly from the operations on X and those on Y . These basic $\mathcal{A}_\infty$ operations $\delta_{1+j}^1 : X \otimes Y \otimes \mathcal{A}^{\otimes j} \rightarrow \mathcal{B} \otimes X \otimes Y$

are defined as follows:

$$\delta_{1+j}^1 := \sum_{\substack{i \in \mathbb{N} \\ |\mathbf{w}| \geq 0}} f_i \circ (\text{id}_{\mathcal{B}^{\otimes i} \otimes X} \otimes m_{i|j}) \circ (\delta_X^i \otimes \text{id}_{Y \otimes \mathcal{A}^{\otimes j}}) \quad (2.57)$$

where $f_i : \mathcal{A}^{\otimes i} \rightarrow \mathcal{A}$ is the i -fold iterate of $\mu_2^{\mathcal{A}}$, i.e. the $\mathcal{A}$ -operation with corolla



and where $\delta_X^1 : X \rightarrow \mathcal{A} \otimes \mathcal{B} \otimes X$ is the basic operation for the DD -bimodule X to be defined in the next section. In terms of corollas, the δ_{1+j}^1 are:

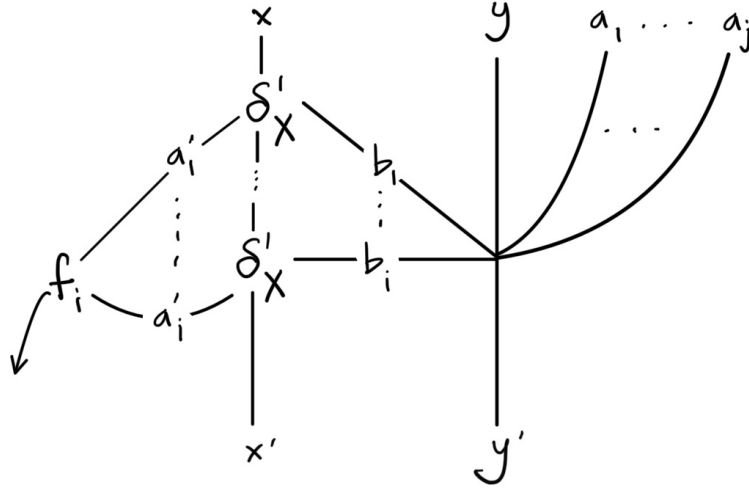


Figure 2.18: A basic operation for $X \boxtimes Y$: j $\mathcal{A}$ -inputs, i applications of δ_X^1 , and one $\mathcal{A}$ -output

2.7 The DD bimodule

The goal of this section is to construct the DD bimodule $\mathcal{A}_0 X^{\mathcal{B}_0}$ over the unweighted algebras which we will use for Theorem 1.0.2, and verify the $\mathcal{A}_\infty$ relations for it. The generators of the DD bimodule X are the complements of generators of Y , that is, writing

$$\bar{\mathbf{x}} = \overline{\{i\}} = \{1, \dots, N\} \setminus \{i\},$$

X is generated by $\{\bar{\mathbf{x}} : \mathbf{x} \text{ a generator of } Y\}$. For the operations, recall that a DD bimodule over $\mathcal{A}$ and $\mathcal{B}$ (e.g. X) is just a type-D module over the (weighted) $\mathcal{A}_\infty$ algebra $\mathcal{A} \otimes \mathcal{B}$, with operations defined as in Section 2.1.6. The operation on X is

$$\delta^1 : X \rightarrow \mathcal{A} \otimes \mathcal{B} \otimes X \quad (2.58)$$

given by

$$\delta^1(\bar{\mathbf{x}}) = U_i \otimes \rho_i \otimes \mathbf{x} + s_i \otimes \sigma_i \otimes \mathbf{y}$$

where $\bar{\mathbf{x}} = \overline{\{i\}}$, $\bar{\mathbf{y}} = \overline{\{i+1\}}$, and $1 \leq i \leq N$. In terms of trees, this looks like

$$\delta^1 : \begin{array}{c} \overline{i} \\ | \\ \rho_i \text{---} \text{---} \sigma_i \\ | \\ \overline{i} \end{array} + \begin{array}{c} \overline{i} \\ | \\ \sigma_i \text{---} \text{---} s_i \\ | \\ \overline{i+1} \end{array}$$

In order to explicitly discuss these operations, we need to state what diagonal we are using to define operations on $\mathcal{A} \otimes \mathcal{B}$. For this section, we are constructing an unweighted diagonal. Therefore, there is only one point in the construction of the diagonal where we are required to make an explicit choice; namely, during the inductive step of the acyclic models construction. We already specified in Section 2.6.1 that in each acyclic models step, we would choose the (unique) right-moving tree. Refer back to the last portion of Section 2.6.1 for further details.

The rest of this section is devoted to verifying (2.11) for this particular choice of δ^1 . The first step will be to identify which summands $(\mu_n \otimes \mathbf{I}) \circ \delta^n$ do not vanish, for a given $\bar{\mathbf{x}} = \overline{\{i\}}$.

Before we do this, however, we need to recall the definition of *total length* of a sequence of algebra elements $\{a_1, \dots, a_n\}$, initially given defined in Section 2.3. Again, length is only defined for a sequence $\{a_1, \dots, a_n\}$ where the initial idempotent of a_i is the final idempotent of a_{i-1} for each i . To get the length of such a sequence, start a_1 , and go along the list adding 1 each time we pass into to a new idempotent.

For instance, U_1, s_1, U_2 has length 2, and $\sigma_1, \sigma_2, \sigma_3, \sigma_1$ has length 4 (where we are assuming $N = 3$ to make this allowable from an idempotent standpoint). Notice again that the notion of length is distinct from the number of basic inputs for a given sequence; so for instance $(\sigma_2 \rho_2 \sigma_1, \sigma_3)$ has length 3, but has 4 basic inputs.

We are now ready to state the main lemma.

Lemma 2.7.1. *Start with $\bar{\mathbf{x}} = \overline{\{i\}}$. Then in the unweighted case, where we are only constructing a DD -bimodule over $\mathcal{A}_0$ and $\mathcal{B}_0$, $(\mu_n \otimes \mathbf{I}) \circ \delta^n \equiv 0$.*

Remark 2.7.2. This lemma will also serve to show that the sum of (2.11) is finite, and verify the $\mathcal{A}_\infty$ -relations (2.11).

Proof. In this proof we are not going to try to determine all the possible operations $\mu_n^{\mathbf{w}}$; since these come from the diagonal, they get very complicated very quickly. Instead, since we are looking at $(\mu_n^{\mathcal{A}_0 \otimes \mathcal{B}_0} \otimes \mathbf{I}) \circ \delta^n$, the trick here is that the input sequence of $\mu_n^{\mathcal{A}_0 \otimes \mathcal{B}_0}$, say $(a_1 \otimes b_1, \dots, a_n \otimes b_n)$, must come from repeated applications of δ_1 , and hence,

1. Each a_i and each b_i is basic (no composite / pre-multiplied inputs);
2. There are the same number of basic inputs on the $\mathcal{A}$ - and $\mathcal{B}$ -sides;
3. The Alexander gradings of the k -th input on the $\mathcal{A}$ -side and the k -th input on the $\mathcal{B}$ -side match, for each $1 \leq k \leq n$;

The key takeaway from these conditions – especially (3) – is that once we determine the $\mathcal{A}_0$ -side of the inputs, we have now determined the $\mathcal{B}_0$ -side of the inputs, as well; each $\mathcal{A}_0$ -input has a unique partner on the $\mathcal{B}_0$ -side, and vice-versa. For instance, if the sequence of $\mathcal{A}_0$ -inputs for $\mu_n \otimes \mathbf{I}$ is s_1, U_2, s_2, s_3 , then we know the sequence of $\mathcal{B}_0$ -inputs *must* be $\sigma_1, \rho_2, \sigma_2, \sigma_3$, and the full sequence is

$$(\bar{\mathbf{x}}, s_1 \otimes \sigma_1, U_2 \otimes \rho_2, s_2 \otimes \sigma_2, s_3 \otimes \sigma_3)$$

Now, notice that if a_1, a_2 are basic elements in $\mathcal{A}_0$ with $a_1 a_2 \neq 0$, then their partner elements b_1, b_2 on the $\mathcal{B}_0$ -side will have $b_1 b_2 = 0$, and vice versa. Indeed,

- $U_i \cdot U_i \neq 0$ and $\rho_i \cdot \rho_i = 0$;
- $s_i \cdot s_{i+1} \neq 0$ and $\sigma_{i+1} \sigma_i = 0$;³
- $\sigma_i \rho_i \neq 0$ and $U_i s_i = 0$;
- $\rho_{i+1} \sigma_i \neq 0$ and $s_i U_{i+1} = 0$;

But we are constructing a bimodule over only $\mathcal{A}_0$ and $\mathcal{B}_0$, which have only μ_2 's. This means that the only pairs of trees $T_1 \otimes T_2$ which correspond to non-zero outputs of $\mu_n^{\mathcal{A}_0 \otimes \mathcal{B}_0}$ are ones where T_1, T_2 are both just compositions of μ_2 's. This means that in order to find a non-zero output of some

³Recall that multiplication in $\mathcal{B}$ goes right to left rather than left to right, which is why we reverse the order. (In this particular case, it is of course also true that $\sigma_i \cdot \sigma_{i+1} = 0$.)

$(\mu_n^{\mathcal{A}_0 \otimes \mathcal{B}_0} \otimes \mathbf{I}) \circ \delta^1$, we would need to have a sequence $(a_1 \otimes b_1, \dots, a_n \otimes b_n)$ of elements in $\mathcal{A}_0 \otimes \mathcal{B}_0$ which satisfy 1 through 3, and such that $a_i a_{i+1}$, $b_{i+1} b_i$ are not zero for each i . But by the previous paragraph, this is impossible. Hence, all the terms of (2.11) are zero. $\square$

We have now additionally verified:

Proposition 2.7.3. *The $\mathcal{A}_\infty$ -relations, (2.11), hold for X , so X is a valid weighted DD -imodule.*

2.8 Koszul Duality

The aim of this section is to prove Theorem 1.0.2. First, we need just a little more background, which will be provided in the following two subsections.

2.8.1 The DA Bimodules

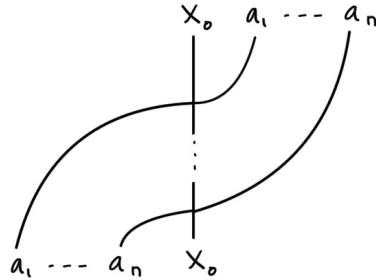
Recall first what ${}^{\mathcal{A}}\text{id}_{\mathcal{A}}$ is (for a given *unweighted* $\mathcal{A}_\infty$ -algebra $\mathcal{A}$). This DA -bimodule has a single generator $\mathbf{x}_0$, and has

$$\delta_2^1 : \text{id} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \text{id},$$

with $\delta_2^1(\mathbf{x}_0 \otimes a) = a \otimes \mathbf{x}_0$. We define $\delta_{1+j}^1 \equiv 0$ for $j > 1$, so $\delta^1 = \delta_2^1$. Non-zero higher operations are iterates of this initial operation, i.e.

$$\delta^{i+1} = (\mathbf{I}_{\mathcal{A}^{\otimes i}} \otimes \delta^1) \circ (\mathbf{I}_{\mathcal{A}^{\otimes (i-1)}} \otimes \delta^1) \circ \dots \circ \delta^1 \circ (\mathbf{I}_{\text{id}} \otimes \triangle)$$

so in terms of trees,



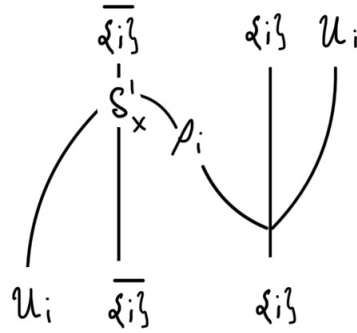
All other higher operations, δ_{j+1}^i with $i \neq j$, are zero.

Now consider $Z = X \boxtimes Y$. Again, we are dealing with the unweighted associative algebras $\mathcal{A}_0$ and $\mathcal{B}_0$, with associated DD - and AA -bimodules, as constructed above, all with ground ring $\mathbb{F}_2$. Therefore, Z is a bona-fide DA -bimodule with a single generator $\bar{z}_0$. We want to apply Proposi-

tions 2.1.1 to produce an $\mathcal{A}_\infty$ -homomorphism $\varphi : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ which has $Z = {}^{\mathcal{B}_0}[\varphi]_{\mathcal{A}_0}$. To do this, we just need to show that $\delta_1^1 = 0$ and δ_2^1 is the identity.

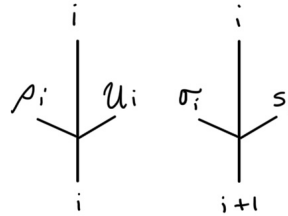
While, as noted in Section 2.6.3, we are not equipped to compute all operations on $X \boxtimes Y$ directly, we *are* equipped to compute those with small numbers of inputs. This is because primitives only come into play when we can partition the inputs into more than one way (this is still true in the weighted case, because we have no diagonal curvature). If there are zero or one algebra inputs, there is only one partition, and hence, the options are much more limited.

First, consider the DA -operation $\delta_1^1 : Z \rightarrow \mathcal{A} \otimes Z$. Any operation on Z must have any $\mathcal{A}$ -inputs, and the Y -component of the $X \boxtimes Y$ -input, feeding into some bona-fide Y -operation, e.g.



But then the only way to form a valid DA -operation is to find a composition of the DD -bimodule operation δ_X^1 that matches the inputs of the suitable AA -bimodule operation, as in Figure 2.18. There is no composition of DD -bimodule operations that has $\mathcal{B}$ -output any V_i , so $\delta_1^1 \equiv 0$.

Next, look at $\delta_2^1 : Z \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes Z$. We are only going to explicitly compute this for short-chord inputs, i.e. U_i and s_i , $1 \leq i \leq N$. For those $\mathcal{A}$ -inputs, there are only two classes of operations on Y , those on the left side of Figure 2.14, that is



The only composition of δ_B^1 's which gives the correct $\mathcal{B}$ -output for either of these two cases is δ_B^1 itself, so we have

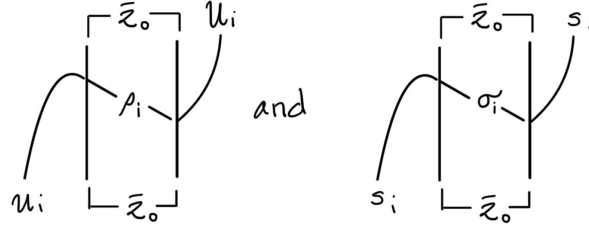


Figure 2.19

where $\bar{z}_0$ denotes the single generator of $Z = X \boxtimes Y$, that is, δ_2^1 is the identity on the short-chords

Now consider $Z = Y \boxtimes X$, also viewed as a DA bimodule, this time over $\mathcal{B}$ on the right and $\mathcal{A}$ on the left (i.e. ${}^{\mathcal{B}}Z_{\mathcal{B}}$), with operations such as

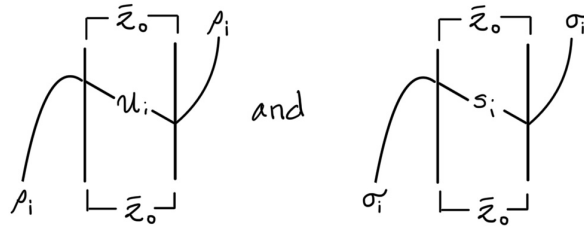


Figure 2.20

By an argument identical to the one for $X \boxtimes Y$, we can show that here as well, $\delta_1^1 \equiv 0$, and δ_2^1 is the identity on the short chords σ_i, ρ_i ($1 \leq i \leq N$).

2.8.2 Proof of Theorem 1.0.2

We are now ready to give the proof of the main duality theorem.

Proof of Theorem 1.0.2. There are three steps in the proof. First, recall that $X \boxtimes Y$ has $\delta_1^1 = 0$, as noted above. Hence, by Proposition 2.1.1, there exists an $\mathcal{A}_{\infty}$ -homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ with

$$X \boxtimes Y = {}^{\mathcal{A}}[\varphi]_{\mathcal{A}} \quad (2.59)$$

Second, we show manually that $\varphi_1 = \text{id}$. Then, for grading reasons, we can show that $\varphi_i = 0$ for each $i > 1$. This shows that $\varphi = \text{id}_{\mathcal{A}}$. Then (1.3) follows by Proposition 2.1.2.

To show that $\varphi_1 = \text{id}$, we note that the operations in $X \boxtimes Y$ with one input are still of the form in Figure 2.19, so $\varphi_1(U_i) = U_i$, $\varphi_1(s_i) = s_i$, for all $1 \leq i \leq N$.

Now, the $\mathcal{A}_\infty$ relation for φ with two inputs is as in Figure 2.2. Since the differential on $\mathcal{A}$ is zero, we get back

$$\begin{array}{c} \begin{array}{c} | \quad | \\ \varphi_1 \quad \varphi_1 \\ \diagdown \quad \diagup \\ | \end{array} + \begin{array}{c} \diagdown \quad \diagup \\ \varphi_1 \\ | \end{array} = 0 \end{array}$$

that is,

$$\varphi_1(a_1 a_2) = \varphi_1(a_1) \varphi_1(a_2),$$

This now means, by induction, φ_1 is the identity on $\mathcal{A}$.

Next, look back at the $\mathcal{A}_\infty$ -relation for two inputs in Figure 2.2. The third and fourth terms vanish immediately, so look at $d\varphi_2$. If, for a given pair of elements $a_1, a_2 \in \mathcal{A}$, we had $\varphi_2(a_1 a_2) \neq 0$, then order for the Maslov gradings to work out in this relation, we would need

$$m(\varphi_2(a_1, a_2)) + 1 = m(a_1 a_2) = 0$$

But there is no such element, so $\varphi_2(a_1, a_2) = 0$. By a similar argument φ_2 's involving V_i 's are zero. Now by induction, if we know that all $\varphi_{k'}$ with $1 < k' < k$ are zero, the $\mathcal{A}_\infty$ -relation for k inputs and one output gives

$$m(\varphi_k(a_1, \dots, a_k)) = m(\mu_k(a_1, \dots, a_k)) + 1 = \begin{cases} k - 1 & \text{this } \mu_k \text{ is nonzero} \\ -1 & \text{otherwise} \end{cases} \quad (2.60)$$

If the μ_k is nonzero, then looking back at Theorem 2.2.1, which tells as what k must be, we see that there is no element in grading $k - 1$. There are also no elements (of $\mathcal{A}$) in grading -1 . Hence, $\varphi_k(a_1, \dots, a_k)$ is zero, too.

It now follows that $\varphi = \text{id}_{\mathcal{A}}$. By Proposition 2.1.2, it now follows that

$$X \boxtimes Y = [\varphi] \simeq [\text{id}_{\mathcal{A}}] \simeq {}^{\mathcal{A}}\text{id}_{\mathcal{A}},$$

as desired.

Now, we want to look at the reverse, (2.7). As above, $\delta_1^1 = 0$ for $Y \boxtimes X$, so there is a map $\varphi : \mathcal{B} \rightarrow \mathcal{B}$ with $Y \boxtimes X = {}^{\mathcal{B}}[\varphi]_{\mathcal{B}}$. We are going to show that, again, $\varphi_1 = \text{id}_{\mathcal{B}}$. As before, φ_1 is the identity on short chords because the only one-input operations on $Y \boxtimes X$ with inputs σ_i 's and ρ_i 's

are the ones pictured in Figure 2.20. For the V_i 's, use the $\mathcal{A}_\infty$ -relation

$$\begin{array}{c} \rho_i \\ | \\ \bullet \\ | \\ \varphi_i \\ | \\ \varphi_i(V_i) \end{array} + \begin{array}{c} \rho_i \\ | \\ \varphi_i \\ | \\ \bullet \\ | \\ V_i \end{array} = \bigcirc$$

Now, note that for e.g. σ_i, ρ_i , the $\mathcal{A}_\infty$ -relation in Figure 2.2 gives

$$\begin{array}{c} \sigma_i \quad \rho_i \\ \diagup \quad \diagdown \\ \cup \\ | \end{array} + \begin{array}{c} \sigma_i \quad \rho_i \\ \diagup \quad \diagdown \\ \cup \\ \varphi_2 \\ | \\ \bullet \\ | \end{array} + \begin{array}{c} \sigma_i \quad V_i \\ \diagup \quad \diagdown \\ \cup \\ \varphi_2 \\ | \end{array} + \begin{array}{c} \sigma_i \quad \rho_i \\ \diagup \quad \diagdown \\ \cup \\ \varphi_1 \\ | \end{array} = 0$$

But $\varphi_2(\sigma_i, \rho_i)$ must vanish for grading reasons, as will $\varphi_2(\sigma_i, V_i)$ (as can be seen by a short computation again using Figure 2.2). This means that $\varphi_1(\sigma_i \rho_i) = \sigma_i \rho_i$, as desired. By a similar (inductive) it follows that $\varphi_1 = \text{id}$ on the other longer chords.

Now we are going to pass to homology: $H(\mathcal{B})$ is an $\mathcal{A}_\infty$ -algebra whose non-zero classes correspond to products of the form

$$\sigma_i, \quad \sigma_{i+1} \rho_{i+1} \sigma_i, \quad \sigma_{i+2} \rho_{i+2} \sigma_{i+1} \rho_{i+1} \sigma_i, \quad \text{etc.},$$

up to but not including $\sigma_{N+i-1} \rho_{N+i-2} \cdots \rho_{i+1} \sigma_i$ (which is not a cycle). The V_i , $1 \leq i \leq N$, are all boundaries. Anything with an exposed ρ_i is not a cycle, and anything including a V_i is a boundary.

We want to look at the $\mathcal{A}_\infty$ -homomorphism $\bar{\varphi} : H(\mathcal{B}) \rightarrow H(\mathcal{B})$ induced by φ . Since $\varphi_1 = \text{id}_{\mathcal{B}}$, $\bar{\varphi}_1 = \text{id}$. We will now show that for grading reasons, $\bar{\varphi}_k$ vanishes for each $k > 1$. Supposing $\bar{\varphi}_{k'} = 0$ for $k' < k$, we get an equation analogous to (2.60), but with homology classes $\bar{b}_j$'s instead of a_j 's in $\mathcal{A}$, that is

$$d\bar{\varphi}_k(\bar{b}_1, \dots, \bar{b}_k) = \mu_k(\bar{\varphi}_1(\bar{b}_1), \dots, \bar{\varphi}_1(\bar{b}_k)) = \mu_k(\bar{b}_1, \dots, \bar{b}_k)$$

Each $\bar{b}_j$ corresponds to an element of $\mathcal{B}$ of the form

$$\sigma_{i_j+r_j} \rho_{i_j+r_j} \cdots \rho_{i+1} \sigma_i$$

which has Maslov index $-(2r_j + 1)$, and is in an odd number of distinct Alexander gradings. This means that $\bar{\varphi}_k(\bar{b}_1, \dots, \bar{b}_k)$ must have Maslov index $-\sum_{j=1}^k (2r_j + 1) + 1$ and is in $\sum_{j=1}^k (2r_j + 1)$ Maslov gradings (possibly counted with multiplicity). Since there is no such element, $\bar{\varphi}_k(\bar{b}_1, \dots, \bar{b}_k) = 0$.

We have therefore shown that $\bar{\varphi} : H(\mathcal{B}) \rightarrow H(\mathcal{B})$ is the identity. Now, we use the quasi-isomorphism $\Phi : H(\mathcal{B}) \rightarrow \mathcal{B}$ described in Keller, [7], to get the following diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\varphi} & \mathcal{B} \\ \downarrow \Phi^{-1} & & \uparrow \Phi \\ H(\mathcal{B}) & \xrightarrow{\bar{\varphi}=\text{id}} & H(\mathcal{B}) \end{array}$$

to get that $\varphi = \text{id}_{\mathcal{B}}$, too. Now, as above, by Proposition 2.1.2, we are done. $\square$

Chapter 3

Bordered Algebras and the Wrapped Fukaya Category

3.1 Introduction

The aim of this chapter is to present the proof of Theorem 1.0.3. Note that in this chapter, we will be working only over the ground ring $R = \mathbb{F}_2[V_0, V_{N+1}]$ and will not discuss weighted operations, or μ_0 operations of any kind.

The first step is to review the algebraic concepts required to prove Theorem 1.0.3. We then briefly restate the definitions of the star algebras used in this chapter. (Since these definitions differ significantly from those in the previous chapter, it is useful to state them again in full here.) In the second half of the chapter, we use Lemma 3.5.2, which says that the cobar algebra of $\mathcal{A}_0$ is $\mathcal{B}_0$, and the cobar algebra of $\mathcal{B}_0$ is $\mathcal{A}_0$, to compute the Hochschild cohomologies of the two associative algebras. This allows us to conclude that $\mathcal{A}$ and $\mathcal{B}$ are the unique $\mathcal{A}_\infty$ -deformations of $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, with certain basic $\mathcal{A}_\infty$ -operations (Proposition 3.5.1). We then finish the proof with model calculations (Lemmas 3.6.1 through 3.6.4) which show that the endomorphism algebras in the Fukaya category can be viewed as a deformation of $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, with the desired higher operations.

As noted above, many of the techniques used in this chapter are related to previous works. The star diagrams used to construct the algebras of [9] are similar to the Pong diagrams of [29] and [30], and are motivated by constructions from bordered knot Floer homology, e.g. [26], [27], and [28]. However, the constructions for the higher multiplications on $\mathcal{A}$ and $\mathcal{B}$ are far less similar to the

strand algebras in these papers, or in e.g. [16] or [11], than they are to the weighted torus algebras constructed in [17]. It has also been suggested, [19], that these algebras can be expressed in terms of the bordered sutured construction, [36]. The specific constructions with the wrapped Fukaya category which are used in this thesis are primarily derived from the work of Auroux, see e.g. [1], [2], or [3].

This chapter should provide the basis for a planned future work which will involve the original Koszul dual *weighted* $\mathcal{A}_\infty$ -algebras discussed in the previous chapter. In particular, the reader will note that the $\mathcal{A}_\infty$ -algebras $\mathcal{A}$ and $\mathcal{B}$ discussed in this thesis are actually stripped down versions of the original weighted algebras constructed and discussed in the previous chapter. These algebras include certain higher operations (see Propositions 3.4.1 and 3.4.3), but no weighted operations. This is by design. There is, at the moment, no well-defined notion of the endomorphism algebras from Theorem 1.0.3 which could be used for the weighted case; therefore a version of Theorem 1.0.3 for the original Koszul dual weighted $\mathcal{A}_\infty$ -algebras from the previous chapter is not currently known. The author expects, however, that by viewing the original algebras from the previous chapter as weighted deformations of the algebras $\mathcal{A}$ and $\mathcal{B}$ discussed in this thesis, it will be possible to modify the construction of the endomorphism algebras from Theorem 1.0.3 so as to analogous equivalence for the original algebras.

The chapter is structured as follows. In Sections 3.2 and 3.3, we review the basic definitions and constructions used in this thesis, namely $\mathcal{A}_\infty$ -algebras, Hochschild cohomology, and the wrapped Fukaya category. In Section 3.4, we review the construction of the underlying associative algebras $\mathcal{A}_0$ and $\mathcal{B}_0$. In Section 3.5, we compute the Hochschild cohomology of $\mathcal{A}_0$ and $\mathcal{B}_0$ in order to verify $\mathcal{A}$ and $\mathcal{B}$ are the unique $\mathcal{A}_\infty$ -deformations of $\mathcal{A}_0$ with the desired basic higher operations. Finally, in Section 3.6, we use model calculations in the wrapped Fukaya category setting to show that we can apply this uniqueness result to conclude Theorem 1.0.3.

3.2 More Algebraic Definitions

The aim of this section is to introduce the basic algebraic concepts which will be used in the following sections. None of the definitions or results cited here are original to this thesis, and relevant sources are cited in each case.

3.2.1 The cobar complex

An *augmented associative algebra* is an associative algebra A over a ring R equipped with an augmentation map $\epsilon : A \rightarrow R$. Let $A_+ = \ker \epsilon$ be the augmentation ideal and $\Pi : A \rightarrow A_+$ the canonical projection.

Given an augmented associative algebra A , define the *reduced bar complex* of A as

$$\text{Bar}(A) = A \otimes A \leftarrow A \otimes A_+ \otimes A \leftarrow \cdots \leftarrow A \otimes A_+^{\otimes n} \otimes A \leftarrow \cdots$$

Here the tensor product is taken over R , but this is viewed as a chain-complex of (A, A) -bimodules with differential $\partial : A \otimes A_+^{\otimes n} \otimes A \rightarrow A \otimes A_+^{\otimes(n-1)} \otimes A$ given by

$$\partial(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{0 \leq i \leq n} a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}.$$

Let R_0 be a finite direct sum of copies of $\mathbb{F}_2$. Usually, we will just have $R_0 = \mathbb{F}_2$. The *reduced cobar complex* $\text{Cob}(A)$ is defined to be $\text{Cob}(A) = \bigoplus_{n=1}^{\infty} (A_+^{\otimes n})^*$, with a differential δ^{Cob} which is the transpose of the map

$$\sum_{m,n \geq 0} \text{id}_{A^{\otimes m}} \otimes \mu_2 \otimes \text{id}_{A^{\otimes n}} : \text{Bar}(A) \rightarrow \text{Bar}(A).$$

We usually view $\text{Cob}(A)$ as an associative algebra with multiplication given by

$$(f_n \star g_m)(a_1 \otimes \cdots \otimes a_{n+m}) = f_n(a_1 \otimes \cdots \otimes a_n) \cdot g_m(a_{n+1} \otimes \cdots \otimes a_{n+m}),$$

for $f_n \in (A_+^{\otimes n})^*$, $g_m \in (A_+^{\otimes m})^*$. Moreover, when A is finite-dimensional over $\mathbb{F}_2$, we can write $(A^{\otimes n})^* = (A^*)^{\otimes n}$ for each n , so that δ^{Cob} has the form

$$\delta^{\text{Cob}}(a_1^* \otimes \cdots \otimes a_n^*) = \sum_{i=1}^n a_1^* \otimes \cdots \otimes \mu_2^*(a_i^*) \otimes \cdots \otimes a_n^*,$$

where $\mu_2^*(a^*)(a_1 \otimes a_2) = a^*(a_1 a_2)$, for each $a^* \in A^*$ and $a_1, a_2 \in A$.

The cobar algebra is particularly interesting in the context of this thesis because of Proposition 3.5.2, which says that each of the two basic augmented associative algebras $\mathcal{A}_0$ and $\mathcal{B}_0$, defined in Section 3.4, is the cobar of the other. In order to correctly formulate Proposition 3.5.2, however, we must first modify the definition of the cobar algebra by introducing a unitality condition.

The first step in doing this is to introduce the notion of an *associative algebra with idempotent elements*. Start with an associative algebra A over a ground ring R , equipped with *idempotent*

elements $\{I_1, \dots, I_k\}$ with the following properties:

(I1) Let I be the R -module generated by $\{I_j\}_{j=1}^k$; then there exists an ideal $A_+ \subseteq A$ with $A = I \oplus I_+$;

$$(I2) \quad I_i I_j = \begin{cases} I_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(I3) For each $a \in A_+$, there is a unique $1 \leq i, j \leq k$ such that $I_i a \neq 0$ and $a I_j \neq 0$; these I_i and I_j are called *the initial and final idempotents of a* , respectively;

(I4) If I_i, I_j are the initial and final idempotents of $a \in A_+$, as above, then $I_i a = a I_j = a$.

(I5) For $a, b \in A_+$, $ab = 0$ if the initial idempotent of b and the final idempotent of a are not the same.

For an associative algebra A over a ground ring R equipped with such idempotent elements, the augmentation map $\epsilon : A \rightarrow R$ used in the definition of $\text{Bar}(A)$ is defined so that $\ker(\epsilon) = A_+$.

Remark 3.2.1. Note that the set of idempotents has the structure of a commutative ring (with unit $\sum_{i=1}^k I_i$). In the literature, e.g. [17], the set of idempotents is therefore referred to as the *ring of idempotents*, or *idempotent ring*.

Now define the modified notion of the cobar complex including the proper unitality condition as follows. Let A be an associative algebra with idempotent elements. Define $\widetilde{\text{Bar}}A$ to be the subcomplex of the usual (reduced) bar complex $\text{Bar}(A)$ consisting of strings $a_1 \otimes \dots \otimes a_n \in A_+^{\otimes n}$ such that for each $1 < i \leq n$, the initial idempotent of a_i is the same as the final idempotent of a_{i-1} . This is clearly a subcomplex. Define $\widetilde{\text{Cob}}(A) = \text{Hom}_{A \otimes A^{\text{op}}}(\widetilde{\text{Bar}}(A), A)$ (which is also a subcomplex of the usual cobar $\text{Cob}(A)$).

Remark 3.2.2. In [17], the authors omit the extra step of defining $\widetilde{\text{Bar}}$ and $\widetilde{\text{Cob}}$ and simply work with Bar and Cob . In their construction, however, the tensor products which make up the chain complex $\{A \otimes A_+^{\otimes n} \otimes A\}$ for Bar are implicitly assumed to be over the ring of idempotents. This has the same effect as our extra construction of $\widetilde{\text{Bar}}$ and $\widetilde{\text{Cob}}$, and the same arguments work. We make this auxiliary construction in order to make it clear that the ground ring is R throughout.

3.2.2 Hochschild cohomology and $\mathcal{A}_\infty$ -deformations

Let A be an augmented associative algebra with augmentation ideal A_+ , as in the previous section.

The *Hochschild cohomology* of A is defined to be the cohomology of the chain complex

$$\mathrm{HC}^*(A) = \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(\mathrm{Bar}(A), A),$$

with differential $\delta : \mathrm{HC}^{n-1}(A) \rightarrow \mathrm{HC}^n(A)$ given by

$$(\delta f)(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}).$$

In Section 3.5, we will compute the Hochschild cohomology of the basic associative algebras $\mathcal{A}_0$ and $\mathcal{B}_0$ and use that to conclude that $\mathcal{A}$ and $\mathcal{B}$ are the unique deformations of $\mathcal{A}_0$ and $\mathcal{B}_0$ given certain properties. In order to be able to argue in this way, we need several facts about Hochschild cohomology.

Proposition 3.2.3. (Proposition 5.7 of [17]) *Let $\mathcal{A}_0$ be an associative algebra and let $\mathcal{A}$ be an A_{n-1} -deformation of $\mathcal{A}_0$. Then there is a Hochschild cochain obstruction class $\mathcal{D}_n \in \mathrm{HC}^{n+1}(\mathcal{A}_0)$ so that the following conditions hold:*

- $\mathcal{D}_n$ is a cocycle;
- $\mathcal{D}_n$ is a coboundary if and only if there is an operation μ_n making $\mathcal{A}$ into an A_n -algebra; indeed the operation $\mu_n \in \mathrm{HC}^n(A)$ is a Hochschild cochain with $\delta(\mu_n) = \mathcal{D}_n$;
- If μ_n, μ'_n are cochains with $\delta(\mu_n) = \delta(\mu'_n) = \mathcal{D}_n$, then $\mu_n - \mu'_n$ is a coboundary such that the respective obstruction cocycles $\mathcal{D}_{n+1}, \mathcal{D}'_{n+1} \in \mathrm{HC}^{n+2}(A)$ are cohomologous;
- If $\mathcal{A}, \mathcal{A}'$ are two A_n deformations of $\mathcal{A}_0$ with $\mu_i = \mu'_i$ for each $i < n$, and with $\mu_n - \mu'_n$ a coboundary, then their respective obstruction cocycles $\mathcal{D}_{n+1}$ and $\mathcal{D}'_{n+1}$ are cohomologous.

The statement for maps (which will be used to verify uniqueness of deformations of our associative algebras) is as follows. Given A_n -deformations $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{A}_0$ and an A_{n-1} -homomorphism $f : \mathcal{A} \rightarrow \mathcal{A}'$ such that $f_1 : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ is the identity map, there is an obstruction class $\mathcal{F}_n \in \mathrm{HC}^n(\mathcal{A}_0)$ such that

- $\mathcal{F}_n$ is a cocycle;
- $\mathcal{F}_n$ is a coboundary if and only if there is an A_n homomorphism extending f ;

Corollary 3.2.4. (Corollary 5.22 of [17]) *Let $\mathcal{A}$ be an A_n structure on the associative algebra $\mathcal{A}_0$. If $\mathrm{HH}^{m+2}(\mathcal{A}_0) = 0$ for each $m \geq n$, then $\mathcal{A}$ extends to an $\mathcal{A}_\infty$ structure on $\mathcal{A}_0$; if $\mathrm{HH}^{m+1}(\mathcal{A}_0) = 0$ for all $m \geq n$, then this extension is unique up to isomorphism.*

We next note that we can compute Hochschild cohomology using the cobar complex: the differential on $\mathrm{Cob}(A)$ is given by

$$\delta^{\mathrm{Cob}}(a_1^* \otimes \cdots \otimes a_n^*) = \sum_{i=1}^n a_1^* \otimes \cdots \otimes \mu_2^*(a_i^*) \otimes \cdots \otimes a_n^*. \quad (3.1)$$

and given this fact, we have:

Proposition 3.2.5. (By induction on the proof of Lemma 5.38 of [17]) *Let A be a filtered $\mathbb{F}_2$ -algebra with $F_n A$ finite dimensional (over $\mathbb{F}_2$) for each n . Then there is an isomorphism of chain complexes*

$$A[V_0, V_n] \otimes \mathrm{Cob}(A) \cong \mathrm{HC}^*(A[V_0, V_n]) \quad (3.2)$$

where the differential on the left hand side of (3.2) is given by

$$\delta(a \otimes \xi) = a \otimes (\delta^{\mathrm{Cob}} \xi) + \sum_{\rho \in R_0} (\rho a \otimes (\xi \otimes \rho^*) + a \rho \otimes (\rho^* \otimes \xi)). \quad (3.3)$$

where R_0 is a set of generators for $F_1 A$, and on the right hand side of (3.2), the V_i are viewed as elements of the ground ring.

Remark 3.2.6. In this thesis, we will only be working with the $R = \mathbb{F}_2[V_0, V_{N+1}]$ because we are not allowing any weighted operations weights (which would force us to use the full ring $\mathbb{F}_2[V_0, \dots, V_{N+1}]$ as our ground ring). See [9] for more details about this.

However, we are only interested in $\mathcal{A}_\infty$ -deformations of associative algebras with idempotent elements which are unital in the sense that:

Property I For each μ_j

$$\mu_j(a_1, \dots, a_j) = 0$$

if for any $1 < i \leq j$, the initial idempotent of a_i is not the same as the final idempotent of a_{i-1} .

Given such an algebra, A , we therefore define a subcomplex C^* of HC^* , satisfy a unitality condition analogous to the one used to define $\widetilde{\mathrm{Cob}}$ at the end of Section 3.2.1. The subcomplex C^* is defined to be the collection of strings $a_0^* \otimes \cdots \otimes a_{n+1}^* \in \mathrm{HC}^*(A)$ such that the initial idempotent of a_i is the

final idempotent of a_{i-1} for each $1 \leq i \leq n+1$. Note that the differential on the Hochschild cochain complex preserves this property, so this is in fact a bona fide subcomplex.

We can now modify Lemma 3.2.3 in the following way:

Proposition 3.2.7. *Let A be an associative algebra equipped with a set of idempotents, with all notation as in the previous paragraph. Let $\mathcal{A}$ be an A_{n-1} -deformation of A , satisfying the unitality condition **Property I**. Then there is a Hochschild cochain obstruction class $\mathcal{D}_n \in C^{n+1}$ so that the following conditions hold:*

- $\mathcal{D}_n$ is a cocycle;
- $\mathcal{D}_n$ is a coboundary if and only if there is an operation μ_n satisfying **Property I** and making $\mathcal{A}$ into an A_n -algebra; indeed the operation $\mu_n \in C^n$ is a Hochschild cochain with $\delta(\mu_n) = \mathcal{D}_n$;
- If μ_n, μ'_n are cochains with $\delta(\mu_n) = \delta(\mu'_n) = \mathcal{D}_n$, then $\mu_n - \mu'_n$ is a coboundary such that the respective obstruction cocycles $\mathcal{D}_{n+1}, \mathcal{D}'_{n+1} \in C^{n+2}$ are cohomologous;
- If $\mathcal{A}, \mathcal{A}'$ are two A_n deformations of $\mathcal{A}_0$ with $\mu_i = \mu'_i$ for each $i < n$, and with $\mu_n - \mu'_n$ a coboundary, then their respective obstruction cocycles $\mathcal{D}_{n+1}$ and $\mathcal{D}'_{n+1}$ are cohomologous in C^* .

Likewise, the statement for maps is as follows. Start with A_n -deformations $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{A}_0$ such that each of the operations μ_i on $\mathcal{A}$ and μ'_i on $\mathcal{A}'$ satisfy **Property I**, and an A_{n-1} -homomorphism $f : \mathcal{A} \rightarrow \mathcal{A}'$ such that $f_1 : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ is the identity map and preserving C^* in each degree. Then there is an obstruction class $\mathcal{F}_n \in C^n$ such that

- $\mathcal{F}_n$ is a cocycle;
- $\mathcal{F}_n$ is a coboundary if and only if there is an A_n homomorphism extending f .

To prove Proposition 3.2.7, one just needs to trace through the proof of Proposition 3.2.3 with the additional assumption that all μ_i with $1 \leq i < n$ satisfy **Property I**. It is then clear that under this condition, the $\mathcal{D}_n$ constructed there also satisfies **Property I**, so when $\mathcal{D}_n$ is a coboundary, the class $\mu_n \in HC^n(\mathcal{A})$ with $\delta\mu_n = \mathcal{D}_n$ can be chosen to also satisfy **Property I**. Likewise, if each of the lower degree maps of f preserve C^* , it is clear from the proof of Proposition 3.2.3 that we can choose $\mathcal{F}_n$ so that the extended f still preserves C^* .

Alternately, Proposition 3.2.7 follows directly from the corresponding result in [17], if we view the ring of idempotents as the ground ring, rather than $\mathbb{F}_2[V_0, V_{N+1}]$ as we do here.

This then gives the following analogue of Corollary 3.2.4.

Corollary 3.2.8. *Let A be an associative algebra with idempotent elements, as above. Let $\mathcal{A}$ be an A_n structure on A . If $H^{r+2}(C^*) = 0$ for each $r \geq n$, then $\mathcal{A}$ extends to an $\mathcal{A}_\infty$ structure on $\mathcal{A}_0$; if $H^{r+1}(C^*) = 0$ for all $r \geq n$, then this extension is unique up to isomorphism.*

The last step is to modify Proposition 3.2.7 and Corollary 3.2.8 to accommodate a $\mathbb{Z}$ -valued Maslov grading $m : A \rightarrow \mathbb{Z}$, in the following way. A map (i.e. Hochschild cochain) which respects the Maslov grading is defined as follows. Let A be an associative algebra and choose $f \in \text{HC}^n(A)$. We say that f respects the Maslov grading if and only if there exists a well defined integer j such that for all $a_1, \dots, a_n \in A$

$$m(f(a_1, \dots, a_n)) = \sum_{i=1}^n m(a_i) + i + j - 1. \quad (3.4)$$

For such f , we define the Maslov grading of f , $m(f)$, to be equal to j . Note that if f respects the Maslov grading, so does δf . Define $\text{HC}_{\mathbb{Z}}^{*,*}(A)$ to be the subcomplex of Hochschild cochains which respect the Maslov grading. The bigrading is defined in the obvious way, i.e. $f \in \text{HC}_{\mathbb{Z}}^{n,j}(A)$ if and only if $f \in \text{HC}(A)$ respects the Maslov grading and has $m(f) = j$. Likewise, we define $C^{*,*}$ to be the subcomplex of C^* consisting of all cochains in C^* which respect m , and $C^{n,j}$ is the collection of $f \in C^n$ respecting m and with $m(f) = j$. We denote the cohomology of $C^{*,*}$ by $H^{*,*}(C^*)$.

Proposition 3.2.9. *Let A be an associative algebra with idempotent elements, equipped with Maslov grading $m : A \rightarrow \mathbb{Z}$. Let $\mathcal{A}$ be an A_{n-1} -deformation of A such that for each $1 \leq j < n$, the operations μ_j satisfy **Property I** as well as*

$$m(\mu_j(a_1, \dots, a_j)) = \sum_{i=1}^j m(a_i) + j - 2, \quad (3.5)$$

and operations $f : \mathcal{T}^*(\mathcal{A}) \rightarrow \mathcal{A}$ satisfy

$$m(f_j(a_1, \dots, a_j)) = \sum_{i=1}^j m(a_i) + j - 1. \quad (3.6)$$

Then the classes $\mathcal{D}_n \in H^{n+1}(C^*)$ and $\mathcal{F}_n \in H^n(C^*)$ both respect the Maslov grading, and are in Maslov gradings -2 and -1 respectively.

This follows directly from (3.4), (3.5), and (3.6). As a result, we have the following condition identifying precisely when we can (uniquely) deform a Maslov graded associative algebra with idempotent elements to get an $\mathcal{A}_\infty$ -algebra where all operations satisfy **Property I**.

Corollary 3.2.10. *Let A be an associative algebra with idempotent elements, equipped with Maslov grading m . Let $\mathcal{A}$ be an A_n -deformation of A equipped with Maslov grading, and such that for each $1 \leq j < n$, the operations μ_j satisfy **Property I** as well as (3.5) from Proposition 3.2.9. Then:*

- (Existence of deformation) *If $H^{r,-2}(C^*) = 0$ for each $r \geq n+1$, then $\mathcal{A}$ extends to an $\mathcal{A}_\infty$ -algebra structure on A such that each μ_j satisfies **Property I** and (3.5);*
- (Uniqueness of deformation) *If $H^{r,-1}(C^*) = 0$ for each $m \geq n$, then any two $\mathcal{A}_\infty$ -algebra structures on A extending $\mathcal{A}$, such that each operation in either structure satisfies **Property I** and (3.5), are isomorphic as $\mathcal{A}_\infty$ -algebras.*

This follows from Proposition 3.2.7 and Proposition 3.2.9 in the same way that Corollary 3.2.4 follows from Proposition 3.2.3.

Finally, in order to effectively compute the relevant cohomology groups $H^{*,*}(C^*)$ in Section 3.5.1, we need to modify Proposition 3.2.5 to involve $H^*(C^*)$ instead of the full Hochschild cochain complex.

With this modification, the desired result is as follows:

Proposition 3.2.11. *Let A be an associative $\mathbb{F}_2$ -algebra with idempotent elements, as above. Suppose A is equipped with a filtration such that $F_n A$ is finite dimensional (over $\mathbb{F}_2$) for each n . Then there is an isomorphism of chain complexes*

$$A[V_0, V_{N+1}] \otimes \widetilde{\text{Cob}}(A) \cong C^*[V_0, V_{N+1}] \quad (3.7)$$

where the tensor product is over $\mathbb{F}_2$, the differential on the left hand side of (3.7) is given by

$$\delta(a \otimes \xi) = a \otimes (\delta^{\widetilde{\text{Cob}}} \xi) + \sum_{\rho \in F_1 A} (\rho a \otimes (\xi \otimes \rho^*) + a \rho \otimes (\rho^* \otimes \xi)). \quad (3.8)$$

and on the right hand side of (3.7), the V_i are viewed as elements of the ground ring.

This is obvious from the definitions. (Note also that the sum in (3.8) is finite by the fact that each $F_n A$ is finite-dimensional.) In Section 3.5.1, we will use this fact, along with Lemma 3.5.2, which states that each of the algebras $\mathcal{A}_0$ and $\mathcal{B}_0$ is quasi-isomorphic to the cobar of the other, to compute Hochschild cohomology of $\mathcal{A}_0$ and $\mathcal{B}_0$.

Finally, note that the Maslov grading on a tensor product is defined in the obvious way, that is, as

$$m(a \otimes b) = m(a) + m(b). \quad (3.9)$$

If A is an associative algebra equipped with Maslov grading $m : A \rightarrow \mathbb{Z}$, then the on $\widetilde{\text{Cob}}(A)$ is given by

$$m(a^*) = -m(a) - 1 \quad (3.10)$$

extended to tensor products via (3.9).

3.3 The Wrapped Fukaya category

The goal of this section is to present the general definitions and results related to the *wrapped Fukaya category* $\mathcal{W}(M)$ of a particular class of symplectic manifold (M, ω) . None of these are original to this thesis – see e.g. [17] and [1] – and the exposition mostly follows Section 6 of [17].

3.3.1 Definitions

Let (M, ω) be a symplectic manifold with conical, cylindrical ends. Let H be a Hamiltonian on M , such that near each boundary component of M , H is of the form $H(r, x) = r^2$, where M is parametrized as $(0, \infty] \otimes S^1$, near the boundary component. Let φ^1 denote the time-one flow map for H . Then $\mathcal{W}(M)$ has:

- **Objects:** Simply connected Lagrangian submanifolds L of (M, ω) , which are conical near the boundary.
- **Homomorphisms:** For simply connected Lagrangians $L_0, L_1 \in \text{Ob}(\mathcal{W}(M))$,

$$\text{Hom}(L_0, L_1) = CF(\varphi^1(L_0), L_1),$$

that is, the set of intersections of the (once) wrapped version of L_0 with L_1 ;

- **$\mathcal{A}_\infty$ -composition map:** The n -fold composition map

$$\text{Hom}(L_0, L_1) \otimes \cdots \otimes \text{Hom}(L_{n-1}, L_n) \rightarrow \text{Hom}(L_0, L_n)$$

is an (oriented) count of $(n + 1)$ -gons with boundary along

$$\varphi^n(L_0), \varphi^{n-1}(L_1), \dots, \varphi^1(L_{n-1}), \text{ and then } L_0.$$

with the usual coherent, right-handed orientations. For examples of such $(n + 1)$ -gons with

proper orientation, see the figures surrounding Lemmas 3.6.1, 3.6.2, 3.6.3, and 3.6.4.

3.3.2 Gradings

The next step is to define a grading for the wrapped Fukaya category of a fixed symplectic manifold (M, ω) . This grading will be constructed in two steps. First, we will construct a groupoid $\mathcal{G}$ using lifts of Lagrangians on M , and show that this can be used to define a *groupoid grading* for $\mathcal{W}(M)$. Then, by fixing base-points will define a bona-fide group G from $\mathcal{G}$, and use the $\mathcal{G}$ -grading on $\mathcal{W}(M)$ to induce a grading by G .

In practice, M will be $S^2 \setminus \{p_0, \dots, p_{N+1}\}$, so the group G will be $\mathbb{Z} \times F_{N+1}$ (where F_{N+1} is the free group on $(N+1)$ elements). In Section 3.6, we will also have $M = S^2 \setminus \{p_0, \dots, p_{N+1}\}$, and algebra elements in $\mathcal{A}_0$ and $\mathcal{B}_0$ will correspond to endomorphisms of $\mathcal{W}(M)$. In this case, the $\mathbb{Z}$ -component of the grading for a given homomorphism of $\mathcal{W}(M)$ will indicate the Maslov grading of the corresponding algebra element.

Again, the exposition here mostly follows Section 6.2 of [17], and general results are cited without proof.

First, we need to define the notion of a grading for an $\mathcal{A}_\infty$ -category by a groupoid. Let $\mathcal{C}$ be an $\mathcal{A}_\infty$ category with composition operations denoted by μ_n , and let $\mathcal{G}$ be a groupoid with central element λ and with its operation written as multiplication. Then a *grading* gr of $\mathcal{C}$ by $\mathcal{G}$ consists of the following data:

- **Grading for objects:** For each $L \in \text{Ob}(\mathcal{C})$, a corresponding object $s(L) \in \mathcal{G}$;
- **Grading for homomorphisms:** For each pair of $L_0, L_1 \in \text{Ob}(\mathcal{C})$, a decomposition of the set of homomorphisms $L_0 \rightsquigarrow L_1$ as

$$\text{Hom}(L_0, L_1) = \bigoplus_{\gamma \in \text{Hom}_{\mathcal{G}}(s(L_0), s(L_1))} \text{Hom}(L_0, L_1; \gamma);$$

With this notation, define $\text{gr}(x) = \gamma$ for each homogeneous element $x \in \text{Hom}(L_0, L_1; \gamma)$;

This data is required to satisfy the added property that for each μ_n and each set of homogeneous elements $x_i \in \text{Hom}(L_i, L_{i+1})$ ($1 \leq i \leq n$), we have

$$\text{gr}(\mu_n(x_1, \dots, x_n)) = \lambda^{n-2} \text{gr}(x_1) \cdots \text{gr}(x_n) \quad (3.11)$$

Note, in particular, that (3.5) is precisely (3.11) for $\mathcal{G} = \mathbb{Z}$, $\lambda = 1$, and with the group operations

written additively.

Now, the specific groupoid $\mathcal{G}$ by which we will grade $\mathcal{W}(M)$ is defined as follows. Let $\mathcal{Lag}(M) \subseteq TM$ denote the space of Lagrangian submanifolds in M . Note that $\mathcal{Lag}(M)$ is a bundle over M with fiber $\text{Lag}(n) \subseteq T_x M = \mathbb{R}^{2n}$ (given $\dim M = 2n$), i.e. the space of Lagrangian subspaces of $\mathbb{R}^{2n}$. In particular, for $M = S^2 \setminus \{p_0, \dots, p_{n+1}\}$, the case which arises when we consider the star diagrams, we get $n = 1$, so $\mathcal{Lag}(M)$ is just a circle bundle. In fact, for $M = S^2 \setminus \{p_0, \dots, p_{n+1}\}$, $\mathcal{Lag}(M)$ is a trivial circle bundle. We choose the trivialization

$$\mathcal{Lag}(M) \simeq S^1 \times \left(\bigvee_{n+1} S^1 \right), \quad (3.12)$$

and additionally fix a basepoint $\tilde{b} \in \mathcal{Lag}(M)$. Then

$$G := \pi_1(\mathcal{Lag}(M), \tilde{b}) = \mathbb{Z} \times F_{N+1}. \quad (3.13)$$

Now, return to a general symplectic manifold (M^{2n}, ω) . Note that given basepoints $b \in M$, $\tilde{b} \in \mathcal{Lag}(M)$, $\pi_1(M, b)$ acts freely on $\pi_1(\mathcal{Lag}(M), \tilde{b})$, so that $\pi_1(\mathcal{Lag}(M), \tilde{b})$ is a $\mathbb{Z}$ -central extension of $\pi_1(M, b)$. We can therefore choose an element $\lambda \in \pi_1(\mathcal{Lag}(M), \tilde{b})$ which projects down to a generator of $\pi_1(\text{Lag}(n))$, an element with Maslov index (in the classical sense) equal to 1.

Define $\mathcal{G}$ to be the groupoid whose elements consist of all the simply connected Lagrangians of M . Morphisms between elements L_0 and L_1 of $\mathcal{G}$ are defined as follows. For each Lagrangian $L \subseteq (M, \omega)$, there is a canonical lift $\tilde{L}$ of L to the bundle $\mathcal{Lag}(M)$. For $L_0, L_1 \in \mathcal{G}$, let $\tilde{L}_0, \tilde{L}_1$ denote the canonical lifts to $\mathcal{Lag}(M)$. Define $S(L_0, L_1)$ to be the collection of paths $\{[0, 1], \{0\}, \{1\}\} \rightarrow \{\mathcal{Lag}(M), \tilde{L}_0, \tilde{L}_1\}$, and define the homomorphisms between L_0 and L_1 in $\mathcal{G}$ as $\text{Hom}(L_0, L_1) = S(L_0, L_1)$. (We will use $S(L_0, L_1)$ in other contexts, so we want independent notation for it.) Composition

$$\text{Hom}(L_0, L_1) \times \text{Hom}(L_1, L_2) \rightarrow \text{Hom}(L_0, L_2)$$

in $\mathcal{G}$ is defined as $(\gamma_1, \gamma_2) \mapsto \gamma_1 * \eta * \gamma_2$, where η is the (canonical, because L_1 is simply connected) path in $\tilde{L}_1$ from $\gamma_1(1)$ to $\gamma_2(0)$.

Now, define the grading gr of $\mathcal{W}(M)$ by $\mathcal{G}$ in the following way. First, since $\text{Ob}(\mathcal{W}(M))$ is the set of (simply connected) Lagrangians on M , we just define $s(L) = L$ for each $L \in \text{Ob}(\mathcal{W}(M)) = \mathcal{G}$. Next, we define the grading for homomorphisms of $\mathcal{W}(M)$; that is, given $L_0, L_1 \in \text{Ob}(\mathcal{W}(M))$, and $x \in \varphi^1(L_0) \cap L_1$, we want to define $\text{gr}(x) \in S(L_0, L_1)$. Define the following intermediate maps:

- η : For $L \in \text{Ob}(\mathcal{W}(M))$, $m, n \in \mathbb{Z}_{\geq 0}$ $x \in \varphi^m(L)$, let $\eta_x^{\varphi^m(L) \rightarrow \varphi^n(L)}$ be the path from $T_x \varphi^m(L)$ to $T \varphi^{n-m}(x) \varphi^n(L)$ induced by the Hamiltonian isotopy H . In practice, we will usually have $m = 0$ and $n = 1$;
- γ : For $L_0, L_1 \in \text{Ob}(\mathcal{W}(M))$ and $x \in L_0 \cap L_1$, let $\gamma_x^{L_0 \rightarrow L_1}$ be the path in $\text{Lag}(T_x M)$ from $T_x L_0$ to $T_x L_1$ which is positively oriented with respect to the underlying almost complex structure; when $\dim M = 2$, this will mean that $\gamma_x^{L_0 \rightarrow L_1}$ turns $T_x L_0$ clockwise in $\text{Lag}(T_x M)$ until it reaches $T_x L_1$;

Now, for each $L_0, L_1 \in \text{Ob}(\mathcal{W}(M))$, and $x \in \varphi^1(L_0) \cap L_1$, we define

$$\text{gr}(x) = \eta_{\varphi^{-1}(x)}^{L_0 \rightarrow \varphi^1(L_0)} * \gamma_x^{\varphi^1(L_0) \rightarrow L_1} \quad (3.14)$$

If (as we usually do) we want to require that for some fixed $q_0 \in L_0$, and $q_1 \in L_1$, and each $x \in \varphi^1(L_0) \cap L_1$, $\text{gr}(x) \in S(L_0, L_1)$ starts at $\tilde{q}_0 := T_{q_0} L_0$ and ends at $\tilde{q}_1 := T_{q_1} L_1$, we make the following additional definitions:

- ϵ : Fix $L \in \text{Ob}(\mathcal{W}(M))$ with lift $\tilde{L} \in \mathcal{Lag}(M)$, and $x, y \in L$. Define $\epsilon_{x \rightarrow y}^L$ to be path in $\tilde{L}$ from $T_x L$ to $T_y L$, which is unique up to reparametrization.

We then define

$$\text{gr}(x) = \epsilon_{q_0 \rightarrow \varphi^{-1}(x)}^{L_0} * \eta_{\varphi^{-1}(x)}^{L_0 \rightarrow \varphi^1(L_0)} * \gamma_x^{\varphi^1(L_0) \rightarrow L_1} * \epsilon_{x \rightarrow q_1}^{L_1} \quad (3.15)$$

Moreover, gr is reasonable in the two following senses:

Proposition 3.3.1. (Proposition 6.9 and Lemma 6.10 from [17])

- The $s : \text{Ob}(\mathcal{W}(M)) \rightarrow \mathcal{G}$ and $\text{gr} : \text{Hom}_{\mathcal{W}(M)}(L_0, L_1) \rightarrow S(L_0, L_1)$ defined above determine a bona-fide grading; in particular, gr satisfies (3.11);*
- gr is well-defined up to Hamiltonian isotopy in the following sense: Let L_0, L_n be two simply connected Lagrangians in M , and choose $x \in \varphi^1(L_0) \cap L_n$. Let x' denote the point of $\varphi^n(L_0) \cap L_n$ corresponding to x under dilation by the Hamiltonian H . Then, using (3.15) to write*

$$\text{gr}(x) = \epsilon_{q_0 \rightarrow \varphi^{-1}(x)}^{L_0} * \eta_{\varphi^{-1}(x)}^{L_0 \rightarrow \varphi^1(L_0)} * \gamma_x^{\varphi^1(L_0) \rightarrow L_n} * \epsilon_{x \rightarrow q_1}^{L_n}$$

and

$$\text{gr}(x') = \epsilon_{q_0 \rightarrow \varphi^{-1}(x)}^{L_0} * \eta_{\varphi^{-1}(x)}^{L_0 \rightarrow \varphi^n(L_0)} * \gamma_{x'}^{\varphi^1(L_0) \rightarrow L_1} * \epsilon_{x' \rightarrow q_1}^{L_1}$$

(using the fact that $\varphi^{-1}(x) = \varphi^{-n}(x')$), we have

$$\text{gr}(x) = \text{gr}(x'). \quad (3.16)$$

This proposition holds in an entirely general setting (provided we only consider simply connected Lagrangians, and that (M, ω) satisfies the conditions from Section 3.3.1). We therefore do not repeat the proof here.

Proposition 3.3.1 shows that the grading gr behaves reasonably. It now remains to define a group G related to $\mathcal{G}$, such that gr induces a G grading on $\mathcal{W}(M)$. As noted above, we define G as

$$G = \pi_1(\mathcal{Lag}(M), \tilde{b}),$$

where $\tilde{b} \in \mathcal{Lag}(M)$ is a fixed basepoint. The G -grading is defined as follows. For each simply connected Lagrangian $L \subseteq M$ with canonical lift $\tilde{L} \in \mathcal{Lag}(M)$, choose a basepoint $\tilde{b}_L \in \tilde{L}$. Choose also a fixed anchor path $\eta_L \subseteq \mathcal{Lag}(M)$ for L , which goes from $\tilde{b}$ to $\tilde{b}_L$. For each $L_0, L_1 \in \text{Ob}(\mathcal{W}(M))$, we can now make the identification:

$$f : S(L_0, L_1) \xrightarrow{\cong} G$$

in the following way: write ν_i ($i = 0, 1$) for paths in $\tilde{L}_i$ from $\tilde{b}_{L_i}$ to $\gamma(i)$, and then define

$$f(\gamma) = \eta_{L_0} * \nu_0 * \gamma * \overline{\nu_1} * \overline{\eta_{L_1}}$$

where as usual $\overline{\nu_1}, \overline{\eta_{L_1}}$ are the reverse of these paths. So $f(\gamma)$ is:

- $\eta_{L_0} : \tilde{b} \rightsquigarrow \tilde{b}_{L_0}$ then
- $\nu_0 : \tilde{b}_{L_0} \rightsquigarrow \gamma(0)$ along $\tilde{L}_0$, then
- $\gamma : \gamma(0) \rightsquigarrow \gamma(1)$, then
- $\overline{\nu_1} : \gamma(1) \rightsquigarrow \tilde{b}_{L_1}$, along $\tilde{L}_1$, then lastly
- $\overline{\eta_{L_1}} : \tilde{b}_{L_1} \rightsquigarrow \tilde{b}$.

Clearly, this defines a bijection between $S(L_0, L_1)$ and G . Moreover, since each L_i is simply connected, f is independent of choice of ν_0, ν_1 . Therefore the $\mathcal{G}$ -grading gr on $\mathcal{W}(M)$ induces a G -

grading, such that for each $x \in \text{Hom}_{\mathcal{W}(M)}(L_0, L_1) = \varphi^1(L_0) \cap L_1$,

$$\text{gr}(x) = \eta_{L_0} * \epsilon_{\tilde{b}_{L_0} \rightarrow \varphi^{-1}(x)}^{L_0} * \eta_{\varphi^{-1}(x)}^{L_0 \rightarrow \varphi^1(L_0)} * \gamma_x^{\varphi^1(L_0) \rightarrow L_1} * \epsilon_{x \rightarrow \tilde{b}_{L_1}}^{L_1} * \overline{\eta_{L_1}}. \quad (3.17)$$

Moreover, by Proposition 3.3.1(a) and (3.14), the $\mathbb{Z}$ -component of the grading of (3.17) satisfies (2.12).

3.4 Constructions with $\mathcal{A}$ and $\mathcal{B}$

The constructions in this section follow those in Sections 3 and 4 of the previous chapter, except for two points. First, neither of the $\mathcal{A}_\infty$ -algebras $\mathcal{A}$ and $\mathcal{B}$ constructed here include *weighted $\mathcal{A}_\infty$ -operations*. Weighted $\mathcal{A}_\infty$ -operations will not be discussed further in this thesis, except to note how the proofs of $\mathcal{A}_\infty$ -relations for $\mathcal{A}$ and $\mathcal{B}$ can be adapted from the ones given in the previous chapter; therefore we do not define them here. The second key difference is exposition that in constructing $\mathcal{A}$ and $\mathcal{B}$, we first define associative algebras $\mathcal{A}_0$ and $\mathcal{B}_0$, and then construct $\mathcal{A}$ and $\mathcal{B}$ as deformations of $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively.

For the remainder of the chapter, fix $N > 2$, and a single ground ring $R = \mathbb{F}_2[V_0, V_{N+1}]$.

3.4.1 The $\mathcal{A}_\infty$ -algebra $\mathcal{A}$

Define $\mathcal{A}_0$ as in Section 2.4. Notice that $\mathcal{A}_0$ is equipped with an idempotent set $\{I_i\}_{i=1}^N$, in the sense of Section 3.2.1; we define $\mathcal{A}_+ = \mathcal{A}_0 \setminus \{I_i\}_{i=1}^N$, and the augmentation map $\epsilon : \mathcal{A}_0 \rightarrow R$ to have $\ker \epsilon = \mathcal{A}_+$.

Next we define $\mathcal{A}$ as a deformation of $\mathcal{A}_0$; that is, as an $\mathcal{A}_\infty$ -algebra which is equal to $\mathcal{A}_0$ as an associative R -algebra (and is equipped with the same Maslov and Alexander grading as $\mathcal{A}_0$) but now has higher operations. Higher operations on $\mathcal{A}$ are defined in Sections 4.2 and 4.3 of [9]. Non-zero higher operations are defined to be in one-to-one correspondence with *allowable unweighted planar graphs*. However, we will not use these graphs further in this thesis, so we simply summarize the key results about $\mathcal{A}_\infty$ -operations on $\mathcal{A}$ below.

To do this, we need a few definitions. For $a \in \mathcal{A}$, suppose

$$A(a) = \sum_{j=1}^{2N} k_j \mathbf{h}_j.$$

Then define the *length*, ℓ , of a as:

$$\ell(a) = \sum_{j=1}^{2N} k_j. \quad (3.18)$$

Note that for each $a \in A$, $\ell(a) > 0$, and that $\ell(a) = 0$ if and only if a is one of the basic elements U_i, s_i . Note in particular that ℓ determines an obvious filtration on $\mathcal{A}$, such that each $F_n A$ (and in particular $F_1 A$) are finite-dimensional over $\mathbb{F}_2$.

Next, fix a sequence $(a_1, \dots, a_n) \in \mathcal{A}$, $n > 2$. Define

$$A(a_1, \dots, a_n) = \sum_{i=1}^n A(a_i). \quad (3.19)$$

This sequence is defined to be *centered*

$$\sum_{i=1}^n \ell(a_i) = j \cdot 2N, \quad (3.20)$$

for some $j \in \mathbb{N}$, *left-extended* if we can write $a_1 = a'_1 a''_1$ such that $(a''_1, a_2, \dots, a_n)$ is a centered sequence, and *right-extended* if we can write $a_n = a''_n a'_n$ such that $(a_1, \dots, a_{n-1}, a''_n)$ is a centered sequence. Note that when $(a_1, \dots, a_n)$ is left-extended, the decomposition $a_1 = a'_1 a''_1$ is unique with the property that $(a''_1, a_2, \dots, a_n)$ is centered. Likewise when $(a_1, \dots, a_n)$ is right-extended, the decomposition $a_n = a''_n a'_n$ is unique with the property that $(a_1, \dots, a_{n-1}, a''_n)$ is centered.

We are now ready to discuss higher operations. The following lemma is adapted from Theorem 4.1 of [9]:

Lemma 3.4.1. *Fix $a_1, \dots, a_n \in \mathcal{A}$, for $n \geq 2$. Then $\mu_n(a_1, \dots, a_n) \neq 0$ if and only if the following conditions hold:*

- (i) (The idempotents match up) *The initial idempotent of a_i is the final idempotent of a_{i-1} for each $1 \leq i \leq n$;*
- (ii) (The Maslov grading works out) *$n = j(2N - 2) + 2$, for $j \in \mathbb{N}$;*
- (iii) (Extension) *The sequence $(a_1, \dots, a_n)$ is either centered, or it is left- or right-extended, but not both;*
- (iv) (The Alexander grading works out) *The condition varies depending on whether the sequence $(a_1, \dots, a_n)$ is centered, left- or right-extended.*

- *If centered: $A(a_1, \dots, a_n) = j \cdot \sum_{i=1}^{2N} \mathbf{h}_i$;*

- *If left-extended:* Write $a_1 = a'_1 a''_1$ such that $(a''_1, a_2, \dots, a_n)$ is centered. Then $A(a_1, \dots, a_n) = A(a'_1) + j \cdot \sum_{i=1}^{2N} \mathbf{h}_i$ (for the same j as in (ii));
- *If right-extended:* Write $a_n = a''_n a'_n$ such that $(a_1, \dots, a_{n-1}, a''_n)$ is centered. Then $A(a_1, \dots, a_n) = A(a'_n) + j \cdot \sum_{i=1}^{2N} \mathbf{h}_i$, (again, for the same j as in (ii));

Lemma 3.4.1 follows from Theorem 4.1 of [9]. It also follows from this theorem that each nonzero higher operation has the following output:

$$\mu_n(a_1, \dots, a_n) = \begin{cases} V_0^j & \text{if } (a_1, \dots, a_n) \text{ is centered;} \\ a'_1 V_0^j & \text{if } (a_1, \dots, a_n) \text{ is left-extended;} \\ V_0^j a'_n & \text{if } (a_1, \dots, a_n) \text{ is right-extended;} \end{cases} \quad (3.21)$$

with $n = j(2N - 2) + 2$, and notation for left- and right-extended sequences of elements as in the statement of Lemma 3.4.1.

Now, we have:

Proposition 3.4.2. (Theorem 4.3 of [9]) *With operations as specified in Lemma 3.4.1, $\mathcal{A}$ satisfies $\mathcal{A}_\infty$ -relations, and is a bona fide $\mathcal{A}_\infty$ -algebra.*

Finally, note that all non-zero operations on $\mathcal{A}$ preserve the Alexander grading, and transform the Maslov grading according to

$$m(\mu_n(a_1, \dots, a_n)) = \sum_{i=1}^n m(a_i) + n - 2. \quad (3.22)$$

3.4.2 The $\mathcal{A}_\infty$ -algebra $\mathcal{B}$

As in the case of $\mathcal{A}$, we define $\mathcal{B}$ as a deformation of the associative algebra $\mathcal{B}_0$ from Section 2.4. We define length ℓ on $\mathcal{B}_0$ exactly we did for $\mathcal{A}_0$, and note that here again, ℓ determines a filtration. Also, we define $\mathcal{B}_+ = \mathcal{B}_0 \setminus \{I_i\}_{i=1}^N$, and the augmentation map $\epsilon : \mathcal{B}_0 \rightarrow R$ to have $\ker \epsilon = \mathcal{B}_+$. We also define the *length* of an element $b \in \mathcal{B}$, and the Alexander grading $A(b_1, \dots, b_n)$ of a sequence $b_1, \dots, b_n \in \mathcal{B}$, analogously to (3.18) and (3.19), respectively.

Next, we define $\mathcal{B}$ as a deformation of $\mathcal{B}_0$. Unlike the definition of $\mathcal{B}$ in Section 5.2 of [9], we only need a single family of higher multiplications, namely:

$$\mu_N(\sigma_{i+N-1}, \dots, \sigma_{i+1}, \sigma_i) = V_{N+1}, \quad (3.23)$$

for all $1 \leq i \leq N$, with all indices counted $\bmod N$. Now,

Proposition 3.4.3. *With operations as above, $\mathcal{B}$ satisfies the $\mathcal{A}_\infty$ -relations and is a bona fide $\mathcal{A}_\infty$ -algebra.*

The deformed algebra $\mathcal{B}$ is not exactly the same one constructed in the first chapter – it has many fewer $\mathcal{A}_\infty$ -operations – and therefore the proof of Proposition 3.4.3 is trivial in this case.

3.5 Uniqueness of $\mathcal{A}$ and $\mathcal{B}$

The goal of this section is to verify the following proposition.

Proposition 3.5.1. (Uniqueness of deformations)

(a) *Up to isomorphism, there is a unique $\mathcal{A}_\infty$ -deformation of $\mathcal{A}_0$ over $R = \mathbb{F}_2[V_0, V_{N+1}]$ satisfying:*

- *The deformation is bigraded by ℓ and m , and for each generating element $U_i, s_i \in \mathcal{A}_0$, the gradings agree with the definitions given in Section 3.4.1;*
- *The deformation has no non-zero operations μ_n for $2 < n < 2N$, and has*

$$\mu_{2N}(U_1, s_1, \dots, U_N, s_N) = V_0, \quad (3.24)$$

and likewise for each cyclic permutation of the set $\{U_1, s_2, \dots, U_N, s_N\}$, and no other non-zero $\mu_{2N}s$.

(b) *Up to isomorphism, there is a unique $\mathcal{A}_\infty$ -deformation of $\mathcal{B}_0$ over R satisfying:*

- *The deformation is bigraded by m and A , and for each generating element $\rho_i, \sigma_i \in \mathcal{B}_0$, the gradings agree with the definitions given in Section 3.4.2;*
- *The deformation has no non-zero operations μ_n for $2 < n < N$, and has*

$$\mu_N(\sigma_N, \sigma_{N-1}, \dots, \sigma_1) = V_{N+1} \quad (3.25)$$

and likewise for each cyclic permutation of the set $\{\sigma_N, \dots, \sigma_1\}$, and no other non-zero μ_Ns .

3.5.1 Computing the cobar algebras of $\mathcal{A}_0$ and $\mathcal{B}_0$

In order to prove Proposition 3.5.1, we are going to compute the cohomology of the subcomplex C^* of the Hochschild cochain complex defined in the second half of Section 3.2.2. First, note that $\mathcal{A}_0$ and $\mathcal{B}_0$ are both associative algebras over the ground ring $R = \mathbb{F}_2[V_0, V_{N+1}]$, equipped with a Maslov grading and each with a set of idempotent elements. Therefore all of the results from Section 3.2.2 apply. By Proposition 3.2.11, the first step is to compute $\widetilde{\text{Cob}}(\mathcal{A}_0)$ and $\widetilde{\text{Cob}}(\mathcal{B}_0)$.

Lemma 3.5.2. *Given $\mathcal{A}_0$ and $\mathcal{B}_0$ as in Sections 3.4.1 and 3.4.2, we have*

$$\widetilde{\text{Cob}}(\mathcal{A}_0) \simeq \mathcal{B}_0, \quad (3.26)$$

and

$$\widetilde{\text{Cob}}(\mathcal{B}_0) \simeq \mathcal{A}_0. \quad (3.27)$$

Proof. Start with the case of $\mathcal{A}_0$, that is, (3.26). We are going to manually define a Maslov-graded quasi-isomorphism

$$\varphi : \widetilde{\text{Cob}}(\mathcal{A}_0) \rightarrow \mathcal{B}_0$$

Recall first that the Maslov grading on $\widetilde{\text{Cob}}(\mathcal{A}_0)$ is determined by (3.10) for each element $a \in \mathcal{A}_0$. Since $m(a) = 0$ for each such a , this means that $m(a^*) = -1$ for each a .

Now define φ as follows:

$$\begin{aligned} \varphi(U_i^*) &= \rho_i \text{ for each } 1 \leq i \leq N; \\ \varphi(s_i^*) &= \sigma_i \text{ for each } 1 \leq i \leq N; \\ \varphi(a^*) &= 0 \text{ if } a \in \mathcal{A} \text{ with } \ell(a) > 1; \end{aligned}$$

Remark 3.5.3. Note that although our usual ground ring is $R = \mathbb{F}_2[V_0, V_{N+1}]$, we are currently working *only* over $\mathbb{F}_2$. V_0 and V_{N+1} do not have any meaning in the context of $\mathcal{A}_0$ and $\mathcal{B}_0$, which are associative algebras over $\mathbb{F}_2$. These variables are *not* elements of these associative algebras, and so we will not be defining $\varphi(V_0)$ or $\varphi(V_{N+1})$. We introduce V_0 and V_{N+1} for the first time, in the context of $\mathcal{A}_0$ and $\mathcal{B}_0$, in Proposition 3.2.11.

We extend φ in the obvious way, i.e.

$$\varphi(a_1^* \otimes \cdots \otimes a_n^*) = \varphi(a_n) \cdot \varphi(a_{n-1}) \cdots \varphi(a_1). \quad (3.28)$$

Now define

$$\psi : \mathcal{B}_0 \rightarrow \widetilde{\text{Cob}}(\mathcal{A}_0)$$

as

$$\psi(\rho_i) = U_i;$$

$$\psi(\sigma_i) = s_i;$$

extended multiplicatively as

$$\psi(b_n \cdots b_1) = \psi(b_1) \otimes \cdots \otimes \psi(b_n) \quad (3.29)$$

where we are using the unique factorization such that b_i is a length-one element of $(\mathcal{B}_0)_+$ (the complement of the set of idempotents in $\mathcal{B}_0$).

Clearly, $\varphi \circ \psi = \text{id}_{\mathcal{B}_0}$. We now need to define a chain homotopy operator $H : \widetilde{\text{Cob}}(\mathcal{A}_0) \rightarrow \widetilde{\text{Cob}}(\mathcal{A}_0)$ such that

$$\delta^{\widetilde{\text{Cob}}} \circ H + H \circ \delta^{\widetilde{\text{Cob}}} = \text{id}_{\widetilde{\text{Cob}}(\mathcal{A}_0)} + \psi \circ \varphi. \quad (3.30)$$

Note that $\psi \circ \varphi = \text{id}_{\widetilde{\text{Cob}}(\mathcal{A}_0)}$ on the complement of $\ker \varphi$. This means we just need to construct H satisfying (3.30) for $\xi \in \widetilde{\text{Cob}}(\mathcal{A}_0)$ with $\varphi(\xi) = 0$. To do this, note first that any element of $\text{im } \psi \subseteq \widetilde{\text{Cob}}(\mathcal{A}_0)$ can be written uniquely in the form $a_1^* \otimes \cdots \otimes a_n^*$ for length-one elements $a_i \in \mathcal{A}_0$ such that $\varphi(a_n) \cdots \varphi(a_1) \in \mathcal{B}_0$ is non-zero. By the multiplication rules for $\mathcal{B}_0$, such a string will always be of the form

$$(A) \quad U_i^* \otimes s_i^* \otimes \cdots \otimes s_{i+n-1}^* \otimes U_{i+n}^*;$$

$$(B) \quad U_i^* \otimes s_i^* \otimes \cdots \otimes U_{i+n}^* \otimes s_{i+n}^*;$$

$$(C) \quad s_{i-1}^* \otimes U_i^* \otimes \cdots \otimes s_{i+n-1}^* \otimes U_{i+n}^*;$$

$$(D) \quad s_{i-1}^* \otimes U_i^* \otimes \cdots \otimes U_{i+n}^* \otimes s_{i+n}^*;$$

where all indices are counted mod N . This means that each element of $\widetilde{\text{Cob}}(\mathcal{A}_0)$ can be written uniquely in the form

$$\xi = a_1^* \otimes \cdots \otimes a_n^* \otimes \tilde{a}_1^* \otimes \cdots \otimes \tilde{a}_m^* \quad (3.31)$$

where each $a_1^* \otimes \cdots \otimes a_n^*$ a string of one of the forms (A)-(D), and $\tilde{a}_1^*$ is the first functional in the sequence such that either $\ell(\tilde{a}_1) > 1$ or $\ell(\tilde{a}_1) = 1$ and $\varphi(\tilde{a}_1)\varphi(a_n) = 0$. A string ξ of the form in (3.31) has $\varphi(\xi) = 0$ if and only if $m > 0$.

Define

$$H(a_1^* \otimes \cdots \otimes a_n^* \otimes \tilde{a}_1^* \otimes \cdots \otimes \tilde{a}_m^*) = \begin{cases} a_1^* \otimes \cdots \otimes a_{n-1}^* \otimes (a_n \tilde{a}_1)^* \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^* & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

(The idempotents make sense in the string above because we are working in $\widetilde{\text{Cob}}(\mathcal{A}_0)$, which only contains strings where the idempotents make sense.) Note first that for ξ as in (3.31), $H(\xi) = 0$ whenever $\varphi(\xi) \neq 0$. It just remains to verify (3.30) for each $\xi \in \widetilde{\text{Cob}}(\mathcal{A}_0)$, written as in (3.31). The cases are as follows:

Case 1: $m = 0$, i.e. $\xi = a_1^* \otimes \cdots \otimes a_n^* \in \text{im } \varphi$. Then $H(\xi) = 0$ and $\delta^{\widetilde{\text{Cob}}} \xi = 0$, and

$$(\psi \circ \varphi)(\xi) = \xi,$$

as in (3.30).

Case 2: $n = 0$, or $m, n > 0$ with $a_n \tilde{a}_1 = 0$. In either of these cases, $H(\xi) = 0$, and $\varphi(\xi) = 0$. We want to show that $H \circ \delta^{\widetilde{\text{Cob}}} = \text{id}_{\widetilde{\text{Cob}}(\mathcal{A}_0)}$.

If $n = 0$, i.e. $\xi = \tilde{a}_1^* \otimes \cdots \otimes \tilde{a}_m^*$ then $\ell(\tilde{a}_1) > 1$, and writing $\tilde{a}_1$ as a product $a_{11} \otimes a_{1\ell}$ of basic elements, we get that

$$\delta^{\widetilde{\text{Cob}}}(\xi) = \sum_{1 \leq i \leq (\ell-1)} (a_{11} \cdots a_{1i})^* \otimes (a_{1(i+1)} \cdots a_{1\ell})^* \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^* + A,$$

where A is a sum of terms with first tensor factor $\tilde{a}_1^*$, and hence, with $H(A) = 0$. This means that

$$\begin{aligned} H\delta^{\widetilde{\text{Cob}}}(\xi) &= \sum_{1 \leq i \leq (\ell-1)} H((a_{11} \cdots a_{1i})^* \otimes (a_{1(i+1)} \cdots a_{1\ell})^* \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^*) \\ &= H(a_{11}^* \otimes (a_{12} \cdots a_{1\ell})^* \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^*) \\ &= \xi. \end{aligned}$$

That is, (3.30) holds.

If $m, n > 0$ with $a_n \tilde{a}_1 = 0$. First note that we cannot have $\ell(\tilde{a}_1) = 0$. Because of the way we chose $\tilde{a}_1$, we would then have $\varphi(\tilde{a}_1)\varphi(a_n) = 0$; but because $\xi \in C^*$, we know the final idempotent of a_n matches the initial idempotent of $\tilde{a}_1$; this means that either $\varphi(\tilde{a}_1)\varphi(a_n) = 0$

or $a_n \tilde{a}_1 = 0$, but not both. We can therefore assume in this case that $\ell(\tilde{a}_1) > 1$. Then the argument is analogous to the one for $n = 0$ above.

Case 3: $m, n > 0$ with $a_n \tilde{a}_1 \neq 0$. Then because we are working with $\xi \in C^*$, we get $\varphi(\tilde{a}_1)\varphi(a_n) = 0$, so $\varphi(\xi) = 0$, and we want to verify that

$$(H \circ \delta^{\widetilde{\text{Cob}}} + \delta^{\widetilde{\text{Cob}}} \circ H)(\xi) = \xi. \quad (3.32)$$

For this, just note that

$$\begin{aligned} \delta^{\widetilde{\text{Cob}}} \xi &= a_1^* \otimes \cdots \otimes a_n^* \otimes \left(\sum_{1 \leq j \leq \ell-1} (a_{11} \cdots a_{1j})^* \otimes (a_{1(j+1)} \otimes a_{1r})^* \right) \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^* \\ &\quad + a_1^* \otimes \cdots \otimes a_n^* \otimes \tilde{a}_1^* \otimes \delta^{\widetilde{\text{Cob}}}(\tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^*) \end{aligned}$$

while

$$H\xi = a_1^* \otimes \cdots \otimes a_{n-1}^* \otimes (a_n \tilde{a}_1)^* \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^*,$$

so that

$$\begin{aligned} H\delta^{\widetilde{\text{Cob}}} \xi &= a_1^* \otimes \cdots \otimes a_{n-1}^* \otimes \left(\sum_{1 \leq j \leq \ell-1} (a_n a_{11} \cdots a_{1j})^* \otimes (a_{1(j+1)} \otimes a_{1r})^* \right) \otimes \tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^* \\ &\quad + a_1^* \otimes \cdots \otimes a_{n-1}^* \otimes (a_n \tilde{a}_1)^* \otimes \delta^{\widetilde{\text{Cob}}}(\tilde{a}_2^* \otimes \cdots \otimes \tilde{a}_m^*) \end{aligned}$$

while $\delta^{\widetilde{\text{Cob}}} H\xi$ is ξ plus the two terms from $H\delta^{\widetilde{\text{Cob}}} \xi$, above. This verifies (3.32)

The proof that $\widetilde{\text{Cob}}\mathcal{B}_0 \simeq \mathcal{A}_0$ is a straightforward modification of the proof that $\widetilde{\text{Cob}}\mathcal{A}_0 \simeq \mathcal{B}_0$ above. $\square$

Remark 3.5.4. The identification of the cobar is related to Koszul duality, but in this case, we can prove it directly, without referring to the duality claim. A result analogous to Lemma 3.5.2 becomes more difficult when we are considering the cobars of bona fide $\mathcal{A}_\infty$ -algebras.

Additionally, a key fact to note is that:

Lemma 3.5.5. *Both $\mathcal{A}_0$ and $\mathcal{B}_0$ are filtered by the length grading ℓ , and $F_n \mathcal{A}_0$, $F_n \mathcal{B}_0$ are finite dimensional for each n .*

Having noted all this, we can apply Proposition 3.2.5 to ascertain the necessary facts about the Hochschild cohomologies of $\mathcal{A}_0$ and $\mathcal{B}_0$. Write $C^{*,*}(\mathcal{A}_0)$ and $C^{*,*}(\mathcal{B}_0)$ for the subcomplexes of

$\mathrm{HC}^{*,*}(\mathcal{A}_0)$ and $\mathrm{HC}^{*,*}(\mathcal{B}_0)$, respectively, which were discussed in Section 3.2.2, and write $H^{*,*}(\mathcal{A}_0), H^{*,*}(\mathcal{B}_0)$ for the (bigraded) cohomology groups corresponding to these chain complexes.

3.5.2 Computing the Hochschild cohomology

The last step before we can apply Corollary 3.2.10 to is to compute $H^{*, -1}$ and $H^{*, -2}$ of $\mathcal{A}_0$ and $\mathcal{B}_0$. In order to make this computation, we are going to combine (3.7) from Proposition 3.2.11 with (3.26) and (3.27) from Lemma 3.5.2 in order to compute $H^{*,*}(\mathcal{A}_0)$ using the chain complex $\mathcal{A}_0[V_0, V_{N+1}] \otimes \mathcal{B}_0$, and compute $H^{*,*}(\mathcal{B}_0)$ using the chain complex $\mathcal{B}_0[V_0, \dots, V_{N+1}] \otimes \mathcal{A}_0$. The result is as follows.

Proposition 3.5.6. *With notation as above,*

$$H^{n, -2}(\mathcal{A}_0) = \begin{cases} R & n = 2N \\ 0 & \text{otherwise} \end{cases} \quad (3.33)$$

and

$$H^{n, -1}(\mathcal{A}_0) = 0 \text{ for all } n \geq 2. \quad (3.34)$$

For $\mathcal{B}_0$, we have

$$H^{n, -2}(\mathcal{B}_0) = \begin{cases} R & n = N \\ 0 & \text{otherwise} \end{cases} \quad (3.35)$$

and

$$H^{n, -1}(\mathcal{B}_0) = 0 \text{ for all } n. \quad (3.36)$$

Proof. We compute $H^{n,j}(\mathcal{A}_0)$ first. Recall that n is the length-grading, and j is the Maslov grading. Also note that we are now working over the full ground ring $R = \mathbb{F}_2[V_0, V_{N+1}]$, with the understanding that for $\rho \in \mathcal{A}_0$, $V_{N+1}\rho = \rho V_{N+1} = 0$. For each $\rho \in \mathcal{A}_0$, we can write ρ uniquely in the form

$$\rho = \rho_0 \cdot V_0^j \quad (3.37)$$

where ρ_0 contains no factors of V_0 . Now, for $\rho \in \mathcal{A}_0$,

$$m(\rho) = j(2N - 2) \quad (3.38)$$

for the j from (3.37).

Now, for $\rho = \rho_0 V_0^j$, any non-zero element $\rho \otimes \tau \in \mathcal{A}_0[V_0, V_{N+1}] \otimes \mathcal{B}_0$ will be of the form

$$\rho_0 V_0^j \otimes U_0^j \tau_0,$$

where $\tau_0 \in \mathcal{B}_0$ is the element dual to $\rho_0 \in \mathcal{A}_0$ according to the procedure from the proof of Lemma 3.5.2. This means that

$$\begin{aligned} m(\rho_0 V_0^j \otimes U_0^j \tau_0) &= 2jN + m(U_0^j) + m(\tau_0) \\ &= j(2N - 2) - 2jN - \ell(\tau_0) \\ &= -2j - \ell(\tau_0) \\ &= -2j - \ell(\rho_0). \end{aligned}$$

Note that we have just shown that there are no $\rho \otimes \tau \in \mathcal{A}_0[V_0, V_{N+1}]$ with the idempotents working out, and with $m(\rho \otimes \tau) = -1$ and length > 2 . This verifies (3.34).

This calculation also means that $m(\rho) = -2$ if and only if $j = 1$ and ρ_0 is trivial, i.e. $\rho = V_0$, or $j = 0$ and $\ell(\rho_0) = 2$. But we are only interested in $\mathrm{HH}_{\mathbb{Z}}^{n,*}(\mathcal{A}_0)$ for $n > 2$, so this latter case is not relevant here. In order to verify (3.33), we just need to show that

- (i) $\partial(V_0 \otimes U_0) = 0$;
- (ii) $V_0 \otimes U_0$ is not a coboundary.

For (i), recall that

$$U_0 = \sum_{i=1}^N \sigma_{i+N-1} \rho_{i+N-1} \cdots \sigma_i \rho_i + \sum_{i=1}^N \rho_i \sigma_{i+N-1} \cdots \rho_{i+1} \sigma_i$$

so using multiplication rules on $\mathcal{B}_0$, we get

$$\begin{aligned} \partial(V_0 \otimes U_0) &= \sum_{i=1}^N (U_i V_0 \otimes \rho_i \sigma_{i+N-1} \cdots \rho_{i+1} \sigma_i \cdot \rho_i + V_0 U_i \otimes \rho_i \cdot \sigma_{i+N-1} \rho_{i+N-1} \cdots \sigma_i \rho_i) \\ &\quad + \sum_{i=1}^N (s_{i-1} V_0 \otimes \sigma_{i+N-1} \rho_{i+N-1} \cdots \sigma_i \rho_i \sigma_{i-1} + V_0 s_i \otimes \sigma_i \cdot \rho_i \sigma_{i+N-1} \cdots \rho_{i+1} \sigma_i) \\ &= 0 \end{aligned}$$

For (ii), note that any $\rho \otimes \tau \in \mathcal{A}_0[V_0, \dots, V_{N+1}] \otimes \mathcal{B}_0$ with $\partial(\rho \otimes \tau) = V_0 \otimes U_0$ would have $m(\rho \otimes \tau) = -3$ and $\ell(\rho) = \ell(\tau) = 2N - 1$. But there are no such elements.

For $H^*(\mathcal{B}_0)$, we note that for each $\rho \in \mathcal{B}_0$, we can write ρ uniquely in the form

$$\rho = \rho_0 V_1^{j_1} \cdots V_{N+1}^{j_{N+1}}.$$

where each j_i is a nonnegative integer and ρ_0 contains no factors of any V_i . If $j_i > 0$ for $1 \leq i \leq N$, then there is not $\tau \in \mathcal{A}$ such that $\rho \otimes \tau$ is a non-zero element of $\mathcal{B}_0[V_0, \dots, V_{N+1}] \otimes \mathcal{A}_0$. Hence, we restrict ourselves to the case of $\rho = \rho_0 V_{N+1}^j$, for j a non-negative integer and ρ_0 not containing any factors of V_i . Then if $\tau \in \mathcal{A}_0$ is such that $\rho \otimes \tau$ is a nonzero element of $\mathcal{B}_0[V_0, \dots, V_{N+1}] \otimes \mathcal{A}_0$, then we have

$$m(\rho \otimes \tau) = -2j - \ell(\rho_0), \quad (3.39)$$

Note that $\ell(\rho) = jN + \ell(\rho_0)$. Hence, for $\ell(\rho) > 2$, we never have $m(\rho \otimes \tau) = -1$. This verifies (3.33). Also, (3.39) means that for $\ell(\rho) > 2$, $m(\rho \otimes \tau) = -2$ implies that $\ell(\rho_0) = 0$ (i.e. ρ_0 is trivial) and $j = 1$? that is, $\rho = V_{N+1}$. Now, the only nonzero element of $\mathcal{B}_0[V_0, \dots, V_{N+1}] \otimes \mathcal{A}_0$ with $\mathcal{B}_0[V_0, \dots, V_{N+1}]$ component V_{N+1} is $V_{N+1} \otimes U_{N+1}$, which has $\ell(V_{N+1} \otimes U_{N+1}) = N$. This verifies (3.5.5). $\square$

We are now ready to prove Proposition 3.5.1, that is, that $\mathcal{A}$ and $\mathcal{B}$ are the unique deformations of $\mathcal{A}_0$ and $\mathcal{B}_0$ with the given μ_{2N} and μ_N operations, respectively.

Proof of Proposition 3.5.1. The proof is a direct application of Corollary 3.2.10 combined with the calculations from Proposition 3.5.6.

The only slightly touchy thing is that the result of Proposition 3.5.6 provides a single Hochschild cochain corresponding to the μ_{2N} for $\mathcal{A}$, and a single Hochschild cochain corresponding to the μ_N for $\mathcal{B}$. This would seem to imply that there is a *single* non-zero μ_{2N} operation on $\mathcal{A}$ and a single μ_N operation on $\mathcal{B}$, rather than the $2N$ non-zero μ_{2Ns} for $\mathcal{A}$ stipulated by (3.24) and the N non-zero μ_{Ns} stipulated by (3.25). However, each of the operations from (3.24) is in fact different components (in separate Alexander gradings) of the single Hochschild cocycle corresponding to the μ_{2N} ; indeed, looking back at the explicit computation of the Hochschild cohomology from the proof of Proposition 3.5.6, it is clear that the cocycle corresponding to the μ_{2N} has one component in each Alexander grading, just like the operations stipulated by (3.24). The case of the μ_N for $\mathcal{B}$ is analogous. $\square$

3.6 Proof of Theorem 1.0.3

The goal of this section is to apply Proposition 3.5.1 in order to verify Theorem 1.0.3. For both $\mathcal{A}$ and $\mathcal{B}$, the proof has the same three steps. Using $\mathcal{A}$ as an example, the first step of the proof is to identify the generators of $\mathcal{A}_+$ (the augmentation ideal in the underlying associative algebra $\mathcal{A}_0$) with the elements of $\text{End}_{\mathcal{W}(M_\alpha)}(\alpha_1 \oplus \cdots \oplus \alpha_N)$. The second step is to show that for each $x \in \text{End}_{\mathcal{W}(M_\alpha)}(\alpha_1 \oplus \cdots \oplus \alpha_N)$ corresponding to an element of $a \in \mathcal{A}_+$, the $\mathbb{Z}$ -component of $\text{gr}(x)$ equals $m(a)$. Finally, we do a series of model calculations which show that $\text{End}_{\mathcal{W}(M_\alpha)}(\alpha_1 \oplus \cdots \oplus \alpha_N)$ satisfies the conditions of Proposition 3.5.1(a); that is, that

- Multiplication preserves the $\mathbb{Z}$ -component of gr ;
- There are no non-zero multiplication operations of order $2 < n < 2N$, and there are μ_{2N} in each Alexander grading, corresponding to the cyclic permutations of (3.24).

Let $C_0, \dots, C_{N+1}$ denote the boundary circles of the star diagram, labelled as in Figure 3.1. Write

$$\tilde{\mathcal{A}} = \text{End}_{\mathcal{W}(M_\alpha)}(\alpha_1 \oplus \cdots \oplus \alpha_N)$$

$$\tilde{\mathcal{B}} = \text{End}_{\mathcal{W}(M_\beta)}(\beta_1 \oplus \cdots \oplus \beta_N)$$

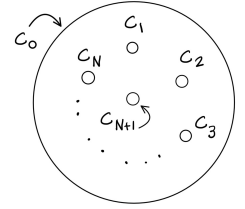


Figure 3.1

Let $\tilde{m}$ denote the $\mathbb{Z}$ -valued gradings induced on both $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ as the projection of gr to the $\mathbb{Z}$ -coordinate. The α - and β -arcs are defined as in Figure 3.2, for arbitrary $N > 2$. (Figure 3.2 has $N = 3$.)

To prove (1.7) from Theorem 1.0.3, we first identify the I_i, U_i and s_i , ($1 \leq i \leq N$), with elements of $\tilde{\mathcal{A}}$. For each $1 \leq i \leq N$, define $\tilde{\alpha}_i$ to be a slight perturbation of α_i such that $\tilde{\alpha}_i \cap C_i$ and $\tilde{\alpha}_i \cap C_{N+1}$ are slightly clockwise of $\alpha_i \cap C_i$ and $\alpha_i \cap C_{N+1}$, respectively. Define $\alpha_i^j = \varphi^j(\tilde{\alpha}_i)$. Then

$$q_{ii}^0 := \alpha_i \cap \tilde{\alpha}_i, \quad (3.40)$$

is a single point on α_i , and

$$\alpha_i \cap \alpha_i^1 = \{q_{ii}^n, r_{ii}^n\}_{n=1}^\infty,$$

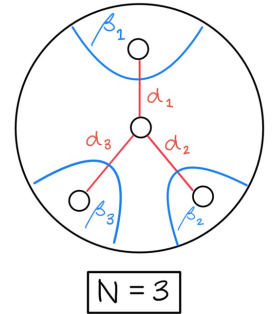


Figure 3.2 (3.41)

where the q_{ii}^0 are a sequence of points radiating inward from q_{ii}^0 along α_i , towards C_{N+1} , and r_{ii}^n are a sequence of points radiating outward from q_{ii}^0 along α_i towards C_i . These points are as on the left

of Figure 3.3. Likewise, for $i \neq j$,

$$\alpha_i \cap \alpha_j^1 = \{q_{ij}^n\}_{n=1}^\infty \quad (3.42)$$

is a sequence of points radiating inward from q_{ii}^0 along α_i , towards C_{N+1} . These points are as on the right of Figure 3.3.

Note also that up to Hamiltonian isotopy, each of these can be viewed as the corresponding intersection point in $\alpha_i^r \cap \alpha_j^{r+1}$ for any $r \in \mathbb{N}$.

The identification between generators of $\tilde{\mathcal{A}}$ and elements of $\mathcal{A}_0$ is as follows:

$$\begin{aligned} I_i &\leftrightarrow q_{ii}^0; & U_i^n &\leftrightarrow r_{ii}^n \\ s_{ii}^n &\leftrightarrow q_{ii}^n & s_{ij}s_{jj}^n &\leftrightarrow q_{ij}^{n+1} \text{ for } i \neq j; \end{aligned}$$

where we recall that s_{ii} is defined as

$$s_{ii} = s_i s_{i+1} \cdots s_{i-1}$$

for each $1 \leq i \leq N$ with all indices counted mod N , and that

$$s_{ii}^n = \underbrace{s_{ii} \cdot s_{ii} \cdots s_{ii}}_{n \text{ times}}.$$

For the rest of the section, we will abuse notation by writing the corresponding element of $\mathcal{A}_0$ instead of the q_{ij}^k and r_{ii}^n .

Define a $\mathbb{Z}_{\geq 0}$ -valued *length grading* ℓ on $\tilde{\mathcal{A}}$ according to the rules:

- (i) $\ell(I_i) = \ell(U_i) = \ell(s_i) = 1$ for each i ;
- (ii) $\ell(V_0) = 2N$
- (iii) $\ell(\mu_n(a_1, \dots, a_n)) = \sum_{i=1}^n \ell(a_i)$.

The next step is to verify that the non-zero μ_2 s on $\tilde{\mathcal{A}}$ are the same as in $\mathcal{A}_0$, and that the μ_2 on $\tilde{\mathcal{A}}$ preserves $\tilde{m}$.

Lemma 3.6.1. (a) *The identifications above define a bijection between $\tilde{\mathcal{A}}$ and $\mathcal{A}_0$;*

(b) *Each U_i, s_i has trivial $\tilde{m}$ -grading;*

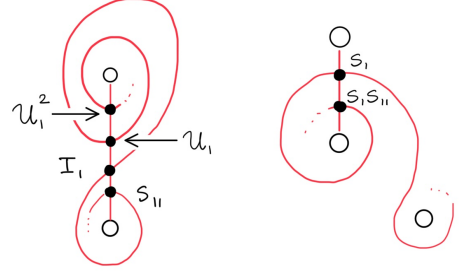


Figure 3.3

(c) The non-zero μ_2 operations on $\tilde{\mathcal{A}}$ are in one-to-one correspondence with the non-zero μ_2 operations on $\mathcal{A}$.

(d) These non-trivial μ_2 operations preserve $\tilde{m}$;

Proof. (a) follows from Figure 3.3. (d) follows directly from Proposition 3.3.1(a), and (b) then follows from Figures 3.4 and 3.5. These pictures depict calculations of the projection of curve used to compute gr onto the S^1 component of $S^1 \times (V_{n+1} S^1) \simeq \mathcal{Lag}(M)$. These figures can therefore be used to calculate which are used to calculate $\tilde{m}$. The point of Figure 3.4 and 3.5 is that the projection of these curves onto the S^1 component is trivial.

There are two further things to note about Figures 3.4 and 3.5. First is that we are grossly abusing notation by denoting by $\tilde{b}_\bullet$ the respective projections in the original manifold M (along with shadows of the corresponding element of $\text{Lag}(2) = S^1$). Second is that the copy of S^1 which forms $\text{Lag}(2)$ is actually the quotient of S^1 under the action of $\{\pm 1\}$; therefore, we should actually be considering the *quotients* of the curves pictured on the right of Figures 3.4 and 3.5 under the $\{\pm 1\}$ -action. However, the curves pictured below convey the same information, and are easier to draw.

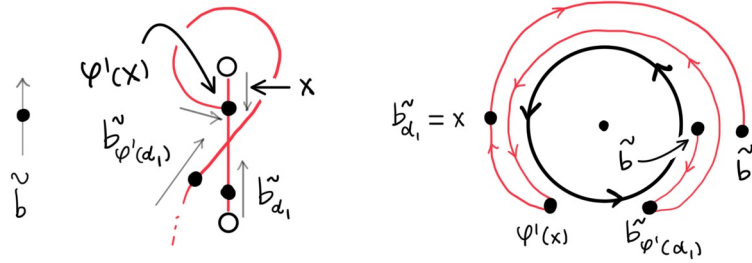


Figure 3.4

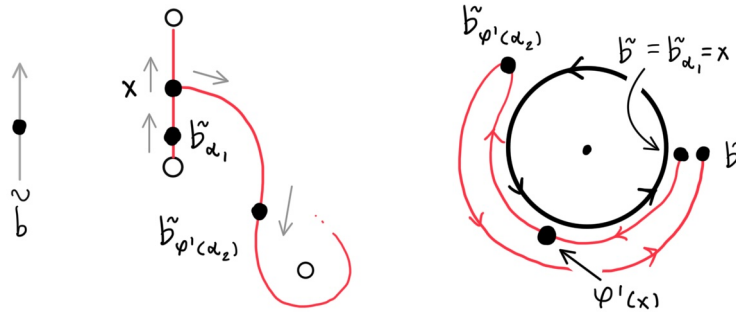


Figure 3.5

For (c), we need to verify first that all the multiplications from $\mathcal{A}_0$ correspond to μ_2 operations

on $\tilde{\mathcal{A}}$, and then the converse. For the first direction, the fact that multiplication on $\tilde{\mathcal{A}}$ is associative implies that we only need to verify that the multiplications match up in the case of the basic multiplications. That is, we need only show that for each $1 \leq i \leq N$, there exist unique (correctly oriented) triangles with

- Vertices U_i, U_i , and U_i^2 , and edges on α_i, α_i^1 , and α^2 ;
- s_i, s_{i+1} and $s_{i(i+2)}$ and edges on α_i, α_{i+1}^1 , and α_{i+2}^2 ;

up to the identifications above, and with all indices counted mod N . These triangles are pictured (for $N = 3, i = 1$) in Figure 3.6.

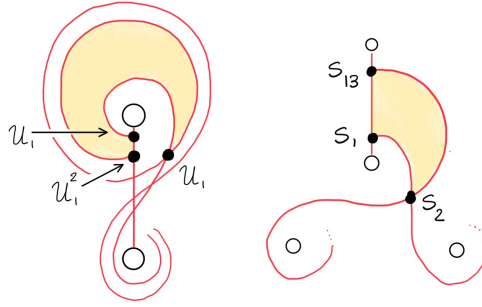


Figure 3.6

Hence, every non-zero μ_2 in $\mathcal{A}_0$ corresponds to a non-zero multiplication on $\tilde{\mathcal{A}}$. The last step is to show that these are the *only* non-zero μ_2 's on $\tilde{\mathcal{A}}$. To do this, note that if $ab \neq 0$ in $\tilde{\mathcal{A}}$, then the final idempotent of a matches the initial idempotent of b . The options (for basic multiplication) which are not already covered are $s_{i-1}U_i$ and $U_i s_i$ (with all indices counted mod N). But these cannot be connected by any Maslov index one, correctly oriented triangle with the correct edges (which are spun correctly), as illustrated in Figure 3.7 (for the generic case $i = 1$). $\square$

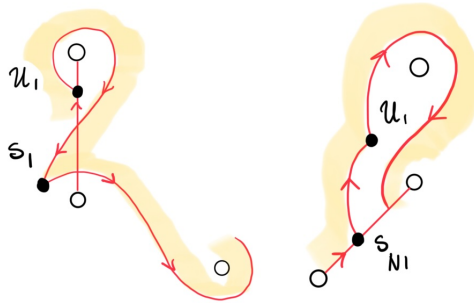


Figure 3.7

The next step is to verify

- Lemma 3.6.2.** (a) *There are no non-trivial operations μ_n on $\tilde{\mathcal{A}}$, with $2 < n < 2N$;*
(b) *There are exactly $2N$ distinct μ_{2N} operations on $\tilde{\mathcal{A}}$, which are precisely those from (3.24) of Proposition 3.5.1;*

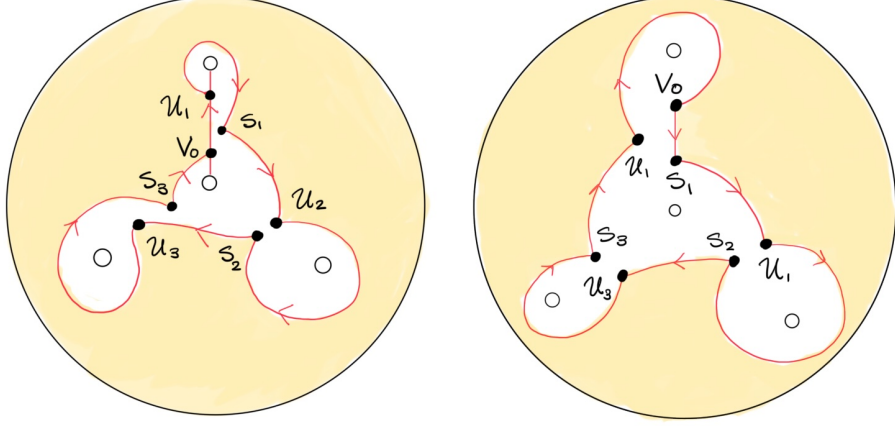


Figure 3.8: (Each shaded disk is not punctured, and covers the whole of S^2 except for the unshaded portion)

Proof. (a) follows for grading reasons as in the verification of the analogous fact for $\mathcal{A}$. To reiterate, we know from Proposition 3.6.1(a) that $\tilde{m}(a) = 0$ for each $a \in \tilde{\mathcal{A}}$; hence, by Proposition 3.3.1(a), $\tilde{m}$ and any operation on $\tilde{\mathcal{A}}$ must satisfy

$$\tilde{m}(\mu_n(a_1, \dots, a_n)) = \sum_{i=1}^n \tilde{m}(a_i) + n - 2 = n - 2.$$

However, no element of $\tilde{\mathcal{A}}$ has grading $n - 2$ for $2 < n < 2N - 2$.

(b) follows directly from the two pictures in Figure 3.8, up to cyclic permutation of labels. After noting that these are in fact the unique $(2N + 1)$ -gons with boundary on the desired α -arcs the proof is complete. $\square$

For $\mathcal{B}_0$, the steps are analogous. For each $1 \leq i \leq N$, let $\tilde{\beta}_i$ be a perturbation of β_i such that the points of $\tilde{\beta}_i \cap C_0$ are slightly clockwise of the corresponding point of $\beta_i \cap C_0$. Define $\beta_i^j = \varphi^j(\tilde{\beta}_i)$, and define

$$\tilde{q}_{ii}^0 := \beta_i \cap \tilde{\beta}_i, \quad (3.43)$$

and

$$\beta_i \cap \beta_i^1 = \{\tilde{q}_{ii}^n, \tilde{r}_{ii}^n\}_{n=1}^\infty, \quad (3.44)$$

and for $i \neq j$,

$$\beta_i \cap \beta_j^1 = \{\tilde{q}_{ij}^n\}_{n=1}^\infty \quad (3.45)$$

The identification between elements of $\tilde{\mathcal{B}}$ and elements of $\mathcal{B}_0$ is as follows

$$\begin{aligned} I_i &\leftrightarrow \tilde{q}_{ii}^0; & \rho_i^n &\leftrightarrow \tilde{r}_{ii}^n \\ s_{ii}^n &\leftrightarrow q_{ii}^n & s_{ij}s_{jj}^n &\leftrightarrow q_{ij}^{n+1} \text{ for } i \neq j; \end{aligned}$$

These points are pictured (labelled by the corresponding element of $\mathcal{B}_0$) in Figure 3.9.

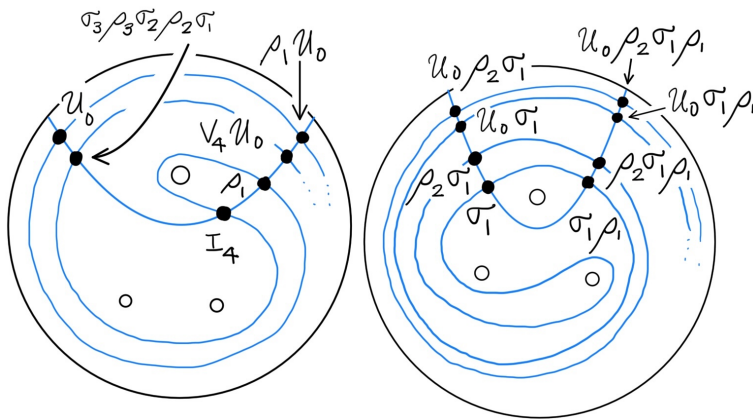


Figure 3.9

Define a $\mathbb{Z}_{>0}$ -valued length grading ℓ on $\tilde{\mathcal{B}}$ according to the same three rules as for $\tilde{\mathcal{A}}$, that is:

- (i) $\ell(I_i) = \ell(\rho_i) = \ell(\sigma_i) = 1$ for each i ;
- (ii) $\ell(V_{N+1}) = N$;
- (iii) We also require that higher operations satisfy

$$\ell(\mu_n(b_1, \dots, b_n)) = \sum_{i=1}^n \ell(b_i).$$

Lemma 3.6.3. (a) *The identifications above define a bijection between $\tilde{\mathcal{B}}$ and $\mathcal{B}_0$;*

- (b) Each $b = \rho_i, \sigma_i$ has $m(b) = -1$;
- (c) The non-zero μ_2 operations on $\tilde{\mathcal{B}}$ are in one-to-one correspondence with the non-zero μ_2 operations on $\mathcal{B}$.

(d) These non-trivial μ_2 operations preserve $\tilde{m}$ in the sense that

$$\tilde{m}(\mu_2(b_1, b_2)) = \tilde{m}(b_1) + \tilde{m}(b_2).$$

Proof. (a) follows from Figure 3.9. (d) again follows from Proposition 3.3.1, and (b) follows from calculations analogous to the ones done in Figures 3.4 and 3.5 for $\tilde{\mathcal{A}}$.

For (c), we want to show that non-zero μ_2 on $\mathcal{B}$ implies non-zero μ_2 on $\tilde{\mathcal{B}}$, and then the reverse. In more detail, the forward direction says that if $b_1, b_2 \in \mathcal{B}$ so that $b_2 b_1 \neq 0$, and we let i_1, i_0 denote the initial and final idempotents of b_1 , respectively, and let i_2 denote the initial idempotent of b_2 , then we want to find a unique, Maslov-grading 1, triangle with vertices b_2, b_1 , and $b_2 b_1$, and with boundary along $\beta_{i_0}, \beta_{i_1}^1$, and $\beta_{i_2}^2$. In fact, because multiplication on $\tilde{\mathcal{B}}$ is associative, we only need to verify it for the basic elements of $\mathcal{B}_+$, that is, the σ_i and ρ_i . For the basic elements

- $b_1 = \rho_1, b_2 = \sigma_1$;
- $b_1 = \sigma_1, b_2 = \rho_2$;

these triangles are pictured in Figure 3.10. For all other multipliable pairs of basic elements in $\mathcal{B}_+$, the triangles are analogous. Now, we just need to verify that given a Maslov index 1 triangle with vertices b_2, b_1 , and b_3 , and boundary along $\beta_{i_0}, \beta_{i_1}^1$, and $\beta_{i_2}^2$, $b_2 b_1 \neq 0$ in $\mathcal{B}$ and is equal to b_3 . The argument that these are the *only* non-zero μ_2 's in $\tilde{\mathcal{B}}$ is analogous to the case of $\tilde{\mathcal{A}}$ (but with order of multiplication reversed). $\square$

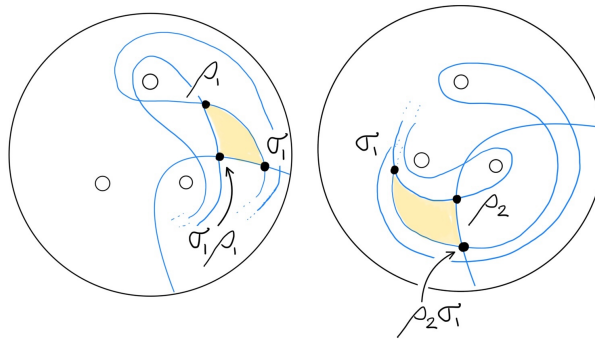


Figure 3.10

Lemma 3.6.4. (a) There are no non-trivial operations μ_n on $\tilde{\mathcal{A}}$, with $2 < n < N$;

(b) There are exactly N distinct non-extended μ_N operations on $\tilde{\mathcal{A}}$, which are precisely those from (3.25) of Proposition 3.5.1;

Proof. (a) follows for grading reasons; indeed, consider $\mu_n(b_1, \dots, b_n) = b_0$ with $2 < n < N$ and $b_i \in \tilde{\mathcal{B}}$ for each i . Then

$$\tilde{m}(b_0) = \sum_{i=1}^n \tilde{m}(b_i) + n - 2 = n - 2 - \sum_{i=1}^n \ell(b_i). \quad (3.46)$$

Moreover, $\ell(b_0) = \sum_{i=1}^n \ell(b_i)$. If $\ell(b_0) < N$, then $\tilde{m}(b_0) = -\ell(b_0)$, so that (3.46) implies $n = 2$, which we are not considering; we can therefore exclude this case. If $\ell(b_0) \geq N$, then b_0 contains some j factors of V_{N+1} , which contribute $-2j$ to $\tilde{m}(b_0)$, and $+j$ to $\ell(b_0)$.

Write $\tau_1, \dots, \tau_k \in \mathcal{B}_+$ for the basic factors of $b_1, \dots, b_n$. Arguing using the fact that $\sum_{i=1}^N \ell(b_i) = \ell(b_0)$, we see that if we temporarily ignore the τ_i which correspond to the factors of b_0 , we are left with jN basic factors. These τ_i contribute $-jN$ to the sum $\sum_{i=1}^n \tilde{m}(b_i)$. The equality of (3.46) now reduces to:

$$-2j = -jN + n - 2. \quad (3.47)$$

We are assuming $n < N$, which means we have

$$j = \frac{n-2}{N-2} < 1,$$

which contradicts $j \in \mathbb{N}$. Hence, (a) follows.

For (b), (3.47) likewise shows that if $n = N$, then we need $j = 1$. Because we are only interested in the generating Hochschild cocycle – that is, the unextended μ_N operation – we want to consider the case when

$$\ell(b_0) = \sum_{i=1}^N \ell(b_i) = N, \quad (3.48)$$

and show that the μ_N must be of the form

$$\mu_N(\sigma_{i+N-1}, \sigma_{i+N-2}, \dots, \sigma_{i+1}, \sigma_i) = V_{N+1}, \quad (3.49)$$

where the indices are counted mod N as usual. In general, because we are only considering the case where (3.48) holds, and all basic elements of $\tilde{\mathcal{B}}$ have length 1, we have $b_0 = V_{N+1}$. It remains to show that

$$(b_1, \dots, b_N) = (\sigma_{i+N-1}, \sigma_{i+N-2}, \dots, \sigma_{i+1}, \sigma_i),$$

for some $1 \leq i \leq N$. Arguing using (3.48) again, we know that each b_i is a length 1 element, and is therefore one of the ρ_j or σ_j . Because we are only looking at sequences $(b_1, \dots, b_n)$ so that the

initial idempotent of b_i is the final idempotent of b_{i-1} for each i , it will suffice to show that each b_i must be *some* σ_j .

We do this by considering the canonical μ_N operation and its corresponding $(N+1)$ -gon. That there exists such a (unique) non-trivial μ_N with the desired input and output follows directly from Figure 3.11 (and the analogous figures for cyclic permutations of $\{1, \dots, N\}$ and arbitrary N).

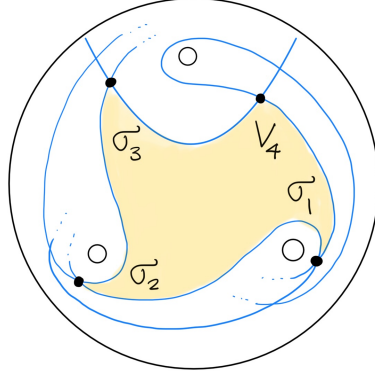


Figure 3.11

Moreover, the calculations in Figures 3.12 and 3.13 show that the F_{N+1} -components of $\text{gr}(\rho_i)$ and $\text{gr}(\sigma_i)$ are $\bar{i}$ and 1, respectively.

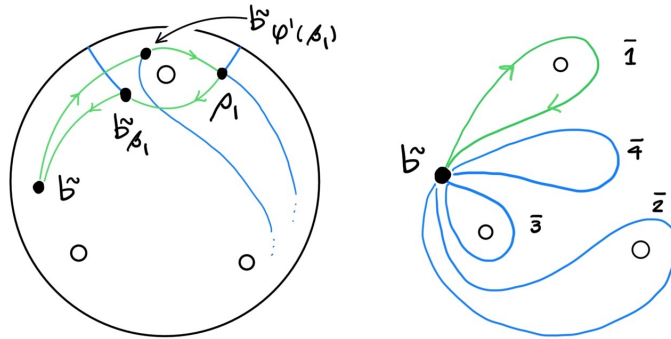


Figure 3.12

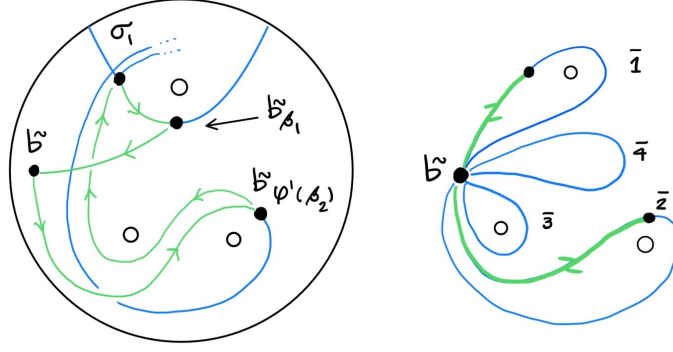


Figure 3.13

Combined with Figure 3.11 and Proposition 3.3.1(a), Figure 3.13 shows that the F_{N+1} -component of $\text{gr}(V_{N+1})$ is equal to 1. Therefore, by Figures 3.12 and 3.13 again, combined with Proposition 3.3.1(a) as before, each b_i must be one of the σ_i . Given the discussion of idempotents above, this suffices to verify that the operation is of the form in (3.49). $\square$

Lemmas 3.6.1 through 3.6.4 confirm that $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ satisfy the conditions of Proposition 3.5.1. Applying Proposition 3.5.1, we can now conclude that Theorem 1.0.3 holds.

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