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Irreducibility of Automorphic Galois Representations of Low Dimensions

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IRREDUCIBILITY OF AUTOMORPHIC GALOIS
REPRESENTATIONS OF LOW DIMENSIONS

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Abstract

Let π be a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where F is a CM field.

We show that for $n \leq 6$, there is a Dirichlet density 1 set $\mathfrak{L}$ of rational primes, such that for all $l \in \mathfrak{L}$, the l -adic Galois representations associated to π are irreducible.

We also show that for any integer $n \geq 1$, in order to show the existence of the aforementioned set $\mathfrak{L}$, it suffices to show that for all but finitely many finite primes λ in a number field determined by π , all the irreducible constituents of the restriction of the corresponding Galois representation $r_{\pi, \lambda}$ to the derived subgroup of the identity component of the Zariski closure of the image, are conjugate self-dual.

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To my parents.

Contents

1	Introduction	1
2	Organization of the manuscript	3
3	Preliminaries	4
3.1	Automorphic representations	4
3.2	Galois representations	4
3.3	Compatible systems of Galois representations	5
4	Reduction to the maximal CM subextension	9
5	An L-function argument	12
6	Reduction to the semi-simple case	17
6.1	Reduction to being polarized with a character of $G_{F_1, \pi}$	17
6.2	Reduction to semi-simple image	19
7	Irreducibility at a set of positive Dirichlet density	21
8	Independence of λ results	23
8.1	Independence of λ of characteristic varieties	23
8.1.1	Serre groups and abelian representations	23
8.1.2	Notes on characteristic varieties	25
8.1.3	A compatible system of λ -adic representations	25
8.1.4	Characteristic varieties and weights of a representation	29
8.2	Semi-simple formal characters and actions by c_λ	31
9	Irreducibility of automorphic representations of $\mathrm{GL}_n, n \leq 6$	36
A	Density of c_λ-semi-simple elements	41
	References	45

1 Introduction

It is conjectured that the Galois representations associated to an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ over a number field F are irreducible. The Fontaine-Mazur-Langlands conjecture suggests that the reducibility of the l -adic representations associated to an automorphic representation π for some rational prime l would show that π cannot be cuspidal.

Previous Works. So far, not much is known in the general case. The conjecture was first proved for classical modular forms ([15]) and Hilbert modular forms ([22]). In [2], the author proved the result for polarizable representations of $\mathrm{GL}_3(\mathbb{A}_F)$ with F totally real. In [23], the authors proved the irreducibility of Galois representations considered in [6], under the assumption that the automorphic representation is square-integrable at some finite prime. In [14], the author proved irreducibility of the l -adic Galois representations associated to polarizable representations of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ for l large enough. The method also extends to the case without the polarizable condition assuming the corresponding Galois representations exist.

Recent Works. In the past decade, good progress has been made in the case of F being a CM field and π satisfies the additional assumptions of regularity and polarizability. In [1], by using potential automorphy theorems (Section 4 of [1]) and an L -function argument (Section 5 of [1]), it was proved that if F is an imaginary CM field and π a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with extremely regular weights, then for a Dirichlet density 1 set of rational primes l , the l -adic Galois representations associated to π are irreducible. Using results from [1] and combining one further idea from [12] (which uses the CM nature of the field of coefficients of the Galois representations), the authors of [13] were able to prove that if F is imaginary CM or totally real and π is a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then for a positive Dirichlet density set of rational primes l , the l -adic Galois representations associated to π are irreducible. They were able to drop the assumption that π is extremely regular, which is a condition that is rather restrictive, but the conclusion is also weakened to give irreducibility only for a positive Dirichlet density set of rational primes. In [4], it was proven that if F is totally real, and π a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with $n \leq 5$, then for a Dirichlet density 1 set of rational primes l , the l -adic Galois representations associated to π are irreducible.

Main Results. In this paper, we let F be a CM field (which includes the case of an imaginary CM field and the case of a totally real field) and let π be a polarizable, regular algebraic, cuspidal automorphic representation

of $\mathrm{GL}_n(\mathbb{A}_F)$. We will prove irreducibility of the associated Galois representations for a Dirichlet density 1 set of rational primes for π of dimension $n \leq 6$.

Strategy. The strategy is to combine the potential automorphy results from [1] and the irreducibility result for a positive Dirichlet density set of primes proved in [13]. The main challenge in applying the potential automorphy theorems comes from showing that each irreducible constituent of the corresponding Galois representation $r_{\pi,\lambda}$ is polarizable and totally odd. We first reduce the problem to showing that each irreducible constituent of $r_{\pi,\lambda}$ restricted to the derived subgroup of the identity component of the Zariski closure of the image is self-dual. We then show there exists a positive Dirichlet density set of primes λ such that $r_{\pi,\lambda}$ is irreducible when restricted to the derived subgroup of the identity component. These steps work for π of any dimension.

We then use arguments similar to the ones in [11] and [8], by looking at the characteristic polynomials of the Frobenius elements, to show that the formal character of $r_{\pi,\lambda}$ restricted to the derived subgroup of the identity component, and the restriction of $r_{\pi,\lambda}$ to the conjugation-invariant subtorus of a maximal torus of the derived subgroup of the identity component of the Zariski closure of the image, are independent of λ in a weakly compatible system of Galois representations. We then list the possibilities for faithful, irreducible and regular representations of small dimensions of semi-simple Lie algebras. Applying the irreducibility result of [13] together with the aforementioned independence of λ results, we show irreducibility for π up to dimension 6.

One might ask whether our methods can be used to prove irreducibility for π of dimension $n \geq 7$. Our method is insufficient for $n = 7$ and we explain the reason here.

The simple Lie group G_2 has a standard representation of dimension 7, which is irreducible, faithful and self-dual. Let l_1, l_2 be two rational primes. Consider two different 7-dimensional representations of the absolute Galois group of $\mathbb{Q}$. The first one is $r_{l_1} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_7(\mathbb{Q}_{l_1})$, where the Zariski closure G_{l_1} of the image is a reductive group of type G_2 and r_{l_1} induces the standard representation, and the second one is $r_{l_2} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_7(\mathbb{Q}_{l_2})$, where the Zariski closure G_{l_2} of the image is a reductive group of type SL_3 and r_{l_2} induces the representation $\mathrm{std} \oplus \mathrm{std}^{\vee} \oplus \mathbb{I}$, with std denoting the standard representation of SL_3 . In this case, the semi-simple ranks of the Zariski closures of the images are the same and so are the formal characters of the two representations. Our techniques cannot rule out the existence of such a case, but r_{l_2} has irreducible constituents that are not self-dual. This renders the potential automorphy results from [1] inapplicable in this case.

2 Organization of the manuscript

The manuscript is organized as follows: in Section 3 we review some background material on automorphic representations and Galois representations. In Section 4, we state our main theorem, and reduce the proof to the case where F is the maximal CM subextension of the Galois extension of F^+ given by the identity component. In Section 5, we recall some potential automorphy results and an L -function argument that we will use. In Section 6, we prove that in order to show the existence of a set of rational primes of Dirichlet density 1, where $r_{\pi,\lambda}$ is irreducible for all λ lying above the set, it suffices to show that for all but finitely many primes λ , each irreducible constituent of the restriction of $r_{\pi,\lambda}$ to the derived subgroup of identity component of the Zariski closure of the image is conjugate self-dual. In Section 7, we show that there exists a set of rational primes of positive Dirichlet density such that for primes λ lying above the set, $r_{\pi,\lambda}$ is irreducible when restricted to the derived subgroup of the identity component of the Zariski closure of the image. In Section 8 we prove some independence of λ results for a weakly compatible system of Galois representations. Finally, in Section 9, we show irreducibility of $r_{\pi,\lambda}$ for π of dimension $n \leq 6$, using results from Section 7 and Section 8.

3 Preliminaries

We follow the setup of [1] and [13]. We will review some definitions and results on automorphic representations, Galois representations and compatible systems of Galois representations. We will not include any proofs in this section. We refer the reader to Section 2.1 of [1], Section 5.1 of [1] and Section 1 of [13] for the proofs and for a more comprehensive review.

By a CM field we mean a number field admitting an automorphism c which coincides with complex conjugation for every embedding of the number field into $\mathbb{C}$. For a CM field F , we let F^+ denote its maximal totally real subfield.

3.1 Automorphic representations

Let F be a CM field. Fix a rational prime l .

Definition 3.1. *By a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ we mean a pair (π, χ) where*

- π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ where π_∞ has the same infinitesimal character as an irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n ,
- $\chi : \mathbb{A}_{F^+}^\times / (F^+)^\times \rightarrow \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1)$ is independent of $v|_\infty$,
- and $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$, where π^c denotes the composition of π with complex conjugation on $\mathrm{GL}_n(\mathbb{A}_F)$.

Definition 3.2. *We call a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ polarizable if there is a character χ of F^+ such that (π, χ) is polarized.*

Definition 3.3. *Let $\tau : F \hookrightarrow \overline{\mathbb{Q}_l}$ be a continuous embedding and fix an isomorphism $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$. We write $(\mathbb{Z}^n)^{\mathrm{Hom}(F, \mathbb{C}), +}$ for the set $a = (a_{\tau, i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F, \mathbb{C})}$ satisfying $a_{\tau, 1} \geq \dots \geq a_{\tau, n}$. Let Ξ_a denote the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}(F, \mathbb{C})}$ which is the tensor product over $\iota \circ \tau$ of the irreducible representations of GL_n with highest weight $a_\tau = (a_{\tau, i})$. We say a regular algebraic, polarized, cuspidal automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ has weight a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$. In this case there must exist $w \in \mathbb{Z}$ such that $a_{\tau, i} + a_{c \circ \tau, n+1-i} = w$ for all τ, i .*

3.2 Galois representations

Let ϵ_l be the l -adic cyclotomic character and $V(m)$ be the Tate twist $V(\epsilon_l^m)$. Let $\mathrm{res}(\lambda)$ denote the residue characteristic of the prime λ . Let F be a CM field. Let $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_l})$ and $\mu : G_{F^+} \rightarrow \overline{\mathbb{Q}_l}^\times$ be continuous

homomorphisms. Let c_v denote the complex conjugation in G_{F^+} associated to a prime v in G_{F^+} . Let δ_{F/F^+} denote the non-trivial character of $\mathbb{A}_{F^+}^\times / F^+ \mathbf{N}_{F/F^+}(\mathbb{A}_F^\times)$.

Definition 3.4. We call (r, μ) polarized if for some infinite place v of F^+ there is $\epsilon_v \in \{\pm 1\}$ and a non-degenerate pairing $\langle, \rangle_v$ on $\overline{\mathbb{Q}_l}^n$ such that

$$\langle x, y \rangle_v = \epsilon_v \langle y, x \rangle_v$$

and

$$\langle r(\sigma)x, r(c_v \sigma c_v)y \rangle_v = \mu(\sigma) \langle x, y \rangle_v$$

for all $x, y \in \overline{\mathbb{Q}_l}^n$ and all $\sigma \in G_F$. If $F \neq F^+$, replacing μ by $\mu \delta_{F/F^+}$ if necessary, we further require $\epsilon_v = -\mu(c_v)$.

Remark. If the conditions in Definition 3.4 are satisfied by one place $v|\infty$, then they will be true for all $v'|\infty$, by taking $\epsilon_{v'} = \mu(c_v c_{v'})\epsilon_v$ and $\langle x, y \rangle_{v'} = \langle x, r(c_v c_{v'})y \rangle_v$.

Definition 3.5. We call a polarized l -adic representation (r, μ) totally odd if $\epsilon_v = 1$ for all $v|\infty$.

Definition 3.6. We call a polarized l -adic representation (r, μ) algebraic if r is unramified at all but finitely many primes of F and de Rham at all primes above l .

Definition 3.7. For $\tau : F \hookrightarrow \overline{\mathbb{Q}_l}$, let $v(\tau)$ be the prime of F induced by τ . We define a multi-set $\text{HT}_\tau(r)$ of integers by letting i have multiplicity

$$\dim_{\overline{\mathbb{Q}_l}}(V(i) \otimes_{\tau, F_{v(\tau)}} \widehat{F_{v(\tau)}})^{G_{F_{v(\tau)}}}.$$

Definition 3.8. We call a polarized l -adic representation (r, μ) regular algebraic if it is algebraic and for all $\tau : F \hookrightarrow \overline{\mathbb{Q}_l}$, the multi-set $\text{HT}_\tau(r)$ has n distinct elements.

3.3 Compatible systems of Galois representations

Let F be a number field, not necessarily CM.

Definition 3.9. By a rank n weakly compatible system of l -adic representations $\mathfrak{R}$ of G_F defined over M we mean a 5-tuple

$$(M, V, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\}),$$

where

- M is a number field,
- V is a finite set of primes of F ,
- for each prime $v \notin V$ of F , $Q_v(X)$ is a monic degree n polynomial in $M[X]$,
- for each prime λ of M , $r_\lambda : G_F \rightarrow \mathrm{GL}_n(\overline{M}_\lambda)$ is a continuous, semi-simple representation such that
 - if $v \notin V$ and $v \nmid \mathrm{res}(\lambda)$ is a prime of F , then r_λ is unramified at v and $r_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$,
 - if $v \mid \mathrm{res}(\lambda)$, then $r_\lambda|_{G_{F_v}}$ is de Rham, and crystalline in the case $v \notin V$,
- for $\tau : F \hookrightarrow \overline{M}$, H_τ is a multi-set of n integers such that for any $\iota : \overline{M} \hookrightarrow \overline{M}_\lambda$ over M we have $\mathrm{HT}_{\iota \circ \tau}(r_\lambda) = H_\tau$.

Let $\mathfrak{R} = (M, V, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ be a weakly compatible system of l -adic representations of G_F .

Definition 3.10. We call $\mathfrak{R}$ pure of weight w if for each $v \notin V$, each root α of $Q_v(X)$ in $\overline{M}$ and each embedding $\iota : \overline{M} \hookrightarrow \mathbb{C}$, we have $|\iota(\alpha)|^2 = (\#k(v))^w$, where $k(v)$ is the residue field at v .

Definition 3.11. We call $\mathfrak{R}$ regular if for each $\tau : F \hookrightarrow \overline{M}$, every element of H_τ has multiplicity 1.

Lemma 3.12 (Lemma 1.2 in [13]). Suppose $\mathfrak{R}$ is pure of weight w . Then

- (1) if c is the restriction to M of any complex conjugation then ${}^c\mathfrak{R} \cong \mathfrak{R}^\vee \otimes \{\epsilon_l^{-w}\}$.
- (2) $\mathfrak{R}$ can be defined over a CM field.
- (3) if $c \in \mathrm{Aut}(\overline{M})$ denotes a complex conjugation then $H_{c\tau} = \{w - h \mid h \in H_\tau\}$.
- (4) suppose that F_{CM} is the maximal CM subfield of F . If $\tau|_{F_{\mathrm{CM}}} = \tau'|_{F_{\mathrm{CM}}}$, then $H_\tau = H_{\tau'}$.

Lemma 3.13 (Lemma 5.3.1(1) in [1]). Let H_λ be the Zariski closure of $r_\lambda(G_F)$ in $\mathrm{GL}_n/\overline{M}_\lambda$ and let H_λ^0 denote its identity component. Then there is a finite Galois extension F_1/F such that for all λ the map $G_F \rightarrow H_\lambda(\overline{M}_\lambda)$ induces an isomorphism $\mathrm{Gal}(F_1/F) \xrightarrow{\sim} H_\lambda/H_\lambda^0$.

Definition 3.14. Let $\mathfrak{R} = (M, V, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ and $\mathfrak{R}' = (M, V', \{Q'_v(X)\}, \{r'_\lambda\}, \{H'_\tau\})$ be two weakly compatible systems of l -adic representations of G_F over M . We say they are equivalent if $Q_v(X) = Q_{v'}(X)$ for a set of v of Dirichlet density 1 in G_F . In this case we write $\mathfrak{R} \equiv \mathfrak{R}'$.

Lemma 3.15 (Lemma 1.3 in [13]). Suppose F is CM, $\mathfrak{R}$ is a weakly compatible system of l -adic representations of G_F pure of weight w and $\mathfrak{M} = (M, V^\mu, \{X - \alpha_v\}, \{\mu_\lambda\}, \{H_\tau^\mu\})$ is a weakly compatible system of l -adic characters of G_{F^+} with

$$\mathfrak{R}^c \equiv \mathfrak{M}|_{G_F} \otimes \mathfrak{R}^\vee.$$

Then for all τ we have $H_\tau^\mu = \{w\}$.

Lemma 3.16 (Lemma 1.4 in [13]). *Let $\mathfrak{R} = (M, V, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ be a regular pure weakly compatible system of l -adic representations. Then we may replace M by a CM field M' such that for all open subgroups H of G_F and all primes λ of M' , all subrepresentations of $r_\lambda|_H$ are defined over M'_λ . In this case we call M' a full field of definition for $\mathfrak{R}$.*

The following theorem states that when F is CM, we can associate l -adic Galois representations to regular algebraic, polarized, cuspidal automorphic representations.

Theorem 3.17 (Theorem 2.1.1 in [1], Theorem 1.1 in [5] and Section 1 of [13]). *Let F be CM. Let (π, χ) be a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ where π has weight a . We define L_π to be the fixed field of the subgroup of $\mathrm{Aut}(\mathbb{C})$ consisting of elements $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that ${}^\sigma \pi^\infty \cong \pi^\infty$, which is a CM field [12]. We let V_π denote the set of finite primes of F where π is ramified. For $v \notin V_\pi$ a finite prime of F , we let $Q_{\pi,v}(X) \in L_\pi[X]$ denote the characteristic polynomial of $\mathrm{rec}_{F_v}(\pi_v | \det|_v^{(1-n)/2})(\mathrm{Frob}_v)$. If $\overline{L_\pi}$ denotes the algebraic closure of L_π in $\mathbb{C}$, then for $\tau : F \hookrightarrow \overline{L_\pi}$ we set*

$$H_{\pi,\tau} = \{a_{\tau,1} + n - 1, a_{\tau,2} + n - 2, \dots, a_{\tau,n}\}.$$

Then there exist a CM field M_π/L_π , an integer w , a finite set V^χ of finite primes of F^+ , a scalar $\alpha_v \in \overline{M_\pi}$ for each prime $v \notin V^\chi$, a finite multi-set H_τ^χ of integers, and for each finite prime λ of M_π there exists a continuous semi-simple representation

$$r_{\pi,\lambda} : G_F \rightarrow \mathrm{GL}_n(\overline{M_{\pi,\lambda}})$$

and a continuous character

$$r_{\chi,\lambda} : G_{F^+} \rightarrow \overline{M_{\pi,\lambda}}^\times$$

such that

$$\mathfrak{R}_\pi := (M_\pi, V_\pi, \{Q_{\pi,v}(X)\}, \{r_{\pi,\lambda}\}, \{H_{\pi,\tau}\})$$

is a regular pure weakly compatible system of l -adic representations of weight w and

$$\mathfrak{R}_\chi := (M_\pi, V^\chi, \{X - \alpha_v\}, \{r_{\chi,\lambda}\}, \{H_\tau^\chi\})$$

is a weakly compatible system of l -adic characters. Moreover, $(r_{\pi,\lambda}, r_{\chi,\lambda})$ is regular algebraic, totally odd and polarized.

Remark. By enlarging M_π if necessary, we will further assume that M_π is a full field of definition of $\mathfrak{R}_\pi$ and

$\mathfrak{A}_\chi$, that $M_\pi/\mathbb{Q}$ is Galois, and it contains the images of all embeddings $F \hookrightarrow \overline{M_\pi}$.

4 Reduction to the maximal CM subextension

Let us first define the compatible system of l -adic representations that we will be working with throughout this manuscript. Let λ be a finite prime of M_π . Define

$$\rho_\lambda = \text{Ind}_{G_F}^{G_{F^+}} r_{\pi,\lambda} \oplus r_{\chi,\lambda},$$

which gives us a system of l -adic representations of dimension $2n + 1$

$$\text{Ind}_{G_F}^{G_{F^+}} \mathfrak{R}_\pi \oplus \mathfrak{R}_\chi.$$

Let G_λ be the Zariski closure of the image of ρ_λ in $\text{GL}_{2n+1}/\overline{M_{\pi,\lambda}}$, G_λ^0 be the identity component of G_λ and $G_\lambda^{0,\text{der}}$ be the derived subgroup. Then by Lemma 3.13, there exists a finite Galois extension $F_{1,\pi}/F^+$ such that for all λ the map $G_{F^+} \rightarrow G_\lambda(\overline{M_{\pi,\lambda}})$ induces an isomorphism

$$G_\lambda/G_\lambda^0 \cong \text{Gal}(F_{1,\pi}/F^+).$$

The goal of the manuscript is to prove the following theorem:

Theorem 4.1. *Suppose F is a CM field and that π is a polarizable, regular algebraic, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ where $n \leq 6$. Then there is a set $\mathfrak{L}$ of rational primes of Dirichlet density 1, such that for all primes λ of M_π dividing some $l \in \mathfrak{L}$, $r_{\pi,\lambda}$ is irreducible.*

In this section we will prove the following:

Proposition 4.2. *Let n_0 be an integer. Suppose for all CM fields F , all integers $n \leq n_0$, and all regular algebraic, cuspidal automorphic representation π of $\text{GL}_n(\mathbb{A}_F)$ such that F is the maximal CM subextension of $F_{1,\pi}/F^+$, we have $r_{\pi,\lambda}$ is irreducible for all primes λ of M_π lying above a Dirichlet density 1 set of rational primes. Then for all CM fields F , all integers $n \leq n_0$, and all regular algebraic, cuspidal automorphic representation π of $\text{GL}_n(\mathbb{A}_F)$, $r_{\pi,\lambda}$ is irreducible for all primes of M_π lying above a Dirichlet density 1 set of rational primes.*

This will allow us to reduce the proof of Theorem 4.1 to the case where F is the maximal CM subextension of $F_{1,\pi}/F^+$. In order to prove the proposition, we need the following result of Patrikis and Taylor:

Theorem 4.3 (Theorem 1.7, Lemma 1.5 and Lemma 1.6 in [13]). *Let F be a CM field and π be a polarizable, regular algebraic, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. Then there is a set $\mathfrak{L}$ of rational primes*

of positive Dirichlet density, such that if a prime λ of M_π divides some $l \in \mathfrak{L}$, then $r_{\pi,\lambda}$ is irreducible. In particular, up to finitely many primes, $\mathfrak{L}$ contains the subset

$$\mathfrak{L}' = \{l \text{ a rational prime} \mid [\text{Frob}_l] = \{c\} \subseteq \text{Gal}(M_\pi/\mathbb{Q}), l \text{ unramified in } M_\pi\}.$$

We will refer to the set of primes of M_π lying above $\mathfrak{L}$ the *Patrikis-Taylor primes*.

We also need the following potential automorphy result from [13]:

Theorem 4.4 (Theorem 2.1 in [13]). *Suppose K/K_0 is a finite Galois extension of CM fields, and that K^{avoid}/K is a finite Galois extension. Suppose also for $i = 1, \dots, r$ that $(\mathfrak{R}_i, \mathfrak{M}_i)$ is a totally odd, polarized weakly compatible system of l -adic representations of G_K with each $\mathfrak{R}_i$ pure and regular. Then there is a finite CM extension K'/K such that K'/K_0 is Galois and K' is linearly disjoint from K^{avoid} over K , with the following property: for each i we have a decomposition*

$$\mathfrak{R}_i = \mathfrak{R}_{i,1} \oplus \dots \oplus \mathfrak{R}_{i,s_i}$$

into weakly compatible systems $\mathfrak{R}_{i,j}$ with each $(\mathfrak{R}_{i,j}, \mathfrak{M}_i)|_{G_{K'}}$ automorphic.

We will now prove Proposition 4.2.

Proof. Let π be a regular algebraic, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. Let F_2/F^+ be the maximal CM subextension of $F_{1,\pi}/F^+$. We can apply Theorem 4.4 to our situation with $K = F_2, K_0 = F^+, K^{\text{avoid}} = F_{1,\pi}$ and $i = 1$ with $(\mathfrak{R}_1, \mathfrak{M}_1) = (\mathfrak{R}_\pi, \mathfrak{R}_\chi)|_{G_{F_2}}$. Then we obtain a finite CM extension F_3/F_2 such that F_3/F^+ is Galois and F_3 is linearly disjoint from $F_{1,\pi}$ over F_2 , and moreover $\mathfrak{R}_\pi|_{G_{F_2}}$ has a decomposition into weakly compatible systems

$$\mathfrak{R}_\pi|_{G_{F_2}} = \mathfrak{R}_1 \oplus \dots \oplus \mathfrak{R}_s, \tag{1}$$

with each $(\mathfrak{R}_i, \mathfrak{R}_\chi)|_{G_{F_3}}$ automorphic.

Let ω be a Patrikis-Taylor prime from Theorem 4.3. In particular, $r_{\pi,\omega}$ is irreducible. Since $r_{\pi,\omega}$ is regular, the irreducible constituents of $r_{\pi,\omega}|_{G_{F_2}}$ are distinct. So we get that $\text{Gal}(F_2/F)$ acts transitively on the irreducible constituents of $r_{\pi,\omega}|_{G_{F_2}}$. From Eq. 1, we have

$$r_{\pi,\omega}|_{G_{F_2}} = r_{1,\omega} \oplus \dots \oplus r_{s,\omega}.$$

Since $r_{i,\omega}$ and $r_{i',\omega}$ have no irreducible constituents in common, $\text{Gal}(F_2/F)$ acts transitively on $\{r_{i,\omega}\}_{i=1}^s$

as well. In particular, all $\{r_{i,\omega}\}_{i=1}^s$ have the same dimension. Let H be the stabilizer of $r_{1,\omega}$ in G_F , which contains G_{F_2} . Then $H = G_{F_4}$ for some extension $F_2/F_4/F$ and $r_{1,\omega}$ extends to a representation of H . We know $\text{Hom}_H(r_{\pi,\omega}|_H, r_{1,\omega}) = \text{Hom}_{G_F}(r_{\pi,\omega}, \text{Ind}_H^{G_F} r_{1,\omega})$ is non-trivial. A non-zero G_F -equivariant homomorphism $\phi : r_{\pi,\omega} \rightarrow \text{Ind}_H^{G_F} r_{1,\omega}$ must be injective because $r_{\pi,\omega}$ is irreducible. We also have $\dim(\text{Ind}_H^{G_F} r_{1,\omega}) = [G_F : H] \cdot \dim(r_{1,\omega}) = s \cdot \dim(r_{1,\omega}) = \dim(r_{\pi,\omega})$ by the orbit-stabilizer theorem. Thus we get ϕ is in fact an isomorphism, which implies $r_{\pi,\omega} \cong \text{Ind}_H^{G_F} r_{1,\omega}$.

So far we only have $r_{\pi,\omega} \cong \text{Ind}_H^{G_F} r_{1,\omega}$ at Patrikis-Taylor primes and we want to extend this to all finite primes of M_π . First, we need to show that $r_{1,\lambda}$ extends to a representation of H at all finite primes λ of M_π . Let V_π be the finite set of bad finite primes of F given in the definition of the weakly compatible system $\mathfrak{R}_\pi$ as in Definition 3.9. Then for $\sigma \in H$, for a prime v of F , $v \notin V_\pi$ and $v \nmid \text{res}(\lambda)$, we have

$$r_{1,\lambda}^\sigma(\text{Frob}_v) \sim r_{1,\omega}^\sigma(\text{Frob}_v) \sim r_{1,\omega}(\text{Frob}_v),$$

where $\sim$ denotes being in the same conjugacy class. Thus we get $r_{1,\lambda}^\sigma \cong r_{1,\lambda}$, which implies $r_{1,\lambda}^\sigma = r_{1,\lambda}$ by regularity of $r_{\pi,\lambda}$. Since $\mathfrak{R}_i|_{G_{F_3}}$ is automorphic, there exists a regular automorphic representation π_i of $\text{GL}_{n_i}(\mathbb{A}_{F_3})$ such that $(\mathfrak{R}_{\pi_i}, \mathfrak{R}_\chi|_{G_{F_3}}) \equiv (\mathfrak{R}_i, \mathfrak{R}_\chi)|_{G_{F_3}}$. Note that $\text{Gal}(F_3/F)$ acts on $\{\pi_i\}_{i=1}^s$, where the action is given by $r_{\gamma(\pi_i),\lambda} = r_{\pi_i,\lambda}^\gamma$ for $\gamma \in \text{Gal}(F_3/F)$ and λ a finite prime in M_π (because M_π is a full field of definition of $\mathfrak{R}_\pi$). We already saw that the action of $\text{Gal}(F_2/F)$ is transitive on $\{r_{i,\omega}\}_{i=1}^s$ at a Patrikis-Taylor prime ω , so the action of G_F on $\{\pi_i\}_{i=1}^s$ is transitive, and thus G_F acts transitively on $\{r_{i,\lambda}\}_{i=1}^s$ for any finite prime λ . Therefore, we get

$$r_{\pi,\lambda} \cong \text{Ind}_H^{G_F} r_{1,\lambda}$$

at all finite primes λ in M_π .

For each i , we have $F_{1,\pi_i} \subseteq F_{1,\pi} F_3$. Thus we get F_3 is the maximal CM subextension of $F_{1,\pi_i}/F^+$ based on the way we chose F_3 . Let λ be any finite prime of M_π . By the assumption in Proposition 4.2, after removing a set of rational primes of Dirichlet density 0, we may assume $r_{1,\lambda}$ is irreducible. For such λ , we have $r_{\pi,\lambda} \cong \text{Ind}_H^{G_F} r_{1,\lambda}$ is irreducible because $r_{\pi,\lambda}$ is regular. $\square$

5 An L -function argument

From now on, we will assume F is the maximal CM subextension of $F_{1,\pi}/F^+$.

In this section we give an overview of the strategy of proving Theorem 4.1, and review the L -function argument in [1] used in the proof of the theorems.

Our approach to proving Theorem 4.1 is to apply the potential automorphy result Theorem 4.5.1 of [1] to the irreducible constituents of $r_{\pi,\lambda}$. Let c be a complex conjugation in G_{F^+} . The arguments in the proof of Theorem 5.5.2 and Proposition 5.4.6 of [1] can be applied as long as we can show:

For all but finitely many primes λ of M_π , and for each irreducible constituent $r \subseteq r_{\pi,\lambda}$, we have $r^c \cong r^\vee \otimes r_{\chi,\lambda}|_{G_F}$.

We now state the analogues of Proposition 5.4.6 and Theorem 5.5.2 in [1] for our situation.

Proposition 5.1 (analogue of Proposition 5.4.6 of [1]). *Suppose F is a CM field and that $(\mathfrak{R}, \mathfrak{M})$ is a totally odd, polarized weakly compatible system of l -adic representations, with $\mathfrak{R}$ regular and pure. Write $r_\lambda = r_{\lambda,1} \oplus \cdots \oplus r_{\lambda,j_\lambda}$ with each $r_{\lambda,\alpha}$ irreducible. Suppose that for all but finitely many λ , the pair $(r_{\lambda,\alpha}, \mu_\lambda)$ is totally odd, polarized for all $\alpha \in \{1, \dots, j_\lambda\}$. Then there is a set of rational primes $\mathfrak{L}$ of Dirichlet density 1 such that if λ is a prime of M_π lying above $l \in \mathfrak{L}$, then there is a finite, CM, Galois extension F'/F such that each $(r_{\lambda,\alpha}|_{G_{F'}}, \mu_\lambda|_{G_{(F')^+}})$ is irreducible and automorphic.*

Proof. The proof is exactly the same as that of Proposition 5.4.6 in [1]. □

Theorem 5.2 (analogue of Theorem 5.5.2 of [1]). *Suppose that F is a CM field and that (π, χ) is a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Suppose also that for all but finitely many primes λ of M_π , and for each irreducible constituent r of $r_{\pi,\lambda}$, we have $r^c \cong r^\vee \otimes r_{\chi,\lambda}|_{G_F}$. Then there is a set of rational primes $\mathfrak{L}$ of Dirichlet density 1 such that if a prime λ of M_π divides some $l \in \mathfrak{L}$, then $r_{\pi,\lambda}$ is irreducible.*

Under the assumptions of Theorem 5.2, let v be an infinite place of F , $l = \mathrm{res}(\lambda)$ and $\langle \cdot, \cdot \rangle_v$ a pairing on $\overline{\mathbb{Q}_l}^n$ as given in Definition 3.4. By regularity, there is only one irreducible constituent of $r_{\pi,\lambda} \cong (r_{\pi,\lambda}^\vee)^c \otimes r_{\chi,\lambda}|_{G_F}$ isomorphic to r . Therefore, $\langle \cdot, \cdot \rangle_v$ restricts to a perfect pairing on the underlying space of r . Then oddity follows because $(r_{\pi,\lambda}, r_{\chi,\lambda})$ is totally odd. So we can conclude there is a set of rational primes $\mathfrak{L}$ of Dirichlet density 1 such that for all primes λ of M_π dividing some $l \in \mathfrak{L}$, the irreducible constituents $(r, r_{\chi,\lambda})$ of $(r_{\pi,\lambda}, r_{\chi,\lambda})$ are totally odd and polarized.

The rest of the proof of Theorem 5.2 is the same as the proof of Theorem 5.5.2 in [1], except we resort to Proposition 5.1 above instead of Proposition 5.4.6 in [1]. We outline the proof of Theorem 5.2, which uses a standard L -function argument. We refer the reader to Section 5.5 of [1] for the details.

We first recall some group theory. Let F be a number field. Fix l a rational prime. Let $\mathrm{GG}_{F,l}$ denote the category of semi-simple, continuous representations of G_F on finite dimensional $\overline{\mathbb{Q}_l}$ -vector spaces which ramify at only finitely many primes. Let $\mathrm{Rep}_{F,l}$ denote the Grothendieck group of $\mathrm{GG}_{F,l}$, which is a commutative ring with 1. For an object V of $\mathrm{GG}_{F,l}$, we denote by $[V]$ its class in $\mathrm{Rep}_{F,l}$. Elements in $\mathrm{GG}_{F,l}$ satisfy a list of functorialities, and we highlight some of the important ones below. The reader can see 5.5 of [1] for the rest.

- (5.5 (4) in [1]) There exists a perfect symmetric $\mathbb{Z}$ -valued pairing $(\ , \)_{F,l}$ on $\mathrm{Rep}_{F,l}$ defined by $([U], [V])_{F,l} = \dim_{\overline{\mathbb{Q}_l}} \mathrm{Hom}_{G_F}(U, V)$.
- (5.5 (7) and (8) in [1]) For F'/F a finite extension, we have a ring homomorphism $\mathrm{res}_{F'/F} : \mathrm{Rep}_{F',l} \rightarrow \mathrm{Rep}_{F,l}$ defined by $\mathrm{res}_{F'/F}[V] = [V|_{G_{F'}}]$ and a $\mathbb{Z}$ -linear map $\mathrm{ind}_{F'/F} : \mathrm{Rep}_{F',l} \rightarrow \mathrm{Rep}_{F,l}$ defined by $\mathrm{ind}_{F'/F}[V] = [\mathrm{Ind}_{G_{F'}}^{G_F} V]$.
- (5.5 (9) in [1]) Let S be a finite set of primes of F including the ones above l . Suppose $A = \sum_i n_i [V_i] \in \mathrm{Rep}_{F,l}$ with each V_i unramified outside S . Then for each $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, we can define

$$L^S(\iota A, s) = \prod_i L^S(\iota V_i, s)^{n_i},$$

at least as a formal Euler product. If for each i the Weil-Deligne representation $\mathrm{WD}(V_i|_{G_{F,v}})$ is pure of some weight w_i for all but finitely many primes $v \notin S$ of F , then the product converges in some right half plane. We also have

$$L^S(\iota(A+B), s) = L^S(\iota A, s) L^S(\iota B, s)$$

and

$$L^S(\iota \mathrm{ind}_{F'/F} A, s) = L^{S'}(\iota A, s),$$

with S' being the set of primes of F' above S .

- (5.5 (8) in [1])

Proposition 5.3 (Brauer's Theorem). *If F'/F is a finite Galois extension then there is a finite collection of intermediate fields $F'/F'_i/F$ with each F'/F'_i soluble, together with characters*

$$\psi_i : \mathrm{Gal}(F'/F'_i) \rightarrow \mathbb{C}^\times$$

and integers n_i such that $1 = \sum_i n_i \mathrm{ind}_{F'_i/F} [\psi_i]$ in the Grothendieck group of finite dimensional representation of $\mathrm{Gal}(F'/F)$ over $\mathbb{C}$.

Remark. In particular, one can show this implies for $A \in \text{Rep}_{F,l}$ and $\iota : \overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$,

$$A = \sum_i n_i \text{ind}_{F'_i/F}([\iota^{-1}\psi_i] \text{res}_{F'_i/F} A).$$

- (Section 5.1 in [1]) For $\mathfrak{R} = (M, V, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ a weakly compatible system of l -adic representations of G_F , that is pure of weight w , and $\iota : M_\pi \hookrightarrow \mathbb{C}$ a continuous embedding, we can define the partial L -function

$$L^V(\iota\mathfrak{R}, s) = \prod_{v \notin V} \frac{\#k(v)^{(\dim \mathfrak{R})s}}{\iota Q_v(\#k(v)^s)},$$

which converges in some right half plane.

Lemma 5.4 (Lemma 2.2.2 of [1]). *Suppose that M/K is a soluble Galois extension of fields and that M is CM. Suppose that (r, μ) is a polarized l -adic representation of G_K with $r|_{G_M}$ irreducible. Then (r, μ) is automorphic if and only if $(r|_{G_M}, \mu|_{G_{M^+}})$ is automorphic.*

Now we let F be a CM field. Let (π, χ) and (π', χ') be regular algebraic, polarized, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ and $\text{GL}_{n'}(\mathbb{A}_F)$ respectively. Then $\mathfrak{R}_\pi$ is pure of some weight w and $\mathfrak{R}_{\pi'}$ is pure of some weight w' . We have (see [18] and [10]):

Theorem 5.5. *Let S be a finite set of finite places of F . The partial L -function*

$$L^S(\pi \times (\pi')^\vee, s + (w - w' + n' - n)/2) = L^S((\pi || \det||_F^{(w+1-n)/2}) \times (\pi' || \det||_F^{(w'+1-n')/2})^\vee, s)$$

is meromorphic, and is holomorphic and non-zero at $s = 1$ unless

$$\pi \cong \pi' || \det||_F^{(w'-w+n-n')/2},$$

in which case it has a simple pole at $s = 1$.

Now we sketch the proof of Theorem 5.2:

Sketch of Proof. Let $\mathfrak{L}$ be the set of rational primes of Dirichlet density 1 provided by applying Proposition 5.1 to $(\mathfrak{R}_\pi, \mathfrak{R}_\chi)$. Let $\lambda|l$ for some $l \in \mathfrak{L}$ and fix some $\iota : \overline{M_{\pi,\lambda}} \xrightarrow{\sim} \mathbb{C}$. Decompose $r_{\pi,\lambda}$ into irreducible constituents

$$r_{\pi,\lambda} = r_1 \oplus \cdots \oplus r_j.$$

Let F'/F be as given in Proposition 5.1 for $\mathfrak{R}_\pi$ and λ . Let S be the finite set of primes of F which divide l or above which either π or F' ramifies. By Theorem 5.5,

$$\text{ord}_{s=1} L^S(\iota(\mathfrak{R}_\pi \otimes \mathfrak{R}_\pi^\vee), s) = \text{ord}_{s=1} L^S(\pi \times \pi^\vee, s) = -1.$$

We will also show

$$\text{ord}_{s=1} L^S(\iota(\mathfrak{R}_\pi \otimes \mathfrak{R}_\pi^\vee), s) = -j.$$

Then the result follows.

Applying Proposition 5.3 to F'/F , we get a collection of items $F'/F'_i/F, n_i$ and ψ_i . Then one can show

$$L^S(\iota(\mathfrak{R}_\pi \otimes \mathfrak{R}_\pi^\vee), s) = \prod_i L^S(\iota(r_{\pi, \lambda}|_{G_{F'_i}} \otimes r_{\pi, \lambda}|_{G_{F'_i}}^\vee) \otimes \psi_i, s)^{n_i}. \quad (2)$$

Fix i . Since F'/F'_i is soluble, Lemma 5.4 implies that for each $\alpha \in \{1, \dots, j\}$, there exists a regular algebraic, polarizable, cuspidal automorphic representation π_α of $\text{GL}_{n_\alpha}(\mathbb{A}_{F'_i})$ such that

$$r_{\pi_\alpha, \lambda} \cong r_\alpha|_{G_{F'_i}},$$

and $r_\alpha|_{G_{F'_i}}$ is irreducible. Note that since M_π is a full field of definition of $\mathfrak{R}_\pi$, $r_{\pi_\alpha, \lambda}$ is well-defined. Then one can show

$$\begin{aligned} & \text{ord}_{s=1} L^S(\iota(r_\alpha|_{G_{F'_i}} \otimes r_\beta|_{G_{F'_i}}^\vee) \otimes \psi_i, s) \\ &= \text{ord}_{s=1} L^S(\pi_\alpha \times \pi_\beta^\vee \times (\psi_i \circ \text{Art}_{F'_i}), s + (n_\beta - n_\alpha)/2) \\ &= -\delta_{\alpha, \beta} \delta_{\psi_i, \mathbb{I}}, \end{aligned} \quad (3)$$

where δ is the Kronecker-delta function and $\text{Art}_{F'_i}$ denotes the Artin character of F'_i .

Using regularity of $r_{\pi, \lambda}$, we get

$$(\text{res}_{F'_i/F}[r_\alpha][\psi_i], \text{res}_{F'_i/F}[r_\beta])_{F'_i, l} = \delta_{\alpha, \beta} \delta_{\psi_i, \mathbb{I}}. \quad (4)$$

Putting together Eq. 2, Eq. 3 and Eq. 4, we get

$$\begin{aligned}
L^S(\iota(\mathfrak{R}_\pi \otimes \mathfrak{R}_\pi^\vee), s) &= \prod_i L^S(\iota(r_{\pi, \lambda}|_{G_{F'_i}} \otimes r_{\pi, \lambda}|_{G_{F_i}}^\vee) \otimes \psi_i, s)^{n_i} \\
&= - \sum_i n_i (\text{res}_{F'_i/F}[r_{\pi, \lambda}][\psi_i], \text{res}_{F'_i/F}[r_{\pi, \lambda}])_{F'_i, l}
\end{aligned} \tag{5}$$

$$= - \sum_i n_i (\text{ind}_{F'_i/F}([\psi_i] \text{res}_{F'_i/F}[r_{\pi, \lambda}]), [r_{\pi, \lambda}])_{F, l} \tag{6}$$

$$= -([r_{\pi, \lambda}], [r_{\pi, \lambda}])_{F, l} \tag{7}$$

$$= -j,$$

where going from Eq. 5 to Eq. 6 uses Frobenius reciprocity and going from Eq. 6 to Eq. 7 follows from the remark after Proposition 5.3. Now the theorem follows. $\square$

6 Reduction to the semi-simple case

In this section, we will prove that the following theorem:

Theorem 6.1. *Suppose that F is a CM field and that (π, χ) is a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Moreover, suppose that $F_{1,\pi}/F^+$ has F as the maximal CM subextension, and that for all but finitely many primes λ of M_π , and for each irreducible constituent r of $r_{\pi,\lambda}|_{G_\lambda^{0,\mathrm{der}}}$, $r^\sigma \cong r^\vee$ for every $\sigma \in G_{F^+} \setminus G_F$. Then there is a set of rational primes $\mathfrak{L}$ of Dirichlet density 1 such that if a prime λ of M_π divides some $l \in \mathfrak{L}$, then $r_{\pi,\lambda}$ is irreducible.*

6.1 Reduction to being polarized with a character of $G_{F_{1,\pi}}$

Fix λ to be a finite prime of M_π . Let $c \in G_{F^+}$ be a complex conjugation. Now we prove that to show the irreducibility of $r_{\pi,\lambda}$, it suffices to show that each irreducible constituent of $r_{\pi,\lambda}|_{G_{F_{1,\pi}}}$ is polarized with respect to a character of $G_{F_{1,\pi}}$. Let

$$r_{\pi,\lambda}|_{G_{F_{1,\pi}}} \cong r_1 \oplus \dots \oplus r_s$$

be the decomposition of $r_{\pi,\lambda}|_{G_{F_{1,\pi}}}$ into irreducible representations of $G_{F_{1,\pi}}$.

Proposition 6.2. *Let $\sigma \in G_{F^+} \setminus G_F$. Suppose for all i , we have*

$$r_i^\sigma \cong r_i^\vee \otimes \delta_{\sigma,i},$$

for some continuous character

$$\delta_{\sigma,i} : G_{F_{1,\pi}} \rightarrow \overline{M_{\pi,\lambda}}.$$

Then

$$r_i^\sigma \cong r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}. \tag{8}$$

Moreover, if Eq. 8 is true for all $\sigma \in G_{F^+} \setminus G_F$ and all i , then we have

$$r^c \cong r^\vee \otimes r_{\chi,\lambda}|_{G_F}$$

for any irreducible constituent r of $r_{\pi,\lambda}$.

Remark. Note that since $F_{1,\pi}/F^+$ is Galois, r_i^σ is an irreducible representation of $G_{F_{1,\pi}}$ for all $\sigma \in G_{F^+} \setminus G_F$.

Proof. Fix a continuous embedding $\tau : F_{1,\pi} \hookrightarrow \overline{M_\pi}$ and $\sigma \in G_{F^+} \setminus G_F$. Since F is the maximal CM subextension of $F_{1,\pi}/F^+$, we can write

$$\delta_{\sigma,i} = \delta|_{G_{F_{1,\pi}}} \mu,$$

where $\delta : G_F \rightarrow \overline{M_{\pi,\lambda}}$ is a continuous Hodge-Tate character and $\mu : G_{F_{1,\pi}} \rightarrow \overline{M_{\pi,\lambda}}$ is a character of finite order (cf. Lemma 2.3.4 in [12]). So we get

$$\begin{aligned} \text{HT}_\tau(\delta_{\sigma,i}^\sigma) &= \text{HT}_{\tau\sigma|_{F_{1,\pi}}}^{-1}(\delta_{\sigma,i}) \\ &= \text{HT}_{\tau\sigma|_{F_{1,\pi}}}^{-1}(\delta|_{G_{F_{1,\pi}}} \mu) \\ &= \text{HT}_{\tau\sigma|_{F_{1,\pi}}}^{-1}(\delta|_{G_{F_{1,\pi}}}) \\ &= \text{HT}_{\tau|_F \sigma|_F^{-1}}(\delta) \\ &= \text{HT}_{\tau|_F c}(\delta) \\ &= \text{HT}_{cc\tau|_F c}(\delta) \\ &= \text{HT}_{c\tau|_F}(\delta) \\ &= 2w - \text{HT}_{\tau|_F}(\delta), \end{aligned} \tag{9}$$

where the last equality follows from Lemma 3.12(3), Lemma 3.15 and the fact that $\mathfrak{R}_\pi$ is pure of weight w .

For $\sigma \in G_{F^+} \setminus G_F$, consider $r_{\pi,\lambda}^{\sigma^2}$ and $r_i^{\sigma^2}$. We have $\sigma = c\sigma'$ for some $\sigma' \in G_F$. Since $\sigma^2 \in G_F$, we have $\delta^{\sigma^2} \cong \delta$ and $r_{\pi,\lambda}^{\sigma^2} \cong r_{\pi,\lambda}$. So we get

$$r_i^{\sigma^2} \subseteq (r_{\pi,\lambda}|_{G_{F_{1,\pi}}})^{\sigma^2} \cong r_{\pi,\lambda}|_{G_{F_{1,\pi}}} \cong r_{\pi,\lambda}|_{G_{F_{1,\pi}}},$$

where the first isomorphism follows from the fact that $F_{1,\pi}/F$ is Galois. Similarly, for any positive integer m , we have $r_i^{\sigma^{2m}} \subseteq r_{\pi,\lambda}^{\sigma^{2m}}|_{G_{F_{1,\pi}}} \cong r_{\pi,\lambda}|_{G_{F_{1,\pi}}}$.

By assumption, we have that

$$\begin{aligned} r_i^{\sigma^2} &\cong (r_i^\vee \otimes \delta_{\sigma,i})^\vee \otimes \delta_{\sigma,i}^\sigma \\ &\cong r_i \otimes \delta_{\sigma,i}^{-1} \otimes \delta_{\sigma,i}^\sigma. \end{aligned} \tag{10}$$

By repeatedly applying the isomorphism in Eq. 10, we get that

$$\begin{aligned} r_i^{\sigma^{2m}} &\cong r_i \otimes \delta_{\sigma,i}^\sigma / \delta_{\sigma,i} \otimes \delta_{\sigma,i}^{\sigma^3} / \delta_{\sigma,i}^{\sigma^2} \otimes \cdots \otimes \delta_{\sigma,i}^{\sigma^{2m-1}} / \delta_{\sigma,i}^{\sigma^{2m-2}} \\ &\cong r_i \otimes ((\delta^\sigma / \delta)|_{G_{F_{1,\pi}}})^m \otimes \mu^\sigma / \mu \otimes \mu^{\sigma^3} / \mu^{\sigma^2} \cdots \otimes \mu^{\sigma^{2m-1}} / \mu^{\sigma^{2m-2}}. \end{aligned}$$

Since $r_i^{\sigma^{2m}}$ is an irreducible constituent of $r_{\pi,\lambda}$ for every positive integer m and $r_{\pi,\lambda}$ is a finite dimensional representation, we see that $(\delta^\sigma/\delta)|_{G_{F_{1,\pi}}}$ has Hodge-Tate weight equal to 0, and hence $\psi := \delta_{\sigma,i}^\sigma/\delta_{\sigma,i}$ is a finite order character of $G_{F_{1,\pi}}$. So we have that

$$\begin{aligned} \text{HT}_\tau(\delta_{\sigma,i}^\sigma) &= \text{HT}_\tau(\delta_{\sigma,i}\psi) \\ &= \text{HT}_\tau(\delta_{\sigma,i}) \\ &= \text{HT}_{\tau|_F}(\delta). \end{aligned} \tag{11}$$

Combining Eq. 9 with Eq. 11, we get $\text{HT}_\tau(\delta_{\sigma,i}) = \text{HT}_{\tau|_F}(\delta) = w$. We therefore have

$$\delta_{\sigma,i} = r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \otimes \eta_i$$

for some finite order character η_i of $G_{F_{1,\pi}}$.

Consider $r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \subseteq (r_{\pi,\lambda}^\vee \otimes r_{\chi,\lambda}|_{G_F})|_{G_{F_{1,\pi}}} \cong r_{\pi,\lambda}^c|_{G_{F_{1,\pi}}}$. Since $F_{1,\pi}/F^+$ is Galois, we have $r_i^\sigma \subseteq r_{\pi,\lambda}|_{G_{F_{1,\pi}}}^\sigma \cong r_{\pi,\lambda}^\sigma|_{G_{F_{1,\pi}}} \cong r_{\pi,\lambda}^c|_{G_{F_{1,\pi}}}$. Since we also know

$$r_i^\sigma \cong r_i^\vee \otimes \delta_{\sigma,i} \cong r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \otimes \eta_i,$$

we must have $r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \otimes \eta_i \cong r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}$ by regularity, which implies $r_i^\sigma \cong r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}$. This concludes the first half of the proof of the proposition.

Now suppose that for each i and each $\sigma \in G_{F^+} \setminus G_F$, we have $r_i^\sigma \cong r_i^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}$. We will show this implies $r^c \cong r^\vee \otimes r_{\chi,\lambda}|_{G_F}$ for any irreducible constituent r of $r_{\pi,\lambda}$. Let $\sigma \in [c] \subseteq \text{Gal}(F_{1,\pi}/F^+)$. Let $r|_{G_{F_{1,\pi}}} = u_1 \oplus \cdots \oplus u_k$ be the decomposition into irreducible constituents. Since $F_{1,\pi}/F^+$ is Galois, we get

$$\begin{aligned} r^c|_{G_{F_{1,\pi}}} &\cong r|_{G_{F_{1,\pi}}}^\sigma \\ &\cong u_1^\sigma \oplus \cdots \oplus u_k^\sigma \\ &\cong u_1^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \oplus \cdots \oplus u_k^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \\ &\cong r^\vee|_{G_{F_{1,\pi}}} \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}. \end{aligned} \tag{12}$$

So we have $r^{c,\vee}|_{G_{F_{1,\pi}}} \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}} \cong r|_{G_{F_{1,\pi}}}$. By regularity of $r_{\pi,\lambda}$, we must have $r^{c,\vee} \otimes r_{\chi,\lambda}|_{G_F} \cong r$. $\square$

6.2 Reduction to semi-simple image

In the rest of this section we will finish the proof of Theorem 6.1.

For a finite prime λ in M_π , We have $(r_{\pi,\lambda}|_{G_\lambda^{0,\text{der}}})^\sigma \cong (r_{\pi,\lambda}|_{G_\lambda^{0,\text{der}}})^\vee$ for $\sigma \in G_{F^+} \setminus G_F$. By Schur's lemma, the elements in the center of G_λ^0 act as scalars, which implies for each irreducible constituent $u \subseteq r_{\pi,\lambda}|_{G_{F_1,\pi}}$, $u|_{G_\lambda^{0,\text{der}}}$ remains irreducible.

We have the following lemma:

Lemma 6.3. *If an irreducible constituent $u \subseteq r_{\pi,\lambda}|_{G_{F_1,\pi}}$ satisfies $(u|_{G_\lambda^{0,\text{der}}})^\sigma \cong (u|_{G_\lambda^{0,\text{der}}})^\vee$ for some $\sigma \in G_{F^+} \setminus G_F$, then there exists a continuous character $\delta_\sigma : G_{F_1,\pi} \rightarrow \overline{M_{\pi,\lambda}}$ such that $u^\sigma \cong u^\vee \otimes \delta_\sigma$.*

Proof. Since $G_\lambda^{0,\text{der}}$ is a normal subgroup of G_λ^0 , $\text{Hom}_{G_\lambda^{0,\text{der}}}((u|_{G_\lambda^{0,\text{der}}})^\sigma, (u|_{G_\lambda^{0,\text{der}}})^\vee)$ is a non-trivial $G_\lambda^0/G_\lambda^{0,\text{der}}$ -module. But since $(u|_{G_\lambda^{0,\text{der}}})^\sigma$ and $(u|_{G_\lambda^{0,\text{der}}})^\vee$ are irreducible $G_\lambda^{0,\text{der}}$ -modules, we get $\text{Hom}_{G_\lambda^{0,\text{der}}}((u|_{G_\lambda^{0,\text{der}}})^\sigma, (u|_{G_\lambda^{0,\text{der}}})^\vee)$ is 1-dimensional. Thus $G_\lambda^0/G_\lambda^{0,\text{der}}$ acts as scalars. So there exists a continuous character $\delta_\sigma : G_{F_1,\pi} \rightarrow \overline{M_{\pi,\lambda}}$ such that $u^\sigma \cong u^\vee \otimes \delta_\sigma$. $\square$

Finally, we are ready to prove Theorem 6.1.

Proof. Let $r \subseteq r_{\pi,\lambda}$ be an irreducible constituent. Consider the decomposition $r|_{G_{F_1,\pi}} = \bigoplus_{i=1}^k u_i$ of $r|_{G_{F_1,\pi}}$ into irreducible representations of $G_{F_1,\pi}$. Since $u_i|_{G_\lambda^{0,\text{der}}}$ is irreducible for each i , the irreducible constituents of $r|_{G_\lambda^{0,\text{der}}}$ are simply $\{u_i|_{G_\lambda^{0,\text{der}}}\}_{i=1}^k$, each of which satisfies $(u_i|_{G_\lambda^{0,\text{der}}})^\sigma \cong (u_i|_{G_\lambda^{0,\text{der}}})^\vee$ for all $\sigma \in G_{F^+} \setminus G_F$ by assumption. By Lemma 6.3, we get there exists a continuous character $\delta_{\sigma,i} : G_{F_1,\pi} \rightarrow \overline{M_{\pi,\lambda}}$ for each σ and each i such that $u_i^\sigma \cong u_i^\vee \otimes \delta_{\sigma,i}$. Since r is an arbitrary irreducible constituent of $r_{\pi,\lambda}$, by Proposition 6.2, we get $r^c \cong r^\vee \otimes r_{\chi,\lambda}|_{G_F}$ for each irreducible constituent r of $r_{\pi,\lambda}$. $\square$

7 Irreducibility at a set of positive Dirichlet density

Keeping the assumptions in Theorem 6.1, in this section we prove there is a set of rational primes of positive Dirichlet density such that for primes λ of M_π lying above the set, $r_{\pi,\lambda}|_{G_{F_{1,\pi}}}$ is irreducible. Recall that we have the decomposition into irreducible constituents

$$r_{\pi,\lambda}|_{G_{F_{1,\pi}}} = r_1 \oplus \cdots \oplus r_s.$$

Let N/M_π be a finite extension containing all images of the embeddings $F_{1,\pi} \hookrightarrow \overline{M_\pi}$. Enlarging N if necessary, we also assume $N/\mathbb{Q}$ is Galois. Let $c \in \text{Gal}(N/\mathbb{Q})$ be a complex conjugation. Let $\mathfrak{L}$ be the set of rational primes defined in Theorem 4.3. We define a set of finite rational primes $\mathfrak{L}''$, which is a set of positive Dirichlet density, to be the following

$$\mathfrak{L}'' = \{l \in \mathfrak{L} \mid [\text{Frob}_l] = [c] \subseteq \text{Gal}(N/\mathbb{Q}), l \text{ unramified in } N\}.$$

Then by Theorem 4.3, removing finitely many rational primes if necessary, we may assume if λ of M_π lies above some prime in $\mathfrak{L}''$, then $r_{\pi,\lambda}$ is irreducible. Let $\mathfrak{f}$ be the set of primes of M_π lying above the rational primes in $\mathfrak{L}''$.

We will prove the following result:

Proposition 7.1. *If $\lambda \in \mathfrak{f}$, then $r^\sigma \cong r^\vee \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}$ for all $\sigma \in [c] \subseteq \text{Gal}(F_{1,\pi}/F^+)$ and all irreducible constituents $r \subseteq r_{\pi,\lambda}|_{G_{F_{1,\pi}}}$.*

Proof. Let $l = \text{res}(\lambda)$, and let $\sigma \in [c] \subseteq \text{Gal}(N/F^+)$. There exists a unique permutation $\tilde{\sigma}$ of $\{1, \dots, s\}$ such that

$$r_{\tilde{\sigma}(i)} \cong r_i^{\sigma, \vee} \otimes r_{\chi,\lambda}|_{G_{F_{1,\pi}}}.$$

Let η be a prime of $F_{1,\pi}$ lying above l such that $\text{Frob}_\eta = \sigma|_{F_{1,\pi}} \in \text{Gal}(F_{1,\pi}/F^+)$. By our choice of λ , $c|_{M_\pi}$ extends to a unique automorphism of $M_{\pi,\lambda}$, and is equal to $\text{Frob}_l|_{M_{\pi,\lambda}}$. Let $\tilde{c} \in \text{Gal}(\overline{M_{\pi,\lambda}}/\mathbb{Q})$ be a lift of $c|_{M_\pi}$. Let $\tau : F_{1,\pi} \hookrightarrow \overline{M_{\pi,\lambda}}$ be a continuous embedding inducing the prime η .

Then we have

$$\mathrm{HT}_\tau(r_{\tilde{\sigma}(i)}) = \mathrm{HT}_\tau(r_i^{\sigma, \vee} \otimes r_{\chi, \lambda}|_{G_{F_{1, \pi}}}) \quad (13)$$

$$= \mathrm{HT}_\tau(\tilde{c}r_i^\sigma) \quad (14)$$

$$= \mathrm{HT}_{\tilde{c}\tau\sigma^{-1}}(r_i)$$

$$= \mathrm{HT}_{\tau\mathrm{Frob}_\eta\sigma^{-1}}(r_i)$$

$$= \mathrm{HT}_\tau(r_i),$$

where going from Eq. 13 to Eq. 14 follows from Lemma 3.12(1), Lemma 3.12(4) and Lemma 3.15. By regularity of $r_{\pi, \lambda}$, we get $r_i = r_{\tilde{\sigma}(i)}$ and thus $\tilde{\sigma}$ is the trivial permutation. $\square$

Corollary 7.2. *If $\lambda \in \mathfrak{f}$, then $r_{\pi, \lambda}|_{G_{F_{1, \pi}}}$ is irreducible.*

Proof. Let $\sigma_1, \sigma_2 \in [c] \subseteq \mathrm{Gal}(F_{1, \pi}/F^+)$. Then by Proposition 7.1, we have $r_i^{\sigma_1\sigma_2} \cong r_i$ for all $i \in \{1, \dots, s\}$. Since F is the maximal CM subextension of $F_{1, \pi}/F^+$, we know the set $\{\sigma_1\sigma_2 | \sigma_1, \sigma_2 \in [c] \subseteq \mathrm{Gal}(F_{1, \pi}/F^+)\}$ generates $\mathrm{Gal}(F_{1, \pi}/F)$. So $\mathrm{Gal}(F_{1, \pi}/F)$ acts trivially on $\{r_i\}_{i=1}^s$ and each r_i extends to a representation of G_F . Since $r_{\pi, \lambda}$ is irreducible as a representation of G_F , we must have $s = 1$. $\square$

Remark. Let $\lambda \in \mathfrak{f}$. Then we know $r_{\pi, \lambda}|_{G_{F_{1, \pi}}}$ is irreducible by Corollary 7.2. So G_λ^0 is a connected reductive group with an irreducible faithful representation into GL_n . By Schur's lemma, the elements in the center of G_λ^0 act as scalars. In particular, $r_{\pi, \lambda}|_{G_\lambda^{0, \mathrm{der}}}$ is irreducible and regular.

8 Independence of λ results

The rest of the manuscript will be dedicated to showing that the conditions of Theorem 6.1 are satisfied by π of dimension less than or equal to 6:

Let F be a CM field and π be a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where $n \leq 6$. Then for all but finitely many finite primes λ of M_π , and for each irreducible constituent $r \subseteq r_{\pi, \lambda}|_{G_\lambda^{0, \mathrm{der}}}$, $r^\sigma \cong r^\vee$ for every $\sigma \in G_{F^+} \setminus G_F$.

In this section we collect some results along the lines of independence of λ in a weakly compatible system of Galois representations, which will be useful in the proof of the statement above.

8.1 Independence of λ of characteristic varieties

8.1.1 Serre groups and abelian representations

To prove our independence of λ results, we first recall some results on Serre groups and abelian representations from [17] and [7]. We will use ZC to denote Zariski closure, Ker to denote the kernel of a map and Im to denote the image of a map.

Let K be a number field. We can associate to K a projective family $\{S_{\mathfrak{m}}\}_{\mathfrak{m}}$ of abelian algebraic groups over $\mathbb{Q}$ called the Serre groups, as detailed in Chap. II 2.1 and 2.2 of [17].

Let S be a finite set of finite primes of K . By a modulus of support of S we mean $\mathfrak{m} = (m_v)_{v \in S}$ where each m_v is a positive integer. Let $K_{\mathfrak{m}} = \{u \in \mathfrak{O}_K^\times \mid \mathrm{val}_v(1 - u) \geq m_v \text{ for all } v \in S\}$, where $\mathfrak{O}_K^\times$ is the group of units in the ring of integers of K . Let $T_{\mathfrak{m}} = \mathrm{Res}_{\mathbb{Q}}^K(\mathbb{G}_{m, K})/\mathrm{ZC}(K_{\mathfrak{m}})$, which is a connected torus. The identity component of $S_{\mathfrak{m}}$ is $T_{\mathfrak{m}}$.

The $S_{\mathfrak{m}}$'s form a projective system. For moduli $\mathfrak{m}_1 \mid \mathfrak{m}_2$ there exists a morphism $S_{\mathfrak{m}_2} \rightarrow S_{\mathfrak{m}_1}$, which has finite kernel and finite cokernel. As a result, all $S_{\mathfrak{m}}$'s have the same dimension for all moduli $\mathfrak{m}$ of K .

Let E be a number field. Let λ be a finite prime in E . We have the following:

Definition 8.1 (Chap. I 2.3 in [17] and Section II in [7]). *Let $\alpha : G_K \rightarrow \mathrm{GL}_n(E_\lambda)$ be a λ -adic representation of K (by which we mean α is a continuous homomorphism). We say α is rational over E if there exists a finite set S of finite primes in K such that α is unramified outside S and if $v \notin S$, the characteristic polynomial of $\alpha(\mathrm{Frob}_v)$ has coefficients in E .*

Definition 8.2 (Chap. I 2.3 in [17] and Section III in [7]). *For each finite prime λ of E , let $\alpha_\lambda : G_K \rightarrow \mathrm{GL}_n(E_\lambda)$ be a rational λ -adic representation of K . The collection $\{\alpha_\lambda\}_\lambda$ is said to form a strictly compatible system if*

- there exists a finite set S of finite primes in K such that for any $v \notin S$ and $v \nmid \text{res}(\lambda)$, α_λ is unramified at v and the characteristic polynomial of $\alpha_\lambda(\text{Frob}_v)$ has coefficients in E , and
- for any pair λ, λ' of finite primes, the characteristic polynomials of $\alpha_\lambda(\text{Frob}_v)$ and $\alpha_{\lambda'}(\text{Frob}_v)$ are the same for $v \notin S$ and $v \nmid \text{res}(\lambda), \text{res}(\lambda')$.

Remark. Note that the notion of ‘strictly compatible’ given by Serre in Definition 8.2 is weaker than the definition of ‘weakly compatible’ given in Definition 3.9. In particular, there is no restriction at the ramified primes in Definition 8.2.

Proposition 8.3 (Chap. I 2.3. in [17]). *For each finite prime λ in E and each modulus $\mathfrak{m}$ of K , there exists a canonical abelian λ -adic representation with values in $S_{\mathfrak{m}}$:*

$$\kappa_{\mathfrak{m},\lambda} : G_K^{\text{ab}} \rightarrow S_{\mathfrak{m}}(E_\lambda).$$

Proposition 8.4 (Chap. I 2.5 in [17]). *Let $\phi : S_{\mathfrak{m},E} \rightarrow \text{GL}_{b,E}$ be a linear representation of $S_{\mathfrak{m}}$ over E , and let ϕ_λ be the corresponding linear representation of $S_{\mathfrak{m},E_\lambda}$ over E_λ . Then the representation*

$$\phi_\lambda \circ \kappa_{\mathfrak{m},\lambda} : G_K^{\text{ab}} \rightarrow \text{GL}_b(E_\lambda)$$

is semi-simple and $\{\phi_\lambda \circ \kappa_{\mathfrak{m},\lambda}\}_\lambda$ is a strictly compatible system in the sense of Definition 8.2.

Proposition 8.5 (Chap. III 2.3 Theorem 2 and Chap. III 2.4 in [17]). *Let $\alpha : G_K^{\text{ab}} \rightarrow \text{GL}_n(E_\lambda)$ be an abelian λ -adic representation. If α is Hodge-Tate and semi-simple, then there exist a modulus $\mathfrak{m}$ of K and a morphism of algebraic groups $\phi_\lambda : S_{\mathfrak{m},E_\lambda} \rightarrow \text{GL}_{n,E_\lambda}$ such that $\alpha = \phi_\lambda \circ \kappa_{\mathfrak{m},\lambda}$.*

Remark.

- Note the original terminology in [17] was ‘locally algebraic’, which is replaced by ‘Hodge-Tate and semi-simple’ in Proposition 8.5 for consistency in terminology throughout the manuscript.
- In the statement of Chap. III 2.3, Theorem 2 (together with the comments in 2.4) of [17], there was the extra assumption of α being rational over E . However, that condition was only used in showing that ϕ_λ is defined over E . Since we are not requiring ϕ_λ to be defined over E , we can drop the condition of α being rational over E in the statement of Proposition 8.5.

8.1.2 Notes on characteristic varieties

Now we review some results on the formal characters of representations of a reductive group. In this section the field of definition of every algebraic variety or group is some field K of characteristic zero, unless specified otherwise. We follow the notations in [11] and define $\text{ch} : \text{GL}_n \rightarrow \mathbb{G}_m \times \mathbb{A}^{n-1}$ to be the morphism that associates a matrix to the coefficients of its characteristic polynomial. We want to study the image of tori under the morphism ch , and the extent to which the image determines the tori.

Let $\mathbb{G}_m^n \subseteq \text{GL}_n$ be a split maximal torus. The Weyl group of GL_n for $\mathbb{G}_m^n$ is the symmetric group S_n . The restriction of ch to $\mathbb{G}_m^n$ is a finite morphism which identifies $\mathbb{G}_m \times \mathbb{A}^{n-1}$ with the scheme-theoretic quotient $\mathbb{G}_m^n / S_n$.

Let $T \subseteq \text{GL}_n$ be a torus and $\rho : T \hookrightarrow \text{GL}_n$ be an inclusion map. For any element $g \in \text{GL}_n(\overline{K})$, we have $\text{ch}(\rho(T)) = \text{ch}(g\rho(T)g^{-1})$. Conjugating by an element of $\text{GL}_n(\overline{K})$ if necessary, we may assume $\rho(T) \subseteq \mathbb{G}_m^n$. Since $\text{ch}|_{\mathbb{G}_m^n}$ is a finite morphism onto its quotient by S_n , we have that $\text{ch}(\rho(T))$ is Zariski-closed. Since T is irreducible, so is $\text{ch}(\rho(T))$.

Let $T_1, T_2 \subseteq \text{GL}_n$ be tori, with inclusion maps $\rho_1 : T_1 \hookrightarrow \text{GL}_n$ and $\rho_2 : T_2 \hookrightarrow \text{GL}_n$ respectively. Suppose $\text{ch}(\rho_1(T_1)) = \text{ch}(\rho_2(T_2))$. We claim there exists $g \in \text{GL}_n(\overline{K})$ such that $g\rho_1(T_1)g^{-1} = \rho_2(T_2)$. First replacing T_1, T_2 by $\text{GL}_n(\overline{K})$ -conjugates if necessary, we may assume $T_1, T_2 \subseteq \mathbb{G}_m^n$. Then once again using the fact that $\text{ch}|_{\mathbb{G}_m^n} : \mathbb{G}_m^n \rightarrow \mathbb{G}_m \times \mathbb{A}^{n-1}$ is a finite Galois morphism with Galois group S_n , we get there exists $\sigma \in S_n$ such that $\sigma(\rho_1(T_1)) = \rho_2(T_2)$.

8.1.3 A compatible system of λ -adic representations

We return to the assumptions of Section 5: F is a CM field and (π, χ) is a regular algebraic, polarized, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$.

We fix some modulus $\mathfrak{m}$ of $F_{1,\pi}$ that is stabilized by elements of $\text{Gal}(F_{1,\pi}/F^+)$. Then by Proposition 8.3, we have a family $\{\kappa_{\mathfrak{m},\lambda} : G_{F_{1,\pi}}^{\text{ab}} \rightarrow S_{\mathfrak{m}}(M_{\pi,\lambda})\}_{\lambda}$ of λ -adic representations with values in $S_{\mathfrak{m}}(M_{\pi,\lambda})$. Choose a faithful representation $\iota : S_{\mathfrak{m}} \rightarrow \text{GL}_{m,\mathbb{Q}}$ over $\mathbb{Q}$ and write ι_{λ} for the morphism $S_{\mathfrak{m}} \times_{\mathbb{Q}} M_{\pi,\lambda} \rightarrow \text{GL}_{m,M_{\pi,\lambda}}$. Let $\epsilon_{\mathfrak{m},\lambda} = \iota_{\lambda} \circ \kappa_{\mathfrak{m},\lambda}$. So $\{\epsilon_{\mathfrak{m},\lambda}\}_{\lambda}$ is a m -dimensional strictly compatible system (as defined in Definition 8.2) of λ -adic representations of $F_{1,\pi}$ that is semi-simple and abelian by Proposition 8.4. Let $f := [F_{1,\pi} : F^+]$. Now we define a new compatible system $\{\beta_{\lambda}\}_{\lambda}$, where

$$\beta_{\lambda} = \rho_{\lambda} \oplus \text{Ind}_{G_{F_{1,\pi}}}^{G_{F^+}} \epsilon_{\mathfrak{m},\lambda} : G_{F^+} \rightarrow \text{GL}_{2n+1+mf}(M_{\pi,\lambda}),$$

with ρ_{λ} defined in Section 5. The Zariski closure of its image is a reductive group, which we call B_{λ} , with identity component B_{λ}^0 and derived subgroup $B_{\lambda}^{0,\text{der}}$. Let O_{λ} denote the Zariski closure of the image

of $(\text{Ind}_{G_{F_{1,\pi}}}^{G_{F^+}} \epsilon_{\mathfrak{m},\lambda})(G_{F^+})$ and let O_λ^0 be the identity component. Let O_λ^1 be the kernel of the surjection $O_\lambda \twoheadrightarrow \text{Gal}(F_{1,\pi}/F^+)$. Since $\mathfrak{m}$ is stabilized by elements in $\text{Gal}(F_{1,\pi}/F^+)$, we get a morphism $S_{\mathfrak{m},M_{\pi,\lambda}} \rightarrow O_\lambda^1$. Moreover, since $S_{\mathfrak{m},M_{\pi,\lambda}}$ is the Zariski closure of $\epsilon_{\mathfrak{m},\lambda}(G_{F_{1,\pi}})$, we in fact have a surjection $S_{\mathfrak{m},M_{\pi,\lambda}} \twoheadrightarrow O_\lambda^1$, which induces the surjection on the identity component $S_{\mathfrak{m},M_{\pi,\lambda}}^0 \rightarrow O_\lambda^0$. So we get $O_\lambda^0 = S_{\mathfrak{m},M_{\pi,\lambda}}^0$, which is a connected torus.

Let $c_\lambda = \beta_\lambda(c)$ be the image of a complex conjugation under β_λ . We first recall some results about semi-simple automorphisms of a linear algebraic group from Chap. 7 of [21].

Definition 8.6. *We call an automorphism σ of a linear algebraic group G semi-simple if it can be realized by conjugation by a semi-simple element of some algebraic group containing G .*

In particular, if σ can be realized as conjugation by a finite order element (which is automatically a semi-simple element) in some algebraic group containing G , then σ is a semi-simple automorphism of G .

Proposition 8.7 (Theorem 7.5 in [21]). *Every semi-simple automorphism σ of an algebraic group G stabilizes some Borel subgroup and some maximal torus thereof.*

We include some results on the density of the twisted conjugacy classes of a semi-simple element in a connected reductive group in the appendix. Since c_λ acts as a semi-simple automorphism of B_λ^0 , we have the following results, which follow from Proposition 8.7 and Lemma A.6 respectively:

Proposition 8.8. *There exist a Borel $N_\lambda \subseteq B_\lambda^0$ and a maximal torus $S_\lambda \subseteq N_\lambda$ of B_λ^0 that are stabilized by c_λ .*

Proposition 8.9. *There exists an open dense subset $X_\lambda \subseteq B_\lambda^0$ such that for each $s \in X_\lambda$, there exist $g \in B_\lambda^0$ and $t \in S_\lambda$ satisfying $t = gs(c_\lambda g)^{-1}$, where ${}^{c_\lambda}g$ denotes $c_\lambda g c_\lambda^{-1}$.*

We fix some notations that we will use throughout this section. For each finite prime λ of M_π , we have the embedding $B_\lambda \hookrightarrow G_\lambda \times O_\lambda$, and B_λ surjects onto both factors with projections p_1 and p_2 . We have the surjection $p_1|_{B_\lambda^0} : B_\lambda^0 \twoheadrightarrow G_\lambda^0$ and $T_\lambda := p_1(S_\lambda)$ is a maximal torus of G_λ^0 . Note that c_λ restricts to an automorphism of G_λ^0 that stabilizes T_λ . Let $D_\lambda := p_1(X_\lambda)$, which is an open dense subset of G_λ^0 such that for each $s \in D_\lambda$, there exist $g \in G_\lambda^0$ and $t \in T_\lambda$ satisfying $t = gs(c_\lambda g)^{-1}$. We also have the surjection $p_2|_{B_\lambda^0} : B_\lambda^0 \twoheadrightarrow O_\lambda^0 = S_{\mathfrak{m},M_{\pi,\lambda}}^0$ and $p_2(S_\lambda) = S_{\mathfrak{m},M_{\pi,\lambda}}^0$ because $S_{\mathfrak{m},M_{\pi,\lambda}}^0$ is a torus.

Let $h_{G,\lambda} : G_\lambda^0 \rightarrow \text{GL}_n$ be the natural representation induced by $r_{\chi,\lambda}|_{G_{F_{1,\pi}}}$, $\mu_{G,\lambda} : G_\lambda^0 \rightarrow \mathbb{G}_m$ be the natural representation induced by $r_{\chi,\lambda}|_{G_{F_{1,\pi}}}$ and $l_{S,\lambda} : S_{\mathfrak{m},M_{\pi,\lambda}} \rightarrow \text{GL}_m$ be the natural representation induced by $\epsilon_{\mathfrak{m},\lambda}$. Let $h_\lambda := h_{G,\lambda} \circ p_1|_{B_\lambda^0} : B_\lambda^0 \rightarrow \text{GL}_n$, $\mu_\lambda := \mu_{G,\lambda} \circ p_1|_{B_\lambda^0} : B_\lambda^0 \rightarrow \mathbb{G}_m$ and $l_\lambda := l_{S,\lambda} \circ p_2|_{B_\lambda^0} : B_\lambda^0 \rightarrow \text{GL}_m$. Let $W_\lambda \subseteq X^\times(S_\lambda)$ be the multi-set of weights of h_λ . Let $U_\lambda \subseteq X^\times(S_\lambda)$ be the multi-set of weights of l_λ . Then $\{\mu_\lambda\} \amalg W_\lambda \amalg U_\lambda$ spans $X^\times(S_\lambda)$.

Define ${}^{1+c_\lambda}S_\lambda := \{t^{c_\lambda}t | t \in S_\lambda\}$, which is a subtorus of S_λ . We have the inclusion ${}^{1+c_\lambda}S_\lambda \hookrightarrow S_\lambda$, which induces the surjection $X^\times(S_\lambda) \twoheadrightarrow X^\times({}^{1+c_\lambda}S_\lambda)$ given by $w \mapsto w|_{{}^{1+c_\lambda}S_\lambda}$.

We define the following characteristic varieties:

$$\mathfrak{X}_\lambda = \text{ZC}\{(\text{ch}(\mu_\lambda(g)), \text{ch}(h_\lambda(g)), \text{ch}(l_\lambda(g))) | g \in B_\lambda^0\} \subseteq \mathbb{G}_m^3 \times \mathbb{A}^{n+m-2}/M_{\pi,\lambda}$$

and

$$\mathfrak{Y}_{c_\lambda} = \text{ZC}\{(\text{ch}(\mu_\lambda(gc_\lambda gc_\lambda)), \text{ch}(h_\lambda(gc_\lambda gc_\lambda)), \text{ch}(l_\lambda(gc_\lambda gc_\lambda))) | g \in B_\lambda^0\} \subseteq \mathbb{G}_m^3 \times \mathbb{A}^{n+m-2}/M_{\pi,\lambda}.$$

For $g \in B_\lambda^0$, we know $c_\lambda g c_\lambda \in B_\lambda^0$, so $gc_\lambda gc_\lambda \in B_\lambda^0$ and $h_\lambda(gc_\lambda gc_\lambda)$ is well-defined.

We will show the varieties $\mathfrak{X}_\lambda$ and $\mathfrak{Y}_{c_\lambda}$ are determined by the restrictions of the representation β_λ to the maximal torus S_λ and the c_λ -invariant subtorus ${}^{1+c_\lambda}S_\lambda$ of S_λ respectively.

Lemma 8.10. *The variety $\mathfrak{Y}_{c_\lambda}$ is equal to*

$$\text{ZC}\{(\text{ch}(\mu_\lambda(gc_\lambda gc_\lambda)), \text{ch}(h_\lambda(gc_\lambda gc_\lambda)), \text{ch}(l_\lambda(gc_\lambda gc_\lambda))) | g \in X_\lambda\}.$$

Proof. The statement follows from the knowledge that the set X_λ is Zariski dense in B_λ^0 and the general fact that for a continuous map $f : X \rightarrow Y$ between two topological spaces X and Y , we have $\overline{f(\overline{A})} = \overline{f(A)}$ for a subspace $A \subseteq X$. $\square$

Lemma 8.11. *The variety $\mathfrak{Y}_{c_\lambda}$ is equal to*

$$\text{ZC}\{(\text{ch}(\mu_\lambda(tc_\lambda tc_\lambda)), \text{ch}(h_\lambda(tc_\lambda tc_\lambda)), \text{ch}(l_\lambda(tc_\lambda tc_\lambda))) | t \in S_\lambda\}.$$

Proof. Let $s \in X_\lambda$. We may assume $S_\lambda \subseteq X_\lambda$ after replacing X_λ by $S_\lambda \cup X_\lambda$. We have $gs({}^{c_\lambda}g)^{-1} = t$ for some $g \in B_\lambda^0$ and $t \in S_\lambda$. So we get

$$\begin{aligned} tc_\lambda tc_\lambda &= gs({}^{c_\lambda}g)^{-1}c_\lambda gs({}^{c_\lambda}g)^{-1}c_\lambda \\ &= gs(c_\lambda gc_\lambda^{-1})^{-1}c_\lambda gs(c_\lambda gc_\lambda^{-1})^{-1}c_\lambda \\ &= gsc_\lambda sc_\lambda g^{-1}. \end{aligned}$$

Thus we get $\text{ch}(h_\lambda(tc_\lambda tc_\lambda)) = \text{ch}(h_\lambda(sc_\lambda sc_\lambda))$, $\text{ch}(\mu_\lambda(tc_\lambda tc_\lambda)) = \text{ch}(\mu_\lambda(sc_\lambda sc_\lambda))$ and $\text{ch}(l_\lambda(tc_\lambda tc_\lambda)) = \text{ch}(l_\lambda(sc_\lambda sc_\lambda))$.

Thus by Lemma 8.10, we have

$$\mathfrak{Y}_{c_\lambda} = \text{ZC}\{(\text{ch}(\mu_\lambda(tc_\lambda tc_\lambda)), \text{ch}(h_\lambda(tc_\lambda tc_\lambda)), \text{ch}(l_\lambda(tc_\lambda tc_\lambda))) | t \in S_\lambda\}.$$

□

Proposition 8.12. *We have*

$$\mathfrak{X}_\lambda = \text{ZC}\{(\text{ch}(\mu_\lambda(t)), \text{ch}(h_\lambda(t)), \text{ch}(l_\lambda(t))) | t \in S_\lambda\}$$

and

$$\mathfrak{Y}_{c_\lambda} = \text{ZC}\{(\text{ch}(\mu_\lambda(s)), \text{ch}(h_\lambda(s)), \text{ch}(l_\lambda(s))) | s \in {}^{1+c_\lambda}S_\lambda\}.$$

Proof. For all $g \in B_\lambda^0$, there exists $h \in B_\lambda^0$ such that $hg^{\text{ss}}h^{-1} \in S_\lambda$, where g^{ss} denotes the semi-simple part of g . So we get

$$\mathfrak{X}_\lambda = \text{ZC}\{(\text{ch}(\mu_\lambda(t)), \text{ch}(h_\lambda(t)), \text{ch}(l_\lambda(t))) | t \in S_\lambda\}.$$

By Lemma 8.11, we get

$$\begin{aligned} \mathfrak{Y}_{c_\lambda} &= \text{ZC}\{(\text{ch}(\mu_\lambda(tc_\lambda tc_\lambda)), \text{ch}(h_\lambda(tc_\lambda tc_\lambda)), \text{ch}(l_\lambda(tc_\lambda tc_\lambda))) | t \in S_\lambda\} \\ &= \text{ZC}\{(\text{ch}(\mu_\lambda(s)), \text{ch}(h_\lambda(s)), \text{ch}(l_\lambda(s))) | s \in {}^{1+c_\lambda}S_\lambda\}. \end{aligned}$$

□

Traditionally, it was observed by Serre that the variety $\mathfrak{X}_\lambda$ is independent of λ (cf. Section 3 in [16]). We will prove an analogous result for both of the varieties $\mathfrak{X}_\lambda$ and $\mathfrak{Y}_{c_\lambda}$.

Proposition 8.13. *The varieties $\mathfrak{X}_\lambda$ and $\mathfrak{Y}_{c_\lambda}$ are defined over M_π and are independent of λ , by which we mean there exist varieties $\mathfrak{Y} \subseteq \mathfrak{X}$ defined over M_π such that $\mathfrak{X} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{X}_\lambda$ and $\mathfrak{Y} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{Y}_{c_\lambda}$ for all finite primes λ of M_π .*

Proof. Let V^χ be the finite set of finite primes of F^+ in the definition of the weakly compatible system $\mathfrak{R}_\chi$ and V_π be the finite set of finite primes of F in the definition of $\mathfrak{R}_\pi$ as given in Theorem 3.17. Let V_ϵ be the finite set of finite primes of $F_{1,\pi}$ in the definition of the strictly compatible system $\{\epsilon_{\mathfrak{m},\lambda}\}_\lambda$ as in Definition 8.2. For a non-empty finite set $\mathfrak{R}$ of rational primes, define the sets

$$R_{\mathfrak{R}} := \{\text{Frob}_v \in G_{F_{1,\pi}} | v \text{ a prime of } F_{1,\pi}, v \notin V_\epsilon, v|_F \notin V_\pi, v|_{F^+} \notin V^\chi \text{ and } v \nmid l \text{ for } l \in \mathfrak{R}\}$$

$$R_{\mathfrak{R},c} := \{\text{Frob}_v \in G_{F^+} \cap cG_{F_{1,\pi}} | v \text{ a prime of } F^+, v \notin V^\chi, v' \notin V_\pi \text{ for any } v'|v \text{ in } F,$$

$$v'' \notin V_\epsilon \text{ for any } v''|v \text{ in } F_{1,\pi}, \text{ and } v \nmid l \text{ for } l \in \mathfrak{R}\}.$$

For λ a finite prime of M_π such that $\text{res}(\lambda) \in \mathfrak{K}$, we define

$$\mathfrak{X}_{\mathfrak{K},\lambda} = \text{ZC}\{(\text{ch}(r_{\chi,\lambda}(g)), \text{ch}(r_{\pi,\lambda}(g)), \text{ch}(\epsilon_{\mathfrak{m},\lambda}(g))) | g \in R_{\mathfrak{K}}\} \subseteq \mathbb{G}_m^3 \times \mathbb{A}^{n+m-2}/M_{\pi,\lambda}$$

and

$$\mathfrak{Y}_{\mathfrak{K},\lambda} = \text{ZC}\{(\text{ch}(r_{\chi,\lambda}(g^2)), \text{ch}(r_{\pi,\lambda}(g^2)), \text{ch}(\epsilon_{\mathfrak{m},\lambda}(g^2))) | g \in R_{\mathfrak{K},c}\} \subseteq \mathbb{G}_m^3 \times \mathbb{A}^{n+m-2}/M_{\pi,\lambda}.$$

Then in fact $\mathfrak{X}_{\mathfrak{K},\lambda}$ is defined over M_π and is independent of λ . So we can drop the λ on the subscript and write $\mathfrak{X}_{\mathfrak{K}}$ instead. For $\text{Frob}_v \in R_{\mathfrak{K},c}$, there exists a prime $v'|v$ of $F_{1,\pi}$ such that $\text{Frob}_v^2 = \text{Frob}_{v'} \in R_{\mathfrak{K}}$. So as in the case of $\mathfrak{X}_{\mathfrak{K},\lambda}$, if $\text{res}(\lambda) \in \mathfrak{K}$, we know $\mathfrak{Y}_{\mathfrak{K},\lambda}$ is defined over M_π and is independent of λ . Thus we drop the λ on the subscript and write $\mathfrak{Y}_{\mathfrak{K}}$ instead.

Since $R_{\mathfrak{K}}$ is dense in $G_{F_{1,\pi}}$ (resp. $R_{\mathfrak{K},c}$ is dense in $cG_{F_{1,\pi}}$), and the Zariski topology is coarser than the l -adic topology, we get for λ a finite prime of M_π lying above some rational prime in $\mathfrak{K}$

$$\mathfrak{X}_{\mathfrak{K}} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{X}_\lambda \quad (15)$$

$$(\text{resp. } \mathfrak{Y}_{\mathfrak{K}} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{Y}_{c_\lambda}). \quad (16)$$

Consider any two non-empty finite sets of rational primes $\mathfrak{K}$ and $\mathfrak{K}'$. We have $R_{\mathfrak{K} \cup \mathfrak{K}'} = R_{\mathfrak{K}} \cap R_{\mathfrak{K}'}$ (resp. $R_{\mathfrak{K} \cup \mathfrak{K}',c} = R_{\mathfrak{K},c} \cap R_{\mathfrak{K}',c}$). For λ such that $\text{res}(\lambda) \in \mathfrak{K} \cup \mathfrak{K}'$, we have $\mathfrak{X}_{\mathfrak{K} \cup \mathfrak{K}'} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{X}_\lambda$ (resp. $\mathfrak{Y}_{\mathfrak{K} \cup \mathfrak{K}'} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{Y}_{c_\lambda}$). But we also have either $\mathfrak{X}_{\mathfrak{K}} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{X}_\lambda$ (resp. $\mathfrak{Y}_{\mathfrak{K}} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{Y}_{c_\lambda}$) or $\mathfrak{X}_{\mathfrak{K}'} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{X}_\lambda$ (resp. $\mathfrak{Y}_{\mathfrak{K}'} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{Y}_{c_\lambda}$), depending on whether $\text{res}(\lambda) \in \mathfrak{K}$ or $\text{res}(\lambda) \in \mathfrak{K}'$. Since $\mathfrak{K}, \mathfrak{K}'$ are non-empty, we must have $\mathfrak{X}_{\mathfrak{K}} = \mathfrak{X}_{\mathfrak{K} \cup \mathfrak{K}'} = \mathfrak{X}_{\mathfrak{K}'}$ (resp. $\mathfrak{Y}_{\mathfrak{K}} = \mathfrak{Y}_{\mathfrak{K} \cup \mathfrak{K}'} = \mathfrak{Y}_{\mathfrak{K}'}$).

Let $\mathfrak{K}$ be any non-empty finite set of rational primes. Let $\mathfrak{X} := \mathfrak{X}_{\mathfrak{K}}$ and $\mathfrak{Y} := \mathfrak{Y}_{\mathfrak{K}}$. Then $\mathfrak{Y} \subseteq \mathfrak{X}$ are defined over M_π and satisfy $\mathfrak{X} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{X}_\lambda, \mathfrak{Y} \times_{M_\pi} M_{\pi,\lambda} = \mathfrak{Y}_{c_\lambda}$ for all finite primes λ of M_π . $\square$

8.1.4 Characteristic varieties and weights of a representation

We will explore to what extent are the weights of the representations β_λ determined by $\mathfrak{X}$ and $\mathfrak{Y}$. In this section the field of definition of every algebraic group or variety is M_π unless otherwise specified. For simplicity of notation, we let ch denote the restriction of $\text{ch} : \text{GL}_{1+n+m} \rightarrow \mathbb{G}_m \times \mathbb{A}^{n+m}$ to $\mathbb{G}_m \times \text{GL}_n \times \text{GL}_m$, where m, n are integers in the definition of the representation β_λ . In other words, ch is the morphism $\text{ch} : \mathbb{G}_m \times \text{GL}_n \times \text{GL}_m \rightarrow \mathbb{G}_m^3 \times \mathbb{A}^{n+m-2}$ that sends a matrix in $\mathbb{G}_m \times \text{GL}_n \times \text{GL}_m$ to the coefficients of its characteristic polynomial.

For a torus S , we write $X^\times(S)$ for the character group of S . Now we apply the results in Section 8.1.2

to our situation. Let $f_j : \mathbb{G}_m^{1+n+m} \rightarrow \mathbb{G}_m$ be the projection onto the j -th factor. Let S be a subtorus of $\mathbb{G}_m^{1+n+m}$ and $\rho : S \hookrightarrow \mathrm{GL}_{1+n+m}$ be the inclusion map. Define $\mu_\rho := f_1 \circ \rho \in X^\times(S)$, the multi-sets $W_{\rho,1} := \{f_j \circ \rho | 2 \leq j \leq 1+n\} \subseteq X^\times(S)$ and $W_{\rho,2} := \{f_j \circ \rho | n+1 < j \leq 1+n+m\} \subseteq X^\times(S)$.

We give the following definition:

Definition 8.14. Consider tuples (X, h, Z_1, Z_2) where X is a finite-rank free abelian group, $h \in X$, and $Z_1, Z_2 \subseteq X$ are finite multi-sets such that $\{h\} \amalg Z_1 \amalg Z_2$ spans X . Then we say two such tuples (X, h, Z_1, Z_2) and (X', h', Z'_1, Z'_2) are isomorphic if there exists an isomorphism $f : X \xrightarrow{\sim} X'$ of free abelian groups such that $f(h) = h'$, $f(Z_1) = Z'_1$ and $f(Z_2) = Z'_2$.

It turns out the weights of β_λ and the restriction of the weights to the c_λ -invariant subtorus are determined up to isomorphisms by $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

Proposition 8.15. There exists a torus S of $\mathbb{G}_m^{1+n+m}/\mathbb{Q}$ such that $\mathrm{ch}(\rho(S)) = \mathfrak{X}$, where $\rho : S \hookrightarrow \mathrm{GL}_{1+n+m}$ is the natural inclusion map. Moreover, the tuple $(X^\times(S), \mu_\rho, W_{\rho,1}, W_{\rho,2})$ is determined by $\mathfrak{X}$ up to isomorphism.

Proof. Fix λ a finite prime of M_π . In Proposition 8.12 and Proposition 8.13, we showed there exist a torus $S_\lambda \subseteq \mathbb{G}_m \times \mathrm{GL}_n \times \mathrm{GL}_m / M_{\pi,\lambda}$ and an inclusion map $i_\lambda : S_\lambda \hookrightarrow \mathbb{G}_m \times \mathrm{GL}_n \times \mathrm{GL}_m$ such that $\mathrm{ch}(i_\lambda(S_\lambda)) = \mathfrak{X}$. After conjugating by an element of $\mathbb{G}_m \times \mathrm{GL}_n \times \mathrm{GL}_m(\overline{M_{\pi,\lambda}})$, we may assume $i_\lambda(S_\lambda) \subseteq \mathbb{G}_m^{1+n+m} / \overline{M_{\pi,\lambda}}$. Since every split torus defined over $M_{\pi,\lambda}$ is obtained from a split torus over $\mathbb{Q}$ via extension of scalars (cf. 4.3 of [11]), there exists a torus $S \subseteq \mathbb{G}_m^{1+n+m}/\mathbb{Q}$ with the natural inclusion map $\rho : S \hookrightarrow \mathrm{GL}_{1+n+m}/\mathbb{Q}$ such that $\mathrm{ch}(\rho(S) \times_{\mathbb{Q}} M_\pi) = \mathrm{ch}(i_\lambda(S_\lambda)) = \mathfrak{X}$.

Suppose there exists another torus $S' \subseteq \mathbb{G}_m \times \mathrm{GL}_n \times \mathrm{GL}_m$ such that $\mathrm{ch}(\rho'(S')) = \mathfrak{X}$, where $\rho' : S' \hookrightarrow \mathrm{GL}_{1+n+m}$ is the natural inclusion. Conjugating by an element of $\mathbb{G}_m \times \mathrm{GL}_n \times \mathrm{GL}_m$, we may assume $\rho'(S') \subseteq \mathbb{G}_m^{1+n+m}$. Since $\mathrm{ch}|_{\mathbb{G}_m^{1+n+m}} : \mathbb{G}_m^{1+n+m} \rightarrow \mathbb{G}_m^3 \times \mathbb{A}^{n+m-2}$ is a finite Galois morphism with Galois group $\{1\} \times S_n \times S_m$, and since $\rho(S) \times_{\mathbb{Q}} M_\pi$ and $\rho'(S')$ are both irreducible components of $\mathrm{ch}^{-1}(\mathfrak{X}) \cap \mathbb{G}_m^{1+n+m}$, there exists some $\sigma \in \{1\} \times S_n \times S_m$ such that $\sigma(\rho(S)) = \rho'(S')$. In particular, σ induces an isomorphism between $(X^\times(S), \mu_\rho, W_{\rho,1}, W_{\rho,2})$ and $(X^\times(S'), \mu_{\rho'}, W_{\rho',1}, W_{\rho',2})$ in the sense of Definition 8.14. $\square$

Proposition 8.16. There exists a subtorus S_c of $\mathbb{G}_m^{1+n+m}/\mathbb{Q}$ such that $\mathrm{ch}(\rho_c(S_c)) = \mathfrak{Y}$ where $\rho_c : S_c \hookrightarrow \mathrm{GL}_{1+n+m}$ is the natural inclusion map. Moreover, the tuple $(X^\times(S_c), \mu_{\rho_c}, W_{\rho_c,1}, W_{\rho_c,2})$ is determined by $\mathfrak{Y}$ up to isomorphism.

Proof. The proof is the same as that of Proposition 8.15, except we replace S by S_c , ρ by ρ_c , S_λ by $^{1+c_\lambda}S_\lambda$ and $\mathfrak{X}$ by $\mathfrak{Y}$. $\square$

Recall the definitions of W_λ, U_λ and μ_λ from the paragraphs after Proposition 8.9. Proposition 8.15 and Proposition 8.16 immediately imply the following corollaries:

Corollary 8.17. *For each pair λ, λ' of finite primes of M_π , there exists an isomorphism $i_{\lambda, \lambda'} : X^\times(S_\lambda) \xrightarrow{\sim} X^\times(S_{\lambda'})$ of free abelian groups such that $i_{\lambda, \lambda'}(W_\lambda) = W_{\lambda'}, i_{\lambda, \lambda'}(\mu_\lambda) = \mu_{\lambda'}$, and $i_{\lambda, \lambda'}(U_\lambda) = U_{\lambda'}$.*

Corollary 8.18. *For each pair λ, λ' of finite primes of M_π , there exists an isomorphism $r_{\lambda, \lambda'} : X^\times({}^{1+c_\lambda}S_\lambda) \xrightarrow{\sim} X^\times({}^{1+c_{\lambda'}}S_{\lambda'})$ of free abelian groups such that*

$$r_{\lambda, \lambda'}(W_\lambda|_{1+c_\lambda S_\lambda}) = W_{\lambda'}|_{1+c_{\lambda'} S_{\lambda'}},$$

$$r_{\lambda, \lambda'}(U_\lambda|_{1+c_\lambda S_\lambda}) = U_{\lambda'}|_{1+c_{\lambda'} S_{\lambda'}},$$

$$r_{\lambda, \lambda'}(\mu_\lambda|_{1+c_\lambda S_\lambda}) = \mu_{\lambda'}|_{1+c_{\lambda'} S_{\lambda'}}.$$

8.2 Semi-simple formal characters and actions by c_λ

Keep the definitions in Section 8.1.3. Let $S_\lambda^{\text{der}} := (S_\lambda \cap B_\lambda^{0, \text{der}})^0$ and $T_\lambda^{\text{der}} := (T_\lambda \cap G_\lambda^{0, \text{der}})^0$. Let $({}^{1+c_\lambda}S_\lambda)^{\text{der}} := ({}^{1+c_\lambda}S_\lambda \cap B_\lambda^{0, \text{der}})^0$ and $({}^{1+c_\lambda}T_\lambda)^{\text{der}} := ({}^{1+c_\lambda}T_\lambda \cap G_\lambda^{0, \text{der}})^0$. Let W_λ^{ss} be the multi-set of weights of $h_\lambda|_{B_\lambda^{0, \text{der}}}$ in $X^\times(S_\lambda^{\text{der}})$, and let $(W_\lambda|_{1+c_\lambda S_\lambda})^{\text{ss}} \subseteq X^\times({}^{1+c_\lambda}S_\lambda^{\text{der}})$ be the multi-set of weights of h_λ restricted to $({}^{1+c_\lambda}S_\lambda)^{\text{der}}$. Note that c_λ restricts to automorphisms on S_λ^{der} and on T_λ^{der} , so we can also define ${}^{1+c_\lambda}(S_\lambda^{\text{der}})$ and ${}^{1+c_\lambda}(T_\lambda^{\text{der}})$ correspondingly.

We will prove the following result:

Proposition 8.19. *For each pair λ, λ' of finite primes of M_π , there exist an isomorphism $i_{\lambda, \lambda'} : X^\times(S_\lambda^{\text{der}}) \xrightarrow{\sim} X^\times(S_{\lambda'}^{\text{der}})$ such that $i_{\lambda, \lambda'}(W_\lambda^{\text{ss}}) = W_{\lambda'}^{\text{ss}}$, and an isomorphism $r_{\lambda, \lambda'} : X^\times({}^{1+c_\lambda}S_\lambda^{\text{der}}) \xrightarrow{\sim} X^\times({}^{1+c_{\lambda'}}S_{\lambda'}^{\text{der}})$ such that $r_{\lambda, \lambda'}((W_\lambda|_{1+c_\lambda S_\lambda})^{\text{ss}}) = (W_{\lambda'}|_{1+c_{\lambda'} S_{\lambda'}})^{\text{ss}}$.*

The method we employ in this section is inspired by Section 3 of [8]. The following result, which was originally proved in [8] (cf. Theorem 3.19 in [8]), now becomes a simple corollary of Proposition 8.19.

Corollary 8.20. *The rank of $B_\lambda^{0, \text{der}}$ is independent of λ .*

Proof. This follows directly from Proposition 8.19 since the rank of $B_\lambda^{0, \text{der}}$ is just the rank of S_λ^{der} as an abelian group. $\square$

We need the following two results in the proof of Proposition 8.19:

Proposition 8.21. *Let $r : G_K \rightarrow \text{GL}_d$ be a semi-simple and Hodge-Tate Galois representation of some number field K and let G be the Zariski closure of the image. Suppose G is connected and let $C := G/G^{\text{der}}$.*

Then there exists a semi-simple representation $z : C \rightarrow \mathrm{GL}_s$ for some integer s with finite kernel, and moreover $z \circ r$ is Hodge-Tate.

Proof. First note that G is a connected reductive group and C is a torus. Let $r = r_1 \oplus \dots \oplus r_s$ be the decomposition into absolutely irreducible constituents, where r_i has dimension d_i . Let $z_i := \det \circ r_i$. Then since r is Hodge-Tate, so is z_i for each i . Let $z := (z_1, \dots, z_s)|_C : C \rightarrow \mathrm{GL}_s$, which is a Hodge-Tate and semi-simple representation of the abelian group C .

We now show that $\mathrm{Ker}(z)$ is finite. Let Z be the center of G . Then we have a surjection $\pi : Z \twoheadrightarrow C$. Let $z' := (z_1, \dots, z_s)|_Z : Z \rightarrow \mathrm{GL}_s$. Since each r_i is irreducible, the action of Z on r_i is given by a continuous character $\chi_i : Z \rightarrow \mathbb{G}_m$. Since r induces a faithful representation of G , we get $\bigcap_i \mathrm{Ker}(\chi_i) = \{1\}$. We also have

$$\mathrm{Ker}(z') = \bigcap_i \mathrm{Ker}(z_i|_Z) = \bigcap_i \mathrm{Ker}(\chi_i^{d_i}),$$

which implies $\mathrm{Ker}(z') \subseteq \prod_i \mathbb{U}_{d_i} \subseteq \mathbb{G}_m^s$, where $\mathbb{U}_{d_i}$ denotes the multiplicative group of d_i -th roots of unity. So we have $\mathrm{Ker}(z) = \pi(\mathrm{Ker}(z'))$ is finite. $\square$

For simplicity of notation, we write M_λ for $M_{\pi, \lambda}$ and $\kappa_{\mathfrak{m}_\lambda}$ for $\kappa_{\mathfrak{m}_\lambda, \lambda}$ in the proofs of the next two propositions.

Proposition 8.22. *We have $G_\lambda^{0, \mathrm{der}} = B_\lambda^{0, \mathrm{der}} \subseteq \mathrm{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m}, M_\lambda}^0)$ is a subgroup of finite index.*

Proof. Let $[\cdot, \cdot]$ be the map that sends two elements to their commutator. Since we have $B_\lambda^0 \hookrightarrow G_\lambda^0 \times S_{\mathfrak{m}, M_\lambda}^0$ and $S_{\mathfrak{m}, M_\lambda}^0$ is abelian, we get $[B_\lambda^0, B_\lambda^0] \subseteq [G_\lambda^0, G_\lambda^0]$, and thus $B_\lambda^{0, \mathrm{der}} \subseteq G_\lambda^{0, \mathrm{der}}$. But since we also have the surjection $B_\lambda^0 \twoheadrightarrow G_\lambda^0$, which implies $[B_\lambda^0, B_\lambda^0] \twoheadrightarrow [G_\lambda^0, G_\lambda^0]$, we can conclude that $G_\lambda^{0, \mathrm{der}} = B_\lambda^{0, \mathrm{der}} \subseteq \mathrm{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m}, M_\lambda}^0)$.

We know $C_\lambda := G_\lambda^0 / G_\lambda^{0, \mathrm{der}}$ is a torus. By Proposition 8.21, we can find an integer n' and a morphism $z_\lambda = (z_1, \dots, z_{n'}) : C_\lambda \rightarrow \mathrm{GL}_{n', M_\lambda}$ of finite kernel such that $z_\lambda \circ \rho_\lambda|_{G_{F_{1, \pi}}} : G_{F_{1, \pi}} \rightarrow \mathrm{GL}_{n', M_\lambda}$ is an abelian, semi-simple and Hodge-Tate representation. Then $\{z_1, \dots, z_{n'}\}$ spans $X^\times(C_\lambda)_\mathbb{Q}$ and $G_\lambda^{0, \mathrm{der}} \subseteq \mathrm{Ker}(z_\lambda)$ has finite index. By Proposition 8.5, there exist a modulus $\mathfrak{m}_\lambda$ of $F_{1, \pi}$ and a linear representation $\phi_\lambda : S_{\mathfrak{m}_\lambda} \times_\mathbb{Q} M_\lambda \rightarrow \mathrm{GL}_{n', M_\lambda}$ such that $\phi_\lambda \circ \kappa_{\mathfrak{m}_\lambda} = z_\lambda \circ \rho_\lambda|_{G_{F_{1, \pi}}}$. Furthermore, by enlarging $\mathfrak{m}_\lambda$, we may assume $\mathfrak{m} | \mathfrak{m}_\lambda$, where $\mathfrak{m}$ is the modulus of $F_{1, \pi}$ in the definition of the representation β_λ .

By Lemma 3.13, there exists a finite Galois extension $F'_{1, \pi} / F^+$ such that for any finite prime λ of M_π we have $B_\lambda / B_\lambda^0 \cong \mathrm{Gal}(F'_{1, \pi} / F^+)$. Let $F''_{1, \pi} / F'_{1, \pi} / F^+$ be a finite Galois extension such that $\kappa_{\mathfrak{m}_\lambda}(G_{F''_{1, \pi}}) \subseteq S_{\mathfrak{m}_\lambda, M_\lambda}^0$.

We have the homomorphisms

$$t_\lambda := (\kappa_{\mathfrak{m}, \lambda}, \kappa_{\mathfrak{m}_\lambda}) : G_{F''_{1, \pi}} \rightarrow S_{\mathfrak{m}, M_\lambda}^0 \times S_{\mathfrak{m}_\lambda, M_\lambda}^0$$

$$\alpha_\lambda := (\kappa_{\mathfrak{m},\lambda}, \phi_\lambda \circ \kappa_{\mathfrak{m},\lambda}) : G_{F_{1,\pi}''} \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda).$$

Since $z_\lambda : G_\lambda^0/G_\lambda^{0,\text{der}} \rightarrow \text{GL}_{n',M_\lambda}$ has finite kernel, we get $B_\lambda^{0,\text{der}} = G_\lambda^{0,\text{der}}$ embeds in $\text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda))$ with finite index. We also have

$$\text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda)) \subseteq \text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0). \quad (17)$$

We want to show the inclusion in Eq. 17 is of finite index. Then we could conclude that $B_\lambda^{0,\text{der}} \subseteq \text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0)$ is a subgroup of finite index.

We have $\text{ZC}(\alpha_\lambda(G_{F_{1,\pi}''}))^0 = \text{Im}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda))$ and we have the surjection $\text{Im}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda)) \twoheadrightarrow S_{\mathfrak{m},M_\lambda}^0$. We also know $\text{ZC}(t_\lambda(G_{F_{1,\pi}''}))^0 \twoheadrightarrow \text{ZC}(\alpha_\lambda(G_{F_{1,\pi}''}))^0$. Putting it all together, we get

$$\text{ZC}(\kappa_{\mathfrak{m},\lambda}(G_{F_{1,\pi}''}))^0 \twoheadrightarrow \text{ZC}(t_\lambda(G_{F_{1,\pi}''}))^0 \twoheadrightarrow \text{ZC}(\alpha_\lambda(G_{F_{1,\pi}''}))^0 = \text{Im}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda)) \twoheadrightarrow S_{\mathfrak{m},M_\lambda}^0. \quad (18)$$

Since $S_{\mathfrak{m},\lambda} \rightarrow S_{\mathfrak{m}}$ has a finite kernel, so does the induced map

$$\text{ZC}(\kappa_{\mathfrak{m},\lambda}(G_{F_{1,\pi}''}))^0 \twoheadrightarrow \text{ZC}(\kappa_{\mathfrak{m},\lambda}(G_{F_{1,\pi}''}))^0 = S_{\mathfrak{m},M_\lambda}^0.$$

Thus the composed map in Eq. 18 has finite kernel, which implies $\text{Im}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda)) \twoheadrightarrow S_{\mathfrak{m},M_\lambda}^0$ has finite kernel. So we get $\text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0 \times z_\lambda(C_\lambda)) \subseteq \text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0)$ has finite index as desired. $\square$

We now turn to proving Proposition 8.19.

Proof. By Proposition 8.22, $B_\lambda^{0,\text{der}} \subseteq \text{Ker}(B_\lambda^0 \rightarrow S_{\mathfrak{m},M_\lambda}^0)$ is a subgroup of finite index. This implies $S_\lambda^{\text{der}} \subseteq J_\lambda := \text{Ker}(S_\lambda \rightarrow S_{\mathfrak{m},M_\lambda}^0)$ and $(^{1+c_\lambda}S_\lambda)^{\text{der}} \subseteq ^{1+c_\lambda}J_\lambda = \text{Ker}(^{1+c_\lambda}S_\lambda \rightarrow S_{\mathfrak{m},M_\lambda}^0)$ are also subgroups with finite indices.

We have the exact sequences

$$0 \rightarrow J_\lambda \rightarrow S_\lambda \rightarrow S_{\mathfrak{m},M_\lambda}^0 \rightarrow 0$$

$$0 \rightarrow ^{1+c_\lambda}J_\lambda \rightarrow ^{1+c_\lambda}S_\lambda \rightarrow ^{1+c_\lambda}S_{\mathfrak{m},M_\lambda}^0 \rightarrow 0,$$

which give the dual exact sequences of the character groups

$$0 \rightarrow X^\times(S_{\mathfrak{m},M_\lambda}^0) \rightarrow X^\times(S_\lambda) \rightarrow X^\times(J_\lambda) \rightarrow 0$$

$$0 \rightarrow X^\times(^{1+c_\lambda}S_{\mathfrak{m},M_\lambda}^0) \rightarrow X^\times(^{1+c_\lambda}S_\lambda) \rightarrow X^\times(^{1+c_\lambda}J_\lambda) \rightarrow 0.$$

Since $S_\lambda^{\text{der}} \subseteq J_\lambda$ and $(^{1+c_\lambda}S_\lambda)^{\text{der}} \subseteq ^{1+c_\lambda}J_\lambda$ have finite indices, there exist finite abelian groups A_1, A_2 such that the following sequences are exact:

$$0 \rightarrow S_\lambda^{\text{der}} \rightarrow J_\lambda \rightarrow A_1 \rightarrow 0$$

$$0 \rightarrow X^\times(A_1) \rightarrow X^\times(J_\lambda) \rightarrow X^\times(S_\lambda^{\text{der}}) \rightarrow 0$$

$$0 \rightarrow (^{1+c_\lambda}S_\lambda)^{\text{der}} \rightarrow ^{1+c_\lambda}J_\lambda \rightarrow A_2 \rightarrow 0$$

$$0 \rightarrow X^\times(A_2) \rightarrow X^\times(^{1+c_\lambda}J_\lambda) \rightarrow X^\times((^{1+c_\lambda}S_\lambda)^{\text{der}}) \rightarrow 0.$$

So we get $X^\times(A_1)$ and $X^\times(A_2)$ are the torsion subgroups of $X^\times(J_\lambda)$ and $X^\times(^{1+c_\lambda}J_\lambda)$ respectively, and $X^\times(S_\lambda^{\text{der}})$ and $X^\times((^{1+c_\lambda}S_\lambda)^{\text{der}})$ are the corresponding torsion free parts. In other words,

$$\begin{aligned} X^\times(S_\lambda^{\text{der}}) &= X^\times(S_\lambda)/(X^\times(S_{\mathfrak{m}, M_\lambda}^0)_\mathbb{Q} \cap X^\times(S_\lambda)) \\ &= X^\times(S_\lambda)/(\text{span}\langle U_\lambda \rangle_\mathbb{Q} \cap X^\times(S_\lambda)) \end{aligned} \tag{19}$$

$$\begin{aligned} X^\times((^{1+c_\lambda}S_\lambda)^{\text{der}}) &= X^\times(^{1+c_\lambda}S_\lambda)/(X^\times(^{1+c_\lambda}S_{\mathfrak{m}, M_\lambda}^0)_\mathbb{Q} \cap X^\times(^{1+c_\lambda}S_\lambda)) \\ &= X^\times(^{1+c_\lambda}S_\lambda)/(\text{span}\langle U_\lambda|_{^{1+c_\lambda}S_\lambda} \rangle_\mathbb{Q} \cap X^\times(^{1+c_\lambda}S_\lambda)). \end{aligned} \tag{20}$$

So we get that the isomorphism $i_{\lambda, \lambda'} : X^\times(S_\lambda) \xrightarrow{\sim} X^\times(S_{\lambda'})$ defined in Corollary 8.17 induces an isomorphism between $X^\times(S_\lambda^{\text{der}})$ and $X^\times(S_{\lambda'}^{\text{der}})$ which we again call $i_{\lambda, \lambda'}$:

$$i_{\lambda, \lambda'} : X^\times(S_\lambda^{\text{der}}) \xrightarrow{\sim} X^\times(S_{\lambda'}^{\text{der}})$$

such that

$$i_{\lambda, \lambda'}(W_\lambda^{\text{ss}}) = W_{\lambda'}^{\text{ss}}.$$

Similarly, the isomorphism $r_{\lambda, \lambda'}$ in Corollary 8.18 induces an isomorphism

$$r_{\lambda, \lambda'} : X^\times((^{1+c_\lambda}S_\lambda)^{\text{der}}) \xrightarrow{\sim} X^\times((^{1+c_{\lambda'}}S_{\lambda'})^{\text{der}})$$

such that

$$r_{\lambda, \lambda'}((W_\lambda|_{^{1+c_\lambda}S_\lambda})^{\text{ss}}) = (W_{\lambda'}|_{^{1+c_{\lambda'}}S_{\lambda'}})^{\text{ss}}. \quad \square$$

Define $W_{c_\lambda}^{\text{ss}} \subseteq X^\times(T_\lambda^{\text{der}})_{\mathbb{Q}}^{c_\lambda=1}$ to be the multi-set:

$$W_{c_\lambda}^{\text{ss}} = \{(w + w^{c_\lambda})|_{T_\lambda^{\text{der}}} | w \in W_\lambda\}.$$

Then we have the following proposition:

Proposition 8.23. *For each pair λ, λ' of finite primes of M_π , there exist an isomorphism $i_{\lambda, \lambda'} : X^\times(T_\lambda^{\text{der}}) \xrightarrow{\sim} X^\times(T_{\lambda'}^{\text{der}})$ such that $i_{\lambda, \lambda'}(W_{c_\lambda}^{\text{ss}}) = W_{c_{\lambda'}}^{\text{ss}}$, and an isomorphism $r_{\lambda, \lambda'} : X^\times(T_\lambda^{\text{der}})_{\mathbb{Q}}^{c_\lambda=1} \xrightarrow{\sim} X^\times(T_{\lambda'}^{\text{der}})_{\mathbb{Q}}^{c_{\lambda'}=1}$ such that $r_{\lambda, \lambda'}(W_{c_\lambda}^{\text{ss}}) = W_{c_{\lambda'}}^{\text{ss}}$.*

Proof. By identifying T_λ^{der} with S_λ^{der} , we can view $i_{\lambda, \lambda'}$ as an isomorphism between $X^\times(T_\lambda^{\text{der}})$ and $X^\times(T_{\lambda'}^{\text{der}})$ that satisfies $i_{\lambda, \lambda'}(W_{c_\lambda}^{\text{ss}}) = W_{c_{\lambda'}}^{\text{ss}}$.

We have $X^\times({}^{1+c_\lambda}(S_\lambda^{\text{der}})) = X^\times({}^{1+c_\lambda}(T_\lambda^{\text{der}}))$. To prove the second part of the proposition, we first show that $({}^{1+c_\lambda}S_\lambda)^{\text{der}} = {}^{1+c_\lambda}(S_\lambda^{\text{der}})$.

Since we have the embeddings ${}^{1+c_\lambda}(S_\lambda^{\text{der}}) \hookrightarrow B_\lambda^{0, \text{der}}, {}^{1+c_\lambda}(S_\lambda^{\text{der}}) \hookrightarrow {}^{1+c_\lambda}S_\lambda$ and that ${}^{1+c_\lambda}(S_\lambda^{\text{der}})$ is connected, we get ${}^{1+c_\lambda}(S_\lambda^{\text{der}}) \subseteq ({}^{1+c_\lambda}S_\lambda)^{\text{der}}$. Now we show the containment in the other direction. Let $s \in ({}^{1+c_\lambda}S_\lambda)^{\text{der}}$, then $s = t^{c_\lambda}t$ for some $t \in S_\lambda$. Then $t = za$, for some $z \in Z(B_\lambda^{0, \text{der}})$, the center of $B_\lambda^{0, \text{der}}$, and $a \in S_\lambda^{\text{der}}$. Thus we have

$$s = t^{c_\lambda}t = z^{c_\lambda}za^{c_\lambda}a.$$

Since $s \in B_\lambda^{0, \text{der}}$ and $a \in B_\lambda^{0, \text{der}}$, we get $z^{c_\lambda}z \in B_\lambda^{0, \text{der}}$ and thus $z^{c_\lambda}z \in Z(B_\lambda^{0, \text{der}})$. As a result, there exists an integer j not depending on s such that $s^j = (a^{c_\lambda}a)^j \in {}^{1+c_\lambda}(S_\lambda^{\text{der}})$. So we have $(({}^{1+c_\lambda}S_\lambda)^{\text{der}})^j = {}^{1+c_\lambda}(S_\lambda^{\text{der}})$. Since both $({}^{1+c_\lambda}S_\lambda)^{\text{der}}$ and ${}^{1+c_\lambda}(S_\lambda^{\text{der}})$ are tori, we get ${}^{1+c_\lambda}(S_\lambda^{\text{der}}) = ({}^{1+c_\lambda}S_\lambda)^{\text{der}}$. Thus we have $X^\times(({}^{1+c_\lambda}S_\lambda)^{\text{der}}) = X^\times({}^{1+c_\lambda}(S_\lambda^{\text{der}})) = X^\times({}^{1+c_\lambda}(T_\lambda^{\text{der}}))$.

We have the surjection $T_\lambda^{\text{der}} \twoheadrightarrow {}^{1+c_\lambda}(T_\lambda^{\text{der}})$, which induces the inclusion $X^\times({}^{1+c_\lambda}(T_\lambda^{\text{der}})) \hookrightarrow X^\times(T_\lambda^{\text{der}})$, and it factors through $\iota : X^\times({}^{1+c_\lambda}(T_\lambda^{\text{der}})) \hookrightarrow X^\times(T_\lambda^{\text{der}})_{\mathbb{Q}}^{c_\lambda=1}$, defined by $\iota(w|_{{}^{1+c_\lambda}(T_\lambda^{\text{der}})}) = w + w^{c_\lambda}|_{T_\lambda^{\text{der}}}$ for $w \in X^\times(T_\lambda)$. Then by identifying $X^\times(T_\lambda^{\text{der}})_{\mathbb{Q}}^{c_\lambda=1}$ with $X^\times({}^{1+c_\lambda}(T_\lambda^{\text{der}}))_{\mathbb{Q}}$ and thus with $X^\times(({}^{1+c_\lambda}S_\lambda)^{\text{der}})_{\mathbb{Q}}$, and using the isomorphism $r_{\lambda, \lambda'}$ from Proposition 8.19, we can view $r_{\lambda, \lambda'}$ as an isomorphism $r_{\lambda, \lambda'} : X^\times(T_\lambda^{\text{der}})_{\mathbb{Q}}^{c_\lambda=1} \xrightarrow{\sim} X^\times(T_{\lambda'}^{\text{der}})_{\mathbb{Q}}^{c_{\lambda'}=1}$ such that $r_{\lambda, \lambda'}(W_{c_\lambda}^{\text{ss}}) = W_{c_{\lambda'}}^{\text{ss}}$. $\square$

9 Irreducibility of automorphic representations of $\mathrm{GL}_n, n \leq 6$

Recall we need to show the following proposition:

Proposition 9.1. *Let F be a CM field and π be a polarizable, regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where $n \leq 6$. Then for all but finitely many finite primes λ of M_π , and for each irreducible constituent $r \subseteq r_{\pi,\lambda}|_{G_\lambda^{0,\mathrm{der}}}$, $r^\sigma \cong r^\vee$ for every $\sigma \in G_{F^+} \setminus G_F$.*

Recall the set $\mathfrak{f}$ of finite primes of M_π from Section 7. By the remark after Corollary 7.2, we know $r_{\pi,\omega}|_{G_\omega^{0,\mathrm{der}}}$ is regular for $\omega \in \mathfrak{f}$. So $r_{\pi,\lambda}|_{G_\lambda^{0,\mathrm{der}}}$ is regular for all primes λ .

Let $\mathfrak{g}_\lambda$ be the Lie algebra of $G_\lambda^{0,\mathrm{der}}$ and let $q_\lambda : \mathfrak{g}_\lambda \rightarrow \mathfrak{gl}_n$ be the Lie algebra representation induced by $r_{\pi,\lambda}$. Then q_λ is a faithful regular representation of a semi-simple Lie algebra. We also use c_λ to denote the Lie algebra automorphism of $\mathfrak{g}_\lambda$ induced by c_λ on $G_\lambda^{0,\mathrm{der}}$. So we have $q_\lambda^{c_\lambda} \cong q_\lambda^\vee$ for every finite prime λ of M_π .

For $\omega \in \mathfrak{f}$, we know q_ω is irreducible by Corollary 7.2. The following proposition (adapted from Proposition 4.5 and the erratum in [4]) lists the possibilities for $\mathfrak{g}_\omega$ and q_ω when the dimension $n \leq 6$.

Proposition 9.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with an irreducible and faithful representation q of dimension $2 \leq n \leq 6$. Then $(\mathfrak{g}, q)$ is one of following. In each case, we also include whether q is self-dual. We use std to denote the standard representation of the given Lie algebra.*

- $n = 2$
 $\mathfrak{sl}_2 : \mathrm{std} \text{ (self-dual)}$
- $n = 3$
 $\mathfrak{sl}_2 : \mathrm{Sym}^2(\mathrm{std}) \text{ (self-dual)}$
 $\mathfrak{sl}_3 : \mathrm{std}, \mathrm{std}^\vee$
- $n = 4$
 $\mathfrak{sl}_2 : \mathrm{Sym}^3(\mathrm{std}) \text{ (self-dual)}$
 $\mathfrak{so}_4 : \mathrm{std} \text{ (self-dual)}$
 $\mathfrak{sp}_4 : \mathrm{std} \text{ (self-dual)}$
 $\mathfrak{sl}_4 : \mathrm{std}, \mathrm{std}^\vee$
- $n = 5$
 $\mathfrak{sl}_2 : \mathrm{Sym}^4(\mathrm{std}) \text{ (self-dual)}$
 $\mathfrak{so}_5 : \mathrm{std} \text{ (self-dual)}$
 $\mathfrak{sl}_5 : \mathrm{std}, \mathrm{std}^\vee$

- $n = 6$
- $\mathfrak{sl}_2 : \text{Sym}^5(\text{std})$ (*self-dual*)
- $\mathfrak{sl}_2 \times \mathfrak{sl}_2 : \text{std} \otimes \text{Sym}^2(\text{std})$ (*self-dual*)
- $\mathfrak{sl}_3 : \text{Sym}^2(\text{std}), \text{Sym}^2(\text{std}^\vee)$
- $\mathfrak{sl}_2 \times \mathfrak{sl}_3 : \text{std} \otimes \text{std}, \text{std} \otimes \text{std}^\vee$
- $\mathfrak{sp}_6 : \text{std}$ (*self-dual*)
- $\mathfrak{sl}_4 : \wedge^2(\text{std})$
- $\mathfrak{sl}_6 : \text{std}, \text{std}^\vee$

Remark. Note that in the $n = 6$ case, we use the Lie algebra isomorphism $D_3 \cong A_3$ to identify the standard representation of $\mathfrak{so}_6$ with $\wedge^2(\text{std})$ of $\mathfrak{sl}_4$.

We will prove the Lie algebra version of Proposition 9.1:

Proposition 9.3. *For $2 \leq n \leq 6$, if $q_\lambda : \mathfrak{g}_\lambda \rightarrow \mathfrak{gl}_n$ is reducible and $r \subseteq q_\lambda$ is an irreducible constituent, then $r^{c_\lambda} \cong r^\vee$.*

In the proof, we will combine Proposition 9.2 with the independence of λ results that we have proved in Proposition 8.23: the tuples $(X^\times(T_\lambda^{\text{der}}), W_\lambda^{\text{ss}})$ and $(X^\times(T_\lambda^{\text{der}})_{\mathbb{Q}}^{c_\lambda=1}, W_{c_\lambda}^{\text{ss}})$ are independent of λ , and with the knowledge that q_ω is faithful and irreducible for $\omega \in \mathfrak{f}$. In our arguments, we will consider all possibilities for $(\mathfrak{g}_\lambda, q_\lambda)$, and reduce the number of cases we need to consider by using the following observations:

- Since $\mathfrak{g}_\lambda$ is semi-simple, the only continuous character of $\mathfrak{g}_\lambda$ is the trivial one $\mathbb{I}$. It must have multiplicity no more than one as an irreducible constituent of q_λ by regularity. So we can rule out any representation with multiple 1-dimensional irreducible constituents.
- Suppose at some prime λ , $\mathfrak{g}_\lambda = \mathfrak{sl}_2$. Then by Corollary 8.20, we must have $h_\lambda = \mathfrak{sl}_2$ at all primes. Since $\mathfrak{sl}_2$ does not have non-trivial exterior automorphisms, c_λ must act as an inner automorphism. So $r^{c_\lambda} \cong r \cong r^\vee$ for each irreducible constituent $r \subseteq q_\lambda$. Thus we will exclude this case in the discussion below and discuss only cases where $\mathfrak{g}_\lambda$ is not $\mathfrak{sl}_2$.
- If q_λ has a unique irreducible constituent r of dimension d for some $d \geq 0$, then the same holds true for $q_\lambda^{c_\lambda}$ and $q_\lambda^\vee$. We have $r^{c_\lambda} \subseteq q_\lambda^{c_\lambda}$ and $r^\vee \subseteq q_\lambda^\vee$. So we must have $r^\vee \cong r^{c_\lambda}$ since $q_\lambda^\vee \cong q_\lambda^{c_\lambda}$. In particular, if $\dim(r) > n/2$ for some irreducible constituent r , then $r^{c_\lambda} \cong r^\vee$. We also do not need to analyze the cases where all irreducible constituents have distinct dimensions, since then it must be the case that all the irreducible constituents satisfy $r^{c_\lambda} \cong r^\vee$.
- Recall that c_λ is a semi-simple automorphism of $G_\lambda^{0, \text{der}}$ and it stabilizes a maximal torus T_λ^{der} and a

Borel N_λ^{der} . If c_λ acts as a trivial exterior automorphism (see Definition A.1), then the action of c_λ is given by conjugation by some element $b \in G_\lambda^{0,\text{der}}$. We have the following result from [9]:

Proposition 9.4. *Let G be a connected reductive group, T a maximal torus of G and B a Borel containing T . Let $N_G(B)$ be the normalizer of B in G and $N_G(T)$ be the normalizer of T in G . Then $N_G(T) \cap N_G(B) = N_G(T) \cap B = T$.*

Then we get $c_\lambda \in T_\lambda^{\text{der}}$ by Proposition 9.4. So $c_\lambda = 1 \in \text{Aut}(X^\times(T_\lambda^{\text{der}})_\mathbb{Q})$ and thus acts trivially on W_λ^{ss} .

Now we fix some notations. We write W_λ for W_λ^{ss} , V_λ for $X^\times(T_\lambda^{\text{der}})_\mathbb{Q}$, and W_{c_λ} for $W_{c_\lambda}^{\text{ss}}$. Without loss of generality, we may assume c_λ stabilizes the Cartan subalgebra of diagonal matrices for each of the semi-simple Lie algebra. Let $E_{i,i}$ denote the matrix that takes the basis vector e_i to itself and sends e_j to 0 for $j \neq i$. We denote by L_i the weight defined by $L_i(E_{j,j}) = \delta_{i,j}$. For a Lie algebra of the form $\mathfrak{sl}_2 \times \dots \times \mathfrak{sl}_2$, we use L^j to indicate the weight that comes from L_1 on the j -th copy of $\mathfrak{sl}_2$. For a direct product of simple Lie algebras, we use p_j to denote the projection onto the j -th factor.

Now we proceed to prove Proposition 9.3.

Proof. We analyze the possibilities for each dimension n .

- $n = 2$

We must have $\mathfrak{g}_\lambda = \mathfrak{sl}_2$ at all primes. So there is no need to analyze this case.

- $n = 3$

Let $\omega \in \mathfrak{f}$. We must have a standard representation of $\mathfrak{sl}_3$ at ω . By Corollary 8.20, at another prime λ , we have $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ or $\mathfrak{sl}_3$. If $\mathfrak{g}_\lambda = \mathfrak{sl}_3$, then ρ_λ is either a sum of three trivial characters, which is not regular, or it is the standard representation, which is irreducible. If $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, then q_λ is not faithful, because the smallest dimension of a faithful representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ is 4 (cf. 2.2 and Proposition 2.9 in [3]).

- $n = 4$

Let $\omega \in \mathfrak{f}$. One possibility is $\mathfrak{g}_\omega = \mathfrak{sl}_4$ with the standard representation or its dual. By Corollary 8.20, at another prime λ , $\mathfrak{g}_\lambda$ must be one of $\mathfrak{sl}_4, \mathfrak{sl}_2 \times \mathfrak{sl}_3, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2, \mathfrak{sp}_6, \mathfrak{so}_6$ and $\mathfrak{so}_7$. None but $\mathfrak{sl}_4$ has faithful representations of dimension ≤ 4 (cf. 2.2 and Proposition 2.9 in [3]). So the only possibilities are the standard representation of $\mathfrak{sl}_4$ or its dual, which are irreducible.

If $\mathfrak{g}_\omega$ is not $\mathfrak{sl}_4$ and $c_\omega \neq 1 \in \text{Aut}(V_\omega)$ then $\mathfrak{g}_\omega = \mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with the standard representation, and c_ω exchanges the two normal copies of $\mathfrak{sl}_2$. The weights W_ω of q_ω are $\{L^1 + L^2, L^1 - L^2, -L^1 + L^2, -L^1 - L^2\}$, and $W_{c_\omega} = \{2(L^1 + L^2), -2(L^1 + L^2), 0, 0\}$.

By Corollary 8.20, if q_λ is not irreducible, then it is one of the following: a direct sum of a trivial character and a 3-d irreducible representation, or a direct sum of two 2-d irreducible representations (where $\mathfrak{g}_\lambda = \mathfrak{sl}_2, \mathfrak{sl}_2 \times \mathfrak{sl}_2$). After excluding cases where $\mathfrak{g}_\lambda$ is $\mathfrak{sl}_2$ or where the irreducible constituents have distinct dimensions, we only need to analyze the case where $q_\lambda = \text{std} \circ p_1 \oplus \text{std} \circ p_2 : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_4$. If we have $c_\lambda = 1 \in \text{Aut}(V_\lambda)$, then we have $(\text{std} \circ p_i)^{c_\lambda} \cong \text{std} \circ p_i \cong (\text{std} \circ p_i)^\vee$ for each i . So we may assume $c_\lambda \neq 1 \in \text{Aut}(V_\lambda)$. Thus $c_\omega \neq 1 \in \text{Aut}(V_\omega)$ by Proposition 8.23. At λ , we have $W_\lambda = \{L^1, -L^1, L^2, -L^2\}$ and $W_{c_\lambda} = \{L^1 + L^2, L^1 + L^2, -(L^1 + L^2), -(L^1 + L^2)\}$. Note that $0 \in W_{c_\omega}$ while $0 \notin W_{c_\lambda}$. Thus by Proposition 8.23, this case cannot happen.

- $n = 5$

At a prime λ , if q_λ is reducible, then the possible dimensions of the irreducible constituents are 1+4, 2+3, 1+2+2 (where $\mathfrak{g}_\lambda = \mathfrak{sl}_2, \mathfrak{sl}_2 \times \mathfrak{sl}_2$). We only need to analyze the last case, since we do not need to consider the cases where $\mathfrak{g}_\lambda = \mathfrak{sl}_2$ or the case where the dimensions of the irreducible constituents are distinct. Let $\omega \in \mathfrak{f}$. By Corollary 8.20, the only possible situation is to have $\mathfrak{g}_\omega = \mathfrak{so}_5$ with $q_\omega = \text{std}$ and $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $q_\lambda = \mathbb{I} \oplus \text{std} \circ p_1 \oplus \text{std} \circ p_2$. Since $\mathfrak{so}_5$ has no non-trivial exterior automorphism, we have $c_\omega = 1 \in \text{Aut}(V_\omega)$. So we must have $c_\lambda = 1 \in \text{Aut}(V_\lambda)$ by Proposition 8.23. Therefore we have $(\text{std} \circ p_i)^{c_\lambda} \cong \text{std} \circ p_i \cong (\text{std} \circ p_i)^\vee$ for $i \in \{1, 2\}$ and $\mathbb{I}^{c_\lambda} \cong \mathbb{I}$.

- $n = 6$

Let λ be a prime where q_λ is reducible. The possible dimensions of the irreducible constituents are 1+5, 3+3 (where $\mathfrak{g}_\lambda = \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{sl}_2 \times \mathfrak{sl}_2, \mathfrak{sl}_2 \times \mathfrak{sl}_3, \mathfrak{sl}_3 \times \mathfrak{sl}_3$), 2+2+2 (where $\mathfrak{g}_\lambda = \mathfrak{sl}_2, \mathfrak{sl}_2 \times \mathfrak{sl}_2, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$), 2+4 and 1+2+3. We do not need to consider any of the cases where $\mathfrak{g}_\lambda$ is $\mathfrak{sl}_2$, or where all the irreducible constituents have distinct dimensions. By Corollary 8.20, we also do not need to consider the case where $h_\lambda = \mathfrak{sl}_3 \times \mathfrak{sl}_3$ since h_ω cannot have rank 4 for $\omega \in \mathfrak{f}$. In the 2+2+2 case, if $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, then q_λ must have duplicate weights, which contradicts regularity. So the possibilities left are $\mathfrak{g}_\lambda = \mathfrak{sl}_3$ with $q_\lambda = \text{std} \oplus \text{std}^\vee$, $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $q_\lambda = \text{Sym}^2(\text{std}) \circ p_1 \oplus \text{Sym}^2(\text{std}) \circ p_2$, $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_3$ with $q_\lambda = \text{Sym}^2(\text{std}) \circ p_1 \oplus \text{std} \circ p_2$ or $q_\lambda = \text{Sym}^2(\text{std}) \circ p_1 \oplus \text{std}^\vee \circ p_2$ and $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $q_\lambda = \text{std} \circ p_1 \oplus \text{std} \circ p_2 \oplus \text{std} \circ p_3$.

Suppose we have $\mathfrak{g}_\lambda = \mathfrak{sl}_3$ with $q_\lambda = \text{std} \oplus \text{std}^\vee$. Then $W_\lambda = \{L_1, L_2, -(L_1 + L_2), -L_1, -L_2, L_1 + L_2\}$. Thus we must also have $W_\omega = -W_\omega$ for $\omega \in \mathfrak{f}$. By Corollary 8.20, the only possibility is $\mathfrak{g}_\omega = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $q_\omega = \text{std} \otimes \text{Sym}^2(\text{std})$. Note we have $L_1 + L_2 \in W_\lambda$ at λ . However, for $\mathfrak{g}_\omega = \mathfrak{sl}_2 \times \mathfrak{sl}_2$, we have $W_\omega = \{L^1 + 2L^2, -(L^1 + 2L^2), L^1, -L^1, L^1 - 2L^2, -(L^1 - 2L^2)\}$, and no two elements in W_ω add to another element in W_ω . This contradicts Proposition 8.23.

Suppose we have $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $q_\lambda = \text{Sym}^2(\text{std}) \circ p_1 \oplus \text{Sym}^2(\text{std}) \circ p_2$. Then $W_\lambda =$

$\{2L^1, 0, -2L^1, 2L^2, 0, -2L^2\}$. In particular, q_λ is not regular, which rules out the existence of this case. Suppose we have $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_3$ with $q_\lambda = \text{Sym}^2(\text{std}) \circ p_1 \oplus \text{std} \circ p_2$ or $q_\lambda = \text{Sym}^2(\text{std}) \circ p_1 \oplus \text{std}^\vee \circ p_2$. Since $q_\lambda^{c_\lambda} \cong q_\lambda^\vee$, and since $\text{Sym}^2(\text{std}) \circ p_1$ is self-dual while $\text{std} \circ p_2$ or $\text{std}^\vee \circ p_2$ is not, we must have c_λ stabilizes each constituent. So this case is okay.

We are left with the case where $\mathfrak{g}_\lambda = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $q_\lambda = \text{std} \circ p_1 \oplus \text{std} \circ p_2 \oplus \text{std} \circ p_3$. If c_λ acts as an inner automorphism, then each constituent satisfies $r^{c_\lambda} \cong r \cong r^\vee$. Otherwise c_λ interchanges two of the three copies of $\mathfrak{sl}_2$. Without loss of generality, we may assume c_λ exchanges the first two copies of $\mathfrak{sl}_2$. Thus $W_\lambda = \{L^1, -L^1, L^2, -L^2, L^3, -L^3\}$ and $W_{c_\lambda} = \{L^1 + L^2, L^1 + L^2, -(L^1 + L^2), -(L^1 + L^2), 2L^3, -2L^3\}$. By Corollary 8.20, the possibilities for $\omega \in \mathfrak{f}$ are $\mathfrak{g}_\omega = \mathfrak{sp}_6, \mathfrak{sl}_2 \times \mathfrak{sl}_3$ or $\mathfrak{sl}_4$. In the first case, we must have c_ω act as an inner automorphism, and thus $c_\omega = 1 \in \text{Aut}(V_\omega)$. In the second case, we have $W_\omega \neq -W_\omega$. So we can rule out the first two cases by Proposition 8.23. The last case left is $h_\omega = \mathfrak{sl}_4$ with $q_\omega = \wedge^2(\text{std})$. If we denote the weights of the standard representation of $\mathfrak{sl}_4$ by $\{L_1, L_2, L_3, -(L_1 + L_2 + L_3)\}$. Then we have $W_\omega = \{L_1 + L_2, L_1 + L_3, -(L_2 + L_3), L_2 + L_3, -(L_1 + L_3), -(L_1 + L_2)\}$. Since $c_\omega \neq 1 \in \text{Aut}(V_\omega)$ stabilizes the diagonal torus and a Borel containing it, c_ω has to be of the form

$$c_\omega(A) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} {}^t A^{-1} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Thus we have $c_\omega(L_1) = L_1 + L_2 + L_3, c_\omega(L_2) = -L_3$ and $c_\omega(L_3) = -L_2$. So we get $W_{c_\omega} = \{L_1 + L_2, -(L_1 + L_2), L_1 + L_3, -(L_1 + L_3), L_2 + L_3, -(L_2 + L_3)\}$, which spans a subspace of V_ω of dimension 3. But W_{c_λ} spans a 2-dimensional subspace of V_λ . Again by Proposition 8.23, this case cannot happen. $\square$

A Density of c_λ -semi-simple elements

We generalize the result in [20] to prove there is an open dense subset of elements in G_λ^0 that is c_λ -semi-simple (see Definition A.3). Even though the result in Lemma 4 in [20] does not directly apply to our situation because the assumptions are not met, we can generalize the argument in the proof to make it apply to our case.

We first recall the assumptions made in [20]: G is a semi-simple and simply connected algebraic group and δ is an exterior automorphism of G , which in particular implies it fixes a pinning (see Definition A.1 for definition). The author of [20] shows in that case there exists an open dense subset of G that consists of δ -semi-simple elements (see Definition A.3). We will generalize the result to the case where G is a connected reductive group, and δ is a semi-simple automorphism, which are weaker assumptions than in [20].

We first recall the definition of an exterior automorphism:

Definition A.1. *Let G be a connected reductive group, T a maximal torus and B a Borel subgroup containing it. For each root $\alpha \in X^\times(T)$ in the root system of G with respect to T , let T_α be the identity component of the kernel of α . Let G_α be the derived subgroup of the centralizer of T_α in G . Then we can associate to α a homomorphism of algebraic groups $x_\alpha : \mathbb{G}_a \rightarrow G_\alpha$ such that $\text{Inn}(t) \circ x_\alpha(u) = x_\alpha(\alpha(t)u)$ for all $t \in T$, where $\text{Inn}(t)$ is the inner automorphism defined by conjugating by t . Let $U_\alpha = \text{Im}(x_\alpha)$. Order the roots so that B is generated by T and $\{U_\alpha\}_{\alpha > 0}$. Let D be the set of simple roots. For each $\beta \in D$, fix $u_\beta \neq 0 \in U_\beta$.*

Let γ be an automorphism of G . If there exist $T \subseteq B, \{u_\beta\}_{\beta \in D}$ such that

$$\gamma(T, B, \{u_\beta\}_{\beta \in D})\gamma^{-1} = (T, B, \{u_\beta\}_{\beta \in D}),$$

then γ is called an exterior automorphism of G .

From now on we fix a reductive group G over an algebraically closed field and a semi-simple automorphism δ of G . Then it will stabilize a maximal torus T and a Borel B containing it by Proposition 8.7. Following the construction in Definition A.1 for T and B , we obtain the tuple $(T, B, \{u_\beta\}_{\beta \in D})$. We have the following well-known result [19]:

Proposition A.2. *There exists $s \in T$ such that $\sigma := \delta \circ \text{Inn}(s^{-1})$ is an exterior automorphism of G stabilizing $(T, B, \{u_\beta\}_{\beta \in D})$.*

The following definitions are from [20].

Definition A.3. *We define the δ -twisted conjugation action of G on itself by $g \star x = gx(\delta g)^{-1}$. Two elements in the same orbit are said to be δ -conjugate. We say an element $g \in G$ is δ -semi-simple if it is δ -conjugate*

to an element of T .

Definition A.4. The δ -centralizer of $g \in G$ is $Z_\delta(g) = \{x \in G | xg(\delta x)^{-1} = g\}$.

Definition A.5. We further assume G is semi-simple and simply connected. Let T^δ be the elements in T that is fixed by δ , which is a connected torus (see 8.2 in [21]). Let ${}^{\delta^{-1}}T \subseteq T$ be the subtorus of the elements of the form $t^{-1}(\delta t)$ where $t \in T$. Let $\Gamma_\delta = {}^{\delta^{-1}}T \cap T^\delta$, which we view as a (possibly non-reduced) subgroup scheme of T . We let Γ_{red} denote the reduced scheme.

Remark. We have the short exact sequence

$$0 \rightarrow T^\delta \rightarrow T \rightarrow {}^{\delta^{-1}}T \rightarrow 0.$$

Since $T = {}^{\delta^{-1}}T \cdot T^\delta$, we get Γ_δ is finite group scheme.

Note that the action of δ on G induces an action on the Weyl group W of G with respect to T . We let W^δ denote the fixed points. We have an action of $\Gamma_{\text{red}} \rtimes W^\delta$ on T^δ given by

$$(\gamma, w) \cdot t = w \cdot (\gamma t).$$

We will prove the following:

Lemma A.6. Let G be a connected reductive group and δ a semi-simple automorphism of G . Then the set of δ -semi-simple elements contains an open dense subset of G .

We first generalize Lemma 3 and Lemma 4 in [20]. For the convenience of the reader, we include the statements of the two lemmas below:

Lemma A.7 (Lemma 3 in [20]). Let H be a simply connected semi-simple group over an algebraically closed field and γ be an exterior automorphism of H . Thus γ stabilizes a maximal torus T of H . Then

- (1) $t, t' \in T^\gamma$ lie in the same γ -orbit if and only if t, t' lie in the same $\Gamma_{\text{red}} \rtimes W^\gamma$ -orbit in T^γ ,
- (2) there exists an open dense subset A of T^γ such that for $t \in A$, we have $Z_\gamma(t) = T^\gamma$.

Lemma A.8 (Lemma 4 in [20]). Keep the assumptions of Lemma A.7. The set of γ -semi-simple elements contains an open dense subset of H .

Let G, T and δ be as given in the paragraph preceding Proposition A.2. Let $s \in T$ be the element given in Proposition A.2 so that $\sigma := \delta \circ \text{Inn}(s^{-1})$ is an exterior automorphism of G . First we will prove a generalization of Lemma A.7.

Lemma A.9. *We further assume G is semi-simple and simply connected. Let a denote $(\sigma s)^{-1}$. Then*

- (1) *$t, t' \in T^\sigma a$ lie in the same δ -orbit if and only if $ta^{-1}, t'a^{-1}$ lie in the same $\Gamma_{\text{red}} \rtimes W^\delta$ -orbit in T^σ ,*
- (2) *there exists an open dense subset I of $T^\sigma a$ such that for $t \in I$, we have $Z_\delta(t) = T^\sigma$.*

Proof. Since G is semi-simple simply connected and σ is an exterior automorphism, Lemma A.7 and Lemma A.8 apply to σ .

1. By Lemma A.7. (1), elements in T^σ lie in the same σ -orbit in G if and only if they lie in the same $\Gamma_{\text{red}} \rtimes W^\sigma$ -orbit in T^σ . We also have the actions of σ and δ are the same on T , so we have $T^\sigma = T^\delta$, $W^\sigma = W^\delta$ and $\Gamma_\sigma = \Gamma_\delta$. Suppose there exist $g \in G$ such that

$$t' = gt(\delta g)^{-1} = gt^\sigma(sgs^{-1})^{-1} = gt(\sigma s)(\sigma g)^{-1}(\sigma s)^{-1},$$

then $t'a^{-1} = t'\sigma s = gt(\sigma s)(\sigma g)^{-1} = g(ta^{-1})(\sigma g)^{-1}$. Note that $ta^{-1}, t'a^{-1} \in T^\sigma$. So $ta^{-1}, t'a^{-1}$ lie in the same σ -orbit in G . Similarly, if $t'a^{-1} = g(ta^{-1})(\sigma g)^{-1}$ for some $g \in G$. Then

$$t' = g(ta^{-1})(\sigma g)^{-1}a = gt(\sigma s)(\sigma g)^{-1}(\sigma s)^{-1} = gt^\sigma(sgs^{-1})^{-1} = gt(\delta g)^{-1}.$$

Thus by applying Lemma A.7. (1) to σ , we get $ta^{-1}, t'a^{-1}$ lie in the same $\Gamma_{\text{red}} \rtimes W^\delta$ -orbit in T^σ .

2. Let A be the set given by Lemma A.7. (2) when applied to σ . Then we have for $t \in A$

$$Z_\delta(ta) = \{x \in G | xta(\delta x)^{-1} = ta\} = \{x \in G | xta(\sigma(sxs^{-1}))^{-1} = ta\}$$

$$= \{x \in G | xta(\sigma s)^\sigma x^{-1}a = ta\} = \{x \in G | xt(\sigma x)^{-1} = t\} = Z_\sigma(t) = T^\sigma.$$

Thus we can let $I = Aa$. □

Now we prove the following lemma generalizing Lemma A.8:

Lemma A.10. *Let G be a semi-simple simply connected group and δ a semi-simple automorphism of G . Then the set of δ -semi-simple elements contains an open dense subset of G .*

Proof. Let $\phi : G \times T^\sigma a \rightarrow G$ be the morphism that sends (g, t) to $gt(\delta g)^{-1}$. Let Z be the closure of the image. Let d_g be the dimension of the generic fiber of ϕ . Then we have $\dim Z \geq \dim G + \dim T^\sigma a - d_g$. Let $t \in I$, and let $o \in G$ be δ -conjugate to t . Then there exists $e \in G$ such that $o = et(\delta e)^{-1}$. For each

$w \in \Gamma_{\text{red}} \rtimes W^\delta$, there exists $e_w \in G$ such that $w(ta^{-1})a = e_w t({}^\delta e_w)^{-1}$. We have

$$\begin{aligned} \phi^{-1}(o) &= \phi^{-1}(et({}^\delta e)^{-1}) \\ &= \{(h, u) \in G \times T^\sigma a | hu({}^\delta h)^{-1} = et({}^\delta e)^{-1}\} \\ &= \{(h, u) \in G \times T^\sigma a | (h^{-1}e)t({}^\delta(h^{-1}e))^{-1} = u\} \end{aligned} \quad (21)$$

$$= \bigcup_{w \in \Gamma_{\text{red}} \rtimes W^\delta} \{(h, w(ta^{-1})a) \in G \times T^\sigma a | e_w^{-1}h^{-1}e \in Z_\delta(t)\}, \quad (22)$$

where going from Eq. 21 to Eq. 22 follows from the fact that u and t lie in the same (finite) δ -orbit in $T^\sigma a$ and from Lemma A.9. (1). Thus by Lemma A.9. (2), we have $\dim(\phi^{-1}(o)) \leq \dim Z_\delta(t) + \dim(\Gamma_{\text{red}} \rtimes W^\delta) = \dim Z_\delta(t) = \dim T^\sigma$. Since this is true for any element that is δ -conjugate to an element in I , we get d_g equals $\dim T^\sigma$. Thus we get $\dim Z \geq \dim G + \dim T^\sigma a - \dim T^\sigma = \dim G$. Therefore we can conclude $Z = G$. $\square$

Lemma A.11. *Let G be a semi-simple group and δ a semi-simple automorphism. Then the set of δ -semi-simple elements contains an open dense subset of G .*

Proof. Let $\tilde{G}$ be the simply connected cover of G . Then we have a (central) isogeny $\eta : \tilde{G} \rightarrow G$ and δ lifts to a semi-simple automorphism $\tilde{\delta}$ on $\tilde{G}$. Moreover, there exists a maximal torus $\tilde{T}$ of $\tilde{G}$ that is stabilized by $\tilde{\delta}$ (cf 2.4 in [19]). Let $\tilde{S}$ be the open dense subset of $\tilde{G}$ we get by applying Lemma A.10 to $\tilde{\delta}$ and $\tilde{T}$. Let $\tilde{s} \in \tilde{S}$, then there exist $\tilde{g} \in \tilde{G}$ and $\tilde{t} \in \tilde{T}$ such that $\tilde{s} = \tilde{g}\tilde{t}(\tilde{\delta}\tilde{g})^{-1}$. We have $\eta(\tilde{s}) = \eta(\tilde{g}\tilde{t}(\tilde{\delta}\tilde{g})^{-1}) = gt^\delta g^{-1}$. Let K be the set of δ -semi-simple elements in G . We have $\eta(\tilde{S}) \subseteq K$ is an open subset of G , and $\overline{\eta(\tilde{S})} \supseteq \overline{\eta(\tilde{S})} = G$. $\square$

We can now prove Lemma A.6:

Proof. Let G' be the derived subgroup of G and $Z(G)^0$ be the identity component of the center of G . Then we have an isogeny $p : G' \times Z(G)^0 \rightarrow G$, and δ restricts to a semi-simple automorphism of G' stabilizing a maximal torus $T' \subset G'$. Note that $T := T' \cdot Z(G)^0$ is a maximal torus G stabilized by δ . Let S' be the open dense subset of G' we get by applying Lemma A.11 to $\delta|_{G'}$. Then we get $p(S' \times Z(G)^0) \subseteq G$ is an open dense subset consisting of δ -semi-simple elements and moreover $\overline{p(S' \times Z(G)^0)} \supseteq p(\overline{S'} \times \overline{Z(G)^0}) = G$. $\square$

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