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Stability for area in codimension one

Stryker, Douglas

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Stability for area in codimension one

Douglas Stryker

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Fernando Codá Marques

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Abstract

We present three results investigating aspects of the stability of minimal hypersurfaces.

First, in joint work with Otis Chodosh, Chao Li, and Paul Minter, we prove that every complete oriented stable minimal hypersurface in $\mathbb{R}^5$ is a hyperplane, which settles the stable Bernstein problem in dimension 5. Some consequences of this classification are curvature estimates for stable minimal hypersurfaces in 5-dimensional Riemannian manifolds.

Second, in joint work with Otis Chodosh and Chao Li, we prove a rigidity result for complete two-sided stable minimal hypersurfaces in 4-dimensional manifolds with bounded and nonnegative sectional curvature and uniformly positive scalar curvature. We use this rigidity result to prove new topological obstructions for complete noncompact 4-dimensional manifolds satisfying these curvature assumptions.

Finally, we present a new localization technique to prove the existence of minimal hypersurfaces in some complete noncompact manifolds. The minimal hypersurfaces produced by these methods are not necessarily stable, but they are quantitatively almost stable (index at most one). This result can be viewed as the sharp generalization to complete noncompact manifolds of the successful min-max theory of minimal hypersurfaces in closed manifolds.

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Chapter 1

Introduction

The variational calculus of area is a fundamental subject in differential geometry. The central characters in this study are *minimal hypersurfaces*. Inside an $(n+1)$-dimensional Riemannian manifold (M, g), a hypersurface $\Sigma \subset M$ (a submanifold of dimension n, or equivalently, codimension 1) is *minimal* if

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(\Sigma_t) = 0$$

for any smooth family of hypersurfaces $\{\Sigma_t\}_{t\in(-1,1)}$ satisfying $\Sigma_0 = \Sigma$ and $\Sigma_t \setminus K = \Sigma \setminus K$ for each $t \in (-1, 1)$ and for some compact set $K \subset M$. In the language of calculus, minimal hypersurfaces are exactly the *critical points* of the area functional.

The Euler-Lagrange equation for critical points of the area functional is the nonlinear geometric elliptic partial differential equation $H_\Sigma = 0$, where H_Σ is the mean curvature of the hypersurface (the trace of the second fundamental form A_Σ). The prototypical nature of this equation merits interest in the study of the existence and behavior of solutions.

An essential tool in the study of critical points of a function is the second derivative. The "second derivative test" from calculus can be used to detect whether a given critical point is a local minimum/maximum, and in general provides insight about

the local behavior of the function near the critical point. For a two-sided[1] minimal hypersurface, the second derivative of the area along a variation of Σ with speed $\phi\nu$, where $\phi \in C^\infty_{\text{cpt}}(\Sigma)$ and ν is a continuous unit normal vector field, is given by

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(\Sigma_t) = \int_\Sigma [|\nabla \phi|^2 - (|A_\Sigma|^2 + \text{Ric}_M(\nu,\nu))\phi^2] \, d\,\text{Area}_\Sigma,$$

where A_Σ is the second fundamental form of the hypersurface Σ and Ric_M is the Ricci curvature of the ambient space.

A main focus for the results in this thesis are stable minimal hypersurfaces. A minimal hypersurface is *stable* if the second derivative of the area is nonnegative for any normal variation. For a two-sided minimal hypersurface, stability is equivalent to the condition that

$$\int_\Sigma [|\nabla \phi|^2 - (|A_\Sigma|^2 + \text{Ric}_M(\nu,\nu))\phi^2] \, d\,\text{Area}_\Sigma \geq 0$$

for any $\phi \in C^\infty_{\text{cpt}}(\Sigma)$.

1.1 Discussion of results

In this thesis, we present three results that investigate different aspects of the stability of critical points for area (and related functionals). Here we provide a short overview of the results of these chapters; a more detailed introduction to the context of these results and the ideas of the proofs can be found at the beginning of each chapter.

[1] A hypersurface is two-sided if it admits a continuous unit normal vector field.

1.1.1 Classification of stable minimal hypersurfaces in Euclidean space

The behavior of complete stable minimal hypersurfaces in Euclidean space is the model for the small-scale behavior of stable minimal hypersurfaces in general Riemannian manifolds, as they arise from zooming in on a stable minimal hypersurface inside a Riemannian manifold. Consequently, the study of stable minimal hypersurfaces in Euclidean space is fundamental.

In Chapter 2, we present joint work with Otis Chodosh, Chao Li, and Paul Minter (see [CLMS24]) resolving the stable Bernstein problem in $\mathbb{R}^5$. This result classifies all complete oriented stable minimal hypersurfaces in $\mathbb{R}^5$. Specifically, we show:

Theorem 1.1.1. *Every complete oriented stable minimal hypersurface in $\mathbb{R}^5$ is a hyperplane.*

This problem has a long and interesting history. Since the 1960s, it was known due to work of [BDGG69] that the identical statement is false in $\mathbb{R}^n$ for $n \geq 8$; namely, there are complete oriented stable minimal hypersurfaces in these dimensions that are not flat. Around 1980, three groups independently proved this classification statement in $\mathbb{R}^3$ [dCP79, FCS80, Pog81]. It was not until the present decade that further conclusive progress was made. In [CL24b], Chodosh and Li proved the classification in $\mathbb{R}^4$; different proofs were subsequently given by [CL23, CMR24]. After our work in $\mathbb{R}^5$ [CLMS24], Mazet adapted our techniques to prove the classification in $\mathbb{R}^6$ [Maz24]. The question in $\mathbb{R}^7$ remains open as of the writing of this thesis.

Besides the interesting history, a good reason to care about this classification result is that it supplies useful curvature estimates for stable minimal hypersurfaces in Riemannian manifolds:

Corollary 1.1.2. *There is a constant $C(K) < \infty$ such if (M^5, g) is a complete 5-dimesional Riemannian manifold with $|\mathrm{sec}_g| \leq K$ and $\Sigma^4 \to (M^5, g)$ is a compact*

two-sided stable minimal hypersurface, then

$$\sup_{q\in\Sigma} |A_\Sigma(q)| \min\{1, d_\Sigma(q, \partial\Sigma)\} \leq C(K).$$

The proof in $\mathbb{R}^5$ makes use of the exciting developments in scalar curvature using μ-bubbles, as introduced by Gromov in [Gro96]. The essential new insight of the proof in $\mathbb{R}^5$ is the central role played by the intrinsic *bi-Ricci* curvature of stable minimal hypersurfaces, as a replacement for the role played by the intrinsic scalar curvature of stable minimal hypersurfaces in $\mathbb{R}^4$.

1.1.2 Rigidity of noncompact stable minimal hypersurfaces in positive curvature

Equipped with constraints on the local behavior of stable minimal hypersurfaces coming from the classification of stable minimal hypersurfaces in Euclidean space, it is natural to investigate how these objects can be used to study the geometry and topology of Riemannian manifolds. A typical goal in Riemannian geometry is to deduce geometric or topological restrictions on Riemannian manifolds with metrics that satisfy curvature restrictions, as a way to implicitly understand the meaning of curvature.

In Chapter 3, we present joint work with Otis Chodosh and Chao Li (see [CLS22]) demonstrating the rigidity of stable minimal hypersurfaces in 4-dimensional manifolds satisfying certain positive curvature assumptions, which we use to deduce new conclusions about the topology of 4-dimensional manifolds satisfying those curvature assumptions. Specifically, we show:

Theorem 1.1.3. *Let (M^4, g) be a complete 4-dimensional Riemannian manifold satisfying $0 \leq \sec \leq C < +\infty$ and $R \geq 1$. Then any complete two-sided stable minimal*

hypersurface Σ *in* M *satisfies*

$$A_\Sigma \equiv 0 \quad \text{and} \quad \mathrm{Ric}_M(\nu,\nu)\mid_\Sigma \equiv 0.$$

An immediate corollary is the nonexistence of complete two-sided stable minimal hypersurfaces under the additional assumption $\mathrm{Ric}_M > 0$. We use this nonexistence conclusion to deduce the following topological consequence:

Corollary 1.1.4. *Suppose* (M^4, g) *is a complete noncompact oriented 4-dimensional Riemannian manifold satisfying* $0 \leq \sec \leq C < +\infty$, $\mathrm{Ric} > 0$, *and* $R \geq 1$. *Then the first betti number of* M *is zero.*

These rigidity results also make use of the developments of Gromov's μ-bubbles [Gro96]. Our argument is the first to study the volume behavior of manifolds with positive scalar curvature using μ-bubbles.

1.1.3 Existence of index one prescribed mean curvature hypersurfaces in noncompact manifolds

Despite the nonexistence results for stable minimal hypersurfaces presented above, the existence question for minimal hypersurfaces (without the stability assumption) is still fundamentally important. One way to avoid nonexistence for stable minimal hypersurfaces is to slightly weaken the demand to find minimal hypersurfaces that have index at most one. A minimal hypersurface has *index* k if the maximal dimension of a subspace of the space of normal variations on which the second derivative of area is negative definite is k. An index zero minimal hypersurface is stable. An index one minimal hypersurface is not stable, but there is only one direction in which the second derivative of area is negative; it is stable in all other directions.

The possibility of nonexistence for stable minimal hypersurfaces also implies that minimization of area cannot always be used to produce nontrivial examples of minimal

hypersurfaces. Instead, a min-max program was developed by [Alm62, Pit81] to produce closed minimal hypersurfaces in closed Riemannian manifolds. By the work of [MN16], it is always possible to find a closed minimal hypersurface with index at most one in any closed Riemmanian manifold of dimension $3 \leq n+1 \leq 7$.

Despite the dramatic success of min-max techniques for producing minimal hypersurfaces in closed manifolds, the existence question in complete noncompact manifolds is still a significant challenge. The problem in the noncompact setting is a consequence of the reliance of min-max techniques on compactness results for submanifolds; in the noncompact setting, sequences of submanifolds may drift off to infinity.

In Chapter 4, we present a new localization technique that can be used to prove the existence of complete minimal hypersurfaces with index at most one in some complete noncompact manifolds (see [Str23, Str24]). For example, we give a new proof of the following result originally due to [CL20] and [Son19]:

Theorem 1.1.5. *Let (M^{n+1}, g) be a complete finite volume Riemannian manifold of dimension $3 \leq n+1 \leq 7$. Then there exists a complete finite area minimal hypersurface Σ in M with index at most one.*

One advantage of the localization technique developed here is its applicability to the prescribed mean curvature functional. For a locally smooth function $h : M \to \mathbb{R}$ and an open subset $\Omega \subset M$ with smooth boundary, we define the *prescribed mean curvature functional* by

$$\mathcal{A}^h(\Omega) = \text{Area}(\partial\Omega) - \int_\Omega h.$$

The Euler-Lagrange equation for critical points of this functional is $H_{\partial\Omega} = h\,|_{\partial\Omega}$. In other words, critical points of $\mathcal{A}^h$ have *prescribed mean curvature* h. For one example, the localization technique developed here can be used to produce hypersurfaces of constant mean curvature in finite volume manifolds:

Theorem 1.1.6. *Let (M^{n+1}, g) be a complete finite volume Riemannian manifold of*

dimension $3 \leq n+1 \leq 7$. *Then there is a positive constant* $\varepsilon(M,g) > 0$, *depending explicitly on the geometry of* (M,g), *so that there exists a complete finite area hypersurface* Σ *in* M *with constant mean curvature* c *and* $\mathcal{A}^c$*-index at most one for any* $|c| \leq \varepsilon(M,g)$.

There are complete finite volume manifolds satisfying $\varepsilon(M,g) = +\infty$.

If (M,g) *is hyperbolic, then* $\varepsilon(M,g) \geq n$.

Chapter 2

Classification of stable minimal hypersurfaces in Euclidean space

2.1 Introduction

A two-sided immersion $M^n \to \mathbb{R}^{n+1}$ is *minimal* if its mean curvature vector vanishes. A minimal immersion is *stable* if

$$\int_M |A_M|^2 \varphi^2 \leq \int_M |\nabla \varphi|^2$$

for all $\varphi \in C_c^\infty(M)$, where A_M is the second fundamental form of the immersion. The stable Bernstein problem asks whether a complete two-sided stable minimal hypersurface in $\mathbb{R}^{n+1}$ must be a flat affine hyperplane. We resolve here the stable Bernstein problem in $\mathbb{R}^5$:

Theorem 2.1.1. *A complete, connected, two-sided stable minimal immersion $M^4 \to \mathbb{R}^5$ is a flat affine hyperplane.*

The stable Bernstein problem was resolved in $\mathbb{R}^3$ by do Carmo–Peng, Fischer-Colbrie–Schoen, and Pogorelov [dCP79, FCS80, Pog81] and recently in $\mathbb{R}^4$ by the

first- and second-named authors [CL24b] (subsequently, two alternative proofs in $\mathbb{R}^4$ were found in [CL23] and [CMR24]). After the first version of this paper appeared, Mazet was able to refine our methods to resolve the stable Bernstein problem in $\mathbb{R}^6$ in the affirmative [Maz24] (see Remark 2.1.7). The stable Bernstein problem remains open $\mathbb{R}^7$ but holds assuming M has (extrinsic) Euclidean volume growth by work of Schoen–Simon and Simons [SS81, Sim67] (for embeddings) and Bellettini [Bel25] (for immersions). On the other hand, non-flat stable (area-minimizing) minimal immersions in $\mathbb{R}^8$ (and beyond) were found by Bombieri–de Giorgi–Giusti [BDGG69].

It is well known that the validity of the stable Bernstein property is equivalent to an *a priori* interior curvature bound for stable minimal hypersurfaces. We recall that a two-sided minimal immersion into a Riemannian manifold $M^n \to (X^{n+1}, g)$ is stable if

$$\int_M (|A_M|^2 + \mathrm{Ric}_g(\nu, \nu))\varphi^2 \leq \int_M |\nabla \varphi|^2$$

for all $\varphi \in C_c^\infty(M)$, where A_M is the second fundamental form of M and $\mathrm{Ric}_g(\nu, \nu)$ is the Ricci curvature of the ambient metric g in the normal direction. As in [CLS22, Corollary 2.5], Theorem 2.1.1 implies interior curvature estimates for stable minimal immersions in 5-manifolds that only depend on a norm for the ambient sectional curvature.

Corollary 2.1.2. *Let (X^5, g) be a complete Riemannian manifold with bounded sectional curvature $|\mathrm{sec}_g| \leq K$. Then any compact, two-sided stable minimal immersion $M^4 \to (X^5, g)$ satisfies*

$$|A_M|(q) \min\{1, d_M(q, \partial M)\} \leq C(K)$$

for all $q \in M$.

To prove Theorem 2.1.1, show that a complete, two-sided, stable minimal immersion $M^4 \to \mathbb{R}^5$ has intrinsic Euclidean volume growth. In fact, as in [CL23], this

can be localized in the spirit of Pogorelov's area bounds [Pog81] for stable minimal surfaces in $\mathbb{R}^3$.

Theorem 2.1.3. *Let $F : M^4 \to \mathbb{R}^5$ be a simply connected, two-sided stable minimal immersion so that $F(x_0) = 0 \in \mathbb{R}^5$ for some $x_0 \in M$, ∂M is connected, and $F : M \to B_{\mathbb{R}^5}(0,1)$ is proper. Then*

$$\mathcal{H}^4(M^*_{\rho_0}) \leq 8\pi^2,$$

where $M^*_{\rho_0}$ is the connected component of $F^{-1}(B_{\mathbb{R}^5}(0,\rho_0))$ containing x_0 and $\rho_0 = e^{-11\pi}$.

Similarly, we can give a geometric characterization of minimal hypersurfaces in $\mathbb{R}^5$ with finite Morse index, generalizing the well-known results of Gulliver, Fischer-Colbrie, and Osserman in $\mathbb{R}^3$ [Gul86, FC85, Oss64]. See [CL24b] for the corresponding result in $\mathbb{R}^4$. Recall that a two-sided minimal immersion $M^4 \to \mathbb{R}^5$ has *finite Morse index* if

$$\sup\{\dim V : V \subset C^\infty_c(M) \text{ is a subspace with } Q(f,f) < 0 \text{ for all } 0 \neq f \in V\} < \infty$$

where $Q(f,f) = \int_M |\nabla f|^2 - \int_M |A_M|^2 f^2$. We additionally recall [Sch83b, Section 2] that an end E of a minimal immersion $M^4 \to \mathbb{R}^5$ is *regular at infinity* if it is contained in the graph of a function w on a hyperplane Π with asymptotics

$$w(x) = b + a|x|^{-2} + \sum_{j=1}^{4} c_j x_j |x|^{-4} + O(|x|^{-4})$$

where $a, b, c_1, \ldots, c_4$ are constants and $x_1, \ldots, x_4$ are the Euclidean coordinates on Π.

Theorem 2.1.4. *A complete, two-sided minimal immersion $M^4 \to \mathbb{R}^5$ has finite Morse index if and only if it has finite total curvature, i.e., $\int_M |A_M|^4 < \infty$, in which case M is properly immersed, has finitely many ends, each end of M is regular at*

infinity, and M has Euclidean volume growth, namely $|M \cap B_{\mathbf{R}^5}(0,\rho)| \leq C\rho^4$ for all $\rho > 0$; here $C = C(M)$.

The results in this paper should be relevant to the study of stable/finite index minimal hypersurfaces in 5-manifolds in a manner similar to [CLS22].

2.1.1 Discussion of results and methods

Let $M^n \to \mathbb{R}^{n+1}$ be a complete, two-sided stable minimal immersion. The main difficulty in resolving the stable Bernstein problem is that the extrinsic and intrinsic geometry of $M^n \to \mathbb{R}^{n+1}$ could be *a priori* very complicated. For example, if g is the induced metric on M, the manifold (M, g) might have exponential volume growth. Furthermore, stability of the immersion does not directly imply any pointwise curvature condition on g, while minimality only implies that g has *non-positive* scalar curvature via the traced Gauss equation.

The strategy used in this article is motivated by the one developed by the first- and second-named authors for $M^3 \to \mathbb{R}^4$ in [CL23]. More precisely, we let g denote the induced metric on M and consider the conformally changed metric $\tilde{g} = r^{-2}g$, where r is the Euclidean distance function from the origin. This conformal change was first considered by Gulliver–Lawson in their study of isolated singularities in stable minimal hypersurfaces [GL86]. Note that if M is a flat hyperplane containing the origin, then $(M, \tilde{g})$ will be the standard round cylinder $\mathbb{R} \times \mathbf{S}^{n-1}$.

A key insight of Gulliver–Lawson is that $\tilde{g}$ has uniformly positive scalar curvature in a weak spectral sense; namely, we have

$$-\tilde{\Delta} + \frac{1}{2}\left(\tilde{R} - \frac{n(n-2)}{2}\right) \geq 0, \tag{2.1.1}$$

where $\tilde{R}$ is the scalar curvature and $\tilde{\Delta}$ is the (nonpositive) Laplace–Beltrami operator of $\tilde{g}$. It is now well known that a 3-manifold N^3 with uniformly positive scalar

curvature has *macroscopic dimension one* in several ways:

- *Distance-sense*: If N is simply connected, then N has bounded Urysohn 1-width, see [Kat88] and [LM23].
- *Area-sense*: N admits equally spaced separating surfaces with uniformly bounded area; see [CL24a] and [CL23].
- *Volume-sense*: If N has nonnegative Ricci curvature, then N has linear volume growth, see [MW22] and [CLS23].

Since each of these results can be proved using μ-bubbles—a tool introduced by Gromov [Gro96] (see also [Gro18])—and since the weaker condition (2.1.1) suffices to carry out the μ-bubble argument, these observations hold for 3-manifolds satisfying (2.1.1). In particular, an appropriate version of the fact that $(M, \tilde{g})$ has *macroscopic dimension one* was used in [CL23] to deduce the stable Bernstein theorem for $M^3 \to \mathbb{R}^4$.

In higher dimensions, the positive scalar curvature property (2.1.1) of $(M, \tilde{g})$ appears to be too weak to deduce the stable Bernstein property. (In particular, $\mathbb{R}^{n-2} \times \mathbf{S}^2$ has uniformly positive scalar curvature.) The first main idea in the proof of Theorem 2.1.1 is to replace scalar curvature with a stronger curvature condition, namely *bi-Ricci curvature*. The bi-Ricci curvature of two orthonormal vectors $v, w \in T_pM$ is defined as

$$\mathrm{BiRic}(v, w) = \mathrm{Ric}(v, v) + \mathrm{Ric}(w, w) - \mathrm{R}(v, w, w, v)$$

where R is the curvature tensor. Alternatively, $\mathrm{BiRic}(v, w)$ is the sum of sectional curvatures of 2-planes intersecting the plane spanned by v and w. We note that the bi-Ricci curvature of a 3-manifold is a multiple of the scalar curvature. Importantly, $\mathbb{R} \times \mathbf{S}^{n-1}$ has uniformly positive bi-Ricci curvature, while $\mathbb{R}^k \times \mathbf{S}^{n-k}$ does not for any $k > 1$.

Remark 2.1.5. The bi-Ricci curvature was introduced by Shen–Ye [SY96, SY97] motivated by the relationship between bi-Ricci curvature and stable minimal hypersurfaces. More recently, Brendle–Hirsch–Johne studied bi-Ricci curvature as part of a general notion of curvatures that interpolate between Ricci and scalar curvature [BHJ24].

In Theorem 2.3.1, we prove that for $M^4 \to \mathbb{R}^5$ a complete, two-sided stable minimal immersion, the Gulliver–Lawson conformal metric $(M, \tilde{g})$ has uniformly positive bi-Ricci curvature in the weak spectral sense; namely, we have

$$-\tilde{\Delta} + (\tilde{\lambda}_{\mathrm{BiRic}} - 1) \geq 0, \tag{2.1.2}$$

where $\tilde{\lambda}_{\mathrm{BiRic}}(x)$ is the smallest bi-Ricci curvature of $\tilde{g}$ at x. This computation follows the general strategy introduced by Gulliver–Lawson [GL86], but is considerably more involved. At this point, we must leverage the improved positivity (2.1.2) to prove "one-dimensionality" of $(M^4, \tilde{g})$, and then use this to conclude the stable Bernstein theorem.

In [Xu25], Xu showed that if N^n has uniformly positive bi-Ricci curvature and $n \leq 5$, then N admits μ-bubbles with uniformly positive Ricci curvature in the weak spectral sense. We generalize his arguments (for $n = 4$) to *spectral* uniformly positive bi-Ricci curvature. This construction produces an exhaustion by (warped) μ-bubbles that satisfy

$$-\Delta^{\Sigma} + \frac{3}{4}\left(\lambda^{\Sigma}_{\mathrm{Ric}} - \frac{1}{2}\right) \geq 0, \tag{2.1.3}$$

where $\lambda^{\Sigma}_{\mathrm{Ric}}(x)$ is the smallest eigenvalue of the Ricci curvature of Σ at x. The spectral condition (2.1.3) can be thought of as a weak form of uniform positivity of the Ricci curvature of the μ-bubble Σ. Indeed, (2.1.3) implies that Σ has uniformly bounded diameter by a result of Shen–Ye [SY97] (see Theorem 2.5.1). In particular, this implies that $(M, \tilde{g})$ has bounded Urysohn 1-width.

The fact that $(M, \tilde{g})$ has bounded Urysohn 1-width does confirm that a complete, two-sided stable minimal immersion $M^4 \to \mathbb{R}^5$ has controlled geometry in a certain sense, but does not seem sufficient to conclude that M is a flat hyperplane. Indeed, in the proof of the stable Bernstein theorem for $M^3 \to \mathbb{R}^4$ in [CL23], the key property of the μ-bubbles is that they have bounded area (in addition to bounded diameter). In this case, when the μ-bubbles are two-dimensional, the area bound follows directly from a spectral condition like (2.1.3) thanks to the Gauss–Bonnet theorem.

In higher dimensions (as considered in this paper) the Bishop–Gromov volume comparison gives uniform volume upper bounds from Ricci curvature lower bounds. We consider a weighted version of Bray's proof [Bra97] of the Bishop theorem via the isoperimetric profile and use this to deduce a sharp volume comparison theorem for 3-dimensional manifolds satisfying spectrally positive Ricci curvature (see Theorem 2.5.1).

Remark 2.1.6. A different spectral version of the Bishop–Gromov theroem was obtained by Carron for manifolds with a Euclidean Sobolev inequality and a (strong) form of non-negative Ricci curvature in a spectral sense [Car19]. Note that this result does not seem applicable to our setting since the μ-bubbles are compact and thus cannot admit a Euclidean Sobolev inequality.

Granted these two ingredients, i.e., spectral positivity of bi-Ricci curvature for the Gulliver–Lawson conformal metric and volume bounds for the 3-dimensional μ-bubbles, we can follow the arguments of [CL23] to deduce the stable Bernstein theorem for $n = 4$.

Remark 2.1.7. There have been several exciting developments in this area that occured after this paper was first posted. Antonelli–Xu have generalized the spectral Bishop–Gromov result to hold in all dimensions [AX24a]. Mazet subsequently combined their spectral Bishop–Gromov result with a delicate refinement of the strat-

egy used here to resolve the stable Bernstein problem in $\mathbf{R}^6$ [Maz24]. (See also [AX24b, Tam24].)

2.1.2 Organization

We review the notation and conventions in Section 2.2. Then, we compute the spectral curvature properties of the Gulliver–Lawson conformal metric in Section 2.3. We discuss μ-bubble existence and stability in Section 2.4 and then use the stability inequality to prove geometric estimates for the μ-bubbles in Section 2.5. Finally, in Section 2.6 we prove the results stated in the introduction.

2.2 Notation and Conventions

Let (M^n, g) be a Riemannian manifold. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for T_pM. We write $\mathrm{R}(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ for the curvature operator with the convention that $\mathrm{R}(e_i, e_j, e_j, e_i)$ is the sectional curvature of the 2-plane spanned by e_i, e_j. At $p \in M$, we define

- *The Ricci curvature tensor*:

$$\mathrm{Ric}(e_1, e_1) = \sum_{i=2}^{n} \mathrm{R}(e_1, e_i, e_i, e_1).$$

- *The minimum Ricci curvature scalar*:

$$\lambda_{\mathrm{Ric}} = \inf_{|v|=1} \mathrm{Ric}(v, v).$$

- *The bi-Ricci curvature*:

$$\mathrm{BiRic}(e_1, e_2) = \sum_{i=2}^{n} \mathrm{R}(e_1, e_i, e_i, e_1) + \sum_{j=3}^{n} \mathrm{R}(e_2, e_j, e_j, e_2).$$

- *The minimum bi-Ricci curvature scalar*:

$$\lambda_{\mathrm{BiRic}} = \inf_{\{v,w\} \text{ orthonormal}} \mathrm{BiRic}(v, w).$$

Let $M^n \to (X^{n+1}, g_X)$ be a two-sided codimension one smooth immersion. Let ν be a smooth unit normal vector field along M. We define

- *The second fundamental form*:

$$A(X, Y) = -\langle \nabla^N_X Y, \nu \rangle.$$

- *The mean curvature*:

$$H = \sum_{i=1}^{n} A(e_i, e_i).$$

In particular, the unit sphere in $\mathbb{R}^{n+1}$ has positive mean curvature with respect to the outward unit normal. Given our conventions, for a smooth family of immersions $\{F_t : M^n \to N^{n+1}\}_t$, we have

$$\frac{d}{dt}\mathcal{H}^n_{F_t^* g_X}(M) = \int_M H_{F_t} \left\langle \nu_{F_t}, \frac{dF_t}{dt} \right\rangle \, d\mathcal{H}^n_{F_t^* g},$$

where $F_t^* g_X$ is the pullback metric under F_t, H_{F_t} is the mean curvature of the immersion F_t, ν_{F_t} is the unit normal vector field to F_t, and $\mathcal{H}^n_h$ is the n-dimensional Hausdorff measure with respect to the metric h.

2.3 Conformal Change for Stable Minimal Hypersurfaces

We study the geometry of stable minimal hypersurfaces in Euclidean space under a conformal change introduced by Gulliver–Lawson [GL86] to study singularities of

stable minimal hypersurfaces.

Let $F : M^n \to \mathbb{R}^{n+1}$ be a complete two-sided stable minimal immersion. Let g be the pullback metric on M. Let r denote the Euclidean distance from 0 in $\mathbb{R}^{n+1}$. Consider the conformal metric $\tilde{g} = r^{-2}g \equiv e^{2\phi}g$ on $N = M \setminus F^{-1}(\{0\})$, as in [GL86]. Note that $(N, \tilde{g})$ is complete. Henceforth, we use tildes to denote quantities with respect to $\tilde{g}$; and otherwise we use the metric g. For instance, we let $d\mu$ and $d\tilde{\mu}$ denote the volume measures on N with respect to g and $\tilde{g}$ respectively.

We prove here the following result.

Theorem 2.3.1. *Suppose $n = 4$. Then there is a smooth function V on $(N, \tilde{g})$ so that*

$$V \geq 1 - \tilde{\lambda}_{\mathrm{BiRic}}$$

and

$$\int_N |\tilde{\nabla}\psi|^2_{\tilde{g}} d\tilde{\mu} \geq \int_N V\psi^2 d\tilde{\mu}$$

for any $\psi \in C^\infty_c(N, \tilde{g})$.

We note that Theorem 2.3.1 shows that $(N, \tilde{g})$ satisfies a weak form of the condition $\widetilde{\mathrm{BiRic}} \geq 1$.

2.3.1 Standard calculations for Euclidean minimal hypersurfaces

We begin with some standard calculations about the Euclidean distance function r on minimal hypersurfaces.

Proposition 2.3.2.

$$\mathrm{Hess}^M r = r^{-1}g - r^{-1}dr \otimes dr - r^{-1}\langle \vec{x}, \nu \rangle A.$$

Proof. We compute

$$\partial_i r = \frac{x_i}{r}, \quad \partial_i \partial_j r = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}$$

Thus

$$\operatorname{Hess}^{\mathbb{R}^{n+1}} r = r^{-1} g_{\mathrm{Euc}} - r^{-1} dr \otimes dr.$$

Hence, for X and Y tangent to M, we have

$$\begin{aligned}
\operatorname{Hess}^M r(X,Y) &= \langle \nabla^M_X \nabla^M r, Y \rangle \\
&= \langle \nabla^{\mathbb{R}^{n+1}}_X \nabla^{\mathbb{R}^{n+1}} r, Y \rangle - \langle \nabla^{\mathbb{R}^{n+1}}_X (\nabla^{\mathbb{R}^{n+1}} r)^\perp, Y \rangle \\
&= \operatorname{Hess}^{\mathbb{R}^{n+1}} r(X,Y) + \left\langle \frac{\vec{x}^\perp}{r}, (\nabla^{\mathbb{R}^{n+1}}_X Y)^\perp \right\rangle \\
&= r^{-1} g(X,Y) - r^{-1} (dr \otimes dr)(X,Y) - r^{-1} \langle \vec{x}, \nu \rangle A(X,Y),
\end{aligned}$$

which concludes the proposition. □

We write $\phi = -\log r$ so that $\tilde{g} = e^{2\phi} g$.

Proposition 2.3.3.

$$\operatorname{Hess}^M (\log r) = r^{-2} g - 2 r^{-2} dr \otimes dr - r^{-2} \langle \vec{x}, \nu \rangle A.$$

Proof. For any $f > 0$, we compute

$$\operatorname{Hess}^M (\log f)(X,Y) = \left\langle \nabla^M_X \left(\frac{\nabla^M f}{f} \right), Y \right\rangle = f^{-1} \operatorname{Hess}^M f(X,Y) - f^{-2} (df \otimes df)(X,Y).$$

We conclude by Proposition 2.3.2. □

W define the tensor

$$T = \operatorname{Hess}^M \phi - d\phi \otimes d\phi + \frac{1}{2} |d\phi|^2 g.$$

This quantitiy appears in the formula for curvature in the conformal metric $\tilde{g}$ (see [Lee18, Theorem 7.30]). We now use Proposition 2.3.3 to compute T.

Proposition 2.3.4.

$$T = -r^{-2}g + r^{-2}dr \otimes dr + \frac{1}{2}r^{-2}|dr|^2 g + r^{-2}\langle \vec{x}, \nu \rangle A.$$

2.3.2 Curvature in the conformal metric

We now compute relevant curvature quantities related to the conformal metric $\tilde{g}$.

First, we set up a convenient orthonormal basis for T_pM with respect to the metrics g and $\tilde{g}$. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for T_pM with respect to g. Then $\{\tilde{e}_i = re_i\}_{i=1}^n$ is an orthonormal basis for T_pM with respect to $\tilde{g}$.

We are now equipped to compute the sectional curvatures of $(N, \tilde{g})$. We will write $R_{ijji} = \mathrm{R}(e_i, e_j, e_j, e_i)$, $A_{ij} = A(e_i, e_j)$, and $T_{ij} = T(e_i, e_j)$. We also denote $R_{ijji} = \tilde{\mathrm{R}}(\tilde{e}_i, \tilde{e}_j, \tilde{e}_j, \tilde{e}_i)$.

Proposition 2.3.5.

$$r^2 R_{ijji} = \tilde{R}_{ijji} - 2 + |dr|^2 + (dr(e_i))^2 + (dr(e_j))^2 + \langle \vec{x}, \nu \rangle (A_{ii} + A_{jj}).$$

Proof. Using the formula for the Riemann curvature tensor under a conformal change [Lee18, Theorem 7.30] and the tensor T as defined in Proposition 2.3.4, we compute

$$\begin{aligned}
\tilde{R}_{ijji} &= \tilde{\mathrm{R}}(\tilde{e}_i, \tilde{e}_j, \tilde{e}_j, \tilde{e}_i) = r^4 \tilde{\mathrm{R}}(e_i, e_j, e_j, e_i) \\
&= r^2 (R_{ijji} - T_{ii} - T_{jj}) \\
&= r^2 R_{ijji} + 2 - |dr|^2 - (dr(e_i))^2 - (dr(e_j))^2 - \langle \vec{x}, \nu \rangle (A_{ii} + A_{jj}),
\end{aligned}$$

which concludes the proposition. □

Taking the appropriate combinations of sectional curvatures, we use Proposition 2.3.5 to compute the bi-Ricci curvatures of $(N, \tilde{g})$.

Proposition 2.3.6.

$$\begin{aligned} r^2 \operatorname{BiRic}(e_1, e_2) = \widetilde{\operatorname{BiRic}}(\tilde{e}_1, \tilde{e}_2) - (4n-6) + (2n-1)|dr|^2 \\ + (n-3)(dr(e_1)^2 + dr(e_2)^2) \\ + (n-3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}). \end{aligned}$$

Proof. Using Proposition 2.3.5, we compute

$$\begin{aligned} r^2 \operatorname{BiRic}(e_1, e_2) &= \sum_{i=2}^{n} r^2 R_{1ii1} + \sum_{j=3}^{n} r^2 R_{2jj2} \\ &= \widetilde{\operatorname{BiRic}}(\tilde{e}_1, \tilde{e}_2) - (4n-6) + (2n-3)|dr|^2 \\ &\quad + 2|dr|^2 + (n-3)(dr(e_1)^2 + dr(e_2)^2) \\ &\quad + 2\langle \vec{x}, \nu \rangle \operatorname{Tr}(A) + (n-3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}) \\ &= \widetilde{\operatorname{BiRic}}(\tilde{e}_1, \tilde{e}_2) - (4n-6) + (2n-1)|dr|^2 \\ &\quad + (n-3)(dr(e_1)^2 + dr(e_2)^2) \\ &\quad + (n-3)\langle \vec{x}, \nu \rangle (A_{11} + A_{22}). \end{aligned}$$

In the last inequality we used that the trace of A is zero for a minimal hypersurface. □

To exploit the stability inequality, we use the Gauss equation to express BiRic in terms of the second fundamental form of M.

Proposition 2.3.7.

$$\operatorname{BiRic}(e_1, e_2) = -\sum_{i=1}^{n} A_{1i}^2 - \sum_{j=2}^{n} A_{2j}^2 - A_{11}A_{22}.$$

Proof. Using the Gauss equation and $\operatorname{Tr} A = 0$, we compute

$$\begin{aligned}\operatorname{BiRic}(e_1, e_2) &= \sum_{i=2}^{n} R_{1ii1} + \sum_{j=3}^{n} R_{2jj2} \\ &= \sum_{i=2}^{n}(A_{11}A_{ii} - A_{1i}^2) + \sum_{j=3}^{n}(A_{22}A_{jj} - A_{2j}^2) \\ &= -\sum_{i=1}^{n} A_{1i}^2 - \sum_{j=2}^{n} A_{2j}^2 - A_{11}A_{22}.\end{aligned}$$

This completes the proof. □

Choose the basis vectors e_1 and e_2 so that $\widetilde{\operatorname{BiRic}}(\tilde{e}_1, \tilde{e}_2) = \tilde{\lambda}_{\operatorname{BiRic}}$. We can now bound $|A|^2$ in terms of $\tilde{\lambda}_{\operatorname{BiRic}}$.

Proposition 2.3.8. *For $n \geq 3$,*

$$r^2|A|^2 \geq \frac{2}{n-2}\Big((3n-3) - (2n-1)|dr|^2 - \tilde{\lambda}_{\operatorname{BiRic}}\Big)$$

Proof. Combining Propositions 2.3.6 and 2.3.7, we estimate

$$\begin{aligned}&r^2\Big(\sum_{i=1}^{n} A_{1i}^2 + \sum_{j=2}^{n} A_{2j}^2 + A_{11}A_{22}\Big) + (n-3)\langle \vec{x}, \nu\rangle(A_{11} + A_{22}) \qquad (2.3.1)\\ &\qquad = (4n-6) - (2n-1)|dr|^2 - (n-3)(dr(e_1)^2 + dr(e_2)^2) - \tilde{\lambda}_{\operatorname{BiRic}}.\end{aligned}$$

Using $\langle \vec{x}, \nu\rangle = r dr(\nu)$ and Young's inequality, we have

$$|(n-3)\langle \vec{x}, \nu\rangle(A_{11} + A_{22})| \leq (n-3)dr(\nu)^2 + \frac{n-3}{4} r^2 (A_{11} + A_{22})^2.$$

Combined with (2.3.1) and the fact that $dr(e_1)^2 + dr(e_2)^2 + dr(\nu)^2 \leq 1$, we find

$$r^2\Big(\sum_{i=1}^{n} A_{1i}^2 + \sum_{j=2}^{n} A_{2j}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11}+A_{22})^2\Big) \geq (3n-3) - (2n-1)|dr|^2 - \tilde{\lambda}_{\operatorname{BiRic}}.$$

Using the fact that $\operatorname{Tr} A = 0$, we now compute

$$\begin{aligned} A_{11}^2 + A_{22}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11}+A_{22})^2 &= \frac{1}{2}(A_{11}^2 + A_{22}^2) + \frac{n-1}{4}(A_{11}+A_{22})^2 \\ &= \frac{1}{2}(A_{11}^2 + A_{22}^2) + \frac{n-1}{4}\sigma(A_{11}+A_{22})^2 + \frac{n-1}{4}(1-\sigma)(A_{33}+\ldots+A_{nn})^2 \\ &\leq \left(\frac{1}{2} + \frac{n-1}{2}\sigma\right)(A_{11}^2 + A_{22}^2) + \frac{(n-1)(n-2)}{4}(1-\sigma)(A_{33}^2+\ldots+A_{nn}^2) \\ &= \frac{n-2}{2}(A_{11}^2 + \ldots + A_{nn}^2), \end{aligned}$$

where we took $\sigma = \frac{n-3}{n-1}$ in the last line. Hence, for $n \geq 3$, we have

$$\begin{aligned} \frac{n-2}{2}r^2|A|^2 &\geq r^2\Big(\frac{n-2}{2}\sum_{i=1}^n A_{ii}^2 + \sum_{i=2}^n A_{1i}^2 + \sum_{j=3}^n A_{2j}^2\Big) \\ &\geq r^2\Big(\sum_{i=1}^n A_{1i}^2 + \sum_{j=2}^n A_{2j}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11}+A_{22})^2\Big) \\ &\geq (3n-3) - (2n-1)|dr|^2 - \tilde{\lambda}_{\mathrm{BiRic}}, \end{aligned}$$

as asserted. □

2.3.3 Stability inequality in the conformal metric

Proposition 2.3.9. *We have* $\tilde{\Delta}_M(\log r) = n - n|dr|^2$.

Proof. By the formula for the Laplace–Beltrami operator under a conformal transformation and Proposition 2.3.3, we have

$$\begin{aligned} \tilde{\Delta}_M(\log r) &= r^2(\Delta_M(\log r) - (n-2)r^{-2}|dr|^2) \\ &= n - 2|dr|^2 - (n-2)|dr|^2 = n - n|dr|^2, \end{aligned}$$

as desired. □

We can now rewrite the stability inequality in the metric $\tilde{g}$.

Proposition 2.3.10. *For any $\psi \in C_c^\infty(N, \tilde{g})$ we have*

$$\int_N |\tilde{\nabla}\psi|_{\tilde{g}}^2 d\tilde{\mu} \geq \int_N \Big(r^2|A|^2 - \frac{n(n-2)}{2} + \Big(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} \Big) |dr|^2 \Big) \psi^2 d\tilde{\mu}.$$

Proof. Using

$$d\tilde{\mu} = r^{-n} d\mu \quad \text{and} \quad |\tilde{\nabla} f|_{\tilde{g}}^2 = r^2 |\nabla f|^2,$$

the stability inequality for M can be written as

$$\int_N r^{n-2} |\tilde{\nabla} f|_{\tilde{g}}^2 d\tilde{\mu} \geq \int_N r^{n-2} (r^2|A|^2) f^2 d\tilde{\mu}$$

for any $f \in C_c^{0,1}(N, \tilde{g})$. We take $f = r^{\frac{2-n}{2}} \psi$ for $\psi \in C_c^{0,1}(N, \tilde{g})$. Then

$$\tilde{\nabla} f = r^{\frac{2-n}{2}} \tilde{\nabla}\psi - \frac{n-2}{2} r^{-\frac{n}{2}} \psi \tilde{\nabla} r.$$

In particular,

$$|\tilde{\nabla} f|_{\tilde{g}}^2 = a + b + c$$

where

$$a := r^{2-n} |\tilde{\nabla}\psi|_{\tilde{g}}^2, \quad b := \frac{(n-2)^2}{4} r^{-n} \psi^2 |\tilde{\nabla} r|_{\tilde{g}}^2, \quad c := -(n-2) r^{1-n} \psi \langle \tilde{\nabla}\psi, \tilde{\nabla} r \rangle_{\tilde{g}}.$$

We have

$$\int_N r^{n-2} a \, d\tilde{\mu} = \int_N |\tilde{\nabla}\psi|_{\tilde{g}}^2 d\tilde{\mu}.$$

Since $r^{-2} |\tilde{\nabla} r|_{\tilde{g}}^2 = |dr|^2$, we have

$$\int_N r^{n-2} b \, d\tilde{\mu} = \int_N \frac{(n-2)^2}{4} |dr|^2 \psi^2 d\tilde{\mu}.$$

Finally, we use integration by parts and Proposition 2.3.9 to compute

$$\begin{aligned}\int_N r^{n-2} c\, d\tilde{\mu} &= -\int_N \frac{n-2}{2}\langle \tilde{\nabla}(\psi^2), \tilde{\nabla}(\log r)\rangle_{\tilde{g}} d\tilde{\mu} \\ &= \int_N \frac{n-2}{2}\tilde{\Delta}(\log r)\psi^2 d\tilde{\mu} \\ &= \int_N \Big(\frac{n(n-2)}{2} - \frac{n(n-2)}{2}|dr|^2\Big)\psi^2 d\tilde{\mu}.\end{aligned}$$

The assertion follows from the above expressions. □

We now rephrase our estimate for $r^2|A|^2$ from Proposition 2.3.8 to suit this form of the stability inequality.

Proposition 2.3.11. *For* $3 \leq n \leq 5$,

$$r^2|A|^2 - \frac{n(n-2)}{2} + \Big(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4}\Big)|dr|^2 \geq \frac{2}{n-2}\Big(\frac{(2-n)(n^2-4n-4)}{8} - \tilde{\lambda}_{\mathrm{BiRic}}\Big).$$

Proof. By Proposition 2.3.8, we have

$$\begin{aligned}&r^2|A|^2 - \frac{n(n-2)}{2} + \Big(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4}\Big)|dr|^2 \\ &\qquad \geq \frac{6(n-1)}{n-2} - \frac{n(n-2)}{2} + \Big(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} - \frac{2(2n-1)}{n-2}\Big)|dr|^2 - \frac{2}{n-2}\tilde{\lambda}_{\mathrm{BiRic}}.\end{aligned}$$

Note that the coefficient of $|dr|^2$ on the right-hand side is negative for $3 \leq n \leq 5$, so we can use $|dr|^2 \leq 1$ to conclude that the left hand side is greater than or equal to

$$\frac{2}{n-2}\Big(\frac{(2-n)(n^2-4n-4)}{8} - \tilde{\lambda}_{\mathrm{BiRic}}\Big),$$

as desired. □

Theorem 2.3.1 follows by plugging $n = 4$ into Proposition 2.3.11 and applying the stability inequality as formulated in Proposition 2.3.10.

2.4 μ-Bubbles in Spectral Positive Bi-Ricci Curvature

We generalize the μ-bubble construction in uniformly positive bi-Ricci curvature of [Xu25] to manifolds with spectral uniformly positive bi-Ricci curvature. Our main tool is the notion of *warped* μ-bubbles (see [CL24a]), extending the standard μ-bubbles used in [Xu25].

Suppose that (N^n, g_N) is a smooth complete noncompact Riemannian manifold that admits a smooth function V so that

$$V \geq 1 - \lambda_{\mathrm{BiRic}}(g_N)$$

and

$$\int_N |\nabla \psi|^2 \geq \int_N V\psi^2$$

for all $\psi \in C_c^\infty(N, g_N)$. Recall from [FCS80, Theorem 1] that there is a positive function u on N satisfying

$$-\Delta_N u = Vu \geq (1 - \lambda_{\mathrm{BiRic}}(g_N))u. \tag{2.4.1}$$

We prove the following theorem in dimension $n = 4$.

Theorem 2.4.1. *Let $X \subset N^4$ be a closed subset with smooth boundary $\partial X = \partial_+ X \sqcup \partial_- X$ for some nonempty smooth hypersurfaces $\partial_\pm X$. Suppose $d_N(\partial_+ X, \partial_- X) \geq 10\pi$. Then there is a connected, relatively open subset $\Omega \subset X$ with smooth boundary $\partial\Omega = \partial_- X \sqcup \Sigma$ so that*

- *$\partial_- X \subset \Omega$,*
- *$\Sigma \subset X \setminus \partial X$ is a closed submanifold*

- $\Omega \subset B_{10\pi}(\partial_- X)$,
- *there is a smooth function* $W \in C^\infty(\Sigma)$ *so that*

$$W \geq \frac{3}{4}\left(\frac{1}{2} - \lambda_{\mathrm{Ric}}(\Sigma)\right)$$

and

$$\int_\Sigma |\nabla \psi|^2 \geq \int_\Sigma W \psi^2$$

for all $\psi \in C^\infty(\Sigma)$.

The strategy to prove Theorem 2.4.1 is to minimize a certain warped prescribed mean curvature functional.

Let w be a smooth positive function on N. Let $\Omega \subset N$ be an open set with smooth boundary (or more generally a set of finite perimeter). Let h be a smooth function defined in a neighborhood of $\partial\Omega$. We study minimizers of the warped prescribed mean curvature functional

$$\mathcal{A}(\Omega) = \int_{\partial\Omega} w d\mathcal{H}^{n-1} - \int_\Omega h w d\mathcal{H}^n.$$

Ultimately, we will take $w = u$ to be our warping function, but we leave w general for most of the calculations. Note that this functional may be viewed as a μ-bubble functional on a warped manifold, see [CL24a, Remark 11].

2.4.1 First variation formula

The first variation formula for $\mathcal{A}$ can be computed as.

Proposition 2.4.2. *Let* $\{\Omega_t\}_{|t|<\varepsilon}$ *be a smooth family of sets of finite perimeter with*

$\Omega_0 = \Omega$ *and variation vector field* V_t*. Then*

$$\frac{d}{dt}\mathcal{A}(\Omega_t) = \int_{\partial\Omega_t} \langle \nabla^N w, V_t^\perp \rangle + wH_t\langle \nu_t, V_t\rangle - wh\langle \nu_t, V_t\rangle d\mathcal{H}^{n-1}.$$

Hence, critical points for $\mathcal{A}$ *satisfy*

$$H = h - w^{-1}\langle \nabla^N w, \nu\rangle.$$

2.4.2 Second variation formula

We now prove the following second variation formula. Although the theorem statement below only holds in dimension $n = 4$, although we carry out the computations for general n until plugging in $n = 4$ at the very end of the proof.

Theorem 2.4.3. *Let* $\Omega \subset N^4$ *be an open set with smooth boundary that's stationary and stable for* $\mathcal{A}$ *with weight function* $w = u$*. Let* $\Sigma = \partial\Omega$*. Let* γ *denote the pullback metric on* Σ*. Then there is a smooth function* W *on* Σ *so that*

$$W \geq \frac{3}{4}\left(\frac{1}{2} - \lambda_{\mathrm{Ric}}(\gamma)\right)$$

and

$$\int_\Sigma |\nabla^\Sigma \psi|^2 \geq \int_\Sigma W\psi^2 + \frac{3}{8}\int_\Sigma \left(1 + h^2 - 2|\nabla^N h|\right)\psi^2$$

for any $\psi \in C_c^\infty(\Sigma)$*.*

Proof. We take a variation $\{\Omega_t\}$ where $\Omega_0 = \Omega$ is a critical point for $\mathcal{A}$. We can choose our variation so that $D_t V_t = 0$ by taking the normal exponential flow of $V_0 = \phi\nu$.

We compute

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{A}(\Omega_t) = &\int_\Sigma \phi^2 \operatorname{Hess}^N w(\nu,\nu) - \phi\langle\nabla^\Sigma w, \nabla^\Sigma\phi\rangle + \phi^2\langle\nabla^N w,\nu\rangle H \\ &- \int_\Sigma w(\phi\Delta^\Sigma\phi + (|A_\Sigma|^2 + \operatorname{Ric}_{g_N}(\nu,\nu))\phi^2) \\ &- \int_\Sigma \phi^2\langle\nabla^N w,\nu\rangle h + \phi^2 w\langle\nabla^N h,\nu\rangle.\end{aligned}$$

We use integration by parts on the $-w\phi\Delta^\Sigma\phi$ term and the formula.

$$\operatorname{Hess}^N f(\nu,\nu) = \Delta^N f - \Delta^\Sigma f - \langle\nabla^N f,\nu\rangle H,$$

This gives

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{A}(\Omega_t) = &\int_\Sigma \phi^2(\Delta^N w - \Delta^\Sigma w) \\ &+ \int_\Sigma w(|\nabla^\Sigma\phi|^2 - (|A_\Sigma|^2 + \operatorname{Ric}_{g_N}(\nu,\nu))\phi^2) \\ &- \int_\Sigma \phi^2\langle\nabla^N w,\nu\rangle h - \int_\Sigma \phi^2 w\langle\nabla^N h,\nu\rangle.\end{aligned}$$

Since Ω is a stable critical point of $\mathcal{A}$, we thus have

$$\begin{aligned}\int_\Sigma w|\nabla^\Sigma\phi|^2 - \phi^2\Delta^\Sigma w \geq &\int_\Sigma (-\Delta^N w + (|A_\Sigma|^2 + \operatorname{Ric}_{g_N}(\nu,\nu))w)\phi^2 \\ &+ \int_\Sigma (h\langle\nabla^N w,\nu\rangle + w\langle\nabla^N h,\nu\rangle)\phi^2.\end{aligned} \tag{2.4.2}$$

Take $\phi = w^{-1/2}\psi$. We compute

$$\nabla^\Sigma\phi = w^{-1/2}\nabla^\Sigma\psi - \frac{1}{2}w^{-3/2}\psi\nabla^\Sigma w.$$

Write

$$w|\nabla^\Sigma\phi|^2 = a + b + c$$

where

$$a := |\nabla^\Sigma \psi|^2, \quad b := -w^{-1}\psi\langle \nabla^\Sigma w, \nabla^\Sigma \psi\rangle, \quad \text{and} \quad \text{and} \quad c := \frac{1}{4} w^{-2}\psi^2 |\nabla^\Sigma w|^2.$$

We have

$$\begin{aligned} \int_\Sigma a &= \int_\Sigma |\nabla^\Sigma \psi|^2, \\ \int_\Sigma c &= \frac{1}{4}\int_\Sigma |\nabla^\Sigma \log w|^2 \psi^2, \end{aligned}$$

and

$$\begin{aligned} \int_\Sigma b - \phi^2 \Delta^\Sigma w &= \int_\Sigma w^{-1}\psi\langle \nabla^\Sigma w, \nabla^\Sigma \psi\rangle - \int_\Sigma \psi^2 |\nabla^\Sigma \log w|^2 \\ &\leq \left(\frac{\varepsilon}{2} - 1\right)\int_\Sigma \psi^2 |\nabla^\Sigma \log w|^2 + \frac{1}{2\varepsilon}\int_\Sigma |\nabla^\Sigma \psi|^2 \end{aligned}$$

for all $\varepsilon > 0$. Taking $\varepsilon = 3/2$, we have

$$\int_\Sigma w|\nabla^\Sigma \phi|^2 - \phi^2 \Delta^\Sigma w \leq \frac{4}{3}\int_\Sigma |\nabla^\Sigma \psi|^2.$$

Combined with (2.4.2) we have

$$\begin{aligned} \frac{4}{3}\int_\Sigma |\nabla^\Sigma \psi|^2 \geq \int_\Sigma \left(-\frac{\Delta^N w}{w} + |A_\Sigma|^2 + \mathrm{Ric}_{g_N}(\nu,\nu) - \frac{1}{2}H^2 - \frac{1}{2}\right)\psi^2 \\ + \int_\Sigma \left(\frac{1}{2} + \frac{1}{2}H^2 + h\langle \nabla^N \log w, \nu\rangle + \langle \nabla^N h, \nu\rangle\right)\psi^2. \end{aligned} \tag{2.4.3}$$

Since Σ is a critical point for $\mathcal{A}$, we have

$$H^2 = h^2 + \langle \nabla^N \log w, \nu\rangle^2 - 2h\langle \nabla^N \log w, \nu\rangle \geq h^2 - 2h\langle \nabla^N \log w, \nu\rangle.$$

Hence, we have

$$\frac{4}{3}\int_\Sigma |\nabla^\Sigma \psi|^2 \geq \int_\Sigma \left(-\frac{\Delta^N w}{w} + |A_\Sigma|^2 + \mathrm{Ric}_{g_N}(\nu,\nu) - \frac{1}{2}H^2 - \frac{1}{2} \right) \psi^2 + \frac{1}{2}\int_\Sigma \left(1 + h^2 - 2|\nabla^N h|\right)\psi^2. \tag{2.4.4}$$

It remains to find a good lower bound for the first integrand on the right-hand side of (2.4.4). Using the Gauss equation, we compute

$$\begin{aligned}
\mathrm{Ric}_\gamma(e_1,e_1) &= \sum_{i=1}^{n-1} \mathrm{R}_\gamma(e_1,e_i,e_i,e_1) \\
&= \sum_{i=1}^{n-1} (\mathrm{R}_{g_N}(e_1,e_i,e_i,e_1) + A_{11}A_{ii} - A_{1i}^2) \\
&= \mathrm{BiRic}_{g_N}(e_1,\nu) - \mathrm{Ric}_{g_N}(\nu,\nu) + A_{11}\sum_{i=2}^{n-1} A_{ii} - \sum_{i=2}^{n-1} A_{1i}^2.
\end{aligned}$$

Moreover, using $\mathrm{Tr}(A) = H$, we have

$$\begin{aligned}
A_{11}\sum_{i=2}^{n-1} A_{ii} = -A_{11}^2 + A_{11}H &= -A_{11}^2 - H\sum_{i=2}^{n-1} A_{ii} + H^2 \\
&\geq -A_{11}^2 - \frac{1}{2\varepsilon}\left(\sum_{i=2}^{n-1} A_{ii}\right)^2 + \left(1 - \frac{\varepsilon}{2}\right)H^2 \\
&\geq -A_{11}^2 - \frac{n-2}{2\varepsilon}\sum_{i=2}^{n-1} A_{ii}^2 + \left(1 - \frac{\varepsilon}{2}\right)H^2 \\
&= -\sum_{i=1}^{n-1} A_{ii}^2 + \frac{6-n}{4}H^2,
\end{aligned}$$

where we took $\varepsilon = \frac{n-2}{2}$ in the last line. Choosing e_1 so that $\lambda_{\mathrm{Ric}}(\gamma) = \mathrm{Ric}_\gamma(e_1,e_1)$, we have

$$|A_\Sigma|^2 + \mathrm{Ric}_{g_N}(\nu,\nu) \geq \lambda_{\mathrm{BiRic}}(g_N) - \lambda_{\mathrm{Ric}}(\gamma) + \frac{6-n}{4}H^2.$$

Taking $n = 4$ and $w = u$, we have

$$-\frac{\Delta^N w}{w} + |A_\Sigma|^2 + \mathrm{Ric}_{g_N}(\nu, \nu) - \frac{1}{2}H^2 - \frac{1}{2} \geq \frac{1}{2} - \lambda_{\mathrm{Ric}}(\gamma),$$

which completes the proof. □

Proof of Theorem 2.4.1. Equipped with the second variation formula from Theorem 2.4.3, the proof now follows by taking h to be the standard μ-bubble prescribing function, chosen precisely so that $1 + h^2 - 2|\nabla^N h| \geq 0$, and then minimizing the functional $\mathcal{A}$ (see [CL23, Lemma 24] for the choice of h and see [CL24a, Proposition 12] or [Zhu21, Proposition 2.1] for the existence theory).

For completeness, we construct the function h. Let φ_0 be a smoothing of the function $d(\partial_- X, \cdot\,)$ so that $|\nabla\varphi_0| \leq 2$ and $\varphi_0|_{\partial_- X} \equiv 0$. Let $\varepsilon \in (0, 1/2)$ so that ε and $4\pi + 2\varepsilon$ are regular values of φ_0. Define

$$\varphi = \frac{\varphi_0 - \varepsilon}{4 + \frac{\varepsilon}{\pi}} - \frac{\pi}{2}.$$

Then $|\nabla\varphi| \leq \frac{1}{2}$, and the set $\Omega_1 = \{-\pi/2 < \varphi < \pi/2\}$ has smooth boundary and satisfies $\Omega_1 \subset B_{10\pi}(\partial_- X)$. On Ω_1, we define

$$h = -\tan(\varphi).$$

Since

$$\nabla h = -(1 + \tan^2(\varphi))\nabla\varphi = -(1 + h^2)\nabla\varphi,$$

it holds that

$$2|\nabla h| \leq 1 + h^2.$$

Hence, h has the desired property; the rest of the proof follows as in [CL23, Lemma 24]. □

2.5 Geometric Estimates for μ-Bubbles

We prove that the μ-bubbles constructed in the previous section have uniformly bounded diameter and volume.

Theorem 2.5.1. *Suppose that (Σ^3, γ) is a connected closed 3-manifold that admits a smooth function W and a constant $\alpha \in (0, 2]$ so that*

$$W \geq \alpha^{-1}(2 - \lambda_{\mathrm{Ric}}(\gamma))$$

and

$$\int_\Sigma |\nabla \psi|^2 \geq \int_\Sigma W \psi^2$$

for all $\psi \in C^\infty(\Sigma)$. Then

$$\mathrm{diam}(\Sigma, \gamma) \leq \pi$$

and

$$\mathrm{Vol}(\Sigma, \gamma) \leq 2\pi^2.$$

Note that both inequalities are sharp for the round 3-sphere (see also Remark 2.5.7 explaining the rigidity statement for the volume estimate).

Proof of diameter bound. A smooth positive first eigenfunction θ of the operator $-\Delta - W$ satisfies

$$-\Delta\theta \geq \alpha^{-1}(2 - \lambda_{\mathrm{Ric}}(\gamma))\theta.$$

Hence,

$$\mathrm{Ric}_\gamma^{(\theta,\alpha)} \equiv \mathrm{Ric}_\gamma - \alpha(\theta^{-1}\Delta\theta)\gamma \geq \mathrm{Ric}_\gamma - \lambda_{\mathrm{Ric}}\gamma + 2\gamma \geq 2\gamma,$$

where we adapt the notation of [SY97]. Since $\alpha \leq \frac{4}{3-1} = 2$, we have

$$\mathrm{diam}(\Sigma, \gamma) \leq \pi,$$

by [SY97, Corollary 1]. This completes the proof of the diameter bound. □

It remains to prove the volume bound for Σ. The strategy is to exploit the concavity properties of a weighted isoperimetric profile. These arguments extend the strategy of Bray's proof of the Bishop volume comparison from [Bra97] (we follow the exposition in [Bre12]).

2.5.1 Weighted isoperimetric profile

For an open set $\Omega \subset \Sigma$ with smooth boundary, we define a weighted area and volume functional by

$$a(\Omega) = \int_{\partial\Omega} \theta^\alpha \, d\mathcal{H}^2_\gamma \text{ and } \; v(\Omega) = \int_\Omega \theta^\alpha \, d\mathcal{H}^3_\gamma,$$

where θ be the unique first eigenfunction of $-\Delta - W$ with $\min \theta = 1$. Namely, θ satisfies

$$-\Delta\theta \geq \alpha^{-1}(2 - \lambda_{\mathrm{Ric}}(\gamma))\theta. \tag{2.5.1}$$

The *weighted isoperimetric profile* is the function $\mathcal{I} : (0, v(\Sigma)) \to \mathbb{R}$ given by

$$\mathcal{I}(v) = \inf\{a(\Omega) \mid v(\Omega) = v\}.$$

By compactness of Caccioppoli sets and [Mor03, §3.10], there's $\Omega \subset \Sigma$ achieving $\mathcal{I}(v)$ for all $v \in (0, v(\Sigma))$.

Proposition 2.5.2. *$\mathcal{I}$ is continuous.*

Proof. By the compactness theory for Caccioppoli sets and the lower semi-continuity of mass, we have

$$\liminf_{v \to v_0} \mathcal{I}(v) \geq \mathcal{I}(v_0).$$

By the existence of a continuous upper barrier function for $\mathcal{I}$ at v_0 for any $v_0 \in$

$(0, v(\Sigma))$ (see §2.5.4 below), we also have

$$\limsup_{v \to v_0} \mathcal{I}(v) \leq \mathcal{I}(v_0),$$

so $\mathcal{I}$ is continuous. □

2.5.2 First variation

We compute the first variation of the functionals a and v. Let $\{\Omega_t\}_{|t|<\varepsilon}$ be a smooth family of open sets with smooth boundary whose variation vector field along $\partial\Omega = \partial\Omega_0$ is $f\nu$, where ν is the unit normal field to $\partial\Omega$ pointing out of Ω. The following computation is standard.

Proposition 2.5.3. *We have*

$$\frac{d}{dt}\Big|_{t=0} a(\Omega_t) = \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle) f\theta^\alpha \quad \textit{and} \quad \frac{d}{dt}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} f\theta^\alpha.$$

2.5.3 Second variation

We compute the second variation of the functionals a and v. We consider the same setup as the previous subsection.

Proposition 2.5.4. *We have*

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} a(\Omega_t) = &\int_{\partial\Omega} |\nabla^{\partial\Omega} f|^2\theta^\alpha - (\mathrm{Ric}_\Sigma(\nu,\nu) + |A_{\partial\Omega}|^2) f^2\theta^\alpha + \alpha(\Delta^\Sigma\theta - \Delta^{\partial\Omega}\theta) f^2\theta^{\alpha-1} \\ &+ \int_{\partial\Omega} \alpha(\alpha-1)\langle\nabla^\Sigma\theta,\nu\rangle^2 f^2\theta^{\alpha-2} + H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle) f^2\theta^\alpha,\end{aligned}$$

and

$$\frac{d^2}{dt^2}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle) f^2\theta^\alpha.$$

Proof. By Propostion 2.5.3, we compute

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} a(\Omega_t) = &\int_{\partial\Omega} (-\Delta^{\partial\Omega} f - (\mathrm{Ric}_\Sigma(\nu,\nu) + |A_{\partial\Omega}|^2) f) f\theta^\alpha \\ &+ \int_{\partial\Omega} \alpha \,\mathrm{Hess}^\Sigma\, \theta(\nu,\nu) f^2\theta^{\alpha-1} - \alpha\langle\nabla^{\partial\Omega}\theta, \nabla^{\partial\Omega} f\rangle f\theta^{\alpha-1} \\ &+ \int_{\partial\Omega} \alpha H\langle\nabla^\Sigma\theta,\nu\rangle f^2\theta^{\alpha-1} + \alpha(\alpha-1)\langle\nabla^\Sigma\theta,\nu\rangle^2 f^2\theta^{\alpha-2} \\ &+ \int_{\partial\Omega} H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle) f^2\theta^\alpha.\end{aligned}$$

Using the formula

$$\mathrm{Hess}^\Sigma\,\phi(\nu,\nu) = \Delta^\Sigma\phi - \Delta^{\partial\Omega}\phi - H\langle\nabla^\Sigma\phi,\nu\rangle$$

and applying integration by parts to the $-f\theta^\alpha\Delta^{\partial\Omega} f$ term, we deduce

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} a(\Omega_t) = &\int_{\partial\Omega} |\nabla^{\partial\Omega} f|^2\theta^\alpha - (\mathrm{Ric}_\Sigma(\nu,\nu) + |A_{\partial\Omega}|^2) f^2\theta^\alpha + \alpha(\Delta^\Sigma\theta - \Delta^{\partial\Omega}\theta) f^2\theta^{\alpha-1} \\ &+ \int_{\partial\Omega} \alpha(\alpha-1)\langle\nabla^\Sigma\theta,\nu\rangle^2 f^2\theta^{\alpha-2} + H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle) f^2\theta^\alpha.\end{aligned}$$

Similarly, by Proposition 2.5.3, we compute

$$\frac{d^2}{dt^2}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle) f^2\theta^\alpha.$$

This completes the proof. □

2.5.4 Differential inequality in the barrier sense

Fix $v_0 \in (0, v(\Sigma))$.

Let Ω be a weighted isoperimetric surface for the problem $\mathcal{I}(v_0)$. Let $\{\Omega_t\}_{|t|<\varepsilon}$ be a smooth family of open sets with smooth boundary with $\Omega_0 = \Omega$ whose variation vector field at $t = 0$ is $\theta^{-\alpha}\nu$, where ν is the outward pointing unit normal vector field

along $\partial\Omega$.

We note that $v(t) := v(\Omega_t)$ is a smooth function. By Proposition 2.5.3, we have

$$v'(0) = \frac{d}{dt}\Big|_{t=0} v(\Omega_t) = \int_{\partial\Omega} 1 > 0.$$

By the inverse function theorem, there is some small $\sigma > 0$ and a smooth function

$$t : (v_0 - \sigma, v_0 + \sigma) \to \mathbb{R}$$

that is the inverse of $v(t)$.

Let $u : (v_0 - \sigma, v_0 + \sigma) \to \mathbb{R}$ be defined by $u(v) = a(t(v))$. Note that $u(v_0) = a(0) = \mathcal{I}(v_0)$. Moreover, since $v(\Omega_{t(v)}) = v$, we have $u(v) \geq \mathcal{I}(v)$ for all $v \in (v_0 - \sigma, v_0 + \sigma)$.

Let primes denote derivatives with respect to v and dots denote derivatives with respect to t.

Proposition 2.5.5. *The function u satisfies*

$$u''(v_0) \leq -\left(2 + \frac{1}{2}u'(v_0)^2\right) u(v_0)^{-1}.$$

Proof. We have

$$t'(v) = \frac{1}{\dot{v}(t(v))} \text{ and } t''(v) = -\frac{\ddot{v}(t(v))}{\dot{v}(t(v))^3}.$$

By Proposition 2.5.3 and 2.5.4, we have

$$t'(v_0) = \left(\int_{\partial\Omega} 1\right)^{-1} \quad \text{and} \quad t''(v_0) = -\left(\int_{\partial\Omega} 1\right)^{-3} \int_{\partial\Omega} (H + \alpha\theta^{-1}\langle\nabla^{\Sigma}\theta, \nu\rangle)\theta^{-\alpha}.$$

We note that

$$\frac{d}{dv}a(t(v)) = \dot{a}(t(v))t'(v)$$

and

$$\frac{d^2}{dv^2}a(t(v)) = \ddot{a}(t(v))t'(v)^2 + \dot{a}(t(v))t''(v).$$

We compute

$$u'(v_0) = H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle,$$

where we use the fact that isoperimetric surfaces satisfy the equation $H+\alpha\theta^{-1}\langle\nabla^\Sigma\theta, \nu\rangle = \lambda$ for some $\lambda \in \mathbb{R}$.

By (2.5.1) and Propositions 2.5.3 and 2.5.4 with $f = \theta^{-\alpha}$, we have

$$\begin{aligned}
u''(v_0)\left(\int_{\partial\Omega} 1\right)^2 &= \left(\ddot{a}(0)(t'(v_0))^2 + \dot{a}(0)t''(v_0)\right)\left(\int_{\partial\Omega} 1\right)^2 \\
&= \int_{\partial\Omega} |\nabla^{\partial\Omega}(\theta^{-\alpha})|^2\theta^\alpha - (\mathrm{Ric}_\Sigma(\nu,\nu) + |A_{\partial\Omega}|^2)\theta^{-\alpha} \\
&\quad + \int_{\partial\Omega} \alpha(\Delta^\Sigma\theta - \Delta^{\partial\Omega}\theta)\theta^{-\alpha-1} + \alpha(\alpha-1)\langle\nabla^\Sigma\theta,\nu\rangle^2\theta^{-\alpha-2} \\
&\quad + \int_{\partial\Omega} H(H + \alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle)\theta^{-\alpha} - (H+\alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle)^2\theta^{-\alpha} \\
&= \int_{\partial\Omega} (\alpha\theta^{-1}\Delta^\Sigma\theta - \mathrm{Ric}_\Sigma(\nu,\nu))\theta^{-\alpha} - \alpha|\nabla^{\partial\Omega}\theta|^2\theta^{-\alpha-2} - |A_{\partial\Omega}|^2\theta^{-\alpha} \\
&\quad + \int_{\partial\Omega} -\alpha H\langle\nabla^\Sigma\theta,\nu\rangle\theta^{-\alpha-1} - \alpha\langle\nabla^\Sigma\theta,\nu\rangle^2\theta^{-\alpha-2} \\
&\le \int_{\partial\Omega} -2\theta^{-\alpha} - \frac{1}{2}H^2\theta^{-\alpha} - \alpha H\langle\nabla^\Sigma\theta,\nu\rangle\theta^{-\alpha-1} - \alpha\langle\nabla^\Sigma\theta,\nu\rangle^2\theta^{-\alpha-2} \\
&= \int_{\partial\Omega} -\left(2 + \frac{1}{2}(H+\alpha\theta^{-1}\langle\nabla^\Sigma\theta,\nu\rangle)^2\right)\theta^{-\alpha} + \frac{1}{2}\alpha(\alpha-2)\langle\nabla^\Sigma\theta,\nu\rangle^2\theta^{-\alpha-2} \\
&\le -\left(2 + \frac{1}{2}u'(v_0)^2\right)\int_{\partial\Omega}\theta^{-\alpha},
\end{aligned}$$

where we used $0 < \alpha \le 2$. In particular, $u''(v_0) \le 0$.

By Hölder's inequality, we have

$$\left(\int_{\partial\Omega} 1\right)^2 \le \int_{\partial\Omega}\theta^\alpha \int_{\partial\Omega}\theta^{-\alpha} = u(v_0)\int_{\partial\Omega}\theta^{-\alpha}.$$

Hence, u satisfies

$$u''(v_0) \leq -\left(2 + \frac{1}{2}u'(v_0)^2\right) u(v_0)^{-1},$$

which concludes the proof. □

We consider a power of $\mathcal{I}$ and u to simplify the corresponding differential inequality. We let $\mathcal{F}(v) = \mathcal{I}(v)^{3/2}$. By Proposition 2.5.5, we have the following result.

Proposition 2.5.6. *For any $v_0 \in (0, V)$, there is a smooth function $U : (v_0 - \sigma, v_0 + \sigma) \to \mathbb{R}$ satisfying*

- $U(v_0) = \mathcal{F}(v_0)$,
- $U(v) \geq \mathcal{F}(v)$ *for all* $v \in (v_0 - \sigma, v_0 + \sigma)$,
- $U''(v_0) \leq -3U(v_0)^{-1/3}$.

Proof. We take $U(v) = u(v)^{3/2}$ as in Proposition 2.5.5, and conclude that

$$U'(v) = \frac{3}{2}u^{1/2}(v)u'(v)$$

and

$$\begin{aligned} U''(v_0) &= \frac{3}{4}u^{-1/2}(v_0)u'(v_0)^2 + \frac{3}{2}u(v_0)^{1/2}u''(v_0) \\ &\leq \frac{3}{4}u^{-1/2}(v_0)u'(v_0)^2 - 3u(v_0)^{-1/2} - \frac{3}{4}u(v_0)^{-1/2}(u'(v_0))^2 = -3U(v_0)^{-1/3}, \end{aligned}$$

as desired. □

2.5.5 Integrating the differential inequality

We study solutions to the ODE

$$f''(v) = -3f(v)^{-1/3}. \tag{2.5.2}$$

By Propositions 2.5.2 and 2.5.6 combined with the fact that $-3f^{-1/3}$ is increasing in f, it follows from a standard ODE comparison that no solution to (2.5.2) can touch $\mathcal{F}(v)$ from below unless they are equal.

Observe that $g : [0, 1) \to [0, \frac{\pi}{4})$ given by

$$g(x) = \frac{1}{3} \int_0^x \frac{dt}{\sqrt{1 - t^{2/3}}}$$

is a diffeomorphism. As such, the map $v \mapsto g^{-1}(\frac{\pi}{4} - v)$ extends to an even smooth map $\tilde{f} : [-\frac{\pi}{4}, \frac{\pi}{4}] \to \mathbb{R}$ so that $\tilde{f}(0) = 1$ and $\tilde{f}''(v) = -3\tilde{f}(v)^{-1/3}$. For $z > 0$ define $f_z : (-\frac{\pi}{4}z, \frac{\pi}{4}z) \to \mathbb{R}$ by

$$v \mapsto z^{\frac{3}{2}} \tilde{f}(z^{-1}v).$$

Note that $f_z(v)$ solves (2.5.2) and has $f_z'(0) = 0$ and $f_z(0) = z^{3/2}$. We set $\beta(z) := \frac{\pi}{4}z$ and observe that $f_z(\pm\beta(z)) = 0$. We extend f_z to all of $\mathbb{R}$ by zero.

Assume for the sake of contradiction that $v(\Sigma) = \int_\Sigma \theta^\alpha > 2\pi^2$.

We claim that there is a $\delta > 0$ so that for $z = 4\pi + \delta$, we have

$$\mathcal{F}(v) \geq f_z(v - \beta(z)) \tag{2.5.3}$$

for all $v \in (0, 2\pi^2)$.

Indeed, let $\delta > 0$ and $\varepsilon > 0$ be sufficiently small so that $\frac{\pi}{2}z + \varepsilon z < v(\Sigma)$ for $z \in (0, 4\pi + \delta)$, which is possible since $v(\Sigma) > 2\pi^2 = 2\beta(4\pi)$. Consider the graph of

$$g_z(v) = f_z(v - \beta(z) - \varepsilon z)$$

for $v \in [\varepsilon z, 2\beta(z) + \varepsilon z]$. Note that

$$g_z(\varepsilon z) = g_z(2\beta(z) + \varepsilon z) = 0 < \min\{\mathcal{F}(\varepsilon z), \mathcal{F}(2\beta(z) + \varepsilon z)\}.$$

Moreover, g_z converges uniformly to zero as $z \to 0$. Hence, if $g_{z^*}(v^*) > \mathcal{F}(v^*)$ for some v^* and z^*, then there must be some $z \in (0, z^*]$ so that g_z touches $\mathcal{F}$ from below, which contradicts Proposition 2.5.6. Therefore, we have $\mathcal{F} \geq g_z$ for every $z \in (0, 4\pi + \delta)$. We send z to $4\pi + \delta$ and then ε to 0. This proves the assertion.

We now study the asymptotic behavior of $\mathcal{F}$ and $f_{4\pi+\delta}(v - \beta(4\pi + \delta))$ as $v \to 0$.

Since $f_{4\pi+\delta}(-\beta(4\pi + \delta)) = 0$, we have $f'_{4\pi+\delta}(-\beta(4\pi + \delta)) = 3\sqrt{4\pi + \delta}$. Hence, we have

$$f_{4\pi+\delta}(v - \beta(4\pi + \delta)) = 3\sqrt{4\pi + \delta}v + o(v) \tag{2.5.4}$$

as $v \to 0$.

On the other hand, we compute an upper bound for $\mathcal{F}$ as $v \to 0$ by studying small geodesic balls. Take x_0 so that $\theta(x_0) = \min \theta = 1$. A straightforward computation gives

$$v(B_r(x_0)) = \frac{4}{3}\pi r^3 + o(r^3)$$

and

$$a(B_r(x_0)) = 4\pi r^2 + o(r^2)$$

as $r \to 0$. Solving the first equation for r and plugging into the second equation, we deduce

$$\mathcal{I}(v) \leq (36\pi)^{1/3} v^{2/3} + o(v^{2/3}),$$

so

$$\mathcal{F}(v) \leq 6\sqrt{\pi}v + o(v). \tag{2.5.5}$$

However, (2.5.3), (2.5.4), and (2.5.5) imply $3\sqrt{4\pi + \delta} \leq 6\sqrt{\pi}$, which yields a contradiction. Therefore, since we normalized so that $\min \theta = 1$, we have

$$\mathrm{Vol}(\Sigma, \gamma) \leq \int_\Sigma \theta^\alpha = v(\Sigma) \leq 2\pi^2, \tag{2.5.6}$$

which concludes the proof of Theorem 2.5.1.

Remark 2.5.7. We remark that the rigidity case of Theorem 2.5.1 follows easily. We proved above that $v(\Sigma) \leq 2\pi^2$. As such, if $\mathrm{Vol}(\Sigma, \gamma) = 2\pi^2$ then (2.5.6) and the normalization of θ gives that $\theta \equiv 1$ on Σ. Thus, (2.5.1) gives $\lambda_{\mathrm{Ric}}(\gamma) \geq 2$. As such, the rigidity case in the classical Bishop–Gromov theorem implies that (Σ, γ) is isometric to the round 3-sphere.

2.6 Stable Bernstein Theorem

We use the estimates of the previous sections to prove the stable Bernstein theorem in $\mathbb{R}^5$ as well as the various consequences indicated in the introduction.

Let $F : M^4 \to \mathbb{R}^5$ be a complete, simply connected, two-sided stable minimal hypersurface, and let g denote the pullback metric on M. On $N = M \setminus F^{-1}(0)$, let $\tilde{g} = r^{-2}g$, where r is the Euclidean distance function from 0.

Combining Theorem 2.3.1, Theorem 2.4.1, and Theorem 2.5.1 (with the appropriate rescaling of the μ-bubble metric), we have the following tool.

Lemma 2.6.1. *Let $X \subset N$ be a closed subset with boundary $\partial X = \partial_+ X \sqcup \partial_- X$. Suppose $d_{\tilde{g}}(\partial_+ X, \partial_- X) \geq 10\pi$. Then there is a connected, relatively open subset $\Omega \subset X$ with smooth boundary $\partial\Omega = \partial_- X \sqcup \Sigma$ so that*

- *$\partial_- X \subset \Omega$,*
- *$\Sigma \subset X \setminus \partial X$ is a closed hypersurface,*
- *$\Omega \subset B_{10\pi}(\partial_- X)$, and*
- *any connected component Σ_0 of Σ has intrinsic diameter at most 2π and volume at most $16\pi^2$.*

We first prove *a priori* that any complete, simply connected, two-sided stable minimal immersion $M^4 \to \mathbb{R}^5$ has Euclidean volume growth. The proof follows the same strategy as [CL23].

Theorem 2.6.2. *Any complete, simply connected, two-sided stable minimal immersion $M^4 \to \mathbb{R}^5$ satisfies*

$$\mathcal{H}^4(B_\rho(x_0) \subset M) \leq 8\pi^2 e^{44\pi} \rho^4$$

for all $\rho > 0$ and $x_0 \in M$.

Proof. Without loss of generality we can assume that $0 \in F(M)$ and that $F(x_0) = 0$. By Lemma 2.6.1, there is a relatively open subset $\tilde{\Omega} \subset N \setminus B_\rho(x_0)$ such that

- $\partial B_\rho(x_0) \subset \tilde{\Omega}$,
- $\tilde{\Omega} \subset \tilde{B}_{10\pi}(\partial B_\rho(x_0))$, and
- any connected component of $\partial\tilde{\Omega} \setminus \partial B_\rho(x_0)$ is smooth with $\tilde{g}$-volume at most $16\pi^2$.

Converting back to the metric g (see [CL23, Lemma 6.2]), we have

$$\tilde{\Omega} \subset \{x \in M : d_M(x, \partial B_\rho(x_0)) < \rho e^{10\pi}\} \subset B_{\rho e^{11\pi}}(x_0).$$

Hence, any connected component of $\partial\tilde{\Omega} \setminus \partial B_\rho(x_0)$ has g-volume at most $16\pi^2 e^{33\pi}\rho^3$. Since M is simply connected and has one end (by [CSZ97]), there is a precompact open set Ω containing $B_\rho(x_0) \cup \tilde{\Omega}$ that has exactly one boundary component, which is one of the components of $\partial\tilde{\Omega} \setminus \partial B_\rho(x_0)$. By the isoperimetric inequality for minimal hypersurfaces in $\mathbb{R}^{n+1}$ [MS73] (we use the sharp version due to Brendle [Bre21] to improve the resulting constant), we have

$$\mathcal{H}^4(B_\rho(x_0) \subset M) \leq c_{\text{iso}}(16\pi^2)^{4/3} e^{44\pi} \rho^4,$$

where $c_{\text{iso}} = (128\pi^2)^{-1/3}$ in this dimension. This concludes the proof. □

Proof of Theorem 2.1.1. Suppose that $M^4 \to \mathbb{R}^5$ is a complete, two-sided stable minimal immersion. Since stability passes to the universal cover, we can assume that M is simply connected. Theorem 2.6.2 implies that M has (intrinsic) Euclidean volume growth. Thus, [SSY75] implies that M is flat. □

Proof of Corollary 2.1.2. Arguing exactly as in [CLS22, Corollary 2.5] if the asserted curvature estimates failed then there would exist a non-flat complete, two-sided stable minimal immersion $M^4 \to \mathbb{R}^5$. This contradicts Theorem 2.1.1. □

Proof of Theorem 2.1.3. By Lemma 2.6.1, there is a relatively open set $\tilde{\Omega} \subset M \setminus F^{-1}(\{0\})$ so that

- $\partial M \subset \tilde{\Omega}$,
- $\tilde{\Omega} \subset \tilde{B}_{10\pi}(\partial M)$, and
- any connected component of $\partial\tilde{\Omega} \setminus \partial M$ is smooth with $\tilde{g}$-volume at most $16\pi^2$.

Since M is simply connected and ∂M is connected, we can find an open subset $M' \Subset \text{Int}(M)$ so that $\partial M'$ is connected and consists of one of the components of $\partial\tilde{\Omega} \setminus \partial M$. Converting between distances in g and distances in $\tilde{g}$ by [CL23, Lemma 25], we have $M^*_{\rho_0} \subset M'$, where $\rho_0 = e^{-11\pi}$ and $M^*_{\rho_0}$ is the connected component of $F^{-1}(B_{\mathbb{R}^5}(0, \rho_0))$ that contains x_0. By the same calculation as in the proof of Theorem 2.6.2, we have

$$\mathcal{H}^4(M^*_{\rho_0}) \leq 8\pi^2.$$

This finishes the proof. □

Proof of Theorem 2.1.4. We follow [CL24b, §6]. By [Tys89, §3], if a complete, two-sided, minimal immersion $M^4 \to \mathbb{R}^5$ has $\int_M |A_M|^4 < \infty$ then M has finite index. As such, it suffices to consider a complete, two-sided, minimal immersion $M^4 \to \mathbb{R}^5$ with

finite Morse index. Fix $x_0 \in M$ and consider $(M, \tilde{g})$ the Gulliver–Lawson conformal metric. Arguing as in [CL24b, Lemma 21], there's $k < \infty$ so that for any $\rho > 0$ sufficiently large so that $M \setminus B_\rho(x_0)$ (where $B_\rho(x_0)$ is the g-metric ball) is stable and if $B_\rho(x_0) \subset \Omega$ is bounded with smooth boundary and $M \setminus \Omega$ has only unbounded components, then $\partial\Omega$ has k components.

The proof of Theorem 2.6.2 directly carries over to prove that

$$\mathcal{H}^4(B_\rho(x_0) \subset M) \leq k\, 8\pi^2 e^{44\pi} \rho^4$$

for all $\rho \gg 0$ sufficiently large. Using the L^4-estimates from [SSY75] on the exterior region of M we thus conclude that $\int_M |A_M|^4 < \infty$. It follows from [And84, Theorem A] (cf. [SZ98, Tys89]) that the ends of M are graphical over hyperplanes with bounded slope (see [And84, p. 22]). Granted this, [Sch83b, Proposition 3] gives they are regular at infinity. This completes the proof. □

Chapter 3

Rigidity of noncompact stable minimal hypersurfaces in positive curvature

3.1 Introduction

Recall that a *two-sided stable minimal hypersurface* $M^{n-1} \to (X^n, g)$ is an immersed hypersurface with vanishing mean curvature and trivial normal bundle satisfying the inequality

$$\int_M (|A_M|^2 + \mathrm{Ric}_g(\nu, \nu))\varphi^2 \leq \int_M |\nabla \varphi|^2$$

for any $\varphi \in C_c^\infty(M \backslash \partial M)$. Here, A_M is the second fundamental form of the immersion, Ric_g is the ambient Ricci curvature, and ν is (any) choice of unit normal. Stable minimal hypersurfaces can be used in a similar manner to stable geodesics to probe the ambient geometry of (X^n, g). The basic results along these lines are as follows. When $M^{n-1} \to (X^n, g)$ is a *closed* (compact without boundary) two-sided stable minimal hypersurface, it holds that

1. If $\mathrm{Ric}_g \geq 0$ then M is totally geodesic and $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M [Sim68]

(cf. [BT73]). In particular, when $\mathrm{Ric} > 0$, there are no closed two-sided stable minimal hypersurfaces.

2. If $R_g \geq 1$ (scalar curvature) then M also admits a metric of positive scalar curvature [SY79a, SY79b]. In particular, when $n = 3$ and X is oriented, each component of M must be a 2-sphere.

The second result is a fundamental tool in the study of manifolds of positive scalar curvature.

When M is now assumed to be complete with respect to the induced metric (in particular, M has no boundary) and *non-compact*, the theory becomes considerably more complicated. As the following results show, it has been well-developed in 3-dimensions. Consider $M^2 \to (X^3, g)$ complete two-sided stable minimal immersion:

1. When $R_g \geq 0$, the induced metric on M is conformal to either the plane or the cylinder [FCS80]. In the latter case, M is totally geodesic, intrinsically flat, and $R_g \equiv 0, \mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M (cf. [CCE16, Proposition C.1]).

2. When $\mathrm{Ric}_g \geq 0$, M is totally geodesic, intrinsically flat, and $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M [SY82] (cf. [FCS80, dCP79, Pog81]).

3. When $R_g \geq 1$, M must be compact [SY83] (cf. [GL83]).

These results have had important applications to comparison geometry. In particular, we refer to the following (incomplete) list of results that rely specifically on (non-compact) stable minimal surfaces in 3-manifolds: [SY82, GL83, AR89, Liu13, Wan24]. For most of these applications, it is essential that no assumption is made related to properness or volume growth of M.

The first- and second- named authors have recently resolved [CL24b] the conjecture of Schoen that a complete two-sided stable minimal immersion $M^3 \to \mathbf{R}^4$ is flat (without any additional assumptions on M); see also the subsequent proofs in

[CL23, CMR24]. It is natural to ask what happens in a curved ambient background, rather than flat Euclidean space. For example, a basic question is to determine which ambient curvature conditions suffice to show that a complete stable minimal hypersurface must be compact, or cannot exist. The methods used in [CL24b] do not seem to extend to the curved setting.[1] (See [SSY75, SY96, CSZ97, LW04] for previous results in this direction.)

In fact, (as was discussed above) in a 3-manifold, $R_g \geq 1$ implies compactness and $\mathrm{Ric}_g > 0$ implies non-existence; on the other hand, in 4-dimensions, neither of these results can hold:

Example 3.1.1 (Non-compact stable minimal hypersurface in 4-manifold with $R_g \geq 1$)**.** Fix an oriented closed 2-dimensional Riemannian manifold (X^2, g) admitting

$$\sigma : \mathbf{R} \to (X^2, g)$$

unit speed stable geodesic. Then, for $\varepsilon > 0$ sufficiently small $(X^2, g) \times \mathbf{S}^2(\varepsilon)$ has $R_g \geq 1$, while $\sigma \times \mathbf{S}^2(\varepsilon)$ is an unbounded two-sided stable minimal immersion.

Example 3.1.2 (Existence of stable minimal hypersurface in a 4-manifold with positive sectional curvature)**.** Consider a rotationally symmetric metric g on $\mathbf{R}^4$ with strictly positive sectional curvatures. There are totally geodesic copies of $\mathbf{R}^3$ in such a metric, and as long as the Ricci curvature of g decays to zero sufficiently fast, these will be (complete) two-sided stable minimal embeddings. Essentially, the point is that on flat Euclidean 3-space $(\mathbf{R}^3, \bar{g})$, any Schrödinger operator with non-negative, rapidly decaying potential[2] will be stable, thanks to a classical Hardy inequality

$$\int_{\mathbf{R}^3} 4|\bar{\nabla}\varphi|^2 d\bar{\mu} \geq \int_{\mathbf{R}^3} r^{-2}\varphi^2 d\bar{\mu}.$$

[1] In particular, the use of the L^3 Schoen–Simon–Yau inequality [SSY75] seems troublesome in most curved ambient manifolds.

[2] Note that a Schrödinger operator with non-negative (but not identically zero) potential cannot be stable on $\mathbf{R}^2$, thanks to the log-cutoff trick.

We explicitly construct such an example and check its stability in Section 3.8.1.

3.1.1 Main results

It turns out that if one *combines* positivity of curvature with strict positivity of scalar curvature, such examples can be avoided. This is the main result of this article.

Theorem 3.1.3. *If (X^4, g) is a complete 4-manifold with weakly bounded geometry, non-negative sectional curvature, and strictly positive scalar curvature $R_g \geq R_0 > 0$, then any complete two-sided stable minimal immersion $M^3 \to (X^4, g)$ is totally geodesic and has $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M.*

Here, weakly bounded geometry means that around any point, there is a *local* diffeomorphism from a ball in $\mathbb{R}^4$ of definite size so that the pullback metric is $C^{1,\alpha}$-close to Euclidean (see Definition 3.2.3). (Note that this allows for collapsing behavior at infinity.)

Remark 3.1.4. All results in this paper assume that (X, g) at least has $\mathrm{Ric}_g \geq 0$, in which case "weakly bounded geometry" could be replaced by the assumption "$\mathrm{sec}_g \leq K$" anywhere it occurs here. It would be interesting to understand if this condition could be removed.

Remark 3.1.5. Shen–Ye [SY96] proved that for $n \leq 4$, if (X^{n+1}, g) satisfies

$$\mathrm{BiRic} \geq k > 0, \tag{3.1.1}$$

then any complete two-sided stable minimal immersion $M^n \to (X^{n+1}, g)$ is compact. Here BiRic is called the bi-Ricci curvature, defined in [SY96] ($\mathrm{BiRic} \geq k$ implies that $R_g \geq c_n k$ for some dimensional constant c_n). (See Appendix 3.7 for precise definitions of these curvature conditions as well as Section 3.1.2 below.)

In particular, it follows that such a minimal immersion does not exist if (X^{n+1}, g) is assumed to be closed with positive sectional curvature. After our paper appeared, the Shen–Ye result was rediscovered and extended to $n = 5$ by Catino–Mastrolia–Roncoroni [CMR24], under the additional assumption of nonnegative sectional curvature (or even just nonnegative 4-intermediate Ricci curvature). The approach in [SY96, CMR24] is very different from that of this paper and is based on a conformal deformation technique first introduced by Fischer-Colbrie [FC85]. An interesting feature of our paper (see Theorem 3.1.10) is that we are able to replace (when $n = 3$) the condition $\mathrm{BiRic} \geq k > 0$ by the weaker assumption on scalar curvature $R_g \geq R_0 > 0$ in (3.1.1). We note that the condition considered in this paper allows for the existence of complete, *non-compact* stable minimal immersions (we prove they must be totally geodesic and have vanishing normal Ricci curvature). For example, one may consider $S^2 \times \mathbf{R} \to S^2 \times \mathbf{R} \times S^1$.

Of course, any closed manifold (X, g) has weakly bounded geometry, so we have the following result:

Corollary 3.1.6. *If (X^4, g) is a closed 4-manifold with non-negative sectional curvature and positive scalar curvature, then a complete two-sided stable minimal immersion $M^3 \to (X^4, g)$ is totally geodesic and $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M.*

This applies (for example) to the standard product metrics on S^4, $S^1 \times S^3$, $S^2 \times S^2$, and $T^2 \times S^2$. We emphasize that the latter example does not have positive bi-Ricci curvature, and is thus not covered by the results in [SY96, CMR24].

Remark 3.1.7. The study of complete (non-compact) stable minimal immersions in a *compact* 3-manifold has played an important role in the study of embedded minimal surfaces with bounded Morse index [CKM17, Car17, LZ16]. One may hope that the Corollary 3.1.6 could lead to similar results in closed 4-manifolds.

Remark 3.1.8. Theorem 3.1.3 fails in 6-dimensional manifolds (and higher). See Appendix 3.8.3 for a counterexample. Gromov's conjecture that a 4-manifold with uniformly positive scalar curvature has macroscopic dimension 2 suggests that the 5-dimensional version of Theorem 3.1.3 would be the "critical" dimension (this is related to the log-cutoff trick for minimal surfaces in 3-manifolds).

3.1.2 Concerning the assumption of non-negative sectional curvature

It is natural to ask whether or not there exist complete two-sided stable minimal immersions $M^3 \to (X^4, g)$ where (X^4, g) has weakly bounded geometry, $\mathrm{Ric}_g \geq 0$ and $R_g \geq 1$. (Note that in Theorem 3.1.3, $\mathrm{Ric}_g \geq 0$ is replaced by non-negative sectional curvature.)

To this end, we have the following example indicating that this is unlikely to be true.

Example 3.1.9 (Stable minimal hypersurface in a 4-manifold with strictly positive Ricci curvature)**.** There exists a closed 4-manifold (X^4, g) (so it automatically has bounded geometry) and a complete two-sided stable minimal immersion $M^3 \to (X^4, g)$ so that $\mathrm{Ric}_g \geq 1$ (so, in particular $R_g \geq 1$) in some δ-neighborhood of the image of M.

This example is constructed in detail in Section 3.8.2, but we briefly describe it here. The minimal immersion $M^3 \to (X^4, g)$ is constructed by taking the universal cover of some embedded compact[3] minimal submanfold $M_0 \subset (X^4, g)$. We choose M_0 so that it is diffeomorphic to $(S^1 \times S^2)\#(S^1 \times S^2)$, so $\pi_1(M_0) = F_2$ is a "non-amenable" group. This implies that the universal cover has positive first eigenvalue. As such, as long as $|A_M|^2 + \mathrm{Ric}_g(\nu, \nu)$ is sufficiently small, the universal cover will be

[3]Note that M_0 is necessarily unstable, since $\mathrm{Ric}_g(\nu, \nu) > 0$; the instability occurs due to the "largeness" of the universal cover.

stable.

We note that it is unclear if one can construct such an example where (X^4, g) is complete and satisfies $\mathrm{Ric}_g > 0$ and $R_g \geq 1$ at all points (not just near the image of M). (We discuss the possibility of doing so further in Section 3.8.2.) However, we emphasize that this example precludes any argument concerning stable minimal immersions in such (X^4, g) that is based purely on the second variation of area.[4]

On the other hand, it is possible to weaken non-negativity of sectional curvature to an intermediate condition. Namely, we say that (X^4, g) has non-negative 2-intermediate Ricci curvature if the sectional curvatures satisfy $\sec_g(\Pi_1)+\sec_g(\Pi_2) \geq 0$ for any two 2-planes $\Pi_1, \Pi_2 \subset T_pX$ intersecting in a line at a right angle (cf. Definition 3.2.1.) We will write $\mathrm{Ric}_2^g \geq 0$ for this condition. (Note that non-negative sectional curvature implies $\mathrm{Ric}_2^g \geq 0$ which implies $\mathrm{Ric}_g \geq 0$.) The results discussed above all hold with non-negative sectional curvature replaced by $\mathrm{Ric}_2^g \geq 0$. More precisely, we have

Theorem 3.1.10. *If (X^4, g) is a complete 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g \geq 0$, and $R_g \geq R_0 > 0$, then any complete two-sided stable minimal immersion $M^3 \to (X^4, g)$ is totally geodesic and has $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M.*

We discuss the topological implications of this result in Section 3.6.

We remark that $\mathrm{Ric}_g \geq 0$ (instead of $\mathrm{Ric}_2^g \geq 0$) suffices for most (but not all) steps of the proof of Theorem 3.1.10, so if one places additional topological assumptions on the minimal hypersurface, then the proof of Theorem 3.1.10 can be adapted in a straightforward way to show the following result.

[4]Note that Li–Wang have used the Busemann function in their analysis of *proper* ends of stable minimal hypersurfaces in non-negatively curved manifolds [LW04]. As such, this argument relies on the ambient geometry outside of just a tubular neighborhood of the immersion. However, it is unclear how to extend such a argument to non-negative Ricci curvature (the Busemann function is subharmonic rather than convex so it is difficult to control the restriction to a minimal hypersurface). Along these lines, we note that the stable minimal immersion $M^3 \to (X^4, g)$ discussed in Example 3.1.9 has infinitely many ends.

Theorem 3.1.11. *Let (X^4, g) be a complete 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g \geq -K$, $\mathrm{Ric}_g \geq 0$, and $R_g \geq R_0 > 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion with finitely many ends and $b_1(M) < \infty$. Then M is totally geodesic and has $\mathrm{Ric}_g(\nu, \nu) \equiv 0$ along M.*

3.1.3 Topological applications

Recall that Schoen–Yau used minimal surfaces to prove that a complete non-compact (X^3, g) with $\mathrm{Ric}_g > 0$ is diffeomorphic to $\mathbb{R}^3$ [SY82] (cf. [AR89, Liu13]). They did this by showing that $\pi_2(X) = 0$ and that X is simply connected at infinity. In this section, we observe that given Theorem 3.1.10, similar techniques can be used to prove new topological restrictions on (X^4, g) with $\mathrm{Ric}_2^g > 0$, $R_g \geq R_0 > 0$, and weakly bounded geometry. (See [Mou] for a collection of results concerning the Ric_k^g curvature condition.)

All homology groups and cohomology groups here will be taken with $\mathbb{Z}$-coefficients.

Definition 3.1.12. Let X be a non-compact manifold, $k \in \mathbb{N}$. Define the k-th homology group of X at infinity as the inverse limit

$$H_k^\infty(X) = \varprojlim_{A \Subset X} H_k(X - A).$$

We say X is H_k-trivial at infinity if the natural homeomorphism $H_k^\infty(X) \to H_k(X)$ is injective.

In section 3.6 we prove the following result.

Theorem 3.1.13. *Suppose (X^4, g) is a complete, non-compact, oriented 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g > 0$, and strictly positive scalar curvature $R_g \geq R_0 > 0$. Then we have:*

1. *$H_1(X)$ only has torsion elements;*

2. X is H_2-trivial at infinity.

It would be interesting to see if the techniques from [AR89, Liu13] (cf. [CCE16, CEM19]) could be adapted to study the $\mathrm{Ric}_2^g \geq 0$ rigidity version of this result.

Remark 3.1.14. If (X^n, g) has $\mathrm{Ric}_g > 0$, then $H_{n-1}(X) = 0$ by [Yau76, SS01].

In light of Examples 3.1.1 and 3.1.2, it would be interesting to know if Theorem 3.1.13 held under weaker curvature conditions (e.g., $\mathrm{Ric}_2^g > 0$ but without $R_g \geq R_0 > 0$).

3.1.4 Idea of the proof and related results

An interesting feature of Theorem 3.1.3 is that it combines different curvature conditions. Indeed, the different conditions come in at rather different places in the proof. As a consequence, the strategy of the proof is quite different from most previous results concerning stability of complete (non-compact) minimal surfaces (including the previous work of the first- and second-named authors on stable minimal immersions $M^3 \to \mathbf{R}^4$ [CL24b]). A key feature of this paper is the use of μ-bubbles, a technique introduced by Gromov [Gro96], allowing one to gain distance control (in certain settings) from positivity of scalar curvature.

The overall strategy is as follows, indicating which curvature conditions enter at which step. Consider $M^3 \to (X^4, g)$ a complete two-sided stable minimal hypersurface.

1. To show that M is totally geodesic and has vanishing normal Ricci curvature, we would like to find a sequence of compactly supported functions φ_i with $\varphi_i \to 1$ and $\int_M |\nabla \varphi_i|^2 = o(1)$. Taking such functions in the stability inequality then gives the desired conclusion (this just uses $\mathrm{Ric}_g \geq 0$). Note that for minimal surfaces in a 3-manifold, this condition is usually guaranteed by proving that the surface is conformal to the plane and then using the log-cutoff trick. However,

in the higher dimensions, this is not usually not possible (e.g., no such functions exist on flat $\mathbf{R}^3$).

2. The non-negativity of sectional curvature (or $\mathrm{Ric}_2^g \geq 0$) ensures that M has at most one nonparabolic end (see Definition 3.3.3). Nonparabolic ends are more difficult to handle, since parabolic ends automatically admit test good functions as described in (1).

3. The remaining issue[5] is to construct a good test function along the nonparabolic end. In general, this is not possible with only a non-negative sectional curvature assumption (cf. Example 3.1.2). This is where the positivity of scalar curvature comes in. One knows (going back to the work of Schoen–Yau [SY79b]) that a stable minimal hypersurface in a positive scalar curvature manifold "acts" as if it itself has positive scalar curvature.

   In particular, a 3-dimensional manifold with positive scalar curvature tends to be macroscopically 1-dimensional in various senses (cf. [GL83, MN12, Son18, LM23]). As such, one might imagine that a single end of M has *linear* volume growth (this is where the parabolic versus nonparabolic distinction from (2) is important). Indeed, a metric tree is 1-dimensional but has exponential length growth - for instance the universal cover of $(S^1 \times S^1)\#(S^1 \times S^2)$. Note that if this end had linear volume growth, then the standard cutoff function that is 1 in B_R and 0 in B_{2R} would have vanishing Dirichlet energy as $R \to \infty$, so this would allow us to construct the desired cutoff function.

   It seems difficult (or potentially impossible)[6] to actually prove that the nonparabolic end E has linear volume growth. Instead we construct an exhaustion

[5]The reality is somewhat more complicated, since the nonparabolic component could have infinitely many parabolic ends attached, but this captures the main idea.

[6]One can imagine that E is a cylinder $S^2 \times (0, \infty)$ connected sum at integer points with spheres S^3 with volume tending to ∞ and scalar curvature ≥ 1. It is unclear if this could be ruled out by stability.

of E by bounded sets $\Omega_1 \subset \Omega_2 \subset \dots$ so that $\partial\Omega_i \cap E$ has controlled diameter. To do so, we use the theory of μ-bubbles (surfaces of carefully prescribed mean curvature) first introduced by Gromov [Gro96] (see also [Gro18]), allowing one to study metric quantities related to scalar curvature. (This application is similar in spirit to the use of μ-bubbles in [CL24a, Gro20, CLL23].) This is the step that uses the assumption of strictly positive scalar curvature.

4. Finally, to construct a good test function, it remains to show that the L-neighborhood of $\partial\Omega_i \cap E$ has bounded volume (for some constant L). To upgrade diameter control (obtained in (3)) to volume control, we need to control (from below) the intrinsic Ricci curvature of the minimal immersion M. This is achieved (thanks to the Gauss equations) by combining a lower bound on the sectional (or Ric_2^g) curvatures with an absolute bound on the second fundamental form of M as follows from [CL24b]. This is where the weakly bounded geometry assumption enters.

We remark that steps (3) and (4) are somewhat reminiscent of a recent result (but not the proof) of Munteanu–Wang proving that a complete 3-manifold with $R_g \geq 1$ and $\mathrm{Ric}_g \geq 0$ has linear volume growth [MW22] (see also [Xu20, MW23, Zhu22]). See our recent work [CLS23] for an approach to this result following the methods of this paper.

3.1.5 Organization of the paper

In Section 3.2 we discuss the $\mathrm{Ric}_2 \geq 0$ and bounded geometry conditions used here. Section 3.3 contains some preliminary results concerning the ends of non-compact manifolds. We then study the nonparabolic ends in further detail in Section 3.4. We discuss the μ-bubble exhaustion result and corresponding volume growth estimates and then prove the main results in Section 3.5. We discuss some topological appli-

cations of the main results in Section 3.6. Appendix 3.7 contains various curvature conditions referred to in the paper. Appendix 3.8 contains some examples relevant to the main results. Finally, Appendix 3.9 describes how to pullback immersions along local diffeomorphisms. Finally, Appendix 3.10 relates sectional curvature bounds to the weakly bounded geometry condition.

3.2 Curvature Preliminaries

If (X, g) is a Riemannian manifold, we denote the Riemann curvature tensor by $R_g(\cdot,\cdot,\cdot,\cdot)$ with 4 arguments, the sectional curvature by $\sec_g$, the Ricci curvature by Ric_g, and the scalar curvature by R_g with no argument. If $M \to (X, g)$ is an immersion, we write Ric_M and R_M to denote respectively the Ricci curvature and scalar curvature of the pullback metric on M. We follow the curvature conventions used in [Pet16, Chapter 3]; in particular, if $\{u, v\}$ is an orthonormal basis for a 2-plane $\Pi \subset T_pX$, then $\sec_g(\Pi) = R_g(v, u, u, v)$.

3.2.1 The $\mathrm{Ric}_2 \geq 0$ Condition

Let (X^4, g) be a complete Riemannian 4-manifold. The following curvature condition is tailored to exploit the Gauss equation for hypersurfaces in 4-manifolds, and sits between non-negative sectional curvature and non-negative Ricci curvature.

Definition 3.2.1. (X^4, g) satisfies $\mathrm{Ric}_2^g \geq 0$ if

$$R_g(v, u, u, v) + R_g(w, u, u, w) \geq 0$$

for any orthonormal vectors $\{u, v, w\}$ in TX.

This condition has been called non-negativity of 2-*intermediate Ricci curvature* in the literature (cf. Appendix 3.7). In particular, we note that there is a metric on

$S^2 \times S^2$ with strictly positive 2-intermediate Ricci curvature [Mue87] (see [Mou22, Example 2.3]).

Lemma 3.2.2. *Let (X^4, g) be a 4-manifold with $\mathrm{Ric}_2^g \geq 0$. If $M^3 \to (X^4, g)$ is a minimal immersion, then*

$$\mathrm{Ric}_M \geq -|A_M|^2.$$

Proof. Pick an orthonormal basis $\{e_l\}$ at a point $p \in M$ with e_4 normal to M. The traced Gauss equation gives

$$R_g(e_2, e_1, e_1, e_2) + R_g(e_3, e_1, e_1, e_3) - \mathrm{Ric}_M(e_1, e_1) = \sum_{j=1}^{3} A_M(e_1, e_j)^2.$$

Rearranging and using $\mathrm{Ric}_2^g \geq 0$ we have

$$\mathrm{Ric}_M(e_1, e_1) \geq R_g(e_2, e_1, e_1, e_2) + R_g(e_3, e_1, e_1, e_3) - |A_M|^2 \geq -|A_M|^2.$$

Since e_1 was an arbitrary unit vector in T_pM, the conclusion follows. □

3.2.2 Weakly Bounded Geometry

Definition 3.2.3. In this paper, we say that a complete Riemannian manifold (X^n, g) has *Q-weakly bounded geometry* if for any $p \in X$, there is a $C^{2,\alpha}$ local diffeomorphism $\Phi : ((B(0, Q^{-1}), 0) \subset \mathbb{R}^n \to (U, p) \subset X$ so that

1. we have $e^{-2Q}\delta \leq \Phi^* g \leq e^{2Q}\delta$ in the sense of bilinear forms, and
2. it holds that $\|\partial_k \Phi^* g_{ij}\|_{C^\alpha} \leq Q$.

We will often say that (X, g) has "weakly bounded geometry" if the above definition holds for some Q.

Note that this condition allows for some collapsing behavior of (X, g) at infinity.[7]

[7]For example, a hyperbolic cusp has weakly bounded geometry but the injectivity radius tends to zero at infinity.

It is well-known that $|\sec_g| \leq K < \infty$ implies Q-weakly bounded geometry for $Q = Q(K)$ (cf. [RST10]). We outline the proof of this fact in Proposition 3.10.1. Note also that $\mathrm{Ric}_g \geq 0$ and $\sec_g \leq K$ imply that $|\sec_g| \leq (n-2)K$, and in this paper the ambient space (X^4, g) is always assumed to have $\mathrm{Ric}_g \geq 0$ (often a stronger condition is assumed).

Alternatively, we recall that $|\mathrm{Ric}_g| \leq K$ and $\mathrm{inj}(X, g) \geq i_0 > 0$ also implies this condition (and actually one can replace "local diffeomorphism" by "diffeomorphism" in this case); this follows from [And90] (cf. [Pet16, Theorem 11.4.3]).

The first- and second-named authors have recently proven that a two-sided complete stable minimal hypersurface $M^3 \to \mathbf{R}^4$ is flat [CL24b]. Using a standard blow-up argument, this yields curvature estimates for stable minimal hypersurfaces in (compact) 4-manifolds. Here, we observe that the same thing holds for non-compact 4-manifolds, assuming the weakly bounded geometry condition. The proof below is similar to the argument used in [RST10] (see also [Coo10]) to prove curvature estimates for stable minimal (or more generally CMC) surfaces in 3-manifolds (cf. [Sch83a]).

Lemma 3.2.4. *Let (X^4, g) be a complete 4-manifold with Q-weakly bounded geometry. Then there is a constant $C = C(Q) < \infty$ such that every compact two-sided stable minimal immersion $M^3 \to (X^4, g)$ satisfies*

$$\sup_{q \in M} |A_M(q)| \min\{1, d_M(q, \partial M)\} \leq C.$$

Proof. This follows by combining a standard point picking argument (cf. [CL24b, Theorem 3]) with the lifting argument described in Appendix 3.9. For the sake of completeness we describe the full argument below.

For contradiction, we assume that the assertion fails. Then there is a sequence of

compact two-sided stable minimal immersions $M_i^3 \to (X_i^4, g_i)$ such that

$$\sup_{q \in M_i} |A_{M_i}(q)| \min\{1, d_{M_i}(q, \partial M_i)\} \to \infty$$

as $i \to \infty$. Since M_i is compact and the argument of the supremum is continuous and vanishes on ∂M_i, there is $p_i \in M_i \setminus \partial M_i$ with

$$|A_{M_i}(p_i)| \min\{1, d_{M_i}(p_i, \partial M_i)\} = \sup_{q \in M_i} |A_{M_i}(q)| \min\{1, d_{M_i}(q, \partial M_i)\} \to \infty.$$

Define $r_i = |A_{M_i}(p_i)|^{-1} \to 0$ and x_i the image of p_i in X_i. By the weakly bounded geometry assumption, there are local diffeomorphisms

$$\Phi_i : (B(0, Q^{-1}), 0) \subset \mathbf{R}^4 \to (X_i, x_i)$$

with $e^{-2Q}\delta \leq \Phi^* g_i \leq e^{2Q}\delta$ and $\|\partial_a \Phi_i^*(g_i)_{bc}\|_{C^\alpha} \leq Q$.

By the pullback operation described in Appendix 3.9, we can find a sequence of pointed 3-manifolds (S_i, s_i), immersions $F_i : (S_i, s_i) \to (B(0, Q^{-1}), 0)$, and local diffeomorphisms

$$\Psi_i : (S_i, s_i) \to (M_i, p_i)$$

so that the following diagram commutes (writing $B = B(0, Q^{-1}) \subset \mathbf{R}^4$)

$$\begin{array}{ccc} S_i & \xrightarrow{F_i} & B \\ \downarrow{\scriptstyle \Psi_i} & & \downarrow{\scriptstyle \Phi_i} \\ M_i & \longrightarrow & X_i \end{array}$$

and so that $F_i : S_i \to (B, \Phi_i^* g_i)$ is a two-sided stable minimal immersion. (Two-sided and minimality follow immediately as they are local properties, while stability follows by lifting a positive first eigenfunction for the stability operator on M_i to S_i.)

Define maps $D_i : B(0, r_i^{-1} Q^{-1}) \to B(0, Q^{-1})$, $x \mapsto r_i x$ and then consider the

metrics $\tilde{g}_i := r_i^{-2} D_i^* \Phi^* g_i$ on $B(0, r_i^{-1} Q^{-1})$. The bounded geometry condition ensures that $\tilde{g}_i$ converges to the flat metric δ on $\mathbf{R}^4$ in $C^{1,\alpha}_{\mathrm{loc}}$ (in sense of $C^{1,\alpha}_{\mathrm{loc}}$ convergence of the metric coefficients in Euclidean coordinates). In particular, the Christoffel symbols of $\tilde{g}_i$ with respect to the Euclidean coordinates on $B(0, r_i^{-1} Q^{-1})$ converge to 0 in $C^{0,\alpha}_{\mathrm{loc}}$.

Consider the rescaled immersions $\tilde{F}_i := D_i^{-1} \circ F_i : S_i \to B(0, r_i^{-1} Q^{-1})$. Observe that $\tilde{F}_i$ is a minimal immersion with respect to $\tilde{g}_i$. Furthermore, by the point picking argument, if $q \in S_i$ has $d(s_i, q) \leq \rho$ (with respect to $\tilde{F}_i^* g_i$) then the second fundamental fortm of $\tilde{F}_i$ with respect to $\tilde{g}_i$ satisfies $|A_{\tilde{F}_i}(q)| \leq C = C(\rho)$.

The bounded curvature condition ensures there is $\mu > 0$ (a numerical constant) so that for any $q \in S_i$, for i sufficiently large, we can write $B^{S_i}_\mu(q)$ as a normal graph (in the Euclidean coordinates) over a subset of $T_q S_i$ of a function f_i with (Euclidean) C^2-norm $\leq \mu$ (cf. [CM11, Lemma 2.4], except one should use that the Christoffel symbols are uniformly controlled). Geometric considerations show that the area of such a graph depends on the metric coefficients of $\tilde{g}_i$ (but not derivatives of the metric). In particular, the first variation formula (and minimality of $\tilde{F}_i$) implies that f_i satisfies a (uniformly) elliptic PDE in divergence form whose coefficients depend on the metric coefficients $\tilde{g}_i$ (but not derivatives). Thus, f_i satisfies a non-divergence form elliptic PDE, whose coefficients depend on the coefficients of $\tilde{g}_i$ and $\partial \tilde{g}_i$ (in Euclidean coordinates). These coefficients are uniformly controlled in C^α, and thus Schauder estimates yield interior $C^{2,\alpha}$-estimates for f_i. In particular, the injectivity radius of the pull back metric on $\tilde{h}_i := \tilde{F}_i^* \tilde{g}_i$ is locally uniformly bounded. The Gauss equations imply that the sectional curvature of $\tilde{h}_i$ is locally uniformly bounded.

In particular, the injectivity radius and curvature bounds imply that we can take a subsequential $C^{1,\alpha}$ limit of $(S_i, \tilde{h}_i, s_i)$ in the pointed Cheeger–Gromov sense (see [Pet16, Theorem 11.4.7]) to a limit (S, h, s). Recall that this means that for any

$s \in \Omega \Subset S$ there is $i_0(\Omega)$ large so that for $i \geq i_0(\Omega)$ there are embeddings

$$G_i : (\Omega, s) \to (S_i, s_i)$$

so that $G_i^* \tilde{h}_i \to h$ in the $C^{1,\alpha}$-topology.

We now verify that we can also pass the immersions $\tilde{F}_i$ to a subsequential $C^{2,\beta}_{\text{loc}}$ limit for any $\beta < \alpha$. To be precise, we claim that up to passing to a subsequence, for $\Omega \Subset S$ the maps $\tilde{F}_i \circ G_i : (\Omega, s) \to B(0, r_i^{-1}Q^{-1}) \subset \mathbf{R}^4$ converge in the $C^{2,\beta}$-topology. It is convenient prove this with respect to the flat metric on $\mathbf{R}^4$; in particular, this is just a statement about $\mathbf{R}^4$ valued functions on (Ω, s) as opposed to a statement about maps between Riemannian manifolds.

To prove this claim, it suffices to obtain $C^{2,\alpha}$ estimates of the functions $\tilde{x}_k := x_k \circ \tilde{F}_i \circ G_i$ for $k = 1, 2, 3, 4$ (with respect to the metrics $G_i^* \tilde{h}_i$). Indeed, this follows from the fact that $D^2_{G_i^* \tilde{F}_i^* \delta} \tilde{x}_k = (\partial_k \cdot \nu_\delta) A_{G_i \circ \tilde{F}_i, \delta}$ and that the (Euclidean) second fundamental form $A_{G_i \circ \tilde{F}_i, \delta}$ is locally uniformly in C^α (this follows in an straightforward way from the fact that the local graphical functions f_i are locally uniformly bounded in $C^{2,\alpha}$).

Thus, up to passing to a subsequence, the immersions $\tilde{F}_i$ limit to $F : (S, s) \to (\mathbf{R}^4, 0)$ in the $C^{2,\beta}_{\text{loc}}$-sense as described above. This (and the $C^{1,\alpha}_{\text{loc}}$ convergence of $\tilde{g}_i$ to δ) implies that F is a complete stable minimal immersion with $|A_F(s)| = 1$. By [CL24b, Theorem 1], such an immersion must be flat. This contradiction completes the proof. □

Proposition 3.10.1 shows that $|\text{sec}_g| \leq K$ implies Q-weakly bounded geometry for $Q = Q(K)$, so we have also proven the following:

Corollary 3.2.5. *Let (X^4, g) be a complete 4-manifold with $|\text{sec}_g| \leq K$. Then there is a constant $C(K) < \infty$ such that every compact two-sided stable minimal immersion*

$M^3 \to (X^4, g)$ *satisfies*

$$\sup_{q \in M} |A_M(q)| \min\{1, d_M(q, \partial M)\} \leq C(K).$$

This corollary will not be used in this paper, but we have included it since it resolves the conjecture of Schoen stated in [CM11, Conjecture 2.13] (the analogous 3-dimensional result was proven by Schoen in [Sch83a], see also [RST10]). As mentioned above, the result in [CL24b, Theorem 3] established such an estimate only for closed (X^4, g) where $C = C(X, g)$ (see also [Whi16, §3]).

3.3 Preliminary Results on Ends

We establish notation and collect some relevant facts about ends.

3.3.1 Ends adapted to geodesic balls

Let M be a complete Riemannian manifold. Fix a point $x \in M$, and take a length scale $L > 0$.

Definition 3.3.1. A collection of open sets $\{E_k\}_{k \in \mathbb{N}}$ is an *end of M adapted to x with length scale L* if each set E_k is an unbounded connected component of $M \setminus \overline{B}_{kL}(x)$ and satisfies $E_{k+1} \subset E_k$.

Proposition 3.3.2. *If M is simply connected and $\{E_k\}$ is an end adapted to x with length scale L, then both $\overline{E}_k \setminus E_{k+1}$ and ∂E_k are connected for all k.*

Proof. Suppose ∂E_k has at least two components. Since $B_{kL}(x)$ and E_k are connected, we can construct a loop in M having nontrivial intersection number with two of the components of ∂E_k, which contradicts that M is simply connected. The connectedness of $\overline{E}_k \setminus E_{k+1}$ follows from the connectedness of ∂E_k by taking segments of radial geodesics from x, which must intersect ∂E_k.

□

3.3.2 Parabolicity and nonparabolicity

We recall the notion of parabolicity for subsets.

Definition 3.3.3. Let M be a complete Riemannian manifold. Let $K \subset M$ be a compact subset of M. Let $E \subset M$ be an unbounded component of $M \setminus K$ with smooth boundary. We say that E is *parabolic* if it does not admit a positive harmonic function f satisfying

$$f|_{\partial E} \equiv 1 \text{ and } f|_E < 1.$$

Otherwise E is *nonparabolic*.

We first state a well-known result about parabolic sets.

Proposition 3.3.4. *Let M be a complete Riemannian manifold. Let $K \subset M$ be a compact subset of M. Let $E \subset M$ be an unbounded component of $M \setminus K$ with smooth boundary. Suppose E is parabolic. Let f_i be harmonic functions on $E \cap B_{R_i}$ satisfying*

$$f_i|_{\partial E} \equiv 1 \text{ and } f_i|_{\partial B_{R_i}(x)} \equiv 0,$$

where $R_i \to \infty$ and $K \subset B_{R_i}(x)$ for all i. Then f_i converges uniformly to 1 on compact subsets, and

$$\lim_{i\to\infty} \int_E |\nabla f_i|^2 = 0.$$

Proof. We follow [Li04, Theorem 10.1]. The maximum principle guarantees that $0 \leq f_i \leq 1$. Thus, up to a subsequence, f_i limits to f locally smoothly, with $\Delta f = 0$, $0 \leq f \leq 1$, and $f|_{\partial E} = 1$. By parabolicity, $f(x) = 1$ for some $x \in E$, so $f \equiv 1$ by the maximum principle. Thus, we see that f_i limits locally smoothly to 1 on E. Finally,

integrating by parts, we see that

$$\int_E |\nabla f_i|^2 = -\int_{\partial E} f_i \nabla_\nu f_i \to 0.$$

This completes the proof. □

We now discuss when nonparabolicity is inherited by subsets.

Proposition 3.3.5. *Let M be a complete Riemannian manifold. Let $K \subset \tilde{K} \subset M$ be compact subsets of M with smooth boundary. Let $E \subset M$ be an unbounded component of $M \setminus K$. If E is nonparabolic, then there is a nonparabolic unbounded component $\tilde{E}$ of $E \setminus \tilde{K}$.*

Proof. Since E is nonparabolic, there is a positive harmonic function f on E such that

$$f|_{\partial E} \equiv 1 \quad \text{and} \quad f|_E < 1.$$

By addition and scaling, we can assume without loss of generality that

$$\lim_{r\to\infty} \inf_{\partial B_r(x)\cap E} f = 0.$$

Since $\tilde{K}$ is compact with smooth boundary, $\partial\tilde{K}$ is a disjoint union of finitely many smooth closed surfaces. Since E is connected, the boundary of each component of $E \setminus \tilde{K}$ is a union of at least one of the components of $\partial\tilde{K}$. Since distinct components of $E \setminus \tilde{K}$ have disjoint boundaries, the number of such components is finite. Hence, there is an unbounded component $\tilde{E}$ of $E \setminus \tilde{K}$ such that

$$\lim_{r\to\infty} \inf_{\partial B_r(x)\cap \tilde{E}} f = 0.$$

We show that $\tilde{E}$ is nonparabolic.

We take a sequence of harmonic functions $\tilde{f}_i$ on $\tilde{E} \cap B_{R_i}(x)$ satisfying

$$\tilde{f}_i|_{\partial\tilde{E}} \equiv 1 \text{ and } \tilde{f}_i|_{\partial B_{R_i}(x)} \equiv 0,$$

where $R_i \to \infty$ such that $\tilde{K} \subset B_{R_i}(x)$ for all i. Then $\tilde{f}_i$ converges locally uniformly to a positive harmonic function $\tilde{f}$ on $\tilde{E}$ with

$$\tilde{f}|_{\partial\tilde{E}} \equiv 1 \text{ and } \tilde{f}|_{\tilde{E}} \leq 1.$$

Let $c = \inf_{\partial\tilde{E}} f > 0$. Then $\tilde{f}_i \leq c^{-1}f$ on $\tilde{E} \cap B_{R_i}(x)$ for all i by the maximum principle, which implies $\tilde{f} \leq c^{-1}f$. In particular,

$$\lim_{r\to\infty} \inf_{\partial B_r(x)\cap\tilde{E}} \tilde{f} = 0,$$

so $\tilde{f} \not\equiv 1$. Hence, $\tilde{f}|_{\tilde{E}} < 1$ by the maximum principle, so $\tilde{E}$ is nonparabolic. □

By Proposition 3.3.5, it makes sense to define nonparabolicity for an end.

Definition 3.3.6. An end $\{E_k\}$ adapted to x with length scale L is *nonparabolic* if there exist connected open sets $F_k \subset E_k$ with smooth boundary such that

$$E_k \setminus \overline{B}_{(k+1)L}(x) \subset F_k \subset E_k$$

and F_k is nonparabolic for all k sufficiently large.

3.4 Nonparabolic Ends of Stable Hypersurfaces

In this section, we observe[8] that a minor modification of the arguments used in [LW04], [Li04, §11] restricts the number of nonparabolic ends of a stable hypersurface in a 4-manifold with $\mathrm{Ric}_2 \geq 0$.

Theorem 3.4.1. *Let (X^4, g) be a complete 4-manifold with $\mathrm{Ric}_2^g \geq 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion. Let $K \subset M$ be a compact subset of M with smooth boundary. Then $M \setminus K$ admits at most one nonparabolic unbounded component. In particular, M has at most one nonparabolic end.*

We give the proof in Section 3.4.3 after establishing some preliminary results.

3.4.1 Schoen-Yau inequality

We use the following inequality of Schoen and Yau, which is known in non-negative sectional curvature. We show that the same proof works under the weaker assumption $\mathrm{Ric}_2 \geq 0$. We include the proof for completeness.

Lemma 3.4.2 ([SY76])**.** *Let (X^4, g) be a complete 4-manifold satisfying $\mathrm{Ric}_2^g \geq 0$. Let $M^3 \to (X^4, g)$ be a complete two-sided stable minimal immersion. Let u be a harmonic function on M. Then*

$$\frac{1}{3}\int_M \phi^2 |A_M|^2 |\nabla u|^2 + \frac{1}{2}\int_M \phi^2 |\nabla |\nabla u||^2 \leq \int_M |\nabla \phi|^2 |\nabla u|^2$$

for any compactly supported, nonnegative $\phi \in W^{1,2}(M)$.

Proof. We choose a favorable test function in the stability inequality, combined with a rearrangement of the Bochner formula. Below, we assume that u is not a constant function (otherwise the desired inequality is trivial).

[8] One may observe that the same result holds (with the same proof) for $M^{n-1} \to (X^n, g)$ when (X^n, g) has $\mathrm{Ric}_{n-2}^g \geq 0$ (meaning that if $\Pi_1, \ldots, \Pi_{n-2}$ is a set of planes in T_pX meeting pairwise orthogonally along a fixed line, then $\sum_{i=1}^{n-2} \sec_g(\Pi_i) \geq 0$).

The favorable test function is $\psi := \phi|\nabla u|$, where ϕ is a compactly supported, non-negative function in $W^{1,2}(M)$. Since $\mathrm{Ric}_2^g \geq 0$ implies non-negative Ricci curvature, the stability inequality gives

$$\int_M \phi^2 |A_M|^2 |\nabla u|^2 \leq \int_M |\nabla\phi|^2 |\nabla u|^2 + 2\int_M \phi|\nabla u|\langle \nabla\phi, \nabla|\nabla u|\rangle + \int_M \phi^2 |\nabla|\nabla u||^2 \tag{3.4.1}$$

$$= \int_M |\nabla\phi|^2|\nabla u|^2 + \frac{1}{2}\int_M \langle \nabla\phi^2, \nabla|\nabla u|^2\rangle + \int_M \phi^2|\nabla|\nabla u||^2 \tag{3.4.2}$$

$$= \int_M |\nabla\phi|^2|\nabla u|^2 - \int_M \phi^2 \left(\frac{1}{2}\Delta|\nabla u|^2 - |\nabla|\nabla u||^2\right).$$

where the divergence theorem is used in the third equality.

We now rearrange the Bochner formula. Recall that the usual Bochner formula gives

$$\frac{1}{2}\Delta|\nabla u|^2 = \mathrm{Ric}_M(\nabla u, \nabla u) + |\mathrm{Hess}_M u|^2. \tag{3.4.3}$$

For now, consider a point p where $\nabla u \neq 0$. Choose a local orthonormal frame around p so that $e_1 = \frac{\nabla u}{|\nabla u|}$. The improved Kato inequality (cf. [Li04, Lemma 7.2]) yields

$$|\mathrm{Hess}_M u|^2 \geq \frac{3}{2}|\nabla|\nabla u||^2. \tag{3.4.4}$$

Now we rearrange the Ricci curvature term. The Gauss equation and $\mathrm{Ric}_2 \geq 0$ give

$$\mathrm{Ric}_M(e_1, e_1) = \sum_{j=2}^{3} R(e_j, e_1, e_1, e_j) - \sum_{j=1}^{3} A(e_1, e_j)^2 \geq -\sum_{j=1}^{3} A(e_1, e_j)^2. \tag{3.4.5}$$

On the other hand, Cauchy-Schwarz and minimality give

$$\begin{aligned} |A_M|^2 &\geq A_M(e_1,e_1)^2 + 2\sum_{j=2}^{3} A_M(e_1,e_j)^2 + \sum_{j=2}^{3} A_M(e_j,e_j)^2 \\ &\geq A_M(e_1,e_1)^2 + 2\sum_{j=2}^{3} A_M(e_1,e_j)^2 + \frac{1}{2}\left(\sum_{j=2}^{3} A_M(e_j,e_j)\right)^2 \\ &= A_M(e_1,e_1)^2 + 2\sum_{j=2}^{3} A_M(e_1,e_j)^2 + \frac{1}{2}A_M(e_1,e_1)^2 \\ &\geq \frac{3}{2}\sum_{j=1}^{3} A_M(e_1,e_j)^2. \end{aligned} \tag{3.4.6}$$

Together, (3.4.5) and (3.4.6) give

$$\mathrm{Ric}_M(\nabla u, \nabla u) \geq -\frac{2}{3}|A_M|^2|\nabla u|^2. \tag{3.4.7}$$

Hence, (3.4.3), (3.4.4), (3.4.7) give the rearrangement

$$\frac{1}{2}\Delta|\nabla u|^2 \geq -\frac{2}{3}|A_M|^2|\nabla u|^2 + \frac{3}{2}|\nabla|\nabla u||^2, \tag{3.4.8}$$

so

$$\frac{1}{2}\Delta|\nabla u|^2 - |\nabla|\nabla u||^2 \geq -\frac{2}{3}|A_M|^2|\nabla u|^2 + \frac{1}{2}|\nabla|\nabla u||^2. \tag{3.4.9}$$

We derived (3.4.9) under the assumption $\nabla u \neq 0$ at the given point.

Now suppose that $|\nabla u|$ vanishes at p but $x \mapsto |\nabla u|(x)$ is differentiable at p. In this case, the right hand side of (3.4.9) is $= 0$ (since p is a local minimum of $|\nabla u|$). On the other hand, the left-hand side is ≥ 0 since p is a local minimum of the smooth function $|\nabla u|^2$. In sum, we find that that (3.4.9) holds anytime that $|\nabla u|$ is differentiable, which is almost everywhere (by Rademacher's theorem). Thus, we can use (3.4.9) in (3.4.1) to complete the proof. □

3.4.2 Multiple nonparabolic ends

Under the assumption of multiple nonparabolic ends with respect to a fixed compact set, we produce a non-constant harmonic function.

Lemma 3.4.3 (cf. [Li04, Theorem 4.3])**.** *Let M be a complete Riemannian manifold. Let $K \subset M$ be a compact subset of M with smooth boundary. Suppose $M \setminus K$ has at least two nonparabolic unbounded components. Then there exists a non-constant harmonic function with finite Dirichlet energy on M.*

Proof. Let E, F denote nonparabolic components of $M \setminus K$. There exists a harmonic function $0 \leq h \leq 1$ on $M \setminus K$ so that $h \equiv 0$ on $M \setminus (K \cup E \cup F)$, $h|_{\partial E} = 1$, $h|_{\partial F} = 0$, $\liminf_{x \in E, x \to \infty} h = 0$, and $\limsup_{x \in F, x \to \infty} h = 1$. By the argument in [Li04, Lemma 3.6], we can assume that h has finite Dirichlet energy on $M \setminus K$.

For $R_i \to \infty$, define f_i to be the harmonic function on $B_{R_i}(x)$ with $f_i|_{\partial B_{R_i}(x)} = h|_{\partial B_{R_i}(x)}$. Note that $0 \leq f_i \leq 1$. In particular, $f_i \leq h$ on $E \cap B_{R_i}(x)$, and $f_i \geq h$ on $F \cap B_{R_i}(x)$. Passing to a subsequence, the f_i limit locally smoothly to a harmonic function f on M with $f \leq h$ on E and $f \geq h$ on F. In particular, f is non-constant.

Finally, observe that

$$\int_{B_{R_{i+1}}(x)} |\nabla f_{i+1}|^2 \leq \int_{B_{R_i}(x)} |\nabla f_i|^2 + \int_{B_{R_{i+1}}(x) \setminus B_{R_i}(x)} |\nabla h|^2.$$

This implies that f has finite Dirichlet energy, completing the proof. □

3.4.3 Proof of Theorem 3.4.1

Proof. Suppose for contradiction that $\{E_k\}$ and $\{\tilde{E}_k\}$ are distinct nonparabolic ends of M adapted to x with length scale L. Let F_k and $\tilde{F}_k$ be open subsets with smooth boundary given by Definition 3.3.6. Since the ends are distinct, there is a $k_0 \in \mathbb{N}$ such that $E_{k_0} \cap \tilde{E}_{k_0} = \varnothing$. Taking k_0 larger if necessary, F_{k_0} and $\tilde{F}_{k_0}$ are disjoint and nonparabolic. Hence, it suffices to rule out two nonparabolic sets.

Suppose E and F are nonparabolic unbounded components of $M \setminus K$. By Lemma 3.4.3, there is a nonconstant harmonic function u with finite Dirichlet energy on M.

We choose a nice cutoff function to use in Lemma 3.4.2. Let ρ_i be a smoothing of $d_M(x, \cdot)$ with $\rho_i \mid_{\partial B_{(k_0+i)L}(x)} = (k_0+i)L$, $\rho_i \mid_{\partial B_{(k_0+2i)L}(x)} = (k_0+2i)L$, and $|\nabla \rho_i| \leq 2$. Then we define

$$\phi_i(y) := \begin{cases} 1 & y \in \overline{B}_{(k_0+i)L}(x) \\ \frac{(k_0+2i)L - \rho_i(y)}{iL} & y \in \overline{B}_{(k_0+2i)L}(x) \setminus B_{(k_0+i)L}(x) \\ 0 & y \in M \setminus B_{(k_0+2i)L}(x). \end{cases}$$

Lemma 3.4.2 and the fact that u has finite Dirichlet energy implies

$$\begin{aligned} \int_{B_{(k_0+i)L}(x)} \frac{1}{3}|A_M|^2|\nabla u|^2 + \frac{1}{2}|\nabla|\nabla u||^2 &\leq \int_M \frac{1}{3}\phi_i^2|A_M|^2|\nabla u|^2 + \frac{1}{2}\phi_i^2|\nabla|\nabla u||^2 \\ &\leq \int_M |\nabla \phi_i|^2|\nabla u|^2 \\ &\leq \frac{C}{i^2}. \end{aligned}$$

Taking $i \to \infty$, we conclude that

$$|A_M|^2|\nabla u|^2 \equiv 0 \quad \text{and} \quad |\nabla|\nabla u||^2 \equiv 0.$$

Since $|\nabla u|$ is constant and u is non-constant, we have $|\nabla u| \neq 0$ everywhere. Then $|A_M| \equiv 0$, which implies $\mathrm{Ric}_M \geq 0$ by Lemma 3.2.2. The assumption of at least two ends implies (by Cheeger-Gromoll splitting) that M is a product $\mathbf{R} \times P$. Therefore, M has infinite volume. However, $|\nabla u|^2$ is a nonzero constant and $\int_M |\nabla u|^2 < \infty$ by Lemma 3.4.3, which implies that M has finite volume. Hence, we reach a contradiction. $\square$

3.5 Stable Hypersurfaces in PSC 4-Manifolds with $\mathrm{Ric}_2 \geq 0$

Our aim in this section is to prove Theorem 3.1.10 (which immediately implies Theorem 3.1.3 and Corollary 3.1.6).

Remark 3.5.1. If M is parabolic, then the conclusion of Theorem 3.1.10 follows immediately, as parabolicity allows us to "plug 1 into the stability inequality." The hard case is therefore the nonparabolic case. In this case, Theorem 3.4.1 implies that M has precisely one nonparabolic end. Hence, the main difficulty of Theorem 3.1.10 is this troublesome end.

Remark 3.5.2. We can assume without loss of generality that M is simply connected. In the general case, we pass to the universal cover, which inherits minimality, stability, two-sidedness, and completeness from the original immersion. The conclusion of Theorem 3.1.10 for the universal cover then descends to the original immersion.

3.5.1 μ-bubbles

We first recall a diameter bound that uses the theory of warped μ-bubbles for stable minimal hypersurfaces in PSC 4-manifolds.

Lemma 3.5.3 (Warped μ-bubble diameter bound)**.** *Let (X^4, g) be a complete 4-manifold with $R_g \geq 1$. Let $N^3 \to (X^4, g)$ be a two-sided stable minimal immersion with compact boundary. There are universal constants $L > 0$ and $c > 0$ such that if there is a $p \in N$ with $d_N(p, \partial N) > L/2$, then in N there is an open set $\Omega \subset B^N_{L/2}(\partial N)$ and a smooth surface Σ^2 such that $\partial\Omega = \Sigma \sqcup \partial N$ and each component of Σ has diameter at most c.*

Proof. This follows from Gromov's band-width estimate technique [Gro18]. For completeness, we sketch the proof here, with references to the relevant statements from

[CL24a]. Choose $L = 20\pi$. Since N is two-sided and stable, we have

$$\int_N |\nabla\psi|^2 - \tfrac{1}{2}(R_g - R_N + |A_N|^2)\psi^2 \geq 0, \quad \forall \psi \in C_c^1(N).$$

Since $R_g \geq 1$, there exists $u \in C^\infty(N)$, $u > 0$ in $\mathring{N}$ such that

$$\Delta_N u \leq -\tfrac{1}{2}(1 - R_N)u. \tag{3.5.1}$$

For instance, see [FCS80, Theorem 1].

Take $\rho_0 \in C^\infty(M)$ to be a smoothing of $d_N(\cdot, \partial N)$, such that $|\operatorname{Lip}(\rho_0)| \leq 2$, and $\rho_0 = 0$ on ∂N. Choose $\varepsilon \in (0, \frac{1}{2})$ such that $\varepsilon,\ 4\pi + \frac{3}{2}\varepsilon,\ 8\pi + 2\varepsilon$ are regular values of ρ_0. Define

$$\rho = \frac{\rho_0 - \varepsilon}{8 + \frac{\varepsilon}{\pi}} - \frac{\pi}{2},$$

$\Omega_1 = \{x \in N : -\frac{\pi}{2} < \rho < \frac{\pi}{2}\}$, and $\Omega_0 = \{x \in N : -\frac{\pi}{2} < \rho \leq 0\}$. Clearly $|\operatorname{Lip}(\rho)| < \frac{1}{4}$.

On Ω_1, define $h(x) = -\frac{1}{2}\tan(\rho(x))$. For Caccioppoli sets Ω in Ω_1 such that $\Omega \Delta \Omega_0$ is compactly contained in Ω_1, consider

$$\mathcal{A}(\Omega) = \int_{\partial\Omega} u \, d\mathcal{H}^2 - \int_{\Omega_1} (\chi_\Omega - \chi_{\Omega_0}) h u \, d\mathcal{H}^3.$$

By [CL24a, Proposition 12], there exists a minimizer $\tilde{\Omega}$ for $\mathcal{A}$ such that $\tilde{\Omega}\Delta\Omega_0$ is compactly contained in Ω_1. Let $\Omega = \{x \in \mathbb{N} : 0 \leq \rho_0(x) \leq \varepsilon\} \cup \tilde{\Omega}$. We claim that Ω satisfies the conclusion with $c = \frac{4\pi}{\sqrt{3}}$.

Indeed, let Σ_0 be a connected component of $\Sigma = \partial\Omega \cap \Omega_1$. The nonnegativity of the second variation for $\mathcal{A}$ on Σ_0 implies that (see [CL24a, Lemma 14])

$$\begin{aligned}\int_{\Sigma_0} &|\nabla_{\Sigma_0}\psi|^2 u - \tfrac{1}{2}\left(R_N - \tfrac{1}{4} - 2K_{\Sigma_0}\right)\psi^2 u + (\Delta_N u - \Delta_{\Sigma_0} u)\psi^2 \\ &- \tfrac{1}{2}u^{-1}\langle \nabla_N u, \nu\rangle^2 \psi^2 - \tfrac{1}{2}\left(\tfrac{1}{4} + h^2 + 2\langle \nabla_N h, \nu\rangle\right)\psi^2 u \geq 0, \quad \forall \psi \in C^1(\Sigma_0). \end{aligned} \tag{3.5.2}$$

The choice of ρ and h guarantees that $\frac{1}{4} + h^2 + 2\langle \nabla_N h, \nu \rangle \geq 0$ pointwise on Ω_1. Combined with (3.5.1), this implies that

$$\int_{\Sigma_0} |\nabla_{\Sigma_0} \psi|^2 u - \tfrac{1}{2}\left(\tfrac{3}{4} - 2K_{\Sigma_0}\right)\psi^2 u - (\Delta_{\Sigma_0} u)\psi^2 \geq 0, \quad \forall \psi \in C^1(\Sigma_0).$$

Thus we apply [CL24a, Lemma 16 and Lemma 18] and conclude that $\operatorname{diam}(\Sigma_0) \leq \frac{4\pi}{\sqrt{3}}$. This completes the proof. □

3.5.2 Almost linear volume growth

We show that every end of a stable hypersurface in a $\mathrm{Ric}_2 \geq 0$ PSC 4-manifold has a core tube with linear volume growth.

Lemma 3.5.4 (Almost linear volume growth of an end)**.** *Let (X^4, g) be a complete 4-manifold with weakly bounded geometry, $\mathrm{Ric}_2^g \geq 0$, and $R_g \geq 1$. Let $M^3 \to (X^4, g)$ be a simply connected complete two-sided stable minimal immersion. Let $\{E_k\}_{k\in\mathbb{N}}$ be an end of M adapted to $x \in M$ with length scale L, where L is the constant from Lemma 3.5.3. Let $M_k := E_k \cap B_{(k+1)L}(x)$. Then there is a constant $C > 0$ such that*

$$\mathrm{Vol}_M(M_k) \leq C$$

for all k.

Proof. We apply Lemma 3.5.3 to E_k, which supplies an open set Ω_k in $B_{L/2}(\partial E_k) \cap E_k$ with $\partial E_k \subset \partial \Omega_k$. Note that $\Omega_k \subset M_k$ and $\partial E_{k+1} \cap \overline{\Omega_k} = \varnothing$ by construction. Let $\Sigma_k^{(i)}$ denote the components of $\partial\Omega_k \setminus \partial E_k$.

First, we claim that $\overline{M}_k$ is connected. Indeed, this claim follows from the connectedness of ∂E_k by taking segments of radial geodesics from x, just as in Proposition 3.3.2.

Second, we claim that some $\Sigma_k^{(i)}$ separates ∂E_k from ∂E_{k+1}. Let γ be any curve

from ∂E_k to ∂E_{k+1} in $\overline{M}_k$, which exists by the connectedness of $\overline{M}_k$. Perturbing γ if necessary, we can guarantee that any intersection with $\bigcup_i \Sigma_k^{(i)}$ is transverse. Since $\partial E_k \subset \overline{\Omega}_k$ and $\partial E_{k+1} \cap \overline{\Omega}_k = \varnothing$, γ intersects some $\Sigma_k^{(i)}$ an odd number of times. Suppose for contradiction that $\Sigma_k^{(i)}$ does not separate ∂E_k from ∂E_{k+1}. Then there is another curve γ' from ∂E_k to ∂E_{k+1} in $\overline{M}_k \setminus \Sigma_k^{(i)}$. Since ∂E_k and ∂E_{k+1} are connected by Proposition 3.3.2, we have constructed a loop with nontrivial intersection number with $\Sigma_k^{(i)}$, contradicting the simple connectedness of M.

Let Σ_k denote the component of $\partial\Omega_k \setminus \partial E_k$ provided by the above claim. By Lemma 3.5.3, we have $\operatorname{diam}(\Sigma_k) \leq c$.

We claim that there is a constant $D > 0$ (independent of k) so that $\operatorname{diam}(M_k) \leq D$. Let y and z be any points in M_k. Take the radial geodesic from x to y. As in the proof of the connectedness of M_k, this curve intersects ∂E_k and ∂E_{k-1}. By the choice of the components $\{\Sigma_k\}$, an arc of this curve of length at most $2L$ joins y to Σ_{k-1}. The same argument applies to z. Hence, the diameter bound for Σ_{k-1} implies

$$d_M(y, z) \leq 4L + c =: D.$$

By Lemma 3.2.4, M has bounded second fundamental form (e.g. use the compact immersion of $B_2^M(q)$ for any $q \in M$). By Lemma 3.2.2, Ric_M is bounded from below. Hence, by Bishop–Gromov volume comparison, there is a constant $C > 0$ so that

$$\operatorname{Vol}(B_D^M(p)) \leq C$$

for all $p \in M$. Since $M_k \subset B_D^M(p)$ for any $p \in M_k$, the lemma follows. □

3.5.3 Decomposition of M

To construct a nice sequence of test functions for the stability inequality, we need to decompose M into convenient building-blocks.

Let M^3 be a complete, simply connected, Riemannian 3-manifold such that the conclusions of Theorem 3.4.1 hold. Let $\{E_k\}_{k\in\mathbb{N}}$ be the nonparabolic end of M adapted to $x \in M$ with length scale L (where L is the universal constant from Lemma 3.5.3).

We now decompose M to handle the nonparabolic end. Let $k_0 \geq 1$ and $i \geq 1$.

- $M_k := E_k \cap B_{(k+1)L}(x)$ for all k.
- $\{P_{k_0}^{(\alpha)}\}_{\alpha=1}^{n_{k_0}}$ are the components of $M \setminus \overline{B}_{k_0 L}(x)$ besides E_{k_0}.
- $\{P_k^{(\alpha)}\}_{\alpha=1}^{n_k}$ are the components of $E_{k-1} \setminus \overline{B}_{kL}(x)$ besides E_k for $k > k_0$.

Note that we have the decomposition

$$M = \overline{B}_{(k_0+i)L}(x) \cup \bigcup_{k=k_0+i}^{k_0+2i-1} \left(\overline{M}_k \cup \bigcup_{\alpha=1}^{n_k} \overline{P}_k^{(\alpha)} \right) \cup (\overline{E}_{k_0+2i-1} \setminus B_{(k_0+2i)L}(x)).$$

By Theorem 3.4.1, $P_k^{(\alpha)}$ is either bounded or $P_k^{(\alpha)} \setminus K$ is parabolic for all compact $K \subset M$ such that $P_k^{(\alpha)} \setminus K$ has smooth boundary.

3.5.4 Proof of Theorem 3.1.10

Proof. Supposing it exists, let $\{E_k\}$ be the nonparabolic end of M. We use the decomposition of M from §3.5.3.

The Core Tube. For each k, let ρ_k be a smooth function on $\overline{M}_k$ such that $|\nabla \rho_k| \leq 2$,

$$\rho_k|_{\partial E_k} \equiv kL, \quad \text{and} \quad \rho_k|_{\partial M_k \setminus \partial E_k} \equiv (k+1)L.$$

For instance, we can take ρ_k to be a smoothing of the distance function from x.

Let $\phi : [(k_0+i)L, (k_0+2i)L] \to [0,1]$ be the linear function satisfying $\phi((k_0+i)L) = 1$ and $\phi((k_0+2i)L) = 0$.

The Extraneous Pieces. By Proposition 3.3.4, there is a compactly supported Lipschitz function $u_{k,\alpha,i}$ on $\overline{P}_k^{(\alpha)}$ such that

$$u_{k,\alpha,i}|_{\partial P_k^{(\alpha)}} \equiv 1 \text{ and } \int_{P_k^{(\alpha)}} |\nabla u_{k,\alpha,i}|^2 < \frac{1}{i^2 n_k}.$$

Note that if $\partial P_k^{(\alpha)}$ is not smooth, we apply Proposition 3.3.4 to $P_k^{(\alpha)} \setminus K$ (where K is compact and $P_k^{(\alpha)} \setminus K$ has smooth boundary) and extend the function by 1 on $P_k^{(\alpha)} \cap K$. Moreover, if $P_k^{(\alpha)}$ is bounded, we can just take $u_{k,\alpha,i} \equiv 1$.

The Test Function. We now construct a test function f_i for the stability inequality as follows:

$$f_i(y) := \begin{cases} 1 & y \in \overline{B}_{(k_0+i)L}(x) \\ 0 & y \in \overline{E}_{k_0+2i-1} \setminus B_{(k_0+2i)L}(x) \\ \phi(\rho_k(y)) & y \in \overline{M}_k,\ k_0+i \le k < k_0+2i \\ \phi(kL)u_{k,\alpha,i} & y \in \overline{P}_k^{(\alpha)},\ k_0+i \le k < k_0+2i. \end{cases}$$

By construction, f_i is compactly supported and Lipschitz. Therefore, f_i is an eligible test function for the stability inequality. Hence, the stability inequality and Lemma 3.5.4 give

$$\begin{aligned} \int_M (\mathrm{Ric}_g(\nu,\nu)+|A_M|^2) f_i^2 &\le \int_M |\nabla f_i|^2 \\ &= \sum_{k=k_0+i}^{k_0+2i-1} \int_{M_k} \phi'(\rho_k)^2 |\nabla \rho_k|^2 + \sum_{k=k_0+i}^{k_0+2i-1} \sum_{\alpha=1}^{n_k} \phi(kL)^2 \int_{P_k^{(\alpha)}} |\nabla u_{k,\alpha,i}|^2 \\ &\le \frac{4C}{iL^2} + \frac{1}{i} = \frac{C'}{i}. \end{aligned}$$

Since f_i converges uniformly to the constant 1 function on compact subsets, the $i \to \infty$ limit yields the desired conclusion. □

3.6 Topology of PSC 4-manifolds with $\mathrm{Ric}_2 > 0$

In this section we prove Theorem 3.1.13. We begin with the following lemma.

Lemma 3.6.1. *Suppose X^n is a non-compact oriented manifold, $\sigma \subset X$ is a closed curve and $[\sigma] \in H_1(X)$ is a nontrivial, non-torsion element. For any pre-compact open set $D \subset X$ containing a neighborhood of σ, there exists a smooth compact submanifold M_0 of dimension $(n-1)$, such that $\partial M_0 \subset X \setminus D$, and the algebraic intersection number of M_0 and σ is 1.*

Proof. By Poincaré duality for non-compact manifolds, $H_1(X)$ is isomorphic to $H_c^{n-1}(X)$. Thus, there exists a connected pre-compact open set A with $D \subset A$ as well as $\alpha \in H^{n-1}(X, X \setminus A)$ with $\alpha \frown \mu_A = [\sigma]$. By assumption, α is not a torsion element of $H^{n-1}(X, X \setminus A)$. Thus, by the universal coefficient theorem, we can find $\beta \in H_{n-1}(X, X \setminus A)$ with $\alpha \frown \beta = 1 \in H_0(X, X \setminus A)$. By the excision theorem and Lefschetz duality, we have

$$H_{n-1}(X, X \setminus A) \simeq H^1(A) \simeq \langle A, S^1 \rangle,$$

where $\langle A, S^1 \rangle$ is the basepoint-preserving homotopy classes of maps to S^1. Take a smooth map f in $\langle A, S^1 \rangle$ corresponding to β, and let $p \in S^1$ be a regular value of f. Then the smooth hypersurface $f^{-1}(p) \subset A$ represents β and hence has algebraic intersection 1 with σ. □

Proof of Theorem 3.1.13. We first prove that $H_1(X)$ consists only of torsion elements. If not, there exists a closed curve σ with $[\sigma] \in H_1(X)$ non-torsion. Take $p \in X$, $R_0 > 0$ with $\sigma \subset B_{R_0}(p)$. By Lemma 3.6.1, for any integer $k > R_0$, there exists a smooth compact submanifold $\tilde{M}_k$ such that $\partial \tilde{M}_k \subset X \setminus B_k(p)$, and the algebraic intersection number of $\tilde{M}_k$ with σ is 1. Choose $\Omega_k \subset X$ a pre-compact region with smooth boundary, with $\tilde{M}_k \subset \Omega_k$. Form a metric g_k so that Ω_k has mean-convex boundary

and g_k agrees with g away from $\partial\Omega_k$.

Consider the area-minimizing problem

$$\inf\{\mathcal{H}^3_{g_k}(M) : \partial M = \partial\tilde{M}_k, M \text{ is homologous to } \tilde{M}_k\}.$$

It is standard to show that a compact smooth two-sided embedded minimizer M_k exists. Moreover, since M_k is homologous to $\tilde{M}_k$, the algebraic intersection number of M_k with σ is 1. In particular, $M_k \cap \sigma \neq \emptyset$.

By Lemma 3.2.4, we have curvature estimates for $\{M_k\}$ on any compact subset of X. Thus, by passing to a subsequence (not relabeled), $\{M_k\}$ converges to a complete two-sided stable minimal immersion $M^3 \to (X, g)$. Note that M is not empty, thanks to the condition that $M_k \cap \sigma \neq \emptyset$. This contradicts Theorem 3.1.10.

Now we prove that X is H_2-trivial at infinity. Suppose the contrary, that there exists $\alpha \in H_2^\infty(X)$ whose image in $H_2(X)$ is zero. By definition, there exist nested bounded open sets $\{A_j\}_{j=1}^\infty$ such that $A_i \subset A_j$ when $i \leq j$ and $\cup A_j = X$, and non-trivial elements $\alpha_j \in H_2(X \setminus A_j)$, such that:

1. When $i \leq j$, $\iota_*(\alpha_j) = \alpha_i \neq 0$, here $\iota : X \setminus A_j \hookrightarrow X \setminus A_i$ is the inclusion.

2. For each j, $\iota_*(\alpha_j) = 0$, here $\iota : X \setminus A_j \to X$ is the inclusion.

Thus, for each j, there exists a 2-cycle $\Sigma_j \subset X \setminus A_j$ such that $[\Sigma_j]$ is null homologous in X and if $\tilde{M}$ is a 3-chain with $\partial\tilde{M} = \Sigma_j$, then $M \cap A_1 \neq \emptyset$ (otherwise $[\Sigma_j]$ would be null-homologous in $X \setminus A_1$, contradiction). Consider the area-minimizing problem (for metrics g_j as before)

$$\inf\{\mathcal{H}^3_{g_j}(\tilde{M}) : \partial\tilde{M} = \Sigma_j\}.$$

A smooth compact two-sided minimizer M_j exists. By construction, $M_j \cap A_1 \neq \emptyset$. Thus, by Lemma 3.2.4, a subsequence converges smoothly to a complete two-sided stable minimal immersion $M^3 \to (X^4, g)$, contradicting Theorem 3.1.10. This

completes the proof. □

3.7 Appendix A: Curvature conditions

Here we review the various curvature conditions referred to in this paper. Fix (X^{n+1}, g) a Riemannian manifold. We recall the convention used here: $\Pi \subset T_pX$ a 2-plane with orthonormal basis $\{u, v\}$ then $\sec_g(\Pi) = R_g(v, u, u, v)$.

For $k \in \{1, \dots, n\}$ and $v_0, \dots, v_k$ set of $k+1$ orthonormal vectors, we define the *k-th intermediate Ricci curvature* by

$$\mathrm{Ric}_k(v_0, \dots, v_k) := \sum_{i=1}^{k} R(v_0, v_k, v_k, v_0).$$

Note that Ric_n is the usual Ricci curvature while Ric_1 is the sectional curvature.

Following [SY96] we define the *bi-Ricci curvature* of an orthonormal set of vectors $\{u, v\}$ by

$$\mathrm{BiRic}(u, v) = \mathrm{Ric}(u, u) + \mathrm{Ric}(v, v) - R(u, v, v, u).$$

Note that when $n+1 = 3$ then $\mathrm{BiRic}(u, v) = R_g/2$, but in higher dimensions $\mathrm{BiRic} \geq 0$ is a stronger curvature condition than $R_g \geq 0$.

3.8 Appendix B: Examples

3.8.1 Positive sectional curvature

Here we give the details of Example 3.1.2 concerning the existence of a complete stable minimal hypersurface in ambient positive sectional curvature.

For $\alpha \in (0, 1)$ to be chosen close to 1 below, define

$$\rho(r) = \alpha r + (1 - \alpha) \int_0^r e^{-s^2} ds.$$

Consider a metric g on $\mathbf{R}^4$ given by

$$g = dr^2 + \rho(r)^2 g_{\mathbf{S}^3}.$$

It is standard to compute (cf. [Pet16, §4.2.3]) that the sectional curvatures of g_α lie between

$$-\frac{\rho''(r)}{\rho(r)} = \frac{2(1-\alpha)re^{-r^2}}{\alpha r + (1-\alpha)\int_0^r e^{-s^2}ds}$$

and

$$\frac{1-\rho'(r)^2}{\rho(r)^2} = \frac{1-(\alpha+(1-\alpha)e^{-r^2})^2}{\left(\alpha r + (1-\alpha)\int_0^r e^{-s^2}ds\right)^2},$$

so $(\mathbf{R}^4, g)$ has positive sectional curvature. On the other hand, we can fix a totally geodesic $\mathbf{R}^3 \to (\mathbf{R}^4, g)$ (corresponding to $[0,\infty)\times\mathbb{S}^2 \subset [0,\infty)\times\mathbb{S}^3$ in the radial coordinates used above). We claim that this is a stable immersion (embedding), at least for α sufficiently close to 1. Indeed, for $\varphi \in C_c^\infty(\mathbf{R}^3)$, we compute (un-barred quantities denote the induced metric $dr^2+\rho(r)^2 g_{\mathbf{S}^2}$ and barred quantities denote the Euclidean metric $dr^2 + r^2 g_{\mathbf{S}^2}$)

$$\begin{aligned}\int_{\mathbf{R}^3} |\nabla\varphi|^2 d\mu &\geq \int_{\mathbf{R}^3} \alpha^2 |\bar{\nabla}\varphi|^2 d\bar{\mu} \\ &\geq \int_{\mathbf{R}^3} \frac{\alpha^2}{4r^2}\varphi^2 d\bar{\mu} \\ &\geq \int_{\mathbf{R}^3} \frac{\alpha^2}{4r^2}\varphi^2 d\mu.\end{aligned}$$

In the first and third lines we used $\alpha r \leq \rho(r) \leq r$, in the second we used a well-known[9]

[9]For any $1 < p < n$ and $\varphi \in C_c^\infty(\mathbf{R}^n)$ it holds that $\int_{\mathbf{R}^n} \frac{u^p}{|x|^p} d\bar{\mu} \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbf{R}^n} |\bar{\nabla}\varphi|^p d\bar{\mu}$. This can be deduced from the usual one-variable Hardy inequality [HLP88, Theorem 327] via symmetrization (cf. [CF08, Section 2]). Alternatively, the $p = 2$ case (all we need here) can be deduced via a simple integration by parts argument (cf. [Eva10, Section 5.8.4]).

Hardy inequality. Finally, we note that

$$\mathrm{Ric}_g(\nu,\nu) \leq 2\left(\frac{1-\alpha^2}{\alpha^2 r^2} + \frac{1-\alpha}{\alpha}e^{-r^2}\right) \leq \frac{\alpha^2}{4r^2},$$

where the second inequality holds for α sufficiently close to 1. Hence, for this choice of α, we find that $\mathbf{R}^3 \to (\mathbf{R}^4, g)$ is stable.

3.8.2 Positive Ricci curvature and strictly positive scalar curvature

The following example should be compared to the example of Schoen indicated in [MR06, Appendix]. We construct a complete two-sided stable minimal hypersurface $M^3 \to (X^4, g)$ so that along M, $R_g \geq 1, \mathrm{Ric}_g \geq 1$, and the sectional curvature is uniformly bounded below.

Remark 3.8.1. We emphasize that (X^4, g) will not be complete, but M^3 will have a tubular neighborhood of uniform diameter. We discuss this point further below.

Remark 3.8.2. The example below can be directly generalized to produce $M^{n-1} \to (X^n, g)$ as above, for any dimension $n \geq 4$.

Fix $M_0 = (S^1 \times S^2)\#(S^1 \times S^2)$. Because $\dim M_0 = 3 > 2$, the Schoen–Yau/Gromov–Lawson surgery construction [SY79b, GL80] and the Kazdan–Warner trichotomy theorem [KW75c, KW75b, KW75a] (see e.g., [KW75c, Theorem 1.3]) imply that there is a scalar flat metric h_0 on M_0.

Write (M, h) for the universal cover of (M_0, h_0). Defining

$$\lambda_1(M,h) = \inf_{u \in C_c^\infty(M)\setminus\{0\}} \frac{\int_M |\nabla u|^2}{\int_M u^2},$$

we claim that $\lambda_1(M, h) > 0$. This is a consequence of a result of Brooks [Bro81] (see also [DK18, §16]) which says that $\lambda_1(M, h) = 0$ if and only if $\pi_1(M_0)$ is amenable. The

definition of an amenable group can be found in [Bro81, §1] (among other places), but all that matters here is that $\pi_1(M_0) = F_2$, the free group on two generators, which is not amenable (see e.g., [Tao09, Proposition 4]).

We now fix $\varepsilon = \frac{\lambda_1(M,h)}{6}$.

For $\delta > 0$ to be fixed below (depending only on h), define a family of metrics h_t on M_0 by:

$$h_t = h_0 + t^2(\mathrm{Ric}_{h_0} - 2\varepsilon h_0), \quad t \in (-\delta, \delta).$$

For δ sufficiently small, h_t is a Riemannian metric. Finally, we define a metric g on $X = M_0 \times (-\delta, \delta)$ by

$$g = h_t + dt^2.$$

We make a few observations. First, since $\partial_t h_t|_{t=0} = 0$, the embedding $M_0 \times \{0\} \subset (X, g)$ is totally geodesic. Moreover, by the Riccati equation (cf. [Gra04, Corollary 3.3]), we have that

$$2\varepsilon h_0 - \mathrm{Ric}_{h_0} = -\tfrac{1}{2}\partial_t^2 h_t|_{t=0} = R_g(\cdot, \partial_t, \partial_t, \cdot). \tag{3.8.1}$$

Let $\{e_i\}_{i=1}^4$ be a local orthonormal frame on M, where $e_4 = \partial_t$. By the Gauss equations for $M_0 \times \{0\}$ and (3.8.1), we have for $i \in \{1, 2, 3\}$

$$\mathrm{Ric}_g(e_i, e_i) = \mathrm{Ric}_{h_0}(e_i, e_i) + R_g(e_i, e_4, e_4, e_i) = 2\varepsilon.$$

Furthermore, by (3.8.1), we have

$$\mathrm{Ric}_g(e_4, e_4) = \sum_{i=1}^{3} R_g(e_i, e_4, e_4, e_i) = 6\varepsilon - R_{h_0} = 6\varepsilon.$$

In particular, $\mathrm{Ric}_g \geq 2\varepsilon$ along $M_0 \times \{0\}$. If we take $\delta > 0$ even smaller, we find that $\mathrm{Ric}_g \geq \varepsilon$ on X.

Now we verify that the immersion $M^3 \to (X^4, g)$ from the universal cover is stable. Note that the potential term in the stability operator for M satisfies $|A_M|^2 + \mathrm{Ric}_g(\nu, \nu) = 6\varepsilon = \lambda_1(M)$. Thus, for any $u \in C_c^\infty(M)$, we have

$$\int_M |\nabla u|^2 \geq \lambda_1(M, h) \int_M u^2 = \int_M (|A_M|^2 + \mathrm{Ric}_g(\nu, \nu))u^2.$$

This completes the proof (after scaling so that $\mathrm{Ric}_g \geq 1$). Note that because the image of M is an embedded submanifold, the sectional curvature is uniformly bounded below along a tubular neighborhood of M.

Remark 3.8.3. It is an interesting question as to whether or not one can find $M^3 \to (X^4, g)$ stable minimal where (X^4, g) is a *closed* (or complete) manifold with $\mathrm{Ric}_g \geq 1$ at all points. It is tempting to try to study the metric of positive Ricci curvature on $(S^2 \times S^2)\#(S^2 \times S^2)$ constructed in [SY91]. If the construction is carried out in the most symmetric way possible, there will be a totally geodesic submanifold diffeomorphic to $M_0 := (S^1 \times S^2)\#(S^1 \times S^2)$. If one could modify the construction so as to keep positivity of the curvature, but to ensure that $0 \leq \mathrm{Ric}_g(\nu, \nu) \to 0$ along M_0, then the proof used above would show that the universal cover of M_0 was eventually stable.

3.8.3 Non-negative sectional curvature and strictly positive scalar curvature in six dimensions

Choose $(\mathbf{R}^4, g)$ as in Appendix 3.8.1 and consider $(X^6, g_X) := (\mathbf{R}^4, g) \times (S^2, g_{S^2})$. Then, (X^6, g_X) will have non-negative sectional curvature, scalar curvature $R_{g_X} \geq 2$, and strictly positive Ricci curvature. On the other hand, if we cross the totally geodesic $\mathbf{R}^3 \to (\mathbf{R}^4, g)$ by S^2, we find $M^5 := \mathbf{R}^3 \times S^2 \to (X^6, g_X)$ two-sided minimal immersion. The immersion $M \to (X, g_X)$ will be stable. To see this, consider $B_\rho(0) \times S^2 \subset M$ for $\rho > 0$. The first eigenfunction of the stability operator on this set will be

S^2-invariant, so the argument used in Appendix 3.8.1 (the Hardy inequality) implies that the first eigenvalue of the stability operator on this set is positive. Letting $\rho \to \infty$, we see that M is stable. On the other hand $\mathrm{Ric}_{g_X}(\nu, \nu) > 0$ along M.

3.9 Appendix C: Pulling back immersions along a local diffeomorphism

Suppose that X, Y, M are smooth manifolds, $\Psi : Y \to X$ is a local diffeomorphism and $F : M \to X$ is an immersion. We describe here how to "pull back[10] F along Ψ." This construction is presumably well-known.

Below, we will use the standard notation $f \pitchfork Z$ to mean that the map f is transverse to the submanifold Z.

Consider the map

$$F \times \Psi : M \times Y \to X \times X.$$

Write $\Delta = \{(x, x) : x \in X\}$ for the diagonal in $X \times X$.

Lemma 3.9.1. $F \times \Psi \pitchfork \Delta$.

Proof. We note that

$$(F \times \Psi)^{-1}(\Delta) = \{(m, y) \in M \times Y : F(m) = \Psi(y)\}.$$

As such, for $(m, y) \in (F \times \Psi)^{-1}(\Delta)$, we find

$$\mathrm{image}\, d(F \times \Psi)_{(m,y)} = \mathrm{image}\, dF_m \times T_{\Psi(y)}X.$$

[10]This is a pullback in the category theoretic sense (in the category of smooth maps/manifolds). Note that the pullback of two maps need not always exist in this category, but it does when the maps are transversal.

Hence, for $(v_1, v_2) \in T_{F(m)}X \times T_{\Psi(y)}X$, we can write

$$(v_1, v_2) = (v_1, v_1) + (0, v_2 - v_1) \in T_{(F(m),\Psi(y))}\Delta + \operatorname{image} d(F \times \Psi)_{(m,y)}.$$

This completes the proof. □

Thus $S := (F \times \Psi)^{-1}(\Delta)$ is a submanifold of $M \times Y$. Recall that

$$T_s S = (d(F \times \Psi)_s)^{-1}(T_{(F(s),\Psi(s))}\Delta)$$

and

$$\dim M + \dim Y - \dim S = \operatorname{codim}(S \subset Y \times M) = \operatorname{codim}(\Delta \subset X \times X) = \dim X,$$

so (because $\dim Y = \dim X$) we have

$$\dim S = \dim M.$$

Write $F_S : S \to Y$ for the restriction of the projection map $M \times Y \to Y$ and similarly for $\Psi_S : S \to M$. In particular, the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{F_S} & Y \\ \downarrow{\scriptstyle \Psi_S} & & \downarrow{\scriptstyle \Psi} \\ M & \xrightarrow{F} & X \end{array}$$

We now check that the maps Ψ_S, F_S have the desired properties.

Lemma 3.9.2. *The map $F_S : S \to Y$ is an immersion and the map $\Psi_S : S \to M$ is a local diffeomorphism.*

Proof. We start with F_S. For $s \in S$, write $s = (m, y) \in Y \times M$. We have that $(dF_S)_s$ is the restriction of the projection onto the second factor map $\pi_2 : T_m M \times T_y Y \to T_y Y$.

As such,

$$\ker(dF_S)_s = (T_m M \times 0) \cap T_s S.$$

Hence, if $(t, 0) \in \ker(dF_S)_s$ then

$$(t, 0) \in (d(F \times \Psi)_s)^{-1}(T_{(F(s),\Psi(s))}\Delta),$$

i.e., $dF_m(t) = 0$. Because F is an immersion, we have $t = 0$. This proves that F_S is an immersion. For Ψ_S, note that $\dim S = \dim M$, so it suffices to prove that Ψ_S is an immersion. The proof is then identical to the one we just gave for F_S. □

3.10 Appendix D: Existence of local covering maps with good regularity

Recall the definition of Q-weakly bounded geometry from Definition 3.2.3. The following result is well-known (cf. [RST10], [NT18, Lemma 2.1], and [Pet16, Exercise 11.6.15]). We sketch the proof below, referring to [RST10, Pet16] for several crucial details.

Proposition 3.10.1. *If (X^n, g) is a complete manifold with $|\text{sec}| \leq K < \infty$, then there is $Q = Q(K)$ so that (X^n, g) has Q-weakly bounded geometry.*

Proof. Fix $x \in X$ and choose an orthonormal basis for $T_x X$. We will identify $T_x X$ with $\mathbb{R}^n$ (so g_x agrees with the standard inner product on $\mathbb{R}^n$). Using Jacobi field estimates (and $|\text{sec}| \leq K < \infty$), there is $r_0 = r_0(K, n) > 0$ so that

$$\exp_x : B(0, 4r_0) \subset \mathbb{R}^n \to X$$

is a local diffeomorphism, and $\tilde{g} := \exp_x^* g$ satisfies $\frac{1}{2}\delta \leq \tilde{g} \leq 2\delta$ on $B(0, 4r_0) \subset \mathbb{R}^n$ as quadratic forms. By [RST10, Lemma 2.2] it holds that $\text{inj}_{\tilde{g}}(v) \geq i_0 = i_0(K, n) > 0$ for

$v \in B(0, r_0)$. The assertion then follows by constructing harmonic coordinates for $\tilde{g}$ in a uniformly big neighborhood of 0 as in [Pet16, Theorem 11.4.3] (see also [And90] and [RST10, Theorem 2.1]) $\square$

Chapter 4

Existence of index one prescribed mean curvature hypersurfaces in noncompact manifolds

4.1 Introduction

Given a smooth function h on a Riemannian manifold (M, g), a hypersurface $\Sigma \subset M$ has *prescribed mean curvature* h (abbreviated h-PMC) if the mean curvature of Σ satisfies $H_\Sigma = h \mid_\Sigma$. The existence question for closed PMC hypersurfaces is a fundamental problem in geometric analysis. Moreover, PMC hypersurfaces are useful tools in Riemannian geometry; for instance, μ-bubbles (which are special PMC hypersurfaces introduced by [Gro96]) have been instrumental in recent breakthroughs in the study of positive scalar curvature (for example, see [Gro20, CLL23, CL24a]).

A fruitful approach to the existence problem relies on the observation that h-PMC hypersurfaces arise as the boundaries of critical points of the functional

$$\mathcal{A}^h(\Omega) = \text{Area}(\partial\Omega) - \int_\Omega h$$

for finite perimeter sets Ω. Hence, h-PMC hypersurfaces can be constructed using minimization techniques (see for instance [Gro96, Mor03, Zhu21]) and min-max techniques (see for instance [ZZ19, ZZ20, Zho20]) for $\mathcal{A}^h$. These minimization and min-max results for $\mathcal{A}^h$ build on the original minimization and min-max programs for the area functional; we refer the reader to [FF60, Fle62, DG65, Alm66, Sim68] and [Alm62, Pit81, MN16, MN21] for some particularly relevant results in these directions.

An essential difficulty arises for these variational techniques in noncompact ambient spaces, since critical points are constructed as subsequential limits of minimizing or min-max sequences, which may escape to infinity in a noncompact manifold. Despite these difficulties, there are many important problems related to the existence of PMC hypersurfaces in noncompact manifolds.

A first example is a well-known question from [Yau82, Problem 59]: for which functions $h : \mathbb{R}^3 \to \mathbb{R}$ does there exist an embedded h-PMC surface with prescribed genus? For some progress towards this problem in the case of genus zero, we refer the reader to [BK74, TW83, Yau97]. It is natural to consider the following modification of Yau's problem, where we remove the restriction that the surface has prescribed genus and weaken the restriction of embedding to almost embedding[1].

Question 4.1.1. *For which locally smooth functions $h : \mathbb{R}^{n+1} \to \mathbb{R}$ does there exist a closed almost embedded h-PMC hypersurface?*

A reason to remove the topological restriction from the variational point of view is the fact that minimization and min-max for $\mathcal{A}^h$ over surfaces of a fixed topological type does not produce smooth solutions in general (see [SS23] and [WZ23]). Moreover, it seems difficult or impossible to rule out hypersurfaces that are almost embedded but not embedded in general (see [ZZ19, ZZ20]), even though embeddedness can be guaranteed in some special situations (see [Whi19, Zho20, BW24]). Progress towards

[1] A codimension 1 immersion is an almost embedding if it can be decomposed locally near any point into a union of embedded sheets that may touch each other but do not cross through each other.

Question 4.1.1 using min-max methods was obtained by [Maz22].

A second example is a natural synthesis of the main theorem of [CL20]—every complete finite volume manifold contains a complete finite area embedded minimal hypersurface—and the main theorem of [ZZ19]—every closed manifold contains a closed almost embedded hypersurface of constant mean curvature c (abbreviated c-CMC) for all $c > 0$.

Question 4.1.2. *Does every complete finite volume manifold contain a complete finite area almost embedded c-CMC hypersurface for every $c > 0$?*

We refer the reader to [Tho78, Ban80, Son19, CL20, Str23, Dey24] for results in the case $c = 0$. The case of nonzero CMC hypersurfaces appears to be mostly unexplored. Even a positive answer for an inexplicit range of small values of $c > 0$ via an implicit function theorem argument (using as a starting point the existence of a finite area embedded minimal hypersurface) seems difficult, since such hypersurfaces may be noncompact in general.

4.1.1 Main results

In this paper, we develop a localized min-max theory for complete finite area PMC hypersurfaces in complete noncompact manifolds. The following statement is a simplified version of our main result, where we postpone the precise min-max definitions to §4.3.4 (see Theorem 4.9.5 for the most general result). For clarity, when we say there is a homotopy class of sweepouts with nontrivial width for $\mathcal{A}^h$, we mean that we are in a situation where Morse theory (or mountain pass) heuristics suggest the existence of a critical point of the functional.

Theorem 4.1.3. *Let (M, g) be a complete manifold of dimension $3 \leq n + 1 \leq 7$. Let $h \in C^\infty_{loc}(M)$ be nonpositive outside a compact set. If there is a homotopy class of*

sweepouts of bounded finite perimeter sets with nontrivial width for $\mathcal{A}^h$, then there is a smooth, complete, finite area, almost embedded h-PMC hypersurface in M.

Theorem 4.1.3 is the first general existence result for PMC hypersurfaces in arbitrary complete noncompact manifolds (meaning we do not require (M, g) to have special structure).

Surprisingly, the fact that Theorem 4.1.3 works (with the sign restriction on the prescribing function h) boils down to the fact that negative real numbers a satisfy $-a > a$ (and nonnegative numbers do not). To see exactly where this simple observation enters the proof, we refer the reader to the outline of the proof below (see equation (4.1.1)).

In the course of the proof of Theorem 4.1.3, we require a min-max theorem for compact manifolds whose boundary is a good barrier. The following statement is a simplified version (see Theorem 4.7.6 for the most general result).

Theorem 4.1.4. *Let $(N, \partial N, g)$ be a compact manifold of dimension $3 \leq n+1 \leq 7$ with smooth boundary. Let $h \in C^\infty(N)$ satisfy $h\,|_{\partial N} < H_{\partial N}$. If there is a homotopy class of sweepouts of finite perimeter sets in $\mathrm{Int}(N)$ with nontrivial width for $\mathcal{A}^h$, then there is a smooth, closed, almost embedded h-PMC hypersurface in $\mathrm{Int}(N)$.*

The main novelty of Theorem 4.1.4 is that we only require a *one-sided* bound on h relative to the mean curvature of the boundary. This setting is more delicate because the possibility of boundary touching depends on the orientation of the hypersurface. For instance, a minimal boundary cannot be touched from the inside by a 1-CMC hypersurface oriented *away* from the boundary, but it *can* be touched from the inside by a 1-CMC hypersurface oriented *towards* the boundary (e.g. any sphere in Euclidean space has nonzero CMC but touches its tangent planes from one side).

We apply Theorems 4.1.3 and 4.1.4 to Questions 4.1.1 and 4.1.2. Towards Question 4.1.1, we have the following result.

Corollary 4.1.5. *If $h \in C^\infty_{loc}(\mathbb{R}^{n+1})$ for $3 \leq n+1 \leq 7$ satisfies*

1. *there exists a nonempty bounded finite perimeter set Λ satisfying $\mathcal{A}^h(\Lambda) \leq 0$,*

2. *$h(x) \leq C|x|^{-1-\alpha}$ for some $\alpha,\ C > 0$,*

then there is a smooth, closed, almost embedded h-PMC hypersurface in $\mathbb{R}^{n+1}$.

We emphasize that condition (2) above is a one-sided bound, meaning that h can be arbitrarily large in magnitude if it is negative.

Corollary 4.1.5 applies to a set of prescribing functions that is disjoint from the set of prescribing functions considered by [Maz22][2]. A nice feature of our result is that, unlike most of the earlier progress towards Question 4.1.1, we do not require any restrictions on the derivatives of h, nor do we require h to limit to a constant at infinity. Moreover, our technique is flexible enough to handle prescribing functions in any complete noncompact ambient space (and not just spaces with nice structure).

Towards Question 4.1.2, we have the following result.

Corollary 4.1.6. *Let (M, g) be a complete finite volume manifold of dimension $3 \leq n+1 \leq 7$. If*

$$0 < c < \mathscr{D}(M, g) := \sup_{v_1 \neq v_2} \left| \frac{\mathscr{I}(v_2) - \mathscr{I}(v_1)}{v_2 - v_1} \right|,$$

where $\mathscr{I}$ is the isoperimetric profile of (M, g), then there is a smooth, complete, finite area, almost embedded c-CMC hypersurface in M.

Since $\mathscr{D}(M, g) > 0$ for all complete finite volume manifolds, we give a positive answer to Question 4.1.2 for a nonempty range of c for every complete finite volume manifold. We emphasize that this conclusion is the first positive result towards Question 4.1.2.

[2]Both results require $\mathcal{A}^h$ to have a nontrivial mountain pass over bounded sets, but [Maz22] requires h to limit to a positive constant at infinity, whereas our result requires h to be nonpositive at infinity.

Moreover, we completely resolve Question 4.1.2 for any manifold with $\mathscr{D}(M,g) = +\infty$. We provide examples (see Example 4.11.10) of complete noncompact manifolds with finite volume satisfying $\mathscr{D}(M,g) = +\infty$, so the class of such manifolds is nontrivial.

Finally, we obtain a clean application to complete finite volume hyperbolic manifolds.

Corollary 4.1.7. *Let (M,g) be a complete finite volume hyperbolic manifold of dimension $3 \leq n+1 \leq 7$. For any $0 < c < n$, there is a smooth, complete, finite area, almost embedded c-CMC hypersurface in M.*

4.1.2 Outline of the proof of Theorem 4.1.3

We use a tubular neighborhood cutting scheme similar to the one developed in [Str23] for the area functional. In that work, the crucial fact is that if we delete a small tubular neighborhood of a two-sided unstable minimal hypersurface, the new boundary components will be strictly mean convex, and therefore provide a good barrier for subsequent min-max constructions. A key difficult for the $\mathcal{A}^h$ functional is that if we delete a tubular neighborhood of a two-sided $\mathcal{A}^h$-unstable h-PMC hypersurface, then one of the new boundary components may not provide a good barrier for subsequent min-max constructions. Instead, we develop a more delicate cutting argument.

Suppose $h \in C^\infty_{\mathrm{loc}}(M)$ is negative outside a compact set K[3] and $\Phi_0 : [0,1] \to \mathcal{C}(M)$ is a sweepout of bounded finite perimeter sets whose homotopy class has nontrivial width for $\mathcal{A}^h$. By enlarging K if necessary, suppose that $\overline{\Phi_0(t)} \subset K$ for all $t \in [0,1]$.

We take an exhaustion of M by compact sets $\{N_i\}$ with smooth boundary satisfying $K \subset N_i$ for all i. We modify h in a small neighborhood of ∂N_i so that $h\,|_{\partial N_i} < H_{\partial N_i}$, making Theorem 4.1.4 applicable. We also perturb h and the metric (to a "good pair" in the sense of [Zho20, §3.3]) so that the hypersurface produced

[3]The case where h is only nonpositive outside K follows by approximation.

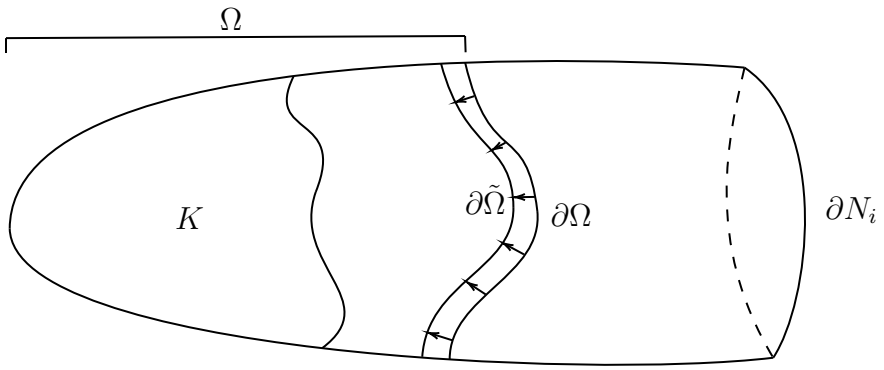


Figure 4.1: $K \subset \Omega$.

by Theorem 4.1.4 is an embedded boundary that is unstable for $\mathcal{A}^h$. We claim that we can find an h-PMC hypersurface intersecting K (with uniform area and index bounds). If so, then the result follows by taking a limit of these hypersurfaces as $i \to \infty$, using the compactness theorem of [Zho20, Theorem 2.8] (which builds on the compactness results of [Sha17]). The limit cannot escape to infinity because it must intersect K.

If the hypersurface $\Sigma = \partial\Omega$ produced by Theorem 4.1.4 intersects K, then we are done. Otherwise, we either have $K \subset \Omega$ or $K \cap \Omega = \varnothing$. In either case, we find a subdomain of N_i that contains K but does not contain Σ and satisfies the hypotheses of Theorem 4.1.4 (see the below paragraphs). So we produce a new h-PMC hypersurface that is distinct from Σ. By choosing h and the metric generically (so that every closed h-PMC hypersurface is nondegenerate), we can only iterate these steps finitely many times, so we eventually find Σ intersecting K.

Case 1: $K \subset \Omega$. Let Σ^* be a component of $\partial\Omega$ that is unstable for $\mathcal{A}^h$. By pushing Σ^* into Ω, the new boundary component $\tilde{\Sigma}^*$ satisfies $h\,|_{\tilde{\Sigma}^*} < H_{\tilde{\Sigma}^*}$. We can similarly modify the other components of $\partial\Omega$ to produce $\tilde{\Omega}$ satisfying $\Sigma^* \cap \tilde{\Omega} = \varnothing$ and $h\,|_{\partial\tilde{\Omega}} < H_{\partial\tilde{\Omega}}$. Hence, $\Sigma \not\subset \tilde{\Omega}$, and we can apply Theorem 4.1.4 in $\tilde{\Omega}$ and iterate. For an illustration of this case, see Figure 4.1.

Case 2: $K \cap \Omega = \varnothing$. We let $\tilde{\Omega}$ be the open set obtained by slightly enlarging Ω

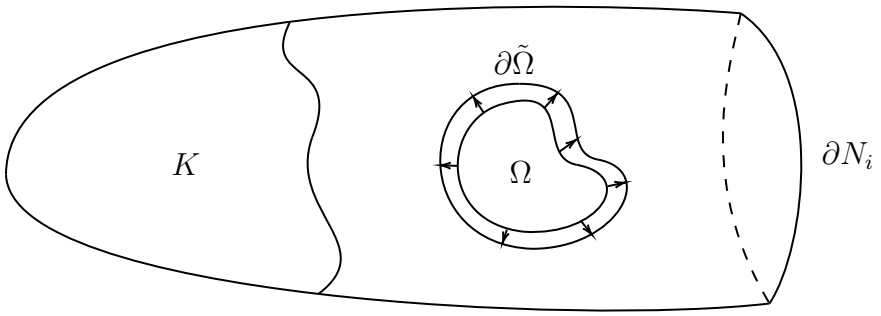


Figure 4.2: $K \cap \Omega = \varnothing$.

along all its boundary components. Then $\tilde{N}_i = N_i \setminus \tilde{\Omega}$ contains K, does not contain Σ, and satisfies

$$H_{\partial \tilde{N}_i} \approx -H_{\partial\Omega} = -h \mid_{\partial\Omega} > h \mid_{\partial\Omega}, \tag{4.1.1}$$

where the last inequality follows from the fact that $h \mid_{N_i \setminus K} < 0$. Hence, we can apply Theorem 4.1.4 in $\tilde{N}_i$ and iterate. For an illustration of this case, see Figure 4.2.

4.1.3 Outline of the proof of Theorem 4.1.4

It is tempting to try to use the maximum principle of [Whi10] after the pull tight step of the min-max program to ensure that every element in the critical set of a minimizing sequence is contained in $\mathrm{Int}(N)$. However, these elements are only known to be varifolds with bounds on the first variation of mass, meaning that we do not have any information about how they are *oriented* relative to ∂N. Since we only require the one-sided bound on h, this argument will not work if $|h| > H_{\partial N}$.

Instead, we carry out all of the steps of the min-max program as if we were doing min-max with an obstacle (as in [Wan22]). After proving the existence of a varifold in the critical set of a pulled tight minimizing sequence that is almost minimizing, we can use the regularity for the obstacle problem (see [Tam82]) to show that this varifold is $C^{1,\alpha}$. This regularity is sufficient to ensure that the hypersurface has the correct orientation at the boundary, at which point the classical Hopf lemma for

elliptic second order operators applies to rule out boundary touching. Once we know that this hypersurface is contained in $\mathrm{Int}(N)$, the local regularity results from [ZZ20] apply unchanged.

4.1.4 Organization of the paper

In §4.2, we establish the basic notations and conventions for the rest of the paper.

In the first half, we develop a localized min-max theory for PMC hypersurfaces in compact manifolds with good boundaries. In §4.3-4.6, we modify the standard min-max machinery to adapt to the setting with boundary. In §4.7 we develop a new localization argument essential for the results in noncompact manifolds.

In the second half, we use the theory from the first half to prove a localized min-max theorem for PMC hypersurfaces in complete noncompact manifolds. In §4.9 we prove the main general theorem of the paper. In §4.10 we apply this theory to the problem of PMC hypersurfaces in Euclidean space. In §4.11 we apply this theory to the problem of CMC hypersurfaces in finite volume manifolds.

4.2 Conventions and notation

Let (M, g) be a complete Riemannian manifold of dimension $n+1$. Let $N \subset M$ be a compact domain with smooth boundary.

4.2.1 Basic notations

We denote

- $B_r(x) = \{y \in M \mid \mathrm{dist}_g(x, y) < r\}$,
- $B_r^N(x) = B_r(x) \cap N$,
- $A_{s,r}(x) = \{y \in M \mid s < \mathrm{dist}_g(x, y) < r\}$,

- $A^N_{s,r}(x) = A_{s,r}(x) \cap N$,
- $\mathcal{H}^k$ is the k-dimensional Hausdorff measure on (M, g),
- $\mathrm{Int}(A)$ for $A \subset M$ is the union of all open sets W in M satisfying $W \subset A$.

We say $U \subset N$ is *relatively open* if there is an open set $\tilde{U} \subset M$ with $\tilde{U} \cap N = U$.

- $\mathrm{int}_{\mathrm{rel}}(A)$ for $A \subset N$ is the union relatively open sets U in N satisfying $U \subset A$.

4.2.2 Varifolds

We denote

- $\mathcal{V}_k(M)$ is the space of k-varifolds in M,
- $\mathcal{V}_k(N)$ is the space of k-varifolds in M with support in N,
- $\mathbf{F}$ is the metric on $\mathcal{V}_k(M)$ given by

$$\mathbf{F}(V_1, V_2) = \sup\{V_1(f) - V_2(f) \mid f \in C^{0,1}_{\mathrm{cpt}}(\mathrm{Gr}_k(M)),\ |f| \leq 1,\ \mathrm{Lip}(f) \leq 1\}.$$

- $\|V\|$ is the associated Radon measure on M for $V \in \mathcal{V}_k(M)$,
- $|\Sigma|$ is the integral k-varifold associated to Σ an immersed k-dimensional submanifold,
- $\|\Sigma\|$ is the associated Radon measure on M for the k-varifold $|\Sigma|$, where Σ is an immersed k-dimensional submanifold,
- $\delta V(X)$ is the first variation of the mass of $V \in \mathcal{V}_k(M)$ under the flow generated by the vector field X,
- $\mathrm{VarTan}(V, x)$ is the set of varifold tangent cones at $x \in M$ for V a k-varifold with bounded first variation.

Let $\bar{\nu}$ denote the unit normal vector field along ∂N pointing into N. Let $U \subset N$ be relatively open. We denote

- $\mathcal{X}_{\text{cpt}}(U)$ is the space of smooth vector fields with compact support in U,
- $\mathcal{X}^+_{\text{cpt}}(U) = \{X \in \mathcal{X}_{\text{cpt}}(U) \mid \langle X, \bar{\nu}\rangle \geq 0\}$.

Fix $c > 0$.

Definition 4.2.1 (Bounded constrained first variation)**.** $V \in \mathcal{V}_k(N)$ has *c-bounded constrained first variation* in U if

$$\delta V(X) \geq -c \int_U |X| \; d\|V\|$$

for all $X \in \mathcal{X}^+_{\text{cpt}}(U)$.

Definition 4.2.2 (Bounded first variation)**.** $V \in \mathcal{V}_k(N)$ has *c-bounded first variation* in U if there is an open set $\tilde{U} \subset M$ satisfying $\tilde{U} \cap N = U$ and

$$|\delta V(X)| \leq c\|V\|(U)\|X\|_{C^0}$$

for all $X \in \mathcal{X}_{\text{cpt}}(\tilde{U})$.

The following lemma is an adaptation of [Wan22, Lemma 2.6].

Lemma 4.2.3. *Let $V \in \mathcal{V}_k(N)$ have c-bounded constrained first variation in N. Then there is a constant $C = C(N, c) < \infty$ so that V has C-bounded first variation in N.*

Proof. Let $r > 0$ small enough so that the normal exponential map along ∂N is a diffeomorphism. Extend $\bar{\nu}$ in $B_r(\partial N)$ by $\nabla_{\bar{\nu}}\bar{\nu} = 0$. Let $\eta \in C^1_{\text{cpt}}(B_r(\partial N), [0, 1])$ so that $\eta \equiv 1$ in a neighborhood of ∂N.

Let $X \in \mathcal{X}_{\text{cpt}}(M)$ satisfy $\|X\|_{C^0} = 1$ (without loss of generality). Set $X_0 = X - \eta\langle X, \bar{\nu}\rangle\bar{\nu}$. By construction, $\pm X_0 \,|_N \in \mathcal{X}^+_{\text{cpt}}(U)$, $1 \pm \langle X, \bar{\nu}\rangle \geq 0$, and $|X_0| \leq |X| \leq 1$.

We compute

$$\begin{aligned}\delta V(X) &= \int (-\mathrm{div}_S(-X_0) + \mathrm{div}_S(\eta\langle X, \bar{\nu}\rangle\bar{\nu}))\ dV(x,S)\\ &\leq c\|V\|(N) - \int (\mathrm{div}_S(\eta(1-\langle X,\bar{\nu}\rangle)\bar{\nu}) + \mathrm{div}_S(\eta\bar{\nu}))\ dV(x,S)\\ &\leq c\|V\|(N) + c\|V\|(N)\|\eta(1-\langle X,\bar{\nu}\rangle)\|_{C^0} + \|\eta\bar{\nu}\|_{C^1}\|V\|(N)\\ &\leq (3c + \|\eta\bar{\nu}\|_{C^1})\|V\|(N),\end{aligned}$$

and

$$\begin{aligned}\delta V(X) &= \int (\mathrm{div}_S X_0 + \mathrm{div}_S(\eta\langle X, \bar{\nu}\rangle\bar{\nu}))\ dV(x,S)\\ &\geq -c\|V\|(N) + \int (\mathrm{div}_S(\eta(1+\langle X,\bar{\nu}\rangle)\bar{\nu}) - \mathrm{div}_S(\eta\bar{\nu}))\ dV(x,S)\\ &\geq -c\|V\|(N) - c\|V\|(N)\|\eta(1+\langle X,\bar{\nu}\rangle)\|_{C^0} - \|\eta\bar{\nu}\|_{C^1}\|V\|(N)\\ &\geq -(3c + \|\eta\bar{\nu}\|_{C^1})\|V\|(N).\end{aligned}$$

The conclusion follows by taking $C = 3c + \|\eta\bar{\nu}\|_{C^1}$. □

4.2.3 Currents

We denote

- $\mathbf{I}_k(M)$ is the space of k-dimensional integral currents in M,
- $\mathbf{I}_k(N)$ is the space of k-dimensional integral currents in M with support in N,
- $[\Sigma]$ is the k-dimensional integral current associated to an oriented immersed k-dimensional submanifold of M,
- $|T|$ is the associated integral k-varifold for $T \in \mathbf{I}_k(M)$,
- $\|T\|$ is the associated Radon measure on M for $T \in \mathbf{I}_k(M)$.

The space of integral currents has various norms and metrics:

- $\mathbf{M}$ is the mass norm on $\mathbf{I}_k(M)$,
- $\mathcal{F}$ is the flat norm on $\mathbf{I}_k(M)$,
- $\mathbf{F}(T,S)$ for $T,S \in \mathbf{I}_k(M)$ is given by

$$\mathbf{F}(T,S) = \mathcal{F}(T-S) + \mathbf{F}(|T|,|S|).$$

4.2.4 Finite perimeter sets

We denote

- $\mathcal{C}(M)$ is the collection of sets of finite perimeter in M,
- $\partial^*\Omega$ is the reduced boundary for $\Omega \in \mathcal{C}(M)$,
- $|\partial^*\Omega|$ is the integral n-varifold associated to $\partial^*\Omega$,
- $[\partial^*\Omega]$ is the n-dimensional integral current associated to $\partial^*\Omega$ for $\Omega \in \mathcal{C}(M)$.

$\mathcal{C}(M)$ naturally inherits 3 metrics from $\mathbf{I}_n(M)$:

- $\mathbf{M}([\partial^*\Omega])$,
- $\mathcal{F}([\partial^*\Omega])$,
- $\mathbf{F}([\partial^*\Omega_1],[\partial^*\Omega_2])$.

By definition, we have

$$\mathcal{F}([\partial^*\Omega]) \leq \mathbf{F}([\partial^*\Omega],0) \text{ and } \mathcal{F}([\partial^*\Omega]) \leq \mathbf{M}([\partial^*\Omega]). \tag{4.2.1}$$

We recall the following fact when M is closed, which can be found for instance in [IM18, Lemma 3.4].

Lemma 4.2.4. *There is a constant $\varepsilon_0 > 0$ (depending on M closed) so that if $\mathcal{F}([\partial^*\Omega]) \leq \varepsilon_0$ and $\mathcal{H}^{n+1}(\Omega) \leq \frac{1}{2}\mathcal{H}^{n+1}(M)$, then $\mathcal{H}^{n+1}(\Omega) = \mathcal{F}([\partial^*\Omega])$.*

By (4.2.1) and Lemma 4.2.4, convergence in any of the above metrics implies L^1 convergence of the indicator functions of the finite perimeter sets.

4.2.5 Hypersurfaces and mean curvature

Let $\phi : \Sigma \to (M, g)$ be a two-sided immersed hypersurface, and let ν be a continuous choice of unit normal vector field along Σ. We let $H_{\Sigma,\nu}$ denote the mean curvature of H with respect to ν. To specify our convention, for $\Sigma = \partial B_1^{\mathbb{R}^{n+1}}(0)$ and $\nu(x) = x$ (the outward pointing unit normal vector field for $B_1^{\mathbb{R}^{n+1}}(0)$), we have $H_{\Sigma,\nu} = n$. In this convention, for a smooth vector field X on M, we have

$$\delta|\Sigma|(X) = \int_\Sigma \langle X, H_{\Sigma,\nu}\nu\rangle d\|\Sigma\|.$$

If $\phi(\Sigma) = \partial\Omega$, we write $H_{\Sigma,\Omega}$ to mean $H_{\Sigma,\nu}$ for ν the outward pointing unit normal vector field along Σ.

Definition 4.2.5 (Almost embedding)**.** Σ is *almost embedded* if for any $p \in \phi(\Sigma)$ where ϕ fails to be an embedding, there is a neighborhood W of p in M so that $\Sigma \cap \phi^{-1}(W) = \sqcup_{i=1}^N \Sigma_i$, $\phi\,|_{\Sigma_i}$ is an embedding for all i, and for each $i \neq j$, $\phi(\Sigma_i)$ lies on one side of $\phi(\Sigma_j)$ in W.

Definition 4.2.6 (Prescribed/constant mean curvature)**.** For $h \in C^\infty_{\text{loc}}(M)$, a two-sided immersion is an *h-PMC hypersurface* if there is a choice of continuous unit normal vector field along Σ satisfying $H_{\Sigma,\nu} = h\,|_\Sigma$. If $h \equiv c$ constant, then Σ is a *c-CMC* hypersurface.

4.2.6 Second variation and index

If $\Sigma = \partial\Omega$ is an h-PMC hypersurface, then Σ is a critical point[4] for the functional

$$\mathcal{A}^h(\Omega') = \mathcal{H}^n(\partial^*\Omega') - \int_{\Omega'} h.$$

Let ν be the unit normal vector field along Σ pointing out of Ω. The second variation of $\mathcal{A}^h$ for Σ for any compactly supported normal variation $\varphi\nu$ is given by (see [Zho20, (1.3)])

$$Q^h_\Sigma(\varphi,\varphi) = \int_\Sigma (|\nabla^\Sigma\varphi|^2 - (\mathrm{Ric}_M(\nu,\nu) + |A_\Sigma|^2 + \partial_\nu h)\varphi^2), \tag{4.2.2}$$

where Ric_M is the Ricci curvature of (M,g) and A_Σ is the second fundamental form of the immersion Σ.

Definition 4.2.7 (Stability)**.** Let $U \subset M$ open. Σ is *$\mathcal{A}^h$-stable* in U if Q^h_Σ is positive semi-definite on $C^\infty_{\mathrm{cpt}}(U)$ (otherwise Σ is *$\mathcal{A}^h$-unstable*). Σ is *strictly $\mathcal{A}^h$-stable* in U if Q^h_Σ is positive definite on $C^\infty_{\mathrm{cpt}}(U)$.

Definition 4.2.8 (Index)**.** Let $U \subset M$ open. We define

$$\mathrm{index}_{\mathcal{A}^h,U}(\Sigma) = \sup\{\dim(L) \mid L \text{ is a subspace of } C^\infty_{\mathrm{cpt}}(U),\ Q^h_\Sigma\,|_L \text{ is negative definite}\}.$$

If $U = M$, we omit U from the subscript.

Definition 4.2.9 (Nondegenerate)**.** Suppose Σ is closed. Σ is *$\mathcal{A}^h$-nondegenerate* if the operator

$$\mathcal{L}^h_\Sigma = -\Delta_\Sigma - (\mathrm{Ric}_M(\nu,\nu) + |A_\Sigma|^2 + \partial_\nu h)$$

has trivial kernel[5].

[4] Just like for the area functional, we can make sense of being a critical point over compactly supported variations even if the functional itself is undefined for Ω.

[5] If Σ is nondegenerate and stable, then Σ is strictly stable.

4.3 Setup for Compact Manifolds with Barrier

In this part of the paper, we develop the tools for a localized min-max theorem in compact manifolds whose boundary provides a good barrier.

Let $(N, \partial N, g)$ be a smooth compact Riemannian manifold with smooth boundary. We always assume that the dimension of N is $3 \leq n+1 \leq 7$.

Let $(\tilde{N}, \tilde{g})$ be a smooth closed Riemannian manifold of dimesion $n+1$ which contains $(N, \partial N, g)$ isometrically.

4.3.1 Space of subsets

We fix a disjoint decomposition by connected components $\tilde{N} \setminus N = E_+ \sqcup E_-$ and let $P_\pm = \partial E_\pm$.

We let $\mathcal{C}_{E_+,E_-}(N)$ denote the set of all $\Omega \in \mathcal{C}(\tilde{N})$ satisfying

$$E_+ \subset \Omega \quad \text{and} \quad E_- \cap \Omega = \varnothing.$$

Note that $\mathcal{C}_{E_+,E_-}(N)$ naturally inherits any metric or topology on $\mathcal{C}(\tilde{N})$.

4.3.2 Functional

Fix $h \in C^\infty(N)$. In an abuse of notation, we also let h denote a fixed smooth extension of h to $\tilde{N}$. We let $c := \sup_{\tilde{N}} |h|$. For $\Omega \in \mathcal{C}_{E_+,E_-}(N)$, we define

$$\mathcal{A}^h(\Omega) = \mathcal{H}^n(\partial^*\Omega) - \int_\Omega h.$$

4.3.3 Barrier

We define what it means for ∂N to be a good barrier for min-max constructions for $\mathcal{A}^h$ over $\mathcal{C}_{E_+,E_-}(N)$.

Definition 4.3.1. ∂N is a *barrier* for (h, P_+, P_-) if

$$H_{P_+,E_+} < h\,|_{P_+} \quad \text{and} \quad H_{P_-,E_-} < -h\,|_{P_-}\ .$$

4.3.4 Min-max setup

Let X^k be a connected cubical complex of dimension k in some $I^m = [0,1]^m$. Let $Z \subset X$ be a nonempty cubical subcomplex.

Let $\Phi_0 : X \to \mathcal{C}_{E_+,E_-}(N)$ be a continuous map in the **F**-topology[6].

Definition 4.3.2 (Homotopy class)**.** Let Π be the set of all sequences of **F**-continuous maps $\{\Phi_i : X \to \mathcal{C}_{E_+,E_-}(N)\}_i$ so that there are $\mathcal{F}$-continuous homotopies $\{\Psi_i : [0,1] \times X \to \mathcal{C}_{E_+,E_-}(N)\}_i$ satisfying $\Psi_i(0,\cdot) = \Phi_i$, $\Psi_i(1,\cdot) = \Phi_0$, and

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(\Psi_i(t,z), \Phi_0(z)) \mid t \in [0,1],\ z \in Z\} = 0.$$

Such a sequence $\{\Phi_i\}$ is called an (X,Z)-*homotopy sequence of mappings to* $\mathcal{C}_{E_+,E_-}(N)$, and Π is called the (X,Z)-*homotopy class of* Φ_0.

Definition 4.3.3 (Width)**.** The *h-width* of Π is

$$\mathbf{L}^h(\Pi) = \inf_{\{\Phi_i\}\in\Pi} \limsup_{i\to\infty} \sup_{x\in X} \mathcal{A}^h(\Phi_i(x)).$$

Definition 4.3.4 (Minimizing sequence)**.** We say $\{\Phi_i\}_i \in \Pi$ is a *minimizing sequence* if

$$\limsup_{i\to\infty} \sup_{x\in X} \mathcal{A}^h(\Phi_i(x)) = \mathbf{L}^h(\Pi).$$

Definition 4.3.5 (Critical set)**.** The *critical set* of a minimizing sequence $\{\Phi_i\}_i \in \Pi$

[6]We can assume, without loss of generality by [Mag12, Remarks 15.2 and 15.3], that $\partial\Phi_0(x) = \overline{\partial^*\Phi_0(x)}$ for all $x \in X$.

is

$$\mathbf{C}(\{\Phi_i\}) = \left\{ V \in \mathcal{V}_n(N) \mid V = \lim_{j\to\infty} |\partial^*\Phi_{i_j}(x_j)|,\ \lim_{j\to\infty} \mathcal{A}^h(\Phi_{i_j}(x_j)) = \mathbf{L}^h(\Phi) \right\}.$$

4.4 Preliminary tools

Here we collect the standard facts required for the min-max construction.

4.4.1 Local regularity for obstacle problem

We recall that the dimension of N is assumed to be $3 \leq n+1 \leq 7$.

Theorem 4.4.1. *Let $U \subset N$ be relatively open. Suppose $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ minimizes $\mathcal{A}^h$ in $\{\Omega' \in \mathcal{C}(\tilde{N}) \mid \Omega \triangle \Omega' \subset\subset U\} \subset \mathcal{C}_{E_+,E_-}(N)$. Then $\partial^*\Omega \cap U = \partial\Omega \cap U$ is a $C^{1,\alpha}$ embedded hypersurface in U.*

Proof. Let $\Omega' \subset \mathcal{C}(\tilde{N})$ satisfy $\Omega \triangle \Omega' \subset B_r(x) \cap N \subset U$. Then

$$\begin{aligned}
\mathcal{H}^n(\partial^*\Omega) - \mathcal{H}^n(\partial^*\Omega') &= \mathcal{A}^h(\Omega) - \mathcal{A}^h(\Omega') + \int_\Omega h - \int_{\Omega'} h \\
&\leq \int_{\Omega\setminus\Omega'} h - \int_{\Omega'\setminus\Omega} h \\
&\leq c\mathcal{H}^{n+1}(B_r(x)) \\
&\leq c\omega r^{n+1},
\end{aligned}$$

where ω is a constant depending on (N, g) and r_0 for all $r < r_0$. In other words, Ω is almost minimizing with respect to the obstacle ∂N. Hence, the arguments of [Tam82, §3] apply to give the desired regularity (working only in the assumed dimensions). □

When ∂N is a barrier for (h, P_+, P_-), we use the maximum principle and the above $C^{1,\alpha}$ regularity to show that local $\mathcal{A}^h$-minimizers lie in $\mathrm{Int}(N)$.

Corollary 4.4.2. *Let $U \subset N$ be relatively open. Suppose $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ minimizes $\mathcal{A}^h$ in $\{\Omega' \in \mathcal{C}(\tilde{N}) \mid \Omega \triangle \Omega' \subset\subset U\} \subset \mathcal{C}_{E_+,E_-}(N)$. Further suppose that ∂N is a barrier for (h, P_+, P_-). Then $\partial^*\Omega \cap U = \partial\Omega \cap U$ is a smooth embedded $\mathcal{A}^h$-stable hypersurface in $U \cap \mathrm{Int}(N)$ with $H_{\partial\Omega,\Omega} = h\,|_{\partial\Omega}$.*

Proof. By the standard interior regularity (see [ZZ20, Theorem 2.2]), $\partial\Omega \cap U \cap \mathrm{Int}(N)$ is a smooth embedded stable h-PMC hypersurface. Suppose for contradiction that a component Σ of $\partial\Omega \cap U$ touches $\partial N \cap U$.

We first claim that Σ must be identically $\partial N \cap U$. Suppose otherwise, so Σ contains a point in $\mathrm{Int}(N) \cap U$. By Theorem 4.4.1, we can find a ball $B \subset \partial N$ and a relatively open set $W \subset U$ so that (in normal Fermi coordinates over ∂N) $\Sigma \cap W$ is the graph of a $C^{1,\alpha}$ function u on $\overline{B}$ satisfying $u\,|_{\mathrm{Int}(B)} > 0$, $u(y) = 0$ for some $y \in \partial B$, and $\nabla u(y) = 0$. Since $\Omega \in \mathcal{C}_{E_+,E_-}(N)$, we have (depending on whether $\partial N \cap W \subset P_+$ or $\partial N \cap W \subset P_-$) that $\overline{\Omega} \cap W$ is the sublevel graph of u (respectively the suplevel graph of u). We now arrive at a contradiction by the interior regularity, the Hopf lemma, and the assumption that ∂N is a barrier. So we conclude that Σ is identically $\partial N \cap U$, and the rest of $\partial\Omega \cap U$ lies in $U \cap \mathrm{Int}(N)$ by embeddedness.

Since ∂N is a barrier and $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ minimizes $\mathcal{A}^h$ in U, the first variation of $\mathcal{A}^h$ for any inward pointing vector field with small compact support in U gives a contradiction, so we conclude that $\partial\Omega \cap U \subset \mathrm{Int}(N)$. □

4.4.2 Unique continuation

Let $\tilde{\mathcal{S}} \subset C^\infty(\tilde{N})$ consist of smooth Morse functions $\tilde{h} \in C^\infty(\tilde{N})$ so that $\tilde{h}^{-1}(0)$ is a smooth closed hypersurface whose mean curvature vanishes to at most finite order. By [ZZ20, Proposition 0.2], $\tilde{\mathcal{S}}$ is an open and dense subset of $C^\infty(\tilde{N})$.

Let $\mathcal{S} \subset C^\infty(N)$ consist of those $h \in C^\infty(N)$ so that there is some $\tilde{h} \in \tilde{\mathcal{S}}$ with $\tilde{h}\,|_N = h$.

Proposition 4.4.3. *$\mathcal{S}$ is a dense subset of $C^\infty(N)$.*

Proof. Take $h \in C^\infty(N)$. Let $\tilde{h}$ be any extension of h to $\tilde{N}$. By the density of $\tilde{\mathcal{S}}$, there is a sequence $\{\tilde{h}_i\}_i \subset \tilde{\mathcal{S}}$ converging smoothly to $\tilde{h}$. By definition, $\{\tilde{h}_i \mid_N\} \subset \mathcal{S}$ converges smoothly to $\tilde{h} \mid_N = h$, as desired. □

By [ZZ20, Theorem 3.11], h-PMC hypersurfaces for functions $h \in \mathcal{S}$ satisfy a unique continuation property required for the proof of regularity of almost minimizers in min-max.

4.5 Min-max with barrier

In this section, we prove the following theorem.

Theorem 4.5.1. *Suppose $h \in \mathcal{S}$ and ∂N is a barrier for (h, P_+, P_-). Assume*

$$\mathbf{L}^h(\Pi) > \sup_{z \in Z} \mathcal{A}^h(\Phi_0(z)).$$

Then there is $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ so that $\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi)$ and $\Sigma = \partial\Omega$ is a smooth, closed, almost embedded hypersurface contained in $\mathrm{Int}(N)$ *satisfying $H_{\Sigma,\Omega} = h \mid_\Sigma$.*

4.5.1 Nontriviality

Here we show that $\partial^*\Omega \neq \varnothing$. If $\partial^*\Omega = \varnothing$, then $\Omega \in \{\varnothing, \tilde{N}\}$[7]. By the isoperimetric inequality in $\tilde{N}$ (see [ZZ20, Theorem 2.3]), $\varnothing$ and $\tilde{N}$ are strict minimizers for $\mathcal{A}^h$ in a small flat neighborhood, so any nontrivial min-max h-width cannot equal $\mathcal{A}^h(\varnothing)$ nor $\mathcal{A}^h(\tilde{N})$ (as X is connected and Z is nonempty).

[7]This scenario is only possible if at least one of E_- or E_+ is empty.

4.5.2 Pull-tight

Let $c = \sup_N |h|$ and $L = 2(\mathbf{L}^h + c\mathrm{Vol}(\tilde{N}))$. Let

$$A^L = \{V \in \mathcal{V}_n(N) \mid \|V\|(N) \leq L\}$$

and

$A_\infty = \{V \in A^L \mid V \text{ has } c\text{-bounded constrained first variation in } N\} \cup \{|\partial^* \Phi_0(z)| \mid z \in Z\}$.

We follow the construction of [Zho20, §1.2], [ZZ20, §5], and [ZZ19, §4] for a map to the space of vector fields $\mathcal{D} : A^L \to \mathcal{X}_{\mathrm{cpt}}(N)$. By the definition of A_∞, if $V \in A^L$ satisfies $2^{-j} \leq \mathbf{F}(V, A_\infty) \leq 2^{-j+1}$, then we can find $X_V \in \mathcal{X}^+_{\mathrm{cpt}}(N)$ so that

$$\|X_V\|_{C^1} \leq 1, \quad \delta V(X_V) + c \int_N |X_V| \, d\|V\| \leq -c_j < 0,$$

where $c_j > 0$ does not depend on V. Since the map to vector fields in [ZZ19, §4.2] is constructed using linear combinations of the vector fields X_V for some V, the identical construction actually gives $\mathcal{D} : A^L \to \mathcal{X}^+_{\mathrm{cpt}}(N)$. Hence, we can use $\mathcal{D}$ to construct the desired tightening map using flows as in [ZZ19, §4.3, §4.4] (the flows stay in N by construction).

To summarize, we obtain an $\mathbf{F}$-continuous map

$$\mathcal{PT} : [0,1] \times \{\Omega \in \mathcal{C}_{E_+,E_-}(N) \mid |\partial^* \Omega| \in A^L\} \to \mathcal{C}_{E_+,E_-}(N)$$

satisfying

- $\mathcal{PT}(0, \Omega) = \Omega$ for all Ω,
- $\mathcal{PT}(t, \Omega) = \Omega$ for all $t \in [0,1]$ if $|\partial^* \Omega| \in A_\infty$,

- there is a continuous function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $L(0) = 0$ and $L(t) > 0$ for $t > 0$ so that

$$\mathcal{A}^h(\mathcal{PT}(1, \Omega)) - \mathcal{A}^h(\Omega) \leq -L(\mathbf{F}(|\partial^*\Omega|, A_\infty)).$$

Lemma 4.5.2 (Pull-tight)**.** *Let $\{\Phi_i^*\}_i \subset \Pi$ be a minimizing sequence. Then there is another minimizing sequence $\{\Phi_i\}_i \subset \Pi$ so that $\mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi_i^*\}) \cap A_\infty$.*

Proof. The proof of [Zho20, Lemma 1.8] applies. □

4.5.3 Existence of almost minimizers

We make the same definition of h-almost minimizing as in [ZZ20, Definition 6.1].

Definition 4.5.3. For $\varepsilon, \delta > 0$ and $U \subset N$ relatively open, we define $\mathscr{A}^h(U; \varepsilon, \delta)$ to be the set of $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ such that if $\Omega = \Omega_0,\ \Omega_1, \ldots, \Omega_m \in \mathcal{C}_{E_+,E_-}(N)$ satisfies

- $\Omega_i \triangle \Omega \subset U$,
- $\mathcal{F}([\partial^*\Omega_i] - [\partial^*\Omega_{i+1}]) \leq \delta$,
- $\mathcal{A}^h(\Omega_i) \leq \mathcal{A}^h(\Omega) + \delta$,

then $\mathcal{A}^h(\Omega_m) \geq \mathcal{A}^h(\Omega) - \varepsilon$.

We say that $V \in \mathcal{V}_n(N)$ is *h-almost minimizing in U* if there are $\varepsilon_i, \delta_i \to 0$ and $\Omega_i \in \mathscr{A}^h(U; \varepsilon_i, \delta_i)$ such that $|\partial^*\Omega_i| \rightharpoonup V$.

Proposition 4.5.4. *If $V \in \mathcal{V}_n(N)$ is h-almost minimizing in U, then V has c-bounded constrained first variation in U for $c = \sup |h|$.*

Proof. The same proof as [ZZ19, Lemma 5.2] applies. □

Definition 4.5.5. We say $V \in \mathcal{V}_n(N)$ is *h-almost minimizing in small annuli* if for any $p \in N$, there exists $r_{\mathrm{am}}(p) > 0$ such that V is h-almost minimizing in $A^N_{s,r}(p)$ for all $0 < s < r \leq r_{\mathrm{am}}(p)$.

Theorem 4.5.6. *For every pulled-tight minimizing sequence $\{\Phi_i\}$ in Π, there is some $V \in \mathbf{C}(\{\Phi_i\})$ that has c-bounded constrained first variation in N and is h-almost minimizing in small annuli.*

Proof. The proof follows from the combinatorial construction of [Pit81, 4.9, 4.10] (see [Zho20, Theorem 1.16]) and the discretization and interpolation constructions of [Zho20, §1.3]. That V has c-bounded constrained first variation in N follows from the same argument as the final paragraph of the proof of [Zho20, Theorem 1.7]. □

4.5.4 Regularity of almost minimizers

The interior regularity theory of [ZZ20, Theorem 7.1] is local, and therefore applies to $\mathrm{Int}(N)$. Hence, we only need to deal with the boundary. Throughout this subsection, we exploit the fact that c-bounded contrained first variation in N implies C-bounded first variation in $\tilde{N}$ for some $C(N, c)$ (by Lemma 4.2.3).

We first recall the construction of replacements for h-almost minimizing varifolds.

Lemma 4.5.7 (Constrained minimization)**.** *Let $\varepsilon, \delta > 0$ and $U \subset N$ relatively open. Take $\Omega \in \mathscr{A}^h(U; \varepsilon, \delta)$. Fix a relatively compact subset $K \subset U$. Let $\mathcal{C}_\Omega$ be the set of $\Lambda \in \mathcal{C}_{E_+,E_-}(N)$ such that there is a sequence $\Omega = \Omega_0, \ \Omega_1, \ldots, \Omega_m = \Lambda$ in $\mathcal{C}_{E_+,E_-}(N)$ satisfying*

- $\Omega_i \triangle \Omega \subset K$,
- $\mathcal{F}([\partial^* \Omega_i] - [\partial^* \Omega_{i+1}]) \leq \delta$,
- $\mathcal{A}^h(\Omega_i) \leq \mathcal{A}^h(\Omega) + \delta$.

Then there exists $\Omega^* \in \mathcal{C}_{E_+,E_-}(N)$ such that

1. *$\Omega^* \in \mathcal{C}_\Omega$ and*

$$\mathcal{A}^h(\Omega^*) = \inf\{\mathcal{A}^h(\Lambda) \mid \Lambda \in \mathcal{C}_\Omega\},$$

2. Ω^* *is locally minimizing for* $\mathcal{A}^h$ *in* $\mathrm{int}_{\mathrm{rel}}(K)$,

3. $\Omega^* \in \mathscr{A}^h(U;\varepsilon,\delta)$.

Proof. The proof follows from the proof of [ZZ20, Lemma 6.7], [ZZ19, Lemma 5.7]. □

Remark 4.5.8. If ∂N is a barrier for (h, P_+, P_-), Lemma 4.5.7 (1) and (2) combined with Corollary 4.4.2 imply that $\partial^*\Omega^* \cap \mathrm{int}_{\mathrm{rel}}(K) = \partial\Omega^* \cap \mathrm{int}_{\mathrm{rel}}(K)$ is an $\mathcal{A}^h$-stable smooth embedded h-PMC hypersurface contained in $\mathrm{Int}(N)$.

Proposition 4.5.9 (Existence of replacements)**.** *Let $V \in \mathcal{V}_n(N)$ be h-almost minimizing in a relatively open set $U \subset N$. Let $K \subset U$ be a compact subset. Then there exists $V^* \in \mathcal{V}_n(N)$, called an* h-replacement *for V in K, so that*

1. $V \llcorner (N \setminus K) = V^* \llcorner (N \setminus K)$,

2. $-c\mathcal{H}^{n+1}(K) \leq \|V\|(N) - \|V^*\|(N) \leq c\mathcal{H}^{n+1}(K)$,

3. V^* *is h-almost minimizing in U,*

4. *there are $\Omega_i^* \in \mathscr{A}^h(U;\varepsilon_i,\delta_i)$ with $\varepsilon_i, \delta_i \to 0$ so that Ω_i^* locally minimizes $\mathcal{A}^h$ in $\mathrm{int}_{\mathrm{rel}}(K)$ and $|\partial^*\Omega_i^*| \rightharpoonup V^*$,*

5. *if V has c-bounded constrained first variation in N, then so does V^*.*

Proof. The proof follows [ZZ20, Proposition 6.8], [ZZ19, Proposition 5.8]. □

Proposition 4.5.10 (Regularity of replacements)**.** *Under the same hypotheses as Proposition 4.5.9, if ∂N is a barrier for (h, P_+, P_-), then $V^* \llcorner \mathrm{int}_{\mathrm{rel}}(K) = |\Sigma|$ for Σ a smooth almost embedded stable h-PMC hypersurface in* $\mathrm{int}_{\mathrm{rel}}(K) \cap \mathrm{Int}(N)$.

Proof. As in [ZZ20, Proposition 6.9], [ZZ19, Proposition 5.9], the result follows from the regularity of local minimizers in Corollary 4.4.2 (which applies because we assumed that ∂N is a barrier) and the compactness theory for stable h-PMC hypersurfaces with uniformly bounded area (see [ZZ20, Theorem 3.6]). The fact that the

limit lies in $\mathrm{Int}(N)$ follows from the fact that ∂N is a barrier and the maximum principle. □

Proposition 4.5.11 (Tangent cones are planar)**.** *Suppose ∂N is a barrier for (h, P_+, P_-). Suppose $V \in \mathcal{V}_n(N)$ has c-bounded constrained first variation in N and is h-almost minimizing in a relatively open set $U \subset N$. For any $y \in \partial N \cap U \cap \mathrm{spt}\|V\|$ and $C \in \mathrm{VarTan}(V, y)$,*

$$C = m|T_y \partial N|$$

for some $m \in \mathbb{N}$.

Proof. The arguments of [ZZ20, Proposition 6.10, 6.11], [ZZ19, Proposition 5.10, 5.11] apply to guarantee that any tangent cone is an integer multiple of a hyperplane in $T_y \tilde{N}$. Since $\mathrm{spt}\|V\| \subset N$, this hyperplane must be contained in the closed halfspace $\overline{T_y^+ N}$ and containing the origin, and therefore must equal $T_y \partial N$. □

We can now use the properties of replacements to exclude boundary touching when we choose $h \in \mathcal{S}$.

Proposition 4.5.12 (Interior for almost minimizing)**.** *Suppose ∂N is a barrier for (h, P_+, P_-) and $h \in \mathcal{S}$. Suppose $V \in \mathcal{V}_n(N)$ has c-bounded constrained first variation in N and is h-almost minimizing in a relatively open set $U \subset N$. Then* $\mathrm{spt}\|V\| \cap U \cap \partial N = \varnothing$.

Proof. Suppose for contradiction that $y \in U \cap \partial N \cap \mathrm{spt}\|V\|$. Let $x \in U \cap \partial N$ with $d(x, y) = d$ small so that $\partial B_d(x)$ intersects ∂N transversally and $B_{2d}(x) \cap N \subset U$.

Let V' be an h-replacement for V in $\overline{B_d^N(x)}$. By Proposition 4.5.11 and the assumption of transversality, $y \in \mathrm{spt}\|V'\|$. Since V' is almost minimizing, Proposition 4.5.11 implies that every tangent cone to V' at y is an integer multiple of $T_y \partial N$. Hence, any varifold V^* that agrees with V' on $A^N_{d/2,d}(x)$ has $y \in \mathrm{spt}\|V^*\|$.

Let V'' be an h-replacement for V' in $\overline{A^N_{d/2,3d/2}(x)}$. By Proposition 4.5.10,

$$\mathrm{spt}\|V''\| \cap A^N_{d/2,3d/2}(x) \cap \partial N = \varnothing,$$

so $y \notin \mathrm{spt}\|V''\|$. However, by the gluing and unique continuation arguments from the proof of [ZZ20, Theorem 7.1] (which apply because they are local and the gluing happens in $\mathrm{Int}(N)$, and we assumed $h \in \mathcal{S}$), V'' agrees with V' on $A^N_{d/2,d}(x)$, so $y \in \mathrm{spt}\|V''\|$ by the above paragraph. Hence, we reach a contradiction. □

We now use a removable singularities result to deduce the same conclusion for varifolds that are h-almost minimizing in annuli.

Theorem 4.5.13. *Suppose ∂N is a barrier for (h, P_+, P_-) and $h \in \mathcal{S}$. Suppose $V \in \mathcal{V}_n(N)$ has c-bounded constrained first variation in N and is h-almost minimizing in small annuli. Then V is the varifold of a smooth almost embedded h-PMC hypersurface in* $\mathrm{Int}(N)$.

Proof. By Proposition 4.5.12, $\mathrm{spt}\|V\| \cap \partial N$ consists of finitely many isolated points. By removable singularities (see [ZZ20, Proof of Theorem 7.1, Step 4]), V is the varifold of a smooth almost embedded h-PMC hypersurface, with finitely many isolated points touching ∂N tangentially. Since ∂N is a barrier, the standard maximum principle implies $\mathrm{spt}\|V\| \cap \partial N = \varnothing$. □

Now that we know that V avoids the boundary, Theorem 4.5.1 follows as in [ZZ20].

4.6 Min-max with barrier in generic metrics

We use special results applicable to a generic set of metrics and prescribing functions to upgrade the conclusions of Theorem 4.5.1.

4.6.1 Good pairs

Let $(N, \partial N)$ be a compact manifold with smooth boundary. We say a pair (g, h), where g is a smooth Riemannian metric on N and $h \in C^\infty(N)$, is a *good pair*, if

- $h \in \mathcal{S}(g)$[8] as defined in §4.4.2
- every closed two-sided almost embedded h-PMC hypersurface in $\mathrm{Int}(N)$ is embedded and $\mathcal{A}^h$-nondegenerate.

The following result is a consequence of [Zho20, Lemma 3.5], building on [ZZ20, Proposition 0.2] and [Whi19, Theorems 34 and 35].

Theorem 4.6.1. *Let N be a compact manifold with smooth boundary. The set of good pairs is C^∞ dense.*

Proof. Let (g, h) be a Riemannian metric and smooth function on N. Let $(\tilde{g}, \tilde{h})$ be any extension of g and h to $\tilde{N}$. By [Zho20, Lemma 3.5], there is a sequence of good pairs $(\tilde{g}_i, \tilde{h}_i)$ for $\tilde{N}$ in the sense of [Zho20, §3.3] converging smoothly to $(\tilde{g}, \tilde{h})$. Since the restriction to N of a good pair on $\tilde{N}$ is a good pair on N by definition, $(\tilde{g}_i \mid_N, \tilde{h}_i \mid_N)$ is a sequence of good pairs converging smoothly to (g, h), as desired. □

4.6.2 Minimality for strict stability

We show, based on the arguments of [IM18], that a strictly $\mathcal{A}^h$-stable hypersurface minimizes $\mathcal{A}^h$ in a small flat neighborhood.

Theorem 4.6.2. *Let (M, g) be a smooth closed Riemannian manifold. Let $\Sigma = \partial\Omega$ be a smooth closed embedded h-PMC hypersurface which is strictly $\mathcal{A}^h$-stable. Then there exists $\varepsilon > 0$ (depending on Ω and M) so that*

$$\mathcal{A}^h(\Omega') > \mathcal{A}^h(\Omega)$$

[8] Here we include explicitly in the notation the dependence of $\mathcal{S}$ on the metric, since we consider varying metrics on N.

for all $\Omega' \in \mathcal{C}(M)$ *with* $0 < \mathcal{F}([\partial\Omega] - [\partial^*\Omega']) < \varepsilon$.

Proof. We follow the proof of [IM18, Proposition 3.1], but we take extra care because $\mathcal{A}^h$ is not necessarily an elliptic parametric functional in the sense of [IM18, §2.2] on all of M.

As observed on [Whi94, Page 205], $\mathcal{A}^h$ on a small tubular neighborhood of Σ is an elliptic parametric functional in the sense of [Whi94, §1] and [IM18, §2.2], so [Whi94, Theorem 2] applies.

We observe that upper bounds on $\mathcal{A}^h$ imply upper bounds on area (since h is smooth and M is compact), so the arguments of [IM18] requiring the compactness theory for bounded area apply. Moreover, since flat convergence implies L^1 convergence of the indicator functions of the finite perimeter sets (as noted in §4.3.1), $\mathcal{A}^h$ is lower semi-continuous with respect to $\mathcal{F}$-convergence.

The arguments of [IM18, §3.2] apply verbatim, with the additional observation that [IM18, Lemma 3.5] for $\mathcal{A}^h$ follows directly from [IM18, Lemma 3.5] for area combined with [IM18, Lemma 3.4] (to handle the integral of h terms).

The arguments of [IM18, §3.3] preceding [IM18, Lemma 3.8] apply to prove [IM18, (3.7)] with $F = \mathcal{A}^h$. Moving the integral of h terms to the right hand side (where we write Λ to be the integral $(n+1)$-current supported in $B_r(x_0)$ with $\partial\Lambda = X$), we get

$$\mathbf{M}(R_{\delta,\lambda}) \leq \mathbf{M}(R_{\delta,\lambda} + X) + \lambda\mathcal{F}(X) + c\mathbf{M}(\Lambda).$$

We can now apply [IM18, Lemma 3.4] (since we can assume $\mathcal{F}(X) \leq \varepsilon_0$ for the almost minimizing condition) to deduce

$$\mathbf{M}(R_{\delta,\lambda}) \leq \mathbf{M}(R_{\delta,\lambda} + X) + (\lambda + c)\mathcal{F}(X).$$

We can now apply [IM18, Lemma 3.8] to conclude that $R_{\delta,\lambda}$ is almost minimizing for mass. Hence, we can apply [IM18, Lemma 3.10] and conclude. □

4.6.3 Generic min-max theorem

The following result upgrades the conclusions of Theorem 4.5.1 for good pairs.

Theorem 4.6.3. *Suppose (g, h) is a good pair and ∂N is a barrier for (h, P_+, P_-). Assume*

$$\mathbf{L}^h(\Pi) > \sup_{z \in Z} \mathcal{A}^h(\Phi_0(z)).$$

Then there is $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ so that $\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi)$ and $\Sigma = \partial\Omega$ is a smooth, closed, embedded hypersurface contained in $\mathrm{Int}(N)$ satisfying $H_{\Sigma,\Omega} = h \mid_\Sigma$ and $1 \leq \mathrm{index}_{\mathcal{A}^h}(\Sigma) \leq k$.

Proof. The only new conclusions beyond Theorem 4.5.1 are embeddedness and the index bounds.

Embeddedness follows immediately from the assumption the (g, h) is a good pair.

Since $\partial\Omega$ is embedded and contained in $\mathrm{Int}(N)$, the index upper bound of [Zho20, Theorem 3.6] (which is a local deformation argument) applies.

The index lower bound follows from Theorem 4.6.2. Indeed, if Σ was $\mathcal{A}^h$-stable, then Σ would be strictly $\mathcal{A}^h$-stable since (g, h) is a good pair. Then by Theorem 4.6.2, Ω strictly minimizes $\mathcal{A}^h$ in a flat neighborhood, so any nontrivial min-max h-width cannot equal $\mathcal{A}^h(\Omega)$ (as X is connected and Z is nonempty). □

4.6.4 Corollaries of generic min-max

We record two consequence of Theorem 4.6.3 combined with the compactness theory for h-PMC hypersurfaces with uniformly bounded area and index (see [Zho20, Theorem 2.8]).

The first corollary is that the index upper bound applies for metrics g that are not bumpy.

Corollary 4.6.4. *Suppose $h \in \mathcal{S}$ and ∂N is a barrier for (h, P_+, P_-). Assume*

$$\mathbf{L}^h(\Pi) > \sup_{z \in Z} \mathcal{A}^h(\Phi_0(z)).$$

Then there is $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ so that $\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi)$ and $\Sigma = \partial\Omega$ is a smooth, closed, almost embedded hypersurface contained in $\mathrm{Int}(N)$ *satisfying $H_{\Sigma,\Omega} = h\,|_\Sigma$ and* $\mathrm{index}_{\mathcal{A}^h}(\Sigma) \leq k$.

The second corollary is that existence (and the index upper bound) applies for g that are not bumpy and h that are not in $\mathcal{S}$.

Corollary 4.6.5. *Suppose ∂N is a barrier for (h, P_+, P_-). Assume*

$$\mathbf{L}^h(\Pi) > \sup_{z \in Z} \mathcal{A}^h(\Phi_0(z)).$$

Then there is a smooth, closed, two-sided, almost embedded h-PMC hypersurface Σ contained in $\mathrm{Int}(N)$ *satisfying $\mathcal{H}^n(\Sigma) \leq \mathbf{L}^h(\Pi) + \int_{\{h>0\}} h$ and* $\mathrm{index}_{\mathcal{A}^h}(\Sigma) \leq k$.

4.7 Localization in compact manifolds with barrier

Using our generic min-max result from Theorem 4.6.3, we develop a cutting argument to localize min-max hypersurfaces.

4.7.1 Setup

We make a few specialized definitions related to the min-max setup considered above. See Figure 4.3 for an illustration.

Definition 4.7.1 (Span)**.** The *span* of Φ_0 is

$$\mathrm{Span}(\Phi_0) = \overline{\bigcup_{x \in X} \partial^* \Phi_0(x)}.$$

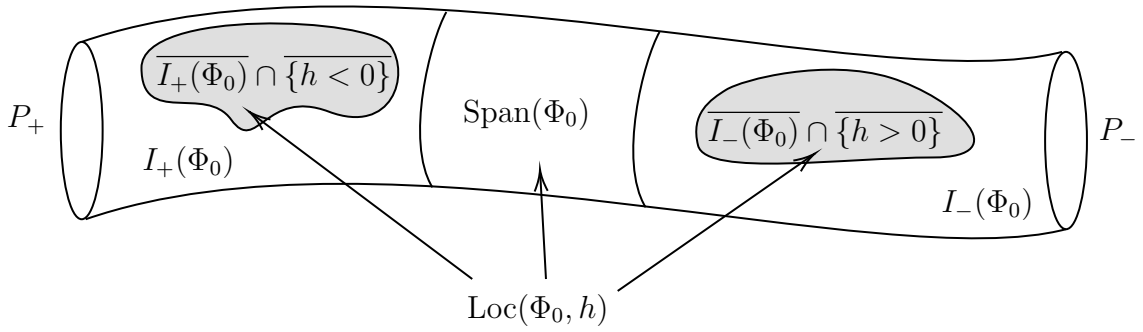


Figure 4.3: Span, unspanned halves, and localizer.

Definition 4.7.2 (Unspanned halves)**.** For any $x_0 \in X$[9], we define the positive and negative *unspanned halves* by

$$I_+(\Phi_0) = (N \cap \mathrm{Int}(\Phi_0(x_0))) \setminus \mathrm{Span}(\Phi_0) \quad \text{and} \quad I_-(\Phi_0) = (N \setminus \overline{\Phi_0(x_0)}) \setminus \mathrm{Span}(\Phi_0).$$

Note that $I_\pm(\Phi_0)$ are well-defined: for any $x \in X$ we have

$$I_+(\Phi_0) \subset \mathrm{Int}(\Phi_0(x)) \quad \text{and} \quad I_-(\Phi_0) \cap \overline{\Phi_0(x)} = \varnothing.$$

Definition 4.7.3 (Localizer)**.** The *localizer* of Φ_0 and h is

$$\mathrm{Loc}(\Phi_0, h) = \mathrm{Span}(\Phi_0) \cup (\overline{I_+(\Phi_0)} \cap \overline{\{h < 0\}}) \cup (\overline{I_-(\Phi_0)} \cap \overline{\{h > 0\}}).$$

4.7.2 Subdomains

Here we discuss how to restrict the min-max setup to special subdomains of the original compact manifold with boundary.

Let $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ be an open set with smooth boundary in $\mathrm{Int}(N)$ satisfying $\partial\Omega \cap \mathrm{Span}(\Phi_0) = \varnothing$.

We consider two possibilities.

[9]Recall that we assume (without loss of generality) that $\partial\Phi_0(x) = \overline{\partial^*\Phi_0(x)}$ for all $x \in X$.

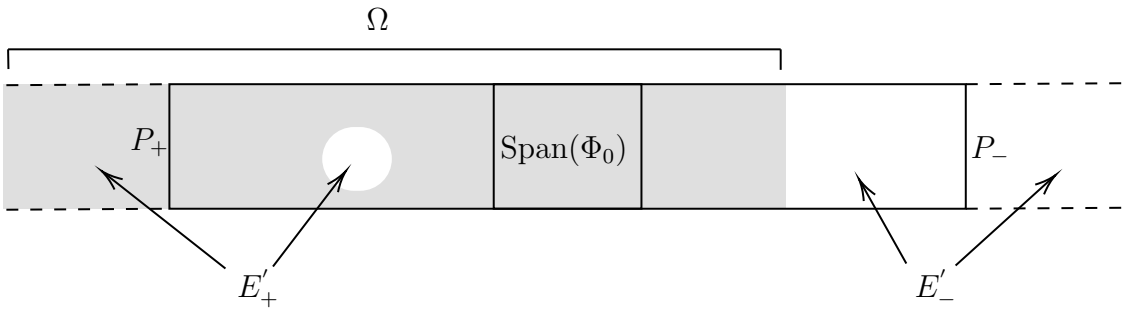


Figure 4.4: Schematic for the construction of N' when $\mathrm{Span}(\Phi_0) \subset \Omega$.

- $\mathrm{Span}(\Phi_0) \subset \Omega$. Consider the subdomain $N' = N \cap \overline{\Omega}$. Then $\partial N' = P_+ \sqcup \partial\Omega$. We define E'_+ to be the union of E_+ and all components of $\tilde{N} \setminus N'$ in I_+. E'_- is defined to be the union of the remaining components of $\tilde{N} \setminus N'$. See Figure 4.4.

- $\mathrm{Span}(\Phi_0) \cap \overline{\Omega} = \varnothing$. Consider the subdomain $N' = N \setminus \Omega$. Then $\partial N' = P_- \sqcup \partial\Omega$. We define E'_- to be the union of E_- and all components of $\tilde{N} \setminus N'$ in I_-. E'_+ is defined to be the union of the remaining components of $\tilde{N} \setminus N'$. See Figure 4.5.

We record a few observations that apply in both cases.

- $\mathcal{C}_{E'_+,E'_-}(N') \subset \mathcal{C}_{E_+,E_-}(N)$. Hence, $\mathcal{A}^h$ (as defined on $\mathcal{C}_{E_+,E_-}(N)$) is well-defined on $\mathcal{C}_{E'_+,E'_-}(N')$.

- By construction, the image of Φ_0 lies in $\mathcal{C}_{E'_+,E'_-}(N')$. Let $\Pi(N')$ denote the (X, Z)-homotopy class of Φ_0 for maps to $\mathcal{C}_{E'_+,E'_-}(N')$, which satisfies $\Pi \supset \Pi(N')$ by definition.

- Let $\mathbf{L}^h(\Pi(N'))$ denote the h-width of $\Pi(N')$. Then $\mathbf{L}^h(\Pi(N')) \geq \mathbf{L}^h(N)$.

4.7.3 Neighborhood cutting

We use the formula for the derivative of mean curvature along a smooth variation to find good hypersurfaces near unstable and strictly stable h-PMC hypersurfaces along which we will cut our manifold.

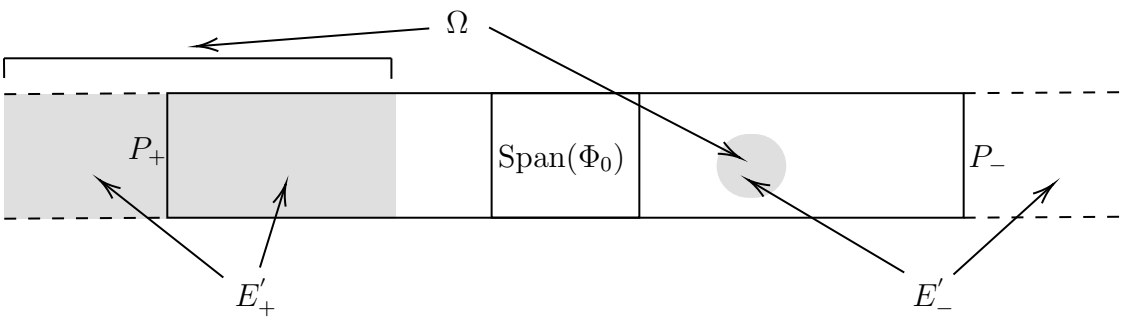


Figure 4.5: Schematic for the construction of N' when $\mathrm{Span}(\Phi_0) \cap \overline{\Omega} = \varnothing$.

Lemma 4.7.4. *Let $(N, \partial N, g)$ be a compact Riemannian manifold with smooth boundary and $h \in C^\infty(N)$. Suppose $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ has smooth closed embedded boundary contained in* $\mathrm{Int}(N)$*. Suppose Σ is a connected component of $\partial\Omega$ satisfying $H_{\Sigma,\Omega} = h\,|_\Sigma$ and Σ is $\mathcal{A}^h$-unstable. Then for any $\varepsilon > 0$, there are sets $\Omega_- \subset \Omega$ and $\Omega_+ \supset \Omega$ satisfying*

- $\Sigma \cap \Omega_- = \varnothing$ *and* $\Sigma \subset \Omega_+$,
- $\Omega \triangle \Omega_\pm \subset B_\varepsilon(\Sigma)$,
- $\Sigma_\pm = (\partial\Omega_\pm) \cap B_\varepsilon(\Sigma)$ *are smooth closed embedded hypersurfaces contained in* $\mathrm{Int}(N)$,
- $H_{\Sigma_-,\Omega_-} > h\,|_{\Sigma_-}$,
- $H_{\Sigma_+,\Omega_+} < h\,|_{\Sigma_+}$.

Proof. Let ν denote the unit normal vector field along Σ pointing out of Ω. Since $\Sigma = \partial\Omega$ is $\mathcal{A}^h$-unstable, there is a negative number $\lambda < 0$ and a smooth positive function $\varphi \in C^\infty(\Sigma)$ so that

$$-\Delta_\Sigma \varphi - (|A_\Sigma|^2 + \mathrm{Ric}(\nu,\nu))\varphi = (\partial_\nu h)\varphi + \lambda\varphi < (\partial_\nu h)\varphi. \tag{4.7.1}$$

Consider a variation of Σ given by $\varphi\nu$ (namely, extend $\varphi\nu$ to a vector field supported in a neighborhood of Σ, then apply the flow diffeomorphisms generated by this vector

field to Σ and Ω). Since φ is positive and Σ is closed and embedded, we obtain a foliation $\{\Sigma_t = \partial\Omega_t\}_{t\in(-\delta,\delta)}$ of a neighborhood of Σ for some small $\delta > 0$, where $\Sigma_0 = \Sigma$ and $\Omega_0 = \Omega$.

Since the derivative of the mean curvature along this foliation at $t = 0$ is given by the left hand side of (4.7.1) and the derivative of h along this foliation at $t = 0$ is given by the right hand side of (4.7.1), we find that for all $\eta > 0$ sufficiently small

$$H_{\Sigma_{-\eta},\Omega_{-\eta}} > h\,|_{\Sigma_{-\eta}}\ ,\quad H_{\Sigma_\eta,\Omega_\eta} < h\,|_{\Sigma_\eta}\ .$$

We can now take $\Omega_\pm = \Omega_{\pm\eta}$ for $\eta > 0$ chosen sufficiently small depending on ε. □

Lemma 4.7.5. *Let $(N, \partial N, g)$ be a compact Riemannian manifold with smooth boundary and $h \in C^\infty(N)$. Suppose $\Omega \in \mathcal{C}_{E_+,E_-}(N)$ has smooth closed embedded boundary contained in* $\mathrm{Int}(N)$. *Suppose Σ is a connected component of $\partial\Omega$ satisfying $H_{\Sigma,\Omega} = h\,|_\Sigma$ and Σ is strictly $\mathcal{A}^h$-stable. Then for any $\varepsilon > 0$, there are sets $\Omega_- \subset \Omega$ and $\Omega_+ \supset \Omega$ satisfying*

- $\Sigma \cap \Omega_- = \varnothing$ *and* $\Sigma \subset \Omega_+$,
- $\Omega \triangle \Omega_\pm \subset B_\varepsilon(\Sigma)$,
- $\Sigma_\pm = (\partial\Omega_\pm) \cap B_\varepsilon(\Sigma)$ *are smooth closed embedded hypersurfaces contained in* $\mathrm{Int}(N)$,
- $H_{\Sigma_-,\Omega_-} < h\,|_{\Sigma_-}$,
- $H_{\Sigma_+,\Omega_+} > h\,|_{\Sigma_+}$.

Proof. The proof is identical to Lemma 4.7.4, where we flip the direction of the inequality (4.7.1) due to strict stability. □

4.7.4 Localized min-max theorem with barrier

We are now equipped to prove a localized version of Theorem 4.5.1.

Theorem 4.7.6. *Suppose ∂N is a barrier for (h, P_+, P_-). Assume*

$$\mathbf{L}^h(\Pi) > \sup_{z \in Z} \mathcal{A}^h(\Phi_0(z)).$$

Then there is a smooth, closed, two-sided, almost embedded h-PMC hypersurface Σ contained in $\mathrm{Int}(N)$ satisfying

- $\mathcal{H}^n(\Sigma) \leq \sup_{x \in X} \mathcal{A}^h(\Phi_0(x)) + \int_{\{h>0\}} h$,
- $\mathrm{index}_{\mathcal{A}^h}(\Sigma) \leq k$,
- $\Sigma \cap \mathrm{Loc}(\Phi_0, h) \neq \varnothing$.

Proof. We give the proof when (g, h) is a good pair satisfying

$$h \mid_{I_+(\Phi_0) \setminus \mathrm{Loc}(\Phi_0, h)} > 0 \quad \text{and} \quad h \mid_{I_-(\Phi_0) \setminus \mathrm{Loc}(\Phi_0, h)} < 0.$$

By Theorem 4.6.1 and [Zho20, Theorem 2.8], the general conclusion follows by approximation[10].

By Theorem 4.6.3, there is $\Omega \in \mathcal{C}_{E_+, E_-}(N)$ so that $\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi)$ and $\Sigma = \partial\Omega$ is a smooth, closed, embedded, h-PMC hypersurface contained in $\mathrm{Int}(N)$, satisfying $1 \leq \mathrm{index}_{\mathcal{A}^h}(\Sigma) \leq k$.

If $\Sigma \cap \mathrm{Loc}(\Phi_0, h) \neq \varnothing$, then we are done.

First, suppose that $\mathrm{Loc}(\Phi_0, h) \subset \Omega$. We modify Ω in a small neighborhood of Σ in the following way. Let Σ^* be a component of Σ, which (by bumpiness) is either unstable or strictly stable for $\mathcal{A}^h$.

[10] To ensure h has the correct sign on the respective sets, we can first add to h the function $\varepsilon\eta$ for $\varepsilon > 0$ small and $\eta \in C^\infty(N)$ any smooth function satisfying $\eta \mid_{\overline{I}_+(\Phi_0)} > 0$ and $\eta \mid_{\overline{I}_-(\Phi_0)} < 0$, and then take $\varepsilon \to 0$.

1. If $\Sigma^* \subset I_+$, then $h\mid_{\Sigma^*} > 0$. In this case, we modify Ω near Σ^* by taking Ω_- from Lemma 4.7.4 or Lemma 4.7.5, choosing the perturbation small enough so that the modified boundary Σ_- has $H_{\Sigma_-,\Omega_-} > 0$.

2. If Σ^* is unstable and $\Sigma^* \subset I_-$, then we modify Ω near Σ^* by taking Ω_- from Lemma 4.7.4.

3. If Σ^* is strictly stable and $\Sigma^* \subset I_-$, then we modify Ω near Σ^* by taking Ω_+ from Lemma 4.7.4.

Let Ω' denote the modification of Ω. We take the subdomain $N' = N \cap \overline{\Omega}'$, with the decomposition $E'_\pm$ given in §4.7.2. By construction, for Σ^* in case (1) above, the modified boundary Σ_- will lie in P'_+. Then we have

$$H_{\Sigma_-,E'_+} < 0 < h\mid_{\Sigma_-},$$

so these boundary components satisfy the barrier property. For Σ^* in cases (2) and (3), the modified boundary will lie in P'_-. Then the barrier property follows directly from Lemmas 4.7.4 and 4.7.5 respectively.

Second, suppose that $\mathrm{Loc}(\Phi_0, h) \cap \overline{\Omega} = \varnothing$. We modify Ω in a small neighborhood of Σ in the following way. Let Σ^* be a component of Σ, which (by bumpiness) is either unstable or strictly stable for $\mathcal{A}^h$.

1. If $\Sigma^* \subset I_-$, then $h\mid_{\Sigma^*} < 0$. In this case, we modify Ω near Σ^* by taking Ω_+ from Lemma 4.7.4 or Lemma 4.7.5, choosing the perturbation small enough so that the modified boundary Σ_+ has $H_{\Sigma_+,\Omega_+} < 0$.

2. If Σ^* is unstable and $\Sigma^* \subset I_+$, then we modify Ω near Σ^* by taking Ω_+ from Lemma 4.7.4.

3. If Σ^* is strictly stable and $\Sigma^* \subset I_+$, then we modify Ω near Σ^* by taking Ω_- from Lemma 4.7.5.

Let Ω' denote the modification of Ω. We take the subdomain $N' = N \setminus \mathrm{Int}(\Omega')$, with the decomposition $E'_\pm$ given in §4.7.2. By construction, for Σ^* in case (1) above, the modified boundary Σ_+ will lie in P'_-. Then we have

$$H_{\Sigma_+, E'_-} < 0 < -h\,|_{\Sigma_-},$$

so these boundary components satisfy the barrier property. For Σ^* in cases (2) and (3), the modified boundary will lie in P'_+. Then the barrier property follows directly from Lemmas 4.7.4 and 4.7.5 respectively.

In both cases, we find a subdomain N' with decomposition $E'_\pm$ so that

- $\mathrm{Span}(\Phi_0) \subset \mathrm{Loc}(\Phi_0, h) \subset N'$,
- $\partial N'$ is a barrier for (h, P'_+, P'_-),
- the unstable components of Σ (of which there is at least one by Theorem 4.6.3) are not contained in N',
- $(g\,|_{N'}, h\,|_{N'})$ is a good pair[11],
- $\sup_{x\in X} \mathcal{A}^h(\Phi_0(x)) \geq \mathbf{L}^h(\Pi(N')) \geq \mathbf{L}^h(\Pi) > \sup_{z\in Z} \mathcal{A}^h(\Phi_0(z))$.

Hence, we can re-apply Theorem 4.6.3 on N', and therefore iterate the above steps. Since each iteration produces a distinct, connected, $\mathcal{A}^h$-unstable, smooth, closed, embedded, h-PMC hypersurface Σ' with

$$\mathcal{H}^n(\Sigma') \leq \sup_{x\in X} \mathcal{A}^h(\Phi_0(x)) + \int_{\{h>0\}} h < \infty$$

and $\mathrm{index}_{\mathcal{A}^h}(\Sigma') \leq k$, the iteration must terminate after at most finitely many steps by the bumpiness of the metric (if there were infinitely many such hypersurfaces, [Zho20, Theorem 2.6(iv)] yields a limit embedded h-PMC hypersurface that is degenerate).

[11] It follows from the definition that the restriction of a good pair to a subdomain is a good pair.

Hence, at some step of the iteration the output of Theorem 4.6.3 will satisfy $\Sigma \cap \mathrm{Loc}(\Phi_0, h) \neq \varnothing$. $\square$

4.8 Setup for Noncompact Manifolds

In this part, we apply the localized min-max result of Theorem 4.7.6 to develop a min-max theory in noncompact manifolds. We also provide applications of the theory.

Let (M, g) be a complete Riemannian manifold of dimension $3 \leq n+1 \leq 7$.

4.8.1 Space of subsets

Let Ω_0 be an open set with smooth closed boundary (we emphasize that Ω_0 need not be bounded).

Let $\mathcal{C}_{\Omega_0}(M)$ be the collection of sets of finite perimeter $\Omega \in \mathcal{C}(M)$ so that $\Omega \triangle \Omega_0$ is bounded.

4.8.2 Functional

Let $h \in C^\infty_{\mathrm{loc}}(M)$. For $\Omega \in \mathcal{C}_{\Omega_0}(M)$, we define

$$\mathcal{A}^h(\Omega) = \mathcal{H}^n(\partial^*\Omega) - \int_{\Omega \setminus \Omega_0} h + \int_{\Omega_0 \setminus \Omega} h.$$

By the definition of $\mathcal{C}_{\Omega_0}(M)$, $\mathcal{A}^h(\Omega)$ is well-defined for all $\Omega \in \mathcal{C}_{\Omega_0}(M)$.

4.8.3 Admissible functions

We let $\mathcal{W}_{\Omega_0} \subset C^\infty_{\mathrm{loc}}(M)$ denote the set of locally smooth functions h with the property that there is a compact set $K_h \subset M$ satisfying

$$h\,|_{\Omega_0 \setminus K_h} \geq 0 \quad \text{and} \quad h\,|_{(M \setminus \Omega_0) \setminus K_h} \leq 0.$$

Note that if $\Omega \in \mathcal{C}_{\Omega_0}(M)$, $h \in \mathcal{W}_{\Omega_0}$, and $\mathcal{A}^h(\Omega) \leq \Lambda < \infty$, then

$$\begin{aligned}\mathcal{H}^n(\partial^*\Omega) &= \mathcal{A}^h(\Omega) + \int_{\Omega\setminus\Omega_0} h - \int_{\Omega_0\setminus\Omega} h \\ &\leq \Lambda + \int_{(\Omega\setminus\Omega_0)\cap K_h} h - \int_{(\Omega_0\setminus\Omega)\cap K_h} h \\ &\leq \Lambda + \mathcal{H}^{n+1}(K_h) \sup_{K_h} |h| < \infty.\end{aligned}$$

Hence, for these functions, uniform $\mathcal{A}^h$ bounds imply uniform area bounds.

4.8.4 Min-max setup

We follow the same definitions as §4.3.4.

Let X^k be a connected cubical complex of dimension k in some $I^m = [0,1]^m$. Let $Z \subset X$ be a nonempty cubical subcomplex.

Let $\Phi_0 : X \to \mathcal{C}_{\Omega_0}(M)$ be a continuous map in the $\mathbf{F}$-topology.

Definition 4.8.1 (Homotopy class)**.** Let Π be the set of all sequences of $\mathbf{F}$-continuous maps $\{\Phi_i : X \to \mathcal{C}_{\Omega_0}(M)\}_i$ so that there are $\mathcal{F}$-continuous homotopies $\{\Psi_i : [0,1] \times X \to \mathcal{C}_{\Omega_0}(M)\}_i$ satisfying $\Psi_i(0,\cdot) = \Phi_i$, $\Psi_i(1,\cdot) = \Phi_0$, and

$$\limsup_{i\to\infty} \sup\{\mathbf{F}(\Psi_i(t,z), \Phi_0(z)) \mid t \in [0,1],\ z \in Z\} = 0.$$

Such a sequence $\{\Phi_i\}$ is called an (X,Z)-*homotopy sequence of mappings to* $\mathcal{C}_{\Omega_0}(M)$, and Π is called the (X,Z)-*homotopy class of* Φ_0.

Definition 4.8.2 (Width)**.** The *h-width* of Π is

$$\mathbf{L}^h(\Pi) = \inf_{\{\Phi_i\}\in\Pi} \limsup_{i\to\infty} \sup_{x\in X} \mathcal{A}^h(\Phi_i(x)).$$

Definition 4.8.3 (Minimizing sequence)**.** We say $\{\Phi_i\}_i \in \Pi$ is a *minimizing sequence*

if

$$\limsup_{i\to\infty}\sup_{x\in X}\mathcal{A}^h(\Phi_i(x)) = \mathbf{L}^h(\Pi).$$

Definition 4.8.4 (Critical set)**.** The *critical set* of a minimizing sequence $\{\Phi_i\}_i \in \Pi$ is

$$\mathbf{C}(\{\Phi_i\}) = \left\{V \in \mathcal{V}_n(M) \mid V = \lim_{j\to\infty} |\partial^*\Phi_{i_j}(x_j)|,\ \lim_{j\to\infty}\mathcal{A}^h(\Phi_{i_j}(x_j)) = \mathbf{L}^h(\Phi)\right\}.$$

4.9 Localization in noncompact manifolds

We use our localized min-max theory in compact manifolds with barriers to deduce a localized min-max theory in noncompact manifolds.

4.9.1 Setup

We follow the same definitions as §4.7.1.

Definition 4.9.1 (Span)**.** The *span* of Φ_0 is

$$\mathrm{Span}(\Phi_0) = \overline{\bigcup_{x\in X}\partial^*\Phi_0(x)}.$$

Definition 4.9.2 (Unspanned halves)**.** For any $x_0 \in X$[12], we define the positive and negative *unspanned halves* by

$$I_+(\Phi_0) = (M \cap \mathrm{Int}(\Phi_0(x_0))) \setminus \mathrm{Span}(\Phi_0) \quad \text{and} \quad I_-(\Phi_0) = (M \setminus \overline{\Phi_0(x_0)}) \setminus \mathrm{Span}(\Phi_0).$$

Note that $I_\pm(\Phi_0)$ are well-defined: for any $x \in X$ we have

$$I_+(\Phi_0) \subset \mathrm{Int}(\Phi_0(x)) \quad \text{and} \quad I_-(\Phi_0) \cap \overline{\Phi_0(x)} = \varnothing.$$

[12]Recall that we can assume (without loss of generality) that $\partial\Phi_0(x) = \overline{\partial^*\Phi_0(x)}$ for all $x \in X$.

Definition 4.9.3 (Localizer)**.** The *localizer* of Φ_0 and h is

$$\mathrm{Loc}(\Phi_0, h) = \mathrm{Span}(\Phi_0) \cup (\overline{I_+(\Phi_0)} \cap \overline{\{h < 0\}}) \cup (\overline{I_-(\Phi_0)} \cap \overline{\{h > 0\}}).$$

Remark 4.9.4. For $h \in \mathcal{W}_{\Omega_0}$, the localizer $\mathrm{Loc}(\Phi_0, h)$ is a compact set.

4.9.2 Compact subdomains

Here we discuss how to restrict the min-max setup to special compact subdomains of the original complete manifold, where we assume $h \in \mathcal{W}_{\Omega_0}$.

Let $N \subset M$ be a compact subdomain with smooth boundary satisfying $\mathrm{Loc}(\Phi_0, h) \subset N$.

We give N a boundary decomposition as in §4.3.1 by

$$P_+ = \partial N \cap I_+(\Phi_0) \quad \text{and} \quad P_- = \partial N \cap I_-(\Phi_0).$$

We also set

$$F_+ = I_+(\Phi_0) \setminus N \quad \text{and} \quad F_- = I_-(\Phi_0) \setminus N.$$

Let $\mathcal{C}_{F_+,F_-}(M)$ be the space of $\Omega \in \mathcal{C}_{\Omega_0}(M)$ satisfying

$$F_+ \subset \Omega \quad \text{and} \quad \Omega \cap F_- = \varnothing.$$

Let $(\tilde{N}, \tilde{g})$ be a closed Riemannian manifold of dimension $n+1$ containing (N, g) isometrically, and so that $\tilde{N} \setminus N$ has a decomposition $E_+ \sqcup E_-$ into unions of connected components of $\tilde{N} \setminus N$ satisfying $\partial E_\pm = P_\pm$.

We are now in the setting of the first half. We make a few observations.

- There is a natural $\mathbf{F}$-homeomorphism from $\mathcal{C}_{F_+,F_-}(M)$ to $\mathcal{C}_{E_+,E_-}(N)$ given by deleting F_+ from Ω and adding E_+.

- Using this homeomorphism, $\mathcal{A}^h$ as defined in §4.8.2 is well-defined on $\mathcal{C}_{E_+,E_-}(N)$. Moreover, this definition of $\mathcal{A}^h$ only differs from §4.3.2 by a fixed constant, which does not affect the theory in the first half (since the difference of $\mathcal{A}^h$ between any two sets is unchanged).

- By construction, the image of Φ_0 under this homeomorphism lies in $\mathcal{C}_{E_+,E_-}(N)$. Let $\Pi(N)$ denote the (X, Z)-homotopy class of Φ_0 for maps to $\mathcal{C}_{E_+,E_-}(N)$, which satisfies $\Pi \supset \Pi(N)$ by definition.

- Let $\mathbf{L}^h(\Pi(N))$ denote the h-width of $\Pi(N)$. Then $\mathbf{L}^h(\Pi(N)) \geq \mathbf{L}^h(N)$.

4.9.3 Localized min-max theorem in noncompact manifolds

In this subsection, we prove the main min-max theorem in noncompact manifolds.

Theorem 4.9.5. *Suppose $h \in \mathcal{W}_{\Omega_0}$. Suppose*

$$\mathbf{L}^h(\Pi) > \sup_{z \in Z} \mathcal{A}^h(\Phi_0(z)).$$

Then there is a smooth, complete, almost embedded, h-PMC hypersurface Σ in M satisfying

- $\mathcal{H}^n(\Sigma) \leq \sup_{x \in X} \mathcal{A}^h(\Phi_0(x)) + \mathcal{H}^{n+1}(K_h) \sup_{K_h} |h| < \infty$,

- $\operatorname{index}_{\mathcal{A}^h}(\Sigma) \leq k$,

- $\Sigma \cap \operatorname{Loc}(\Phi_0, h) \neq \varnothing$.

Proof. Let $\{N_i\}_i$ be a nested compact exhaustion of M with smooth boundary satisfying $\operatorname{Loc}(\Phi_0, h) \subset N_1$.

Fix any i (henceforth we drop the subscript). We follow the constructions of §4.9.2 for N, which produces a boundary decomposition $P_\pm$.

We modify h to $\tilde{h}$ near ∂N so that $\tilde{h} \mid_{P_+} > H_{P_+, M\setminus N}$, $-\tilde{h} \mid_{P_-} > H_{P_-, M\setminus N}$, and $\int_{\tilde{N}} |h - \tilde{h}|$ is arbitrarily small.

Then ∂N is a barrier for $(\tilde{h}, P_+, P_-)$ and $\mathbf{L}^{\tilde{h}}(\Pi(N)) > \sup_{z\in Z} \mathcal{A}^{\tilde{h}}(\Phi_0(z))$.

We can now apply Theorem 4.7.6, producing a smooth closed almost embedded $\tilde{h}$-PMC hypersurface Σ in $\mathrm{Int}(N)$ satisfying the conclusions of Theorem 4.7.6[13].

The theorem follows by taking the limit over the compact exhaustion and applying the compactness theory from [Zho20, Theorem 2.8]. □

4.10 Application I: PMC hypersurfaces in Euclidean space

We use the theory developed in Theorem 4.5.1 (and Corollary 4.6.5) to provide new answers to Question 4.1.1.

Theorem 4.10.1. *If $h \in C^\infty_{loc}(\mathbb{R}^{n+1})$ for $3 \leq n+1 \leq 7$ satisfies*

1. *there is a compact set with smooth boundary $N \subset \mathbb{R}^{n+1}$ satisfying $h < H_{\partial N, N}$,*
2. *there is an $\mathbf{F}$-continuous map $\Phi_0 : [0,1] \to \mathcal{C}_\varnothing(\mathbb{R}^{n+1})$ satisfying $\Phi_0(t) \subset N$ for all t and*

$$\mathbf{L}^h(\Pi_{\Phi_0}(N)) > \max\{\mathcal{A}^h(\Phi_0(0)),\ \mathcal{A}^h(\Phi_0(1))\}.$$

then there exists a smooth closed almost embedded h-PMC hypersurface contained in $\mathrm{Int}(N)$ *with $\mathcal{A}^h$-index at most 1.*

Remark 4.10.2. We emphasize that condition (1) is only a 1-sided bounded; h is allowed to be arbitrarily negative.

Proof. Apply Theorem 4.7.6 (and Corollary 4.6.5 for the index bound) on N with $P_+ = \varnothing$ and $P_- = \partial N$. □

[13]The difference in the area bounds is simply a reflection of the fact that the definition of $\mathcal{A}^h$ here differs from the definition of $\mathcal{A}^h$ in the first half by a fixed constant.

Corollary 4.10.3. *If $h \in C^\infty_{loc}(\mathbb{R}^{n+1})$ for $3 \leq n+1 \leq 7$ satisfies*

1. *there exists a nonempty bounded set of finite perimeter Λ satisfying $\mathcal{A}^h(\Lambda) \leq 0$,*
2. *$h(x) \leq C|x|^{-1-\alpha}$ for some $\alpha,\ C > 0$,*

then there exists a smooth closed almost embedded h-PMC hypersurface in $\mathbb{R}^{n+1}$ with $\mathcal{A}^h$-index at most 1.

Proof. Choose R so large that $\Lambda \subset B_R(0)$ and $h\ |_{\partial B_R(0)} \leq CR^{-1-\alpha} < n/R$. Set $N = \overline{B_R(0)}$, and note that $H_{\partial N, N} = n/R > h$.

Let $\Phi_0(t) = \Lambda \cap B_{Rt}(0)$. Then

$$\max\{\mathcal{A}^h(\Phi_0(0)), \mathcal{A}^h(\Phi_0(1))\} = \max\{\mathcal{A}^h(\varnothing), \mathcal{A}^h(\Lambda)\} = 0.$$

By the Euclidean isoperimetric inequality (and the fact that h is bounded on N), we have $\mathbf{L}^h(\Phi_0(N)) > 0$.

Hence, the conclusion follows from Theorem 4.10.1. □

4.11 Application II: CMC hypersurfaces in finite volume

We use the theory developed in Theorem 4.9.5 to provide new answers to Question 4.1.2.

4.11.1 Isoperimetric profile

Let (M, g) be a complete finite volume manifold.

Definition 4.11.1 (Isoperimetric profile)**.** For $v \in (0, \mathrm{Vol}(M))$, we define

$$\mathscr{I}(v) = \inf\{\mathcal{H}^n(\partial^*\Omega) \mid \Omega \in \mathcal{C}(M),\ \mathcal{H}^{n+1}(\Omega) = v\}.$$

Since we assume (M, g) has finite volume, $\mathscr{I}(v) > 0$ for all $v \in (0, \mathrm{Vol}(M))$ and the infimum is achieved by a finite perimeter set (by [NR21, Theorem 2.2]), and $\mathscr{I}$ is continuous (by [NR21, Corollary 2.3]).

Definition 4.11.2. We define

$$\mathscr{D}(M, g) := \mathrm{Lip}(\mathscr{I}) = \sup_{v_1 \neq v_2} \left| \frac{\mathscr{I}(v_2) - \mathscr{I}(v_1)}{v_2 - v_1} \right|.$$

Remark 4.11.3. Since $\mathscr{I}(v) > 0$ for finite volume manifolds, $\mathscr{D}(M, g) > 0$.

4.11.2 Existence of CMC hypersurfaces

We now prove the main result of the subsection.

Theorem 4.11.4. *Let (M, g) be a complete finite volume manifold of dimension $3 \leq n+1 \leq 7$. If $0 < c < \mathscr{D}(M, g)$, then there is a smooth complete almost embedded c-CMC hypersurface in M with $\mathcal{A}^c$-index at most 1.*

Proof. Since $\mathscr{I}$ is symmetric under $v \mapsto \mathrm{Vol}(M) - v$, we can find $v_1 < v_2$ so that

$$\frac{\mathscr{I}(v_1) - \mathscr{I}(v_2)}{v_2 - v_1} > c.$$

By Lemma 4.11.5 below, we can find $\Omega_1 \in \mathcal{C}_\varnothing(M)$ with $\mathcal{H}^{n+1}(\Omega_1)$ approximately v_2 and

$$\frac{\mathscr{I}(v_1) - \mathcal{H}^n(\partial^* \Omega_1)}{\mathcal{H}^{n+1}(\Omega_1) - v_1} \geq c + \delta \tag{4.11.1}$$

for some $\delta > 0$ small.

Let $\Phi_0 : [0, 1] \to \mathcal{C}_\varnothing(M)$ be any $\mathbf{F}$-continuous path from $\varnothing$ to Ω_1. Take $h = -c$. Then $\varnothing$ is the unique minimizer of $\mathcal{A}^h$. For any $\Omega \in \mathcal{C}_\varnothing(M)$ with $\mathcal{H}^{n+1}(\Omega) = v_1$, we

have by (4.11.1)

$$\begin{aligned}\mathcal{A}^h(\Omega) &= \mathcal{H}^n(\partial^*\Omega) + c\mathcal{H}^{n+1}(\Omega)\\ &\geq \mathscr{I}(v_1) + cv_1\\ &\geq \mathcal{H}^n(\partial^*\Omega_1) + c\mathcal{H}^{n+1}(\Omega_1) + \frac{\delta}{2}(v_2 - v_1)\\ &= \mathcal{A}^h(\Omega_1) + \frac{\delta}{2}(v_2 - v_1).\end{aligned}$$

Hence, we have

$$\mathbf{L}^h(\Pi_{\Phi_0}) > \max\{\mathcal{A}^h(\Phi_0(0)),\ \mathcal{A}^h(\Phi_0(1))\}.$$

The theorem now follows from Theorem 4.9.5. □

Lemma 4.11.5. *For any $v \in (0, \mathrm{Vol}(M))$ and any $\varepsilon > 0$, there is a bounded set $\Omega \in \mathcal{C}(M)$ satisfying*

$$|\mathcal{H}^{n+1}(\Omega) - v| < \varepsilon \quad \textit{and} \quad |\mathscr{I}(v) - \mathcal{H}^n(\partial^*\Omega)| < \varepsilon.$$

Proof. Since M has finite volume, there is $\{K_i\}_i$ a nested exhaustion by compact sets with smooth boundary satisfying $\limsup_{i\to\infty} \mathcal{H}^n(\partial K_i) = 0$.

Let $\Omega \in \mathcal{C}(M)$ satisfy $\mathcal{H}^{n+1}(\Omega) = v$ and $\mathcal{H}^n(\partial^*\Omega) \leq \mathscr{I}(v) + \varepsilon/2$. Let $\Omega_i = \Omega \cap K_i$. Then

$$\limsup_{i\to\infty} \mathcal{H}^n(\partial^*\Omega_i) \leq \mathcal{H}^n(\partial^*\Omega) + \limsup_{i\to\infty} \mathcal{H}^n(\partial K_i) = \mathcal{H}^n(\partial^*\Omega),$$

$$\liminf_{i\to\infty} \mathcal{H}^n(\partial^*\Omega_i) \geq \liminf_{i\to\infty} \mathcal{H}^n((\partial^*\Omega) \cap K_i) = \mathcal{H}^n(\partial^*\Omega).$$

Moreover,

$$\limsup_{i\to\infty} \mathcal{H}^{n+1}(\Omega_i) \leq \mathcal{H}^{n+1}(\Omega),$$

$$\liminf_{i\to\infty} \mathcal{H}^{n+1}(\Omega_i) \geq \mathcal{H}^{n+1}(\Omega) - \limsup_{i\to\infty} \mathcal{H}^{n+1}(M \setminus K_i) = \mathcal{H}^{n+1}(\Omega).$$

Hence, we can take Ω_i for i sufficiently large (depending on ε). □

4.11.3 Lower bounds for Lipschitz constant of isoperimetric profile

In the rest of the section, we develop tools to compute lower bounds for the invariant $\mathscr{D}$. We conclude the subsection with a few examples.

Proposition 4.11.6. *Let $\delta \in (0, \mathrm{Vol}(M))$. Suppose there are locally smooth functions*

$$\eta_1 : (0, \delta) \to (0, +\infty), \quad \eta_2 : (0, \delta) \to (0, +\infty)$$

satisfying $\lim_{t\to 0} \eta_j(t) = 0$ and $\mathscr{I}(v) \geq \min\{\eta_1(v), \eta_2(v)\}$ for all $v \in (0, \delta)$. Then

$$\mathscr{D}(M, g) \geq \min\left\{\liminf_{t\to 0} \eta_1'(t),\ \liminf_{t\to 0} \eta_2'(t)\right\}.$$

Proof. Set $m = \min\{\liminf_{t\to 0} \eta_1'(t),\ \liminf_{t\to 0} \eta_2'(t)\}$ (unless the right hand side is $+\infty$, in which case we set $m = 1$ for convenience). If $m = 0$, then the conclusion is trivial, so we assume $m > 0$.

Fix $\varepsilon \in (0, m)$. We claim that

$$\min\{\eta_1(v), \eta_2(v)\} \geq (m - \varepsilon)v.$$

for all v sufficiently small. Indeed, by the definition of m, there is some $\sigma > 0$ so that for all $t \in (0, \sigma)$, we have $\eta_j'(t) \geq m - \varepsilon$ for $j = 1, 2$. By the fundamental theorem of calculus, we have $\eta_j(t) \geq (m - \varepsilon)t$ for all $t \in (0, \sigma)$ for $j = 1, 2$, so the desired conclusion holds.

Hence, we have $\mathscr{I}(v) \geq (m - \varepsilon)v$ for any $v \in (0, \sigma)$. Let $v_2 \in (0, \sigma)$. Since $\lim_{v\to 0} \mathscr{I}(v) = 0$, there is a sequence of positive real numbers $\{v_1^i\}_i$ converging to 0

with $\mathscr{I}(v_1^i) < \mathscr{I}(v_2)$. We have

$$\begin{aligned}\mathscr{D}(M,g) &\geq \liminf_{i\to\infty} \left| \frac{\mathscr{I}(v_2) - \mathscr{I}(v_1^i)}{v_2 - v_1^i} \right| \\ &= \liminf_{i\to\infty} \frac{\mathscr{I}(v_2) - \mathscr{I}(v_1^i)}{v_2 - v_1^i} \\ &\geq \liminf_{i\to\infty} \frac{\mathscr{I}(v_2) - \mathscr{I}(v_1^i)}{v_2} \\ &\geq m - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows. □

Suppose $E \subset M$ is the complement of a precompact open subset with smooth boundary in M. We define

$$\mathscr{I}_E(v) = \inf\{\mathcal{H}^n(\partial^*\Omega) \mid \Omega \in \mathcal{C}(M),\ \Omega \subset E,\ \mathcal{H}^{n+1}(\Omega) = v\}.$$

Proposition 4.11.7. *Suppose $E \subset M$ is the complement of a precompact open subset with smooth boundary in M. Let $\delta \in (0, \mathcal{H}^{n+1}(E))$. Suppose there is a locally smooth function*

$$\eta : (0,\delta) \to (0,+\infty)$$

satisfying $\lim_{t\to 0} \eta(t) = 0$ and $\mathscr{I}_E(v) \geq \eta(v)$ for all $v \in (0,\delta)$. Then

$$\mathscr{D}(M,g) \geq \liminf_{t\to 0} \eta'(t).$$

Proof. If $\liminf_{t\to 0} \eta'(t) = 0$, then the conclusion is trivial, so we assume otherwise.

Since ∂E is smooth, compact, and separating, we can find a compact set $K \subset M$ with smooth boundary so that

- $E \cup K = M$,

- there is a diffeomorphism

$$\phi : \partial E \times [0, \varepsilon] \to E \cap K$$

satisfying $\phi(\partial E, t) = \{x \in E \mid d(x, \partial E) = t\}$.

Since K is compact with smooth boundary, by [Rit23, Theorem 9.6] there are constants $C_K,\ v_K > 0$ so that for any $\Lambda \in \mathcal{C}(M)$ with $\mathcal{H}^{n+1}(\Lambda \cap K) \leq v_K$, we have

$$\mathcal{H}^n((\partial^* \Lambda) \cap K) \geq C_K (\mathcal{H}^{n+1}(\Lambda \cap K))^{\frac{n}{n+1}}. \tag{4.11.2}$$

For $t \in [0, \varepsilon]$, we let $S_t = \phi(\partial E, t)$, $K_t = K \setminus \phi(\partial E, [t, \varepsilon])$, and $E_t = E \setminus \phi(\partial E, [0, t])$.

Fix $v \in (0, \delta)$. By [NR21, Corollary 2.2] (since we assume (M, g) has finite volume), we can find $\Omega \in \mathcal{C}(M)$ satisfying

$$\mathcal{H}^{n+1}(\Omega) = v \ \text{ and } \ \mathcal{H}^n(\partial^* \Omega) = \mathscr{I}(v).$$

Case 1: Suppose $\mathcal{H}^{n+1}(\Omega \cap K) \geq v^{\frac{n+2}{n+1}}$. By (4.11.2), we have (for v sufficiently small)

$$\mathscr{I}(v) = \mathcal{H}^n(\partial^* \Omega) \geq \mathcal{H}^n((\partial^* \Omega) \cap K) \geq C_K v^{\frac{n+2}{n+1}\frac{n}{n+1}} = C_K v^{\frac{n^2+2n}{n^2+2n+1}} =: \eta_1(v). \tag{4.11.3}$$

Case 2: Suppose $\mathcal{H}^{n+1}(\Omega \cap K) \leq v^{\frac{n+2}{n+1}}$. By the coarea formula in $E \cap K$, there is a constant $c_1 = c_1(E, K) > 0$ so that for some $t \in [0, \varepsilon]$ (depending on Ω), we have

$$\mathcal{H}^n(\Omega \cap S_t) \leq c_1 v^{\frac{n+2}{n+1}}. \tag{4.11.4}$$

By assumption and the definition of $\mathscr{I}_E$, we have

$$\mathcal{H}^n((\partial^* \Omega) \cap E_t) + \mathcal{H}^n(\Omega \cap S_t) = \mathcal{H}^n(\partial^*(\Omega \cap E_t)) \geq \eta(\mathcal{H}^{n+1}(\Omega \cap E_t)). \tag{4.11.5}$$

Since $\liminf_{t\to 0}\eta'(t) > 0$, η is increasing for v sufficiently small. So for sufficiently small v, by (4.11.4) and (4.11.5), we have

$$\mathscr{I}(v) = \mathcal{H}^n(\partial^*\Omega) \geq \mathcal{H}^n((\partial^*\Omega)\cap E_t) \geq \eta(v - v^{\frac{n+2}{n+1}}) - c_1 v^{\frac{n+2}{n+1}} =: \eta_2(v). \tag{4.11.6}$$

Hence, for v sufficiently small, we have

$$\mathscr{I}(v) \geq \min\{\eta_1(v), \eta_2(v)\},$$

where η_1 and η_2 are as defined in (4.11.3) and (4.11.6). Since

$$\liminf_{t\to 0}\eta_1'(t) = +\infty \quad \text{and} \quad \liminf_{t\to 0}\eta_2'(t) = \liminf_{t\to 0}\eta'(t),$$

the conclusion follows by Proposition 4.11.6. □

We now compute lower bounds for $\mathscr{D}$ for a large class of finite volume manifolds with special structure outside a compact set.

Proposition 4.11.8. *Let (M^{n+1}, g) be a complete manifold with finite volume and dimension $3 \leq n+1 \leq 7$. Suppose there is a compact set $K \subset M$ so that $M \setminus K$ is isometric to*

$$((A,\infty)\times N,\ \psi^2(t)dt^2 \oplus \rho^2(t)g_N)$$

for a fixed closed Riemannian manifold (N^n, g_N) of dimension n and locally smooth positive functions ψ and ρ. If $\rho' < 0$ and $\frac{\rho'}{\rho\psi}$ is nonincreasing, then

$$\mathscr{D}(M,g) \geq -n\lim_{t\to\infty}\frac{\rho'(t)}{\rho(t)\psi(t)}.$$

Proof. In the provided coordinates, we let $E = [A+1,\infty)\times N$.

Take any $v \in (0, \mathcal{H}^{n+1}(E))$. Let $\Omega \in \mathcal{C}(M)$ with $\Omega \subset E$ satisfy (by [NR21,

Corollary 2.2])

$$\mathcal{H}^{n+1}(\Omega) = v \quad \text{and} \quad \mathcal{H}^n(\partial^*\Omega) = \mathscr{I}_E(v).$$

Let $\Omega_t = \Omega \cup ([t, \infty) \times N)$. Since Ω has finite perimeter and M has finite volume, we have

$$\lim_{t\to\infty} \mathcal{H}^n(\partial^*\Omega_t) = \mathcal{H}^n(\partial^*\Omega) \quad \text{and} \quad \lim_{t\to\infty} \mathcal{H}^{n+1}(\Omega_t) = \mathcal{H}^{n+1}(\Omega_t).$$

Let O_t be the subset of the form $[\tau_t, \infty) \times N$ so that $\mathcal{H}^{n+1}(O_t) = \mathcal{H}^{n+1}(\Omega_t)$.

Let

$$\omega = \rho^n \mathrm{vol}_N.$$

Then

$$d\omega = n\rho^{n-1}\rho' dt \wedge \mathrm{vol}_N = n\frac{\rho'}{\rho\psi}\mathrm{vol}_M.$$

By Stokes' theorem, we have

$$\begin{aligned}
\mathcal{H}^n(\partial O_t) - \mathcal{H}^n(\partial^*\Omega_t) &\leq \int_{[\partial O_t]} \omega - \int_{[\partial^*\Omega_t]} \omega \\
&= \int_{[O_t]-[\Omega_t]} d\omega \\
&= \int_{O_t\setminus\Omega_t} n\frac{\rho'}{\rho\psi} - \int_{\Omega_t\setminus O_t} n\frac{\rho'}{\rho\psi}
\end{aligned}$$

Since O_t has the same volume as Ω_t, we have $\mathcal{H}^{n+1}(O_t\setminus\Omega_t) = \mathcal{H}^{n+1}(\Omega_t\setminus O_t)$. Moreover, by assumption,

$$n\frac{\rho'}{\rho\psi}\Big|_{O_t\setminus\Omega_t} \leq n\frac{\rho'}{\rho\psi}\Big|_{\Omega_t\setminus O_t}.$$

Hence, we have

$$\mathcal{H}^n(\partial O_t) \leq \mathcal{H}^n(\partial^*\Omega_t).$$

Taking $t \to \infty$, we have

$$\mathcal{H}^n(\partial O) \leq \mathcal{H}^n(\partial^*\Omega),$$

where O is the subset of the form $[\tau, \infty) \times N$ with $\mathcal{H}^{n+1}(O) = \mathcal{H}^{n+1}(\Omega) = v$, so

$$\mathscr{I}_E(v) = \mathcal{H}^n(\partial O).$$

In other words, the areas and volumes of the subsets of the form $[\tau, \infty) \times N$ provide a lower bound for $\mathscr{I}_E$.

Let $a(t) = \mathcal{H}^n(\{t\} \times N)$ and $v(t) = \mathcal{H}^{n+1}([t, \infty) \times N)$. If $t(v)$ denotes the inverse function of $v(t)$, we have

$$\eta(v) = a(t(v))$$

is a lower bound for $\mathscr{I}_E$. By the coarea formula and the formula for the derivative of an inverse function, we have

$$\eta'(v) = \frac{n\rho^{n-1}\rho'}{-\rho^n\psi} = -n\frac{\rho'}{\rho\psi}.$$

The conclusion follows by Proposition 4.11.7. □

We now compute lower bounds for $\mathscr{D}$ for a few examples.

Example 4.11.9 (Finite volume hyperbolic manifolds)**.** Let (M, g) be a complete hyperbolic manifold with finite volume. By the structure theorem for these manifolds (see [Thu97, Chapter 4.5]), there is a compact set $K \subset M$, a constant $A > 0$, and a closed Riemannian manifold (N, g_N) (a union of flat tori and flat Klein bottles) so that $M \setminus K$ is isometric to

$$((A, \infty) \times N, t^{-2}dt^2 \oplus t^{-2}g_N).$$

In this case, $\psi(t) = \rho(t) = t^{-1}$. We see that

$$\rho' = -t^{-2} < 0 \text{ and } \frac{\rho'}{\rho\psi} = -1 \text{ is nonincreasing.}$$

By Proposition 4.11.8, we have

$$\mathscr{D}(M, g) \geq n.$$

By Theorem 4.11.4, we have Corollary 4.1.7.

Example 4.11.10 ($\mathscr{D} = +\infty$)**.** Let (N, g_N) be a closed Riemannian manifold. Let (M, g) be a Riemannian manifold isometric to

$$((A, \infty) \times N, dt^2 \oplus e^{-2t^{1+\alpha}} g_N)$$

outside a compact set, where $\alpha > 0$. Completeness follows from construction, and the fact that M has finite volume follows because

$$\int_1^\infty e^{-nt^{1+\alpha}} dt \leq \int_1^\infty e^{-nt} dt = \frac{-1}{n} e^{-nt}\Big|_1^\infty = \frac{e^{-n}}{n}.$$

In this case, we have $\psi(t) = 1$ and $\rho(t) = e^{-t^{1+\alpha}}$. We compute

$$\rho' = -(1+\alpha)t^\alpha e^{-t^{1+\alpha}} < 0$$

and

$$\frac{\rho'}{\rho\psi} = -(1+\alpha)t^\alpha \text{ is nonincreasing.}$$

By Proposition 4.11.8, we have

$$\mathscr{D}(M, g) \geq \lim_{t\to\infty} n(1+\alpha)t^\alpha = +\infty.$$

By Theorem 4.11.4, we can find a smooth complete almost embedded finite area c-CMC hypersurface in M for any $0 < c < \infty$.

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