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Combinatorial Methods in Bordered Heegaard Floer Homology

Zhan, Bohua

Princeton, NJ : Princeton University

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Combinatorial Methods in Bordered Heegaard Floer Homology

Bohua Zhan

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

In this thesis we give several combinatorial constructions and proofs in bordered Heegaard Floer homology.

In the first part, we give an explicit description of a rank-1 model of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$, the type AA invariant associated to the identity diffeomorphism. This leads to a combinatorial proof of the quasi-invertibility of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$.

In the second part, we use the newly constructed $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ to describe the type DA invariant $\widehat{CFDA}(\tau)$ for any arcslide τ. Using this, we give a combinatorial construction of $\widehat{HF}(Y)$ for any 3-manifold Y, and prove that it is a 3-manifold invariant. Along the way, we prove combinatorially that bordered Floer theory gives a linear-categorical representation of the (strongly-based) mapping class groupoid.

In the third part, we develop the theory of local type DA bimodules and extension by identity, and use it to give another construction of $\widehat{CFDA}(\tau)$ for arcslides, which leads to a faster algorithm to compute $\widehat{HF}$ for an arbitrary 3-manifold.

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Contents

Abstract . . . iii

Acknowledgements . . . iv

1 Introduction **1**

1.1 Overview . . . 1

1.2 Algebraic preliminaries . . . 3

1.3 Heegaard Floer homology . . . 8

1.4 Pointed matched circles and strand algebras . . . 10

1.5 Bordered invariants and Heegaard diagrams . . . 12

1.6 Gradings on bordered invariants . . . 15

1.7 The strongly-based mapping class groupoid . . . 18

1.8 Invariants of the identity diffeomorphism . . . 19

1.9 Invariants of arcslides . . . 20

2 $\widehat{CFA}A(\mathbb{I}_{\mathcal{Z}})$ and Koszul duality **24**

2.1 Introduction . . . 24

2.2 Descriptions of the large chain complex . . . 26

2.3 Description of the homotopy . . . 30

2.3.1 Multiplicity one . . . 30

2.3.2 Other generators . . . 35

2.4 Verification of the homotopy . . . 37

2.4.1 Multiplicity one . . . 37

2.4.2 Other generators . . . 45

2.5 A rank-1 model of $\widehat{CFA}A(\mathbb{I}_{\mathcal{Z}})$ . . . 49

2.6 Examples of Koszul duality . . . 51

2.7 Appendix: cancellation diagrams . . . 55

3 Construction and proof of invariance of $\widehat{HF}$ **69**

3.1 Introduction . . . 69

3.2 Combinatorial construction . . . 70

3.3 Representation of the mapping class groupoid . . . 72

3.3.1 Cancellation lemmas . . . 75

3.3.2 Characterization of the identity bimodule . . . 81

3.3.3 Certain arrows in $C\widehat{FD}A$ of arcslides . . . 85

3.3.4 Arcslides and their compositions . . . 97

3.3.5 The involution relation . . . 101

3.3.6 Other relations . . . 106

3.4 The 3-manifold invariant . . . 112

4 Local type DA bimodules and computations **121**

4.1 Introduction . . . 121

4.1.1 Notes . . . 122

4.2 Local PMCs and algebras . . . 122

4.2.1 Local PMCs without smearing . . . 122

4.2.2 Strand algebra associated to a local PMC without smearing . . . 123

4.2.3 Restriction and Join maps . . . 125

4.2.4 Local PMCs and algebras with smearing . . . 126

4.3 Extension by identity . . . 128

4.3.1 Type DA bimodules . . . 128

4.3.2 No-smearing case . . . 130

4.3.3 Smearing case . . . 134

4.4 Type DA structure of arcslides . . . 140

4.5 An algorithm for computing $\widehat{HF}$ . . . 148

4.6 Results of computation . . . 151

Chapter 1

Introduction

1.1 Overview

Heegaard Floer homology is a 3-manifold invariant introduced by Ozsváth and Szabó in [22] and [21]. It is used to construct an invariant for smooth 4-manifolds in [20]. The invariants are used to give alternative, often easier, ways to prove topological results about 3 and 4-manifolds formerly proven using gauge theory. Indeed, the 3-manifold invariant is recently shown to be equivalent to Seiberg-Witten Floer homology (see [10] for an overview of this result).

The definitions of Heegaard Floer invariants involve counting pseudo-holomorphic curves in certain symplectic manifolds. As such, they are difficult to compute directly from the definition. Much work has been done in finding combinatorial methods to compute the invariants. This is most successful for the hat version $\widehat{HF}$ of Heegaard Floer homology. Sarkar and Wang gave in [24] the first combinatorial algorithm to compute $\widehat{HF}$ for an arbitrary 3-manifold, by giving a systematic way to change any Heegaard diagram into a *nice* diagram, for which the identification of holomorphic disks is combinatorial. The first combinatorial proof of invariance of $\widehat{HF}$ is obtained by Ozsváth, Stipsicz, and Szabó in [19], using a special kind of nice diagrams called convenient diagrams.

Much of this thesis is based on another method to compute $\widehat{HF}$, using bordered Heegaard Floer homology (bordered Floer for short). Bordered Floer homology is introduced by Lipshitz, Ozsváth, and Thurston in [13] and [14], as an extension of $\widehat{HF}$ to invariants of 3-manifolds with boundary. To a 3-manifold Y with connected boundary, it assigns a type D invariant $\widehat{CFD}(Y)$ and a type A invariant $\widehat{CFA}(Y)$. To a 3-manifold Y with two boundary components, it assigns a type DD invariant $\widehat{CFDD}(Y)$, a type DA invariant $\widehat{CFDA}(Y)$, and a type AA invariant $\widehat{CFAA}(Y)$.

Throughout bordered Floer theory, the ground ring has characteristic 2, so there are no sign issues. The same will be true in this thesis.

There are pairing theorems which express the invariants of glued 3-manifolds in terms of invariants associated to the constituent pieces. Thus, one can hope to compute $\widehat{HF}(Y)$ for an arbitrary 3-manifold Y by cutting Y into simpler pieces. This is carried out in [16]. We will take a moment to describe this construction in detail.

The starting point is to consider the 3-manifold Y as two genus g handlebodies Y_1 and Y_2 glued along their boundary. If we take standard parametrizations of Y_1 and Y_2, all the information about Y is contained in the gluing map ψ, which is a surface diffeomorphism of $\partial Y_1 = -\partial Y_2$. The 3-manifold Y can be decomposed into three parts: Y_1 and Y_2 at the ends, and the mapping cylinder of ψ in the middle. The map ψ can be further factored into *arcslides*, which are particularly simple types of surface diffeomorphisms that generate the (strongly-based) mapping class groupoid. This gives the decomposition of Y into Y_1, Y_2, and the mapping cylinders of arcslides $\tau_1, \dots, \tau_n$.

Among the bordered invariants, the type D and type DD invariants are usually the simplest to describe combinatorially. The type D invariant for the standard handlebody is particularly simple. The type DD invariants for arcslides are computed in [16, Section 4]. These results can be combined with the following gluing theorems to compute $\widehat{HF}(Y)$ for any Y.

Theorem 1. *[18, Theorem 1, Corollary 8] Let Y_1 and Y_2 be bordered 3-manifolds with boundary parametrized by the same surface $F(\mathcal{Z})$, then*

$$\widehat{HF}(-Y_1 \cup_\partial Y_2) \simeq H_*(\mathrm{Mor}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2))). \tag{1.1}$$

Let Y_1 be a strongly bordered 3-manifold with boundaries parametrized by $F(\mathcal{Z}_1)$ and $F(\mathcal{Z}_2)$, and Y_2 be a bordered 3-manifold with boundary parametrized by $F(\mathcal{Z}_2)$, then

$$\widehat{CFD}(-Y_2 \cup_\partial Y_1) \simeq \mathrm{Mor}(\widehat{CFD}(Y_2), \widehat{CFDD}(Y_1)). \tag{1.2}$$

When using equations (1.1) and (1.2), taking the morphism complex produces chain complexes or modules with many generators, which then need to be reduced using the cancellation lemma. This is manageable by computers (and indeed produces a very practical algorithm to compute $\widehat{HF}$), but makes theoretical analysis difficult.

One common theme of this thesis is giving combinatorial descriptions to the other invariants (of types A, AA, and DA) in bordered Floer theory, which allows the use of pairing theorems involving

the box tensor product, which does not dramatically increase the number of generators. We now describe each part of the thesis in turn.

In the remaining portions of this chapter, we review the basics of Heegaard Floer and bordered Heegaard Floer theory. The subsequent three chapters are the three parts of the thesis.

In the first part, we give a combinatorial description of a rank-1 model of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$, the type AA invariant associated to (the mapping cylinder of) the identity diffeomorphism. Using known pairing theorems with the box tensor product, this allows us to produce combinatorial descriptions of type A, DA, or AA invariants from any available type D or type DD invariants for the same 3-manifold, without increasing the number of generators (though admittedly the actions on the module become much more complicated). We also use it to give a combinatorial proof of the quasi-invertability of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$. As another application, we give some concrete examples of the Koszul duality on the strand algebras $\mathcal{A}(\mathcal{Z})$, by computing some explicit representatives of homology classes in the cobar resolution of $\mathcal{A}(\mathcal{Z})$.

In the second part, we describe the type DA invariant for arcslides, using the type DD invariants from [16] and the $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ that is just computed. Using this, we give a combinatorial description of $\widehat{HF}(Y)$ for an arbitrary 3-manifold Y, using box tensor product rather than taking morphism complexes. We prove combinatorially that this description gives an invariant of 3-manifolds. An intermediate result is a combinatorial proof that bordered Floer theory gives a linear-categorical representation of the (strongly-based) mapping class groupoid.

In the third part, we develop the theory of local type DA bimodules and extension by identity. This allows us to give another construction of the type DA invariant for arcslides, by constructing a local type DA invariant for each of the finite number of local cases of arcslides, then extending it by the identity to the usual type DA invariant for the full arcslide. This version of the invariant is particularly easy for the computer to work with, giving a faster algorithm to compute $\widehat{HF}$ for an arbitrary 3-manifold.

1.2 Algebraic preliminaries

In this section, we review some algebraic concepts used in this thesis. More details can be found in [13, Chapter 2] and [14, Section 2].

A *chain complex* C is a vector space over $\mathbb{F}_2$ equipped with a differential d satisfying $d^2 = 0$. In all situations that we will consider here, there is a canonical set of generators $\mathcal{G}$ of the vector space. Given $\mathbf{x}, \mathbf{y} \in \mathcal{G}$, we define $\mathbf{x} \to \mathbf{y}$ to be the linear map that takes $\mathbf{x}$ to $\mathbf{y}$ and all other generators to

zero. We call such a map an *arrow*. Given a choice of generators, we can write d uniquely as a sum of arrows.

A *dg-algebra* A is a chain complex equipped with a multiplication that satisfies the Leibniz rule, and a grading map from generators of A to a (possibly non-commutative) group G, satisfying the usual relations

$$\mathrm{gr}(ab) = \mathrm{gr}(a)\mathrm{gr}(b) \quad \text{and} \quad \mathrm{gr}(da) = \lambda^{-1}\mathrm{gr}(a),$$

where λ is a fixed central element of G.

Given a dg-algebra A, a right A_∞-module $\mathcal{M}$ over A consists of a vector space M over $\mathbb{F}_2$, together with multilinear maps:

$$m_{1,i} : M \otimes A^{\otimes i} \to M,$$

for all $i \geq 0$, where $A^{\otimes i}$ denotes the tensor product of i copies of A, and $A^{\otimes 0}$ is just $\mathbb{F}_2$. These maps satisfy the A_∞ structure equation:

$$\begin{aligned} 0 = &\sum_{i+j=n} m_{1,i}(m_{1,j}(\mathbf{x}, a_1, \dots, a_i), a_{i+1}, \dots, a_n) \\ &+ \sum_{i=1}^{n} m_{1,n}(\mathbf{x}, a_1, \dots, da_i, \dots, a_n) \\ &+ \sum_{i=1}^{n-1} m_{1,n-1}(\mathbf{x}, a_1, \dots, a_i a_{i+1}, \dots, a_n) \end{aligned} \tag{1.3}$$

If we ignore those maps $m_{1,i}$ with $i > 0$, we get a chain complex, which is called the chain complex underlying $\mathcal{M}$. We call the totality of $m_{1,i}$ the A_∞-*action* on $\mathcal{M}$. With a choice of generators in M and A, the A_∞-action can be written uniquely as a sum of arrows of the form

$$m_{1,i}(\mathbf{x}, a_1, \dots, a_i) \to \mathbf{y}.$$

A left type D structure $\mathcal{N}$ over A (we will also call them modules) consists of a vector space N over $\mathbb{F}_2$, together with a map:

$$\delta^1 : N \to A \otimes N,$$

satisfying the type D structure equation:

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta^1 \to A \otimes N. \tag{1.4}$$

Here μ_1 and μ_2 denote differential and multiplication on the dg-algebra A (in keeping with the

notation for the more general case, where A is an A_∞-algebra).

The map δ^1 is called the *type D action* of $\mathcal{N}$. With a choice of generators on N and A, the type D action can be written uniquely as a sum of arrows of the form

$$\mathbf{x} \to a \otimes \mathbf{y}.$$

In bordered Floer theory, we will assign an idempotent generator of A to each generator of an A_∞-module or type D structure, so that the tensor products over $\mathbb{F}_2$ in (1.3) and (1.4) can be replaced by tensor products over $\mathbf{k}$, the vector space over $\mathbb{F}_2$ generated by the idempotent generators of A. Intuitively, idempotents are required to match in all algebra actions and in generators of tensor products.

Given a right A_∞-module $\mathcal{M}_A$ and a left type D structure ${}^A\mathcal{N}$, the box tensor product $\mathcal{M}_A \boxtimes {}^A\mathcal{N}$ is a chain complex whose underlying vector space is $M \otimes_{\mathbf{k}} N$, and whose differential is given by:

$$\partial(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (m_{1,k} \otimes \mathbb{I}_N)(\mathbf{x} \otimes \delta^k(\mathbf{y})). \tag{1.5}$$

Here $\delta^k : N \to A^{\otimes k} \otimes N$ is given by applying δ^1 repeatedly on the N factor k times. The sum in (1.5) is finite under certain boundedness conditions on $\mathcal{M}_A$ and ${}^A\mathcal{N}$. In bordered Floer theory, they correspond to admissability conditions on the Heegaard diagrams (see [13, Section 2.4 and 4.4]).

The differential can be viewed in the following way in terms of arrows: for each integer $k \geq 0$, arrows

$$\begin{aligned} \mathbf{y_0} &\to a_1 \otimes \mathbf{y_1} \\ \mathbf{y_1} &\to a_2 \otimes \mathbf{y_2} \\ &\vdots \\ \mathbf{y_{k-1}} &\to a_k \otimes \mathbf{y_k} \end{aligned}$$

in the type D action of $\mathcal{N}$, and arrow

$$\delta_k^1(\mathbf{x_0}, a_1, \ldots, a_k) \to \mathbf{x_1}$$

in the action of $\mathcal{M}$, there is an arrow

$$(\mathbf{x_0}, \mathbf{y_0}) \to (\mathbf{x_1}, \mathbf{y_k})$$

in the differential of $\mathcal{M} \boxtimes \mathcal{N}$.

The A_∞ and type D actions can be combined in various ways to give definitions for bimodules. A left-left type DD bimodule over dg-algebras A and A' has action that is a sum of arrows of the form

$$\mathbf{x} \to a \otimes a' \otimes \mathbf{y},$$

where $\mathbf{x}, \mathbf{y}$ are generators of the bimodule, and a, a' are generators of A, A', respectively.

A left-right type DA bimodule over A and A' has action that is a sum of arrows of the form

$$\delta^1_{k+1}(\mathbf{x}, a'_1, \ldots, a'_k) \to a \otimes \mathbf{y},$$

where $\mathbf{x}, \mathbf{y}$ are generators of the bimodule, each a'_i is a generator of A', and a is a generator of A.

Finally, a right-right type AA bimodule over A and A' has action that is a sum of arrows of the form

$$m_{1,i,j}(\mathbf{x}, a_1, \ldots, a_i, a'_1, \ldots, a'_j) \to \mathbf{y}.$$

In each case, the action needs to satisfy a structure equation, analogous to (1.3) and (1.4).

There are analogous definitions of box tensor product between bimodules. In general, we can take box tensor product between any left type D action and any right A_∞-action, defined in a way analogous to (1.5), with similar interpretation in terms of arrows. We also note that each generator of the bimodules will have two idempotents, one for each algebra action. Finally, there are well-defined concepts of *morphism* and *homotopy equivalence* for each of the above kinds of structures. For these and more details, see [14, Section 2].

For dg-algebras A and B with ring of idempotents $\mathbf{k}$, we say a bimodule M (of type DD, DA, or AA) over A and B has *rank-1* if its underlying module over $\mathbf{k}$ is free of rank 1. There is a correspondence between rank-1 type DA bimodules ${}^B M_A$ with $\delta^1_1 = 0$ and A_∞-morphisms $\phi : A \to B$ ([14, Lemma 2.2.50]). Given $\phi : A \to B$, the corresponding bimodule is denoted ${}^B[\phi]_A$. It has type DA actions $\delta^1_1 = 0$ and

$$\delta^1_{k+1}(\mathbf{1}, a_1, \ldots, a_k) = \phi(a_1, \ldots, a_k) \otimes \mathbf{1}.$$

We conclude this section with a review of cobar resolutions.

Definition 2. *An* augmentation *of a dg-algebra A is a surjective dg-algebra homomorphism $\epsilon : A \to \mathbf{k}$ (in particular $\epsilon(1) = 1$ and $\epsilon(a_1 a_2) = \epsilon(a_1)\epsilon(a_2)$). Given an augmentation, we let $A_+ = \ker(\epsilon)$ be the* augmentation ideal *of A.*

The strand algebra $\mathcal{A}$ is augmented with the augmentation map ϵ sending idempotents to themselves and other generators to zero. So the augmentation ideal $\mathcal{A}_+$ is generated by the non-idempotents.

Definition 3. *Given an augmented dg-algebra A, the cobar resolution* $\mathrm{Cob}(A)$ *is defined as $T^*(A_+[1]^*)$, the tensor algebra of the dual of the augmentation ideal. This can be given the structure of a dg-algebra, with product being the one on the tensor algebra, and the differential consisting of the following arrows:*

- $a_1^* \otimes \cdots \otimes b^* \otimes \cdots \otimes a_k^* \to a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_k^*$, *for each i and term b in da_i,*
- $a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_k^* \to a_1^* \otimes \cdots \otimes b^* \otimes b'^* \otimes \cdots \otimes a_k^*$, *for each i and generators b, b' such that $bb' = a_i$.*

Furthermore, we denote $\widehat{\mathrm{Cob}}(A)$ *to be the completion of* $\mathrm{Cob}(A)$ *with respect to the length filtration (that is, an element of* $\widehat{\mathrm{Cob}}(A)$ *is a formal sum of elements in $A_+[1]^{\otimes i}$ for possibly infinitely many i).*

With this definition, the category of A_∞-modules over A is equivalent to the category of type D modules over $\widehat{\mathrm{Cob}}(A)$, where the arrow

$$m_{1,i}(\mathbf{x}, a_1, \ldots, a_i) \to \mathbf{y}$$

in the action of an A_∞-module M corresponds to the arrow

$$\delta^1(\mathbf{x}) \to a_1^* \otimes \cdots \otimes a_i^* \otimes \mathbf{y},$$

in the action of the type D module corresponding to M ([14, Remark 2.2.35]). Likewise, the category of type DA bimodules over A' on the D-side and A on the A-side is equivalent to the category of type D modules over $\widehat{\mathrm{Cob}}(A) \otimes A'$, where the arrow

$$\delta^1_{i+1}(\mathbf{x}, a_1, \ldots, a_i) \to c \otimes \mathbf{y}$$

in the action of a type DA bimodule M corresponds to the arrow

$$\delta^1(\mathbf{x}) \to (a_1^* \otimes \cdots \otimes a_i^*) \otimes c \otimes \mathbf{z}$$

in the action of the type D module corresponding to M. In both cases, if M is bounded, in the sense that $m_{1,i} = 0$ or $\delta^1_{i+1} = 0$ for sufficiently large i, then $\widehat{\mathrm{Cob}}(A)$ can be replaced by $\mathrm{Cob}(A)$.

1.3 Heegaard Floer homology

We now give an overview of the hat version of Heegaard Floer homology. See [22], [21] for details.

Given a 3-manifold Y. Take a Heegaard splitting $Y = H_1 \cup_\Sigma H_2$ of Y, where H_1 and H_2 are handlebodies, and Y is formed by gluing them along their common boundary Σ. Such a splitting can be expressed as a Heegaard diagram $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$, where

- Σ is a surface of genus g.
- $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ is a g-tuple of circles that are homologically independent and mutually non-intersecting on Σ. Likewise for $\vec{\beta} = \{\beta_1, \ldots, \beta_g\}$. The α and β circles intersect each other transversely.
- z is a point on $\Sigma \setminus \{\vec{\alpha}, \vec{\beta}\}$.

The 3-manifold Y can be recovered by attaching disks to the α-circles on one side of Σ, to the β-circles on the other side, and filling in the resulting boundary with two balls. Given a Heegaard diagram $\mathcal{H}$ of Y, the chain complex $\widehat{CF}(Y, \mathcal{H})$ is generated by g-tuples of points $\mathbf{x} = \{x_1, \ldots, x_g\} \subset \vec{\alpha} \cap \vec{\beta}$, where each α-curve and each β-curve contains exactly one point of $\mathbf{x}$.

The differential on $\widehat{CF}(Y, \mathcal{H})$ is specified using a suitable almost complex structure on $\mathrm{Sym}^g(\Sigma)$, the gth symmetric product of Σ. Briefly, the α and β curves give raise to two Lagrangian tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in $\mathrm{Sym}^g(\Sigma)$, and each generator $\mathbf{x}$ corresponds to an intersection $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For any two generators $\mathbf{x}$ and $\mathbf{y}$, the coefficient of $\mathbf{y}$ in $\partial\mathbf{x}$ is defined by counting holomorphic disks in $\mathrm{Sym}^g(\Sigma)$ connecting the two intersections $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. One can show that the homology of $\widehat{CF}(Y, \mathcal{H})$ is independent of the choice of Heegaard diagram and almost complex structure, so it is an invariant of the 3-manifold Y. This invariant is written as $\widehat{HF}(Y)$. We will also write $\widehat{CF}(Y)$ to denote the homotopy equivalence class of $\widehat{CF}(Y, \mathcal{H})$.

An alternative definition, given by Robert Lipshitz in [12], involves choosing an almost complex structure on $\Sigma \times [0,1] \times \mathbb{R}$, and counting certain holomorphic curves. This is actually the approach

that will be extended to the bordered case.

For our purposes, it is most useful to know how arrows in the differential of $\widehat{CF}(Y,\mathcal{H})$ correspond to domains in $\mathcal{H}$. Given a Heegaard diagram $\mathcal{H}=(\Sigma,\vec{\alpha},\vec{\beta},z)$, a *region* in $\mathcal{H}$ is a connected component of $\Sigma\setminus\{\vec{\alpha},\vec{\beta}\}$, and a *domain* of $\mathcal{H}$ is an integral linear combination of regions in $\mathcal{H}$. For any two generators $\mathbf{x},\mathbf{y}$, let $\pi_2(\mathbf{x},\mathbf{y})$ be the set of domains satisfying $\partial(\partial^{\alpha}(B))=\mathbf{y}-\mathbf{x}$, where $\partial^{\alpha}(B)$ denotes the parts of boundary of B that are pieces of α-circles. Each arrow $\mathbf{x}\to\mathbf{y}$ comes from a domain $B\in\pi_2(\mathbf{x},\mathbf{y})$, such that all coefficients of regions in B are non-negative, and the coefficient of the region containing the basepoint z is zero.

The generators can be grouped into *spin-c classes*, where $\pi_2(\mathbf{x},\mathbf{y})$ is non-empty only if $\mathbf{x}$ and $\mathbf{y}$ are in the same spin-c class. This shows

$$\widehat{CF}(Y,\mathcal{H})=\bigoplus_{\mathfrak{s}}\widehat{CF}(Y,\mathcal{H},\mathfrak{s}),$$

where $\widehat{CF}(Y,\mathcal{H},\mathfrak{s})$ is the sub-complex of $\widehat{CF}(Y,\mathcal{H})$ with generators in the spin-c class $\mathfrak{s}$.

For each $\mathbf{x},\mathbf{y}$ and domain $D\in\pi_2(\mathbf{x},\mathbf{y})$, there is an index $\mathrm{ind}(D)$ defined by

$$\mathrm{ind}(D)=e(D)+n_{\mathbf{x}}(D)+n_{\mathbf{y}}(D),$$

where $e(D)$ is the Euler measure of D, and $n_{\mathbf{x}}(D),n_{\mathbf{y}}(D)$ are the sums of average multiplicities of D at the four corners of points in $\mathbf{x}$ and $\mathbf{y}$, respectively. In each spin-c class $\mathfrak{s}$, there is a relative grading by $\mathbb{Z}/n\mathbb{Z}$, where $n=\mathrm{div}(c_1(\mathfrak{s}))$. This means there is a function μ from generators to $\mathbb{Z}/n\mathbb{Z}$, well-defined up to translation in $\mathbb{Z}/n\mathbb{Z}$, such that for each $D\in\pi_2(\mathbf{x},\mathbf{y})$,

$$\mathrm{ind}(D)=\mu(\mathbf{x})-\mu(\mathbf{y})\pmod{n}.$$

The differential d on $\widehat{CF}$ decreases μ by 1, so in particular, only domains with index 1 can contribute to the differential.

The relative grading on generators and the correspondence of arrows to domains are very useful in restricting what kind of arrows can appear in the differential, especially since they only uses the combinatorial features of the Heegaard diagram. However, in general a choice of almost complex structure is needed to determine which of the potential domains do contribute to the differential, making $\widehat{HF}(Y)$ difficult to compute directly using the definition.

1.4 Pointed matched circles and strand algebras

In bordered Floer theory, the connected, compact, orientable surfaces that serve as boundary components of 3-manifolds are specified using pointed matched circles. A pointed matched circle is a quadruple $\mathcal{Z} = (Z, z, \mathbf{a}, M)$, consisting of a circle Z, a point $z \in Z$, a set of $4k$ points $\mathbf{a} \subset Z \setminus \{z\}$, and a two-to-one map M from $\mathbf{a}$ to $\{1, 2, \ldots, 2k\}$, pairing the points in $\mathbf{a}$, that satisfies the following condition: if we thicken the circle Z to an annulus and attach a 1-handle to the outside boundary of the annulus joining each pair of points in $\mathbf{a}$, then the new outside boundary must be a single circle. Given this requirement, we may glue a disk onto that boundary, obtaining a genus k surface $F^\circ(\mathcal{Z})$ with one boundary component and a basepoint z on the boundary. We say that the pointed matched circle $\mathcal{Z}$ *parametrizes* $F^\circ(\mathcal{Z})$.

Let $F(\mathcal{Z})$ be the result of filling the boundary of $F^\circ(\mathcal{Z})$ with a disk. Then $F(\mathcal{Z})$ is a closed surface of genus k, marked with a homotopically trivial circle Z and a basepoint $z \in Z$. We will also say $F(\mathcal{Z})$ is parametrized by $\mathcal{Z}$.

An example of a pointed matched circle for $k = 2$ is shown in Figure 1.1.

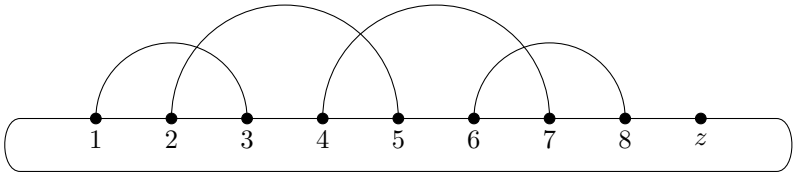


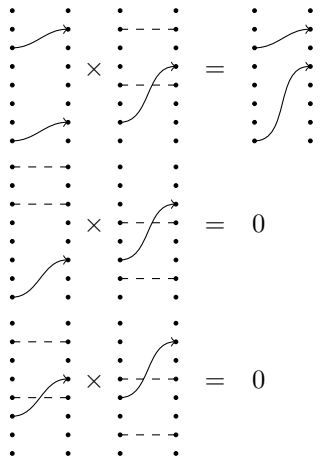
Figure 1.1: Linear pointed matched circle with $k = 2$.

To each pointed matched circle $\mathcal{Z}$, bordered Floer theory associates a combinatorially defined dg-algebra $\mathcal{A}(\mathcal{Z})$. The algebra is generated by "strand diagrams". They are best visualized as follows: cut the circle Z at the basepoint z and unwind it into a vertical line, so that the points $\mathbf{a}$ are labeled in increasing order from bottom to top. We will denote the vertical line $Z \backslash z$ by Z'. Make a copy of Z' and shift it to the right. A strand is an arc connecting the two sides at points in $\mathbf{a}$, going either upwards or horizontally. A strand diagram is a collection of strands satisfying the following conditions:

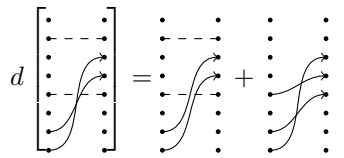
- Horizontal strands must come in matched pairs. That is, if there is a horizontal strand at $p \in \mathbf{a}$ and q is paired with p, then there must be a horizontal strand at q.
- Apart from pairs of horizontal strands, no two strands can start at points in the same pair or end at points in the same pair.

The algebra $\mathcal{A}(\mathcal{Z})$ is then generated over $\mathbb{F}_2$ by the set of strand diagrams. Two generators v

and w are multiplied as follows: place the strand diagram for v to the left of the strand diagram for w, and try to match the right endpoints of strands in v with the left endpoints of strands in w. If the endpoint of some non-horizontal strand or both horizontal strands in a pair are not matched, the product equals zero. If exactly one of the horizontal strands in a pair is matched, the other is dropped. If there are two strands that cross each other twice in the resulting strand diagram, the product is also set to zero. Otherwise the product vw is the generator represented by the strand diagram formed by joining the strand diagrams for v and w. Some examples for the linear pointed matched circle given in Figure 1.1 are:



The differential of a generator v is defined as follows: for each crossing in v, that is, for each pair of strands going from a to b and from c to d such that $a < c < d < b$, the crossing is "resolved", so that the two strands are replaced by the two strands from a to d and from c to b. If in the resulting diagram two strands cross each other more than once, we do not consider this term. Otherwise the resulting strand diagram gives one term in the differential dv. For example,



One may check that $d^2v = 0$ and the Leibniz rule $d(vw) = v \cdot dw + dv \cdot w$ are satisfied.

A single, non-horizontal strand is also called a "chord". We denote a chord from points i to j in $\mathbf{a}$ by ρ_{ij}. Given such a strand ρ_{ij} we can associate an algebra element $a(\rho_{ij})$, which is the sum of all strand diagrams whose only non-horizontal strand is ρ_{ij}. Such elements will be important in describing modules over $\mathcal{A}(\mathcal{Z})$ later.

Given a collection of pairs of points $\mathbf{s}$ in $\mathbf{a}$, there is a generator $I(\mathbf{s})$ which has exactly horizontal strands at $\mathbf{s}$. This is an idempotent in $\mathcal{A}(\mathcal{Z})$ (indeed such generators are all minimal idempotents in the algebra).

Given a pointed matched circle $\mathcal{Z}$, let $-\mathcal{Z}$ be the pointed matched circle with the reversed orientation. Then $F^\circ(-\mathcal{Z})$ (resp. $F(-\mathcal{Z})$) is the orientation reversal of $F^\circ(\mathcal{Z})$ (resp. $F(\mathcal{Z})$). If the points in $\mathcal{Z}$ are labeled $a_{\mathcal{Z},1},\dots,a_{\mathcal{Z},4k}$ as above, and the points in $-\mathcal{Z}$ are labeled $a_{-\mathcal{Z},1},\dots,a_{-\mathcal{Z},4k}$, then there is a natural identification from $a_{\mathcal{Z},i}$ to $a_{-\mathcal{Z},4k+1-i}$. In this way, we can identify chords, idempotents, and algebra elements on $\mathcal{Z}$ to those on $-\mathcal{Z}$. For chords and algebra elements, the direction of the strands is reversed. It is not difficult to see that this produces a natural anti-homomorphism from $\mathcal{A}(\mathcal{Z})$ to $\mathcal{A}(-\mathcal{Z})$. For a chord ρ, a collection of pairs of points $\mathbf{s}$, and an algebra element a on $\mathcal{Z}$, we denote the corresponding objects on $-\mathcal{Z}$ by $-\rho, \bar{\mathbf{s}}$, and $\bar{a}$, respectively. For example, in this notation we have $a(-\rho) = \bar{a}(\rho)$.

For any generator $a \in \mathcal{A}(\mathcal{Z})$, the *multiplicity* of a, denoted $\mathrm{mult}(a)$, is an element in $H_1(Z\backslash\{z\}, \mathbf{a})$, recording how many times the strands in a cover each non-basepoint interval on Z. The *length* of a is the sum of coefficients in $\mathrm{mult}(a)$. Equivalently, it is the sum of lengths of strands in a.

For an idempotent $i \in \mathcal{A}(\mathcal{Z})$, the idempotent *complementary* to i is the idempotent that has horizontal strands exactly where i does not. We denote $o(i) \in \mathcal{A}(\mathcal{Z})$ to be the idempotent complementary to i.

1.5 Bordered invariants and Heegaard diagrams

Given a 3-manifold Y with one boundary component ∂Y, a parametrization of ∂Y by a pointed matched circle $\mathcal{Z} = (Z, z, \mathbf{a}, M)$ is a diffeomorphism $\phi : F(\mathcal{Z}) \to \partial Y$. This marks ∂Y with a circle and a basepoint on the circle, which by abuse of notation we will also call Z and z. Bordered Floer theory associates two invariants to a 3-manifold Y with boundary ∂Y parametrized by $\mathcal{Z}$:

- Type A invariant $\widehat{CFA}(Y)_{\mathcal{A}(\mathcal{Z})}$, a right A_∞-module over $\mathcal{A}(\mathcal{Z})$.
- Type D invariant ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y)$, a left type D module over $\mathcal{A}(-\mathcal{Z})$.

They are invariants of Y up to homotopy equivalence of A_∞-modules or type D modules. We follow the standard convention in expressing types of actions on the module: each algebra is written on the side it acts on. Subscripts indicate A_∞-actions, and superscripts indicate type D actions. These may be omitted when there is no danger of confusion.

These invariants satisfy the following pairing theorem: let Y_1 and Y_2 be two 3-manifolds with boundaries parametrized by $\mathcal{Z}$ and $-\mathcal{Z}$, respectively. Let $Y = Y_1 \cup_\partial Y_2$ be the closed 3-manifold obtained by gluing them along their boundaries (with the gluing map induced by the parametrizations). Then

$$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(Y_2), \tag{1.6}$$

([13, Theorem 1.3]).

The theory extends to 3-manifolds with two boundary components as follows. Given Y with two boundary components $\partial_L Y$ and $\partial_R Y$. Fix parametrizations $\phi_1 : F(\mathcal{Z}_1) \to \partial_L Y$ and $\phi_2 : F(\mathcal{Z}_2) \to \partial_R Y$. This induces circles Z_1 and Z_2 on $\partial_L Y$ and $\partial_R Y$, and basepoints $z_1 \in Z_1, z_2 \in Z_2$. We further fix a map γ from the framed cylinder $(S^1, z) \times [0,1]$ into Y, so that $(S^1, z) \times \{0\}$ and $(S^1, z) \times \{1\}$ map to (Z_1, z_1) and (Z_2, z_2), respectively. We call the totality of the data $(Y, \partial_L Y, \partial_R Y, \phi_1, \phi_2, \gamma)$ a *strongly bordered 3-manifold with two boundary components*. From now on whenever we mention a 3-manifold Y with two boundary components, we mean a strongly bordered 3-manifold, omitting the other data when they are clear from context. To a 3-manifold Y with two boundary components, bordered Floer theory associates the following invariants:

- Type AA invariant $\widehat{CFAA}(Y)_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}$, a right A_∞-bimodule over $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$.
- Type DD invariant ${}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(Y)$, a left type D bimodule over $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(-\mathcal{Z}_2)$.
- Type DA invariant ${}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z}_2)}$, a left type D, right A_∞-bimodule over $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$.
- Type AD invariant ${}^{\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFAD}(Y)_{\mathcal{A}(\mathcal{Z}_1)}$, a right A_∞, left type D bimodule over $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(-\mathcal{Z}_2)$.

The bimodules satisfy similar pairing theorems, as described in [14, Section 7.1]. The general rule is that tensor product can be taken between a right A_∞-action and a left type D action over the same algebra $\mathcal{A}(\mathcal{Z})$. Taking this tensor product corresponds to gluing two boundaries parametrized by $\mathcal{Z}$ and $-\mathcal{Z}$.

Just as 3-manifolds are represented by Heegaard diagrams when computing Heegaard Floer homology, bordered 3-manifolds are represented by bordered Heegaard diagrams in order to compute the bordered invariants. A bordered Heegaard diagram $\mathcal{H}$ for a 3-manifold with one boundary consists of the data $(\Sigma, \vec{\alpha}^o, \vec{\alpha}^c, \vec{\beta})$ where:

- Σ is a surface of genus g with one circle boundary Z, and a basepoint $z \in Z$.
- $\vec{\alpha}^o$ is a set of $g-k$ circles in Σ (the α circles).
- $\vec{\alpha}^c$ is a set of $2k$ arcs in Σ, whose boundary points are at distinct points of $Z \backslash z$ (the α arcs).
- $\vec{\beta}$ is a set of g circles in Σ (the β circles).

We also denote $\vec{\alpha} = \vec{\alpha}^o \coprod \vec{\alpha}^c$. We require that circles in $\vec{\beta}$ be disjoint, so are circles and arcs in $\vec{\alpha}$. Moreover, objects in $\vec{\beta}$ and objects in $\vec{\alpha}$ intersect each other transversely. Finally, $\vec{\alpha}$ and $\vec{\beta}$ are both homologically independent, in the sense that $H \backslash \vec{\alpha}$ and $H \backslash \vec{\beta}$ are both connected. Let $\mathbf{a}$ be the $4k$ points of intersection between α arcs and Z. The data $(Z, z, \mathbf{a})$, together with the natural pairing on $\mathbf{a}$ given by the α arcs, gives a pointed matched circle $\mathcal{Z}$. Similar to the construction of a closed 3-manifold from a closed Heegaard diagram, there is a standard way to construct an oriented 3-manifold Y with boundary parametrized by $F(\mathcal{Z})$ from the bordered Heegaard diagram $\mathcal{H}$.

Given a bordered diagram $\mathcal{H}$ for a 3-manifold Y with boundary, the invariant ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y, \mathcal{H})$ is defined as follows. The generators of the module are g-tuples of intersection points between β-circles and α-circles or arcs, with one intersection point on each circle, and at most one on each arc. Let $\mathcal{G}(\mathcal{H})$ be the set of generators. The idempotent associated to a generator is the set of pairs of points in Z that are ends of α-arcs *not* occupied by the generator.

The type D action on $\widehat{CFD}(Y, \mathcal{H})$ is defined by choosing a compatible almost complex structure on the symplectic manifold $Y \times [0,1] \times \mathbb{R}$, and counting certain holomorphic curves. As in the closed case, we define a region to be a connected component of $F \setminus \{\vec{\alpha}, \vec{\beta}\}$, and a domain as an integral linear combination of regions. Given $\mathbf{x}, \mathbf{y} \in \mathcal{G}(\mathcal{H})$, let $\pi_2(\mathbf{x}, \mathbf{y})$ be the set of domains satisfying $\partial(\partial^\alpha B) = \mathbf{y} - \mathbf{x}$, where $\partial^\alpha B$ denotes the parts of boundary of B that are pieces of α circles or arcs, and the $\partial(\cdot)$ on the outside excludes points on the boundary of F. Then each arrow $\mathbf{x} \to a \otimes \mathbf{y}$ comes from a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$, such that all coefficients of regions in B are non-negative, and the coefficient of the region containing the basepoint z is zero (each domain can give raise to multiple arrows). Moreover, for each interval in $\mathcal{Z}$, the multiplicity of a at the interval equals the coefficient in B of the region of $\mathcal{H}$ neighboring that interval.

The case for $\widehat{CFA}(Y, \mathcal{H})_{\mathcal{A}(\mathcal{Z})}$ is similar. The generators are the same. Each arrow $m_{1,k}(\mathbf{x}, a_1, \ldots, a_k) \to \mathbf{y}$ comes from a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$, such that the sum of multiplicities of $a_1, \ldots, a_k$ equals the multiplicity of B at the intervals on the boundary.

It can be proved that the homotopy type of $\widehat{CFD}(Y, \mathcal{H})$ (resp. $\widehat{CFA}(Y, \mathcal{H})$) is independent of the choice of $\mathcal{H}$ and the almost complex structure, so we may write $\widehat{CFD}(Y)$ (resp. $\widehat{CFA}(Y)$) to

denote the module up to homotopy equivalence that is associated to Y.

This construction generalizes to 3-manifolds Y with two boundary components, which are assigned Heegaard diagrams with two circle boundaries. The main modifications are that the basepoint z is now an arc connecting the two boundaries, with its intersection with the boundaries being the basepoint of the pointed matched circles. Each α-arc can only end at one of the two boundaries of F. An arrow in the algebra action of the bimodule invariant comes from a domain B that may intersect both boundaries, and there are similar correspondences between the multiplicities of B at the two boundaries and the multiplicity of algebra inputs/outputs.

1.6 Gradings on bordered invariants

In this section, we give a brief overview of gradings on bordered invariants. For details, see [13, Chapter 10] and [14, Section 6.5].

We begin with gradings on the dg-algebra $\mathcal{A}(\mathcal{Z})$. There are two kinds of gradings, by a larger group $G'(\mathcal{Z})$ and a "refined" grading by a smaller group $G(\mathcal{Z})$. Both $G(\mathcal{Z})$ and $G'(\mathcal{Z})$ are non-commutative, equipped with a distinguished central element λ.

An element of $G'(\mathcal{Z})$ is specified by a pair (k, α), where $k \in \frac{1}{2}\mathbb{Z}$ and $\alpha \in H_1(Z', \mathbf{a})$. With points of $\mathbf{a}$ labeled $1, \dots, 4k$, we can write α as a sequence of integers $\alpha_i, 1 \leq i \leq 4k-1$, where α_i is the multiplicity of α at the interval $[i, i+1]$. Then multiplication on $G'(\mathcal{Z})$ is defined by

$$(k, \alpha) \cdot (l, \beta) = (k + l + L(\alpha, \beta), \alpha + \beta),$$

where

$$L(\alpha, \beta) = \sum_{i=1}^{4k-2} \frac{1}{2}(\alpha_i \beta_{i+1} - \alpha_{i+1}\beta_i).$$

Actually, the grading lies in an index 2 subgroup of $G'(\mathcal{Z})$, but we will not make use of this here.

For later use, we define an anti-homomorphism

$$R : G'(\mathcal{Z}) \to G'(-\mathcal{Z})$$

given by

$$R(k, \alpha_1, \dots, \alpha_{4k-1}) = (k, -\alpha_{4k-1}, \dots, -\alpha_1).$$

To define the grading of a generator of $\mathcal{A}(\mathcal{Z})$, we first define a map

$$m : H_1(Z', \mathbf{a}) \times H_0(\mathbf{a}) \to \frac{1}{2}\mathbb{Z}.$$

For an interval α and a point p, let $m(\alpha, p) = 1$ if p is in the interior of α, $\frac{1}{2}$ if p is on the boundary, and 0 otherwise. This is then extended bilinearly to all of $H_1(Z', \mathbf{a}) \times H_0(\mathbf{a})$ to define m.

Given a generator $a \in \mathcal{A}(\mathcal{Z})$, let $\boldsymbol{\rho}$ be the non-horizontal strands of a. Let $\mathrm{inv}(\boldsymbol{\rho})$ be the number of inversions in $\boldsymbol{\rho}$, $S \in H_0(\mathbf{a})$ be the starting points of $\boldsymbol{\rho}$, and $[a] \in H_1(Z', \mathbf{a})$ be the multiplicity of a. Then

$$\mathrm{gr}'(a) = (\mathrm{inv}(\boldsymbol{\rho}) - m([a], S), [a]).$$

Next, we consider gradings on the type D invariant. Fix a bordered Heegaard diagram $\mathcal{H}$. Let $\mathbf{x}, \mathbf{y}$ be generators and $B \in \pi_2(\mathbf{x}, \mathbf{y})$, define $g'(B) \in G'(\mathcal{Z})$ as

$$g'(B) = (-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), \partial^{\partial}(B)).$$

Here $e(B)$ is the Euler measure of B, and $n_{\mathbf{x}}(B), n_{\mathbf{y}}(B)$ are multiplicities of B at $\mathbf{x}, \mathbf{y}$ as before, and $\partial^{\partial}(B)$ is the boundary of B on $H_1(Z', \mathbf{a})$.

There is a grading set for each spin-c class on $\mathcal{H}$ (in most bordered cases we consider here, there is just one spin-c class). The grading set $S'_D(\mathcal{H}, \mathfrak{s})$ for the Heegaard diagram $\mathcal{H}$ and spin-c class $\mathfrak{s}$ is defined as follows. Choose a base generator $\mathbf{x_0}$ with spin-c class $\mathfrak{s}$. Let $P'(\mathbf{x_0})$ be the set of $g'(P)$ for all $P \in \pi_2(\mathbf{x_0}, \mathbf{x_0})$ (the domains in $\pi_2(\mathbf{x_0}, \mathbf{x_0})$ are called *periodic domains*). Then set

$$S'_D(\mathcal{H}, \mathfrak{s}) = G'(-\mathcal{Z})/R(P'(\mathbf{x_0})).$$

This grading set has an obvious left action by $G'(\mathcal{Z})$. For another generator $\mathbf{x}$, choose a domain $B_0 \in \pi_2(\mathbf{x_0}, \mathbf{x})$ (unlike in the closed case, such a domain generally exists), and set

$$\mathrm{gr}'(\mathbf{x}) = R(g'(B_0)) \cdot R(P'(\mathbf{x_0})).$$

The type D action respects the grading in the sense that, for each arrow $\mathbf{x} \to a \otimes \mathbf{y}$ in the action, we have

$$\lambda^{-1}\mathrm{gr}'(\mathbf{x}) = \mathrm{gr}'(a)\mathrm{gr}'(\mathbf{y}).$$

Gradings on type A invariants are similar. The grading set is

$$S'_A(\mathcal{H}, \mathfrak{s}) = P'(\mathbf{x_0})\backslash G'(\mathcal{Z}).$$

This carries a natural right action by $G'(\mathcal{Z})$. For any generator $\mathbf{x}$, choose a domain $B_0 \in \pi_2(\mathbf{x_0}, \mathbf{x})$ and set

$$\mathrm{gr}'(\mathbf{x}) = P'(\mathbf{x_0}) \cdot g'(B_0).$$

The A_∞-action respects the grading in the sense that, for each arrow

$$m_{1,k}(\mathbf{x}, a_1, \ldots, a_k) \to \mathbf{y},$$

we have

$$\lambda^{k-1}\mathrm{gr}'(\mathbf{x})\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_k) = \mathrm{gr}'(\mathbf{y}).$$

Gradings on bimodules are defined similarly. In particular, a domain in a Heegaard diagram with two boundaries parametrized by $\mathcal{Z}_1$ and $\mathcal{Z}_2$ give rise to an element of $G'(\mathcal{Z}_1) \times_\lambda G'(\mathcal{Z}_2) = G'(\mathcal{Z}_1) \times G'(\mathcal{Z}_2)/(\lambda_1 = \lambda_2)$. The grading set is a certain coset of the tensor product of two grading groups.

Now we briefly discuss refined gradings, which contain essentially the same information, but are cleaner to work with theoretically.

The group $G(\mathcal{Z})$ can be considered as a subgroup of $G'(\mathcal{Z})$, generated by λ and elements of the form $(0, [p, q])$, where p, q is a pair of matched points, and $[p, q]$ denotes the interval in $H_1(Z', \mathbf{a})$ between p and q. An element (k, α) of $G'(\mathcal{Z})$ is in $G(\mathcal{Z})$ if and only if $M_*(\partial\alpha) = 0$ where $\partial : H_1(Z', \mathbf{a}) \to H_0(\mathbf{a})$ is the boundary operator and $M_* : H_0(\mathbf{a}) \to \mathbb{Z}^{2k}$ is a map sending each matched pair of points to the same basis element of $\mathbb{Z}^{2k}$.

To construct the refined grading on $\mathcal{A}(\mathcal{Z})$, we first choose a base idempotent s_0 in $\mathcal{A}(\mathcal{Z})$. Then for every idempotent s, choose a grading element $\psi(s) = (k, \alpha) \in G'(\mathcal{Z})$ such that $M_*(\partial\alpha) = s - s_0$. For an algebra element a with left idempotent s and right idempotent t, we set

$$\mathrm{gr}(a) = \psi(s)\mathrm{gr}'(a)\psi(t)^{-1}.$$

It is easy to check that this element lies in $G(\mathcal{Z})$ and that the two conditions on the grading are satisfied.

Similarly, we can "refine" the grading on the bordered invariants to use $G(\mathcal{Z})$ rather than $G'(\mathcal{Z})$.

We will omit the details here.

We will not perform any detailed grading computations in this thesis, but will simply note that all such computations can be done combinatorially from the Heegaard diagram. Given a module (or bimodule) M of any type, after the gradings of generators on M are fixed, grading imposes a constraint on what kind of terms can appear in the algebra action of M. In particular, it imposes the following constraint that is otherwise only available from the Heegaard diagram: if a region in a diagram with two boundaries touches the boundaries at intervals i and i', respectively, then for all arrows in the algebra action, the total multiplicity at i and at i' must be the same. Such constraints are crucial to establishing uniqueness properties of bimodule invariants, to be discussed in the following sections and in the rest of the thesis.

1.7 The strongly-based mapping class groupoid

Now we turn our attention to bimodule invariants associated to diffeomorphisms of surfaces, in particular the identity and arcslide diffeomorphisms.

The strongly-based mapping class groupoid is a category whose objects are pointed matched circles $\mathcal{Z}$ with $4g$ points. These should be considered as surfaces $F^\circ(\mathcal{Z})$ with standard parametrizations by $\mathcal{Z}$. The morphisms from $\mathcal{Z}_1$ to $\mathcal{Z}_2$ in the category are isotopy classes of diffeomorphisms $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, sending basepoint $z_1 \in F^\circ(\mathcal{Z}_1)$ to basepoint $z_2 \in F^\circ(\mathcal{Z}_2)$. Identity and composition in the category correspond to the identity diffeomorphism and composition of diffeomorphisms, respectively.

If we fix a pointed matched circle $\mathcal{Z}$ and only consider morphisms from $\mathcal{Z}$ to itself, we obtain the strongly-based mapping class group of $F_{g,1}$ (genus g surfaces with one circle boundary). This is simply the group of isotopy classes of boundary-preserving self-diffeomorphisms of $F_{g,1}$.

Given a diffeomorphism $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, we can construct its mapping cylinder $Y(\phi) = F(\mathcal{Z}_2) \times [0,1]$ as a strongly bordered 3-manifold with two boundary components, parametrized by $-\mathcal{Z}_1$ and $\mathcal{Z}_2$. The left boundary $\partial_L Y(\phi) = F(\mathcal{Z}_2) \times \{0\}$ is parametrized by the induced map $\phi_* : -F(\mathcal{Z}_1) \to -F(\mathcal{Z}_2)$ (reverse orientation and extend over the disk filling the boundary), while the right boundary is parametrized by the identity map on $F(\mathcal{Z}_2)$. The map γ is simply the inclusion $(Z, z) \times [0,1] \to F(\mathcal{Z}_2) \times [0,1]$.

This establishes an one-to-one correspondence between strongly bordered 3-manifolds that are topologically $F_g \times [0,1]$, and morphisms in the strongly-based mapping class groupoid with genus g. For a morphism $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, we denote $\widehat{CFAA}(\phi)_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}$ to be the type AA

invariant $\widehat{CFAA}(Y(\phi))_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}$ associated to the mapping cylinder of ϕ. Likewise we have invariants ${}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}_2)}$ and ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\phi)$ corresponding to ϕ.

For future reference, we write down the pairing theorems involving the DA invariants. For morphisms $\phi_1 : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ and $\phi_2 : F^\circ(\mathcal{Z}_2) \to F^\circ(\mathcal{Z}_3)$, the DA invariant for $\phi_2 \circ \phi_1$ is given by:

$$ {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi_2 \circ \phi_1)_{\mathcal{A}(\mathcal{Z}_3)} = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi_1)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDA}(\phi_2)_{\mathcal{A}(\mathcal{Z}_3)}. \tag{1.7} $$

For a morphism $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ and a 3-manifold Y with boundary parametrized by $\psi : F(-\mathcal{Z}_2) \to \partial Y$, let Y' be the same manifold with boundary parametrized by $\psi \circ \phi_* : F(-\mathcal{Z}_1) \to \partial Y$, then

$$ {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFD}(Y') = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFD}(Y). \tag{1.8} $$

1.8 Invariants of the identity diffeomorphism

Let $\mathbb{I}_{\mathcal{Z}}$ be the identity morphism $F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z})$. All bimodule invariants associated to $\mathbb{I}_{\mathcal{Z}}$ have special significance in the theory. First, it can be shown that ([14, Section 8.1])

$$ \widehat{CFDA}(\mathbb{I}_{\mathcal{Z}}) \simeq {}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}, $$

where the latter denotes the identity type DA bimodule over $\mathcal{A}(\mathcal{Z})$. This bimodule is generated over $\mathbb{F}_2$ by idempotents of $\mathcal{A}(\mathcal{Z})$, and the algebra action on the bimodule is given by:

$$ \delta_2^1(i, a) = a \otimes j, $$

for any generator $a \in \mathcal{A}(\mathcal{Z})$, where i and j are the left and right idempotents of a.

The type DD invariant ${}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ and type AA invariant $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}$ relate the type A and type D invariants through taking the tensor product. For any 3-manifold Y with one boundary component parametrized by $\mathcal{Z}$, the relations are:

$$ \widehat{CFA}(Y)_{\mathcal{A}(\mathcal{Z})} = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y) \tag{1.9} $$

$$ {}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y) = \widehat{CFA}(Y)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}). \tag{1.10} $$

One implication is that $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$ contain the same information about Y. Likewise, there are relations among the bimodule invariants, showing that all bimodule invariants also contain

the same information. For any 3-manifold Y with two boundary components parametrized by $\mathcal{Z}_1$ and $\mathcal{Z}_2$:

$$^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z}_2)} = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_2})_{\mathcal{A}(-\mathcal{Z}_2),\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(Y) \tag{1.11}$$

$$\widehat{CFAA}(Y)_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)} = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z}_2)} \tag{1.12}$$

$$^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z}_2)} = {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_1}) \boxtimes \widehat{CFAA}(Y)_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)} \tag{1.13}$$

$$^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(Y) = {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_2}) \boxtimes {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z}_2)}. \tag{1.14}$$

The above equations are special cases of the pairing theorems (where one side is a cylinder). It indicates the importance of finding combinatorial descriptions of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ and $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$.

The type DD invariant $^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ is computed in [18, Theorem 14] and [16, Section 3]. It is generated over $\mathbb{F}_2$ by the set of pairs of complementary idempotents $i \otimes i'$, with $i \in \mathcal{A}(\mathcal{Z})$ and $i' = \overline{o(i)} \in \mathcal{A}(-\mathcal{Z})$ (notation from end of section 1.4). The type DD action is given by:

$$\delta^1(i \otimes i') = \sum_{\xi \in \mathcal{C}, ia(\xi) = a(\xi)j, i'\overline{a(\xi)} = \overline{a(\xi)}j'} (a(\xi) \otimes \overline{a(\xi)}) \otimes (j \otimes j'), \tag{1.15}$$

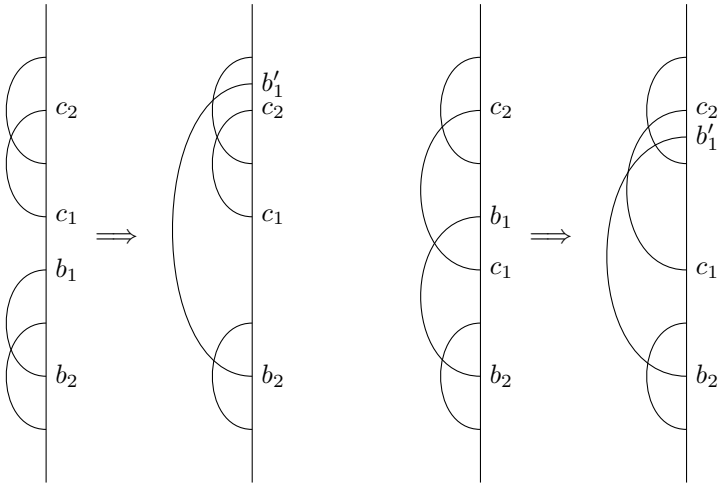
where $\mathcal{C}$ is the set of chords on $\mathcal{Z}$ whose two endpoints are not matched.

1.9 Invariants of arcslides

The strongly-based mapping class groupoid is generated by a particularly simple class of morphisms called *arcslides*. We will now review their definitions and the invariants associated to them.

Given a pointed matched circle $\mathcal{Z}_1$, and two matched pairs of points $B = (b_1, b_2)$ and $C = (c_1, c_2)$ in $\mathbf{a} \subset Z_1$, such that b_1 and c_1 are adjacent in $\mathbf{a}$. An arcslide of b_1 over c_1 moves b_1 to be adjacent to c_2, on the side opposite to its original position with respect to c_1. This results in a new pointed matched circle $\mathcal{Z}_2$. Such a move corresponds to a certain diffeomorphism $F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, which we will also call an arcslide. See Figure 1.2 for two examples of arcslides. The first example is an *overslide* meaning b_1 is outside the interval $[c_1, c_2]$. The second example is an *underslide* meaning b_1 is inside that interval.

Given an arcslide $\tau : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, the invariant $\widehat{CFDD}(\tau)$ is a left type DD bimodule over $A(\mathcal{Z}_1)$ and $A(-\mathcal{Z}_2)$, or equivalently, a left-right type DD bimodule over $A(\mathcal{Z}_1)$ and $A(\mathcal{Z}_2)$. This bimodule can be calculated from a standard Heegaard diagram for the mapping cylinder $Y(\tau)$. For the two arcslides in Figure 1.2, these standard Heegaard diagrams are shown in Figure 1.3. The tiny



(1.16)

Figure 1.2: Two examples of arcslides.

circles in the diagrams are 1-handle attachment points, paired according to their vertical positions. The larger circles are β circles, and all other arcs inside the boundary are α-arcs. Later on, we will draw more schematic versions of the diagrams, omitting the β circles and some of the attaching points of 1-handles.

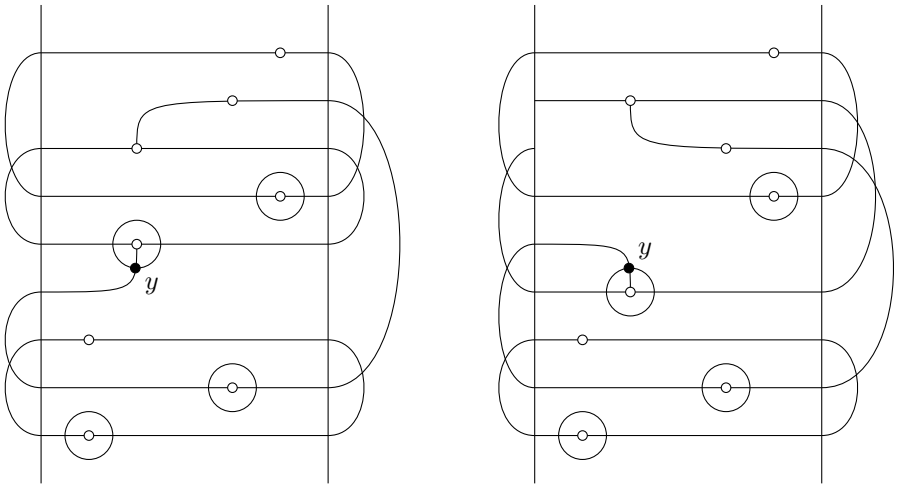


Figure 1.3: Examples of Heegaard diagrams of arcslides.

Each generator of $\widehat{CFDD}(\tau)$ corresponds to a set of intersection points between α and β curves, with each β circle containing exactly one point, and each α-arc containing at most one point. The (type D) idempotent of the generator specifies which pairs of α-arcs are *not* occupied by the generator. In the case of arcslides here, each generator is uniquely specified by its idempotents on the two sides.

There is an obvious identification between pairs of points on the two sides, under which we can talk about two idempotents on different sides being complementary, etc. There are two types of generators. A generator of type X has complementary idempotents, and a generator of type Y has idempotents that are complementary except for both containing the C pair and neither containing the B pair. The type X generators are those that do not occupy the intersection point y in Figure 1.3, while the type Y generators do.

The type DD action on the bimodule can be described as follows. Given a pointed matched circle $\mathcal{Z}$, let $C(\mathcal{Z})$ denote the collection of sets of chords in $\mathcal{Z}$. For some $\xi \in C(\mathcal{Z})$, let $a(\xi) \in \mathcal{A}(\mathcal{Z})$ denote the sum of all generators of $\mathcal{A}(\mathcal{Z})$ produced by adding horizontal strands to ξ (this definition extends the case where ξ is a chord). For any arcslide $\tau : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, there is a collection of pairs $C_\tau = \{(\xi_i, \eta_i)\}$ where $\xi_i \in C(\mathcal{Z}_1), \eta_i \in C(\mathcal{Z}_2)$ such that

$$\delta^1(i \otimes i') = \sum_{\substack{(\xi_k,\eta_k)\in C_\tau \\ ia(\xi_k)=a(\xi_k)j \\ i'\overline{a(\eta_k)}=\overline{a(\eta_k)}j' \\ j\otimes j' \text{is a generator}}} (a(\xi_k) \otimes \overline{a(\eta_k)}) \otimes (j \otimes j'), \tag{1.17}$$

where generators are represented by their idempotents at the two sides.

Intuitively, there is a term in the type DD action whenever the idempotent agrees, and the moving strands part of the algebra coefficients match one of the fixed patterns. Depending on whether the arcslide is an underslide or overslide, there are 6 or 8 types of elements in C_τ. See Figure 17 and 22 in [16] for diagrams of these patterns. Note that not all pairs in Figure 22 are actually in C_τ — there is an additional choice involved. In the following we will only use some of the simpler pairs, involving algebra elements that have small total length. In particular we will not need to consider any pair where a choice is necessary.

The following properties $\widehat{CFDD}(\tau)$, which follow from its definition in terms of holomorphic curve counts, can be directly verified for the above description:

- Every arrow in the type DD action corresponds to a domain away from the basepoint in the Heegaard diagram. If $\mathbf{x} \to a_1 \otimes a_2 \otimes \mathbf{y}$ corresponds to the domain B, then $\partial(\partial^\alpha B) = \mathbf{y} - \mathbf{x}$, and the multiplicities of a_1, a_2 equal the multiplicities of B at the two boundaries of the Heegaard diagram.

- There is a grading on the generators of $\widehat{CFDD}(\tau)$ with a particular grading set S. There are left-right actions of $G(\mathcal{Z}_1)$ and $G(\mathcal{Z}_2)$ on S. Both actions on S are free and transitive, which means S induces a group isomorphism $G(\mathcal{Z}_1) \to G(\mathcal{Z}_2)$. This group isomorphism is an

invariant of τ, up to composing by inner automorphisms of $G(\mathcal{Z}_1)$ and $G(\mathcal{Z}_2)$. In other words, τ induces an element in the set of outer isomorphisms $\mathrm{Out}(G(\mathcal{Z}_1), G(\mathcal{Z}_2))$. In fact, this outer isomorphism can be found directly from the actions of τ on the homology of the surface [16, Section 5.2].

- For any arcslide $\tau : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, let $\bar{\tau} : F^\circ(-\mathcal{Z}_1) \to F^\circ(-\mathcal{Z}_2)$ be the arcslide with reversed orientation. Then ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\tau)$ and ${}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\bar{\tau})$ are *dual* to each other. For a type DD bimodule M, its dual $\overline{M}$ is the type DD bimodule over the same generators, and each term $\mathbf{x} \to a_1 \otimes a_2 \otimes \mathbf{y}$ in the type DD action of M corresponds to a term $\mathbf{y} \to a_1 \otimes a_2 \otimes \mathbf{x}$ in the type DD action of $\overline{M}$. So the left actions by $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(-\mathcal{Z}_2)$ on $\widehat{CFDD}(\tau)$ become right actions by the same algebras, or equivalently left actions by $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$ on $\overline{M}$.

Chapter 2

$\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ and Koszul duality

2.1 Introduction

The main aim of this chapter is to describe combinatorially the type AA invariant $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$. More precisely, we construct an explicit rank 1 A_∞-bimodule $\mathcal{N}$ with the following property:

Theorem 4. *$\mathcal{N} \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ has rank 1. Furthermore, $\mathcal{N}$ is quasi-invertible, in the sense that there exists a type DA bimodule $\mathcal{N}'$, such that $\mathcal{N} \boxtimes \mathcal{N}'$ is homotopy equivalent to the identity bimodule ($\mathcal{N}'$ is called the quasi-inverse of $\mathcal{N}$).*

This quickly leads to a combinatorial proof that $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ itself is quasi-invertible. By construction, the bimodule $\mathcal{N}$ is in the homotopy class of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$, so we know from analysis that it is the actual quasi-inverse of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$. This stronger statement will be proved in the next chapter.

Considering $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ as a left-right type DD bimodule ${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})^{\mathcal{A}(\mathcal{Z})}$, its quasi-invertibility also implies a kind of Koszul self-duality of $\mathcal{A}(\mathcal{Z})$. See [18, Section 8] for details. One consequence of this Koszul duality is the existence of an A_∞-morphism from $\mathcal{A}(\mathcal{Z})$ to the cobar resolution of $\mathcal{A}(\mathcal{Z})$, inducing an isomorphism on homology. We will give some explicit computations of this map as an application.

In addition to giving a more concrete understanding of Koszul duality in $\mathcal{A}(\mathcal{Z})$, an explicit description of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ can be useful in various computations. In addition to $\widehat{CFD}$ and $\widehat{CFA}$, there are also bimodule invariants $\widehat{CFDD}, \widehat{CFDA}$ and $\widehat{CFAA}$ associated to any 3-manifold with two boundaries, in particular mapping cylinders of surface diffeomorphisms. In general, it is easier to compute $\widehat{CFD}$ and $\widehat{CFDD}$, since it involves counting simpler holomorphic curves, and there are

known methods to exploit the type DD structure equations. These methods are used to compute $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ in [18, 16], and $\widehat{CFDD}(\tau)$ for any arcslide τ. With an explicit description of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$, we can then obtain descriptions of type A and type DA invariants with the same number of generators whenever the type D and type DD invariants can be computed. This is used in the next chapter to give a combinatorial construction and proof of invariance for $\widehat{HF}$ using bordered Floer theory.

The construction of the rank 1 model of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ begins with a previously known model $\mathcal{M}$ of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$. The chain complex M underlying $\mathcal{M}$ has far more generators than is necessary for $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$. The tool used to reduce the number of generators is *homological perturbation theory*. To use this theory, we find a smaller chain complex N that is homotopy equivalent to M. Since the theory is over $\mathbb{F}_2$, we may take N to be the homology of M. Homological perturbation theory will construct an A_∞-bimodule $\mathcal{N}$ homotopy equivalent to $\mathcal{M}$, whose underlying chain complex is N.

To construct $\mathcal{N}$, we need the additional data that verifies the homotopy equivalence between M and N. That is, morphisms $f : M \to N$ and $g : N \to M$ such that $g \circ f = \mathbb{I}_N$, and a homotopy map $H : M \to M$ such that $\mathbb{I}_M + f \circ g = d \circ H + H \circ d$. Both f and g will become obvious after we describe the chain complex and its homology. So the focus of this paper will be on constructing H and verifying that it is indeed a homotopy.

When $\mathcal{Z}$ is the (unique) genus 1 pointed matched circle, the size of the chain complex M is small enough that H can be found directly. This is done in [16, Section 8.4]. The computation here works for pointed matched circles of any genus, and one can easily check that it agrees with the previous computation in the genus 1 case.

Throughout this chapter, we will fix a pointed matched circle $\mathcal{Z}$. We write $\mathcal{A}$ and $\mathcal{A}'$ for the dg-algebras $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$. A generator of $\mathcal{A}(\mathcal{Z})$ will be represented by upward-veering strands, and a generator of $\mathcal{A}(-\mathcal{Z})$ by downward-veering ones. Paired horizontal strands in the generator will be shown using dashed lines. Later on, when we are dealing solely with generators of $\mathcal{A}(\mathcal{Z})$, we will also omit the direction markers on strands.

We now describe plans for the rest of this chapter. In section 2.2, we describe the initial, larger model $\mathcal{M}$ of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$. In section 2.3 we describe the homotopy map H, and in section 2.4 verify that it satisfies the homotopy equation. In section 2.5, we apply homological perturbation theory to describe the rank-1 model $\mathcal{N}$ of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$, and prove the quasi-invertibility of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$. Finally, in the section 2.6 we give an application calculating some homology classes in the cobar resolution of $\mathcal{A}(\mathcal{Z})$.

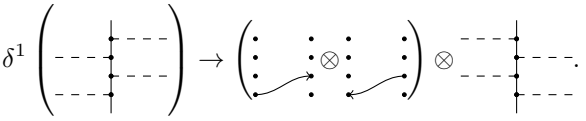
2.2 Descriptions of the large chain complex

In this section, we describe in detail the initial model $\mathcal{M}$ of the A_∞-bimodule $\widehat{CFAA}(\mathbb{I}_\mathcal{Z})$, using the formula from [14, Proposition 9.2]:

$$\begin{aligned}\widehat{CFAA}(\mathbb{I}_\mathcal{Z})_{\mathcal{A},\mathcal{A}'} &= \mathrm{Mor}^{\mathcal{A}}({}_{\mathcal{A}'}^{\mathcal{A}}\widehat{CFDD}(\mathbb{I}_\mathcal{Z}), {}^{\mathcal{A}}\mathbb{I}_{\mathcal{A}}) \\ &= \mathrm{Mor}^{\mathcal{A}}({}_{\mathcal{A}'}\mathcal{A}'_{\mathcal{A}'} \boxtimes {}^{\mathcal{A},\mathcal{A}'}\widehat{CFDD}(\mathbb{I}_\mathcal{Z}), {}^{\mathcal{A}}\mathbb{I}_{\mathcal{A}}) \\ &= \overline{\widehat{CFDD}(\mathbb{I}_\mathcal{Z})}^{\mathcal{A}',\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}_{\mathcal{A}'}\overline{\mathcal{A}'}_{\mathcal{A}'} \end{aligned} \tag{2.1}$$

We begin by describing each of the three factors in the last line of (2.1).

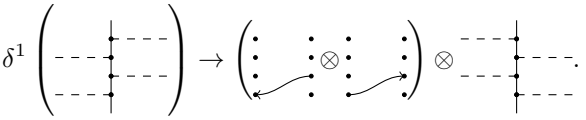
The type DD structure ${}^{\mathcal{A},\mathcal{A}'}\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$ is described in the introductory chapter. As an example, an arrow in the genus 1 case is:



(2.2)

Note that the generators of $\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$ are in one-to-one correspondence with idempotents of $\mathcal{A}$. So $\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$ has rank 1 as a module over $\mathbf{k}$. We call such bimodules rank-1 bimodules.

On taking the dual, the directions of the arrows are reversed while the coefficients are kept the same. For later convenience we will also reverse the order of the two algebra actions, so that the action by $\mathcal{A}'$ comes first. As an example, the arrow in (2.2) gives raise to the following arrow in $\overline{\widehat{CFDD}(\mathbb{I}_\mathcal{Z})}^{\mathcal{A}',\mathcal{A}}$:



(2.3)

This is shown graphically in Figure 2.1.

The bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is a left-right A_∞-bimodule with the same generators as $\mathcal{A}$. The A_∞-bimodule actions simply come from the dg-algebra actions on $\mathcal{A}$. That is:

$$m_{0,1,0}(a) \to b, \text{ for each generator } b \text{ in } da.$$

$$m_{1,1,0}(b;a) = ba, \text{ whenever } ba \neq 0.$$

$$m_{0,1,1}(a;b) = ab, \text{ whenever } ab \neq 0.$$

Here $m_{i,1,k}$ is the part of the A_∞-bimodule action with i algebra inputs on the left and k algebra

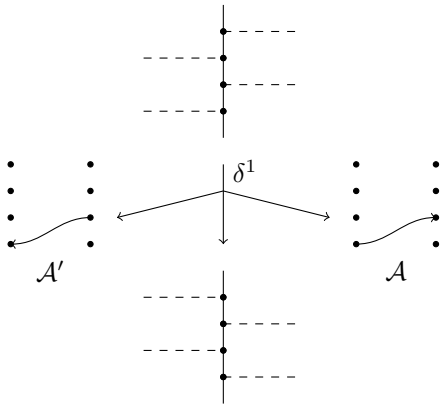


Figure 2.1: An example of an arrow in $\overline{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}^{\mathcal{A}',\mathcal{A}}$. Note strands in $\mathcal{A}$ go upward and strands in $\mathcal{A}'$ go downward. For later convenience, we put $\mathcal{A}'$ on the left and $\mathcal{A}$ on the right, but keep in mind that both are right type D actions.

inputs on the right.

The bimodule ${}_{\mathcal{A}'}\mathcal{A}'_{\mathcal{A}'}$ is defined the same way, based on the dg-algebra $\mathcal{A}'$. By taking duals, the directions of the arrows are reversed. The algebra coefficients stay the same, but acting on the opposite side. So the actions on ${}_{\mathcal{A}'}\overline{\mathcal{A}'}_{\mathcal{A}'}$ can be written as:

$$\begin{aligned} m_{0,1,0}(b) &\to a, \text{ for each generator } b \text{ in } da. \\ m_{0,1,1}(ba; b) &\to a, \text{ whenever } ba \neq 0. \\ m_{1,1,0}(b; ab) &\to a, \text{ whenever } ab \neq 0. \end{aligned}$$

We can now describe the bimodule $\mathcal{M}$ in (2.1). The underlying vector space is generated by triples $[a', i' \otimes i, a]$, such that $i' = \overline{o(i)}$, i' agrees with the right idempotent of a', and i agrees with the left idempotent of a. Since i' and i are determined by a' and a, we will omit them and simply write the pair $[a', a]$. The condition on idempotents becomes that the right idempotent of a' is complementary to the left idempotent of a (using the identification between idempotents of $\mathcal{A}$ and $\mathcal{A}'$).

The differentials in ${}_{\mathcal{A}'}\overline{\mathcal{A}'}_{\mathcal{A}'}$ and ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ give raise to the following arrows in $\mathcal{M}$:

$$\begin{aligned} [a', a] &\to [a', b], \text{ for each generator } b \text{ in } da. \\ [b', a] &\to [a', a], \text{ for each generator } b' \text{ in } da'. \end{aligned}$$

There is a third type of arrow, coming from the type DD action on $\overline{\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})}^{\mathcal{A}',\mathcal{A}}$. In general,

they are of the form:

$$[a'\overline{a(\xi)}, a] \to [a', a(\xi)a],$$

for each chord ξ, and $a \in \mathcal{A}, a' \in \mathcal{A}'$ such that $a(\xi)a \neq 0$ and $a'\overline{a(\xi)} \neq 0$.

As an example, we show how one such arrow follows from the definition of the box tensor product in Figure 2.2, which involves the type DD arrow in Figure 2.1.

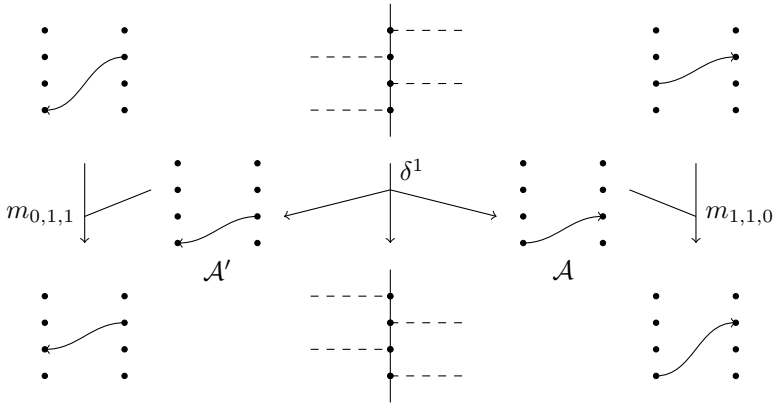


Figure 2.2: Example of the formation of an arrow in $\mathcal{M}$. From left to right, the three arrows composing it come from the A_∞-bimodule action $m_{0,1,1}$ on $\overline{\mathcal{A}'}$, the type DD action δ^1 on $\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$, and the A_∞-bimodule action $m_{1,1,0}$ on $\mathcal{A}$.

The right A_∞-actions on $\mathcal{M}$ are simply the ones inherited from that on $\mathcal{A}'$ and $\mathcal{A}$ (this is because ${}_\mathcal{A}\mathcal{A}_\mathcal{A}$ and ${}_{\mathcal{A}'}\overline{\mathcal{A}'}_{\mathcal{A}'}$ have no A_∞ actions with both left and right algebra inputs). They are given by the arrows:

$$m_{1,1,0}([a', a]; b;\ ) \to [a', ab], \text{ whenever } ab \neq 0.$$
$$m_{1,0,1}([b'a', a];\ ; b') \to [a', a], \text{ whenever } b'a' \neq 0.$$

Here $m_{1,j,k}$ is the part of the A_∞-bimodule action with j inputs from $\mathcal{A}$ and k inputs from $\mathcal{A}'$. The first of the two equations comes from the action on $\mathcal{A}$, and the second from the action on $\mathcal{A}'$.

For simplicity of the discussion later, we will write everything in terms of elements of $\mathcal{A}$, so that the pair $[a', a]$ is written as the pair $[a_1, a_2]$, where $a_1 = \overline{a'}$ and $a_2 = a$. Translating the differential and the condition on idempotents, we arrive at the following statement:

Proposition 5. *The A_∞-bimodule $\mathcal{M}$ is generated by pairs $[a_1, a_2]$, where a_1 and a_2 are generators of $\mathcal{A}$, such that the left idempotent of a_1 is complementary to the left idempotent of a_2. There are*

three types of arrows in the differential:

$$\begin{aligned}
[a_1, a_2] &\to [a_1, b], \text{ for each generator } b \text{ in } da_2 \\
[b, a_2] &\to [a_1, a_2], \text{ for each generator } b \text{ in } da_1 \\
[a(\xi)c, a] &\to [c, a(\xi)a], \text{ for each chord } \xi \text{ and } a, c \in \mathcal{A} \text{ such that } a(\xi)c \neq 0 \text{ and } a(\xi)a \neq 0.
\end{aligned}$$

The A_∞-bimodule action consists of the following arrows:

$$\begin{aligned}
m_{1,1,0}([a_1, a_2]; a) &\to [a_1, a_2 a], \text{ whenever } a_2 a \neq 0 \\
m_{1,0,1}([a_1 \overline{a'}, a_2]; a') &\to [a_1, a_2], \text{ whenever } a_1 \overline{a'} \neq 0
\end{aligned}$$

In the following sections, we will draw a generator of $\mathcal{M}$ by drawing a_1 and a_2 side by side. All strands are then going upward, so we omit directions on strands. Using this convention, examples of the three types of arrows in the differential are shown in Figure 2.3.

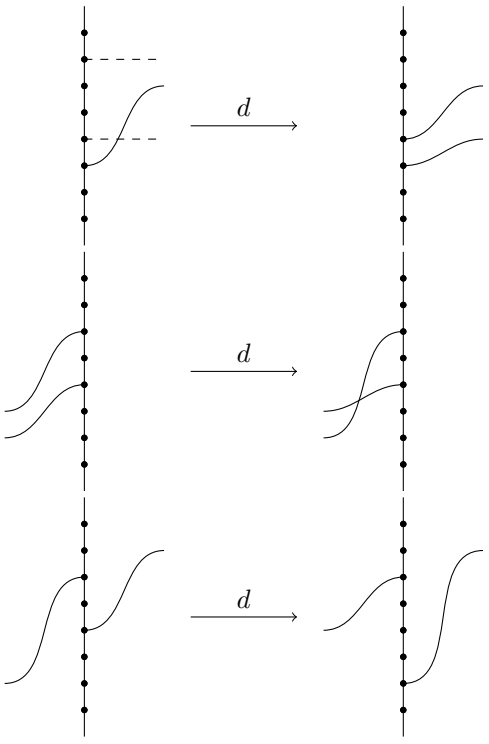


Figure 2.3: Examples of arrows in the chain complex M underlying $\mathcal{M}$.

2.3 Description of the homotopy

Let (M, d) be the chain complex underlying $\mathcal{M}$, computed in the previous section. There is a distinguished set of generators of M, consisting of $[a_1, a_2]$ where both a_1 and a_2 are idempotents (which are necessarily complementary). By abuse of notation we will also call these generators *idempotents*. It is clear that there are no arrows in the differential in or out of these generators. Let $(N, d' = 0)$ be the subcomplex of M generated by the idempotents. We will show that N is the homology of M, so that N and M are homotopy equivalent. There are obvious chain maps $f : N \to M$ and $g : M \to N$, where f is the inclusion map and g is the map sending the idempotents to themselves and all other generators to zero. It remains to find a homotopy $H : M \to M$ satisfying

$$d \circ H + H \circ d = \mathbb{I}_M + f \circ g.$$

We set $H(\mathbf{x}) = 0$ for all idempotents $\mathbf{x}$. Then the above equation holds trivially on idempotents. On the other generators, we have $f \circ g = 0$, so the equation reduces to $d \circ H + H \circ d = \mathbb{I}_M$.

We now describe H on the non-idempotents, and verify the required relation in the next section. From now on all generators that we mention are assumed to be a non-idempotent. We will call the arrows in the differential of M the d-arrows, and the arrows in the map H the H-arrows.

For a generator $[a_1, a_2]$ in C, we define the multiplicity $\mathrm{mult}([a_1, a_2]) \in H_1(Z \setminus z, \mathbf{a})$ to be the sum of multiplicities of a_1 and a_2. Note this is invariant under the differential. So the chain complex splits into disjoint parts according to multiplicity, and to specify the homotopy, it is sufficient to do so on each part. We say an element $[a_1, a_2]$ has multiplicity one if its multiplicity on each component of $Z \setminus \mathbf{a}$ is at most one. The construction of H for generators of multiplicity one is different from that for other generators.

2.3.1 Multiplicity one

We begin with generators of multiplicity one. Let S be the set of components of $Z \setminus \mathbf{a}$ not containing the basepoint z (that is, the generators of $H_1(Z \setminus z, \mathbf{a})$). There is an ordering $<_{\mathcal{Z}}$ on S, depending on the pointed matched circle $\mathcal{Z}$, given as follows. Attach handles to Z according to the matching on $\mathbf{a}$. Then, traverse the boundary of Z with handles attached, starting from the top (with Z oriented so that strands in $\mathcal{A}(\mathcal{Z})$ go upwards). Order S according to when we encounter each element. Note we will encounter each element of S exactly once before reaching the bottom due to the condition that must be satisfied for the matching on a pointed matched circle. An example is shown in Figure

2.4.

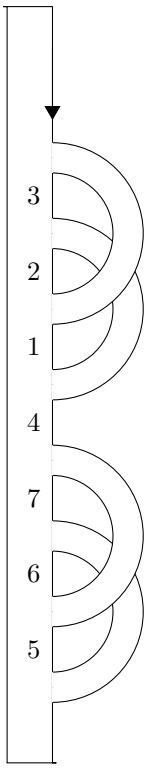


Figure 2.4: Example of traversal on the split pointed matched circle with genus 2. The numbers denote the ordering $<_{\mathcal{Z}}$ on S. Note strands in $\mathcal{A}(\mathcal{Z})$ go upwards.

Let $(p, p+1)$ be an element of S (with $p+1$ above p). The segment immediately before $(p, p+1)$ in the ordering $<_{\mathcal{Z}}$ can be found as follows. Let q be the point paired with $p+1$ in $\mathcal{Z}$. If q is the topmost point of $\mathbf{a} \subset Z$, then $(p, p+1)$ is the initial segment. Otherwise $(q, q+1)$ is the segment prior to $(p, p+1)$. See Figure 2.5 for a demonstration.

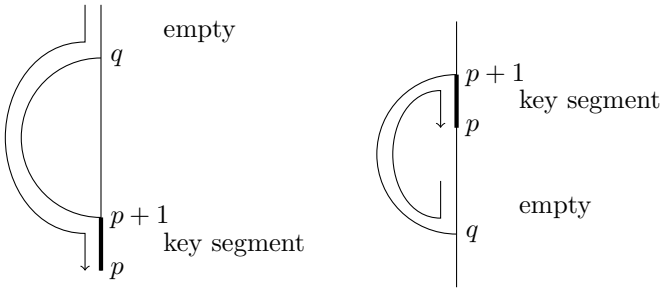


Figure 2.5: Identifying the segment prior to $(p, p+1)$. In the first case the segment $(q, q+1)$ may in fact be the component of Z containing the basepoint.

Given a generator $\mathbf{x} = [a_1, a_2]$ with multiplicity one, we define the *key segment* of $\mathbf{x}$ to be the first segment in S, according to the ordering $<_{\mathcal{Z}}$, at which $\text{mult}(\mathbf{x})$ is one. Let $(p, p+1)$ be the key segment, and q be the point paired with $p+1$. Define the pair $\{p+1, q\}$ to be the *key pair* of $\mathbf{x}$.

The main property that results from this construction is that while $\mathbf{x}$ has multiplicity 1 at $(p, p+1)$, it must have multiplicity zero at $(q, q+1)$ (one possibility is that q is the topmost point).

In the following, given a generator $[a_1, a_2]$, we say that the key segment is occupied on the left if it is covered by a_1, and on the right if it is covered by a_2. Since the left idempotents of a_1 and a_2 are complementary, one of them contains the key pair. We say the key pair is occupied on the left if it is contained in the left idempotent of a_1, and on the right if it is contained in the left idempotent of a_2. We also distinguish whether the algebra element realizes the key pair as a double-horizontal or as the start of a non-horizontal (moving) strand. Note in the latter case, since the multiplicity at $(q, q+1)$ is zero, the strand must start at $p+1$.

The above classification divides all (non-idempotent) generators into eight types. We will define $H(\mathbf{x})$ based on the type of $\mathbf{x}$.

In the following, let $a \to b$ denote the strand starting at a and ending at b (so we always have $a < b$). By *moving the strand $a \to b$ to the left*, we mean that starting from $[a_1, a_2]$, construct a new generator $[a_1', a_2']$, with the strands in a_2' obtained from that in a_2 by either removing the strand $a \to b$, or by factoring $a \to b$ from the end of a longer strand in a_2; and with the strands in a_1' obtained from that in a_1 by adding the strand $a \to b$. If rather than adding the strand $a \to b$, we wish to attach it to an existing strand in a_1, we will state it explicitly. Similarly we have the notion of moving the strand $a \to b$ to the right. Sometimes we will make two moves at the same time (with the intermediate state possibly not valid strand diagrams). The location of double horizontals on a_1' and a_2' is usually clear (noting that the left idempotents of a_1' and a_2' must be complementary). We will clarify it when it is ambiguous.

In all figures illustrating multiplicity one cases, we will use a dotted parenthesis to denote the key pair. We will also show q to be above rather than below $p+1$, so the key segment is just below the dotted parenthesis, but the definition of H is the same in both cases.

For exactly four types of $\mathbf{x}$, we have $H(\mathbf{x}) \neq 0$. The value of $H(\mathbf{x})$ in these four types are as follows (see Figure 2.6):

1. If the key segment is occupied on the left and the key pair is occupied on the left as a double-horizontal, then there is a strand $a \to b$ on the left with $a < p+1 < b$. Set H to resolve the crossing involving the horizontal strand at $p+1$.

2. If the key segment is occupied on the right and the key pair is occupied on the right by a moving strand, then there must be strands $i \to p+1$ and $p+1 \to j$ on the right. Set H to replace these two strands with the strand $i \to j$ and the double-horizontal at $\{p+1, q\}$.

If there is a strand ending at q on the left, then there is an additional term in H, moving the strand ending at q to the right and the strand $i \to p+1$ to the left.

3. If the key segment is occupied on the right by a strand $i \to j$, and the key pair is occupied on the left as a double-horizontal, then H factors the strand $i \to p+1$ from the right and moves it to the left.

4. If the key segment is occupied on the right, and the key pair is occupied on the left by a moving strand, then there must be a strand $p+1 \to j$ on the left and a strand $i \to p+1$ on the right. Set H to move the strand $i \to p+1$ to the left attaching it to the $p+1 \to j$ strand, and leaving a double horizontal at $\{p+1, q\}$ at right.

   There are two special cases: first, if there is a strand $j \to q$ with $j \neq p+1$ on the left, then H contains an additional term moving $j \to q$ to the right and $i \to p+1$ to the left.

   Second, if there is a strand $p+1 \to q$ on the left (possible only if $q > p+1$), then H contains an additional term moving $p+1 \to q$ to the right and $i \to p+1$ to the left.

The overall picture is as follows: we partition all generators of multiplicity one into ordered pairs, such that for each ordered pair $(\mathbf{x_i}, \mathbf{y_i})$, there is a d-arrow $\mathbf{x_i} \to \mathbf{y_i}$. The part of H not including the three special cases maps each $\mathbf{y_i}$ to $\mathbf{x_i}$ and $\mathbf{x_i}$ to zero. In verifying the relation $d \circ H + H \circ d = \mathbb{I}_M$, the compositions $d \circ H : \mathbf{y_i} \to \mathbf{x_i} \to \mathbf{y_i}$ and $H \circ d : \mathbf{x_i} \to \mathbf{y_i} \to \mathbf{x_i}$ account for the identity. If $(\mathbf{x_i}, \mathbf{y_i})$ is an ordered pair, we say $\mathbf{x_i}$ is on the *d-side* and $\mathbf{y_i}$ is on the *H-side*. The three special cases in H are additional maps from the H-side to the d-side (that is, mapping $\mathbf{y_i}$ to $\mathbf{x_j}$ for some $i \neq j$). Intuitively, the part of H mapping each $\mathbf{y_i}$ to $\mathbf{x_i}$ performs the inverse of a d-arrow around $p+1$. The special arrows perform a different move around $p+1$, along with moving a strand ending at q to the right.

Whether a generator is on the d-side or the H-side depends solely on the type of the generator. The eight types are summarized in Table 2.1. The numbers after d or H indicate under which case they will be covered in the proof in the next section, and if labeled H, also the case of H in the above description.

key seg. \ key pair	left, double-hor.	right, double-hor.	left, moving	right, moving
left	H-1	d-7	d-5	d-8
right	H-3	d-6	H-4	H-2

Table 2.1: Summary of cases for multiplicity one generators.

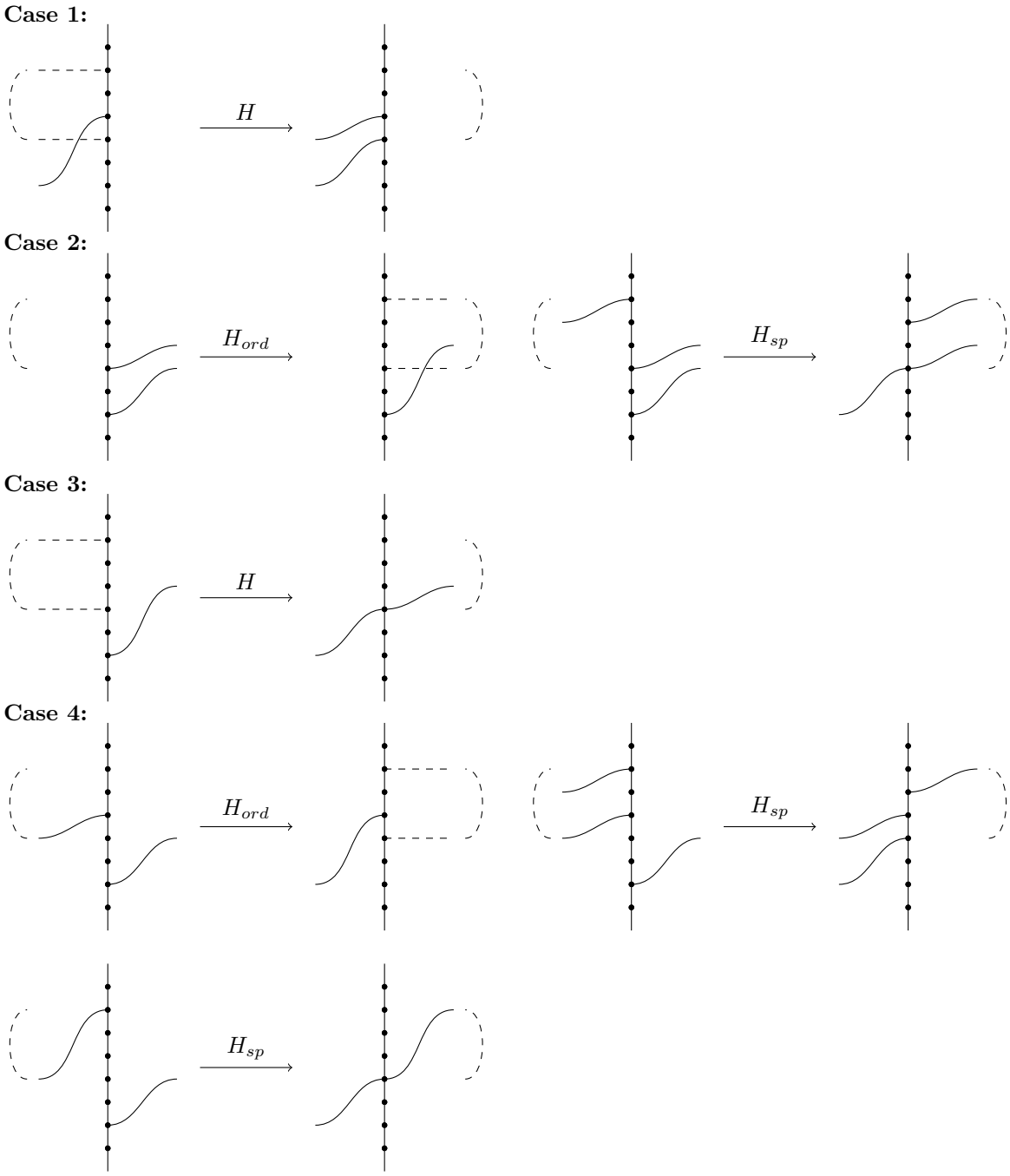
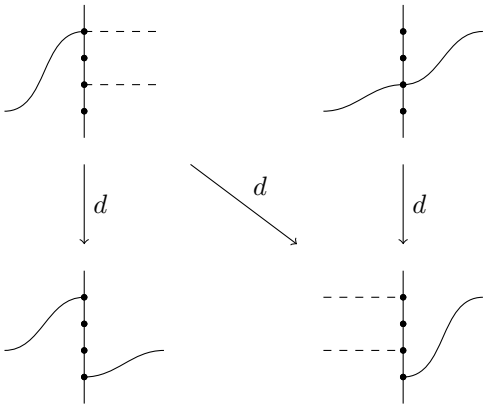


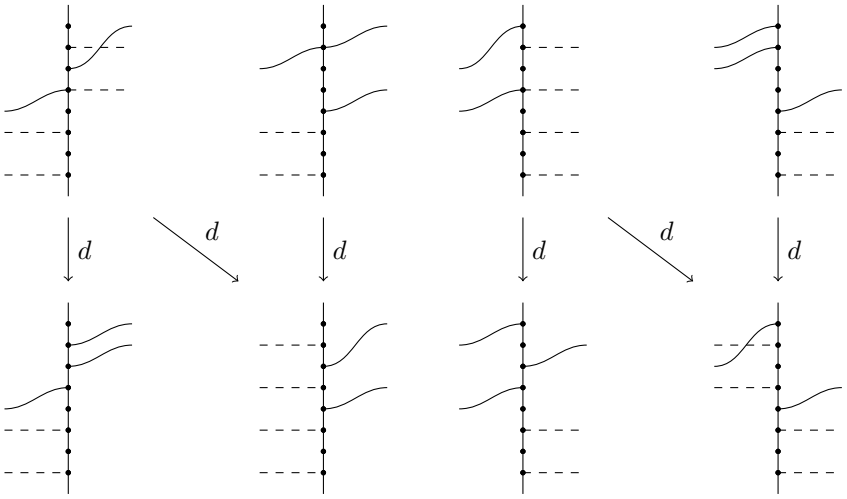
Figure 2.6: Diagrams for the homotopy map in multiplicity-one cases.

It would be simpler if there were no special cases in H. That is, if the only H-arrows were those from the H-side to the d-side of the same pair. However, this is impossible as demonstrated in Figures 2.7 and 2.8. Each figure shows four generators $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ such that $d\mathbf{x} = \mathbf{y} + \mathbf{w}$ and $d\mathbf{z} = \mathbf{w}$, with no other d-arrows involving these generators. This forces the homotopy to be $H\mathbf{y} = \mathbf{x} + \mathbf{z}$ and $H\mathbf{w} = \mathbf{z}$.



Case 4, second special H

Figure 2.7: Examples of why special cases of homotopy are needed, part 1.



Case 2, special H Case 4, first special H

Figure 2.8: Examples of why special cases of homotopy are needed, part 2.

2.3.2 Other generators

We now consider generators that do not satisfy the multiplicity one condition. Let $\mathbf{x} = [a_1, a_2]$ be such a generator. This means $\text{mult}(\mathbf{x})$ is greater than 1 at some segment in S. Let $[i, i+1]$ be the bottom-most segment with multiplicity greater than 1 (note we are no longer using the ordering $<_{\mathcal{Z}}$ on S). Since two strands cannot start from the same point i, this segment must have multiplicity 2, and the segment $[i-1, i]$ must have multiplicity 1. There is a unique strand starting at i (call it the i strand) and a unique strand that covers $[i-1, i]$. Let j be the starting point of the strand covering $[i-1, i]$, and call that strand the j strand. Note i, but not j, is unchanged in any d-arrow.

The definition of H consists of the following cases (see Figure 2.9):

1. If both the i strand and the j strand are on the left, and they cross each other, the H-arrow

uncrosses the two strands.

2. If both the i strand and the j strand are on the right, and they do not cross, the H-arrow crosses the two strands.

3. If the i strand is on the left and the j strand is on the right, the H-arrow factors the strand $j \to i$ from the right and moves it to the left attaching to the i strand.

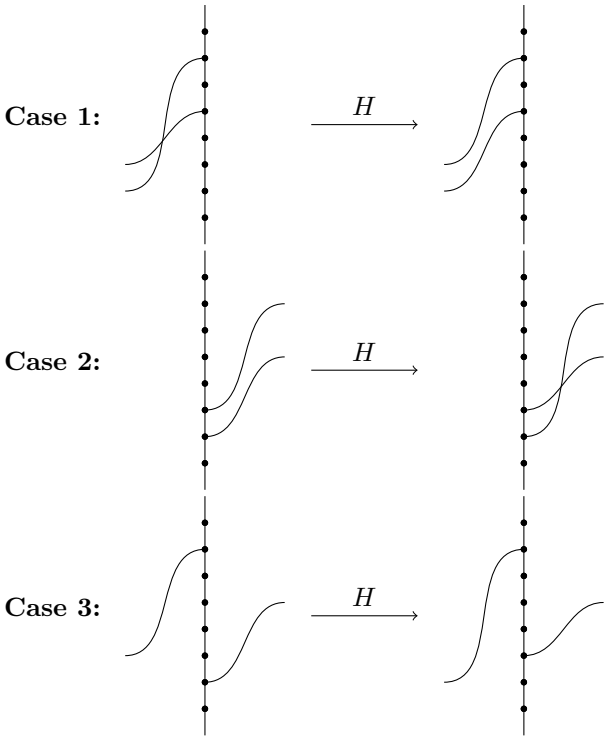


Figure 2.9: Diagrams for the homotopy map in the non-multiplicity-one cases.

On all other generators H is zero. The cases are summarized in Table 2.2. There is again a pairing of generators $(\mathbf{x_i}, \mathbf{y_i})$, so that for each pair there is a d-arrow $\mathbf{x_i} \to \mathbf{y_i}$. In this case, there are no special cases in H, so H is exactly the map sending $\mathbf{y_i}$ to $\mathbf{x_i}$ and $\mathbf{x_i}$ to 0 in each pair $(\mathbf{x_i}, \mathbf{y_i})$.

i strand \ j strand	left	right
left	(d,H)-1	H-3
right	d-4	(d,H)-2

Table 2.2: Summary of cases for generators not of multiplicity one.

2.4 Verification of the homotopy

2.4.1 Multiplicity one

In this section we verify that the map H defined above satisfies the equation $d \circ H + H \circ d = \mathbb{I}_M$ on the non-idempotent part of M, beginning with the multiplicity one case. We do so by checking the equation

$$d \circ H + H \circ d = \mathbb{I}_M \tag{2.4}$$

on each of the eight types of generators. Depending on the type, the generator appears on one side of some pair $(\mathbf{x_i}, \mathbf{y_i})$. For every such pair, there is a d-arrow $\mathbf{x_i} \to \mathbf{y_i}$ and an H-arrow $\mathbf{y_i} \to \mathbf{x_i}$. So there is a natural identity term in $d \circ H + H \circ d$, and it remains to check that all other terms sum to zero.

Write $H = H_{ord} + H_{sp}$, where H_{sp} contains arrows coming from the three special cases of H, and H_{ord} is the remaining part of H. So for each pair $(\mathbf{x_i}, \mathbf{y_i})$ we have $H_{ord}\mathbf{y_i} = \mathbf{x_i}$. It is also not difficult to determine $\mathbf{y_i}$ from $\mathbf{x_i}$ - simply take the obvious differential around $p+1$. We call a generator *special* if it appears on either side of a pair $(\mathbf{x_i}, \mathbf{y_i})$ such that $H_{sp}\mathbf{y_i} \neq 0$. One easily checks that a generator is special if and only if there is a strand ending at q on the left. Generators of type H-1, H-3, D-5 and D-8 can never be special (the first two due to double horizontal on the left, the last two due to a strand ending at $p+1$ on the left).

The strategy for verifying (2.4) is different for generators on the H side and the d side, so we will describe them separately. For a generator $\mathbf{y}$ on the H-side, we define three sets of generators of M. Let G_1 be the set of terms in $d\mathbf{y}$ that are on the H side, G_2 be the set of all terms in $d(H_{ord}\mathbf{y})$ excluding $\mathbf{y}$, and G_3 be the set of all terms in $d(H_{sp}\mathbf{y})$. For a non-special $\mathbf{y}$, G_3 is of course empty. We will show that in this case all generators in G_1 are also non-special, and that H_{ord} induces a bijection between G_1 and G_2. For special $\mathbf{y}$ we will show the following:

- There is a generator $\mathbf{y_o}$ common to both G_2 and G_3.
- All generators in G_1 are also special.
- The map H_{ord} induces a bijection between G_1 and $G_2 \setminus \{\mathbf{y_o}\}$.
- The map H_{sp} induces a bijection between G_1 and $G_3 \setminus \{\mathbf{y_o}\}$.

These imply that the terms in $d \circ H + H \circ d$ other than the natural identity term sum to zero.

A large part of the bijection can be considered "trivial", since they involve d-arrows that do not modify strands around $p+1$ or q. They can be carried out in the "same" way on either side of an

H-arrow. So a term in $d\mathbf{y}$ coming from a d-arrow away from $p+1$ and q corresponds to the term obtained by performing the "same" move in $d(H_{ord}\mathbf{y})$ or $d(H_{sp}\mathbf{y})$. Since H-arrows can introduce double horizontals at $\{p+1, q\}$ on the right side, it can affect whether a strand ending at q on the left can be moved to the right in a d-arrow. This is essentially the reason why special cases are needed.

The main checks are therefore for differentials involving strands around $p+1$ and q. The cases are described below, and the reader is advised to follow along using the illustrations in the appendix. In each cancellation diagram in the appendix, the top-left generator is $\mathbf{y}$, the top-right generator is either $H_{ord}\mathbf{y}$ or $H_{sp}\mathbf{y}$ (as indicated on the top edge). The bottom-left generator is an element in G_1, and the bottom-right generator is the corresponding element in G_2 or G_3. For generators that are special, there is one more diagram showing the cancellation involving $\mathbf{y_o}$.

We now turn to the case of a generator $\mathbf{x}$ on the d-side. The first term in $d \circ H + H \circ d$ is zero. There is a distinguished term $\mathbf{y}$ in $d\mathbf{x}$ - the generator paired with $\mathbf{x}$. If $\mathbf{x}$ is non-special, then $\mathbf{y}$ is also non-special, and we show that this is the only term in $d\mathbf{x}$ on the H side. If $\mathbf{x}$ is special, then $\mathbf{y}$ is also special, and we show there is exactly one other, non-special, term $\mathbf{y}'$ in $d\mathbf{x}$ on the H side, and that $H_{sp}\mathbf{y} = H\mathbf{y}'$. This verifies the equation for $\mathbf{x}$. In the appendix, we will show a diagram for each type of the last cancellation, with $\mathbf{x}$ on the top-left, $\mathbf{y}$ on the bottom-left, $\mathbf{y}'$ on the top-right, and $H_{sp}\mathbf{y} = H\mathbf{y}'$ on the bottom-right.

We will assume throughout that in the key pair $\{p+1, q\}$ the point q is above $p+1$. The reader may check that in the other case the situation is the same or simpler.

Case 1

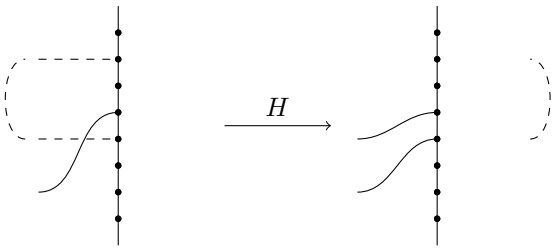


Figure 2.10: Homotopy, Case 1

The H-arrow is shown in Figure 2.10. There are no special generators. d-arrows starting at $\mathbf{y}$ involving the one strand around $p+1$ include:

- Merging with a strand above (case 1.1)

- Merging with a strand below (case 1.2)
- Moving a part not containing $p+1$ to the right (case 1.3)
- Moving a part containing $p+1$ to the right (case 1.4).

d-arrows starting at $H_{ord}\mathbf{x}$ involving the two strands shown include:

- Merging upper strand with a strand above (case 1.1)
- Merging lower strand with a strand below (case 1.2)
- Moving a part of the lower strand (case 1.3)
- Moving a part of the upper strand (case 1.4).

Crossing the two strands gets back to $\mathbf{y}$.

Case 2

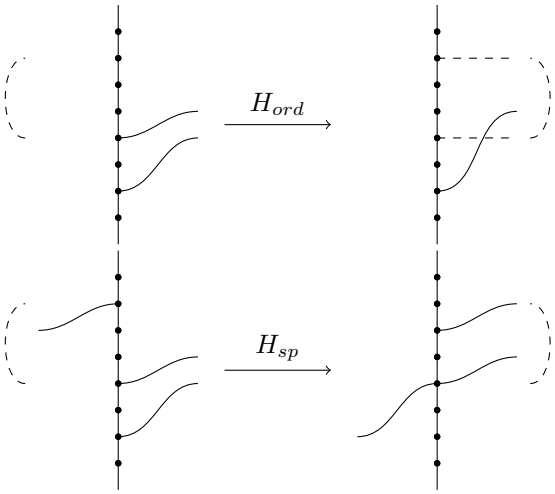


Figure 2.11: Homotopy, Case 2

The H-arrow is shown in Figure 2.11. First suppose the generator $\mathbf{y}$ is non-special (that is, no strands of $\mathbf{y}$ end at q). d-arrows starting at $\mathbf{y}$ involving the two strands shown include:

- Splitting the lower strand (case 2.1)
- Splitting the upper strand (case 2.2)
- Moving a strand from left to attach to lower strand (case 2.3).

d-arrows starting at $H_{ord}\mathbf{y}$ involving the single strand shown include:

- Splitting the single strand below $p+1$ (case 2.1)
- Splitting the single strand above $p+1$ (case 2.2)
- Moving a strand from left to attach to the single strand (case 2.3).

Uncrossing the single strand gets back to $\mathbf{y}$.

Now suppose $\mathbf{y}$ is special. There is one more d-arrow starting at $H_{ord}\mathbf{y}$. This is because the homotopy produces a double-horizontal at $\{p+1, q\}$, so a strand ending at q on the left can now move to the right. This term cancels against a special d-arrow starting at $H_{sp}\mathbf{y}$, as shown in case 2.4.

For the other cases, there is one more relevant strand - the strand ending at q. d-arrows starting at $\mathbf{y}$ involving this strand include:

- Merging with a strand below (case 2.5)
- Moving part of the strand to the right (case 2.6).

d-arrows starting at $H_{ord}\mathbf{y}$ involving this strand are exactly the same. The cancellations are shown in cases 2.5a and 2.6a. d-arrows starting at $H_{sp}\mathbf{y}$ involving this strand include:

- Attaching with a strand moved from the left (case 2.5)
- Splitting the strand (case 2.6).

The cancellations are shown in cases 2.5b and 2.6b. d-arrows starting at $H_{sp}\mathbf{y}$ involving the two strands around $p+1$ include:

- Moving part of the lower strand to the right (case 2.7b)
- Splitting the upper strand (case 2.8b)
- Mergine the lower strand with a strand below (case 2.9b).

These cancel against d-arrows starting at $\mathbf{y}$ in cases 2.1, 2.2, and 2.3, respectively.

Case 3

The H-arrow is shown in Figure 2.12. Note if the single strand on the H side ends exactly at $p+1$, then a double horizontal is produced on the right side (however, since no strand can end at q on the left, there are no special cases). d-arrows starting at $\mathbf{y}$ involving the single strand include:

- Splitting at a point above $p+1$ (case 3.1)

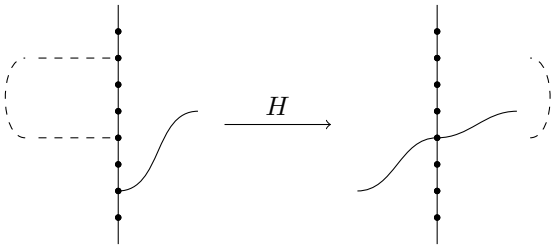


Figure 2.12: Homotopy, Case 3

- Splitting at a point below $p+1$ (case 3.2)
- Attaching with a strand moved from the left (case 3.3).

d-arrows starting at $H_{ord}\mathbf{y}$ include:

- Splitting the right strand (case 3.1)
- Moving part of the left strand to the right (case 3.2)
- Mergine the left strand with a strand below (case 3.3).

Moving all of the left strand to the right gets back to $\mathbf{y}$.

Case 4

The H-arrow is shown in Figure 2.13. This is the most complicated case, with two different special cases. Note the conditions for the two special cases are mutually exclusive, so there can be at most two terms in $H\mathbf{y}$.

We first consider the case where $\mathbf{y}$ is non-special. d-arrows starting at $\mathbf{y}$ involving the two strands shown include:

- Splitting the right strand (case 4.1)
- Attaching the right strand with a strand moved from the left (case 4.2)
- Moving part of left strand to the right (case 4.3)
- Merging the left strand with a strand above (case 4.4).

d-arrows starting at $H_{ord}\mathbf{y}$ involving the single strand shown include:

- Moving to the right a part not containing $p+1$ (case 4.1)
- Moving to the right a part strictly containing $p+1$ (case 4.3)

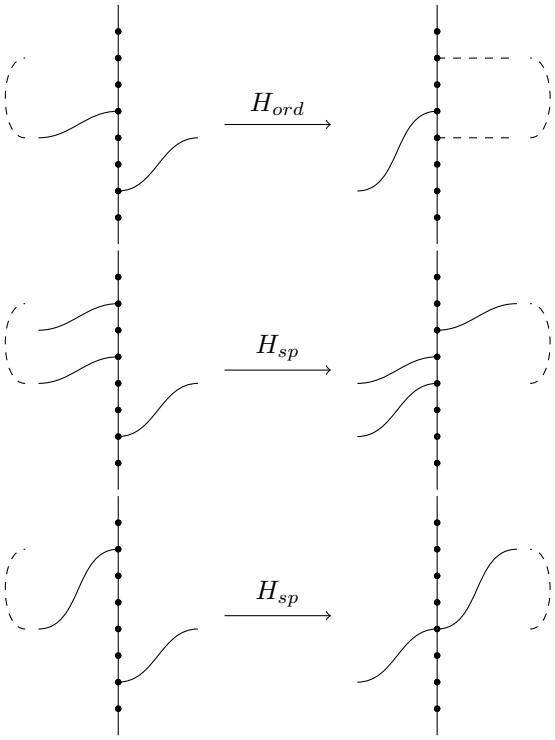


Figure 2.13: Homotopy, Case 4

- Merging with a strand below (case 4.2)
- Merging with a strand above (case 4.4).

Moving a part containing $p+1$ on the boundary gets back to $\mathbf{y}$.

Now suppose $\mathbf{y}$ is special. Since H creates a double horizontal at $\{p+1, q\}$, there is one more d-arrow starting at $H_{ord}\mathbf{y}$, moving the strand ending at q to the right. This strand may or may not extend to $p+1$, giving the two special cases. This cancels against a special d-arrow starting at $H_{sp}\mathbf{y}$ in both cases, as shown in cases 4.5 and 4.6.

For the other cases, there is again one more relevant strand, ending at q. d-arrows starting at $\mathbf{y}$ involving this strand include:

- Merging with a strand below, resulting in a strand starting above $p+1$ (case 4.7)
- Moving part of the strand to the right (case 4.8)
- Merging with a strand below, resulting in the strand $p+1 \to q$ (case 4.9)

d-arrows starting at $H_{ord}\mathbf{y}$ are exactly the same for cases 4.7 and 4.8. For case 4.9 the strand ending at q is merged with a longer strand (shown in case 4.9a). d-arrows starting at $H_{sp}\mathbf{y}$ involving this strand include:

- Attaching a strand moved from the left, resulting in a strand starting above $p+1$ (case 4.7b)
- Splitting the strand (case 4.8b)
- Attaching a strand moved from the left, resulting in the strand $p+1 \to q$ (case 4.9b).

Finally, we consider d-arrows starting at $H_{sp}\mathbf{y}$ involving the two strands around $p+1$. They include:

- Moving part of the lower strand to the right (case 4.10b,c)
- Joining the lower strand with a strand below (case 4.11b,c)
- Moving part of the upper strand to the right (first special case, case 4.12b)
- Splitting the upper strand (second special case, case 4.12c)
- Joining the upper strand with a strand above (first special case, case 4.13b).

These cancel against d-arrows starting at $\mathbf{y}$ in cases 4.1 through 4.4.

Case 5

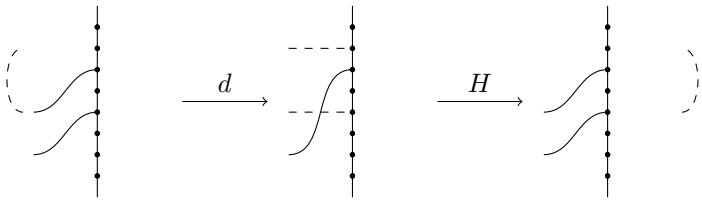


Figure 2.14: Identity term in Case 5.

In this and the next three cases we will consider a generator $\mathbf{x}$ on the d side. There are no special generators in this case (due to a strand ending at $p+1$ on the left). The only d-arrow to the H side, giving the identity term, is shown in Figure 2.14.

The only other d-arrow that changes the type of the generator is shown in Figure 2.15, but we are still on the d side.

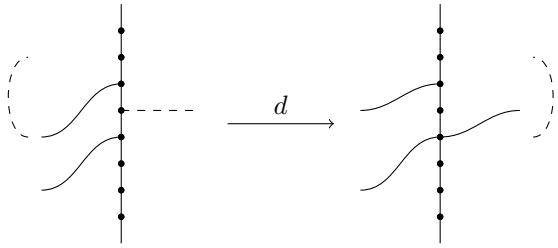


Figure 2.15: A d-arrow that moves to a different type, but still on the d side.

Case 6

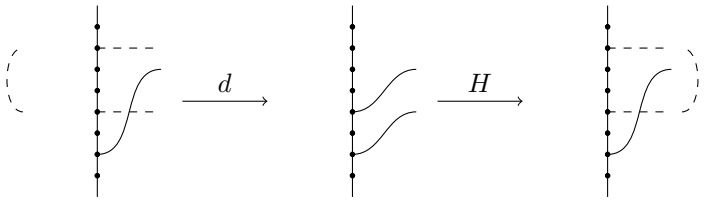


Figure 2.16: Identity term in Case 6.

If $\mathbf{x}$ is not special, Figure 2.16 shows the only d-arrow to the H side. If $\mathbf{x}$ is special - that is, if there is a strand ending at q on the left, then the d-arrow moving that strand to the right will move the double horizontal to the left, making a generator on the H side. This corresponds to the special case in case 2 (Case 6.1).

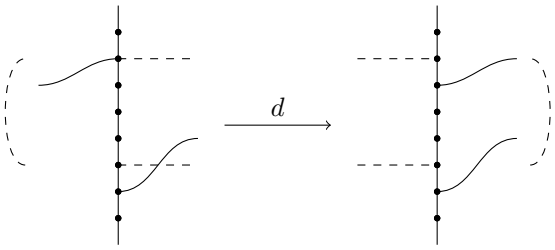


Figure 2.17: A d-arrow that maps to the H side in case 6, forcing a special case.

Case 7

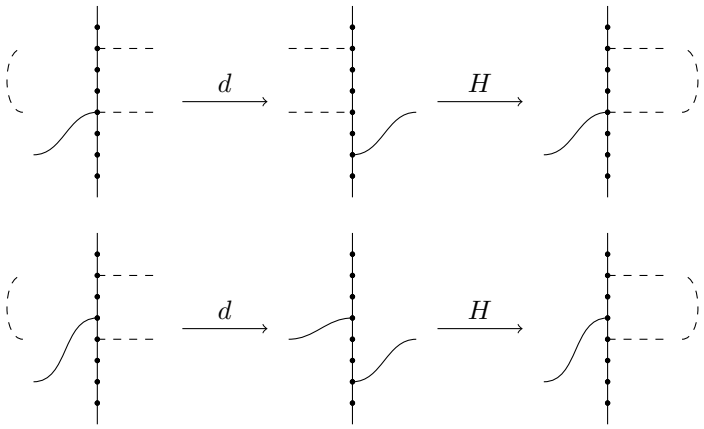


Figure 2.18: Identity term in Case 7.

Figure 2.18 shows the obvious d-arrow to the H side. If there is a strand ending at q (only possible in the second of the two cases in Figure 2.18), this strand can be moved to the right, moving the

double horizontal to the left and changing the type of the generator. There are two ways this can happen, as shown in Figure 2.19. They correspond to the special cases in case 4 (Case 7.1 and 7.2).

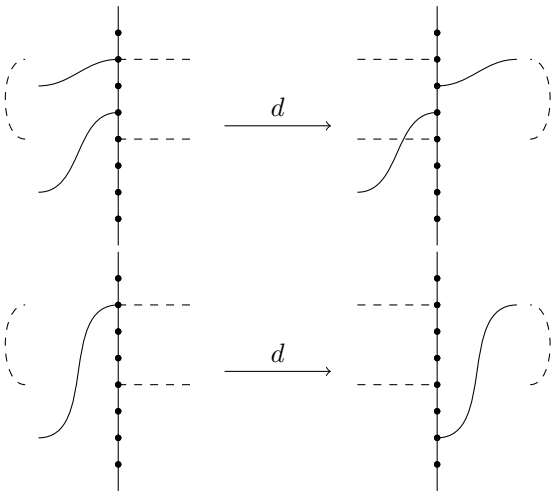


Figure 2.19: d-arrows that map to the H side in case 7, forcing two special cases.

Case 8

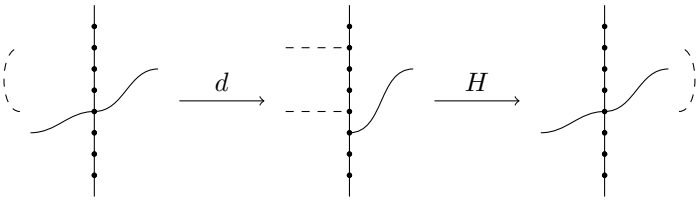


Figure 2.20: Identity term in Case 8.

The only d-arrow to the H side is shown in Figure 2.20. There are no special generators.

This concludes the check for generators of multiplicity one.

2.4.2 Other generators

We now consider generators that do not have multiplicity one. They can be classified into four types as in Table 2.2. We will again check each type separately. The strategy is even simpler than before, as there are no special cases. For a generator on the H side, we check that there is a bijection between the corresponding sets G_1 and G_2. For a generator on the d side, we check that there is exactly one differential to the H side, carrying it to the generator in the same pair.

The checks for cases 1 and 2 in the definition of H have a common ingredient – checking those terms where both the d-arrow and the H-arrow are restricted to the same side. For case 2, the

argument is the same as the one used to show $H_*(\mathcal{A}(\mathcal{Z})) = 0$ on generators with multiplicity greater than one. This is given in [14, Section 4]. While the proof there uses a slightly different homotopy, the argument easily adapts to the homotopy we give here. For case 1 we use the dual of that argument. With these cases covered, we will only consider terms in $d \circ H + H \circ d$ that involve both sides.

Case 1

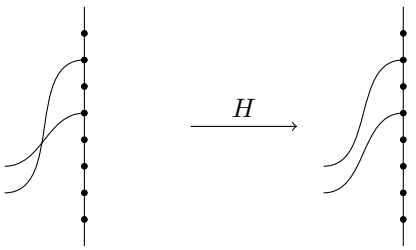


Figure 2.21: Homotopy in Case 1.

In this case both the i strand and the j strand are on the left. The H-arrow is shown in Figure 2.21.

We begin with a generator $\mathbf{y}$ on the H side. d-arrows starting at $\mathbf{y}$ include:

- Moving a part of the j strand to the right. The end of that part can be:

  Lower than i (case 9.1)

  Higher than the endpoint of the i strand (case 9.2)

  Coinciding with the endpoint of the i strand (case 9.2')

  In between i and the endpoint of the i strand (case 9.3)

Note that if part of the shorter strand is moved, we remain on the d side (Figure 2.22). As stated in the beginning of this section, we may ignore d-arrows within the left side.

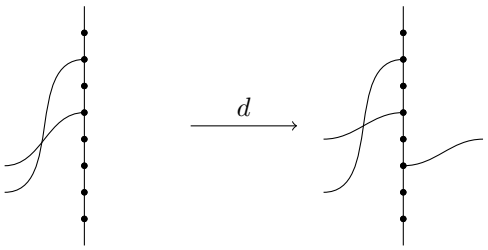


Figure 2.22: A term in the differential that remains on the d side.

d-arrows starting at $H\mathbf{y}$ include:

- Moving part of the j strand not containing i (case 9.1)
- Moving part of the i strand. The end of that part can be:

  Above the endpoint of the j strand (case 9.2)

  Coinciding with the endpoint of the j strand (case 9.2')

  Below the endpoint of the j strand (case 9.3).

Note it is impossible to move part of lower strand containing j, due to the multiplication rule that double crossing gives zero.

Now consider a generator $\mathbf{x}$ on the d side. Crossing the two strands shown is the only way to go to a generator of type 1 on the H side (note we already covered the case where the d-arrow happens within the left side). It is impossible to reach type 2 (both strands on the right). To reach type 3, we need to arrange that the new j strand is at right. This means we will need to move a part of the current j strand containing i to the right. However, this will produce a double crossing.

Case 2

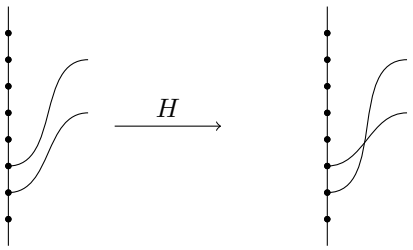


Figure 2.23: Homotopy in Case 2.

The H-arrow is shown in Figure 2.23.

For generators on the H side, the d-arrows that involve only the right side are again covered by the computation of $H_*(A(\mathcal{Z}))$. There is only one other case that involves the displayed strands: moving a strand from the left to attach to the j strand (for both $d\mathbf{y}$ and $d(H\mathbf{y})$). This is shown in case 10.1. Note the interval $[i, j]$ cannot be occupied on the left side.

For generators on the d side, there is no way to reach the H side other than by d-arrows involving only the right side.

Case 3

The H-arrows is shown in Figure 2.24. All generators of this type are on the H side.

d-arrows starting at $\mathbf{y}$ include:

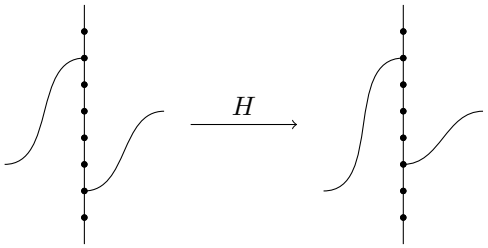


Figure 2.24: Homotopy in Case 3.

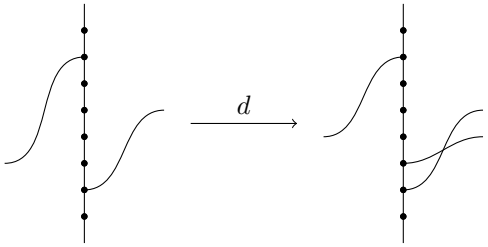


Figure 2.25: A term in the differential that remains on the d side.

- Merging the i strand with a strand above (case 11.1)
- Splitting the j strand below i (case 11.2)
- Splitting the j strand above i (case 11.3)
- Attaching the j strand with a strand moved from left (case 11.4)
- Move part of the i strand to the right, so that the new strand on the right does not cross the j strand (case 11.5).

If the new strand on the right does cross the i strand, we remain on the d side (Figure 2.25).

d-arrows starting at $H\mathbf{y}$ include:

- Merging the j strand with a strand above (case 11.1)
- Merging the j strand with a strand below (case 11.4)
- Splitting the i strand (case 11.3)
- Moving part of j strand not containing i to the right (case 11.2)
- Moving part of j strand to the right, so that the new strand on the right crosses the i strand (case 11.5).

Note if the new strand on the right interleaves, and does not cross the i strand, the term is zero by double crossing (Figure 2.26).

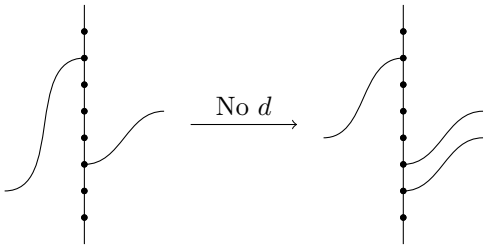


Figure 2.26: An example of a term that is not in the differential of $H\mathbf{x}$ due to double crossing.

Case 4

Beside the obvious d-arrow to the H side in the same pair, the only other possible way to reach the H side is by moving part of the j strand containing i to the right. However, this is impossible due to double crossing (same example as in Figure 2.26).

This concludes the check for generators with multiplicity greater than one, and the verification that H is indeed a homotopy.

2.5 A rank-1 model of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$

Having found a homotopy H on the chain complex underlying the A_∞-bimodule $\mathcal{M}$, we can use homological perturbation theory to describe the smaller bimodule $\mathcal{N}$. We refer the reader to [16, Section 8.2] on how to construct an A_∞-bimodule action using homological perturbation theory. The result is summarized in the following theorem.

Theorem 6. *The A_∞-bimodule $\mathcal{N}_{\mathcal{A}',\mathcal{A}}$, described below, is homotopy equivalent to the bimodule $\mathcal{M}$ given in (2.1), and is therefore a model of the invariant $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}',\mathcal{A}}$.*

The vector space N underlying $\mathcal{N}$ is generated over $\mathbb{F}_2$ by the indecomposable idempotents of $\mathcal{A}$ (using the definitions in section 1.2, $\mathcal{N}$ is a rank-1 bimodule). The arrows in the A_∞-bimodule action correspond one-to-one with sequences $[a_{1,1}, a_{1,2}], \dots, [a_{2n,1}, a_{2n,2}]$ of generators of $\mathcal{A}$, that satisfies the following three conditions:

1. *$[a_{1,1}, a_{1,2}] = [i', i]$ and $[a_{2n,1}, a_{2n,2}] = [j', j]$ for some idempotents $i, j \in A$, and $i' = o(i)$, $j' = o(j)$.*

2. *Each $[a_{2k,1}, a_{2k,2}]$ is obtained from $[a_{2k-1,1}, a_{2k-1,2}]$ by either factoring out some $\overline{b'} \in \mathcal{A}$ from $a_{2k-1,1}$ on the right (so that $a_{2k-1,1} = a_{2k,1}\overline{b'}$ and $a_{2k-1,2} = a_{2k,2}$), or by multiplying $a_{2k-1,2}$ with some $b \in \mathcal{A}$ on the right (so that $a_{2k,2} = a_{2k-1,2}b$ and $a_{2k,1} = a_{2k-1,1}$). The elements b' and b are not necessarily equal between steps.*

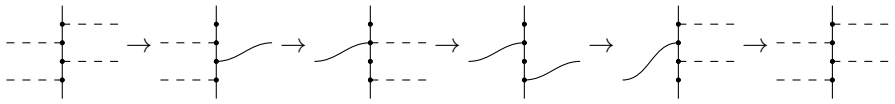
3. *Each* $[a_{2k+1,1}, a_{2k+1,2}]$ *is obtained from* $[a_{2k,1}, a_{2k,2}]$ *by one of the* H *arrows described in section 2.3.*

Let $b'_1, \ldots, b'_p$ *be the sequence of* $b' \in \mathcal{A}'$ *and* $b_1, \ldots, b_q$ *be the sequence of* $b \in \mathcal{A}$ *used in condition 2. The arrow corresponding to this sequence is:*

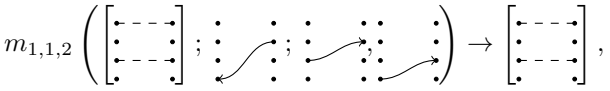
$$m_{1,p,q}([i]; b'_1, \ldots, b'_p; b_1, \ldots, b_q) \to [j].$$

Here $m_{1,p,q}$ *is the part of the* A_∞*-bimodule action on* $\mathcal{N}$ *with* p *inputs on the* $\mathcal{A}'$ *side and* q *inputs on the* $\mathcal{A}$ *side. The idempotent of a generator* $[i]$ *of this* A_∞*-bimodule is* i *on the* $\mathcal{A}$ *side and* $\overline{o(i)}$ *on the* $\mathcal{A}'$ *side.*

For example, let $\mathcal{Z}$ be the genus 1 pointed matched circle. The following sequence of $[a_{i,1}, a_{i,2}]$



gives raise to the following arrow in the action:



where $[i]$ for an idempotent $i \in \mathcal{A}$ denotes the corresponding generator of $\mathcal{N}$.

Since $\mathcal{A}'$ and $\mathcal{A}$ are opposite algebras, we can also consider $\mathcal{N}$ as a left-right A_∞-bimodule ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{A}}$. We will use this form of $\mathcal{N}$ in the next section.

The bimodule $\mathcal{N}$ inherits a relative grading from the larger model $\mathcal{M}$. Since the generators in $\mathcal{N}$ are those in $\mathcal{M}$ composed of idempotents, we may choose the relative grading so that the grading of all generators in $\mathcal{N}$ are zero (this is also the grading one would obtain from the standard Heegaard diagram for the identity diffeomorphism, starting with a zero grading for any one of the generators).

We now prove the main results of this chapter:

Proof of Theorem 4. Let

$$ {}^{\mathcal{A}}\mathcal{L}_{\mathcal{A}} = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}',\mathcal{A}} \boxtimes {}^{\mathcal{A},\mathcal{A}'}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}) \simeq \mathcal{N}_{\mathcal{A}',\mathcal{A}} \boxtimes {}^{\mathcal{A},\mathcal{A}'}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}). \tag{2.5}$$

Since the vector spaces underlying both $\mathcal{N}$ and $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ are generated by idempotents, the vector space underlying $\mathcal{L}$ is also generated by idempotents. So $\mathcal{L}$ is a rank-1 bimodule. Furthermore, the type DA-bimodule action in $\mathcal{L}$ satisfies $\delta_1^1 = 0$ since all arrows in $\mathcal{N}$ involve non-idempotent algebra

inputs from both $\mathcal{A}'$ and $\mathcal{A}$. Hence Lemma 2.2.50 in [14] applies, showing ${}^{\mathcal{A}}\mathcal{L}_{\mathcal{A}} = {}^{\mathcal{A}}[\phi]_{\mathcal{A}}$ for some A_∞-algebra morphism $\phi : \mathcal{A} \to \mathcal{A}$.

From the grading on $\mathcal{N}$ and $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$, there is a relative grading on $\mathcal{L}$ with all generators having grading zero. This implies that the A_∞-morphism ϕ respects the $G(\mathcal{Z})$-grading on $\mathcal{A}$. By classifying arrows in $\mathcal{N}$ involving only one length-1 chord on each side, it is clear that $\phi_1(\xi) = \xi$ for any length-1 chord ξ. By Proposition 4.11 in [14], we conclude that ϕ_1 induces the identity map on homology. So ϕ is a quasi-isomorphism and $\mathcal{L}$ is quasi-invertible. □

Corollary 7. *$\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ is quasi-invertible, hence the functor $\cdot \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ induces an equivalence of categories between the category of right A_∞-modules over $\mathcal{A}(\mathcal{Z})$, and the category of left type D structures over $\mathcal{A}(-\mathcal{Z})$.*

Proof. With $\mathcal{L}$ as in (2.5), there is $\mathcal{L}'$ such that

$$ {}^{\mathcal{A}}\mathbb{I}_{\mathcal{A}} \simeq \mathcal{L}' \boxtimes \mathcal{L} \simeq \mathcal{L}' \boxtimes \mathcal{N} \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}), $$

which means $\mathcal{L}' \boxtimes \mathcal{N}$ is the quasi-inverse to $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$. □

Remark: The stronger statement, that $\mathcal{N}$ is the quasi-inverse of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$, will be proved combinatorially in the next chapter.

2.6 Examples of Koszul duality

In this section, we use our description of $\mathcal{N} \simeq \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ to give an explicit A_∞-morphism from $\mathcal{A} = \mathcal{A}(\mathcal{Z})$ to $\mathrm{Cob}(\mathcal{A})$, inducing an isomorphism on homology. We will assume in this section that $\mathcal{N}$ is in fact the quasi-inverse of $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$. First, we review some material from [14] and [18, Section 8].

For any augmented dg-algebra A, there is a type DD bimodule ${}^{\mathrm{Cob}(A)}K^{A}$, of rank 1 over $\mathbf{k}$, and with type DD action given by:

$$ \delta^1(\mathbf{1}) = \sum_i a_i^* \otimes \mathbf{1} \otimes a_i, $$

where the sum is over a set of generators of A_+ (the sides of the action are reversed in comparison to [18], for ease of computation later). One may check that this satisfies the structure equation for type DD bimodules. The fact that this bimodule is always quasi-invertible shows that A is Koszul dual to $\mathrm{Cob}(A)$ (see [18, Proposition 8.12]).

The following is contained in the proof of [18, Proposition 8.11]:

Proposition 8. *Let ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{A}}$ be a rank-1 representative of ${}_{\mathcal{A}}\widehat{CFA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}}$, then*

$$ {}^{\mathrm{Cob}(\mathcal{A})}K^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{N}_{\mathcal{A}}$$

is of the form ${}^{\mathrm{Cob}(\mathcal{A})}[\phi]_{\mathcal{A}}$, where ϕ is an A_∞-morphism $\mathcal{A} \to \mathrm{Cob}(\mathcal{A})$ that induces isomorphism on homology.

In particular ϕ_1 (the part of ϕ taking one input) maps representatives of homology classes of $\mathcal{A}$ to representatives of homology classes of $\mathrm{Cob}(\mathcal{A})$. Unwinding all the definitions, we obtain the following rule for computing this map:

Proposition 9. *Let $a \in \mathcal{A}$ be a generator of $\mathcal{A}$. Then terms in $\phi_1(a) \in \mathrm{Cob}(A)$ correspond to arrows in the A_∞-action of ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{A}}$ of the form*

$$m_{p,1,1}(a_1, \dots, a_p, [i], a) \to [j],$$

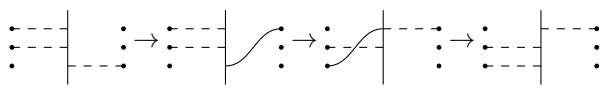
with the above arrow giving rise to the term

$$a_p^* \otimes \cdots \otimes a_1^*$$

in $\phi_1(a)$.

We now give some examples. In each case, there is exactly one algebra input $a \in \mathcal{A}$ on the right. So under the notations of Theorem 6, the sequence must start with $a_{2,2} = a$, then alternate between applying an H-arrow and factoring from $a_{2k-1,1}$.

For a length-1 strand $a \in \mathcal{A}$, we get $\phi_1(a) = a^*$. For the first non-trivial case, we consider a length-2 strand, when none of the three points are paired. There are two possible orderings of the two intervals in $<_{\mathcal{Z}}$. If the upper interval comes first, the sequence

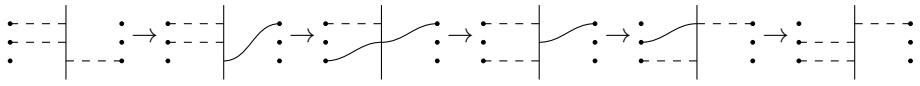


gives

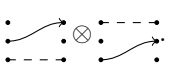
$$\phi_1\left(\right) = , \tag{2.6}$$

where we used the same diagram to represent a and a^*.

If the lower interval comes first, the sequence



gives

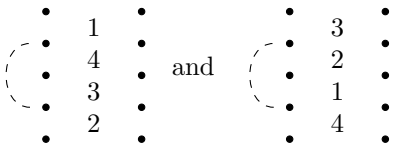
$$\phi_1\left(\right) =$$  (2.7)

The right side of equations (2.6) and (2.7) represent the same homology class in $\mathrm{Cob}(\mathcal{A})$, since their difference is given by

$$d\left(\right).$$

Now we consider a more complicated example, which shows the possible complications that can arise in such a computation.

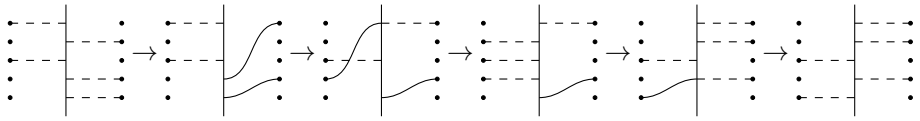
Consider a length 4 interval in a larger pointed matched circle, where the second and fourth of the five points are paired. There are two possibilities for the relative orderings of the four intervals in $<_{\mathcal{Z}}$. This is because the second interval (counting from below) must immediately precede the third interval, and the fourth interval must immediately precede the first interval. The two possible relative orderings are given as follows. We will simply refer to them as 1-4-3-2 and 3-2-1-4 from now on.



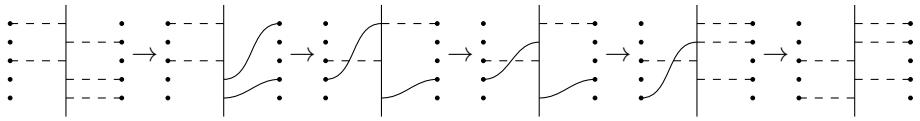
The following two generators a_1, a_2 represent the same homology class in $\mathcal{A}$:

$$a_1 =$$  $$, \quad a_2 =$$  (2.8)

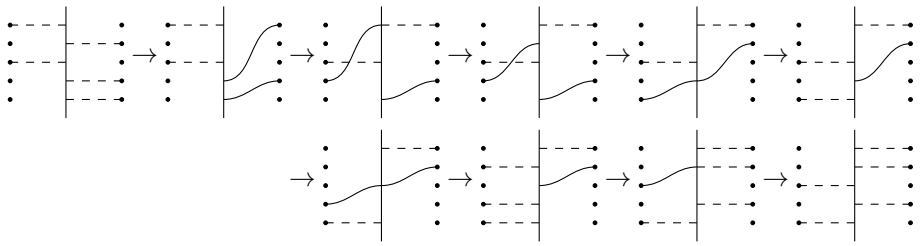
Consider first the ordering 1-4-3-2, and using the representative a_1. There are three terms in $\phi_1(a_1)$. The first term comes from the sequence:



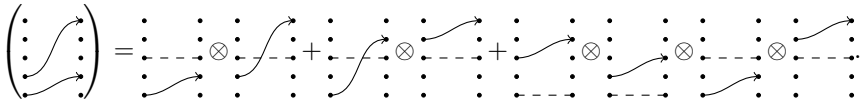
The second term comes from



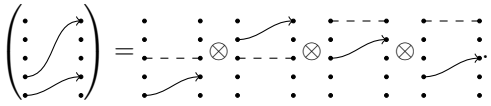
The third term comes from



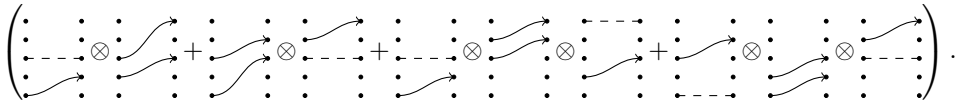
Note the use of a special H-arrow in the fourth step of the last sequence. This produces

ϕ_1  (2.9)

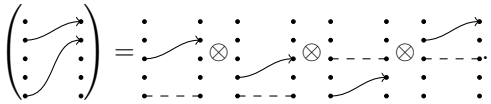
If the ordering is 3-2-1-4, a straightforward sequence gives

ϕ_1  (2.10)

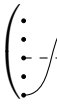
The difference between the right sides of equations (2.9) and (2.10) is:

d .

Now we consider representative a_2 of the same homology class of $\mathcal{A}$. If the ordering is 1-4-3-2, the sequence is straightforward, giving

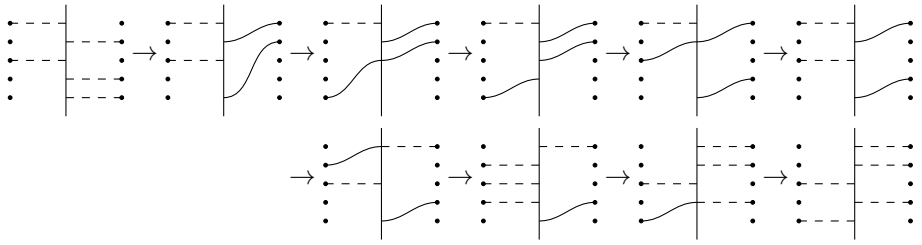
ϕ_1  (2.11)

The difference between the right sides of equations (2.11) and (2.9) is:

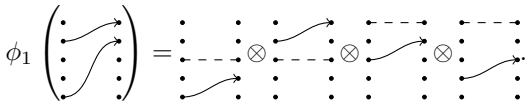
d .

Finally, if the ordering is 3-2-1-4, one can check there is a single sequence, which uses a special

H-arrow in the fourth step:



This produces



(2.12)

The right side is the same as that in equation (2.10). This finishes the consistency check that the same homology class in $\mathrm{Cob}(\mathcal{A})$ is obtained using any valid local ordering of the intervals, and any representative of the same homology class in $\mathcal{A}$. In general, one can expect the computation to be more complicated as the length of elements in the homology class increases.

2.7 Appendix: cancellation diagrams

In these diagrams, d-arrows are labeled d, ordinary H-arrows are labeled H, and special H-arrows are labeled H_{sp}.

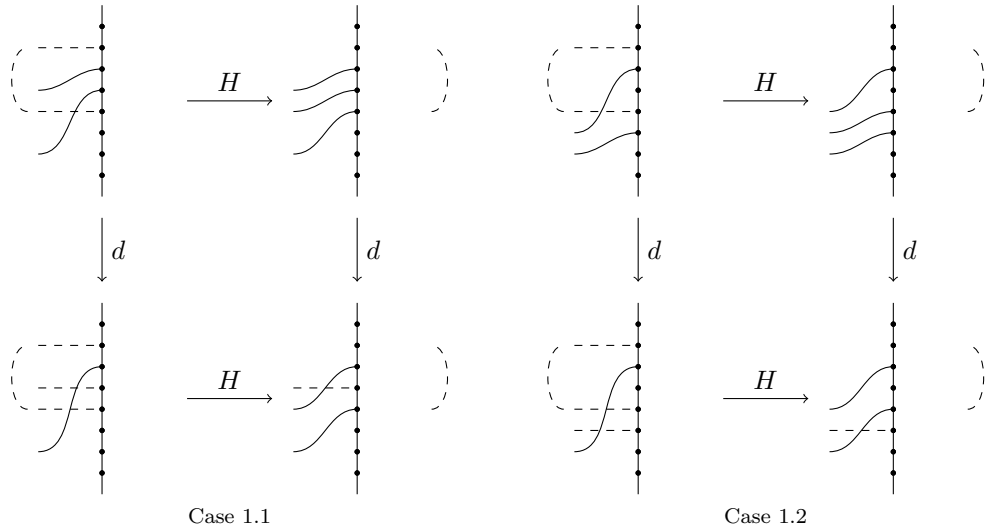
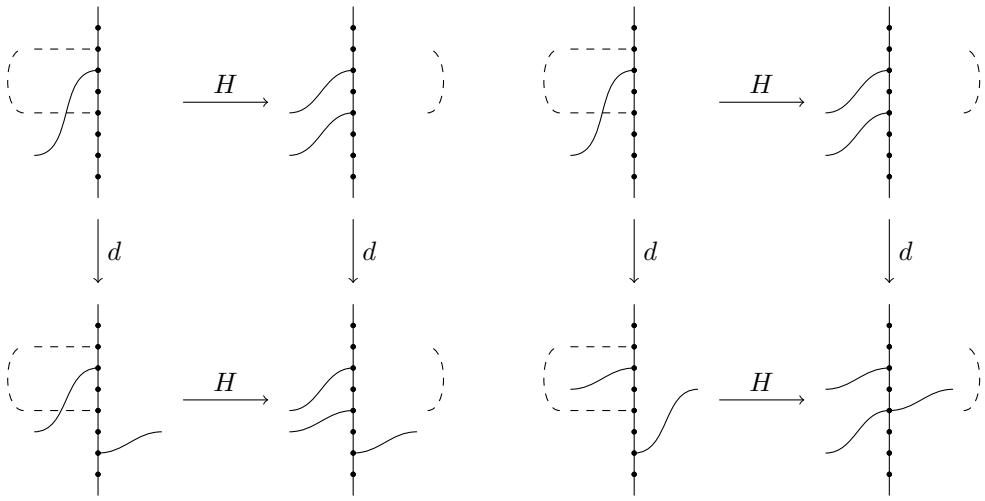
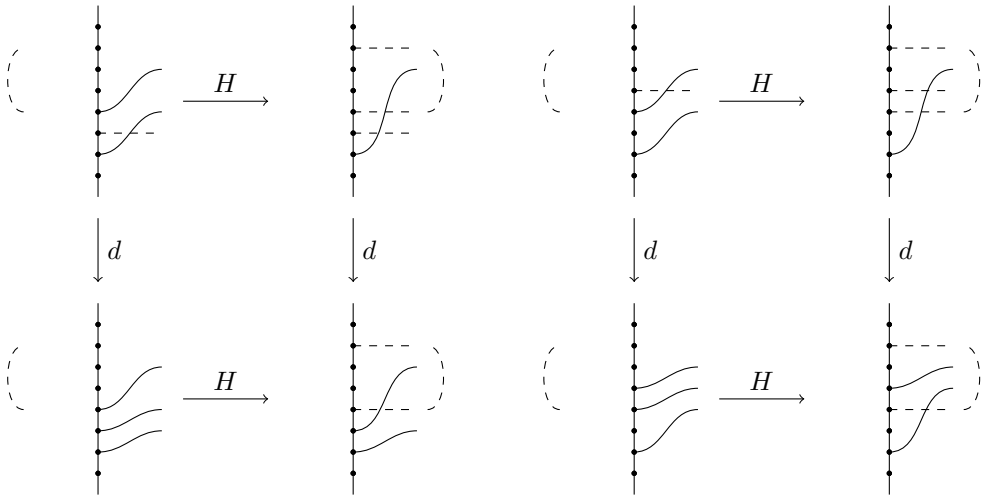


Figure 2.27: Cancellations in $d \circ H + H \circ d = I$.



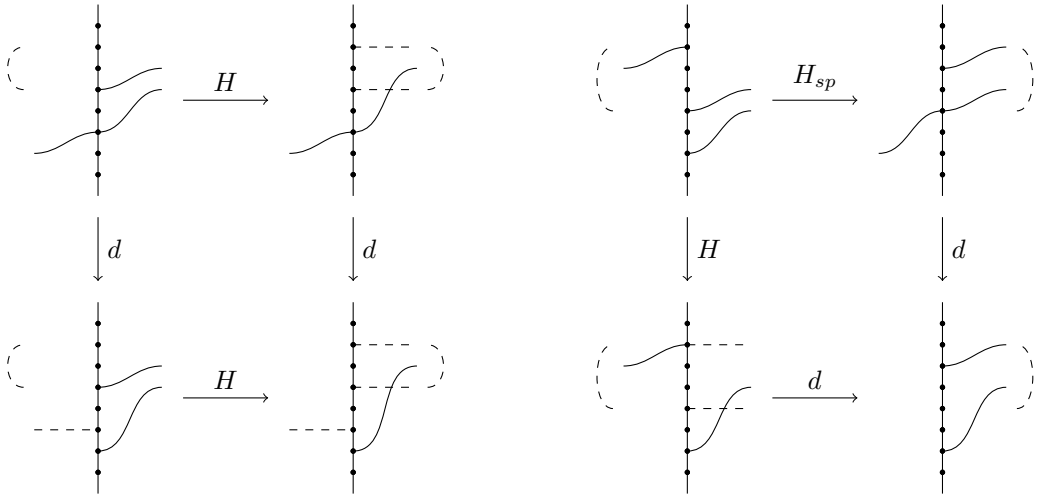
Case 1.3 Case 1.4

Figure 2.28: Cancellations in $d \circ H + H \circ d = I$.



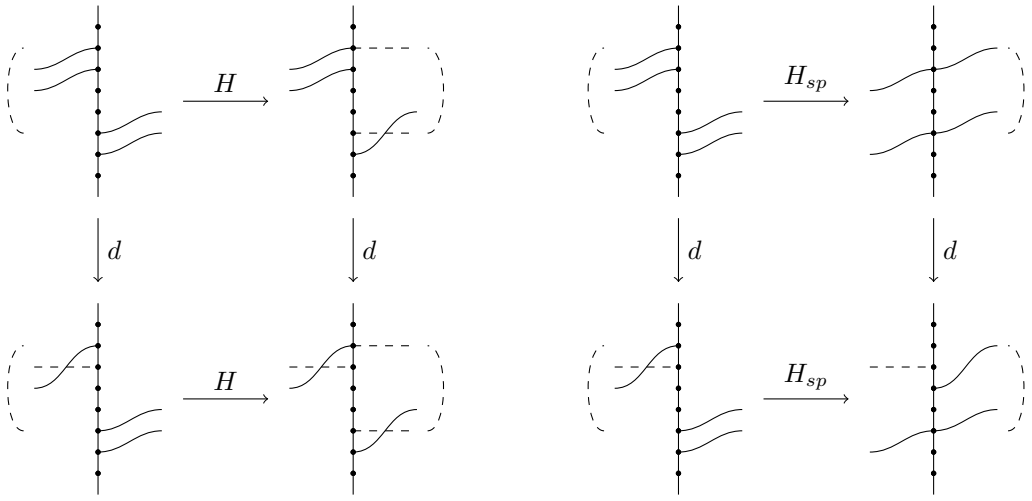
Case 2.1 Case 2.2

Figure 2.29: Cancellations in $d \circ H + H \circ d = I$.



Case 2.3 Case 2.4

Figure 2.30: Cancellations in $d \circ H + H \circ d = I$.



Case 2.5a Case 2.5b

Figure 2.31: Cancellations in $d \circ H + H \circ d = I$.

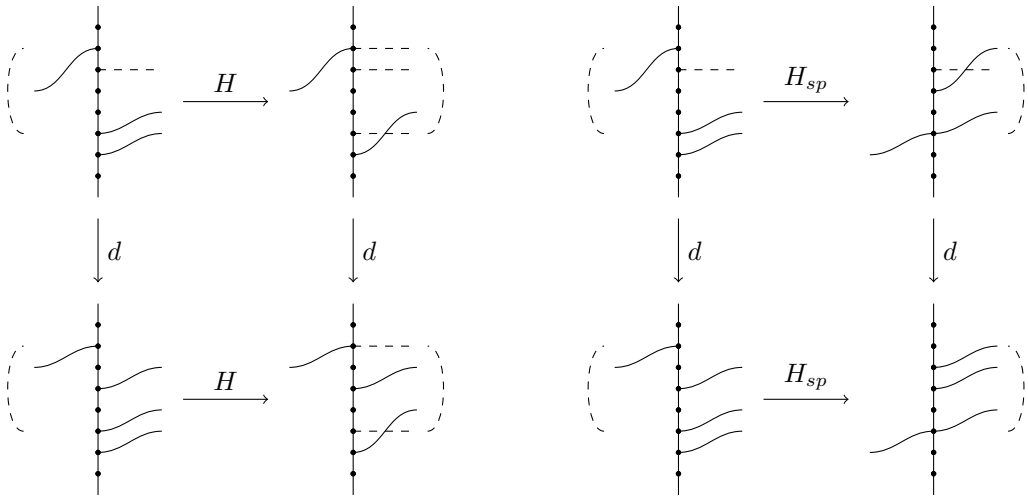


Figure 2.32: Cancellations in $d \circ H + H \circ d = I$.

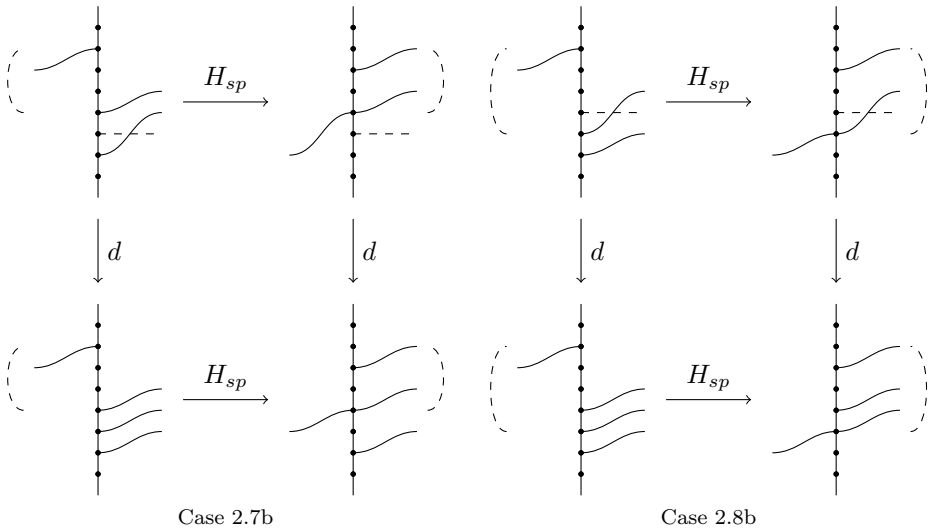
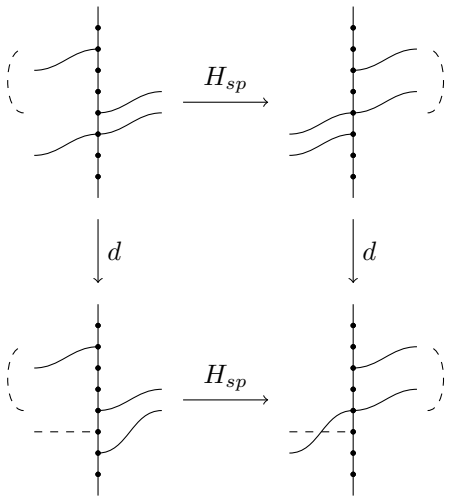
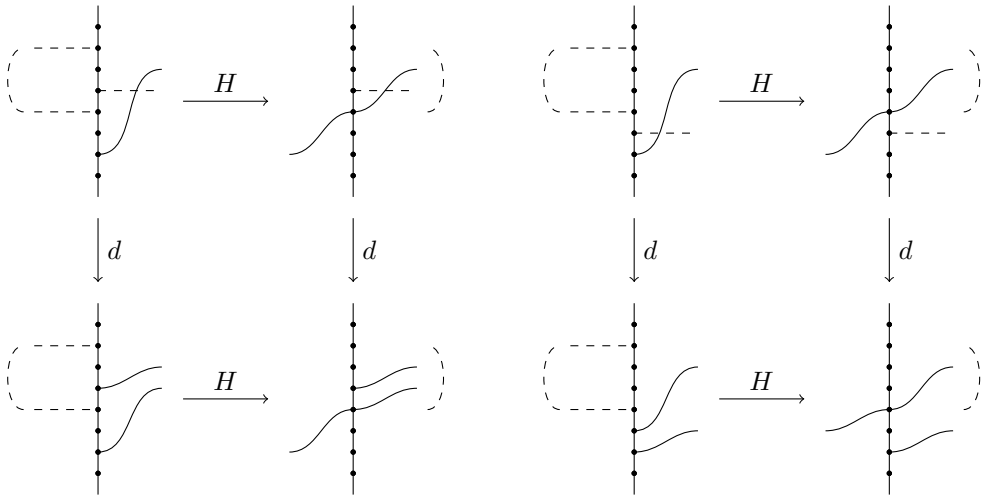


Figure 2.33: Cancellations in $d \circ H + H \circ d = I$. Cases 2.7a and 2.8a are exactly the same as Cases 2.1 and 2.2.



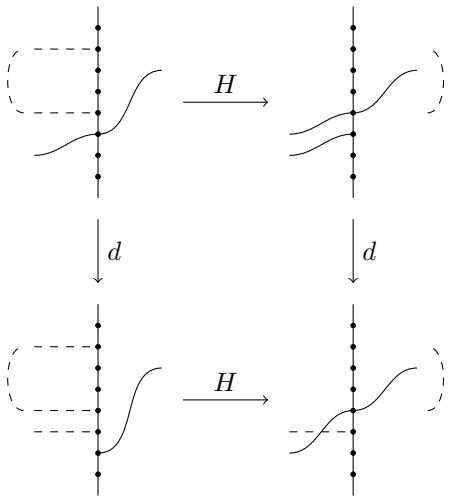
Case 2.9b

Figure 2.34: Cancellations in $d \circ H + H \circ d = I$. Case 2.9a are exactly the same as Case 2.3.



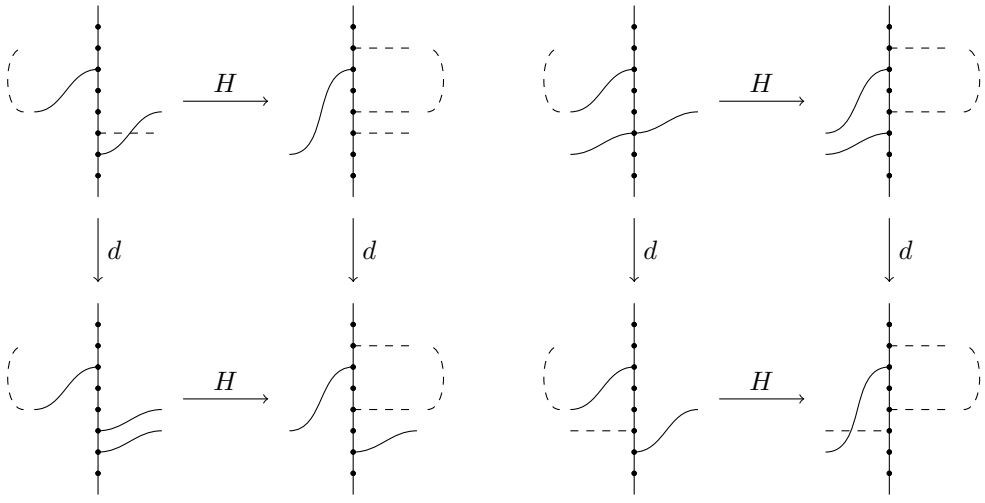
Case 3.1 Case 3.2

Figure 2.35: Cancellations in $d \circ H + H \circ d = I$.



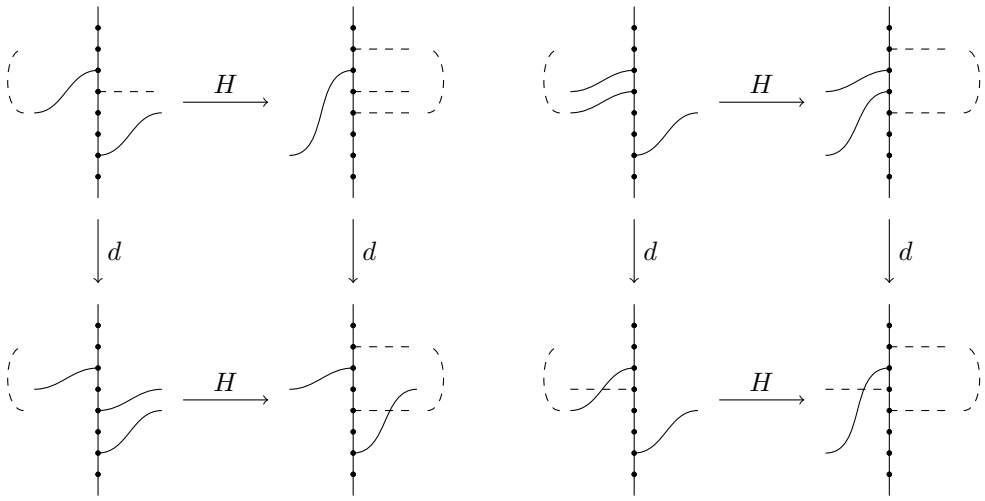
Case 3.3

Figure 2.36: Cancellations in $d \circ H + H \circ d = I$.



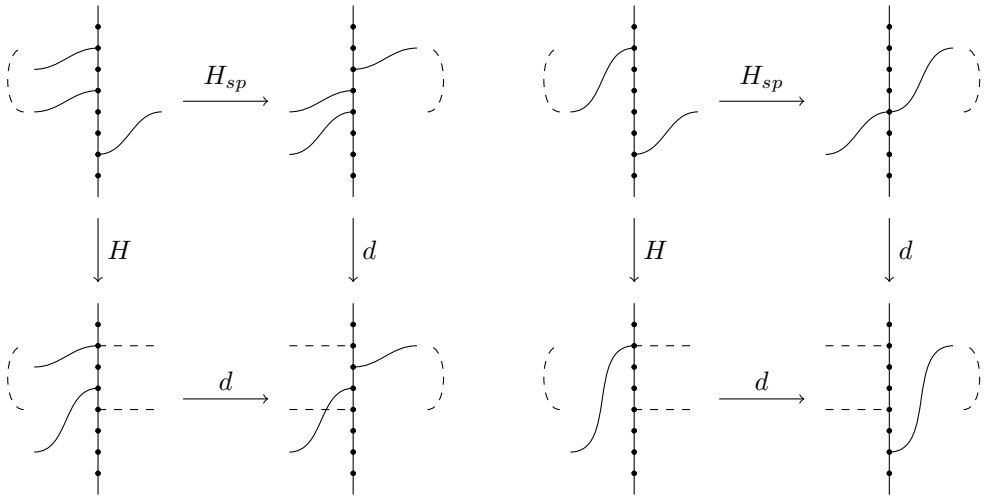
Case 4.1 Case 4.2

Figure 2.37: Cancellations in $d \circ H + H \circ d = I$.



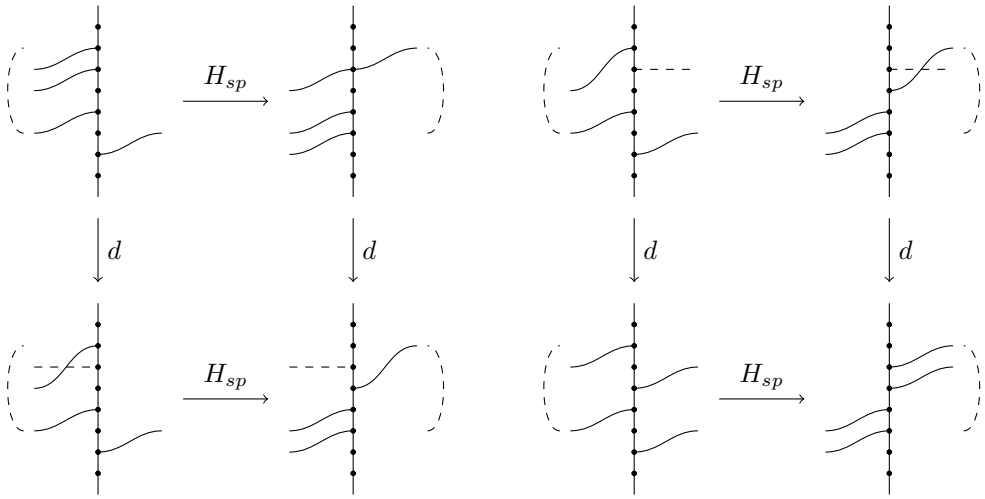
Case 4.3 Case 4.4

Figure 2.38: Cancellations in $d \circ H + H \circ d = I$.



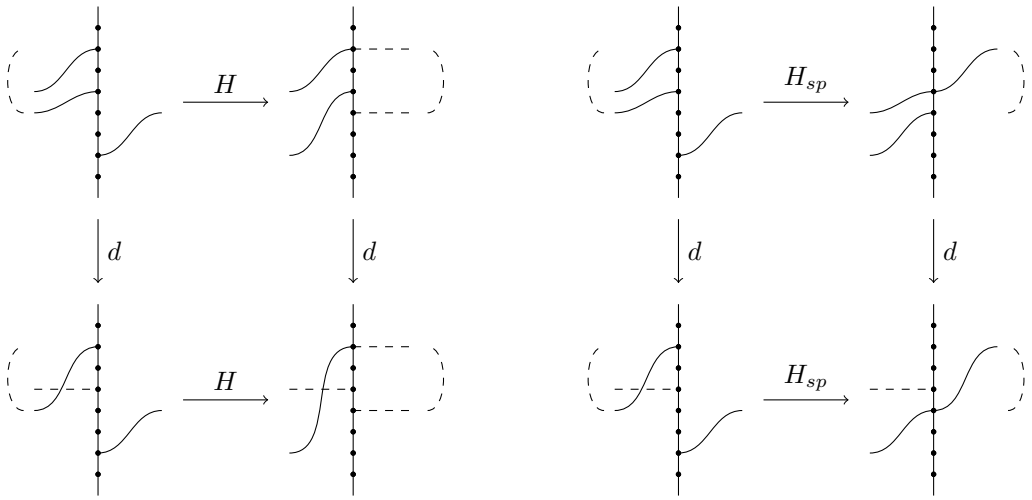
Case 4.5 Case 4.6

Figure 2.39: Cancellations in $d \circ H + H \circ d = I$.



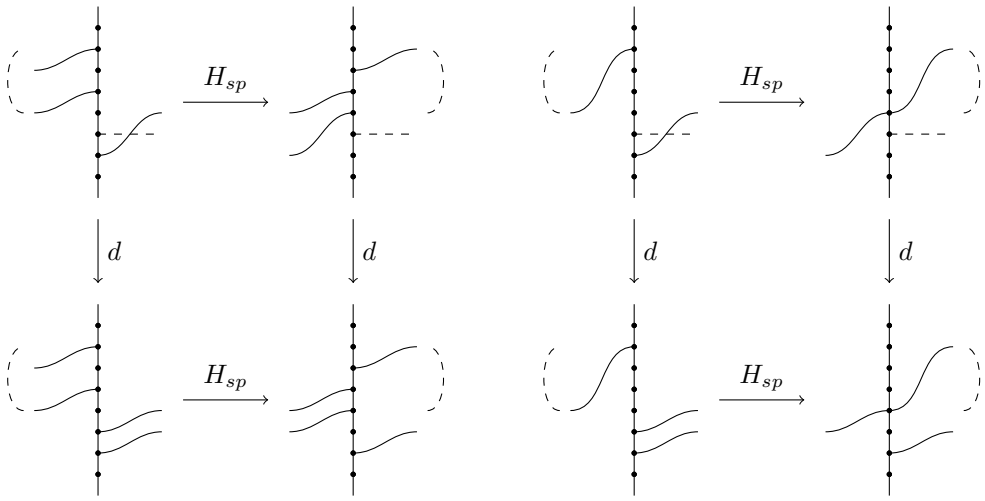
Case 4.7b Case 4.8b

Figure 2.40: Cancellations in $d \circ H + H \circ d = I$. Cases 4.7a and 4.8a are similar to Cases 2.5a and 2.6a.



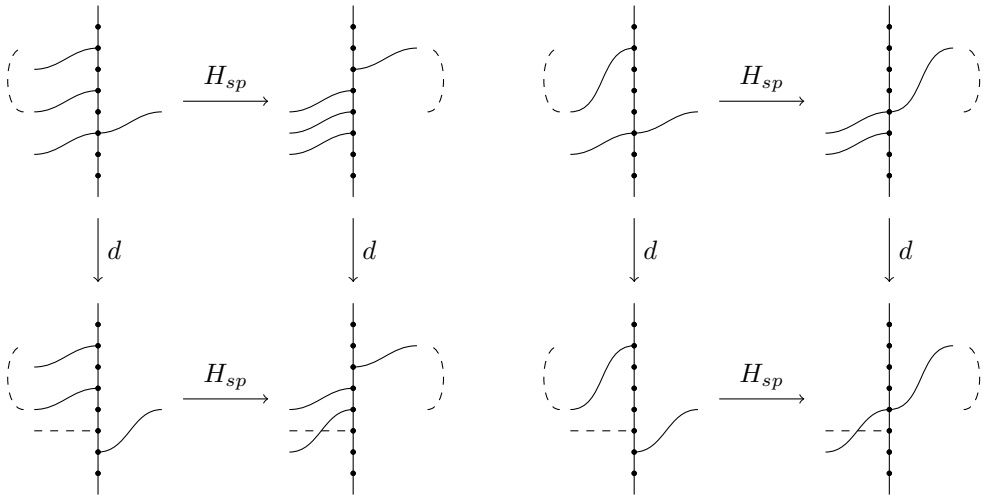
Case 4.9a Case 4.9b

Figure 2.41: Cancellations in $d \circ H + H \circ d = I$.



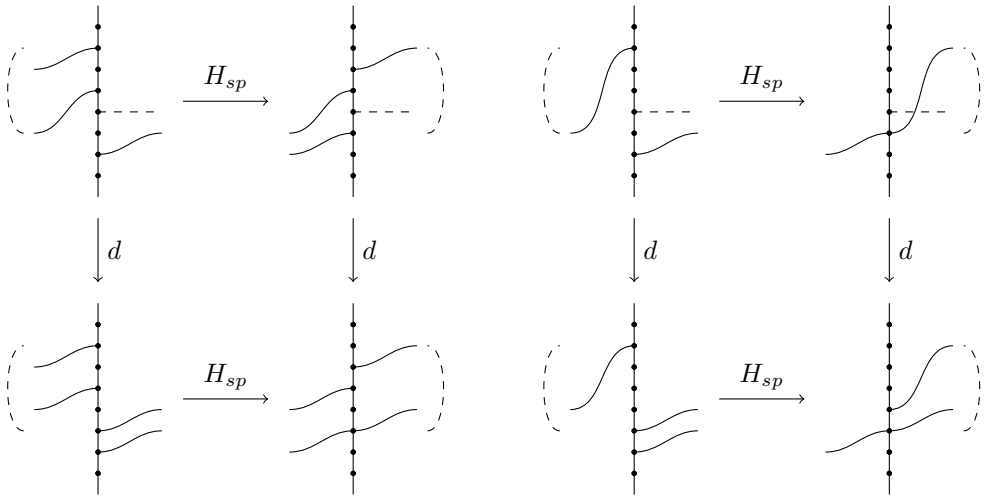
Case 4.10b Case 4.10c

Figure 2.42: Cancellations in $d \circ H + H \circ d = I$. Case 4.10a is same as Case 4.1.



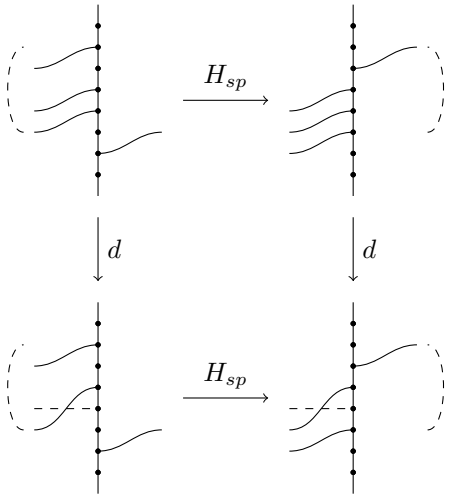
Case 4.11b Case 4.11c

Figure 2.43: Cancellations in $d \circ H + H \circ d = I$. Case 4.11a is same as Case 4.2.



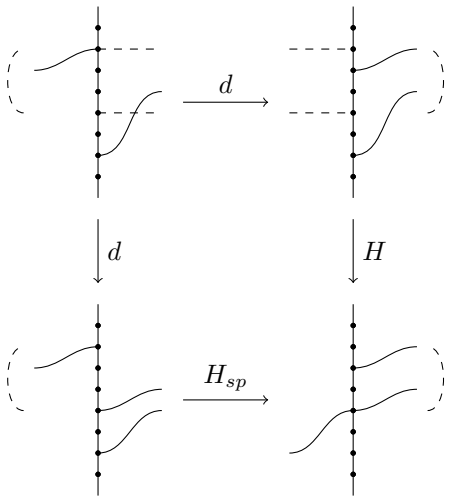
Case 4.12b Case 4.12c

Figure 2.44: Cancellations in $d \circ H + H \circ d = I$. Case 4.12a is same as Case 4.3.



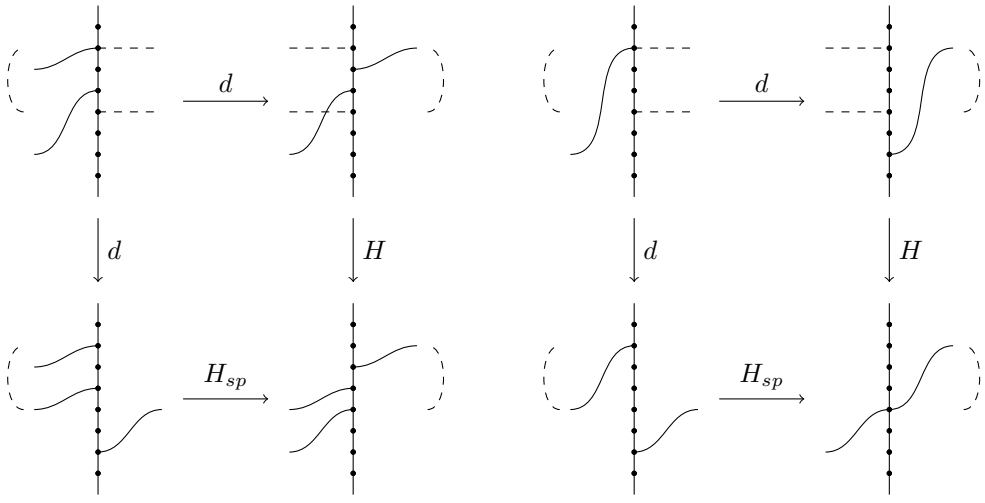
Case 4.13b

Figure 2.45: Cancellations in $d \circ H + H \circ d = I$. Case 4.13a is same as Case 4.4. Case 4.13c, with the second special case, cannot occur.



Case 6.1

Figure 2.46: Cancellations in $d \circ H + H \circ d = I$.



Case 7.1 Case 7.2

Figure 2.47: Cancellations in $d \circ H + H \circ d = I$.

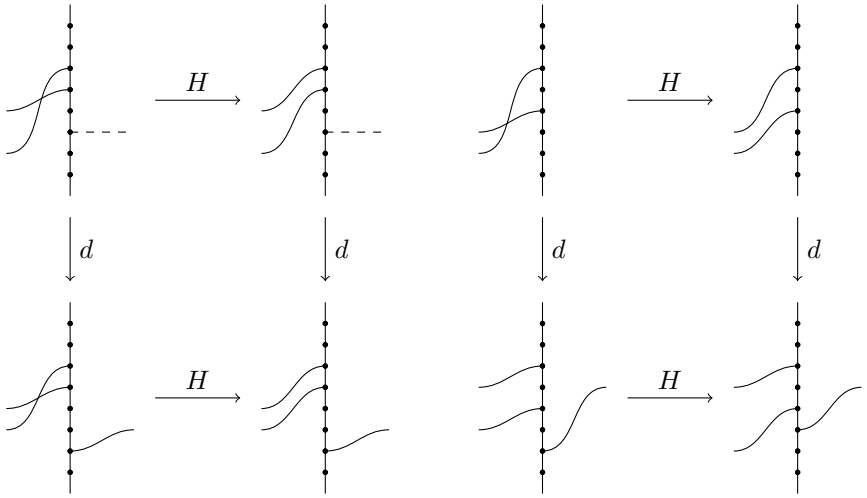


Figure 2.48: Cancellations in $d \circ H + H \circ d = I$.

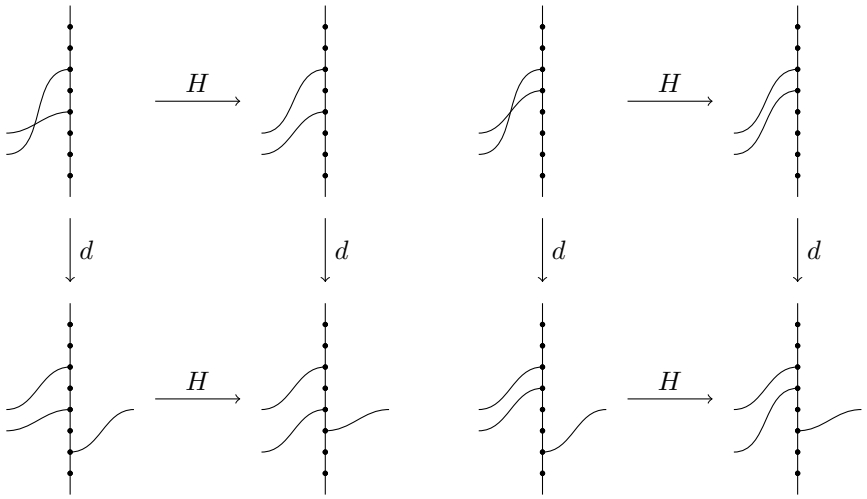
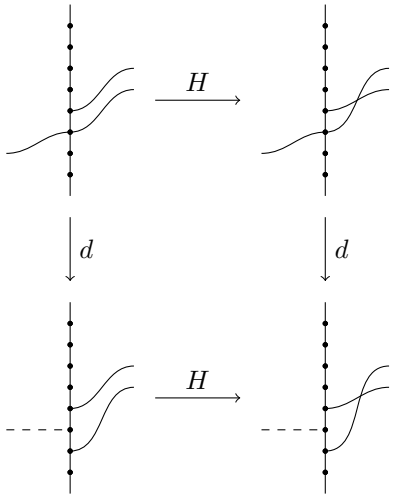
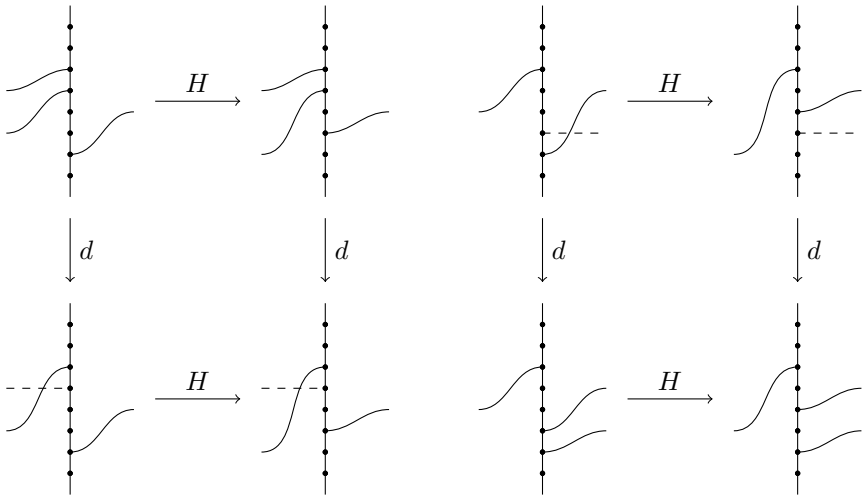


Figure 2.49: Cancellations in $d \circ H + H \circ d = I$.



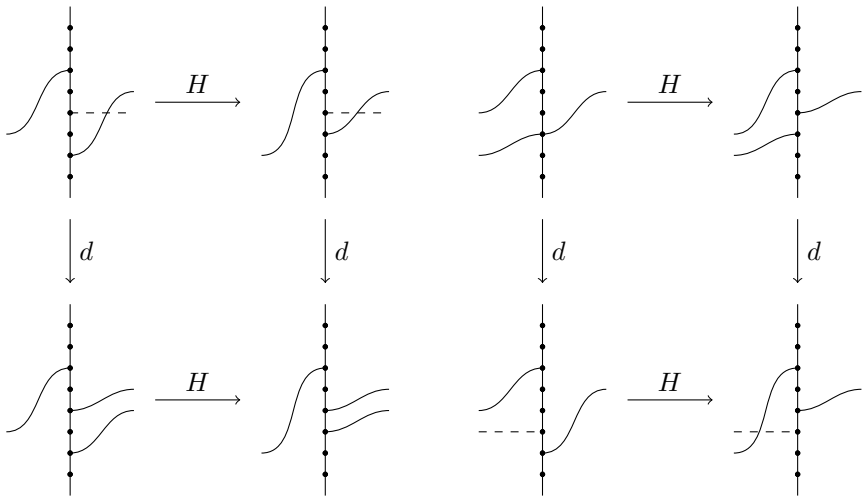
Case 10.1

Figure 2.50: Cancellations in $d \circ H + H \circ d = I$.



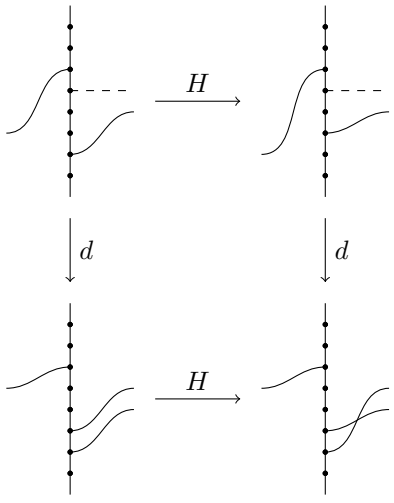
Case 11.1 Case 11.2

Figure 2.51: Cancellations in $d \circ H + H \circ d = I$.



Case 11.3 Case 11.4

Figure 2.52: Cancellations in $d \circ H + H \circ d = I$.



Case 11.5

Figure 2.53: Cancellations in $d \circ H + H \circ d = I$.

Chapter 3

Construction and proof of invariance of $\widehat{HF}$

3.1 Introduction

The aim for this chapter is to give a combinatorial description and proof of invariance for $\widehat{HF}$. One main result we use is the alternative description of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ given in the previous chapter. This allows us to use a combinatorial construction that is easier to reason about.

An intermediate statement in the proof, which may be of independent interest, is that bordered Floer theory gives a linear-categorical representation of the strongly-based mapping class groupoid (which contains the strongly-based mapping class group). By a linear-categorical representation of a group or groupoid, we mean assigning homotopy equivalence classes of bimodules to each element of the group (resp. groupoid), in such a way that composition in the group (resp. groupoid) corresponds to taking an appropriate tensor product of bimodules.

Linear-categorical representations of important groups in topology have been investigated before. Bordered Floer homology actually gives a family of representations of the strongly-based mapping class groupoid. For a given genus g, the representations are indexed by an integer w, called the weight, between $-g$ and g. The representation that is relevant for 3-manifold invariants, and which we will focus on in this paper, corresponds to $w = 0$. The cases $w = \pm g$ are trivial. The cases $w = \pm(g-1)$ are described combinatorially by Lipshitz, Ozsváth, and Thurston in [17], and a combinatorial proof of invariance is given by Siegel in [27]. Linear-categorical representations of other groups occuring in topology have also been studied. See the introduction in [9] for a review

and a list of references. One major example is linear-categorical representations of the braid group, studied in, for example, [8], [26], [2], [25], and [7].

We now give an overview of this chapter. In section 3.2, we give in detail the combinatorial description of $\widehat{HF}$ to be considered here. In section 3.3, we prove the intermediate statement on the linear-categorical representation of the strongly-based mapping class groupoid. Using it, we complete the proof of invariance for closed 3-manifolds in section 3.4.

3.2 Combinatorial construction

Using the available description of $\widehat{CFDD}(\tau)$ for an arbitrary arcslide $\tau : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, we may define $\widehat{CFDA}(\tau)$ combinatorially as the tensor product

$$ {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\tau)_{\mathcal{A}(\mathcal{Z}_2)} = {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\tau) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_2})_{\mathcal{A}(-\mathcal{Z}_2),\mathcal{A}(\mathcal{Z}_2)} \tag{3.1} $$

We should clarify that while equation (1.11) is a statement about analytically defined invariants, proved using analysis, equation (3.1) is a part of the combinatorial construction of $\widehat{HF}$, for which we will prove invariance using combinatorial methods. Similar distinctions should be made for the other definitions in this section.

Given this, we can define $\widehat{CFDA}(\phi)$ for an arbitrary element ϕ of the strongly-based mapping class groupoid, by factoring ϕ into arcslides. The precise statement is:

Construction 10. *Given an element $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_{n+1})$ of the strongly-based mapping class groupoid, let $\phi = \tau_n \circ \cdots \circ \tau_1$, where $\tau_i : F^\circ(\mathcal{Z}_i) \to F^\circ(\mathcal{Z}_{i+1})$ be a factorization of ϕ. Write $\vec{\tau_i}$ for the sequence $\tau_1, \ldots, \tau_n$. Define*

$$ {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi, \vec{\tau_i})_{\mathcal{A}(\mathcal{Z}_{n+1})} = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\tau_1)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes \cdots \boxtimes {}^{\mathcal{A}(\mathcal{Z}_n)}\widehat{CFDA}(\tau_n)_{\mathcal{A}(\mathcal{Z}_{n+1})}. $$

Theorem 11. *The homotopy type of $\widehat{CFDA}(\phi, \vec{\tau_i})$ does not depend on the choice of factorization $\vec{\tau_i}$. Hence, $\widehat{CFDA}(\phi, \vec{\tau_i})$ is an invariant of ϕ up to homotopy equivalence.*

The analytical version of Theorem 11 is implied by the invariance of $\widehat{CFDA}$ for 3-manifolds with two boundary components. We will prove it combinatorially in section 3.3. Knowing this, we may use the notation $\widehat{CFDA}(\phi)$ to represent $\widehat{CFDA}(\phi, \vec{\tau_i})$ for some factorization $\vec{\tau_i}$ of ϕ. The pairing theorem implies that this definition of $\widehat{CFDA}(\tau)$ agrees with the analytic definition.

The other bimodule invariants $\widehat{CFDD}(\phi)$, $\widehat{CFAA}(\phi)$, and $\widehat{CFAD}(\phi)$ for a general morphism

$\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ can then be defined as:

$$^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\phi) = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_2}) \tag{3.2}$$

$$\widehat{CFAA}(\phi)_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)} = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}_2)} \tag{3.3}$$

$$^{\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFAD}(\phi)_{\mathcal{A}(-\mathcal{Z}_1)} = \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\phi). \tag{3.4}$$

Since $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ is the quasi-inverse of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ (as will be shown later this chapter), this new definition of $\widehat{CFDD}(\phi)$ for a general ϕ is consistent (up to homotopy equivalence) with the previous definition of $\widehat{CFDD}(\tau)$ for arcslides. Also, expanding out the definitions, we see $\widehat{CFAD}(\phi) \simeq \widehat{CFAA}(\phi) \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_2})$.

To construct invariants of closed 3-manifolds, we need one more building block: $\widehat{CFD}$ of the 0-framed handlebody $\mathbf{H}^g$. This is a 3-manifold with one boundary component, given by the Heegaard diagram in Figure 3.1.

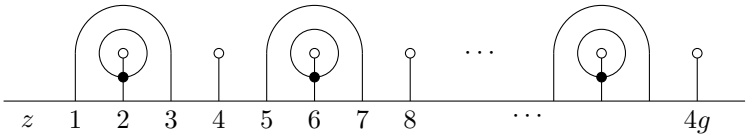


Figure 3.1: Heegaard diagram for the 0-framed handlebody. The numbers at bottom label points in $\mathcal{Z}^g$. Points in $-\mathcal{Z}^g$ will be labeled in the reverse order.

As in the diagram for arcslides, the small circles are 1-handle attachment points, paired consecutively. The larger circles are β circles, and all other arcs inside the boundary are α arcs. From the way the α arcs meet the boundary, we see that the boundary of $\mathbf{H}^g$ is parametrized by the split pointed matched circle of genus g. This is the pointed matched circle with matching

$$(1,3),(2,4),(5,7),(6,8),\ldots,(4g-3,4g-1),(4g-2,4g).$$

From now on we will denote it by $\mathcal{Z}^g$. While it is true that $-\mathcal{Z}^g = \mathcal{Z}^g$, we will usually distinguish them in order to emphasize orientation changes.

The orientation reversal $-\mathbf{H}^g$ is called the ∞-framed handlebody. Its boundary is also parametrized by $\mathcal{Z}^g$ (using the identification $-\mathcal{Z}^g = \mathcal{Z}^g$). The Heegaard diagram for the ∞-framed handlebody is the reflection of that for the 0-framed handlebody (so for instance, the location of 1-handle attachment points are above the odd points in $\mathcal{Z}^g$ rather than the even points).

The invariant $\widehat{CFD}(\mathbf{H}^g)$ has left type D action by $\mathcal{A}(-\mathcal{Z}^g)$. It is generated by a single generator $\mathbf{x}$, corresponding to the set of intersection points indicated in Figure 3.1. The idempotent of $\mathbf{x}$

contains pairs $(2,4),(6,8),\dots$ in $-\mathcal{Z}^g$ (pairs corresponding to α-arcs not occupied by $\mathbf{x}$; note the labeling of points for $-\mathcal{Z}^g$ is reversed). The type D action is

$$\delta^1\mathbf{x} = \sum_{\xi\in\mathcal{D}} a(\xi)\cdot\mathbf{x},$$

where $\mathcal{D}$ is the collection of chords $2\to 4, 6\to 8$, etc.

As usual, we denote $\overline{\widehat{CFD}(\mathbf{H}^g)}$ to be the dual of $\widehat{CFD}(\mathbf{H}^g)$. This is a right type D structure over $\mathcal{A}(-\mathcal{Z}^g)$, or equivalently a left type D structure over $\mathcal{A}(\mathcal{Z}^g)$. From the definition using holomorphic curves, it is clear that $\overline{\widehat{CFD}(\mathbf{H}^g)} \simeq \widehat{CFD}(-\mathbf{H}^g)$. For our purposes we can simply *define* this to be the case.

We now give the combinatorial construction of $\widehat{HF}(Y)$ for a closed 3-manifold Y, following the spirit of the construction in [16].

Construction 12. *Let Y be a closed 3-manifold. Choose a Heegaard splitting $Y_1\cup_\phi Y_2$ of Y. Fix a basepoint z on the gluing boundary $Y_1\cap Y_2$. Fix basepoint-preserving diffeomorphisms $f_1:\mathbf{H}^g\to Y_1$ and $f_2:-\mathbf{H}^g\to Y_2$ from Y_1, Y_2 to the standard handlebodies. Let f_{1*} and f_{2*} be the restrictions of f_1 and f_2 to the boundary. Let $\psi = f_{2*}^{-1}\circ\phi\circ f_{1*}$ be the induced gluing map between standard handlebodies. This is an element of the strongly-based mapping class group $F^\circ(\mathcal{Z}^g)\to F^\circ(\mathcal{Z}^g)$. Define:*

$$\widehat{HF}(Y,Y_1,Y_2,f_1,f_2) = \widehat{CFAA}(\psi)_{\mathcal{A}(-\mathcal{Z}^g),\mathcal{A}(\mathcal{Z}^g)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}^g)}\widehat{CFD}(\mathbf{H}^g) \boxtimes {}^{\mathcal{A}(\mathcal{Z}^g)}\overline{\widehat{CFD}(\mathbf{H}^g)}.$$

Theorem 13. *The homotopy type of $\widehat{HF}(Y,Y_1,Y_2,f_1,f_2)$ does not depend on the choice of Heegaard splitting $Y=Y_1\cup_\phi Y_2$ or the parametrizations f_1,f_2. Therefore it gives an invariant of Y up to homotopy equivalence.*

We will prove Theorem 13 combinatorially in section 3.4. Given this theorem, we can simplify notation by writing $\widehat{HF}(Y)$ for $\widehat{HF}(Y,Y_1,Y_2,f_1,f_2)$, for some choice of Heegaard splitting and parametrizations. From the construction, it is clear that this is equivalent to the definition of $\widehat{HF}(Y)$ using holomorphic curves.

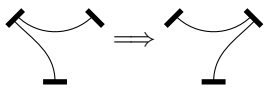
3.3 Representation of the mapping class groupoid

The goal of this section is to prove Theorem 11, showing that the combinatorial construction of $\widehat{CFDA}(\phi)$, for an element ϕ of the mapping class groupoid, does not depend on the choice of

factorization of ϕ into arcslides.

For this purpose, it is necessary to understand relations among arcslides. This is studied in detail in [1] and [23]. The notions of pointed matched circles and arcslides correspond to linear chord diagrams and chord slides in these papers. We now give a summary of the results.

Locally, an arcslide can be viewed as one end of the B pair sliding along the C pair:

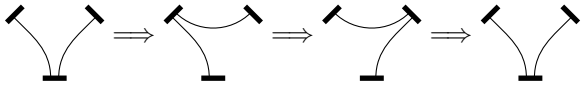


(3.5)

In this diagram, the three short segments denote portions of the straight line in the pointed matched circle. The upper, stationary arc denotes the C pair; and the lower, moving arc denotes the B pair.

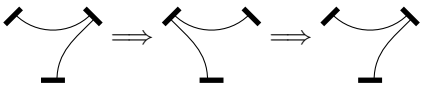
There are five types of relations on arcslides, and together they exhaust all relations. The local diagrams for the five relations are as follows (see [1, Theorem 6.2, Figure 6.1]):

- Triangle



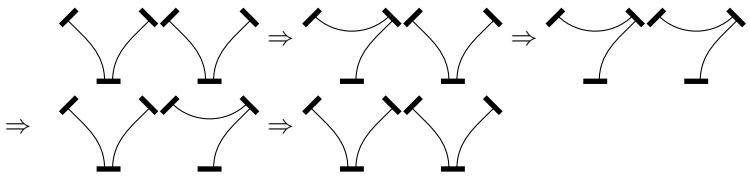
(3.6)

- Involution



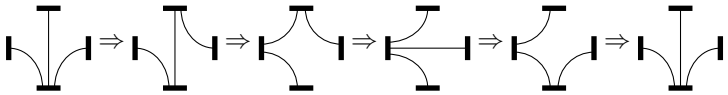
(3.7)

- Commutativity



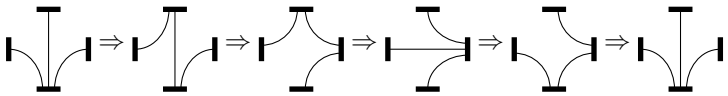
(3.8)

- Left Pentagon



(3.9)

- Right Pentagon



(3.10)

Each relation gives one way to factor the identity morphism $\mathbb{I}_{\mathcal{Z}}$ starting and ending at some

pointed matched circle $\mathcal{Z}$. For proving Theorem 11, it suffices to check that for each such factorization

$$\mathbb{I}_{\mathcal{Z}} = \tau_n \circ \cdots \circ \tau_1,$$

the corresponding homotopy equivalence

$$\widehat{CFDA}(\mathbb{I}_{\mathcal{Z}}) \simeq \widehat{CFDA}(\tau_1) \boxtimes \cdots \boxtimes \widehat{CFDA}(\tau_n) \tag{3.11}$$

holds. Note that in general, the starting and ending pointed matched circles of τ_i may be different from $\mathcal{Z}$. This is the main reason why we need to consider strongly-based mapping class groupoids, even if we are only interested in statements about strongly-based mapping class groups.

We now sketch the overall strategy for verifying equation (3.11). With our combinatorial definition of $\widehat{CFDA}(\tau)$, we can readily enumerate the set of generators on the right side of the equation. There are, however, more generators than on the left side. The *cancellation lemma* for type DA bimodules describes situations in which we can prove that a bimodule is homotopy equivalent to one with 2 fewer generators. Using it, we can remove generators from the right side in pairs, so that the set of remaining generators matches that on the left side. The cancellation lemma is stated and proved in section 3.3.1.

It turns out that a type DA bimodule with the same set of generators as ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$, and that has a few more properties in common with ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$, must be homotopy equivalent to ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$. We prove two lemmas of this kind, which we call *rigidity lemmas*, in section 3.3.2. The first lemma will be used to prove the involution relation, and the second lemma will be used for all other relations. The idea here is that once the involution relation is proved, we can show that $\widehat{CFDA}(\tau)$ is quasi-invertible for any arcslide τ, which implies that any tensor product of such bimodules is quasi-invertible. This means checking the quasi-invertibility condition in the second lemma becomes trivial, and we can avoid checking the more involved condition in the first lemma that it replaces. We note here that the rigidity lemmas depend on specific properties of $\mathcal{A}(\mathcal{Z})$, and is not applicable to dg-algebras in general.

To apply the cancellation lemma, and in the case of the involution relation, the rigidity lemma, we need to compute certain arrows in the type DA action of the bimodule on the right side. To prepare for this, we compute in section 3.3.3 some arrows in the type DA action of $\widehat{CFDA}(\tau)$ for arcslides τ (the components in the tensor product).

In section 3.3.4, we describe how to enumerate the set of generators of a tensor product of

$\widehat{CFDA}(\tau_i)$, where τ_i are arcslides, and how properties of $\widehat{CFDD}(\tau_i)$ carry over to the tensor product. With all these preparations in place, we prove equation (3.11) for the involution relation in section 3.3.5, and for the other relations in section 3.3.6.

3.3.1 Cancellation lemmas

In this section we prove cancellation lemmas for type D and type DA structures over a dg-algebra. Everything will be done over the field $\mathbb{F}_2$. Both proofs are generalizations of the proof in the case of chain complexes. This result is well-known (see, for example, [11, Section 2.6]). The only things the reader needs to know for later sections are Definition 15, Theorem 16, and the remarks at the end.

For a type D structure M over a dg-algebra A, with a fixed set of generators $\mathcal{G}$. For each $g \in \mathcal{G}$ let $i(g)$ be the idempotent of g. For any $x, y \in \mathcal{G}$, let c_{xy} the coefficient of y in $\delta^1(x)$. That is, if:

$$\delta^1 x = \sum_i a_i \otimes g_i,$$

with each $a_i \in A$, and g_i distinct elements of $\mathcal{G}$, such that $a_i \cdot i(g) = a_i$, then $c_{x,g_i} = a_i$.

We say that an element $a \in A$ is *invertible* if there exists $a' \in A$ such that $aa' = a'a = i$ for some idempotent $i \in A$. We choose such an inverse and denote it by a^{-1}. Now suppose that for some $x, y \in \mathcal{G}$, the coefficient c_{xy} is invertible in A, and $d(c_{xy}) = 0$. Then there is a new type D structure M', generated by $\mathcal{G}' = \mathcal{G} \setminus \{x, y\}$ and with the type D action

$$\delta^{1'} a = \sum_{b \in \mathcal{G}'} (c_{ab} + c_{ay} c_{xy}^{-1} c_{xb}) \otimes b, \tag{3.12}$$

for any $a \in \mathcal{G}'$. The first part of each term in the sum is simply the original δ^1 (excluding terms involving x and y). The second term is as follows: for each "zigzag" as shown in Figure 3.2 in M, the term $c_{ay} c_{xy}^{-1} c_{xb} \otimes b$ is added to $\delta^{1'} a$. The coefficient can be read out by "following the arrows" from a to b, treating a reversed arrow as taking inverse.

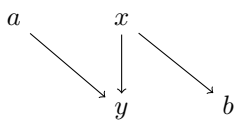


Figure 3.2: Standard example of a cancellation.

The *cancellation lemma* states that $\delta^{1'}$ satisfies the type D structure equations, and the resulting type D structure M' is homotopy equivalent to M. The homotopy equivalence is given by type D

morphisms $f : M \to M'$ and $g : M' \to M$ where:

$$f(x) = 0, f(y) = \sum_{b \in \mathcal{G}'} c_{xy}^{-1} c_{xb} \otimes b, \text{ and } f(a) = 1 \otimes a \text{ for } a \in \mathcal{G}'. \tag{3.13}$$

$$g(a) = 1 \otimes a + c_{ay} c_{xy}^{-1} \otimes x. \tag{3.14}$$

For the zigzag in Figure 3.2, we follow the arrows from y to b in the map f, and from a to x in the map g.

The homotopy $h : M \to M$ is given by $h(y) = c_{xy}^{-1} \otimes x$ and $h(a) = 0$ for any generator $a \neq y$. Then the precise statement of the cancellation lemma is:

Theorem 14 (Cancellation Lemma for Type D structures)**.** *With the above definitions,*

1. *The action $\delta^{1'}$ on M' satisfies the type D structure equation.*
2. *The* $\mathbf{k}$*-module maps f and g satisfy the structure equation for type D morphisms.*
3. *$f \circ g = \mathbb{I}$ and $g \circ f = \mathbb{I} + h \circ \delta^1 + \delta^1 \circ h$.*

and therefore M' is homotopy equivalent to M.

Proof. 1. The type D structure equation on M is:

$$(\mu_2 \otimes \mathbb{I}_M) \circ (\mathbb{I}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_M) \circ \delta^1 = 0.$$

Applied to an arbitrary $a \in \mathcal{G}$, this is,

$$dc_{ac} + \sum_{b \in \mathcal{G}} c_{ab} c_{bc} = 0. \tag{3.15}$$

for any $c \in \mathcal{G}$. Likewise, we need to prove:

$$dc'_{ac} + \sum_{b \in \mathcal{G}'} c'_{ab} c'_{bc} = 0. \tag{3.16}$$

for any $a, c \in \mathcal{G}'$, where the coefficients are

$$c'_{ab} = c_{ab} + c_{ay} c_{xy}^{-1} c_{xb}. \tag{3.17}$$

Expanding (3.16), our goal is to show:

$$\begin{aligned} & dc_{ac} + (dc_{ay})c_{xy}^{-1}c_{xc} + c_{ay}(dc_{xy}^{-1})c_{xc} + c_{ay}c_{xy}^{-1}(dc_{xc}) \\ + & \sum_{b\in\mathcal{G}'} c_{ab}c_{bc} + c_{ay}c_{xy}^{-1}c_{xb}c_{bc} + c_{ab}c_{by}c_{xy}^{-1}c_{xc} + c_{ay}c_{xy}^{-1}c_{xb}c_{by}c_{xy}^{-1}c_{xc} \\ = & \ 0. \end{aligned} \tag{3.18}$$

Using (3.15) on a and c, we have:

$$dc_{ac} + \sum_{b\in\mathcal{G}'} c_{ab}c_{bc} = c_{ax}c_{xc} + c_{ay}c_{yc}. \tag{3.19}$$

Using the same equation on a and y, we have:

$$dc_{ay} + \sum_{b\in\mathcal{G}'} c_{ab}c_{by} = c_{ax}c_{xy} + c_{ay}c_{yy}. \tag{3.20}$$

Using the invertibility of c_{xy}, we get:

$$c_{ax} = \left(dc_{ay} + c_{ay}c_{yy} + \sum_{b\in\mathcal{G}'} c_{ab}c_{by} \right) c_{xy}^{-1}. \tag{3.21}$$

And so

$$c_{ax}c_{xc} = dc_{ay}c_{xy}^{-1}c_{xc} + c_{ay}c_{yy}c_{xy}^{-1}c_{xc} + \sum_{b\in\mathcal{G}'} c_{ab}c_{by}c_{xy}^{-1}c_{xc}. \tag{3.22}$$

Following a similar procedure on x and c, we get:

$$c_{yc} = c_{xy}^{-1} \left(dc_{xc} + c_{xx}c_{xc} + \sum_{b\in\mathcal{G}'} c_{xb}c_{bc} \right). \tag{3.23}$$

And then,

$$c_{ay}c_{yc} = c_{ay}c_{xy}^{-1}dc_{xc} + c_{ay}c_{xy}^{-1}c_{xx}c_{xc} + \sum_{b\in\mathcal{G}'} c_{ay}c_{xy}^{-1}c_{xb}c_{bc} \tag{3.24}$$

Using (3.15) on x and y, we get:

$$c_{xx}c_{xy} + c_{xy}c_{yy} + \sum_{b\in\mathcal{G}'} c_{xb}c_{by} = dc_{xy}, \tag{3.25}$$

and so

$$d(c_{xy}^{-1}) = c_{xy}^{-1}(dc_{xy})c_{xy}^{-1} = c_{xy}^{-1}c_{xx} + c_{yy}c_{xy}^{-1} + \sum_{b\in\mathcal{G}'} c_{xy}^{-1}c_{xb}c_{by}c_{xy}^{-1}. \tag{3.26}$$

Combining (3.19), (3.22), (3.24) and (3.26), we see equation (3.18) holds.

2. We want to prove that f and g satisfy the structure equation for type D morphisms, which is, for a $\mathbf{k}$-module map $\phi : M \to A \otimes N$:

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta_N^1) \circ \phi + (\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \phi) \circ \delta_M^1 + (\mu_1 \otimes \mathbb{I}_N) \circ \phi = 0. \tag{3.27}$$

For a map ϕ and a, b generators of M and N, respectively, let ϕ_{ab} be the coefficient of b in $\phi(a)$. Then, applying the above equation to an arbitrary generator a of M, we see that the structure equation is equivalent to

$$d\phi_{ac} + \sum_{b\in\mathcal{G}(N)} \phi_{ab}c_{bc,N} + \sum_{b'\in\mathcal{G}(M)} c_{ab',M}\phi_{b'c} = 0 \tag{3.28}$$

for any $c \in \mathcal{G}(N)$.

For the map $f : M \to A \otimes M'$, we have:

$$f_{yb} = c_{xy}^{-1}c_{xb}, \quad f_{bb} = 1 \text{ for } b \in \mathcal{G}', \tag{3.29}$$

and all other coefficients zero. In (3.28), if $a = x$, then only the third term remains, and

$$c_{xy}f_{yc} + \sum_{b'\in\mathcal{G}'} c_{xb'}f_{b'c} = c_{xy}c_{xy}^{-1}c_{xc} + c_{xc} = 0,$$

as desired. If $a = y$:

$$\begin{aligned} & df_{yc} + \left(c_{yx}f_{xc} + c_{yy}f_{yc} + \sum_{b\in\mathcal{G}'} f_{yb}c'_{bc} \right) + \sum_{b\in\mathcal{G}'} c_{yb}f_{bc} \\ = \quad & d(c_{xy}^{-1}c_{xc}) + c_{yy}c_{xy}^{-1}c_{xc} + c_{yc} + \sum_{b\in\mathcal{G}'} c_{xy}^{-1}c_{xb}(c_{bc} + c_{by}c_{xy}^{-1}c_{xc}). \end{aligned}$$

Using (3.23) for c_{yc} and (3.25) for $\sum_{b\in\mathcal{G}'} c_{xb}c_{by}$, we can check this is indeed zero. Finally, if $a \in \mathcal{G}'$, then (3.28) reduces to:

$$0 = c'_{ac} + c_{ay}f_{yc} + c_{ac} = c'_{ac} + c_{ay}c_{xy}^{-1}c_{xc} + c_{ac},$$

which follows by the definition of c'_{ac}.

For $g : M' \to A \otimes M$, we have:

$$g_{aa} = 1, \quad g_{ax} = c_{ay}c_{xy}^{-1} \text{ for } a \in \mathcal{G}', \tag{3.30}$$

and all other coefficients zero. This time we go over cases for c in (3.28). If $c = x$, then:

$$\begin{aligned} & dg_{ax} + g_{ax}c_{xx} + g_{ay}c_{yx} + \sum_{b\in\mathcal{G}'} g_{ab}c_{bx} + \sum_{b\in\mathcal{G}'} c'_{ab}g_{bx} \\ = \quad & d(c_{ay}c_{xy}^{-1}) + c_{ay}c_{xy}^{-1}c_{xx} + c_{ax} + \sum_{b\in\mathcal{G}'}(c_{ab} + c_{ay}c_{xy}^{-1}c_{xb})c_{by}c_{xy}^{-1} \end{aligned}$$

As in the calculations for f, we use (3.21) to replace c_{ax} and (3.25) to replace $\sum_{b\in\mathcal{G}'} c_{xb}c_{by}$, to get this is zero.

If $c = y$, then only the second term remains, with

$$g_{ax}c_{xy} + \sum_{b\in\mathcal{G}'} g_{ab}c_{by} = c_{ay}c_{xy}^{-1}c_{xy} + c_{ay} = 0.$$

If $c \in \mathcal{G}'$, then

$$0 = g_{ax}c_{xc} + c_{ac} + c'_{ac} = c_{ay}c_{xy}^{-1}c_{xc} + c_{ac} + c'_{ac},$$

which follows by the definition of c'_{ac}.

3. In general, the composition of two maps $\phi : M \to A \otimes N$ and $\psi : N \to A \otimes P$ is given by:

$$\psi \circ \phi = (\mu_2 \otimes \mathbb{I}_P) \circ (\mathbb{I}_A \otimes \psi) \circ \phi.$$

In terms of coefficients, this is:

$$(\psi \circ \phi)_{ab} = \sum_{c\in\mathcal{G}(N)} \phi_{ac}\psi_{cb}, \tag{3.31}$$

for any $a \in \mathcal{G}(M)$ and $b \in \mathcal{G}(P)$.

We first check that $f \circ g : M' \to A \otimes M'$ is the identity morphism. That is, $(f \circ g)_{aa} = 1$ for all $a \in \mathcal{G}'$ and other coefficients are zero. We have:

$$(f \circ g)_{ab} = \sum_{c\in\mathcal{G}} g_{ac}f_{cb}.$$

However, from the expressions for f and g we see neither $c = x$ and $c = y$ contributes. Then the only contributing term is $c = a = b$, with $f_{aa}g_{aa} = 1$.

Next, we show $g \circ f = \mathbb{I} + h \circ \delta^1 + \delta^1 \circ h$. For $h : M \to A \otimes M$, we have:

$$h_{yx} = c_{xy}^{-1},$$

and all other coefficients zero. On the left side,

$$(g \circ f)_{ab} = \sum_{c \in \mathcal{G}'} f_{ac} g_{cb}.$$

If $a, b \in \mathcal{G}'$, then the only nonzero term is $a = b = c$ giving $(g \circ f)_{aa} = 1$ for $a \in \mathcal{G}'$. If $a = y$ and $b \in \mathcal{G}'$, then $(g \circ f)_{yb} = f_{yb} = c_{xy}^{-1} c_{xb}$. If $b = x$ and $a \in \mathcal{G}'$, then $(g \circ f)_{ax} = g_{ax} = c_{ay} c_{xy}^{-1}$. Finally, if $a = y$ and $b = x$, then

$$(g \circ f)_{yx} = \sum_{c \in \mathcal{G}'} f_{yc} g_{cx} = \sum_{c \in \mathcal{G}'} c_{xy}^{-1} c_{xc} c_{cy} c_{xy}^{-1} = d(c_{xy}^{-1}) + c_{xy}^{-1} c_{xx} + c_{yy} c_{xy}^{-1}.$$

Now on the right side, the identity term contributes $\mathbb{I}_{xx} = 1, \mathbb{I}_{yy} = 1$ and $\mathbb{I}_{aa} = 1$ for $a \in \mathcal{G}'$. The term $h \circ \delta^1$ contributes

$$(h \circ \delta^1)_{ab} = \sum_{c \in \mathcal{G}} c_{ac} h_{cb}.$$

For this to be nonzero we must have $b = x$ and $c = y$, then this is $(h \circ \delta^1)_{ax} = c_{ay} h_{yx} = c_{ay} c_{xy}^{-1}$. The $a = x$ part gives the identity term for x. Setting $a = y$ gives $c_{yy} c_{xy}^{-1}$, and $a \in \mathcal{G}'$ gives $c_{ay} c_{xy}^{-1}$. Likewise, the term $\delta^1 \circ h$ contributes:

$$(\delta^1 \circ h)_{ab} = \sum_{c \in \mathcal{G}} h_{ac} c_{cb}.$$

For this to be nonzero we must have $a = y$ and $c = x$, then this is $(\delta^1 \circ h)_{yb} = h_{yx} c_{xb} = c_{xy}^{-1} c_{xb}$. The $b = y$ part gives the identity term for y. Setting $b = x$ gives $c_{xy}^{-1} c_{xx}$, and $b \in \mathcal{G}'$ gives $c_{xy}^{-1} c_{xb}$. Checking all the terms, we see the two sides are the same as long as $d(c_{xy}^{-1}) = 0$, or equivalently $d(c_{xy}) = 0$.

□

The cancellation lemma for type DA bimodules follows from that for type D modules, by viewing type DA bimodules over A' and A as type D modules over $\widehat{\mathrm{Cob}}(A) \otimes A'$. Using this correspondence,

we can define "coefficients" on a type DA bimodule:

Definition 15. *Given two generators x, y of M, define the coefficient C_{xy} to be the formal sum, in $\widehat{\mathrm{Cob}}(A') \otimes A$, of $(a_1^* \otimes \cdots \otimes a_i^*) \otimes c$ over all arrows of the form $\delta^1_{1+i}(x; a_1, \ldots, a_i) \to c \otimes y$.*

This allows us to state the cancellation lemma for type DA bimodules:

Theorem 16 (Cancellation Lemma for Type DA structures)**.** *Let M be a type DA bimodule with type D action by A' on the left and A_∞ action by A on the right. Let $\mathcal{G}$ be the set of generators of M. Suppose there are $x, y \in \mathcal{G}$ such that $C_{xy} = 1 \otimes a$ with a invertible in A and $da = 0$. Set $C_{xy}^{-1} = 1 \otimes a^{-1}$. Then there are:*

1. *A type DA bimodule M', generated by $\mathcal{G}' = \mathcal{G} \backslash \{x, y\}$, with coefficients $C'_{ab} = C_{ab} + C_{ay} C_{xy}^{-1} C_{xb}$.*
2. *Two type DA morphisms $f : M \otimes A'^{\otimes} \to A \otimes M'$ and $g : M' \otimes A'^{\otimes} \to A \otimes M$ with coefficients $f_{yb} = C_{xy}^{-1} C_{xb}, f_{bb} = 1$ for $b \in \mathcal{G}'$, and $g_{aa} = 1, g_{ax} = C_{ay} C_{xy}^{-1}$ for $a \in \mathcal{G}$, and all other coefficients zero.*
3. *A map $h : M \otimes A'^{\otimes} \to A \otimes M$ with coefficient $h_{yx} = C_{xy}^{-1}$, and all other coefficients zero, such that $f \circ g = \mathbb{I}$ and $g \circ f = \mathbb{I} + h \circ \delta^1 + \delta^1 \circ h$.*

These imply that M' is homotopy equivalent to M.

We end this section with a remark on gradings. If M is graded by a grading set S_M, and if every generator being cancelled is homogeneous in grading, then M' is also graded by S_M, with the grading of each generator in M' equal to the grading of the corresponding generator in M. The arrows that are added to M' satisfy the grading constraints, because they come from traversing a zig-zag as in Figure 3.2, where each of the three arrows in the zig-zag satisfy the grading constraints. The homogeneity condition of cancelled generators will be automatically satisfied in our case.

3.3.2 Characterization of the identity bimodule

In this section, we give two lemmas on certain conditions under which we can assert a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$ is homotopy equivalent to the identity bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$. The main result we use is the characterization of the type DD invariant of identity diffeomorphism, given in [16]. We will start by reviewing that result here.

Definition 17 ([16, Definition 3.1])**.** *The* diagonal subalgebra *of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$ is the algebra generated by $a \otimes b$, where a and b satisfy the following conditions:* $\mathrm{mult}(a) = \mathrm{mult}(\bar{b})$*, the left idempotents of a and b are complementary, and the right idempotents of a and b are complementary.*

Theorem 18 ([16, Proposition 3.8, Proof of Theorem 1])**.** *Let M be a left type DD bimodule over $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$, where $\mathcal{Z}$ has genus greater than one. Suppose M satisfies the following conditions, then M is isomorphic to $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$.*

1. *The generators of M are in one-to-one correspondence with the idempotents of $\mathcal{A}(\mathcal{Z})$, so that the generator corresponding to idempotent i has (type D) idempotents i and $\overline{o(i)}$.*
2. *For any arrow $\mathbf{x} \to a \otimes b \otimes \mathbf{y}$ in the differential of M, the element $a \otimes b$ lies in the diagonal subalgebra.*
3. *M is graded by a λ-free grading set S, with a left-right $G(\mathcal{Z})$-$G(\mathcal{Z})$ action.*
4. *The differential in M contains all arrows of the form*

$$\mathbf{x} \to a(\xi) \otimes \overline{a(\xi)} \otimes \mathbf{y},$$

where ξ is a length-1 chord.

In the case where $\mathcal{Z}$ has genus one, if M satisfies an additional stability condition, in the sense of [16, Definition 1.10], then we can still conclude that $M = \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$.

The following result will be used in the proof of the second lemma:

Proposition 19. *Let $M = {}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$ be a left-right type DA bimodule over $\mathcal{A}(\mathcal{Z})$-$\mathcal{A}(\mathcal{Z})$. Suppose M satisfies the following two conditions:*

1. *M is homotopy equivalent to the identity bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$.*
2. *The generators of M are in one-to-one correspondence with the idempotents of $\mathcal{A}(\mathcal{Z})$, so that the generator corresponding to idempotent i has both left (type D) and right (type A) idempotent equal to i.*

Then the type DA action on M contains all arrows of the form

$$\delta_2^1(\mathbf{x}, a(\xi)) \to a(\xi) \otimes \mathbf{y}, \tag{3.32}$$

where ξ is a length-1 chord.

Proof. Consider generators $\mathbf{x}, \mathbf{y}$ corresponding to idempotents $i, j \in \mathcal{A}(\mathcal{Z})$, and ξ a length-1 chord, such that the idempotent matches in the arrow (3.32). We want to show that (3.32) does exist as an arrow.

Let T_D be a type D structure over $\mathcal{A}(\mathcal{Z})$ with two generators $\mathbf{x}_D$ and $\mathbf{y}_D$, whose idempotents are i and j, such that $\delta^1(\mathbf{x}_D) = a(\xi) \otimes \mathbf{y}_D$ and $\delta^1(\mathbf{y}_D) = 0$. It is clear that δ^1 satisfies the type D structure equations.

Likewise, let T_A be the A_∞-module over $\mathcal{A}(\mathcal{Z})$ with two generators $\mathbf{x}_A$ and $\mathbf{y}_A$ whose idempotents are i and j, such that $m_{1,1}(\mathbf{x}_A, a(\xi)) \to \mathbf{y}_A$ is the only arrow in the A_∞ action.

Consider the tensor product $T_A \boxtimes N \boxtimes T_D$, with $N = M$ or $N = \mathbb{I}$. This is a chain complex with two generators $\mathbf{x}_A \otimes \mathbf{x} \otimes \mathbf{x}_D$ and $\mathbf{y}_A \otimes \mathbf{y} \otimes \mathbf{y}_D$, and there is an arrow between these two if and only if the arrow (3.32) exists in N for the given $\mathbf{x}, \mathbf{y}$ and $a(\xi)$. In particular, $T_A \boxtimes \mathbb{I} \boxtimes T_D$ has zero homology. By assumption, $M \simeq \mathbb{I}$, so $T_A \boxtimes M \boxtimes T_D$ must also have zero homology. This shows the arrow (3.32) exists in M. □

We now state and prove the lemmas on the characterization of $\widehat{CFDA}(\mathbb{I}_\mathcal{Z})$.

Lemma 20. *Let $M = {}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$ be a left-right type DA bimodule over $\mathcal{A}(\mathcal{Z})$-$\mathcal{A}(\mathcal{Z})$. Suppose M satisfies the following properties, then M is homotopy equivalent to the identity bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$.*

- *(ID-1). The generators of M are in one-to-one correspondence with the idempotents of $\mathcal{A}(\mathcal{Z})$, so that the generator corresponding to idempotent i has both left (type D) and right (type A) idempotent equal to i.*
- *(ID-2). M can be graded by a principal left-right $G(\mathcal{Z})$-$G(\mathcal{Z})$ set, such that the induced map $\phi \in \mathrm{Out}(G(\mathcal{Z}), G(\mathcal{Z}))$ (as in [16, Lemma 5.5]) is the identity map, and there is a choice of relative grading with every generator having grading zero. (The choice of grading refinement for $G(\mathcal{Z})$ is arbitrary but must be the same on both sides).*
- *(ID-3). The type DA action on M contains all arrows of the form*

$$\delta_2^1(\mathbf{x}, a(\xi)) \to a(\xi) \otimes \mathbf{y},$$

  where ξ is a length-1 chord.

- *(ID-4). M is stable in the sense of [16, Definition 1.10] (this condition is only necessary when $\mathcal{Z}$ is the unique genus-1 pointed matched circle).*

Proof. Consider the type DD bimodule $M_{DD} = M \boxtimes \widehat{CFDD}(\mathbb{I}_\mathcal{Z})$. We check that M_{DD} satisfies all the conditions of Proposition 18, which would show that M_{DD} is isomorphic to $\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$. Since $\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$ is quasi-invertible, we obtain $M \simeq \mathbb{I}$.

Using the fact that relative grading can be chosen on $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ so that every generator has grading zero, Condition (ID-2) on the grading of M implies a similar condition on the grading of M_{DD}. The constraint that the type DD operation must respect the grading implies the following: for each arrow

$$\mathbf{x} \to a \otimes b \otimes \mathbf{y},$$

the multiplicities of a and $\bar{b}$ must be the same. The idempotent conditions on the diagonal subalgebra follow from the constraints on idempotents on each arrow, and the fact that both $\mathbf{x}$ and $\mathbf{y}$ have complementary idempotents. This verifies condition (2) in Proposition 18.

The other deductions are trivial. (ID-1), (ID-2), and (ID-3) imply conditions 1, 3, and 4, respectively. Condition (ID-4) implies the stability of M_{DD}, needed for the genus 1 case. □

The condition (ID-3) can still be difficult to check in actual computations. It is possible to replace it as follows.

Lemma 21. *With the same notation as in Lemma 20, if M satisfies the conditions (ID-1), (ID-2), (ID-4), and the following condition, then it is homotopy equivalent to $\mathbb{I}$.*

- *(ID-3') M is invertible, with a quasi-inverse M' that satisfies conditions (ID-1) and (ID-2).*

Proof. It suffices to show that (ID-3'), together with the other conditions, imply (ID-3). We first show that $\delta_1^1 = 0$ on M or M'. That is, there are no arrows of the form

$$\delta_1^1(\mathbf{x}) \to a \otimes \mathbf{y}.$$

By the grading constraints on any arrow, the algebra generator a must have multiplicity zero. That is, it must be an idempotent in $\mathcal{A}(\mathcal{Z})$. However, this would mean that the grading of $\mathbf{x}$ and $\mathbf{y}$ differ by λ, contradicting the assumption that all generators in M (or M') have grading zero.

Both M and its quasi-inverse M' also satisfy condition (ID-1), so by [14, Lemma 2.2.50], they can be represented as ${}^{\mathcal{A}(\mathcal{Z})}[\phi]_{\mathcal{A}(\mathcal{Z})}$ and ${}^{\mathcal{A}(\mathcal{Z})}[\phi']_{\mathcal{A}(\mathcal{Z})}$ respectively, for A_∞-algebra morphisms $\phi, \phi' : \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z})$. Then $M \boxtimes M'$ is represented by ${}^{\mathcal{A}(\mathcal{Z})}[\phi' \circ \phi]_{\mathcal{A}(\mathcal{Z})}$.

Since $M \boxtimes M'$ satisfies the grading condition (ID-2), the map $\phi' \circ \phi$ must preserve gradings. This means that for $a = a(\xi)$ where ξ is a length-1 chord, the only possible term in $(\phi' \circ \phi)(a)$ is a. Since $M \boxtimes M'$ is homotopy equivalent to identity, by Proposition 19, we have $(\phi' \circ \phi)(a) = a$, which implies $\phi(a) \neq 0$. By the same grading argument, either $\phi(a) = 0$ or $\phi(a) = a$. So we must have $\phi(a) = a$. This shows ϕ is the identity map on length-1 chords, which implies condition (ID-3). □

3.3.3 Certain arrows in $\widehat{CFDA}$ of arcslides

In this section we will compute some of the arrows in $\widehat{CFDA}(\tau)$ for a general arcslide τ, using equation (3.1). From the description in the previous chapter, one can expect arrows in $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ to be extremely complicated in general. The same would then be true for arrows in $\widehat{CFDA}(\tau)$. We will manage this complexity by focusing only on arrows whose algebra coefficients have a small total length (say length 1 or 2 on each side). It turns out that these are sufficient to prove the needed properties of a tensor product of $\widehat{CFDA}$ of arcslides.

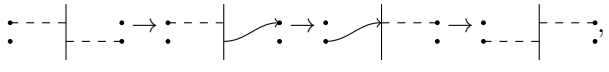
Since the algebra coefficients have small total length, the arrow is restricted to a small part of the Heegaard diagram. For arcslides, the parts of the Heegaard diagram that we are particularly interested in are the differences with the Heegaard diagram for the identity diffeomorphism — that is, around the points b_1, c_1, c_2 and b_1'.

One source of complexity comes from the fact that the definition of the homotopy map H in the construction of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ is non-local, in that it depends on an ordering $<_{\mathcal{Z}}$ on the intervals of the pointed matched circle. In a local situation, if we cannot tell which interval comes first in the ordering, we will need to cover all possible cases. Note only the restriction of $<_{\mathcal{Z}}$ to the intervals covered by the algebra coefficients matter for determining the arrows. Whether an arrow exists may depend on the ordering, but the overall result of the calculation will not.

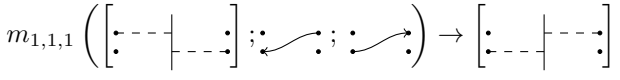
When we show a set of local arrows in a given local situation and restriction of the ordering $<_{\mathcal{Z}}$, we intend to make the following assertions:

- There is an arrow for every way of extending the local arrow by completing the pointed matched circle and adding the appropriate number of horizontal lines to the algebra coefficients.
- Every arrow in the bimodule action whose algebra coefficients lie within the area shown can be obtained by extending one of the local arrows.

We now begin with the simplest case: arrows in the type AA action on $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ where the algebra coefficients have length 1 on either side. The coefficients must then cover the same interval. The sequence of pairs $[a_{i,1}, a_{i,2}]$ is:



giving the arrow:



(3.33)

From (3.33), we obtain a simple method of deriving arrows in $\widehat{CFDA}(Y)$ from arrows in $\widehat{CFDD}(Y)$, in cases where the second coefficient of the type DD arrow has length 1 (the second algebra action is the one that is involved in the tensor product). For each type DD arrow $\mathbf{x} \to a_1 \otimes a_2 \otimes \mathbf{y}$, where a_2 has length 1, there corresponds a type DA arrow $\delta_2^1(\mathbf{x}, \bar{a_2}) \to a_1 \otimes \mathbf{y}$, where by abuse of notation we used the same letter to denote corresponding generators of $\widehat{CFDA}(Y)$ and $\widehat{CFDD}(Y)$.

As an application, we give a combinatorial proof of the following:

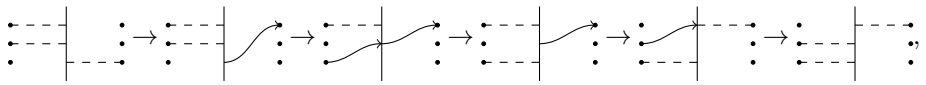
Corollary 22. *The tensor product $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ is homotopy equivalent to $\mathbb{I}$.*

Proof. Directly check each of the conditions in Lemma 20. For Condition (ID-2), we use the relative grading on $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ with all generators having grading zero (see section 2.5). For Condition (ID-3), use the type AA arrows computed here. For (ID-4), we note that attaching a pointed matched circle above or below $\mathcal{Z}$ does not affect the ordering of the intervals within $\mathcal{Z}$ that is used to describe the homotopy, so the type AA action restricted to $\mathcal{Z}$ is the same as before. □

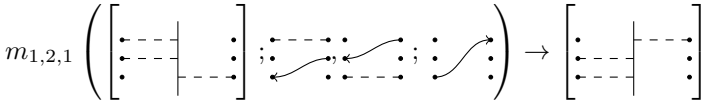
Type AA on a size 2 interval

The next simplest case for type AA arrows is the size 2 interval. We assume that no two of the three points are paired with each other. There are four subcases, depending on whether the middle idempotent is occupied on the left or on the right, and whether the lower or the upper interval comes first in the ordering $<_{\mathcal{Z}}$.

Middle idempotent on the left, lower interval first in ordering. The only sequence covering the size 2 interval is:

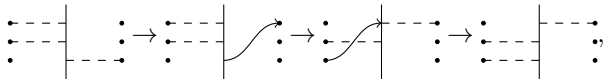


giving the arrow:

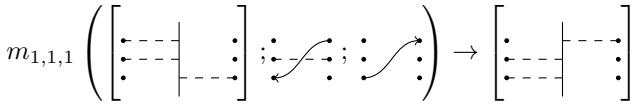


(3.34)

Middle idempotent on the left, upper interval first in ordering. The only sequence covering the size 2 interval is:

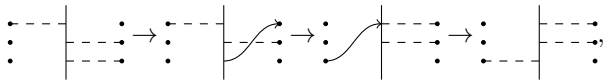


giving the arrow:

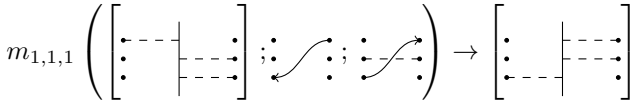


(3.35)

Middle idempotent on the right, upper interval first in ordering. The only sequence covering the size 2 interval is:

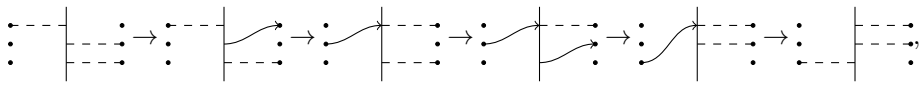


giving the arrow:

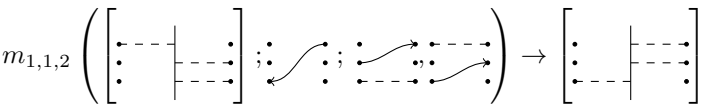


(3.36)

Middle idempotent on the right, lower interval first in ordering. The only sequence covering the size 2 interval is:



giving the arrow:

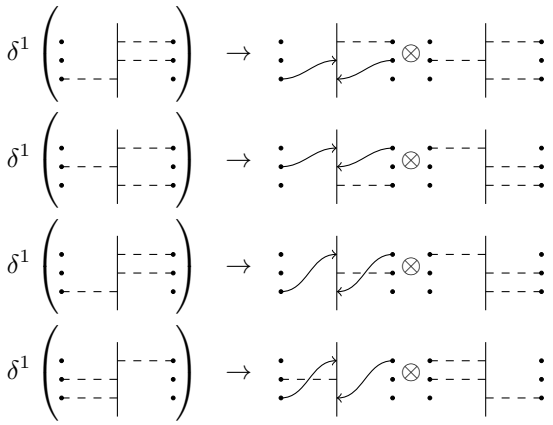


(3.37)

Type DA on a size 2 interval

We now compute the type DA arrows in $\widehat{CFDA}(\mathbb{I}_{\mathcal{Z}})$ that cover a size 2 interval. While the results here will not be directly used in the next two sections, it serves as a model for the calculations of similar arrows in $\widehat{CFDA}(\tau)$ for an arcslide τ.

To compute the type DA arrows, we combine the results of the previous section with what is known about type DD arrows in $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$. On the size 2 interval, the possibilities are given below. On each line, $\delta^1(\mathbf{x}) \to (a, a') \otimes \mathbf{y}$ represents the arrow $\delta^1(\mathbf{x}) \to a \otimes a' \otimes \mathbf{y}$, where $a \in \mathcal{A}(\mathcal{Z})$ and $a' \in \mathcal{A}(-\mathcal{Z})$.



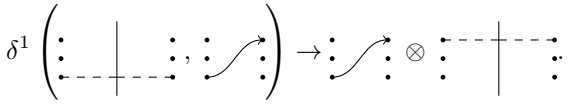
It is now a matter of combining these following the rules of the (box) tensor product. In the figures below, for both type DD and type AA bimodules, we will show the first algebra action on the left and the second algebra action on the right. This is purely for ease of visualization, and does not indicate which side the algebras act on. Indeed, both actions on the type DD bimodule are on the left, and both actions on the type AA bimodule are on the right. Nevertheless, we will often talk about "left action" or "left idempotents" to match how the figures are drawn.

Each type DA arrow comes from a single type AA arrow and zero or more type DD arrows. The left outputs (in $\mathcal{A}(\mathcal{Z})$) of the type DD arrows are multiplied together to give the overall type D output, while the right outputs (in $\mathcal{A}(-\mathcal{Z})$) are given as the left inputs to the type AA arrow. The overall type A inputs in $\mathcal{A}(\mathcal{Z})$ are given as the right inputs to the type AA arrow.

The right idempotent of the DD generator must agree with the left idempotent of the AA generator. The left idempotent of the DD generator and the right idempotent of the AA generator then combine to form the idempotent of the resulting DA generator.

We now look at each of the four cases in the previous section.

Middle idempotent on the left, lower interval first in ordering. The combination is shown as Figure 3.3. The resulting arrow is:



(3.38)

Middle idempotent on the left, upper interval first in ordering. The combination is shown in Figure 3.4. The resulting arrow is the same as in (3.38). So in this case the order of the two intervals already does not matter at the DA level.

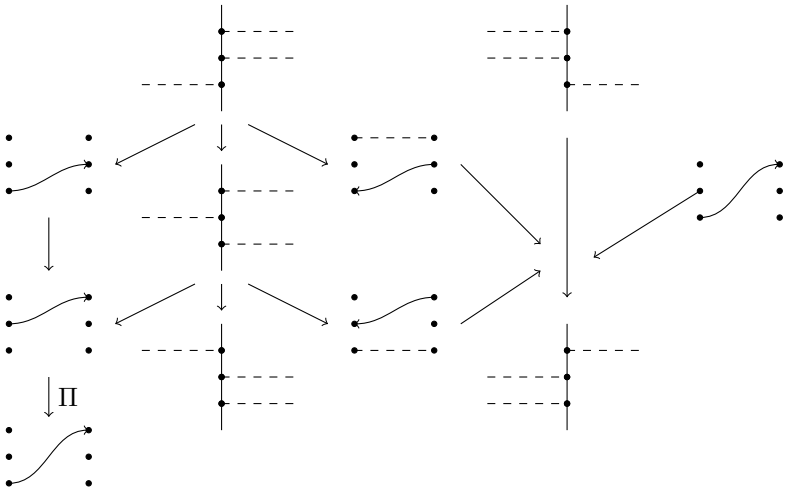


Figure 3.3: Formation of type DA operation, case 1.

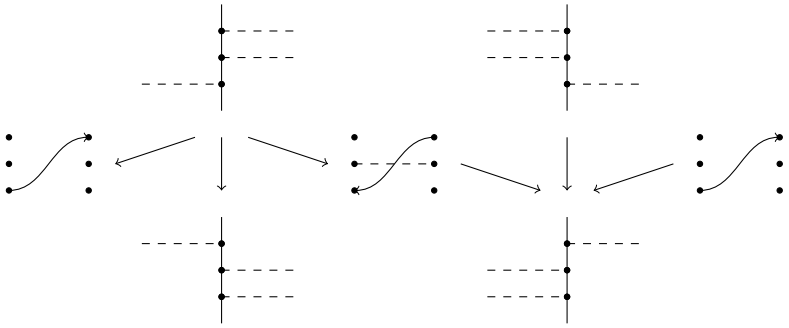
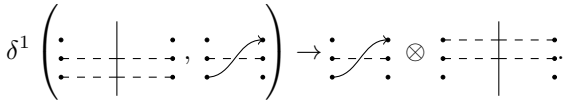


Figure 3.4: Formation of type DA operation, case 2.

Middle idempotent on the right, upper interval first in ordering. The combination is shown in Figure 3.5. The resulting arrow is:



(3.39)

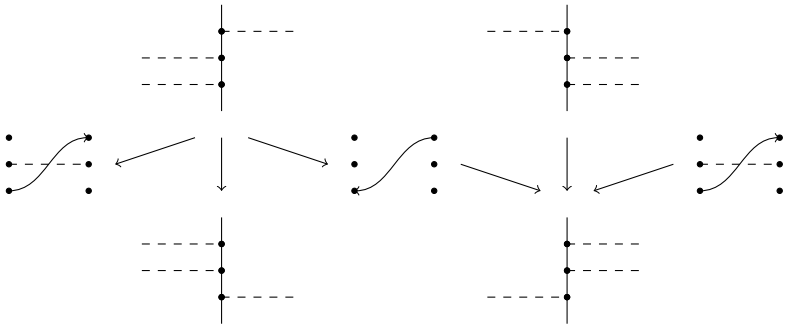
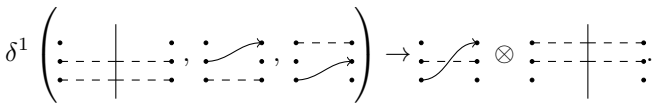


Figure 3.5: Formation of type DA operation, case 3.

Middle idempotent on the right, lower interval first in ordering. The combination is shown in Figure 3.6. The resulting arrow is:



(3.40)

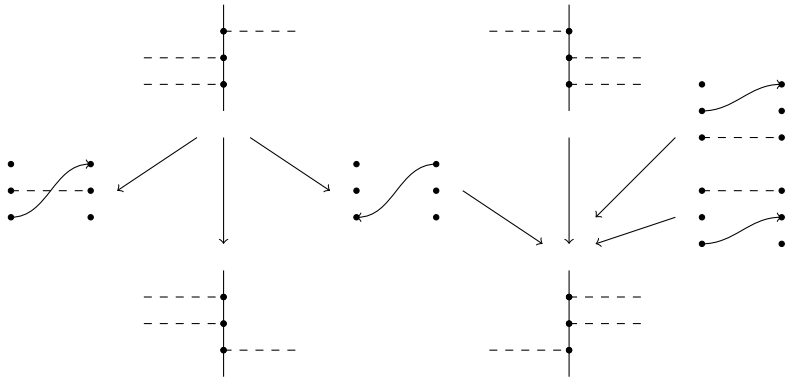


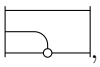
Figure 3.6: Formation of type DA operation, case 4.

This arrow shows that the $\widehat{CFDA}(\mathbb{I}_{\mathcal{Z}})$ we compute here is not exactly the same, but only homotopy equivalent to $\mathbb{I}$.

Pieces of Arcslide

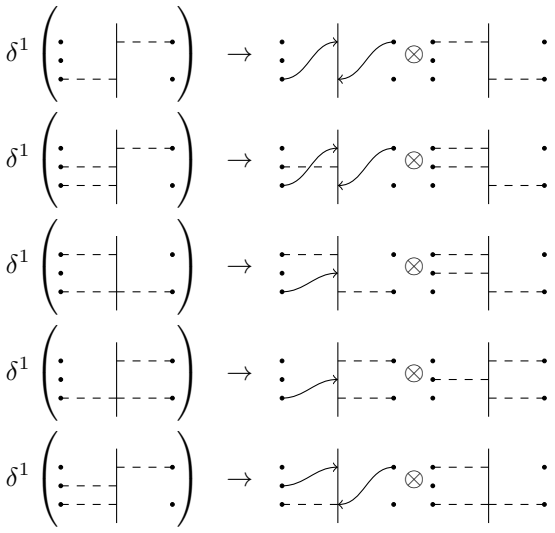
We now compute some simple arrows in $\widehat{CFDA}(\tau)$ for an arcslide τ. Since the method used here is similar to that in previous sections, we will show only the results.

The first local situation we consider is



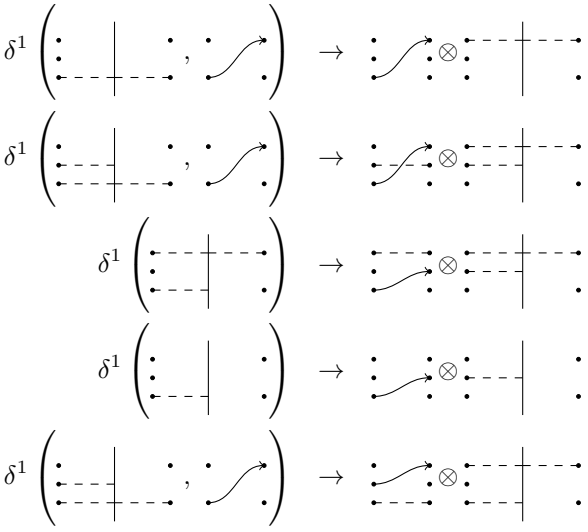
around the point b_1 on the left pointed matched circle.

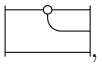
The possible type DD arrows are:



This comes directly from [16]. The only potentially tricky part is figuring out the possible locations of idempotents. For example, in the arrow $\delta^1(\mathbf{x}) \to (a(\sigma), 1) \otimes \mathbf{y}$ (third and fourth arrows above; (σ) is the chord $c_1 \to b_1$), the left idempotent of $\mathbf{y}$ must be occupied at the pair B and unoccupied at the pair C. Since generators of $\widehat{CFDD}$ of an arcslide either have complementary idempotents or idempotents that are both occupied at C, the right idempotent of $\mathbf{y}$ must be occupied at the pair C (so $\mathbf{y}$ is of type X). From the idempotent of $\mathbf{y}$, we can deduce that of $\mathbf{x}$, and see that $\mathbf{x}$ is of type Y. Similar arguments are used to list possible idempotents in the other cases.

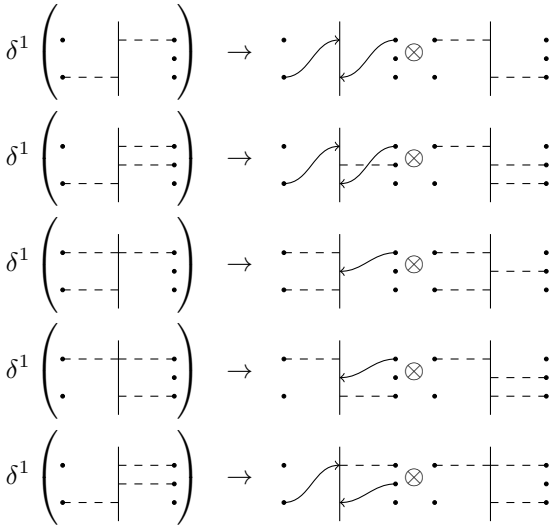
Computing the type DA arrows in this case is relatively straightforward, as we are combining with AA arrows on a size 1 interval (so we use the rule given at the beginning of this section). The results are:



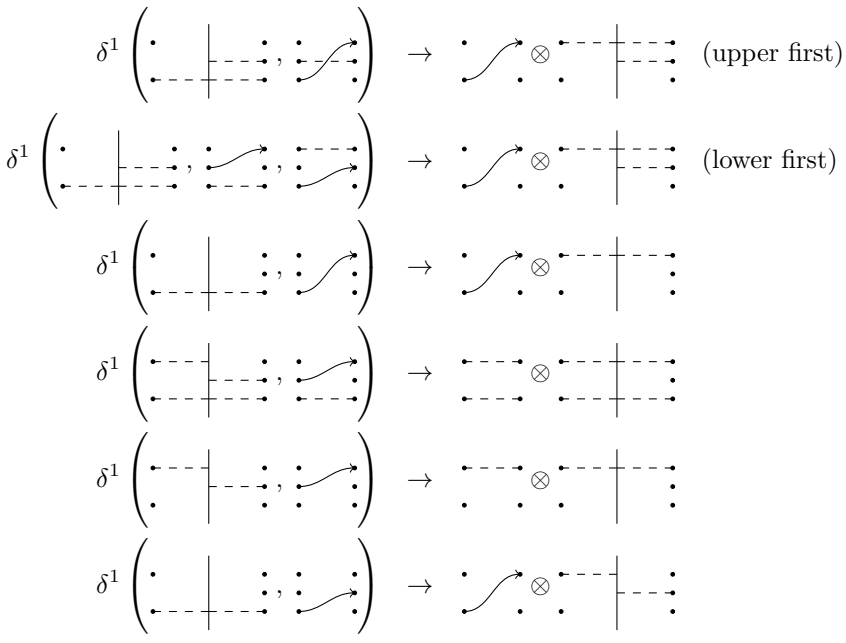
A more difficult case is ,

around b_1' on the right pointed matched circle.

The type DD operations are:



This time we will need to combine with type AA arrows on a size 2 interval, emulating the method in section 3.3.3. The results are:



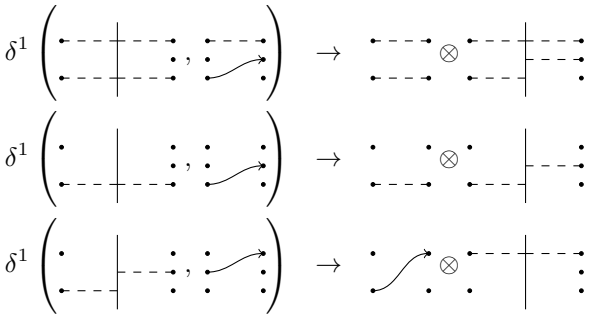
The first arrow exists only if the upper interval comes first in the ordering $<_{\mathcal{Z}'}$ for the right

pointed matched circle. The second arrow exists only if the lower interval comes first in the ordering. The third arrow does not depend on ordering. However, it is formed in different ways for the two orderings — there is one step on the DD side if the upper interval comes first, and two steps otherwise. The last three arrows use only a size 1 interval on the AA side, so they do not depend on ordering at all.

The case for



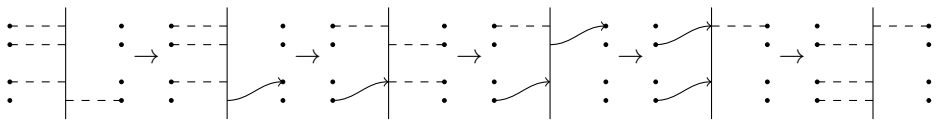
is very similar. In fact the first three arrows are identical to the previous case. We record the other three arrows here for reference.



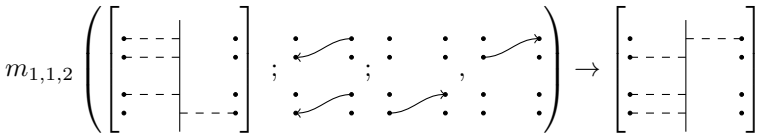
Type AA on two separated intervals with pairing

To consider more local situations for the arcslide, we will need type AA arrows on two separated intervals, such that either the two inner positions or the two outer positions are paired. These two cases are very similar, so we will only write out the first case here.

In this case, the upper interval immediately precedes the lower interval in the ordering $<_{\mathcal{Z}}$. If the middle idempotent (consisting of the two paired inner points) is occupied on the left, then it is not possible to multiply both intervals to the right as the first step. So the only possible sequence of $[a_{i,1}, a_{i,2}]$ is:



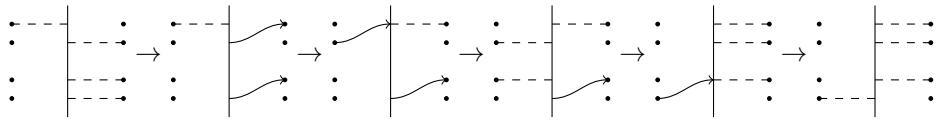
giving the arrow:



$$m_{1,1,2}\left(\left[\quad\right];\ ;\ ,\right)\to\left[\quad\right]. \tag{3.41}$$

In these diagrams, the two middle positions are paired, and there can be an arbitrary number of points between them in the full pointed matched circle. Since no arrow in $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ gives off an algebra element with two separate strands, this cannot be used to form a type DA arrow for the identity.

If the middle idempotent is occupied on the right, it is possible to multiply both intervals as the first step, but not possible to multiply only the lower interval. So the only sequence is:



This gives the type AA arrow:

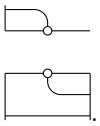
$$m_{1,2,1}\left(\left[\quad\right]\;;\;,\;;\;\right)\rightarrow\left[\quad\right]. \tag{3.42}$$

This leads to the following type DA arrow for identity:

$$\delta^1\left(\quad,\quad\right)\rightarrow\quad\otimes\quad. \tag{3.43}$$

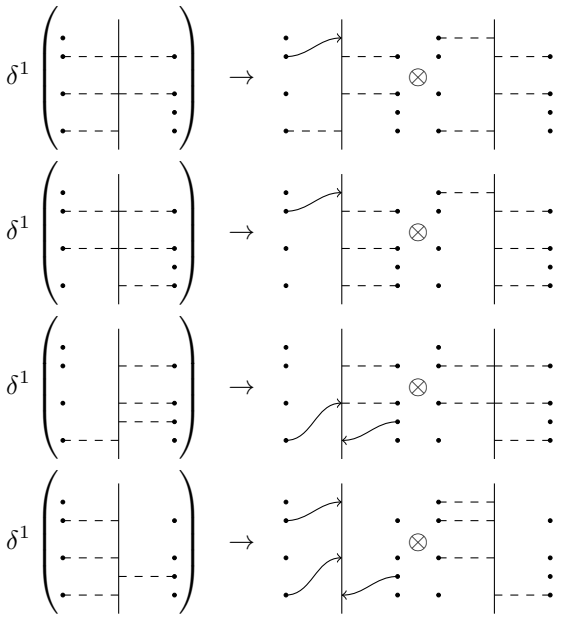
More Pieces of Arcslide

We now finish our analysis of various pieces of the arcslide. The next case we consider is

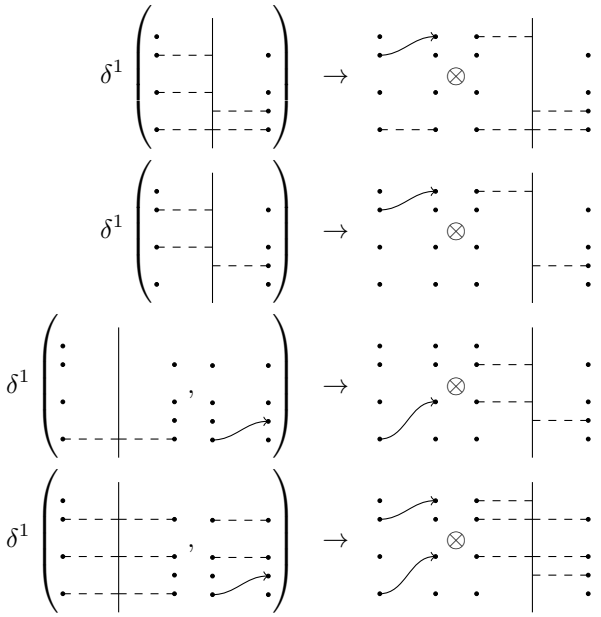


Here the two horizontal lines where the 1-handle is attached contain the α-arcs on the C pair. Immediately above and below these two lines are the points b_1 on the left and b_1' on the right.

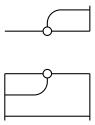
The type DD arrows are:



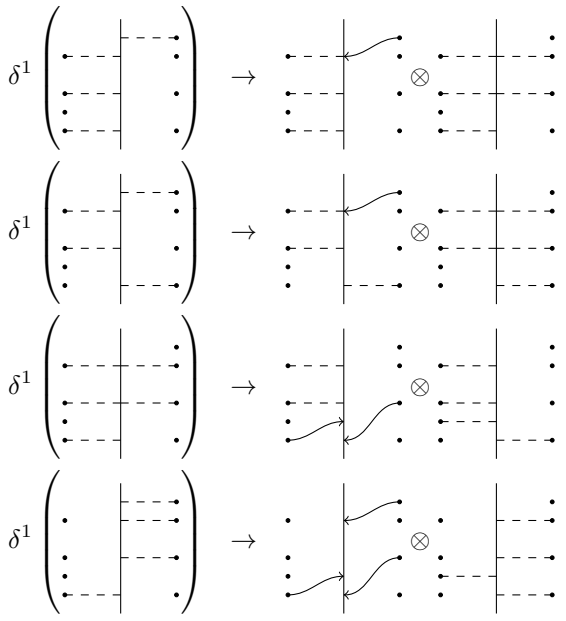
These give rise to type DA arrows



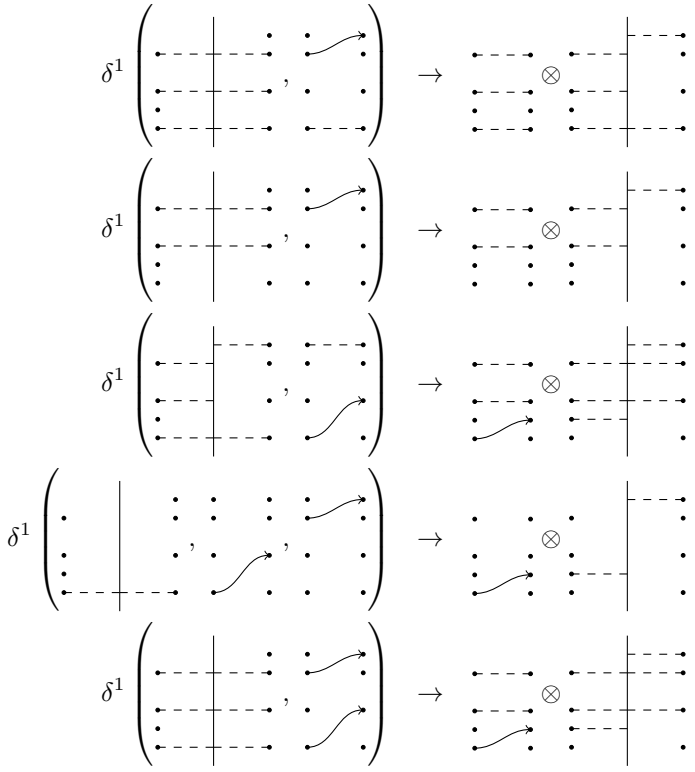
The symmetric case



is again more complicated. The type DD arrows are:



The resulting type DA arrows are:



The derivation of the last arrow has two steps on the DD side, using the first and third type DD arrow above.

3.3.4 Arcslides and their compositions

Given an arcslide τ, the description of the set of generators for $\widehat{CFDA}(\tau)$ follows from that for $\widehat{CFDD}(\tau)$ and $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$. The generators are classified by their idempotents on the two sides (type D idempotent on the left and type A idempotent on the right). As before, we use the canonical identification of pairs of points between the pointed matched circles on the two sides. There are two types of generators in $\widehat{CFDA}(\tau)$:

- Type X, where the idempotents on the two sides contain the same pairs.
- Type Y, where the idempotents on the two sides differ at exactly one pair, with the C pair occupied on the left and B pair occupied on the right.

Using this, and the definition of the (box) tensor product, we can enumerate the set of generators of

$$\widehat{CFDA}(\tau_1) \boxtimes \cdots \boxtimes \widehat{CFDA}(\tau_n)$$

for a sequence of arcslides $\tau_1, \ldots, \tau_n$. We now describe the procedure in detail.

For a single arcslide, there is an obvious identifification the pairs of points on the starting and ending pointed matched circle (identifying the B pairs with each other, and the C pairs with each other). By combining these identifications, we obtain an identification of pairs on all pointed matched circles appearing in the sequence. Note that even if the starting and ending pointed matched circle of a sequence is the same, the identification of pairs between the two, induced by the sequence of arcslides, may not be the identity. See the triangle relation for an example.

With this identification of pairs throughout a sequence, we can talk about a pair of points in the sequence, which are pairs of points, one for each pointed matched circle in the sequence, that are identified to be the same. We assign a number from 1 to d to each pair of points in the sequence that served as either the B pair or the C pair of some arcslide, where d is the total number of such pairs.

Each generator of the tensor product $\widehat{CFDA}(\tau_1) \boxtimes \cdots \boxtimes \widehat{CFDA}(\tau_n)$ is of the form $\mathbf{x}_1 \boxtimes \cdots \boxtimes \mathbf{x}_n$, where $\mathbf{x}_i$ is a generator of $\widehat{CFDA}(\tau_i)$ for each $1 \leq i \leq n$, and the right idempotent of $\mathbf{x}_i$ agrees with the left idempotent of $\mathbf{x}_{i+1}$ for each $1 \leq i < n$. A generator $\mathbf{x}_1 \boxtimes \cdots \boxtimes \mathbf{x}_n$ is determined by the set of occupied pairs at the starting and ending pointed matched circles, and at each pointed matched circles in the middle. It is clear that each unnumbered pair must be either occupied throughout or unoccupied throughout. For the numbered pairs, the only possible changes are as follows: suppose that for a certain arcslide τ_i in the sequence, the B pair is numbered b_i and the C pair is numbered

c_i, then it is possible to have c_i, but not b_i, occupied in the left idempotent of $\mathbf{x}_i$, and b_i, but not c_i, occupied in the right idempotent, with all other pairs staying the same. This corresponds to choosing $\mathbf{x}_i$ to have type Y.

We can therefore specify a type of generators by specifying which of the numbered pairs are occupied at each pointed matched circle. At each arcslide, the generator is either type X or type Y. In the first case, the occupied pairs must be the same before and after, and in the second case, the C pair for that arcslide occurs before and is replaced by the B pair. To choose a specific generator of a given type, it remains to choose which unnumbered pairs to occupy throughout, so that the total number of occupied pairs is g (half of the total $2g$ pairs).

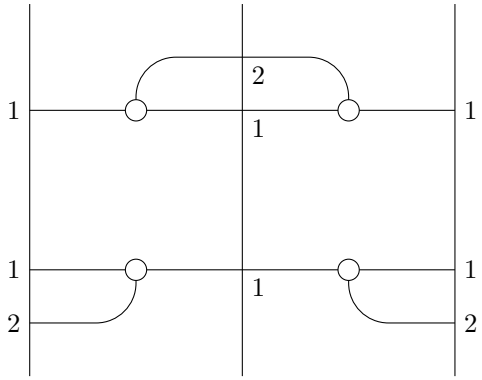


Figure 3.7: Heegaard diagram for involution relation.

We now study the involution relation as an example. This is the simplest case, which nevertheless illustrates most of the reasoning required. As shown in Figure 3.7, there are two numbered pairs ($d = 2$). Pair 1 served as the C pair and pair 2 served as the B pair for both arcslides. The possible types of generators are:

- ${}_{()}X_{()}X_{()}$
- ${}_{(1)}X_{(1)}X_{(1)}$
- ${}_{(2)}X_{(2)}X_{(2)}$
- ${}_{(12)}X_{(12)}X_{(12)}$
- ${}_{(1)}X_{(1)}Y_{(2)}$
- ${}_{(1)}Y_{(2)}X_{(2)}$

For generators with X at both positions, any combinations of occupying pairs are possible. For generators with X at first position and Y at second position, pair 1 and not pair 2 are occupied in

the middle, so 1 can be replaced by 2 at the end. This implies pair 1 and not pair 2 are occupied at the beginning as well. The same reasoning can be used for type YX, and for showing that type YY is not possible.

Later, we may use $(*)$ to denote an arbitrary subset of the numbered pairs, that stays the same for a given generator. So for example, we may collect the first four types above into ${}_{(*)}X_{(*)}X_{(*)}$.

We now state several general facts about $\widehat{CFDA}(\tau)$ and tensor products of such bimodules. These can be proved using the corresponding facts about $\widehat{CFDD}(\tau)$ in section 1.9, and the definition of $\widehat{CFDA}(\tau)$ as the tensor product of $\widehat{CFDD}(\tau)$ with $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$.

Remark:

1. Just as in the type DD case, each generator of $\widehat{CFDA}(\tau)$ corresponds to a tuple of points in the Heegaard diagram for the arcslide, with its left (type D) idempotent the set of unoccupied α-arcs on the left, and its right (type A) idempotent the set of occupied α-arcs on the right. When Heegaard diagrams of arcslides are glued side-by-side along their boundaries, the result is a larger Heegaard diagram that now contains α-circles. Note the boundaries that are glued along are removed from the resulting diagram. Each generator $\mathbf{x}_1 \boxtimes \cdots \boxtimes \mathbf{x}_n$ of the tensor product corresponds to a tuple of points, with each α and β circles containing exactly one point, and each α-arc containing at most one point.

   As in the type DD case, each arrow in $\widehat{CFDA}(\tau)$ corresponds to a domain away from the basepoint in the Heegaard diagram of the arcslide. Likewise, each arrow in the tensor product $\widehat{CFDA}(\tau_1) \boxtimes \cdots \boxtimes \widehat{CFDA}(\tau_n)$ corresponds to a domain in the Heegaard diagram obtained by gluing the diagrams for $\tau_1, \ldots, \tau_n$ in sequence. The multiplicity of the domain on the left (resp. right) boundary equals the total multiplicity of the algebra coefficients on the left (resp. right) of the arrow. The relation $\partial(\partial^{\alpha} B) = \mathbf{y} - \mathbf{x}$ when domain B represents an arrow from $\mathbf{x}$ to $\mathbf{y}$ still holds.

2. The remarks on the grading set of $\widehat{CFDD}(\tau)$ for an arcslide τ generalizes to the case of an arbitrary element ϕ of the strongly-based mapping class groupoid. Given $\phi : F^{\circ}(\mathcal{Z}_1) \to F^{\circ}(\mathcal{Z}_2)$, the bimodules $\widehat{CFDD}(\phi)$ and $\widehat{CFDA}(\phi)$ are both graded by a set S having free and transitive left-right actions by $G(\mathcal{Z}_1)$ and $G(\mathcal{Z}_2)$. The grading set induces an element of $\mathrm{Out}(G(\mathcal{Z}_1), G(\mathcal{Z}_2))$. If ϕ starts and ends at the same pointed matched circle $\mathcal{Z}$, then it induces an element of the outer automorphism group $\mathrm{Out}(G(\mathcal{Z}), G(\mathcal{Z}))$. That element can be found from the action of ϕ on the homology of the surface. In particular, the identity morphism on

$F^\circ(\mathcal{Z})$ induces the identity outer isomorphism on $G(\mathcal{Z})$.

3. For any $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, let $\widehat{CFDD}(\phi)$ be given by equation (3.2). Let $\bar{\phi} : F^\circ(-\mathcal{Z}_1) \to F^\circ(-\mathcal{Z}_2)$ be the map with reversed orientation. Then there is a duality

$$ {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\overline{\widehat{CFDD}(\phi)} \simeq {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\bar{\phi}). $$

We prove this next via an intermediate lemma.

Given a right type AA bimodule M over dg-algebras A and A' (with M finite-dimensional over $\mathbf{k}$), its dual $\overline{M}$ is a left type AA bimodule over A and A' with the same generators and all arrows reversed (this is analogous to [14, Definition 2.2.31, 2.2.53], called opposite structures there). With similar definitions of dual on other types of modules, taking dual commutes with box tensor product. That is,

$$ \overline{M_A \boxtimes {}^A N} = \overline{N}^A \boxtimes {}_A\overline{M}, $$

where M and N may have additional actions.

Lemma 23. *There is a self-duality in $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$:*

$$ \overline{\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})}_{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})} \simeq \widehat{CFAA}(\mathbb{I}_{-\mathcal{Z}})_{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})} \tag{3.44} $$

Here, when taking the dual of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}$, we took the opposite of the algebras, so they still act from the right side.

Proof. Both sides of the equation, when tensored with $\widehat{CFDD}(\mathbb{I}_{-\mathcal{Z}})$, produces a bimodule that can be shown to be homotopy equivalent to ${}^{\mathcal{A}(-\mathcal{Z})}\mathbb{I}_{\mathcal{A}(-\mathcal{Z})}$ using Lemma 20. Since $\widehat{CFDD}(\mathbb{I}_{-\mathcal{Z}})$ is quasi-invertible, this shows that the two sides of the equation are homotopy equivalent. □

Lemma 24. *Given a morphism $\phi : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, and $\bar{\phi} : F^\circ(-\mathcal{Z}_1) \to F^\circ(-\mathcal{Z}_2)$ be the morphism with reversed orientation. Then*

$$ {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\overline{\widehat{CFDD}(\phi)} \simeq {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\bar{\phi}) \tag{3.45} $$

Again, in taking the dual of ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\phi)$, we took the opposite of algebras so they still act from the left side.

Proof. We show the case where ϕ can be factored into two arcslides. The general case is similar.

Let $\phi = \tau_2 \circ \tau_1$, where $\tau_1 : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_m)$ and $\tau_2 : F^\circ(\mathcal{Z}_m) \to F^\circ(\mathcal{Z}_2)$. Then $\bar{\phi} = \bar{\tau_2} \circ \bar{\tau_1}$. We have

$$
\begin{aligned}
&{}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\phi) = \\
&\qquad {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_m)}\widehat{CFDD}(\tau_1) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_m})_{\mathcal{A}(-\mathcal{Z}_m),\mathcal{A}(\mathcal{Z}_m)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_m),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(\tau_2)
\end{aligned}
\tag{3.46}
$$

Taking the dual of both sides, and using duality for $\widehat{CFDD}$ of arcslides and Lemma 23 to rewrite the right side, we get:

$$
\begin{aligned}
&{}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\overline{\widehat{CFDD}(\phi)} = \\
&\qquad {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_m)}\widehat{CFDD}(\bar{\tau_1}) \boxtimes \widehat{CFAA}(\mathbb{I}_{-\mathcal{Z}_m})_{\mathcal{A}(\mathcal{Z}_m),\mathcal{A}(-\mathcal{Z}_m)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}_m),\mathcal{Z}(\mathcal{Z}_2)}\widehat{CFDD}(\bar{\tau_2}),
\end{aligned}
\tag{3.47}
$$

which is the same as the expansion of ${}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\bar{\phi})$. □

3.3.5 The involution relation

In this section we will verify the involution relation. Figure 3.7 shows one of the four possible cases. The computations for the other three cases are similar.

Recall that the tensor product is generated by three types of generators:

- ${}_{(*)}X_{(*)}X_{(*)}$
- ${}_{(1)}X_{(1)}Y_{(2)}$
- ${}_{(1)}Y_{(2)}X_{(2)}$

For each type XY generator, there is a corresponding type YX generator that occupies the same unnumbered pairs. The plan is to cancel out pairs of XY and YX generators using this correspondence, and show that the resulting bimodule satisfies the four conditions in Lemma 20.

There are five domains that contribute type DA arrows of interest. They are shown in Figures 3.8 and 3.9.

Domain C contributes an arrow with one idempotent coefficient from any XY generator to the corresponding YX generator. This allows us to cancel all XY and YX generators using the cancellation lemma. We now focus on the resulting bimodule, with the type XX generators.

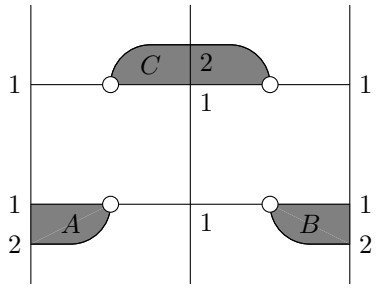


Figure 3.8: Domains that contributes type DA operations of interest.

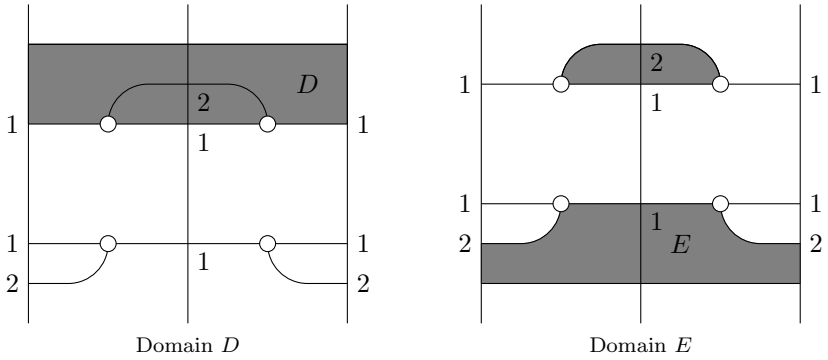
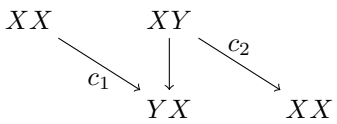


Figure 3.9: Domains that contributes type DA operations of interest.

This bimodule clearly satisfies (ID-1). Condition (ID-2) can be checked by explicit grading computations, which uses only the combinatorial features of the Heegaard diagram. In particular, the fact that the induced map $\phi \in \mathrm{Out}(G(\mathcal{Z}), G(\mathcal{Z}))$ is identity is equivalent to the fact that the action of this composition of arcslides on $H_1(F(\mathcal{Z}))$ is the identity. To verify the stability condition, (ID-4), adjoin a genus one pointed matched circle and Heegaard diagram for identity diffeomorphism to the bottom of the diagram. See Figure 3.10. Since the ordering $<_{\mathcal{Z}}$ among the original intervals is unchanged by the addition, the type DD and type DA arrows covering only the original part of the diagram are the same as before. So the stability condition is clear.

It remains to verify (ID-3). For this, we need to classify all arrows whose coefficients have length at most one on both boundaries. Such an arrow either exists before applying the cancellation lemma, or is produced via a zig-zag. In the first case, they correspond to one connected domain between type XX generators. They include trivial horizontal strips in regions away from the slide, domain D, and domain E. In the second case, the zig-zag must be of the form:



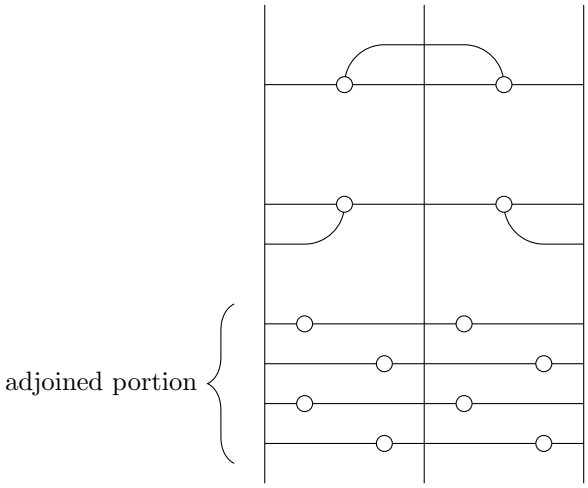
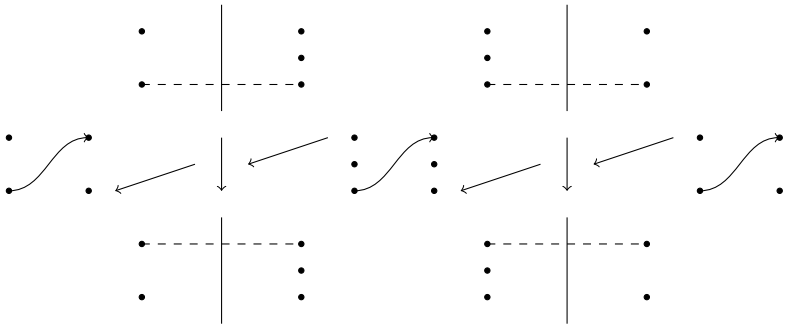


Figure 3.10: Adjoining a genus one PMC at the bottom.

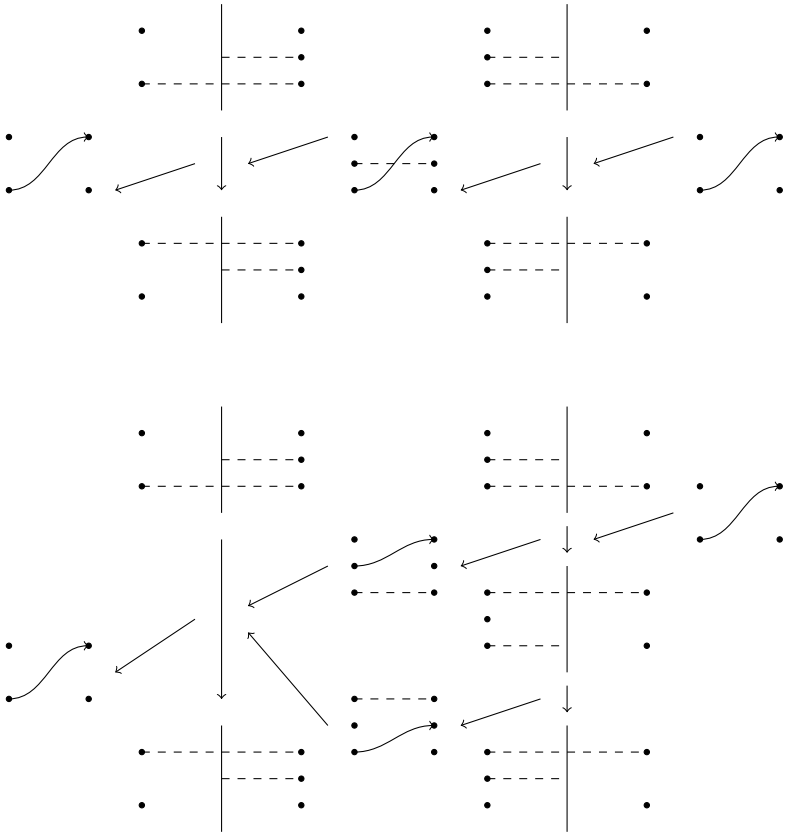
The coefficient c_1 must have length one on the left boundary and zero on the right, and c_2 must have length zero on the left and one on the right, or vice-versa. Looking at the Heegaard diagram, the only possibility is that c_1 is produced by domain A and c_2 by domain B.

In what follows, we show that for each of the domains A, B, D, E and starting and ending generators with the correct idempotents, there is exactly one arrow. These arrows, together with the ones coming from simple horizontal strips, cover each length-one interval exactly once, which verifies (ID-3).

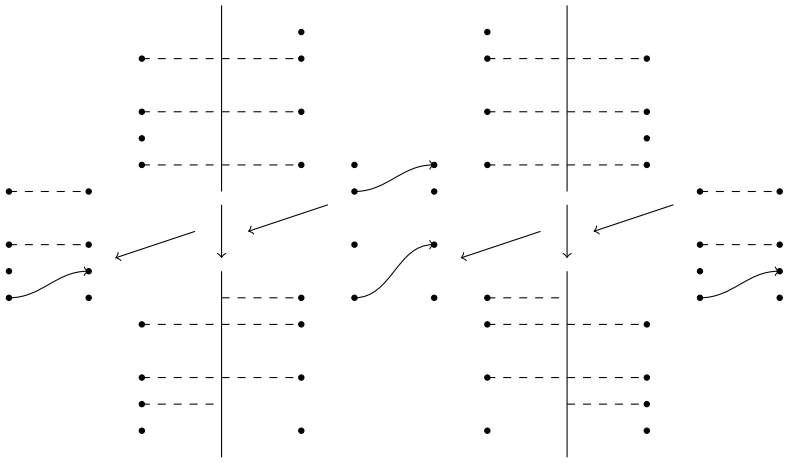
Domain A and B are straightforward since they involve only length one coefficients. The next case is domain D. If the B pair is not occupied, the arrow is produced by:



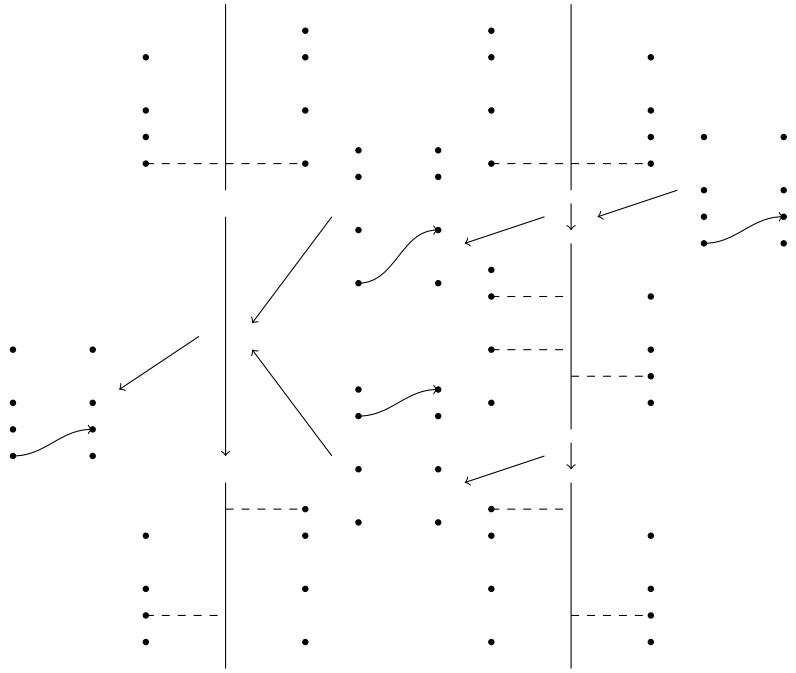
If the B pair is occupied, then the type DA arrow on the left side depends on the ordering. However, in either case we get one arrow after tensoring:



Now for domain E: if the C pair is occupied, the arrow is produced by:



and otherwise:



This finishes the verification of the involution relation.

Corollary 25. *The type DA bimodule for an arcslide is quasi-invertible, and the same is true for any tensor product of such bimodules.*

Proof. For a single arcslide, the quasi-inverse is given by $\widehat{CFDA}$ of the inverse arcslide. It is clear that tensor products of quasi-invertible bimodules are quasi-invertible. □

The computations here allow us to prove an uniqueness statement on $\widehat{CFDD}(\tau)$. A similar statement is proved in [16].

Corollary 26. *Let $\tau : F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ be an arcslide. If a bimodule ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}M$ is stable, has the same generators and gradings as $\widehat{CFDD}(\tau)$, and its type DD action matches that of $\widehat{CFDD}(\tau)$ on all arrows with sum of lengths of algebra coefficients at most 3, then M is homotopy equivalent to $\widehat{CFDD}(\tau)$.*

Proof. Let $M_{DA} = M \boxtimes \widehat{CFAA}(\mathbb{I})$. Since we only used type DD arrows whose algebra coefficients have sum of lengths at most 3 in this section, we can perform the same computations on M_{DA} as on $\widehat{CFDA}(\tau)$, showing that:

$$M_{DA} \boxtimes \widehat{CFDA}(\tau^{-1}) = \mathbb{I} = \widehat{CFDA}(\tau) \boxtimes \widehat{CFDA}(\tau^{-1}).$$

Since $\widehat{CFDA}(\tau^{-1})$ is quasi-invertible, we see M_{DA} is homotopy equivalent to $\widehat{CFDA}(\tau)$. Since $\widehat{CFAA}(\mathbb{I})$ is also quasi-invertible, we see M is homotopy equivalent to $\widehat{CFDD}(\tau)$. □

3.3.6 Other relations

For each of the other relations, we check the conditions in Lemma 21. (ID-2) is checked by explicit grading computations as before. (ID-3) follows from Corollary 25, with the inverse M' being the tensor product of inverse arcslides (in the opposite order). (ID-4) is shown by adjoining a genus one pointed matched circle as before. So it remains to show (ID-1) (with the same technique as given here we can show (ID-1) for the inverse M').

We now go through the cases.

Triangle

For the triangle relation, the Heegaard diagram for one of the possible cases is shown in Figure 3.11. The other cases differ from this one only by switching the ordering of the points and underslides with overslides. The enumeration of generators, and which pairs of generators can be cancelled, remain the same.

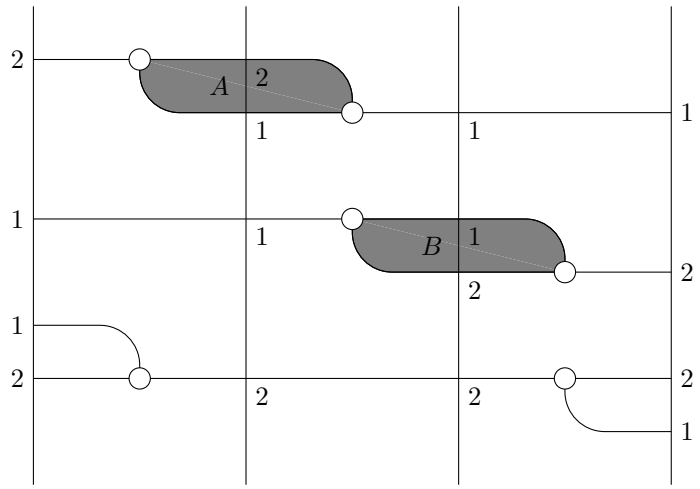


Figure 3.11: Heegaard diagram for triangle relation.

The roles of the numbered pairs are as follows:

- Arcslide 1: $C = 2, B = 1$.
- Arcslide 2: $C = 1, B = 2$.
- Arcslide 3: $C = 2, B = 1$.

Only the sequence YXY is forbidden. For that sequence, pair 1 must be occupied after the first arcslide, and therefore after the second arcslide, so type Y is not possible at the third arcslide. The possible types are:

- ${}_{(*)}X_{(*)}X_{(*)}X_{(*)}$
- ${}_{(2)}Y_{(1)}X_{(1)}X_{(1)}$
- ${}_{(1)}X_{(1)}Y_{(2)}X_{(2)}$
- ${}_{(2)}X_{(2)}X_{(2)}Y_{(1)}$
- ${}_{(2)}Y_{(1)}Y_{(2)}X_{(2)}$
- ${}_{(1)}X_{(1)}Y_{(2)}Y_{(1)}$
- ${}_{(2)}Y_{(1)}Y_{(2)}Y_{(1)}$
- ${}_{(1)}X_{(1)}Y_{(2)}X_{(2)}$

There are two domains that give rise to cancellable arrows: domain A and B as shown in the figure.

Domain A gives rise to arrows from $YY*$ to $XX*$, and domain B gives rise to arrows from $*YY$ to $*XX$. So the cancellable arrows are:

- $YYY \to XXY$
- $YYY \to YXX$
- ${}_{(1)}XYY_{(1)} \to {}_{(1)}XXX_{(1)}$
- ${}_{(2)}YYX_{(2)} \to {}_{(2)}XXX_{(2)}$

We choose to cancel everything except the second set $YYY \to YXX$ (cancelling the first set of arrows eliminates the option of cancelling the second). The remaining generators are:

$$ {}_{(2)}YXX_{(1)}, \quad {}_{(1)}XYX_{(2)}, \quad {}_{()}XXX_{()} \text{ and } {}_{(12)}XXX_{(12)}. $$

Since each type of idempotents at the two ends occurs exactly once, we have checked (ID-1). Note the 1-pair at the left corresponds to the 2-pair at the right, and vice versa, under the bijection of pairs coming from the equality of pointed matched circles at the two ends.

Commutativity

The Heegaard diagram for one of the cases is shown in Figure 3.12 (as in the triangle case, the other possibilities are similar).

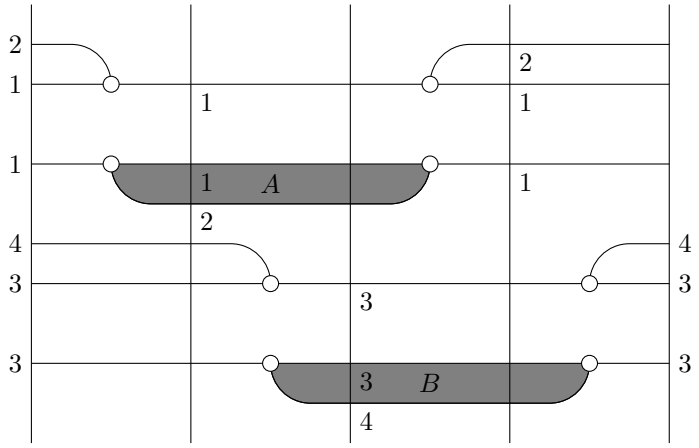


Figure 3.12: Heegaard diagram for commutativity relation.

The role of the numbered pairs are as follows:

- Arcslide 1: $C = 1, B = 2$.
- Arcslide 2: $C = 3, B = 4$.
- Arcslide 3: $C = 1, B = 2$.
- Arcslide 4: $C = 3, B = 4$.

The restriction on the types is that at most one of the types at arcslides 1 and 3 can be Y, and at most one at arcslides 2 and 4 can be Y. The possibilities are:

- ${}_{(*)}X_{(*)}X_{(*)}X_{(*)}X_{(*)}$
- ${}_{(1)}Y_{(2)}X_{(2)}X_{(2)}X_{(2)}$ with pairs 3 and/or 4 possibly added to each idempotent (similarly in the next three types).
- ${}_{(1)}X_{(1)}X_{(1)}Y_{(2)}X_{(2)}$ with 3 and/or 4 possibly added.
- ${}_{(3)}X_{(3)}Y_{(4)}X_{(4)}X_{(4)}$ with 1 and/or 2 possibly added.
- ${}_{(3)}X_{(3)}X_{(3)}X_{(3)}Y_{(4)}$ with 1 and/or 2 possibly added.
- ${}_{(13)}Y_{(23)}Y_{(24)}X_{(24)}X_{(24)}$

- ${}_{(13)}Y_{(23)}X_{(23)}X_{(23)}Y_{(24)}$
- ${}_{(13)}X_{(13)}Y_{(14)}Y_{(24)}X_{(24)}$
- ${}_{(13)}X_{(13)}X_{(13)}Y_{(23)}Y_{(24)}$

The two domains giving rise to cancellable arrows are labelled A and B in the figure.

Domain A gives rise to arrows from $YaXb$ to $XaYb$, for any valid choice of $a, b \in \{X, Y\}$. Likewise, domain B gives rise to arrows from $aYbX$ to $aXbY$. So the cancellable arrows are:

- ${}_{(1)}YXXX_{(2)} \to {}_{(1)}XXYX_{(2)}$ with 3 and 4 arbitrary
- ${}_{(3)}XYXX_{(4)} \to {}_{(3)}XXXY_{(4)}$ with 1 and 2 arbitrary

and

$$\begin{array}{ccc} {}_{(13)}YYXX_{(24)} & \longrightarrow & {}_{(13)}YXXY_{(24)} \\ \downarrow & & \downarrow \\ {}_{(13)}XYYX_{(24)} & \longrightarrow & {}_{(13)}XXYY_{(24)} \end{array}$$

The first two arrows cancel all generators with one Y. For generators with two Y's, we can either cancel both horizontal arrows or both vertical arrows in the square above. In the end, only generators of type ${}_{(*)}XXXX_{(*)}$ remain, which checks (ID-1).

Left and Right Pentagon

The Heegaard diagram for one of the cases of the left pentagon relation is shown in Figure 3.13. Other cases of the left and right pentagon relation are similar.

The role of the numbered pairs are as follows:

- Arcslide 1: $C = 2, B = 3$.
- Arcslide 2: $C = 1, B = 2$.
- Arcslide 3: $C = 2, B = 3$.
- Arcslide 4: $C = 1, B = 3$.
- Arcslide 5: $C = 1, B = 2$.

The possible types are:

- ${}_{(*)}X_{(*)}X_{(*)}X_{(*)}X_{(*)}$

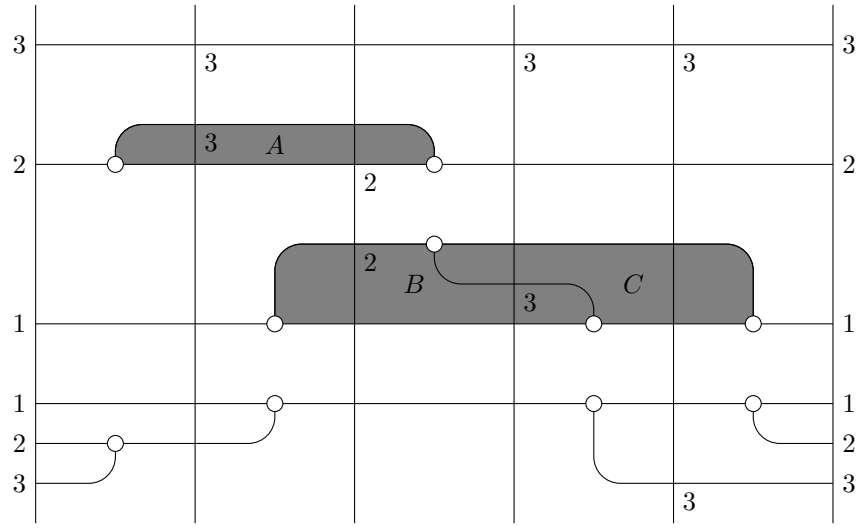


Figure 3.13: Heegaard diagram for pentagon relation.

- ${}_{(12)}Y_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}$
- ${}_{(12)}X_{(12)}X_{(12)}Y_{(13)}X_{(13)}Y_{(23)}$
- ${}_{(12)}Y_{(13)}X_{(13)}X_{(13)}X_{(13)}Y_{(23)}$
- ${}_{(1)}X_{(1)}Y_{(2)}Y_{(3)}X_{(3)}X_{(3)}$
- ${}_{(2)}Y_{(3)}X_{(3)}X_{(3)}X_{(3)}X_{(3)}$, ${}_{(12)}Y_{(13)}X_{(13)}X_{(13)}X_{(13)}X_{(13)}$
- ${}_{(1)}X_{(1)}Y_{(2)}X_{(2)}X_{(2)}X_{(2)}$, ${}_{(13)}X_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}$
- ${}_{(2)}X_{(2)}X_{(2)}Y_{(3)}X_{(3)}X_{(3)}$, ${}_{(12)}X_{(12)}X_{(12)}Y_{(13)}X_{(13)}X_{(13)}$
- ${}_{(1)}X_{(1)}X_{(1)}X_{(1)}Y_{(3)}X_{(3)}$, ${}_{(12)}X_{(12)}X_{(12)}X_{(12)}Y_{(23)}X_{(23)}$
- ${}_{(1)}X_{(1)}X_{(1)}X_{(1)}X_{(1)}Y_{(2)}$, ${}_{(13)}X_{(13)}X_{(13)}X_{(13)}X_{(13)}Y_{(23)}$

The main difficulty here is to analyze the domain $B+C$ and see when it contributes a cancellable arrow. This involves tensoring together four type DA bimodules as shown in Figure 3.14.

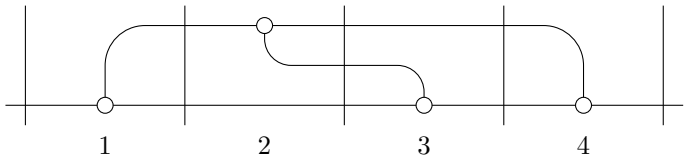
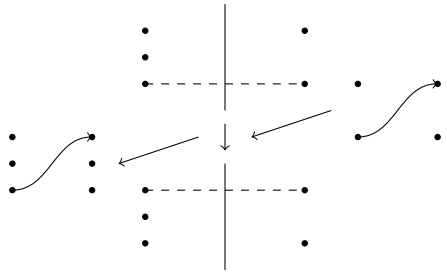
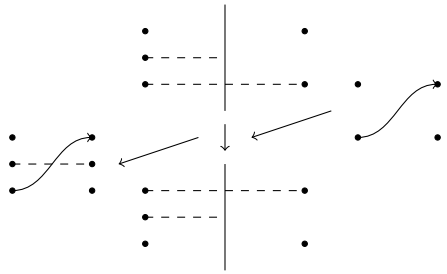


Figure 3.14: Domain $B+C$.

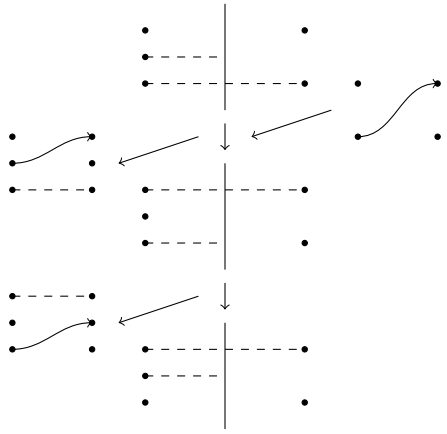
The arrow needed in the fourth piece is simple. For the third piece, there are several ways to cover the domain. First, if pair 3 is unoccupied in the middle pieces:



If pair 3 is occupied, there are two different ways, one of which involves two steps:



and



Now looking at the possible arrows in the second piece, we see that whether or not pair 3 is occupied, there is always exactly one way to continue forming the arrow in the tensor product to the second piece (and then trivially to the first piece). If pair 3 is occupied, which of the two ways shown above is used depends on the ordering $<_{\mathcal{Z}}$. This shows that the domain $B + C$ gives rise to arrows:

- $*_{(1)}X_{(1)}X_{(1)}X_{(1)}Y_{(2)} \rightarrow *_{(1)}Y_{(2)}X_{(2)}X_{(2)}X_{(2)}$
- $*_{(13)}X_{(13)}X_{(13)}X_{(13)}Y_{(23)} \rightarrow *_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}$

The other domains are:

- Domain A: $XXY** \to YXX**$
- Domain B: $*XXY* \to *YYX*$
- Domain C: $**YXY \to **XYX$

So the cancellable arrows are:

- ${}_{(12)}YXXXY_{(23)} \to {}_{(12)}YYXXX_{(23)}$
- ${}_{(1)}XXXXY_{(2)} \to {}_{(1)}XYXXX_{(2)}$
- ${}_{(13)}XXXXY_{(23)} \to {}_{(13)}XYXXX_{(23)}$
- ${}_{(12)}XXYXY_{(23)} \to {}_{(12)}YXXXY_{(23)}$
- ${}_{(2)}XXYXX_{(3)} \to {}_{(2)}YXXXX_{(3)}$
- ${}_{(12)}XXYXX_{(13)} \to {}_{(12)}YXXXX_{(13)}$
- ${}_{(1)}XXXYX_{(3)} \to {}_{(1)}XYYXX_{(3)}$
- ${}_{(12)}XXYXY_{(23)} \to {}_{(12)}XXXYX_{(23)}$

If all arrows, except for the fourth, are cancelled, only generators of type ${}_{(*)}XXXXX_{(*)}$ remain, which verifies (ID-1).

3.4 The 3-manifold invariant

In this section, we prove Theorem 13, showing that the homotopy type of the chain complex $\widehat{HF}$ given in Construction 12 does not depend on the choices made. There are two main components of the proof, given by the two lemmas below. We let $\mathrm{MCG}_0(\mathcal{Z}^g)$ denote the strongly-based mapping class group on $F_{g,1}$, parametrized by the genus g split pointed matched circle $\mathcal{Z}^g$.

Lemma 27 (Stabilization)**.** *Let ψ be an element of $\mathrm{MCG}_0(\mathcal{Z}^g)$. Consider $F_{g+1,1}$, parametrized by $\mathcal{Z}^{g+1}$, as the surface obtained from $F_{g,1}$ by adding a handle in a neighborhood of the basepoint. Let $\mathring{\psi}$ be the element of $\mathrm{MCG}_0(\mathcal{Z}^{g+1})$ that fixes the new handle and acts as ψ elsewhere. Then*

$$\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAA}(\psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)} \simeq \widehat{CFD}(\mathbf{H}^{g+1}) \boxtimes \widehat{CFAA}(\mathring{\psi}) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^{g+1})}. \tag{3.48}$$

Definition 28. *Define the group* $\mathrm{MCG}_0^\beta(\mathcal{Z}^g)$ *to be the subgroup of* $\mathrm{MCG}_0(\mathcal{Z}^g)$ *that extends to an automorphism of the 0-framed handlebody* $\mathbf{H}^g$*. Likewise, define* $\mathrm{MCG}_0^\alpha(\mathcal{Z}^g)$ *to be the corresponding subgroup for the* ∞*-framed handlebody* $-\mathbf{H}^g$*.*

Equivalently, $\mathrm{MCG}_0^\beta(\mathcal{Z}^g)$ is the subgroup of $\mathrm{MCG}_0(\mathcal{Z}^g)$ that are restrictions of automorphisms of $\mathbf{H}^g$ to its boundary, and similarly for $\mathrm{MCG}_0^\alpha(\mathcal{Z}^g)$.

Lemma 29 (Reparametrization of the 0-framed handlebody)**.** *For each element* $\phi \in \mathrm{MCG}_0^\beta(\mathcal{Z}^g)$*, we have*

$$\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAD}(\phi) \simeq \widehat{CFD}(\mathbf{H}^g). \tag{3.49}$$

We first show that these two lemmas imply Theorem 13.

Proof of Theorem 13. There are two choices made in Construction 12: the choice of Heegaard splitting $Y = Y_1 \cup Y_2$, and choice of parametrization of Y_1 and Y_2 by standard handlebodies. It is well-known that any two Heegaard splittings become isotopic after a finite number of stabilizations. Also, any stabilization can be isotopied to the standard one, adding a handle in a neighborhood of the basepoint. If ψ is a valid choice of element in $\mathrm{MCG}_0(\mathcal{Z}^g)$ in the second stage of the construction, then $\mathring{\psi}$ is a valid choice of element in $\mathrm{MCG}_0(\mathcal{Z}^{g+1})$ after a standard stabilization. So Lemma 27 implies that Construction 12 is invariant under stabilizations.

Now we consider choice of parametrization of Y_1 and Y_2. Any two choices of parametrization of Y_1 differs by an element $\phi_1 \in \mathrm{MCG}_0^\beta(\mathcal{Z}^g)$. Hence, changing the parametrization of Y_1 changes ψ to $\psi' = \psi \circ \phi_1$ for some $\phi_1 \in \mathrm{MCG}_0^\beta(\mathcal{Z}^g)$. Likewise, changing the parametrization of Y_2 changes ψ to $\psi' = \phi_2 \circ \psi$ for some $\phi_2 \in \mathrm{MCG}_0^\alpha(\mathcal{Z}^g)$.

It remains to show the following:

$$\begin{aligned}
\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAA}(\psi \circ \phi_1) &\simeq \widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAA}(\psi), \\
\widehat{CFAA}(\phi_2 \circ \psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)} &\simeq \widehat{CFAA}(\psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)}
\end{aligned}$$

for $\phi_1 \in \mathrm{MCG}_0^\beta(\mathcal{Z}^g)$ and $\phi_2 \in \mathrm{MCG}_0^\alpha(\mathcal{Z}^g)$.

The first equation follows directly from Lemma 29:

$$\begin{aligned}
& \widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAA}(\psi \circ \phi_1) \\
\simeq\ & \widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAD}(\phi_1) \boxtimes \widehat{CFAA}(\psi) \\
\simeq\ & \widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAA}(\psi).
\end{aligned}$$

For the second equation, we use the fact that reversing orientation in $\mathcal{Z}^g$ switches the standard positions of α and β circles. This means $\phi_2 \in \mathrm{MCG}_0^\alpha(\mathcal{Z}^g)$ if and only if $\overline{\phi_2} \in \mathrm{MCG}_0^\beta(\mathcal{Z}^g)$. We will also use the formula $\overline{\widehat{CFDD}(\psi)} = \widehat{CFDD}(\overline{\psi})$ stated earlier, and the fact that $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ is dual to $\widehat{CFAA}(\mathbb{I}_{-\mathcal{Z}})$. Then we have:

$$\begin{aligned}
& \widehat{CFAA}(\phi_2 \circ \psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)} \\
\simeq\ & \widehat{CFAA}(\psi) \boxtimes \widehat{CFDA}(\phi_2) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)} \\
\simeq\ & \widehat{CFAA}(\psi) \boxtimes \widehat{CFDD}(\phi_2) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}_g}) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)} \\
\simeq\ & \widehat{CFAA}(\psi) \boxtimes \overline{\widehat{CFDD}(\bar{\phi}_2)} \boxtimes \overline{\widehat{CFAA}(\mathbb{I}_{-\mathcal{Z}_g})} \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)} \\
\simeq\ & \widehat{CFAA}(\psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAA}(\mathbb{I}_{-\mathcal{Z}_g}) \boxtimes \widehat{CFDD}(\bar{\phi}_2)} \\
\simeq\ & \widehat{CFAA}(\psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAD}(\bar{\phi}_2)} \\
\simeq\ & \widehat{CFAA}(\psi) \boxtimes \overline{\widehat{CFD}(\mathbf{H}^g)}
\end{aligned}$$

□

We now conclude by proving the two lemmas, starting with stabilization invariance.

Proof of Lemma 27. Let $\tau : F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$ be an arcslide between parametrizations of $F_{g,1}$. We can extend the definition of stabilization to τ as follows. Let $\mathring{\mathcal{Z}}_1 = \mathcal{Z}_1 \# \mathcal{Z}^1$ and $\mathring{\mathcal{Z}}_2 = \mathcal{Z}_2 \# \mathcal{Z}^1$ (here $\mathcal{Z}^1$ is the genus 1 pointed matched circle, and $\cdot \# \cdot$ denotes the connect sum of pointed matched circles). From the parametrizations of $F_{g,1}$ by $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we obtain standard parametrizations of $F_{g+1,1}$ by $\mathring{\mathcal{Z}}_1$ and $\mathring{\mathcal{Z}}_2$, where $F_{g+1,1}$ is considered as the result of adding a handle to $F_{g,1}$ near the basepoint. Then define $\mathring{\tau} : F(\mathring{\mathcal{Z}}_1) \to F(\mathring{\mathcal{Z}}_2)$ as the arcslide acting as the identity on the new handle, and as τ elsewhere.

We will use Heegaard diagrams for the standard handlebody and arcslides to understand the invariants in our construction, noting that generators and arrows in the invariants correspond respectively to tuples of intersection points and domains in the diagram away from the basepoint (see the first Remark in section 3.3.4). The standard Heegaard diagram for $\mathring{\tau}$ is obtained from that for τ by adjoining the diagram for the identity diffeomorphism of the genus 1 surface to the top. For an arbitrary element ψ of the strongly-based mapping class groupoid, and a factorization $\psi = \tau_n \circ \cdots \circ \tau_1$, we obtain a Heegaard diagram for ψ as the result of gluing the standard diagrams of $\tau_1, \ldots, \tau_n$ in sequence. Then $\mathring{\psi}$ has factorization $\mathring{\tau}_n \circ \cdots \circ \mathring{\tau}_1$, and its Heegaard diagram can be obtained from that for ψ by adding to the top a "strip" of diagrams for the identity diffeomorphism

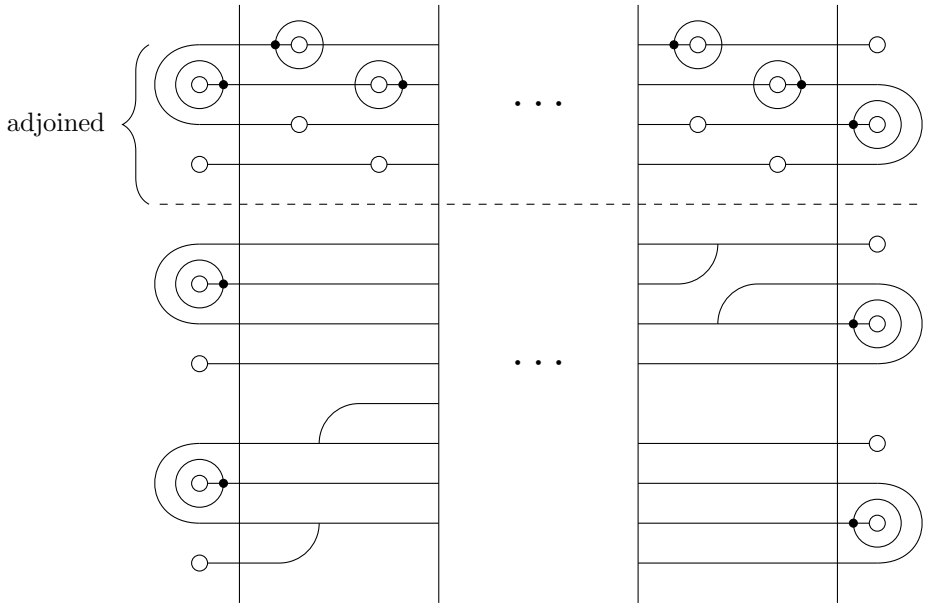


Figure 3.15: Proof of invariance under stabilization.

of the genus 1 surface.

For the handlebodies $\mathbf{H}^g$ and $-\mathbf{H}^g$, we see that the diagrams of $\mathbf{H}^{g+1}$ and $-\mathbf{H}^{g+1}$ are obtained from that of $\mathbf{H}^g$ and $-\mathbf{H}^g$ by adjoining diagrams of $\mathbf{H}^1$ and $-\mathbf{H}^1$ to the top. These constructions are illustrated in Figure 3.15.

Given a genus g Heegaard diagram and a choice $\psi \in \mathrm{MCG}_0(\mathcal{Z}^g)$ for that diagram, the chain complex

$$\widehat{CFD}(-\mathbf{H}^g) \boxtimes \widehat{CFAA}(\psi) \boxtimes \widehat{CFD}(\mathbf{H}^g) \tag{3.50}$$

can be seen to come from the Heegaard diagram in Figure 3.15 below the dotted line, in the sense that generators of the chain complex correspond to tuples of intersection points in the diagram, and arrows in the differential correspond to certain domains. Likewise, the chain complex

$$\widehat{CFD}(-\mathbf{H}^{g+1}) \boxtimes \widehat{CFAA}(\mathring{\psi}) \boxtimes \widehat{CFD}(\mathbf{H}^{g+1}) \tag{3.51}$$

comes from the entire Heegaard diagram in Figure 3.15.

The choice of intersection points in the adjoined portion of the diagram is forced (as marked in the figure), which means it is the same for all generators. So there is a one-to-one correspondence between generators of the old and new complexes. Moreover, since all areas above the dotted line are in the same region as the basepoint, no domain giving raise to an arrow in the differential can contain parts above the line. This shows the chain complexes in (3.50) and (3.51) are exactly the

same, proving Lemma 27 (note we used the equivalence between $\overline{\widehat{CFD}(\mathbf{H}^g)}$ and $\widehat{CFD}(-\mathbf{H}^g)$). $\square$

For Lemma 29, we need to show

$$\widehat{CFDA}(\phi) \boxtimes \widehat{CFD}(\mathbf{H}^g) \simeq \widehat{CFD}(\mathbf{H}^g),$$

for any $\phi \in \mathrm{MCG}_0^\beta(\mathcal{Z}_g)$. It suffices to verify the equation for a set of generators of $\mathrm{MCG}_0^\beta(\mathcal{Z}_g)$.

We find generators for strongly-based mapping class groups by appealing to results on the usual mapping class group. Let F_g be the genus g surface with a basepoint. Let $\mathrm{MCG}(F_g)$ be the group of isotopy classes of diffeomorphisms on F_g that fixes the basepoint, with isotopies also required to fix the basepoint. It is related to $\mathrm{MCG}_0(\mathcal{Z}_g)$ by a short exact sequence (see [5, Section 4.2.5]):

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\tau_\partial} \mathrm{MCG}_0(\mathcal{Z}_g) \longrightarrow \mathrm{MCG}(F_g) \longrightarrow 0$$

Here τ_∂ maps the generator of $\mathbb{Z}$ to the *boundary Dehn twist* in $\mathrm{MCG}_0(\mathcal{Z}_g)$. This is the element that performs a Dehn twist along a loop parallel to the boundary of $F_{g,1}$.

There is likewise a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\tau_\partial} \mathrm{MCG}_0^\beta(\mathcal{Z}_g) \longrightarrow \mathrm{MCG}^\beta(F_g) \longrightarrow 0$$

where $\mathrm{MCG}^\beta(F_g)$ is the subgroup of $\mathrm{MCG}(F_g)$ that are restrictions of automorphisms of the 0-framed handlebody $\mathbf{H}^g$. This exact sequence shows that a generating set of $\mathrm{MCG}_0^\beta(\mathcal{Z}_g)$ can be obtained by adding the boundary Dehn twist to the lifting of a generating set of $\mathrm{MCG}^\beta(F_g)$.

A generating set of $\mathrm{MCG}^\beta(F_g)$ is given in [28] (the corresponding notation in that paper is $\mathrm{MCG}^*(F_g)$). We reproduce the list of generators, together with the action of each generator on $\pi_1(F_g)$ here. For an element $\psi \in \mathrm{MCG}^\beta(F_g)$, let $\psi_\sharp : \pi_1(F_g) \to \pi_1(F_g)$ be its action on $\pi_1(F_g)$. We let $a_1, b_1, \ldots, a_g, b_g$ be a set of standard generators of $\pi_1(F_g)$, with each b_i contractible in the handlebody, and each a_i intersecting b_i once. Let $s_i = a_i^{-1} b_i^{-1} a_i b_i$, so that $s_n \ldots s_2 s_1 = 1$ is a relation in $\pi_1(F_g)$. In [28], a genus g surface is considered as a sphere with g handles attached. Each handle, together with its immediate base, is called a *knob*. We refer to that paper for diagrams and geometric description of these generators.

Theorem 30 ([28])**.** *The group $\mathrm{MCG}^\beta(F_g)$ is generated by $\rho, \omega_1, \tau_1, \rho_{12}, \theta_{12}$ and ξ_{12}, whose actions on $\pi_1(F_g)$ are the following:*

- *Cyclic translation of handles: $\rho_\sharp : a_i \to a_{i+1},\ b_i \to b_{i+1}$, where indices are taken modulo g.*
- *Twisting a knob: $\omega_{1\sharp} : a_1 \to a_1^{-1} s_1^{-1},\ b_1 \to a_1^{-1} b_1^{-1} a_1,\ a_j \to a_j,\ b_j \to b_j$ for $2 \le j \le n$.*

- *Twisting a handle, or Dehn twist:* $\tau_{1\sharp} : a_1 \to a_1 b_1^{-1},\ b_1 \to b_1,\ a_j \to a_j,\ b_j \to b_j$ *for* $2 \le j \le n$.
- *Interchanging two knobs:* $\rho_{12\sharp} : a_1 \to s_1^{-1} a_2 s_1,\ a_2 \to a_1,\ b_1 \to s_1^{-1} b_2 s_1,\ b_2 \to b_1,\ a_j \to a_j,\ b_j \to b_j$ *for* $3 \le j \le n$.
- *Sliding along* a_2*:* $\theta_{12\sharp} : a_1 \to a_1(b_2^{-1} a_2^{-1} b_2),\ a_j \to a_j$ *for* $j \ne 1$, $b_2 \to a_2 b_2 (a_1^{-1} b_1 a_1)(b_2^{-1} a_2^{-1} b_2),\ b_j \to b_j$ *for* $j \ne 2$.
- *Sliding along* b_2*:* $\xi_{12\sharp} : a_1 \to b_1 a_1 b_2^{-1} s_2 (a_1^{-1} b_1^{-1} a_1),\ a_2 \to a_2 b_2 (a_1^{-1} b_1^{-1} a_1) b_2^{-1},\ a_j \to a_j$ *for* $j \ne 1, 2,\ b_i \to b_i$ *for* $1 \le i \le g$.

Of these, only ρ is non-local in the sense that it is not restricted to a part of the surface with fixed genus. All other generators are restricted to a genus 1 or 2 part of the surface. We can remove ρ in favor of other local generators, by writing:

$$\rho^{-1} = \rho_{12} \circ \rho_{23} \circ \cdots \circ \rho_{g-1,g} \circ (\omega_{g\sharp})^{-2},$$

where $\omega_{g\sharp}$ is similar to $\omega_{1\sharp}$, except acting on the gth handle, and $\rho_{i,i+1}$ interchanges the ith and $(i+1)$th knobs. The equation can be verified by comparing the actions of two sides on $\pi_1(F_g)$: the initial $(\omega_{g\sharp})^{-2}$ has the effect of conjugating a_g and b_g by s_g^{-1}. After interchanging the knobs in succession, the action of the right side on $\pi_1(F_g)$ is $a_1 \to s_1^{-1} s_2^{-1} \cdots s_g^{-1} a_g s_g \cdots s_2 s_1$, $a_2 \to a_1$, $a_3 \to a_2$, and so on, and similarly for the b_i's. We then apply the relation $s_g \cdots s_2 s_1 = 1$.

Since the boundary Dehn twist equals ρ^g, the gth power of the cyclic translation of handles, the same generators also generate the group $\mathrm{MCG}_0^\beta(\mathcal{Z}_g)$.

So we have proved the following:

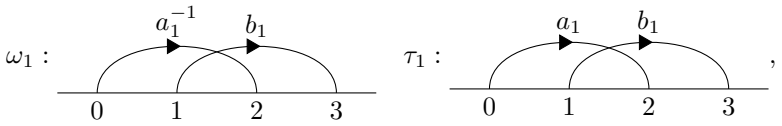
Corollary 31. *The group* $\mathrm{MCG}_0^\beta(F_g)$ *is generated by* $\omega_1, \omega_g, \tau_1, \theta_{12}, \xi_{12}$ *and* $\rho_{i,i+1}$ *for* $1 \le i \le g-1$.

Each of the generators in Corollary 31 is confined to one or two knobs on the surface. Our strategy will be to check Equation 3.49 on a surface of the corresponding genus (1 or 2), then extend the result to the general case. First, we compute a decomposition of these generators into arcslides. An arcslide with B pair (b_1, b_2) and C pair (c_1, c_2), with b_1 sliding over c_1, is denoted $b_1 \to c_1$. The

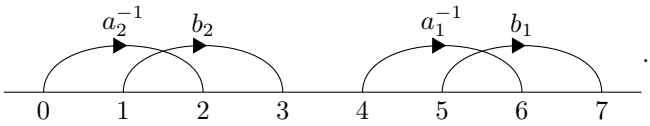
points are always labeled 0 to $4g-1$ from left to right. The results are:

$$\begin{aligned}
\rho_{12}: &\quad 3\to 4\ \ 6\to 7\ \ 5\to 6\ \ 4\to 5\ \ 2\to 3\ \ 5\to 6\ \ 4\to 5\ \ 3\to 4\\
&\quad 1\to 2\ \ 4\to 5\ \ 3\to 4\ \ 2\to 3\ \ 0\to 1\ \ 3\to 4\ \ 2\to 3\ \ 1\to 2\\
\theta_{12}: &\quad 4\to 3\ \ 1\to 0\ \ 1\to 2\ \ 5\to 4\ \ 6\to 5\\
\xi_{12}: &\quad 0\to 1\ \ 3\to 4\ \ 6\to 7\ \ 6\to 5\ \ 2\to 3\ \ 1\to 2\ \ 3\to 2\\
\omega_1: &\quad 2\to 3\ \ 1\to 2\ \ 2\to 3\ \ 1\to 2\ \ 2\to 3\ \ 1\to 2\\
\tau_1: &\quad 2\to 3
\end{aligned}$$

To verify these decompositions, we compute their actions on $\pi_1(F^\circ(\mathcal{Z}^g))$. For any pointed matched circle $\mathcal{Z}$, recall that the surface with circle boundary $F^\circ(\mathcal{Z})$ is formed by attaching 1-handles to Z along the matched pairs of points in $\mathbf{a}\subset Z$, then gluing in a solid disk on the other side. Choosing $z\in Z$ as the basepoint, the fundamental group of $F^\circ(\mathcal{Z})$ is generated freely by paths through the 1-handles. We choose the following orientation for the generators of the fundamental group. For the genus 1 cases:



and for all genus 2 cases:



An arcslide $\tau:\mathcal{Z}_1\to\mathcal{Z}_2$ induces an action

$$\tau_*:\pi_1(F^\circ(\mathcal{Z}_1))\to\tau_*:\pi_1(F^\circ(\mathcal{Z}_2))$$

on the fundamental groups. We describe this action by expressing each generator of $\pi_1(F^\circ(\mathcal{Z}_2))$ (corresponding to a pair of matched points in $\mathcal{Z}_2$) in terms of the images under τ_* of generators of $\pi_1(F^\circ(\mathcal{Z}_1))$. This can be computed from the definition of arcslide (for example, see Figure 3 in [16]). The results are shown in Figures 3.16 and 3.17. For example, the first diagram means that if the two displayed handles in the starting pointed matched circle correspond to generators a_1 and

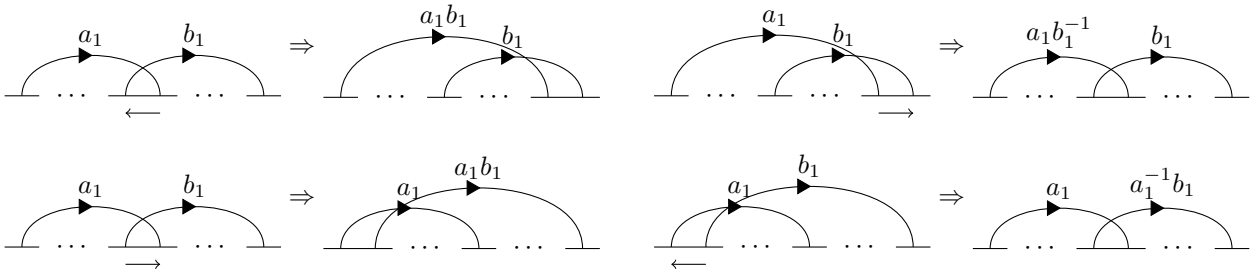


Figure 3.16: Actions of arcslides on the fundamental group (underslide).

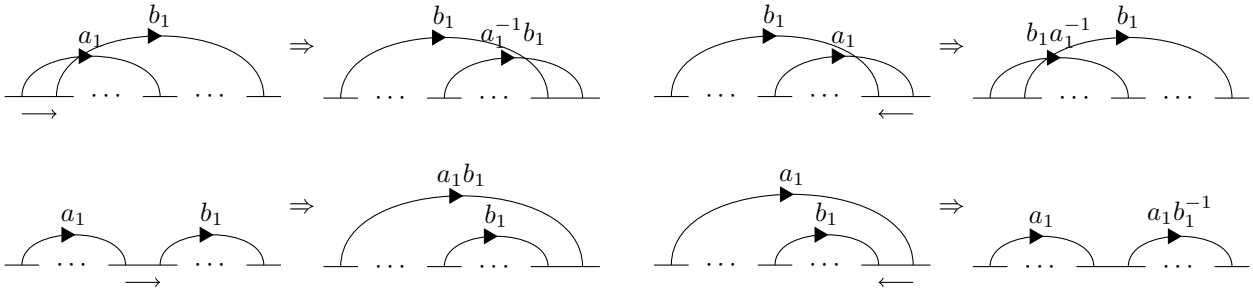


Figure 3.17: Actions of arcslides on the fundamental group (overslide).

b_1, then the two displayed handles in the ending pointed matched circle correspond to $\tau_*(a_1b_1)$ and $\tau_*(b_1)$ (the relation for handles unaffected by the arcslide is clear).

As an example, we verify the decomposition of θ_{12} into arcslides. Only the two middle pairs, corresponding to generators b_2 and a_1^{-1}, are moved during this sequence of arcslides. We follow what happened to these two pairs in the following table. We identify pairs of points in a sequence of arcslides as before. Each line in the table writes the generator corresponding to pairs identified with the initial b_2 and a_1^{-1} pairs in terms of $\tau_*(\cdot)$ of the initial generators.

Arcslide	b_2	a_1^{-1}
$4 \to 3$	b_2	$b_2a_1^{-1}$
$1 \to 0$	b_2	$a_2b_2a_1^{-1}$
$1 \to 2$	$b_2^{-1}a_2b_2a_1^{-1}$	$a_2b_2a_1^{-1}$
$5 \to 4$	$b_2^{-1}a_2b_2a_1^{-1}$	$a_2b_2a_1^{-1}b_1$
$6 \to 5$	$b_2^{-1}a_2b_2a_1^{-1}$	$a_2b_2a_1^{-1}b_1a_1b_2^{-1}a_2^{-1}b_2$

After this sequence of arcslides, the two middle pairs have switched positions. So the action is $a_1^{-1} \to b_2^{-1}a_2b_2a_1^{-1}$ and $b_2 \to a_2b_2a_1^{-1}b_1a_1b_2^{-1}a_2^{-1}b_2$. The first equation can be rewritten as

$a_1 \to a_1 b_2^{-1} a_2^{-1} b_2$. This agrees with the fundamental group action given in Theorem 30.

Proof of Lemma 29. By the same argument as in the proof of Lemma 27, we have

$$\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAD}(\phi) \simeq \widehat{CFD}(\mathbf{H}^{g+1}) \boxtimes \widehat{CFAD}(\mathring{\phi}),$$

where $\mathring{\phi}$ is the element of $\mathrm{MCG}_0(\mathcal{Z}^{g+1})$ that fixes the new handle and acts as ϕ elsewhere. The evaluation in the adjoined portion is equivalent to the evaluation of $\widehat{CFD}(\mathbf{H}^1) \boxtimes \widehat{CFAD}(\mathbb{I}_{\mathcal{Z}}) \simeq \widehat{CFD}(\mathbf{H}^1)$ on the genus 1 pointed matched circle. This works whether $\mathcal{Z}^{g+1}$ is formed as $\mathcal{Z}^1 \# \mathcal{Z}^g$ or as $\mathcal{Z}^g \# \mathcal{Z}^1$. From this, we see that it is sufficient to verify Equation 3.49 for each of the generators of $\mathrm{MCG}_0^{\beta}(F_g)$ in its respective minimum genus (1 or 2) case.

To do so, we decompose each generator ϕ of $\mathrm{MCG}_0^{\beta}(F_g)$ (for $g = 1$ or 2 depending on ϕ) into arcslides $\tau_n \circ \cdots \circ \tau_1$, as given above. Then we compute:

$$\widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAD}(\phi) \simeq \widehat{CFD}(\mathbf{H}^g) \boxtimes \widehat{CFAD}(\tau_1) \boxtimes \cdots \boxtimes \widehat{CFAD}(\tau_n).$$

This reduces to a finite computation, which we performed on a computer using a Python program (which implements the description of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ and the box tensor product). The entire computation took less than 20 seconds. □

Chapter 4

Local type DA bimodules and computations

4.1 Introduction

In this chapter we describe a new algorithm for computing the Heegaard Floer homology $\widehat{HF}(Y)$ of a general three-manifold Y. This algorithm uses bordered Floer homology, and is based on a previous method of computation using this theory (see [16]). However, it offers significant improvements in running time, especially in the more complicated cases.

The main new result is an explicit construction of the type DA invariant of arcslide diffeomorphisms. The construction uses the following idea: cut the Heegaard diagram for the arcslide into two parts: the *inside* part where the arcslide happens, and the *outside* part that looks like the diagram for the identity diffeomorphism. The inside part is fixed in size (regardless of the genus of the surface on which the arcslide acts), and there are only finitely many cases. We find a *local* type DA bimodule M for each of the several cases, and then *extend M by identity* to obtain a type DA bimodule $\widetilde{M}$ that will be the type DA invariant of the arcslide.

Section 4.2 defines local pointed matched circles and the strand algebras on them. Section 4.3 discusses extension by identity of type DA bimodules. Section 4.4 constructs the local type DA bimodules for different cases of the arcslide. Section 4.5 describes in detail the algorithm for computing $\widehat{HF}$. Section 4.6 shows some of the results of the computation.

4.1.1 Notes

The main definitions and theorems in Sections 4.2 and 4.3 are based on an unpublished preprint of Robert Lipshitz, Peter Ozsváth, and Dylan Thurston.

4.2 Local PMCs and algebras

In this section, we define local pointed matched circles (local PMCs) and the dg-algebra of strand diagrams on them. These definitions are modeled after that of pointed matched circles and strand algebras in [13]. We assume the reader is familiar with these previous definitions.

4.2.1 Local PMCs without smearing

A *local PMC without smearing* consists of the following data:

- A finite set $\{Z_i, 1 \leq i \leq s\}$ of oriented segments.
- On each Z_i, a finite set of points $\mathbf{a_i} = \{a_{ij}, 1 \leq j \leq k_i\}$, such that the total number of points $k = \sum k_i$ is even. Let $\mathbf{a} = \coprod_i \mathbf{a_i}$.
- A two-to-one map $\mathbf{a} \to \{1, \ldots, k/2\}$, matching the set of all points into pairs.
- For each $1 \leq i \leq s$, a specification of each end of Z_i as either *open* or *closed*.
- A labeling of the open ends by integers $\{1, \ldots, l\}$, where l is the total number of open ends.

A pointed matched circle from [13] can be considered as a special case of local PMCs without smearing, with a single segment $Z_1 = Z \setminus \{z\}$, points $\mathbf{a} \subset Z$ with the same matching on them, and each end of Z_1 specified as closed. For emphasis, we will call the pointed matched circles in the original definition the *full* PMCs.

Two examples of local PMCs without smearing are shown in Figure 4.1. We denote an open end by an arrow (with its label besides it), and closed end without arrows.

For our purposes, a local PMC without smearing is obtained by *restricting* a full PMC $\mathcal{Z}$ to a subset $\mathbf{s}$ of its points, where $\mathbf{s}$ satisfies the condition that for every matched pair $\{p_1, p_2\}$ of points in $\mathcal{Z}$, we have $p_1 \in \mathbf{s}$ if and only if $p_2 \in \mathbf{s}$. The open ends are then labeled in order according to the orientation on $\mathcal{Z}$. For example, the two local PMCs $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in Figure 4.1 can be formed by restricting the full PMC in Figure 4.2 to points $\{3, 4, 5, 6\}$ and $\{1, 2, 7, 8\}$, respectively.

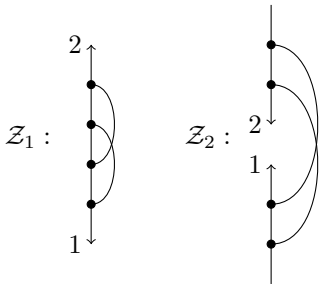


Figure 4.1: Two examples of local PMCs without smearing.

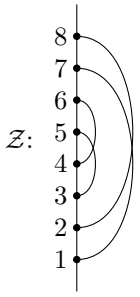


Figure 4.2: Example of a full PMC.

Suppose $\mathbf{s_1}$ and $\mathbf{s_2}$ is a partition of points on a full PMC $\mathcal{Z}$ such that every matched pair of points in $\mathcal{Z}$ are either both in $\mathbf{s_1}$ or both in $\mathbf{s_2}$, and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the restriction of $\mathcal{Z}$ to $\mathbf{s_1}$ and $\mathbf{s_2}$, respectively. Then $\mathcal{Z}_1$ and $\mathcal{Z}_2$ joins together according to the labeling on their respective open ends to form $\mathcal{Z}$. This is denoted $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$. We also say $\mathcal{Z}_1$ and $\mathcal{Z}_2$ is a *splitting* of $\mathcal{Z}$ without smearing.

4.2.2 Strand algebra associated to a local PMC without smearing

To each local PMC $\mathcal{Z}$ without smearing, we associate a dg-algebra called the *strand algebra* of that local PMC, denoted $\mathcal{A}(\mathcal{Z})$. This is a generalization of the strand algebra in the full PMC case.

A *strand diagram* on $\mathcal{Z}$ consists of a set of *strands* and *double horizontals*, where

- A strand lies in a particular segment Z_i, and is specified by its starting and ending points on Z_i. The starting and ending points can be one of a_{ij} or one of the open ends of Z_i, with the condition that the starting point must come before the ending point according to the orientation on Z_i.
- A double horizontal is specified by choosing a matched pair of points on $\mathcal{Z}$ (it is so named because it is drawn as two dashed horizontal strands starting and ending at each of the two points in the pair).

The set of strands and double horizontals must satisfy the following condition: for each matched pair P in $\mathcal{Z}$, if there is a double horizontal at P, then no strand can start or end at either point of P. Otherwise, there can be at most one strand starting at points of P, and at most one strand ending at points of P.

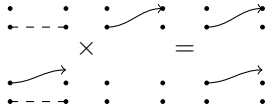
To each strand diagram we can associate a *multiplicity*, which assigns to each component of $\coprod_i Z_i \backslash \mathbf{a_i}$ that are not closed ends the number of times a strand in the diagram covers that component. We say that a strand diagram has *multiplicity-one* if its multiplicity at each component is at most one. We define $\mathcal{A}(\mathcal{Z})$ to be generated as a vector space over $\mathbb{F}_2$ by multiplicity-one strand diagrams.

A generator $a \in \mathcal{A}(\mathcal{Z})$ is idempotent ($a \cdot a = a$) if and only if its diagram contains only double horizontals (including the empty diagram). We will call the idempotent generators simply *idempotents*. It is clear that an idempotent is specified by choosing a subset of pairs of points on $\mathcal{Z}$. Every generator $a \in \mathcal{A}(\mathcal{Z})$ has a unique left idempotent $i_L(a)$ satisfying $i_L(a) \cdot a = a$, and a unique right idempotent $i_R(a)$ satisfying $a \cdot i_R(a) = a$.

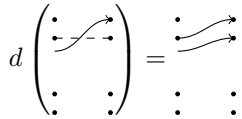
Remark: The main difference with the strand algebra for a full PMC introduced in [13] is that we allow strands to come from or go out into an open end of a segment. We also do not fix the number of pairs in the starting and ending idempotents of a strand diagram. In fact due to the open ends, there are strand diagrams for which they are different. We can restrict to multiplicity-one strand diagrams because the same can be done in the full PMC case (the non-multiplicity-one part of the algebra is contractible).

Multiplication and differentiation on $\mathcal{A}(\mathcal{Z})$ are defined as a straightforward extension of the full PMC case. Roughly speaking, the product $a_1 \cdot a_2$ is nonzero only if the diagrams for a_1 and a_2 can be joined together. Ends of strands that go to an open end of a segment do not need to be joined. Differentials resolve crossings as before.

Two examples of multiplication and differentiation on $\mathcal{A}(\mathcal{Z}_2)$ (where $\mathcal{Z}_2$ is as in Figure 4.1) are shown below. We use the same method for drawing strand diagrams as in [13]. Solid dots denote points on the local PMC. Strands starting or ending at an open end of a segment are indicated by ends of strands that are not on solid dots.



(4.1)



(4.2)

There is a grading of $\mathcal{A}(\mathcal{Z})$ by $\mathbb{Z}^l$, denoted $g_{\text{open}}(\cdot)$ and called the *open end grading*. For each strand diagram a, define $g_{\text{open}}(a)$ to be the multiplicity of a at the l open ends of a. This grading is unchanged by the differential and adds under multiplication. Since we imposed the multiplicity-one condition, actually $g_{\text{open}}(a) \in \{0,1\}^l$ for all a.

Remark: The grading $g_{\text{open}}(\cdot)$ is only one part of the full grading on $\mathcal{A}(\mathcal{Z})$, which we will not discuss in this paper.

4.2.3 Restriction and Join maps

Let $\mathcal{Z}'$ be a restriction of a full PMC $\mathcal{Z}$. Then there is a linear map

$$r_{\mathcal{Z}\to\mathcal{Z}'} : \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z}')$$

defined on generators by restriction of strand diagrams. Note this does not commute with the differential, and so is not a map of differential algebras.

Let $\mathcal{Z}$ be a full PMC and $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$ be a splitting of $\mathcal{Z}$ without smearing. If $a, b \in \mathcal{A}(\mathcal{Z}_1)$ have the same open end grading, they can be joined to form a generator of $\mathcal{A}(\mathcal{Z})$, which we denote by $a * b$.

This can be formalized as follows:

Definition 32. *Let A and B be two dg-algebras graded by $\mathbb{Z}^l$ for the same integer l, where the grading is preserved by the differential and adds under multiplication. For each $s \in \mathbb{Z}^l$, let A_s (resp. B_s) be the part of A (resp. B) with grading s. Define the* graded product *$A * B$ of A and B to be the dg-algebra*

$$\bigoplus_{s\in\mathbb{Z}^l} A_s \otimes B_s.$$

We write $a \otimes b \in A_s \otimes B_s$ as $a * b$. It follows that the differential and multiplication on $A * B$ are given on generators by:

$$d(a * b) = da * b + a * db \tag{4.3}$$

$$(a_1 * b_1)(a_2 * b_2) = (a_1 a_2) * (b_1 b_2) \tag{4.4}$$

Then the following statement can be easily verified:

Proposition 33. *Let $\mathcal{Z}$ be a full PMC and $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$ be a splitting of $\mathcal{Z}$ without smearing. Then*

$$\mathcal{A}(\mathcal{Z}_1) * \mathcal{A}(\mathcal{Z}_2) = \mathcal{A}(\mathcal{Z}).$$

4.2.4 Local PMCs and algebras with smearing

A *local PMC with smearing* consists of the following data:

- A finite set $\{Z_i\}, 1 \leq i \leq s$ of oriented segments.
- On each Z_i, a finite set of points $\mathbf{a_i} = \{a_{ij}, 1 \leq j \leq k_i\}$, such that the total number of points $k = \sum k_i$ is odd. Let $\mathbf{a} = \coprod_i \mathbf{a_i}$.
- A distinguished point $p_o \in \mathbf{a}$, and a two-to-one map $\mathbf{a} \setminus \{p_o\} \to \{1, \ldots, (k-1)/2\}$, matching all points in $\mathbf{a}$ except p_o into pairs.
- For each $1 \leq i \leq s$, a specification of each end of Z_i as either *open* or *closed*.
- A labeling of the open ends by integers $\{1, \ldots, l\}$, where l is the total number of open ends.

Let $\mathcal{Z}$ be a full PMC. Fix a matched pair $P = (p_1, p_2)$ of points in $\mathcal{Z}$. Let $\mathbf{s_1}$ and $\mathbf{s_2}$ be a partition of points in $\mathcal{Z}$ such that $p_1 \in \mathbf{s_1}$, $p_2 \in \mathbf{s_2}$, and every other pair of points on $\mathcal{Z}$ are either both in $\mathbf{s_1}$ or both in $\mathbf{s_2}$. Then the restriction of $\mathcal{Z}$ to $\mathbf{s_1}$ and $\mathbf{s_2}$ are both local PMCs with smearing. For example, the two local PMCs with smearing shown in Figure 4.3 are obtained from the full PMC in Figure 4.2 by restricting to subsets $\{4, 5, 6\}$ and $\{1, 2, 3, 7, 8\}$, respectively. Again, the local PMCs $\mathcal{Z}_1$ and $\mathcal{Z}_2$ can be joined together according to the labeling on their open ends to form $\mathcal{Z}$. We write this as $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$, and say $\mathcal{Z}_1$ and $\mathcal{Z}_2$ is a splitting of $\mathcal{Z}$ with smearing (on the pair P).

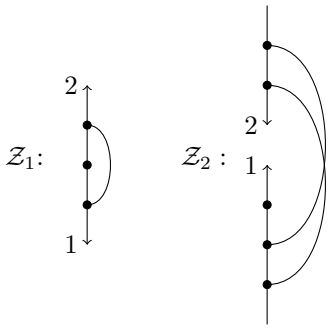


Figure 4.3: Two examples of local PMCs with smearing.

To a local PMC $\mathcal{Z}$ with smearing, we define the associated dg-algebra $\mathcal{A}(\mathcal{Z})$ as follows. A strand diagram on $\mathcal{Z}$ consists of a set of strands and double horizontals, together with a choice of whether there is a *single horizontal* at p_o (so named because it is drawn as a single dashed horizontal strand at p_o). In addition to the conditions in the case without smearing, a strand diagram must satisfy the following: if there is a single horizontal at p_o, then no strand may start or end at p_o (otherwise at most one strand can start and at most one strand can end at p_o). The differential and multiplication on the algebra are defined as a straightforward extension of the case without smearing.

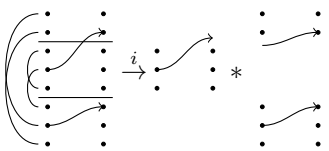
A generator $a \in \mathcal{A}(\mathcal{Z})$ is idempotent if and only if its diagram contains only double and single horizontals. An idempotent is specified by choosing a subset of paired points on $\mathcal{Z}$ and the single point p_o. As before, every generator $a \in \mathcal{A}(\mathcal{Z})$ has a unique left idempotent $i_L(a)$ and a unique right idempotent $i_R(a)$.

We also define a linear map $u : \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z})$ as follows. If a generator $a \in \mathcal{A}(\mathcal{Z})$ has a single horizontal at p_o, let $u(a)$ be the generator with the same strand diagram, except without the single horizontal. Otherwise, let $u(a) = 0$.

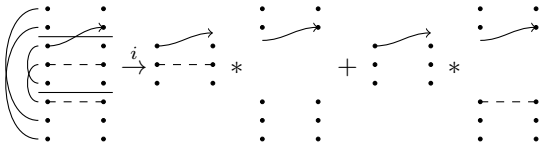
Let $\mathcal{Z}$ be a full PMC and $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$ be a splitting of $\mathcal{Z}$ with smearing on a pair $P = \{p_1, p_2\}$. Let $\mathcal{A}_P(\mathcal{Z}) = \mathcal{A}(\mathcal{Z}_1) * \mathcal{A}(\mathcal{Z}_2)$. Then $\mathcal{A}_P(\mathcal{Z})$ does not equal $\mathcal{A}(\mathcal{Z})$, but is a slightly modified version of it, where the two points of P are no longer considered to be paired (we can think of $\mathcal{A}_P(\mathcal{Z})$ as lying between the algebras $\mathcal{A}(n)$ and $\mathcal{A}(\mathcal{Z})$, in the notation of [13], where n is the number of points in $\mathcal{Z}$). There is a canonical inclusion of dg-algebras:

$$i : \mathcal{A}(\mathcal{Z}) \to \mathcal{A}_P(\mathcal{Z})$$

given as follows: on a generator $a \in \mathcal{A}(\mathcal{Z})$ without a double horizontal at P, the image $i(a)$ is the "same" strand diagram in $\mathcal{A}_P(\mathcal{Z})$. If $a \in \mathcal{A}(\mathcal{Z})$ has a double horizontal at P, let a_{p_1} (resp. a_{p_2}) be the strand diagram in $\mathcal{A}_P(\mathcal{Z})$ where the double horizontal in a at P is replaced by a single horizontal at p_1 (resp. p_2). Then let $i(a) = a_{p_1} + a_{p_2}$ (this is analogous to the canonical inclusion $i : \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(n)$ in [13]). Two examples are shown below.



(4.5)



(4.6)

4.3 Extension by identity

In this section, we define extension by identity of local type DA bimodules. The basic setup is as follows:

- $\mathcal{Z}$ and $\mathcal{Z}'$ are two full PMCs with the same genus g,
- $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$ is a splitting of $\mathcal{Z}$,
- $\mathcal{Z}'_1 * \mathcal{Z}_2 = \mathcal{Z}'$ is a splitting of $\mathcal{Z}'$.

Note the second local PMC is the same between the two splittings. We write $\mathcal{A}(\mathcal{Z}, g)$ and $\mathcal{A}(\mathcal{Z}', g)$ for the part of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$ whose diagrams contain g strands and double horizontals (that is, those with weight g. This is actually the algebra used in most of [13]). We will start with the case where both splittings are without smearing since it is simpler, although the ultimate focus is on the case where both splittings are with smearing.

4.3.1 Type DA bimodules

We begin with by recalling some algebraic preliminaries on type DA bimodules (see section 1.2 and [14, Section 2]).

Given augmented dg-algebras A and A', over rings of idempotents $\mathbf{k}$ and $\mathbf{k}'$, respectively. The type DA bimodule ${}^{A}M_{A'}$ is a bimodule over $\mathbf{k}$ on the left and $\mathbf{k}'$ on the right, together with $\mathbf{k}$-$\mathbf{k}'$-linear maps

$$\delta^1_{M,k+1} : M \otimes A'[1]^{\otimes k} \to A \otimes M \tag{4.7}$$

called *type DA actions*. Here the tensor products in the inputs are taken over $\mathbf{k}'$, and the tensor product in the output is taken over $\mathbf{k}$.

Let $T^*(A) = \oplus_{k\geq 0} A[1]^{\otimes k}$ be the tensor algebra of A. Then $\delta^1_{M,k}$ can be collected together into a map $\delta^1_M : M \otimes T^*(A') \to A \otimes M'$. We define a differential

$$D : T^*(A') \to T^*(A')$$

as follows: $D(a_1 \otimes \cdots \otimes a_k)$ is the sum of all possible ways to differentiate one of a_i, or multiply

together two consecutive a_i. We also define

$$\Delta : T^*(A') \to T^*(A') \otimes T^*(A')$$

on the generators of $T^*(A')$ as the sum of all possible ways to split the input into two (possibly empty) groups. Let d and μ_2 denote differential and multiplication on a dg-algebra. Then the type DA action δ^1_M must satisfy the following *structure equation*:

$$\delta^1_M \circ (\mathbb{I}_M \otimes D) + (d \otimes \mathbb{I}_M) \circ \delta^1_M + (\mu_2 \otimes \mathbb{I}_M) \circ (\mathbb{I}_A \otimes \delta^1_M) \circ (\delta^1_M \otimes \mathbb{I}_{T^*(A')}) \circ (\mathbb{I}_M \otimes \Delta) = 0 \quad (4.8)$$

as a map $M \otimes T^*(A') \to A \otimes M$.

In all cases that we will encounter here, the ground rings $\mathbf{k}$ and $\mathbf{k}'$ are vector spaces over $\mathbb{F}_2$, generated by idempotent generators of some strand algebra. In addition, there is always a canonical choice of generators of M over $\mathbb{F}_2$, so that each generator $\mathbf{x}$ has a D-side idempotent $i_D(\mathbf{x}) \in \mathbf{k}$ and an A-side idempotent $i_A(\mathbf{x}) \in \mathbf{k}'$. Part of the type DA action on M is

$$\delta^1_{M,2}(\mathbf{x}, i_A(\mathbf{x})) = i_D(\mathbf{x}) \otimes \mathbf{x}.$$

In all other cases, $\delta^1_{M,k} = 0$ when one of the algebra inputs from A' is an idempotent.

Since there is a canonical choice of generators on M and on the strand algebras, we can write the type DA action as a sum of *arrows*. We write

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{z}, \quad (4.9)$$

where $\mathbf{x}, \mathbf{z}, c$, and each a_i are generators, to mean that $c \otimes \mathbf{z}$ is a term in $\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k)$, when the latter is completely expanded in terms of the canonical generators. We note here that $\delta^1_{M,k+1}$ being $\mathbf{k}$-$\mathbf{k}'$-linear, and tensor products in (4.7) being taken over $\mathbf{k}$ and $\mathbf{k}'$ imply the following:

$$i_D(\mathbf{x}) = i_L(c),\ i_R(\mathbf{x}) = i_L(a_1),\ i_R(a_i) = i_L(a_{i+1}),\ i_R(a_k) = i_A(\mathbf{z}),\ i_R(c) = i_D(\mathbf{z})$$

if $k > 0$, and

$$i_D(\mathbf{x}) = i_L(c),\ i_A(\mathbf{x}) = i_A(\mathbf{z}),\ i_R(c) = i_D(\mathbf{z})$$

if $k = 0$.

An alternative way to think of a type DA bimodule ${}^A M_{A'}$ is as a type D module over $\widehat{\mathrm{Cob}}(A') \otimes A$.

The arrow in (4.9) corresponds to the arrow

$$\delta^1(\mathbf{x}) \to (a_1^* \otimes \cdots \otimes a_k^*) \otimes c \otimes \mathbf{z},$$

in the type D action by $\widehat{\mathrm{Cob}}(A') \otimes A$ (the idempotent actions are omitted). When M is bounded, in the sense that $\delta^1_{M,k} = 0$ for sufficiently large k, there is no need to take completion, so the algebra can be replaced by $\mathrm{Cob}(A') \otimes A$. This will be the case in our computations in section 4.4.

4.3.2 No-smearing case

Suppose the splittings are without smearing. Given a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z}_1)}M_{\mathcal{A}(\mathcal{Z}_1')}$. We assume M respects the open-end grading on $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_1')$. That is, for each arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{y}$$

in the type DA action of M, we have

$$\sum_{i=1}^{k} g_{\mathrm{open}}(a_i) = g_{\mathrm{open}}(c).$$

We will define a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z},g)}\widetilde{M}_{\mathcal{A}(\mathcal{Z}',g)}$ called the *extension by identity* of M.

Let I_2 be the set of idempotents of $\mathcal{A}(\mathcal{Z}_2)$, and $\mathcal{G}(M)$ be the set of generators of M. Let $i_D(\cdot)$ and $i_A(\cdot)$ denote the D-side and A-side idempotents of bimodule generators. The bimodule $\widetilde{M}$ is generated by pairs $(\mathbf{x}, j)$ for $\mathbf{x} \in \mathcal{G}(M)$ and $j \in I_2$, such that the idempotents $i_D((\mathbf{x}, j)) = i_D(\mathbf{x}) * j$ and $i_A((\mathbf{x}, j)) = i_A(\mathbf{x}) * j$ both contain g pairs of points (i.e. have weight g).

Given $a \in \mathcal{A}(\mathcal{Z}_1), b \in \mathcal{A}(\mathcal{Z}_2)$ with the same open-end grading, such that $a * b$ has weight g, and a generator $\mathbf{x} \in M$ such that $i_R(a) = i_D(\mathbf{x})$, we define:

$$(a \otimes \mathbf{x}) \star b = (a * b) \otimes (\mathbf{x}, i_R(b)).$$

Note that since $i_R(a) = i_D(\mathbf{x})$, we have $i_R(a * b) = i_D((\mathbf{x}, i_R(b)))$. In particular, since $a * b$ has weight g, we are assured that $i_D((\mathbf{x}, i_R(b)))$ has weight g, as is required for generators of M.

Extending by linearity, and by zero on unsuitable choices of a, b and $\mathbf{x}$, we get a map

$$\cdot \star \cdot : (\mathcal{A}(\mathcal{Z}_1) \otimes M) \times \mathcal{A}(\mathcal{Z}_2) \to \mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M}.$$

Using the isomorphism $\mathcal{A}(\mathcal{Z}') = \mathcal{A}(\mathcal{Z}'_1) * \mathcal{A}(\mathcal{Z}_2)$, we may write any generator of $\mathcal{A}(\mathcal{Z}', g)$ in the form $a * b$ for $a \in \mathcal{A}(\mathcal{Z}'_1)$ and $b \in \mathcal{A}(\mathcal{Z}_2)$. In this way, the type DA action on $\widetilde{M}$ is given by:

$$\delta^1_{\widetilde{M},k+1}((\mathbf{x}, j), a_1 * b_1, \ldots, a_k * b_k) = \delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \star (b_1 \cdots b_k). \tag{4.10}$$

We check that in fact the extension by zero part of $\cdot \star \cdot$ is never used. Let $c \otimes \mathbf{y}$ be any term in $\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k)$. Then $g_{\text{open}}(a_i) = g_{\text{open}}(b_i)$ implies

$$g_{\text{open}}(c) = \sum_{i=1}^{k} g_{\text{open}}(a_i) = \sum_{i=1}^{k} g_{\text{open}}(b_i) = g_{\text{open}}(b_1 \cdots b_k).$$

Moreover, $i_L(c) = i_D(\mathbf{x})$ and $i_L(b_1 \cdots b_k) = i_L(b_1) = i_L(j)$, so that $i_L(c * (b_1 \cdots b_k)) = i_L((\mathbf{x}, j))$. This shows $c * (b_1 \cdots b_k)$ is defined and has weight g. The condition $i_R(c) = i_D(\mathbf{y})$ holds because the output of $\delta^1_{M,k}$ lies in $\mathcal{A}(\mathcal{Z}_1) \otimes_{\mathbf{k}} M$.

The main result of this subsection is:

Theorem 34. *The map $\delta^1_{\widetilde{M},k}$ defined above satisfies the type DA structure equations.*

Proof. Let $d^2_N : N \otimes T^*(A') \to A \otimes N$ denote the left side of (4.8).

We need to show

$$d^2_{\widetilde{M}}((\mathbf{x}, j), a_1 * b_1, \ldots, a_k * b_k) = 0$$

for all possible inputs $(\mathbf{x}, j), a_1 * b_1, \ldots, a_k * b_k$. The strategy is to classify the four types of terms in $d^2_{\widetilde{M}}$, according to the terms in (4.8) (we separate the first term into two types, for differentiating one input and multiplying together two inputs).

Type I: Differentiate one of the algebra inputs, then apply δ^1.

For any $1 \leq i \leq k$ and arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, da_i, \ldots, a_k) \to c \otimes \mathbf{z},$$

which produces a term $c \otimes \mathbf{z}$ in $d^2_M(\mathbf{x}, a_1, \ldots, a_k)$, there is an arrow

$$\delta^1_{\widetilde{M},k+1}((\mathbf{x}, j), a_1 * b_1, \ldots, da_i * b_i, \ldots, a_k * b_k) \to (c * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)),$$

giving a term

$$(c * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)) \tag{Ia}$$

in $d^2_{\widetilde{M}}$.

Also, for any $1 \leq i \leq k$ and arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c' \otimes \mathbf{z}',$$

there is an arrow

$$\delta^1_{\widetilde{M},k+1}((\mathbf{x}, j), a_1 * b_1, \ldots, a_i * db_i, \ldots, a_k * b_k) \to (c' * (b_1 \cdots db_i \cdots b_k)) \otimes (\mathbf{z}', i_R(b_k)),$$

giving a term

$$(c' * (b_1 \cdots db_i \cdots b_k)) \otimes (\mathbf{z}', i_R(b_k)) \tag{Ib}$$

in $d^2_{\widetilde{M}}$. Note we used the identity $d(a_i * b_i) = da_i * b_i + a_i * db_i$ on each of the algebra inputs.

Type II: Multiply together two algebra inputs, then apply δ^1.

For any $1 \leq i < k$ and arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_i a_{i+1}, \ldots, a_k) \to c \otimes \mathbf{z},$$

which produces a term $c \otimes \mathbf{z}$ in $d^2_M(\mathbf{x}, a_1, \ldots, a_k)$, there is an arrow

$$\delta^1_{\widetilde{M},k+1}((\mathbf{x}, j), a_1 * b_1, \ldots, a_i a_{i+1} * b_i b_{i+1}, \ldots, a_k * b_k) \to (c * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)),$$

giving a term

$$(c * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)) \tag{II}$$

in $d^2_{\widetilde{M}}$. Note we used $a_i a_{i+1} * b_i b_{i+1} = (a_i * b_i)(a_{i+1} * b_{i+1})$.

Type III: Apply δ^1 and then differentiate the algebra output.

For each arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{z},$$

which produces a term $dc \otimes \mathbf{z}$ in $d^2_M(\mathbf{x}, a_1, \ldots, a_k)$, there is an arrow

$$\delta^1_{\widetilde{M},k+1}((\mathbf{x}, j), a_1 * b_1, \ldots, a_k * b_k) \to (c * (b_1 \cdots, b_k)) \otimes (\mathbf{z}, i_R(b_k)),$$

giving terms

$$(dc * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)) \tag{IIIa}$$

and

$$(c * (b_1 \cdots db_i \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)),\ 1 \leq i \leq k \tag{IIIb}$$

in $d^2_{\widetilde{M}}$.

Type IV: Apply δ^1 twice and multiply the algebra outputs. For each $0 \leq i \leq k$ and pair of arrows

$$\delta^1_{M,i+1}(\mathbf{x}, a_1, \ldots, a_i) \to c_1 \otimes \mathbf{y}$$
$$\delta^1_{M,k-i+1}(\mathbf{y}, a_{i+1}, \ldots, a_k) \to c_2 \otimes \mathbf{z}$$

which produces a term $c_1c_2 \otimes \mathbf{z}$ in $d^2_M(\mathbf{x}, a_1, \ldots, a_k)$, we have arrows

$$\delta^1_{\widetilde{M},i+1}((\mathbf{x}, j), a_1 * b_1, \ldots, a_i * b_i) \to (c_1 * (b_1 \cdots b_i)) \otimes (\mathbf{y}, i_R(b_i))$$
$$\delta^1_{\widetilde{M},k-i+1}((\mathbf{y}, j'), a_{i+1} * b_{i+1}, \ldots, a_k * b_k) \to (c_2 * (b_{i+1} \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k))$$

which produces the term

$$((c_1c_2) * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)) \tag{IV}$$

in $d^2_{\widetilde{M}}$.

This accounts for all terms in $d^2_{\widetilde{M}}$. Terms of type (Ia), (II), (IIIa), and (IV) cancel each other since they correspond to terms in $d^2_M(\mathbf{x}, a_1, \ldots, a_k) = 0$. Terms of type (Ib) and (IIIb) cancel each other since both correspond to terms in $\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k)$. □

Remark:

1. Extension by identity commutes with the box tensor product. In detail: let $\mathcal{Z}, \mathcal{Z}', \mathcal{Z}''$ be full PMCs, and $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$, $\mathcal{Z}'_1 * \mathcal{Z}_2 = \mathcal{Z}'$, and $\mathcal{Z}''_1 * \mathcal{Z}_2 = \mathcal{Z}''$ be splittings without smearing. Let ${}^{\mathcal{A}(\mathcal{Z}_1)}M_{\mathcal{A}(\mathcal{Z}'_1)}$ and ${}^{\mathcal{A}(\mathcal{Z}'_1)}N_{\mathcal{A}(\mathcal{Z}''_1)}$ be type DA bimodules, then

$$\widetilde{M \boxtimes N} = \widetilde{M} \boxtimes \widetilde{N}.$$

   In fact, both sides have the same generators and type DA actions.

2. Extension by identity extends to morphisms between type DA bimodules. The extension

respects differential and composition of morphisms, and therefore homotopy equivalence (in short, it is a *dg*-functor between categories of type DA bimodules).

3. The constructions in this paper may be related to cornered Floer homology (see [3], [4]). In the case where the local PMC has only one open end, the strand algebra with its open-end grading can be viewed as an algebra-module over the multiplicity-1 sequential nilCoxeter 2-algebra, and the gluing map is the vertical tensor product. We note that since the part of the sequential nilCoxeter 2-algebra with multiplicity greater than 1 is contractible, the category of algebra-modules over the full sequential 2-algebra is equivalent to that over the multiplicity-1 sequential 2-algebra.

4.3.3 Smearing case

We now consider the case with smearing. Let $\mathcal{Z}_1 * \mathcal{Z}_2 = \mathcal{Z}$ be a splitting of $\mathcal{Z}$ with smearing on the pair $P = (p_1, p_2)$, and let $\mathcal{Z}'_1 * \mathcal{Z}_2 = \mathcal{Z}'$ be a splitting of $\mathcal{Z}'$ with smearing on the pair $P' = (p'_1, p'_2)$. We label the points so that p_2 and p'_2 restrict to $\mathcal{Z}_2$, and p_1 (resp. p'_1) restricts to $\mathcal{Z}_1$ (resp. $\mathcal{Z}'_1$). By abuse of notation, we also name the unpaired points in $\mathcal{Z}_1, \mathcal{Z}'_1$, and $\mathcal{Z}_2$ as p_1, p'_1, and p_2, respectively. Given a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z}_1)}M_{\mathcal{A}(\mathcal{Z}'_1)}$. If M satisfies some additional properties to be described below, making it a *smearable* type DA bimodule, then we can extend M to a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z},g)}\widetilde{M}_{\mathcal{A}(\mathcal{Z}',g)}$.

This is completed in two steps. In the first step, we define an intermediate type DA bimodule $\widetilde{M}_0$, using essentially the same method as in the previous section. In the second step, we *smear* $\widetilde{M}_0$ to produce the end-product $\widetilde{M}$.

Let $\mathcal{A}_P(\mathcal{Z}, g)$ and $\mathcal{A}'_P(\mathcal{Z}', g)$ be the part of $\mathcal{A}_P(\mathcal{Z})$ and $\mathcal{A}'_P(\mathcal{Z}')$ with weight g (that is, limit to strand diagrams whose total number of strands, double horizontals, and single horizontals is g). Since $\mathcal{A}_P(\mathcal{Z}, g)$ is a sub-dg-algebra of $\mathcal{A}_P(\mathcal{Z}) = \mathcal{A}(\mathcal{Z}_1) * \mathcal{A}(\mathcal{Z}_2)$, and $\mathcal{A}_P(\mathcal{Z}', g)$ is a sub-dg-algebra of $\mathcal{A}'_P(\mathcal{Z}') = \mathcal{A}(\mathcal{Z}'_1) * \mathcal{A}(\mathcal{Z}_2)$, by the same argument as in the previous section, we obtain a type DA bimodule ${}^{\mathcal{A}_P(\mathcal{Z},g)}\widetilde{M}_0{}_{\mathcal{A}_{P'}(\mathcal{Z}',g)}$.

We now define smearable type DA bimodules.

Definition 35. *Let $\mathcal{Z}_1$ and $\mathcal{Z}'_1$ be two local PMCs with smearing. Let p_o (resp. p'_o) be the unpaired point on $\mathcal{Z}_1$ (resp. $\mathcal{Z}'_1$). A* smearable *type DA bimodule is a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z}_1)}M_{\mathcal{A}(\mathcal{Z}'_1)}$, such that no generator $\mathbf{x}$ of M has $p_o \in i_D(\mathbf{x})$ and $p'_o \notin i_A(\mathbf{x})$, together with the following data:*

- *A bijection* $u_b : S_1 \to S_0$, *where*

$$S_1 = \{\mathbf{x} \in \mathcal{G}(M) \mid p_o \in i_D(\mathbf{x}) \text{ and } p'_o \in i_A(\mathbf{x})\}$$

and

$$S_0 = \{\mathbf{x} \in \mathcal{G}(M) \mid p_o \notin i_D(\mathbf{x}) \text{ and } p'_o \notin i_A(\mathbf{x})\}.$$

such that $i_D(u_b(\mathbf{x})) = u(i_D(\mathbf{x}))$ *and* $i_A(u_b(\mathbf{x})) = u(i_A(\mathbf{x}))$.

Define a linear map u *on the vector space underlying* M *by setting* $u = u_b$ *on generators in* S_1, *and* $u = 0$ *for all other generators. Define* u *on* $\mathcal{A}(\mathcal{Z}_1) \otimes_{\mathbf{k}} M$ *by* $u(c \otimes \mathbf{x}) = u(c) \otimes u(\mathbf{x})$. *Then the type DA action on* M *must satisfy:*

$$u(\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k)) = \delta^1_{M,k+1}(u(\mathbf{x}), u(a_1), \ldots, u(a_k)). \tag{4.11}$$

For intuition, we translate (4.11) into a pairing among a subset of arrows in δ^1_M. For any arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{z}, \tag{4.12}$$

where c contains a single horizontal at p_o (that is, $u(c) \neq 0$), and $\mathbf{z} \in S_1$, the following holds:

- $\mathbf{x} \in S_1$, and each a_i contain a single horizontal at p'_o.
- There is a corresponding arrow

$$\delta^1_{M,k+1}(u_b(\mathbf{x}), u(a_1), \ldots, u(a_k)) \to u(c) \otimes u_b(\mathbf{z}). \tag{4.13}$$

Conversely, for any arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{z}, \tag{4.14}$$

where $\mathbf{x} \in S_0$, and $p'_o \notin i_L(a_i), p'_o \notin i_R(a_i)$ for each i, the following holds:

- $\mathbf{z} \in S_0$, and $p_o \notin i_L(c), p_o \notin i_R(c)$.
- There is a corresponding arrow

$$\delta^1_{M,k+1}(\bar{\mathbf{x}}, \bar{a_1}, \ldots, \bar{a_k}) \to \bar{c} \otimes \bar{\mathbf{z}} \tag{4.15}$$

such that $u_b(\bar{\mathbf{x}}) = \mathbf{x}, u_b(\bar{\mathbf{z}}) = \mathbf{z}, u(\bar{c}) = c$, and $u(\bar{a_i}) = a_i$ for each $1 \leq i \leq k$.

The main theorem of this section is:

Theorem 36. *With the notations above, suppose M is smearable, then the bimodule ${}^{\mathcal{A}_P(\mathcal{Z},g)}\widetilde{M}_{0\mathcal{A}_{P'}(\mathcal{Z}',g)}$ induces a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z},g)}\widetilde{M}_{\mathcal{A}(\mathcal{Z}',g)}$.*

First, we discuss the construction of $\widetilde{M}$. The vector space underlying $\widetilde{M}$ is a subspace of that underlying $\widetilde{M}_0$. The generators of the subspace (which will be considered generators of $\widetilde{M}$) are as follows:

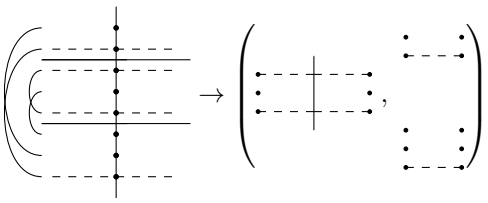
- Each $(\mathbf{x}, j)$ where $\mathbf{x} \in S_0$ and $p_2 \notin j$. The idempotents are $i_D((\mathbf{x}, j)) = i_D(\mathbf{x}) * j \in \mathcal{A}(\mathcal{Z}, g)$ and $i_A((\mathbf{x}, j)) = i_A(\mathbf{x}) * j \in \mathcal{A}(\mathcal{Z}', g)$.
- Each $(\mathbf{x}, j)$ where $p_1 \notin i_D(\mathbf{x})$, $p_1' \in i_A(\mathbf{x})$, and $p_2 \notin j$. Here $i_A(\mathbf{x}) * j \notin \mathcal{A}(\mathcal{Z}', g)$. However, there is a unique idempotent $i' \in \mathcal{A}(\mathcal{Z}', g)$ such that $i_A(\mathbf{x}) * j$ is a term in the inclusion of i' into $\mathcal{A}_{P'}(\mathcal{Z}', g)$. We let $i_A((\mathbf{x}, j)) = i'$ and $i_D((\mathbf{x}, j)) = i_D(\mathbf{x}) * j \in \mathcal{A}(\mathcal{Z}, g)$.
- Each $(\mathbf{x}, j)_{\text{smeared}} = (\mathbf{x}, u(j)) + (u_b(\mathbf{x}), j)$ where $p_1 \in i_D(\mathbf{x}), p_1' \in i_A(\mathbf{x})$, and $p_2 \in j$. We have

$$i_D((\mathbf{x}, j)_{\text{smeared}}) = i_D(\mathbf{x}) * u(j) + i_D(u_b(\mathbf{x})) * j \in \mathcal{A}(\mathcal{Z}, g),$$

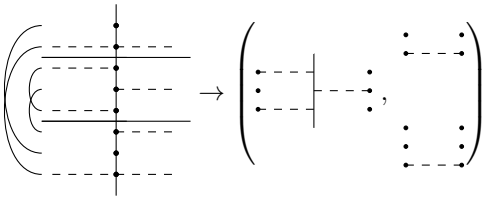
  and

$$i_A((\mathbf{x}, j)_{\text{smeared}}) = i_A(\mathbf{x}) * u(j) + i_A(u_b(\mathbf{x})) * j \in \mathcal{A}(\mathcal{Z}', g).$$

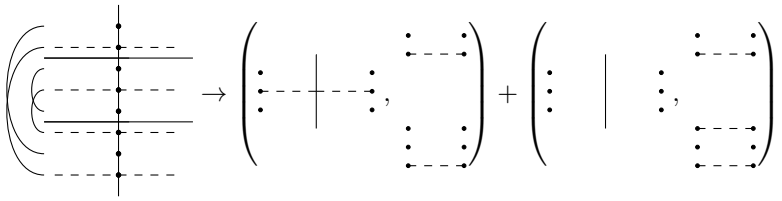
Examples for the three cases are shown below, with $\mathcal{Z} = \mathcal{Z}'$ in Figure 4.2 and $\mathcal{Z}_1 = \mathcal{Z}_1'$ and $\mathcal{Z}_2$ in Figure 4.3. On each line, the left side is a generator $\mathbf{x}$ in $\widetilde{M}$ (with $i_D(\mathbf{x})$ on the left and $i_A(\mathbf{x})$ on the right). The right side is the image of $\mathbf{x}$ in $\widetilde{M}_0$, with each term being a pair $(\mathbf{x}', j)$.



(4.16)



(4.17)



(4.18)

We can think of $\widetilde{M}$ as being obtained from $\widetilde{M_0}$ by removing some generators (those $(\mathbf{x}, j)$ with $\{p_1, p_2\} \subset i_D((\mathbf{x}, j))$ or $\{p_1', p_2\} \subset i_A((\mathbf{x}, j))$), and combining some of its generators in pairs (in the case of $(\mathbf{x}, j)_{\text{smeared}}$). The latter is analogous to the generators of $\mathcal{A}(\mathcal{Z})$ with a double horizontal at pair P. In both cases, we call those generators that become a sum of two generators under inclusion the *smeared* generators.

There is a natural inclusion

$$i_D : \mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M} \to \mathcal{A}_P(\mathcal{Z}, g) \otimes \widetilde{M_0}$$

coming from inclusion on two sides of the tensor product. We emphasize that the tensor products are taken over the idempotents.

Likewise, there are inclusions:

$$i_A : \widetilde{M} \otimes T^*(\mathcal{A}(\mathcal{Z}', g)) \to \widetilde{M_0} \otimes T^*(\mathcal{A}_{P'}(\mathcal{Z}', g)).$$

The type DA action of $\widetilde{M}$,

$$\delta^1_{\widetilde{M},k} : \widetilde{M} \otimes T^*(\mathcal{A}(\mathcal{Z}', g)) \to \mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M},$$

is defined as the composition of inclusion and the type DA action on $\widetilde{M_0}$:

$$\delta^1_{\widetilde{M_0},k} \circ i_A : \widetilde{M} \otimes T^*(\mathcal{A}(\mathcal{Z}', g)) \to \widetilde{M_0} \otimes T^*(\mathcal{A}_{P'}(\mathcal{Z}', g)) \to \mathcal{A}_P(\mathcal{Z}, g) \otimes \widetilde{M_0}.$$

Proposition 37. *The image of $\delta^1_{\widetilde{M_0},k} \circ i_A$ lies in $\mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M}$ under the inclusion i_D.*

Proof. We can consider $\mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M}$ as a subspace of $\mathcal{A}_P(\mathcal{Z}, g) \otimes \widetilde{M_0}$, formed from the latter in two steps:

- Remove any $c' \otimes \mathbf{z}'$ such that one of $i_L(c'), i_R(c')$ and $i_D(\mathbf{z}')$ contains $\{p_1, p_2\}$, or $i_A(\mathbf{z}')$ contains $\{p_1', p_2\}$.

- Combine terms of the form

$$(u(c) * b) \otimes (u_b(\mathbf{z}), i_R(b)) \text{ and } (c * u(b)) \otimes (\mathbf{z}, i_R(u(b))),$$

  into pairs.

Hence, we can verify the proposition in two steps. First, show any term in the image is not one of those that are removed (by checking idempotents). Second, show that terms that are paired when forming $\mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M}$ must occur together in the image.

Step I: Given input

$$\mathbf{x}' \otimes a_1' \otimes \cdots \otimes a_k' \in \widetilde{M} \otimes T^*(\mathcal{A}(\mathcal{Z}', g)).$$

Let

$$(\mathbf{x}, j) \otimes (a_1 * b_1) \otimes \cdots \otimes (a_k * b_k) \in \widetilde{M_0} \otimes T^*(\mathcal{A}_{P'}(\mathcal{Z}', g))$$

be a term in $i_A(\mathbf{x}' \otimes a_1' \otimes \cdots \otimes a_k')$, and let

$$(c * (b_1 \cdots b_k)) \otimes (\mathbf{z}, i_R(b_k)) \in \mathcal{A}_P(\mathcal{Z}, g) \otimes \widetilde{M_0}$$

be a term in its image under $\delta^1_{\widetilde{M_0}, k+1}$, coming from the arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{z}.$$

Since $\delta^1_{M,k+1}$ respects idempotents, we have $i_L(c) = i_D(\mathbf{x})$. Also, $j = i_L(b_1) = i_L(b_1 \cdots b_k)$. So

$$i_L(c * (b_1 \cdots b_k)) = i_L(c) * i_L(b_1 \cdots b_k) = i_D(\mathbf{x}) * j.$$

Since $(\mathbf{x}, j)$ is in the image of some $\mathbf{x}' \in \widetilde{M}$, it is impossible to have $\{p_1, p_2\} \subset i_L(c * (b_1 \cdots b_k))$. Next,

$$i_R(c * (b_1 \cdots b_k)) = i_D((\mathbf{z}, i_R(b_k))) = i_D(\mathbf{z}) * i_R(b_k).$$

If $p_2 \in i_R(b_k)$, then $p_1' \notin i_R(a_k) = i_A(\mathbf{z})$. By the smearability condition on the generators of M, we have $p_1 \notin i_D(\mathbf{z})$. So again it is impossible to have $\{p_1, p_2\} \subset i_R(c * (b_1 \cdots b_k))$.

We have already verified $\{p_1, p_2\} \notin i_D((\mathbf{z}, i_R(b_k)))$. Then since

$$i_A((\mathbf{z}, i_R(b_k))) = i_R(a_k * b_k),$$

we see $\{p_1', p_2\} \notin i_A((\mathbf{z}, i_R(b_k)))$ as well.

Step II: We use the same setup as in Step I. Suppose $(c * u(b)) \otimes (\mathbf{z}, i_R(u(b)))$ is a term in the image. It must come from an arrow in the type DA action of M:

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to c \otimes \mathbf{z}, \tag{4.19}$$

where c has a single horizontal at p_1 and $\mathbf{z} \in S^1$. By the smearability condition, there is also an arrow

$$\delta^1_{M,k+1}(u_b(\mathbf{x}), u(a_1), \ldots, u(a_k)) \to u(c) \otimes u_b(\mathbf{z}). \tag{4.20}$$

Moreover, each a_i has a single horizontal at p_1', and $\mathbf{x} \in S_1$. This means $\mathbf{x}'$ and a_i' are all smearable generators. The term $(c * u(b)) \otimes (\mathbf{z}, i_R(u(b)))$ comes from choosing the horizontals at p_1' in the inclusion, and using arrow (4.19). Choosing instead the horizontals at p_2 in the inclusion and using (4.20), we see $(u(c) * b) \otimes (u_b(\mathbf{z}), i_R(b))$ is also a term in $\delta^1_{\widetilde{M_0},k+1} \circ i_A(\mathbf{x}' \otimes a_1' \otimes \cdots \otimes a_k')$.

Now we show the converse. Suppose $(u(c) * b) \otimes (u_b(\mathbf{z}), i_R(b))$ is in the image of $\delta^1_{\widetilde{M_0},k+1} \circ i_A$. It comes from an arrow

$$\delta^1_{\widetilde{M_0},k+1}((\mathbf{x}, j), a_1 \otimes b_1, \ldots, a_k \otimes b_k) \to (u(c) * b) \otimes (u_b(\mathbf{z}), i_R(b))$$

in $\delta^1_{\widetilde{M_0}}$. In particular $b = b_1 \cdots b_k$. Since b has single horizontal at p_2, each b_i must also have a single horizontal at p_2, so $p_1' \notin i_A(a_i)$ and $p_1' \notin i_D(a_i)$ for all i. In addition, $p_1 \notin i_D(\mathbf{x}) = i_L(u(c))$ and $p_1' \notin i_A(\mathbf{x}) = i_L(a_1)$ (or $i_A(u_b(\mathbf{z}))$ if $k = 0$), showing $\mathbf{x} \in S_0$.

So the arrow

$$\delta^1_{M,k+1}(\mathbf{x}, a_1, \ldots, a_k) \to u(c) \otimes u_b(\mathbf{z}) \tag{4.21}$$

implies, by the smearability condition an arrow

$$\delta^1_{M,k+1}(\bar{\mathbf{x}}, \bar{a_1}, \ldots, \bar{a_k}) \to c \otimes \mathbf{z}, \tag{4.22}$$

where $u_b(\bar{\mathbf{x}}) = \mathbf{x}$, and $u(\bar{a_i}) = a_i$ for each $1 \leq i \leq k$. This produces the term $(c * u(b)) \otimes (\mathbf{z}, i_R(u(b)))$ in the image of $\delta^1_{\widetilde{M_0},k+1} \circ i_A$. □

Proof of Theorem 36. We have shown that the type DA action on $\widetilde{M}$ is well-defined as a map

$$\widetilde{M} \otimes T^*(\mathcal{A}(\mathcal{Z}', g)) \to \mathcal{A}(\mathcal{Z}, g) \otimes \widetilde{M}.$$

Observe that $\delta^1_{M,k}$ is simply a restriction of $\delta^1_{\widetilde{M},k}$ to inputs from $\widetilde{M}$ and $\mathcal{A}(\mathcal{Z}', g)$. So the equation $d^2_{\widetilde{M}} = 0$ is a special case of the equation $d^2_{\widetilde{M_0}} = 0$. □

Remark:

1. We may define box tensor product between two local type DA bimodules with smearing. Extension by identity still commutes with the box tensor product.

2. To extend a morphism by identity, we need corresponding smearability conditions on the morphism. Extension by identity then gives a dg-functor between categories of type DA bimodules, where we restrict to smearable morphisms.

3. It is unclear what is the corresponding notion of the smearability condition in cornered Floer homology.

4.4 Type DA structure of arcslides

We now apply the results in the previous section to construct type DA bimodules for arcslides.

The definition of arcslides, Heegaard diagrams for them, and the type DD invariants associated to them are given in [16]. We assume the reader is familiar with the main constructions in that paper.

Let $\mathcal{Z}$ and $\mathcal{Z}'$ be the starting and ending pointed matched circles (full PMCs) of an arcslide. There are two distinguished pairs of points in $\mathcal{Z}$ and $\mathcal{Z}'$: the B pair and the C pair. We label the points in the distinguished pairs as follows. In $\mathcal{Z}$, the B pair is (b_1, b_2) and the C pair is (c_1, c_2). In $\mathcal{Z}'$, the B pair is (b'_1, b'_2) and the C pair is (c'_1, c'_2). The points b_1 and c_1 are adjacent, and so are b'_1 and c'_2.

We now explain the general ideas of the construction. The Heegaard diagram for an arcslide looks like that of the identity diffeomorphism outside a region that borders $\mathcal{Z}$ at points b_1, c_1, and c_2, and $\mathcal{Z}'$ at points b'_1, c'_1, and c'_2. Hence, we can attempt to find a smearable type DA bimodule for that region, then extending it by identity to obtain the type DA bimodule for the full Heegaard diagram. This means we will choose a splitting $\mathcal{Z} = \mathcal{Z}_1 * \mathcal{Z}_2$ using $\mathbf{s_1} = \{b_1, c_1, c_2\}$, and a splitting $\mathcal{Z}' = \mathcal{Z}'_1 * \mathcal{Z}'_2$ using $\mathbf{s'_1} = \{b'_1, c'_1, c'_2\}$ (with $\mathbf{s_2}$ and $\mathbf{s'_2}$ being their complements). From the definition of arcslide, one can check that $\mathcal{Z}_2 = \mathcal{Z}'_2$ as is required in the previous section. On both sides, the smearing is on the B pair.

So our goal is to find a smearable type DA bimodule ${}^{\mathcal{A}(\mathcal{Z}_1)}M_{\mathcal{A}(\mathcal{Z}_1')}$. Its extension by identity is ${}^{\mathcal{A}(\mathcal{Z},g)}\widetilde{M}_{\mathcal{A}(\mathcal{Z}',g)}$.

In constructing M, we will use the region of the Heegaard diagram near $\mathcal{Z}_1$ and $\mathcal{Z}_1'$ as a guide, for finding the set of generators and some basic arrows in the type DA action. However, the correctness of M is eventually only verified by checking that $\widetilde{M}$ is homotopy equivalent to the type DA invariant of the arcslide (in Theorem 40). We emphasize that there is currently no formal notion of local Heegaard diagrams, and no definitions of invariants on them using holomorphic curves.

Let τ be an arcslide. There are several cases, of which two are representative of all others. These two cases are underslides where $|c_1 - c_2| = 2$, and underslides where $|c_1 - c_2| > 2$. We will begin with the latter case. The Heegaard diagram locally looks like Figure 4.4.

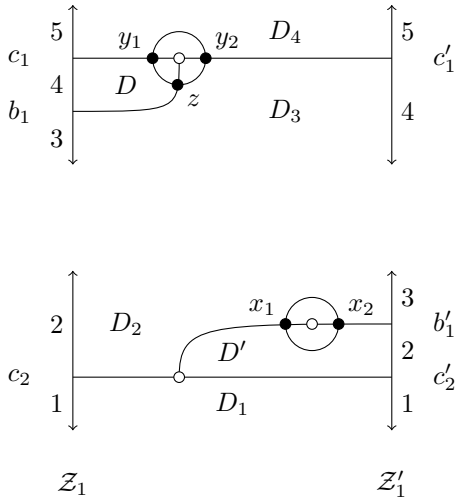


Figure 4.4: Local Heegaard diagram for an underslide with $b_1 = c_1 - 1$ and $|c_1 - c_2| > 2$. Numbers label the intervals of the local PMC. Lower case variables label points, and upper case variables label regions in the Heegaard diagram. The left local PMC is $\mathcal{Z}_1$ and the right local PMC is $\mathcal{Z}_1'$.

We now give the generators of M. We specify an idempotent as the set of pairs and unpaired point it contains. In both $\mathcal{Z}_1$ and $\mathcal{Z}_1'$ there is a C pair and the unpaired point is from the B pair. We will simply name them C and B, respectively. The generators are as follows (the corresponding choice of intersection points in the local Heegaard diagram is shown in parenthesis).

- $\mathbf{x_1} = (x_1, y_1)$ with $i_D(\mathbf{x_1}) = \emptyset$ and $i_A(\mathbf{x_1}) = \emptyset$.
- $\mathbf{x_2} = (x_2, y_1)$ with $i_D(\mathbf{x_2}) = \{B\}$ and $i_A(\mathbf{x_2}) = \{B\}$.
- $\mathbf{x_3} = (x_1, y_2)$ with $i_D(\mathbf{x_3}) = \{C\}$ and $i_A(\mathbf{x_3}) = \{C\}$.
- $\mathbf{x_4} = (y_1, y_2)$ with $i_D(\mathbf{x_4}) = \{B, C\}$ and $i_A(\mathbf{x_4}) = \{B, C\}$.

- $\mathbf{x_5} = (x_2, z)$ with $i_D(\mathbf{x_5}) = \{C\}$ and $i_A(\mathbf{x_5}) = \{B\}$.

The map u_b from Definition 35 is given by $u_b(\mathbf{x_2}) = \mathbf{x_1}$ and $u_b(\mathbf{x_4}) = \mathbf{x_3}$. We note that there are no generators $\mathbf{x}$ with $B \in i_D(\mathbf{x})$ but $B \notin i_A(\mathbf{x})$.

Let ρ_i (resp. ρ_i') be the ith interval or open end on the local PMC $\mathcal{Z}_1$ (resp. $\mathcal{Z}_1'$), as labeled in Figure 4.4. Let $a(\rho_i)$ (resp. $a(\rho_i')$) be the sum of generators in $\mathcal{A}(\mathcal{Z}_1)$ (resp. $\mathcal{A}(\mathcal{Z}_1')$) covering only the interval ρ_i (resp. ρ_i'). Let I and I' be the identity element of $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_1')$ (the sum of all idempotents). From the local Heegaard diagram, we make the assumption that the following arrows must be in the type DA action of M:

- $A_i : \delta_1^1(\mathbf{x_i}, I') \to I \otimes \mathbf{x_i}$ for $1 \leq i \leq 5$ (idempotents)
- $B_1 : \delta_0^1(\mathbf{x_2}) \to a(\rho_4) \otimes \mathbf{x_5}$ (region D)
- $B_2 : \delta_1^1(\mathbf{x_3}, a(\rho_2')) \to I \otimes \mathbf{x_5}$ (region D')
- $E_1 : \delta_1^1(\mathbf{x_2}, a(\rho_1')) \to a(\rho_1) \otimes \mathbf{x_4}$ (region D_1)
- $E_1' : \delta_1^1(\mathbf{x_1}, a(\rho_1')) \to a(\rho_1) \otimes \mathbf{x_3}$ (region D_1)
- $E_2 : \delta_1^1(\mathbf{x_5}, a(\rho_3')) \to a(\rho_2) \otimes \mathbf{x_1}$ (region D_2)
- $E_3 : \delta_1^1(\mathbf{x_5}, a(\rho_4')) \to a(\rho_3) \otimes \mathbf{x_4}$ (region D_3)
- $E_4 : \delta_1^1(\mathbf{x_4}, a(\rho_5')) \to a(\rho_5) \otimes \mathbf{x_2}$ (region D_4)
- $E_4' : \delta_1^1(\mathbf{x_3}, a(\rho_5')) \to a(\rho_5) \otimes \mathbf{x_1}$ (region D_4)

Note the smearability condition is satisfied with these set of arrows, with E_1 corresponding to E_1' and E_4 corresponding to E_4'. Of course, the type DA structure equation is not yet satisfied.

We claim the following:

Theorem 38. *With the above setup, and local PMCs shown in Figure 4.4, there is a (non-unique) smearable type DA bimodule M with generators $\mathbf{x_1}$ through $\mathbf{x_5}$, and whose type DA action contains arrows A_1 through E_4'.*

Remark: It is possible that the smearable type DA bimodule in Theorem 38 is unique up to homotopy equivalence. We do not claim this here. Later on, in Theorem 40 we will essentially prove that its extension by identity is unique up to homotopy equivalence.

Computer assistance is used to verify this theorem, by finding an explicit example of type DA action on M (with 197 non-idempotent arrows in total). We now describe the technique used to find these arrows.

For this purpose, it is better to think of the type DA action on M as a type D action over $\mathrm{Cob}(\mathcal{A}(\mathcal{Z}_1')) \otimes \mathcal{A}(\mathcal{Z}_1)$. We have the additional advantage that the generators of M are distinguished by their idempotents, so that each arrow in the action simply corresponds to a generator of $\mathrm{Cob}(\mathcal{A}(\mathcal{Z}_1')) \otimes \mathcal{A}(\mathcal{Z}_1)$ (since the starting and ending generator of the arrow can be inferred from the idempotents). The type D action then corresponds to an element $A \in \mathrm{Cob}(\mathcal{A}(\mathcal{Z}_1')) \otimes \mathcal{A}(\mathcal{Z}_1)$, and the type D structure equation becomes:

$$dA + A \cdot A = 0. \tag{4.23}$$

Thus, we have reduced the problem to solving (4.23) for A, subject to smearability conditions and the condition that A contains the basic arrows A_1 through E_4'.

We solve for A in four steps. They can be thought of as "adding" in sequence the four regions D_1 through D_4 in Figure 4.4, or equivalently the four types of terms $E_1^{(')}$ through $E_4^{(')}$.

At each step i, for $1 \leq i \leq 4$, we begin with an initial solution $A_{i-1} \in \mathrm{Cob}(\mathcal{A}(\mathcal{Z}_1')) \otimes \mathcal{A}(\mathcal{Z}_1)$ satisfying (4.23). We choose a certain subspace $S_0^i \subset \mathrm{Cob}(\mathcal{A}(\mathcal{Z}_1')) \otimes \mathcal{A}(\mathcal{Z}_1)$ such that for any $A', A'' \in S_0^i$, we have $A' \cdot A'' = 0$. We choose another subspace $S^i \subset S_0^i$ such that adding any $A' \in S^i$ to A_{i-1} would maintain the smearability condition. The goal is then to find $A_i' \in S^i$, containing certain arrows, so that $A_{i-1} + A_i'$ continues to satisfy (4.23). The equation to be satisfied is:

$$\begin{aligned} 0 &= d(A_{i-1} + A_i') + (A_{i-1} + A_i') \cdot (A_{i-1} + A_i') \\ &= dA_{i-1} + dA_i' + A_{i-1} \cdot A_{i-1} + A_{i-1} \cdot A_i' + A_i' \cdot A_{i-1} + A_i' \cdot A_i' \\ &= dA_i' + A_{i-1} \cdot A_i' + A_i' \cdot A_{i-1}. \end{aligned}$$

On the last line, we used the assumptions $A_i' \cdot A_i' = 0$ and $dA_{i-1} + A_{i-1} \cdot A_{i-1} = 0$.

Thus, define a linear map $d' : S^i \to S^i$ by:

$$d'(A') = dA' + A_{i-1} \cdot A' + A' \cdot A_{i-1},$$

and solve the equation $d'(A_i') = 0$ for A_i', with the only remaining condition being that A_i' must

contain some initial arrows. Then set $A_i = A_{i-1} + A'_i$, with A_4 being the final solution of A.

Remark: The map d' actually satisfies $(d')^2 = 0$, so it may be possible to treat this problem by considering the homology of S^i under the differential d'. One way to approach this is through the homology of $\mathrm{Cob}(\mathcal{A}(\mathcal{Z}'_1)) \otimes \mathcal{A}(\mathcal{Z}_1)$ under the differential d. However, this does have many generators, including ones in the correct grading. So a solution along this path is not immediate.

For the first step, we set $A_0 \in \mathrm{Cob}(\mathcal{A}(\mathcal{Z}'_1)) \otimes \mathcal{A}(\mathcal{Z}_1)$ to contain the arrows B_1, and B_2 (note the arrows A_i do not corresponds to terms in $\mathrm{Cob}(\mathcal{A}(\mathcal{Z}'_1)) \otimes \mathcal{A}(\mathcal{Z}_1)$). We can easily verify that A_0 satisfies the structure equation and the smearability condition.

Choose $S^1_0 \in \mathrm{Cob}(\mathcal{A}(\mathcal{Z}'_1)) \otimes \mathcal{A}(\mathcal{Z}_1)$ to be the subspace spanned by $a_1^* \otimes \cdots \otimes a_k^* \otimes b$, with $a_i \in \mathcal{A}(\mathcal{Z}'_1)$ and $b \in \mathcal{A}(\mathcal{Z}_1)$, such that:

- b has multiplicity zero on the intervals $\{2, 3, 5\}$.
- b has multiplicity one on the interval 1.

By the multiplicity-one condition on the interval 1, we see $A' \cdot A'' = 0$ for any $A', A'' \in S^1_0$.

Next, we choose $S^1 \subset S^1_0$ to be the subspace spanned by the following:

- $a_1^* \otimes \cdots \otimes a_k^* \otimes b$ if b does not have a single horizontal at point b_1.
- $a_1^* \otimes \cdots \otimes a_k^* \otimes b + u(a_1)^* \otimes \cdots \otimes u(a_k)^* \otimes u(b)$, if each a_i has a single horizontal at point b'_1, b has a single horizontal at point b_1, and the map u removes the single horizontals (as defined in the previous section).

Then adding any $A' \in S^1$ preserves the smearability condition. Finally, we require the solution $A'_i \in S^1$ to contain the two arrows E_1 and E'_1.

In solving for A'_i, we only need to consider generators of $\mathrm{Cob}(\mathcal{A}(\mathcal{Z}'_1)) \otimes \mathcal{A}(\mathcal{Z}_1)$ that are linked to E_1 and E'_1 through the map d' and the smearability condition (directly or through a chain of other generators). The number of such generators is finite, so this reduces to a computation in linear algebra. Considering only such terms has the added benefit that the grading condition on M (and the eventual $\widetilde{M}$) is automatically satisfied.

The principles for the next steps are the same:

- Second step: consider $a_1^* \otimes \cdots \otimes a_k^* \otimes b$ such that b has multiplicity zero on the intervals $\{3, 5\}$, and multiplicity one on the interval 2. The solution A'_2 must contain E_2.

- Third step: consider $a_1^* \otimes \cdots \otimes a_k^* \otimes b$ such that b has multiplicity zero on the interval 5, and multiplicity one on the interval 3. The solution A_3' must contain E_3.

- Fourth step: consider $a_1^* \otimes \cdots \otimes a_k^* \otimes b$ such that b has multiplicity one on the interval 5. The solution A_4' must contain E_4 and E_4'.

This finishes the construction of M satisfying all conditions in Theorem 38. There are choices to be made when solving $d'(A') = 0$, hence the type DA action on M is not unique (even the number of arrows in the action is not unique. The number 197 is simply a particular choice, with no claim that it is minimal).

The other essentially different case is underslides where $|c_1 - c_2| = 2$. The Heegaard diagram locally looks like Figure 4.5.

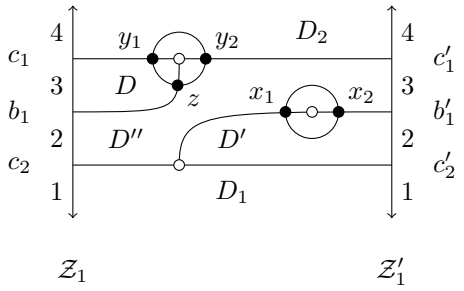


Figure 4.5: Local Heegaard diagram for an underslide with $b_1 = c_1 - 1 = c_2 + 1$. Numbers label the intervals of the local PMC. Lower case variables label points, and upper case variables label regions in the Heegaard diagram. The left local PMC is $\mathcal{Z}_1$ and the right local PMC is $\mathcal{Z}_1'$.

The set of generators is the same as before. The following are arrows that must be in M. Here $a(\rho_{ij})$ with $j = i + 1$ denotes the sum of generators in $\mathcal{Z}_1$ that contain a single strand covering the intervals i and j.

- $A_i : \delta_1^1(\mathbf{x_i}, I') \to I \otimes \mathbf{x_i}$ for $1 \leq i \leq 5$ (idempotents)

- $B_1 : \delta_0^1(\mathbf{x_2}) \to a(\rho_3) \otimes \mathbf{x_5}$ (region D)

- $B_2 : \delta_1^1(\mathbf{x_3}, a(\rho_2')) \to I \otimes \mathbf{x_5}$ (region D')

- $B_3 : \delta_1^1(\mathbf{x_5}, a(\rho_3')) \to a(\rho_{23}) \otimes \mathbf{x_3}$ (region $D + D''$)

- $B_4 : \delta_1^1(\mathbf{x_3}, a(\rho_{23}')) \to a(\rho_{23}) \otimes \mathbf{x_3}$ (region $D + D' + D''$)

- $B_4' : \delta_1^1(\mathbf{x_4}, a(\rho_{23}')) \to a(\rho_{23}) \otimes \mathbf{x_4}$ (region $D + D' + D''$)

- $B_5 : \delta_1^1(\mathbf{x_4}, da(\rho_{23}')) \to da(\rho_{23}) \otimes \mathbf{x_4}$ (region $D + D' + D''$)

- $B_6 : \delta_2^1(\mathbf{x_5}, a(\rho_3'), a(\rho_2')) \to a(\rho_2) \otimes \mathbf{x_2}$ (region $D' + D''$)
- $B_7 : \delta_2^1(\mathbf{x_3}, a(\rho_{23}'), a(\rho_2')) \to a(\rho_2) \otimes \mathbf{x_2}$ (region $2D' + D''$)
- $E_1 : \delta_1^1(\mathbf{x_1}, a(\rho_1')) \to a(\rho_1) \otimes \mathbf{x_3}$ (region A)
- $E_1' : \delta_1^1(\mathbf{x_2}, a(\rho_1')) \to a(\rho_1) \otimes \mathbf{x_4}$ (region A)
- $E_2 : \delta_1^1(\mathbf{x_3}, a(\rho_4')) \to a(\rho_4) \otimes \mathbf{x_1}$ (region B)
- $E_2' : \delta_1^1(\mathbf{x_4}, a(\rho_4')) \to a(\rho_4) \otimes \mathbf{x_2}$ (region B)

Theorem 39. *For the local PMCs shown in Figure 4.5, there is a (non-unique) smearable type DA bimodule M with generators $\mathbf{x_1}$ through $\mathbf{x_5}$, and whose type DA action contains the terms A_1 through E_2'.*

This is verified by computer as in the previous case. The computation has two steps rather than four this time. Begin with A_0 containing the arrows B_1 through B_7 (one checks that this satisfies the type DA structure equation and the smearability conditions).

- First step: consider $a_1^* \otimes \cdots \otimes a_k^* \otimes b$ such that b has multiplicity zero on the interval 4, and multiplicity one on the interval 1. The solution A_1' must contain E_1 and E_1'.
- Second step: consider $a_1^* \otimes \cdots \otimes a_k^* \otimes b$ such that b has multiplicity one on the interval 4. The solution A_2' must contain E_2 and E_2'.

The number of non-idempotent arrows in the action is 77 for one particular choice.

The two other underslide cases, with $b_1 = c_1 + 1$ and $c_2 > b_1$, can be treated similarly. The overslide cases are obtained by exchanging the two segments of the local PMCs in the underslide case with $|c_1 - c_2| > 2$. If some of the ends of the segments in the local PMC are in fact closed (as when they touch the boundary of the full PMC), we simply discard all arrows in the action that involve algebra generators that go off to those ends.

Thus, we are able to construct the type DA bimodule for all local cases of arcslides. Using extension by identity, we can construct a candidate for the type DA bimodule for all arcslides.

Theorem 40. *For any arcslide τ, the bimodule $\widetilde{M}$ constructed above using extension by identity from M is in the homotopy type of $\widehat{CFDA}(\tau)$.*

Proof. Given arcslide $\tau : \mathcal{Z} \to \mathcal{Z}'$ and let $\widetilde{M}$ be the type DA bimodule constructed above for τ. Let $M_{DD} = \widetilde{M} \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$, where $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ is the type DD bimodule for the identity

diffeomorphism (described in [16]). We aim to show that $M_{DD} \simeq \widehat{CFDD}(\tau)$ as type DD bimodules. Then,

$$\begin{aligned}
\widetilde{M} &\simeq \widetilde{M} \boxtimes \mathbb{I} \\
&\simeq \widetilde{M} \boxtimes \widehat{CFDD}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}) \\
&\simeq M_{DD} \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}) \\
&\simeq \widehat{CFDD}(\tau) \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}) \\
&\simeq \widehat{CFDA}(\tau).
\end{aligned}$$

To show $M_{DD} \simeq \widehat{CFDD}(\tau)$, we use the uniqueness theorem of $\widehat{CFDD}(\tau)$ proved in [16]. To use it, we need to show the following conditions are satisfied by M_{DD} (this is slightly different from the version in [16], but it can be shown that they are equivalent).

1. M_{DD} has the same generators as $\widehat{CFDD}(\tau)$, with the same idempotents.
2. The same grading can be placed on the generators of M_{DD} as on $\widehat{CFDD}(\tau)$, so that the type DD action on M_{DD} respects that grading.
3. The set of arrows in the type DD action of the form

$$\delta^1(\mathbf{x}) \to a_1 \otimes a_2 \otimes \mathbf{y},$$

   where each of a_1 and a_2 is either an idempotent or a length-1 chord, is the same between M_{DD} and $\widehat{CFDD}(\tau)$.
4. M_{DD} is *stable* in the sense of [16, Definition 1.10].

Condition (1) is clear from the definition of the set of generators in $\widetilde{M}$ (which is in one-to-one correspondence with generators in M_{DD}).

For condition (2), we verify that the extensions of the initial arrows A_i, B_i, and E_i respect grading on $\widehat{CFDD}(\tau)$. Since all other added arrows are related to the inital arrows via the map d' or the smearability condition, extensions of them must also respect grading on $\widehat{CFDD}(\tau)$.

For condition (3), we see that the initial arrows A_i, B_i, and E_i in $\widetilde{M}$, when extended and combined with the type DD action on $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$, already produce all arrows in $\widehat{CFDD}(\tau)$ whose algebra outputs are either idempotents or length-1 chords. All added arrows involve algebra inputs or outputs of length greater than 1. Since combining with type DD action on $\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}})$ does not

change the total length of algebra outputs, the added arrows cannot change the set of arrows in M_{DD} of this form.

Condition (4) is clear since for any other pointed matched circle $\mathcal{Z}_0$ of genus g', the same local type DA bimodule M can be used to produce a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z}\#\mathcal{Z}_0,g+g')}\widetilde{M'}_{\mathcal{A}(\mathcal{Z}'\#\mathcal{Z}_0,g+g')}$ that induces $\widetilde{M}$. □

4.5 An algorithm for computing $\widehat{HF}$

In this section, we describe the algorithm to compute $\widehat{HF}$ in more detail.

As in [16], we consider a three-manifold Y as the union of two standard handlebodies: the 0-framed handlebody $\mathbf{H}^g$ and the ∞-framed handlebody $-\mathbf{H}^g$ of genus g, glued along a diffeomorphism $\psi : F_g \to F_g$ of the genus g surface. We may factor ψ into a product of arcslides

$$\psi = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_1,$$

so that the Heegaard Floer homology of Y is given by

$$\widehat{HF}(Y) = \widehat{CFA}(-\mathbf{H}^g) \boxtimes \widehat{CFDA}(\tau_n) \boxtimes \cdots \boxtimes \widehat{CFDA}(\tau_1) \boxtimes \widehat{CFD}(\mathbf{H}^g).$$

We use the bimodule given in the previous section as the representative for $\widehat{CFDA}(\tau_i)$. Both $\widehat{CFA}(-\mathbf{H}^g)$ and $\widehat{CFD}(\mathbf{H}^g)$ can be described simply. Thus, the main focus is on quickly evaluating the box tensor product

$$\widehat{CFDA}(\tau) \boxtimes N,$$

where τ is any arcslide, and N is any type D module.

The definition of box tensor product between a type DA bimodule and a type D module is given in [14]. From the computational point of view, the definition is as follows:

Definition 41. *Let M be a type DA bimodule and N be a type D module, then $M \boxtimes N$ is a type D module generated by pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ is a generator of M and $\mathbf{y}$ is a generator of N, such that*

$i_A(\mathbf{x}) = i(\mathbf{y})$. *The action on* $M \boxtimes N$ *is as follows: for each integer* $k \geq 0$, *arrows*

$$\delta_N^1(\mathbf{y_0}) \to a_1 \otimes \mathbf{y_1}$$
$$\delta_N^1(\mathbf{y_1}) \to a_2 \otimes \mathbf{y_2}$$
$$\vdots$$
$$\delta_N^1(\mathbf{y_{k-1}}) \to a_k \otimes \mathbf{y_k}$$

in the type D *action of* N, *and an arrow*

$$\delta_{M,k+1}^1(\mathbf{x_0}, a_1, \ldots, a_k) \to b \otimes \mathbf{x_1}$$

in the type DA *action of* M, *there is an arrow*

$$(\mathbf{x_0}, \mathbf{y_0}) \to b \otimes (\mathbf{x_1}, \mathbf{y_k})$$

in the type D *action of* $M \boxtimes N$.

Thus, the arrows in the action of $\widehat{CFDA}(\tau) \boxtimes N$ can be found by traversing the "graph" of N, with generators of N as vertices and arrows in the action of N as edges. There are two kinds of queries we wish to ask about $\widetilde{M} = \widehat{CFDA}(\tau)$:

- Given a generator $\mathbf{x}$ of $\widehat{CFDA}(\tau)$ and a sequence $a_1, \ldots, a_k$ of algebra inputs, what is the value of $\delta^1_{\widetilde{M},k+1}(\mathbf{x}, a_1, \ldots, a_k)$?
- Given a generator $\mathbf{x}$ of $\widehat{CFDA}(\tau)$ and a sequence $a_1, \ldots, a_i$ of algebra inputs, can there possibly exist $k > i$ and further inputs $a_{i+1}, \ldots, a_k$ such that $\delta^1_{\widetilde{M},k+1}(\mathbf{x}, a_1, \ldots, a_i, a_{i+1}, \ldots, a_k)$ is nonzero?

For the second kind of query, we do not need a precise answer of whether further inputs exist, but only need to discover enough cases where it does not exist in order to cut down the size of the search tree.

With $\widehat{CFDA}(\tau)$ as constructed in the previous section, both kinds of queries can be answered in a constant amount of time for a fixed genus g (and the time only increases linearly with g, as the amount of information needed to record a strand diagram is proportional to g). For the first kind of query, we wish to evaluate

$$\delta^1_{\widetilde{M},k+1}(\mathbf{x}, a_1, \ldots, a_k)$$

where $\mathbf{x} \in \widetilde{M}$ and $a_1, \ldots, a_k \in \mathcal{A}(\mathcal{Z}', g)$ are generators. Recall that each a_i can be considered as an element of $\mathcal{A}_{P'}(\mathcal{Z}')$, where P' is the smeared pair in $\mathcal{Z}'$ (the B pair of the arcslide). As an element of $\mathcal{A}_{P'}(\mathcal{Z}')$, it is either a generator or a sum of two generators, corresponding to choosing single horizontals at one of the two points in P'. Likewise, $\mathbf{x}$ can be considered as an element of $\widetilde{M}_0$, that is either a generator or a sum of two generators.

Thus, to reduce the evaluation of $\delta^1_{\widetilde{M},k}$ to the evaluation of $\delta^1_{\widetilde{M_0},k}$, we need to choose one of the two generators for each $\mathbf{x}$ and a_i that can be written as a sum of two generators. However, since the tensor products are over the idempotents, there are at most two consistent choices. These can be enumerated by making one choice at $\mathbf{x}$ or some a_i, then repeatedly enforcing the idempotent restrictions.

We have now reduced the problem of evaluating $\delta^1_{\widetilde{M},k}$ to the problem of evaluating $\delta^1_{\widetilde{M_0},k}$ at most twice. For the latter problem, we restrict each input to $\mathcal{Z}'_1$ and $\mathcal{Z}_2$, then apply Equation (4.10).

The second type of query is similar. After choosing generators consistent with idempotents, we restrict the algebra inputs to $\mathcal{A}(\mathcal{Z}'_1)$ and $\mathcal{A}(\mathcal{Z}_2)$. In $\mathcal{A}(\mathcal{Z}_2)$ we check whether their product is nonzero. In $\mathcal{A}(\mathcal{Z}_1)$ we check whether the current list of inputs is a prefix of any of the finite number of arrows in the type DA action of M.

This completes the discussion of the computation of $\widehat{CFDA}(\tau) \boxtimes N$. We apply this computation to the formula for $\widehat{HF}(Y)$, starting at the end. At each step, the type D structure is simplified using the cancellation lemma. This continues until we reach the equation

$$\widehat{HF}(Y) = \widehat{CFA}(-\mathbf{H}^g) \boxtimes N,$$

for some type D structure N. The evaluation of the last box tensor product can be done by a similar traversal of the graph for N.

Remark: Another important case for which it may be interesting to find local type DA bimodules is Dehn twists on the linear pointed matched circle (see [15]). When used to compute $\widehat{HF}$ of branched double cover along a knot K with a given knot diagram, each Dehn twist corresponds to a crossing in the knot diagram, and the generators of the local bimodule correspond to local Kauffman states at that crossing. With a slight extension of the above theory to splittings with multiple smeared pairs, we can compute the local type DA bimodule for a Dehn twist by taking the box tensor product of bimodules for arcslides (in fact box tensor product of two arcslides is sufficient for any Dehn twist of the needed form). For the use of Kauffman states to compute $\widehat{HF}$ of branched double cover along

a knot, see also [6].

4.6 Results of computation

In this section, we describe the results of using the algorithm stated in the previous section to compute $\widehat{HF}(\Sigma(K))$, the Heegaard Floer homology of the branched double cover of a knot K.

Given a knot K, we first choose a *bridge presentation* for K. An n-bridge presentation of K is a representation of K as the result of closing off a braid with $2n$ strands by joining pairs of strands on the top and bottom in a way that does not introduce additional crossings. There is a special closure, called the *plat closure*, that is especially convenient for our computations (plat closures are used for similar purposes in [15]). The plat closure of $2n$ strands joins the pair $(1, 2n)$ and each pair $(2b, 2b+1)$ for $1 \leq b \leq n-1$. Figure 4.6 shows a 2-bridge presentation of the trefoil with plat closures at the two ends.

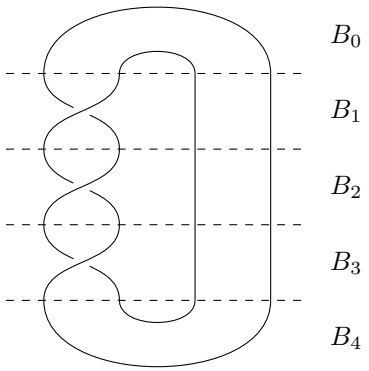


Figure 4.6: A representation of the trefoil knot as the plat closure of a braid.

We will assume that the n-bridge presentation has plat closures (in practice, we find n-bridge presentations with any closure, then transform it into one with plat closure by adding additional crossings at the two ends of the braid). As indicated in Figure 4.6, given an n-bridge presentation of a knot K with k crossings, we can slice K into $k+2$ parts $B_i, 0 \leq i \leq k+1$, with B_0 and B_{k+1} being the plat closures, and each $B_i, 1 \leq i \leq k$ containing one of the crossings in the braid.

The branched double cover $\Sigma(B_i)$ for each B_i can be considered as a bordered 3-manifold. Each boundary is a surface of genus $n-1$ (being the branched double cover of S^2, branching over $2n$ points). To simplify the descriptions of $\Sigma(B_i)$ later, we parametrize each boundary using the *linear* pointed matched circle of genus $n-1$. Here a linear pointed matched circle of genus g is defined by

the matched pairs

$$(1,3),(2,5),(4,7),\ldots,(2g-4,2g-1),(2g-2,2g).$$

Then $\Sigma(B_0)$ and $\Sigma(B_{k+1})$ are *plat handlebodies*, and each $\Sigma(B_i), 1 \leq i \leq k$ is a mapping cylinder of some Dehn twist on the surface. The latter have specific decompositions into arcslides (see [16] and [15] for details).

As such, we can write $\widehat{HF}(\Sigma(K))$ as the box tensor product of the (already known) type A and type D invariants of the plat handlebody, and type DA invariants of arcslides. This is the formula used to compute the results below.

Remark: There are some minor modifications for efficiency in the actual implementation. In particular, we work with bridge presentations with arbitrary closure, and compute $\widehat{CFD}(\Sigma(B))$ for any closure B that occurs, by transforming B into the plat closure. The computation repeatedly evaluates the box tensor product of the current type D module (beginning with $\widehat{CFD}(\Sigma(B))$ where B is the top closure) and the type DA bimodule of some arcslide, simplifying the type D module using the cancellation lemma at each step. At the end, the chain complex is found by evaluating

$$N \boxtimes \widehat{CFAA}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{CFD}(\Sigma(B')),$$

where N is the invariant for the branched double cover of all but the bottom closure, and B' is the bottom closure. This is evaluated using the description of $\widehat{CFAA}(\mathbb{I}_{\mathcal{Z}})$ given before. Since $\widehat{CFD}(\Sigma(B'))$ usually has very few generators, the last step is not a major factor in the running time of the algorithm.

For some of the cases given below, we will compare the running time with that of the algorithm described in [16]. This algorithm uses type DD invariants of arcslides and pairing theorems that involve the morphism complex. The number of generators in the morphism complex is proportional to the number of generators in the strand algebra, which increases exponentially with genus. With the usual implementation it is only practical up to genus 3 (for 4-bridge presentations).

The running times of both algorithms depend on two factors: the bridge number and the number of crossings in the knot. The bridge number affects the number of generators in the strand algebra, and the time needed to process each generator. The number of crossings affects the number of arcslides in the computation and the size of type D modules encountered (the size of type D modules, rather than number of crossings, is really the key factor, but that may be difficult to predict

beforehand).

First, we compare the algorithm on some 11-12 crossing knots with 3 or 4-bridge presentations in Table 4.1. We emphasize that the efficiency comparisons here are only rough estimates. Both implementations are written in Python and neither is heavily optimized. There is a more efficient implementation of the first algorithm written in C++, making it practical up to 4-bridge presentations of 14 crossing knots.

Knot	Presentation	Result	Time (Mor)	Size (Mor)	Time (Tensor)	Size (Tensor)
$11n6$	3-bridge	$15 \cdot [1] + 2 \cdot [1,2]$	72s	95	4.7s	95
$11n9$	3-bridge	$2 \cdot [1] + 3 \cdot [2,1]$	96s	111	6.2s	111
$11n24$	3-bridge	$23 \cdot [1]$	71s	98	4.5s	98
$11a14$	4-bridge	$133 \cdot [1]$	3300s	649	42s	649
$12n55$	4-bridge	$111 \cdot [1]$	5700s	872	59s	872
$12n56$	4-bridge	$5 \cdot [1] + 4 \cdot [2,1]$	4200s	391	32s	391

Table 4.1: Comparison of two algorithms on 11 and 12-crossing knots. In the results column, $n \cdot [r_1, \ldots, r_k]$ means there are n spin-c classes for which there are r_1 generators in the lowest relative grading, r_2 generators in the next relative grading, etc, until r_k generators in the highest relative grading. Size indicates the number of generators in the largest type D structure encountered.

For knots where only 5-bridge presentations are known, the method using morphism complexes becomes infeasible. There are no such knots in 13 crossings and below, and 11 such knots with 14 crossings (we do not claim that the bridge numbers of these knots are 5). The results for these knots are shown in Table 4.2. The running time is approximately 25 minutes for each knot.

Knot	Result	Size
$14n5631$	$77 \cdot [1] + 4 \cdot [2,1]$	5946
$14n5632$	$77 \cdot [1] + 4 \cdot [2,1]$	4159
$14n5633$	$133 \cdot [1] + 2 \cdot [1,2]$	3694
$14n5634$	$77 \cdot [1] + 4 \cdot [2,1]$	6444
$14n5635$	$133 \cdot [1] + 2 \cdot [1,2]$	6429
$14n5637$	$77 \cdot [1] + 4 \cdot [1,2]$	3381
$14n5643$	$19 \cdot [1] + 6 \cdot [2,1] + 2 \cdot [2,1,1,1]$	4254
$14n5644$	$188 \cdot [1] + 1 \cdot [1,2]$	4334
$14n5645$	$19 \cdot [1] + 6 \cdot [1,2] + 2 \cdot [1,1,1,2]$	3255
$14n6285$	$18 \cdot [1] + 8 \cdot [1,2] + 1 \cdot [2,2,1,2]$	2398
$14n6302$	$135 \cdot [1]$	7037

Table 4.2: Results on the 11 knots of 14 crossings with only 5-bridge presentations known. See Table 4.1 for the meaning of the columns. The running time is approximately 25 minutes each.

Among these 11 knots, only $14n6302$ is an L-space (rank of $\widehat{HF}(\Sigma(K))$ equals the determinant of K).

Finally, we ran the algorithm on the first few torus knots $T(m,n)$. The results are shown in Table 4.3. For each $T(m,n)$, we use the standard braid presentation of n half-twists on an m-strand

braid, which is also an m-bridge presentation (so the computations for $T(m,n)$ and $T(n,m)$ uses completely different bridge presentations).

m	n	Result	Size	Time
3	1	$1 \cdot [1]$	5	0.51s
3	2	$3 \cdot [1]$	6	0.55s
3	4	$3 \cdot [1]$	13	1.3s
3	5	$1 \cdot [1]$	16	1.7s
3	7	$1 \cdot [2,1]$	23	2.8s
3	8	$2 \cdot [1] + 1 \cdot [2,1]$	26	3.5s
3	10	$2 \cdot [1] + 1 \cdot [2,1]$	33	4.9s
3	20	$2 \cdot [1] + 1 \cdot [4,3]$	66	16s
3	50	$2 \cdot [1] + 1 \cdot [9,8]$	166	95s
3	100	$2 \cdot [1] + 1 \cdot [17,16]$	333	420s
4	1	$1 \cdot [1]$	16	6.0s
4	3	$3 \cdot [1]$	28	7.1s
4	5	$4 \cdot [1] + 1 \cdot [2,1]$	66	17s
4	7	$6 \cdot [1] + 1 \cdot [2,1]$	106	31s
4	9	$6 \cdot [1] + 2 \cdot [2,1] + 1 \cdot [3,2]$	168	58s
4	11	$8 \cdot [1] + 2 \cdot [2,1] + 1 \cdot [3,2]$	232	87s
4	13	$8 \cdot [1] + 2 \cdot [2,1] + 2 \cdot [3,2] + 1 \cdot [4,3]$	318	130s
4	15	$10 \cdot [1] + 2 \cdot [2,1] + 2 \cdot [3,2] + 1 \cdot [4,3]$	406	180s
4	17	$10 \cdot [1] + 2 \cdot [2,1] + 2 \cdot [3,2] + 2 \cdot [4,3] + 1 \cdot [5,4]$	516	250s
4	19	$12 \cdot [1] + 2 \cdot [2,1] + 2 \cdot [3,2] + 2 \cdot [4,3] + 1 \cdot [5,4]$	628	340s
4	21	$12 \cdot [1] + 2 \cdot [2,1] + 2 \cdot [3,2] + 2 \cdot [4,3] + 2 \cdot [5,4] + 1 \cdot [6,5]$	762	440s
4	23	$14 \cdot [1] + 2 \cdot [2,1] + 2 \cdot [3,2] + 2 \cdot [4,3] + 2 \cdot [5,4] + 1 \cdot [6,5]$	898	570s
5	1	$1 \cdot [1]$	54	48s
5	2	$5 \cdot [1]$	51	33s
5	3	$1 \cdot [1]$	90	59s
5	4	$4 \cdot [1] + 1 \cdot [2,1]$	112	89s
5	6	$4 \cdot [1] + 1 \cdot [2,1]$	266	240s
5	7	$1 \cdot [3,2]$	341	350s
5	8	$4 \cdot [1] + 1 \cdot [3,2]$	346	440s
5	9	$1 \cdot [3,2]$	392	570s
6	1	$1 \cdot [1]$	184	610s
6	5	$4 \cdot [1] + 1 \cdot [2,1]$	526	3200s

Table 4.3: Results for torus knots $T(m,n)$. Uses the standard braid presentation of n half-twists on an m-strand braid, which is also an m-bridge presentation. See Table 4.1 for the meaning of the columns.

We note that on the torus knots, the algorithm is able to handle much more crossings than arbitrary knots. This is because the size (in number of generators) of the type D module encountered increases more slowly with torus knots than with arbitrary knots, presumably due to the regularities in the braid for the torus knots. For arbitrary knots, we expect the algorithm to be practical for knots in the 14 crossings range, but not much more.

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