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Topics on algebraic varieties in characteristic p

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TOPICS ON ALGEBRAIC VARIETIES IN
CHARACTERISTIC p

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A DISSERTATION
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Abstract

In this thesis, which consists of two parts, we study some questions relating to the geometry of algebraic varieties in characteristic p , where p is 0 or a prime number. The two parts may be read independently of one other; we give a separate introduction and background section for each one.

In the first part, we prove the Noether–Lefschetz theorem for divisor class groups of normal varieties in arbitrary characteristic. Our proof does not use any Hodge theory, monodromy, or cohomological arguments. The main ingredients are based on techniques and ideas that were available during M. Noether’s lifetime. As an addendum to the first part, we give an alternative argument for showing the injectivity statement in Noether–Lefschetz using alterations, following ideas of Ravindra and Srinivas.

The second part is joint work with Joe Waldron and is in positive characteristic $p > 0$. We study pathologies that can arise from the failure of Bertini’s theorem, in particular geometric non-reducedness, and show a structural result that has applications to Fano varieties including Mori fiber spaces.

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Part I

Noether–Lefschetz for divisor class groups in arbitrary characteristic

Chapter 1

Introduction

In this part of the thesis we will work over an algebraically closed field of arbitrary characteristic unless otherwise specified.

1.1 History and related results

Given a smooth complex projective variety X of dimension n and a smooth ample hypersurface $Y \hookrightarrow X$, the Lefschetz hyperplane theorem says that many topological and algebro-geometric invariants of Y are determined by X . One such invariant is the Picard group of isomorphism classes of line bundles. This is the focus of Noether–Lefschetz theory, which studies when the restriction map $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y)$ is an isomorphism. In general this is true if $\dim X \geq 4$, false if $\dim X = 2$ (for example $X = \mathbb{P}^2$), and more delicate when $\dim X = 3$. We give a brief overview of some results in Noether–Lefschetz theory and related questions here; we largely follow the excellent survey of Brevik and Nolle [BN14] and encourage the interested reader to refer to their article for more detail (especially over $\mathbb{C}$).

The starting point is M. Noether’s classical theorem, which he stated in 1882 and Lefschetz proved in the 1920s:

Theorem 1.1.1 (Noether 1882 [Noe82], Lefschetz 1921 [Lef21]). *If $S \subset \mathbb{P}_{\mathbb{C}}^3$ is a very general surface of degree $d \geq 4$, then the restriction map*

$$\mathrm{Pic}(\mathbb{P}_{\mathbb{C}}^3) \longrightarrow \mathrm{Pic}(S)$$

is an isomorphism.

Here very general means away from a countable union of proper closed subsets of $|\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^3}(d)|$. This countable union is referred to as the Noether–Lefschetz locus.

In fact, Lefschetz proved an even more general statement: he showed that if $S \subset \mathbb{P}_{\mathbb{C}}^n$ is a very general complete intersection of hypersurfaces, where $n \geq 3$, then $\mathrm{Pic}(\mathbb{P}_{\mathbb{C}}^n) \rightarrow \mathrm{Pic}(S)$ is an isomorphism unless S is an intersection of quadric threefolds in $\mathbb{P}_{\mathbb{C}}^4$ or S is a degree ≤ 3 surface in $\mathbb{P}_{\mathbb{C}}^3$. His proof uses a monodromy argument. He showed that if the restriction map is not surjective, the one can find a Hodge class in $H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{Z})$ representing a curve that is not a complete intersection, and that under deformation along a Lefschetz pencil it moves to $H^{0,2}(Y, \mathbb{C})$ and becomes non-algebraic.

One can also ask what happens when $\mathbb{P}_{\mathbb{C}}^3$ is replaced by a different threefold. In 1967, Moishezon adapted Lefschetz’s methods and characterized the smooth projective complex threefolds and very ample line bundles $\mathcal{L} = \mathcal{O}_X(1)$ for which the Noether–Lefschetz theorem holds:

Theorem 1.1.2 (Moishezon 1967 [Moi67]). *Let $X \subset \mathbb{P}_{\mathbb{C}}^N$ be a smooth threefold and $Y \subset X$ a very general hyperplane section. Then $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y)$ is an isomorphism if and only if*

1. *the Betti numbers $b_2(Y) = b_2(X)$ are equal, and*
2. *$h^{2,0}(Y) > h^{2,0}(X)$.*

He also notes that the second condition will hold if $\mathcal{L}$ is taken to be a sufficiently high multiple.

Modern study of Noether–Lefschetz theory began in 1983 when Carlson, Green, Griffiths, and Harris introduced infinitesimal variations of Hodge structures and proved the infinitesimal Noether theorem for smooth complex varieties (see [CGGH83] 3.a.16 for the precise statement):

Theorem 1.1.3 (Carlson–Green–Griffiths–Harris, 1983 [CGGH83]). *Let X be a smooth projective variety of dimension n and $\mathcal{L}$ a sufficiently ample line bundle on X . Then for any smooth $Y \in |\mathcal{L}|$ the infinitesimally fixed part of the middle cohomology groups of Y is equal to the fixed cohomology coming from X .*

Many other results related to the Noether–Lefschetz problem soon followed. In their paper they also made a conjecture about the codimension of the components of the Noether Lefschetz locus. In 1984 Green [Gre84] used Koszul cohomology to show that each component of the Noether–Lefschetz locus in $|\mathcal{O}_{\mathbb{P}^3}(d)|$ has codimension between $d - 3$ and $\binom{d-1}{3}$.

In 1985, Griffiths and Harris introduced another approach to the problem. Instead of primarily relying on Hodge theory, they proved the Noether–Lefschetz theorem (for $\mathbb{P}_{\mathbb{C}}^3$) by degenerating a general surface S_d to a reducible one with two components: a smooth one of degree $d - 1$ and a plane. After taking a resolution of the family, they used a monodromy argument to compute the Picard group of the central fiber and deduce Pic of a very general member.

Joshi used another approach in 1995, using ideas of Green and Nori from a set of unpublished notes of Mohan Kumar and Srinivas. The proof considers the universal family of hypersurfaces in $|\mathcal{L}|$ and uses formal completions.

Theorem 1.1.4 (Joshi 1995 [Jos95]). *Let X be a smooth threefold over $\mathbb{C}$ and $\mathcal{L}$ a very ample line bundle. Assume*

1. $H^1(X, \Omega_X^2 \otimes \mathcal{L}) = 0$, and
2. $H^1(X, M \otimes K_X \otimes \mathcal{L}) = 0$, where $M = \text{Ker}(H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \rightarrow \mathcal{L})$.

Then the restriction map $\text{Pic}(X) \rightarrow \text{Pic} Y$ is an isomorphism for the very general surface $Y \in |\mathcal{L}|$.

For sufficiently high multiples of the line bundle, Joshi showed that Noether–Lefschetz also holds for very general singular surfaces Y :

Theorem 1.1.5 (Joshi 1995 [Jos95]). *Let X be a smooth projective threefold over $\mathbb{C}$ and $\mathcal{L}$ a very ample line bundle on X . Then there exists an integer $n_0(X, \mathcal{L}) > 0$ such that for all $n \geq n_0(X, \mathcal{L})$, the Noether–Lefschetz locus of the linear system $|\mathcal{L}^n|$ has codimension ≥ 2 .*

In particular, the restriction $\text{Pic}(X) \rightarrow \text{CH}^1(Y)$ is an isomorphism for a very general singular surface $Y \in |\mathcal{L}^n|$.

Recently Brevik and Nollet proved another generalization of Moishezon’s theorem. Previous results in this direction assumed either global generation of $K_X \otimes \mathcal{L}$ or surjectivity of the multiplication map $H^0(K_X \otimes \mathcal{L}) \otimes H^0(\mathcal{L}) \rightarrow H^0(K_X \otimes \mathcal{L} \otimes \mathcal{L})$, but their result relaxes the assumption to $H^0(K_X \otimes \mathcal{L}) \neq 0$ and the decomposition of the line bundle $\mathcal{L}$ as a tensor product. Their proof is mostly algebraic but still requires a monodromy argument, and the characteristic 0 assumption is also needed for Kawamata–Viehweg vanishing and some results on the relative Picard scheme.

Theorem 1.1.6 (Brevik–Nollet 2020 [BN20]). *Let X be a smooth projective threefold over $\mathbb{C}$, and let $\mathcal{L}_1, \mathcal{L}_2$ be very ample line bundles on X such that $H^0(X, K_X \otimes \mathcal{L}_1 \otimes \mathcal{L}_2) \neq 0$. Then the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism for a very general $Y \in |\mathcal{L}_1 \otimes \mathcal{L}_2|$.*

1.1.1 Noether–Lefschetz problem over $k \neq \mathbb{C}$

In positive characteristic, the analogue of Lefschetz’s theorem is also known. Deligne, Grothendieck, and Katz used l -adic cohomology to study monodromy groups during the 1967–1969 Séminaire de Géométrie Algébrique du Bois Marie. After they developing the framework, Deligne showed:

Theorem 1.1.7 (Deligne 1973 [DK73, Exposé XIX, Théorème 1.2]). *Let k be an algebraically closed field of arbitrary characteristic. Let $Y \subset \mathbb{P}_k^{2+d}$ be a complete intersection surface of multidegree $\underline{a}$, and assume $\underline{a} \neq (2), (3), (2, 2)$. Then the restriction map $\text{Pic}(\mathbb{P}_k^{2+d}) \rightarrow \text{Pic}(Y)$ is an isomorphism for Y outside of a countable union of proper closed subvarieties of the parameter space.*

If the ground field k is uncountable (as is the case when $k = \mathbb{C}$), then the complement of the Noether–Lefschetz locus automatically contains a point. It could a priori happen that over a countable field the Noether–Lefschetz locus covers everything. However, subsequent work showed that this is not the case as long as $k \neq \overline{\mathbb{F}_p}$. Using a Hilbert irreducibility argument, Terasoma showed that the complement contains a point over $\overline{\mathbb{Q}}$. Over $\overline{\mathbb{F}_p}$ the problem becomes more complicated. Via explicit computations for a particular hypersurface, Shioda showed that for degree d surfaces in $\mathbb{P}_k^3$, the complement of the Noether–Lefschetz locus contains a k -point when $p \equiv 1 \pmod{(d-1)^3 + 1}$ [Shi83, Theorem 5.1]. However, he also notes that if the degree d of the surface is even, then the Tate conjecture implies that every smooth S_d has even Picard number, so the Noether–Lefschetz locus covers $|\mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}_p}}^3}(d)|(\overline{\mathbb{F}_p})$. In particular this happens for $m = 4$ and $p \geq 5$ [Cha13].

In higher dimensions, the results are much stronger. Grothendieck proved the following theorem, known as the Grothendieck–Lefschetz Theorem.

Theorem 1.1.8 (Grothendieck 1968 [Gro05, Exposés XI and XII], [Har70, Chapter IV]). *Let X be a smooth projective variety of dimension n over a field k of charac-*

teristic 0 and Y an ample effective divisor on X .

1. If $n \geq 3$ then $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is injective, and
2. If $n \geq 4$ then $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is an isomorphism.

Grothendieck's strategy was to consider the restriction maps

$$\text{Pic}(X) \longrightarrow \text{Pic}(U) \longrightarrow \text{Pic}(\widehat{X}) \longrightarrow \text{Pic}(Y) ,$$

where U is an open neighborhood of Y and $\widehat{X}$ is the formal completion of X along Y . He showed that each map is an isomorphism; the first map $\text{Pic}(X) \rightarrow \text{Pic}(U)$ is an isomorphism by ampleness of Y and local factoriality of X , and the hardest part is $\text{Pic}(U) \rightarrow \text{Pic}(\widehat{X})$, for which he formulated and proved Lefschetz conditions. The final restriction map $\text{Pic}(\widehat{X}) \rightarrow \text{Pic}(Y)$ is studied by considering the infinitesimal neighborhoods $Y_m = (Y, \mathcal{O}_X/\mathcal{I}_Y^{m+1})$. From the exponential map we have for each m an exact sequence

$$H^1(Y, \mathcal{I}_m/\mathcal{I}_{m+1}) \longrightarrow \text{Pic}(Y_m) \longrightarrow \text{Pic}(Y_{m-1}) \longrightarrow H^2(Y, \mathcal{I}_m/\mathcal{I}_{m+1}) .$$

By Kodaira vanishing each $H^i(Y, \mathcal{I}_m/\mathcal{I}_{m+1}) = 0$.

His approach gives results in positive characteristic as well. Each $H^i(Y, \mathcal{I}_m/\mathcal{I}_{m+1})$ is a p -torsion abelian group since it's a finite-dimensional k vector space, and each map $\text{Pic}(Y_m) \rightarrow \text{Pic}(Y_{m-1})$ has p -torsion kernel and cokernel. Now by Serre duality, Serre vanishing, and [Gro05, Expose XI, Proposition 1.1] there exists $c \in \mathbb{N}$ such that the map $\text{Pic}(\widehat{X}) \rightarrow \text{Pic}(Y_m)$ is injective for $m \geq c$ when $n \geq 3$, and an isomorphism when $n \geq 4$. Therefore the kernel of $\text{Pic}(\widehat{X}) \rightarrow \text{Pic}(Y)$ is p^c -torsion if $n \geq 3$, and the cokernel is also p^c -torsion if $n \geq 4$. So in characteristic p , the statement becomes the existence of an integer c such that

1. if $n \geq 3$ then the kernel of $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is p^c -torsion, and

2. if $n \geq 4$ then the cokernel of $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ is p^c -torsion.

Note that for complete intersections in projective space, the statement holds without torsion [Gro05, Expose XII, Corollary 3.7] [Har70, Chapter IV, Corollary 3.2].

1.1.2 Singular varieties

Another direction in which one can generalize the Noether–Lefschetz problem is to ask the question for singular varieties. For singular X , the Noether–Lefschetz theorem and Grothendieck–Lefschetz theorem for the Picard group are false, as one can see by taking the cone over a hypersurface. Kleiman [Kle66, Chapter I, §4, Corollary 2] showed that injectivity still holds, but the isomorphism statement fails in general. However, instead of considering Cartier divisors, one can consider Weil divisors. In this direction, Ravindra and Srinivas proved Grothendieck–Lefschetz for the class group of a normal variety:

Theorem 1.1.9 (Ravindra–Srinivas 2006 [RS06]). *Let X be a normal variety over an algebraically closed field k of characteristic 0, $\mathcal{L}$ an ample line bundle on X , and $|V| \subset |\mathcal{L}|$ a basepoint free linear system. Then for a general $Y \in |V|$, the restriction map $\text{Cl}(X) \rightarrow \text{Cl}(Y)$ is*

1. *injective with finitely-generated cokernel if $\dim X \geq 3$, and*
2. *an isomorphism if $\dim X \geq 4$.*

Their proof passes to a resolution of singularities $\tilde{X} \rightarrow X$ and follows Grothendieck’s ideas there. They formulate and prove versions of the effective Lefschetz conditions for big and nef line bundles on the resolution, and deduce the result on X by pushing down from $\tilde{X}$. Their proof also makes use of vanishing theorems, so it does not immediately generalize to positive characteristic. Using different methods suggested by the referee, Ravindra and Srinivas also show a result that holds in any characteristic (see §3.0.1 for definitions).

Theorem 1.1.10 (Ravindra–Srinivas 2006 [RS06, Theorem 5]). *Let X be an irreducible projective variety of dimension ≥ 3 over an algebraically closed field k , and assume X is regular in codimension 1. Let $f: X \rightarrow \mathbb{P}_k^N$ be an embedding. Let $\mathcal{Y} \subset X \times |\mathcal{O}_{\mathbb{P}_k^N}(1)|$ be the universal family of hyperplane sections of X , K the function field of the dual projective space $|\mathcal{O}_{\mathbb{P}_k^N}(1)|$, $\overline{K}$ the algebraic closure of K , and $Y_K \subset X_K$ the generic hyperplane section. Then*

1. $\mathrm{Cl}^0(X)_K \rightarrow \mathrm{Cl}^0(Y_K)$ is an isomorphism, and
2. the composition $\mathrm{Cl}^{\mathrm{ns}}(X) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(X \otimes_k \overline{K}) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(Y_K \otimes_K \overline{K})$ is injective.

Using their Grothendieck–Lefschetz result as a starting point, Ravindra and Srinivas also proved a Noether–Lefschetz result for the class group of a singular threefold. Their proof uses Kumar and Srinivas’s ideas to pass to the universal family, study a formal completion, and prove an infinitesimal Noether–Lefschetz result on the family.

Theorem 1.1.11 (Ravindra–Srinivas 2009 [RS09]). *Let X be a normal threefold over an algebraically closed field of characteristic 0, $\mathcal{L}$ an ample line bundle on X , and $|V| \subset |\mathcal{L}|$ a basepoint free linear system. If $K_X \otimes \mathcal{L}$ is globally generated, then the restriction map $\mathrm{Cl}^{\mathrm{ns}}(X) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(Y)$ is an isomorphism for a very general $Y \in |V|$.*

1.1.3 Specialization of Néron–Severi groups

Another closely related problem is the specialization of Néron–Severi groups; this relates to the Noether–Lefschetz problem by considering the universal family of divisors. If $X \rightarrow S$ is a smooth proper morphism of varieties over an algebraically closed field k , then the specialization map $\mathrm{NS}(X_{\overline{\eta}}) \rightarrow \mathrm{NS}(X_s)$ from the geometric generic fiber to a closed fiber is always injective in characteristic 0, and injective up to p -torsion in positive characteristic [MP12, Proposition 3.6]. A Hilbert scheme argument shows that the locus where the Picard rank ρ (i.e. the rank of the Néron–Severi group)

jumps is a countable union of proper subvarieties of S . Over a countable field k , it can happen that this jumping locus covers the entire base S .

Theorem 1.1.12 ([And96] [MP12] [Amb18] [Chr18]). *Let $X \rightarrow S$ be a smooth proper morphism of varieties over an algebraically closed field $k \neq \overline{\mathbb{F}_p}$ of characteristic ≥ 0 . Then there exists $s \in S(k)$ such that $\rho(X_{\overline{\eta}}) = \rho(X_s)$.*

André first proved the result in characteristic 0 in 1996 using Hodge theory. Maulik and Poonen’s result is also in characteristic 0 but they use different methods, embedding a suitable finitely-generated subfield of k into a p -adic field and showing that for a family of varieties “with good reduction,” the jumping locus is p -adically nowhere dense. In positive characteristic, Ambrosi and Christensen independently proved the result. Ambrosi’s proof is inspired by André’s approach and uses crystalline cohomology and other p -adic cohomology theories. Christensen proof uses different methods, and uses Morrow’s crystalline Lefschetz (1, 1) Theorem as a key input.

1.1.4 Local Lefschetz theorems

Another variant of the Grothendieck–Lefschetz theorem is to consider a local version. Instead of considering a projective variety, one can consider the germ of a singularity ($x \in X$) and study the local Picard group $\text{Pic}(X \setminus \{x\})$. Grothendieck proved that if $\text{depth}(O_{X,x}) \geq 4$ then the restriction map on local Picard groups from X to a Cartier divisor containing x is injective [Gro05, Exposé XI 3.16]. Kollár conjectured the following strengthening and proved it in the lc case.

Theorem 1.1.13 (Kollár, Bhatt–de Jong [Kol13] [BdJ14] [Kol16]). *Let $(x \in X)$ be an excellent local scheme of pure dimension ≥ 4 over a field such that $\text{depth}(O_{X,x}) \geq 3$, and let $x \in D \subset X$ be a Cartier divisor. Then the restriction map $\text{Pic}(X \setminus \{x\}) \rightarrow \text{Pic}(D \setminus \{x\})$ is an isomorphism.*

Bhatt–de Jong proved the result in the normal case (modulo p -torsion in positive characteristic), and Kollár then proved it in general. Note that the local statement does not immediately give the global Grothendieck–Lefschetz theorem since the vertex of a cone over a smooth projective variety usually will only have depth 2, but the assumption of the theorem requires $\text{depth} \geq 3$.

Chapter 2

Main result of Part I:

Noether–Lefschetz theorem

For a normal variety X , let $\mathrm{Cl}(X)$ be its Weil divisor class group. The main result of this section of the thesis is the Noether–Lefschetz theorem for class groups of normal varieties in arbitrary characteristic:

Theorem 2.0.1 (Proposition 3.2.1, Proposition 3.4.1). *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p \geq 0$, and assume it has infinite transcendence degree over the prime subfield. Let X be a normal variety of dimension $n \geq 3$ over $\mathbb{K}$, $\mathcal{L}$ a very ample line bundle on X , and d an integer. For a very general $Y \in |\mathcal{L}^d|$ the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

is

1. *is injective for any $d \geq 2$,*
2. *is surjective up to torsion for any integer $d = 4$ or ≥ 6 if $n = 3$,*
3. *is surjective up to torsion for any $d \geq 2$ if $n \geq 4$.*

Some of the assumptions can be relaxed for the injectivity statement; see Proposition 3.2.1 and Corollary 4.2.5 in Chapter 4.

2.0.1 Outline

We prove Theorem 2.0.1 by decomposing the class group Cl into the subgroup Cl^0 of algebraically trivial divisors and the quotient Cl^{ns} (see §3.0.1 for definitions). For injectivity modulo $\text{Cl}^0(X)$ (Section 3.2) we first restrict from X to a complete intersection curve C and use Néron's specialization theorem 3.0.15. Then we factor this through a divisor $Y \in |\mathcal{L}|$.

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & \text{Lem 3.2.3/}p\text{-power torsion} & & & & \\
\text{Cl}^0(X) & \hookrightarrow & \text{Cl}^0(Y) & \longrightarrow & \text{Cl}^0(S) & \longrightarrow & \text{Pic}^0(C) \\
& \searrow & \text{Prop 3.4.2} & \nearrow & & & \\
& \text{Cl}(X) & \longrightarrow & \text{Cl}(Y) & & & \\
\downarrow & & \text{Lem 3.2.7} & & & & \\
\text{Cl}^{\text{ns}}(X) & \longrightarrow & \text{Cl}(Y)/\text{Cl}^0(X)|_Y & \longrightarrow & \text{Cl}(S)/\text{Cl}^0(X)|_S & \longrightarrow & \text{Pic}(C)/\text{Cl}^0(X)|_C \\
\downarrow & & \downarrow & & \parallel & \text{Prop 3.4.2} & \\
0 & & \text{Cl}^{\text{ns}}(Y) & \longrightarrow & \text{Cl}^{\text{ns}}(S) & &
\end{array}$$

To relate $\text{Cl}(Y)/\text{Cl}^0(X)$ to $\text{Cl}^{\text{ns}}(Y)$, we need a result for Cl^0 . We show this by factoring through a complete intersection surface S in $|\mathcal{L}| \cap \dots \cap |\mathcal{L}|$. To lift sections from S , we fiber X by curves, consider the relative Jacobian, and use a result of [GS13] for extending sections of an abelian scheme to conclude $\text{Cl}^0(X) \cong \text{Cl}^0(S)$ in Section 3.4.

For surjectivity on the rest of the class group, we restrict from X to a different kind of complete intersection surface T : one in $|\mathcal{L}^2| \cap |\mathcal{L}| \cap \dots \cap |\mathcal{L}|$. We show that

$\text{Cl}(X) \rightarrow \text{Cl}(T)$ is surjective by specializing to a reducible “ $|\mathcal{L}| \cap \cdots \cap |\mathcal{L}| + |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$ ” surface where we can lift sections using [GS13]. Then if $\dim X \geq 4$ we factor the restriction through $H \in |\mathcal{L}|$.

$$\begin{array}{ccccc} \text{Cl}(X) & \longrightarrow & \text{Cl}(H) & \longrightarrow & \text{Cl}(T) \\ & & \searrow & \nearrow & \\ & & \text{Prop 3.4.7} & & \end{array}$$

If $\dim X = 3$ then $T \subset X$ is a divisor, so we need a different argument for odd multiples of $\mathcal{L}$. Instead of specializing to a member of “ $|\mathcal{L}| + |\mathcal{L}|$ ” we specialize to a member $S_{2A} + S_B$ of “ $|\mathcal{L}^2| + |\mathcal{L}_B|$ ” where $\mathcal{L}_B$ is another line bundle. Then we can lift sections from the $S_{2A} \in |\mathcal{L}^2|$ member.

These are the components of the argument. Here $\mathcal{L} = \mathcal{L}_0^m$ in Theorem 2.0.1.

1. Corollary 3.2.8: If $\dim X \geq 2$ and $k \neq \overline{\mathbb{F}_p}$, then there is a dense set of complete intersection curves C of $|\mathcal{L}|$ such that $\text{Cl}^{\text{ns}}(X) \rightarrow \text{Cl}(S)/\text{Cl}^0(X)$ is injective for any complete intersection $S \supset C$.
2. Corollary 3.4.2: If $\dim X \geq 3$ and $S \in |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$ is a general c.i. surface, then $\text{Cl}^0(X) \rightarrow \text{Cl}^0(S)$ is an isomorphism.
 - (a) Corollary 3.4.3: If $\dim X \geq 4$ then $\text{Cl}^0(X) \rightarrow \text{Cl}^0(Y)$ is an isomorphism for general $Y \in |\mathcal{L}|$.
 - (b) Corollary 3.4.4 (Corollary 3.2.8 + Corollary 3.4.2): If $\dim X \geq 3$ and $k \neq \overline{\mathbb{F}_p}$, then $\text{Cl}^{\text{ns}}(X) \rightarrow \text{Cl}^{\text{ns}}(Y)$ is injective for infinitely many $Y \in |\mathcal{L}|$.
3. Proposition 3.4.7: If $\mathbb{K}$ has infinite transcendence degree, $\dim X \geq 3$, and $T \in |\mathcal{L}^2| \cap |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$ is a very general c.i. surface, then $\text{Cl}(X) \rightarrow \text{Cl}(T)$ is surjective up to torsion.
 - (a) Corollary 3.4.10: If $\dim X \geq 4$ then $\text{Cl}(X) \rightarrow \text{Cl}(H)$ is surjective up to torsion for $H \in |\mathcal{L}|$ very general.

4. Corollary 3.4.11: If $\mathbb{K}$ has infinite transcendence degree, $\dim X = 3$, $T_{2A+B} \in |\mathcal{L}^2 \otimes \mathcal{L}_{B,0}^{m_B}| \cap |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$ is a very general c.i. surface, where $\mathcal{L}_{B,0}$ is ample and basepoint-free and $m_B \geq 2$, and either

(a) X is smooth,

(b) $\text{char } \mathbb{K} = 0$, or

(c) $\mathcal{L}^2 \otimes \mathcal{L}_{B,0}^{m_B} \cong \mathcal{M}^m$ for some very ample $\mathcal{M}$,

then $\text{Cl}(X) \rightarrow \text{Cl}(T_{2A+B})$ is surjective up to torsion.

2.0.2 Comment on the assumptions on the ground field

We assume that $k \neq \overline{\mathbb{F}_p}$ to apply Néron's Theorem 3.0.15. Tate's Conjecture implies that this assumption is necessary already for surfaces of even degree in $\mathbb{P}_{\mathbb{F}_p}^3$. The infinite transcendence degree assumption is used in various places to ensure that a countable union of proper closed subvarieties is not everything (in particular in Chow variety arguments to ensure that there is a component of the Chow variety dominating the base, and in Voisin's argument Lemma 3.4.6). We will elaborate more on this in §3.0.3.

Chapter 3

Preliminaries

3.0.1 Divisors

Let k be a field, $\bar{k}$ an algebraic closure of k , and X a proper, geometrically normal, geometrically integral variety over k . When we say divisor without specifying further, we are referring to Weil divisors.

Definition 3.0.1 ([Kol18, Definition 15]). Let $\mathrm{Cl}^{\mathrm{ns}}(X)$ denote the group of divisors modulo linear equivalence on a proper, geometrically normal, geometrically integral variety X . Denote by $\mathrm{Cl}^0(X)$ the subgroup of divisors that are algebraically equivalent to 0. The *Néron–Severi class group* of X is the quotient $\mathrm{Cl}^{\mathrm{ns}}(X) = \mathrm{Cl}(X)/\mathrm{Cl}^0(X)$.

The Picard group $\mathrm{Pic}(X)$ is the group of line bundles (or equivalently Cartier divisors modulo linear equivalence) on X , $\mathrm{Pic}^0(X)$ is the subgroup of Cartier divisors algebraically equivalent to 0, and the quotient $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is the Néron–Severi group of X . There are natural inclusions

$$\mathrm{Pic}^0(X) \subset \mathrm{Cl}^0(X),$$

$$\mathrm{NS}(X) \subset \mathrm{Cl}^{\mathrm{ns}}(X)$$

that are both isomorphisms if and only if every Weil divisor is Cartier (for example

if X is locally factorial).

Definition 3.0.2. If X is proper, then the group $\mathrm{Pic}(X \otimes_k \bar{k})$ has a natural underlying k -scheme structure. We denote this scheme by $\mathbf{Pic}_{X/k}$.

In characteristic 0 the Picard scheme is reduced. The connected component of the identity $\mathbf{Pic}_{X/k}^0$ is a commutative algebraic group, and if X is projective and geometrically normal over k then $\mathbf{Pic}_{X/k}^0$ is projective. So $\mathbf{Pic}_{X/k}^0$ is an abelian variety in characteristic 0, and in general $\mathrm{red} \mathbf{Pic}_{X/k}^0$ is an abelian variety [Kle05, Theorem 5.4, Remark 5.25].

If X is proper, then there is a natural injection $\mathrm{Pic}^0(X) \hookrightarrow \mathbf{Pic}_{X/k}(k)$. If $X(k) \neq \emptyset$ and $H^0(X, \mathcal{O}_X) \cong k$ then this injection is an isomorphism [Gro62, V.2.1], and in general the quotient is a torsion group.

The class group can also be given a scheme structure via the Albanese variety. Note that here, following [Kol20], we use Albanese variety to refer to the classical, pre-Grothendieck notion of the Albanese variety [Ser97]. Since we work over an algebraically closed field, we will only consider the case when X has a smooth k -point.

Definition 3.0.3 ([Kol20]). Let X be a proper, geometrically normal, geometrically integral variety over a field k , and assume that X has a smooth k -point. The classical Albanese variety is the target of the universal rational map from X to an abelian variety

$$\mathrm{alb}_X: X \dashrightarrow \mathbf{Alb}(X).$$

$\mathbf{Alb}(X)$ is a birational invariant, and alb_X is a morphism on the smooth locus of X .

If X does not have a smooth k -point, then the Albanese is formulated using abelian torsors, but we will not consider such settings.

Theorem 3.0.4 ([Kol20], [Kol18, Theorem 17]). *Let X be a proper, geometrically normal, geometrically integral variety over a field k , and assume X has a smooth*

k -point. Then

$$\mathbf{Cl}^0(X) = \mathbf{Alb}(X)^*.$$

In the above $\mathbf{Alb}(X)^*$ is the dual of the Albanese variety. In the literature, $\mathbf{Cl}^0$ has been referred to as the “Picard variety in the sense of Weil.”

The following is a basic structural result, the latter part of which is known as the Theorem of the Base.

Theorem 3.0.5 ([Nér52], [LN59], [Kol20]). *Let X be a normal and proper variety over an algebraically closed field k .*

1. *There is a natural isomorphism $\mathbf{Cl}^0(X) \cong \mathbf{Pic}^0(\mathbf{Alb}(X))(k)$.*
2. *$\mathbf{Cl}^{\text{ns}}(X) = \mathbf{Cl}(X) / \mathbf{Cl}^0(X)$ is a finitely-generated abelian group.*

The class groups of varieties can be related via birational morphisms in the following way. So if X admits a resolution of singularities, then we can understand the Weil divisors on X by studying its resolution.

Lemma 3.0.6 ([Kol18, Lemma 18]). *Let $\pi: X' \rightarrow X$ be a proper birational morphism between normal varieties over a field k . Then pushforward by π induces isomorphisms*

$$\begin{aligned} \mathbf{Cl}^0(X') &\cong \mathbf{Cl}^0(X), \\ \mathbf{Cl}^{\text{ns}}(X') / \bigoplus_i \mathbb{Z}[E_i] &\cong \mathbf{Cl}^{\text{ns}}(X) \end{aligned}$$

where the E_i are the exceptional divisors of π .

3.0.2 Chow K/k -trace and image

The material on this section is mostly based on [Con06], and we refer the reader to Conrad’s article for more detail.

Definition 3.0.7. A field extension K/k is said to be *regular* if it is separable and k is algebraically closed in K .

Throughout this section, we will consider regular extensions of fields. For example, if k is perfect, then any K/k is regular. In general, if K/k is finitely generated, then regularity is the same as K being the function field of some smooth, geometrically connected scheme B over k .

Definition 3.0.8. Let A be an abelian variety over K . A K/k -trace is a final pair $(\mathrm{Tr}_{K/k}(A), \tau)$ of an abelian k -variety $\mathrm{Tr}_{K/k}(A)$ and a morphism $\tau: \mathrm{Tr}_{K/k}(A) \otimes_k K \rightarrow A$ of abelian K -varieties.

The Chow trace can also be formulated for an abelian scheme $\mathcal{A} \rightarrow B$, as a final pair $(\mathrm{Tr}_{B/k}(\mathcal{A}), \tau)$ of an abelian k -variety $\mathrm{Tr}_{B/k}(\mathcal{A})$ and a morphism $\tau: \mathrm{Tr}_{B/k}(\mathcal{A}) \times B \rightarrow \mathcal{A}$ of abelian B -varieties. A basic result is that the Chow trace exists.

Theorem 3.0.9. *Let $\mathcal{A} \rightarrow B$ be an abelian scheme over a smooth, integral, quasi-projective k -scheme. Then a Chow B/k -trace of $\mathcal{A}$ exists.*

By the universal property, it is unique up to unique isomorphism.

One can think of the Chow B/k -trace as the constant part of the family $\pi: \mathcal{A} \rightarrow B$ of abelian varieties. By Poincaré reducibility, each fiber of π has a decomposition (up to isogeny) as a product of simple abelian varieties, and the Chow trace can be heuristically thought of as the part that is the same in all the fibers. The image of $\mathrm{Tr}_{B/k}(\mathcal{A})$ in $\mathcal{A}$ is a sub-abelian scheme over B .

Theorem 3.0.10 ([Con06, Theorem 6.4]). *Let K/k be a regular extension of fields and A an abelian variety over K . Let $(\mathrm{Tr}_{K/k}(A), \tau)$ be the Chow trace.*

1. *If $A = X \otimes_k K$ for an abelian variety X over k , then τ is an isomorphism.*
2. *The canonical map $\tau: \mathrm{Tr}_{K/k}(A) \otimes_k K \rightarrow A$ has finite kernel.*

3. *There exists a unique abelian subvariety $A' \subset A$, called the K/k -maximal abelian subvariety of A , such that $\mathrm{Tr}_{K/k}(A/A') = 0$ and $\mathrm{Tr}_{K/k}(A') \otimes_k K \rightarrow A'$ is an isogeny. In particular $\mathrm{Tr}_{K/k}(A') \rightarrow \mathrm{Tr}_{K/k}(A)$ is an isomorphism.*

The Chow B/k -trace of an abelian scheme only depends on the function field of B and the generic fiber of $\mathcal{A} \rightarrow B$:

Lemma 3.0.11 ([GS13, Remark 3.3]). *Let $\mathcal{A} \rightarrow B$ be an abelian scheme over a smooth, integral, quasi-projective k -scheme, and let K be the function field of B . Then the Chow B/k -trace of $\mathcal{A}$ depends only on the field extension K/k and the generic fiber A , which is an abelian variety over K .*

Lang–Néron used this theory to prove a generalization of the Mordell–Weil theorem for rational points on abelian varieties.

Theorem 3.0.12 ([LN59, Theorem 1]). *Let K/k be a finitely-generated regular extension of fields and A an abelian variety over K . Then $A(K)/\mathrm{Tr}_{K/k}(A)(k)$ is a finitely generated group.*

Dual to the theory of the Chow K/k -trace is the Chow K/k -image.

Definition 3.0.13. Let A be an abelian variety over K . A K/k -image is an initial pair of a k -variety $(\mathrm{Im}_{K/k}(A), \lambda)$ and a morphism $\lambda: A \rightarrow \mathrm{Im}_{K/k}(A) \otimes_k K$ of abelian K -varieties.

Just as the Chow K/k -trace can be viewed roughly as the largest abelian subvariety of A that can be defined over k , the Chow K/k -image may be thought of as the largest quotient of A that can be defined over k . The Chow K/k -image is obtained as the dual of the Chow trace $(\mathrm{Tr}_{K/k}(A^*), \tau_{A^*, K/k})$ of the dual abelian variety A^* .

3.0.3 Thin sets, Néron's specialization theorem, and very general points

Definition 3.0.14 ([Ser97, §9.1], [FJ08, Chapter 12]). Let k_0 be a field that is finitely generated over its prime subfield. A subset $T \subset \mathbb{P}_{k_0}^n(k_0)$ is *thin* if $T = f(X(k_0))$ for some k_0 -variety X and separable generically finite morphism $f: X \rightarrow \mathbb{P}_{k_0}^n$ admitting no rational section.

Every thin subset of $\mathbb{P}_{k_0}^n(k_0)$ is contained in a finite union of the following two types of thin sets:

1. Thin subsets contained in proper subvarieties, and
2. $f(X(k_0))$ for some separable dominant morphism $f: X \rightarrow \mathbb{P}_{k_0}^n$ with $\dim X = n$ and $\deg(f) \geq 2$.

Following [Kol20], for an arbitrary field k we say that $T \subset \mathbb{P}_k^n(k)$ is *field-locally thin* if $T \cap \mathbb{P}^n(k_0)$ is thin for every finitely-generated subfield $k_0 \subset k$.

If k_0 is a finitely-generated field not contained in $\overline{\mathbb{F}_p}$, then $\mathbb{P}_{k_0}^n(k_0)$ is not thin for $n \geq 1$ [FJ08, Theorem 13.4.2] [CTS87, Corollary 7.14].

Theorem 3.0.15 ([Nér52, Theorem 6]). *Let k be a field, $U \subset \mathbb{P}_k^m$ an open subset, and $\mathcal{A} \rightarrow U$ an abelian scheme. Assume the Mordell–Weil group $\mathcal{A}_\eta(K)$ is finitely generated, where K is the function field of $\mathbb{P}_k^m$ and $\mathcal{A}_\eta$ is the generic fiber of $\mathcal{A}$.*

Then there exists a subset $N \subset U(k)$ containing the complement of a field-locally thin set and such that the specialization map

$$\mathcal{A}_\eta(K) \longrightarrow \mathcal{A}_s(k)$$

is injective for $s \in N$.

When we say *very general*, we will mean a contained in a nonempty subset whose complement is the countable union of proper closed subsets. In the latter section

of this article, in order to make our “very general” choices more concrete, we will assume our algebraically closed field $\mathbb{K}$ has infinite transcendence degree instead of assuming that it is uncountable. (In Weil’s terminology, this is a “universal domain” [Wei46, Chapter I, §1].) Although in practice the only countable fields this adds are the algebraic closures of $\mathbb{Q}(t_i \mid i \in \mathbb{N})$ and $\mathbb{F}_p(t_i \mid i \in \mathbb{N})$, it clarifies some instances when we need to make simultaneously very general choices.

Definition 3.0.16 ([Wei46, Chapter I]). Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree over the prime subfield $\mathbb{Q}$ or $\mathbb{F}_p$. If X is a variety over $\mathbb{K}$, then it can be defined over a field that is finitely generated over the prime subfield. When we say a *field of definition* k^X for X , we mean an algebraically closed (in $\mathbb{K}$) field over which X is defined and such that k^X has finite transcendence degree over the prime subfield. Note that such a field k^X is the algebraic closure (in $\mathbb{K}$) of a finitely-generated field and in particular is countable.

The Chow variety of X is defined over k^X , and a subvariety Y of X is given by a $\mathbb{K}$ -point of $\text{Chow}(X)$. If $[Y]$ denotes the image of this morphism $\text{Spec } \mathbb{K} \rightarrow \text{Chow}(X)$, and k^Y is the algebraic closure (in $\mathbb{K}$) of the residue field at this point, then Y is defined over k^Y . When we say Y is *very general* we will mean that k^Y is not algebraic over k^X . Since k^X is countable, this is consistent with the definition of very general as outside of a countable union of proper closed subvarieties.

A very general r -tuple $(Y_1, \dots, Y_r)$ will mean that the transcendence degree over k^X of the compositum of the k^{Y_i} in $\mathbb{K}$ is equal to the sum of the transcendence degrees of the k^{Y_i} over k^X . (In Weil’s language, the fields $k^{Y_1}, \dots, k^{Y_r}$ are “independent” or “free with respect to each other” [Wei46, Chapter I, §2].)

If X is a variety over $\mathbb{K}$ and $k = k^X$ is a field of definition for X , then the “bad sets” in the Bertini theorems (§3.1) will be defined over k , so a very general choice will also avoid them.

3.1 Bertini-type results

In this section we recall some Bertini theorems for very ample divisors. We also show that if a line bundle is sufficiently positive, then its reducible locus is small (Lemma 3.1.6), and this will be the assumption we put on the line bundle $\mathcal{L}$ throughout this part of the thesis.

Theorem 3.1.1 ([FOV99, Theorem 3.4.10]). *Let X be a variety over an infinite field k , D a Cartier divisor on X , and $|V| \subset |D|$ a finite-dimensional linear system. If*

1. *$|V|$ is not composed with a pencil (meaning the image of X under the morphism $|V|$ defines has dimension ≥ 2), and*
2. $\text{codim}_X \text{Bs}(|V|) \geq 2$,

then a general member of $|V|$ is irreducible.

Lemma 3.1.2 ([FOV99, Theorem 3.4.14]). *Let $X \subset \mathbb{P}_k^n$ be a projective scheme over an infinite field k . If X is regular (resp. normal, reduced, regular in codimension c), then for a general hyperplane $H \subset \mathbb{P}_k^N$, the intersection $H \cap X$ has the same property.*

Lemma 3.1.3 ([GK19, Corollary 3.4]). *Let k be an infinite field, X a normal equidimensional quasi-projective k -scheme of dimension ≥ 1 , and $X \subset \mathbb{P}_k^n$ a locally closed embedding. Then a general hypersurface $H \subset \mathbb{P}_k^n$ satisfies*

1. $X \cap H$ is normal, and
2. $X^{\text{reg}} \cap H$ is regular.

Lemma 3.1.4 (Grothendieck's connectedness lemma [Gro05, Exposé XIII, Theorem 2.1]). *Let (x, X) be a local, excellent, S_2 scheme of pure dimension ≥ 3 and $x \in D := (t = 0) \subset X$ a Cartier divisor. Then $D \setminus \{x\}$ is connected.*

The following lemmas will ensure that for the linear systems we work with, the locus of reducible members will have codimension ≥ 2 . This is not always the case; for example the closure of the reducible plane conics is a divisor (the discriminant) in $|\mathcal{O}_{\mathbb{P}^2}(2)|$.

Lemma 3.1.5. *Let X be a variety, $\mathcal{L}$ a line bundle and $\Gamma \subset H^0(X, \mathcal{L})$ a nonempty subspace, and $\mathcal{B}$ a line bundle defining a birational map. Then the linear system $|\Gamma + \mathcal{B}| = \{r \otimes s \mid r \in \Gamma, s \in H^0(X, \mathcal{B})\} \subset H^0(X, \mathcal{L} \otimes \mathcal{B})$ defines a birational map.*

Proof. Let $s_0, \dots, s_N$ be a basis of $H^0(X, \mathcal{B})$ and $r_0, \dots, r_m$ a basis of Γ . Then $\{r_i \otimes s_j \mid 0 \leq i \leq m, 0 \leq j \leq N\}$ is a basis of $|\Gamma| + |\mathcal{B}|$ and so the rational map it defines is given (up to isomorphism) by

$$\phi: x \mapsto [r_0 s_0(x) : r_0 s_1(x) : \dots : r_0 s_N(x) : r_1 s_0(x) : \dots : r_m s_N(x)] \in \mathbb{P}^{(m+1)(N+1)}.$$

Denote the coordinates of $\mathbb{P}^{(m+1)(N+1)}$ by $[y_{00} : y_{01} : \dots : y_{mN}]$. For each i , define $\pi_i: \mathbb{P}^{(m+1)(N+1)} \dashrightarrow \mathbb{P}^N$ to be the projection onto the coordinates $[y_{i0} : y_{i1} : \dots : y_{iN}]$. Then $\pi_i \circ \phi$ agrees with the rational map defined by $|\mathcal{B}|$ (away from $(s_i = 0)$) so its image has dimension n . So $\dim \phi(X) = n$. $\square$

Lemma 3.1.6. *Let X be a variety of dimension $n \geq 2$ over an algebraically closed field k , and assume that either*

1. X is normal, or
2. $\dim \text{Sing}(X) = 0$.

Let $\mathcal{L}$ be a very ample line bundle on X . Then the set of reducible divisors in $|\mathcal{L}^m|$ is contained in a closed subscheme of $|\mathcal{L}^m|$ of codimension ≥ 2 for

1. $m \geq 2$ if the pair is not $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, and
2. $m \geq 3$ if $(X, \mathcal{L}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

Proof. The case where $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ for $n \geq 2$ follows from an explicit computation, so we assume that $(X, \mathcal{L}) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. For each smooth (closed) point $x \in X$ and line bundle $\mathcal{M}$ on X , define $|B_x^{\mathcal{M}}| = \{D \in |\mathcal{M}| \mid x \in \text{Sing } D\}$. First we claim that a general member of the linear system $|B_x^{\mathcal{L}^m}|$ is irreducible. Then $|B_x^{\mathcal{L}}| = \{H \cap X \mid H \in |\mathcal{O}_{\mathbb{P}^{h^0(X, \mathcal{L})-1}}(1)| \text{ and } T_{\phi|_{\mathcal{L}}(X), x} \subset H\} \neq \emptyset$ since $(X, \mathcal{L}) \neq (\mathbb{P}^n, \mathcal{O}(1))$, so by Lemma 3.1.5 the linear system $|B_x^{\mathcal{L}} + \mathcal{L}^{m-1}|$ defines a birational morphism to its image. Since $|B_x^{\mathcal{L}}| + |\mathcal{L}^{m-1}| \subset |B_x^{\mathcal{L}^m}|$, the span $|B_x^{\mathcal{L}} + \mathcal{L}^{m-1}|$ of these reducible divisors is also contained in $|B_x^{\mathcal{L}^m}|$ and so the morphism $|B_x^{\mathcal{L}^m}|$ defines is birational to its image. In particular its image has dimension $n \geq 2$. Since the base locus of $|B_x^{\mathcal{L}^m}|$ is x , which has codimension $n \geq 2$ in X , we apply Theorem 3.1.1 and see that a general member of $|B_x^{\mathcal{L}^m}|$ is irreducible.

Now replacing X by its image under the closed embedding $\phi|_{\mathcal{L}^m}$ defined by $\mathcal{L}^m$, we may assume that $\mathcal{L}^m = \mathcal{O}_{\mathbb{P}^M}(1)|_X$, where $\mathbb{P}^M = |\mathcal{L}^m|$, and that X is not contained in a hyperplane. Let $\mathcal{C} = \overline{\{(x, H) \mid x \in X \setminus \text{Sing}(X), T_{X,x} \subset H\}} \subset X \times (\mathbb{P}^M)^*$ be the conormal variety of X , with projections $p_1: X \times (\mathbb{P}^M)^* \rightarrow X$ and $p_2: X \times (\mathbb{P}^M)^* \rightarrow (\mathbb{P}^M)^*$. The dual variety $p_2(\mathcal{C})$ is irreducible since $\mathcal{C}$ is irreducible of dimension $M - 1$.

Now consider the locus $S_{\text{reducible}}$ of hyperplanes such that $H \cap X$ is reducible. This is a constructible subset of the closed subvariety of $|\mathcal{L}^m|$ parametrizing hyperplanes not regular in codimension 0 [FOV99, Theorem 3.3.14], and $S_{\text{reducible}} \subsetneq |\mathcal{L}^m|$ by Theorem 3.1.1. Note that by the Enriques–Severi–Zariski lemma $H \cap X$ is singular if it's reducible. So if X is smooth, then $S_{\text{reducible}} \subset p_2(\mathcal{C})$ and by the above discussion $\text{codim}_{|\mathcal{L}^m|}(\overline{S_{\text{reducible}}}) \geq 2$ under the assumptions in the lemma.

Suppose first that $\text{Sing}(X)$ has dimension 0. For a closed point $x \in \text{Sing}(X)$ consider the linear system $|\mathcal{L}_x|$ of divisors passing through x . By Lemma 3.1.5 $|\mathcal{L}_x + \mathcal{L}^{m-1}|$ is not composed of a pencil, and hence neither is $|\mathcal{L}_x^m|$. By Theorem 3.1.1 a general member of $|\mathcal{L}_x^m|$ is reducible, and so the reducible members R_x of $|\mathcal{L}_x^m|$ form a closed subset of $p_1^{-1}(x)$ of dimension $\leq M - 2$.

Let $H \in S_{\text{reducible}}$. If the intersection of two components of $H \cap X$ contains a smooth point of X , then $H \in p_2(\mathcal{C})$. If not, then let $x \in \text{Sing}(X)$ be a closed point in the intersection of two components of $H \cap X$. Then $H \in p_2(R_x)$. Since $\text{Sing}(X)$ is a finite set, $\bigcup_{x \in \text{Sing}(X)} B_x$ has dimension at most $M - 2$, so its image in $(\mathbb{P}^M)^*$ has dimension at most $M - 2$. Thus, in the case where $\dim \text{Sing}(X) = 0$ we have shown that $\text{codim}_{|\mathcal{L}^m|}(\overline{S_{\text{reducible}}}) \geq 2$.

Now assume X is normal. If $n = 2$ then normality implies that $\dim \text{Sing}(X) \leq 0$, so we may assume that $n \geq 3$. Then R is the union of the two sets $\{H \mid H \cap X \text{ is reducible and } H \supset T_{X,x} \text{ for some closed } x \in X_{\text{reg}}\} \cup \{H \mid H \cap X \text{ is reducible and } \text{Sing}(H \cap X) \subset \text{Sing}(X)\}$. The closure of the first set has codimension at least 2 in $|\mathcal{L}^m|$ by the argument in the smooth case, so it remains to consider the second set.

Let H be such that $H \cap X$ is reducible and $\text{Sing}(H \cap X) \subset \text{Sing}(X)$, and denote the irreducible components by $Z_1, \dots, Z_e$, then each intersection $Z_i \cap Z_j \subset \text{Sing}(H \cap X)$. For each codimension 3 point $x \in X$ contained in $Z_i \cap Z_j$, $\mathcal{O}_{X,x} \setminus \{x\}$ is connected by Lemma 3.1.4, so $Z_i \cap Z_j$ has codimension 2 in X . Since X is regular in codimension 1, and $Z_i \cap Z_j \subset \text{Sing}(X)$ both have codimension 2 in X , we have that $Z_i \cap Z_j$ is a component of $\text{Sing}(X)$. So $H \cap X$ must contain an $(n - 2)$ -dimensional irreducible component of $\text{Sing}(X)$. By assumption $n - 2 \geq 1$, so requiring that H contains a irreducible component of $\text{Sing}(X)$ imposes a codimension ≥ 2 condition on the elements of $|\mathcal{L}^m|$. So in this case we again have that $\text{codim}_{|\mathcal{L}^m|}(\overline{S_{\text{reducible}}}) \geq 2$. $\square$

Corollary 3.1.7. *Let $X \neq \mathbb{P}_k^2$ be a variety of dimension $n \geq 2$ over an algebraically closed field k , and assume that either*

1. X is normal, or
2. $\dim \text{Sing}(X) = 0$.

Fix $r \leq n - 1$, let $\mathcal{L}$ be a very ample line bundle on X , and let $m \geq 2$ be an integer. Then the locus of reducible complete intersections of r members of $|\mathcal{L}^m|$, is contained

in a closed subset of $|\mathcal{L}^m|$ of codimension ≥ 2 .

Proof. Induct on r : the $r = 1$ case is Lemma 3.1.6. If $r \geq 2$, let $\pi_1: |\mathcal{L}^m| \times \cdots \times |\mathcal{L}^m|$ denote the first projection from the r -fold product, and let $R \subset |\mathcal{L}^m| \times \cdots \times |\mathcal{L}^m|$ denote the set of r -tuples $(H_1, \dots, H_r)$ such that $H_1 \cap \cdots \cap H_r$ is reducible. First assume X is normal. For each fixed $H_1 \in |\mathcal{L}^m|$, we have that $\text{codim}_{\pi_1^{-1}(H_1)}(R \cap \pi_1^{-1}(H_1)) \geq 2$ if H_1 is normal by hypothesis, and $\text{codim}_{\pi_1^{-1}(H_1)}(R \cap \pi_1^{-1}(H_1)) \geq 1$ if H_1 is not normal. In the latter case, the locus of non-normal H_1 is contained in a codimension 1 subset of $|\mathcal{L}^m|$ by Bertini's theorem (Lemma 3.1.3). So by dimension counting we have that $\text{codim}_{|\mathcal{L}^m| \times \cdots \times |\mathcal{L}^m|}(R) \geq 2$. The $\dim \text{Sing}(X) = 0$ case follows by the same argument. $\square$

3.2 Injectivity of the restriction map

In this section, we will show an ingredient for the following injectivity result:

Proposition 3.2.1 (Corollary 3.4.2, Corollary 3.4.3, and Corollary 3.4.4). *Let X be a normal variety of dimension $n \geq 3$ over an algebraically closed field $k \neq \overline{\mathbb{F}}_p$, and let $\mathcal{L}$ be a very ample line bundle on X such that $\mathcal{L} = \mathcal{L}_0^m$ for some $m \geq 2$. For infinitely many $Y \in |\mathcal{L}|$, the restriction map*

$$\text{Cl}(X) \longrightarrow \text{Cl}(Y)$$

is injective.

From the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Cl}^0(X) & \longrightarrow & \text{Cl}(X) & \longrightarrow & \text{Cl}^{\text{ns}}(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Cl}^0(Y) & \longrightarrow & \text{Cl}(Y) & \longrightarrow & \text{Cl}^{\text{ns}}(Y) \longrightarrow 0 \end{array}$$

it suffices to show that the induced maps $\mathrm{Cl}^0(X) \rightarrow \mathrm{Cl}^0(Y)$ and $\mathrm{Cl}^{\mathrm{ns}}(X) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(Y)$ have torsion kernels. In this section we will show injectivity for $\mathrm{Cl}^0(X)$ (Corollary 3.2.4) up to torsion and for $\mathrm{Cl}^{\mathrm{ns}}(X) \rightarrow \mathrm{Cl}(Y)/\mathrm{Cl}^0(X)$ (Corollary 3.2.8), but we will need surjectivity for Cl^0 to finish the injectivity statement for $\mathrm{Cl}^{\mathrm{ns}}$ (Corollary 3.4.4).

We note here that a different approach gives injectivity up to torsion over any algebraically closed field (Corollary 4.2.5); we show this in Chapter 4.

Overview of Section 3.2

The first part §3.2.1 of this section recalls known results about injectivity modulo torsion for Cl^0 ; this is not new and follows from the Albanese variety after cutting down to a curve. Although the statements in this section have torsion and do not appear in the statement of Theorem 2.0.1, they hold for ample linear systems, whereas the later results in Section 3.4 (using cutting to surfaces) apply to very ample linear systems. We also show that the restriction map from Cl^0 to a curve is not surjective; this result is used later in Section 3.4.

For injectivity modulo Cl^0 (Lemma 3.2.7, Corollary 3.2.8), the idea is to fiber X by curves, consider the relative Jacobian, and use Néron’s specialization theorem. We briefly introduce some useful results about the relative Jacobian in §3.2.2 before showing the $\mathrm{Cl}^{\mathrm{ns}}$ result.

Before considering Cl^0 and $\mathrm{Cl}^{\mathrm{ns}}$ separately, we first note that any torsion in the kernel of the restriction map on Picard groups of order coprime to the characteristic can be killed by étale covers. So for $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y)$ the kernel can only contain p -power torsion. By passing to an appropriate cover we get the same result for Cl . The proof is essentially the same as the one for the local Picard group [Kol17, 2.91].

Lemma 3.2.2. *Let X be a normal variety of dimension $n \geq 3$ over a perfect field k , let $\mathcal{L}$ be a line bundle on X , $|V| \subset |\mathcal{L}|$ a basepoint free linear system not composed*

with a pencil, and $Y \in |V|$ a general member. Let D be an effective Weil divisor on X such that

1. $D|_Y \sim 0$, and
2. there is a minimal positive integer l such that $lD \sim 0$.

Then $l = p^e$ for some $e = e(X, |V|) \in \mathbb{Z}_{\geq 0}$ if $\text{char } k = p > 0$, and $l = 1$ if $\text{char } k = 0$.

That is, the kernel of the restriction map $\text{Cl}(X) \rightarrow A^1(Y)$ contains no l -torsion for any l coprime to the characteristic of the ground field.

Proof. Assume otherwise, and let l be the smallest positive integer such that $lD \sim 0$. Replacing l by $\frac{l}{p^c}$ and D by $p^c D$, where p^c is the higher power of p dividing l , we may assume $l \geq 2$ and p are coprime.

First assume we have a resolution of singularities $\pi: \tilde{X} \rightarrow X$, and let $\tilde{Y} \in \pi^*|V|$ be the divisor on $\tilde{X}$ corresponding to $Y \in |\mathcal{L}|$ with inclusion $\tilde{i}: \tilde{Y} \hookrightarrow \tilde{X}$. The sequence

$$0 \longrightarrow \langle E_i \rangle \longrightarrow \text{Pic}(\tilde{X}) \xrightarrow{\pi_*} \text{Cl}(\tilde{X} \setminus \{E_i\}) = \text{Cl}(X \setminus S) = \text{Cl}(X) \longrightarrow 0$$

is exact, where $\{E_i\}$ is the set of irreducible exceptional divisors of π and $S \subset X$ is the codimension ≥ 2 closed subset over which π is not an isomorphism. Let $\mathcal{O}_{\tilde{X}}(\tilde{D})$ be a Cartier divisor on $\tilde{X}$ such that $\pi_* \tilde{D} \sim D$. Since the E_i are all in the kernel of π_* , we may assume the support of $\tilde{D}$ does not contain any E_i . Then $\mathcal{O}_{\tilde{X}}(\tilde{D})^{\otimes m} \cong \mathcal{O}_{\tilde{X}}$.

Next, note that $\tilde{Y}$ need not be normal but it is irreducible by Lemma 3.1.1, and the normalization $\tilde{Y}^\nu \rightarrow \tilde{Y}$ is a universal homeomorphism. This follows from considering the universal family of $\pi^*|V|$ over $\tilde{X}$ and taking a base change, maximal reduced subscheme, and normalization so that the generic fiber is geometrically normal [Sta20, Tag 0C3N]. Then $\tilde{Y}^\nu \rightarrow \tilde{Y}$ factors through some power of Frobenius on $\tilde{Y}$, so $\text{Pic}(\tilde{Y}) \rightarrow \text{Pic}(\tilde{Y}^\nu)$ is an isomorphism up to p^e -torsion for some $e = e(\pi^*|V|)$.

We know that $\tilde{i}^*\mathcal{O}_{\tilde{X}}(\tilde{D}) \in \text{Ker}(\text{Pic}(\tilde{Y}) \rightarrow \text{Cl}(Y))$. Since the sequence

$$0 \longrightarrow \langle E_i \cap \tilde{Y} \rangle \longrightarrow \text{Cl}(\tilde{Y}) \longrightarrow \text{Cl}(\tilde{Y} \setminus \{E_i \cap \tilde{Y}\}) = A^1(Y \setminus (S \cap Y)) = A^1(Y) \longrightarrow 0$$

is exact, where $\langle E_i \cap \tilde{Y} \rangle$ denotes the subgroup generated by the irreducible divisorial components of $E_i \cap \tilde{Y}$, and $\tilde{D}$ is not supported on the E_i , we conclude that $\tilde{i}^*\mathcal{O}_{\tilde{X}}(p^e\tilde{D}) \cong \mathcal{O}_{\tilde{Y}}$.

Let $\phi: \tilde{X}' \rightarrow \tilde{X}$ be the degree l cyclic cover corresponding to the line bundle $\mathcal{O}_{\tilde{X}}(p^e D)$ and a nowhere vanishing section s of $\mathcal{O}_{\tilde{X}}(lp^e D)$. $\tilde{X}'$ is connected since $\mathcal{O}_{\tilde{X}}(p^e D) \not\cong \mathcal{O}_{\tilde{X}}$. Since the restriction $\tilde{i}^*\mathcal{O}_{\tilde{X}}(p^e D)$ is trivial, then $\phi^{-1}(\tilde{Y})$ is disconnected. This is a contradiction by Grothendieck's connectedness lemma 3.1.4.

In general, we can find a regular alteration $\pi: \tilde{X} \rightarrow X$ where the Stein factorization $g: Z \rightarrow X$ is separable and has degree p^f [Tem17, Theorem 4.3.1]. Let $Y_Z \in g^*|\mathcal{L}| \subset |g^*\mathcal{L}|$ correspond to Y . By miracle flatness and normality of Z and X , there are dense open subsets $U_Z \subset Z$ and $U_X \subset X$ whose complements have codimension ≥ 2 and such that $g: U_Z \rightarrow U_X$ is a proper flat morphism. Since flat pullback is defined over U_X and $\text{Cl}(Z) = \text{Cl}(U_Z)$, we can consider the Weil divisor g^*D on Z . Since the composition $A_*(U_X) \xrightarrow{g^*} A_*(U_Z) \xrightarrow{g_*} A_*(U_X)$ is multiplication by p^f [Ful98, Example 1.7.4] and D is not p -power torsion by assumption, we have that $g^*D \not\sim 0$ but $lg^*D \sim 0$ on Z . Since g is finite, the codimension 1 cycle $g^*D|_{Y_Z}$ on Y_Z is rationally equivalent to 0.

Now $g^*[V]$ is a basepoint free linear system not composed with a pencil on the normal variety Z , and $\pi: \tilde{X} \rightarrow Z$ is a resolution of singularities. So we are in the first case again.

□

3.2.1 Injectivity for Cl^0

First we recall the following result about the connected component of the identity, which holds for geometrically normal, geometrically integral varieties over any field.

Lemma 3.2.3 ([Kol20, Corollary 131]). *Let X be a normal projective variety and $i: C \hookrightarrow X$ an irreducible curve contained in the smooth locus of X such that C is the set-theoretic complete intersection of ample Cartier divisors. Let $\tilde{C}$ denote the normalization of C . Then the restriction map $\mathrm{Cl}^0(X) \rightarrow \mathbf{Jac}(\tilde{C})$ is obtained as the dual of the surjection $\mathrm{alb}_i: \mathbf{Alb}(\tilde{C}) = \mathbf{Alb}(C) \rightarrow \mathbf{Alb}(X)$ and therefore has finite kernel.*

Applying Lemma 3.2.2 we see that the torsion can only come from $\mathrm{char} k$:

Corollary 3.2.4. *Let X be a normal projective variety of dimension $n \geq 2$ over an algebraically closed field k and $\mathcal{L}$ an ample line bundle on X . Let $|\Lambda| \subset |\mathcal{L}|$ be a general $(n - 2)$ -dimensional linear system, and let S be a complete intersection of members of $|\Lambda|$. Then the restriction map*

$$\mathrm{Cl}^0(X) \longrightarrow A^1(S)$$

has torsion kernel, and if $\dim X \geq 3$ is

1. *injective if $\mathrm{char} k = 0$, and*
2. *injective up to p^e -torsion for some $e = e(|\Lambda|)$ if $\mathrm{char} k = p > 0$.*

Proof. The kernel of $\mathrm{Cl}^0(X) \rightarrow \mathbf{Jac}(C)$ is finite by Lemma 3.2.3, where C is the base curve of $|\Lambda|$ and is contained in the smooth locus of X by generality. Let m be its cardinality. Then m kills the kernel of $\mathrm{Cl}^0(X) \rightarrow A^1(Y)$ for any Cartier divisor $Y \in |\Lambda|$, and the kernel of $\mathrm{Cl}^0(X) \rightarrow A^1(S)$ for any complete intersection S of

members of $|\Lambda|$. Assume $\dim X \geq 3$. If $\text{char } k = 0$ then $\text{Cl}^0(X) \rightarrow A^1(Y)$ is injective by Lemma 3.2.2, so the restriction map to $\text{Cl}^0(S)$ is as well. If $\text{char } k = p > 0$, let p^e be the largest power of p dividing m . If a divisor $D \not\sim 0$ is in the kernel of $\text{Cl}^0(X) \rightarrow A^1(S)$, then it is also in the kernel of the restriction to Y . Then $\frac{m}{p^e}D \not\sim 0$ by Lemma 3.2.2, so D must be p^e -torsion. $\square$

In a later section (Corollary 3.4.2) we will see that in fact injectivity holds without torsion if $\dim X = 3$.

We will use the following corollary of Lemma 3.2.3 in a later section, when we show surjectivity up to torsion for Cl .

Corollary 3.2.5. *Let X be a normal projective variety of dimension $n \geq 2$ over an algebraically closed field k and $\mathcal{L}_0$ an ample and basepoint-free line bundle on X . For any smooth complete intersection curve D of members of $|\mathcal{L}_0^m|$ such that D is contained in the smooth locus of X , the cokernel of the restriction map*

$$\mathbf{Cl}^0(X) \longrightarrow \mathbf{Jac}(D)$$

has positive dimension if

1. $m \geq 2$, unless $X = \mathbb{P}^2$ and $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^2}(1)$, and
2. $m \geq 3$ in general.

Proof. Let $C \subset X^{\text{reg}}$ be a smooth complete intersection curve of members of $|\mathcal{L}_0|$. Since the morphism $\mathbf{Jac}(C) \rightarrow \mathbf{Alb}(X)$ is surjective [Kol20, Lemma 130] we have $p_a(C) \geq \dim \mathbf{Alb}(X) = \dim \mathbf{Cl}^0(X)$. So we need to show that $p_a(D) > p_a(C)$ for D in the statement of the lemma.

First suppose X is a surface. The arithmetic genus of any member of $|\mathcal{L}_0^m|$ is equal

to that of $\sum_{i=1}^m C_i$ for curves $C_i \in |\mathcal{L}_0|$, which is equal to

$$p_a\left(\sum_{i=1}^m C_i\right) = \sum_{i=1}^m p_a(C_i) + (1 - m) + \frac{m(m-1)}{2} \mathcal{L}_0^2.$$

If $m = 2$ then $p_a(C_1 + C_2) = p_a(C_1) + p_a(C_2) + \mathcal{L}_0^2 - 1$, and this is strictly larger than $p_a(C_i)$ unless $p_a(C_i) = 0$ and $\mathcal{L}_0^2 = 1$. In this case, $X = \mathbb{P}^2$ and $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^2}(1)$, and any plane curve of degree $m \geq 3$ has positive genus.

If $n \geq 3$, let D be a complete intersection curve of members of $|\mathcal{L}_0^m|$. Similarly one computes that

$$p_a(D) = mp_a(C) + 1 - m + m \frac{(m^{n-1} - 1)(n - 1)}{2} \mathcal{L}_0^n$$

where C is a complete intersection curve of members of $|\mathcal{L}_0|$. Since $n \geq 3$ and $\mathcal{L}_0$ is ample we have

$$p_a(D) - mp_a(C) = 1 - m + m \frac{(m^{n-1} - 1)(n - 1)}{2} \mathcal{L}_0^n > m^n - 2m + 1,$$

which is positive for $m \geq 2$.

□

We next state some results about the relative Jacobian of a relative curve that will be used in the proof of the injectivity for Cl^{ns} .

3.2.2 Relative Jacobians

Let $f: X \rightarrow S$ be a projective and flat morphism of relative dimension one that is locally of finite presentation. If the fibers of f are geometrically reduced and irreducible, then $\mathbf{Pic}_{X/S}$ is representable as a smooth separated S -scheme, and if f is

smooth over $U \subset S$, then $\mathbf{Pic}_{X|U/U}^0$ is an abelian U -scheme [BLR90, Theorem 9.3.1, Proposition 9.4.4].

We will consider the relative Jacobian in the following setting. Let X be a normal variety of dimension $n \geq 2$ over an algebraically closed field k , $\mathcal{L}$ a very ample line bundle on X , and $|V| \subset |\mathcal{L}|$ a general linear system of dimension $n - 1$. Assume that $\mathcal{L} = \mathcal{L}_0^m$ for $m \geq 2$ and a very ample line bundle $\mathcal{L}_0$ on X (or $m \geq 3$ if $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^2}(1)$). The linear system $|V|$ defines a rational map $\phi: X \dashrightarrow \mathbb{P}^{n-1}$, and the closures $C_b = \overline{\phi^{-1}(b)}$ of the fibers of ϕ are complete intersection curves of members of $|V|$. Since the conditions of Lemma 3.1.6 are satisfied, we have that the locus R of reducible fibers has codimension ≥ 2 in $|V|$. Let U_i denote the complement of the closure of R .

Let $\phi': X' \rightarrow \mathbb{P}^{n-1}$ be the normalization of the closure of the graph of ϕ ; by Lemma 3.1.3 this is a smooth morphism over a dense open subset $U_{\text{sm}} \subset \mathbb{P}^{n-1} \setminus \phi'(\text{Sing}(X')) \subset \mathbb{P}^{n-1}$.

- $\mathbf{Pic}_{X'|U_i/U_i}^0$ is a variety and smooth over U_i , and
- $\mathbf{Pic}_{X'|U_{\text{sm}}/U_{\text{sm}}}^0 \cong \mathbf{Pic}_{X'|U_i/U_i}^0 \times_{U_i} U_{\text{sm}}$ with the choice of a zero section is an abelian scheme over U_{sm} .

Frequently we will use $\mathbf{Jac}(X'/\mathbb{P}^{n-1})$ to denote the abelian U_{sm} -scheme $\mathbf{Pic}_{X'|U_{\text{sm}}/U_{\text{sm}}}^0$.

Pick a distinguished base point $x \in \text{Bs}|V|$. Then the section of ϕ' given by the exceptional divisor E_x over x defines a rational map $X' \dashrightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1})$, which is a morphism on the locus U_{sm} of smooth fibers. So for any $U_0 \subset U_i$ we have an isomorphism

$$\text{Pic}^0(X'|_{U_0}) \cong \mathbf{Pic}_{X'|U_i/U_i}^0(U_0)$$

between equivalence classes of algebraically trivial line bundles and rational sections [BLR90, Proposition 8.1.4].

An algebraically trivial Cartier divisor on X' defines a section of $\mathbf{Pic}_{X'|U_i/U_i}^0 \rightarrow U_i$

by pullback. An algebraically trivial Weil divisor D on X' defines a *rational* section σ_D of $\mathbf{Pic}_{X'|U_i/U_i}^0 \rightarrow U_i$ over U_{sm} by restricting first to the regular locus of X' and then pulling back the Cartier divisor $\mathcal{O}_{X'^{\text{reg}}}(D|_{X'^{\text{reg}}})$. Note that this rational map does not in general extend over all of U_i .

Lemma 3.2.6. *The map*

$$\begin{aligned} \text{Cl}(X') \rightarrow \text{Jac}(C_\eta) &= \{\text{sections of } \mathbf{Jac}(X'/\mathbb{P}^{n-1}) \rightarrow U_{\text{sm}}\} \\ D' &\mapsto (D' - \deg(D'|_{C_\eta})E)|_{C_\eta} \end{aligned} \quad (3.1)$$

is a surjection with kernel generated by E and $\phi'^\mathcal{O}_{\mathbb{P}^{n-1}}(d)$ for some $d \in \mathbb{Z}$. In particular, its restriction to $\text{Cl}(X) \subset \text{Cl}(X')$ is injective.*

Proof. The morphism ϕ' admits a section, so a section σ of $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \rightarrow U_{\text{sm}}$ is the same as a divisor class in $\text{Pic}^0(X'|_{U_{\text{sm}}})$ whose restriction to C_b agrees with $\sigma(b) \in \text{Jac}(C_b)$. So surjectivity follows from surjectivity of $\text{Cl}(X') \rightarrow \text{Cl}(X'|_{U_{\text{sm}}})$.

For the kernel, we first consider the kernel of the restriction $\text{Cl}(X') \rightarrow \text{Cl}(X'|_{U_{\text{sm}}})$. The codimension 1 irreducible components $\{\gamma_i\}_i$ of $\mathbb{P}^{n-1} \setminus U_{\text{sm}}$ are degree d_i hypersurfaces in $\mathbb{P}^{n-1}$. By Lemma 3.1.7, each preimage $\phi'^{-1}(\gamma_i)$ is irreducible and thus is linearly equivalent to $d_i\phi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, so the kernel of $\text{Cl}(X') \rightarrow \text{Cl}(X'|_{U_{\text{sm}}}) \cong \text{Pic}(C_\eta)$ is generated by $\phi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(d)$ where $d = \gcd\{d_i\}$. So for a divisor D' to be in the kernel of the map (3.1), it must satisfy $D' - \deg(D'|_{C_\eta})E \in \langle \phi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(d) \rangle$. Then D' must be in the subgroup of $\text{Cl}(X')$ generated by $\phi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(d)$ and the exceptional divisors. By Lemma 3.2.3 we know $\#\text{Bs}|V| > 1$, so E is not the only exceptional divisor of π , and for any other exceptional divisor E' the restrictions $E|_{C_\eta}$ and $E'|_{C_\eta}$ are independent in $\text{Pic}(C_\eta)$. So the kernel of (3.1) is generated by E and $\phi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(d)$. Note that this subgroup is disjoint from the image of $\text{Cl}(X)$ since there is more than one exceptional divisor, so the restriction of (3.1) to $\text{Cl}(X)$ is injective.

□

Remark. [LN59] observed in their simplified proof of the Theorem of the Base that $\mathrm{Cl}^0(X')$ can be interpreted in terms of the Chow trace. For an abelian scheme $\mathcal{A} \rightarrow B \rightarrow \mathrm{Spec} k$ we denote by $\mathrm{Tr}_{B/k}(\mathcal{A})$ and $\mathrm{Im}_{B/k}(\mathcal{A})$ the Chow B/k -trace and Chow B/k -image, respectively. These depend only on the field extension $k(B)/k$ and the abelian variety $\mathcal{A} \times_B \mathrm{Spec} k(B)$ over $k(B)$ [GS13, Remark 3.3]. See [Con06] and [GS13, Section 3] for more details.

We consider the abelian scheme $(\mathbf{Jac}(X'/\mathbb{P}^{n-1}), \sigma_0)$ with the zero section $\sigma_0: b \mapsto \mathcal{O}_{C_b}$. Then there is a canonical isomorphism

$$\mathbf{Cl}^0(X') \cong \mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})).$$

In one direction, from the rational maps

$$X' \dashrightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1}) \xrightarrow{\cong} \mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee \longrightarrow \mathrm{Im}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee) \times U_{\mathrm{sm}}$$

we get a map $X' \dashrightarrow \mathrm{Im}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee)$ inducing a homomorphism $\mathbf{Alb}(X') \rightarrow \mathrm{Im}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee)$. This dualizes to a homomorphism

$$\mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})) \longrightarrow \mathbf{Cl}^0(X') .$$

Conversely, the map (3.1) defines a morphism $\mathbf{Cl}^0(X') \times U_{\mathrm{sm}} \rightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1})$ of abelian U_{sm} -schemes and hence a homomorphism of abelian varieties $\mathbf{Cl}^0(X') \rightarrow \mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1}))$ by the universal property of the Chow trace.

3.2.3 Injectivity for $\mathrm{Cl}^{\mathrm{ns}}$

We will now assume $k \neq \overline{\mathbb{F}_p}$. We first show injectivity modulo Cl^0 for the restriction map from X to a complete intersection curve.

Lemma 3.2.7. *Let X be a normal variety of dimension $n \geq 2$ over an algebraically closed field $k \neq \overline{\mathbb{F}_p}$. Let $\mathcal{L}$ a very ample line bundle on X such that $\mathcal{L} = \mathcal{L}_0^m$ for some very ample $\mathcal{L}_0$ and $m \geq 2$. Let $|V| \subset |\mathcal{L}|$ a general linear system of dimension $n - 1$ inducing a rational map $\phi: X \dashrightarrow \mathbb{P}^{n-1}$. Then there exist infinitely many k -points $s \in |V|$ such that the map*

$$\mathrm{Cl}^{\mathrm{ns}}(X) \longrightarrow \mathrm{Pic}(C_s)/i_s^* \mathrm{Cl}^0(X)$$

is injective, where C_s is the complete intersection curve corresponding to $s \in |V|$, C_s is contained in the smooth locus of X , and the map is induced by the restriction $i_s^: \mathrm{Cl}(X) \rightarrow \mathrm{Pic}(C_s)$.*

Proof. If $X = \mathbb{P}^n$ then $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(C)$ for any curve, so we may assume X is not $\mathbb{P}^n$. Let $\phi': X' \rightarrow \mathbb{P}^{n-1}$ be the normalization of the closure of the graph of ϕ . Then $\pi: X' \rightarrow X$ is birational, so $\pi_*: \mathrm{Cl}^0(X') \xrightarrow{\cong} \mathrm{Cl}^0(X)$. The morphism ϕ' is smooth over the open subset $U_{\mathrm{sm}} = \mathbb{P}^{n-1} \setminus \phi'(\mathrm{Sing}(X')) \subset \mathbb{P}^{n-1}$. Denote $X'|_{U_{\mathrm{sm}}} = X' \times_{\mathbb{P}_k^{n-1}} U_{\mathrm{sm}}$.

The fibers of ϕ' are precisely the complete intersection curves of members of $|V|$. For a general such curve, C_s is contained in the regular locus of X , and the restriction map $\mathrm{Cl}(X) \rightarrow \mathrm{Pic}(C_s)$ factors

$$\mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(X') \longrightarrow \mathrm{Cl}(X'|_{U_{\mathrm{sm}}}) = \mathrm{Pic}(X'|_{U_{\mathrm{sm}}}) \longrightarrow \mathrm{Pic}(C_s)$$

so we first consider $\mathrm{Cl}(X') \rightarrow \mathrm{Pic}(X'|_{U_{\mathrm{sm}}})$.

Let $\gamma_1, \dots, \gamma_m$ be the codimension 1 irreducible components of $\mathbb{P}^{n-1} \setminus U_{\mathrm{sm}}$, as in the proof of Lemma 3.2.6. Then $X'|_{U_{\mathrm{sm}}}$ is the union of $X' \setminus \bigcup_{i=1}^m \phi'^{-1}(\gamma_i)$ together with a codimension ≥ 2 subset of X' , so the class groups of $X'|_{U_{\mathrm{sm}}}$ and $X' \setminus \bigcup_{i=1}^m \phi'^{-1}(\gamma_i)$ are the same. Again by Corollary 3.1.7 each $\phi'^{-1}(\phi'(\gamma_i))$ is irreducible, and since the γ_i are hypersurfaces in $\mathbb{P}^{n-1}$, then each $\phi'^{-1}(\gamma_i) \sim \phi'^* \mathcal{O}_{\mathbb{P}_k^{n-1}}(d_i)$ for some integer $d_i \geq 1$.

Then

$$\mathrm{Pic}(X'|_{U_{\mathrm{sm}}}) = \mathrm{Cl}(X'|_{U_{\mathrm{sm}}}) \cong \mathrm{Cl}(X') / \langle \phi'^* \mathcal{O}_{\mathbb{P}_k^{n-1}}(d) \rangle$$

where $d = \gcd\{d_i \mid 1 \leq i \leq m\}$. Since $\phi'^* \mathcal{O}_{\mathbb{P}_k^{n-1}}(d)$ is not in the image of $\mathrm{Cl}(X)$ (it differs from $\pi^* \mathcal{L}^d$ by a nonzero exceptional divisor) we have that $\mathrm{Cl}(X) \rightarrow \mathrm{Pic}(X'|_{U_{\mathrm{sm}}})$ is injective.

We next consider the specialization of $\mathrm{Pic}(X'|_{U_{\mathrm{sm}}}) / \mathrm{Cl}^0(X')$ to a curve, by looking at $\mathrm{Pic}^0(X'|_{U_{\mathrm{sm}}}) / \mathrm{Cl}^0(X')$ and $\mathrm{NS}(X'|_{U_{\mathrm{sm}}})$ separately. There is a morphism of abelian U_{sm} -schemes $\mathbf{Cl}^0(X') \times U_{\mathrm{sm}} \rightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1})$ given by the map (3.1) of §3.4.1. Consider the quotient

$$\mathcal{A} := \mathbf{Jac}(X'/\mathbb{P}^{n-1}) / (\mathrm{Cl}^0(X') \times U_{\mathrm{sm}})$$

and let $\mathcal{A}_\eta$ denote the generic fiber of $\mathcal{A} \rightarrow U_{\mathrm{sm}}$. Since $\mathrm{Cl}^0(X')$ is the Chow trace of $\mathbf{Jac}(X'/\mathbb{P}^{n-1})$ over U , then by the Lang–Néron Theorem 3.0.12 $\mathcal{A}_\eta(k(|V|))$ is a finitely-generated abelian group. We have that $\mathcal{A}_\eta(k|V|) = \mathrm{Pic}^0(X'_\eta) / \mathrm{Cl}^0(X') = \mathrm{Pic}^0(X'|_{U_{\mathrm{sm}}}) / \mathrm{Cl}^0(X')$ (since $U_{\mathrm{sm}} \subsetneq \mathbb{P}^n$ is an open subset).

Now let $k_0 \subset k$ be a finitely-generated subfield over which $\mathrm{Pic}^0(X'|_{U_{\mathrm{sm}}}) / \mathrm{Cl}^0(X')$ is defined. By Néron’s Theorem 3.0.15 there exist infinitely many points $s \in U_{\mathrm{sm}}(k_0)$ such that

$$\mathrm{Pic}^0(X'|_{U_{\mathrm{sm}}}) / \mathrm{Cl}^0(X') \longrightarrow \mathrm{Pic}^0(C_s) / i_s^* \mathrm{Cl}^0(X')$$

is injective. The specialization on Néron–Severi groups $\mathrm{NS}(X'_\eta) \rightarrow \mathrm{NS}(C_s)$ is also injective since X'_η and C_s are curves, so we have that $\mathrm{Pic}(X'|_{U_{\mathrm{sm}}}) / \mathrm{Cl}^0(X') \rightarrow \mathrm{Pic}(C_s) / i_s^* \mathrm{Cl}^0(X')$ is injective for such s .

□

By factoring the restriction map, we get injectivity modulo $\mathrm{Cl}^0(X)$ for complete intersections of higher dimension, in particular for divisors in $|\mathcal{L}|$.

Corollary 3.2.8. *Let X be a normal variety of dimension $n \geq 3$ over an algebraically*

closed field $k \neq \overline{\mathbb{F}_p}$, and let $\mathcal{L}$ be a very ample line bundle on X such that $\mathcal{L} = \mathcal{L}_0^m$ for some $m \geq 2$. Then there exist infinitely many complete intersection curves C of members of $|\mathcal{L}|$ such that the kernel of the restriction map

$$\mathrm{Cl}(X) \longrightarrow A^1(S)$$

is contained in $\mathrm{Cl}^0(X)$ for any complete intersection S of members of $|\mathcal{L}|$ containing C .

Proof. The restriction map $\mathrm{Cl}(X) \rightarrow \mathrm{Pic}(C)$ factors through the Gysin map $\mathrm{Cl}(X) \rightarrow A^1(S)$ on codimension 1 cycles modulo rational equivalence [Ful98, Chapter 6], so apply Lemma 3.2.7. $\square$

After proving surjectivity for Cl^0 up to torsion in §3.4.1, we will be able to conclude from Corollary 3.2.8 that the kernel of $\mathrm{Cl}^{\mathrm{ns}}(X) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(Y)$ is torsion for sufficiently general $Y \in |\mathcal{L}|$.

3.3 Sections of abelian schemes

Throughout this section we work over an algebraically closed field $\neq \overline{\mathbb{F}_p}$ of characteristic ≥ 0 . We will use $[m]_{\mathcal{A}/B}$ to denote multiplication by $m \in \mathbb{Z}$ on an abelian scheme $\mathcal{A} \rightarrow B$. The main result of this section is Lemma 3.3.5, which we will apply to get surjectivity of the restriction maps on $\mathbf{Cl}^0$. We show a strictly weaker version of the following result of [GS13].

Theorem 3.3.1 ([GS13, Theorem 1.3]). *Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree over the prime subfield, B a normal variety of dimension ≥ 2 , and $f: B \rightarrow \mathbb{P}^{\dim B}$ a generically finite, generically unramified morphism. Let $\mathcal{A} \rightarrow B$ be an abelian scheme. Then for a very general pair of incident lines $L_1 \cup L_2 \subset$*

$\mathbb{P}_k^{\dim B}$, the restriction map

$$\{\text{sections of } \mathcal{A} \rightarrow B\} \rightarrow \{\text{sections of } \mathcal{A}|_{f^{-1}(L_1 \cup L_2)} \rightarrow f^{-1}(L_1 \cup L_2)\}$$

is a bijection.

Note that the result in their paper is stated over an uncountable field, but their proof only needs the transcendence degree to be infinite.

Although Lemma 3.3.5 is a strictly weaker version of a special case of Graber–Starr’s result, we include it because the proof uses some of the same themes as the rest of this article.

First, note that a purely inseparable section of an abelian scheme gives a section after multiplication by a p -power.

Lemma 3.3.2. *Let B be a variety over a perfect field k , $\pi: \mathcal{A} \rightarrow B$ an abelian scheme, and $\sigma: B \rightarrow \mathcal{A}$ a morphism such that the composition $\pi \circ \sigma$ agrees with $F_{B/k}^e$ for some $e \in \mathbb{Z}_{\geq 0}$. Then $[p^{e(\dim \mathcal{A} - \dim B)}]_{\mathcal{A}/B} \circ \sigma$ is a section of π .*

Proof. By Weil’s extension [Kol96a, Theorem VI.1.9.3] it suffices to consider the generic fiber of π . So let A be an abelian variety over an imperfect field K and K'/K a finite, purely inseparable extension. We would like to show that the cokernel of $A(K) \rightarrow A(K')$ is p -power torsion. Fixing an algebraic closure of K , we have that $K' \subset K^{1/p^e}$ for some e so we may assume $K' = K^{1/p^e}$, and by iterating Frobenius it suffices to consider the case $K' = K^{1/p}$.

Now $A(K^{1/p}) = (A \otimes_K K^{1/p})(K^{1/p})$ and $A(K) = A^{1/p}(K^{1/p})$, where $A^{1/p}$ denotes the source of absolute Frobenius on A , so by the push-pull formula for Chow groups applied to the relative Frobenius morphism $F_{A/K}: A^{1/p} \rightarrow A \otimes_K K^{1/p}$ we obtain that the kernel and cokernel of $A(K) \rightarrow A(K^{1/p})$ are p -power torsion.

So after multiplication by $p^{\dim \mathcal{A} - \dim B}$ we obtain a rational section of π , which again extends uniquely to a section over B . □

Lemma 3.3.3. *Let $k \neq \overline{\mathbb{F}_p}$ be an algebraically closed field, $U \subset \mathbb{P}_k^n$ an open subscheme, and $\pi: \mathcal{A} \rightarrow U$ an abelian scheme with $\mathrm{Tr}_{U/k}(\mathcal{A}) = 0$. Let $\mathcal{L}$ be the universal family of lines in $\mathbb{P}^n$ not contained in $\mathbb{P}^n \setminus U$. Then for a very general $(p, L) \in \mathcal{L}$, the specialization map*

$$\{\text{sections of } \pi: \mathcal{A} \rightarrow U \text{ over } L\} \rightarrow \mathcal{A}_b(k)$$

is injective. That is, the set of (p, L) for which injectivity holds is a dense subset of $\mathcal{L}$ and contains a G_δ set. (Here we are using L to denote $L \cap U$ for a line $L \subset \mathbb{P}^n$ that meets U .)

Proof. Let $W \subset \mathrm{Gr}(1, \mathbb{P}^n)$ denote the open subscheme parametrizing lines not contained in $\mathbb{P}^n \setminus U$, and let $\tau: \Sigma \rightarrow W$ be the parameter space for sections over such lines. Consider the set

$$\Omega = \{(q, L, s_1, s_2) \mid q \in s_1(L) \cap s_2(L)\} \subset \mathcal{A} \times W \times \Sigma \times \Sigma.$$

This is a countable union of locally closed subsets Ω_i , since Chow varieties have a countable number of components, and it maps to $\mathcal{L}$ under $(q, L, s_1, s_2) \mapsto (\pi(q), L)$. So its image Δ is also a countable union of locally closed subsets Δ_i , and Δ is exactly the pairs (b, L) for which the specialization map is not injective. The complement

$$\mathcal{L} \setminus \Delta = \mathcal{L} \setminus \bigcup_i \Delta_i = \bigcap_i (\mathcal{L} \setminus \Delta_i) \supset \bigcap_i (\mathcal{L} \setminus \overline{\Delta_i})$$

contains a countable intersection of open subsets of $\mathcal{L}$.

To show that $\mathcal{L} \setminus \Delta$ is dense in $\mathcal{L}$, let $p_2: \mathcal{L} \rightarrow W \subset \mathrm{Gr}(1, \mathbb{P}^n)$ be the first projection, and consider the fibers $p_2^{-1}(L) \cap \Delta$.

Assume k is uncountable. The sections over L form a finitely-generated group by the Lang–Néron Theorem 3.0.12, so there are countably many, and so they have

countably many intersection points. So $p_2^{-1}(L) \cap \Delta$ is a countable subset of $p_2^{-1}(L) = L \times \{L\}$. Then $p_2^{-1}(L) \cap (\mathcal{L} \setminus \Delta)$ is dense in $p_2^{-1}(L)$, so the closure $\overline{\mathcal{L} \setminus \Delta}$ contains $p_2^{-1}(L)$. Since this holds for any L , then $\mathcal{L} \setminus \Delta$ is dense in $\mathcal{L}$.

Now assume $k \neq \overline{\mathbb{F}_p}$ is countable. Then $p_2^{-1}(L) \cap \Delta$ is contained inside a field locally-thin subset of $p_2^{-1}(L)$ by Néron's Theorem 3.0.15, i.e. for every finitely-generated field $\overline{\mathbb{F}_p} \neq k_0 \subset k$

$$(p_2^{-1}(L) \setminus (p_2^{-1}(L) \cap \Delta))(k_0) \subset p_2^{-1}(L)(k_0)$$

is dense. Then $\overline{\mathcal{L} \setminus \Delta}(k_0)$ contains the k_0 -points of every fiber $p_2^{-1}(L)$, so again the k_0 -points of $\mathcal{L} \setminus \Delta$ are dense in $\mathcal{L}(k_0)$.

If $\bigcap_i (\mathcal{L} \setminus \overline{\Delta_i}) = \emptyset$, then $\mathcal{L} \setminus \Delta = \bigcap_i (\overline{\Delta_i} \setminus \Delta_i)$ is dense in $\mathcal{L}$. This is impossible, since each Δ_i is a locally closed subset of $\mathcal{L}$. Thus Δ is contained in a countable union $\bigcup_i \overline{\Delta_i}$ of proper closed subvarieties of $\mathcal{L}$. $\square$

Lemma 3.3.4. *Let k be an algebraically closed field and $\pi: \mathcal{A} \rightarrow \mathbb{P}_k^n$ an abelian scheme such that $\mathrm{Tr}_{\mathbb{P}_k^n/k}(\mathcal{A}) = 0$. Then for a general line $L \subset \mathbb{P}_k^n$, the group of sections of $\mathcal{A}|_L \rightarrow L$ is finitely generated. In particular there are only countably many sections over a general L .*

This also holds if $\mathbb{P}_k^n$ is replaced by a quasi-finite, generically unramified cover from a normal variety.

Proof. We first note that $\mathrm{Tr}_{\mathbb{P}_k^n/k}(\mathcal{A}) \cong \mathrm{Tr}_{L/k}(\mathcal{A}|_L)$ for a general line L . For any line $L \subset \mathbb{P}_k^n$, we can restrict the morphism $\mathrm{Tr}_{B/k}(\mathcal{A}) \times B \rightarrow \mathcal{A}$ of abelian B -schemes to L . By the universal property we get $\mathrm{Tr}_{B/k}(\mathcal{A}) \rightarrow \mathrm{Tr}_{L/k}(\mathcal{A}|_L)$. For a map in the opposite direction, pick $b \in B$ general and consider the family of lines $L_\lambda \ni b$. For each λ let $\mathrm{Tr}_{L_\lambda/k}(\mathcal{A}|_{L_\lambda})$ be the trace of the abelian scheme $\mathcal{A}_{L_\lambda} \rightarrow L_\lambda$. Then the fibers over b of

the maximal abelian k -subschemes of $\mathcal{A}|_{L_\lambda} \rightarrow L_\lambda$

$$\{\tau_\lambda(\mathrm{Tr}_{L_\lambda/k}(\mathcal{A}|_{L_\lambda}) \times L_\lambda)_b \mid L_\lambda \ni b\}$$

are a family of abelian subvarieties of $\mathcal{A}_b$. So for a general λ must be constant (see [Cho55, §2]) and equal to some abelian k -variety A_0 , which is isogenous to $\mathrm{Tr}_{L_\lambda/k}(\mathcal{A}|_{L_\lambda}) \times L_\lambda$, so the traces for these λ are isogenous. Pick one and call it T . Then we have $T \otimes_k k(\mathbb{P}_k^n) \rightarrow \mathcal{A}_\eta$, so by the universal property of the trace we get a morphism of abelian varieties $T \rightarrow \mathrm{Tr}_{\mathbb{P}_k^n/k}(\mathcal{A})$.

So if $\mathcal{A} \rightarrow \mathbb{P}_k^n$ is traceless, then so is $\mathcal{A}|_L \rightarrow L$ for a general line L . Then $\mathcal{A}|_L(L)$ is finitely generated by the Lang–Néron Theorem 3.0.12. $\square$

We now show a weaker version of a special case of [GS13, Theorem 1.3]. As in their paper we reduce to the cases of a constant abelian scheme and one “without too many sections” [GS13, Corollary 3.7].

Lemma 3.3.5. *Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree over the prime subfield, let $U \subset \mathbb{P}_{\mathbb{K}}^n$ be an open subscheme, and let $\pi: \mathcal{A} \rightarrow U$ an abelian scheme. Let L_1 be a very general line in $\mathbb{P}^n$, $b \in L_1$ a very general point, and L_2 a very general line through b . Let s_i be a section of $\pi|_{\mathcal{A}|_{L_i}}: \mathcal{A}|_{L_i} \rightarrow L_i$, and assume that $s_1(b) = s_2(b)$. Then there exists a section σ of π such that $s_i = ([m]_{\mathcal{A}/U} \circ \sigma)|_{L_i}$, where*

- $m = 1$ if $\mathrm{char} k = 0$, and
- $m = p^f$ for some $f \in \mathbb{Z}_{\geq 0}$ if $\mathrm{char} k = p > 0$.

Proof. Let $k = k^{\mathcal{A}}$ be a field of definition for $\mathcal{A}$. First note that by [Con06, Theorem 6.4] and Poincaré reducibility, after replacing U by a smaller open set we can find

isogenies of abelian U -schemes

$$(\mathrm{Tr}_{U/k}(\mathcal{A}) \times U) \times \mathcal{Q} \longrightarrow \mathcal{A} \longrightarrow (\mathrm{Tr}_{U/k}(\mathcal{A}) \times U) \times \mathcal{Q}$$

where $\mathcal{Q} = \mathcal{A}/\tau(\mathrm{Tr}_{U/k}(\mathcal{A}) \times U)$ is the quotient and $\mathrm{Tr}_{U/k}(\mathcal{Q}) = 0$. If the abelian U -scheme $(\mathrm{Tr}_{U/k}(\mathcal{A}) \times U) \times \mathcal{Q}$ satisfies the property that sections over a very general pair of incident lines extend uniquely, then $\mathcal{A}$ satisfies this property as well. So it suffices to consider the two separate cases of a constant abelian scheme and a traceless abelian scheme.

If $\mathcal{A} = A_0 \times U$ is constant, then a section over any line L is determined by where it sends a point, since if s_1, s_2 are two sections over L agreeing at a point, then composing $s_1 - s_2$ with the projection to A_0 defines a map from $\mathbb{P}^1$ to an abelian variety, so it must be constant. So the restriction map from sections to sections over $L_1 \cup L_2$ is surjective for any incident pair of lines, and by the same argument it must be injective as well since any two points in U can be joined by a line.

Now assume $\mathcal{A}$ is traceless. Let $\Delta \subset \mathcal{L}$ be the subset of Lemma 3.3.3, let $(b, L_1) \in \mathcal{L} \setminus \Delta$, and let s_1 be the given section over L_1 . Let $k^b \supsetneq k$ be an algebraically closed field of definition for (b, L_1) . Let $\mathcal{L}_b$ denote the fiber of $\mathcal{L}$ over $b \in U$; this is the set of lines containing b and so identifies with a closed subset of $\mathrm{Gr}(1, \mathbb{P}^n)$ (since $b \in U$). Consider the family $\tau: \Sigma = \Sigma_b \rightarrow \mathcal{L}_b$ of sections over lines containing b ; this is contained in the Chow variety of $\mathcal{A} \times U$ so it has a countable number of irreducible components. Let Σ° be the subset of Σ defined as the union of all irreducible components Σ_i of Σ whose image is dense in $\mathcal{L}_b$. Then each Σ_i is defined over k^b . By assumption $(b, L_2) \in \mathcal{L}_b$ is in the image of Σ for a very general line L_2 , so since L_2 is algebraically independent over k^b we have $\Sigma^\circ \neq \emptyset$. Also note that if L_2 is very general, then the fiber $\tau^{-1}((b, L_2))$ meets every irreducible component Σ_i of Σ° . Let Ω denote the subset of $\mathcal{A}$ defined by the images of the sections in Σ° , and for each i let $\Omega_i \subset \mathcal{A}$ denote the image of

$\Sigma_i \subset \Sigma^\circ$.

For any $b' \neq b$ in U there is a unique line $L_{b,b'}$ containing b, b' , and for a general b' the abelian scheme $\mathcal{A}|_{L_{b,b'}} \rightarrow L_{b,b'}$ is traceless. Then the sections over $L_{b,b'}$ are finitely generated by the Lang–Néron Theorem 3.0.12, so the fiber of τ over $(b, L_{b,b'})$ meets each component Σ_i in at most finitely many points. If b' is very general then this intersection $\tau^{-1}((b, L_{b,b'})) \cap \Sigma_i = \{s_1^i, \dots, s_{r_i}^i\}$ will be nonempty. Then $\pi^{-1}(b') \cap \Omega_i = \{s_1^i(b'), \dots, s_{r_i}^i(b')\}$ is also finite. So we see that each $\pi|_{\Omega_i}: \Omega_i \rightarrow U$ is quasi-finite onto its image.

By the assumption that $(b, L_1) \notin \Delta$, the specialization map

$$\{\text{sections of } \pi: \mathcal{A} \rightarrow U \text{ over } L_1\} \rightarrow \mathcal{A}_b(k)$$

is injective, so there is a unique Ω_i containing $s_1(L_1)$ and $s_1(b)$ is not contained in the branch locus of $\pi|_{\Omega_i}: \Omega_i \rightarrow U$. Let $\overline{\Omega_i}$ denote its closure in $\mathcal{A}$. For any point $b' \neq b$ on L_1 , the fiber of π over b' meets Ω_i in a single point (namely at $s_1(b')$). Since L_1 is not contained in the branch locus of $\Omega_i \rightarrow U$, we see that $\Omega_i \rightarrow U$ maps one-to-one onto its image.

We can factor $\Omega_i \rightarrow U$ through some iteration of Frobenius $U^{1/p^e} \xrightarrow{\sigma} \Omega_i \xrightarrow{\pi|_{\Omega_i}} U$, so by Lemma 3.3.2

$$[p^{e(\dim \mathcal{A} - n)}]_{\mathcal{A}/U} \circ \sigma$$

is a section of π . On topological spaces $\sigma|_{L_i}$ and s_i agree as maps from $L_i \rightarrow \mathcal{A}|_{L_i}$. So since they are both sections we have that $[p^{e(\dim \mathcal{A} - n)}]_{\mathcal{A}/U} \circ \sigma|_{L_i} = s_i$ for each i . $\square$

Remark. It's not enough to take a section over a very general line in Lemma 3.3.5. Indeed, a smooth cubic threefold is an abelian scheme over a dense open subset $U \subset \mathbb{P}^2$ that admits 27 sections over a general line $L \cap U$ but no sections over U . In this example $\Sigma^\circ = \emptyset$ for general $b \in U$.

Remark. If $\mathbb{K}$ is uncountable and $\dim \mathcal{L}_b = 1$ (i.e. if $U \subset \mathbb{P}^2$), then the statement of

Lemma 3.3.5 still holds with very general L_2 replaced by uncountably many $L_2 \ni b$. In this setting $\Sigma^\circ \neq \emptyset$ because the image of each Σ_i is either dense in $\mathbb{P}^2$ or is contained in a curve. So if uncountably many lines $L_2 \ni b$ are in the image of Σ , some Σ_i has to be dense.

3.4 Surjectivity of the restriction map

The main result of this section is surjectivity for the restriction map.

Proposition 3.4.1. *Let $\mathbb{K}$ be a field of infinite transcendence degree over the prime subfield, X a normal variety of dimension $n \geq 3$ over $\mathbb{K}$, $\mathcal{L}_0$ a very ample line bundle, and d an integer. For a very general $Y \in |\mathcal{L}_0^d|$, the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

is

1. surjective for $d = 4$ or ≥ 6 if $n = 3$ (Corollary 3.4.12), and
2. surjective for $d \geq 2$ if $n \geq 4$ (Corollary 3.4.10).

Overview of Section 3.4

The rough idea of this section is to show surjectivity by fibering X by curves, considering the relative Jacobian, and extending sections from pairs of lines using the extension results of [GS13] as described in §3.3. We will first show that a general complete intersection surface (corresponding to a line in $\mathbb{P}^{n-1}$) has the same Cl^0 as X (Corollary 3.4.2), since the trace of an abelian scheme is the same when restricted to a general line. For the rest of the class group we lift sections from reducible complete intersection surfaces $|\mathcal{L}| \cap \cdots \cap |\mathcal{L}| + |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$ consisting of two components (Proposition 3.4.5). By specializing we get the result for the sufficiently general

surface $|\mathcal{L}^2| \cap |\mathcal{L}| \cdots \cap |\mathcal{L}|$ (Proposition 3.4.7). Finally, using a different type of specialization we show that under certain assumptions we get a result for odd multiples as well (Proposition 3.4.11).

We first fix notation. Later in this section we will assume our ground field has infinite transcendence degree in order to use Theorem 3.3.1 ([GS13, Theorem 1.3]) and Lemma 3.4.6, but the other assumptions are the same as those in §3.2.2. As before, the assumptions on the line bundle are to ensure that the reducible locus is small (Corollary 3.1.7).

Notation

- Let X be a normal projective variety of dimension $n \geq 3$ over an algebraically closed field k of characteristic ≥ 0 .
- Let $\mathcal{L}$ be a very ample line bundle on X such that $\mathcal{L} = \mathcal{L}_0^m$ for some very ample line bundle $\mathcal{L}_0$ and $m \geq 2$.
- Let $|V| \subset |\mathcal{L}|$ a general $(n-1)$ -dimensional linear system defining a rational map $\phi: X \dashrightarrow \mathbb{P}^{n-1}$.
- For $b \in \mathbb{P}^{n-1}$ we denote the closure $C_b = \overline{\phi^{-1}(b)}$ of a general fiber. This maps isomorphically to its image in X .
- Let $\phi': X' \rightarrow \mathbb{P}^{n-1}$ be the normalization of the closure of the graph of ϕ and $\beta: X' \rightarrow X$ the induced birational morphism, which defines an isomorphism $\beta_*: \text{Cl}^0(X') \xrightarrow{\cong} \text{Cl}^0(X)$.
- Fix a base point $x \in \text{Bs}|V|$, with corresponding exceptional divisor E_x giving a section of ϕ' .
- For a complete intersection surface S of members of $|V|$, we denote its strict transform on X' by S' . Then $\phi'(S') = \overline{\phi(S)}$ is a line in $\mathbb{P}^{n-1}$.

3.4.1 Isomorphism for Cl^0

Corollary 3.4.2. *In the setting of §3.4, for a general complete intersection surface S of members of $|V|$, the restriction map $\mathrm{Cl}^0(X) \rightarrow \mathrm{Cl}^0(S)$ is an isomorphism.*

Proof. Let $L \subset \mathbb{P}^{n-1}$ be a general line corresponding to a complete intersection surface $S \subset X$. Then the abelian schemes $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \rightarrow U_{\mathrm{sm}}$ and $\mathbf{Jac}(X'/\mathbb{P}^{n-1})|_{L \cap U_{\mathrm{sm}}} \rightarrow L \cap U_{\mathrm{sm}}$ have the same trace (see the proof of Lemma 3.3.4), and these are the same as $\mathrm{Cl}^0(X)$ and $\mathrm{Cl}^0(S)$, respectively, as noted in the remark after Lemma 3.2.6. $\square$

For a divisor $Y \in |V|$ containing S , the restriction map factors through $\mathrm{Cl}^0(Y)$. So combining the above with Corollary 3.2.4 applied to $\mathrm{Cl}^0(Y) \rightarrow \mathrm{Cl}^0(S)$, we get the following result for restrictions to divisors:

Corollary 3.4.3. *Let X be a normal variety of dimension $n \geq 4$ over an algebraically closed field k . Let $\mathcal{L}$ be a very ample line bundle on X such that $\mathcal{L} = \mathcal{L}_0^m$ for some $\mathcal{L}_0$ very ample and $m \geq 2$. Let $|\Lambda| \subset |\mathcal{L}|$ be a general $(n-2)$ -dimensional linear system. Then for a general $Y \in |\Lambda|$, the restriction map*

$$\mathrm{Cl}^0(X) \longrightarrow \mathrm{Cl}^0(Y)$$

is an isomorphism.

Proof. Let $Y \in |\Lambda|$ be general, and let $S \subset Y$ be a general complete intersection surface of members of $|\Lambda|$. The isomorphism $\mathrm{Cl}^0(X) \rightarrow \mathrm{Cl}^0(S)$ factors through $\mathrm{Cl}^0(Y)$, and $\mathrm{Cl}^0(Y) \rightarrow \mathrm{Cl}^0(S)$ is also an isomorphism by Corollary 3.4.2. So $\mathrm{Cl}^0(X) \rightarrow \mathrm{Cl}^0(Y)$ is an isomorphism. $\square$

Combining the above with Corollary 3.2.8 immediately yields injectivity for $\mathrm{Cl}^{\mathrm{ns}}$:

Corollary 3.4.4. *In the setting of §3.4, assume $k \neq \overline{\mathbb{F}_p}$. Then the restriction map*

$$\mathrm{Cl}^{\mathrm{ns}}(X) \longrightarrow \mathrm{Cl}^{\mathrm{ns}}(Y)$$

is injective for infinitely many $Y \in |\Lambda|$.

3.4.2 Surjectivity for Cl

We use the notation of §3.4 and assume moreover that X is defined over an algebraically closed field $\mathbb{K}$ of infinite transcendence degree over the prime subfield. “Very general” will be as in Definition 3.0.16, and $k = k^X$ will denote a field of definition for X . We will lift divisors by extending sections of the relative Jacobian from sections over pairs of lines. We first consider divisors coming from sums of members of $|\mathcal{L}|$ and show that we can lift divisors from their union. Then we will specialize from a very general member of $|\mathcal{L}^2|$ to such a reducible member.

Recall that the homomorphism (3.1)

$$\begin{aligned} \mathrm{Cl}(X') \rightarrow \mathrm{Jac}(C_\eta) &= \{\text{sections of } \mathbf{Jac}(X'/\mathbb{P}^{n-1}) \rightarrow U_{\mathrm{sm}}\} \\ D' &\mapsto (D' - \deg(D'|_{C_\eta})E)|_{C_\eta} \end{aligned}$$

of Lemma 3.2.6 is a surjection with kernel generated by E and $\phi'^* \mathcal{O}_{\mathbb{P}^{n-1}}(d)$ for some $d \in \mathbb{Z}$.

Proposition 3.4.5. *In the setting of §3.4, assume X is defined over $\mathbb{K}$ of infinite transcendence degree. Let $S_1, S_2 \in \mathrm{Gr}(1, |V|)$ be very general complete intersection surfaces of members of $|V|$. Let D_i be a Weil divisor on each S_i such that $D_1|_{S_1 \cap S_2} \sim D_2|_{S_1 \cap S_2}$. Then there is a Weil divisor D on X , uniquely determined up to linear*

equivalence, such that $D|_{S_i} \sim D_i$ for each i . That is, the restriction map

$$\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(S_1) \times_{\mathrm{Pic}(S_1 \cap S_2)} \mathrm{Cl}(S_2)$$

is an isomorphism.

Proof. Let S'_i denote the strict transform of S_i , let L_i be the line such that $S'_i = \phi'^{-1}(L_i)$, and consider the fiber product $\mathrm{Cl}(S'_1) \times_{\mathrm{Pic}(S_1 \cap S_2)} \mathrm{Cl}(S'_2)$, where the restriction maps $\mathrm{Cl}(S'_i) \rightarrow \mathrm{Cl}(S_i) \rightarrow \mathrm{Pic}(S_1 \cap S_2)$ are pushforward by $\pi|_{S'_i}$ composed with the Gysin homomorphisms [Ful98, Chapter 6]. Let D_i be divisors on S_i whose restrictions agree in $\mathrm{Pic}(S_1 \cap S_2)$. For each i let D'_i be the pullback of D_i ; this is defined because the base points of the net are smooth points of S_i by generality.

Then the restrictions $D'_i|_{C_s}$ have the same degree d for general C_s , and by the procedure of Lemma 3.2.6 this defines a section of $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \rightarrow U_{\mathrm{sm}}$ over $(L_1 \cup L_2) \cap U_{\mathrm{sm}}$. By Theorem 3.3.1 [GS13, Theorem 1.3] this extends uniquely to a section σ of $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \rightarrow U_{\mathrm{sm}}$. Then by Lemma 3.2.6 we can find a divisor D' on X' that defines the same sections over L_i as D'_i , and we may choose D' so that $\deg(D'|_{C_\eta}) = d$. Pushing D' forward to X we get a divisor whose restriction to S_i agrees with D_i for each i . $\square$

Next we specialize from a sufficiently general member of $\mathcal{L}^2$ to a reducible member. To do this we will need the following result of Voisin.

Lemma 3.4.6 ([Kol20, Conjecture 15]). *Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree over the prime subfield. Let C be a smooth projective curve over $\mathbb{K}$ and L a very ample line bundle on C . Let $A \subsetneq \mathbf{Pic}^0(C)$ be an abelian subvariety and $[L] \in G \subset \mathrm{Pic}(C)$ a finitely-generated subgroup.*

For a divisor $D \in |L|$ write $\mathrm{Supp} D = \bigcup_i p_i(D)$. Then for very general $D \in |L|$

we have injections

$$\bigoplus_i \mathbb{Z}[p_i(D)] \hookrightarrow \text{Pic}(C)$$

$$\bigoplus_i \mathbb{Z}[p_i(D)] / \sum_i [p_i(D)] \hookrightarrow \text{Pic}(C) / \langle A(k), G \rangle$$

Argument of Voisin. Let k be a field of definition for C . Let $d = \deg L$, let $Z \subset C \times |L|$ be the universal family, and let $U \subset |L|$ be the open subset parametrizing divisors consisting of d distinct points. Let $Z^{(d)} \rightarrow |L|$ be the universal family of divisors in $|L|$ with an ordering of the d (not necessarily distinct) points.

First we show that $Z_U^{(d)}$ is irreducible, or equivalently that the monodromy group of $Z_U \rightarrow U$ is the full symmetric group S_d . It suffices to show that the monodromy group is 2-transitive and contains a transposition. 2-transitivity is equivalent to irreducibility of $Z_U^{(2)}$, which can be identified with the set of pairs $\{(P_1, P_2, D) \mid P_1, P_2 \in \text{Supp } D\} \subset C \times C \times U$. The fiber over a point $(P_1, P_2) \in C \times C$ is the linear system $|L - P_1 - P_2|$, so since L is very ample the dimension of this linear system is independent of the P_i and $Z_U^{(2)} \rightarrow U$ is a projective bundle and thus irreducible. Deforming a section with a double zero gives a transposition.

Next, for $\gamma \in G$ and integers $n_1, \dots, n_d, m \in \mathbb{Z}$ such that $m \neq 0$ and $(n_1, \dots, n_d) \neq (0, \dots, 0)$, define the subset

$$Z(n_1, \dots, n_d, m, \gamma) = \left\{ (P_1, \dots, P_d, \sum_{i=1}^d P_i) \in Z_U^{(d)} \mid m(\gamma - \sum_{i=1}^d n_i [P_i]) \in A \right\}$$

of $Z_U^{(d)} \subset C \times \dots \times C \times |L|$. Then

$$\bigcup_{(n_1, \dots, n_d, m, \gamma)} Z(n_1, \dots, n_d, m, \gamma) \subset Z_U^{(d)}$$

is a countable union of k -subvarieties, and to show that the complement contains $\mathbb{K}$ -points, it suffices to show that each $Z(n_1, \dots, n_d, m, \gamma) \subsetneq Z_U^{(d)}$.

By contradiction, assume that some $Z(n_1, \dots, n_d, m, \gamma) = Z_U^{(d)}$. Then $m(\gamma - \sum_{i=1}^d n_i [P_i]) \in A$ for every $(P_1, \dots, P_d, \sum_{i=1}^d P_i) \in Z_U^{(d)}$, and since the monodromy group is the entire symmetric group S_d , this also holds for any permutation of the P_i . Now the permutation representation on $\mathbb{Q}^d$ is the sum of two irreducible representations: the diagonal and its complement, which is spanned by differences of the basis vectors. So we have that either $n_1 = \dots = n_d$, or that $m(\gamma - n[P_1] + n[P_2]) \in A$ for any $P_1, P_2 \in C$. In the second case we get that $\mathbf{Pic}^0(C)/A$ is torsion since the differences $[P_1] - [P_2]$ generate $\mathbf{Pic}^0$, which is a contradiction. So we are in the first case, i.e. a multiple of $\sum_{i=1}^d P_i$. $\square$

We are now ready to show surjectivity for $\mathcal{L}^2$.

Proposition 3.4.7. *Let X be a normal variety of dimension $n \geq 3$ over an algebraically closed field $\mathbb{K}$ of infinite transcendence degree, and let $\mathcal{L}$ be a very ample line bundle on X such that $\mathcal{L} = \mathcal{L}_0^m$ for a very ample line bundle $\mathcal{L}_0$ and integer $m \geq 2$. Let $|V| \subset |\mathcal{L}|$ be a general $(n-1)$ -dimensional linear system. Then for a very general complete intersection surface T_g of $|\mathcal{L}^2| = |\mathcal{L}_0^{2m}|$ with $n-3$ members of $|V|$, the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T_g)$$

has torsion cokernel.

Proof. Let k be a field of definition for X . Throughout the proof we will use H to refer to members of $|\mathcal{L}|$ and Y to refer to members of $|\mathcal{L}^2|$. S will denote complete intersection surfaces of members of $|\mathcal{L}|$, and T will denote complete intersection surfaces of a member of $|\mathcal{L}^2|$ with $n-3$ members of $|\mathcal{L}|$. So in the case $\dim X = 3$, the H 's are the same as S 's, and the Y 's are the same as T 's.

Let $T_s = (H_{s,0} + H_{s,1}) \cap H_{s,2} \cap \dots \cap H_{s,n-2}$ where the $H_{s,i} \in |V| \subset |\mathcal{L}|$ are very general divisors. Then $T_s = S_0 + S_1$ is a reducible complete intersection surface, given by intersecting a reducible member of $|\mathcal{L}^2|$ with $n-3$ members of $|V|$. Each

$S_i = H_{s,i} \cap H_{s,2} \cap \cdots \cap H_{s,n-2}$ is a normal surface with $S_i \cap X^{\text{reg}}$ is regular by Lemma 3.1.3. The two components meet in a smooth curve $C = S_0 \cap S_1$ contained in X^{reg} .

Let $|\Delta| \subset |\mathcal{L}^2|$ and $|\Lambda_j| \subset |V|$ for $j = 2, \dots, n-2$ be very general pencils such that $H_{s,0} + H_{s,1} \in |\Delta|$ and $H_{s,j} \in |\Lambda_j|$ for each $j = 2, \dots, n-2$. Let $s \in |\Delta| \times |\Lambda_2| \times \cdots \times |\Lambda_{n-2}|$ be the point $(H_{s,0} + H_{s,1}, H_{s,2}, \dots, H_{s,n-2})$ corresponding to T_s .

Now take a very general line in $|\Delta| \times |\Lambda_2| \times \cdots \times |\Lambda_{n-2}|$ containing s , and let ℓ denote its image in the parameter space for complete intersection surfaces of $|\mathcal{L}^2|$ with $n-3$ members of $\mathcal{L}$. Let $\mathcal{T} \rightarrow \ell$ be the universal family of complete intersection surfaces over ℓ . The Chow variety $\text{WDiv}(\mathcal{T}/\ell)$ parametrizing Weil divisors on the fibers of $\mathcal{T} \rightarrow \ell$ [Kol17, 3.21.3] is defined over k and has countably many components not dominating the base ℓ . The image of each is a k -point in ℓ , so a very general $g \in \ell$ will not be contained in the union of their images.

Recall that we are using very general as in Definition 3.0.16. More explicitly, for $i = 0, 1$ let k^{S_i} be a field of definition of S_i , so that S_i is given by a k^{S_i} -point inside the parameter space for complete intersection surfaces of members of $|\mathcal{L}|$, and k^{S_i} contains a field of definition $k^{H_{s,j}}$ for each $H_{s,j}$. Now pick a very general $g \in \ell$ so that for the corresponding complete intersection surface $T_g = Y \cap \cdots \cap H_{g,n-2}$, where $Y \in |\mathcal{L}^2|$ and $H_{g,j} \in |V|$, the coordinates of each $[S_i]$ in the parameter space are algebraically independent over k^{T_g} .

Let D_g be an integral (Weil) divisor on T_g , and consider the component Ω of $\text{WDiv}(\mathcal{T}/\ell)$ containing $[D_g]$. By assumption Ω dominates ℓ . Let $\Gamma \subset \Omega$ be a curve passing through the point $[D_g]$ and dominating ℓ . Then the base change $\text{WDiv}(\mathcal{T}/\ell) \times_{\ell} \Gamma = \text{WDiv}(\mathcal{T} \times_{\ell} \Gamma/\Gamma) \rightarrow \Gamma$ admits a section whose image contains the divisor $[D_g]$ on the fiber $(\mathcal{T} \times_{\ell} \Gamma)_{g_0} = T_g$ for any point $g_0 \in \Gamma$ mapping to g . That is, we have a divisor $\mathcal{D}^\circ$ on $\mathcal{T} \times_{\ell} \Gamma$ whose restriction to the fiber over g_0 is D_g , and its restriction to any fiber is a 1-cycle on the corresponding complete intersection

surface.

Let $s_0 \in \Gamma$ be a point mapping to s , let r be the ramification index of $\Gamma \rightarrow \ell$ at s_0 , and consider the fiber over s_0 . We have $(\mathcal{T} \times_\ell \Gamma)_{s_0} \xrightarrow{\cong} \mathcal{T}_s = S_0 \cup S_1$. Denote the components of this reducible fiber on $\mathcal{T} \times_\ell \Gamma$ by $\tilde{S}_0$ and $\tilde{S}_1$, so that each $\tilde{S}_i \xrightarrow{\cong} S_i$, and let $\{\tilde{P}_i\}$ be the points in $\tilde{S}_0 \cap \tilde{S}_1 \xrightarrow{\cong} C$ corresponding to $\{P_i\} = C \cap Y \in |\mathcal{L}^2|_C|$. We have a diagram

$$\begin{array}{ccccc} \tilde{S}_i & \xrightarrow{\cong} & \alpha(\tilde{S}_i) & \xrightarrow{\cong} & S_i \\ \mathcal{T} \times_\ell \Gamma & \xrightarrow{\alpha} & \mathcal{T} & \xrightarrow{\pi} & \pi(\mathcal{T}) \subset X \\ \downarrow & & \downarrow & \swarrow \text{dashed} & \\ \Gamma & \longrightarrow & \ell & & \end{array}$$

where the projection π of $\mathcal{T} \subset X \times \ell$ onto its first factor is a birational morphism onto its image $\pi(\mathcal{T})$. The rational map $\pi(\mathcal{T}) \dashrightarrow \ell$ is defined away from the reducible curve $T_s \cap T_g = (S_0 + S_1) \cap T_g$, and there is an irreducible exceptional divisor E_i over each curve $S_i \cap T_g$.

The restriction of $\mathcal{D}_{s_0}^\circ$ to each $\tilde{S}_i$ determines a unique Weil divisor class $[D_i^\circ]$ in each $\text{Cl}(\tilde{S}_i)$. By Lemma 3.4.8 we know that on $(\mathcal{T} \times_\ell \Gamma) \setminus \{\tilde{P}_i\}$ every Weil divisor is locally Cartier along $\tilde{S}_1 \cap \tilde{S}_2 \setminus \{\tilde{P}_i\}$ after multiplication by r . So the difference $D_0^\circ|_{\tilde{S}_0 \cap \tilde{S}_1} - D_1^\circ|_{\tilde{S}_0 \cap \tilde{S}_1}$ is a divisor on $\tilde{S}_0 \cap \tilde{S}_1$ supported on $\{\tilde{P}_i\}$. The local computation also shows that the local class group is $\mathbb{Z} = \langle (\tilde{S}_0)_{\tilde{P}_i} \rangle = \langle (-\tilde{S}_1)_{\tilde{P}_i} \rangle$ at each $\tilde{P}_i$.

We now apply Lemma 3.4.6 to the very ample line bundle $\mathcal{L}^2|_C$ on C and to the subgroups $A \subset \text{Pic}^0(C)$ and $G \subset \text{Pic}(C)$ generated by the images of the restriction maps

$$\text{Cl}(\tilde{S}_i) \xrightarrow{(\alpha \circ \pi)_*} \text{Cl}(S_i) \longrightarrow \text{Pic}(C)$$

for $i = 0, 1$. By Lemma 3.2.5 and the Theorem of the Base 3.0.5, these A and G satisfy the hypotheses of Lemma 3.4.6, so we conclude that $(\pi_* \alpha_* D_0^\circ)|_C - (\pi_* \alpha_* D_1^\circ)|_C = a \sum P_i \in |\mathcal{L}^{2a}|_C|$ for some $a \in \mathbb{Z}$.

Now we replace the divisor $\mathcal{D}^\circ$ by on $\mathcal{T} \times_\ell \Gamma$ by $\mathcal{D}^\circ - a\tilde{S}_0$ and consider the divisor

$$\mathcal{D}^\circ - a\tilde{S}_0$$

on $\mathcal{T} \times_\ell \Gamma$. Then

$$(\mathcal{D}^\circ - a\tilde{S}_0)|_{g_0} = \mathcal{D}^\circ|_{g_0} = D_g,$$

and the restriction of $\mathcal{D}^\circ - a\tilde{S}_0$ to each $\tilde{S}_i \subset (\mathcal{T} \times_\ell \Gamma)_{s_0}$ determines a unique Weil divisor class D_i in $\text{Cl}(\tilde{S}_i)$. The divisor $\mathcal{D}^\circ - a\tilde{S}_0$ is $\mathbb{Q}$ -Cartier along C (it becomes trivial in the local class group at each $\tilde{P}_i$), so the restrictions of $\pi_*\alpha_*D_0$ and $\pi_*\alpha_*D_1$ to C agree in $\text{Pic}(C)$.

The divisors $\pi_*\alpha_*D_0$ on S_0 and $\pi_*\alpha_*D_1$ on S_1 satisfy the hypotheses of Proposition 3.4.5, and there is a unique Weil divisor class $[D^X] \in \text{Cl}(X)$ such that $D|_{S_i} \sim \pi_*\alpha_*D_i$ for $i = 0, 1$.

We now have two divisors $D^X|_{T_g}$ and D_g on T_g , and we claim that they are linearly equivalent on T_g .

Let $(\pi \circ \alpha)^*D^X$ denote the divisor $(D^X \times \Gamma)|_{\mathcal{T} \times_\ell \Gamma}$ on the base changed pencil $\mathcal{T} \times_\ell \Gamma$. To show the claim, the idea is that $(\pi \circ \alpha)^*D^X - (\mathcal{D}^\circ - a\tilde{S}_0)$ is a divisor on $\mathcal{T} \times_\ell \Gamma$ whose specialization to the reducible fiber s_0 is trivial, and we would like to show that, up to torsion, its specialization to the very general fiber g_0 is also trivial. Replacing X by the image $\pi(\mathcal{T})$ of the pencil we may assume X is a normal threefold.

Notation:

- $\mathcal{T}_\eta$ denotes the generic fiber of the pencil $\mathcal{T} \rightarrow \ell$. This is a surface over the function field of $\ell = \mathbb{P}^1$.
- $(\mathcal{T} \times_\ell \Gamma)_\xi$ denotes the generic fiber of the base change $\mathcal{T} \times_\ell \Gamma \rightarrow \Gamma$. This is a surface over the function field of Γ . The geometric generic fiber is denoted $(\mathcal{T} \times_\ell \Gamma)_{\bar{\xi}}$.

1. If X is smooth:

The generic fiber $\mathcal{T}_\eta$ is a smooth surface over the function field of $\ell = \mathbb{P}^1$, so $(\mathcal{T} \times_\ell \Gamma)_\xi$ is also smooth over the function field of Γ and the divisor $(\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)$ on $\mathcal{T} \times_\ell \Gamma$ is Cartier on the generic fiber. It's also $\mathbb{Q}$ -Cartier on the reducible fiber $\tilde{S}_0 + \tilde{S}_1$ over s_0 because D^X is Cartier on X and $\mathcal{D}^\circ - a\tilde{S}_0$ was chosen to be $\mathbb{Q}$ -Cartier (with Cartier index r) along $\tilde{S}_0 \cap \tilde{S}_1$. Then $r((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0))_\xi$ is in the kernel of

$$\mathrm{Pic}((\mathcal{T} \times_\ell \Gamma)_\xi) \longrightarrow \mathrm{NS}((\mathcal{T} \times_\ell \Gamma)_\xi) \longrightarrow \mathrm{NS}((\mathcal{T} \times_\ell \Gamma)_{\bar{\xi}}) \xrightarrow{\mathrm{sp}_{s_0}} \mathrm{NS}((\mathcal{T} \times_\ell \Gamma)_{s_0}).$$

The specialization map on Néron-Severi groups is injective up to torsion, so some multiple $lr((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0))_{\bar{\xi}}$ is in $\mathrm{Pic}^0((\mathcal{T} \times_\ell \Gamma)_{\bar{\xi}}) = \mathrm{Cl}^0((\mathcal{T} \times_\ell \Gamma)_{\bar{\xi}})$. Then the specialization to the general fiber g_0 is in $\mathrm{Cl}^0((\mathcal{T} \times_\ell \Gamma)_{g_0})$, i.e. $lr(D^X|_{T_g} - D_g)$ is in $\mathrm{Pic}^0(T_g)$. Since $\mathrm{Cl}^0(X) = \mathrm{Cl}^0(T_g)$ by Corollary 3.4.2 we have $lrD_g \sim (lrD^X - N)|_{T_g}$ for some $N \in \mathrm{Cl}^0(X)$.

2. If X is singular:

- (a) If $\mathrm{char} \mathbb{K} = 0$, let $r: X^+ \rightarrow X$ be a resolution of singularities.
- (b) If $\mathrm{char} \mathbb{K} = p > 0$, we will define a purely inseparable alteration $r: X^+ \rightarrow X$ as follows.

Let $|\Lambda|$ be a general pencil in $|\mathcal{L}_0^m|$, and let $\mathcal{Y} \rightarrow |\Lambda|$ be the total space of the pencil. The generic fiber Y is a geometrically normal variety over the function field K of $|\Lambda| \cong \mathbb{P}_{\mathbb{K}}^1$, and after a finite purely inseparable field extension the base change $Y \otimes_K K^{1/p^e}$ admits a resolution of singularities Y^+ that is smooth over K^{1/p^e} . Spreading out Y^+ and $Y \otimes_K K^{1/p^e}$ over an open subset U^{1/p^e} of $|\Lambda|^{1/p^e}$ we get a smooth family $\mathcal{Y}^+ \rightarrow U^{1/p^e}$ of

surfaces, a family $\mathcal{Y}|_U \times_U U^{1/p^e}$ of normal surfaces, and a diagram

$$\begin{array}{ccccccc}
\mathcal{Y}^+ & \xrightarrow{\text{bir}} & \mathcal{Y}|_U \times_U U^{1/p^e} & \xrightarrow{\text{univ homeo}} & \mathcal{Y}|_U & \xrightarrow[\text{Bl}_{\text{Bs}|\Lambda|}]{\text{bir}} & p_1(\mathcal{Y}|_U) \subset X \\
& & \downarrow & & \downarrow & & \swarrow \\
& & U^{1/p^e} & \xrightarrow{\quad\quad\quad} & U & &
\end{array}$$

where $p_1(\mathcal{Y}|_U)$ is a dense open subset of X whose complement is the union of finitely many divisors $Y_1, \dots, Y_l$ in $|\Lambda|$. By Bertini's theorem (after possibly removing finitely many points from U) the birational morphism $\mathcal{Y}^+ \rightarrow \mathcal{Y}|_U \times_U U^{1/p^e}$ is an isomorphism over the regular locus of X .

Let $\clubsuit: X^\clubsuit \rightarrow X$ be the normalization of X in K^{1/p^e} . Then $\clubsuit^{-1}(X^{\text{reg}}) \rightarrow X^{\text{reg}}$ is a purely inseparable cover of $X^{\text{reg}} \setminus \text{Bs}|\Lambda|$ that agrees with $\mathcal{Y}^+$ over the intersection $X^{\text{reg}} \cap p_1(\mathcal{Y}|_U) = X^{\text{reg}} \setminus (X^{\text{reg}} \cap (Y_1 \cup \dots \cup Y_l))$, so we may glue $\clubsuit^{-1}(X^{\text{reg}})$ to $\mathcal{Y}^+$ to get a $\mathbb{Q}$ -factorial variety with a morphism

$$\mathcal{Y}^+ \cup \clubsuit^{-1}(X^{\text{reg}}) \longrightarrow p_1(\mathcal{Y}|_U) \cup (X^{\text{reg}})$$

that factors as the composition of a purely inseparable cover and a birational morphism. Note that $p_1(\mathcal{Y}|_U) \cup (X^{\text{reg}}) = X \setminus (\text{Sing}(X) \cap (Y_1 \cup \dots \cup Y_l))$ is an open subset of X whose complement consists of finitely many points $Q_1, \dots, Q_r$.

Let X^+ be a projective closure of $\mathcal{Y}^+ \cup \clubsuit^{-1}(X^{\text{reg}} \setminus \text{Bs}|\Lambda|)$. Over $X \setminus \{Q_1, \dots, Q_r\}$ the morphism $r: X^+ \rightarrow X$ agrees with $\mathcal{Y}^+ \cup \clubsuit^{-1}(X^{\text{reg}}) \rightarrow X$, so X^+ has the following properties:

- $r^{-1}(X \setminus \{Q_1, \dots, Q_r\})$ is $\mathbb{Q}$ -factorial,
- $r: X^+ \rightarrow X$ is finite and purely inseparable over X^{reg} , and
- the pullback of a general member of $|\Lambda|$ to X^+ is smooth except possibly over $\text{Bs}|\Lambda|$ (and in particular is smooth over the singular locus

of X).

In particular $(S_0 + S_1) \times_X X^+$ is smooth away from $(S_0 \cap S_1) \times_X X^+ = \text{Bs}|\Lambda| \times_X X^+$. Moreover, the pullback of a general member of $|\mathcal{L}_0^m|$ is also smooth along the exceptional divisors of $r: X^+ \rightarrow X$, and this holds for the sum of a general members of $|\mathcal{L}_0^m|$ as well, so by openness of smoothness

- the pullback of a general member of $|\mathcal{L}_0^{bm}|$ is smooth along the exceptional divisors of $r: X^+ \rightarrow X$ for any $b \in \mathbb{Z}_{\geq 1}$. In particular the pullback is $\mathbb{Q}$ -factorial.

Now let X^+ be as defined above. We have a divisor $(\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)$ on $\mathcal{T} \times_\ell \Gamma$, and we now claim that the pullback $((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)) \times_X X^+$ to $\mathcal{T} \times_\ell \Gamma \times_X X^+$ is $\mathbb{Q}$ -Cartier near the reducible fiber $(\tilde{S}_0 + \tilde{S}_1) \times_X X^+$.

- $(\pi \circ \alpha)^* D^X \times_X X^+$ is $\mathbb{Q}$ -Cartier near $(\tilde{S}_0 + \tilde{S}_1) \times_X X^+$. If $\text{char } \mathbb{K} = 0$ this is because X^+ is smooth. If $\text{char } \mathbb{K} = p$ this is because X^+ is $\mathbb{Q}$ -factorial except over $\{Q_j\}$, and S_0, S_1 do not contain any of the points Q_j by generality.
- The divisor $\mathcal{D}^\circ - a\tilde{S}_0$ on $\mathcal{T} \times_\ell \Gamma$ is $\mathbb{Q}$ -Cartier along $C = \tilde{S}_0 \cap \tilde{S}_1$, and the base change $(S_0 + S_1) \times_X X^+ \rightarrow S_0 + S_1$ resolves the singularities of $(S_0 + S_1) \setminus (S_0 \cap S_1)$. Furthermore, a general member of $|\mathcal{L}_0^{2m}|$ will become $\mathbb{Q}$ -factorial after base change $\times_X X^+$ since smoothness is an open condition. So after base changing the family, the divisor $(\mathcal{D}^\circ - a\tilde{S}_0) \times_X X^+$ is $\mathbb{Q}$ -Cartier along $(\tilde{S}_0 + \tilde{S}_1) \times_X X^+$.

Then $((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)) \times_X X^+$ is also $\mathbb{Q}$ -Cartier on the geometric generic fiber of $\mathcal{T} \times_\ell \Gamma \times_X X^+ \rightarrow \Gamma$. The exceptional divisors of $(\tilde{S}_0 + \tilde{S}_1) \times_X X^+ \rightarrow \tilde{S}_0 + \tilde{S}_1$ are specializations of the exceptional divisors of $(\mathcal{T} \times_\ell \Gamma \times_X X^+)_{\bar{\xi}} \rightarrow (\mathcal{T} \times_\ell \Gamma)_{\bar{\xi}}$, and so there is some (Cartier) divisor $\mathcal{E}$ on $(\mathcal{T} \times_\ell \Gamma \times_X X^+)_{\bar{\xi}}$ with

exceptional support such that the divisor $((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)) \times_X X^+ \big|_{\bar{\mathcal{E}}} + \mathcal{E}$ becomes trivial after specializing to $(\tilde{S}_0 + \tilde{S}_1) \times_X X^+$. Since intersection numbers specialize, then $((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)) \times_X X^+ \big|_{\bar{\mathcal{E}}} + \mathcal{E}$ is numerically trivial, so some multiple $l((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)) \times_X X^+ + \mathcal{E} \big|_{\tilde{T}_g \times_X X^+}$ is algebraically trivial on the very general fiber. Pushing forward again, we have that

$$\pi_* \alpha_* (lr((\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ - a\tilde{S}_0)) \big|_{\tilde{T}_g}) = D^X|_{T_g} - D_g \in \text{Cl}^0(T_g).$$

Since $\text{Cl}^0(X) \cong \text{Cl}^0(T_g)$ by generality, we may find $N \in \text{Cl}^0(X)$ such that $lrD_g \sim (lrD^X - N)|_{T_g}$.

□

Remark. In the above proof, to conclude that $D^X|_{T_g} \sim D_g$ on T_g , one might try to apply Lemma 3.2.7 to a component of the intersection $T_g \cap (S_0 + S_1)$. However, if $\Gamma \rightarrow \ell = \mathbb{P}^1$ is a nontrivial cover then it need not be the case a priori that the restrictions of $D_g = (\mathcal{D}^\circ - a\tilde{S}_0)_{g_0}$ and $(\mathcal{D}^\circ - a\tilde{S}_0)_{s_0}$ agree in $T_g \cap (S_0 + S_1)$, since the curves on $\mathcal{T} \times_\ell \Gamma$ are connected by a $(T_g \cap (S_0 + S_1)) \times \Gamma$.

Lemma 3.4.8. *In the proof of Proposition 3.4.7, $\mathcal{T} \times_\ell \Gamma$ is locally $\mathbb{Q}$ -factorial over $C \setminus \{P_i\}$ with Cartier index equal to the ramification index r of $\Gamma \rightarrow \ell$ at s_0 .*

Proof. Note that this computation applies for any collection of Cartier divisors $H_{s,0} \in |\mathcal{L}_0|$, $H_{s,1} \in |\mathcal{L}_1|$, $Y \in |\mathcal{L}_0 \otimes \mathcal{L}_1|$ and $H_{s,j}, H_{g,j} \in |\mathcal{L}_j|$ for $j \geq 2$ as long as each one is smooth at x .

Let $x \in C = S_0 \cap S_1 \subset X$ be a closed point. Then $x \in X^{\text{reg}}$ and $H_{s,j}^{\text{reg}}$ for each $j = 0, \dots, n-2$, so we may choose local parameters $y_1, \dots, y_n$ at x such that $H_{s,j}$ is

locally defined by y_{j+1} at x . Then

$$\begin{aligned}\widehat{\mathcal{O}}_{S_0,x} &\cong k[[y_1, \dots, y_n]]/(y_1, y_3, \dots, y_{n-1}) \cong k[[y_2, y_n]], \\ \widehat{\mathcal{O}}_{S_1,x} &\cong k[[y_1, \dots, y_n]]/(y_2, y_3, \dots, y_{n-1}) \cong k[[y_1, y_n]], \\ \widehat{\mathcal{O}}_{T_s,x} &\cong k[[y_1, \dots, y_n]]/(y_1 y_2, y_3, \dots, y_{n-1}) \cong k[[y_1, y_2, y_n]]/(y_1 y_2), \\ \widehat{\mathcal{O}}_{C,x} &\cong k[[y_n]].\end{aligned}$$

Let q be a local equation for Y at x , and for $j = 2, \dots, n-2$ let f_j be a local equation for $H_{g,j}$ at x . Then $\widehat{\mathcal{O}}_{T_g,x} \cong k[[y_1, \dots, y_n]]/(q f_2 \cdots f_{n-2})$ and the total space of the family is locally

$$\begin{aligned}\widehat{\mathcal{O}}_{\mathcal{T},x} &\cong k[[y_1, \dots, y_n, s]]/(y_1 y_2 - sq, y_3 - s f_2, y_4 - s f_3, \dots, y_{n-1} - s f_{n-2}) \\ &\cong k[[y_1, y_2, y_n]]/(y_1 y_2 - sq)\end{aligned}$$

where s is a local parameter for $\mathcal{O}_{\ell,s}$. We have that $x \in Y$ if and only if q is in the maximal ideal of $k[[y_1, \dots, y_n]]$, i.e. not a unit.

Let t be a local parameter at $\mathcal{O}_{\Gamma,s_0}$. Then at a point x_0 in $\mathcal{T} \times_{\ell} \Gamma$ mapping to x

$$\widehat{\mathcal{O}}_{\mathcal{T} \times_{\ell} \Gamma, x_0} \cong k[[y_1, y_2, y_n, t]]/(y_1 y_2 - u t^r q)$$

for some unit u .

If $x \notin Y$, then $\widehat{\mathcal{O}}_{\mathcal{T} \times_{\ell} \Gamma, x_0} \cong k[[y_1, y_2, y_n, t]]/((uq)^{-1} y_1 y_2 - t^r)$ is an A_{r-1} -singularity, so its class group is $\mathbb{Z}/r\mathbb{Z}$.

If $x \in Y$ then q is in the maximal ideal of $k[[y_1, y_2, y_n]]$. Since $Y \in |\mathcal{L}^2|$ is very general, x is not in its singular locus and Y not cut out by any of the same equations as T_s , so we may assume that our coordinates were chosen so that $q = y_n$. Then $\widehat{\mathcal{O}}_{\mathcal{T} \times_{\ell} \Gamma, x_0} \cong k[[y_1, y_2, y_n, t]]/(y_1 y_2 - u t^r y_n) \cong k[[y_1, y_2, y_n, t]]/(y_1 y_2 - (u y_n) t^r)$. This ring has trivial Picard group and its class group is $\mathbb{Z} = \langle (y_1 = t = 0) \rangle$. $\square$

Noether–Lefschetz for even multiples ≥ 4 immediately follows from Corollary 3.4.2, Corollary 3.4.4, and Proposition 3.4.7:

Corollary 3.4.9. *Let X be a normal threefold over an algebraically closed field $\mathbb{K}$ of infinite transcendence degree and $\mathcal{L}_0$ a very ample line bundle on X . Then for any $m \geq 2$, the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T)$$

to a very general divisor $T \in |\mathcal{L}_0^{2m}|$ is injective and is an isomorphism up to torsion.

In higher dimensions, we can factor the restriction map from X to a complete intersection surface in $|\mathcal{L}^2| \cap |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$ through a divisor in $|\mathcal{L}|$ to obtain the following result for all multiples ≥ 2 :

Corollary 3.4.10. *Let X be a normal variety of dimension $n \geq 4$ over $\mathbb{K}$, $\mathcal{L}_0$ a very ample line bundle on X , and $m \geq 2$ an integer. For a very general $H \in |\mathcal{L}_0^m|$, the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(H)$$

is an isomorphism.

Proof. Let T be a complete intersection surface obtained as the intersection of H with a very general member of $|\mathcal{L}_0^{2m}|$ and $n - 4$ very general members of $|\mathcal{L}_0^m|$. Then the result follows since the restriction map $\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(T)$ factors

$$\mathrm{Cl}(X) \xrightarrow{\text{Prop 3.2.1 + 3.4.7}} \mathrm{Cl}(H) \xrightarrow[\text{Prop 3.2.1 + 3.4.7}]{\text{Prop 3.2.1 + 3.4.7}} \mathrm{Cl}(T) .$$

□

Next we show surjectivity for odd multiples when $\dim X = 3$. The idea will be to specialize to a reducible member $S_{2A} + S_B$ where Noether–Lefschetz holds for S_{2A} . The proof is similar to that of Proposition 3.4.7. The argument also works if

$\dim X \geq 4$ by considering complete intersection surfaces, but it is clearer to state in the $\dim X = 3$ case, and the result for higher dimensions can also be obtained by first cutting down to the threefold case.

Proposition 3.4.11. *Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree and X a normal projective threefold over $\mathbb{K}$. Let $\mathcal{L}_A$ be a very ample line bundle on X such that $\mathcal{L}_A = \mathcal{L}_{A,0}^{m_A}$ for some very ample line bundle $\mathcal{L}_{A,0}$ and integer $m_A \geq 2$. Let $\mathcal{L}_B = \mathcal{L}_{B,0}^{m_B}$ where $\mathcal{L}_{B,0}$ is an ample and basepoint-free line bundle and $m_B \geq 2$. Assume that one of the following holds:*

1. X is smooth,
2. $\text{char } \mathbb{K} = 0$, or
3. $\mathcal{L}_A^{\otimes 2} \otimes \mathcal{L}_B \cong \mathcal{M}^m$ for some very ample line bundle $\mathcal{M}$.

Then for a very general $T_{2A+B} \in |\mathcal{L}_A^{\otimes 2} \otimes \mathcal{L}_B|$, the restriction map

$$\text{Cl}(X) \longrightarrow \text{Cl}(T_{2A+B})$$

is surjective up to torsion.

Proof. Let k be a field of definition for X . Let $S_{2A} \in |\mathcal{L}_A^2|$ and $S_B \in |\mathcal{L}_B|$ be very general. Then $S_{2A} + S_B$ is a reducible member of $|\mathcal{L}_A^{\otimes 2} \otimes \mathcal{L}_B|$. Let $\ell \subset |\mathcal{L}_A^{\otimes 2} \otimes \mathcal{L}_B|$ be a very general pencil containing $S_{2A} + S_B$, let T_{2A+B} be a very general member of ℓ , and let s, g denote the points on ℓ corresponding to $S_{2A} + S_B$ and T_{2A+B} . Since T_{2A+B} is very general, g is not in the image of any component of the Chow variety $\text{WDiv}(\mathcal{T}/\ell)$ not dominating ℓ .

Let D_g be an integral divisor on T_{2A+B} . As in the proof of Proposition 3.4.7, $\text{WDiv}(\mathcal{T}/\ell) \times_{\ell} \Gamma \rightarrow \Gamma$ admits a section after taking some finite base change $\Gamma \rightarrow \ell$. So we have a divisor $\mathcal{D}^\circ$ on $\mathcal{T} \times_{\ell} \Gamma$ that restricts to D_g on the fiber over a point $g_0 \in \Gamma$ mapping to g .

Let $s_0 \in \Gamma$ be a point mapping to s . Then $(\mathcal{T} \times_\ell \Gamma)_{s_0} \xrightarrow{\cong} \mathcal{T}_s = S_{2A} + S_B$, and we denote the components of the reducible fiber $(\mathcal{T} \times_\ell \Gamma)_{s_0}$ by $\tilde{S}_{2A}$ and $\tilde{S}_B$. As in the proof of Proposition 3.4.7, there is an integer a such that $\mathcal{D}^\circ + \tilde{S}_B$ is $\mathbb{Q}$ -Cartier along $\tilde{S}_{2A} \cap \tilde{S}_B$, and so we consider the divisor $\mathcal{D}^\circ + \tilde{S}_B$ instead. This defines a unique Weil divisor class $D_{2A} \in \text{Cl}(\tilde{S}_{2A})$.

By Proposition 3.2.1 and Proposition 3.4.7 the restriction $\text{Cl}(X) \rightarrow \text{Cl}(S_{2A})$ is an isomorphism, so there is a unique divisor class D^X on X such that $D|_{S_{2A}} \sim \pi_* \alpha_* D_{2A}$. Let $(\pi \circ \alpha)^* D^X$ denote $(D^X \times \Gamma)|_{\mathcal{T} \times_\ell \Gamma}$.

We have a divisor $(\pi \circ \alpha)^* D^X - (\mathcal{D}^\circ + a\tilde{S}_B)$ on $\mathcal{T} \times_\ell \Gamma$ that is trivial on the reducible fiber $\tilde{S}_{2A} + \tilde{S}_B$. If X is smooth or if $\text{char } \mathbb{K} = 0$, then the same specialization argument in Proposition 3.4.7 implies a multiple of this divisor is trivial when restricted to $\tilde{T}_{2A+B}$, and using the same argument we conclude that $\text{Cl}(X) \rightarrow \text{Cl}(T_{2A+B})$ is surjective up to torsion.

If X is singular and $\text{char } \mathbb{K} = p > 0$, construct a purely inseparable alteration $X^+ \rightarrow X$ as in Proposition 3.4.7 using a general pencil in the very ample linear system $|\mathcal{M}|$. Then $((S_{2A} + S_B) \setminus (S_{2A} \cap S_B)) \times_X X^+$ and $T_{2A+B} \times_X X^+$ are $\mathbb{Q}$ -factorial, and so the same argument as in Proposition 3.4.7 works.

□

In particular we obtain

Corollary 3.4.12. *Let X be a normal threefold over $\mathbb{K}$ and $\mathcal{L}_0$ a very ample line bundle on X . Then for any integer $d \in \{4\} \cup \mathbb{Z}_{\geq 6}$ the restriction map*

$$\text{Cl}(X) \longrightarrow \text{Cl}(T)$$

is surjective for very general $T \in |\mathcal{L}_0^d|$.

In particular we recover the statement for $X = \mathbb{P}_{\mathbb{C}}^3$ except for the case of degree 5 surfaces.

Chapter 4

Another approach to injectivity using alterations

In this chapter we give another argument for showing injectivity up to p -power torsion for the restriction map on class groups. Following ideas of Ravindra and Srinivas [RS06], we pass to a regular alteration $\tilde{X} \rightarrow X$ and show a result for $\tilde{Y} \hookrightarrow \tilde{X}$ on the alteration. This is the main result of this chapter:

Proposition 4.0.1 (Corollary 4.2.5). *Let X be a normal variety of dimension $n \geq 3$ over an algebraically closed field k of characteristic $p \geq 0$, $\mathcal{L}_0$ a very ample line bundle on X , and $m \geq 2$ an integer. Then for a general $Y \in |\mathcal{L}|$, the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

has p -power torsion kernel if $p > 0$, and is injective if $p = 0$.

The argument is a completely different flavor from the previous chapter. It follows the general theme of adapting characteristic 0 arguments to characteristic p using alterations. We note that it may be possible to adapt the surjectivity arguments of [RS06] using [KW21], but we have not tried to do this.

The starting point of this part is Section 3.1 from the previous chapter. First we fix some notation.

Let X be a normal variety over a field k and $\mathcal{L}$ a globally generated line bundle on X . The universal family of $\mathcal{L}$ is the subvariety

$$\mathcal{Y} = \{(x, Y_s) \in X \times |\mathcal{L}| \mid x \in Y_s\} \subset X \times |\mathcal{L}| = \mathbb{P}(H^0(X, \mathcal{L})^\vee \otimes \mathcal{O}_X)$$

where $|\mathcal{L}| = \mathbb{P}(H^0(X, \mathcal{L})^\vee)$ denotes the parameter space of the complete linear system of $\mathcal{L}$. The kernel $\mathcal{V}$ of the evaluation map $H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \rightarrow \mathcal{L}$ is a locally free sheaf, and $\mathcal{Y}$ has the structure of a projective bundle $\mathcal{Y} = \mathbb{P}_X(\mathcal{V}^\vee) \rightarrow X$ via the first projection. The dual $\mathcal{O}_{\mathbb{P}_X(\mathcal{V}^\vee)}(1)$ of the tautological bundle agrees with the pullback of $\mathcal{O}_{|\mathcal{L}|}(1)$ from the second projection, since locally over X they are both defined by a basis of $H^0(X, \mathcal{L})^\vee$, so $\text{Pic } \mathcal{Y} \cong \text{Pic } X \oplus \text{Pic } |\mathcal{L}|$ and the restriction map $\text{Pic}(X \times |\mathcal{L}|) \rightarrow \text{Pic } \mathcal{Y}$ is an isomorphism.

If $\text{char } k = p > 0$, a general $Y_s \in |\mathcal{L}|$ need not be normal or even reduced, but the normalization of its reduced subscheme is a universal homeomorphism (of finite height); that is, the normalization factors through some power of Frobenius on Y_s . This is because Y_s^ν is obtained by taking enough Frobenius base changes of $\mathcal{Y}$ over $|\mathcal{L}|$ so that the generic fiber of $(\mathcal{Y} \times_{|\mathcal{L}|} |\mathcal{L}|^{1/p^e})^\nu$ is geometrically normal and taking the fiber over $s \in |\mathcal{L}|^{1/p^e}$; this is a universal homeomorphism [Sta20, Tag 0C3N].

4.1 The map $\text{NS}(\tilde{X}) \rightarrow \text{Cl}^{\text{ns}}(\tilde{Y}^\nu)$

The following hypotheses on our line bundle are the same as those from Lemma 3.1.6 of the previous part of this thesis.

Corollary 4.1.1. *Let X be a normal variety of dimension $n \geq 2$ over $k = \bar{k}$, let $\mathcal{L}$ be a very ample line bundle on X , and let*

1. $m \geq 2$ if the pair $(X, \mathcal{L})$ is not $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, or

2. $m \geq 3$.

Let $\pi: \tilde{X} \rightarrow X$ be a surjective proper morphism from a normal variety of dimension n , and set $\mathcal{M} = \pi^* \mathcal{L}$. Let $\mathcal{Y} \subset X \times S$ be the universal family of $|\mathcal{L}^m|$ with projections $p_1: \mathcal{Y} \rightarrow X$ and $p_2: \mathcal{Y} \rightarrow S$, let Y_η be the generic fiber of p_2 , let $\tilde{\mathcal{Y}} \subset \tilde{X} \times S$ be the universal family of $\pi^* |\mathcal{L}^m| \subset |\mathcal{M}^m|$ with projections $q_1: \tilde{\mathcal{Y}} \rightarrow \tilde{X}$ and $q_2: \tilde{\mathcal{Y}} \rightarrow S$, and let $\tilde{Y}_\eta$ be the generic fiber of q_2 . Then the following hold:

1. The sequence

$$0 \longrightarrow I_{\mathcal{Y}} + \text{Cl } S \longrightarrow \text{Cl } \tilde{\mathcal{Y}} \longrightarrow \text{Cl } \tilde{Y}_\eta \longrightarrow 0$$

where $I_{\mathcal{Y}}$ is the subgroup of $\text{Cl } \tilde{\mathcal{Y}}$ generated by $\{\tilde{\mathcal{Y}} \cap (E \times S) \mid E \in \text{Cl } \tilde{X} \text{ and } \dim \pi(E) = 0\}$.

2. The restriction map on generic fibers

$$\text{Pic}(\tilde{X})_\eta \longrightarrow \text{Pic}(\tilde{Y})_\eta$$

is surjective. If $\tilde{X}$ is locally factorial, the kernel is generated by irreducible exceptional divisors E on $\tilde{X}$ with 0-dimensional image under π .

Proof. Denote the induced morphism $\psi: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$. We have a commutative diagram

$$\begin{array}{ccccccc} & & \text{Cl } S & \longrightarrow & \text{Cl } \tilde{\mathcal{Y}} & \longrightarrow & \text{Cl } \tilde{Y}_\eta \\ & & \parallel & & \downarrow \psi_* & & \downarrow \\ 0 & \longrightarrow & \text{Cl } S & \longrightarrow & \text{Cl } \mathcal{Y} & \longrightarrow & \text{Cl } Y_\eta \longrightarrow 0 \end{array}$$

where the vertical maps are proper pushforwards of cycles. We first show that the bottom row is exact.

For left exactness, we have injections $\mathrm{Cl} S = \mathrm{Pic} S \hookrightarrow \mathrm{Pic} \mathcal{Y} \hookrightarrow \mathrm{Cl} \mathcal{Y}$ since S is projective space and $\mathcal{Y}$ is a projective bundle over the normal variety X . Right exactness comes from taking the closure of any divisor on Y_η . For middle exactness, let $\mathcal{D}$ be an integral divisor representing a class in the kernel of $\mathrm{Cl} \mathcal{Y} \rightarrow \mathrm{Cl}(Y)_\eta$. Then $p_2(\mathcal{D})$ is a proper subvariety of S , and if $\dim S = M$ then the general fibers of $p_2|_{\mathcal{D}}$ have dimension $\dim \mathcal{D} - \dim p_2(\mathcal{D}) = n + M - 2 - (M - \mathrm{codim}_S p_2(\mathcal{D})) = n - 2 + \mathrm{codim}_S p_2(\mathcal{D}) \leq n - 1$. So $\mathrm{codim}_S p_2(\mathcal{D}) = 1$ and $p_2(\mathcal{D})$ is a divisor on S . We have $p_2^{-1}(p_2(\mathcal{D})) = a\mathcal{D} + \mathcal{D}'$ for some effective divisor $\mathcal{D}'$ not supported on $\mathcal{D}$. Now $\mathcal{D}' = 0$ by Lemma 3.1.6, so $a\mathcal{D}$ pulls back from S . Since S is projective space and the pullback of a hyperplane on S is an integral divisor on $\mathcal{Y}$, then $\mathcal{D}$ pulls back from S as well. So the bottom row is exact.

Now let $\tilde{\mathcal{D}}$ represent a class in the kernel of $\mathrm{Cl} \tilde{\mathcal{Y}} \rightarrow \mathrm{Cl} \tilde{Y}_\eta$. If $\tilde{\mathcal{D}}$ is not in the image of $\mathrm{Cl} S$, then it's in the kernel of $\mathrm{Cl} \tilde{\mathcal{Y}} \rightarrow \mathrm{Cl} \mathcal{Y}$. Consider the following diagram, where the vertical arrows are proper pushforwards and the horizontal arrows are flat pullbacks:

$$\begin{array}{ccc} \mathrm{Cl} \tilde{X} & \xrightarrow{q_1^*} & \mathrm{Cl} \tilde{\mathcal{Y}} \\ \downarrow \pi_* & & \downarrow \psi_* \\ \mathrm{Cl} X & \xrightarrow{p_1^*} & \mathrm{Cl} \mathcal{Y} \end{array}$$

By dimension counting $\mathrm{codim}_{\tilde{X}} q_1(\tilde{\mathcal{D}}) \leq 1$, and if $q_1(\tilde{\mathcal{D}}) = \tilde{X}$ then $p_1(\psi(\tilde{\mathcal{D}})) = \pi(q_1(\tilde{\mathcal{D}})) = X$ and so $\tilde{\mathcal{D}}$ cannot be in the kernel of $\mathrm{Cl} \tilde{\mathcal{Y}} \rightarrow \mathrm{Cl} \mathcal{Y}$. So $q_1(\tilde{\mathcal{D}})$ is a divisor on $\tilde{X}$, and $\tilde{\mathcal{D}} = q_1^*(q_1(\tilde{\mathcal{D}}))$ since the latter is a projective bundle over $q_1(\tilde{\mathcal{D}})$ and hence irreducible.

Since $q_1(\tilde{\mathcal{D}})$ is in the kernel of π_* we have that $\pi(q_1(\tilde{\mathcal{D}}))$ has codimension ≥ 2 in X . By assumption $q_1(\tilde{\mathcal{D}})$ does not meet $\tilde{Y}_s$ for a general $s \in S$ and so $\pi(q_1(\tilde{\mathcal{D}}))$ does not meet a general Y_s . This implies $\pi(q_1(\tilde{\mathcal{D}}))$ is a point, and so $\tilde{\mathcal{D}} \in I_{\mathcal{Y}}$. Conversely, any divisor on $\tilde{X}$ mapping to a point in X defines a class in the kernel of $\mathrm{Cl} \tilde{\mathcal{Y}} \rightarrow \mathrm{Cl} \tilde{Y}_\eta$, so (1) is shown.

For (2), applying the snake lemma to the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Pic} S & \longrightarrow & \mathrm{Pic}(\tilde{X} \times S) = \mathrm{Pic}(\tilde{X}) \oplus \mathbb{Z} & \longrightarrow & \mathrm{Pic}(\tilde{X})_\eta = \mathrm{Pic}(\tilde{X}) \longrightarrow 0 \\
& & \downarrow & & \parallel & & \downarrow \\
0 & \longrightarrow & \mathrm{Pic} \tilde{\mathcal{Y}} \cap (I_{\mathcal{Y}} + \mathrm{Pic} S) & \longrightarrow & \mathrm{Pic} \tilde{\mathcal{Y}} = \mathrm{Pic}(\tilde{X}) \oplus \mathbb{Z} & \longrightarrow & \mathrm{Pic}(\tilde{Y})_\eta \longrightarrow 0
\end{array}$$

yields that $\mathrm{Pic}(\tilde{X})_\eta \rightarrow \mathrm{Pic}(\tilde{Y})_\eta$ is surjective, with kernel isomorphic to $(\mathrm{Pic} \tilde{\mathcal{Y}} \cap (I_{\mathcal{Y}} + \mathrm{Pic} S)) / \mathrm{Pic} S$. If $\tilde{X}$ is locally factorial, this quotient is $I_{\mathcal{Y}}$. $\square$

Proposition 4.1.2. *In the situation of Corollary 4.1.1, if $\tilde{X}$ is locally factorial then, up to torsion, for a general $\tilde{Y} \in \pi^*|\mathcal{L}^m|$, the kernel of $\mathrm{NS}(\tilde{X}) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(\tilde{Y}^\nu)$ is generated by the irreducible exceptional divisors on $\tilde{X}$ with 0-dimensional image under π .*

Proof. $\tilde{Y}_\eta$ is a regular but not necessarily smooth or geometrically normal variety over the function field K of S . There exists an integer f such that the normalization of the base change $(\tilde{Y}_\eta) \otimes_K K^{1/p^f}$ is geometrically normal over K^{1/p^f} and that admits a regular alteration $Y' \rightarrow ((\tilde{Y}_\eta) \otimes_K K^{1/p^f})^\nu$ with Y' smooth over K^{1/p^f} . Spreading Y' out to a smooth family $\mathcal{Y}'_U$ over some open subset $U \subset S^{1/p^f}$, we get the following diagram:

$$\begin{array}{ccccccccc}
\mathcal{Y}'_U & \longrightarrow & (\tilde{\mathcal{Y}} \times_S U)^\nu & \longrightarrow & \tilde{\mathcal{Y}} \times_S S^{1/p^f} & \longrightarrow & \tilde{\mathcal{Y}} \hookrightarrow & \tilde{X} \times S & \longrightarrow & \tilde{X} \\
& & \downarrow & & \downarrow & & \downarrow & \swarrow & & \\
& & U & \longrightarrow & S^{1/p^f} & \longrightarrow & S & & &
\end{array}$$

On generic fibers this becomes

$$\begin{array}{ccccccccc}
Y' & \longrightarrow & (\tilde{Y}_\eta \otimes_K K^{1/p^f})^\nu & \longrightarrow & \tilde{Y}_\eta \otimes_K K^{1/p^f} & \longrightarrow & \tilde{Y}_\eta \hookrightarrow & \tilde{X} \otimes_k K & \longrightarrow & \tilde{X} \\
& & \downarrow & & \downarrow & & \downarrow & \swarrow & & \\
& & \mathrm{Spec} K^{1/p^f} & \longrightarrow & \mathrm{Spec} K^{1/p^f} & \longrightarrow & \mathrm{Spec} K & & &
\end{array} \tag{4.1}$$

Let $t \in S$ be a general point, so that it is contained in the image of U . By abuse of

notation, since Frobenius is a universal homeomorphism we also denote its preimage $t \in S^{1/p^f}$. We get a diagram of closed fibers

$$\begin{array}{ccccccccc}
 Y'_t & \longrightarrow & \tilde{Y}_t^\nu & \longrightarrow & \tilde{Y}_t & \longrightarrow & \tilde{Y}_t^\subset & \longrightarrow & \tilde{X} \otimes_k \kappa(t) & \longrightarrow & \tilde{X} \\
 & \searrow & \downarrow & & \downarrow & & \downarrow & & \swarrow & & \\
 & & \text{Spec } \kappa(t) & \longrightarrow & \text{Spec } \kappa(t) & \longrightarrow & \text{Spec } \kappa(t) & & & &
 \end{array} \quad (4.2)$$

There is a specialization map $\text{Pic}(Y' \otimes_{K^{1/p^f}} \bar{K}) \rightarrow \text{Pic}(Y'_t)$ defined by choosing a DVR R with closed point mapping to t and generic point mapping to the generic point of U and taking the direct limit of the diagram

$$\text{Pic}(\mathcal{Y}'_U \times_U \text{Spec } L) \xleftarrow{\cong} \text{Pic}(\mathcal{Y}'_U \times_U \text{Spec } R'_L) \longrightarrow \text{Pic}(Y'_t)$$

over finite extensions L of K^{1/p^f} , where R'_L is the DVR obtained by localizing the integral closure of R in L (see [MP12, Proposition 3.3] for more details).

The pullback along (4.1) gives a map

$$\text{Pic}(\tilde{X})^\subset \longrightarrow \text{Pic}(\tilde{X} \otimes_k K) \longrightarrow \text{Pic}(\tilde{Y}_\eta) \longrightarrow \text{Pic}(Y')^\subset \longrightarrow \text{Pic}(Y' \otimes_{K^{1/p^f}} \bar{K}) \longrightarrow \text{Pic}(Y'_t) \quad (4.3)$$

that agrees with the pullback via $Y'_t \rightarrow \tilde{Y}_t \hookrightarrow \tilde{X}$ by how the specialization map $\text{Pic}(Y' \otimes_{K^{1/p^f}} \bar{K}) \rightarrow \text{Pic}(Y'_t)$ is defined. We also have $\text{Pic}(Y'_t) \rightarrow \text{Cl}(\tilde{Y}_t^\nu)$ from pushing forward along the regular alteration $Y'_t \rightarrow \tilde{Y}_t^\nu$. Let $Y'_t \xrightarrow{\rho_t} Z_t \xrightarrow{\theta_t} \tilde{Y}_t^\nu$ be the Stein factorization.

We claim that the composition of (4.3) with $(\rho_t \circ \theta_t)_*$ agrees with the map

$$\text{Pic}(\tilde{X}) \longrightarrow \text{Pic}(\tilde{Y}_t) \xrightarrow{i^*} \text{Pic}(\tilde{Y}_t^\nu)^\subset \xrightarrow{\nu^*} \text{Cl}(\tilde{Y}_t^\nu) \xrightarrow{\cdot \deg(\theta_t)} \text{Cl}(\tilde{Y}_t^\nu)$$

or equivalently that the diagram

$$\begin{array}{ccc}
\mathrm{Pic}(\tilde{X}) & \xrightarrow{(\nu \circ i)^*} & \mathrm{Pic}(\tilde{Y}_t^\nu) \\
(\nu \circ i)^* \downarrow & \nearrow & \downarrow \\
\mathrm{Pic}(\tilde{Y}_t^\nu) & & \mathrm{Cl}(\tilde{Y}_t^\nu) \\
\theta_t^* \downarrow & & \downarrow \cdot \deg(\theta_t) \\
\mathrm{Pic}(Z_t) & & \mathrm{Cl}(Z_t) \\
\rho_t^* \downarrow & \nearrow & \downarrow \\
\mathrm{Pic}(Y'_t) & \xrightarrow{\rho_{t*}} \mathrm{Cl}(Z_t) \xrightarrow{\theta_{t*}} & \mathrm{Cl}(\tilde{Y}_t^\nu)
\end{array}$$

commutes. The bottom left triangle commutes because ρ_t is a resolution of singularities of a normal variety. The rest of the diagram commutes because Z_t and $\tilde{Y}_t^\nu$ both become regular away from closed subsets of codimension ≥ 2 , and on these open sets θ_t is finite and flat so the composition $\theta_{t*} \circ \theta_t^*$ is multiplication by $\deg(\theta_t)$ [Ful98, Example 1.7.4 and Theorem 6.2].

Therefore to show the claim it suffices to show that up to torsion the kernel of the composition

$$\begin{array}{ccccccc}
\mathrm{NS}(\tilde{X} \otimes_k K) & \longrightarrow & \mathrm{NS}(\tilde{Y}_\eta) & \longrightarrow & \mathrm{NS}(Y') & \longrightarrow & \mathrm{NS}(Y' \otimes_{K^{1/p^f}} \bar{K}) \longrightarrow \mathrm{NS}(Y'_t) \\
\uparrow & & & & & & \downarrow \\
\mathrm{NS}(\tilde{X}) & & & & & & \mathrm{Cl}^{\mathrm{ns}}(\tilde{Y}_t)
\end{array}$$

induced by (4.3) is generated by the irreducible exceptional divisors on $\tilde{X}$ with 0-dimensional image under π .

1. $\mathrm{NS}(\tilde{X}) \hookrightarrow \mathrm{NS}(\tilde{X} \otimes_k K)$ since these are the k -points and K -points of the component groups of the group schemes $\mathbf{Pic}_{\tilde{X}/k}^0$ and $\mathbf{Pic}_{\tilde{X}/k}^0 \otimes_k K = \mathbf{Pic}_{\tilde{X} \otimes_k K/K}^0$, respectively.
2. By Corollary 4.1.1 (2) the kernel of $\mathrm{NS}(\tilde{X} \otimes_k K) \rightarrow \mathrm{NS}(\tilde{Y}_\eta)$ is generated by π -exceptional divisors on $\tilde{X}$ with 0-dimensional image in X .

3. $(\tilde{Y}_\eta \otimes_K K^{1/p^f})^\nu \rightarrow \tilde{Y}_\eta$ is a universal homeomorphism so their Picard groups are isomorphic after $\otimes \mathbb{Z}[\frac{1}{p}]$.

Let $Y' \xrightarrow{\rho} Z \xrightarrow{\theta} \tilde{Y}_\eta^\nu$ be the Stein factorization. Since $\rho: Y' \rightarrow Z$ is a resolution of singularities of a normal variety we have that the composition $\mathrm{NS}(Z) \xrightarrow{\rho^*} \mathrm{NS}(Y') \xrightarrow{\rho_*} \mathrm{Cl}^{\mathrm{ns}}(Z)$ agrees with the injection $\mathrm{NS}(Z) \hookrightarrow \mathrm{Cl}^{\mathrm{ns}}(Z)$, so the map $\mathrm{NS}(Z) \xrightarrow{\rho^*} \mathrm{NS}(Y')$ is injective.

Z and $\tilde{Y}_\eta^\nu$ are both normal varieties, so by restricting to the locus where are both regular, as in the argument for $Z_t \rightarrow \tilde{Y}_t^\nu$, we see that the kernel of $\mathrm{NS}(Z) \rightarrow \mathrm{NS}(\tilde{Y}_\eta^\nu)$ is $\deg(\theta)$ -torsion. So $\mathrm{NS}(\tilde{Y}_\eta) \rightarrow \mathrm{NS}(Y')$ has torsion kernel.

4. The map from $\mathrm{NS}(Y')$ to the K^{1/p^f} -points of the component group of $\mathbf{Pic}_{Y'/K^{1/p^f}}$ has torsion kernel. The K^{1/p^f} of this group scheme inject into the $\overline{K}$ points of the component group of $\mathbf{Pic}_{Y'/K^{1/p^f}} \otimes_{K^{1/p^f}} \overline{K} = \mathbf{Pic}_{Y' \otimes_{K^{1/p^f}} \overline{K}/\overline{K}}$, and this is the same as $\mathrm{NS}(Y' \otimes_{K^{1/p^f}} \overline{K})$ since $Y' \otimes_{K^{1/p^f}} \overline{K}$ has a $\overline{K}$ -point. Therefore the map $\mathrm{NS}(Y') \rightarrow \mathrm{NS}(Y' \otimes_{K^{1/p^f}} \overline{K})$ has torsion kernel.
5. $\mathrm{NS}(Y' \otimes_{K^{1/p^f}} \overline{K}) \rightarrow \mathrm{NS}(Y'_t)$ is injective with torsion-free kernel after $\otimes \mathbb{Z}[\frac{1}{p}]$ by [MP12, Proposition 3.6 (b)]
6. $\mathrm{NS}(Y'_t) \rightarrow \mathrm{NS}(\tilde{Y}_t)$ is surjective with kernel generated by the exceptional divisors of $Y'_t \rightarrow \tilde{Y}_t$ by [Kol18, Lemma 18].

□

4.2 The map $\mathrm{Cl}^{\mathrm{ns}}(X) \rightarrow \mathrm{Cl}^{\mathrm{ns}}(Y)$

The main result of this section is Corollary 4.2.5, which uses Proposition 3.4.3 from the previous chapter. First we fix notation.

Notation throughout this section

Let X be a normal variety over an algebraically closed field k of characteristic $p \geq 0$ and $\mathcal{L}$ a very ample line bundle on X . Let $\pi: \tilde{X} \rightarrow X$ be a separable, generically finite proper morphism from a smooth variety, let E_X denote the exceptional locus (of the Stein factorization), and assume $\pi(E_X)$ has codimension ≥ 2 in X . Set $\mathcal{M} = \pi^*\mathcal{L}$.

Let $Y \in |\mathcal{L}|$ be a general member and $\tilde{Y} \in |\mathcal{M}|$ its pullback. Then Y is normal, and we have a diagram

$$\begin{array}{ccccccc} \mathrm{Pic}(\tilde{X}) & \longrightarrow & \mathrm{Pic}(\tilde{Y}) & \longrightarrow & \mathrm{Pic}(\tilde{Y})^\nu & \hookrightarrow & \mathrm{Cl}(\tilde{Y}^\nu) \\ \downarrow \pi_* & & & & & & \downarrow \\ \mathrm{Cl} X & \longrightarrow & & & & & \mathrm{Cl}(Y) \end{array} \quad (4.4)$$

where the normalization $\nu: \tilde{Y}^\nu \rightarrow \tilde{Y}$ is a universal homeomorphism, and hence a height b purely inseparable morphism for some b (which remains the same for a general choice of Y). Since ν is finite, the kernel of $\pi|_{\tilde{Y}^\nu} \circ \nu_*: \mathrm{Cl}(\tilde{Y}^\nu) \rightarrow \mathrm{Cl}(Y)$ is generated by $\{(\nu^*F)_{\mathrm{red}} \mid F \text{ is a codimension 1 component of } \tilde{Y} \cap E_X\}$.

First we describe how the exceptional divisors of $\pi: \tilde{X} \rightarrow X$ meet $\tilde{Y}$, as in the proof of [RS06, Corollary 3.16]. Let E be an irreducible exceptional divisor on $\tilde{X}$.

- (i) If $\dim \pi(E) = 0$, then $\pi(E) \cap Y = \emptyset$ by generality of Y , so $E \cap \tilde{Y} = \emptyset$.
- (ii) If $\dim \pi(E) = 1$, then let $E \xrightarrow{\psi} C \xrightarrow{\theta} \pi(E)$ be the Stein factorization of $\pi|_E$.

Since Y is general and $\mathcal{L}$ is very ample we have that $\pi(E) \cap Y$ is reduced. Now $C \rightarrow \pi(E)$ factors as $C \rightarrow C' \rightarrow \pi(E)$ where $C' \rightarrow \pi(E)$ is separable and $C \rightarrow C'$ is a purely inseparable map of smooth curves, hence some power e of relative Frobenius on C [Sta20, Tag 0CCZ]. So $\theta^*(\pi(E) \cap Y) = p^e \sum_{i=1}^s Q_i$ where the Q_i are distinct points on C . Finally, ψ has reduced fibers because it is a fibration over a curve [ML39, Theorem 2]. So in summary, $E \cap \tilde{Y} = p^e \sum_{i=1}^s F_j$ where F_j are distinct connected components and are distinct from the connected

components of $E' \cap \tilde{Y}$ for any other exceptional E' on $\tilde{X}$.

- (iii) If $\dim \pi(E) \geq 2$, then $E \cap \tilde{Y}$ is irreducible and $E \cap \tilde{Y} = p^f(E \cap \tilde{Y})_{\text{red}}$ by applying [Zar58, Theorem I.6.3] to the normalization of E . (For each fixed E , this exponent f is the same for general $\tilde{Y}$.)

For an integer n , we define the following property:

Definition 4.2.1. We say that $\mathcal{E}_n$ holds if for every $X, \mathcal{L}$, and $\pi: \tilde{X} \rightarrow X$ as in the situation of 4.2 with $\dim X = \dim \tilde{X} = n$, there exists an integer $r = r(X, \mathcal{L}, \pi)$ such that the following holds:

Let $\tilde{Y} \in |\mathcal{M}|$ be general with inclusion $i: \tilde{Y} \hookrightarrow \tilde{X}$, and let $\nu: \tilde{Y}^\nu \rightarrow \tilde{Y}$ be the normalization of the reduced subscheme of $\tilde{Y}$. If $\mathcal{N}$ is a line bundle on $\tilde{Y}$ contained in $\text{Im}(\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})) \cap \text{Ker}(\text{Pic}(\tilde{Y}) \rightarrow \text{Cl}(Y))$, then there is a divisor L' on $\tilde{X}$ with π -exceptional support such that $\nu^* \mathcal{N}^{p^r} \cong \nu^* i^* \mathcal{O}_{\tilde{X}}(L')$.

Proposition 4.2.2. *Let $n \geq 2$ be an integer such that $\mathcal{E}_n$ holds, and let X be a normal variety of dimension n over an algebraically closed field k of characteristic $p \geq 0$, $\mathcal{L}_0$ a very ample line bundle on X , and $m \geq 2$ an integer. For a general $Y \in |\mathcal{L}_0^m|$, if the map $\text{Cl}^0(X) \rightarrow \text{Cl}^0(Y)$ is surjective, then the kernel of $\text{Cl}^{\text{ns}}(X) \rightarrow \text{Cl}^{\text{ns}}(Y)$ is p -power torsion.*

Proof. If X is smooth then the result is classical from the Grothendieck–Lefschetz theorem for Picard groups, noting that a general $Y \in |\mathcal{L}_0^m|$ is normal so $\text{Pic}(Y) \hookrightarrow \text{Cl}(Y)$. So we may assume the pair $(X, \mathcal{L}_0)$ is not $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Let $\pi: \tilde{X} \rightarrow X$ be a regular alteration and $\tilde{X} \rightarrow Z \xrightarrow{g} X$ the Stein factorization. By [Tem17, Theorem 1.2.9] we may choose π so that g is separable and $\deg g$ is a p^{th} power.

Let D be a Weil divisor on X , not algebraically trivial on X , whose algebraic equivalence class is in the kernel of the restriction $\text{Cl}^{\text{ns}}(X) \rightarrow \text{Cl}^{\text{ns}}(Y)$. If the restriction of D to Y lands in $\text{Cl}^0(Y)$, then it comes from a nonzero class in the cokernel

of $\mathrm{Cl}^0(X) \rightarrow \mathrm{Cl}^0(Y)$, which contradicts our assumption. So D is in the kernel of $\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(Y)$.

Using [Ful98, Example 1.7.4 and Theorem 6.2] and the fact that X is normal, $\deg g \cdot D$ is the pushforward of a divisor D_Z on Z . Let $\tilde{D}$ be a divisor on $\tilde{X}$ mapping to D_Z . We chase diagram (4.4).

By $\mathcal{E}_n$ there exists a π -exceptional divisor L' on $\tilde{X}$ such that $\nu^* i^* \mathcal{O}_{\tilde{X}}(p^r \tilde{D}) \cong \nu^* i^* \mathcal{O}_{\tilde{X}}(L')$. Then $\mathcal{O}_{\tilde{X}}(p^r \tilde{D} - L')$ is in the kernel of $\mathrm{Pic}(\tilde{X}) \rightarrow \mathrm{Pic}(\tilde{Y}) \rightarrow \mathrm{Pic}(\tilde{Y})^\nu$, so by Proposition 4.1.2 and the fact that ν is a height b universal homeomorphism, we have that $\mathcal{O}_{\tilde{X}}(p^{b+r} \tilde{D} - p^b L')$ is in the subgroup of $\mathrm{Pic}(\tilde{X})$ generated by the irreducible π -exceptional divisors mapping to points in X . Since $\mathcal{O}_{\tilde{X}}(p^{b+r} \tilde{D} - p^b L')$ and $\mathcal{O}_{\tilde{X}}(p^b L')$ are both in the kernel of the pushforward $\mathrm{Pic}(\tilde{X}) \rightarrow \mathrm{Cl} Z$, we conclude that $p^{b+r} D_Z \sim 0$ on Z . Therefore $\deg g \cdot p^{b+r} D \sim 0$ on X . $\square$

Now it remains to show that $\mathcal{E}_n$ holds for $n \geq 3$. The following is a lemma of [RS06], just with some p 's added:

Lemma 4.2.3 ([RS06] Lemma 3.17). $\mathcal{E}_3$ holds.

Proof, completely following the proof of [RS06] Lemma 3.17. Since $\dim \tilde{X} = 3$, $\tilde{Y}$ is an irreducible, generically reduced surface and case (iii) above does not occur. By assumption, $\nu^* \mathcal{N} = \mathcal{O}_{\tilde{Y}^\nu}(L)$ with $L = \sum_{i=1}^t \sum_{j=1}^{s_i} a_{ij} (\nu^* F_{ij})_{\mathrm{red}}$ where the i index the exceptional divisors E_i on $\tilde{X}$ and $E_i \cap \tilde{Y} = p^{e_i} \sum_{j=1}^{s_i} F_{ij}$ for each i . Note that if $\nu^* \mathcal{N} \cong \mathcal{O}_{\tilde{Y}^\nu}$, then $\mathcal{N}^{p^b} \cong \mathcal{O}_{\tilde{Y}} = i^* \mathcal{O}_{\tilde{X}}$ where b is height of the purely inseparable morphism $\nu: \tilde{Y}^\nu \rightarrow \tilde{Y}$ (b is the same for general choice of $\tilde{Y}$). So we may assume $\nu^* \mathcal{N} = \mathcal{O}_{\tilde{Y}^\nu}(L)$ is nontrivial.

For illustration, first consider the case where $\#\{E_i \mid \text{some } a_{ij} \neq 0\}$ and write $E = E_i$, $E \cap \tilde{Y} = p^e \sum_{j=1}^s F_j$, and $L = \sum_j a_j (\nu^* F_j)_{\mathrm{red}}$. From the discussion in (ii) the $p^e F_j$ are algebraically equivalent 1-cycles on E and hence on $\tilde{X}$, so the intersection number $D \cdot_{\tilde{X}} p^e F_j$ is independent of j for any divisor D on $\tilde{X}$. By the projection formula [Ful98,

Proposition 2.3 (c)] $D \cdot_{\tilde{X}} p^e i_* \nu_* ((\nu^* F_j)_{\text{red}}) = i_* \nu_* ((\nu^* i^* D) \cdot_{\tilde{Y}^\nu} p^e (\nu^* F_j)_{\text{red}})$ and so

$$D' \cdot_{\tilde{Y}^\nu} p^e (\nu^* F_j)_{\text{red}} \quad (4.5)$$

is independent of j for any Cartier divisor D' on $\tilde{Y}^\nu$ in the image of $\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})^\nu$.

In particular $\nu^* i^* E = p^e \sum_j \nu^* F_j$ and L satisfy (4.5).

The Stein factorization of $\tilde{Y}^\nu \rightarrow Y$ is a birational morphism of normal projective surfaces, and each $(\nu^* F_j)_{\text{red}}$ is an exceptional Weil divisor. So the intersection matrix $\left((\nu^* F_j)_{\text{red}} \cdot_{\tilde{Y}^\nu} (\nu^* F_{j'})_{\text{red}} \right)$ is negative definite [Sak84, Theorem 1.2] and the $\{(\nu^* F_j)_{\text{red}} \mid 1 \leq j \leq s\}$ are linearly independent in $(\text{Cl}(\tilde{Y}^\nu)/\equiv_{\text{num}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Define the $\mathbb{Q}$ -vector space

$$\mathbb{V} := \text{span}\{(\nu^* F_j)_{\text{red}} - (\nu^* F_{j'})_{\text{red}} \mid 1 \leq j < j' \leq s\}.$$

Then $\dim \mathbb{V} = s - 1$ so its orthogonal complement $\mathbb{V}^\perp$ in the span of $\{(\nu^* F_j)_{\text{red}} \mid 1 \leq j \leq s\}$ has dimension 1. By (4.5) we know that $\nu^* i^* E = p^e \sum_j \nu^* F_j$ and L are both in $\mathbb{V}^\perp$, so they are integral divisors on $\tilde{Y}^\nu$ that are $\mathbb{Q}$ -multiples of one another up to numerical equivalence. Since they are both supported on exceptional curves $(\nu^* F_j)_{\text{red}}$, we conclude that $a_j/v_{(\nu^* F_j)_{\text{red}}}(\nu^* F_j)$ is independent of j , and each $v_{(\nu^* F_j)_{\text{red}}}(\nu^* F_j)$ is $p^{b'}$ for some $b' \leq b$, so $p^{b+e}L$ is an integer multiple of $\nu^* i^* E$.

In the general case, for each $1 \leq i \leq t$, equation (4.5) still holds for $D' \cdot_{\tilde{Y}^\nu} p^{e_i} (\nu^* F_{ij})_{\text{red}}$, and again $\{(\nu^* F_{ij})_{\text{red}} \mid 1 \leq i \leq t, 1 \leq j \leq s_i\}$ is a linearly independent set in $(\text{Cl}(\tilde{Y}^\nu)/\equiv_{\text{num}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. The subspace

$$\mathbb{V}' := \text{span}\{(\nu^* F_{ij})_{\text{red}} - (\nu^* F_{ij'})_{\text{red}} \mid 1 \leq i \leq t, 1 \leq j < j' \leq s_i\}$$

has dimension $\sum_{i=1}^t (s_i - 1)$, so its orthogonal complement $\mathbb{V}'^\perp$ in the span of $\{(\nu^* F_{ij})_{\text{red}} \mid 1 \leq i \leq t, 1 \leq j \leq s_i\}$ has dimension t . Let e be the least common multiple of the e_i .

Then L and $\nu^*i^*E_i$ are $t+1$ elements in $\mathbb{V}'^\perp$, and the $\nu^*i^*E_i$ are linearly independent, so as above we conclude that $p^{b+e}L$ must be contained in their $\mathbb{Z}$ -span. $\square$

Lemma 4.2.4. *If $n \geq 4$ and $\mathcal{E}_{n-1}$ holds, then $\mathcal{E}_n$ holds.*

Proof. Let T be a general hypersurface of sufficiently high degree in some projective embedding of $\tilde{X}$ such that the pullback of $\mathcal{O}_{\tilde{X}}(T)$ to $\tilde{Y}^\nu$ is an m^{th} power of a very ample line bundle for some $m \geq 2$. Then $\tilde{X} \cap T$ is a smooth threefold and $\nu^{-1}(\tilde{Y} \cap T)$ is an irreducible, generically reduced surface. We have morphisms

$$\begin{array}{ccccc}
 & & (\tilde{Y} \cap T)^\nu & & \\
 & \searrow & \downarrow & & \\
 \nu_T \left(\begin{array}{c} \nu^{-1}(\tilde{Y} \cap T) \hookrightarrow \tilde{Y}^\nu \\ \downarrow \nu|_{\tilde{Y} \cap T} \end{array} \right. & & & & \\
 & \swarrow & \tilde{Y} \cap T \xrightarrow{j_Y} \tilde{Y} & & \\
 & & \downarrow i & & \\
 & & \tilde{X} \cap T \xrightarrow{j} \tilde{X} & &
 \end{array}$$

where $\nu^{-1}(\tilde{Y} \cap T) \rightarrow \tilde{Y} \cap T$ is a partial normalization, and a diagram

$$\begin{array}{ccccccc}
 \text{Pic}(\tilde{X}) & \longrightarrow & \text{Pic}(\tilde{Y}) & \longrightarrow & \text{Pic}(\tilde{Y})^\nu & \xrightarrow{\pi|_{\tilde{Y}^*} \circ \nu_*} & \text{Cl}(Y) \\
 \downarrow j^* & & \downarrow j_Y^* & & \downarrow & & \\
 \text{Pic}(\tilde{X} \cap T) & \longrightarrow & \text{Pic}(\tilde{Y} \cap T) & \xrightarrow{\nu_T^*} & \text{Pic}((\tilde{Y} \cap T)^\nu) & \longrightarrow & \text{Cl}(\pi(\tilde{Y} \cap T)^\nu)
 \end{array}$$

If $\mathcal{N} \in \text{Pic}(\tilde{Y})$ is in $\text{Im}(\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})) \cap \text{Ker}(\text{Pic}(\tilde{Y}) \rightarrow \text{Cl}(Y))$, then $j_Y^*\mathcal{N} \in \text{Im}(\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y} \cap T)) \cap \text{Ker}(\text{Pic}(\tilde{Y} \cap T) \rightarrow \text{Cl}(\pi(\tilde{Y} \cap T)^\nu))$. By $\mathcal{E}_{n-1}$ there is a divisor L'_T on $\tilde{X} \cap T$ with $\pi|_{\tilde{X} \cap T}$ -exceptional support such that $\nu_T^*j_Y^*\mathcal{N}^{p^e} \cong \nu_T^*i_T^*\mathcal{O}_{\tilde{X} \cap T}(L'_T)$. By Grothendieck–Lefschetz in the smooth case we have $\mathcal{O}_{\tilde{X} \cap T}(p^e L'_T) = j^*\mathcal{O}_{\tilde{X}}(L')$ for some L' on $\tilde{X}$ with π -exceptional support. So $\nu_T^*j_Y^*i^*\mathcal{O}_{\tilde{X}}(L') \cong \nu_T^*j_Y^*\mathcal{N}^{p^{c+e}} =$

$\nu_T^* j_Y^* \mathcal{N}^{p^{c+e}}|_{\tilde{Y} \cap T}$. So

$$\mathcal{N}^{p^{c+e}} \otimes i^* \mathcal{O}_{\tilde{X}}(-L') \in \text{Ker}(\nu_T^* \circ j_Y^*: \text{Pic}(\tilde{Y}) \rightarrow \text{Pic}(\tilde{Y} \cap T)^\nu).$$

Now by Proposition 4.2.2 and the fact that ν_T is a universal homeomorphism, the kernel of $\nu_T^* \circ j_Y^*$ is p^d -torsion for some d . Therefore $\mathcal{N}^{p^{c+d+e}} \cong i^* \mathcal{O}_{\tilde{X}}(p^d L')$. Note that c, d, e did not depend on the choice of $\mathcal{N} \in \text{Pic}(\tilde{Y})$. So we have shown that $\mathcal{E}_n$ holds. $\square$

Corollary 4.2.5. *Let X be a normal variety of dimension $n \geq 3$ over an algebraically closed field k of characteristic $p > 0$, $\mathcal{L}_0$ a very ample line bundle on X , and $m \geq 2$ an integer. Then for a general $Y \in |\mathcal{L}|$, the restriction map*

$$\text{Cl}(X) \longrightarrow \text{Cl}(Y)$$

has p -power torsion kernel.

Proof. First, by Proposition 3.4.2 or Corollary 3.4.3 the restriction map $\text{Cl}^0(X) \rightarrow \text{Cl}^0(Y)$ is an isomorphism. Since $\mathcal{E}_n$ holds by Lemma 4.2.3 and Lemma 4.2.4, we can apply Proposition 4.2.2 to conclude that $\text{Cl}^{\text{ns}}(X) \rightarrow \text{Cl}^{\text{ns}}(Y)$ has p -power torsion kernel. $\square$

Part II

Geometrically non-reduced varieties

Chapter 5

Introduction to Part II

Everything in this part of the thesis is joint work with Joe Waldron. We now work over fields of positive characteristic $p > 0$. In contrast to the last part, many of the fields in this part are *not* algebraically closed.

Given a fibration $f: \mathcal{X} \rightarrow S$ of smooth varieties over an algebraically closed field k , the generic fiber X is a regular variety over the function field $K = K(S)$. In characteristic 0 this implies that X is smooth over K . On the other hand, a key difficulty in positive characteristic comes from the existence of regular varieties X over K for which $X \otimes_K \overline{K}$ is singular. This corresponds to a fibration f with every fiber singular; the most familiar examples of such objects are the quasi-elliptic fibrations which occur in the Bombieri–Mumford classification of smooth surfaces in characteristics 2 and 3. Furthermore, Example 11.2.1 gives regular Fano varieties in characteristic $p < \dim X + 2$ for which $X \otimes_K \overline{K}$ is non-reduced. Such a variety is *geometrically non-reduced*.

These fibrations cause difficulties when applying standard methods of characteristic zero birational geometry, for example fiber space adjunction [Eji19] and semi-positivity [Pat14]. Therefore a key part of extending birational geometry to higher dimensions in positive characteristic will be to understand and control the occurrence

of these fibrations.

One way to do this is to compare a geometrically non-reduced variety X over K to the variety Y over $\overline{K}$ which one obtains by taking the normalization of the maximal reduced subscheme of $X \otimes_K \overline{K}$ with natural morphism $\phi: Y \rightarrow X$.

The earliest result of this kind was the genus change formula of Tate [Tat52], which implies that if X is a regular curve and $X \otimes_K \overline{K}$ is reduced, there is an effective Weil divisor C such that

$$K_Y + (p - 1)C \sim \phi^* K_X.$$

This formula was recently extended to arbitrary dimensional normal varieties, including the geometrically non-reduced case, by Patakfalvi and Waldron [PW20]. This enabled them to prove for example that regular del Pezzo surfaces over a field of characteristic $p \geq 11$ are smooth, and hence techniques such as those of [Pat14] can be applied to del Pezzo fibrations of relative dimension 2 in these characteristics. Further applications of this formula to del Pezzo surfaces have appeared in [Tan19b], [BT20], and [Tan19a]. We discuss applications of the present article to Fano varieties (including higher dimensional ones) in Section 6.0.2.

Chapter 6

Main result: Canonical bundle formula

One of the main aims of this part of the thesis is to reveal the geometric significance of the Weil divisor C which appeared in the [PW20] formula above. That is, we determine a canonical linear system $\mathfrak{C} \subseteq |C|$ which is intricately connected with the non-normal singularities of $X \otimes_K K^{1/p}$.

Theorem 6.0.1. *Let K be the function field of a variety over a perfect field of characteristic $p > 0$. Let X be a projective normal variety over K , and suppose that $H^0(X, \mathcal{O}_X) = K$. Let L be a field lying between K and $K^{1/p}$, and let Y be the normalization of $(X \otimes_K L)_{\text{red}}$ with morphism $\phi: Y \rightarrow X$.*

Then there is a canonically determined linear system $\mathfrak{C}$ of Weil divisors with fixed part $\mathfrak{F}$ and movable part $\mathfrak{M}$ such that

$$K_Y + (p-1)\mathfrak{C} \sim \phi^* K_X.$$

Furthermore, $\mathfrak{F}$ and $\mathfrak{M}$ satisfy the following properties:

1. $\mathfrak{F} = 0$ if and only if $(X \otimes_K L)_{\text{red}}$ is regular in codimension 1.

2. $\mathfrak{F}$ has support equal to that of the conductor divisor of $Y \rightarrow (X \otimes_K L)_{\text{red}}$.

3. $\mathfrak{M} = 0$ if and only if $X \otimes_K L$ is reduced.

Let $f: X \dashrightarrow V$ be the Stein factorization of the rational map induced by $\mathfrak{M}$ and let W be the normalization of $(V \otimes_K L)_{\text{red}}$.

4. If X_ξ is the generic fiber of f then $X_\xi \otimes_{K(V)} K(W)$ is reduced.

5. If $g: X \dashrightarrow Z$ is a rational map with generic fiber X_η such that $X_\eta \otimes_{K(Z)} K((Z \otimes_K L)_{\text{red}})$ is reduced, then there is a factorization $X \dashrightarrow Z \dashrightarrow V$.

6. If $\mathfrak{M}_W$ is the linear system on W associated to the base change $V \otimes_K L$, then there is a natural map $g: Y \dashrightarrow W$ and $\mathfrak{M} = g^*\mathfrak{M}_W$.

In summary, the first two points tell us that the fixed part $\mathfrak{F}$ measures the failure of normality of $(X \otimes_K L)_{\text{red}}$, while the next three tell us that the movable part $\mathfrak{M}$ measures the failure of reducedness of $X \otimes_K L$. In particular, the last two points give a universal property satisfied by f : it is the deepest of all morphisms whose generic fibers stay reduced under the base change L/K .

It should be noted that $\mathfrak{C}$ may not be, and often is not, equal to the complete linear system $|C|$. However, we can use it to read some consequences from knowledge of $|C|$. In the following corollaries, let X be a normal variety over an arbitrary field K with $H^0(X, \mathcal{O}_X) = K$, and $f: Y \rightarrow X$ be the normalized base change by L/K for an arbitrary field extension L .

Corollary 6.0.2. *If C is fixed, then $X \otimes_K L$ is reduced.*

Corollary 6.0.3. *$|C| = \emptyset$ if and only if $X \otimes_K L$ is normal.*

Note that Corollary 6.0.3 appeared recently in [Tan19b], as a consequence of [PW20], but we mention it here for completeness.

6.0.1 Essential part of a base change

On the way to Theorem 6.0.1 we construct a particular subextension $L/L'/K$ which we call the *essential part* of the base change. This is the universally smallest subextension which reveals all the non-reducedness of $X \otimes_K L$, in the following precise sense:

Theorem 6.0.4 (Proposition 10.2.4). *Let K be the function field of a variety over a perfect field of characteristic $p > 0$. Let X be a projective normal variety over K , and suppose that $H^0(X, \mathcal{O}_X) = K$. Let L be a field extension such that $K^{1/p} \supseteq L \supseteq K$.*

There exists a unique subextension $L/L'/K$ such that the varieties

$$\begin{array}{ccccc} Z = (X \otimes_K L)_{\text{red}} & \longrightarrow & Z' = (X \otimes_K L')_{\text{red}} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(L) & \longrightarrow & \text{Spec}(L') & \longrightarrow & \text{Spec}(K) \end{array}$$

satisfy:

1. *All of the following properties, which are equivalent for any such diagram:*

- (a) *$Z \rightarrow Z'$ and $\text{Spec}(L) \rightarrow \text{Spec}(L')$ are morphisms of the same degree.*
- (b) *$Z' \otimes_{L'} L$ is reduced.*
- (c) *$Z' \otimes_{L'} L \cong Z$.*

2. *If $L/L''/K$ is a subextension satisfying the properties (1), then $L' \subseteq L''$.*

3. *$H^0(Z', \Omega_{Z'/X}^\vee) = 0$.*

Furthermore, L' is characterized as the unique subextension satisfying both (1) and (2), and also as the unique subextension satisfying both (1) and (3)

Note that it is possible to define the essential part for a field extension of arbitrary height by iteratively applying the above construction.

6.0.2 Fano varieties

Our investigation of the structure of geometrically non-reduced varieties is ultimately motivated by an ongoing investigation of the structure of geometric singularities of Fano varieties over imperfect fields begun in [PW20]. Over a fixed algebraically closed field of characteristic zero, it is known that regular Fano varieties of fixed dimension form a bounded family by [KMM92]. Furthermore, it is known that if one allows certain controlled singularities (so called ε -lc for fixed ε) then the same remains true by a theorem of Birkar [Bir16] (once known as the Borisov–Alexeev–Borisov conjecture). It is an interesting question whether these statements extend to positive characteristic, and to varieties over non-closed fields. A consequence of this would be that regular Fano varieties of dimension n should exhibit pathologies of positive characteristic such as geometric non-reducedness only in small characteristics depending on n . Conversely, ruling out such pathologies directly can be a useful step towards proving boundedness results. Of particular interest are those Fano varieties over non-algebraically closed fields which satisfy $\rho = 1$, for these occur as generic fibers of Mori fiber spaces over algebraically closed fields.

The following theorem is a result in this direction. It implies that if X is a generic fiber of a Mori fiber space, in a characteristic which is large compared to $\dim(X)$, X can only fail to be geometrically reduced due to the base of some fibration.

Theorem 6.0.5. *Let K be the function field of a variety over a perfect field of characteristic $p > 0$. Let X be a variety over K , and assume the following:*

- *X is normal, $\mathbb{Q}$ -factorial and $\rho(X) = 1$,*
- *X is geometrically irreducible over K ,*
- *there is an effective divisor B such that $K_X + B \equiv 0$, and*
- *if $n = \dim(X)$ then $p > 2n + 1$.*

Then there is a birational morphism $\phi: \tilde{X} \rightarrow X$ and a contraction $\tilde{X} \rightarrow V$ of relative dimension at least 1, such that $(\tilde{X} \otimes_K K^{1/p})_{\text{red}}$ is birational to $\tilde{X} \times_V (V \otimes_K K^{1/p})_{\text{red}}$.

Note that there are no assumptions on the coefficients of B . In characteristic zero, if one assumes additionally that X is of Fano type, one would be tempted to prove that V is also of Fano type and proceed by induction on dimension using the LMMP. However, note that this approach is more difficult in positive characteristic, see the example in [Tan16, Theorem 1.4].

We can also apply our results to give restrictions on the existence of geometrically non-reduced del Pezzo surfaces in small characteristics, in the style of [PW20, Theorem 4.1]. In the following corollaries, X will be a regular del Pezzo surface over a field $K = H^0(X, \mathcal{O}_X)$ of characteristic $p > 0$, which is geometrically non-reduced over K . Y will be the normalization of $(X \otimes_K \overline{K})_{\text{red}}$. In these results we refer to the classification of the possibilities for $(Y, (p-1)C)$ obtained in [PW20, Theorem 4.1], and the notation used there.

Corollary 6.0.6. *$(Y, (p-1)C)$ cannot be of the type (H_d, D) .*

Corollary 6.0.7. *If $p = 3$ then $(X \otimes_K \overline{K})_{\text{red}}$ is normal.*

Corollary 6.0.8. *Suppose that $(Y, (p-1)C)$ is of the type $(\mathbb{P}^1 \times \mathbb{P}^1, F)$. Then there exists a contraction $X \rightarrow V$ to a geometrically non-reduced curve V such that over the generic point of V*

$$Y = X \times_V ((V \otimes_K \overline{K})_{\text{red}})^\nu.$$

Note that these consequences have been chosen for illustration, and the list is far from exhaustive. For example it is possible to combine the ideas of the above corollaries to give more precise descriptions of what happens when X is geometrically non-reduced and $(X \otimes_K \overline{K})_{\text{red}}$ is non-normal. We leave this to the interested reader.

6.0.3 Outline

We first review background material in Section 7. Then in Section 8 we recall the setting and some of the results of [PW20] and begin to describe the linear system $\mathfrak{C}$. In Section 9 we study the fixed part $\mathfrak{F}$ and prove Theorem 6.0.1 (1) and (2). Then in Section 10 we define the movable part $\mathfrak{M}$ and prove Theorem 6.0.4 and Theorem 6.0.1 (3). In Section 10.3 we describe the map $X \dashrightarrow V$ and prove Theorem 6.0.1 (4), (5), and (6). Finally, in Section 11.1 we give geometric applications and prove Theorem 6.0.5, and we give some concrete examples in Section 11.2.

In summary, the parts of Theorem 6.0.1 are proven in the following locations:

- (1) Proposition 9.0.4,
- (2) Proposition 9.0.4,
- (3) Proposition 10.2.5,
- (4) Proposition 10.3.3,
- (5) Proposition 10.3.4,
- (6) Proposition 10.3.4.

Chapter 7

Preliminaries

In this part of the thesis, a variety over a field K will be an integral separated scheme of finite type over K . Unless stated otherwise, we may work over imperfect fields, and all varieties will be quasi-projective.

7.0.1 Fields and geometric singularities

Notation

Let K be a field.

1. We consider a fixed algebraic closure $\overline{K}$ of K .
2. By K^{1/p^n} we mean the target of the composition of n Frobenius homomorphisms, that is $F^n: K \hookrightarrow K^{1/p^n}$. This appears naturally as a subring of $\overline{K}$, in which we have adjoined the $(p^n)^{\text{th}}$ root of every element of K .
3. $K^{1/p^\infty} = \cup_n K^{1/p^n}$ denotes the perfect closure of K inside $\overline{K}$.
4. If L/K is a finite purely inseparable extension, the minimal n such that $K \subseteq L \subseteq K^{1/p^n}$ is said of be the *height* of L/K .

If X is a variety over a field K , and L is a field extension, denote the normalization of the maximal reduced subscheme of $X \otimes_K L$ by $(X \otimes_K L)_{\text{red}}^\nu$. To obtain geometric

consequences, we are usually interested in the singularities of $(X \otimes_K K^{1/p^\infty})_{\text{red}}^\nu$. The assumptions on the field K and the extension L/K in Theorem 6.0.1 may at first glance appear restrictive. However, this is not the case, as general field extensions can be reduced to this case using the following lemmas. Firstly, a single Frobenius base change is sufficient to reveal geometric singularities of our variety:

Lemma 7.0.1 ([Sta20, Tag 035X, Tag 038O, Tag 038V],[Tan19b, Proposition 2.10]).

Let X be a scheme of finite type over a field K . The following are equivalent:

- *X is geometrically reduced (resp. geometrically normal, smooth) over K .*
- *$X \otimes_K K^{1/p}$ is reduced (resp. normal, regular).*
- *$X \otimes_K K^{1/p^\infty}$ is reduced (resp. normal, regular).*
- *$X \otimes_K \overline{K}$ is reduced (resp. normal, regular).*

Next, we may usually reduce questions about varieties over general fields to those over function fields using the following:

Lemma 7.0.2. *Let X be a variety over a field k . Then there is a field $K \subset k$ that is the function field of a variety over a perfect field and a variety X_K over K such that $X \cong X_K \otimes_K k$.*

Furthermore, we also have

$$(X_K \otimes_K K^{1/p^\infty})_{\text{red}}^\nu \otimes_{K^{1/p^\infty}} k^{1/p^\infty} \cong (X \otimes_k k^{1/p^\infty})_{\text{red}}^\nu.$$

Proof. Let $X_i = \text{Spec}(A_i)$ where $A_i = k[x_1^i, \dots, x_{n_i}^i]/I_i$ is a finite set of affine charts of X , $X_{ij} = X_i \cap X_j \subset X_i$, and $\phi_{ij}: X_{ij} \rightarrow X_{ji}$ are the gluing maps. Furthermore, for each i, j , fix an affine cover of X_{ij} of the form $\{U_{ijk} = \text{Spec}(A_i[\frac{1}{f_{ijk}}])\}$.

Let $\{a_m\} \subset k$ be a finite set consisting of

- the k -coefficients of a finite set of generators of $I(X_i) \subset k[x_1^i, \dots, x_{n_i}^i]$ for all i ,

- the k -coefficients of a finite set of generators of $I(X_i \setminus X_{ij}) \subset A_i$ for all i and j ,
- the k -coefficients of a polynomial representative of $f_{ijk} \in A_i$ for all i, j and k ,
and
- the k -coefficients of a set of polynomial representatives of a set of functions defining $\phi_{ij}|_{U_{ijk}}$, for all i, j and k .

Let B be the normalization of the finitely generated $\mathbb{F}_p$ -algebra $\mathbb{F}_p[\{a_m\}] \subset k$. Then we can define varieties $\mathcal{X}_i \subset \mathbb{A}_B^{n_i}$, open subsets $\mathcal{X}_{ij} \subset \mathcal{X}_i$, affine covers $\{U_{ijk}\}$ of $\mathcal{X}_{ij}$, and morphisms $\psi_{ij}: \mathcal{X}_{ij} \rightarrow \mathcal{X}_{ji}$ using the same equations. If we tensor these up to $K = \text{Frac}(B)$, they then glue to form the required variety X_K . The necessary identities of the gluing maps hold over K because they hold after applying $\otimes_K k$, and field extensions are faithfully flat.

For the furthermore, we know that $(X_K \otimes_K K^{1/p^\infty})_{\text{red}}^\nu$ is a normal variety over a perfect field K^{1/p^∞} , and so is geometrically normal over K^{1/p^∞} by Lemma 7.0.1. Thus $(X_K \otimes_K K^{1/p^\infty})_{\text{red}}^\nu \otimes_{K^{1/p^\infty}} k^{1/p^\infty}$ is already normal. $\square$

Finally, the singularities stop becoming worse after performing finitely many Frobenius base changes:

Lemma 7.0.3. *Let X be a variety over a field K . There is a finite purely inseparable extension L/K such that*

$$(X \otimes_K L)_{\text{red}}^\nu \otimes_L K^{1/p^\infty} \cong (X \otimes_K K^{1/p^\infty})_{\text{red}}^\nu.$$

In particular, we could take $L = K^{1/p^n}$ for some n .

Proof. First, as X is of finite type, there exists a finite affine cover, and so by taking the compositum of a set of field extensions satisfying the condition for each affine chart we may assume that X is affine. Note that each purely inseparable field extension $L \supset$

K embeds uniquely into K^{1/p^∞} . So we assume that X is affine. Then $(X \otimes_K K^{1/p^\infty})_{\text{red}}^\nu$ is also affine, and so take an embedding $Z := (X \otimes_K K^{1/p^\infty})_{\text{red}}^\nu \subset \mathbb{A}_{K^{1/p^\infty}}^n$. Let L be the finite extension of K obtained by adjoining all coefficients of equations of a generating set of the ideal of Z . Then there is $Z_L \subset \mathbb{A}_L^n$ defined by the same equations such that $Z_L \otimes_L K^{1/p^\infty} \cong (X \otimes_K K^{1/p^\infty})_{\text{red}}^\nu$, and hence Z_L is geometrically normal by Lemma 2.2.

After further extending L we may assume that the morphism $Z \rightarrow X \otimes_K K^{1/p^\infty}$ descends to L , that is we obtain $Z_L \rightarrow X \otimes_K L$, and again after further extending L we may assume that there is an open subset such that the inverse of $Z \rightarrow (X \otimes_K K^{1/p^\infty})_{\text{red}}$ descends, i.e. so that $Z_L \rightarrow (X \otimes_K L)_{\text{red}}$ is birational. We also know that this is finite, so then as Z_L is normal, $Z_L \cong (X \otimes_K L)_{\text{red}}^\nu$, and so we are done.

□

Given the finite extension L/K in Lemma 7.0.3, we can canonically decompose it into a sequence of height one extensions $L_i = L \cap K^{1/p^i}$, so that L_i/K has height i . Our results can be applied to each of the extensions L_i/L_{i-1} sequentially, or to a further decomposition of this sequence.

Remark. It is likely that most of our arguments can be applied directly with the assumption on the ground field relaxed to *differentially finite over a perfect subfield*. However, due to the above reduction to the general case and the fact that the function field situation is the one which arises naturally in practical applications, we have not verified this.

7.0.2 Properties of sheaves and purely inseparable Galois theory

Saturated subsheaves

Definition 7.0.4. Let X be a scheme, and let $\mathcal{G} \subset \mathcal{F}$ be coherent sheaves on X . $\mathcal{G}$ is *saturated* in $\mathcal{F}$ if the quotient $\mathcal{F}/\mathcal{G}$ is torsion free, or equivalently if a section of $\mathcal{F}$ is in $\mathcal{G}$ if and only if its image in $\mathcal{F}_\xi$ is in $\mathcal{G}_\xi$ where ξ is the generic point of X . For any inclusion of sheaves $\mathcal{G} \subset \mathcal{F}$, there is a unique *saturation* of $\mathcal{G}$ in $\mathcal{F}$.

Lemma 7.0.5. *Let $f: X \rightarrow Y$ be a flat morphism of integral schemes, and let $\mathcal{G} \subset \mathcal{F}$ be coherent sheaves on Y .*

1. *If $\mathcal{G}$ is saturated in $\mathcal{F}$, then $f^*\mathcal{G}$ is saturated in $f^*\mathcal{F}$.*

2. $f^*\mathcal{G}^{\text{sat}} = (f^*\mathcal{G})^{\text{sat}}$.

Proof. We may assume that $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, $\mathcal{G} = \tilde{M}$ and $\mathcal{F} = \tilde{N}$.

1. If M is saturated in N , then N/M is torsion free. By [Sta20, Tag 0AXM]

$(N/M) \otimes_A B \cong (N \otimes_A B)/(M \otimes_A B)$ is also torsion free.

2. Since B is flat over A we have an inclusion $M \otimes_A B \hookrightarrow M^{\text{sat}} \otimes_A B$ and so

$(M \otimes_A B)^{\text{sat}} \hookrightarrow (M^{\text{sat}} \otimes_A B)^{\text{sat}} = M^{\text{sat}} \otimes_A B$. If $\sum n_i \otimes b_i \in M^{\text{sat}} \otimes_A B$, then

for each i there is some $a_i \in A$ with $a_i n_i \in M$. So letting $a = \prod a_i$ we have

$$a \sum n_i \otimes b_i = \sum a n_i \otimes b_i \in M \otimes_A B.$$

□

Reflexive sheaves

We record here some properties of reflexive sheaves; see [Har80] for more details. We denote the dual of a coherent sheaf $\mathcal{F}$ on X by $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Definition 7.0.6. A coherent sheaf $\mathcal{F}$ on a Noetherian integral scheme X is said to be *reflexive* if the natural morphism

$$\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism.

If $\mathcal{F}$ is any coherent sheaf then $\mathcal{F}^\vee$ is reflexive, so in particular for any morphism $X \rightarrow Y$, the sheaf $\mathcal{T}_{X/Y} := \Omega_{X/Y}^\vee$ is reflexive. Note that we generally denote $\mathcal{T}_{X/k}$ by just $\mathcal{T}_X$.

If X is normal, there is a bijective correspondence between linear equivalence classes of Weil divisors D and rank 1 reflexive sheaves $\mathcal{O}_X(D)$, where addition of Weil divisors corresponds to reflexivized tensor product. When we need to refer to the class of Weil divisors corresponding to such a sheaf $\mathcal{F}$, we will use the notation $[\mathcal{F}]$. Open brackets $|\mathcal{F}|$ will refer to the complete linear system of effective divisors in $[\mathcal{F}]$.

An open subset $j: U \rightarrow X$ is called *big* if $\text{codim}_X(X \setminus U) \geq 2$.

Lemma 7.0.7 ([Sta20, Tag 0AY6]). *The following conditions are equivalent for a coherent sheaf $\mathcal{F}$ on a normal variety X :*

1. $\mathcal{F}$ is reflexive,
2. $\mathcal{F}$ is torsion free and has property S_2 , and
3. there is a big open subset $j: U \rightarrow X$ such that $j^*\mathcal{F}$ is locally free (of finite rank) and $\mathcal{F} \cong j_*j^*\mathcal{F}$.

Determinants

The following result will allow us to define the determinant of a torsion-free sheaf that is not necessarily reflexive.

Lemma 7.0.8 ([Ish18] Proposition 5.1.7). *Let X be a normal variety over k and $\mathcal{F}$ a torsion-free coherent sheaf on X . Then there is a big open subset $j: U \hookrightarrow X$ such that $\mathcal{F}|_U$ is locally free.*

Definition 7.0.9. Given a coherent sheaf $\mathcal{F}$ on a normal variety X , we define a reflexive sheaf $\det \mathcal{F}$ called the *determinant* in increasing generality:

1. If $\mathcal{F}$ is reflexive then $\det \mathcal{F} = (\bigwedge^{\text{rank } \mathcal{F}} \mathcal{F})^{\vee\vee}$.
2. If $\mathcal{F}$ is torsion free then $\det \mathcal{F} = i_* \det(\mathcal{F}|_U)$, where U is a big open subset of X on which $\mathcal{F}$ is locally free by Lemma 7.0.8
3. If $0 \rightarrow \mathcal{G} \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow 0$ is an exact sequence of sheaves with $\mathcal{G}$ and $\mathcal{F}'$ torsion free, we define

$$\det \mathcal{F} = (\det \mathcal{F}' \otimes \det \mathcal{G}^{-1})^{\vee\vee}.$$

Remark.

- It follows from Lemma 7.0.7 that (2) makes sense and agrees with (1) when $\mathcal{F}$ is reflexive. [Sta20, Tag 0B38] together with Lemma 7.0.7 and Lemma 7.0.8 implies that (3) agrees with (2) when $\mathcal{F}$ is torsion free.
- These definitions agree with those which would be obtained by restricting to the big open subset on which all the torsion free sheaves which appear are locally free, then reflexivising.

Purely inseparable Galois theory

In this subsection we describe a Galois theory for certain purely inseparable morphisms. First we clear up some notation surrounding Frobenius.

Definition 7.0.10. Let X be a scheme over a field of characteristic $p > 0$. The *absolute Frobenius* morphism on X is the endomorphism $F_X: X \rightarrow X$ induced by the p^{th} power map on rings.

Now suppose X comes with a morphism $f: X \rightarrow S$. Let $X' = X \times_S S$ the base change of X by $F_S: S \rightarrow S$. The *relative Frobenius* morphism of X over S is the induced S -morphism $F_{X/S}: X \rightarrow X'$.

$$\begin{array}{ccccc}
 & & F_X & & \\
 & \nearrow & & \searrow & \\
 X & \xrightarrow{F_{X/S}} & X' & \xrightarrow{\phi} & X \\
 & \searrow & \downarrow & & \downarrow \\
 & & S & \xrightarrow{F_S} & S
 \end{array}$$

If k is a perfect field and $S = \operatorname{Spec} k$ then $F_k: \operatorname{Spec} k \rightarrow \operatorname{Spec} k$ is an isomorphism, so $\phi: X' \rightarrow X$ is also an isomorphism of schemes (but not of k -schemes). Throughout this article, there will usually be a fixed perfect ground field k , where K is the function field of a variety over k . In this situation, we denote the relative Frobenius and its target by $F: X \rightarrow X^p$. This is a k -morphism and agrees with the global Frobenius F_X up to the isomorphism described above.

Definition 7.0.11. A finite purely inseparable morphism $f: X \rightarrow Y$ is said to be of *height one* if there is a finite purely inseparable morphism $g: Y \rightarrow X$ such that $g \circ f = F_X$.

For field extensions this agrees with height as defined at the beginning of §7.0.1. In the case where k is perfect and $f: X \rightarrow Y$ is a k -morphism, there is a k -morphism $Y \rightarrow X^p$ such that the composition is relative Frobenius of X over k .

Let Y be a variety over a perfect field k of characteristic $p > 0$. A $(k\text{-})$ *derivation* is (locally) a k -linear map $\Delta: \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ satisfying the Leibniz rule. It can be composed with itself p times, giving a function Δ^p which we call the p^{th} power of Δ . In characteristic $p > 0$, Δ^p also satisfies the Leibniz rule and so is also a derivation. The tangent sheaf $\mathcal{T}_{Y/k}$ identifies with the sheaf of k -derivations on Y .

Definition 7.0.12. A *foliation* on a variety Y over a perfect field of characteristic p is a coherent saturated subsheaf $\mathcal{F} \subset \mathcal{T}_Y$ that is closed under Lie brackets and p^{th}

powers.

Remark. The condition of closedness under Lie brackets is in fact redundant by [Ger64]. Therefore it is enough to check saturation and p -closedness.

A foliation $\mathcal{F}$ of rank r on a variety Y defines a height one purely inseparable k -morphism $\phi: Y \rightarrow Z$ of degree p^r by taking $Z = \underline{\text{Spec}}_{Y^p} \mathcal{A}$, where $\mathcal{A}$ is the subsheaf of $\mathcal{O}_Y$ killed by all sections of $\mathcal{F}$, which satisfies $\mathcal{O}_Y^p \subset \mathcal{A} \subset \mathcal{O}_Y$. Z is called the quotient of Y by $\mathcal{F}$.

On the other hand, given a height one purely inseparable morphism $Y \rightarrow Z$ of k -varieties, the sheaf $\mathcal{T}_{Y/Z}$ of derivations on Y that vanish on $\mathcal{O}_Z \subset \mathcal{O}_Y$ is a subsheaf of $\mathcal{T}_Y$ that is closed under p^{th} powers. If Y is normal, then $\mathcal{T}_{Y/Z} \subset \mathcal{T}_Y$ is saturated and hence a foliation on Y . Moreover, in the normal case, these procedures give the following correspondence:

Lemma 7.0.13 ([Eke87, Proposition 2.4]). *Let Y be a normal variety over a perfect field k of characteristic $p > 0$. There is a one-to-one correspondence between the following three sets:*

1. *Finite purely inseparable k -morphisms $\phi: Y \rightarrow Z$ of height one to a normal variety,*
2. *intermediate fields $K(Y) \supset L \supset K(Y)^p$, and*
3. *foliations $\mathcal{F} \subset \mathcal{T}_Y$.*

There is a canonical bundle formula for quotients by foliations:

Proposition 7.0.14 ([Eke87, Corollary 3.4], [PW20, Proposition 2.10]). *Let $Y \rightarrow X$ be a finite purely inseparable morphism of height one of normal varieties over a perfect field k of characteristic $p > 0$, and let $\mathcal{F}$ be the corresponding foliation. Then*

$$\omega_{Y/X} \cong \det \mathcal{F}^{\otimes [p-1]}.$$

7.0.3 Grassmannians and linear systems

Let S be a variety and $\mathcal{E}$ a vector bundle on S . We define the Grassmannian $\mathrm{Gr}_S(r, \mathcal{E})$ of r -planes on $\mathcal{E}$ to be the variety over S which parametrizes r -dimensional subbundles of $\mathcal{E}$. That is, for a morphism $f: X \rightarrow S$, giving a diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Gr}_S(r, \mathcal{E}) \\ & \searrow & \downarrow \\ & & S \end{array}$$

is the same as giving an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow f^*\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathrm{rank} \mathcal{F} = r$. Furthermore, to give this data, it is sufficient to give the surjection

$$f^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$$

with $\mathrm{rank} \mathcal{Q} = \mathrm{rank} \mathcal{E} - r$, for then the kernel is automatically a subbundle.

Furthermore, if we are given a torsion free sheaf $\mathcal{Q}$ and a surjection $f^*\mathcal{E} \rightarrow \mathcal{Q}$ then we obtain a rational map $X \dashrightarrow \mathrm{Gr}_S(r, \mathcal{E})$ which is a morphism away from the points where $\mathcal{Q}$ is not locally free. This has codimension at most 2 by Lemma 7.0.8.

Let X be a normal variety with morphism $f: X \rightarrow S = \mathrm{Spec}(A)$. Suppose $H^0(X, \mathcal{O}_X) = A$, and let $\mathcal{L}$ be a torsion free sheaf of rank one, together with a surjection

$$\mathcal{O}_X^{\oplus d} \xrightarrow{\alpha} \mathcal{L} \longrightarrow 0.$$

Then we can define a linear system of Weil divisors associated to the map α by restricting to the complement of the codimension 2 locus on X where $\mathcal{L}$ fails to be locally free, and then taking the closure of the resulting Weil divisors. This is

equivalent to taking the double dual of the above surjection. Doing this, we obtain a linear system $\mathfrak{M} \subset |\mathcal{O}_X(\mathcal{L}^{\vee\vee})|$. We will denote the corresponding subspace of $H^0(X, \mathcal{O}_X(\mathcal{L}^{\vee\vee}))$ by $V(\mathfrak{M})$. We denote the dimension of $V(\mathfrak{M})$ by $\dim_A V(\mathfrak{M})$ and the dimension of $\mathfrak{M}$ by $\dim \mathfrak{M}$. So by definition $\dim \mathfrak{M} = \dim_A V(\mathfrak{M}) - 1$.

7.0.4 Flatness and Stein factorization

This subsection includes some lemmas which we need later, which are presumably well known but for which we could find no satisfactory reference.

Lemma 7.0.15. *Let $g: Y \rightarrow T$ be a projective dominant morphism of normal varieties. Then there is an open subset $W \subseteq T$ with $\text{codim}(T \setminus W) \geq 2$ such that Y_W is flat over W .*

Proof. By the Raynaud–Gruson flatification ([RG71] Theorem 5.2.2) there exists a modification $\mu: T' \rightarrow T$ such that the proper transform of g under μ is a flat morphism. This is defined by the following diagram:

$$\begin{array}{ccc}
 Y \times_T T' & & \\
 \uparrow & & \\
 \overline{Y \times_T U} & \xrightarrow{\quad} & Y \\
 \downarrow \text{flat} & & \downarrow g \\
 T' & \xrightarrow[\text{birational}]{\mu} & T \\
 \uparrow & & \uparrow \\
 \mu^{-1}(U) & \xrightarrow{\mu|_{\mu^{-1}(U)}} & U
 \end{array}$$

where U is an open dense subscheme of T over which μ is an isomorphism.

Let W be the largest open subset of T over which μ has finite fibers. Then μ is finite over W and moreover by normality of W , μ is an isomorphism over W . Furthermore, as T' is irreducible, $T \setminus W$ must have codimension at least 2. But as μ is an isomorphism over W , then $Y_W \times_W T'_W \rightarrow Y_W$ is an isomorphism. Thus $Y_W \rightarrow W$

is flat. □

Lemma 7.0.16. *Let $f: X \rightarrow S$ be a proper morphism of irreducible Noetherian schemes with X normal. The following conditions are equivalent:*

1. $f_*\mathcal{O}_X = \mathcal{O}_S$,
2. S is integrally closed in X , and
3. S is normal and $K(S)$ is integrally closed in $K(X)$.

Proof. Let $X \rightarrow S'$ be the Stein factorization of f . Assume (1), then $S' = S$ and so (2) follows from [Sta20, Tag 03H0]. Conversely, assuming (2) then we also have $S' = S$ by [Sta20, Tag 03H0] and so (1) follows. (3) implies (2) is clear, so we must prove the converse.

Suppose S is integrally closed in X , but $K(S)$ is not integrally closed in $K(X)$. Then there is some finite morphism $T \rightarrow S$ such that $K(S) \rightarrow K(T) \rightarrow K(X)$. We then get a rational map $g: X \dashrightarrow T \rightarrow S$ factoring $X \rightarrow S$.

Let $Y \subset X \times T$ be the closure of the graph of g . It is sufficient to show that $\phi: Y \rightarrow X$ is an isomorphism. Suppose that P is a fundamental point of ϕ^{-1} , so that ϕ^{-1} is not defined at P . By Zariski's main theorem, the total transform has dimension at least 1, and contains a curve C . But then consider the composition $Y \rightarrow Y_S \subset X \times S \rightarrow X$ will also contract the image of C , and so $X \dashrightarrow S$ is not a morphism. We have a contradiction. □

Chapter 8

Canonical bundle formula

The following notation will be used throughout the remainder of the thesis.

8.0.1 Notational conventions

- $f: \mathcal{X} \rightarrow S$ will be a projective morphism of normal varieties over a perfect field k of characteristic $p > 0$, satisfying $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$.
- The function field of S will be K , and the generic fiber of f will be X . These are the varieties which appear in Theorem 6.0.1.
- T will be a normal variety with a purely inseparable height one k -morphism $\tau: T \rightarrow S$ (Definition 7.0.11).
- We will denote $(\mathcal{X} \times_S T)_{\text{red}}$ by $\mathcal{Z}$, and $\mathcal{Y}$ will be the normalization of $\mathcal{Z}$. Often we will refer to $\mathcal{Y}$ as the normalization of $\mathcal{X} \times_S T$, with the understanding that this does not imply that $\mathcal{X} \times_S T$ is reduced.
- Note that the scheme $\mathcal{X} \times_S T$ is reduced if and only if it satisfies the conditions S_1 and R_0 . However, the S_1 condition is preserved under flat base change by [Mat89, Theorems 15.1 and 23.3], so if $T \rightarrow S$ is flat, reducedness of $\mathcal{X} \times_S T$ is equivalent to regularity at its generic point.

- We will always remove a subset of S of codimension at most 2 to ensure that $\mathcal{X} \rightarrow S$ and $\mathcal{Y} \rightarrow T$ are flat, using Lemma 7.0.15. Furthermore, we can similarly remove a codimension 2 set to ensure that $\mathcal{T}_{T/S}$ is locally free. The codimension condition ensures this makes no difference in determining Weil divisors.
- Denote the ideal sheaf of $\mathcal{Z}$ in $\mathcal{X} \times_S T$ by $I_{\mathcal{Z}}$, or just I if no confusion can arise.
- Denote the morphisms between these schemes as follows:

$$\begin{array}{ccccccc}
 & & & \phi & & & \\
 & & & \curvearrowright & & & \\
 \mathcal{Y} & \xrightarrow{\nu} & \mathcal{Z} & \xrightarrow{i} & \mathcal{X} \times_S T & \xrightarrow{\quad} & \mathcal{X} \\
 & \searrow & \downarrow h & & \downarrow f_T & & \downarrow f \\
 & & & g & & T & \xrightarrow{\tau} S
 \end{array}$$

- Let $\mathcal{F}_{\mathcal{Y}/\mathcal{X}} \subset \mathcal{T}_{\mathcal{Y}}$ be the foliation corresponding to ϕ , and let $\mathcal{T}_{T/S} \subset \mathcal{T}_{T/k} = \mathcal{T}_{T/T^p}$ be that corresponding to τ (see Lemma 7.0.13). We drop the subscripts whenever no confusion can arise.
- The dimension of $\mathcal{X}$ will be denoted by n , and the rank of $\mathcal{F}$ will be denoted by r .
- In addition to denoting the function field of S by K , denote that of T by L , and drop the calligraphic fonts to denote generic fibers, giving a diagram of schemes over function fields as follows:

$$\begin{array}{ccccccc}
 & & & \phi & & & \\
 & & & \curvearrowright & & & \\
 Y & \xrightarrow{\nu} & Z & \xrightarrow{i} & X \otimes_K L & \xrightarrow{\quad} & X \\
 & \searrow & \downarrow h & & \downarrow f_L & & \downarrow f \\
 & & & g & & L & \xrightarrow{\tau} K
 \end{array}$$

In particular, we will abuse notation by dropping Spec in e.g. $\text{Spec } K$, and use $X \otimes_K L$ in place of $X \times_{\text{Spec } K} \text{Spec } L$.

- Recall that the Frobenius morphisms considered will all be the relative Frobenius over $\mathrm{Spec} k$, which matches absolute Frobenius up to a canonical isomorphism of the target.
- The objects defined in the remainder of this section will also be used without further comment throughout the article.

8.1 Construction

Our construction begins where the proof of [PW20, Theorem 3.1] ends, so we first summarize the constructions made in that proof. We start with the cotangent sequence

$$g^*\Omega_{T/k} \longrightarrow \Omega_{\mathcal{Y}/k} \longrightarrow \Omega_{\mathcal{Y}/T} \longrightarrow 0.$$

Recall that we assume throughout that we have already replaced T and S by big open subsets so that g is flat, so we have that $(g^*\Omega_{T/k})^\vee \cong g^*\mathcal{T}_{T/k}$. Thus dualising the left-hand map gives a map $\mathcal{T}_{\mathcal{Y}/k} \rightarrow g^*\mathcal{T}_{T/k}$. It is shown in the proof of point (b) of [PW20, Theorem 3.1] that, if $\mathcal{F} := \mathcal{F}_{\mathcal{Y}/\mathcal{X}}$ is the foliation corresponding to $\mathcal{Y} \rightarrow \mathcal{X}$, the composition

$$\mathcal{F} \hookrightarrow \mathcal{T}_{\mathcal{Y}/k} \rightarrow g^*\mathcal{T}_{T/k}$$

is injective. Furthermore, the proof of point (c) of [PW20, Theorem 3.1] also shows that the image of $\mathcal{F}$ lands inside $g^*\mathcal{T}_{T/S}$, which after shrinking S and T to ensure flatness, is a subsheaf of $g^*\mathcal{T}_{T/k}$. We have already shrunk the base to a big open subset in §8.0.1 to ensure that $\mathcal{T}_{T/S}$ is locally free. This gives the key injection

$$\mathcal{F} \hookrightarrow g^*\mathcal{T}_{T/S}.$$

Definition 8.1.1. Define $\mathcal{F}'_{T/S}$ to be the saturation of $\mathcal{F}$ inside $g^*\mathcal{T}_{T/S}$, where again

we drop the subscript if no confusion can arise.

We have an inclusion $\mathcal{F} \hookrightarrow \mathcal{F}'$ between sheaves of equal rank. This gives a global element of $\mathrm{Hom}_{\mathcal{Y}}(\det \mathcal{F}, \det \mathcal{F}')$, which after restricting to a suitable big open set of $\mathcal{Y}$ and then reflexivising, gives a canonical global section in $|\det \mathcal{F}' - \det \mathcal{F}| = |\det(\mathcal{F}'/\mathcal{F})|$.

Definition 8.1.2. Define the fixed part of the linear system to be this canonical section, denoted

$$\mathfrak{F}_{T/S} \in |\det(\mathcal{F}'/\mathcal{F})|.$$

Remark. If $\mathcal{X} \times_S T$ is reduced, then $\mathfrak{F}_{T/S}$ agrees with the canonical choice of C given in (c) of [PW20, Theorem 3.1], and in that case $\mathcal{F}' = g^*\mathcal{T}_{T/S}$.

Definition 8.1.3. Now define a torsion free sheaf $\mathcal{Q}_{T/S}$ by

$$0 \longrightarrow \mathcal{F}'_{T/S} \longrightarrow g^*\mathcal{T}_{T/S} \xrightarrow{\alpha} \mathcal{Q}_{T/S} \longrightarrow 0.$$

Remark. Let U be an open affine subset of T on which $\mathcal{T}_{T/S}$ is free. Then after restricting $\mathcal{Y}$ to $g^{-1}(U)$, $\det \mathcal{Q}_{T/S} = (\bigwedge^{n-r} \mathcal{Q}_{T/S})^{\vee\vee}$ is globally generated by $\wedge^r \alpha$ away from some codimension 2 points by Lemma 7.0.7.

Finally, let $h: \mathcal{Y} \dashrightarrow \mathrm{Gr}(r, \mathcal{T}_{T/S})$ be the rational map induced by $g^*\mathcal{T}_{T/S} \rightarrow \mathcal{Q}_{T/S} \rightarrow 0$ (see Section 7.0.3), and let $\mathcal{W}_{\mathrm{Im}}$ be the closure of the image. As $\mathcal{Y} \rightarrow \mathcal{X}$ is of height one (Definition 7.0.11), there is a rational map $\mathcal{X} \rightarrow \mathcal{Y}^p \dashrightarrow \mathcal{W}_{\mathrm{Im}}^p$. Let $\mathcal{V}$ be the normalization of $\mathcal{W}_{\mathrm{Im}}^p$ inside $K(\mathcal{X})$. By definition of normalization $K(\mathcal{V})$ is integrally closed in $K(\mathcal{X})$, so it must contain $K(S)$ (for it certainly contains $K(S)^p$). Therefore we get a factorization $\mathcal{X} \dashrightarrow \mathcal{V} \rightarrow S$.

Definition 8.1.4. Here we define the intermediate varieties in the main result Theorem 6.0.1. The variety $\mathcal{V}$ was defined in the previous paragraph. Let $\mathcal{W}$ be the

normalization of the maximal reduced subscheme of $\mathcal{V} \times_S T$. By universal properties there is a map $\mathcal{Y} \dashrightarrow \mathcal{W}$.

Example 8.1.5. In the above setting, $\mathcal{W}$ and $\mathcal{W}_{\text{lm}}$ may be different; see Example 11.2.1 with $m > 1$.

The following canonical bundle formula is an extension of [PW20, Theorem 3.1], which we think should be of independent interest. The first two terms in the relative canonical divisor correspond to the decomposition of the linear system $\mathfrak{C}$ described in Theorem 6.0.1. Note that if we allow shrinking S , $\det \mathcal{Q}_{T/S}$ becomes effective and so we recover the formula of [PW20, Theorem 3.1].

Theorem 8.1.6. *In the situation of §8.0.1, recalling that this involved shrinking S and T to big open subsets, we have a linear equivalence of Weil divisors:*

$$K_{\mathcal{Y}} + (p - 1)(\mathfrak{F}_{T/S} + \det \mathcal{Q}_{T/S} - g^* \det \mathcal{T}_{T/S}) \sim \phi^* K_{\mathcal{X}}.$$

Proof. Firstly note that it makes sense to discuss the pullback of Weil divisors by ϕ because it is a finite morphism. To be precise, we define such a pullback by restricting to the smooth loci of base and target and then taking the closure in $\mathcal{Y}$ of the resulting Cartier divisor pullback. Note that we are also able to remove arbitrary closed subvarieties of codimension at most 2 from $\mathcal{X}$ prior to calculating this pullback. Finally, by equidimensionality, the preimage of a big open subset of S in $\mathcal{X}$ is also a big open subset. Therefore we are able to remove such a subset without affecting the computation of Weil divisor pullbacks. Recall that the assumptions of §8.0.1 restricted to big open subsets of S and T to assume that g is flat and $\mathcal{T}_{T/S}$ is locally free.

To obtain the canonical bundle formula, we use the exact sequences:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'/\mathcal{F} \longrightarrow 0 ,$$

$$0 \longrightarrow \mathcal{F}' \longrightarrow g^* \mathcal{T}_{T/S} \longrightarrow \mathcal{Q}_{T/S} \longrightarrow 0 .$$

So we have

$$[-\det \mathcal{F}] \sim [\det(\mathcal{F}'/\mathcal{F})] - [\det \mathcal{F}'] \sim [\det(\mathcal{F}'/\mathcal{F})] + [\det \mathcal{Q}_{T/S}] - [\det g^* \mathcal{T}_{T/S}]$$

which together with (7.0.14) gives the required canonical bundle formula.

□

Remark. Note that if the original f has fibers of constant dimension, then the same formula holds over all of T , as Weil divisors extend uniquely from the big open subsets of §8.0.1.

Chapter 9

The fixed part of the linear system

In this section we prove Theorem 6.0.1 (1) and (2). First we give a reinterpretation of $\mathcal{F}$ and $\mathcal{F}'$.

Proposition 9.0.1. *In the situation of Section 8, we have*

$$\mathcal{F} = \mathcal{H}om_{\mathcal{Y}}(\Omega_{\mathcal{Y}/\mathcal{X}}, \mathcal{O}_{\mathcal{Y}})$$

and on the locus where $g: \mathcal{Y} \rightarrow T$ is flat

$$\mathcal{F}' = \mathcal{H}om_{\mathcal{Y}}(\nu^* \Omega_{\mathcal{Z}/\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}).$$

Proof. The first part of the proposition holds because

$$\mathcal{F} = \mathcal{D}er_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}}) = \Omega_{\mathcal{Y}/\mathcal{X}}^{\vee}.$$

It remains to prove the statement about $\mathcal{F}'$.

Consider the conormal sequence of $\mathcal{Z} \rightarrow \mathcal{X} \times_S T \rightarrow \mathcal{X}$:

$$I/I^2 \longrightarrow i^* \Omega_{\mathcal{X} \times_S T/\mathcal{X}} \cong h^* \Omega_{T/S} \longrightarrow \Omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow 0 .$$

Pulling back the above sequence to $\mathcal{Y}$,

$$\nu^* I/I^2 \longrightarrow g^* \Omega_{T/S} \longrightarrow \nu^* \Omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow 0. \quad (9.1)$$

Finally we have the cotangent sequence of $\mathcal{Y} \rightarrow \mathcal{Z} \rightarrow \mathcal{X}$:

$$\nu^* \Omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow \Omega_{\mathcal{Y}/\mathcal{X}} \longrightarrow \Omega_{\mathcal{Y}/\mathcal{Z}} \longrightarrow 0. \quad (9.2)$$

We break the proof into several claims which together imply the result:

1. $\mathcal{F} \hookrightarrow (\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee$:

Taking the dual of (9.2) gives

$$0 \longrightarrow \Omega_{\mathcal{Y}/\mathcal{Z}}^\vee \longrightarrow \mathcal{F} \longrightarrow (\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee.$$

But $\Omega_{\mathcal{Y}/\mathcal{Z}}$ is a torsion $\mathcal{O}_{\mathcal{Y}}$ -module, as it is supported where ν is not an isomorphism by (9.2). So $(\Omega_{\mathcal{Y}/\mathcal{Z}})^\vee = 0$, giving the required injection.

2. $(\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee / \mathcal{F}$ is torsion:

This is because $\mathcal{F} = (\Omega_{\mathcal{Y}/\mathcal{X}})^\vee \rightarrow (\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee$ is an isomorphism where ν is an isomorphism.

3. $(\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee \hookrightarrow g^* \mathcal{T}_{T/S}$:

Since g is flat, we have $g^* \mathcal{T}_{T/S} = (g^* \Omega_{T/S})^\vee$. To show the required injectivity, it is enough to show surjectivity of $g^* \Omega_{T/S} \rightarrow \nu^* \Omega_{\mathcal{Z}/\mathcal{X}}$, which we see from (9.1).

4. $(\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee$ is saturated in $g^* \mathcal{T}_{T/S}$:

Define $\mathcal{K}_{\mathcal{Y}}$ to be the kernel in the following exact sequence:

$$0 \longrightarrow \mathcal{K}_{\mathcal{Y}} \longrightarrow g^* \Omega_{T/S} \longrightarrow \nu^* \Omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow 0$$

This gives

$$0 \longrightarrow (\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee \longrightarrow g^* \mathcal{T}_{T/S} \longrightarrow \mathcal{K}_{\mathcal{Y}}^\vee.$$

The cokernel of $(\nu^* \Omega_{\mathcal{Z}/\mathcal{X}})^\vee \hookrightarrow g^* \mathcal{T}_{T/S}$ is a submodule of the torsion-free $\mathcal{O}_{\mathcal{Y}}$ -module $\mathcal{K}_{\mathcal{Y}}^\vee$. So it is also torsion-free.

□

Next we use this to give an alternate description of $\mathfrak{F}$.

Proposition 9.0.2. *For an integral Weil divisor D , denote the generic point of D by ξ_D . Then we have:*

$$\mathfrak{F} = \sum_D \text{length}((\Omega_{\mathcal{Y}/\mathcal{Z}})_{\xi_D}) D$$

Proof. Recall that the divisor $\mathfrak{F}$ is defined as the element of $\theta \in \text{Hom}(\det \mathcal{F}, \det \mathcal{F}')$ which arises from the inclusion $\mathcal{F} \hookrightarrow \mathcal{F}'$. We can state this as

$$\mathfrak{F} = \sum_D \text{length}((\det \mathcal{F}' / \theta(\det \mathcal{F}))_{\xi_D}) D,$$

where we are using that D is a codimension one point, to ensure that both $\det \mathcal{F}'$ and $\det \mathcal{F}$ are Cartier divisors there.

On the other hand, in Proposition 9.0.1 we saw that the inclusion $\mathcal{F} \hookrightarrow \mathcal{F}'$ is dual to the map ψ in the exact sequence

$$\nu^* \Omega_{\mathcal{Z}/\mathcal{X}} \xrightarrow{\psi} \Omega_{\mathcal{Y}/\mathcal{X}} \longrightarrow \Omega_{\mathcal{Y}/\mathcal{Z}} \longrightarrow 0.$$

On the locus where $\mathcal{Y}$ and $\mathcal{X}$ are both regular, the sheaf $\Omega_{\mathcal{Y}/\mathcal{X}}$ is locally free (Theorem of [KN82] and 38.A of [Mat80]) and hence torsion free. So the result follows by applying Lemma 9.0.3 to the map ψ .

□

Lemma 9.0.3. *Let $\theta: M_1 \rightarrow M_2$ be a morphism of finitely-generated modules over a Noetherian normal integral domain A such that*

$$(\theta \otimes 1): M_1 \otimes_A K(A) \longrightarrow M_2 \otimes_A K(A)$$

is an isomorphism, and assume that M_2 is torsion free in codimension one. Then

$$\text{length}(M_2/\theta(M_1))_{\mathfrak{p}} = \text{length} \left(\bigwedge^r M_1^\vee / (\bigwedge^r \theta^\vee)(\bigwedge^r M_2^\vee) \right)_{\mathfrak{p}}$$

for every height one prime ideal $\mathfrak{p}$ of A , where r is the rank of the modules M_i .

Proof. Let $\mathfrak{p}$ be a height 1 prime ideal of A . Then $A_{\mathfrak{p}}$ is a DVR with valuation $v_{\mathfrak{p}}$, and by the structure theorem for finitely-generated modules over a PID we may write $M_{1\mathfrak{p}} \cong \widetilde{M}_1 \oplus T_1$ and $M_{2\mathfrak{p}} \cong A_{\mathfrak{p}}^{\oplus f_2}$, where $\widetilde{M}_1 \cong A_{\mathfrak{p}}^{\oplus f_1}$ and $T_1 \subset M_{1\mathfrak{p}}$ is the torsion submodule. Since by hypothesis $M_1 \otimes_A K(A) \cong M_2 \otimes_A K(A)$, we have $f_1 = f_2$, so let $r = f_i$.

We may choose bases of $\widetilde{M}_1$ and $M_{2\mathfrak{p}}$ so that $\theta_{\mathfrak{p}}|_{\widetilde{M}_1}$ is represented by an $r \times r$ matrix in Smith normal form, which is diagonal with non-zero entries a_i . The quotient $M_{2\mathfrak{p}}/\theta(M_{1\mathfrak{p}})$ has length $\sum_{i=1}^r v_{\mathfrak{p}}(a_i)$.

On the other hand, the dual map $\theta^\vee: (M_{2\mathfrak{p}})^\vee \rightarrow (\widetilde{M}_1)^\vee = (M_{1\mathfrak{p}})^\vee$ is represented by the same matrix with respect to the dual bases, and the map $\bigwedge^r(\theta^\vee)$ is multiplication by the determinant of this matrix, so is multiplication by $\prod_{i=1}^r a_i$. It follows that the length of $\bigwedge^r M_{1\mathfrak{p}}^\vee / ((\bigwedge^r \theta^\vee)(\bigwedge^r M_{2\mathfrak{p}}^\vee))$ is $v_{\mathfrak{p}}(\prod_{i=1}^r a_i) = \sum_{i=1}^r v_{\mathfrak{p}}(a_i)$ as required. □

Finally we reach the proof of Theorem 6.0.1 (1) and (2).

Proposition 9.0.4 (Theorem 6.0.1 (1), (2)). *The support of $\mathfrak{F}$ is equal to the codimension one part of the locus on which ν is not an isomorphism.*

In particular, $\mathfrak{F} = 0$ if and only if $(\mathcal{X} \times_S T)_{\text{red}}$ is R_1 .

Proof. We have seen in Proposition 9.0.2 that the support of $\mathfrak{F}$ is equal to the codimension one part of the support of $\Omega_{Y/Z}$. Now the result follow from the more general Lemma 9.0.5. $\square$

Lemma 9.0.5. *Let $(A, \mathfrak{m}_A, \kappa_A)$ be a reduced local ring of characteristic $p > 0$. Suppose that the normalization $f: A \rightarrow B$ is purely inseparable and finite (the latter holds for instance if A is excellent [Sta20, Tag 03GH]).*

Then $\Omega_{B/A} = 0$ if and only if A is normal.

Proof. If $\Omega_{B/A} \neq 0$ then f is not an isomorphism and hence A is not normal. Conversely, suppose that f is not an isomorphism and we will show that $\Omega_{B/A} \neq 0$. By hypothesis κ_B/κ_A is a finite purely inseparable extension, where κ_B denotes the residue field of the local ring B . If $[\kappa_B : \kappa_A] > 1$ then we are done by the surjection

$$\Omega_{B/A} \longrightarrow \Omega_{\kappa_B/\kappa_A} \longrightarrow 0.$$

So assume $\kappa_A \rightarrow \kappa_B$ is an isomorphism, and consider $\Omega_{(B/\mathfrak{m}_A B)/\kappa_A}$. If this module is 0 then $B/\mathfrak{m}_A B$ is an étale κ_A -algebra and hence is isomorphic to a finite product of finite separable extensions of κ_A . Since $\kappa_A \cong \kappa_B$ it must be finite product of copies of κ_A , and since $B/\mathfrak{m}_A B$ is local we conclude that $B/\mathfrak{m}_A B \cong \kappa_A$. But this is a contradiction by Nakayama, as we would have $B \cong A$. Therefore $\Omega_{(B/\mathfrak{m}_A B)/\kappa_A} \neq 0$ and so we get the result from the surjection

$$\Omega_{B/A} \longrightarrow \Omega_{(B/\mathfrak{m}_A B)/\kappa_A} \longrightarrow 0.$$

$\square$

Chapter 10

The movable part of the linear system

In this section we define and deal with the movable part $\mathfrak{M}$.

10.0.1 Assumption

In addition to the assumptions in §8.0.1, in order to define $\mathfrak{M}$ we must now add:

- The locally free sheaf $\mathcal{T}_{T/S}$ must be free, so we will replace S and T by dense open subsets to assume this.

10.1 Construction

Let U be the big open subset of $\mathcal{Y}$ on which $\mathcal{Q}$ and $\mathcal{F}'$ are locally free.

We have an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow g^* \mathcal{T}_{T/S} \longrightarrow \mathcal{Q} \longrightarrow 0 .$$

Dualising this gives

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow g^*\Omega_{T/S} \longrightarrow (\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee}.$$

Furthermore, this is right exact after restriction to U since $\mathcal{E}xt^1(\mathcal{Q}^\vee|_U, \mathcal{O}_U) = 0$. Recall that $r = \text{rank } \mathcal{F} = \text{rank } \nu^*\Omega_{\mathcal{Z}/X}$.

As $\mathcal{F}'|_U$ is locally free, $\mathcal{F}'^\vee|_U = (\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee}|_U$ is also locally free, and we have $\det(\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee}|_U \cong (\bigwedge^r(\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee})|_U$. Note also that to determine a linear system of Weil divisors, it is enough to define it after restriction to the big open subset U .

Definition 10.1.1. The movable linear system $\mathfrak{M}_{T/S} \subset |\det((\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee})|$ of Weil divisors on $\mathcal{Y}$ is defined by the map

$$\bigwedge^r g^*\Omega_{T/S}|_U \longrightarrow \bigwedge^r (\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee}|_U \longrightarrow 0.$$

As U is a big open set, each divisor in the linear equivalence class extends uniquely to a divisor on $\mathcal{Y}$, and the linear equivalence class is equal to

$$\det((\nu^*\Omega_{\mathcal{Z}/X})^{\vee\vee}) = -\det(\mathcal{Q}^\vee) = \det \mathcal{Q}.$$

We now ensure that this is consistent with our earlier description of the varieties $\mathcal{W}$ and $\mathcal{V}$ (Definition 8.1.4).

Lemma 10.1.2. *Let $U \subseteq \mathcal{Y}$ be the big open subset on which $\mathcal{Q}$ and $\mathcal{F}'$ are locally free. Then the rational maps induced by $g^*\mathcal{T}_{T/S} \rightarrow \mathcal{Q} \rightarrow 0$ and $\bigwedge^r g^*\Omega_{T/S}|_U \rightarrow (\bigwedge^r \Omega_{\mathcal{Y}/X})^{\vee\vee}|_U \rightarrow 0$ have isomorphic images.*

Proof. We may replace $\mathcal{Y}$ with U to assume that $\mathcal{Q}$ is locally free. Then the dual of the sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow g^*\mathcal{T}_{T/S} \longrightarrow \mathcal{Q} \longrightarrow 0$$

is exact:

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow g^*\Omega_{T/S} \longrightarrow (\nu^*\Omega_{\mathcal{Z}/\mathcal{X}})^{\vee\vee} \longrightarrow 0.$$

The map $\bigwedge^r \mathcal{F}' \rightarrow \bigwedge^r g^*\mathcal{T}_{T/S}$ determines composition of $\mathcal{Y} \dashrightarrow \mathcal{W}_{\text{Im}}$ with the Plücker embedding of $\text{Gr}_T(r, \mathcal{T}_{T/S})$. If we then compose this with the dual Grassmannian isomorphism, then using the identifications

$$\begin{aligned} \left(\bigwedge^r g^*\Omega_{T/S} \rightarrow \bigwedge^r \nu^*\Omega_{\mathcal{Z}/\mathcal{X}} \right)^\vee &= \left(\left(\bigwedge^r \nu^*\Omega_{\mathcal{Z}/\mathcal{X}} \right)^\vee \rightarrow \left(\bigwedge^r g^*\mathcal{T}_{T/S} \right) \right) \\ &= \left(\bigwedge^r \mathcal{F}' \rightarrow \bigwedge^r g^*\mathcal{T}_{T/S} \right) \end{aligned}$$

the resulting map $\mathcal{Y} \dashrightarrow \text{Gr}_T\left(\binom{d}{r} - 1, \bigwedge^r \Omega_{T/S}\right)$ is exactly the one given by $\mathfrak{M}_{T/S}$. $\square$

Note that the map $g^*\Omega_{T/S} \rightarrow (\nu^*\Omega_{\mathcal{Z}/\mathcal{X}})^{\vee\vee}$ matches the map $h^*\Omega_{T/S} \rightarrow \Omega_{\mathcal{Z}/\mathcal{X}}$ on the non-empty open subset of $\mathcal{Y}$ on which $\mathcal{Y} \rightarrow \mathcal{Z}$ is an isomorphism. Therefore, as we are interested in a movable linear system, it is enough to study it on this open subset, for there is a unique extension to all of $\mathcal{Y}$. Therefore we now shift attention to $\mathcal{Z}$.

10.2 The essential part of a base change

In this subsection we prove Theorem 6.0.1 (3) via Theorem 6.0.4. We begin with some preparatory lemmas.

First note that in one direction of Theorem 6.0.1 (3) it is easy to obtain a stronger statement:

Lemma 10.2.1. *If $\mathcal{X} \times_S T$ is regular in codimension zero, then $\mathcal{Q}_{T/S} = 0$.*

Proof. The exact sequence

$$\nu^*I/I^2 \longrightarrow g^*\Omega_{T/S} \longrightarrow \nu^*\Omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow 0$$

dualizes to

$$0 \longrightarrow \mathcal{F}' \longrightarrow g^* \mathcal{T}_{T/S} \longrightarrow (\nu^* I/I^2)^\vee.$$

So recalling that we have assumed $g: \mathcal{Y} \rightarrow T$ to be flat, we get the natural inclusion

$$\mathcal{Q}_{T/S} \subseteq (\nu^* I/I^2)^\vee.$$

The statement follows because if $\mathcal{X} \times_S T$ is R_0 then I/I^2 is torsion and so the right-hand side is zero. $\square$

Next is a useful criterion for when a base change is non-reduced.

Lemma 10.2.2. *In the situation of §8.0.1, $\mathcal{X} \times_S T$ is non-reduced if and only if $\deg(\mathcal{Y} \rightarrow \mathcal{X}) < \deg(T \rightarrow S)$.*

Proof. First recall that reducedness is equivalent to having properties R_0 and S_1 . Since $T \rightarrow S$ is a universal homeomorphism, to show that Serre's condition S_k is preserved we may reduce to the local case. Since $\mathcal{X} \times_S T \rightarrow \mathcal{X}$ is flat, then [Mat89, Theorems 15.1 and 23.3] imply that $\mathcal{X} \times_S T$ is S_1 because $\mathcal{X}$ is. So it remains to show that $\mathcal{X} \times_S T$ is not R_0 if and only if the degree inequality is strict. Since the extension $K(T)/K(S)$ is purely inseparable, $K(\mathcal{X}) \otimes_{K(S)} K(T)$ is an Artinian local algebra, and its dimension as a $K(\mathcal{X})$ -module is exactly $p^{\deg(T \rightarrow S)}$ because $K(T)$ is a flat $K(S)$ -module. Since the residue field of this algebra is $K(\mathcal{Y})$, then it contains a copy of $K(\mathcal{Y})$ by the Cohen structure theorem, and the containment is strict if and only if the base change is non-reduced. $\square$

Finally, we prove that $\mathcal{Z} \rightarrow T$ is its own Stein factorization:

Lemma 10.2.3. *In the situation of §8.0.1, $h_* \mathcal{O}_{\mathcal{Z}} = \mathcal{O}_T$.*

Proof. We may assume that S and T are affine. Since $f_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$ and $\mathcal{X} \rightarrow S$ is assumed flat, it follows that $f_{T*} \mathcal{O}_{\mathcal{X} \times_S T} = \mathcal{O}_T$ by [Sta20, Tag 03GY]. So if A is a

homogeneous coordinate ring of $\mathcal{X} \times_S T$, then in the zeroth graded piece, $A_0 = \mathcal{O}_T$ is reduced. Now $\mathcal{Z}$ is realized as $\text{Proj}(A/\sqrt{0})$, and the quotient map $A \rightarrow A/\sqrt{0}$ is a graded homomorphism that induces an isomorphism in grading 0. Therefore $(A/\sqrt{0})_0 = \mathcal{O}_T$; that is, $H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = H^0(T, \mathcal{O}_T)$. $\square$

We now prove a version of Theorem 6.0.4 for fibrations, from which Theorem 6.0.4 follows immediately by restriction to generic fibers. The purpose of Theorem 6.0.4 is to reduce the study of geometrically non-reduced varieties to the case where $H^0(\mathcal{Z}, \Omega_{\mathcal{Z}/\mathcal{X}}^\vee) = 0$. We see from the sequence

$$0 \longrightarrow \Omega_{\mathcal{Z}/\mathcal{X}}^\vee \longrightarrow h^* \mathcal{T}_{T/S} \longrightarrow (I/I^2)^\vee$$

that on such a variety we also have

$$H^0(\mathcal{Z}, h^* \mathcal{T}_{T/S}) \hookrightarrow H^0(\mathcal{Z}, (I/I^2)^\vee).$$

Proposition 10.2.4 (Theorem 6.0.4). *With notation as in §8.0.1, after possibly replacing S and T by dense open subschemes there exist normal k -varieties and k -morphisms fitting into the following commutative diagram*

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T' & \longrightarrow & S \end{array}$$

such that the following hold:

1. $\mathcal{Y}'$ is the normalization of $\mathcal{Z}' := (\mathcal{X} \times_S T')_{\text{red}}$.
2. $\mathcal{Y} \rightarrow \mathcal{Y}'$ and $T \rightarrow T'$ have the same degree.
3. If $T \rightarrow T'' \rightarrow S$ satisfies property (2) with respect to $\mathcal{Y}'' = (\mathcal{X} \times_S T'')_{\text{red}}^\nu$, then

there is a factorization

$$T \rightarrow T'' \rightarrow T' \rightarrow T.$$

4. $\mathcal{Y}' \times_{T'} T$ is reduced and $\mathcal{Y}$ is its normalization.

5. $h'_*(\Omega_{\mathcal{Z}'/X}^\vee) = 0$, where $h': \mathcal{Z}' \rightarrow T'$.

Furthermore, T' is the unique subextension satisfying both (2) and (5), and the unique subextension satisfying both (2) and (3).

Proof. We may replace S and T by open subsets to assume that $T = \operatorname{Spec} A$ is affine, $\Omega_{T/S} \cong \mathcal{O}_T^{\oplus d}$ is free, and $g: \mathcal{Z} \rightarrow T$ is flat. We will show that, after shrinking the base, $h_*(\Omega_{\mathcal{Z}/X}^\vee)$ is a foliation on T , and it induces a foliation $\mathcal{H}$ on $\mathcal{Y}$. We then show that the varieties obtained by taking the quotients of T and $\mathcal{Y}$ by these respective foliations satisfy the desired properties.

Pushing forward the inclusion $\Omega_{\mathcal{Z}/X}^\vee \hookrightarrow h^*\mathcal{T}_{T/S}$ yields an injection

$$h_*(\Omega_{\mathcal{Z}/X}^\vee) \hookrightarrow h_*h^*\mathcal{T}_{T/S} \cong \mathcal{T}_{T/S}.$$

where the isomorphism uses the fact that $\mathcal{T}_{T/S}$ is free, and $h_*\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_T$ by Lemma 10.2.3.

Step 1: $\mathcal{G} := h_*(\Omega_{\mathcal{Z}/X}^\vee) \subset \mathcal{T}_{T/S}$ is a foliation, inducing a morphism $T \rightarrow T'$.

From the sequence

$$0 \longrightarrow \Omega_{\mathcal{Z}/X}^\vee \longrightarrow h^*\mathcal{T}_{T/S} \longrightarrow (I/I^2)^\vee$$

we see that $\Omega_{\mathcal{Z}/X}^\vee \subset h^*\mathcal{T}_{T/S}$ is a saturated subsheaf, because its cokernel is a subsheaf of a torsion free sheaf. Therefore $h_*(\Omega_{\mathcal{Z}/X}^\vee)$ is saturated in $h_*h^*\mathcal{T}_{T/S}$. Since closedness under Lie brackets follows from p -closedness [Ger64], we only need to show the p -closedness of $\mathcal{G}$. Recall that we assume that $T = \operatorname{Spec} A$.

Let $\Delta \in H^0(T, \mathcal{G}) \subset H^0(T, h_*h^*\mathcal{T}_{T/S}) = H^0(\mathcal{Z}, h^*\mathcal{T}_{T/S})$. We first unravel what it means to say $\Delta \in H^0(T, \mathcal{G})$. Let $\{U_i = \operatorname{Spec} B_i\}$ be an affine open cover of $\mathcal{Z}$. Then

Δ corresponds to a compatible collection of

$$\Delta_i \in \Gamma(U_i, \text{Hom}_{B_i}(\Omega_{\mathcal{Z}/X}|_{U_i}, B_i)) = \text{Der}_{\mathcal{O}_X}(B_i, B_i)$$

and since they glue, the images of the Δ_i when applied to a fixed element of $A = H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \subset B_i$ also glue to give an element of $H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = A$. Now each $\Delta_i^p \in \text{Der}_{\mathcal{O}_X}(B_i, B_i)$, and the collection $\{U_i, \Delta_i^p \in \Gamma(U_i, \text{Hom}_{B_i}(\Omega_{\mathcal{Z}/X}|_{U_i}, B_i))\}$ still glues since the following diagram commutes.

$$\begin{array}{ccccccc} \mathcal{O}_{U_i} & \xrightarrow{\Delta_i} & \mathcal{O}_{U_i} & \xrightarrow{\Delta_i} & \cdots & \xrightarrow{\Delta_i} & \mathcal{O}_{U_i} \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{O}_{U_i \cap U_j} & \xrightarrow{\Delta_i|_{\mathcal{O}_{U_i \cap U_j}}} & \mathcal{O}_{U_i \cap U_j} & \xrightarrow{\Delta_i|_{\mathcal{O}_{U_i \cap U_j}}} & \cdots & \xrightarrow{\Delta_i|_{\mathcal{O}_{U_i \cap U_j}}} & \mathcal{O}_{U_i \cap U_j} \end{array}$$

So we obtain a section of $H^0(T, \mathcal{G})$ which agrees with Δ^p as a function on $\mathcal{O}_T$.

Step 2: The image $\mathcal{H}$ of $h^*h_*\Omega_{\mathcal{Z}/X}^\vee \rightarrow \Omega_{\mathcal{Z}/X}^\vee$ is p -closed.

Let $U = \text{Spec}(B)$ be an affine open subset of $\mathcal{Z}$. Then $\Omega_{\mathcal{Z}/X}^\vee(U) = \text{Der}_{\mathcal{O}_X}(B, B)$. As in the proof of the previous lemma, a section coming from $h^*h_*(\Omega_{\mathcal{Z}/X}^\vee)$ is a compatible system of derivations on charts, which glue to give a global section. We saw above that this means their p -powers also glue to give a global section. It follows from this that if Δ is in the image of $h^*h_*(\Omega_{\mathcal{Z}/X}^\vee) \rightarrow \Omega_{\mathcal{Z}/X}^\vee$ then so is Δ^p .

Step 3: Obtaining a variety $\mathcal{Y}'$ fitting into the following diagram:

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T' & \longrightarrow & S. \end{array}$$

Let U be the open subset on which $\mathcal{Y} \cong \mathcal{Z}$. Then $\mathcal{H}$ induces a morphism $U \rightarrow U' \rightarrow \mathcal{X}$, from which we can take normalizations of $\mathcal{X}$ in the corresponding function fields, to get a normal variety $\mathcal{Y}'$ that factors the morphism $\mathcal{Y} \rightarrow \mathcal{Y}' \rightarrow \mathcal{X}$. The middle vertical arrow exists because $\mathcal{O}_{T'}$ is killed by $\mathcal{G}$, and so its image in $\mathcal{O}_U$ is also killed

by the image of $h^*\mathcal{G}$ in $\Omega_{\mathcal{Z}/\mathcal{X}}^\vee(U)$.

Step 4: $\deg(\mathcal{Y} \rightarrow \mathcal{Y}') = \deg(T \rightarrow T')$.

Consider the composition

$$h^*h_*\Omega_{\mathcal{Z}/\mathcal{X}}^\vee \twoheadrightarrow \mathcal{H} \hookrightarrow \Omega_{\mathcal{Z}/\mathcal{X}}^\vee \hookrightarrow h^*\mathcal{T}_{T/S}.$$

As $h^*h_*\Omega_{\mathcal{Z}/\mathcal{X}}^\vee \rightarrow h^*\mathcal{T}_{T/S}$ is injective by flatness, we find that $h^*h_*\Omega_{\mathcal{Z}/\mathcal{X}}^\vee \cong \mathcal{H}$, and so $\deg(\mathcal{Y} \rightarrow \mathcal{Y}') = \deg(T \rightarrow T') = p^{\text{rank } \mathcal{H}}$.

Step 5: We claim that both squares in the diagram of Step 3 are normalized base changes and $\mathcal{Y}' \times_{T'} T$ is reduced.

Let $\tilde{\mathcal{Y}}'$ be the normalization of $\mathcal{X} \times_S T'$. By universal properties this fits into a similar diagram as above, where both squares are normalized base changes, so it suffices to show that $\mathcal{Y}' = \tilde{\mathcal{Y}}'$. By universal properties there is a morphism $\mathcal{Y}' \rightarrow \tilde{\mathcal{Y}}'$, but as $\mathcal{Y}$ is the normalized base change of $\tilde{\mathcal{Y}}'$ by T , the morphism $\mathcal{Y} \rightarrow \tilde{\mathcal{Y}}'$ can have degree at most that of $T \rightarrow T'$. Therefore $\mathcal{Y}' \rightarrow \tilde{\mathcal{Y}}'$ is an isomorphism as both are normal and the morphism between them is forced to have degree at most 1. That $\mathcal{Y}' \times_{T'} T$ is reduced follows from Lemma 10.2.2.

Step 6: Our varieties satisfy part (3).

Suppose

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}'' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T'' & \longrightarrow & S \end{array}$$

is a diagram of normalized base changes such that $\deg(\mathcal{Y} \rightarrow \mathcal{Y}'') = \deg(T \rightarrow T'')$.

Then let $\mathcal{G}''$ be the foliation determining $T \rightarrow T''$ and $\mathcal{H}''$ be the foliation determining $\mathcal{Y} \rightarrow \mathcal{Y}''$. We claim that $\mathcal{G}'' \subseteq \mathcal{G}$ and $\mathcal{H}'' \subseteq \mathcal{H}$.

Let

$$\begin{array}{ccccc} \mathcal{Z} & \longrightarrow & \mathcal{Z}'' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T'' & \longrightarrow & S \end{array}$$

be the corresponding diagram of reduced base changes. Note that it follows from the assumptions on degrees that $\mathcal{Z} = \mathcal{Z}'' \times_{T''} T$, that is to say this base change is already reduced. This means $\Omega_{\mathcal{Z}/\mathcal{Z}''} = h^* \Omega_{T/T''}$, which is trivial (after shrinking the base). The surjection $\Omega_{\mathcal{Z}/\mathcal{X}} \rightarrow \Omega_{\mathcal{Z}/\mathcal{Z}''}$ dualizes to give an injection $\Omega_{\mathcal{Z}/\mathcal{Z}''}^\vee \hookrightarrow \Omega_{\mathcal{Z}/\mathcal{X}}^\vee$.

This means that $h_*(\Omega_{\mathcal{Z}/\mathcal{Z}''}^\vee) = \mathcal{G}'' \subset \mathcal{G}$, and hence we also have that $\mathcal{H}'' \subset \mathcal{H}$ (after restricting to a suitable open subset and taking saturations).

So that there is a factorization

$$\begin{array}{ccccccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}'' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T'' & \longrightarrow & T' & \longrightarrow & S \end{array}$$

from which we get

$$\begin{array}{ccccccc} \mathcal{Z} & \longrightarrow & \mathcal{Z}'' & \longrightarrow & \mathcal{Z}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & T'' & \longrightarrow & T' & \longrightarrow & S \end{array}$$

where the left and middle squares are Cartesian. By a similar argument to above, this means that $H^0(\mathcal{Z}'', \Omega_{\mathcal{Z}''/\mathcal{X}}^\vee) \neq 0$ so long as $\mathcal{Z}'' \neq \mathcal{Z}'$, as the morphism $\mathcal{Z}'' \rightarrow \mathcal{Z}'$ determines such a global section.

Step 7: Our varieties satisfy part (5).

Suppose that $h_*(\Omega_{\mathcal{Z}'/\mathcal{X}}^\vee) \neq 0$. Then by applying steps 1-5, we obtain a morphism $T' \rightarrow T''$ with positive degree such that $\mathcal{Y}'' \times_{T''} T$ is reduced. But by step 6 this implies that there is a morphism $T'' \rightarrow T'$ such that the composition factors into $T \rightarrow S$, and so $T'' = T'$, giving a contradiction.

Step 8: The “furthermore.”

It is clear that there can be at most one morphism satisfying both (2) and (3). We have shown the existence of this already. Now suppose we have another morphism $T \rightarrow T''$ satisfying (2) and (5). Then by (2) we obtain a factorization $T \rightarrow T'' \rightarrow T'$. But then $\deg(\mathcal{Y}'' \rightarrow \mathcal{Y}') = \deg(T'' \rightarrow T')$, and as in the argument of step 6, $h''_*(\Omega_{\mathcal{Z}''/X}^\vee) \neq 0$ unless this degree is 1, as required.

□

Proposition 10.2.5 (Theorem 6.0.1 (3)). *In the situation of §8.0.1, after possibly replacing S and T by open subschemes, we have $\mathfrak{M}_{T/S} = 0 \iff \mathcal{X} \times_S T$ is reduced.*

Proof. If $\mathcal{X} \times_S T$ is reduced, then it follows from Lemma 10.2.1 that $\mathfrak{M} = 0$. So suppose that $\mathcal{X} \times_S T$ is non-reduced. Apply Proposition 10.2.4 to obtain $\mathcal{Y}'$ and T' such that $\mathcal{Y}' \times_{T'} T$ is reduced. The linear system $\mathfrak{M}_{T/S} \subset |\det((\nu^* \Omega_{\mathcal{Z}/X})^{\vee\vee})| = |-\det \mathcal{F}'_{T/S}|$ being 0 is equivalent to the line bundle $-\det \mathcal{F}'_{T/S}|_U$ being trivial, where U is the big open subset from §10.1, and similarly for T'/S . Now $-\det \mathcal{F}'_{T/S} \geq -\psi^* \det \mathcal{F}'_{T'/S}$ by Lemma 10.2.6, so $\mathfrak{M}_{T'/S} \neq 0$ implies that $\mathfrak{M}_{T/S} \neq 0$. So we may replace $\mathcal{Y}$ and T with $\mathcal{Y}'$ and T' to assume that $H^0(\mathcal{Z}, \Omega_{\mathcal{Z}/X}^\vee) = 0$.

Let $\mathcal{R}$ be the cokernel of $0 \rightarrow \Omega_{\mathcal{Z}/X}^\vee \rightarrow h^* \mathcal{T}_{T/S}$. The map $h^* \mathcal{T}_{T/S} \rightarrow \mathcal{R}$ matches the map $g^* \mathcal{T}_{T/S} \rightarrow \mathcal{Q}$ on the open subset U on which $\mathcal{Y} \cong \mathcal{Z}$. Note that the generic rank of $\mathcal{R}$ is less than that of $\mathcal{T}_{T/S}$.

We see that $H^0(\mathcal{Z}, h^* \mathcal{T}_{T/S}) \rightarrow H^0(\mathcal{Z}, \mathcal{R})$ is injective, and $\mathcal{R}$ is torsion free because it is a subsheaf of $(I/I^2)^\vee$. Therefore for any open subset U of $\mathcal{Z}$, the map $h^* \mathcal{T}_{T/S}|_U \rightarrow \mathcal{R}|_U \rightarrow 0$ cannot be a quotient by a subsheaf that pulled back from T . Therefore the rational map $\mathcal{Z} \dashrightarrow \mathcal{W}'_{\text{lm}}$ induced by $h^* \mathcal{T}_{T/S} \rightarrow \mathcal{R} \rightarrow 0$ is non-trivial. This matches the rational map given by $g^* \mathcal{T}_{T/S} \rightarrow \mathcal{Q} \rightarrow 0$ as these maps match on an open subset, which in turn matches the rational map induced by $\mathfrak{M}$ by Lemma 10.1.2. Therefore $\mathfrak{M} \neq 0$.

□

Proposition 10.2.6. *Let $f: \mathcal{X} \rightarrow S$ be a morphism of normal varieties with equidimensional fibers, and let $T \rightarrow T' \rightarrow S$ be a height one composition of morphisms of normal varieties. Let the following be the corresponding normalized base changes and their associated foliations:*

$$\begin{array}{ccccc}
 & & \phi & & \\
 & \swarrow & \mathcal{F}_{T/S} & \searrow & \\
 \mathcal{Y} & \xrightarrow[\mathcal{F}_{T/T'}]{\psi} & \mathcal{Y}' & \xrightarrow[\mathcal{F}_{T'/S}]{} & \mathcal{X} \\
 \downarrow g & & \downarrow g' & & \downarrow f \\
 T & \longrightarrow & T' & \longrightarrow & S
 \end{array}$$

Let $\mathcal{F}'_{T/T'}$ denote the saturation of $\mathcal{F}_{T/T'}$ in $g^*\mathcal{T}_{T/T'}$, and similarly for $\mathcal{F}'_{T/S}$ and $\mathcal{F}'_{T'/S}$. Then

$$|\det \mathcal{F}'_{T/T'} + \psi^* \det \mathcal{F}'_{T'/S} - \det \mathcal{F}'_{T/S}| \neq \emptyset.$$

Proof. Note that we are free to remove a closed subvariety of codimension at most two from S , T and T' , and also from $\mathcal{Y}$, $\mathcal{Y}'$, and $\mathcal{X}$ (by equidimensionality) to assume that g , g' , and ψ are flat. The flatness of ψ is attainable because $\mathcal{Y}$ becomes regular and $\mathcal{F}_{T/T'}$ locally free after removing a closed subset of codimension at least 2, and in this situation ψ is flat by Proposition 2.1.4 of [Mad16].

We claim there is a diagram of exact sequences as follows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{F}_{T/T'} & \xrightarrow{a_1} & \mathcal{F}_{T/S} & \xrightarrow{b_1} & \psi^* \mathcal{F}_{T'/S} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{F}'_{T/T'} & \xrightarrow{a_2} & \mathcal{F}'_{T/S} & \xrightarrow{b_2} & \psi^* \mathcal{F}'_{T'/S} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & g^* \mathcal{T}_{T/T'} & \longrightarrow & g^* \mathcal{T}_{T/S} & \longrightarrow & \psi^* g'^* \mathcal{T}_{T'/S} \longrightarrow 0
 \end{array}$$

The proof of Theorem 3.11 in [PW20] gives exactness of the top and bottom rows of the diagram, so we only need to show that the middle row fits in.

$\psi^* \mathcal{F}'_{T'/S}$ is saturated in $\psi^* g'^* \mathcal{T}_{T'/S}$ by Lemma 7.0.5, and therefore $\psi^* \mathcal{F}'_{T'/S}$ is the

saturation of $\psi^* \mathcal{F}_{T'/S}$ in $\psi^* g'^* \mathcal{T}_{T'/S}$. The left and middle exactness of the second row then follow from the exactness of the first:

- (Left exactness) If $\delta \in \text{Ker}(\mathcal{F}'_{T/T'} \rightarrow \mathcal{F}'_{T/S})$ then $f\delta \in \text{Ker}(\mathcal{F}_{T/T'} \rightarrow \mathcal{F}_{T/S})$ for some nonzero $f \in \mathcal{O}_{\mathcal{Y}}$, and so $\delta = 0$ because $\mathcal{F}'_{T/S}$ is torsion free.
- (Middle exactness) If $\delta \in \text{Im}(\mathcal{F}'_{T/T'} \rightarrow \mathcal{F}'_{T/S})$ then $f\delta \in \text{Im}(\mathcal{F}_{T/T'} \rightarrow \mathcal{F}_{T/S})$ for some nonzero $f \in \mathcal{O}_{\mathcal{Y}}$, and so $0 = b_1(f\delta) = f \cdot b_2(\delta)$.

Conversely, if $\Delta \in \text{Ker}(b_2)$ then $g\Delta \in \text{Ker}(b_1) = \text{Im}(\mathcal{F}_{T/T'} \rightarrow \mathcal{F}_{T/S})$ for some nonzero $g \in \mathcal{O}_{\mathcal{Y}}$. Then there is an element $\delta \in \mathcal{F}_{T/T'}$ such that $a_1(\delta) = g\Delta$. But $\delta \in \mathcal{F}'_{T/T'}$ and so $a_2(\delta) = g\Delta$. But $\mathcal{F}'_{T/T'}$ is saturated in $g^* \mathcal{T}_{T/S}$, and so is saturated in $\mathcal{F}'_{T/S}$ too. That means there must be δ' such that $a_2(\delta') = \Delta$.

Let $\mathcal{G}$ denote the image of $\mathcal{F}'_{T/S}$ in $\psi^* \mathcal{F}'_{T'/S}$. Since $\psi^* \mathcal{F}_{T'/S} \hookrightarrow \mathcal{G} \hookrightarrow \psi^* \mathcal{F}'_{T'/S}$, the ranks of $\mathcal{G}$ and $\psi^* \mathcal{F}'_{T'/S}$ are the same, and so the inclusion $\mathcal{G} \hookrightarrow \psi^* \mathcal{F}'_{T'/S}$ gives an element of $\text{Hom}_{\mathcal{Y}}(\det \mathcal{G}, \psi^* \det \mathcal{F}'_{T'/S}) \cong H^0(\mathcal{Y}, \psi^* \det \mathcal{F}'_{T'/S} - \det \mathcal{G})$. Therefore

$$\det \mathcal{F}'_{T/S} = \det \mathcal{F}'_{T/T'} + \det \mathcal{G} \leq \det \mathcal{F}'_{T/T'} + \psi^* \det \mathcal{F}'_{T'/S}.$$

□

10.3 The universal property

In this section we prove (4), (5), and (6) of Theorem 6.0.1. First we verify that we can replace $\mathcal{X}$ by a birational model, and in particular pass to a resolution of indeterminacies.

Recall, using Definition 8.1.4 and Lemma 10.1.2, that the varieties $\mathcal{V}$ and $\mathcal{W}$ are defined as follows. The movable linear system $\mathfrak{M}_{T/S}$ on $\mathcal{Y}$ of Section 10 defines a rational map $\mathcal{Y} \dashrightarrow \mathcal{W}_{\text{Im}}$, and $\mathcal{V}$ is defined as the Stein factorization of $\mathcal{X} \dashrightarrow \mathcal{W}_{\text{Im}}^p$. Then $\mathcal{W}$ is defined as the normalization of the maximal reduced subscheme of $\mathcal{V} \times_S T$.

Proposition 10.3.1. *The birational equivalence class of the rational map $\mathcal{X} \dashrightarrow \mathcal{V}$ and linear system $\mathfrak{M}$ is independent of the choice of models $\mathcal{X}/S$ within its birational equivalence class.*

Proof. The construction is local on the base, so it is clear that the birational equivalence class of $\mathcal{V}$ is unaffected by birational changes in S . Now given any two birational varieties $\mathcal{X}_1$ and $\mathcal{X}_2$ over S , we can let $\tilde{\mathcal{X}}$ be the normalization of the graph of $\mathcal{X}_1 \dashrightarrow \mathcal{X}_2$, to produce a normal variety with birational morphisms to both. Therefore it is enough to fix a model $\tilde{\mathcal{X}}$ of $\mathcal{X}$, which is a resolution of indeterminacies of $\mathcal{X} \dashrightarrow \mathcal{V}$, and show that applying the construction of Theorem 8.1.6 to $\tilde{\mathcal{X}} \rightarrow S$ results in the same rational map $\tilde{\mathcal{X}} \dashrightarrow \mathcal{V} \rightarrow S$ up to birational equivalence of $\mathcal{V}$. It is harmless to shrink S again to assume that $\tilde{\mathcal{X}} \rightarrow S$ (and its normalized base change) is still flat.

The inclusion of function fields $K(\mathcal{Y}) = K(\tilde{\mathcal{Y}}) \supset K(\mathcal{X}) \supset K(\tilde{\mathcal{Y}})^p$ determines a unique height one purely inseparable morphism $\tilde{\phi}: \tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{X}}$ fitting into the diagram below, and the induced map $\beta: \tilde{\mathcal{Y}} \rightarrow \mathcal{W}_{\text{Im}}$ is a morphism since $\tilde{\mathcal{X}}$ resolves the indeterminacies of $\mathcal{X} \dashrightarrow \mathcal{V}$. Denote by $\tilde{\mathcal{F}}$ the foliation corresponding to $\tilde{\phi}$.

$$\begin{array}{ccccc}
 \tilde{\mathcal{Y}} & \xrightarrow{\quad / \tilde{\mathcal{F}} \quad} & \tilde{\mathcal{X}} & \longrightarrow & \tilde{\mathcal{Y}}^p \\
 \searrow \pi & & \searrow & & \searrow \\
 & \mathcal{Y} & \xrightarrow{\quad / \mathcal{F} \quad} & \mathcal{X} & \\
 \beta \swarrow & \downarrow \alpha & & \downarrow & \searrow \\
 & \mathcal{W}_{\text{Im}} & \longrightarrow & \mathcal{V} & \longrightarrow \mathcal{W}_{\text{Im}}^p \\
 & \downarrow & & \downarrow & \downarrow \\
 & T & \longrightarrow & S & \longrightarrow T^p
 \end{array}$$

Let $\tilde{\mathcal{F}}'$ denote the saturation of $\tilde{\mathcal{F}}$ in $(g \circ \pi)^* \mathcal{T}_{T/S}$. Then define $\tilde{\mathcal{Q}}_{T/S}$ as the quotient. Let $U \subset \mathcal{Y}$ be the open set on which $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$ is an isomorphism. Then $\tilde{\mathcal{F}}'|_{\pi^{-1}(U)} \cong \mathcal{F}'|_U$, and the maps $\pi^* g^* \mathcal{E} \rightarrow \det \mathcal{Q}_{\tilde{\mathcal{F}}}$ and $g^* \mathcal{E} \rightarrow \det \mathcal{Q}_{T/S}$ agree on their restriction to U . So $\alpha \circ \pi: \tilde{\mathcal{Y}} \rightarrow \mathcal{W}_{\text{Im}}$ agrees with $\beta: \tilde{\mathcal{Y}} \rightarrow \mathcal{W}_{\text{Im}}$ on $\pi^{-1}(U)$ as maps to $\text{Gr}_T(r, \mathcal{T}_{T/S})$, and so it follows that these morphisms are the same.

□

In view of Proposition 10.3.1 we may assume that $\mathcal{Y} \rightarrow \mathcal{W} \rightarrow T$ and $\mathcal{X} \rightarrow \mathcal{V} \rightarrow S$ are morphisms. We denote by $\mathfrak{M}_{\mathcal{W}/\mathcal{V}}$ the movable part of the canonical linear system produced by the normalized base change $\mathcal{Y} = (\mathcal{X} \times_{\mathcal{V}} \mathcal{W})_{\text{red}}^{\nu}$ over a suitable open subset of $\mathcal{W}$, or on the generic fiber of $\mathcal{Y} \rightarrow \mathcal{W}$, as constructed in §10.1. We require $\mathcal{Y} \rightarrow \mathcal{W}$ to be a morphism in order to make sense of $\mathfrak{M}_{\mathcal{W}/\mathcal{V}}$, but this is not strictly necessary, as we could define $\mathfrak{M}_{\mathcal{W}/\mathcal{V}}$ to be the birational transform of that on a higher model.

Lemma 10.3.2. $\mathfrak{M}_{\mathcal{W}/\mathcal{V}} = 0$ (over the generic point of $\mathcal{W}$).

Proof. Denote

$$\mathcal{Y} \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{g_1} \mathcal{W} \xrightarrow{g_2} T \end{array}$$

By Proposition 10.3.1 we may assume that $\mathcal{Y} \rightarrow \mathcal{W} \rightarrow T$ are morphisms. We are then free to shrink $\mathcal{V}$ and $\mathcal{W}$ as much as necessary. In particular we may assume that $\mathcal{T}_{\mathcal{W}/\mathcal{V}} \hookrightarrow g_2^* \mathcal{T}_{T/S}$ is saturated and that $\mathcal{Y} \rightarrow \mathcal{W}$ is flat. Then $\mathcal{F}$ has the same saturation in $g_1^* \mathcal{T}_{\mathcal{W}/\mathcal{V}}$ and in $g^* \mathcal{T}_{T/S}$, so we have the following exact diagram of sheaves on $\mathcal{Y}$.

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}'_{\mathcal{W}/\mathcal{V}} & \longrightarrow & g_1^* \mathcal{T}_{\mathcal{W}/\mathcal{V}} & \longrightarrow & \mathcal{Q}_{\mathcal{W}/\mathcal{V}} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}'_{T/S} & \longrightarrow & g^* \mathcal{T}_{T/S} & \longrightarrow & \mathcal{Q}_{T/S} \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & \tilde{\mathcal{Q}} \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

We know that $\det \mathcal{Q}_{T/S}$ is trivial on the generic fiber over $\mathcal{W}_{\text{lm}}$, and so from the factorization $\mathcal{Y} \rightarrow \mathcal{W} \rightarrow \mathcal{W}_{\text{lm}}^p$ we know that $(\det \mathcal{Q}_{T/S})^{\otimes p}$ is trivial on the generic

fiber $Y_{\mathcal{W}}$ of $\mathcal{Y} \rightarrow \mathcal{W}$. But then $\det \mathcal{Q}_{T/S}|_Y$ is a globally generated and numerically trivial $\mathbb{Q}$ -Cartier divisor, and hence trivial as $Y_{\mathcal{W}} \rightarrow \operatorname{Spec} K(\mathcal{W})$ is projective.

Therefore on a big open subset U' of $Y_{\mathcal{W}}$, we have two globally generated vector bundles $\mathcal{Q}_{\mathcal{W}/\mathcal{V}}|_{U'}$ and $\tilde{\mathcal{Q}}|_{U'}$ with $\det \mathcal{Q}_{\mathcal{W}/\mathcal{V}}|_{U'} \otimes \det \tilde{\mathcal{Q}}|_{U'} \cong \mathcal{O}_{U'}$. Therefore $\det \mathcal{Q}_{\mathcal{W}/\mathcal{V}}|_{U'} \cong \det \tilde{\mathcal{Q}}|_{U'} \cong \mathcal{O}_{U'}$.

□

Proposition 10.3.3 (Theorem 6.0.1(4)). *Let ξ be the generic point of $\mathcal{V}$. Then $\mathcal{X}_{\xi} \otimes_{K(\mathcal{V})} K(\mathcal{W})$ is reduced.*

Proof. This follows immediately from Lemma 10.3.2 and Proposition 10.2.5. □

Remark. $\mathcal{X}_{\xi}$ need not be geometrically reduced; see Example 11.2.2.

Next we prove points (5) and (6) of Theorem 6.0.1.

Proposition 10.3.4 (Theorem 6.0.1 (5), (6)). *Let $\mathcal{U} \rightarrow S$ be a variety with a projective morphism $\alpha: \mathcal{X} \rightarrow \mathcal{U}$ such that $\mathcal{X} \times_{\mathcal{U}} \operatorname{Spec}(K((\mathcal{U} \times_S T)_{\text{red}}))$ is reduced, where $K((\mathcal{U} \times_S T)_{\text{red}})$ is the function field of the variety $(\mathcal{U} \times_S T)_{\text{red}}$. Then there is a rational map $(\mathcal{U} \times_S T)_{\text{red}} \dashrightarrow \mathcal{W}_{\text{Im}}$ factoring the natural map from $\mathcal{Y}$.*

Furthermore, if $\alpha_ \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{U}}$, then there is a rational map $\mathcal{U} \dashrightarrow \mathcal{V}$ factoring $\mathcal{X} \dashrightarrow \mathcal{V}$.*

Proof. Since we only require rational maps, we will shrink $\mathcal{U}$ as needed throughout the proof. We have a diagram

$$\begin{array}{ccccccc}
 \mathcal{Z} & \xrightarrow{i'} & \mathcal{X} \times_{\mathcal{U}} ((\mathcal{U} \times_S T)_{\text{red}}) & \xrightarrow{i} & \mathcal{X} \times_S T & \longrightarrow & \mathcal{X} \\
 & \searrow & \downarrow \alpha_r & & \downarrow \alpha_T & & \downarrow \alpha \\
 & & (\mathcal{U} \times_S T)_{\text{red}} & \xrightarrow{i_{\mathcal{U}}} & \mathcal{U} \times_S T & \longrightarrow & \mathcal{U} \\
 & & & \searrow h_{\mathcal{U}} & \downarrow & & \downarrow \\
 & & & & T & \longrightarrow & S.
 \end{array}$$

$\swarrow h$

As $i_{\mathcal{U}}$ is a closed immersion, so is i , but then by assumption $\mathcal{X} \times_{\mathcal{U}} ((\mathcal{U} \times_S T)_{\text{red}})$ is reduced, and so i' is an isomorphism.

The conormal sequence on $\mathcal{U}$ gives

$$I_{\mathcal{U}}/I_{\mathcal{U}}^2 \longrightarrow h_{\mathcal{U}}^* \Omega_{T/S} \longrightarrow \Omega_{(\mathcal{U} \times_S T)_{\text{red}}/\mathcal{U}} \longrightarrow 0 .$$

But we have $\alpha_r^* \Omega_{(\mathcal{U} \times_S T)_{\text{red}}/\mathcal{U}} \cong \Omega_{\mathcal{Z}/\mathcal{X}}$, and so the pullback of the right-hand map is exactly

$$h^* \Omega_{T/S} \longrightarrow \Omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow 0 ,$$

whose top wedge defines the map $\mathcal{Y} \dashrightarrow \mathcal{W}_{\text{Im}}$ on the open set on which $\mathcal{Z} \cong \mathcal{Y}$ and $\mathcal{Y}$ is smooth, by Lemma 10.1.2. Therefore α_r factors into the map $\mathcal{Z} \dashrightarrow \mathcal{W}$.

Similarly, taking top wedge of this map commutes with pullback, so we obtain equality of linear systems $\alpha_r^* \mathfrak{M}_{\mathcal{U}} = \mathfrak{M}$.

This gives a factorization $\mathcal{X} \dashrightarrow (\mathcal{U} \times_S T)_{\text{red}}^p \dashrightarrow \mathcal{W}_{\text{Im}}^p$, so we get a rational map between the normalizations of $(\mathcal{U} \times_S T)_{\text{red}}^p$ and $\mathcal{W}_{\text{Im}}^p$ in $K(\mathcal{X})$. The latter is $\mathcal{V}$ by definition, so if moreover $\alpha_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{U}}$ we obtain $\mathcal{X} \rightarrow \mathcal{U} \dashrightarrow \mathcal{V}$. $\square$

Chapter 11

Applications and examples

11.1 Fano varieties

In this section we prove some consequences of our main results to Fano varieties over imperfect fields. We first prove Theorem 6.0.5, whose statement we recall below:

Theorem 11.1.1 (Theorem 6.0.5). *Let K be the function field of a variety over a perfect field of characteristic $p > 0$. Let X be a variety over K , and assume the following:*

- *X is normal, $\mathbb{Q}$ -factorial and $\rho(X) = 1$,*
- *X is geometrically irreducible over K ,*
- *there is an effective divisor B such that $K_X + B \equiv 0$, and*
- *if $n = \dim(X)$ then $p > 2n + 1$.*

Then there is a birational morphism $\phi: \tilde{X} \rightarrow X$ and a contraction $\tilde{X} \rightarrow V$ of relative dimension at least 1, such that $(\tilde{X} \otimes_K K^{1/p})_{\text{red}}$ is birational to $\tilde{X} \times_V (V \otimes_K K^{1/p})_{\text{red}}$.

Proof of Theorem 6.0.5. Let $Y^{1/p} = (X \otimes_K K^{1/p})_{\text{red}}^\nu$ and ϕ_1 be the map $Y^{1/p} \rightarrow X$. By [Tan18, Proposition 2.4], $\rho(Y^{1/p}) = \rho(X) = 1$, and by [Tan18, Lemma 2.5] $Y^{1/p}$ is $\mathbb{Q}$ -factorial. By Theorem 6.0.1 we obtain a formula

$$\phi_1^*(K_X + B) \sim K_{Y^{1/p}} + (p-1)(\mathfrak{F} + \mathfrak{M}) + \phi_1^*B \equiv 0$$

where $\text{Bs}(\mathfrak{M})$ has codimension at most 2. In our situation $\mathfrak{F}, \mathfrak{M}$ are effective Weil divisors and ϕ_1^*B an effective $\mathbb{Q}$ -divisor on a $\mathbb{Q}$ -factorial variety of Picard rank 1, so they are each nef $\mathbb{Q}$ -Cartier divisors. Let $g: Y^{1/p} \dashrightarrow W$ be the rational map from Theorem 6.0.1, which matches that given by the linear system $\mathfrak{M}$ up to a finite morphism. Denote the rational map induced by $\mathfrak{M}$ as $g_{\mathfrak{M}}$.

Suppose that $g_{\mathfrak{M}}$ is generically finite, and we aim to find a contradiction. The key is that if we fix a point P of $Y^{1/p}$ where $g_{\mathfrak{M}}$ is defined as a morphism and is finite, and a curve Γ through P , this assumption allows us to choose a representative of $\mathfrak{M}$ which passes through P but does not contain Γ .

Let $Y^{\text{perf}} = (Y^{1/p} \otimes_{K^{1/p}} K^{1/p^\infty})_{\text{red}}^\nu$ and let ϕ_∞ be the map $Y^{\text{perf}} \rightarrow Y^{1/p}$. By [PW20, Theorem 1.1] we can fix an effective Weil divisor C on Y^{perf} such that

$$\phi_\infty^* \phi_1^*(K_X + B) = K_{Y^{\text{perf}}} + (p-1)(\phi_\infty^*(\mathfrak{F} + \mathfrak{M}) + C) + \phi_\infty^* \phi_1^*B.$$

As before Y^{perf} is $\mathbb{Q}$ -factorial and $\rho(Y^{\text{perf}}) = 1$, and so C is nef.

Finally let $Y = Y^{\text{perf}} \otimes_{K^{1/p^\infty}} \overline{K}$, which is an irreducible normal variety because X was geometrically irreducible, and let $\overline{\phi}$ be the map $Y \rightarrow Y^{\text{perf}}$. We have $K_Y = \overline{\phi}^* K_{Y^{\text{perf}}}$, and so

$$\begin{aligned} \overline{\phi}^*(K_{Y^{\text{perf}}} + (p-1)(\phi_\infty^*(\mathfrak{F} + \mathfrak{M}) + C) + \phi_\infty^* \phi_1^*B) \\ = K_Y + (p-1)((\mathfrak{F}_Y + \mathfrak{M}_Y) + C_Y) + B_Y \equiv 0 \end{aligned}$$

where $\mathfrak{F}_Y$, $\mathfrak{M}_Y$, B_Y and C_Y are nef $\mathbb{Q}$ -Cartier Weil divisors obtained by pulling back the corresponding divisors from Y^{perf} .

Cut Y by general hyperplanes to obtain a smooth curve D contained within the smooth locus of Y . Fix a point P of Y which lies on D but is not contained in the support of $\mathfrak{F}_Y$, B_Y or C_Y (we can do that by generality of the hyperplanes). Furthermore, we can choose P to be over the locus on which the map $g_{\mathfrak{M}}$ is defined as a morphism and is finite.

Now apply bend and break [Kol96b, II.5.8] to the nef $\mathbb{Q}$ -Cartier divisor $-K_Y$ and the curve D to obtain a curve Γ through the point P with $-K_Y \cdot \Gamma \leq 2n$. Then

$$(p-1)\mathfrak{M}_Y \cdot \Gamma \leq ((p-1)(\mathfrak{F}_Y + \mathfrak{M}_Y + C_Y) + B_Y) \cdot \Gamma = -K_Y \cdot \Gamma \leq 2n.$$

But P is chosen in such a way that we could have chosen an element $M_P \in \mathfrak{M}$ through P which does not contain the curve Γ . This implies that $\mathfrak{M}_Y \cdot \Gamma$ is greater than a positive integer, and in particular $\mathfrak{M}_Y \cdot \Gamma \geq 1$. The inequality from bend and break now gives $p-1 \leq 2n$, that is $p \leq 2n+1$.

It follows that if $p > 2n+1$, the rational map $Y^{1/p} \dashrightarrow W$ is not birational, and Theorem 6.0.1 gives the required property.

□

Remark. Note that Theorem 6.0.5 applies equally to the pair obtained by base change to $K^{1/p}$, that is $(Y^{1/p}, (p-1)(\mathfrak{F} + \mathfrak{M}) + \phi_1^* B)$. So in fact we inductively obtain a sequence of contractions as in Theorem 6.0.5 at each Frobenius base change as we continue towards K^{1/p^∞} .

In the next corollaries we may reduce to the case where the ground field is a function field using the process outlined in Section 7.0.1, and so apply Theorem 6.0.1.

Proof of Corollary 6.0.6. As we assume X is geometrically non-reduced, we deduce from Theorem 6.0.1 and Section 7.0.1 that the linear system $|C|$ has nonzero movable

part. But the exceptional section D on the Hirzebruch surface is fixed so this cannot occur. $\square$

Proof of Corollary 6.0.7. If $p = 3$ then $(Y, C) = (\mathbb{P}^2, L)$ or (S_d, F) . In both of these situations, any linear system of $\dim \geq 1$ inside $|C|$ has no fixed part. $\square$

Proof of Corollary 6.0.8. Since we assume that $(Y, (p-1)C)$ is of the type $(\mathbb{P}^1 \times \mathbb{P}^1, F)$ (see [PW20, Theorem 4.1]), C is a prime divisor and $X \otimes_K K^{1/p}$ is non-reduced, and we have $Y \cong (X \otimes_K K^{1/p})_{\text{red}} \otimes_{K^{1/p}} \overline{K}$. That is, $(X \otimes_K K^{1/p})_{\text{red}}$ is geometrically normal. The linear system $\mathfrak{M}_{K^{1/p}/K}$ is 1-dimensional so V is a curve, and by universal properties $X \times_V K((V \otimes_K K^{1/p})_{\text{red}})$ is isomorphic to $(X \otimes_K K^{1/p})_{\text{red}}$. So over the generic point of V

$$Y \cong (X \times_V (V \otimes_K K^{1/p})_{\text{red}}^\nu) \otimes_{K^{1/p}} \overline{K} \cong X \times_V (V \otimes_K \overline{K})_{\text{red}}^\nu$$

where the second isomorphism holds because $X \times_V K((V \otimes_K K^{1/p})_{\text{red}})$ is geometrically normal over the generic point of V . $\square$

11.2 Examples

In this section k will always denote a perfect field of characteristic $p > 0$.

Our first example is the archetypal example of a geometrically non-reduced variety. We compute our linear system explicitly in this case.

Example 11.2.1. Let $S = \mathbb{A}_k^{n+1}$ with coordinates $s_0, \dots, s_n$, and let

$$\mathcal{X} = \left(\left(\sum_{i=0}^n s_i x_i^{p^m} \right) + x_{n+1}^{p^m} = 0 \right) \subset \mathbb{P}_S^{n+1}.$$

Let $\tau: T \rightarrow S$ be relative Frobenius over k , so that $\Omega_{T/S} = \Omega_{T/k}$. Then $(\mathcal{X} \times_S T)_{\text{red}}$ is normal, so $\mathfrak{F}_{T/S} = 0$ and

$$\mathcal{Y} = \left(\left(\sum_{i=0}^n s_i^{1/p} x_i^{p^{m-1}} \right) + x_{n+1}^{p^{m-1}} = 0 \right) \subset \mathbb{P}_T^{n+1}.$$

We compute $\mathfrak{M}_{T/S}$ explicitly on the affine cover $\{(x_j \neq 0) \mid 0 \leq j \leq n\}$ of $\mathcal{Y}$. We have

$$\Omega_{\mathcal{Y}/\mathcal{X}}(x_j \neq 0) = \Omega_{k[\{s_l^{1/p} \mid l \neq j\}, x_0, \dots, x_{n+1}] / k[\{s_l \mid l \neq j\}, x_0, \dots, x_{n+1}]} = \langle d_{\mathcal{Y}/\mathcal{X}} s_l^{1/p} \mid l \neq j \rangle$$

because $d_{\mathcal{Y}/\mathcal{X}} s_j^{1/p} = -\sum_{l \neq j} (\frac{x_l}{x_j})^{p^{m-1}} d_{\mathcal{Y}/\mathcal{X}} s_l^{1/p}$ on $(x_j \neq 0)$.

The differentials $\{d_{T/S} s_i^{1/p} \mid 0 \leq i \leq n\}$ form a basis of $\Omega_{T/S}$, and for each $0 \leq i \leq n$, the image of $d_{T/S} s_i^{1/p} \otimes 1$ in $H^0(\mathcal{Y}, \Omega_{\mathcal{Y}/\mathcal{X}})$ in terms of the bases on the affine charts described above, is given by

$$\begin{cases} d_{\mathcal{Y}/\mathcal{X}} s_i^{1/p} & \text{on } (x_j \neq 0) \text{ for all } j \neq i, \\ -\sum_{l \neq i} (\frac{x_l}{x_i})^{p^{m-1}} d_{\mathcal{Y}/\mathcal{X}} s_l^{1/p} & \text{on } (x_i \neq 0). \end{cases}$$

So

$$\bigwedge_{k \neq i} d_{T/S} s_k^{1/p} \mapsto \begin{cases} -(\frac{x_i}{x_j})^{p^{m-1}} \bigwedge_{l \neq j} d_{\mathcal{Y}/\mathcal{X}} s_l^{1/p} & \text{on } (x_j \neq 0) \text{ for all } j \neq i, \\ \bigwedge_{k \neq i} d_{\mathcal{Y}/\mathcal{X}} s_k^{1/p} & \text{on } (x_i \neq 0). \end{cases}$$

and therefore $\det \Omega_{\mathcal{Y}/\mathcal{X}} \cong \mathcal{O}_{\mathbb{P}_T^n}(p^{m-1})|_{\mathcal{Y}}$ and

$$\mathfrak{M}_{T/S} = \langle x_i^{p^{m-1}} \mid 0 \leq i \leq n \rangle \subset \left| \mathcal{O}_{\mathbb{P}_T^n}(p^{m-1})|_{\mathcal{Y}} \right|$$

and defines a map to $\text{Proj } k[s_0^{1/p}, \dots, s_n^{1/p}][x_0^{p^{m-1}}, \dots, x_n^{p^{m-1}}]$.

So the factorization from Theorem 6.0.1 is trivial; that is, $\mathcal{X} = \mathcal{V}$ and $\mathcal{Y} = \mathcal{W}$.

In particular if $m = 1$, then for any n and $p < n + 2$ the generic fiber X is an

n -dimensional regular, geometrically non-reduced Fano variety over $k(s_0, \dots, s_n)$.

The following is an example which illustrates that the generic fiber of $\mathcal{X} \rightarrow \mathcal{V}$ need not be geometrically reduced.

Example 11.2.2. Let $S = \mathbb{A}_{k,(s,t)}^2$ and

$$\mathcal{X} = (sx^{p^m} + ty^{p^m} + z^{p^m} = xu^{p^n} + yv^{p^n} + zw^{p^n} = 0) \subset \mathbb{P}_{S,[x:y:z]}^2 \times_S \mathbb{P}_{S,[u:v:w]}^2.$$

If $\tau: T \rightarrow S$ is relative Frobenius over k then $\Omega_{T/S} = \Omega_{T/k}$ and the normalized base change is

$$\mathcal{Y} = (\mathcal{X} \times_S T)_{\text{red}} = (s^{1/p}x^{p^{m-1}} + t^{1/p}y^{p^{m-1}} + z^{p^{m-1}} = xu^{p^n} + yv^{p^n} + zw^{p^n} = 0)$$

in $\mathbb{P}_{T,[x:y:z]}^2 \times_T \mathbb{P}_{T,[u:v:w]}^2$. So $\mathfrak{F}_{T/S} = 0$ and the same computation as in Example 11.2.1 shows that $\mathfrak{M}_{T/S} = \langle x^{p^{m-1}}, y^{p^{m-1}} \rangle \subsetneq |\mathcal{O}_{\mathbb{P}_T^2 \times_T \mathbb{P}_T^2}(p^{m-1}, 0)|_{\mathcal{Y}}$. Therefore the factorization of Theorem 6.0.1 is

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad\quad\quad} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{W} = \text{Proj} \frac{k[s^{1/p}, t^{1/p}][x, y, z]}{(s^{1/p}x^{p^{m-1}} + t^{1/p}y^{p^{m-1}} + z^{p^{m-1}})} & \longrightarrow & \mathcal{V} = \text{Proj} \frac{k[s, t][x, y, z]}{(sx^{p^m} + ty^{p^m} + z^{p^m})} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad\quad\quad} & S \end{array}.$$

$\mathcal{X} \times_{\mathcal{V}} \mathcal{W}$ is reduced, but the generic fiber $\mathcal{X}_{\xi}$ of $\mathcal{X} \rightarrow \mathcal{V}$ is not geometrically reduced over $K(\mathcal{V})$; indeed $\mathcal{X}_{\xi}$ is given by $(xu^p + yv^p + zw^p = 0) \subset \mathbb{P}_{K(\mathcal{V})}^2$ and becomes non-reduced upon base change to $K(\mathcal{V})^{1/p}$.

By taking products of p^m -Fermat hypersurfaces, we obtain examples where both $\mathfrak{F}$ and $\mathfrak{M}$ are nonzero.

Example 11.2.3. Let $S = \mathbb{A}_{k,(r,s,t)}^3$ and

$$\mathcal{X} = (sx^{p^m} + ty^{p^m} + z^{p^m} = ru^{p^n} + sv^{p^n} + w^{p^n} = 0) \subset \mathbb{P}_{S,[x:y:z]}^2 \times_S \mathbb{P}_{S,[u:v:w]}^2.$$

If $T = \operatorname{Spec} k[r, s^{1/p}, t^{1/p}] \rightarrow S$ then $\Omega_{T/S}$ is freely generated by $d_{T/S}s^{1/p}$ and $d_{T/S}t^{1/p}$, and the normalized base change is

$$\mathcal{Y} = (s^{1/p}x^{p^{m-1}} + t^{1/p}y^{p^{m-1}} + z^{p^{m-1}} = r^{1/p}u^{p^{n-1}} + s^{1/p}v^{p^{n-1}} + w^{p^{n-1}} = 0)$$

inside $\mathbb{P}_{S^p,[x:y:z]}^2 \times_{S^p} \mathbb{P}_{S^{(1)],[u:v:w]}^2$. One can compute on the affine open subsets $\{(x \neq 0) \cap (u \neq 0), (y \neq 0) \cap (u \neq 0), (y \neq 0) \cap (v \neq 0)\}$ of $\mathcal{Y}$ that

$$\begin{aligned} \mathfrak{F}_{T/S} &= u^{p^{n-1}} \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, p^{n-1})|_{\mathcal{Y}}, \\ \mathfrak{M}_{T/S} &= \langle x^{p^{m-1}}, y^{p^{m-1}} \rangle \subsetneq |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(p^{m-1}, 0)|_{\mathcal{Y}}. \end{aligned}$$

Indeed, by adjunction $(p-1)(\mathfrak{M}_{T/S} + \mathfrak{F}_{T/S}) = (p-1)\mathfrak{C}_{T/S}$ is in the linear equivalence class of $\phi^*K_{\mathcal{X}} - K_{\mathcal{Y}} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(p^m - p^{m-1}, p^n - p^{n-1})|_{\mathcal{Y}}$.

$\mathfrak{M}_{T/S}$ defines a map to $\operatorname{Proj} k[r, s^{1/p}, t^{1/p}][x^{p^{m-1}}, y^{p^{m-1}}]$ and so the factorization of Theorem 6.0.1 is

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{W} = \operatorname{Proj} \frac{k[r, s^{1/p}, t^{1/p}][x, y, z]}{(s^{1/p}x^{p^{m-1}} + t^{1/p}y^{p^{m-1}} + z^{p^{m-1}})} & \longrightarrow & \mathcal{V} = \operatorname{Proj} \frac{k[r, s, t][x, y, z]}{(sx^{p^m} + ty^{p^m} + z^{p^m})} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad} & S. \end{array}$$

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