From the holdings of the Department of Special Collections , Princeton University Library, Princeton, NJ

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyrightlaw. We do not charge any permission or usage fees for the publication of images of material inour collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff.For more information regarding copyright, credit, and citation please review the information onthe Copyright, Credit and Citation Guidelines page of our website.
https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. https://library.princeton.edu/special-collections/ask-us

---

Doctoral Dissertations, 2011-2026 Mathematics

2024

Geometric Variations of an Allen-Cahn Type Energy

Silva, Erico

Princeton, NJ : Princeton University

---

http://arks.princeton.edu/ark:/88435/dsp015m60qw271

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. https://www.copyright.gov/title17/title17.pdf

Geometric Variations of an Allen-Cahn Type Energy

Erico Silva

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Fernando Coda Marques

May 2024

© Copyright by Erico Silva, 2024.

All Rights Reserved

Abstract

In this thesis (adapted from joint work with Jared Marx-Kuo) a new Allen–Cahn type functional, BE_ε, is introduced, which defines an energy on separating hypersurfaces, Y, of closed Riemannian Manifolds. In Chapter 2 we establish Γ-convergence of BE_ε to the area functional as $\varepsilon \to 0$.

In Chapter 3 we compute first and second variations of this functional under hypersurface pertrubations. We then compute an explicit expansion for the variational formula as $\varepsilon \to 0$. A key component of this proof is the invertibility of the linearized Allen–Cahn equation about a solution, on the space of functions vanishing on Y.

In Chapter 4 we also study the index and nullity of BE_ε and relate it to the usual Allen–Cahn index of a corresponding solution vanishing on Y. We apply the second variation formula and index theorems to show that the family of $2p$-dihedrally symmetric solutions to Allen–Cahn on S^1 have index $2p-1$ and nullity 1.

Acknowledgements

Many people contributed to the completion of this work, and for their constant support I will be forever grateful.

I thank my advisor Fernando Codá Marques for his generosity and patience.

I owe a great deal to all the teachers who put me in contact with the beauty of mathematics, and I especially want to thank those who helped me through my undergraduate years: André Neves, Wilhelm Schlag, Takis Souganidis, Benson Farb, Danny Calegari, Marco Mendez Guaraco, and Amie Wilkinson.

I thank Jared Marx-Kuo for his invaluable mathematical companionship, from which I learned much.

I am grateful to all my friends and fellow mathematicians who made my time at Princeton always enlightening, especially Thomas Massoni, Ryan Unger, Nina Zubrilina. I also want to especially thank my fellow students: Ben Lowe, Akashdeep Dey, Yangyang Li, Zhihan Wang, Alejandra Ramirez-Luna, Lorenzo Sarnatoro, and Doug Stryker.

I want to thank the Princeton mathematics department staff who have been very helpful to me throughout the years, and especially in my fifth year at Princeton, Jill Leclair deserves a special mention for always going above and beyond to support graduate students.

I want to thank Mike Miller Eismeier and Maggie Eismeier Miller for their always intellectually invigorating friendship.

I thank Bailey Dean and Michael Severson for their ability to always uplift my spirit.

I thank Wen Darr for countless things, for being my dear friend, and for officiating my wedding in the final year of my time at Princeton.

I thank my mothers-in-law Pam Prior and Deb Reinhard for taking me in during my first year, and creating a home for me.

I thank my parents, Nelson and Ronilda, my brother Bruno, and my aunt Rogeria, for believing in me relentlessly.

I thank my wife Lindsey, for everything.

To Lindsey.

Contents

Abstract iii

Acknowledgements iv

1 Introduction. 3

1.1 Results and Overview 5

1.1.1 Paper Organization 10

1.2 Notation and Preliminaries. 11

1.2.1 BE_ε energies 14

2 Γ-convergence of BE_ε to the Perimeter Functional. 16

3 First Variation and Second Variation. 22

3.1 First Variation. 22

3.1.1 Set up 23

3.1.2 First variation computation 24

3.2 Second Variation. 25

3.3 Indices and Nullity 27

3.3.1 Proof of Corollary 1.1.13 29

3.3.2 Proof of 1.1.4 32

3.4 Second Variation for $\varepsilon \to 0$ 34

3.4.1 Kernel of $L_{\varepsilon,u}$ 35

3.5 Proof of Theorem 1.1.16 40

3.5.1 Set up 40

3.5.2 Computing the bound 40

3.5.3 Computing second variation 42

4 Applications of Second Variation Formula **47**

4.0.1 Allen–Cahn Index of symmetric solutions 47

4.0.2 Application: Allen–Cahn Index of D_{2p}-symmetric solutions on S^1 49

4.0.3 Classification of stable BE critical points in noncompact M^3 56

4.1 Appendix 58

4.1.1 Smooth dependence of solutions on domains 58

4.1.2 Computing $\dot{u}_x(0)$ on S^1 60

4.1.3 Elliptic Functions 67

4.1.4 Reducing $\mathrm{BE}''(f,f)$ 69

4.1.5 Absolute Minimizers of BE_ε 74

Chapter 1

Introduction.

Let (M^n, g) be a Riemannian manifold and $u \in W^{1,2}(M)$. The Allen–Cahn energy is given by

$$E_\varepsilon(u) = \int_M \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u) \tag{1.0.1}$$

where $W(u) = \frac{(1-u^2)^2}{4}$ is a double-well potential. Critical points of this functional satisfy the Allen–Cahn equation

$$\varepsilon^2 \Delta_g u = W'(u) \tag{1.0.2}$$

There is a well-known correspondence between zero sets of solutions to (1.0.2) and minimal surfaces. Modica and Mortola ([22] [23]) showed that the Allen–Cahn energy functional Γ-converges to perimeter. In particular a large body of literature has emerged in recent years using min-max properties of the Allen–Cahn energy to construct min-max minimal hypersurfaces due to Guaraco, Gaspar, Chodosh, Mantoulidis, and many others ([11] [8] [5]). Under certain geometric constraints, Wang and Wei ([[25], Thm 1.1]) showed that the level sets of a sequence of stable solutions to (1.0.2), $\{u_{\varepsilon_i}\}$, converge to a minimal surface with good regularity.

One drawback of the Allen–Cahn energy (1.0.1) is that a priori, it is difficult to control the topology of the zero set of minimizers, and thus the topology of the limiting minimal surfaces. This motivates us to define an Allen–Cahn energy functional *on hypersurfaces themselves*. For $Y^{n-1} \subseteq M^n$, separating with $M = M^+ \sqcup_Y M^-$, let $u_\varepsilon^\pm$ be the non-negative (positive) minimizers of equation (1.0.1) on $M^\pm$ over all functions vanishing on $Y = \partial M^\pm$. For Y satisfying some geometric constraints, $u_\varepsilon^\pm$

are non-zero. We then define a **Balanced Energy** functional on hypersurfaces as

$$\mathrm{BE}_\varepsilon(Y) := E_\varepsilon(u_\varepsilon^+, M^+) + E_\varepsilon(u_\varepsilon^-, M^+)$$

And investigate the variational properties of BE_ε on the space of separating hypersurfaces. This includes explicit formula for the first and second variation, as well as connections between the Morse index and nullity of this function to the normal Allen–Cahn index and nullity on solutions.

In section §3, we show that if Y is critical for BE_ε, then $u_\varepsilon^\pm$ can be glued together to form a solution to (1.0.2) on M. In particular, we have the existence of critical points for BE_ε from the following theorem of Pacard and Ritore (See figure 1.1 for a visualization):

Theorem 1.0.1 (Pacard–Ritoré, Theorem 1.1)**.** *Assume that (M, g) is an n-dimensional closed Riemannian manifold and $Y^{n-1} \subseteq M^n$ is a two-sided, nondegenerate minimal hypersurface. Then there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon < \varepsilon_0$, there exist solutions, u_ε, to equation* (1.0.2) *such that u_ε converges to $+1$ (resp. -1) on compact subsets of $(\Omega^+)^o$ (resp. $(\Omega^c)^o$) and*

$$E_\varepsilon(u_\varepsilon) \xrightarrow{\varepsilon \to 0} \frac{1}{\sqrt{2}} A(Y)$$

for $A(Y)$ the $n-1$-dimensional area of Y

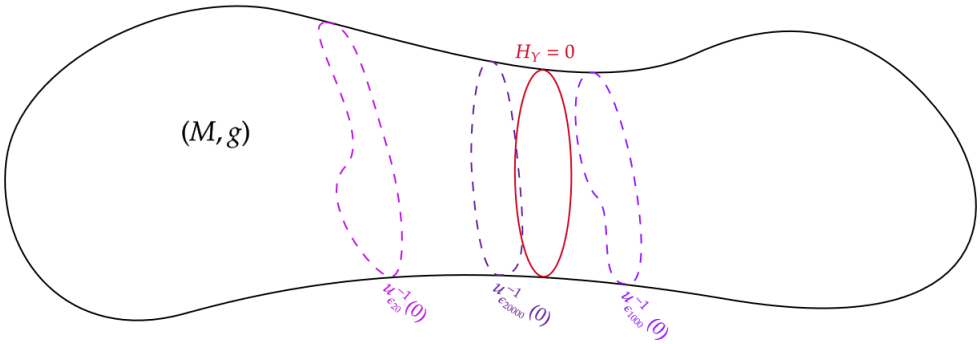


Figure 1.1: Illustration of level set convergence to a minimal hypersurface

We are inspired by [10], Ex. 19, in which the author uses energy minimization and symmetry to glue solutions on 3 subdomains of S^n together. Energy minimization is also considered in the research

of "ground states" of the Allen–Cahn energy in [4]. We also draw inspiration from an analogous approach in the theory of extremal eigenvalue problems for the Laplacian, as presented in Henrot [12] on variations of the first eigenvalue.

We also mention a similar theory of variations of the Allen–Cahn energy using "inner variations" of Allen–Cahn solutions as developed by Le [17] and Gaspar [7], among others. In this setting, variations of the form $u_\varepsilon \circ \Phi_t$ are considered, for Φ_t a one parameter family of diffeomorphisms. Such variations are geometrically similar to variations of BE_ε, since they will move the nodal set of u_ε. However, variations of BE_ε move the nodal set *and* simultaneously re-minimizes the Allen–Cahn energy on both sides, so the settings are different in both the setup and formulae. In particular, the first and second variations of BE_ε can be framed as Dirichlet-to-Neumann problems and expanded asymptotically in ε to acquire precise estimates on subleading terms.

1.1 Results and Overview

Given $Y^{n-1} \subseteq M$, sufficiently regular and separating so that

$$M = \Omega^+ \sqcup_Y \Omega^-$$

we define $u^\pm_{Y,\varepsilon}$ to be the positively (negatively) signed minimizers of (1.0.1) on $\Omega^\pm$ with 0 dirichlet condition on Y, and $U_{\varepsilon,Y}$ to be the combined, pasted function defined on all of M. We then informally define the "*Balanced Energy*" on Y as

$$\mathrm{BE}_\varepsilon(Y) := E_\varepsilon(u^+_{Y,\varepsilon}, \Omega^+) + E_\varepsilon(u^-_{Y,\varepsilon}, \Omega^-)$$

(see 1.2.13 for full details). This is one way to associate an energy to a large class of hypersurfaces in M. We will often suppress the $\{Y, \varepsilon\}$ subscripts and just write $u^\pm$.

We note that $\mathrm{BE}_\varepsilon \xrightarrow{\Gamma} \mathcal{P}$,the perimeter functional for sets of finite perimeter, which agrees with the area functional on smooth boundaries. This uses the standard Γ-convergence result for the Allen–Cahn energy due to Modica-Mortola [23].

Theorem 1.1.1. *Let Ω be a set of finite perimeter. Then, we have*

1. *If $\Omega_n \xrightarrow{L^1} \Omega$ and $\partial\Omega_n$ is a smooth hypersurface, then $\mathrm{BE}_\varepsilon(\partial\Omega_n) \to \sigma_0 \mathcal{P}(\partial\Omega)$.*

2. There is a sequence of smooth hypersurfaces Ω_n such that $\Omega_n \xrightarrow{L^1} \Omega$ with

$$\limsup_{\varepsilon\to 0^+} \mathrm{BE}_\varepsilon(\partial\Omega_n) = \sigma_0 \mathcal{P}(\partial\Omega).$$

For a more precise statement see §2. We then compute the first variation of BE_ε as a function of the hypersurface Y.

Theorem 1.1.2. *Let $\{Y_t\}$ a one-parameter family of hypersurfaces in (M^n, g) with*

$$Y_t = \exp_p(f(p)\nu(p))$$

and $Y_0 = Y$. Then

$$\frac{d}{dt}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} = \mathrm{BE}'_\varepsilon(f\nu)\Big|_Y = \frac{\varepsilon}{2}\int_Y f[(u_\nu^+)^2 - (u_\nu^-)^2] \tag{1.1.1}$$

Remark: The above shows that critical points of (1.1.1) correspond to solutions of (1.0.2) on all of M, since $u_{\varepsilon,Y}$ is now C^1 and hence smooth across its nodal set $u_{\varepsilon,Y}^{-1}(0) = Y$.

If we use basic results from [21] we can show

Corollary 1.1.1. *For Y a $C^{3,\alpha}$ surface with bounded geometry (1.2.1), there exists an ε_0 such that for all $\varepsilon < \varepsilon_0$*

$$\mathrm{BE}'_\varepsilon(f\nu)\Big|_Y = \frac{1}{2\sqrt{2}}\langle H_Y + E, f\rangle$$

where $|E| \le K\varepsilon$ for K dependent on Y.

Remark 1. The above can be thought of as an extension of Γ-convergence of the Allen–Cahn energy to the area functional, with the first variation converging to the mean curvature.

Having derived a notion of first variation for BE_ε, we can formulate the second variation. We first prove the following theorems, relating the AC-index and AC-nullity to the BE_ε-index and BE_ε-nullity.

Theorem 1.1.3. *There exists an ε_0 so that for all $\varepsilon < \varepsilon_0$, if Y is a critical point for BE_ε with bounded geometry, then*

$$\mathrm{Ind}_{AC_\varepsilon}(u_{Y,\varepsilon}) = \mathrm{Ind}_{\mathrm{BE}_\varepsilon}(Y) \tag{1.1.2}$$

Theorem 1.1.4. *There exists an ε_0 so that for all $\varepsilon < \varepsilon_0$, if Y is a critical point for BE_ε with bounded geometry, then*

$$\mathrm{Null}_{AC_\varepsilon}(u_{Y,\varepsilon}) = \mathrm{Null}_{\mathrm{BE}_\varepsilon}(Y) \tag{1.1.3}$$

Thus, for solutions which are energy minimizers, the index and nullity can be computed over the space of *geometric* variations, i.e. variations of the null set.

We can also explicitly state the second variation formula. We consider solutions $\dot{u}^{\pm}_{\varepsilon,Y} : \Omega^{\pm} \to \mathbf{R}$ to the linearized Allen–Cahn equation at u:

$$\begin{cases} \varepsilon^2 \Delta_g \dot{u}^{\pm}_{\varepsilon,Y} &= W''(u_{\varepsilon,Y})\dot{u}^{\pm}_{\varepsilon,Y} \text{ in } \Omega^{\pm}, \\ \dot{u}^{\pm}_{\varepsilon,Y}\Big|_Y &= -f u_\nu. \end{cases} \tag{1.1.4}$$

We will often suppress the $\{\varepsilon, Y\}$ subscripts out of convenience. The existence and uniqueness of a solution to the above will be a consequene of proposition 3.4.1. With this, we can state the formula for the second derivative of BE_ε at a critical point.

Theorem 1.1.5. *Let $\{Y_t\}$ a one-parameter family of hypersurfaces in (M^n, g) with $Y_0 = Y$ as in theorem 1.1.2. Further assume that Y is a critical point. Then*

$$\frac{d^2}{dt^2}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} = \varepsilon \int_Y f u_\nu [\dot{u}^+_\nu - \dot{u}^-_\nu] \tag{1.1.5}$$

for $\dot{u}^{\pm}$ as in (1.1.4).

We then asymptotically expand $\dot{u}^{\pm}_\nu$ as a function of ε and prove that the equation (1.1.5) actually reflects a Jacobi operator.

Theorem 1.1.6. *Let $\{Y_t\}$ a one-parameter family of hypersurfaces in (M^n, g) with $Y_0 = Y$ as in theorem 1.1.2. Assume that Y is a a $C^{2,\alpha}$ critical point with bounded geometry (1.2.1). Then*

$$\begin{aligned} \frac{d^2}{dt^2}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} &= \frac{\sqrt{2}}{3}\int_Y f u_\nu [-2(\Delta_Y + \dot{H}_0)](f) + E(f) \\ &= \frac{2\sqrt{2}}{3}\int_Y |\nabla f|^2 - \dot{H}_Y f^2 + E \\ &= \frac{2\sqrt{2}}{3}[D^2 A(f) + E(f)] \end{aligned} \tag{1.1.6}$$

where

$$|E(f)| \leq K\varepsilon^{1/2}||f||^2_{H^1}$$

Remark 2. The above second variation formula is a further extension of Γ-convergence of the Allen–Cahn energy to the area functional, but now with the second variation converging to the second variation of area. This contrasts with the results of Gaspar [7] and Le [17], where there is an extra term that obstructs the convergence of the second inner variation of the Allen–Cahn energy to the second variation of the area functional. Our result corresponds to the situation where the extra term in the second inner variation vanishes, and theorem 1.1.6 additionally shows a precise control of the size of the error in ε.

To prove theorem 1.1.6 we establish the following theorem on the linearized operator $\varepsilon^2\Delta_g - W''(u)$ (the content of which is contained in proposition 3.4.1):

Theorem 1.1.7. *The operator $L_{\varepsilon,u} = \varepsilon^2\Delta_g - W''(u) : H^1_{\varepsilon,0}(M^{\pm}) \to H^{-1}_{\varepsilon}(M^{\pm})$ has a bounded inverse.*

In particular, theorem 1.1.6 tells us that the second variation of BE_ε for a critical point is $O(\varepsilon)$ away from the second variation of area for a minimal surface. This leads to the following applications using the geometry of Y

Corollary 1.1.2. *Suppose that Y_ε are a sequence of critical points to BE_ε for all $\varepsilon < \varepsilon_0$ sufficiently small. Let Y_ε a $C^{2,\alpha}$ hypersurface and suppose that there exists an ε_0 such that for all $\varepsilon < \varepsilon_0$ either one of the following hold*

- *The Ric curvature is bounded from below, i.e.*

$$\exists c_0 > 0 \quad s.t. \qquad Ric_g(\nu,\nu)\Big|_Y > c_0$$

- *The second fundamental form has bounded L^2 norm and $Vol(Y)$ is finite*

$$\exists c_1, c_2 > 0, \quad s.t. \quad ||A_Y||^2_{L^2(Y)} \geq c_1, \quad \exists Vol(Y) \leq c_2 \tag{1.1.7}$$

Then Y is unstable.

Similarly, we produce a parallel result to that of minimal surfaces, proved by Fischer-Colbrie–Schoen:

Theorem 1.1.8. *Let M^3 be a complete oriented 3-manifold of non-negative scalar curvature. Let Y^2 be a $C^{2,\alpha}$, oriented complete, **compact**, stable critical point for BE_ε in M . Let S denote the*

scalar curvature of M at any point. Then Y is conformally equivalent to a (S^2, g_{round}) or Y is "almost" a totally geodesic flat torus T^2, i.e.

$$||A_Y||^2_{L^2(Y)} \leq K\varepsilon^{1/2}$$

for some $K > 0$ independent of ε for K sufficiently small. If $S > 0$ on M, then Y is conformally equivalent to (S^2, g_{round}). If both:

$$S \geq \kappa_0 > 0$$
$$Vol(Y) \geq v_0 > 0$$

both independent of ε, then Y must be conformally equivalent to S^2.

We then produce results on the Morse index of Allen–Cahn level sets. We first prove that

Corollary 1.1.3. *Suppose $\{u_\varepsilon\}$ is a sequence of solutions with $Y = u_\varepsilon^{-1}(0)$ for all ε. If Y is minimal and u_ε are each critical points of (1.2.1), then there exists an $\varepsilon_0 > 0$ sufficiently small, such that for all $\varepsilon < \varepsilon_0$*

$$\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) = \mathrm{Ind}_{\mathrm{BE}_\varepsilon}(u_\varepsilon) \geq \mathrm{Ind}_{Min}(Y) \tag{1.1.8}$$

Separately, we use the BE frame work and variation formula to compute the Morse index of solutions to (1.0.2) on S^1. We first define $u_{\varepsilon,2p}$ to be the unique solution vanishing at $Y_{2p} := \{i/(2p)\}_{i=0}^{2p-1}$ on $S^1 \cong [0,1]$ (see §4.0.2)

Proposition 1.1.1. *Let $p > 0$, and $u_{\varepsilon,2p}$ the solution vanishing on Y_{2p}. Consider a variation of the form*

$$a_i(t) := \frac{i}{2p} + f(i/(2p))t$$

for $f : \{0, 1, \ldots, 2p-1\} \to \mathbf{R}$. Then

$$BE''(f)\Big|_{Y_{2p}} = \varepsilon c^2 v(\varepsilon) \sum_{i=0}^{2p-1} [f(i/(2p)) - f((i+1)/2p)]^2 \tag{1.1.9}$$

for $v(\varepsilon) < 0$ for all $\varepsilon > 0$

The above allows us to classify the Morse index of all solutions to (1.0.2) on S^1 as $\varepsilon \to 0$.

Theorem 1.1.9. *Fix $p > 0$. There exists an ε_p such that $\forall \varepsilon < \varepsilon_p$, the function, $u_{\varepsilon,2p}$, has Allen–Cahn Morse index $2p-1$ and nullity 1. The nullity is given by rotations of S^1 and every other*

non-trivial variation produces a strictly negative variation.

We note that formulas (1.1.1), (1.1.5) (1.1.6) are local to Y, thought of as the nodal set of a solution, u_ε, to (1.0.2). The first and second variation for (1.0.1) are integrals over all of M, despite most of the geometry of u_ε being contained in a neighborhood of Y. The locality to BE_ε can be an advantage that gives more information about Allen–Cahn solutions, as in theorem 1.1.9. In this way, we feel that the direct connection of BE_ε to the surface gives a new, geometric way to investigate a smaller class of Allen–Cahn functions and variations.

Finally, we note that while our visuals and notation seem to be adapted for Y connected (and $M^\pm$ each connected), our results apply to Y with finitely many components. As long as Y is separating, one can make sense of the decomposition $M = M^+ \sqcup_Y M^-$ and define energy minimizers on each component of $M^\pm$ which are positively (resp. negatively) signed on the interior of each component. Non-zero minimizers are guaranteed with a bound on λ_1 on each component of $M^\pm$, as in theorem 1.2.1. This can also be achieved by bounding the distance between components of Y from below. Assuming this, we can reproduce our expansions of $u_\varepsilon(s,t)$, $\dot{u}(s,t)$ locally on each component of Y and prove the same theorems.

1.1.1 Paper Organization

The paper is organized as follows:

- Section §1.2 sets up the machinery of BE_ε.
- In section §2, we prove that the BE_ε functional Γ-converges to the area functional up to a scaling factor. This closely mimics the original proof by Modica–Mortola in [23]
- In section §4.1.5 we show that *absolute minimizers* of BE_ε do *not* exist on compact manifold (M^n, g) for ε sufficiently small. In fact $\inf_Y \mathrm{BE}_\varepsilon(Y)$ over all hypersurfaces is 0 when $n \geq 2$. On S^1, $\inf_Y \mathrm{BE}_\varepsilon(Y)$ has a non-trivial lower bound.
- In §3, we prove theorem 1.1.2. In §3.2, we prove theorem 1.1.5 similarly
- In §3.3, we prove that if $Y \leftrightarrow u_{\varepsilon,Y}$ is a critical point of $\mathrm{BE}_\varepsilon \leftrightarrow E_\varepsilon(M)$, then the BE_ε Morse index and nullity of Y equals the Allen–Cahn Morse index and nullity of $u_{\varepsilon,Y}$. The equalities, along with an explicit expansion of the dirichlet-to-neumann operator for the linearized Allen–Cahn system, allows us to compute the Morse index and nullity of all solutions to (1.0.2) on S^1

- In §3.4, §3.5, we compute an asymptotic expansion of (1.1.5) in terms of ε. In particular, this shows that the second variation is $O(\varepsilon)$-close to the second variation of area for minimal surfaces

- In §4, we give some applications of equation (1.1.5), in particular showing that no stable BE_ε-hypersurfaces exist when there is a Ricci lower bound, or when there are bounds on $||A_Y||_{L^2(Y)}$ and $\mathrm{Vol}(Y)$. We also prove an analogous version of Fischer-Colbrie-Schoen's theorem on stable minimal surfaces in 3-dimensional PSC manifolds, our theorem 1.1.8, for BE_ε-stable in 3-dimensional PSC manifolds. Finally, we show that for Y minimal with $\{u_\varepsilon\}$ a sequence of Allen–Cahn solutions and $u_\varepsilon^{-1}(0) = Y$ that the index of Y as a minimal hypersurface equals the index Allen–Cahn index of u_ε for all ε sufficiently small.

- In section §4.0.2, we prove theorem 1.1.9 and compute the Morse index of *all* Allen–Cahn solutions on S^1 for all ε sufficiently small. This culminates in equation (1.1.9) which explicitly reflects the geometry of the variations. In particular, this shows a rigidity result, i.e. all variations lead to non-positive second variation, with null directions given by rotations.

1.2 Notation and Preliminaries.

Throughout we will denote by (M, g) an arbitrary closed Riemannian manifold. We will write $\Omega \subset M$ to denote an arbitrary set of finite perimeter which has smooth boundary, $\partial\Omega$, and we will write $E \subset M$ for sets which may not be smooth, and we identify sets which differ except up to a set of measure 0

- If $\Omega \subset M$ is a smooth domain, then $\lambda_1(\Omega)$ denotes the first eigenvalue of the Laplacian Δ_g on Ω with dirichlet conditions on $\partial\Omega$

- $E_\varepsilon(u; E)$ denotes the ε-Allen–Cahn energy, (1.0.1), of the function $u \in W^{1,2}(M)$ restricted to the Borel set E.

- $\mathcal{C}(M)$ denotes the collection of Borel sets of finite perimeter.

- $\mathcal{C}^\infty(M)$ are those sets of finite perimeter with smooth boundary.

- $\mathcal{C}^\infty_\varepsilon(M) \coloneqq \{\Omega \in \mathcal{C}^\infty(M) \big| \min\{\lambda_1(\Omega)^{-\frac{1}{2}}, \lambda_1(\Omega^c)^{-\frac{1}{2}}\} > \varepsilon\}$. Note that $\bigcup_{\varepsilon>0} \mathcal{C}^\infty_\varepsilon(M) = \mathcal{C}^\infty(M)$.

- We denote by $|\Omega|$ the measure of a set with the Borel measure induced by the volume form on (M, g).

- $|\Omega \Delta \Omega'|$ denotes the measure of the symmetric difference of two sets, i.e. $|\Omega \Delta \Omega'| = |(\Omega \backslash \Omega') \cup (\Omega' \backslash \Omega)|$.
- If Ω is a smooth domain, ν will denote the outer unit normal to $\partial\Omega$.
- Let $\{E_j\} \subset \mathcal{C}(M)$. We say that $E_j \underset{L^1}{\to} E$ if $|E_j \Delta E| \to 0$ as $j \to \infty$.
- Let $\{\Omega_j\} \subset \mathcal{C}^\infty(M)$ we will say that $\Omega_j \underset{C^\infty}{\to} \Omega$ if $\partial\Omega_j \to \partial\Omega$ as graphs in C^∞.
- $\mathrm{BE}_\varepsilon(Y)$ denotes the **Balanced Energy** (see definition 1.2.13) of a hypersurface $Y^{n-1} \subseteq M^n$ when well defined

Basic properties of sets of finite perimeter are developed in [18]. In particular it will be important to note that for any $E \in \mathcal{C}(M)$ there is a sequence $\{\Omega_j\} \subset \mathcal{C}^\infty(M)$ so that $\Omega_j \underset{L^1}{\to} E$.

We will denote by $Y \subset M$ a 2-sided, C^2 (potentially more regular when noted), closed hypersurface. Y divides M into two (possibly not connected) sets, which we will denote by $\Omega^+ \sqcup_Y \Omega^-$ or $M^+ \sqcup M^-$. When needed, we'll assume uniform bounds on the geometry of Y of the following form: there exists a $C(k) > 0$ (independent of ε) such that if Y is a $C^{k,\alpha}$ surface, then

$$||A_Y||_{C^{k-2,\alpha}} \leq C(k) \tag{1.2.1}$$

For convenience we also define

$$\dot{H}_Y = |A_Y|^2 + \mathrm{Ric}_g(\nu, \nu)$$

where Ric_g is the ricci tensor of the ambient metric on M.

Let $g(t) = \tanh(t/\sqrt{2})$ denote the heteroclinic solution to (1.0.2) on $\mathbf{R}$. We define $\dot{g}(t) = g'(t)$ and so on for higher derivatives. Similarly, let $W(u) = \frac{(1-u^2)^2}{4}$. We will denote $w(t), \rho(t), \kappa(t), \tau(t)$ the following functions on $[0, \infty)$ which solve the below ODEs

$$\begin{aligned} \ddot{w} - W''(g)w &= \dot{g} \\ \ddot{\rho} - W''(g)\rho &= \dot{w} \\ \ddot{\tau} - W''(g)\tau &= t\dot{g} \\ \ddot{\kappa} - W''(g)\kappa &= gw \end{aligned} \tag{1.2.2}$$

all subject to the condition that

$$f : [0, \infty) \to \mathbf{R}, \qquad f(0) = 0, \qquad \lim_{t \to \infty} f(t) = 0$$

The functions, $w(t), \rho(t), \kappa(t), \tau(t)$, are then uniquely defined (see [21], Appendix 7.6).

We also define the following constants

$$\sigma_0 := \int_0^\infty \dot{g}^2 = \frac{\sqrt{2}}{3} \tag{1.2.3}$$

$$\sigma := \dot{g}(0) = \sqrt{2}^{-1} \tag{1.2.4}$$

Let

$$||f||_{C^{k,\alpha}_\varepsilon} := \sum_{i=0}^{k} \varepsilon^i ||D^i f||_\infty + \varepsilon^{k+\alpha}[D^k f]_\alpha \tag{1.2.5}$$

$$||f||^2_{H^k_\varepsilon} := \sum_{i=0}^{k} \varepsilon^{2i} \varepsilon ||D^i f||^2_{L^2} \tag{1.2.6}$$

$$||f||_{H^{-k}_\varepsilon} = \sup_{g \in H^k_{\varepsilon,0}} \frac{1}{||g||_{H^k_\varepsilon}} \int_M fg \tag{1.2.7}$$

be the rescaled Hölder and Sobolev norms, respectively. Here $H^k_{\varepsilon,0}$ refers to functions in H^k_ε with 0 trace on the boundary (in context, this will be $M^\pm$ so that $\partial M^\pm = Y$). In the same vein, we notate a rescaled function as

$$f : \mathbf{R}^+ \to \mathbf{R}$$

$$f_\varepsilon(t) := f(t/\varepsilon)$$

We also define

$$\overline{g}(t) := \chi_{-\omega \ln(\varepsilon)}(t) g(t) + [1 - \chi_{-\omega \ln(\varepsilon)}(t)]$$

where $\chi_{-\omega \ln(\varepsilon)}$ goes from 1 to 0 on $[-\omega \ln(\varepsilon), -2\omega \ln(\varepsilon)]$ where $\omega > 5$. Then we note that

$$[\partial_t^2 - W''(\overline{g}(t))]\overline{g}(t) = E \qquad ||E||_{H^k} \leq C(k)\varepsilon^\omega \tag{1.2.8}$$

Note that we'll use the same notation of $\overline{w}, \overline{\tau}, \overline{\kappa}$ to denote cutoffs of w, τ, κ but to 0 since those functions decay exponentially naturally. E.g.

$$\overline{w}(t) := [1 - \chi_{-\omega \ln(\varepsilon)}(t)]g(t)$$
$$[\partial_t^2 - W''(\overline{g}(t))]\overline{w}(t) = E \qquad ||E||_{H^k} \leq C(k)\varepsilon^{\omega}$$

We recall the following result from [21], Theorem 1.5 (and implicitly [20]): For Y a $C^{4,\alpha}$ hypersurface with bounded geometry (equation 1.2.1), we have that $u_\varepsilon^+ : M^+ \to \mathbf{R}$ expands as

$$u_\varepsilon^+(s,t) = \overline{g}_\varepsilon(t) + \varepsilon H_Y(s)\overline{w}_\varepsilon(t) + \phi \tag{1.2.9}$$
$$||\phi||_{C_\varepsilon^{2,\alpha}(M)} \leq K\varepsilon^2$$

where s denotes Fermi coordinates on Y and t is the signed distance to Y. If Y is $C^{5,\alpha}$, the expansion becomes

$$u_\varepsilon^+(s,t) = \overline{g}_\varepsilon(t) + \varepsilon H_Y(s)\overline{w}_\varepsilon(t) + \varepsilon^2\left[\dot{H}_Y(s)\overline{\tau}_\varepsilon(t) + H_Y^2(s)\left(\overline{\rho}_\varepsilon(t) + \frac{1}{2}\overline{\kappa}_\varepsilon(t)\right)\right] + \phi \tag{1.2.10}$$
$$||\phi||_{C_\varepsilon^{2,\alpha}(M)} \leq K\varepsilon^3$$

Finally for $u : M \to \mathbf{R}$, we define $L_{\varepsilon,u} := \varepsilon^2\Delta_g - W''(u)$.

1.2.1 $\mathbf{BE}_\varepsilon$ energies

In [1], Brezis and Oswald showed the following result.

Theorem 1.2.1 (Brezis-Oswald [1])**.** *Suppose $\Omega \subset M$ is an open domain with smooth boundary. If $\varepsilon < \lambda_1(\Omega)^{-\frac{1}{2}}$, then*

$$\begin{cases} \Delta_g u = \frac{W'(u)}{\varepsilon^2} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.11}$$

has a unique solution.

Therefore, for any $\Omega \in \mathcal{C}_\varepsilon^\infty(M)$, one has the existence of a unique positive solution $u_{\Omega,\varepsilon}^+$ to the

ε-Allen–Cahn equation and a unique negative solution $u^-_{\Omega^c,\varepsilon}$ on Ω^c. We define

$$u_{\Omega,\varepsilon} = \begin{cases} u^+_{\Omega,\varepsilon} & \text{on } \overline{\Omega}, \\ u^-_{\Omega^c,\varepsilon} & \text{on } \Omega^c. \end{cases} \tag{1.2.12}$$

We call $u_{\Omega,\varepsilon}$ a **broken ε-phase transition** (or simply a broken phase transition if reference to ε is unimportant). We may also reference $u_{\Omega,\varepsilon}$ as $u_{Y,\varepsilon}$ when $Y = \partial\Omega$ plays a more important role. It follows by construction that $u_{\Omega,\varepsilon} \in C^0(M) \cap W^{1,2}(M)$. In fact, it solves the Allen–Cahn equation outside of $\partial\Omega$. It can be seen that elliptic regularity implies that $u_{\Omega,\varepsilon}$ is a solution to the ε-Allen–Cahn equation if and only if $\nabla_\nu u^-_{\Omega,\varepsilon} = \nabla_\nu u^+_{\Omega,\varepsilon}$.

Definition 1.2.2. If $\Omega \in \mathcal{C}^\infty_\varepsilon(M)$ we define the balanced energy of Ω as

$$\mathrm{BE}_\varepsilon(\Omega) \coloneqq E_\varepsilon(u_{\Omega,\varepsilon}) \tag{1.2.13}$$

We will often have cause to write $\mathrm{BE}_\varepsilon(\partial\Omega)$ to denote the same thing. We will also write $\mathrm{BE}^+_\varepsilon(\Omega) \coloneqq E_\varepsilon(u^+_{\Omega,\varepsilon})$ and similarly $\mathrm{BE}^-_\varepsilon(\Omega) = E_\varepsilon(u^-_{\Omega,\varepsilon})$.

Chapter 2

Γ-convergence of BE_ε to the Perimeter Functional.

The goal of this section is to show that the BE_ε functional defined on $\mathcal{C}^\infty_\varepsilon(M)$ converges to the perimeter functional, $\mathcal{P}$, on $\mathcal{C}(M)$ as $\epsilon \to 0$.

Theorem 2.0.1 (Γ-convergence)**.** *Let $\Omega \in \mathcal{C}(M)$ be an arbitrary Caccioppoli set.*

1. *(Pre-compactness of BE_ε-bounded sets): Let $\{\Omega_{\varepsilon_k}\}$ be a sequence of Cacciopoli sets in $\mathcal{C}^\infty_{\varepsilon_k}(M)$ such that $\mathrm{BE}_{\epsilon_k}(\Omega_{\epsilon_k})$ is bounded and $\varepsilon_k \to 0$ as $k \to \infty$. Then there is some Ω' so that $\Omega_{\epsilon_k} \underset{L^1}{\longrightarrow} \Omega'$.*

2. *(lim-inf inequality): If $\Omega_{\varepsilon_k} \underset{L^1}{\to} \Omega$, then*

$$\liminf_{\varepsilon_k \to 0} \mathrm{BE}_{\varepsilon_k}(\Omega_{\varepsilon_k}) \geq \sigma_0 \mathcal{P}(\Omega).$$

3. *(Recovery sequence): For every $\Omega \in \mathcal{C}(M)$ there is a sequence $\Omega_{\varepsilon_k}(M) \in \mathcal{C}^\infty_{\varepsilon_k}(M)$ so that $\Omega_{\epsilon_k} \underset{L^1}{\to} \Omega$ and*

$$\limsup_{\epsilon_k \to 0} \mathrm{BE}_{\epsilon_k}(\Omega_{\epsilon_k}) \leq \sigma_0 P(\Omega).$$

Strictly speaking what we show is the following: for any set of finite perimeter $\Omega \in \mathcal{C}(M)$, set $\overline{\mathrm{BE}_\varepsilon}(\Omega) = +\infty$ if $\Omega \notin \mathcal{C}^\infty_\varepsilon(M)$ and $\overline{\mathrm{BE}_\varepsilon}(\Omega) = \mathrm{BE}_\varepsilon(\Omega)$ otherwise. Then the functional $\overline{\mathrm{BE}_\varepsilon}$, which is now defined on all of $\mathcal{C}(M)$, Γ-converges to the perimeter functional under the usual topology

induced by convergence of sets of finite perimeter (identified up to a set of measure 0). This does not affect the argument otherwise, so we will proceed without distinguishing $\overline{\mathrm{BE}_\varepsilon}$ and BE_ε. To prove this, we rely fundamentally on the Γ-convergence theory of Modica-Mortola [23]. We recall these below:

Proposition 2.0.2 (BV-compactness)**.** *Let $u_k \in W^{1,2}(M)$ be a sequence of functions so that $E_{\epsilon_k}(u_k)$ is uniformly bounded in k. Then up to subsequence*

$$u_k \underset{L^1}{\to} u \in \mathrm{BV}(M; \{-1, 1\})$$

Theorem 2.0.3 (Γ-convergence for the Allen–Cahn energies)**.** *Let $\Omega \in \mathcal{C}(M)$ be an arbitrary Caccioppoli set.*

1. *(liminf inequality): Suppose $u_k \in W^{1,2}(M)$ and $E_{\varepsilon_k}(u_k)$ are uniformly bounded. If $u_k \underset{L^1}{\to} u \in BV(M; \{-1, 1\})$ then*

$$\liminf_{\epsilon_k \to 0} E_{\epsilon_k} u_k \geq \sigma_0 \mathcal{P}(\{u = 1\}).$$

2. *(limsup inequality): There is a sequence $u_k \in W^{1,2}(M)$ so that*

$$u_k \underset{L^1}{\to} \chi_\Omega - \chi_{\Omega^c},$$

satisfying,

$$\limsup_{\epsilon_k} E_{\epsilon_k}(u_k) \leq \sigma \mathcal{P}(\Omega).$$

Moreover, if we assume further that if $\Omega \in \mathcal{C}^\infty(M)$, then we can construct the u_k so that they satisfy $\{u_k = 0\} = \partial\Omega$.

To prove theorem 2.0.1 we first need the following basic lemma.

Lemma 2.0.4. *If $f_n \underset{L^1}{\to} f$. Let $f_n^+ = \max(f_n, 0)$, then $f_n^+ \underset{L^1}{\to} f^+$.*

Proof. Write $f_n = f_n^+ - f_n^-$ and $f = f^+ - f^-$. Then if $f_n^+ > 0$ we have two cases.

1. $f^+ > 0$, in which case

$$|f_n - f| = |f_n^+ - f^+|.$$

2. $f^- > 0$, in which case

$$|f_n - f| = f_n^+ + f^- \geq f_n^+ = |f_n^+ - f^+|,$$

since $f^+ = 0$ whenever $f^- > 0$.

In either case we find

$$|f_n - f| \geq |f_n^+ - f^+|$$

Note that the above also holds if $f_n^+ = 0$, so

$$\int |f_n^+ - f^+| \leq \int |f_n - f| \to 0.$$

□

Proof of Theorem 2.0.1. The result follows from the Γ-convergence result of Modica-Mortola.

1. (Pre-compactness): Let Ω_k be a bounded sequence for BE_ϵ in $\mathcal{C}^\infty(M)$. That is to say there is a constant $C > 0$

   $$\sup_k \mathrm{BE}_{\epsilon_k}(\Omega_k) \leq C$$

   We wish to construct a Caccioppoli set $\Omega \in \mathcal{C}(M)$ so that $|\Omega_k \Delta \Omega| \to 0$ up to possibly taking a subsequence in k. For each k, Let $u_k \in W^{1,2}(M)$ denote the broken ε_k phase-transition associated to Ω_k. That is to say,

   $$u_k = \begin{cases} u^+_{\Omega_k} \text{ in } \Omega_k, \\ u^-_{\Omega_k} \text{ in } \Omega_k^c. \end{cases}$$

   Note that by taking a subsequence without relabeling, we may assume ε_k is sufficiently small that this broken phase-transition because $\varepsilon_k \to 0$ as $k \to \infty$.

   It follows from the definition of Ω_k that

   $$E_{\epsilon_k}(u_k) = \mathrm{BE}_{\epsilon_k}(\Omega_k).$$

   Boundedness of Ω_k with respect to the Balanced energies BE_ϵ implies that the Allen–Cahn energies for u_k controlled uniformly in k. Therefore theorem 2.0.2 applies to the sequence $\{u_k\}$

and we find a function $u \in \mathrm{BV}(M; \{-1, 1\})$ so that up to taking a subsequence.

$$u_k \underset{L^1}{\to} u.$$

Set $\Omega' = \{u > 0\}$. We wish to show that $|\Omega_k \Delta \Omega'| \to 0$, so we will estimate the measures of $\Omega' \backslash \Omega_k$ and $\Omega_k \backslash \Omega$. Observe that $\Omega_k = \{u_k > 0\}$ by definition of the functions u_k and that $u = \chi_{\Omega'} - \chi_{(\Omega')^c}$.

By Lemma 2.0.4, Theorem 2.0.3(2), and the fact that $u = \chi_{\Omega'} - \chi_{(\Omega')^c}$ it follows that $u_k^+ \underset{L^1}{\to} u^+$, i.e.

$$\int_M |u_k^+ - \chi_{\Omega'}| \to 0.$$

So moreover by restricting the above to Ω_k^c,

$$\int_{\Omega_k^c} \chi_{\Omega'} = |\Omega_k^c \cap \Omega'| = |\Omega' \backslash \Omega_k| \to 0.$$

To show that $|\Omega_k \backslash \Omega'| \to 0$ as $k \to \infty$, we repeat the same with u_k^- and $\chi_{(\Omega')^c}$:

$$\int_M |u_k^- - \chi_{(\Omega')^c}| \to 0.$$

by the same lemma applied to the negative part. In particular

$$\int_M |u_k^- - \chi_{(\Omega')^c}| \geq \int_{\Omega_k} |u_k^- - \chi_{(\Omega')^c}| = \int_{\Omega_k} |\chi_{(\Omega')^c}| = |\Omega_k \cap (\Omega')^c| = |\Omega_k \backslash \Omega'|$$

It follows that $|\Omega_k \backslash \Omega'| \to 0$.

2. (liminf inequality): Suppose we have a sequence $\Omega_k \in \mathcal{C}^\infty_{\varepsilon_k}(M)$ of Caccioppoli sets in M which converge to $\Omega \in \mathcal{C}(M)$. Let u_k be the piecewise Brezis-Oswald solutions as before. Then because

$$E_{\epsilon_k}(u_k) = \mathrm{BE}_{\epsilon_k}(\Omega_k) = O(1)$$

we can apply BV-compactness to the sequence u_k to find a function u as before. By the exact same arguments as the previous it follow that

$$|\Omega \Delta \{u > 0\}| = 0.$$

By the Modica-Mortola liminf inequality

$$\liminf_{k\to\infty} \mathrm{BE}_{\epsilon_k}(\Omega_k) = \liminf_{k\to\infty} E_{\epsilon_k}(u_k) \geq \sigma_0 \mathcal{P}(\Omega).$$

3. (limsup inequality): Suppose that $\Omega \in \mathcal{C}^\infty_\varepsilon(M)$ for some $\varepsilon > 0$. Let $\Omega_k = \Omega$ for each k; as $\varepsilon_k \to 0$, for sufficiently large k we have $\varepsilon_k < \varepsilon$. Then for k large enough the broken phase transitions $\{u_k\}$ form a sequence of functions in $C^0(M) \cap W^{1,2}(M)$ which vanish precisely along $\partial\Omega$. For the sake of comparison we shall construct functions g_k whose Allen–Cahn energy is strictly greater than the u_k and which also vanish along $\partial\Omega_k$. Our task will be accomplished by comparing their energies using the variational characterization of Brezis-Oswald solutions.

Let $\rho > 0$ be a constant taken sufficiently small so that on the tubular neighborhood $B_{\rho_0}(\partial\Omega)$ the signed distance function $\mathrm{d} : B_{\rho_0}(\partial\Omega) \to \mathbb{R}$ is smooth. Let $\chi : \mathbb{R} \to \mathbb{R}$ denote a smooth, even cutoff function satisfying $0 \leq \chi \leq 1$, with $\chi \equiv 1$ on $(-\rho_0/4, \rho_0/4)$ and $\chi \equiv 0$ outside of $(-\rho_0/2, \rho_0/2)$. Define $\tilde{g}_k : \mathbb{R} \to \mathbb{R}$ to be

$$\tilde{g}_k(t) = g_{\varepsilon_k}(t)\chi(t) + (1 - \chi(t))\mathrm{sgn}(t).$$

And now we define $g_k : M \to \mathbb{R}$ by

$$g_k(x) = \tilde{g}_k(\mathrm{d}(x)),$$

extending smoothly to 1 and -1 outside of $B_{\rho_0(M)}$. Now we show that the functions g_k satisfy the limsup inequality. Note that by the coarea formula and the fact that $|\nabla \mathrm{d}(x)| \equiv 1$ where g_k is nonzero

$$\begin{aligned}
E_\varepsilon(g_k) &= \int_M \frac{\varepsilon_k}{2}|\nabla g_k|^2 + \frac{W(g_k)}{\varepsilon_k} \\
&= \int_{B_{\rho_0(M)}} \frac{\varepsilon_k}{2}|\nabla g_k|^2 + \frac{W(g_k)}{\varepsilon_k} \\
&= \int_{|r|\leq \rho_0/4} \left(\frac{\varepsilon_k |\tilde{g}_k'(r)|^2}{2} + \frac{W(\tilde{g}_k(r))}{\varepsilon_k} \right) \mathcal{H}^{n-1}(\{\mathrm{d}(x) = r\}) dr \\
&\quad + \int_{M \setminus B_{\rho_0/4}(M)} \frac{\varepsilon_k}{2}|\nabla g_k|^2 + \frac{W(g_k)}{\varepsilon_k} \\
&= I_1 + I_2.
\end{aligned}$$

$I_2 = o(1)$ as $k \to \infty$ by the exponential decay of the heteroclinic to the values -1 and 1 at infinity. Note that, for the heteroclinic g_{ε_k}, the energy density

$$\left(\frac{\varepsilon_k |g'_{\varepsilon_k}|^2}{2} + \frac{W(g_{\varepsilon_k}}{\varepsilon_k}\right) dr \to \delta_0,$$

where δ_0 denotes the Dirac mass centered at 0, and since $\tilde{g}_k$ agrees with the heteroclinic in the region where $|r| \leq \rho_0/4$, it follows that

$$I_1 = \int_{|r| \leq \rho_0/4} \left(\frac{\varepsilon_k |\tilde{g}'_{\varepsilon_k}|^2}{2} + \frac{W(\tilde{g}_{\varepsilon_k}}{\varepsilon_k}\right) \mathcal{H}^{n-1}(\{\mathrm{d}(x) = r\}) dr \to \mathcal{H}^{n-1}(\partial\Omega),$$

as $k \to \infty$.

To conclude, we note that

$$E_{\epsilon_k}(u_k) = E_{\epsilon_k}(u_k; \Omega) + E_{\epsilon_k}(u_k; \Omega^c) \leq E_{\epsilon_k}(g_k; \Omega) + E_{\epsilon_k}(g_k; \Omega^c) = E_{\epsilon_k}(g_k).$$

The result now follows because

$$\limsup_{k\to\infty} \mathrm{BE}_{\epsilon_k}(\Omega) = \limsup_{k\to\infty} E_{\epsilon_k}(u_k) \leq \limsup_{k\to\infty} E_{\epsilon_k}(g_k) \leq \sigma_0 \mathcal{P}(\Omega).$$

If we now suppose that $\Omega \in \mathcal{C}(M)$ is arbitrary and make no assumption on regularity, by theorem 13.8 in [18] we may find a sequence Ω^j of smooth Caccioppoli sets so that $\Omega^j \to \Omega$ in the flat topology which moreover satisfies

$$\mathcal{P}(\Omega^j) = \mathcal{P}(\Omega) + o(1), \text{ as } j \to \infty.$$

Then if we set $\Omega_k = \Omega^k$ we see that

$$\limsup_{k\to\infty} \mathrm{BE}_{\epsilon_k}(\Omega_k) \leq \sigma_0 \mathcal{P}(\Omega^k) + o(1) = \sigma_0 \mathcal{P}(\Omega) + o(1) \text{ as } k \to \infty.$$

□

Chapter 3

First Variation and Second Variation.

3.1 First Variation.

We consider a smooth family of hypersurfaces $\{Y_t\}$ and corresponding domains, $\{\Omega_t\}$ such that $\Omega =: \Omega_0$ and $\partial\Omega = Y$. Let $\{u_t\}$ be minimizers of (1.0.1) on Ω_t for some $\varepsilon > 0$. Doing the analogous for $\{\Omega_t^c\}$ and $\{u_t^c = u_t\}$, we are interested in the quantity

$$\mathrm{BE}(t) = \int_{\Omega_t} \varepsilon\frac{|\nabla u_t|^2}{2} + \frac{1}{\varepsilon}W(u_t) + \int_{\Omega_t^c} \frac{|\nabla u_t^-|^2}{2} + \frac{1}{\varepsilon}W(u_t^c) \tag{3.1.1}$$

as a function of t. We will use the labels $u_t = u_t^+$ and $u_t^c = u_t^-$ interchangeably throughout. Similarly, $\nu = \nu^+$ and $\nu^c = \nu^- := -\nu$. The goal is to prove theorem 1.1.2

Theorem 1.1.2. *Let $\{Y_t\}$ a one-parameter family of hypersurfaces in (M^n, g) with*

$$Y_t = \exp_p(f(p)\nu(p))$$

and $Y_0 = Y$. Then

$$\frac{d}{dt}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} = \mathrm{BE}'_\varepsilon(f\nu)\Big|_Y = \frac{\varepsilon}{2}\int_Y f[(u_\nu^+)^2 - (u_\nu^-)^2] \tag{1.1.1}$$

3.1.1 Set up

We decompose $M = \Omega_t \sqcup_{Y_t} \Omega_t^c$. Let $u_t : \Omega_t \to \mathbf{R}$ be the minimizers of (1.0.1). We can define

$$u : N_\rho(\Omega) \times (-\delta, \delta) \to \mathbf{R}$$

where

$$u(p,t) = u_t(p)$$

Note that the right hand side is not defined for all $p \in N_\rho(\Omega)$, but since u_t satisfies a dirichlet condition on $\partial\Omega_t$, and if we assume some uniform regularity of $\partial\Omega_t$, then we can define tubular neighborhoods for each $\partial\Omega_t$ and define odd extensions for u_t on $N_{\rho_t}(\Omega_t)$. By uniform regularity assumptions (e.g. (1.2.1) for $k \geq 2$) we have that for $t < t_0$ sufficiently small there exists an $\varepsilon_0 > 0$ so that

$$\bigcap_{0 \leq t < t_0} N_{\rho_t}(\Omega_t) \supseteq N_{\rho_0}(\Omega)$$

so that $u(\cdot, t)$ can be defined $N_{\rho_0}(\Omega)$ for all t. For example, this occurs if

$$\begin{aligned} F_t : \partial\Omega &\to \partial\Omega_t \\ F_t(p) &= \exp_p(f(p)t\nu(p)) \\ ||f||_{C^{2,\alpha}} &\leq K \end{aligned}$$

Then we have that

$$\begin{aligned} \varepsilon^2 \Delta_g u(p,t) &= W'(u(p,t)) \\ p_t \in \partial\Omega_t \implies u(p_t,t) &\equiv 0 \\ \implies \varepsilon^2 \Delta_g \dot{u}(p,0) &= W''(u(p,0))\dot{u}(p,0) \\ \dot{u}(p,0) &= -f u_\nu(p,0) \end{aligned}$$

where ν is the normal pointing to the interior of Ω. Note that u is differentiable on the interior by pulling back $Y_t \to Y$ and expanding this diffeomorphism locally. See §4.1.1 for details.

3.1.2 First variation computation

We define

$$G_\varepsilon(S, f, X) := \int_S \frac{\varepsilon}{2}|X|^2 + \frac{1}{\varepsilon}W(f)$$

So that

$$E_\varepsilon(u_t, \Omega_t) = G_\varepsilon(\Omega_t, u_t, \nabla u_t)$$

which frames

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(u_t, \Omega_t) &= \frac{d}{dt}G_\varepsilon(\Omega_t, u_t, \nabla u_t) \\ &= \int_{\partial\Omega} -f\left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right] \\ &+ \int_\Omega \varepsilon\langle\nabla u, \nabla\dot{u}\rangle + \frac{1}{\varepsilon}W'(u)\dot{u} \\ &= A + B\end{aligned}$$

here the sign choice of $-f$ as opposed to f is because ν points inwards to Ω. We evaluate

$$A = \int_{\partial\Omega} -f\left[\frac{\varepsilon}{2}u_\nu^2 + \frac{1}{4\varepsilon}\right]$$

And for B, we have

$$\begin{aligned}B &= \int_\Omega \varepsilon\langle\nabla u, \nabla\dot{u}\rangle + \frac{1}{\varepsilon}W'(u)\dot{u} \\ &= \int_{\partial\Omega} \varepsilon u_{\nu^c}\dot{u} \\ &= \int_{\partial\Omega} \varepsilon u_\nu^2 f\end{aligned}$$

having noted that $\dot{u}\Big|_{\partial\Omega} = -fu_\nu$ and $u_{\nu^c} = -u_\nu$. Thus

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(u_t, \Omega_t) &= A + B \\ &= \int_{\partial\Omega}\left[\frac{\varepsilon}{2}u_\nu^2 - \frac{1}{4\varepsilon}\right]f\end{aligned}$$

Repeating this on Ω^c, noting that $f \to -f$ with the choice of ν^c as the inward normal to Ω^c, we get that

$$\frac{d}{dt}\mathrm{BE}(t) = \frac{\varepsilon}{2}\int_{\partial\Omega}\left[u_\nu^2 - (u_\nu^c)^2\right]f$$

ending the proof of theorem 1.1.2. □

Remark 3. While we've formulated BE_ε for (M, g) compact, note that the above formula is defined for $Y = \partial\Omega$ compact and M non-compact. Moreover, while we assume that $f \in C^{2,\alpha}(Y)$, we can extend the first variation formula to $f \in L^2(Y)$ by the final formula.

We note that critical points must have $u_\nu = u_{\nu^c}$ everywhere on the boundary. From equation (1.2.10), we have

$$u_\nu = \frac{1}{\varepsilon\sqrt{2}} + \dot{w}(0)H_Y + \varepsilon[\dot{\tau}(0)\dot{H}_Y + H_Y^2\alpha] + O(\varepsilon^2) \tag{3.1.2}$$
$$\implies u_\nu - u_\nu^c = 2\dot{w}(0)H_Y + \varepsilon H_Y^2\alpha + O(\varepsilon^2)$$

which assuming $H_Y = O(1)$ and computing $\dot{w}(0)$ gives corollary 1.1.1

Corollary 1.1.1. *For Y a $C^{3,\alpha}$ surface with bounded geometry (1.2.1), there exists an ε_0 such that for all $\varepsilon < \varepsilon_0$*

$$\mathrm{BE}'_\varepsilon(f\nu)\Big|_Y = \frac{1}{2\sqrt{2}}\langle H_Y + E, f\rangle$$

where $|E| \leq K\varepsilon$ for K dependent on Y.

We further note:

Corollary 3.1.1. *Let Y be a critical point of BE_ε. If $Y \in C^{3,\alpha}$, then $||H_Y||_{C^0} \leq K\varepsilon$ and $[H_Y]_\alpha \leq K\varepsilon^{1-\alpha}$. If Y is $C^{4,\alpha}$, then $||H_Y||_{C^\alpha} = O(\varepsilon^{2-\alpha})$*

Alternatively, we see that if Y is a critical point of BE_ε, then $u_{\Omega,\varepsilon}$ is C^1 across $\partial\Omega$ and hence a smooth solution to (1.0.2) on all of M, as remarked in §1.2.1. Again such solutions exist near minimal surfaces by theorem 1.0.1, and these solutions are exactly the energy minimizers with 0 dirichlet condition on Y by theorem 1.2.1. This gives the following

Corollary 3.1.2. *Suppose $Y^{n-1} \subseteq M^n$ a nondegenerate minimal hypersurface. Then there exist critical points for BE_ε, Y_ε, with $Y_\varepsilon \to Y$ converging graphically as $\varepsilon \to 0$*

In general, the Pacard–Ritore solutions are prime examples of critical points of BE_ε on (M, g) generic, though the set of critical points of BE_ε may be larger.

3.2 Second Variation.

Consider the linearized system, $\dot{u}$, as in (1.1.4) to a family of $\{u_t\}$. These linearizations are defined on both Ω and Ω^c and are labelled $\dot{u}, \dot{u}^c$ respectively. We'll use $\dot{u}$ interchangeably with $\dot{u}^+$ and $\dot{u}^c$

interchangeably with $\dot{u}^-$.

In this section we aim to show the following theorem 1.1.5

Theorem 1.1.5. *Let $\{Y_t\}$ a one-parameter family of hypersurfaces in (M^n, g) with $Y_0 = Y$ as in theorem 1.1.2. Further assume that Y is a critical point. Then*

$$\frac{d^2}{dt^2}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} = \varepsilon \int_Y f u_\nu [\dot{u}^+_\nu - \dot{u}^-_\nu] \tag{1.1.5}$$

for $\dot{u}^\pm$ as in (1.1.4).

Proof: The first variation formula actually holds for any t:

$$\frac{d}{dt}\mathrm{BE}(t)\Big|_{t=s} = \frac{\varepsilon}{2}\int_{\partial\Omega_s} \left[u_{\nu_s}(p,s)^2 - u^c_{\nu_s}(p,s)^2\right] f(p,s) dA$$

where ν_t is an inward normal to $\partial\Omega_t$ and $f(p,t)$ is the map such that

$$F_{s,r} : \partial\Omega_s \xrightarrow{\cong} \partial\Omega_r$$
$$p \in \partial\Omega_s \implies F_{s,r}(p) = \exp_{p,\partial\Omega_s}(f(p,s) r \nu_s(p))$$

and each $f(p,s)$ is defined on $\partial\Omega_s$, or equivalently $f : \partial\Omega \times (-\delta, \delta)$ by pulling back each $\partial\Omega_s$ to $\partial\Omega$ by a diffeomorphism. We take the first variation formula and pull it back by F_s:

$$\frac{d}{dt}\mathrm{BE}(t)\Big|_{t=s} = \frac{\varepsilon}{2}\int_{\partial\Omega} F_s^*\left(\left[u_{\nu_s}(\cdot,s)^2 - u^c_{\nu_s}(\cdot,s)^2\right]\right)(p) F_s^*(f(\cdot,s))(p) F_s^*(dA)$$

for $p \in \partial\Omega$ a parameter. With $t = 0$ a critical point, the derivative of the above is given by

$$\frac{d^2}{dt^2}\mathrm{BE}(t)\Big|_{t=0} = \frac{\varepsilon}{2}\int_{\partial\Omega} \frac{d}{ds} F_s^*\left(\left[u_{\nu_s}(\cdot,s)^2 - u^c_{\nu_s}(\cdot,s)^2\right]\right)(p)\Big|_{s=0} f(p) dA$$

We can write the differentiated part as

$$\begin{aligned} F_s^*\left(\left[u_{\nu_s}(\cdot,s)^2 - u^c_{\nu_s}(\cdot,s)^2\right]\right)(p) &= u_{\nu_s}(F_s(p),s)^2 - u^c_{\nu_s}(F_s(p),s)^2 \\ &= \langle (\nabla u)(F_p(s),s), \nu_s\rangle^2 - \langle (\nabla u^c)(F_p(s),s), \nu_s\rangle^2 \end{aligned}$$

differentiating the individual terms, we get

$$\frac{d}{ds}\langle(\nabla u)(F_p(s),s),\nu_s\rangle^2\Big|_{s=0} = 2u_\nu\left[\dot\nu(u)+\dot u_\nu + f u_{\nu\nu}\right]$$
$$\implies \frac{d^2}{dt^2}\mathrm{BE}(t)\Big|_{t=0} = \varepsilon\int_{\partial\Omega} f u_\nu[\dot u_\nu - \dot u^c_\nu]$$

where $u_\nu = u_\nu(p,0)$ for $p\in\partial\Omega$. In the first line, we've noted that

$$\begin{aligned}\frac{d}{ds}\langle(\nabla u)(F_p(s),s),\nu_s\rangle &= \frac{d}{ds}\nu_s^i(\partial_i u)(F_p(s),s)\\ &= \dot\nu^i(\partial_i u)(p,0)+\nu^i[\nabla(\partial_i u)\cdot\dot F_p]+\nu^i(\partial_i\dot u)(p,0)\\ &= \dot\nu(u)+\nu^i[\nabla(\partial_i u)\cdot(f\nu)]+\nu(\dot u)\end{aligned}$$

if we choose Fermi coordinates off of $\partial\Omega$ so that $\nu=\partial_t$, the middle expression becomes

$$f u_{tt} = f u_{\nu\nu}$$

But note that $u_{\nu\nu} = u^c_{\nu\nu}$, since at a critical point, we have a solution to Allen–Cahn which is necessarily smooth. Thus $f u_{\nu\nu} - f u^c_{\nu\nu} = 0$. Similarly

$$\dot\nu(u) - \dot\nu(u^c)\Big|_{\partial\Omega} = 0$$

Since u is smooth across the boundary. This gives our final expression

$$\implies \frac{d^2}{dt^2}\mathrm{BE}(t)\Big|_{t=0} = \varepsilon\int_{\partial\Omega} f u_\nu[\dot u_\nu - \dot u^c_\nu]$$

this concludes the proof of theorem 1.1.5. □

Remark: Again, we note that the second variation formula can also be defined when M is non-compact or when $\partial\Omega$ non-compact, assuming that $f\in C^\infty_c(\partial\Omega)$.

3.3 Indices and Nullity

Recall that Y is a critical point for BE_ε, then $u_{Y,\varepsilon}$ is a solution to Allen–Cahn and an energy minimizer on each component of $M\backslash Y$. In this section we prove the following two theorems

Theorem 1.1.3. *There exists an ε_0 so that for all $\varepsilon<\varepsilon_0$, if Y is a critical point for BE_ε with*

bounded geometry, then

$$\mathrm{Ind}_{AC_\varepsilon}(u_{Y,\varepsilon}) = \mathrm{Ind}_{\mathrm{BE}_\varepsilon}(Y) \tag{1.1.2}$$

Theorem 1.1.4. *There exists an ε_0 so that for all $\varepsilon < \varepsilon_0$, if Y is a critical point for BE_ε with bounded geometry, then*

$$\mathrm{Null}_{AC_\varepsilon}(u_{Y,\varepsilon}) = \mathrm{Null}_{\mathrm{BE}_\varepsilon}(Y) \tag{1.1.3}$$

To prove the theorems, we first establish two propositions

Proposition 3.3.1. *Let Y a critical point for BE_ε with bounded geometry. If $u_r = u_{\varepsilon,Y} + r\phi$ for $\phi \in C^1(M)$, then there exists an $r_0 > 0$ such that for all $r < r_0$, $Y_r := u_r^{-1}(0)$ is graphical over Y.*

Proof: We argue as follows: we want to solve

$$0 = u_{\varepsilon,Y} + r\phi$$

without loss of generality, assume $||\phi||_{C^0} = 1$. Let s be a coordinate for Y and t denote the signed distance from Y. Parameterize each u_r as

$$u_r = u(r, s, t) = u_{\varepsilon,Y}(s, t) + r\phi(s, t)$$

We want to solve for

$$F(s, r) \quad \text{s.t.} \quad u(r, s, t = F(s, r)) = 0$$

with $F(s, 0) = 0$. First note that

$$u(r = 0, s = s_0, t = 0) = 0$$

for any s_0. Further note that

$$\partial_t u\Big|_{r=0,s=s_0,t=0} = \partial_t u_{\varepsilon,Y}(s, t)\Big|_{s=s_0,t=0} = \frac{1}{\varepsilon\sqrt{2}} + O(1) \neq 0$$

by (1.2.9). By the implicit function theorem, we get $\delta > 0$ and $F(s, r)$ which is differentiable in s and r such that

$$\mathrm{dist}_Y(s, s_0) + |r| < \delta \implies u(r, s, t = F(s, r)) = 0$$

and this is the unique zero in this neighborhood. Note that F is defined locally, but we can repeat

this argument for every $s_0 \in Y$. Taking a finite cover of Y, we find $\delta_0 > 0$ and F, a function, such that

$$\exists F : Y \times (-\delta_0, \delta_0) \text{ s.t. } u(r, s, F(s, r)) = 0$$

and F is continuous in both s and r. This finishes the proof of graphicality. Note that for large r, the level set of u_r may have more than one component. However, using that $||\phi||_{C^0(M)} = 1$ and taking $r < \delta_0^*$ for a potentially smaller $\delta_0^* \leq \delta_0$, we can guarantee that u_r only has one component and it is given by $\{(s, t = F(s, r)\}$. This follows because $u_\varepsilon(s, t)$ is exponentially converging to ± 1 away from $t = 0$. □

Let $f \in C^2(Y)$. For $Y_r = \exp_Y(fr\nu)$, let $w_r(f)$ be the corresponding BE-phase transition for this family of surfaces. We can define

$$\dot{w} = \partial_r w_r(f)\Big|_{r=0}$$

which solves (1.1.4) with the corresponding boundary condition.

Lemma 3.3.2. *Let $Y_{1,r} = \exp_Y(f_1 r\nu)$, $Y_{2,r} = \exp_Y(f_1 r\nu)$, and $w_r(f_1)$, $w_r(f_2)$ the corresponding* BE*-phase transitions. Then*

$$\alpha\dot{w}(f_1) + \beta\dot{w}(f_2) = \dot{w}(\alpha f_1 + \beta f_2)$$

Proof: This follows by the form of (1.1.4) as $\alpha\dot{w}(f_1) + \beta\dot{w}(f_2)$ and $\dot{w}(\alpha f_1 + \beta f_2)$ both satisfy the linearized PDE and have the same boundary conditions. The solution is unique by proposition 3.4.1 (or [21], Thm 1.4). □

3.3.1 Proof of Corollary 1.1.13

Note that any variation of surfaces is a variation in $H^1(M)$, and so

$$\frac{d^2}{dr^2}\mathrm{BE}_\varepsilon(Y_r)\Big|_{r=0} = \frac{d^2}{dr^2}E_\varepsilon(w_r)\Big|_{r=0}$$

where w_r the BE_ε-phase transition corresponding to Y_r. This means that any BE_ε variation with negative second derivative is an AC variation with negative second derivative. To see that

$$\mathrm{Ind}_{\mathrm{BE}_\varepsilon}(Y) \leq \mathrm{Ind}_{AC_\varepsilon}(u_{\varepsilon,Y})$$

we need to show that the map from variations of Y (to first order) is injective in the space of $H^1(M)$ functions which are first order variations of $u_{\varepsilon,Y}$. However, given $f_1, f_2 : Y \to \mathbf{R}$ two variations of Y, the corresponding variations of $u_{\varepsilon,Y}$ are $\dot{w}(f_1), \dot{w}(f_2)$. Since we assume $f_1 \neq f_2$ and $\dot{w}(f_i)\Big|_Y = -f_i u_\nu$ by the dirichlet condition in equation (1.1.4), we see that $\dot{w}(f_1) \neq \dot{w}(f_2)$.

To see equality of indices, let $u_r = u_{\varepsilon,Y} + r\phi$ a $C^1(M)$ family of functions. From proposition 3.3.1, we can write

$$u_r = w_r + \psi_r$$

where w_r is the BE-phase transition on $Y_r = u_r^{-1}(0)$, and $\psi_r\Big|_{Y_r} = 0$ for all r. Both w_r and ψ_r are C^1 so we can define

$$\dot{w} := \partial_r w_r\Big|_{r=0} : M \to \mathbf{R}$$
$$\dot{\psi} := \partial_r \psi_r\Big|_{r=0} : M \to \mathbf{R}$$

and it suffices to consider

$$u_r = u_{\varepsilon,Y} + r[\dot{w} + \dot{\psi}]$$

We then compute

$$\begin{aligned}
\frac{d^2}{dr^2} E_\varepsilon(u_r)\Big|_{r=0} &:= Q(\dot{w} + \dot{\psi}, \dot{w} + \dot{\psi}) \\
&= \int_M \varepsilon |\nabla(\dot{w} + \dot{\psi})|^2 + \frac{W''(u_{\varepsilon,Y})}{\varepsilon}(\dot{w} + \dot{\psi})^2 \\
&= Q(\dot{w}, \dot{w}) + Q(\dot{\psi}, \dot{\psi}) + 2\int_M \varepsilon \langle \nabla \dot{w}, \nabla \dot{\psi} \rangle + \frac{W''(u_{\varepsilon,Y})}{\varepsilon} \dot{w}\dot{\psi}
\end{aligned}$$

We note that

$$p \in Y, \;\; p(r) \in Y_r \implies \psi_r(p(r)) = 0 \implies \dot{\psi}(p) = -\nabla \psi_0 \cdot \dot{p} = 0$$

for any path $p(r) \in Y_r$ with $p(0) = p$. The last line follows since $\psi_0|_Y \equiv 0$. Then because $\dot{w}$ satisfies (1.1.4), we have

$$\begin{aligned}
\int_M \varepsilon \langle \nabla \dot{w}, \nabla \dot{\psi} \rangle + \frac{W''(u_{\varepsilon,Y})}{\varepsilon} \dot{w}\dot{\psi} &= \int_Y \varepsilon \dot{\psi}[\dot{w}_{\nu^+} - \dot{w}_{\nu^-}] - \int_M \dot{\psi}\varepsilon^{-1}[\varepsilon^2 \Delta_g \dot{w} - W''(u_{\varepsilon,Y})\dot{w}]\dot{\psi} \\
&= 0
\end{aligned}$$

Thus

$$\begin{aligned}\frac{d^2}{dr^2}E_\varepsilon(u_r)\Big|_{r=0} &= Q(\dot w,\dot w)+Q(\dot\psi,\dot\psi)\\ &= \frac{d^2}{dr^2}\mathrm{BE}_\varepsilon(Y_r)\Big|_{r=0}+Q(\dot\psi,\dot\psi)\end{aligned}$$

We now note that

$$E_\varepsilon(u_{\varepsilon,Y}+r\dot\psi) = Q(\dot\psi,\dot\psi) \tag{3.3.1}$$

$$\geq E_\varepsilon(u_{\varepsilon,Y}) \tag{3.3.2}$$

$$\implies Q(\dot\psi,\dot\psi)\geq 0$$

precisely because $u_{\varepsilon,Y}$ consists of piecewise energy minimizers vanishing on Y and $u_{\varepsilon,Y}+r\dot\psi$ vanishes on Y. This means that

$$\begin{aligned}\frac{d^2}{dr^2}E_\varepsilon(u_r)\Big|_{r=0}-\frac{d^2}{dr^2}\mathrm{BE}_\varepsilon(Y_r)\Big|_{r=0} &= Q(\dot\psi,\dot\psi)\\ &\geq 0\end{aligned}$$

In particular

$$\frac{d^2}{dr^2}E_\varepsilon(u_r)\Big|_{r=0}<0 \implies \frac{d^2}{dr^2}\mathrm{BE}_\varepsilon(Y_r)\Big|_{r=0}<0$$

Which shows that any variation of u_r contributing AC-index gives rise to a variation of Y that contributes BE-index. In particular, if $\dot u_1,\dots,\dot u_k\in C^1(M)$ span a vector space of dimension k, equal to the AC-Morse index of $u_{\varepsilon,Y}$, then we have

$$\begin{aligned}\forall i,\quad \dot u_i &= \dot w_i+\dot\psi_i\\ \frac{d^2}{dr^2}E_\varepsilon(u+rc^i\dot u_i) &= Q(c^i\dot w_i+c^i\dot\psi_i,c^i\dot q_i+c^i\dot\psi_i)\\ &= Q(c^i\dot w_i,c^j\dot w_j)+Q(c^i\dot\psi_i,c^j\dot\psi_j)\\ &= \frac{d^2}{dr^2}\mathrm{BE}_\varepsilon(Y_r)\Big|_{r=0}+Q(c^i\dot\psi_i,c^j\dot\psi_j)\end{aligned}$$

Here, $\dot w_i$ are again the first order variations of the BE_ε phase transitions on $Y_r^i=(u_{\varepsilon,Y}+r\dot u_i)^{-1}(0)$ and $\dot\psi_i$ are the remainders vanishing on Y. Note that we've adapted Einstein notation and noted that $c^i\dot u_i$ solves the linearized Allen–Cahn equation. Moreover Y_r is the family of surfaces coming

from the combined perturbation

$$f = c^i \dot{u}_i|_Y \in C^1(Y)$$

Since $c^i \dot{\psi}_i|_Y = 0$, we again see that

$$Q(c^i \dot{\psi}_i, c^j \dot{\psi}_j) \geq 0$$
$$\frac{d^2}{dr^2} E_\varepsilon(u + r c^i \dot{u}_i) < 0 \implies \frac{d^2}{dr^2} \mathrm{BE}_\varepsilon(Y_r)\Big|_{r=0}$$

This verifies

$$\mathrm{Ind}_{\mathrm{BE}_\varepsilon}(Y) \geq \mathrm{Ind}_{AC_\varepsilon}(u_{\varepsilon,Y})$$

ending the proof. □

3.3.2 Proof of 1.1.4

To prove this theorem, we first define the corresponding symmetric, bilinear form to the second variation. For any $f \in C^2(Y)$, let $\dot{w}(f)$ denote the C^0 and piecewise C^1 function:

$$\dot{w}(f)(p) := \begin{cases} \dot{w}^+(f)(p) & p \in M^+ \\ \dot{w}^-(f)(p) & p \in M^- \\ -f(p)u_\nu(p) & p \in Y \end{cases}$$
$$\varepsilon^2 \Delta_g \dot{w}^\pm(f) = W''(u)\dot{w}^\pm(f) \quad \text{on } M^\pm$$

Note that if Y is a critical point of BE_ε, then u_ν is well defined on Y. We now define

$$f, h \in C^2(Y)$$
$$Q_{\mathrm{BE}_\varepsilon}(f,h) = -\int_Y \dot{w}(f)\varepsilon[\dot{w}^+(h)_\nu - \dot{w}^+(h)_\nu]$$
$$= \varepsilon \int_Y f u_\nu [\dot{w}^+(h)_\nu - \dot{w}^+(h)_\nu]$$

So that the second variation of BE along f is $Q_{\mathrm{BE}_\varepsilon}(f,f)$. To see that is bilinear, note that equation (1.1.4) is linear in its dirichlet condition. To see symmetry, we write

$$\begin{aligned}
Q_{\mathrm{BE}_\varepsilon}(f,h) &= -\int_Y \dot{w}(f)\varepsilon[\dot{w}^+(h)_\nu - \dot{w}^+(h)_\nu] \\
&= -\int_Y \dot{w}^+(f)\varepsilon\dot{w}^+(h)_\nu - \dot{w}^-(f)\varepsilon\dot{w}^-(h)_\nu \\
&= -\int_Y \dot{w}^+(f)\varepsilon\dot{w}^+(h)_{\nu^+} + \dot{w}^-(f)\varepsilon\dot{w}^-(h)_{\nu^-} \\
&= \varepsilon\int_{M^+} \dot{w}^+(f)\Delta_g\dot{w}^+(h) - \langle\nabla\dot{w}^+(f),\dot{w}^+(h)\rangle \\
&\quad + \varepsilon\int_{M^-} \dot{w}^-(f)\Delta_g\dot{w}^-(h) - \langle\nabla\dot{w}^-(f),\dot{w}^-(h)\rangle \\
&= \varepsilon\int_M -\langle\nabla\dot{w}(f),\nabla\dot{w}(h)\rangle + W''(u)\dot{w}(f)\dot{w}(h)
\end{aligned}$$

The second line follows since $\dot{w}^\pm$ coincide on Y, and in the third line, we recalled $\nu = \nu^+$ points towards M^+ while $\nu^- = -\nu$ points towards M^-. Clearly, the final expression is symmetric in f and h. With this, the associated linearized operator to the second variation of BE_ε is

$$\begin{aligned}
\delta\dot{w}_\nu(\cdot) &: C^2(Y) \to \mathbf{R} \\
\delta\dot{w}_\nu(f) &:= \dot{w}^+(f)_\nu - \dot{w}^-(f)_\nu
\end{aligned}$$

With this, we recall/define

$$\begin{aligned}
\mathrm{Null}_{AC_\varepsilon}(u_{\varepsilon,Y}) &= \dim\ker[\varepsilon^2\Delta_g - W''(u)] \\
\mathrm{Null}_{\mathrm{BE}_\varepsilon}(Y) &= \dim\ker[\delta\dot{w}_\nu]
\end{aligned}$$

where both kernels are considered over C^2.

Proof. As before, any variations of the form $u_r^i = u_{\varepsilon,Y} + r\phi^i$ can be rewritten as

$$u_r^i = u_{\varepsilon,Y} + r[\dot{w}^i + \dot{\psi}^i]$$

for $i = 1,2$, with $\dot{w}^i$ a solution to (1.1.4) and $\dot{\psi}^i|Y = 0$. Moreover, define f^i such that $\dot{w}^i\Big|_Y = -f^i u_\nu$.

We compute

$$\begin{aligned} Q_{AC_\varepsilon}(\dot{w}^1+\dot{\psi}^1,\dot{w}^2+\dot{\psi}^2) &= Q_{AC_\varepsilon}(\dot{w}^1,\dot{w}^2)+Q_{AC_\varepsilon}(\dot{\psi}^1,\dot{w}^2)+Q_{AC_\varepsilon}(\dot{w}^1,\dot{\psi}^2)+Q_{AC_\varepsilon}(\dot{\psi}^1,\dot{\psi}^2) \\ &= Q_{AC_\varepsilon}(\dot{w}^1,\dot{w}^2)+Q_{AC_\varepsilon}(\dot{\psi}^1,\dot{\psi}^2) \\ &= Q_{\mathrm{BE}_\varepsilon}(f^1,f^2)+Q_{AC_\varepsilon}(\dot{\psi}^1,\dot{\psi}^2) \end{aligned}$$

In the second line, we've noted as in the proof of theorem 1.1.3 that

$$Q_{AC_\varepsilon}(\dot{\psi}^1,\dot{w}^2)=Q_{AC_\varepsilon}(\dot{w}^1,\dot{\psi}^2)=0$$

by integration by parts and $\dot{\psi}^i\Big|_Y=0$.

If $\dot{u}^1=\dot{w}^1+\dot{\psi}^1$ is AC-null, then setting $\dot{\psi}^2=0$, we see that f^1 is BE_ε-null. This means that

$$\mathrm{Null}_{AC_\varepsilon}(u_{\varepsilon,Y})-\mathrm{Null}_{\mathrm{BE}_\varepsilon}(Y)\geq 0$$

Similarly, if $f^2=0$, then $\dot{w}(f^2)=0$ by proposition 3.4.1 (or [21], Thm 1.4) and so we have that

$$\begin{aligned} &Q_{AC_\varepsilon}(\dot{\psi}^1,\cdot):C_0^2(M^+)\cup C_0^2(M^-)\to\mathbf{R} \\ &Q_{AC_\varepsilon}(\dot{\psi}^1,\cdot)=0 \end{aligned}$$

where $C_0^k(M^\pm)$ denotes C^k functions vanishing on $Y=\partial M^\pm$. In particular, this means that $\dot{\psi}^1$ satisfies the linearized allen-cahn equation on $M^\pm$, but again vanishing on Y. Again, by proposition 3.4.1, we have that $\dot{\psi}^1\equiv 0$. Thus, if $\dot{u}^1$ lies in the nullity of Q_{AC_ε}, we have that $\dot{u}^1=\dot{w}(f^1)$ for some f^1 in the nullity of $Q_{\mathrm{BE}_\varepsilon}$. This finishes the proof. □

3.4 Second Variation for $\varepsilon\to 0$

In this section we aim to prove theorem 1.1.6 on the asymptotic expansion of the second variation formula in terms of ε.

Theorem 1.1.6. *Let $\{Y_t\}$ a one-parameter family of hypersurfaces in (M^n,g) with $Y_0=Y$ as in*

theorem 1.1.2. Assume that Y is a a $C^{2,\alpha}$ critical point with bounded geometry (1.2.1). Then

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} &= \frac{\sqrt{2}}{3}\int_Y f u_\nu[-2(\Delta_Y + \dot{H}_0)](f) + E(f) \\ &= \frac{2\sqrt{2}}{3}\int_Y |\nabla f|^2 - \dot{H}_Y f^2 + E \\ &= \frac{2\sqrt{2}}{3}[D^2A(f) + E(f)]\end{aligned} \tag{1.1.6}$$

where

$$|E(f)| \leq K\varepsilon^{1/2}||f||^2_{H^1}$$

3.4.1 Kernel of $L_{\varepsilon,u}$

To prove the Sobolev version of the second variation formula, we first prove theorem 1.1.7, i.e. $L_{\varepsilon,u} : H^1_{\varepsilon,0} \to H^{-1}_\varepsilon$ is invertible. This amounts to proving that there is no kernel:

Proposition 3.4.1. *For $\phi \in H^1_{\varepsilon,0}(M^+)$ we have for all $\varepsilon < \varepsilon_0$ sufficientl small that*

$$||\phi||_{H^1_\varepsilon(M^+)} \leq K||L_{\varepsilon,u}(\phi)||_{H^{-1}_\varepsilon(M^+)}$$

for some $K > 0$ independent of ε

An analogous result in the Hölder setting was proved in ([21], Theorem 1.4). The above immediately gives invertibility of $L_{\varepsilon,u}$ (see e.g. [9], Theorem 6.15), as well as solvability of the linearized system. (1.1.4), for $f \in H^1(Y)$. To show this proposition, we first establish the following mimic of elliptic regularity:

Proposition 3.4.2. *For $\phi : M^+ \to \mathbf{R}$ with $\phi \in H^1_{0,\varepsilon}(M^+)$, we have that for all $\varepsilon > 0$*

$$\varepsilon||\nabla\phi||_{L^2(M^+)} \leq 2(||L_{\varepsilon,u}(\phi)||_{H^{-1}_\varepsilon(M^+)} + ||\phi||_{L^2(M^+)})$$

Proof: We have

$$\begin{aligned}
\varepsilon^2 \int_{M^+} |\nabla \phi|^2 &= -\int_{M^+} (\varepsilon^2 \Delta_g \phi)\phi \\
&= -\int_{M^+} (L_{\varepsilon,u}\phi)\phi - \int_{M^+} W''(u)\phi^2 \qquad (3.4.1) \\
\Big| \int_{M^+} (L_{\varepsilon,u}\phi)\phi \Big| &\leq ||L_{\varepsilon,u}\phi||_{H_\varepsilon^{-1}(M^+)} ||\phi||_{H_\varepsilon^1(M^+)} \\
&\leq \frac{1}{2}\left(||L_{\varepsilon,u}\phi||^2_{H_\varepsilon^{-1}(M^+)} + ||\phi||^2_{H_\varepsilon^1(M^+)} \right) \\
\implies \frac{\varepsilon^2}{2} ||\nabla \phi||^2_{L^2(M^+)} &\leq \frac{1}{2} ||L_{\varepsilon,u}\phi||^2_{H_\varepsilon^{-1}(M^+)} + 2||\phi||^2_{L^2(M^+)}
\end{aligned}$$

having used that $|W''(u)| \leq 2$. Taking a square root finishes this. □

Corollary 3.4.3. We have

$$||\phi||_{H_\varepsilon^1} \leq K(||L_{\varepsilon,u}(\phi)||_{H_\varepsilon^{-1}(M^+)} + ||\phi||_{L^2(M^+)})$$

To prove 3.4.1, we see it suffices to show

Proposition 3.4.4. *There exists an $\varepsilon_0 > 0$, such that for all $\varepsilon < \varepsilon_0$ and $\phi \in H^1_{\varepsilon,0}(M^+)$*

$$||\phi||_{L^2(M)} \leq K||L_{\varepsilon,u}(\phi)||_{H_\varepsilon^{-1}(M)}$$

for some $K > 0$ independent of ε

First note the following poincare inequality near the boundary:

Lemma 3.4.5. *Let $\phi \in H^1_\varepsilon(Y \times [0, \kappa\varepsilon))$ and $\phi(s, 0) \equiv 0$ in the trace sense. Then for all $\beta > 0$, there exists an $\varepsilon_0(\beta)$ such that for $\varepsilon < \varepsilon_0(\beta)$, we have*

$$||\phi||_{L^2(Y\times[0,\kappa\varepsilon))} \leq \kappa^2\varepsilon^2(1+\beta)||\nabla\phi||_{L^2(Y\times[0,\kappa\varepsilon))}$$

Proof: WLOG, suppose that $\phi \in C^1(Y \times [0, \kappa\varepsilon))$ and $\phi(s, 0) \equiv 0$ since C^1 is dense in H^1_ε for ε

fixed. Then we have

$$\begin{aligned}
|\phi(s,t)| &= \Big| \int_0^t \partial_r \phi(s,r) dr \Big| \\
&\leq \int_0^t |\nabla^g \phi|(s,r) dr \\
&\leq \int_0^{\kappa\varepsilon} |\nabla^g \phi|(s,r) dr \\
\implies |\phi(s,t)|^2 &\leq \left(\int_0^{\kappa\varepsilon} |\nabla^g \phi|(s,r) dr \right)^2 \\
&\leq \kappa\varepsilon \int_0^{\kappa\varepsilon} |\nabla^g \phi|^2(s,r) dr
\end{aligned}$$

The last line follows by a change of variable $t \to \frac{t}{\kappa\varepsilon}$ and then Jensen's inequality. Now note that

$$\begin{aligned}
\sqrt{\det g(s,t)} dt ds &= [\sqrt{\det g|_Y(s)} + O(\varepsilon)] dt ds \\
|\sqrt{\det g(s,t)} - \sqrt{\det g|_Y(s)}| &\leq K\varepsilon \sqrt{\det g|_Y(s)} dt ds
\end{aligned}$$

here, $\det g|_Y(s) ds = dA_Y$ is the volume form on Y and the $O(\varepsilon)$ expansion follows from being in fermi coordinates and $0 \leq t \leq \kappa\varepsilon$. The second line comes from compactness of Y and assuming that its riemannian (i.e. $\sqrt{\det g|_Y} \geq c_0 > 0$ for some c_0). This gives

$$\begin{aligned}
\int_0^{\kappa\varepsilon} |\phi(s,t)|^2 \sqrt{\det g(s,t)} dt &\leq [(\kappa\varepsilon)^2 + 2K\varepsilon^3] \int_0^{\kappa\varepsilon} |\nabla^g \phi|^2(s,r) \sqrt{\det g(s,t)} dr \\
\implies ||\phi||^2_{L^2(Y\times[0,\kappa\varepsilon))} &\leq [\kappa^2\varepsilon^2 + 2K\varepsilon^3] ||\nabla\phi||^2_{L^2(Y\times[0,\kappa\varepsilon))}
\end{aligned}$$

Taking ε sufficiently small we have the lemma. □

We are interested in applying lemma 3.4.5 with $\kappa < 1$. In particular, we note that

$$\kappa_0 := (3g(t)^2 - 1)^{-1} \cap \mathbf{R}^+ = 0.93123\cdots = 1 - \gamma, \qquad \gamma > 0$$

i.e. the positive root of $W''(g) = 3\tanh(t/\sqrt{2})^2 - 1$ is less than 1. In particular, let

$$\begin{aligned}
&\kappa = 1 - \gamma/2 < 1 \\
&\delta = 3g(\kappa)^2 - 1 > 0, \qquad \delta < 1
\end{aligned} \tag{3.4.2}$$

Then we see that for ε sufficiently small, because of uniform bounds on $||A_Y||$ and (1.2.9), we have

$$W''(u(s,\kappa\varepsilon)) > \frac{\delta}{2}$$

Proof of 3.4.4 We first assume that most of the mass is on $t > \kappa\varepsilon$ and bound as follows: let $0 < \alpha = \alpha(\delta) < 1$ to be determined and assume

$$||\phi||^2_{L^2(t>\kappa\varepsilon)} \geq (1-\alpha)||\phi||^2_{L^2(M)}$$

Then we have

$$\frac{\delta}{2}||\phi||^2_{L^2(t>\kappa\varepsilon)} \leq \int_{t>\kappa\varepsilon} W''(u)\phi^2 \leq 2||\phi||^2_{L^2(t>\kappa\varepsilon)}$$

since $t > \kappa\varepsilon \implies W''(u) > \frac{\delta}{2}$. Note also that

$$\Big|\int_Y \int_0^{\kappa\varepsilon} W''(u)\phi^2\Big| \leq ||\phi||^2_{L^2(t<\kappa\varepsilon)} \leq \alpha||\phi||^2_{L^2(M)}$$

and so

$$\int_M W''(u)\phi^2 \geq \left(1 - \frac{2\alpha}{\delta(1-\alpha)}\right)\int_{t>\kappa\varepsilon} W''(u)\phi^2$$

In particular

$$\begin{aligned}
||\phi||^2_{L^2(M)} &\leq \frac{2}{\delta(1-\alpha)}\int_{t>\kappa\varepsilon} W''(u)\phi^2 \\
\implies ||\phi||^2_{L^2(M)} &\leq \frac{2}{\delta - \alpha\delta - 2\alpha}\int_M W''(u)\phi^2 \\
&= \frac{2}{\delta - \alpha\delta - 2\alpha}\left[-\int_M \phi L_{\varepsilon,u}(\phi) - \varepsilon^2\int_M |\nabla\phi|^2\right] \\
&= \frac{2}{\delta - \alpha\delta - 2\alpha}\left[-\int_M \phi L_{\varepsilon,u}(\phi)\right] \\
&\leq \frac{2}{\delta - \alpha\delta - 2\alpha}\left[\frac{\lambda}{2}||\phi||^2_{H^1_\varepsilon} + \frac{1}{2\lambda}||L_{\varepsilon,u}\phi||^2_{H^{-1}_\varepsilon}\right] \\
\implies ||\phi||^2_{L^2(M)} &\leq K||L_{\varepsilon,u}(\phi)||^2_{H^{-1}_\varepsilon(M)}
\end{aligned}$$

In the last line, we noted that

$$||\phi||^2_{H^1_\varepsilon} \leq K[||\phi||^2_{L^2(M)} + ||L_{\varepsilon,u}(\phi)||^2_{H^{-1}_\varepsilon(M)}]$$

from corollary 3.4.3, and also chose $\alpha < \delta/4$ so that

$$\delta - \alpha(2+\delta) > \frac{\delta}{2}$$

and finally chose λ sufficiently small independent of ε so that we can move the $K\lambda||\phi||_{L^2(M)}$ term over to the left side. This tells us that if $||\phi||^2_{L^2(t>\kappa\varepsilon)} \geq (1-\alpha)||\phi||^2_{L^2(M)}$ for $\alpha < \delta/8$, then bound holds.

Now suppose the contrary:

$$||\phi||^2_{L^2(Y\times[0,\kappa\varepsilon)} > \frac{\delta}{8}||\phi||^2_{L^2(M)}$$

We bound on $||\phi||_{L^2(Y\times[0,\kappa\varepsilon))}$ for κ as in (3.4.2), using explicitly that $\kappa < 1$:

$$\begin{aligned}
||\phi||^2_{L^2(Y\times[0,\kappa\varepsilon)} &\leq \varepsilon^2\kappa^2(1+\beta)||\nabla\phi||^2_{L^2(Y\times[0,\kappa\varepsilon))} \\
&\leq \varepsilon^2\kappa^2(1+\beta)||\nabla\phi||_{L^2(M)} \\
&\leq \kappa^2(1+\beta)\left[-\int_M \phi L_{\varepsilon,u}(\phi) - \int_M W''(u)\phi^2\right]
\end{aligned}$$

explicitly using (3.4.1) from proposition 3.4.2. Now we note that

$$-\int_M W''(u)\phi^2 \leq -\int_{t=0}^{t=\kappa\varepsilon} W''(u)\phi^2 \leq \int_{t=0}^{t=\kappa\varepsilon} \phi^2 = ||\phi||^2_{L^2(Y\times[0,\kappa\varepsilon))}$$

By definition of $W''(u)$ and since $W''(u) > 0$ on $t \geq \kappa\varepsilon$ and $|W''(u)| \leq 1$ on $[0,\kappa\varepsilon]$. Now we have

$$\begin{aligned}
||\phi||^2_{L^2(Y\times[0,\kappa\varepsilon)}(1-\kappa^2(1+\beta)) &\leq -\int_M \phi L_{\varepsilon,u}(\phi) \\
&\leq \frac{\lambda}{2}||\phi||^2_{H^1_\varepsilon} + \frac{1}{2\lambda}||L_{\varepsilon,u}(\phi)||^2_{H^{-1}_\varepsilon}
\end{aligned}$$

Note that for β sufficiently small, $1-\kappa^2(1+\beta) > 0$. If λ is sufficiently small (indepedent of ε) and we use corollary 3.4.3 again, the above gives

$$||\phi||^2_{L^2(Y\times[0,\kappa\varepsilon))} \leq K||L_{\varepsilon,u}(\phi)||^2_{H^{-1}_\varepsilon(M)} \implies ||\phi||^2_{L^2(M)} \leq \frac{8K}{\delta}||L_{\varepsilon,u}(\phi)||^2_{H^{-1}_\varepsilon(M)}$$

which is the desired bound. □

Remark 4. Our proof relies heavily on the fact that $\kappa_0 < 1$, and hence, heavily on the choice of $W(x) = \frac{(1-x^2)^2}{4}$ as our double well potential. We expect a similar statement about $\kappa_0 < 1$ to hold

for all double well potentials with $W(0) = \frac{1}{4}$ and $W(\pm 1) = 0$. The authors currently lack a proof for this.

3.5 Proof of Theorem 1.1.16

3.5.1 Set up

Recall that we have $Y^{n-1} \subseteq M^n$ a closed separating hypersurface with $M = M^+ \sqcup_Y M^-$. Let $u_\varepsilon^\pm$ be the Dirichlet minimizer of Allen–Cahn on $M^\pm$ which is positive (resp. negative) away from Y the boundary. Recall the linearized system for $\dot{u}$ as in (1.1.4):

$$\begin{aligned} \varepsilon^2 \Delta_g \dot{u} &= W''(u)\dot{u} \\ \dot{u}\Big|_Y &= -f(s)u_\nu(s) = h(s) \end{aligned}$$

where s is a coordinate on Y. We posit

$$\dot{u} = h(s)\dot{\bar{g}}_\varepsilon(t) + \phi \tag{3.5.1}$$

so that $\phi : M^+ \to \mathbf{R}$ and $\phi\Big|_Y \equiv 0$. We want to produce H^1_ε bounds on ϕ. In the following sections, we'll be working with one-sided solutions and often write M in place of $M^\pm$, noting that our results can be applied to either M^+ or M^-.

3.5.2 Computing the bound

In this section, we apply proposition 3.4.1 and show

Lemma 3.5.1. *There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $\phi : M \to \mathbf{R}$ as in equation (3.5.1), we have*

$$||\phi||_{H^1_\varepsilon(M)} \leq K\varepsilon[||\nabla f||_{L^2(Y)} + \sqrt{\varepsilon}||f||_{L^2(Y)}] \tag{3.5.2}$$

for K independent of ε. In particular $||\phi||_{H^1_\varepsilon(M)} \leq K\varepsilon||f||_{H^1(Y)}$.

Proof: First note

$$L_{\varepsilon,u}(\phi) = -L_{\varepsilon,u}(h\dot{\bar{g}}_\varepsilon(t))$$

and

$$h(s) = -f(s)\left[\frac{1}{\varepsilon\sqrt{2}} + O(1)\right] \tag{3.5.3}$$

from the expansion of $\partial_t u|_{t=0} = u_\nu$ implicit in (1.2.9) for surfaces with bounded geometry. We compute bounds on $L_{\varepsilon,u}(\phi)$

$$\begin{aligned}||L_{\varepsilon,u}(\phi)||_{H_\varepsilon^{-1}(M)} &= || - \varepsilon^2 \Delta_t(h)\dot{\overline{g}}_\varepsilon + \varepsilon H_t h \ddot{\overline{g}}_\varepsilon - [W''(u) - W''(\overline{g}_\varepsilon) + E]h(s)\dot{\overline{g}}_\varepsilon||_{H_\varepsilon^{-1}} \\ &\leq \Big[\varepsilon||\Delta_t(f)\dot{\overline{g}}_\varepsilon||_{H_\varepsilon^{-1}(M)} + ||H_t f\ddot{\overline{g}}_\varepsilon||_{L^2(M)} \\ &\quad + \varepsilon^{-1}||[W''(u) - W''(\overline{g}_\varepsilon) + E]f\dot{\overline{g}}_\varepsilon||_{L^2(M)}\Big]\end{aligned}$$

by the expansion of $h(s)$ in (3.5.3). Note that

$$H_t = H_Y + \varepsilon \dot{H}_Y(t/\varepsilon) + O(\varepsilon^2)$$

Recall the expansion of u as in (1.2.10)

$$u(s,t) = \overline{g}_\varepsilon(t) + \varepsilon H_Y(s)\overline{w}_\varepsilon(t) + \varepsilon^2\left[\dot{H}_Y(s)\overline{\tau}_\varepsilon(t) + H_Y(s)^2\overline{\rho}_\varepsilon(t) + \frac{1}{2}H_Y(s)^2\overline{\rho}_\varepsilon(t)\right] + \phi(s,t)$$

$$||\phi||_{C_\varepsilon^{2,\alpha}} = O(\varepsilon^3)$$

For critical points of BE, we know that $u_\nu = u_\nu^c$ so from (3.1.2), we have

$$H_Y = O(\varepsilon^2) \tag{3.5.4}$$

We then have the following in C^0 and L^2:

$$\begin{aligned}u(s,t) &= \overline{g}_\varepsilon(t) + \varepsilon^2 \dot{H}_Y(s)\overline{\tau}_\varepsilon(t) + O(\varepsilon^3) \\ \implies W''(u) - W''(\overline{g}_\varepsilon) &= 3(u - \overline{g}_\varepsilon)(u + \overline{g}_\varepsilon) \\ &= 3\varepsilon^2 \dot{H}_Y \overline{\tau}_\varepsilon(2\overline{g}_\varepsilon) + O(\varepsilon^3) \\ &= 6\varepsilon^2 \dot{H}_Y \overline{\tau}_\varepsilon \overline{g}_\varepsilon + O(\varepsilon^3)\end{aligned} \tag{3.5.5}$$

Furthermore, since $H_Y = H_0 = O(\varepsilon^2)$ and $W''(u) - W''(\overline{g}_\varepsilon) = O(\varepsilon^2)$, we have that

$$\begin{aligned}||H_t f\ddot{\overline{g}}_\varepsilon||^2_{L^2(M)} &= \int_Y \int_0^{-2\omega\varepsilon\ln(\varepsilon)} H_t^2 f^2 \ddot{\overline{g}}_\varepsilon^2 \\ &\leq K\varepsilon^3 ||f||^2_{L^2(B_2)} \\ \implies ||H_t f\ddot{\overline{g}}_\varepsilon||_{L^2(M)} &\leq K\varepsilon^{3/2}||f||_{L^2(Y)}\end{aligned}$$

where the extra factor of ε comes from integrating $\ddot{\overline{g}}_\varepsilon^2$. Similarly, we have

$$\frac{1}{\varepsilon\sqrt{2}}||[W''(u) - W''(\overline{g}_\varepsilon) + E]f\dot{\overline{g}}_\varepsilon||_{L^2(M)} \leq K\varepsilon^{3/2}||f||_{L^2(B_2)}$$

So it suffices to compute $||\Delta_t(f)\dot{\overline{g}}_\varepsilon||_{H_\varepsilon^{-1}}$. Note that

$$\begin{aligned}
& v \in H^1_{0,\varepsilon}(M) \\
\implies \int_M \Delta_t(f)\dot{\overline{g}}_\varepsilon v &= \int_Y \int_0^{-2\omega\varepsilon\ln(\varepsilon)} \Delta_t(f)\dot{\overline{g}}_\varepsilon v dA \\
&= \int_0^{-2\omega\varepsilon\ln(\varepsilon)} \dot{\overline{g}}_\varepsilon \int_Y \Delta_t(f) v\sqrt{\det g(s,t)} ds dt \\
&= \int_0^{-2\omega\varepsilon\ln(\varepsilon)} \dot{\overline{g}}_\varepsilon \left(\int_Y g|_Y \left(\nabla^t \left(v(s,t)\sqrt{\det g(s,t)} \right), \nabla^t(f) \right) ds \right) dt \\
\implies \left| \int_M \Delta_t(f)\dot{\overline{g}}_\varepsilon v \right| &\leq K\varepsilon ||v||_{H^1} ||\nabla f||_{L^2} \\
&\leq K||v||_{H^1_\varepsilon} ||\nabla f||_{L^2}
\end{aligned}$$

which tells us that

$$||L_{\varepsilon,u}(\phi)||_{H_\varepsilon^{-1}(M)} \leq K\varepsilon||\nabla f||_{L^2(Y)} + K\varepsilon^{3/2}||f||_{L^2(Y)}$$

Applying proposition 3.4.1 we have

$$||\phi||_{H^1_\varepsilon(M)} \leq K\varepsilon[||\nabla f||_{L^2(Y)} + \sqrt{\varepsilon}||f||_{L^2(Y)}] \tag{3.5.6}$$

completing the lemma. □

3.5.3 Computing second variation

We have that for $f \in C^2$

$$\mathrm{BE}''(f\nu)\Big|_Y = \int_Y f u_\nu [\dot{u}_\nu - \dot{u}_\nu^c]$$

We've shown that

$$\dot{u}_\nu(s) = \partial_t \left(h(s)\dot{\overline{g}}_\varepsilon(t) + \phi \right)(s,0) = \phi_t(s,0)$$

and also $u = \overline{g}_\varepsilon(t) + O(\varepsilon^2)$ in $C^{1,\alpha}_\varepsilon$ and so

$$\begin{aligned}\mathrm{BE}''(f\nu)\Big|_Y &= \int_Y f u_\nu [\phi_t - \phi^c_t] \\ &= \int_Y f\left[(\varepsilon\sqrt{2})^{-1} + O(\varepsilon)\right][\phi_t - \phi^c_t]\end{aligned}$$

We write

$$\begin{aligned}-\varepsilon^2\phi_t(s,0)\sqrt{\det g(s,0)} &= \sqrt{2}\int_0^{-2\omega\varepsilon\ln(\varepsilon)} (\varepsilon^2\partial_t^2 - W''(\overline{g}_\varepsilon))\phi\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dt + R \qquad (3.5.7)\\ &= \sqrt{2}\int_0^{-2\omega\varepsilon\ln(\varepsilon)} L_{\varepsilon,u}(\phi)\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dt \\ &\quad - \sqrt{2}\int_0^{-2\omega\varepsilon\ln(\varepsilon)} [\varepsilon^2\Delta_t(\phi) - H_t\varepsilon^2\phi_t]\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dt \\ &\quad + \sqrt{2}\int_0^{-2\omega\varepsilon\ln(\varepsilon)} [W''(u) - W''(\overline{g}_\varepsilon) + E]\phi\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dt + R\end{aligned}$$

using that $\dot{g}(0) = 2^{-1/2}$ and

$$\Big|\int_Y fR\Big| \le K\varepsilon^3||f||^2_{H^1(Y)} \qquad (3.5.8)$$

(see appendix §4.1.4). From before, we have

$$\begin{aligned}L_{\varepsilon,u}(\phi) &= -L_{\varepsilon,u}(h(s)\dot{\overline{g}}_\varepsilon) \\ &= -\varepsilon^2\Delta_t(h)\dot{\overline{g}}_\varepsilon + \varepsilon H_t h\ddot{\overline{g}}_\varepsilon + [W''(u) - W''(\overline{g}_\varepsilon) + E]h(s)\dot{\overline{g}}_\varepsilon\end{aligned}$$

So that from (3.5.3) and (3.5.7)

$$\begin{aligned}\int_Y f\phi_t &= -\varepsilon^{-2}\sqrt{2}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} \Big(-\varepsilon^2\Delta_t(h)\dot{\overline{g}}^2_\varepsilon + \varepsilon H_t h\ddot{\overline{g}}_\varepsilon\dot{\overline{g}}_\varepsilon + [W''(u) - W''(\overline{g}_\varepsilon) + E]h(s)\dot{\overline{g}}^2_\varepsilon \\ &\qquad - \sqrt{2}[\varepsilon^2\Delta_t(\phi) - H_t\varepsilon^2\phi_t - (W''(u) - W''(\overline{g}_\varepsilon) + E)\phi]\dot{\overline{g}}_\varepsilon\Big)\sqrt{\det g(s,t)}dtds \\ &= \sqrt{2}\int_0^{-2\omega\varepsilon\ln(\varepsilon)}\int_{Y_t}\Big[f\Delta_t(h)\dot{\overline{g}}^2_\varepsilon - \dot{H}_0 f^2(t/\varepsilon)\ddot{\overline{g}}_\varepsilon\dot{\overline{g}}_\varepsilon - 6\dot{H}_0 fh\dot{\tilde{\tau}}_\varepsilon(t)\overline{g}_\varepsilon(t)\dot{\overline{g}}^2_\varepsilon + f\Delta_t(\phi)\dot{\overline{g}}_\varepsilon\Big]dA_{Y_t}dt \\ &\quad + \tilde{R}\end{aligned}$$

where

$$|\tilde{R}| \le K\varepsilon||f||^2_{H^1}$$

by grouping the higher order terms in the expansion of H_t and $W''(u) - W''(\overline{g}_\varepsilon)$. In the appendix (§3.5.2), we integrate by parts and compute integrals of our ODE solutions and show

$$\int_Y f u_\nu \dot{u}_\nu = \frac{\sqrt{2}}{3}\int_Y (|\nabla f|^2 - \dot{H}_Y f^2) dA_Y + E \tag{3.5.9}$$
$$|E| \leq K\varepsilon^{1/2}||f||^2_{H^1(Y)}$$

Noting that $|\nabla f|^2$ and $\dot{H}_Y$ are invariant under the choice of normal, we see that

$$\begin{aligned}
\mathrm{BE}''(f\nu)\Big|_Y &= \int_Y f u_\nu [\dot{u}^+_\nu - \dot{u}^-_\nu] \\
&= \int_Y f u_\nu [\dot{u}^+_\nu + \dot{u}^-_{\nu^-}] \\
&= 2\frac{\sqrt{2}}{3}\int_Y (|\nabla f|^2 - \dot{H}_Y f^2) dA_Y + E \\
|E| &\leq K\varepsilon^{1/2}||f||^2_{H^1(Y)}
\end{aligned}$$

where $\nu^- = -\nu$ is the normal to Y pointing towards M^-.

Remark 3.5.2. Using theorem 1.1.6 it is straightfoward to see that there are no stable critical points of the BE_ε functional on Manifolds with positive Ricci curvature. This is identical to the situation for the usual Allen–Cahn energies, as well as for the case of minimal surfaces. To see this suppose that

$$\mathrm{Ric} \geq C\mathrm{Id}$$

for some $C > 0$. With theorem 1.1.6, we see that if we take $f \equiv 1$ then

$$BE''(\nu) = -\frac{2\sqrt{2}}{3}\int_Y [\dot{H}_0 + O(\varepsilon)]$$

We recall that

$$\dot{H}_0 = |A_Y|^2 + \mathrm{Ric}_g(\nu, \nu)$$

So we can say prove the following result, corollary 1.1.2

Corollary 1.1.2. *Suppose that Y_ε are a sequence of critical points to* BE_ε *for all $\varepsilon < \varepsilon_0$ sufficiently small. Let Y_ε a $C^{2,\alpha}$ hypersurface and suppose that there exists an ε_0 such that for all $\varepsilon < \varepsilon_0$ either one of the following hold*

- *The Ric curvature is bounded from below, i.e.*

$$\exists c_0 > 0 \quad s.t. \qquad Ric_g(\nu,\nu)\Big|_Y > c_0$$

- *The second fundamental form has bounded L^2 norm and Vol(Y) is finite*

$$\exists c_1, c_2 > 0, \quad s.t. \quad ||A_Y||^2_{L^2(Y)} \geq c_1, \quad \exists Vol(Y) \leq c_2 \tag{1.1.7}$$

Then Y is unstable.

Remark 3.5.3. Though we initially assumed $f \in C^2$, we can define the second variation formula for any $f \in H^1(Y)$ by using the final formula.

Remark 3.5.4. One can also prove a Hölder version of this theorem and show that

$$\dot{u}^+_\nu - \dot{u}^-_\nu = 2\frac{\sqrt{2}}{3}[\Delta_Y + \dot{H}_0](f) + E(f)$$
$$||E(f)||_{C^\alpha(Y)} \leq K\varepsilon||f||_{C^{2,\alpha}(Y)}$$

To prove this, one can use (1.2.10) to expand u and fermi coordinates to expand $\varepsilon^2\Delta_g$. This is enough to expand $\dot{u}$ and compute the neumann difference to sufficiently high order in ε

BE_ε eigenvalues

We clearly see that for ε sufficiently small,

$$\frac{d^2}{dt^2}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} \geq \frac{\sqrt{2}}{3}||\nabla f||^2_{L^2(M)} - K||f||^2_{L^2(M)}$$

with K independent of ε. With this, we can minimize

$$\frac{d^2}{dt^2}\mathrm{BE}_\varepsilon(Y_t)\Big|_{t=0} = D^2\mathrm{BE}_Y(f)$$

with respect to f for $f \in H^1(Y)$. This allows us to use the Rayleigh quotient to define eigenvalues for $\mathrm{BE}''_{\varepsilon,Y}$ at a critical point Y:

$$\lambda_i := \inf_{\substack{f \in E \\ E \subset H^1(Y)}} \frac{\mathrm{BE}''_{\varepsilon,Y}(f)}{||f||^2_{L^2(Y)}},$$

$$\lambda_1 \leq \cdots \leq \lambda_m \leq 0 = \lambda_{m+1} = \cdots = \lambda_{m+n} \leq \cdots .$$

Here E ranges over i-dimensional subspaces of $H^1(Y)$, n is the BE_ε nullity, and m is the BE_ε Morse index of Y. We recall from section §3.3 that $m = \mathrm{Ind}_{AC_\varepsilon}(u_{\varepsilon,Y})$ and $n \leq \mathrm{Null}_{AC_\varepsilon}(u_{\varepsilon,Y})$.

Chapter 4

Applications of Second Variation Formula

4.0.1 Allen–Cahn Index of symmetric solutions

For $Y^{n-1} \subseteq M^n$ minimal, let $\mathrm{Ind}_{\mathrm{Min}}(Y)$ denote the index of Y as a minimal surface. We prove the following corollary of theorem 1.1.6, corollary 1.1.3

Corollary 1.1.3. *Suppose $\{u_\varepsilon\}$ is a sequence of solutions with $Y = u_\varepsilon^{-1}(0)$ for all ε. If Y is minimal and u_ε are each critical points of (1.2.1), then there exists an $\varepsilon_0 > 0$ sufficiently small, such that for all $\varepsilon < \varepsilon_0$*

$$\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) = \mathrm{Ind}_{\mathrm{BE}_\varepsilon}(u_\varepsilon) \geq \mathrm{Ind}_{Min}(Y) \tag{1.1.8}$$

Proof: The first equality is shown in theorem 1.1.3. For the third equality, we look at the formula:

$$\mathrm{BE}''_\varepsilon(f\nu)\Big|_Y = -\frac{2\sqrt{2}}{3}\int_Y f[J_Y + E](f)$$

for $J_Y = \Delta_Y + \dot{H}_Y$. Let $V \subseteq C_c^\infty(Y)$ denote the finite dimensional vector space such that

$$f \in V,\ f \neq 0 \implies D^2A(f\nu)\Big|_Y = -\int_Y fJ_Y(f) < 0$$

By self-adjointness of J_Y, we can diagonalize V and get an orthogonal basis $\{f_1, \ldots, f_m\}$. WLOG

we can normalize each so that $||f_i||_{C^2(Y)} = 1$. Now we compute

$$\begin{aligned}
\mathrm{BE}''_\varepsilon(a^i f_i) &= -\frac{2\sqrt{2}}{3}\int_Y (a^i f_i)(J_Y + E)(a^j f_j) \\
&= -\frac{2\sqrt{2}}{3}\left[\int_Y (a^i f_i) J_Y(a^j f_j) + (a^i f_i)E(a^j f_j)\right] \\
&= \frac{2\sqrt{2}}{3} D^2 A(a^i f_i)\Big|_Y - 2\sigma_0 \int (a^i f_i)E(a^j f_j) \\
&= \frac{2\sqrt{2}}{3}\left[(a^i)^2 D^2 A\Big|_Y(f_i)\right] - 2\sigma_0 \int (a^i f_i)E(a^j f_j)
\end{aligned}$$

Now note that we can bound

$$\begin{aligned}
\Big|\int (a^i f_i)E(a^j f_j)\Big| &= \Big|a^i a^j \int f_i E(f_j)\Big| \\
&\leq \frac{(a^i)^2 + (a^j)^2}{2}\Big|\int f_i E(f_j)\Big| \\
&\leq \frac{(a^i)^2 + (a^j)^2}{2}\int |f_i|\,|E(f_j)| \\
&\leq \frac{(a^i)^2 + (a^j)^2}{2} K\varepsilon
\end{aligned}$$

for K independent of i and ε. Summing over both i and j, we get

$$\begin{aligned}
\mathrm{BE}''_\varepsilon(a^i f_i) &\leq \frac{2\sqrt{2}}{3}\sum_i (a^i)^2 D^2 A\Big|_Y(f_i) + K\varepsilon \sum_i (a^i)^2 \\
&\leq \frac{2\sqrt{2}}{3}\sum_i (a^i)^2 [D^2 A|_Y(f_i) + K\varepsilon]
\end{aligned}$$

Define

$$K_i := -D^2 A(f_i \nu) > 0$$

Note that K_i are independent of ε, so as long as

$$\varepsilon < \frac{1}{2K}\sup_i K_i$$

we have that

$$\mathrm{BE}''_\varepsilon(a^i f_i) \leq \frac{\sqrt{2}}{3}\sum_i (a^i)^2 D^2 A|_Y(f_i) < 0$$

this finishes the proof. □

This inequality can be used to reprove the index of a family of symmetric solutions to Allen–Cahn vanishing on the clifford torus in S^3

Corollary 4.0.1. *Let $Y^2 \subseteq S^3$ be the clifford torus and let $u_\varepsilon : S^2 \to \mathbf{R}$ be the sequence of solutions vanishing along Y constructed in [4], Prop 9.1. There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$*

$$\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) = 5$$

Remark 5. This result, and a stronger rigidity result, was originally proved by Hiesmayr in theorem 1 [13]. A separate proof was given in [3], theorem 1.1

Proof: Note that the clifford torus, $\mathcal{T}$, is minimal inside of S^3, and that the solutions $\{u_\varepsilon\}$ are equal to the energy minimizers on either side of $\mathcal{T}$ inside of S^3 (in fact, such solutions are unique by uniqueness of solutions with the dirichlet condition). This tells us that

$$\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) = \mathrm{Ind}_{\mathrm{BE}_\varepsilon}(\mathcal{T})$$

Now apply corollary 1.1.3 and Urbano's original proof that

$$\mathrm{Ind}_{\mathrm{Min}}(\mathcal{T}) = 5$$

(see [24], Theorem) to conclude that

$$\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) \geq 5$$

To bound from above, we apply Chodosh-Mantoulidis Theorem 5.11 [5] to get that

$$\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) + \mathrm{Null}_{AC_\varepsilon}(u_\varepsilon) \leq \mathrm{Ind}_{\mathrm{Min}}(\mathcal{T}) + \mathrm{Null}_{\mathrm{Min}}(\mathcal{T}) \tag{4.0.1}$$

and note that $\mathrm{Null}_{\mathrm{Min}}(\mathcal{T}) = 4$ by [15]. In fact, the nullity of $\mathcal{T}$ is generated by the subspace $V \subseteq \mathfrak{so}(4)$ of dimension 4 which *does not* fix $\mathcal{T}$. The same subspace of killing fields give rise to diffeomorphisms of S^3 which preserve $E_\varepsilon(u_\varepsilon, S^3) = \mathrm{BE}_\varepsilon(\mathcal{T})$ and so $\mathrm{Null}_{AC_\varepsilon}(u_\varepsilon) \geq 4$. Immediately by equation (4.0.1), we conclude $\mathrm{Ind}_{AC_\varepsilon}(u_\varepsilon) = 5$. □

4.0.2 Application: Allen–Cahn Index of D_{2p}-symmetric solutions on S^1

Identify $S^1 \cong [0,1]$. Let $p \in \mathbf{N}^+$. There exists an $\varepsilon_p > 0$, such that for any $\varepsilon_p > \varepsilon > 0$, there exists a positive energy minimizer of (1.0.1) on $[0, 1/(2p)]$ with dirichlet boundary conditions, $u^+_{\varepsilon,2p}$. Now

define

$$u_{\varepsilon,2p}(x) = (-1)^i u^+_{\varepsilon,2p}(x \mod 1/(2p))$$
$$x \in [i/2p, (i+1)/2p]$$

i.e. we reflect and repeat $u^+_{\varepsilon,2p}$ at each subsequent interval of length $1/(2p)$. By symmetry, this is a solution to Allen–Cahn. We recall that all solutions to Allen–Cahn on S^1 with a finite number of zeros are of the above form. This is done in [19], proposition 2.1, among other places

Theorem 4.0.1 ([19], Prop 2.1)**.** *Let $\varepsilon > 0$. Given u a solution to* (1.0.2) *on S^1 with $|u^{-1}(0)| > 0$ and finite, $u = u_{\varepsilon,2p}$ for some $p \geq 0$*

Proof: Lift u to a solution on $\mathbf{R}$. Using the Poincaré-Bendixson theorem (§10.5 in [14]) on the dynamical system given by (u, u_x), we see that u is either constant, periodic, or heteroclinic. If u is constant, then $p = 0$ and $|u^{-1}(0)| = 0$. If $|u^{-1}(0)| > 0$, then u cannot be heteroclinic, as this is not 1-periodic and u is lifted from S^1 with at least 1 zero. Thus u must be periodic on $\mathbf{R}$.

WLOG, $u(0) = 0$ and let $\alpha > 0$ be the first positive 0 of u. Now construct

$$\tilde{u}(x) := (-1)^i u(x \bmod \alpha)$$
$$x \in [i\alpha, (i+1)\alpha]$$

The two functions agree on $[0, \alpha]$, so by existence and uniqueness, they are equal everywhere. Now note that because u is lifted from S^1, we have $u(0) = u(1) = 0$, which means that $n\alpha = 1$ for some n, i.e. $\alpha = 1/n$. Finally, note that u changes sign at $k\alpha$, and $u_x\Big|_{x=0}$ is only defined if n is even. Thus $\alpha = 1/(2p)$ for some $p \in \mathbf{N}^+$. □

With this, our goal is to compute the Morse index of all solutions on S^1 with $|u^{-1}(0)|$ finite. We'll first show proposition 1.1.1

Proposition 1.1.1. *Let $p > 0$, and $u_{\varepsilon,2p}$ the solution vanishing on Y_{2p}. Consider a variation of the form*

$$a_i(t) := \frac{i}{2p} + f(i/(2p))t$$

for $f : \{0, 1, \ldots, 2p-1\} \to \mathbf{R}$. *Then*

$$BE''(f)\Big|_{Y_{2p}} = \varepsilon c^2 v(\varepsilon) \sum_{i=0}^{2p-1} [f(i/(2p)) - f((i+1)/2p)]^2 \tag{1.1.9}$$

for $v(\varepsilon) < 0$ *for all* $\varepsilon > 0$

and then conclude theorem 1.1.9

Theorem 1.1.9. *Fix* $p > 0$. *There exists an* ε_p *such that* $\forall \varepsilon < \varepsilon_p$, *the function,* $u_{\varepsilon,2p}$, *has Allen–Cahn Morse index* $2p-1$ *and nullity* 1. *The nullity is given by rotations of* S^1 *and every other non-trivial variation produces a strictly negative variation.*

Remark 6. By theorem 1.1.3 (adapted for a one dimensional manifold), theorem 1.1.9 shows that the Allen–Cahn index of $u_{\varepsilon,2p}$ equals $2p-1$ for all $\varepsilon > 0$ sufficiently small.

Reformulating second variation

For general p, we reframe

$$q_i = \frac{i}{2p} + f\left(\frac{i}{2p}\right) t \tag{4.0.2}$$

so that we can write the second variation in a more convenient way

Lemma 4.0.2. *Let* $u_{\varepsilon,2p} : S^1 \to \mathbf{R}$ *be the solution to* (1.0.2) *with* $2p$ *symmetric nodes and let* $\dot{u}_{\varepsilon,i}$ *be the solution to* (1.1.4) *on each subinterval of* $\left[\frac{i}{2p}, \frac{i+1}{2p}\right]$. *Then*

$$BE'' = \varepsilon c \sum_{i=0}^{2p-1} \left[f\left(\frac{i}{2p}\right) \dot{u}_{i,x}\left(\frac{i}{2p}\right) + f\left(\frac{i+1}{2p}\right) \dot{u}_{i,x}\left(\frac{i+1}{2p}\right)\right]$$

Proof: Using (1.1.5) and letting $\dot{u}_x^{\pm}$ denote the x derivative from the $\pm$ direction, we compute:

$$\begin{aligned} BE''(f) &= \varepsilon \sum_{i=0}^{2p-1} (-1)^i f\left(\frac{i}{2p}\right) u_x\left(\frac{i}{2p}\right) \left[\dot{u}_x^+\left(\frac{i}{2p}\right) - \dot{u}_x^-\left(\frac{i}{2p}\right)\right] \\ &= \varepsilon \sum_{i=0}^{2p-1} (-1)^i \left[f\left(\frac{i}{2p}\right) u_x\left(\frac{i}{2p}\right) \dot{u}_{i,x}\left(\frac{i}{2p}\right) + f\left(\frac{i+1}{2p}\right) u_x\left(\frac{i+1}{2p}\right) \dot{u}_{i,x}\left(\frac{i+1}{2p}\right)\right] \end{aligned}$$

where on the interval of $\left[\frac{i}{2p}, \frac{i+1}{2p}\right]$, we have defined

$$\begin{aligned}
\varepsilon^2 \dot{u}_{i,xx} &= W''(u)\dot{u} \\
\dot{u}_i\left(\frac{i}{2p}\right) &= f\left(\frac{i}{2p}\right) u_x\left(\frac{i}{2p}\right) \\
\dot{u}_i\left(\frac{i+1}{2p}\right) &= f\left(\frac{i+1}{2p}\right) u_x\left(\frac{i+1}{2p}\right)
\end{aligned}$$

Letting $c = u_x(0) > 0$, and noting that

$$u_x\left(\frac{i}{2p}\right) = (-1)^i c$$

our second variation formula becomes

$$BE'' = \varepsilon c \sum_{i=0}^{2p-1} \left[f\left(\frac{i}{2p}\right) \dot{u}_{i,x}\left(\frac{i}{2p}\right) + f\left(\frac{i+1}{2p}\right) \dot{u}_{i,x}\left(\frac{i+1}{2p}\right)\right]$$

and our system for $\dot{u}_i$ becomes

$$\dot{u}_i : \left[\frac{i}{2p}, \frac{i+1}{2p}\right] \to \mathbf{R} \tag{4.0.3}$$

$$\varepsilon^2 \dot{u}_{i,xx} = W''(u)\dot{u}_i \tag{4.0.4}$$

$$\begin{aligned}
\dot{u}_i\left(\frac{i}{2p}\right) &= cf\left(\frac{i}{2p}\right) \\
\dot{u}_i\left(\frac{i+1}{2p}\right) &= -cf\left(\frac{i+1}{2p}\right)
\end{aligned}$$

proving the lemma. □

Isometric reduction of Allen–Cahn solutions on S^1

We now observe that we can shift the boundary conditions of each individual subinterval $[i/2p, (i+1)/2p]$ and get the same energy, and hence variations of energy. If we let $Y_t = \{q_i(t)\}$ as in (4.0.2), then we can write

$$\mathrm{BE}(Y_t) = \sum_{i=0}^{2p-1} \mathrm{E}_\varepsilon([q_i(t), q_{i+1}(t)])$$

where $q_{2p}(t) = q_0(t)$. Here, $\mathrm{E}([a,b])$ denotes the energy of the minimizer of (1.0.1) on $[a,b]$ at scale ε. Note that for $a(t) \le b(t)$, we have

$$\mathrm{E}_\varepsilon([a(t), b(t)]) = \mathrm{E}_\varepsilon([0, b(t) - a(t)]) = \mathrm{E}_\varepsilon([a(t) - b(t), 0])$$

This can be thought of as a change of reference frame to one of the endpoints. See picture 4.1. We

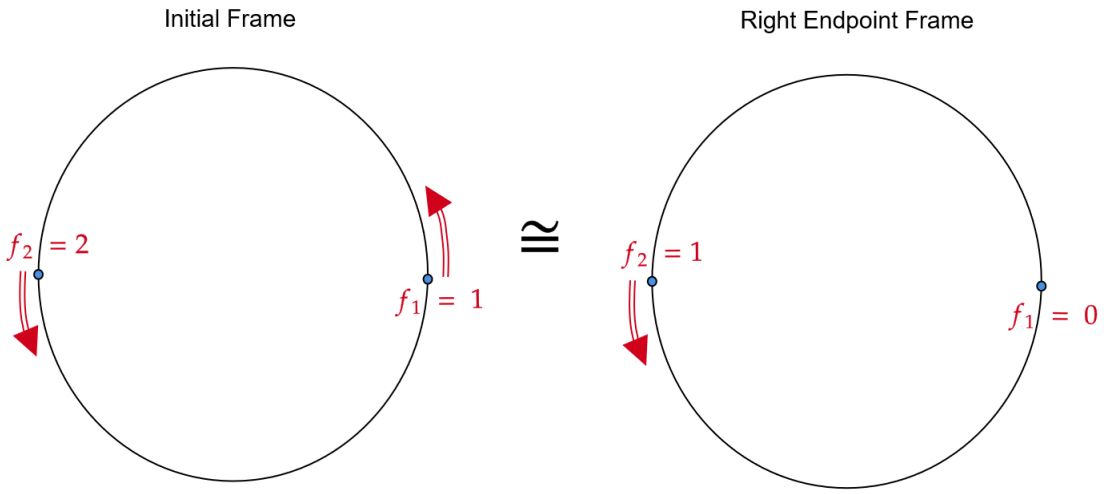


Figure 4.1:

can also take the "center of momentum frame" and get

$$\mathrm{E}_\varepsilon([a(t), b(t)]) = \mathrm{E}_\varepsilon\left(\left[\frac{a(t) - b(t)}{2}, \frac{b(t) - a(t)}{2}\right]\right)$$

See picture 4.2. If we further define

$$\tilde{f}(0) = \frac{\dot{q}_0 - \dot{q}_1}{2}, \qquad \tilde{f}(1/2p) = \frac{\dot{q}_1 - \dot{q}_0}{2} = -\tilde{f}(0)$$

Then it is clear that

$$\begin{aligned} \mathrm{E}_\varepsilon([q_0(t), q_1(t)]) &= \mathrm{E}_\varepsilon\left(\left[\frac{q_0(t) - q_1(t)}{2}, \frac{q_1(t) - q_0(t)}{2}\right]\right) \\ \implies \frac{d^2}{dt^2}\mathrm{E}_\varepsilon([q_0(t), q_1(t)])\Big|_{t=0} &= \frac{d^2}{dt^2}\mathrm{E}_\varepsilon\left(\left[\frac{q_0(t) - q_1(t)}{2}, \frac{q_1(t) - q_0(t)}{2}\right]\right)\Big|_{t=0} \\ &= \varepsilon c\left[\tilde{f}(0)\dot{\tilde{u}}_x(0) + \tilde{f}(1/2p)\dot{\tilde{u}}_x(1/2p)\right] \end{aligned} \tag{4.0.5}$$

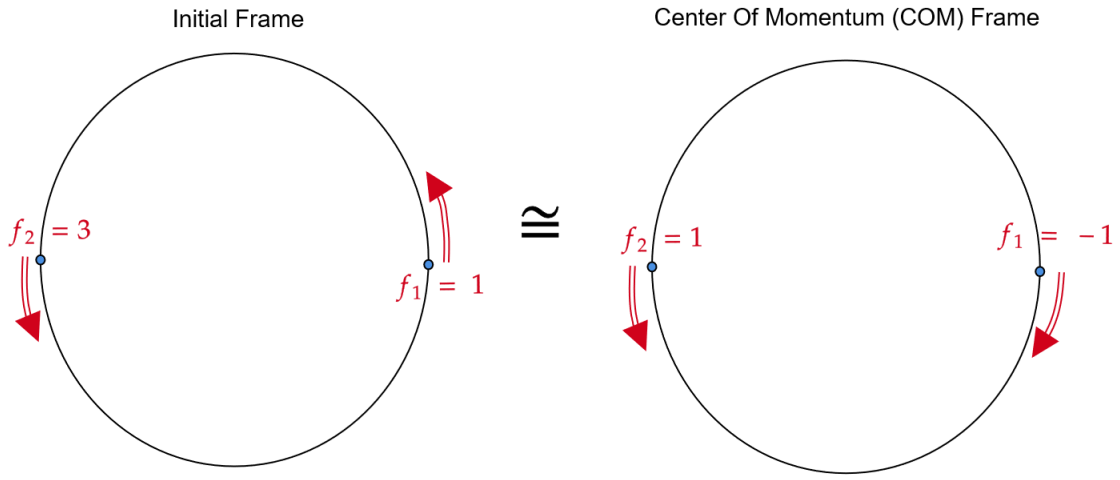


Figure 4.2:

where instead of the system (4.0.3), we have

$$\begin{aligned} \dot{\tilde{u}} : \left[0, \frac{1}{2p}\right] &\to \mathbf{R} \\ \varepsilon^2 \dot{u}_{xx} &= W''(u)\dot{\tilde{u}} \\ \dot{\tilde{u}}\,(0) &= c\tilde{f}\,(0) \\ \dot{\tilde{u}}_i\left(\frac{1}{2p}\right) &= -c\tilde{f}\left(\frac{1}{2p}\right) \\ &= c\tilde{f}(0) \end{aligned} \tag{4.0.6}$$

In the appendix (§4.1.2), we show that for the above system (4.0.6), we can compute

$$\dot{\tilde{u}}_x(0) = c\tilde{f}(0)v, \qquad v = v(\varepsilon) < 0 \tag{4.0.7}$$

With this, we can now prove proposition proposition 1.1.1

Proposition 1.1.1. *Let $p > 0$, and $u_{\varepsilon,2p}$ the solution vanishing on Y_{2p}. Consider a variation of the form*

$$a_i(t) := \frac{i}{2p} + f(i/(2p))t$$

for $f : \{0, 1, \ldots, 2p-1\} \to \mathbf{R}$. *Then*

$$BE''(f)\Big|_{Y_{2p}} = \varepsilon c^2 v(\varepsilon) \sum_{i=0}^{2p-1} [f(i/(2p)) - f((i+1)/2p)]^2 \tag{1.1.9}$$

for $v(\varepsilon) < 0$ *for all* $\varepsilon > 0$

Proof: Breaking up the energy into all of its intervals and using the center-of-momentum frame, we have

$$\begin{aligned}
\mathrm{BE}''(t)\Big|_{t=0} &= \sum_{i=0}^{2p-1} \frac{d^2}{dt^2} \mathrm{E}_\varepsilon \left([q_i(t), q_{i+1}(t)]\right) \\
&= \sum_{i=0}^{2p-1} \frac{d^2}{dt^2} \mathrm{E}_\varepsilon \left(\left[\frac{q_i(t) - q_{i+1}(t)}{2}, \frac{q_{i+1}(t) - q_i(t)}{2} \right] \right) \Big|_{t=0} \\
&= \varepsilon c \sum_{i=0}^{2p-1} \frac{\dot{q}_i - \dot{q}_{i+1}}{2} \left[\dot{\tilde{u}}_{i,x} \left(\frac{i}{2p} \right) - \dot{\tilde{u}}_{i,x} \left(\frac{i+1}{2p} \right) \right]
\end{aligned}$$

from (4.0.5). By equation (4.0.7), and the symmetry of the system 4.0.6, we have that

$$\dot{\tilde{u}}_{i,x} \left(\frac{i}{2p} \right) = -\dot{\tilde{u}}_{i,x} \left(\frac{i+1}{2p} \right) = cv \left(\frac{\dot{q}_i - \dot{q}_{i+1}}{2} \right)$$

so that

$$\begin{aligned}
\mathrm{BE}''(t)\Big|_{t=0} &= \varepsilon c^2 v(\varepsilon) \sum_{i=0}^{2p-1} 2 \left(\frac{\dot{q}_i - \dot{q}_{i+1}}{2} \right)^2 \\
&= \varepsilon c^2 v(\varepsilon) \sum_{i=0}^{2p-1} 2 \left(\frac{f(i/(2p)) - f((i+1)/2p)}{2} \right)^2
\end{aligned}$$

This finishes the proof of proposition 1.1.1. □

The formula for second variation shows:

$$BE'' \geq 0 \iff |f(i/(2p)) - f((i+1)/2p)| = 0 \qquad \forall i = 0, \ldots, 2p-1$$

which geometrically means that our variation either fixes the D_{2p} nodal sets, or rotates all of them at the same rate. This gives the following **rigidity** result

Corollary 4.0.3. Any variation of the $u_{2p,\varepsilon}$ solution produces a non-positive second variation of BE,

i.e.

$$\frac{d^2}{dt^2} BE(t)\Big|_{t=0} \leq 0$$

With equality if and only if the variation rotates or fixes all nodes.

As a result, we also prove theorem 1.1.9

Theorem 1.1.9. *Fix $p > 0$. There exists an ε_p such that $\forall \varepsilon < \varepsilon_p$, the function, $u_{\varepsilon,2p}$, has Allen–Cahn Morse index $2p-1$ and nullity 1. The nullity is given by rotations of S^1 and every other non-trivial variation produces a strictly negative variation.*

Proof: WLOG, the space of variations can be chosen to fix $q = 0$ and hence is spanned by $\{f_i\}$ with f_i moving the point $\frac{i}{2p}$ for $i = 1, \ldots, 2p-1$. Clearly these are linearly independent and this space is dimension $2p-1$. Moreover, any element of their span is of the form:

$$V = \sum_{i=1}^{2p-1} a_i f_i$$

and gives a negative variation by equation (1.1.9). If one of the a_i are non-trivial, then this is a non-trivial variation among all of the points $\{\frac{i}{2p}\}_{i=1}^{2p-1}$ that fixes $q = 0$, and hence is not a rotation of all points. □

4.0.3 Classification of stable BE critical points in noncompact M^3

In this section we prove a result analogous to the following theorems of Fischer-Colbrie-Schoen ([6], Theorem 3)

Theorem 4.0.4 (Fischer-Colbrie-Schoen)**.** *Let M^3 be a complete oriented 3-manifold of non-negative scalar curvature. Let Y^2 be an oriented complete stable minimal surface in M. Let S denote the scalar curvature of M at any point. Then*

1. *If Y compact, then Y is conformally equivalent to S^2 or Y is a totally geodesic flat torus T^2. If $S > 0$ on M, then Y is conformally equivalent to S^2.*
2. *If Y is not compact, then Y is conformally equivalent to the complex plane $\mathbb{C}$, or the cylinder A. If Y is a cylinder, then Y is flat and totally geodesic. If the scalar curvature of M is everywhere positive, then Y cannot be a cylinder with finite total curvature*

If $Ric_M \geq 0$, then the assumption of finite total curvature in 2 can be removed

We prove the following analogous to the first half of the above, theorem 1.1.8

Theorem 1.1.8. *Let M^3 be a complete oriented 3-manifold of non-negative scalar curvature. Let Y^2 be a $C^{2,\alpha}$, oriented complete,* ***compact****, stable critical point for* BE_ε *in M . Let S denote the scalar curvature of M at any point. Then Y is conformally equivalent to a (S^2, g_{round}) or Y is "almost" a totally geodesic flat torus T^2, i.e.*

$$||A_Y||^2_{L^2(Y)} \leq K\varepsilon^{1/2}$$

for some $K > 0$ independent of ε for K sufficiently small. If $S > 0$ on M, then Y is conformally equivalent to (S^2, g_{round}). If both:

$$S \geq \kappa_0 > 0$$
$$Vol(Y) \geq v_0 > 0$$

both independent of ε, then Y must be conformally equivalent to S^2.

Remark: The authors hope that the "almost" descriptor in the above can be removed and we can get obtain a rigidity statement. The authors also hope to mimic the non-compact part of the original theorem.

Proof: Again using the asymptotic expansion of the second variation formula:

$$\begin{aligned} BE''(f\nu)\Big|_Y &= \frac{2\sqrt{2}}{3}\int_Y [|\nabla f|^2 - \dot{H}_0 f^2] + E \\ &= \frac{2\sqrt{2}}{3}\int_Y \left[|\nabla f|^2 - \left(S - K + \frac{1}{2}\sum_{i,j=1}^{2} A_{ij}^2\right) f^2 + E\right] \end{aligned}$$

where $||E||_{L^\infty} \leq K_0\varepsilon^{1/2}$ for some $K_0 > 0$ and we've decomposed

$$\begin{aligned} \dot{H}_0 &= |A_Y|^2 + \mathrm{Ric}_g(\nu,\nu) \\ &= S - K + \frac{1}{2}\sum_{i,j=1}^{2} A_{ij}^2 \end{aligned}$$

here S is the ambient scalar curvature of M restricted to M and K is the sectional/Gaussian

curvature of $Y \subseteq M$. Using stability, taking $f = 1$, we have

$$\begin{aligned}\int_Y K dA &\geq \int_Y \left(S + \frac{1}{2} \sum_{i,j=1}^{2} A_{i,j}^2 + E \right) \\ &\geq -K\varepsilon\end{aligned}$$

Gauss bonnet for a closed manifold now tells us that

$$\int_Y K dA = 2\pi\chi(Y)$$

Now our lower bound gives $\chi(Y) \in \{0, 2\}$. If Y is a torus (i.e. $\chi(Y) = 0$) then we see that

$$\begin{aligned}0 &\geq \int_Y S + \frac{1}{2} \sum_{i,j} A_{ij}^2 + E \\ \implies 0 &\geq \int_Y \sum_{i,j} A_{ij}^2 + O(\varepsilon)\end{aligned}$$

Since $S \geq 0$ everywhere. This tells us that

$$||A_Y||_{L^2}^2 \leq K\varepsilon$$

i.e. Y is "almost" geodesic in an L^2-sense. If Y is a topological sphere, then uniformization tells us that it is conformally equivalent.

If M is PSC and Y has volume bounded from below by a positive constant, then we see that

$$\int_Y S + \frac{1}{2} \sum_{i,j} A_{ij}^2 + E > 0$$

ruling out the torus case. □

4.1 Appendix

4.1.1 Smooth dependence of solutions on domains

In this section, we show that for two domains Ω and $\varphi(\Omega)$, the solutions are continuous under pull-back to Ω.

Throughout, we will assume that Ω is such that it supports a positive Dirichlet phase transition. Let $k > 2$ be a given integer. Suppose that Ω is a domain with $C^{k,\alpha}$ boundary (i.e., the inclusion map $i_\Omega \in C^{k,\alpha}(\Omega; M)$ is $C^{k,\alpha}$), we take a neighborhood $U \subset C^{k,\alpha}(\Omega; M)$ consisting only of $C^{k,\alpha}$ embeddings of Ω. Suppose that $\varphi \in U$ and let v be the unique positive solution, afforded by theorem 1.2.1, to the problem

$$\begin{cases} \varepsilon^2 \Delta v = W'(v) & \text{in } \varphi(\Omega), \\ v = 0 & \text{on } \partial\varphi(\Omega). \end{cases}$$

The $\varphi^* v : \Omega \to \mathbf{R}$ satisfies $\varphi^* v = 0$ along $\partial\Omega$, and moreover $\varphi^* v$ satisfies the PDE

$$(\varphi^* \Delta (\varphi^{-1})^*) \varphi^* v = \varphi^* \Delta v = \varphi^* W'(v) = W'(\varphi^* v).$$

In other words, given v as above, and possibly shrinking U, we have for every $\varphi \in U$ a unique solution to the auxiliary problem

$$\begin{cases} \varepsilon^2 (\varphi^* \Delta (\varphi^{-1})^*) u = W'(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Define the operator $L_{\varphi,\varepsilon} : C_0^{k,\alpha}(\Omega) \to C^{k-2,\alpha}(\Omega)$ by $L_{\varphi,\varepsilon} u \coloneqq \varepsilon^2 (\varphi^* \Delta (\varphi^{-1})^*) u - W'(u)$. We may now define $\Phi : U \times C_0^{k,\alpha}(\Omega) \to C^{k-2,\alpha}(\Omega)$ by

$$\Phi(\varphi, u) \coloneqq L_{\varphi,\varepsilon} u.$$

Proposition 4.1.1. *If u is the positive Dirichlet phase transition on Ω, then the map $D_2\Phi(i_\Omega, u) : C_0^{k,\alpha}(\Omega) \to C^{k-2,\alpha}(\Omega)$ is an isomorphism.*

Proof. The map in question is the linearized Allen–Cahn operator. That is, if $f \in C_0^{k,\alpha}(\Omega)$

$$D_2\Phi(i_\Omega, u) f = \varepsilon^2 \Delta f - W''(u) f.$$

But this operator is invertible by proposition 3.4.1 in the Sobolev setting or [21], Theorem 1.4 in the Hölder setting. □

After possibly shrinking U once more, the implicit function theorem [16] and the above proposition

together imply that there a unique C^k map $\varphi \mapsto u_\varphi$ satisfying

$$\Phi(\varphi, u_\varphi) = 0.$$

In other words, the phase transitions u_Ω^+ depend on the domain Ω in an appropriately smooth way.

4.1.2 Computing $\dot{u}_x(0)$ on S^1

In this section, we consider

$$\begin{aligned} u_\varepsilon &: [0,1] \to \mathbf{R} \\ u_\varepsilon(0) &= u_\varepsilon(1) = 0 \\ \varepsilon^2 u_{\varepsilon,xx} &= W'(u_\varepsilon) \\ \dot{u}_\varepsilon &: [0,1] \to \mathbf{R} \\ \dot{u}_\varepsilon(0) &= \dot{u}_\varepsilon(1) = 1 \\ \varepsilon^2 \dot{u}_{\varepsilon,xx} &= W'(u_\varepsilon) \end{aligned} \tag{4.1.1}$$

and the goal is to show

Theorem 4.1.2. *For $\dot{u}$ as in the system 4.1.1, there exists an $\varepsilon_0 > 0$ such that*

$$\forall \varepsilon_0 > \varepsilon > 0, \qquad sgn(\dot{u}_{\varepsilon,x}(0)) < 0$$

Remark: While we've shown that the geometry of the level set determines the Neumann data (see remark 3.5.4) when it is at least 1 dimensional, there is no geometry for a 0 dimensional level set. Thus, this theorem requires a more careful ODE analysis

Expanding $u = u_\lambda(t)$

First note that u and $\dot{u}$ as in 4.1.1 can be extended to functions on $\mathbf{R}$ by odd (resp. even) reflection about the boundary points of 0 and 1, and then repeating this at every $n \in \mathbf{Z}$. Note that $\dot{u}$ is C^0

but *not* C^1 on $\mathbf{R}$ a priori. From here, we define

$$\tilde{u}_\varepsilon : \mathbf{R} \to \mathbf{R}$$
$$\tilde{u}_\varepsilon(x) := u(\varepsilon^{-1}x)$$
$$\implies \tilde{u}(0) = \tilde{u}(\varepsilon^{-1}) = 0$$
$$\tilde{u}_{\varepsilon,xx} = W'(\tilde{u})$$
$$\dot{\tilde{u}} : \mathbf{R} \to \mathbf{R}$$
$$\dot{\tilde{u}}(x) = \dot{\tilde{u}}(\varepsilon^{-1}x)$$
$$\implies \dot{\tilde{u}}(0) = \dot{\tilde{u}}(\varepsilon^{-1}) = 1$$
$$\dot{\tilde{u}}_{xx} = W''(\tilde{u})\dot{\tilde{u}}$$

We first establish a lemma about the shape of $\tilde{u}_\varepsilon$:

Lemma 4.1.3. *Let $\tilde{u} = \tilde{u}_\varepsilon$ be a solution of the above form. Then on its fundamental domain of $[0, \varepsilon^{-1}]$, $\tilde{u}$ is concave down with a unique critical point and maximum at $x = \varepsilon^{-1}/2$*

Proof: The fact that $\tilde{u}$ is concave down follows because $0 < \tilde{u} < 1$ on $(0, \varepsilon^{-1})$ by the maximum principle compared to $v = 1$. By considering the symmetric reflection of u about $x = \varepsilon^{-1}/2$, we see that $\tilde{u}$ is even about $x = \varepsilon^{-1}/2$ due to its uniqueness from energy minimization, so clearly there is a local max there by concavity. To see that there are no other critical points, we use concavity. If there exists $x^* \in (0, \varepsilon^{-1}/2)$, a critical point, then by concavity, it is another local max. This implies there exists a local min $x^{**} \in (x^*, \varepsilon^{-1}/2)$, so that $\tilde{u}_{xx}(x^{**}) \geq 0$ at that point, a contradiction to the Allen–Cahn equation and bounds on $\tilde{u}$. Thus the only critical point is the maximum that $\tilde{u}$ achieves at $x = \varepsilon^{-1}/2$. □

By multiplying (1.0.2) by $\tilde{u}_x$ and integrating, every solution with $\varepsilon = 1$ on $\mathbf{R}$ satisfies

$$\frac{\tilde{u}_x^2}{2} = W(\tilde{u}) - \lambda \tag{4.1.2}$$

for some $\lambda \in [0, 1/4]$. We now show that for all $\varepsilon < \varepsilon_0$ sufficiently small, there is a one to one correspondence between ε and λ. We now recall continuity of the $\{\tilde{u}_\varepsilon\}$ with respect to ε:

Lemma 4.1.4. *The family $\{\tilde{u}\} = \{\tilde{u}_\varepsilon\}$ are continuous with respect to ε, and in fact*

$$\lim_{\varepsilon \to \varepsilon^*} ||\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon^*}||_{C^0(\mathbf{R})} = 0$$

for all $\varepsilon^* < \varepsilon_0$ postive with ε_0 sufficiently small.

Proof: We present 2 proofs. First, choose ε_0 sufficiently small so that $\tilde{u}_\varepsilon$ is non-zero and periodic for all $\varepsilon < \varepsilon_0$. Now suppose the above did not hold for some $\varepsilon^* < \varepsilon_0$. In particular, there exists a $\delta > 0$ and (WLOG by periodicity) a sequence of points and scales, $\{x_i, \varepsilon_i\}$ with $\varepsilon_i^{-1} < 2(\varepsilon^*)^{-1}$ and $x_i \in [0, 2(\varepsilon^*)^{-1}]$ such that

$$|\tilde{u}_{\varepsilon_i}(x_i) - \tilde{u}_{\varepsilon^*}(x_i)| > \delta \tag{4.1.3}$$

However, by the Allen–Cahn equation, we have uniform C^2, and hence $C^{1,\alpha}$ estimates on $\{\tilde{u}_{\varepsilon_i}\}$ up to a subsequence. This gives convergence in C^1 to $\overline{u}$ along a subsequence. Note that since $\varepsilon_i \to \varepsilon^*$, we also have that

$$0 = \tilde{u}_{\varepsilon_i}(\varepsilon_i^{-1}) \to \overline{u}((\varepsilon^*)^{-1})$$

so $\overline{u}$ vanishes at $x = (\varepsilon^*)^{-1}$. By lemma 4.0.1, it is periodic with period $(\varepsilon^*)^{-1}$. But now by using the uniqueness of periodic solutions to Allen–Cahn on $\mathbf{R}$ with respect to their period (a consequence of rescaling back to $[0,1]$ and uniqueness of energy minimizers on $[0,1]$), we have that $\overline{u} = \tilde{u}_{\varepsilon^*}$, contradicting (4.1.3). □

2nd Proof: from §4.1.3, we know that

$$\tilde{u}_\varepsilon = g(x, k(\varepsilon)) = k(\varepsilon)\sqrt{\frac{2}{1+k(\varepsilon)^2}}\,\mathrm{sn}\left(\frac{x}{\sqrt{1+k(\varepsilon)^2}} \;\middle|\; k(\varepsilon)\right)$$

where $\mathrm{sn}(x \mid k)$ is the Jacobi elliptic integral as defined in [2]. The above function is a solution to Allen–Cahn with fundamental period

$$\tau = 2K'(k) = 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}}$$

setting this equal to ε^{-1} and differentiating, one can see that K' is locally invertible for $k < 1$, corresponding to $\varepsilon > 0$. Now $\tilde{u}_\varepsilon$ is a continuous function of $k(\varepsilon)$ and hence of ε, so the uniform convergence holds by compactness and periodicity. □

Now we prove a correspondence between λ and ε. First note that by evaluating at $x = \varepsilon^{-1}/2$, we have that

$$\lambda = W(\tilde{u}_\varepsilon(\varepsilon^{-1}/2)) = W(||\tilde{u}_\varepsilon||_{C^0})$$

Proposition 4.1.5. *The map from $\varepsilon \to \lambda$ is injective and continuous (and hence monotone) for all $\varepsilon < \varepsilon_0$ sufficiently small. It is surjective from $(0, \varepsilon_0)$ onto $(0, \delta_0)$ for some $\delta_0 = \delta_0(\varepsilon_0)$*

Proof: The map from $\varepsilon \to \lambda$ is clearly defined via the above procedure. To show that it is injective, suppose $\tilde{u}_{\varepsilon_1}$ and $\tilde{u}_{\varepsilon_2}$ had the same λ. Then looking at $x = 0$ and (4.1.2), we have

$$\tilde{u}_{\varepsilon_i}(0) = 0 \implies \tilde{u}_{\varepsilon_i, x} = \sqrt{\frac{1}{4} - \lambda}$$

i.e. $\tilde{u}_{\varepsilon_1}, \tilde{u}_{\varepsilon_2}$ have the same dirichlet and neumann data at $x = 0$, and they're both solutions to Allen–Cahn with $\varepsilon = 1$ on $\mathbf{R}$. By ODE theory, we have $\tilde{u}_{\varepsilon_1} = \tilde{u}_{\varepsilon_2}$, which by period considerations means that $\varepsilon_1 = \varepsilon_2$. Continuity of this map follows from lemma 4.1.4 since

$$|\lambda(\varepsilon_1) - \lambda(\varepsilon_2)| = |W(\tilde{u}_{\varepsilon_1}(\varepsilon_1^{-1}/2) - W(\tilde{u}_{\varepsilon_2}(\varepsilon_2^{-1}/2)|$$

And $\tilde{u}_{\varepsilon_1}(\varepsilon_1^{-1}/2) \to \tilde{u}_{\varepsilon_2}(\varepsilon_2^{-1}/2)$ as $\varepsilon_1 \to \varepsilon_2$ by lemma 4.1.4. To see surjectivity, note that $\lim_{\varepsilon \to 0} ||\tilde{u}_\varepsilon||_{C^0(\mathbf{R})} = 1$, which can be seen by convergence of the family of functions to the heteroclinic on $\mathbf{R}$. But then

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \lim_{\varepsilon \to 0} W(\tilde{u}_\varepsilon(\varepsilon^{-1}/2) = \lim_{\varepsilon \to 0} W(||\tilde{u}_\varepsilon||_{C^0(\mathbf{R})}) = 0$$

since $||\tilde{u}_\varepsilon|| = \tilde{u}_\varepsilon(\varepsilon^{-1}/2)$. By monotonicity (from injectivity and continuity), we get the surjectivity result. □

We can now call each $\tilde{u}_\varepsilon : \mathbf{R} \to \mathbf{R}$ a $u_{\lambda(\varepsilon)} = u_\lambda : \mathbf{R} \to \mathbf{R}$, and similarly $\dot{\tilde{u}}_\varepsilon = \dot{\tilde{u}}_\lambda$

Expansion in λ

In this section, we remove the $\tilde{\cdot}$ notation for convenience. From equation (4.1.2), we have

$$\int_0^{u_\lambda(t)} \frac{dx}{\sqrt{W(x) - \lambda}} = t$$

we can differentiate **with respect to** λ and get

$$(\partial_\lambda u_\lambda)(t) \frac{1}{\sqrt{W(u_\lambda) - \lambda}} = -\frac{1}{2} \int_0^{u_\lambda(t)} \frac{dx}{[W(x) - \lambda]^{3/2}}$$

evaluating at $\lambda = 0$, we have that

$$\begin{aligned}(\partial_\lambda u_\lambda|_{\lambda=0})(t) &= -4(1-g^2)\int_0^g \frac{dx}{(1-x^2)^3} \\ &= -\frac{1-g^2}{4}\left[\frac{2g(5-3g^2)}{(g^2-1)^2} + 3\log\left(\frac{1+g}{1-g}\right)\right] \\ &=: \kappa(t)\end{aligned}$$

where $g = g(t)$ is the heteroclinic and we've used that

$$\int \frac{dx}{(1-x^2)^3} = \frac{1}{16}\left(\frac{10x-6x^3}{(1-x^2)^2} + 3\log\left(\frac{1+x}{1-x}\right)\right)$$

Note that $\kappa(t) < 0$ for all $t > 0$. Let $L = \partial_t^2 - W''(g)$. We now want to solve

$$L(\tau(t)) = \kappa(t)$$

For

$$u_\lambda(t) = g(t) + \lambda\kappa(t) + \phi$$

We compute

$$\begin{aligned} u_{\lambda,tt} &= \ddot{g} + \lambda\ddot{\kappa} + \phi_{tt} \\ W'(u_\lambda) &= W'(g) + W''(g)[\lambda\kappa + \phi] + O(\lambda^2, \phi^2, \lambda\phi) \\ \implies u_{\lambda,tt} - W'(u_\lambda) &= \phi_{tt} - W''(g)\phi + O(\lambda^2, \phi^2, \lambda\phi) \\ \implies L(\phi) &= O(\lambda^2, \phi^2, \lambda\phi)\end{aligned}$$

We now show:

Lemma 4.1.6. *For $\delta = \delta(\lambda)$ sufficiently small, we have*

$$||\phi||_{C^{2,\alpha}([0,\delta))} \leq K\lambda^2$$

for K independent of λ (and hence ε)

Proof: Note that $\phi(0) = 0$ by nature of the Dirichlet condition on $u_\lambda(t)$ and definition of $\kappa(t)$. Let $\delta = \delta(\lambda)$ be such that

$$||\phi||_{C^0([0,\delta])} \leq \lambda^2$$

Then we have that

$$\begin{aligned}\phi_{tt} &= W''(g)\phi + (3g)[\lambda^2\kappa^2 + 2\lambda\kappa\phi + \phi^2] \\ &\quad + [\lambda^3\kappa^3 + 3\lambda^2\kappa^2\phi + 3\lambda\kappa\phi^2 + \phi^3] \\ \implies ||\phi_{tt}||_{C^0} &\leq 2\lambda^2 + 3[\lambda^2 K_0^2 + 2\lambda^3 K_0 + \lambda^4] + [\lambda^3 K_0^3 + 3K_0^2\lambda^4 + 3K_0\lambda^5 + \lambda^6]\end{aligned}$$

where

$$K_0 = ||\kappa(t)||_{C^0(\mathbf{R}^+)}$$

Then for λ sufficiently small, we see that

$$||\phi_{tt}||_{C^0([0,\delta])} \leq 3(1 + K_0)^2\lambda^2$$

and so by interpolation

$$||\phi||_{C^2([0,\delta])} \leq 3(1 + K_0)^2\lambda^2$$

□

Similarly, we now want to show there's a next order expansion for $\dot{u}$ on $[0, \delta(\lambda)]$ at least. We write

$$\begin{aligned}W''(u) &= 3(g(t) + \lambda\tau(t) + \phi)^2 - 1 \\ &= 3g^2 + 6\lambda g(t)\tau(t) - 1 + O(\lambda^2) \\ &= W''(g) + 6\lambda g(t)\tau(t) + O(\lambda^2) \\ f(t) &:= 6g(t)\tau(t)\end{aligned}$$

where the above holds in $C^2([0, \delta))$. Take $\omega(t)$ the solution to

$$\begin{aligned}\omega &: [0, \infty) \to \mathbf{R} \\ \omega(0) &= \lim_{t\to\infty} \omega(t) = 0 \\ L(\omega(t)) &= f(t)\dot{g}(t)\end{aligned}$$

and expand

$$\dot{u} = \dot{g}(t) + \lambda\omega(t) + \varphi$$

Using our prior

$$\begin{aligned}[\partial_t^2 - W''(u)]\dot{u} &= [\partial_t^2 - W''(g) - \lambda f(t) + O(\lambda^2)][\dot{g} + \lambda\omega + \varphi] \\ &= L(\lambda\omega) + L(\varphi) - \lambda f\dot{g} + O(\lambda\varphi, \lambda^2) \\ &= 0 \\ \implies L(\varphi) &= O(\lambda\varphi, \lambda^2)\end{aligned}$$

By the same argument as in 4.1.6, we have

$$||\varphi||_{C^2([0,\delta^*))} \leq K\lambda^2$$

for some $\delta^* = \delta^*(\lambda)$ and K independent of λ. Furthermore, it is clear that

$$\begin{aligned}\dot{u}_t(0) &= \ddot{g}(0) + \lambda\omega_t(0) + \varphi_t(0) \\ &= \lambda\omega_t(0) + O(\lambda^2)\end{aligned}$$

so

$$\text{sign}(\dot{u}_t(0)) = \text{sign}(\omega_t(0))$$

and it suffices to compute $\omega_t(0)$.

Solving for $\omega_t(0)$

Recall that one can solve for $\omega(t) = v(t)\dot{g}(t)$ as in ([21], §7.6 or [20]), so that

$$\begin{aligned}\implies w'(0) &= v'(0)\dot{g}(0) \\ v'(0) &= \frac{a_0}{\dot{g}(0)^2} \\ a_0 &= -\int_0^\infty (f\dot{g})\dot{g}\end{aligned}$$

We check the sign of f. Write

$$\begin{aligned}&f(t) = 6g(t)\tau(t) \\ &\tau(t) \text{ s.t. } L(\tau) = \kappa(t) \\ &\kappa(t) = -\frac{1-g^2}{4}\left[\frac{2g(5-3g^2)}{(1-g^2)^2} + 3\log\left(\frac{1+g}{1-g}\right)\right]\end{aligned}$$

So $\kappa < 0$ for all $t > 0$ and vanishes at $t = 0$. Recall that

$$\begin{aligned}
\tau(t) &= r(t)\dot{g}(t) \\
r(t) &= \int_0^t \dot{g}(s)^{-2} \left[a_0 + \int_0^s \kappa(u)\dot{g}(u)du \right] ds \\
a_0 &= -\int_0^\infty \kappa(u)\dot{g}(u)du \\
&> 0 \\
\implies r(t) &> 0 \qquad \forall t > 0 \\
\implies \tau(t) &> 0 \qquad \forall t > 0 \\
\implies f(t) &> 0 \qquad \forall t > 0 \\
\implies w'(0) &< 0
\end{aligned}$$

i.e.

$$\begin{aligned}
\dot{u} &= \dot{g}(t) + \lambda\omega(t) + \varphi(t) \\
\dot{u}'(0) &= \lambda w'(0) + O(\lambda^2) \\
&< 0
\end{aligned}$$

for all λ, and hence all ε, sufficiently small. This is our desired result.

4.1.3 Elliptic Functions

This section provides a brief introduction to elliptic functions, which describe the solutions to Allen-Cahn on $\mathbf{R}$. Let

$$u(y_1, k) = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\sin^{-1}(y_1)} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\sin^{-1}(y_1), k)$$

be the incomplete elliptic integral of the first kind (see [2]), and let

$$y_1 = \mathrm{sn}(u, k)$$

be the sin amplitude function. From hereon, we'll notate the right as $\mathrm{sn}(x, k)$ and think of $\{\mathrm{sn}(\cdot, k)\}$ as a family of functions on $\mathbf{R}$.

We also recall the definitions and relations

$$\begin{aligned}
\mathrm{cn}(x,k) &= \cos(\sin^{-1}(y_1)) = \sqrt{1-\mathrm{sn}(x,k)^2} \\
\mathrm{dn}(x,k) &= \sqrt{1-k^2\mathrm{sn}(x,k)^2} \\
\mathrm{sn}_x &= \mathrm{cn}\cdot\mathrm{dn} \\
\mathrm{cn}_x &= -\mathrm{sn}\cdot\mathrm{dn} \\
\mathrm{dn}_x &= -k^2\mathrm{sn}\cdot\mathrm{cn}
\end{aligned}$$

One can compute

$$\mathrm{sn}_{xx}(x,k) = -(1+k^2)\mathrm{sn} + 2k^2\mathrm{sn}^3$$

and define

$$g(x,k) := k\sqrt{\frac{2}{1+k^2}}\mathrm{sn}\left(\frac{x}{\sqrt{1+k^2}}\;\middle|\; k\right) \tag{4.1.4}$$

Each of the $g(\cdot,k)$ are solutions to (1.0.2) and from [2] we see that $\mathrm{sn}(u,k) = \mathrm{sn}(u)$ has period equal to $4K(k) = 4F(\pi/2,k)$ which tends to ∞ as $k \to 1$. We know that any $\tilde{u}_\varepsilon : \mathbf{R} \to \mathbf{R}$, a solution to (1.0.2) with period $\tau = \varepsilon^{-1}$, is of the form $\tilde{u}_\varepsilon(x) = g(x,k)$ for k so that

$$\varepsilon^{-1} = 4K(k)\sqrt{1+k^2}$$

From the integral expression for K, it is clear that as $k \to 1$, $\varepsilon^{-1}(k)$ is monotone and increasing to infinity and moreover differentiable so we can write

$$k = F(\varepsilon)$$

for some ε. NOte that we are interested in the regime of when $k = 1$ since

$$u(y_1,1) = \int_0^{y_1} \frac{dt}{1-t^2} = \tanh^{-1}(y_1) \implies y_1 = \tanh(u) = \mathrm{sn}(u,1)$$

and so

$$g(x,k=1) = \mathrm{sn}(x/\sqrt{2},k=1) = \tanh(x/\sqrt{2}) = g(x)$$

the heteroclinic.

4.1.4 Reducing $\mathrm{BE}''(f,f)$

In this section, we first prove

Lemma 4.1.7. *For $\phi : M \to \mathbf{R}$ as in section §3.5.2, we have*

$$-\int_Y f\phi_t = \sigma^{-1}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} -\Delta_t(h)\dot{\overline{g}}_\varepsilon^2 + \varepsilon^{-1}H_t h\ddot{\overline{g}}_\varepsilon\dot{\overline{g}}_\varepsilon + \varepsilon^{-2}[W''(u)-W''(\overline{g}_\varepsilon)]h\dot{\overline{g}}_\varepsilon^2 \\ -[\Delta_t(\phi)-H_t\phi_t+\varepsilon^{-2}(W''(u)-W''(\overline{g}_\varepsilon)+R)\phi]\dot{\overline{g}}_\varepsilon\Big)\sqrt{\det g(s,t)}dtds$$

where

$$\int_Y\int_0^{-2\omega\varepsilon\ln(\varepsilon)} fR\phi\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dts \le K\varepsilon^{7/2}||f||^2_{H^1(Y)}$$

Proof: We first note that by integration by parts

$$-\phi_t\sqrt{\det g(s,0)} = \sigma^{-1}\int_0^{-2\omega\varepsilon\ln(\varepsilon)}(\partial_t^2\phi)\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)} - \sigma^{-1}\int_0^{-2\omega\varepsilon\ln(\varepsilon)}\phi\partial_t^2(\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)})dt \tag{4.1.5}$$

for $\sigma = \sqrt{2}^{-1}$ as in (1.2.3) having noted that

$$\phi(s,0) = 0 = \partial_t^k(\overline{g}_\varepsilon)(-2\omega\varepsilon\ln(\varepsilon)) \qquad \forall k \ge 1$$

so that all of the remaining boundary terms vanish in (4.1.5). This tells us that

$$\begin{aligned}
-\int_Y f\phi_t\sqrt{\det g(s,0)} &= \sigma^{-1}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)}\Big[(\partial_t^2\phi)\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)} - \phi\partial_t^2(\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)})\Big]\,dt \\
&= \sigma^{-1}\varepsilon^{-2}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)}[\varepsilon^2\partial_t^2 - W''(\overline{g}_\varepsilon)+E](\phi)\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)} - \phi[2\varepsilon\ddot{\overline{g}}_\varepsilon\partial_t\sqrt{\det g(s,t)} \\
&\qquad -\varepsilon^2\dot{\overline{g}}_\varepsilon(t)\partial_t^2\sqrt{\det g(s,t)}] - E\phi\sqrt{\det g(s,t)}dtds \\
&= \sigma^{-1}\varepsilon^{-2}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} L_{\varepsilon,u}(\phi)\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)} \\
&\qquad -\phi\left(E+2\varepsilon\ddot{\overline{g}}_\varepsilon\partial_t\sqrt{\det g(s,t)}+\varepsilon^2\dot{\overline{g}}_\varepsilon(t)\partial_t^2\sqrt{\det g(s,t)}\right) \\
&\qquad +[-\varepsilon^2\Delta_t(\phi)+\varepsilon^2H_t\phi_t+[W''(u)-W''(\overline{g}_\varepsilon)]\phi]\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dtds \\
&= \sigma^{-1}\varepsilon^{-2}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} -L_{\varepsilon,u}(h\dot{\overline{g}}_\varepsilon)\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)} \\
&\qquad +[-\varepsilon^2\Delta_t(\phi)+H_t\phi_t+[W''(u)-W''(\overline{g}_\varepsilon)+R]\phi]\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dtds \\
&= \sigma^{-1}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} -\Delta_t(h)\dot{\overline{g}}_\varepsilon^2 + \varepsilon^{-1}H_t h\ddot{\overline{g}}_\varepsilon\dot{\overline{g}}_\varepsilon + \varepsilon^{-2}[W''(u)-W''(\overline{g}_\varepsilon)]h\dot{\overline{g}}_\varepsilon^2 \\
&\qquad -[\Delta_t(\phi)-\varepsilon^{-2}H_t\phi_t+\varepsilon^{-2}(W''(u)-W''(\overline{g}_\varepsilon)+R)\phi]\dot{\overline{g}}_\varepsilon\Big)\sqrt{\det g(s,t)}dtds
\end{aligned}$$

for E as in (1.2.8) and having used $L_{\varepsilon,u}(\phi) = -L_{\varepsilon,u}(h\dot{\bar{g}}_\varepsilon)$. Here

$$R = -[E + 2\varepsilon\ddot{\bar{g}}_\varepsilon(\partial_t\sqrt{\det g(s,t)}) + \varepsilon^2\dot{\bar{g}}_\varepsilon(\partial_t^2\sqrt{\det g(s,t)})]$$

In particular, note that

$$|R| \leq K\varepsilon^2 \tag{4.1.6}$$

holds pointwise. This follows because $\partial_t\sqrt{\det g(s,t)} = H_Y + O(t)$ so that $|\ddot{\bar{g}}_\varepsilon\partial_t\sqrt{\det g(s,t)}| = O(\varepsilon)$ because $H_Y = O(\varepsilon)$ by nature of being a critical point for BE_ε. Now using 3.5.1, we get

$$\begin{aligned}\Big|\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} R\phi\dot{\bar{g}}_\varepsilon\sqrt{\det g(s,t)}dtds\Big| &\leq ||f\dot{\bar{g}}_\varepsilon R||_{L^2(M)}||\phi||_{L^2(M)} \\ &\leq K\varepsilon^{5/2}||f||_{L^2(Y)}||\phi||_{L^2(M)} \\ &K\varepsilon^{7/2}||f||^2_{H^1(Y)}\end{aligned}$$

by equation (3.5.6) and hence verifying equation (3.5.8). This ends the proof of the lemma. □

We now verify equation (3.5.9):

$$\int_Y fu_\nu\dot{u}_\nu = \int_Y fu_\nu\phi_t = \frac{\sqrt{2}}{3}\int_Y(|\nabla f|^2 - \dot{H}_Y f^2)dA_Y + E$$

where $|E| \leq K\varepsilon^{1/2}||f||_{H^1(Y)}$. We analyze the components in our expression for $\int f\phi_t$:

$$\int f\phi_t = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

In the below we often employ proposition 3.4.1. We'll first bound the lower order error terms:

$$\begin{aligned}I_4 &= -\sigma^{-1}\int_0^{-2\omega\varepsilon\ln(\varepsilon)}\dot{\bar{g}}_\varepsilon\int_Y\Delta_t(\phi)fdA_{Y_t} \\ &= \sigma^{-1}\int_0^{-2\omega\varepsilon\ln(\varepsilon)}\dot{\bar{g}}_\varepsilon\int_{Y_t} g(\nabla^t\phi,\nabla^t f) \\ &= \sigma^{-1}\int_M g(\nabla^t\phi, -\dot{\bar{g}}_\varepsilon\nabla^t f) \\ \implies |I_4| &\leq K\varepsilon^{1/2}||f||^2_{H^1(Y)}\end{aligned}$$

having used (3.5.6) in the last line. We can also bound

$$\begin{aligned} I_5 &= -\sigma^{-1}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} H_t\phi_t\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dA_{Y_t}dt \\ &= -\sigma^{-1}\int_M \phi_t f H_t\dot{\overline{g}}_\varepsilon dA \\ \implies |I_5| &\leq K\varepsilon^{3/2}||f||_{L^2}||\nabla\phi||_{L^2} \\ &\leq K\varepsilon^{3/2}||f||^2_{H^1(Y)} \end{aligned}$$

We also have

$$\begin{aligned} I_6 &= \sigma^{-1}\int_Y f\int_0^{-2\omega\varepsilon\ln(\varepsilon)} \phi[W''(u)-W''(\overline{g}_\varepsilon)+R]\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dtds \\ &= \sigma^{-1}\int_M \phi[W''(u)-W''(\overline{g}_\varepsilon)+R]f\dot{\overline{g}}_\varepsilon dA \\ \implies |I_6| &\leq K||\phi||_{L^2(M)}\left|\left|[W''(u)-W''(\overline{g}_\varepsilon)+R]f\dot{\overline{g}}_\varepsilon\right|\right|_{L^2(M)} \\ &\leq K\varepsilon^{7/2}||f||^2_{H^1(Y)} \end{aligned}$$

from equation (3.5.6) and our C^0 bound, equation (4.1.6), on R.

For the remaining terms, we get something on the order of $O(1)$:

$$\begin{aligned} I_1 &= \sigma^{-1}\int_Y\int_0^{-2\omega\varepsilon\ln(\varepsilon)} f\Delta_t(h)\dot{\overline{g}}^2_\varepsilon\sqrt{\det g(s,t)}dsdt \\ &= -\sigma^{-1}\int_Y\int_0^{-2\omega\varepsilon\ln(\varepsilon)} g(\nabla^t f,\nabla^t h)\dot{\overline{g}}^2_\varepsilon\sqrt{\det g(s,t)}dsdt \\ &= \varepsilon^{-1}(1+O(\varepsilon))\int_Y\int_0^{-2\omega\varepsilon\ln(\varepsilon)} |\nabla^t f|^2\dot{\overline{g}}^2_\varepsilon\sqrt{\det g(s,t)}dsdt \\ &= \sigma_0(1+O(\varepsilon))\int_Y|\nabla f|^2 \end{aligned}$$

having used the expansion for h as in (3.5.3) and also noted that $|\nabla^t f|^2 = |\nabla^Y f|^2 + E$ where

$|E| \leq t|\nabla^Y f|^2$. We further used σ_0, σ as in (1.2.3). We also have

$$\begin{aligned}
I_2 &= -\sigma^{-1}\varepsilon^{-1}\int_Y\int_0^{-2\omega\varepsilon\ln(\varepsilon)} H_t f h\ddot{\overline{g}}_\varepsilon\dot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dtds \\
&= \varepsilon^{-2}(1+O(\varepsilon))\Big[\int_Y f^2\varepsilon\dot{H}_Y\left(\int_0^{-2\omega\varepsilon\ln(\varepsilon)}(t/\varepsilon)\dot{\overline{g}}_\varepsilon\ddot{\overline{g}}_\varepsilon\sqrt{\det g(s,t)}dtds + O(\varepsilon^2)\right)\Big] \\
&= [\sigma_1 + O(\varepsilon)]\int_Y f^2\dot{H}_Y\sqrt{\det g(s,0)}ds \\
\sigma_1 &:= \int_0^\infty t\dot{g}(t)\ddot{g}(t)dt
\end{aligned}$$

having used $H_Y = O(\varepsilon^2)$ as in 3.5.4. We also have

$$\begin{aligned}
I_3 &= \varepsilon^{-3}(1+O(\varepsilon))\int_Y f^2\int_0^{-2\omega\varepsilon\ln(\varepsilon)}[W''(u) - W''(\overline{g}_\varepsilon)]\dot{\overline{g}}_\varepsilon^2\sqrt{\det g(s,t)}dtds \\
&= 6\varepsilon^{-1}\int_Y f^2\int_0^{-2\omega\varepsilon\ln(\varepsilon)}[\dot{H}_Y\overline{\tau}_\varepsilon\overline{g}_\varepsilon + O(\varepsilon)]\dot{\overline{g}}_\varepsilon^2\sqrt{\det g(s,t)}dtds \\
&= \sigma_2\int_Y f^2[\dot{H}_Y + O(\varepsilon)]\sqrt{\det g(s,0)}ds \\
\sigma_2 &:= 6\int_0^\infty \tau g\dot{g}^2dt
\end{aligned}$$

using the expansion in equation (3.5.5). In sum

$$\int_Y f\phi_t dA = \int_Y\left(\sigma_0(1+O(\varepsilon))|\nabla f|^2 + [(\sigma_1+\sigma_2)\dot{H}_Y + O(\varepsilon)]f^2\right)dA_Y \tag{4.1.7}$$

We now compute the constants

Computing the constants

Recall that

$$\sigma_0 = \int_0^\infty \dot{g}^2dt = \frac{\sqrt{2}}{3}$$

We compute σ_1

$$\sigma_1 = \int_0^\infty t\dot{g}\ddot{g} = -\frac{1}{3\sqrt{2}}$$

For σ_2, we now compute

$$\int_0^\infty \tau g\dot{g}^2$$

To do this, we note that we can solve

$$\begin{aligned}\alpha : [0,\infty) &\to \mathbf{R}\\ \alpha(0) = \lim_{t\to\infty}\mu(t) &= 0\\ [\partial_t^2 - W''(g)]\alpha &= g\dot{g}^2\end{aligned}$$

In particular,

$$\alpha = -\frac{1}{3\sqrt{2}}g(t)\dot{g}(t)$$

Moreover, $L = \partial_t^2 - W''(g)$ is self-adjoint on the space of smooth functions on $\mathbf{R}^+$ with exponential decay and dirichlet condition at 0 and ∞. This means that

$$\begin{aligned}\int_0^\infty \tau g\dot{g}^2 &= \int \tau L(\alpha)\\ &= -\frac{1}{3\sqrt{2}}\int L(\tau)g\dot{g}\\ &= -\frac{1}{3\sqrt{2}}\int tg\dot{g}^2\\ &= -\frac{1}{18\sqrt{2}}\end{aligned}$$

so that

$$\sigma_2 = 6\int_0^\infty \tau g\dot{g}^2 = -\frac{1}{3\sqrt{2}}$$

and so

$$\sigma_1 + \sigma_2 = -\frac{\sqrt{2}}{3} = -\sigma_0$$

and so we conclude that (4.1.7) becomes

$$\begin{aligned}\int_Y f\phi_t dA &= \sigma_0\int_Y \left((1+O(\varepsilon))|\nabla f|^2 + \left([-1+O(\varepsilon)]\dot{H}_Y + O(\varepsilon)\right)f^2\right)dA_Y\\ &= \sigma_0[D^2A(f)\Big|_Y + R]\\ |R| &\le K\varepsilon^{1/2}||f||_{H^1}^2\end{aligned}$$

Note that this expression is quadratic in f and that $\dot{H}_Y$ does not depend on the choice of normal. We see that we'll get the same contribution from $-\int_Y f\phi_t^-$, meaning that the second variation of BE_ε will be twice the above.

Thus if Y is a critical point, the second variation of BE corresponds to the second variation of area plus an $O(\varepsilon)$ error term.

4.1.5 Absolute Minimizers of BE_ε

M^n compact for $n \geq 2$

In this section, we show that absolute minimizers of BE do not exist on closed riemannian manifolds with $n \geq 2$:

Theorem 4.1.1. *For M^n a compact riemannian manifold with $n \geq 2$, and any $\varepsilon > 0$*

$$\inf_{\substack{Y \text{ separating} \\ \text{hypersurface}}} \mathrm{BE}_\varepsilon(Y) = 0$$

in particular, there is no $Y^{n-1} \subseteq M^n$ separating hypersurface achieving the above.

To prove this, we construct a $W^{1,2}$ function vanishing on a small sphere about a point, with energy bounded above by the $n-1$-dimensional hausdorff area of the sphere. In the small ε limit, this converges to 0 for each $r_0 > 0$ (see 4.3).

Proof: Let ε fixed, $p \in M^n$, and $B_\delta(p)$, $B_{2\delta}(p)$ for $\delta > 0$ small. Consider a diffeomorphism

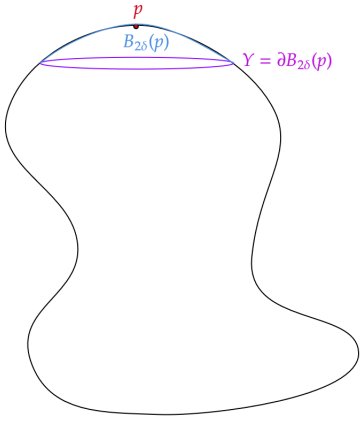


Figure 4.3:

$$\psi : B_{2\delta}(p) \to B_{2\delta}(0)$$

for δ sufficienty small such that the pull back metric on $B_{2\delta}(p)$ is approximately the pull-back metric on $\mathbf{R}^n$. Now define

$$f(r) = \begin{cases} 0 & 0 \leq r \leq \delta \\ \frac{\ln(r/\delta)}{\ln(k)} & \delta \leq r \leq k\delta \\ 1 & r \geq k\delta \end{cases}$$

Then we have that

$$f'(r) = \begin{cases} 0 & 0 \leq r \leq \delta \\ \frac{1}{r\ln(k)} & \delta \leq r \leq k\delta \\ 0 & r \geq k\delta \end{cases}$$

And so

$$\begin{aligned} E_\varepsilon(f) &= \int_M \frac{\varepsilon}{2}|\nabla f|^2 + \frac{1}{\varepsilon}W(f)dA \\ &= \int_{B_{2\delta}(p)} \frac{\varepsilon}{2}|\nabla f|^2 + \frac{1}{\varepsilon}W(f)dA \\ &\leq \int_{B_{2\delta}(p)} \frac{\varepsilon}{2}|f'(r)|^2 dA + 2\mathrm{Vol}(B_{2\delta}(p)) \cdot \frac{1}{4\varepsilon} \\ &= \varepsilon\pi \int_\delta^{k\delta} \frac{1}{r^2\ln(k)^2} r dr + \frac{C\delta^n}{\varepsilon} \\ &\leq \frac{\varepsilon\tilde{C}}{\ln(k)^2} \int_\delta^{k\delta} \frac{1}{r} dr + \frac{C\delta^n}{\varepsilon} \\ &= \begin{cases} \frac{\varepsilon\tilde{C}}{\ln(k)} + \frac{C\delta}{\varepsilon} & n = 2 \\ \frac{\varepsilon\tilde{C}\delta^{n-2}}{\ln(k)^2} + \frac{C\delta^n}{\varepsilon} & n \geq 3 \end{cases} \end{aligned}$$

here, C and $\tilde{C}$ can be taken to be 2 times the volume of the balls of radius 1 in $\mathbf{R}^n$ and surface area of the sphere of radius 1 in $\mathbf{R}^n$ respectively.

If $n \geq 3$, then we can simply send $\delta \to 0$ to prove the theorem. If $n = 2$, set $\delta = (k\ln(k))^{-1}$ (so that δk is bounded in the definition of f) and send $k \to \infty$ so that

$$\inf_Y \mathrm{BE}_\varepsilon(Y) \leq \frac{\varepsilon\tilde{C}}{\ln(k)} + \frac{C}{\varepsilon k\ln(k)} \to 0$$

which tells us that **on M compact, no absolute minimizer of** BE **exists**.

For manifolds which are 2 dimensional or larger, this leads us to look into minimizers which are

only local critical points, in particular critical points which are Morse index 1 or larger.

Absolute minimizers on S^1

Identify $S^1 \cong [0,1]$ with $0 \sim 1$ identified. For any $\varepsilon > 0$, let $u_{0,\varepsilon}$ denote the absolute minimizer of (1.0.1) on $[0,1]$ vanishing at 0 and 1. In this section, we'll prove the following analogous theorem:

Theorem 4.1.2. *For any $\varepsilon > 0$*

$$\inf_{p \neq 0 \in S^1} \mathrm{BE}_\varepsilon(\{0,p\}) = E_\varepsilon(u_{0,\varepsilon}) > 0$$

but there is no 2-node solution which achieves the above.

To prove this, we first prove the following lemma

Lemma 4.1.8. *Let N be a closed manifold with boundary $\partial Y = N$. Let $\{u_\varepsilon\}$ be the minimizers of (1.0.1) on N restricted to functions in $W_0^{1,2}(N)$. Then there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the function*

$$g(\varepsilon) = E_\varepsilon(u_\varepsilon) = \inf_{v \in W_0^{1,2}(N)} E_\varepsilon(v)$$

is continuous in ε. In fact, it is lipschitz away from $\varepsilon = 0$.

Proof: Let $\gamma, \delta < \varepsilon_0$ arbitrary and positive. Let

$$e_\varepsilon(p) := \frac{\varepsilon}{2}|\nabla u_\varepsilon(p)|^2 + \frac{1}{\varepsilon}W(u_\varepsilon(p))$$

Then

$$g(\gamma) - g(\delta) = \int_N [e_\gamma - e_\delta]$$

We rewrite

$$\begin{aligned}
e_\gamma &= \left[\frac{\delta}{2}|\nabla u_\gamma(p)|^2 + \frac{1}{\delta}W(u_\gamma(p))\right] + \left[\frac{\gamma-\delta}{2}|\nabla u_\gamma(p)|^2 + (\gamma^{-1}-\delta^{-1})W(u_\gamma(p))\right] \\
\implies E_\gamma(u_\gamma) &= E_\delta(u_\gamma) + \int_N \frac{\gamma-\delta}{2}|\nabla u_\gamma(p)|^2 + (\gamma^{-1}-\delta^{-1})W(u_\gamma(p)) \\
&\geq E_\delta(u_\gamma) - \left(\frac{|\gamma-\delta|}{\gamma} + \frac{|\delta-\gamma|}{\delta}\right) E_\gamma(u_\gamma) \\
&\geq E_\delta(u_\gamma) - K|\gamma-\delta| \\
&\geq E_\delta(u_\delta) - K|\gamma-\delta|
\end{aligned}$$

Since u_δ is the minimizer of E_δ on N. Here K depends on $E_\gamma(u_\gamma)$ which we know is bounded as long as $\gamma > c_0 > 0$ – take 0 as a competitor to get a bound of $\mathrm{Vol}(N)/4\gamma < \mathrm{Vol}(N)/4c_0$ as a competing energy. This shows that

$$E_\gamma(u_\gamma) - E_\delta(u_\delta) \geq -K|\gamma - \delta|$$

but by the same argument with $\gamma,\ \delta$ switched, we get

$$\left|E_\gamma(u_\gamma) - E_\delta(u_\delta)\right| \leq K|\gamma - \delta|$$

This gives lipschitzness away from 0, and hence continuity. □

We now prove theorem 4.1.2:

Proof: The argument is as follows: Take $\{p_i\}$ such that

$$\lim_{i\to\infty} \mathrm{BE}(\{0, p_i\}) = \inf_{p\neq 0\in S^1} \mathrm{BE}_\varepsilon(\{0, p\})$$

Then, potentially up to a subsequence, we have $\{p_i\}$ converges to some $p \in [0, 1]$. If $p \neq 0, 1$, then $E_\varepsilon(u) > E_\varepsilon(u_{0,\varepsilon})$ as $u_{0,\varepsilon}$ is the unique absolute minimizer with dirichlet conditions on $\{0, 1\}$. However, let $u_{\delta,\varepsilon}$ be the function

$$u_{\delta,\varepsilon}(x) = \begin{cases} u_{[0,1-\delta],\varepsilon}(x) & x \in [0, 1-\delta] \\ 0 & x \in [1-\delta, 1] \end{cases}$$

where $u_{[1-\delta],\varepsilon}$ is the minimizer of (1.0.1) on $[0, 1-\delta]$ with dirichlet conditions at the endpoints. Then we have

$$\lim_{\delta\to 0} E_\varepsilon(u_{\delta,\varepsilon}, [0, 1]) = \lim_{\delta\to 0} E_\varepsilon(u_{[0,1-\delta],\varepsilon}, [0, 1-\delta]) + \frac{\delta}{4\varepsilon}$$

where we know that

$$\int_{1-\delta}^{1} e_\varepsilon(u_{\delta,\varepsilon}) = \int_{1-\delta}^{1} \left(0 + \frac{1}{4\varepsilon}\right) = \frac{\delta}{4\varepsilon}$$

Moreover for δ sufficiently small, we have that

$$\tilde{u}_{\varepsilon,\delta}(x) := u_{[0,1-\delta],\varepsilon}((1-\delta)x)$$

is a well defined solution to (1.0.2) with $\tilde{\varepsilon} = \frac{\varepsilon}{1-\delta}$, and hence the unique solution to (1.0.2) corre-

sponding to $\tilde{\varepsilon}$. In particular, it is the minimizer of $E_{\tilde{\varepsilon}}$ on $[0,1]$, and so by lemma 4.1.8, we have that

$$\lim_{\delta\to 0} E_\varepsilon(u_{[0,1-\delta],\varepsilon}, [0,1-\delta]) = \lim_{\delta\to 0} E_{\varepsilon/(1-\delta)}(\tilde{u}_{\varepsilon,\delta}) = E_\varepsilon(u_{0,\varepsilon})$$

In particular, this tells us that for δ sufficiently small, we have

$$E_\varepsilon(u) > E_\varepsilon(u_{\delta,\varepsilon}) > \mathrm{BE}_\varepsilon(\{0, 1-\delta\})$$

by our construction and the definition of BE_ε on hypersurfaces. Thus, u cannot be an absolute minimizer when $|Y| = 2$ since we've found a competitor. This is a contradiction, and tells us that $p = 0, 1$, so our minimizing solution must be $u_{0,\varepsilon}$, which is only a 1-node function on S^1.

Bibliography

[1] Haïm Brezis and Luc Oswald. "Remarks on sublinear elliptic equations". In: *Nonlinear Analysis: Theory, Methods & Applications* 10.1 (1986), pp. 55–64.

[2] Paul F Byrd and Morris D Friedman. *Handbook of elliptic integrals for engineers and physicists*. Vol. 67. Springer, 2013.

[3] Rayssa Caju and Pedro Gaspar. "Solutions of the Allen-Cahn equation on closed manifolds in the presence of symmetry". In: *arXiv preprint arXiv:1906.05938* (2019).

[4] Rayssa Caju et al. "Ground states of semilinear elliptic equations". In: *arXiv preprint arXiv:2006.10607* (2020).

[5] Otis Chodosh and Christos Mantoulidis. "Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates". In: *Annals of Mathematics* 191.1 (2020), pp. 213–328.

[6] Doris Fischer-Colbrie and Richard Schoen. "The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature". In: *Communications on Pure and Applied Mathematics* 33.2 (1980), pp. 199–211.

[7] Pedro Gaspar. "The second inner variation of energy and the Morse index of limit interfaces". In: *The Journal of Geometric Analysis* 30.1 (2020), pp. 69–85.

[8] Pedro Gaspar and Marco AM Guaraco. "The Allen–Cahn equation on closed manifolds". In: *Calculus of Variations and Partial Differential Equations* 57 (2018), pp. 1–42.

[9] David Gilbarg et al. *Elliptic partial differential equations of second order*. Vol. 224. 2. Springer, 1977.

[10] Marco Guaraco. "Min-Max for the Allen-Cahn Equation and Other Topics (Princeton, 2019)". In: (2019).

[11] Marco AM Guaraco. "Min–max for phase transitions and the existence of embedded minimal hypersurfaces". In: *Journal of Differential Geometry* 108.1 (2018), pp. 91–133.

[12] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Springer Science & Business Media, 2006.

[13] Fritz Hiesmayr. "Rigidity of low index solutions on S^3 via a Frankel theorem for the Allen-Cahn equation". In: *arXiv preprint arXiv:2007.08701* (2020).

[14] Morris W Hirsch, Stephen Smale, and Robert L Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012.

[15] Wu-yi Hsiang and H Blaine Lawson Jr. "Minimal submanifolds of low cohomogeneity". In: *Journal of Differential Geometry* 5.1-2 (1971), pp. 1–38.

[16] Serge Lang. *Differential manifolds*. Springer, 1985.

[17] Nam Q Le. "On the second inner variation of the Allen-Cahn functional and its applications". In: *Indiana University Mathematics Journal* (2011), pp. 1843–1856.

[18] Francesco Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*. 135. Cambridge University Press, 2012.

[19] Christos Mantoulidis. "Double-well phase transitions are more rigid than minimal hypersurfaces". In: *arXiv preprint arXiv:2208.00582* (2022).

[20] Christos Mantoulidis. "Variational aspects of phase transitions with prescribed mean curvature". In: *Calculus of Variations and Partial Differential Equations* 61.2 (2022), p. 43.

[21] Jared Marx-Kuo. "A Dirichlet-to-Neumann Map for the Allen-Cahn Equation on Manifolds with Boundary". In: *arXiv preprint arXiv:2301.08286* (2023).

[22] Luciano Modica. "A gradient bound and a Liouville theorem for nonlinear Poisson equations". In: *Communications on pure and applied mathematics* 38.5 (1985), pp. 679–684.

[23] L Modica-S Mortola. "Un esempio di Γ–convergenza". In: *Boll. UMI* 14 (1977), pp. 285–299.

[24] Francisco Urbano. "Minimal surfaces with low index in the three-dimensional sphere". In: *Proceedings of the American Mathematical Society* (1990), pp. 989–992.

[25] Kelei Wang and Juncheng Wei. "Second order estimate on transition layers". In: *Advances in Mathematics* 358 (2019), p. 106856.