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Two regularity results in the theory of minimal hypersurfaces

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TWO REGULARITY RESULTS IN THE THEORY OF MINIMAL HYPERSURFACES

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Abstract

Questions about the existence of minimal submanifolds in compact Riemannian manifolds occupy a prominent place in the theory of geometric variational problems, and have been addressed using a variety of tools coming from topology, partial differential equations, and the calculus of variations. Geometric measure theory methods have proved particularly successful in providing a weak formulation of this variational problem, and proving the existence of weak solutions under very general assumptions. Thanks to geometric measure theory techniques, a variety of regularity results have also been obtained, showing that in many important situations these weak solutions are indeed smooth minimal submanifolds.

In this thesis, we will discuss two regularity results with applications to the theory of minimal hypersurfaces.

In Part I, which is based on joint work with Douglas Stryker, we develop a sharp regularity theory for minimisers of the prescribed mean curvature functional in isotopy classes of surfaces in 3-manifolds, which enables us to construct prescribed mean curvature embedded spheres in $\mathbb{S}^3$. Minimisation over isotopies also plays a central role in min-max constructions of minimal surfaces with prescribed (or controlled) topology.

In Part II, which is based on joint work with Martin Li and Davide Parise, we describe a regularity theory for singular limits of solutions of the Allen–Cahn equation with a homogeneous Neumann boundary condition on a compact manifold with boundary. In particular, we show that these limit-interfaces are measure-theoretic free boundary minimal hypersurfaces, opening up the possibility of developing a min-max construction of free boundary minimal hypersurfaces based on the Allen–Cahn equation.

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Chapter 1

Introduction

Minimal hypersurfaces are among the most natural and well-studied objects in Riemannian geometry. Given a compact n -dimensional Riemannian manifold (M, g) , a hypersurface $\Sigma \subset M$ is said to be *minimal* if it is a *critical point* of the $(n - 1)$ -area functional¹. More formally, we say Σ is a minimal hypersurface in (M, g) if

$$\left. \frac{d}{dt} \right|_{t=0} \text{Area}_g(\phi_t(\Sigma)) = 0 \quad (1.0.1)$$

for all one-parameter families $(\phi_t)_{t \in (-\varepsilon, \varepsilon)}$ of diffeomorphisms, generated by compactly supported vector fields X on M . By the well-known first variation formula, a smooth hypersurface Σ turns out to be minimal if and only if its mean curvature H_Σ vanishes identically. The equation $H_\Sigma \equiv 0$ can be thought of as an analogue of the Euler–Lagrange equation for the variational problem (1.0.1).

Intuitively, minimal hypersurfaces generalise the classical notion of geodesics on surfaces, and are characterised by the property of *locally* minimising area. Much like the study of geodesics plays a central role in the classical theory of surfaces in Euclidean space, minimal hypersurfaces turn out to be a powerful tool in studying the

¹This can be defined, for example, by $\text{Area}_g(\Sigma) := \mathcal{H}^{n-1}(\Sigma)$, where $\mathcal{H}^{n-1}$ denotes the $(n - 1)$ -dimensional Hausdorff measure coming from the metric space structure induced by the Riemannian metric g on M .

geometry and topology of the Riemannian manifold in which they live. More broadly, they have applications in many areas of mathematics—from three-dimensional topology to general relativity—but also in material science and engineering, and the study of their properties remains an active field of research.

Another important class of hypersurfaces, which is intimately related to minimal hypersurfaces and plays a central role in this thesis, is that of *prescribed mean curvature* hypersurfaces. Given a smooth function $h : M \rightarrow \mathbb{R}$, a hypersurface $\Sigma \subset M$ is said to be *h -prescribed mean curvature* if its (scalar) mean curvature is given by h , i.e.

$$H_\Sigma = h|_\Sigma.$$

Similarly to minimal hypersurfaces, h -prescribed mean curvature hypersurfaces can also be described variationally as the critical points of a functional—denoted by $\mathcal{A}^h$ —involving area and volume (see Chapter 2 for a precise definition of $\mathcal{A}^h$). Therefore, *variational methods* play a central role in the study of both minimal and prescribed mean curvature hypersurfaces.

A fundamental question one can ask about minimal and prescribed mean curvature hypersurfaces in a compact Riemannian manifold (M, g) is whether they always exist. This turns out to be a subtle problem, which has motivated the development of many innovative techniques in geometric measure theory and the calculus of variations throughout the last century.

As in many other variational problems, a very natural idea to prove existence of solutions is that of devising a weak formulation², which allows for a larger class of admissible objects in such a way that this larger class enjoys better compactness properties. Solutions of the weak formulation of the problem are called *weak solutions*, and do not a priori enjoy strong regularity properties. In the specific case of

²Roughly, this corresponds to turning a differential equation into an integral equation, which makes sense for less regular objects.

finding critical points of area and $\mathcal{A}^h$, the appropriate classes in which to look for weak solutions are the measure-theoretic generalisations of the notion of submanifold studied by *geometric measure theory* (for example, those of *sets of finite perimeter*, *currents* and *varifolds*).

However, in order for these solutions to be used as tools in geometry and topology, one needs to be able to work with smooth objects—not just measure theoretic generalisations—and this relies on the development of a successful *regularity theory*. The goal of such a theory is to show that weak solutions of the variational problem at hand are in fact genuine solutions of the original formulation of the problem, i.e. smooth minimal or prescribed mean curvature hypersurfaces in M , away perhaps from a small singular set.

The aim of this thesis is to describe two regularity results based on geometric measure theory techniques, which can be applied to several questions about existence of minimal hypersurfaces. The next section provides an overview of these results and their main applications.

1.1 Main results and organisation

This thesis is divided into two parts, each dealing with a different regularity result. The two parts are mostly independent, and have somewhat different flavours, the unifying theme being that both results have applications in the existence theory of minimal hypersurfaces.

The results in Part I are joint work with Douglas Stryker, and appeared in preprint form in [SS23]. The results in Part II are joint work with Martin Li and Davide Parise, and appeared in preprint form in [LPS23].

We shall now discuss the content of each part separately, and provide some context for the main results and their applications.

1.1.1 Part I: Regularity of minimisers over isotopies

The simplest and most basic technique to produce critical points of a functional is to minimise it over some nontrivial class. This approach produces a special type of solutions, namely *minimisers*, which often enjoy stronger regularity properties than generic critical points. However, since minimal and prescribed mean curvature hypersurfaces are not, in general, global minimisers of the respective functionals, the methods involved in proving their existence are often those of *min-max theory*, which relates the number of general critical points of a functional to the topology of the underlying space. When the functional is defined on a Hilbert or Banach manifold, and satisfies certain well-known conditions (e.g. the so-called Palais–Smale condition), this relation is well-understood. However, neither the area functional nor $\mathcal{A}^h$ satisfy these assumptions, so different ideas are required.

A very general min-max theory for the area functional, which makes heavy use of the tools of geometric measure theory, was initiated almost 60 years ago by F. Almgren and J. Pitts. In the past decade, starting with the work of F.C. Marques and A. Neves, which extended and improved the Almgren–Pitts theory³, important new results have been proved on the existence of minimal hypersurfaces thanks to the joint efforts of a number of mathematicians, culminating in a proof of Yau’s conjecture [Yau82, Problem 88] on the existence of infinitely many closed embedded minimal hypersurfaces in closed Riemannian manifolds of dimension between 3 and 7 (see [Son23a]).

However, due to the weak notions of convergence involved, this framework gives very little control on the topology of the resulting critical points. So far, information about the topological type of the Almgren–Pitts min-max hypersurfaces has only been obtained indirectly, by first controlling the Morse index (via [MN16]), and then

³Very recently, X. Zhou and J. Zhu developed a similar min-max theory for the prescribed mean curvature functional [ZZ19, ZZ20].

appealing to topological bounds in terms index upper bound (see [Son23b]), which tend to be far from sharp. Thus, Almgren–Pitts min-max is generally unsuited to proving the existence of minimal hypersurfaces of a prescribed topological type.

In the case where the ambient dimension is 3, a variant of the theory, called *Simon–Smith min-max*, was developed by F.R. Smith in the early 1980s, following ideas of L. Simon (see also [CDL03] for a more modern treatment). Despite being less flexible than Almgren–Pitts, and having the disadvantage of being restricted to the case of 2-dimensional surfaces in 3-manifolds, Simon–Smith min-max allows for more control on the topology of the resulting minimal surfaces. Roughly speaking, while in the Almgren–Pitts approach one considers homotopy classes of sweepouts⁴ of the manifold by flat hypercycles (a measure-theoretic generalisation of the notion of closed hypersurface), Simon–Smith theory uses smooth sweepouts and their images under ambient isotopies.

The regularity theory for Almgren–Pitts has at its heart the regularity theory for area minimisers over the very large class of *integral currents* (the smooth elements of which can have arbitrarily complicated topology), whereas Simon–Smith regularity is based on minimisation in a much smaller class with controlled topology, and relies on the regularity results of [AS79] and [MSY82] on area minimisers over isotopies.

As a first step towards a Simon–Smith min-max for the prescribed mean curvature functional in closed 3-manifolds, in Part I of this thesis, which is based on joint work with Douglas Stryker, we generalise the ideas of [AS79] and [MSY82] to the functional $\mathcal{A}^h$.

The main difference between isotopy minimisers of the area functional and of $\mathcal{A}^h$ is the regularity theory for the local minimisation problem. In the latter case, the optimal regularity of minimisers is just $C^{1,1}$. The following is a simplified statement of the main regularity result in Part I (see Theorem 2.0.1 for a precise statement).

⁴i.e. topologically nontrivial families of generalised hypersurfaces that *sweep out* the manifold.

Theorem. *Let B be a sufficiently small ball in (M^3, g) , and $U \subset B \subset M$ an open set with C^1 boundary. If $\Omega_0 \subset M$ is an open set such that $\Sigma_0 := \partial\Omega_0 \cap U$ is a smooth embedded surface, and $\{\phi_k\}_{k \in \mathbb{N}}$ an $\mathcal{A}^h$ minimising sequence of isotopies supported in U , i.e.*

$$\lim_{k \rightarrow \infty} \mathcal{A}^h(\phi_k(1, \Omega_0)) = \inf\{\mathcal{A}^h(\phi(1, \Omega_0)) \mid \phi \text{ is an isotopy of } M \text{ supported in } U\},$$

then, there is a $C^{1,1}$ almost embedding⁵ of a surface $\psi : \Sigma^2 \rightarrow M$ such that, up to passing to a subsequence

$$\partial\Omega_k \cap U := \Sigma_k \rightarrow \psi(\Sigma)^6.$$

Moreover, around each point where ψ is an embedding, $\psi(\Sigma)$ is a smooth h -prescribed mean curvature surface, and around each point in the interior of the set where ψ fails to be an embedding $\psi(\Sigma)$ is a smooth minimal surface (with multiplicity).

In particular, local minimisers need not be embedded and can have a self-touching set of positive $\mathcal{H}^2$ -measure, the interior of which is a smooth minimal surface. This regularity is sharp and we provide examples where a large touching set does indeed arise. As an application of our regularity theory, we develop a min-max approach with genus bounds for the $\mathcal{A}^h$ functional along the lines of Simon–Smith, which allows us to prove the following existence result.

Theorem 2.1.3. *There is an open dense set $\mathfrak{P} \subset \{h \in C^\infty(\mathbb{S}^3) \mid |h| \leq 0.547\}$ so that there is an embedded sphere with prescribed mean curvature h with respect to the round metric for any $h \in \mathfrak{P}$. In particular, $\{h \in C^\infty(\mathbb{S}^3) \mid 0 < h \leq 0.547\} \subset \mathfrak{P}$.*

After our work [SS23] was completed, Z. Wang and X. Zhou [WZ23] used the theory developed here to prove the existence of at least 4 embedded minimal spheres

⁵Almost embedding means that ψ is an immersion, and around each point where it fails to be an embedding, it decomposes into finitely many connected embedded components which touch but do not cross. See the statement of Theorem 2.0.1 for a more precise characterisation.

⁶in the sense of varifolds.

in $\mathbb{S}^3$ with a generic metric, by proving a multiplicity one theorem for the Simon–Smith min-max. Their proof follows the strategy used by X. Zhou in his proof of the Almgren–Pitts multiplicity one conjecture [Zho20], and uses the prescribed mean curvature functional to approximate the area functional, thus relying on many of the ideas developed in [SS23].

1.1.2 Part II: Boundary behaviour of Allen–Cahn solutions

As previously discussed, the min-max theory for the area functional is made significantly more difficult by the lack of a Hilbert space structure or a Palais-Smale condition. However, it turns out that it is possible to approximate area by a family of functionals—the ε -Allen–Cahn energies—on the Hilbert space $H^1(M)$, each satisfying the Palais-Smale condition. In particular, M. Guaraco [Gua18] developed an alternative min-max construction of closed minimal hypersurfaces based on sequences of uniformly bounded critical points of the ε -Allen–Cahn energy with a priori energy and index bounds, and their singular limits as $\varepsilon \rightarrow 0$, building on the regularity theory developed by Padilla–Tonegawa [PT98], Hutchinson–Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa–Wickramasekera [TW12]. This approach to min-max has been successful in recovering many of the results first established in the Almgren–Pitts framework, but was also instrumental to some key recent advances in the theory, such as O. Chodosh and C. Mantoulidis’s proof of the *multiplicity one conjecture* in dimension 3 [CM20], and their computation of the *p-widths* of the 2-sphere [CM23].

In the context of a compact Riemannian manifold with boundary $(M, \partial M)$, it is natural to consider compact hypersurfaces with boundary $(\Sigma, \partial\Sigma)$ such that $\partial\Sigma \subset \partial M$ which are critical points of the area functional with respect to variations that constrain $\partial\Sigma$ to ∂M . They are called *free boundary minimal hypersurfaces*.

Despite the additional difficulties posed by the presence of a boundary, both the Almgren–Pitts and the Simon–Smith min-max schemes have been successfully

adapted to the free boundary setting. In particular, free boundary counterparts of many of the spectacular results obtained via min-max methods in the closed hypersurface setting have been established using the Almgren–Pitts framework (see e.g. [LZ21, Wan20, SWZ20]). It is therefore only natural to assume that there should be an Allen–Cahn min-max construction of free boundary minimal hypersurfaces in compact manifolds with boundary.

In Part II of this thesis, which is based on joint work with Martin Li and Davide Parise, we study the singular limits of sequences of uniformly bounded critical points of the ε -Allen–Cahn energy with Neumann boundary conditions, subject to an a priori energy bound. In [MT15], M. Mizuno and Y. Tonegawa have studied a singular limit problem for uniformly bounded solutions of the analogous parabolic problem in convex domains of Euclidean space, with Neumann boundary conditions and initial data of uniformly bounded energy. Using techniques adapted from [HT00], they have managed to prove convergence of the limit energy measures to Brakke’s weak mean curvature flow with a generalised orthogonality condition at the boundary (see also [Kag19] for the non-convex case). This strongly suggests that, for the associated elliptic problem, a Neumann boundary condition is the right condition to impose in order to obtain free boundary minimal hypersurfaces in the singular limit.

In this thesis, we describe the necessary modifications of Hutchinson–Tonegawa’s varifold convergence theory [HT00] in the presence of a boundary. The following is a simplified statement of the main regularity result in Part II (see Theorem 7.3.3 for a precise statement).

Theorem. *Let (M^{n+1}, g) be a closed smooth oriented Riemannian manifold with boundary ∂M , and let (u_k) be a sequence of smooth solutions of the ε_k -Allen–Cahn equations with a homogeneous Neumann boundary condition and $\sup_M |u_k|, E_{\varepsilon_k}(u_k)$ are uniformly bounded. Then, as $k \rightarrow \infty$, up to a subsequence, the level sets of u_k converge (in the varifold sense) to an integer rectifiable n -varifold V , which is stationary*

with respect to variations which preserve ∂M .

Morally, this result shows that one can use the Allen–Cahn equation with a homogeneous Neumann boundary condition to construct measure-theoretic (i.e. weak) free boundary minimal hypersurfaces, and that these weak solutions are integer rectifiable⁷ (i.e. they have some degree of regularity). Although an integer rectifiable varifold with free boundary can be a rather singular object, this result can be seen as the first step of a long term project to develop a more sophisticated regularity theory, which would allow for a min-max construction of free boundary minimal hypersurfaces on compact manifolds with boundary based on the Allen–Cahn equation.

⁷See Section 7.2.4 for a definition of an integer rectifiable varifold with free boundary

Part I

Regularity of minimisers over isotopies

Chapter 2

Introduction

Given a Riemannian manifold (M^n, g) and a function $h : M \rightarrow \mathbb{R}$, a two-sided immersion $\Sigma^{n-1} \rightarrow M$ is said to have *prescribed mean curvature* h if the mean curvature of the immersion satisfies $H_\Sigma = h|_\Sigma \cdot \nu_\Sigma$, where ν_Σ is a continuous unit normal vector field along Σ . The problem of the existence of prescribed mean curvature hypersurfaces has been a rich source of motivation for work in geometric analysis.

An essential observation is that h -prescribed mean curvature hypersurfaces arise as critical points of the *h -prescribed mean curvature functional*

$$\mathcal{A}^h(\Omega) := \mathcal{H}^{n-1}(\partial\Omega) - \int_{\Omega} h \, d\mathcal{H}^n,$$

where $\Omega \subset M$ is an open set with smooth boundary. As in the study of minimal hypersurfaces and the area functional, this observation enables the utilization of variational techniques to produce h -prescribed mean curvature hypersurfaces.

The simplest variational technique to produce critical points of a functional is to minimise the functional over some nontrivial class. In ambient dimensions 3 through 7, classical work of [FF60], [Fle62], [DG65], [Alm66], [Sim68], and [HS79] guarantees that any minimiser of area in a Riemannian manifold over integral currents with a fixed boundary is a smooth embedded minimal hypersurface. In the same dimen-

sions, these results have been adapted by [Mor03] (see [ZZ20, Theorem 2.2]) to the prescribed mean curvature setting, where the minimisation of $\mathcal{A}^h$ occurs over Caccioppoli sets that are allowed to differ from a fixed initial Caccioppoli set in a sufficiently small region $U \subset M$. In this setting, the boundary of any minimiser in U is a smooth embedded h -prescribed mean curvature hypersurface.

While this regularity theory is very powerful, it is important to remember that the classes over which the minimisation occurs are very large. In particular, elements in the same class can have arbitrarily complicated topology. Hence, we cannot draw any general conclusions about the topology of minimisers, which would be a natural hope for any geometric problem.

In the case of the area functional (i.e. $h = 0$) in 3-manifolds, there has been significant work studying the regularity of minimisers of area over smaller classes with controlled topology. A major breakthrough was the work of [AS79], where minimisers of the area functional in $\mathbb{R}^3$ over embedded discs with certain fixed boundaries are shown to be smooth embedded minimal surfaces. By an ingenious procedure to simplify the topology of elements in a minimising sequence, [MSY82] used the regularity of [AS79] to show that minimisers of the area functional in a closed 3-manifold over an isotopy class¹ are smooth embedded minimal surfaces.

Despite the successful application of the regularity theory of [AS79] and [MSY82] in producing minimal surfaces with controlled topology in 3-manifolds (see for example [Smi83], [CDL03], [DLP10], [HK19], [Ket22]), there has been no generalization² of these techniques to the functional $\mathcal{A}^h$.

In this work, we generalise the ideas of [AS79] and [MSY82] to the $\mathcal{A}^h$ functional. For an open set $U \subset M$, we let $\mathcal{I}(U)$ denote the set of isotopies $\phi : [0, 1] \times M \rightarrow M$ satisfying:

¹We emphasise that isotopies preserve the topology of any submanifold.

²A result along these lines is claimed in [Yau01]; see later in the introduction for a discussion of this paper.

- $\phi(t, \cdot) : M \rightarrow M$ is a smooth diffeomorphism,
- $\phi(0, \cdot) = \text{id}$,
- and $\phi(t, x) = x$ for $x \notin U$.

Our goal is to deduce the regularity of minimisers for the problem

$$\inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^h(\phi(1, \Omega_0)).$$

We deduce the following regularity statement.

Theorem 2.0.1. *Let $h : M \rightarrow \mathbb{R}$ be a smooth function, and let $c := \sup_M |h|$. Let $U \subset M$ be a sufficiently small (see §4.3 for a precise definition) open set with C^1 boundary, and let $\Omega_0 \subset M$ be an open subset with smooth boundary having tranverse intersection with ∂U . Let $\{\phi_k\}_{k \in \mathbb{N}} \subset \mathcal{I}(U)$, $\Omega_k := \phi_k(1, \Omega_0)$, and $\Sigma_k := \partial\Omega_k \cap U$ be a sequence satisfying*

$$\mathcal{A}^h(\Omega_k) \leq \inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^h(\phi(1, \Omega_0)) + \varepsilon_k$$

for $\varepsilon_k \rightarrow 0$. Then there is a varifold V , an open set $\Omega \subset U$, and a subsequence (not relabeled) so that

$$\mathbf{1}_{\Omega_k \cap U} \xrightarrow{L^1} \mathbf{1}_\Omega, \quad D\mathbf{1}_{\Omega_k \cap U} \xrightarrow{*} D\mathbf{1}_\Omega, \quad \mathbf{v}(\Sigma_k) \rightharpoonup V.$$

V is an integer rectifiable varifold with c -bounded first variation. For every $x \in \text{spt}\|V\|$, we have $\Theta^2(\|V\|, x) = n_x \in \mathbb{N}$. If $h(x) \neq 0$, then $n_x \in \{1\} \cup 2\mathbb{N}$. For every $x \in \text{spt}\|V\|$, there is a neighbourhood W_x of x so that the following hold.

- $V \llcorner G(W_x, 2) = \sum_{l=1}^{n_x} \mathbf{v}(N_l, 1)$, where $N_l \subset W_x$ is a $C^{1,1}$ surface with c -bounded first variation. Moreover, each N_l is on one side of $N_{l'}$ intersecting tangentially at x for any $l, l' \in \{1, \dots, n_x\}$.
- If $n_x = 1$, then

- $\|V\|_{\perp W_x} = |D\mathbf{1}_\Omega|_{\perp W_x}$,
- $V_{\perp G}(W_x, 2) = \mathbf{v}(N, 1)$, where $N \subset W_x$ is a smooth stable surface with prescribed mean curvature h with respect to the set $\Omega \cap W_x$.
- If $h(x) \neq 0$ and $n_y = 2n$ for all $y \in \text{spt}\|V\| \cap W'_x$ for an open set $W'_x \subset W_x$ containing x , then $V_{\perp G}(W'_x, 2) = \mathbf{v}(N, 2n)$, where $N \subset W'_x$ is a smooth stable minimal surface. Moreover, $\Omega \cap W'_x = W'_x \setminus N$ if $h(x) > 0$, and $\Omega \cap W'_x = \emptyset$ if $h(x) < 0$.
- If $h(x) \neq 0$, $n_x = 2n$, and there is a sequence $x_j \rightarrow x$ with $x_j \in \text{spt}\|V\|$ and $n_{x_j} \neq 2n$, then $n = 1$ and the surfaces $N_1, N_2 \subset W_x$ additionally satisfy that the generalised mean curvature of N_1 points towards N_2 and vice versa. Moreover, the set $\Gamma = \partial(N_1 \cap N_2) \cap W_x$ is locally contained in a $C^{1,\alpha}$ curve and satisfies $\mathcal{H}^1(\Gamma) < +\infty$.

Suppose additionally that U is a sufficiently small geodesic ball. Then there is a neighbourhood Y of ∂U with $V_{\perp G}(Y, 2) = \mathbf{v}(N, 1)$, where $N \subset Y$ is a smooth stable surface of prescribed mean curvature h with respect to Ω satisfying $\partial N = \partial \Sigma$.

In §2.6, we provide the details of examples that show that this regularity is sharp. We note that an adaptation of [MSY82] to the constant mean curvature setting with a stronger regularity conclusion than Theorem 2.0.1 was claimed in [Yau01, Theorem 2.1], but without a detailed proof. It appears that the example in §2.6.2 supplies a counterexample to this statement. In any case, this example illustrates an essential obstruction to a proof along the lines of [AS79] and [MSY82].

Remark 2.0.2. In groundbreaking work, [WZ23] prove the existence of 4 embedded minimal spheres in a generic metric on $\mathbb{S}^3$, which solves an important and longstanding conjecture in the theory of minimal surfaces. The proof of [WZ23] uses extensively Theorem 2.0.1 as well as the new ingredients of the proof we develop in this work.

2.1 Application to existence in the 3-sphere

Before unpacking the ideas of our proof of Theorem 2.0.1, we present a modest application of the theory to the existence of embedded h -prescribed mean curvature 2-spheres in some metrics on $\mathbb{S}^3$. Since any embedded 2-sphere in $\mathbb{S}^3$ can be moved by isotopies to a constant map, minimisation techniques fail to produce nontrivial examples. Hence, we require a min-max variational approach.

To motivate our result, we recall the following question of Yau.

Question 2.1.1 ([Yau82, Problem 59]). *For which smooth functions $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ does there exist an embedded sphere with prescribed mean curvature h (with respect to the flat metric)?*

For progress towards Question 2.1.1 (and related problems) relying more on PDE techniques, we refer to [BK74], [TW83], [Ger98], [Ye91], [PX09], and [EFP23]. There is also partial progress outside the min-max framework in [Yau97]. For progress towards 2.1.1 without topological control, we refer to [Maz22].

Since noncompact ambient spaces present unique difficulties for min-max theory, we consider the analogous question in $\mathbb{S}^3$.

Question 2.1.2. *For which smooth functions $h : \mathbb{S}^3 \rightarrow \mathbb{R}$ does there exist an embedded sphere with prescribed mean curvature h (with respect to the round metric)?*

For the area functional, a min-max variational framework was developed by [Alm65], [Pit81], and [SS81] to produce a closed embedded minimal hypersurface in any closed Riemannian manifold of dimension between 3 and 7. These techniques have been refined to produce an abundance of closed minimal hypersurfaces (see [MN17], [LMN18], [IMN18], [MNS19], and [Son23a]). This approach was successfully adapted to the functional $\mathcal{A}^h$ in [ZZ19] and [ZZ20]. In these works, min-max is carried out over similarly large classes of hypersurfaces without controlled topology, and the regularity

theory relies on the classical regularity theory for minimisers discussed earlier in the introduction. Hence, we cannot draw any general conclusions about the topology of the resulting minimal hypersurfaces.

An alternative min-max approach in the minimal hypersurfaces case was developed by Guaraco ([Gua18]) and Gaspar–Guaraco ([GG16], [GG19]) using phase transitions for the Allen–Cahn equation, building on the regularity theory of Hutchinson–Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa–Wickramasekera [TW12]. (We refer the reader to [Dey22] for a comparison between the Almgren–Pitts theory and the Allen–Cahn theory.) Similarly, [BW20] developed an analogue of the Allen–Cahn min-max approach for prescribed mean curvature hypersurfaces, building on the regularity theory developed in [BW18] and [BW19]. We emphasise that [BW20] only applies to nonnegative prescribing functions. As above, the topology of the produced hypersurfaces from these techniques is uncontrolled.

A min-max approach for the area functional in closed 3-manifolds that controls the genus of the produced minimal surface, called Simon–Smith min-max, was developed by [Smi83] (see [CDL03] and [DLP10]), building on the regularity theory of [AS79] and [MSY82]. In particular, it follows from these works that every metric on $\mathbb{S}^3$ admits an embedded minimal sphere. For progress towards the existence of more embedded minimal spheres using this approach, we refer the reader to [HK19].

As an application of our regularity theory, we develop a min-max approach for the $\mathcal{A}^h$ functional along the lines of Simon–Smith. In particular, we give the following partial answer to Question 2.1.2.

Theorem 2.1.3. *There is an open dense set $\mathfrak{P} \subset \{h \in C^\infty(\mathbb{S}^3) \mid |h| \leq 0.547\}$ so that there is an embedded sphere with prescribed mean curvature h with respect to the round metric for any $h \in \mathfrak{P}$. In particular, $\{h \in C^\infty(\mathbb{S}^3) \mid 0 < h \leq 0.547\} \subset \mathfrak{P}$.*

The restriction to an open dense set is the same restriction from [ZZ20]. We recall that unique continuation is an important ingredient in the regularity theory for min-

max surfaces. It follows from [ZZ20, Proposition 3.8 and Corollary 3.18] that unique continuation holds for an open dense set of prescribing functions that includes all positive functions.

The restriction to functions with L^∞ norm bounded by 0.547 is the more significant assumption. This restriction is a consequence of the more delicate regularity theory for minimisers of the functional (2) over isotopies, as compared to the cleaner regularity conclusion for the area functional. We emphasise that our regularity theory is sharp, so it is not clear how to extend the existence theory beyond this class of prescribing functions using the techniques developed here. Moreover, we are not aware of any counterexamples for large prescribing functions in this setting, so the question remains open.

Outside the setting of the round metric on $\mathbb{S}^3$, it is natural to consider the following related problem.

Question 2.1.4. *For which metrics g on $\mathbb{S}^3$ and which constants $c \in \mathbb{R}_{>0}$ does there exist an embedded sphere with constant mean curvature c (with respect to the metric g)?*³

Outside the min-max framework, there has been considerable progress on the existence of constant mean curvature immersed spheres in homogeneous 3-manifolds (see [DM13], [Mee13], [MMPR21], and [MMPR22]).

A min-max approach using the Dirichlet energy was developed in [SU81] to produce branched immersed minimal spheres. This approach was adapted to the constant mean curvature setting by [CZ20], where they show that every metric on $\mathbb{S}^3$ admits a branched immersed sphere of constant mean curvature c for almost every $c \in \mathbb{R}$. However, it is unclear if the branched immersions produced by these techniques are genuine immersions, let alone embeddings.

³As a point of reference, it is conjectured in the introduction of [RS20] that there is a positive answer to 2.1.4 for any c and any metric g with positive sectional curvature.

Using our regularity theory, we provide the following partial answer to (2.1.4).

Theorem 2.1.5. *There are explicit constants $\varepsilon_0 > 0$ and $c_0 > 0$ so that if $(\mathbb{S}^3, g)$ is the induced metric of a graph over the round 3-sphere $\mathbf{S}^3 \subset \mathbb{R}^4 \subset \mathbb{R}^N$ with C^2 norm bounded by ε_0^4 and $0 \leq c \leq c_0$, then $(\mathbb{S}^3, g)$ admits an embedded sphere with constant mean curvature c .*

We remark that a key feature of Theorems 2.1.3 and 2.1.5 is that the restrictions on the prescription function and metric are explicit. Indeed, we expect that versions of these theorems hold with inexplicit constants by an implicit function theorem argument, using only the well-established existence theory for minimal spheres. Unlike such a result, our work provides actual testable criteria for existence.

2.2 Idea of the proof of regularity

The proof of 2.0.1 follows the same strategy as [AS79] and [MSY82]. For clarity, we outline the approach of [AS79] and [MSY82], and then highlight the key differences.

1. *Reduce the problem of minimising area in an isotopy class to the problem of minimising area over discs.* In [MSY82], a procedure called γ -reduction is introduced. In essence, γ -reduction is used to delete small necks that either generate nontrivial topology or separate large components. By applying the γ -reduction procedure a maximal number of times to a minimising sequence for area over isotopies (and checking that the limit surface does not change), [MSY82] use the γ -irreducibility property to show that the new sequence is a minimising sequence for area over discs in any small ball.
2. *Reduce the problem of minimising area over discs to the problem of minimising area over “stacked discs” in a cylinder.* By zooming in along a minimising

⁴By this, we mean that $(\mathbb{S}^3, g)$ is the induced metric of $\{x + f(x) \mid x_1^2 + x_2^2 + x_3^2 = 1, x_4, \dots, x_N = 0\}$ where $N \geq 3$ and $f : \mathbb{S}^3 \rightarrow \mathbb{R}^N$ satisfies $\|f\|_{C^2} \leq \varepsilon_0$.

sequence for area over discs at a point where the limit has a tangent plane, [AS79] use area comparison estimates to throw away components that are not close to the tangent plane in a small ball, without changing the limit. Therefore, it suffices to study a minimising sequence for area over disjoint unions of discs with homotopically nontrivial boundary in a fixed cylinder. We call discs in this arrangement “stacked discs”.

3. *Reduce the problem of minimising area over many stacked discs in a cylinder to the problem of minimising area over one stacked disc in a cylinder.* To isolate one disc from many in the stacked discs arrangement, [AS79] develop a simple procedure to disentangle intersecting discs without increasing area. It then suffices to study a minimising sequence for area over stacked discs with only one disc.
4. *Conclude by Allard’s regularity.* For a minimising sequence for area over stacked discs with one disc, it is easy to obtain good lower and upper bounds for area. These estimates allow [AS79] to deduce the smooth regularity of the limit using the regularity theorem of [All72].

Steps (1) and (2) work for $\mathcal{A}^h$ with only technical modifications. For example, we make frequent use of the following basic idea. In the $\mathcal{A}^h$ case, inequalities will have an extra volume term as compared to the area functional case. We absorb the volume term into the area terms using the isoperimetric inequality. The resulting inequalities resemble the analogous inequalities in the case of the area functional, only with slightly worse coefficients.

The essential difficulty is adapting steps (3) and (4). To highlight the key difference from the area functional, consider the case illustrated in Figure 2.1. In this case, we have a stack of two discs in a cylinder which is the boundary of the disjoint union of the set above the top disc and the set below the bottom disc. If we

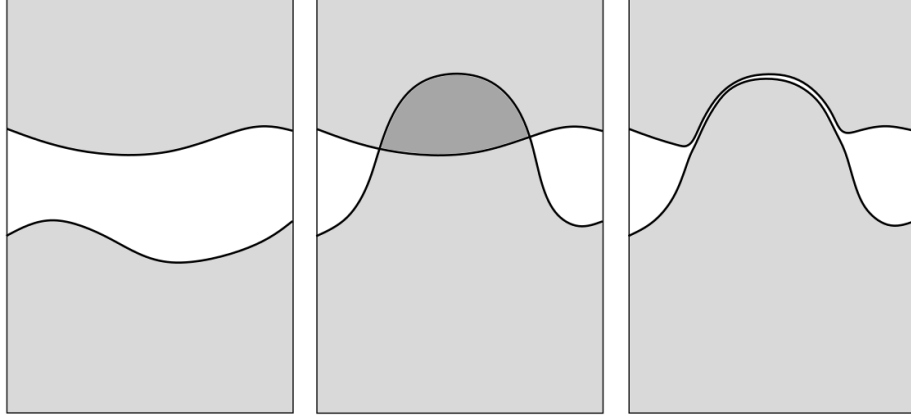


Figure 2.1: The left image depicts the initial stack of two discs. The middle image depicts a replacement for the bottom disc that intersects the top disc. The right image depicts a possible disentanglement of the discs.

replace the bottom disc with a disc that passes through the top disc, the volume between the two discs is now covered twice. So, no matter how we disentangle these intersecting discs, the volume that was covered twice can now only be covered once. Hence, any disentangling procedure will affect $\mathcal{A}^h$ in a complicated way, unlike the area functional.

In the course of our proof, we develop two weaker versions of steps (3) and (4). Together, the weak versions of these steps can be used to deduce Theorem 2.0.1 with sub-optimal $C^{1,\alpha}$ regularity.

First weak version. In this version of steps (3) and (4), we deduce the $C^{1,\alpha}$ regularity of the limit of each disc in the stacked disc arrangement.

(3') *Each disc in a stacked disc arrangement minimising $\mathcal{A}^h$ is itself minimising for $\mathcal{A}^{\tilde{h}}$.* Using a delicate disentanglement procedure, we find that the limit of each disc is minimising for $\mathcal{A}^{\tilde{h}}$, where $\tilde{h}$ is a measurable function bounded by $\sup |h|$ that depends on the limit. We emphasise that $\tilde{h}$ cannot be chosen to be continuous in general.

(4') *Conclude by Allard's regularity.* Using the same area estimates as step (4), we can apply the theorem of [All72]. However, the first variation of the limit is

only known to be L^∞ by step (3'), so the limit of each disc is $C^{1,\alpha}$.

Second weak version. In this version of steps (3) and (4), we determine where the limit is a smooth multiplicity one h -prescribed mean curvature surface and where is it a smooth even multiplicity minimal surface, in the case where h is positive.

(3'') *Reduce the problem of minimising $\mathcal{A}^h$ over many stacked discs in a cylinder to the problem of minimising $\mathcal{A}^h$ over stacked discs with only one or two discs.*

When h has a sign, it is easier to keep track of volumes under disentanglement. Similar in spirit to step (3) from [AS79], we use this observation in combination with our disentanglement procedure to decompose large stacks of discs into units consisting of only one or two discs (depending on orientations).

(4'') *Conclude by Allard's regularity.* The case of only one stacked disc now follows exactly as in step (4) from [AS79], where we find that the limit is a smooth embedded h -prescribed mean curvature surface. In the two stack case, if the limits of the two discs agree on some open set, we see that the limit is stationary. We then follow step (4) from [AS79] to find that the limit is a smooth embedded minimal surface with multiplicity two.

After deducing the $C^{1,\alpha}$ version of Theorem 2.0.1, we use some tools from the theory of free boundary problems to upgrade the regularity of our minimisers to $C^{1,1}$. Indeed, we can show that minimisers locally solve an n -membrane problem for a quasilinear elliptic operator (see [VC74], [Sil05] and [SY19] for more background). While the optimal regularity for the n -membrane problem for arbitrary n is known in the linear case by [SY19], the optimal regularity in the quasilinear case is only proved in the literature for $n = 2$ ([Sil05]). We fill this gap by finding the optimal $C^{1,1}$ regularity for the quasilinear n -membrane problem, allowing us to upgrade our regularity theorem to $C^{1,1}$. The essential idea of our extension to the n -membrane problem is to rewrite the equation for the sum of the graph of the discs in the stack

in a carefully chosen way that enables an induction on the number of discs. This rearrangement is a somewhat nontrivial generalization of a similar idea in [Sil05] for the 2-membrane problem.

2.3 Idea of the proof of existence in the 3-sphere

There are a few key difficulties in carrying over the min-max program of [CDL03] and [DLP10] to our setting, though much remains unchanged. Unique continuation and strong compactness are the two major obstacles.

In the classical min-max settings, unique continuation for surfaces with zero or prescribed mean curvature is used in an essential way to patch together iterated replacements to prove regularity. However, unique continuation does not hold for surfaces with the regularity of Theorem 2.0.1 (e.g. a multiplicity two minimal surface can suddenly split into two prescribed mean curvature surfaces).

Ignoring the full power of our regularity theory, we overcome the unique continuation issue by finding situations in which min-max solutions and replacement varifolds have density one everywhere. In this case, the regularity of Theorem 2.0.1 is the same as in the Almgren–Pitts prescribed mean curvature setting (e.g. smooth stable surfaces with prescribed mean curvature). Unique continuation then applies (assuming the prescribing function is sufficiently nice).

To achieve density one, we make two observations about varifolds in the round 3-sphere. First, a varifold limit of a min-max sequence for the $\mathcal{A}^h$ functional has a mass upper bound (in terms of a bound on the prescribing functional) by comparing with the optimal sweepout for the area functional. Second, a monotonicity formula for the Willmore energy in Euclidean space due to [Sim93] and [Top98] (suitably modified for the varifold setting) gives a lower bound for the mass of a varifold in $\mathbb{S}^3 \subset \mathbb{R}^4$ with a point of density at least two (in terms of a bound on the first variation). These

two bounds contradict when the prescribing function is bounded in absolute value by 0.547.

Another essential ingredient in both the regularity theory and genus bounds is a strong compactness theory for minimisers. In the more classical settings, this strong compactness follows from curvature estimates for stable minimal surfaces. Since we only expect $C^{1,1}$ regularity in general, this approach requires special attention. In fact, there is an important and subtle step in the standard point picking contradiction argument for the curvature estimates of stable minimal surfaces that uses the Schauder estimates to upgrade from $C^{1,\alpha}$ convergence to C^2 convergence. This approach manifestly fails in our setting, because minimisers are not even C^2 . Instead, we sharpen the regularity improvement from the n -membrane problem to obtain a $C^{1,1}$ estimate that allows us to close the usual point picking argument. This approach supplies a sufficiently strong compactness theorem for the rest of the regularity theory and genus bounds.

2.4 Outline

In §2.5, we establish notations and conventions for the rest of Part I of this thesis. In §2.6, we carry out the details of two illustrative examples. In chapter 5, we modify the arguments of [AS79] (e.g. minimising area over discs) to the prescribed mean curvature setting. In chapter 4, we modify the arguments of [MSY82] (e.g. minimising over isotopy classes) to the prescribed mean curvature setting, and prove Theorem 2.0.1. In chapter 5, we develop a strong compactness theory for minimisers with only $C^{1,1}$ regularity. In chapter 6, we apply the above regularity theory in the Simon–Smith min-max program to prove Theorems 2.1.3 and 2.1.5.

2.5 Notation and conventions

- $\mathbb{B}$ denotes $B_1^{\mathbb{R}^3}(0)$.
- $\mathbb{D}$ denotes $\overline{B}_1^{\mathbb{R}^2}(0)$.
- $G(U, 2)$ denotes the Grassmannian of tangent 2-planes over the set $U \subset M$.
- $\mathcal{I}(U)$ is the set of isotopies $\phi : [0, 1] \times M \rightarrow M$ satisfying

$$\phi(0, \cdot) = \text{id} \quad \text{and} \quad \phi(t, \cdot)|_{M \setminus U} = \text{id}.$$

- $\mathcal{H}^n$ denotes the n -dimensional Hausdorff measure on (M, g) .
- $\mathcal{A}^h$ denotes the h -prescribed mean curvature functional; namely, for $h : M \rightarrow \mathbb{R}$ and a set of finite perimeter $\Omega \subset M$, we define

$$\mathcal{A}^h(\Omega) := \mathcal{H}^2(\partial\Omega) - \int_{\Omega} h \, d\mathcal{H}^3.$$

- $\mathbf{v}(N, m)$ denotes the varifold associated to a C^1 surface $N \subset M$ with multiplicity $m \in \mathbb{N}$.
- $\delta V(X)$ is the first variation of a varifold V with respect to the vector field X . In the case V has bounded first variation, we use the convention

$$\delta V(X) = \int_{G(M, 2)} \text{div}_{\pi} X \, dV(x, \pi) = \int_M X \cdot H_V \, d\|V\|,$$

where H_V is the generalised mean curvature vector of V . We note that with this convention, the unit round sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ has mean curvature pointing out of the unit ball with length 2.

- $V_k \rightharpoonup V$ denotes convergence in the sense of varifolds.

- $\mu_k \xrightarrow{*} \mu$ denotes weak star convergence in the sense of [Mag12, §4.3].
- $[\Sigma]$ denotes the integer 2-current associated to an oriented surface Σ .
- $[\Omega]$ denotes the integer 3-current associated to a set of finite perimeter Ω , oriented so that $\partial[\Omega]$ is the integer 2-current associated to $\partial\Omega$ with orientation given by the outward pointing unit normal.
- $\mathbb{M}(T)$ denotes the mass of an integral current.
- d_U denotes the distance function from U in M .
- $U(\theta)$ denotes the set $\{x \in M \mid d_U(x) < \theta\}$.
- U_ξ denotes the set $\{x \in U \mid d(x, \partial U) > \xi\}$.

2.6 Examples

2.6.1 Failure of prescribed mean curvature

Here we supply a general class of examples where the limit of any minimising sequence does not have prescribed mean curvature.

Let $M^3 = S^1 \times \Sigma$ where Σ is a closed surface. Let g be any metric on M , and $h : M \rightarrow \mathbb{R}$ any positive function. Let $\Omega_0 := I \times \Sigma \subset M$, where I is an open interval in the S^1 factor. Let Σ_1 and Σ_2 be the components of $\partial\Omega_0$, which are embedded surfaces in the same isotopy class as $\{0\} \times \Sigma \subset M$. We define

$$m_0 := \inf_{\phi \in \mathcal{I}(M)} \mathcal{H}^2(\phi(1, \Sigma_1)) = \inf_{\phi \in \mathcal{I}(M)} \mathcal{H}^2(\phi(1, \Sigma_2)).$$

By treating the area and volume terms separately, we see that

$$\inf_{\phi \in \mathcal{I}(M)} \mathcal{A}^h(\Omega_0) \geq 2m_0 - \int_M h.$$

Moreover, this lower bound can be achieved by taking a sequence satisfying

$$\mathbf{1}_{\phi_k(1, \Omega_0)} \xrightarrow{L^1} \mathbf{1}_M$$

and

$$\lim_{k \rightarrow \infty} \mathcal{H}^2(\phi_k(1, \Sigma_1)) = \lim_{k \rightarrow \infty} \mathcal{H}^2(\phi_k(1, \Sigma_2)) = m_0.$$

By [MSY82], the varifold limit of the boundaries is a multiplicity 2 stable minimal surface, and therefore does not have prescribed mean curvature $h > 0$.

2.6.2 Failure of smoothness

We give an example where the limit of any minimising sequence is not C^2 . This example is a prototype of the “stacked disc” minimisation problem used to prove regularity, which is outlined in the introduction. The failure of smoothness in this example therefore illustrates the essential reason for the failure of smoothness in general.

Let $U := B_1^{\mathbb{R}^2}(0) \times (-1, 1) \subset \mathbb{R}^3$. Let

$$\Omega_1 := B_1^{\mathbb{R}^2}(0) \times (-1, -\varepsilon) \quad \text{and} \quad \Omega_2 := B_1^{\mathbb{R}^2}(0) \times (\varepsilon, 1).$$

Proposition 2.6.1. *The problem*

$$\inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^c(\phi(1, \Omega_1))$$

has a unique minimiser whose boundary in U is the upward curved spherical cap of radius $2/c$ with boundary $\partial U \cap \{x_3 = -\varepsilon\}$.

Proof. We use the calibration technique, see [CM11, Lemma 1.1.1].

Let Σ^* denote the surface of the upward curved spherical cap of radius $2/c$ with

boundary $\partial U \cap \{x_3 = -\varepsilon\}$. Let $\Omega_1^* \subset U$ be the set below Σ^* in U . It is clear that $\Omega_1^* = \phi^*(1, \Omega_1)$ for some $\phi^* \in \mathcal{I}(U)$.

Note that Σ^* is the graph of a function u over $B_1^{\mathbb{R}^2}(0)$. Moreover, u satisfies the mean curvature equation

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = -c.$$

Let ω be the 2-form given by

$$\omega(X, Y) = \det(X, Y, \nu),$$

where

$$\nu := \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

By the computation from [CM11, Lemma 1.1.1], we have

$$\omega = \frac{dx \wedge dy - u_x dy \wedge dz - u_y dz \wedge dx}{\sqrt{1 + |\nabla u|^2}}$$

and

$$d\omega = \left[\left(\frac{-u_x}{\sqrt{1 + |\nabla u|^2}} \right)_x + \left(\frac{-u_y}{\sqrt{1 + |\nabla u|^2}} \right)_y \right] dx \wedge dy \wedge dz = c \, dx \wedge dy \wedge dz.$$

Note that $|\omega(X, Y)| \leq 1$ for any pair of orthogonal unit vectors, with equality if and only if $X, Y \in T\Sigma^*$.

Now let $\Omega'_1 = \phi(1, \Omega_1)$ be any competitor with $\phi \in \mathcal{I}(U)$. Let $\Sigma' = \partial\Omega'_1 \cap U$.

Suppose $\Sigma' \neq \Sigma^*$. Then by Stokes' theorem, we have

$$\begin{aligned}
\mathcal{A}^c(\Omega^*) &= \mathcal{H}^2(\Sigma^*) - c\mathcal{H}^3(\Omega^*) \\
&= \int_{\Sigma^*} \omega - \int_{\Omega^*} d\omega \\
&= \int_{\Sigma'} \omega - \int_{\Omega'} d\omega \\
&< \mathcal{H}^2(\Sigma') - c\mathcal{H}^3(\Omega') = \mathcal{A}^c(\Omega').
\end{aligned}$$

The conclusion follows. $\square$

By vertical reflection, the analogous statement holds for Ω_2 . Fix $c \leq 1$. Then for ε sufficiently small, we shall see that any minimiser for the problem

$$\inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^c(\phi(1, \Omega_1 \cup \Omega_2))$$

must have a minimal interface, and therefore cannot be C^2 .

We give a more precise description of minimisers.

Proposition 2.6.2. *Any minimiser for the problem*

$$\inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^c(\phi(1, \Omega_1 \cup \Omega_2)) \tag{2.6.1}$$

is graphical, symmetric under vertical reflection, and rotationally symmetric.

Proof. We first show that minimisers must be graphical. Let m denote the infimum for the above problem (2.6.1). Let $\Omega^* \subset U$ and V^* be the Caccioppoli limit and varifold limit respectively of a minimising sequence. Let Ω_1^* and Ω_2^* respectively be the limits of Ω_1 and Ω_2 under the isotopies of the minimising sequence. Then $\Omega^* = \Omega_1^* \cup \Omega_2^*$, and

$$m = \|V^*\|(U) - c\mathcal{H}^3(\Omega^*) \geq P(\Omega_1^*; U) + P(\Omega_2^*; U) - c\mathcal{H}^3(\Omega_1^*) - c\mathcal{H}^3(\Omega_2^*).$$

Suppose for contradiction that at least one of $\partial\Omega_1^* \cap U$ or $\partial\Omega_2^* \cap U$ is not graphical over $B_1^{\mathbb{R}^2}(0)$. For $\mathbf{x} \in B_1^{\mathbb{R}^2}(0)$, let $l_{\mathbf{x}}$ denote the vertical line $\{(\mathbf{x}, z) \mid z \in \mathbb{R}\}$. Consider new sets

$$\begin{aligned}\Omega'_1 &:= \{(\mathbf{x}, z) \mid -1 < z < \mathcal{H}^1(\overline{\Omega^*} \cap l_{\mathbf{x}})/2 - 1\}, \\ \Omega'_2 &:= \{(\mathbf{x}, z) \mid 1 - \mathcal{H}^1(\overline{\Omega^*} \cap l_{\mathbf{x}})/2 < z < 1\}.\end{aligned}$$

By Steiner symmetrization (see [Mag12, Theorem 14.4]), we have

$$\mathcal{H}^3(\Omega'_1 \cup \Omega'_2) = \mathcal{H}^3(\Omega^*) \quad \text{and} \quad P(\Omega'_1; U) + P(\Omega'_2; U) \leq P(\Omega_1^*; U) + P(\Omega_2^*; U),$$

where equality holds for the perimeters if and only if $\Omega_i^* \cap l_{\mathbf{x}}$ is connected for all $\mathbf{x}$. Hence, we have

$$P(\Omega'_1; U) + P(\Omega'_2; U) - c\mathcal{H}^3(\Omega'_1) - c\mathcal{H}^3(\Omega'_2) < m.$$

Since $\partial\Omega'_i \cap U$ is graphical over $B_1^{\mathbb{R}^2}(0)$ and $\partial\Omega'_i \setminus U = \partial\Omega_i^* \setminus U$ by construction, $\Omega' := \Omega'_1 \cup \Omega'_2$ is in the closure of the isotopy class of Ω (e.g. use a smooth kernel to approximate the boundary of Ω'_i by smooth discs with the same boundary), which yields a contradiction.

The proofs of symmetry under reflection and rotation follow similarly by the natural symmetrization procedures. Indeed, these steps are easier, as one can exploit the fact that minimisers are graphical and use the convexity of the function $\sqrt{1 + |x|^2}$.

□

In MATLAB, we programmed a discrete version of the $\mathcal{A}^1$ functional for rotationally and vertically symmetric graphs. Using MATLAB's `fmincon`, we computed minima for different values of ε , and observed that the minimisers were robust under random initial conditions for `fmincon`. In Figures 2.2, 2.3, and 2.4, we provide graph-

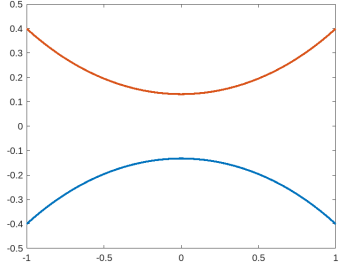


Figure 2.2: $\varepsilon = 0.4$

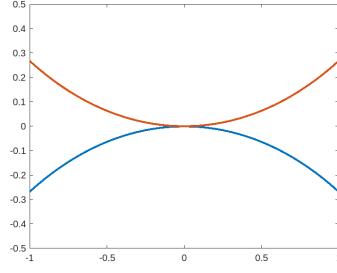


Figure 2.3: $\varepsilon = 2 - \sqrt{3}$

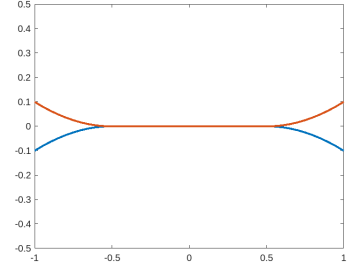


Figure 2.4: $\varepsilon = 0.1$

ics of the solution for values of ε where the two sheets are disjoint ($\varepsilon = 0.4$), where the two sheets touch at one point ($\varepsilon = 2 - \sqrt{3}$), and where a minimal interface forms ($\varepsilon = 0.1$).

Chapter 3

Minimisation over discs

We follow the argument of [AS79], with appropriate modifications to the manifold setting as in [MSY82].

3.1 Preliminaries

Let (M^3, g) be a connected closed¹ Riemannian 3-manifold. Let $h : M \rightarrow \mathbb{R}$ be a smooth function with $c := \sup_M |h|$. Before we formulate the minimisation problem, we need some preliminary facts about the prescribed mean curvature functional and discs in manifolds.

3.1.1 Local control

Since M is compact, there are constants $\rho_0 > 0$, $\mu > 0$, and $\beta_0 \geq 1$ so that the following hold:

- For each $x \in M$, the restriction of the domain and range of $\exp_x$ gives a diffeomorphism

$$\phi_x : \overline{B}_{\rho_0}^{T_x M}(0) \rightarrow \overline{B}_{\rho_0}^M(x).$$

¹As in [MSY82], we only need to assume that M is homogeneously regular. In this case, we should also assume that h is in L^∞ .

- For each $\xi \in \overline{B}_{\rho_0}^{T_x M}(0)$ and $y \in \overline{B}_{\rho_0}^M(x)$, we have

$$\|d_\xi \phi_x\|, \|d_y \phi_x^{-1}\| \leq 2. \quad (3.1.1)$$

- In geodesic normal coordinates x^1, x^2, x^3 for $B_{\rho_0}^M(x)$, we have

$$\sup_{\overline{B}_{\rho_0}^{T_x M}(0)} \left| \frac{\partial g_{ij}}{\partial x^k} \right| \leq \mu/\rho_0, \quad \sup_{\overline{B}_{\rho_0}^{T_x M}(0)} \left| \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} \right| \leq \mu/\rho_0^2.$$

- For any $\rho \in (0, \rho_0)$ and $x \in M$, we have

$$d_{B_\rho(x)} := \text{dist}(B_\rho(x), \cdot) \text{ is convex}$$

on $B_{\rho_0}(x) \setminus \overline{B_\rho(x)}$.

- For any $\rho \in (0, \rho_0)$ and $x \in M$, we have

$$\Delta d_{B_\rho(x)} \geq c$$

on $B_{\rho_0}(x) \setminus \overline{B_\rho(x)}$.

- For any $\rho \in (0, \rho_0)$ and $x \in M$, we have

$$\min\{\mathcal{H}^2(E), \mathcal{H}^2(\partial B_\rho(x) \setminus E)\} \leq \beta_0(\mathcal{H}^1(\partial E))^2$$

for any set of finite perimeter $E \subset \partial B_\rho(x)$.

- For any $0 < \rho_1 < \rho_2 \leq \rho_0$ and $x \in M$, we have

$$\mathcal{H}^2(\partial B_{\rho_1}(x)) \leq \mathcal{H}^2(\partial B_{\rho_2}(x)).$$

Remark 3.1.1. By rescaling the metric and prescribing function, we can assume $\rho_0 = 1$. We henceforth make this assumption for ease of notation.

3.1.2 Thin tube isoperimetric inequality

We make frequent use of the following Lemma from [MSY82].

Lemma 3.1.2 ([MSY82, Lemma 1]). *There is a constant $\delta \in (0, 1)$ independent of M and g so that if Σ is a smooth embedded surface in M and satisfies*

$$\mathcal{H}^2(\Sigma \cap B_1(x)) < \delta^2$$

for all $x \in M$, then there is a unique compact $K_\Sigma \subset M$ with $\partial K_\Sigma = \Sigma$ and

$$\mathcal{H}^3(K_\Sigma \cap B_1(x)) \leq \delta^3$$

for all $x \in M$, and

$$\mathcal{H}^3(K_\Sigma) \leq c_1 \mathcal{H}^2(\Sigma)^{3/2},$$

where $c_1 = c_1(\mu)$. Moreover, if Σ is diffeomorphic to $\mathbb{S}^2$, then K_Σ is diffeomorphic to $\overline{\mathbb{B}}^2$.

3.1.3 Area comparison

The following lemma is the appropriate modification of [MSY82, Lemma 3] for our setting. Essentially, we use the divergence theorem to show that moving pieces of a set inside a c -mean convex domain decreases the $\mathcal{A}^c$ functional.

Lemma 3.1.3. *Let $U \subset B_{1/2}(x_0)$ and $\theta \in (0, 1/2)$ satisfy*

- $x_0 \in U$,

²The exponent of δ in (3.1.2) is written as 2 in the statement of [MSY82, Lemma 1], but a quick examination of the proof reveals that it should be 3.

- $\overline{U}$ is diffeomorphic to $\overline{\mathbb{B}}$,
- d_U is convex on $U(\theta) \setminus \overline{U}$,
- $\Delta d_U \geq c$ on $U(\theta) \setminus \overline{U}$.

Let $\beta \geq 1$ be a constant so that, for each $s \in (\theta/2, \theta)$,

$$\min\{\mathcal{H}^2(E), \mathcal{H}^2(\partial U(s) \setminus E)\} \leq \beta(\mathcal{H}^1(\partial E))^2$$

whenever E is a surface in $\partial U(s)$. Let

$$\delta_1 = \min \left\{ \delta, \frac{\theta}{(1 + 128c_1)\beta^{1/2}}, \frac{\theta}{9\beta^{1/2}(1/32 + cc_1)^{1/2}}, \frac{\mathcal{H}^3(M)^{1/3}}{(4c_1)^{1/3}}, 1 \right\},$$

(where c_1 and δ are as in Lemma 3.1.2). Let Σ be a smooth embedded disc intersecting ∂U transversally with $\partial \Sigma \subset M \setminus U$. Suppose

$$\mathcal{H}^2(\partial U) + \mathcal{H}^2(\Sigma \setminus U) \leq \delta_1^2/32.$$

If Λ is any component of $\Sigma \setminus U$ with $\partial \Sigma \cap \Lambda = \emptyset$, then there is a unique compact $K_\Lambda \subset M \setminus U$ so that

$$\mathcal{H}^3(K_\Lambda) \leq c_1 \delta_1^3, \quad \partial K_\Lambda = \Lambda \cup F,$$

where $F \subset \partial U$ is a surface with $\partial F = \partial \Lambda$ satisfying

$$\mathcal{H}^2(F) < \mathcal{H}^2(\Lambda \cap U(\theta)) - c\mathcal{H}^3(K_\Lambda).$$

Proof. Let $F_0 \subset \partial U$ satisfy $\partial F_0 = \partial \Lambda$, which exists because ∂U is diffeomorphic to a 2-sphere. Then by (3.1.3) and (3.1.3), we have

$$\mathcal{H}^2(F_0) + \mathcal{H}^2(\Lambda) < \delta^2.$$

By Lemma 3.1.2, there is a compact set $W \subset M$ with

$$\mathcal{H}^3(W) \leq c_1 \delta_1^3, \quad \partial W = F_0 \cup \Lambda.$$

Let $K_\Lambda = W \setminus U$, in which case F is either F_0 (if $W \cap U = \emptyset$) or $\partial U \setminus F_0$ (if $U \subset W$).

Since K_Λ must satisfy $\mathcal{H}^3(K_\Lambda) \leq \mathcal{H}^3(M)/4$ by the definition of δ_1 , K_Λ is unique.

Define $F_t = K_\Lambda \cap \partial U(t)$, $E_t = \Lambda \cap U(t)$, and $K_t = K_\Lambda \cap U(t)$. We compute

$$\begin{aligned} c\mathcal{H}^3(K_{t_2} \setminus \overline{K}_{t_1}) &\leq \int_{K_{t_2} \setminus \overline{K}_{t_1}} \Delta d_U \\ &= \mathcal{H}^2(F_{t_2}) - \mathcal{H}^2(F_{t_1}) + \int_{\overline{E}_{t_2} \setminus E_{t_1}} \langle \nu, \nabla d_U \rangle \\ &\leq \mathcal{H}^2(F_{t_2}) - \mathcal{H}^2(F_{t_1}) + \mathcal{H}^2(\overline{E}_{t_2}) - \mathcal{H}^2(E_{t_1}). \end{aligned}$$

Equality in (3.1.3) requires $E_{t_1} = E_{t_2}$, because the identity $\nu(y) = \nabla d_U(y)$ for almost every $y \in \overline{E}_{t_2} \setminus E_{t_1}$ implies $\overline{E}_{t_2} \setminus E_{t_1} \subset \partial U(s)$ for some s .

By setting $t_1 = 0$ and $t_2 = t > 0$ in (3.1.3), we get

$$\mathcal{H}^2(F) < \mathcal{H}^2(F_t) + \mathcal{H}^2(\overline{E}_t) - c\mathcal{H}^3(K_t).$$

If $K_\theta = K_\Lambda$, then we obtain (3.1.3) by setting $t = \theta$. Henceforth, we assume $K_\theta \neq K_\Lambda$.

By the coarea formula, we have

$$\int_0^\theta \mathcal{H}^2(F_s) ds \leq \mathcal{H}^3(K_\theta) \leq c_1 \delta_1^3.$$

Hence,

$$\mathcal{H}^2(F_t) \leq 4c_1 \theta^{-1} \delta_1^3$$

for a set of $t \in [0, \theta]$ of measure at least $\frac{3}{4}\theta$. Since $\delta_1 < (128c_1)^{-1}\theta$, we have

$$\mathcal{H}^2(F_t) \leq \delta_1^2/32$$

on a set of measure at least $\frac{3}{4}\theta$. Then there is a $t^* \in [\frac{3}{4}\theta, \theta]$ so that (3.1.3) holds at t^* . Then (3.1.3) and (3.1.3) imply

$$\mathcal{H}^2(F_t) \leq \delta_1^2/32 + \delta_1^2/32 = \delta_1^2/16$$

for all $t \leq \frac{3}{4}\theta$.

Since $B_{\theta/2}(x_0) \subset U(\theta/2)$ and $U(\theta) \subset B_1(x_0)$, (3.1.1) and the isoperimetric inequality in $\mathbb{R}^3$ (since the $\mathbb{R}^3$ isoperimetric constant is $\sim 4.8 \geq 4$) imply

$$\mathcal{H}^2(\partial U(t)) \geq \frac{1}{4}\mathcal{H}^2(\partial(\phi_{x_0}^{-1}(U(t)))) \geq (\mathcal{H}^3(\phi_{x_0}^{-1}(B_{\theta/2}(x_0))))^{2/3} \geq \theta^2/8$$

for $t \in [\theta/2, \theta]$. Since $\delta_1 \leq \theta$, (3.1.3) implies

$$\mathcal{H}^2(F_t) \leq \frac{1}{2}\mathcal{H}^2(\partial U(t))$$

for all $t \in [\theta/2, 3\theta/4]$. Then by (3.1.3) and the coarea formula, we have for almost every $t \in [\theta/2, 3\theta/4]$

$$\mathcal{H}^2(F_t) \leq \beta(\mathcal{H}^1(\partial F_t))^2 \leq \beta \left(\frac{d}{dt} \mathcal{H}^2(E_t) \right)^2.$$

Hence, (3.1.3) implies

$$\mathcal{H}^2(F) - \mathcal{H}^2(E_t) + c\mathcal{H}^3(K_\Lambda) \leq \beta \left(\frac{d}{dt} (\mathcal{H}^2(F) - \mathcal{H}^2(E_t) + c\mathcal{H}^3(K_\Lambda)) \right)^2$$

for almost every $t \in [\theta/2, 3\theta/4]$. By integration, we have

$$\sqrt{\mathcal{H}^2(F) - \mathcal{H}^2(E_{\theta/2}) + c\mathcal{H}^3(K_\Lambda)} - \sqrt{\mathcal{H}^2(F) - \mathcal{H}^2(E_{3\theta/4}) + c\mathcal{H}^3(K_\Lambda)} \geq \beta^{-1/2}\theta/8$$

provided $\mathcal{H}^2(F) - \mathcal{H}^2(E_{3\theta/4}) + c\mathcal{H}^3(K_\Lambda) > 0$. By (3.1.3), (3.1.3), and (3.1.3), we have

$$\begin{aligned} \sqrt{\mathcal{H}^2(F) + c\mathcal{H}^3(K_\Lambda)} &\leq \sqrt{\delta_1^2/32 + cc_1\delta_1^3} \\ &\leq \delta_1\sqrt{1/32 + cc_1} \\ &\leq \beta^{-1/2}\theta/9, \end{aligned}$$

so we reach a contradiction. Hence, we have

$$\mathcal{H}^2(F) \leq \mathcal{H}^2(E_{3\theta/4}) - c\mathcal{H}^3(K_\Lambda) < \mathcal{H}^2(E_\theta) - c\mathcal{H}^3(K_\Lambda),$$

as desired. □

3.2 Minimisation Problem

In this section, we rigorously formulate the problem of minimising the prescribed mean curvature functional over discs in a manifold.

3.2.1 Setup

Let c_1 and δ as in Lemma 3.1.2. As in the statement of Lemma 3.1.3, we define

$$\delta_2 := \min \left\{ \delta, \frac{\theta}{(1 + 128c_1)\beta_0^{1/2}}, \frac{\theta}{9\beta_0^{1/2}(1/32 + cc_1)^{1/2}}, \frac{\mathcal{H}^3(M)^{1/3}}{(4c_1)^{1/3}}, 1 \right\},$$

where β_0 is defined in (3.1.1).

Let $U \subset B_{1/2}(x_0)$ and $\theta \in (0, 1/2)$ satisfy

- $x_0 \in U$,
- $\overline{U}$ is diffeomorphic to $\overline{\mathbb{B}}$ (denoted by $\overline{U} \approx \overline{\mathbb{B}}$),
- d_U is convex on $U(\theta) \setminus \overline{U}$,
- $\Delta d_U \geq c$ on $U(\theta) \setminus \overline{U}$,
- $\mathcal{H}^2(\partial U) < \delta_2^2/32$.

Let β and δ_1 as in Lemma 3.1.3. Note that by (3.1.1), there are valid sets U so that $\delta_2 \leq \delta_1$ (e.g. we can take small geodesic balls).

We define the following notation.

- $\mathcal{D}$ is the set of oriented smooth embedded 2-discs $\Sigma \subset M$ with smooth boundary.
- $[\Sigma]$ is the integral 2-current corresponding to $\Sigma \in \mathcal{D}$.
- $\mathcal{D}[\Sigma]$ is the set of discs $\Sigma' \in \mathcal{D}$ with $\partial[\Sigma'] = \partial[\Sigma]$.
- $\mathcal{D}(U)$ is the set of discs $\Sigma \in \mathcal{D}$ so that
 - $\partial\Sigma \cap U = \emptyset$,
 - Σ has transverse (or empty) intersection with ∂U ,
 - there is an open set $\Omega \subset U$ so that $\partial[\Omega] \llcorner U = [\Sigma] \llcorner U$.
- $\mathcal{D}^*(U)$ is the set of discs $\Sigma \in \mathcal{D}(U)$ so that
 - $\mathcal{H}^2(\partial U) + \mathcal{H}^2(\Sigma \setminus U) \leq \delta_1^2/32$,
 - $\partial\Sigma \setminus \partial U$ is not contained in K_Λ for any component Λ of $\Sigma \setminus U$ with $\partial\Sigma \cap \Lambda = \emptyset$, where K_Λ is as in Lemma 3.1.3.
- $\mathcal{D}(U)[\Sigma]$ is the set of $\Sigma' \in \mathcal{D}(U) \cap \mathcal{D}[\Sigma]$ satisfying
 - $\Sigma' \setminus U \subset \Sigma \setminus U$,

- if $x \in \Sigma' \setminus U$, then Σ and Σ' have the same orientation at x .

We note that if $\Sigma \in \mathcal{D}^*(U)$ then $\mathcal{D}(U)[\Sigma] \subset \mathcal{D}^*(U)$.

3.2.2 Fill-ins

The prescribed mean curvature functional requires integration over a region. We formulate these regions here.

Proposition 3.2.1. *Let $\Sigma \in \mathcal{D}^*(U)$ and $\Sigma_1, \Sigma_2 \in \mathcal{D}(U)[\Sigma]$. There is a unique integral 3-current $T(\Sigma_1, \Sigma_2)$ satisfying*

$$\partial T(\Sigma_1, \Sigma_2) = [\Sigma_1] - [\Sigma_2] \quad \text{and} \quad \text{spt}(T(\Sigma_1, \Sigma_2)) \subset \overline{U} \cup \bigcup_{\Lambda} K_{\Lambda},$$

where Λ are the components of $\Sigma \setminus U$ with $\Lambda \cap \partial \Sigma = \emptyset$, and K_{Λ} is the corresponding set from Lemma 3.1.3.

Proof. Let $K := \overline{U} \cup \bigcup_{\Lambda} K_{\Lambda}$. Note that there is a 3-current S with $\text{spt}(S) \subset K$ obtained by adding $\pm[K_{\Lambda}]$ for each $\Lambda \subset \Sigma_i$ for some i with the appropriate sign so that

$$\text{spt}([\Sigma_1] - [\Sigma_2] - \partial S) \subset \overline{U}.$$

Since $\overline{U}$ is diffeomorphic to $\overline{\mathbb{B}}$ and $H_2(\overline{\mathbb{B}}, \mathbb{Z}) = 0$, there is a current T with support in $\overline{\mathbb{B}}$ so that

$$[\Sigma_1] - [\Sigma_2] = \partial(S + T).$$

We take $T(\Sigma_1, \Sigma_2) := S + T$. The uniqueness follows by the fact there are no nonzero closed 3-currents in M with support in K . $\square$

Definition 3.2.2. We call $T(\Sigma_1, \Sigma_2)$ from Proposition 3.2.1 the *fill-in* of Σ_1 and Σ_2 .

We let $F : I_3(M, \mathbb{Z}) \rightarrow L^1(M, \mathbb{Z})$ denote the canonical isomorphism of normed linear

spaces, and we define the *fill-in integral* as

$$T_h(\Sigma_1, \Sigma_2) := \int_M h F(T(\Sigma_1, \Sigma_2)) d\mathcal{H}^3.$$

Proposition 3.2.3. *Let $\Sigma \in \mathcal{D}^*(U)$. For $\Sigma_1, \Sigma_2, \Sigma_3 \in \mathcal{D}(U)[\Sigma]$, we have*

$$T(\Sigma_1, \Sigma_2) = -T(\Sigma_2, \Sigma_1),$$

$$T_h(\Sigma_1, \Sigma_2) = -T_h(\Sigma_2, \Sigma_1),$$

$$T(\Sigma_1, \Sigma_2) + T(\Sigma_2, \Sigma_3) = T(\Sigma_1, \Sigma_3),$$

$$T_h(\Sigma_1, \Sigma_2) + T_h(\Sigma_2, \Sigma_3) = T_h(\Sigma_1, \Sigma_3).$$

Proof. We compute

$$\partial(-T(\Sigma_2, \Sigma_1)) = -[\Sigma_2] + [\Sigma_1].$$

By the uniqueness of $T(\Sigma_1, \Sigma_2)$, (3.2.3) follows.

We compute

$$\partial(T(\Sigma_1, \Sigma_2) + T(\Sigma_2, \Sigma_3)) = [\Sigma_1] - [\Sigma_2] + [\Sigma_2] - [\Sigma_3] = [\Sigma_1] - [\Sigma_3].$$

By the uniqueness of $T(\Sigma_1, \Sigma_3)$, (3.2.3) follows. □

3.2.3 Problem formulation

In this part of the thesis, we consider the following question.

Question 3.2.4. *If $\{\Sigma_k\}_{k \in \mathbb{N}} \subset \mathcal{D}^*(U)$ satisfies*

$$\mathcal{H}^2(\Sigma_k) \leq \mathcal{H}^2(\Sigma'_k) + T_h(\Sigma_k, \Sigma'_k) + \varepsilon_k$$

for all $\Sigma'_k \in \mathcal{D}(U)[\Sigma_k]$ with $\varepsilon_k \rightarrow 0$, does (a subsequence of) Σ_k converge to a regular limit with prescribed mean curvature h ?

3.3 Limits of minimising sequences

We apply the standard theory for sets of finite perimeter to deduce some basic preliminary conclusions about the limit in Question 3.2.4. In later sections we upgrade these basic observations to stronger regularity results.

Theorem 3.3.1. *Suppose $\{\Sigma_k\}_{k \in \mathbb{N}} \subset \mathcal{D}^*(U)$ satisfies*

$$\mathcal{H}^2(\Sigma_k) \leq \mathcal{H}^2(\Sigma'_k) + T_h(\Sigma_k, \Sigma'_k) + \varepsilon_k$$

for all $\Sigma'_k \in \mathcal{D}(U)[\Sigma_k]$ with $\varepsilon_k \rightarrow 0$. Let $\Omega_k \subset U$ be the open set satisfying

$$\partial[\Omega_k] \llcorner U = [\Sigma_k] \llcorner U.$$

If $\sup_k \mathcal{H}^2(\Sigma_k \cap U) < \infty$, then there is a subsequence (not relabeled), a set of finite perimeter $\Omega \subset U$, and a varifold V in U so that

$$\mathbf{1}_{\Omega_k} \xrightarrow{L^1} \mathbf{1}_\Omega, \quad D\mathbf{1}_{\Omega_k} \xrightarrow{*} D\mathbf{1}_\Omega, \quad \mathbf{v}(\Sigma_k \cap U) \rightharpoonup V.$$

Furthermore,

- Ω and V satisfy

$$\|V\|(U) - \int_\Omega h \, d\mathcal{H}^3 \leq \|\phi_\# V\|(U) - \int_{\phi(\Omega)} h \, d\mathcal{H}^3$$

for any diffeomorphism ϕ of U equal to the identity outside a compact subset of U .

- the first variation of V satisfies

$$\delta V(X) = \int_{\partial^* \Omega} X \cdot h \nu_{\partial^* \Omega} d|D\mathbf{1}_\Omega|$$

for any C^1 vector field X with $\text{spt}(X) \subset\subset U$.

- V has c -bounded first variation, in the sense that

$$\|\delta V\|(B_r(x)) \leq c\|V\|(B_r(x))$$

for all $B_r(x) \subset U$.

- The function

$$r \mapsto f(r) \frac{\|V\|(B_r(x))}{\pi r^2}$$

is monotone increasing in r , where f is a continuous function depending only on c and (M, g) satisfying $f(r) \rightarrow 1$ as $r \rightarrow 0$.

Proof. The subsequential convergence follows by the standard compactness theorems (see [Mag12, Theorem 12.26] and [Sim83, chapter 1, Theorem 4.4]).

Let ϕ be a diffeomorphism of U equal to the identity outside a compact subset. The assumption of transverse intersection implies

1. $\lim_k \mathcal{H}^2(\Sigma_k \cap U) = \|V\|(U),$
2. $\lim_k \int_{\Omega_k} h d\mathcal{H}^3 = \int_{\Omega} h d\mathcal{H}^3,$
3. $\lim_k \mathcal{H}^2(\phi(\Sigma_k) \cap U) = \|\phi_{\#} V\|(U),$
4. $\lim_k \int_{\phi(\Omega_k)} h d\mathcal{H}^3 = \int_{\phi(\Omega)} h d\mathcal{H}^3.$

Let $\delta > 0$. Then there exists k sufficiently large so that

$$\begin{aligned}
\|V\|(U) - \int_{\Omega} h \, d\mathcal{H}^3 &\leq \mathcal{H}^2(\Sigma_k \cap U) - \int_{\Omega_k} h \, d\mathcal{H}^3 + 2\delta & (1) + (2) \\
&\leq \mathcal{H}^2(\phi(\Sigma_k) \cap U) - \int_{\phi(\Omega_k)} h \, d\mathcal{H}^3 + 3\delta & (3.3.1) \\
&\leq \|\phi_{\#}V\|(U) - \int_{\phi(\Omega)} h \, d\mathcal{H}^3 + 5\delta & (3) + (4).
\end{aligned}$$

Since δ is arbitrary, (3.3.1) follows.

For a C^1 vector field X with $\text{spt}(X) \subset\subset U$, (3.3.1) and [Mag12, Proposition 17.8] imply

$$\begin{aligned}
\delta V(X) &= \left. \frac{d}{dt} \right|_{t=0} \|(\phi_t)_{\#}V\|(U) \\
&= \left. \frac{d}{dt} \right|_{t=0} \int_{\phi_t(\Omega)} h \, d\mathcal{H}^3 \\
&= \int_{\partial^* \Omega} X \cdot h \nu_{\partial^* \Omega} \, d|D\mathbf{1}_{\Omega}|,
\end{aligned}$$

where ϕ_t is the flow of X . Hence, (3.3.1) holds.

(3.3.1) follows from (3.3.1) and [Mag12, Proposition 4.30].

Finally, the monotonicity of (3.3.1) follows from (3.3.1) by [Sim83, chapter 8, section 40]³. □

3.4 Resolution of overlaps

We prove the appropriate modification of [AS79, Lemma 2 and Corollary 1] in our setting, to disentangle overlapping discs. The key difference is that we must keep track of volumes in addition to area.

³We note that [Sim83] only finds a monotonicity formula for varifolds in $\mathbb{R}^N$. We can isometrically embed (M, g) in some $\mathbb{R}^N$ and use this monotonicity formula, in which case the function f will depend on the norm of the second fundamental form of the embedding.

3.4.1 Lipschitz discs

As in [AS79], we define a specific class of Lipschitz discs. Fix an open set $U \subset M$.

Definition 3.4.1. Consider n disjoint copies $\{F_i\}_{i=1}^n$ of $\mathbb{D}$ and define $\mathcal{Y}$ to be the collection of Lipschitz maps $\chi : \bigsqcup_{i=1}^n F_i \rightarrow U$ such that:

1. there are finitely many (or zero) pairwise disjoint C^2 Jordan curves $\Gamma_1, \dots, \Gamma_m \subset U$ such that for each $k = 1, \dots, m$,

$$\chi^{-1}(\Gamma_k) = \gamma_k^1 \cup \gamma_k^2,$$

where $\{\gamma_k^l : k = 1, \dots, m; l = 1, 2\}$ is a collection of $2m$ pairwise disjoint C^2 Jordan curves in $\bigsqcup_{i=1}^n (F_i \setminus \partial F_i)$, and $\chi|_{\gamma_k^l}$ is a C^2 parametrization of Γ_k .

2. the restriction of χ to $(\bigsqcup_{i=1}^n F_i) \setminus \bigcup_{k=1}^m (\gamma_k^1 \cup \gamma_k^2)$ is a C^2 embedding into U .
3. for each $k = 1, \dots, m$, there exist disjoint tubular neighbourhoods W_k^1, W_k^2 of γ_k^1, γ_k^2 respectively, such that $\chi(W_k^1) \cap \chi(W_k^2) = \Gamma_k$ and exactly one of the following holds:

(3a) $\chi(W_k^1)$ lies on one side of $\chi(W_k^2)$,

(3b) $\chi(W_k^1)$ intersects $\chi(W_k^2)$ transversely along Γ_k .

For future convenience, let us introduce the following notation. For each $k = 1, \dots, m$ and $l = 1, 2$, let $W_k^{l,+} = (W_k^l \cap \text{int}(\gamma_k^l)) \setminus \gamma_k^l$ and $W_k^{l,-} = W_k^l \setminus (W_k^{l,+} \cup \gamma_k^l)$, so that $W_k^l = W_k^{l,+} \cup \gamma_k^l \cup W_k^{l,-}$.

Finally, define $\mathcal{Y}_0$ to be the collection of Lipschitz maps $\chi \in \mathcal{Y}$ such that only (3a) occurs for all $k = 1, \dots, m$. These are the “disentangled” discs, so our goal will be to change a map in $\mathcal{Y}$ into a map in $\mathcal{Y}_0$.

Remark 3.4.2. As in [AS79], notice that $\chi(\bigsqcup_{i=1}^n F_i \setminus \partial F_i) \setminus \bigcup_{i=1}^m \Gamma_i$ is a 2-dimensional open submanifold of M , each component of which has closure of the form $\chi(\overline{R})$, where R is a component of $(\bigsqcup_{i=1}^n F_i) \setminus \bigcup_{k=1}^m (\gamma_k^1 \cup \gamma_k^2)$.

We will say a finite collection $\mathcal{S}$ of Lipschitz discs is in $\mathcal{Y}$ (or $\mathcal{Y}_0$), and write $\mathcal{S} \in \mathcal{Y}$ (or $\mathcal{S} \in \mathcal{Y}_0$), if it is the image of some map $\chi \in \mathcal{Y}$ (or $\mathcal{Y}_0$, respectively).

3.4.2 Resolution procedure

We develop a procedure to disentangle intersecting discs by swapping pieces without changing the image of the union of the discs.

Lemma 3.4.3 (Resolution of Overlaps). *Let $U \subset M$ be a bounded connected open set with C^1 boundary. Let $\mathcal{S} = \{D_i\}_{i=1}^n \in \mathcal{Y}$. Then there are Lipschitz discs $\tilde{\mathcal{S}} = \{\tilde{D}_i\}_{i=1}^n \in \mathcal{Y}_0$ with $\tilde{D}_i \subset U$ and $\partial \tilde{D}_i = \partial D_i$ satisfying*

$$\bigcup_{i=1}^n D_i = \bigcup_{i=1}^n \tilde{D}_i.$$

Proof. We use notation as in Definition 3.4.1.

For each intersection curve Γ_k , exactly one of (3a) and (3b) in Definition 3.4.1 holds. Let

$$\mathcal{B}(\mathcal{S}) = \{1 \leq k \leq m \mid (3b) \text{ occurs}\}.$$

If $\mathcal{B}(\mathcal{S}) = \emptyset$, then $\mathcal{S} \in \mathcal{Y}_0$ and we are done. Hence, it suffices by induction to furnish a procedure which replaces the collection $\mathcal{S}$ of discs with another collection $\tilde{\mathcal{S}} \in \mathcal{Y}$ such that:

- $\bigcup_i D_i = \bigcup_i \tilde{D}_i$,
- $\partial \tilde{D}_i = \partial D_i$,
- $|\mathcal{B}(\tilde{\mathcal{S}})| < |\mathcal{B}(\mathcal{S})|$.

Let $q \in \mathcal{B}(\mathcal{S})$. There are (without loss of generality) two cases to consider.

Case 1: $\text{int}\gamma_q^1 \cap \text{int}\gamma_q^2 = \emptyset$. Fix a C^2 diffeomorphism

$$\psi_q : \overline{\text{int}\gamma_q^1} \rightarrow \overline{\text{int}\gamma_q^2}$$

so that $(\chi \circ \psi_q)|_{\gamma_q^1} = \chi|_{\gamma_q^2}$. Then we define a bijection $\Psi_q : \bigsqcup_i F_i \rightarrow \bigsqcup_i F_i$ by

$$\Psi_q(\xi) := \begin{cases} \psi_q(\xi) & \xi \in \overline{\text{int}\gamma_q^1} \\ \psi_q^{-1}(\xi) & \xi \in \overline{\text{int}\gamma_q^2} \\ \xi & \text{otherwise.} \end{cases}$$

Case 2: $\gamma_q^2 \subset \text{int}(\gamma_q^1)$. Fix a C^2 diffeomorphism

$$\psi_q : \overline{\text{int}\gamma_q^1} \setminus \text{int}\gamma_q^2 \rightarrow \overline{\text{int}\gamma_q^1} \setminus \text{int}\gamma_q^2$$

so that $(\chi \circ \psi_q)|_{\gamma_q^1} = \chi|_{\gamma_q^{l+1}}$ (where the superscripts are taken mod 2 for convenience).

Then we define a bijection $\Psi_q : \bigsqcup_i F_i \rightarrow \bigsqcup_i F_i$ by

$$\Psi_q(\xi) := \begin{cases} \psi_q(\xi) & \xi \in \overline{\text{int}\gamma_q^1} \setminus \text{int}\gamma_q^2 \\ \xi & \text{otherwise.} \end{cases}$$

We define $\tilde{\chi} := \chi \circ \Psi_q : \bigsqcup_i F_i \rightarrow M$. By the boundary assumption, in either case, $\tilde{\chi}$ is a Lipschitz map. Since Ψ_q is a bijection, $\tilde{\chi}$ has the same image as χ . Moreover, $\tilde{\chi}$ agrees with χ in a neighbourhood of the boundaries of the F_i . Note that the restriction

$$\Psi_q : \left(\bigsqcup_i F_i \right) \setminus (\gamma_q^1 \cup \gamma_q^2) \rightarrow \left(\bigsqcup_i F_i \right) \setminus (\gamma_q^1 \cup \gamma_q^2)$$

is a C^2 diffeomorphism. We check each component of Definition 3.4.1:

1. Since Ψ_q is a bijection, we have $\tilde{\chi}^{-1}(\Gamma_k) = \tilde{\gamma}_k^1 \cup \tilde{\gamma}_k^2$, where $\tilde{\gamma}_k^l := \Psi_q^{-1}(\gamma_k^l)$ are disjoint sets. Since Ψ_q is a C^2 diffeomorphism away from $\gamma_q^1 \cup \gamma_q^2$, $\tilde{\gamma}_k^l$ is a C^2 Jordan curve for all $k \neq q$ and $\tilde{\chi}|_{\tilde{\gamma}_k^l}$ is a C^2 parametrization. Finally, it follows from the construction that $\tilde{\gamma}_q^l = \gamma_q^{l+1}$ and $\tilde{\chi}|_{\tilde{\gamma}_q^l}$ is a C^2 parametrization.
2. Since Ψ_q is a C^2 diffeomorphism away from $\tilde{\gamma}_q^1 \cup \tilde{\gamma}_q^2$, the restriction of $\tilde{\chi}$ to any component of $(\bigsqcup_i F_i) \setminus (\bigsqcup_k \tilde{\gamma}_k^1 \cup \tilde{\gamma}_k^2)$ is a C^2 embedding.
3. For $k \neq q$, the desired tubular neighbourhoods are $\tilde{W}_k^l = \Psi_q^{-1}(W_k^l)$. Since $\tilde{\chi}(\tilde{W}_k^l) = \chi(W_k^l)$, (3a) (resp. (3b)) holds at $k \neq q$ for $\tilde{\chi}$ if and only if (3a) (resp. (3b)) holds at k for χ . We now define $\tilde{W}_q^l$, which differ in the above cases.

- In Case 1, we have

$$\tilde{W}_q^l = \Psi_q^{-1}(W_q^{l,-} \cup \gamma_q^{l+1} \cup W_q^{l+1,+}).$$

Then

$$\tilde{\chi}(\tilde{W}_q^l) = \chi(W_q^{l,-}) \cup \Gamma_q \cup \chi(W_q^{l+1,+})$$

- In Case 2, we have

$$\tilde{W}_q^1 = \Psi_q^{-1}(W_q^{1,-} \cup \gamma_q^2 \cup W_q^{2,-})$$

$$\tilde{W}_q^2 = \Psi_q^{-1}(W_q^{2,+} \cup \gamma_q^1 \cup W_q^{1,+}).$$

Then

$$\tilde{\chi}(\tilde{W}_q^1) = \chi(W_q^{1,-}) \cup \Gamma_q \cup \chi(W_q^{2,-})$$

$$\tilde{\chi}(\tilde{W}_q^2) = \chi(W_q^{2,+}) \cup \Gamma_q \cup \chi(W_q^{1,+}).$$

Since (3b) holds at q for χ , we see that (3a) holds at q for $\tilde{\chi}$.

Hence, $\tilde{\chi} \in \mathcal{Y}$. Moreover, it follows from (3) above that $|\mathcal{B}(\tilde{\mathcal{S}})| < |\mathcal{B}(\mathcal{S})|$. $\square$

Remark 3.4.4. The procedure described in the proof of Lemma 3.4.3 is not uniquely determined by the initial collection $\mathcal{S}$. However, we will see that this is not an issue in our applications.

After applying the resolution procedure, we will make small perturbations to produce genuinely embedded surfaces. We collect the required statements in the following lemma.

Lemma 3.4.5. *Let $U \subset M$ be an open set with $\overline{U} \approx \overline{\mathbb{B}}$. Let $\Sigma \in \mathcal{D}(U)$. Let $\{D_i\}_{i=1}^n$ be a set of smooth embedded discs with $D_i \subset U$, $\partial D_i \subset \partial U$, and*

$$\Sigma \cap U = \bigcup_{i=1}^n D_i.$$

Let $\{\tilde{D}_i\} \in \mathcal{Y}_0$ satisfy $\tilde{D}_i \subset U$ and $\partial[\tilde{D}_i] = \partial[D_i]$. Then there is an open set $\tilde{\Omega} \subset U$ satisfying

$$\partial[\tilde{\Omega}] \llcorner U = \sum_{i=1}^n [\tilde{D}_i].$$

Moreover, for any $\varepsilon > 0$ there is a $\Sigma' \in \mathcal{D}(U)[\Sigma]$ satisfying

$$\Sigma \Delta \Sigma' \subset U,$$

$$\mathcal{H}^2(\Sigma' \cap U) \leq \sum_{i=1}^n \mathcal{H}^2(\tilde{D}_i) + \varepsilon,$$

and

$$\mathbb{M}(T(\Sigma, \Sigma') - ([\Omega] - [\tilde{\Omega}])) \leq \varepsilon.$$

Proof. The existence of $\tilde{\Omega}$ follows from the existence of Ω and the fact that the boundary orientations of the discs agree.

The remaining conclusions follow from making small perturbations along the intersection curves of the discs $\{\tilde{D}_i\}$. $\square$

3.4.3 Application 1: Collapsing to the boundary

As a first application of the resolution procedure, we would like to ignore some components of a minimising sequence in a ball by collapsing those components to the boundary of the ball.

Lemma 3.4.6. *Let $U \subset M$ be an open set with $\bar{U} \approx \bar{\mathbb{B}}$. Let $\Sigma \in \mathcal{D}(U)$. Let $\mathcal{S}$ be a set of smooth embedded discs D with $D \subset U$, $\partial D \subset \partial U$, and*

$$\Sigma \cap U = \bigcup_{D \in \mathcal{S}} D.$$

Let $\mathcal{S}_0 \subset \mathcal{S}$. For $D \in \mathcal{S}_0$, let $F_D \subset \partial U$ be the isoperimetric region for ∂D , and let Λ_D be the region in U bounded by $D \cup F_D$. For any $\varepsilon > 0$, there is a $\Sigma' \in \mathcal{D}(U)[\Sigma]$ satisfying

$$\Sigma \Delta \Sigma' \subset U,$$

$$\mathcal{H}^2(\Sigma' \cap U) \leq \sum_{D \in \mathcal{S}_0} \mathcal{H}^2(F_D) + \sum_{D \in \mathcal{S} \setminus \mathcal{S}_0} \mathcal{H}^2(D) + \varepsilon,$$

and

$$\mathbb{M}(T(\Sigma, \Sigma')) \leq \sum_{D \in \mathcal{S}_0} \mathcal{H}^3(\Lambda_D) + \varepsilon.$$

Proof. To apply Lemma 3.4.3, we must first perturb the discs F_D . Let $\eta > 0$ so that $\mathcal{H}^3(B_\eta(\partial U)) < \varepsilon/2$. We define discs F'_D for $D \in \mathcal{S}_0$ so that the following hold:

- $F'_{D_1} \cap F'_{D_2} = \emptyset$ for $D_1 \neq D_2 \in \mathcal{S}_0$,
- F_{D_1} has transverse (or empty) intersection with D_2 for any $D_1 \in \mathcal{S}_0$ and $D_2 \in \mathcal{S} \setminus \mathcal{S}_0$,
- $F'_D \subset U$ and $\partial F'_D = \partial D$,

- $|\mathcal{H}^2(F_D) - \mathcal{H}^2(F'_D)| < \varepsilon/(2|\mathcal{S}_0|),$
- $F'_D \subset B_\eta(\partial U) \cap \Lambda_D.$

We can then apply Lemma 3.4.3 to the discs

$$\mathcal{S}' := (\mathcal{S} \setminus \mathcal{S}_0) \cup \{F'_D\}_{D \in \mathcal{S}_0},$$

producing a collection $\tilde{\mathcal{S}}$.

By Lemma 3.4.5 and the choice of η , it then suffices show (up to a set of $\mathcal{H}^3$ measure zero)

$$(\Omega \Delta \tilde{\Omega}) \setminus B_\eta(\partial U) \subset \bigcup_{D \in \mathcal{S}_0} \Lambda_D.$$

Let $x \in (\Omega \Delta \tilde{\Omega}) \setminus B_\eta(\partial U)$. Take a point

$$x_0 \in \partial U \setminus \left(\bigcup_{D \in \mathcal{S}_0} F_D \cup \bigcup_{D \in \mathcal{S}} \partial D \right).$$

We also assume x is not in any $D \in \mathcal{S}$ or $\tilde{D} \in \tilde{\mathcal{S}}$, which is a set of $\mathcal{H}^3$ measure zero. Let $\gamma : (0, 1] \rightarrow U$ be a smooth path having transverse intersection with any $D \in \mathcal{S}$ or $\tilde{D} \in \tilde{\mathcal{S}}$, satisfying $\lim_{t \rightarrow 0} \gamma(t) = x_0$ and $\gamma(1) = x$. Since x_0 and x are in the complement of $B_\eta(\partial U) \cap (\bigcup_{D \in \mathcal{S}_0} \bar{\Lambda}_D)$, which contains the discs F'_D , we conclude that γ intersects each F'_D for $D \in \mathcal{S}_0$ an even number of times. Since $x \in \Omega \Delta \tilde{\Omega}$ and $x_0 \notin \overline{\Omega \Delta \tilde{\Omega}}$, we conclude that γ intersects some $D \in \mathcal{S}_0$ an odd number of times. Since $x_0 \notin \bar{\Lambda}_D$, we have $x \in \Lambda_D$ for some $D \in \mathcal{S}_0$. $\square$

3.4.4 Application 2: Isolating one stacked disc

Our goal in this application is to isolate individual discs in a stacked disc minimisation problem to deduce some (albeit weaker) regularity for each sheet of the limit varifold.

We first establish some notation.

Let $U \approx \mathbb{B}$ satisfy $\partial U = \overline{C} \cup \overline{D^+} \cup \overline{D^-}$, where C is a smoothly embedded annulus and D^+ and D^- are smoothly embedded discs. Let $\Sigma \in \mathcal{D}(U)$, in the sense that $\Sigma \cap \partial C = \emptyset$. Moreover, assume $\Sigma \cap U$ is a disjoint union of smooth embedded oriented discs $\{D_1, \dots, D_n\}$, each of which has homotopically nontrivial boundary in C .

For $i \in \{1, \dots, n\}$, let $B_i \subset U$ be the open subset satisfying

$$D^- \subset \partial B_i \setminus D_i \subset \partial U.$$

If $B_i \subset B_j$ for $i \neq j$, we say $D_i < D_j$. Without loss of generality, we suppose

$$D_1 > D_2 > \dots > D_n.$$

Let $\Omega_i \subset U$ be the open subset satisfying

$$\partial[\Omega_i] \lrcorner U = [D_i] \lrcorner U.$$

If $B_i = \Omega_i$, we say D_i is oriented up. Otherwise, we say D_i is oriented down. By the ordering of the D_i and the fact that $\Sigma \in \mathcal{D}(U)$, D_i is oriented up if and only if D_{i-1} and D_{i+1} are oriented down (i.e. the orientation alternates with respect to the ordering).

Lemma 3.4.7. *Fix some $1 \leq j \leq n$. Let $D'_j \in \mathcal{D}(U)[D_j]$ so that D'_j has transverse intersections with each D_i for $i \neq j$. For any $\varepsilon > 0$, there is a $\Sigma' \in \mathcal{D}(U)[\Sigma]$ satisfying*

$$\Sigma \Delta \Sigma' \subset U,$$

$$\mathcal{H}^2(\Sigma' \cap U) \leq \mathcal{H}^2(D'_j) + \sum_{i \neq j} \mathcal{H}^2(D_i) + \varepsilon,$$

and

$$\mathbb{M}(T(\Sigma, \Sigma') - [\Omega^+] + [\Omega^-]) \leq \varepsilon,$$

where Ω^+ and Ω^- are disjoint sets contained in $\Omega_j \triangle \Omega'_j$.

Moreover, instead of (3.4.7), we can construct Σ' satisfying

$$\mathbb{M}(T(\Sigma, \Sigma') - ([\Omega_j] - [\Omega'_j]) - [(\Omega'_j \setminus \Omega_j) \cap \Omega] + G) \leq \varepsilon.$$

where G is a nonnegative integral current, in the sense that $\int F(G)f \, d\mathcal{H}^3 \geq 0$ for all nonnegative f .

Proof. Step 1: We apply Lemma 3.4.3 to the discs $\mathcal{S}' := \{D'_j\} \cup \{D_i\}_{i \neq j}$, producing new discs $\tilde{\mathcal{S}} = \{\tilde{D}_i\}_{i=1}^n$.

We claim

$$[\Omega] - [\tilde{\Omega}] = [\Omega_j] - [\Omega'_j] + 2[(\Omega'_j \setminus \Omega_j) \cap \Omega] - 2[(\Omega_j \setminus \Omega'_j) \setminus \Omega],$$

which implies (3.4.7) by Lemma 3.4.5 and taking

$$\Omega^+ := (\Omega_j \triangle \Omega'_j) \cap \Omega, \quad \Omega^- := (\Omega_j \triangle \Omega'_j) \setminus \Omega.$$

We first observe

$$[\Omega] - [\tilde{\Omega}] = \sum_{i=1}^n [\Omega_i] - \sum_{i=1}^n [\tilde{\Omega}_i] = ([\Omega_j] - [\Omega'_j]) + \left([\Omega'_j] + \sum_{i \neq j} [\Omega_i] - \sum_{i=1}^n [\tilde{\Omega}_i] \right).$$

Let

$$\tilde{N} := \sum_{i=1}^n \mathbf{1}_{\tilde{\Omega}_i}, \quad N' := \mathbf{1}_{\Omega'_j} + \sum_{i \neq j} \mathbf{1}_{\Omega_i}.$$

To show (3.4.4), it suffices to show for $\mathcal{H}^3$ -a.e. $x \in U$ that

$$N'(x) - \tilde{N}(x) = \begin{cases} 2 & x \in (\Omega'_j \setminus \Omega_j) \cap \Omega \\ -2 & x \in (\Omega_j \setminus \Omega'_j) \setminus \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Let R be the component of $U \setminus \bigcup_i D_i$ containing x . Let $x_0 \in C \cap \overline{R}$. Let $\gamma : (0, 1] \rightarrow U$ be a path in R so that

- γ avoids the intersection curves $\{\Gamma_k\}$,
- γ has transverse intersection with D'_j ,
- $\lim_{t \rightarrow 0} \gamma(t) = x_0$ and $\gamma(1) = x$.

We have $N'(x_0) = \tilde{N}(x_0)$. Along subsequent intersections of γ with D'_j , each of N' and $\tilde{N}$ will increase and decrease by 1 in alternating order.

If γ has an even number of intersections with D'_j , then we see $N'(x) = \tilde{N}(x)$. Hence, we only need to consider the case of an odd number of intersections, which is equivalent to $x \in \Omega_j \triangle \Omega'_j$. In this case, N' increases at the first intersection if and only if $x \in \Omega'_j$. Note that $\tilde{N}$ decreases at the first intersection if and only if $R \subset \Omega$. Hence, (3.4.4) follows.

Step 2: We further modify the discs $\tilde{D}_i$ to $\tilde{D}'_i$ so that (up to a current with arbitrarily small mass)

$$[\tilde{\Omega}] - [\tilde{\Omega}'] = -[(\Omega'_j \setminus \Omega_j) \cap \Omega] - G,$$

where G is a nonnegative integral current, which implies (3.4.7) by (3.4.4) and Lemma 3.4.5.

Let R be a component of $(\Omega'_j \setminus \Omega_j) \cap \Omega$. We collect the following facts.

1. $R \cap \tilde{\Omega} = \emptyset$: a direct consequence of (3.4.4).
2. $\partial R \cap \partial U = \emptyset$: follows from $\partial \Omega_j \cap \partial U = \partial \Omega'_j \cap \partial U$.
3. R is a component of $U \setminus \bigcup_i \tilde{D}_i$: Let $x, y \in R$. By definition, there is a path α from x to y contained in both Ω and in Ω'_j , so α does not intersect any of the discs $\{D_i\}$ nor D'_j . Moreover, since $D_j \cap \overline{\Omega \setminus \Omega_j} = \emptyset$, we have $\partial R \subset \bigcup_i \tilde{D}_i$.

Let $\tilde{\chi} : \bigcup_{i=1}^n F_i \rightarrow M$ be the Lipschitz map corresponding to the resolved discs $\tilde{\mathcal{S}}$. Let $\{\sigma_l\}_l \subset \{\Gamma_k\}_k$ be the components of $\partial(\tilde{\chi}^{-1}(\partial R))$. Choose some $\sigma \in \{\sigma_l\}$ so that $\sigma \not\subset \text{int}(\sigma_l)$ for any l . By Definition 3.4.1, there is a unique distinct curve $\sigma' \in \{\Gamma_k\}_k$ so that $\tilde{\chi}(\sigma) = \tilde{\chi}(\sigma')$. There are two cases.

Case 1: $\text{int}\sigma \cap \text{int}\sigma' = \emptyset$. Since $d_1 := \tilde{\chi}(\overline{\text{int}\sigma})$ and $d_2 := \tilde{\chi}(\overline{\text{int}\sigma'})$ are Lipschitz discs with a common boundary and that do not cross through each other, there is a unique open set $R' \subset U$ with $\partial R' = d_1 \cup d_2$.

We claim that the discs $\{\tilde{D}_i\}$ do not intersect R' . Suppose otherwise for contradiction. Then for some i there is an $x \in F_i \setminus (\overline{\text{int}\sigma \cup \text{int}\sigma'})$ satisfying $\tilde{\chi}(x) \in R'$. Let α be a path in $F_i \setminus (\overline{\text{int}\sigma \cup \text{int}\sigma'})$ from x to ∂F_i . Then we have $\tilde{\chi}(\alpha(0)) \in R'$ and $\tilde{\chi}(\alpha(1)) \notin R'$. Since $\partial R' = d_1 \cup d_2$, we conclude that $\tilde{D}_i$ crosses through one of d_1 or d_2 , which contradicts $\tilde{\chi} \in \mathcal{V}_0$.

We claim that $R \subset R'$. We consider the local picture along $\tilde{\chi}(\sigma)$, where the 4 pieces $\tilde{\chi}(\text{int}(\sigma))$, $\tilde{\chi}(\text{ext}(\sigma))$, $\tilde{\chi}(\text{int}(\sigma'))$, and $\tilde{\chi}(\text{ext}(\sigma'))$ intersect with an X cross-section. We have

- $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{ext}(\sigma))$ are adjacent: follows from $\tilde{\chi} \in \mathcal{V}_0$.
- $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{int}(\sigma'))$ are adjacent: follows from the definition of R' and the fact that $\{\tilde{D}_i\}$ do not intersect R' .

R' contains the component of $U \setminus \bigcup_i \tilde{D}_i$ contained between $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{int}(\sigma'))$ along this intersection. Since R touches $\tilde{\chi}(\text{int}(\sigma))$ and does not touch $\tilde{\chi}(\text{ext}(\sigma))$, fact

(3) above implies that R is the component of $U \setminus \bigcup_i \tilde{D}_i$ contained between $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{int}(\sigma'))$ along this intersection. Hence, $R \subset R'$.

We claim that $R' \cap \tilde{\Omega} = \emptyset$. Since $\tilde{\chi} \in \mathcal{Y}_0$, each component of R' touches the same side of each of d_1 and d_2 . By the definition of $\tilde{\Omega}$, we have either $R' \subset \tilde{\Omega}$ or $R' \cap \tilde{\Omega} = \emptyset$. By fact (1) above, we have $R \cap \tilde{\Omega} = \emptyset$, so $R' \cap \tilde{\Omega}^c \neq \emptyset$. Hence, $R' \cap \tilde{\Omega} = \emptyset$.

Collapse procedure: Relabel σ and σ' as σ_1 and σ_2 so that

$$\mathcal{H}^2(\tilde{\chi}(\text{int}\sigma_1)) \geq \mathcal{H}^2(\tilde{\chi}(\text{int}\sigma_2)).$$

Fix a C^2 diffeomorphism

$$\psi : \overline{\text{int}\sigma_1} \rightarrow \overline{\text{int}\sigma_2}$$

so that $(\tilde{\chi} \circ \psi)|_{\sigma_1} = \tilde{\chi}|_{\sigma_2}$. Then we define the map

$$\tilde{\chi}'(\xi) := \begin{cases} \tilde{\chi}(\psi(\xi)) & \xi \in \overline{\text{int}\sigma_1} \\ \tilde{\chi}(\xi) & \text{otherwise.} \end{cases}$$

We make an arbitrarily small perturbation of the map $\tilde{\chi}'$ on $\text{int}\sigma_1$ so that $\tilde{\chi}' \in \mathcal{Y}_0$.

Case 2: $\sigma' \subset \text{int}\sigma$. Since $A := \tilde{\chi}(\overline{\text{int}\sigma \setminus \text{int}\sigma'})$ is a Lipschitz annulus with a common boundary and that does not cross through itself, there is a unique open set $R' \subset U$ with $\partial R' = A$.

We claim that the discs $\{\tilde{D}_i\}$ do not intersection R' . Suppose otherwise for contradiction. By the argument from Case 1, we need only suppose that $\tilde{\chi}(\text{int}\sigma') \cap R' \neq \emptyset$. Since $\tilde{\chi} \in \mathcal{Y}_0$, we must have $\tilde{\chi}(\text{int}\sigma') \subset R'$ in this case. Yet we observe that $\tilde{\chi}(\sigma')$ is a nontrivial element in $H_1(\overline{R'})$, a contradiction.

We claim that $R \subset R'$. We consider the local picture along $\tilde{\chi}(\sigma)$, where the 4 pieces $\tilde{\chi}(\text{int}(\sigma))$, $\tilde{\chi}(\text{ext}(\sigma))$, $\tilde{\chi}(\text{int}(\sigma'))$, and $\tilde{\chi}(\text{ext}(\sigma'))$ intersect with an X cross-section. We have

- $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{ext}(\sigma))$ are adjacent: follows from $\tilde{\chi} \in \mathcal{Y}_0$.
- $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{ext}(\sigma'))$ are adjacent: follows from the definition of R' and the fact that $\{\tilde{D}_i\}$ do not intersect R' .

R' contains the component of $U \setminus \bigcup_i \tilde{D}_i$ contained between $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{ext}(\sigma'))$ along this intersection. Since R touches $\tilde{\chi}(\text{int}(\sigma))$ and does not touch $\tilde{\chi}(\text{ext}(\sigma))$, fact (3) above implies that R is the component of $U \setminus \bigcup_i \tilde{D}_i$ contained between $\tilde{\chi}(\text{int}(\sigma))$ and $\tilde{\chi}(\text{ext}(\sigma'))$ along this intersection. Hence, $R \subset R'$.

We claim that $R' \cap \tilde{\Omega} = \emptyset$. Since $\tilde{\chi} \in \mathcal{Y}_0$, each component of R' touches the same side of A . By the definition of $\tilde{\Omega}$, we have either $R' \subset \tilde{\Omega}$ or $R' \cap \tilde{\Omega} = \emptyset$. By fact (1) above, we have $R \cap \tilde{\Omega} = \emptyset$, so $R' \cap \tilde{\Omega}^c \neq \emptyset$. Hence, $R' \cap \tilde{\Omega} = \emptyset$.

Collapse procedure: Fix a C^2 diffeomorphism

$$\psi : \overline{\text{int}\sigma} \rightarrow \overline{\text{int}\sigma'}$$

so that $(\tilde{\chi} \circ \psi)|_{\sigma} = \tilde{\chi}|_{\sigma'}$. Then we define the map

$$\tilde{\chi}'(\xi) := \begin{cases} \tilde{\chi}(\psi(\xi)) & \xi \in \overline{\text{int}\sigma} \\ \tilde{\chi}(\xi) & \text{otherwise.} \end{cases}$$

We make an arbitrarily small perturbation of the map $\tilde{\chi}'$ in a neighbourhood of σ so that $\tilde{\chi}' \in \mathcal{Y}_0$.

Conclusion. In either case, we have

$$\mathcal{H}^2(\text{im}\tilde{\chi}') \leq \mathcal{H}^2(\text{im}\tilde{\chi}) + \varepsilon$$

and

$$\mathcal{H}^3((\tilde{\Omega} \cup R') \setminus \tilde{\Omega}') \leq \varepsilon.$$

Let $\tilde{R}$ be a different component of $(\Omega'_j \setminus \Omega_j) \cap \Omega$. Then we claim $\partial\tilde{R} \cap \bar{R}' = \emptyset$. Indeed, if not $\tilde{R}$ would be on the opposite side of a disc in $\tilde{\mathcal{S}}$ from R' . Since $R' \cap \tilde{\Omega} = \emptyset$ and (by fact (1) above) $\tilde{R} \cap \tilde{\Omega} = \emptyset$, we reach a contradiction. Hence, we can iterate the above procedure and conclude by induction. $\square$

3.4.5 Application 3: Decomposing stacks

In the case of positive prescribing function, we will decompose stacked disc minimisation problems into small stacked disc minimisation problems each with 1 or 2 discs.

We first separate stacks consisting of 1 disc.

Lemma 3.4.8. *Suppose D_1 is oriented up. Let $D'_1 \in \mathcal{D}(U)[D_1]$ so that D'_1 has transverse intersections with each D_i . For any $\varepsilon > 0$, there is a $\Sigma' \in \mathcal{D}(U)[\Sigma]$ satisfying*

$$\Sigma \Delta \Sigma' \subset U,$$

$$\mathcal{H}^2(\Sigma' \cap U) \leq \mathcal{H}^2(D'_1) + \sum_{i=2}^n \mathcal{H}^2(D_i) + \varepsilon,$$

and

$$\mathbb{M}(T(\Sigma, \Sigma') - ([\Omega_1] - [\Omega'_1]) + 2[\Omega_1 \setminus (\Omega \cup \Omega'_1)]) \leq \varepsilon.$$

Proof. We apply Lemma 3.4.3 to $\{D'_1\} \cup \{D_i\}_{i=2}^n$. The resulting discs satisfy (3.4.4) as in the first step of the proof of Lemma 3.4.7. However, we have $[(\Omega'_1 \setminus \Omega_1) \cap \Omega] = 0$ because D_1 is oriented up. Hence, the result follows from Lemma 3.4.5. $\square$

Now we separate stacks consisting of 2 discs.

Lemma 3.4.9. *Suppose D_j is oriented down and D_{j+1} is oriented up. Let $D'_j \in \mathcal{D}(U)[D_j]$ and $D'_{j+1} \in \mathcal{D}(U)[D_{j+1}]$ be mutually disjoint and have transverse intersections with each D_i . For any $\varepsilon > 0$, there is a $\Sigma' \in \mathcal{D}(U)[\Sigma]$ satisfying*

$$\Sigma \Delta \Sigma' \subset U,$$

$$\mathcal{H}^2(\Sigma' \cap U) \leq \mathcal{H}^2(D'_j) + \mathcal{H}^2(D'_{j+1}) + \sum_{i \notin \{j, j+1\}} \mathcal{H}^2(D_i) + \varepsilon,$$

and

$$\mathbb{M}(T(\Sigma, \Sigma') - (([\Omega_j] + [\Omega_{j+1}]) - ([\Omega'_j] + [\Omega'_{j+1}])) + 2[\Omega_j \cup \Omega_{j+1} \setminus (\Omega \cup \Omega'_j \cup \Omega'_{j+1})]) \leq \varepsilon.$$

Proof. We apply Lemma 3.4.3 to the discs $\{D'_j, D'_{j+1}\} \cup \{D_i\}_{i \neq j, j+1}$, producing new discs $\{\tilde{D}_i\}_{i=1}^n$.

We compute

$$\begin{aligned} [\Omega] - [\tilde{\Omega}] &= \sum_{i=1}^n [\Omega_i] - \sum_{i=1}^n [\tilde{\Omega}_i] \\ &= ([\Omega_j] + [\Omega_{j+1}]) - ([\Omega'_j] - [\Omega'_{j+1}]) \\ &\quad + \left([\Omega'_j] + [\Omega'_{j+1}] + \sum_{i \neq j, j+1} [\Omega_i] - \sum_{i=1}^n [\tilde{\Omega}_i] \right). \end{aligned}$$

Let

$$\tilde{N} := \sum_{i=1}^n \mathbf{1}_{\tilde{\Omega}_i}, \quad N' := \mathbf{1}_{\Omega'_j} + \mathbf{1}_{\Omega'_{j+1}} + \sum_{i \neq j, j+1} \mathbf{1}_{\Omega_i}.$$

To show (3.4.9), it suffices to show for $x \in U \setminus \bigcup_{i=1}^n \tilde{D}_i$ that

$$N'(x) - \tilde{N}(x) = \begin{cases} -2 & x \in (\Omega_j \cup \Omega_{j+1}) \setminus (\Omega \cup \Omega'_j \cup \Omega'_{j+1}) \\ 0 & \text{otherwise.} \end{cases}$$

Let R be the component of $U \setminus \bigcup_i D_i$ containing x . Let $x_0 \in C \cap \bar{R}$. Let $\gamma : (0, 1] \rightarrow U$ be a path in R so that

- γ avoids Γ_k ,
- γ has transverse intersection with D'_j and D'_{j+1} ,
- $\lim_{t \rightarrow 0} \gamma(t) = x_0$ and $\gamma(1) = x$.

We have $N'(x_0) = \tilde{N}(x_0)$. Along subsequent intersections of γ with $D'_j \cup D'_{j+1}$, each of N' and $\tilde{N}$ will increase and decrease by 1 in alternating order.

If γ has an even number of intersections with $D'_j \cup D'_{j+1}$, then we see $N'(x) = \tilde{N}(x)$. Hence, we only need to consider the case of an odd number of intersections, which is equivalent to $x \in (\Omega_j \cup \Omega_{j+1}) \Delta (\Omega'_j \cup \Omega'_{j+1})$. In this case, N' increases at the first intersection if and only if $x \in \Omega'_j \cup \Omega'_{j+1}$. Note that $\tilde{N}$ decreases at the first intersection if and only if $R \subset \Omega$. We also note that if $x \notin \Omega_j \cup \Omega_{j+1}$, then $x \notin \Omega$ (by the specification of the orientations of D_j and D_{j+1}). Hence, (3.4.5) follows.

The result now follows from Lemma 3.4.5. $\square$

3.5 Replacement

We adapt the replacement theorem [AS79, Theorem 1] to the prescribed mean curvature setting. This lemma allows us to bring components of a minimising surface outside a small ball into the ball.

To prove the replacement theorem, we first require an isoperimetric inequality.

Let δ and c_1 be the constants from Lemma 3.1.2, and let

$$\eta := \min \left\{ \frac{\mathcal{H}^3(M)}{2}, \frac{1}{8c^3c_1^2}, c_1\delta^3 \right\}.$$

Proposition 3.5.1. *If $\Omega \subset M$ is a set of finite perimeter with $\mathcal{H}^3(\Omega) \leq \eta$, then*

$$\mathcal{H}^3(\Omega) \leq c_1 \mathcal{H}^2(\partial\Omega)^{3/2}, \tag{3.5.1}$$

and

$$\mathcal{A}^h(\Omega) \geq \mathcal{A}^c(\Omega) \geq \frac{1}{2} \mathcal{H}^2(\partial\Omega).$$

Proof. If $\mathcal{H}^2(\partial\Omega) \geq \delta^2$, then (3.5.1) holds trivially by the definition of η . On the other hand, if $\mathcal{H}^2(\partial\Omega) < \delta^2$, Lemma 3.1.2 directly implies (3.5.1). Then (3.5.1) follows, as

we have

$$\mathcal{A}^c(\Omega) = \mathcal{H}^2(\partial\Omega) - c\mathcal{H}^3(\Omega) \geq \mathcal{H}^2(\partial\Omega)(1 - cc_1^{2/3}\mathcal{H}^3(\Omega)^{1/3}) \geq \frac{1}{2}\mathcal{H}^2(\partial\Omega).$$

The bound $\mathcal{A}^h(\Omega) \geq \mathcal{A}^c(\Omega)$ holds because $h \leq c$. $\square$

We are now equipped to prove the replacement theorem.

Lemma 3.5.2 (Replacement). *Let $\xi > 0$. Suppose $\mathcal{H}^3(U) \leq \eta$. Let $\Sigma \in \mathcal{D}^*(U)$. Then there exists $\tilde{\Sigma} \in \mathcal{D}(U)[\Sigma]$ and an open subset $B \subset U$ with $\bar{B} \cap \tilde{\Sigma} = \emptyset$ such that*

$$(\tilde{\Sigma} \cup \partial B) \cap U_\xi = \Sigma \cap U_\xi,$$

$$\mathcal{H}^2(\tilde{\Sigma}) + T_h(\Sigma, \tilde{\Sigma}) \leq \mathcal{H}^2(\tilde{\Sigma}) + T_h(\Sigma, \tilde{\Sigma}) + \mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) \leq \mathcal{H}^2(\Sigma),$$

and there is a set of disjoint oriented discs $\mathcal{S}$ with

$$[\tilde{\Sigma}] \llcorner U = \sum_{D \in \mathcal{S}} [D].$$

Suppose additionally that

$$\mathcal{H}^2(\Sigma) \leq \mathcal{H}^2(\Sigma') + T_h(\Sigma, \Sigma') + \varepsilon$$

for all $\Sigma' \in \mathcal{D}(U)[\Sigma]$. Then

$$\mathcal{H}^2(\tilde{\Sigma}) \leq \mathcal{H}^2(\Sigma') + T_h(\tilde{\Sigma}, \Sigma') + \varepsilon$$

for all $\Sigma' \in \mathcal{D}(U)[\tilde{\Sigma}]$, and

$$\mathcal{H}^2((\Sigma \Delta \tilde{\Sigma}) \cap U_\xi) \leq 2\varepsilon.$$

Let $P \subset \Sigma \cap U$ be a union of components of $\Sigma \cap U$. Let $\mathcal{S}_P \subset \mathcal{S}$ be the collection

of discs $D \in \mathcal{S}$ with $D \cap U_\xi \subset P$, and let $\tilde{P} := \bigcup_{D \in \mathcal{S}_P} D$. Then

$$\mathcal{H}^1(\tilde{P} \cap \partial U) \leq \mathcal{H}^1(\bar{P} \cap \partial U)$$

and

$$\mathcal{H}^2(P) \leq \mathcal{H}^2(\tilde{P}) + 2\varepsilon.$$

Proof. We use the same replacement procedure as in the proof of [AS79, Theorem 1]. However, we have to keep track of relative volumes, so we include the details of the procedure before the smoothing, perturbation, and induction steps.

Let Λ be any component of $\Sigma \setminus U$ such that $\Lambda \cap \partial \Sigma = \emptyset$. By Lemma 3.1.3, there is a unique compact set $K_\Lambda \subset M \setminus U$ satisfying (3.1.3) and (3.1.3). Given any component Λ' of $\Sigma \setminus U$ with $\Lambda' \subset K_\Lambda$, we have $\Lambda' \cap \partial \Sigma = \emptyset$ and $K_{\Lambda'} \subset K_\Lambda$ (by the assumption on U and Σ). Hence, we can select a component Λ of $\Sigma \setminus U$ such that $\Lambda \cap \partial \Sigma = \emptyset$ and $\text{int}(K_\Lambda) \cap \Sigma = \emptyset$.

Let χ be a diffeomorphism of $\mathbb{D}$ onto Σ , and consider the set $H := \chi^{-1}(\Lambda)$. H is a compact connected subset of $\mathbb{D}$ bounded by a finite collection of pairwise disjoint smooth Jordan curves $\gamma_1, \dots, \gamma_n$. Suppose without loss of generality that γ_1 is the outermost of the curves. By the connectedness of H , we can write

$$H = \overline{\text{int}\gamma_1} \setminus \bigcup_{j=2}^n \text{int}\gamma_j.$$

Recall that $F = \partial K_\Lambda \cap \partial U$. Define $F_1 \subset F$ to be the component of F which contains the Jordan curve $\chi(\gamma_1)$, and $F_2 := F \setminus F_1$. Define $I := \bigcup \text{int}\gamma_j$, where the union is over those j such that $\chi(\gamma_j) \subset F_2$. Let $\Lambda_2 := \chi(I)$. Then $\Lambda_1 := \Lambda \cup \Lambda_2$ is diffeomorphic to the connected subset $H \cup I$ of $\mathbb{D}$, and $\partial \Lambda_1 = \partial F_1$. By [AS79, Lemma 1], we can construct a diffeomorphism α of Λ_1 onto F_1 such that α coincides with the

identity on $\partial\Lambda_1 = \partial F_1$. We define a Lipschitz map $\hat{\chi} : \mathbb{D} \rightarrow M$ by

$$\hat{\chi}(x) := \begin{cases} (\alpha \circ \chi)(x) & x \in H \cup I \\ \chi(x) & \text{otherwise.} \end{cases}$$

We consider the following two cases:

Case 1: $\partial\Sigma \cap (F_1 \setminus \partial F_1) = \emptyset$. Since $\text{int}(K_\Lambda) \cap \Sigma = \emptyset$, $\hat{\chi}$ is injective, and we define

$$\tilde{\Sigma} := \hat{\chi}(\mathbb{D}) = (\Sigma \setminus \Lambda_1) \cup F_1.$$

Let $V_\Lambda \subset U \cup \bigcup_{\Lambda'} K_{\Lambda'}$ be the unique open set bounded by $\Lambda_1 \cup F_1$. Note that $\text{int}(K_\Lambda) \subset V_\Lambda$, so V_Λ is on the outside of F_1 . Let $B := V_\Lambda \cap U$, $V_\Lambda^o := V_\Lambda \setminus \overline{U}$, $\Lambda_1^i := \Lambda_1 \cap U$, $\Lambda_1^o = \Lambda_1 \setminus \overline{U}$, and $G := V_\Lambda \cap \partial U$. Since V_Λ is on the outside of F_1 , we have $\partial B = \Lambda_1^i \cup G$. Our goal is to show

$$\mathcal{H}^2(\tilde{\Sigma}) + c\mathcal{H}^3(V_\Lambda) + \mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) < \mathcal{H}^2(\Sigma).$$

We have $\partial V_\Lambda^o = \Lambda_1^o \cup F_1 \cup G$, so (3.1.3) implies

$$\mathcal{H}^2(F_1) + \mathcal{H}^2(G) < \mathcal{H}^2(\Lambda_1^o) - c\mathcal{H}^3(V_\Lambda^o).$$

By the definition of $\tilde{\Sigma}$, we have

$$\mathcal{H}^2(\tilde{\Sigma}) = \mathcal{H}^2(\Sigma) - \mathcal{H}^2(\Lambda_1) + \mathcal{H}^2(F_1).$$

Hence, we have

$$\begin{aligned} \mathcal{H}^2(\tilde{\Sigma}) + c\mathcal{H}^3(V_\Lambda) + \mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) &= \mathcal{H}^2(\Sigma) - \mathcal{H}^2(\Lambda_1^o) + \mathcal{H}^2(F_1) + c\mathcal{H}^3(V_\Lambda^o) + \mathcal{H}^2(G) \\ &< \mathcal{H}^2(\Sigma). \end{aligned}$$

Case 2: $\partial\Sigma \cap (F_1 \setminus \partial F_1) \neq \emptyset$. Since $\text{int}(K_\Lambda) \cap \Sigma = \emptyset$, we have $\partial\Sigma \subset F_1 \setminus \partial F_1$ in this case. Hence, $\hat{\chi}^{-1}(\partial\Sigma) = \partial\mathbb{D} \cup \gamma$, where γ is a smooth Jordan curve in $\mathbb{D} \setminus \partial\mathbb{D}$. We define

$$\tilde{\Sigma} := \hat{\chi}(\text{int}\gamma) \subset F_1.$$

Let $V_\Lambda \subset U \cup \bigcup_{\Lambda'} K_{\Lambda'}$ be the unique open set bounded by Σ and $\tilde{\Sigma}$. Note that $\text{int}(K_\Lambda) \subset V_\Lambda$, so V_Λ is on the outside of $\tilde{\Sigma}$. Let $B := V_\Lambda \cap U$, $V_\Lambda^o := V_\Lambda \setminus \bar{U}$, $\Sigma^i := \Sigma \cap U$, $\Sigma^o := \Sigma \setminus \bar{U}$, and $G := V_\Lambda \cap \partial U$. Since V_Λ is on the outside of $\tilde{\Sigma}$, we have $\partial B = \Sigma^i \cup G$. Our goal is to show

$$\mathcal{H}^2(\tilde{\Sigma}) + c\mathcal{H}^3(V_\Lambda) + \mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) < \mathcal{H}^2(\Sigma).$$

We have $\partial V_\Lambda^o = \Sigma^o \cup \tilde{\Sigma} \cup G$, so (3.1.3) implies

$$\mathcal{H}^2(\tilde{\Sigma}) + \mathcal{H}^2(G) < \mathcal{H}^2(\Sigma^o) - c\mathcal{H}^3(V_\Lambda^o).$$

Hence, we have

$$\begin{aligned} \mathcal{H}^2(\tilde{\Sigma}) + c\mathcal{H}^3(V_\Lambda) + \mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) &= \mathcal{H}^2(\tilde{\Sigma}) + c\mathcal{H}^3(V_\Lambda^o) + \mathcal{H}^2(\Sigma^i) + \mathcal{H}^2(G) \\ &< \mathcal{H}^2(\Sigma). \end{aligned}$$

The smoothing, slight perturbation, and induction steps now follow exactly as in the proof of [AS79, Theorem 1] (with the additional observation that small perturbations make small changes to relative volumes). Taking B to be the union of the perturbations of the sets B from each iteration applied to the perturbed $\tilde{\Sigma}$ from the previous step, we obtain (3.5.2) and (3.5.2) immediately. Moreover, by (3.5) and (3.5), we have

$$\mathcal{H}^2(\tilde{\Sigma}) + c|T(\Sigma, \tilde{\Sigma})| + \mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) \leq \mathcal{H}^2(\Sigma).$$

Since

$$T_h(\Sigma, \tilde{\Sigma}) \leq c|T(\Sigma, \tilde{\Sigma})|,$$

the second inequality in (3.5.2) holds. The first inequality in (3.5.2) follows from (3.5.1) and the fact that $B \subset U$.

(3.5.2) follows from (3.5.2), (3.5.2), Proposition 3.2.3, and the fact that $\mathcal{D}(U)[\tilde{\Sigma}] \subset \mathcal{D}(U)[\Sigma]$.

To prove (3.5.2), we note that

$$\Sigma \Delta \tilde{\Sigma} \cap U_\xi = \partial B \cap U_\xi.$$

By (3.5.2) and (3.5.2) with $\Sigma' = \tilde{\Sigma}$, we have

$$\mathcal{H}^2(\partial B) - c\mathcal{H}^3(B) \leq \varepsilon.$$

By (3.5.1), we have

$$\mathcal{H}^2(\partial B) \leq 2\varepsilon,$$

which implies (3.5.2).

(3.5.2) follows from the construction of $\tilde{\Sigma}$; namely, for each component of $\Sigma \cap U$, either (1) it is thrown away in ∂B , or (2) each component of its intersection with ∂U is either (a) kept unchanged, or (b) pushed away from ∂U in the perturbation step.

(3.5.2) follows from the construction of $\tilde{\Sigma}$ and (3.5.2); namely, each component of $\Sigma \cap U$ is either (1) thrown away in ∂B , whose area is at most 2ε by (3.5.2), or (2) kept and given potentially larger area (by adding pieces of ∂U). $\square$

3.6 Filigree

We adapt the filigree lemma [AS79, Lemma 3] to the prescribed mean curvature setting. This lemma allows us to ignore small area components of a minimising sequence in small balls.

Lemma 3.6.1 (Filigree). *Suppose $\mathcal{H}^3(U(\theta)) \leq \eta$. Let $\Sigma \in \mathcal{D}^*(U(\theta))$ satisfy*

$$\mathcal{H}^2(\partial U(\theta)) + \mathcal{H}^2(\Sigma \setminus U) \leq \delta_1^2/32$$

and

$$\mathcal{H}^2(\Sigma) \leq \mathcal{H}^2(\Sigma') + T_h(\Sigma, \Sigma') + \varepsilon$$

for all $\Sigma' \in \mathcal{D}(U)[\Sigma]$. Let P be a union of some components of $\Sigma \cap U(\theta)$. If

$$\mathcal{H}^2(P) \leq \frac{\theta^2}{48\beta}$$

then

$$\mathcal{H}^2(P \cap U) \leq 6\varepsilon.$$

Proof. For almost every $\sigma \in [0, \theta]$, we have $\Sigma \in \mathcal{D}^*(U(\sigma))$. Choose any such $\sigma = t\theta$ for $t \in [0, 1]$. We apply Lemma 3.5.2 to Σ in $U(\sigma)$ with some small constant ξ , yielding $\tilde{\Sigma}$.

Let $\mathcal{S}$ be the set of discs composing $\tilde{\Sigma}$ in $U(\sigma)$. Let $\mathcal{S}_P \subset \mathcal{S}$ be the set of discs $D \in \mathcal{S}$ so that $D \cap U(\sigma)_\xi \subset P$. Let $\tilde{P} := \bigcup_{D \in \mathcal{S}_P} D$. Take a disc $D \in \mathcal{S}_P$. Let F_D denote the isoperimetric region in $\partial U(\sigma)$ for ∂D . Let Λ_D denote the region in $U(\sigma)$ bounded by $D \cup F_D$.

Consider the competitor Σ' for $\tilde{\Sigma}$ given by replacing each disc $D \in \mathcal{S}_P$ with F_D ,

and then resolving the overlaps and perturbing using Lemma 3.4.6. Then

$$T_h(\tilde{\Sigma}, \Sigma') \leq c \sum_{D \in \mathcal{S}_P} \mathcal{H}^3(\Lambda_D) + \varepsilon/2.$$

By (3.5.1), we have

$$\mathcal{H}^3(\Lambda_D)^{2/3} \leq c_1^{2/3}(\mathcal{H}^2(D) + \mathcal{H}^2(F_D)).$$

Hence, we have

$$\begin{aligned} T_h(\tilde{\Sigma}, \Sigma') &\leq c c_1^{2/3} (\mathcal{H}^3(U(\sigma)))^{1/3} \sum_{D \in \mathcal{S}_P} (\mathcal{H}^2(D) + \mathcal{H}^2(F_D)) + \varepsilon/2 \\ &\leq \frac{1}{2} \sum_{D \in \mathcal{S}_P} (\mathcal{H}^2(D) + \mathcal{H}^2(F_D)) + \varepsilon/2. \end{aligned}$$

By Lemma 3.5.2(3.5.2), we have

$$\begin{aligned} \sum_{D \in \mathcal{S}} \mathcal{H}^2(D) &= \mathcal{H}^2(\tilde{\Sigma} \cap U(\sigma)) \\ &\leq \mathcal{H}^2(\Sigma' \cap U(\sigma)) + T_h(\tilde{\Sigma}, \Sigma') + \varepsilon \\ &\leq \sum_{D \in \mathcal{S} \setminus \mathcal{S}_P} \mathcal{H}^2(D) + \sum_{D \in \mathcal{S}_P} \mathcal{H}^2(F_D) + \frac{1}{2} \sum_{D \in \mathcal{S}_P} (\mathcal{H}^2(D) + \mathcal{H}^2(F_D)) + 2\varepsilon, \end{aligned}$$

so

$$\mathcal{H}^2(\tilde{P}) = \sum_{D \in \mathcal{S}_P} \mathcal{H}^2(D) \leq 3 \sum_{D \in \mathcal{S}_P} \mathcal{H}^2(F_D) + 4\varepsilon.$$

By the isoperimetric inequality on $\partial U(\sigma)$, we have

$$\mathcal{H}^2(\tilde{P}) \leq 3\beta(\mathcal{H}^1(\partial \tilde{P}))^2 + 4\varepsilon.$$

By Lemma 3.5.2(3.5.2), we have

$$\mathcal{H}^1(\partial\tilde{P}) \leq \mathcal{H}^1(P \cap \partial U(\sigma)).$$

By Lemma 3.5.2(3.5.2), we have

$$\mathcal{H}^2(P \cap U(\sigma)) \leq \mathcal{H}^2(\tilde{P}) + 2\varepsilon.$$

Hence, we have

$$\mathcal{H}^2(P \cap U(t\theta)) \leq 3\beta(\mathcal{H}^1(P \cap \partial U(t\theta)))^2 + 6\varepsilon.$$

Let $f(t) := \mathcal{H}^2(P \cap U(t\theta)) - 6\varepsilon$. Then by the coarea formula we have

$$f \leq 3\beta\theta^{-2}(f')^2.$$

Suppose without loss of generality that $f(1) > 0$. Let

$$t_0 := \inf\{t \in [0, 1] \mid f(t) > 0\}.$$

Integrating (3.6) gives

$$(1 - t_0)\frac{\theta}{2\sqrt{3\beta}} \leq \sqrt{f(1)} - \sqrt{f(t_0)} \leq \sqrt{\mathcal{H}^2(P)}.$$

Then our area assumption gives

$$t_0 \geq 1 - \frac{2\sqrt{3\beta}}{\theta}\sqrt{\mathcal{H}^2(P)} \geq \frac{1}{2}.$$

Hence, $\mathcal{H}^2(P \cap U) \leq \mathcal{H}^2(P \cap U(\theta/2)) \leq 6\varepsilon$. □

3.7 Interior regularity

In this section we prove the interior regularity for $\mathcal{A}^h$ minimising sequences of discs, following the strategy of [AS79].

We first apply Lemma 3.6.1 to deduce rectifiability of the limit varifold.

Lemma 3.7.1 (Rectifiability). *Let $\{\Sigma_k\}_{k \in \mathbb{N}} \subset \mathcal{D}^*(U)$ satisfy $\mathcal{H}^2(\Sigma_k) \leq \delta_2^2/64$ and*

$$\mathcal{H}^2(\Sigma_k) \leq \mathcal{H}^2(\Sigma'_k) + T_h(\Sigma_k, \Sigma'_k) + \varepsilon_k$$

for all $\Sigma'_k \in \mathcal{D}(U)[\Sigma_k]$ with $\varepsilon_k \rightarrow 0$. Let V be the subsequential varifold limit of Σ_k in U provided by Theorem 3.3.1. Then

$$\Theta^2(\|V\|, x) \geq \frac{1}{1000\beta} > 0$$

for all $x \in \text{spt}\|V\|$. Moreover, V is rectifiable.

Proof. Rectifiability follows from the positive density lower bound by [All72, Theorem 5.5].

Let $x \in \text{spt}\|V\|$. Choose $r_0 \leq 1/2$ small enough so that $\mathcal{H}^3(B_{r_0}(x)) \leq \eta$ and $\mathcal{H}^2(\partial B_r(x)) \leq \delta_2^2/64$ for all $r \leq r_0$. Let $r_i \rightarrow 0$ with $r_i \leq r_0$, $B_{r_i}(x) \subset U$, and $\Sigma_k \in \mathcal{D}^*(B_{r_i}(x))$ for all k and i , which is guaranteed by Sard's theorem. By Lemma 3.6.1 with $\theta = r_i/2$, if

$$\liminf_k \mathcal{H}^2(\Sigma_k \cap B_{r_i}(x)) \leq \frac{r_i^2}{192\beta},$$

then

$$\|V\|(B_{r_i/2}(x)) \leq \liminf_k \mathcal{H}^2(\Sigma_k \cap B_{r_i/2}(x)) \leq \liminf_k 6\varepsilon_k = 0,$$

which contradicts $x \in \text{spt}\|V\|$. Hence, by the assumption of transversality, we have

$$\|V\|(B_{r_i}(x)) = \lim_k \mathcal{H}^2(\Sigma_k \cap B_{r_i}(x)) \geq \frac{r_i^2}{192\beta}.$$

By the monotonicity of (3.3.1) for V , we have

$$\Theta^2(\|V\|, x) \geq \frac{1}{1000\beta},$$

as desired. $\square$

We now have all the tools to use the strategy of [AS79] to deduce interior regularity.

Theorem 3.7.2. *Let $\{\Sigma_k\}_{k \in \mathbb{N}} \subset \mathcal{D}^*(U)$ satisfy $\sup_k \mathcal{H}^2(\Sigma_k \cap U) \leq \delta_2^2/64$ and*

$$\mathcal{H}^2(\Sigma_k) \leq \mathcal{H}^2(\Sigma'_k) + T_h(\Sigma_k, \Sigma'_k) + \varepsilon_k$$

for all $\Sigma'_k \in \mathcal{D}(U)[\Sigma_k]$ with $\varepsilon_k \rightarrow 0$. Let V and Ω respectively be the subsequential varifold limit of $\Sigma_k \cap U$ and the subsequential Caccioppoli set limit of Ω_k provided by Theorem 3.3.1. Let $\alpha \in (0, 1)$.

V is an integer rectifiable varifold with c -bounded first variation. For every $x \in \text{spt}\|V\|$, we have $\Theta^2(\|V\|, x) = n_x \in \mathbb{N}$. If $h(x) \neq 0$, then $n_x \in \{1\} \cup 2\mathbb{N}$. For every $x \in \text{spt}\|V\|$, there is a neighbourhood W_x of x so that the following hold.

- *$V \llcorner G(W_x, 2) = \sum_{l=1}^{n_x} \mathbf{v}(N_l, 1)$, where $N_l \subset W_x$ is a $C^{1,\alpha}$ surface with c -bounded first variation. Moreover, each N_l is on one side of $N_{l'}$ intersecting tangentially at x for any $l, l' \in \{1, \dots, n_x\}$.*

- *If $n_x = 1$, then*

$$- \quad \|V\| \llcorner W_x = |D\mathbf{1}_\Omega| \llcorner W_x,$$

- *$V \llcorner G(W_x, 2) = \mathbf{v}(N, 1)$, where $N \subset W_x$ is a smooth stable surface with prescribed mean curvature h with respect to the set $\Omega \cap W_x$.*

- *If $h(x) \neq 0$ and $n_y = 2n$ for all $y \in \text{spt}\|V\| \cap W'_x$ for an open set $W'_x \subset W_x$ containing x , then $V \llcorner G(W'_x, 2) = \mathbf{v}(N, 2n)$, where $N \subset W'_x$ is a smooth stable*

minimal surface. Moreover, $\Omega \cap W'_x = W'_x \setminus N$ if $h(x) > 0$, and $\Omega \cap W'_x = \emptyset$ if $h(x) < 0$.

- If $h(x) \neq 0$, $n_x = 2n$, and there is a sequence $x_j \rightarrow x$ with $x_j \in \text{spt}\|V\|$ and $n_{x_j} \neq 2n$, then $n = 1$ and the surfaces $N_1, N_2 \subset W_x$ additionally satisfy that the generalised mean curvature of N_1 points towards N_2 and vice versa.

Proof. By Lemma 3.7.1, V is rectifiable and has a uniform positive density lower bound in its support. By Theorem 3.3.1(3.3.1), V has c -bounded first variation.

Integer Density. We show that the density of V is an integer at every $x \in \text{spt}\|V\|$. Since this result concerns the infinitesimal behavior of V near x , we push objects to $T_x M$ using the exponential map ϕ_x . In $T_x M$, we use the standard Euclidean metric in the geodesic normal coordinates, which is the C^∞ limit of the pullback metric under a blow up rescaling.

First, let $x \in \text{spt}\|V\|$ be such that there is a tangent cone $C \in \text{VarTan}(V, x)$ with $\text{spt}\|C\| \subset P_C$, where P_C is a 2-plane in $T_x M$ (by rectifiability, this holds $\|V\|$ -almost everywhere).

Fix any $\xi > 0$.

Without loss of generality, assume $P_C \supset \text{spt}\|C\|$ is the $x_1 x_2$ -plane in $T_x M$.

Let μ_t denote the transformation of $T_x M$ given by $\mu_t(y) = ty$. By the definition of C , there is a sequence $\{r_j\} \rightarrow \infty$ so that

$$(\mu_{r_j} \circ \phi_x^{-1})_\# V \rightarrow C$$

as $j \rightarrow \infty$. We always assume $r_j \geq 1$.

By the lower density bound from Lemma 3.7.1 and the monotonicity of (3.3.1), there is a sequence $\{\sigma_j\} \rightarrow 0$ so that

$$(B_1^{\mathbb{R}^2}(0) \times (-2, 2)) \cap \text{spt}\|(\mu_{r_j} \circ \phi_x^{-1})_\# V\| \subset B_1^{\mathbb{R}^2}(0) \times (-\sigma_j/2, \sigma_j/2).$$

We establish some notation. To distinguish objects in $T_x M$ from objects in M , we use lower case letters here. Let

- $v^j := (\mu_{r_j} \circ \phi_x^{-1})_{\#} V$,
- $s_{k,j} := (\mu_{r_j} \circ \phi_x^{-1})(\Sigma_k)$,
- $b_\rho := B_\rho^{T_x M}(0)$,
- $c_{\rho,\sigma} := \partial b_\rho \cap \{x_3 \in [-\sigma, \sigma]\}$,
- $d_{\rho,\sigma}^+ := \partial b_\rho \cap \{x_3 > \sigma\}$,
- $d_{\rho,\sigma}^- := \partial b_\rho \cap \{x_3 < -\sigma\}$.

By (3.7), we have

$$\limsup_k \mathcal{H}^2(s_{k,j} \cap (B_1^{\mathbb{R}^2}(0) \times ([-1, 1] \setminus (-\sigma_j/2, \sigma_j/2)))) = 0.$$

By (3.7) and the coarea formula, for almost every $\rho < 1$ we have

$$\limsup_k \mathcal{H}^1(s_{k,j} \cap (d_{\rho,\sigma_j/2}^+ \cup d_{\rho,\sigma_j/2}^-)) = 0$$

for every j . Fix some $\rho_1 \in (3/4, 1)$ satisfying (3.7) so that $s_{k,j}$ has transverse intersection with ∂b_{ρ_1} for every k and j , which is guaranteed by Sard's theorem.

Fix some $\zeta > 0$. For $k \geq k_1(j, \zeta, \rho_1)$, we have

$$\mathcal{H}^1(s_{k,j} \cap (d_{\rho_1,\sigma_j/2}^+ \cup d_{\rho_1,\sigma_j/2}^-)) < \zeta.$$

For $\zeta \leq \zeta_1(M)$, the coarea formula gives

$$s_{k,j} \cap (\partial d_{\rho_1,\sigma_{k,j}}^+ \cup \partial d_{\rho_1,\sigma_{k,j}}^-) = \emptyset,$$

for a sequence $\sigma_{k,j} \in (\sigma_j/2, \sigma_j)$.

We apply Lemma 3.5.2 to $s_{k,j} \in \mathcal{D}^*(b_{\rho_1})$ with constant ξ . We obtain the replacement $\tilde{s}_{k,j} \in \mathcal{D}(b_{\rho_1})[s_{k,j}]$ and a collection of discs $\mathfrak{S}_{k,j}$ composing $\tilde{s}_{k,j}$ in b_{ρ_1} .

Let $\xi' \in [\xi, 2\xi]$ so that

$$\|v^j\|(\partial b_{\rho_1-\xi'}) = 0$$

for all j .

By Lemma 3.5.2(3.5.2) and (3.5.2) (also using (3.7)), we have

$$v^j \lrcorner G(b_{\rho_1-\xi'}, 2) = \lim_k \sum_{d \in \mathfrak{S}_{k,j}} \mathbf{v}(d \cap b_{\rho_1-\xi'}).$$

Step 1: We show that we can throw away discs $d \in \mathfrak{S}_{k,j}$ with boundary in the caps $d_{\rho_1, \sigma_{k,j}}^\pm$. Let $\mathfrak{C}_{k,j}$ be the set of all such discs in $\mathfrak{S}_{k,j}$. For every $d \in \mathfrak{C}_{k,j}$, let f_d be the isoperimetric region for ∂d in $d_{\rho_1, \sigma_{k,j}}^+ \cup d_{\rho_1, \sigma_{k,j}}^-$. By (3.7) and Lemma 3.5.2(3.5.2), we have

$$\sum_{d \in \mathfrak{C}_{k,j}} \mathcal{H}^1(\partial d) \leq \zeta$$

for $k \geq k_1(j, \zeta, \rho_1)$. By the isoperimetric inequality in the round sphere, we have

$$\mathcal{H}^2(f_d) \leq \beta_1 \mathcal{H}^1(\partial d)^2$$

for all $d \in \mathfrak{C}_{k,j}$. Hence, we have

$$\sum_{d \in \mathfrak{C}_{k,j}} \mathcal{H}^2(f_d) \leq \beta_1 \sum_{d \in \mathfrak{C}_{k,j}} \mathcal{H}^1(\partial d)^2 \leq \beta_1 \left(\sum_{d \in \mathfrak{C}_{k,j}} \mathcal{H}^1(\partial d) \right)^2 \leq \beta_1 \zeta^2$$

for $k \geq k_1(j, \zeta, \rho_1)$. Consider the competitor given by replacing each $d \in \mathfrak{C}_{k,j}$ by f_d (and then resolving the overlaps without increasing area by more than ε_k by [AS79, Corollary 1]). By (3.7.2), Lemma 3.5.2(3.5.2), and the fact that $\|d\phi_x\|_{B_{\rho/r_j}^M(x)} \leq$

$1 + o_j(1)$, we have

$$r_j^{-2} \sum_{d \in \mathfrak{S}_{k,j}} \mathcal{H}^2(d) \leq r_j^{-2} \sum_{d \in \mathfrak{S}_{k,j} \setminus \mathfrak{C}_{k,j}} \mathcal{H}^2(d) + r_j^{-2} \sum_{d \in \mathfrak{C}_{k,j}} \mathcal{H}^2(d) + 2cr_j^{-3} \mathcal{H}^3(b_{\rho_1}) + 2\varepsilon_k + o(r_j^{-2}).$$

Rewriting, we have

$$\sum_{d \in \mathfrak{C}_{k,j}} \mathcal{H}^2(d) \leq \beta_1 \zeta^2 + \frac{2c\mathcal{H}^3(b_1)}{r_j} + 2r_j^2 \varepsilon_k + o_j(1)$$

for $k \geq k_1(j, \zeta, \rho_1)$. Taking $j \geq j_2(M, \xi)$ and $\zeta \leq \zeta_2(M, \xi)$, and then taking $k \geq k_2(j, \zeta, \rho_1, \xi)$, we can apply Lemma 3.6.1 to conclude that

$$\lim_k \sum_{d \in \mathfrak{C}_{k,j}} \mathbf{v}(d \cap b_{\rho_1 - \xi}) = 0$$

for all $j \geq j_2(M, \xi)$.

Step 2: We show that we can throw away the discs with nullhomotopic boundary in $c_{\rho_1, \sigma_{k,j}}$. Let $\mathfrak{N}_{k,j}$ be the set of all such discs in $\mathfrak{S}_{k,j}$. By the monotonicity of (3.3.1), there is a uniform upper bound for the mass of v^j in b_1 for all j depending on V and c .

Let $\mathfrak{N}_{k,j}^- \subset \mathfrak{N}_{k,j}$ be the set of discs $d \in \mathfrak{N}_{k,j}$ with $\mathcal{H}^2(d) \leq \frac{\xi^2}{48\beta_1}$. By the uniform mass upper bound, there is an $m = m(V, c, \xi) \in \mathbb{N}$ such that

$$\sum_{d \in \mathfrak{N}_{k,j}^-} \mathcal{H}^2(d) \leq m \frac{\xi^2}{48\beta_1}$$

for all j . By a standard counting argument, we can partition $\mathfrak{N}_{k,j}^-$ into $2m$ disjoint sets $\{\mathfrak{N}_{k,j,i}^-\}_{i=1}^{2m}$ (some potentially empty) such that

$$\sum_{d \in \mathfrak{N}_{k,j,i}^-} \mathcal{H}^2(d) \leq \frac{\xi^2}{48\beta_1}.$$

By Lemma 3.6.1, we have

$$\lim_k \sum_{d \in \mathfrak{N}_{k,j}^-} \mathbf{v}(d \cap b_{\rho_1 - \xi}) = \sum_{i=1}^{2m} \lim_k \sum_{d \in \mathfrak{N}_{k,j,i}^-} \mathbf{v}(d \cap b_{\rho_1 - \xi}) = 0.$$

Let $\mathfrak{N}_{k,j}^+$ denote the set of disc $d \in \mathfrak{N}_{k,j}$ with $\mathcal{H}^2(d) > \frac{\xi^2}{48\beta_1}$. By the uniform mass upper bound, we have $\#\mathfrak{N}_{k,j}^+ \leq m$. Let f_d denote the isoperimetric region in $c_{\rho_1, \sigma_{k,j}}$ for ∂d . Consider the competitor given by replacing d with f_d for all $d \in \mathfrak{N}_{k,j}^+$ (and then resolving overlaps without increasing area by more than ε_k by [AS79, Corollary 1]). As in step 1, (3.7.2) and Lemma 3.5.2(3.5.2) imply

$$r_j^{-2} \sum_{d \in \mathfrak{S}_{k,j}} \mathcal{H}^2(d) \leq r_j^{-2} \sum_{d \in \mathfrak{S}_{k,j} \setminus \mathfrak{N}_{k,j}^+} \mathcal{H}^2(d) + r_j^{-2} \sum_{d \in \mathfrak{N}_{k,j}^+} \mathcal{H}^2(f_d) + 2cr_j^{-3} \mathcal{H}^3(b_{\rho_1}) + 2\varepsilon_k + o(r_j^{-2}).$$

Rewriting, we have

$$\sum_{d \in \mathfrak{N}_{k,j}^+} \mathcal{H}^2(d) \leq m \mathcal{H}^2(c_{\rho_1, \sigma_j}) + \frac{2c \mathcal{H}^3(b_1)}{r_j} + 2r_j^2 \varepsilon_k + o_j(1).$$

Assuming $j \geq j_3(c, \rho_1, M, V, \xi)$ and then taking $k \geq k_3(j, \rho_1, M, \xi)$, we have $\#\mathfrak{N}_{k,j}^+ = 0$.

Step 3: We obtain lower and upper bounds for the area of the discs whose boundary is homotopically nontrivial in $c_{\rho_1, \sigma_{k,j}}$. Let $\mathfrak{L}_{k,j}$ denote the set of such discs in $\mathfrak{S}_{k,j}$.

Let $d \in \mathfrak{L}_{k,j}$. Since ∂d is homotopically nontrivial in $c_{\rho_1, \sigma_{k,j}}$, (3.7) implies

$$\mathcal{H}^2(d \cap b_\rho) \geq \pi(\rho - \xi)^2$$

for $\rho \in [\rho_1 - 2\xi, \rho_1]$, $j \geq j_4(\rho_1, M, V, \xi)$, and $k \geq k_4(j, \rho_1, M, \xi)$.

By the uniform mass bound on the limit, there is an $m = m(c, V)$ so that $\#\mathfrak{L}_{k,j} \leq$

m for all $j \geq j_4(c, \rho_1, M, V, \xi)$ and $k \geq k'_4(j, c, \rho_1, M, V, \xi)$.

For each $d \in \mathfrak{L}_{k,j}$, let m_d be the disc with $\partial m_d = \partial d$ so that $m_d \setminus b_{\rho_1} \subset c_{\rho_1, \sigma_{k,j}}$ and $m_d \cap b_{\rho_1}$ is the flat disc $d_{k,j}^* \subset b_{\rho_1}$ with boundary $\partial d_{k,j}^+$. Note that

$$\mathcal{H}^2(m_d) \leq \mathcal{H}^2(c_{\rho_1, \sigma_{k,j}}) + \mathcal{H}^2(d_{k,j}^*).$$

Consider the competitor given by replacing each $d \in \mathfrak{L}_{k,j}$ with m_d (and then resolving overlaps using [AS79, Corollary 1]). As in steps 1 and 2, (3.7.2) implies

$$r_j^{-2} \sum_{d \in \mathfrak{S}_{k,j}} \mathcal{H}^2(d) \leq r_j^{-2} \sum_{d \in \mathfrak{S}_{k,j} \setminus \mathfrak{L}_{k,j}} \mathcal{H}^2(d) + r_j^{-2} \sum_{d \in \mathfrak{L}_{k,j}} \mathcal{H}^2(m_d) + 2cr_j^{-3} \mathcal{H}^3(b_{\rho_1}) + 2\varepsilon_k + o(r_j^{-2}).$$

Rewriting, we have

$$\sum_{d \in \mathfrak{L}_{k,j}} \mathcal{H}^2(d) \leq (\#\mathfrak{L}_{k,j})(\mathcal{H}^2(d_{k,j}^*) + \mathcal{H}^2(c_{\rho_1, \sigma_j})) + \frac{2c\mathcal{H}^3(b_1)}{r_j} + 2r_j^2\varepsilon_k + o_j(1).$$

Assuming $j \geq j_5(c, \rho_1, M, V, \xi)$ and then taking $k \geq k_5(j, c, \rho_1, M, V, \xi)$, we have for every $d \in \mathfrak{L}_{k,j}$ (by (3.7))

$$\mathcal{H}^2(d) \leq \pi(\rho_1 + \xi)^2.$$

Step 4: We compute the density of V at x . By (3.7) and the above work, we have

$$\|v^j\|(b_{\rho_1 - \xi'}) = \lim_k \sum_{d \in \mathfrak{L}_{k,j}} \mathcal{H}^2(d \cap b_{\rho_1 - \xi'}).$$

By (3.7) and (3.7), we have

$$(\limsup_k \#\mathfrak{L}_{k,j})\pi(\rho_1 - \xi)^2 \leq \|v^j\|(b_{\rho_1 - \xi', j}) \leq (\liminf_k \#\mathfrak{L}_{k,j})\pi(\rho_1 + \xi)^2.$$

Assuming ξ is sufficiently small (depending on universal quantities), we have $\lim_k \#\mathfrak{L}_{k,j}$ exists, so we have $\#\mathfrak{L}_{k,j} = n_j \in \mathbb{N}$ for all $j \geq j_5(c, \rho_1, M, V, \xi)$ and $k \geq k_6(j, c, \rho_1, M, V, \xi)$.

By the monotonicity of (3.3.1), n_j is decreasing for $j \geq j_6(c, \rho_1, M, V, \xi)$. By Lemma 3.7.1, we have $n_j \geq 1$. Hence, we have $n_j \equiv n \geq 1$ for all $j \geq j'_6(c, \rho_1, M, V, \xi)$. Therefore, we have

$$n - o_\xi(1) \leq \Theta^2(\|V\|, x) \leq n + o_\xi(1).$$

Since ξ is arbitrary, we conclude that $\Theta^2(\|V\|, x) = n \in \mathbb{N}$, i.e. V has integer density at all points x in its support such that some tangent cone $C \in \text{VarTan}(V, x)$ is supported in a 2-plane $P_C \subset T_x M$.

Flatness of Tangent Cones. We now show that for all $x \in \text{spt}\|V\|$, every tangent cone $C \in \text{VarTan}(V, x)$ is supported in some 2-plane $P_C \subset T_x M$.

Lemma 3.7.3 (Flatness). *Let U, Σ_k, V be as in the statement of Theorem 3.7.2. If $x \in \text{spt}\|V\|$ and $C \in \text{VarTan}(V, x)$, then $\text{spt}\|C\| \subset P_C$ for some 2-plane P_C .*

Proof. Let $x \in \text{spt}\|V\|$, and let $C \in \text{VarTan}(V, x)$. There is a subsequence (not relabeled) and radii $r_k \rightarrow \infty$ so that $v^k := (\mu_{r_k} \circ \phi_x^{-1})_\# V \rightarrow C$. By selecting a further subsequence (not relabeled), the surfaces $s_k := (\mu_{r_k} \circ \phi_x^{-1})(\Sigma_k)$ satisfy

$$\mathbf{v}(s_k) \rightarrow C.$$

We can choose the subsequence so that $r_k^2 \varepsilon_k \rightarrow 0$.

By Sard's theorem, we can find a radius $r > 0$ so that $b_r := b_r^{T_x M}(0)$ satisfies $s_k \in \mathcal{D}^*(b_r)$ for all k . Note that by (3.7.2) (and the same argument as step 1 above), we have

$$\mathcal{H}^2(s_k) \leq \mathcal{H}^2(s'_k) + r_k^{-1} T_h(s_k, s'_k) + r_k^2 \varepsilon_k + o_k(1)$$

for all $s'_k \in \mathcal{D}(b_r)[s_k]$.

We apply Lemma 3.5.2 to s_k in b_r with constant $\theta < r/2$, producing new discs $\tilde{s}_k$ so that

$$\mathcal{H}^2(\tilde{s}_k) \leq \mathcal{H}^2(s'_k) + r_k^{-1} T_h(\tilde{s}_k, s'_k) + r_k^2 \varepsilon_k + o_k(1)$$

for all $s'_k \in \mathcal{D}(b_r)[\tilde{s}_k]$. By (3.5.2), we have

$$\lim_{k \rightarrow \infty} \mathbf{v}(\tilde{s}_k \cap b_{r/2}) = C \llcorner G(b_{r/2}, 2).$$

Since $\tilde{s}_k \triangle s'_k \subset b_r$ for all $s'_k \in \mathcal{D}(b_r)[\tilde{s}_k]$ by Lemma 3.5.2, we have $T_h(\tilde{s}_k, s'_k) \leq c\mathcal{H}^3(b_r) < \infty$. Hence,

$$\varepsilon'_k := r_k^{-1} T_h(s_k, s'_k) + r_k^2 \varepsilon_k + o_k(1)$$

satisfies $\varepsilon'_k \rightarrow 0$, and

$$\mathcal{H}^2(\tilde{s}_k) \leq \mathcal{H}^2(s'_k) + \varepsilon'_k$$

for all $s'_k \in \mathcal{D}(b_r)[\tilde{s}_k]$.

Since we have shown the discs $\tilde{s}_k$ are a minimising sequence for area with respect to competitors s'_k in the class $\mathcal{D}(b_r)[\tilde{s}_k]$, we can now conclude by following the proof strategy of [AS79].

Indeed, we observe that [AS79, Theorem 3 and Corollary 2] hold for the area minimisation problem restricted to $\mathcal{D}(b_r)[\tilde{s}_k]$. Case I and Case II in the proof of [AS79, Theorem 3] follow directly from the argument in [AS79]. As for Case III, note that the relevant step in the proof only uses discs obtained from the replacement procedure (which lie in our restricted set of competitors) and comparison to isoperimetric regions (also in the restricted set of competitors). The rest of the proofs are unchanged. $\square$

Lemma 3.7.3, together with the previous step of the proof, shows that V has positive integer density at every point of its support.

Decomposition of V . To study the local regularity of V , we return to work in M . We adapt the set up from the proof of integer density. Let $x \in \text{spt}\|V\|$, $\xi > 0$, $r_j \rightarrow \infty$, $\sigma_j \rightarrow 0$, $\zeta > 0$, $\sigma_{k,j} \in (\sigma_j/2, \sigma_j)$ as above. We define

- $B_{\rho,j} := B_{\rho/r_j}^M(x),$

- $C_{\rho,\sigma,j} := (\mu_{r_j} \circ \phi_x^{-1})^{-1}(c_{\rho,\sigma}) \subset B_{\rho,j},$
- $D_{\rho,\sigma,j}^{\pm} := (\mu_{r_j} \circ \phi_x^{-1})^{-1}(d_{\rho,\sigma}^{\pm}) \subset B_{\rho,j}.$

We apply Lemma 3.5.2 to $\Sigma_k \in \mathcal{D}^*(B_{\rho_1,j})$ with constant ξ , producing new discs $S_{k,j}$. By (3.5.2), we have

$$\mathcal{H}^2(S_{k,j}) \leq \mathcal{H}^2(S'_{k,j}) + T_h(S_{k,j}, S'_{k,j}) + \varepsilon_k$$

for all $S'_{k,j} \in \mathcal{D}(B_{\rho_1,j})[S_{k,j}]$. Let $\mathcal{S}_{k,j}$ be the set of discs composing $S_{k,j}$ in $B_{\rho_1,j}$. Let $\mathcal{L}_{k,j} \subset \mathcal{S}_{k,j}$ be the discs with homotopically nontrivial boundary in $C_{\rho_1,\sigma_{k,j},j}$. We translate the estimates from the proof of integer density to M .

Henceforth we assume $j \geq j'_6(c, \rho_1, M, V, \xi)$ and $k \geq k_6(j, c, \rho_1, M, V, \xi)$ as above. Namely, we have $\#\mathcal{L}_{k,j} = n$ for all such j and k .

Label the discs $D \in \mathcal{L}_{k,j}$ as $\{D_{k,j,1}, \dots, D_{k,j,n}\}$. There is a subsequence (not relabeled, after shuffling the 1 through n labels) so that $\mathbf{v}(D_{k,j,i})$ converges in the varifold sense to a limit $V_{j,i}$ for $i = 1, \dots, n$ as $k \rightarrow \infty$, satisfying

$$\|V_{j,i}\|(B_{\rho_1-2\xi,j}) \geq \pi(\rho_1 - 4\xi)^2 r_j^{-2} \quad \text{and} \quad \|V_{j,i}\|(B_{\rho_1,j}) \leq \pi(\rho_1 + 2\xi)^2 r_j^{-2},$$

$$\text{spt}\|V_{j,i}\| \subset (\mu_{r_j} \circ \phi_x^{-1})^{-1}(B_{\rho_1}^{T_x M}(0) \cap \{-\sigma_j \leq x_3 \leq \sigma_j\}),$$

and

$$V \llcorner G(B_{\rho_1-2\xi,j}, 2) = \sum_{i=1}^n V_{j,i} \llcorner G(B_{\rho_1-2\xi,j}, 2).$$

Reduction to Stacked disc minimisation. We now reduce the problem to a stacked disc minimisation problem.

Let $\mathcal{E}_{k,j} := \mathcal{S}_{k,j} \setminus \mathcal{L}_{k,j}$. For $D \in \mathcal{E}_{k,j}$ let $F_D \subset \partial B_{\rho_1,j}$ be the isoperimetric region for ∂D . Let $\Lambda_D \subset B_{\rho_1,j}$ be the open set bounded by $D \cup F_D$.

Steps 1 and 2 from the proof of integer density above imply

$$\limsup_k \sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^2(D \cap B_{1/2,j}) = 0.$$

In particular, (3.7) and (3.7) imply

$$\mathcal{H}^2(D) + \mathcal{H}^2(F_D) \leq A(\xi, \zeta, j, k)$$

with $A(\xi, \zeta, j, k) \rightarrow 0$ as $(\xi, \zeta) \rightarrow 0$ and $j, k \rightarrow \infty$. By (3.5.1), we have

$$\mathcal{H}^3(\Lambda_D) \leq c_1 A(\xi, \zeta, j, k)^{3/2}.$$

Taking ξ and ζ small and j and k large, the coarea formula implies

$$\limsup_k \mathcal{H}^2(\Lambda_D \cap \partial B_{\rho,j}) \leq \frac{1}{4} \mathcal{H}^2(\partial B_{1/4,j}) \leq \frac{1}{4} \mathcal{H}^2(\partial B_{\rho,j})$$

for ρ in a set of positive measure in $(1/4, 1/2)$. By the coarea formula and (3.7), we have

$$\limsup_k \sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^1(D \cap \partial B_{\rho,j}) = 0$$

for almost every $\rho \in (1/4, 1/2)$. Take $\rho \in (1/4, 1/2)$ satisfying (3.7) and (3.7). By the coarea formula and (3.7), we have

$$\limsup_k \sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^0(D \cap \partial C_{\rho,\sigma,j}) = 0$$

for σ in a set of positive measure in $(\sigma_0/4, \sigma_0/2)$.

For $D \in \mathcal{E}_{k,j}$, let B_D be the unique component of $D \setminus B_{\rho,j}$ containing ∂D . Note that B_D is a punctured disc with boundary $\partial D \cup \bigcup_l \alpha_l$, where α_l are smooth Jordan curves in $\partial B_{\rho,j}$. Let N_D be a smooth disc in $\overline{B}_{\rho_1,j} \setminus \overline{B}_{\rho,j}$ obtained by gluing F_{α_l} (the

isoperimetric region in $\partial B_{\rho,j}$ for α_l) to B_D and then smoothing. This procedure is well-defined as long as F_{α_l} is glued and perturbed before $F_{\alpha_{l'}}$ if $F_{\alpha_l} \subset F_{\alpha_{l'}}$. Moreover, the resulting surface is a disc by construction.

We construct a competitor $\mathbf{S}_{k,j} \in \mathcal{D}(B_{\rho_1,j})[S_{k,j}]$ by replacing each $D \in \mathcal{E}_{k,j}$ by N_D . Since the surfaces B_D are disjoint, we can choose the smoothing so that the surfaces N_D are also disjoint, so $\mathbf{S}_{k,j}$ is well-defined.

We claim that we can replace $S_{k,j}$ by $\mathbf{S}_{k,j}$ in (3.7) after replacing ε_k by some $\varepsilon_{k,j}$ satisfying $\lim_k \varepsilon_{k,j} = 0$. Note that N_D is obtained from D by (a) applying Lemma 3.1.3 to each component of $D \setminus B_{\rho,j}$ except B_D , (b) deleting everything in $\overline{B}_{\rho,j}$, and (c) turning B_D into N_D as described above. By Lemma 3.1.3 and (3.7), the $\mathcal{A}^h$ functional does not increase after (a). Since we only delete portions of the surface in (b), the area does not increase from (b). By (3.5.1), (3.1.1), and (3.7), we have (for k large)

$$\begin{aligned} \mathcal{H}^3(\Lambda_D \cap B_{\rho,j}) &\leq c_1(\mathcal{H}^2(D \cap B_{\rho,j}) + \mathcal{H}^2(\Lambda_D \cap \partial B_{\rho,j}))^{3/2} \\ &\leq c_1(\mathcal{H}^2(D \cap B_{\rho,j}) + \beta_0 \mathcal{H}^1(D \cap \partial B_{\rho,j})^2)^{3/2} \end{aligned}$$

Hence, we have

$$\begin{aligned} \sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^3(\Lambda_D \cap B_{\rho,j}) &\leq c_1 \sum_{D \in \mathcal{E}_{k,j}} (\mathcal{H}^2(D \cap B_{\rho,j}) + \beta_0 \mathcal{H}^1(D \cap \partial B_{\rho,j})^2)^{3/2} \\ &\leq c_1 \left(\sum_{D \in \mathcal{E}_{k,j}} (\mathcal{H}^2(D \cap B_{\rho,j}) + \beta_0 \mathcal{H}^1(D \cap \partial B_{\rho,j})^2) \right)^{3/2} \\ &\leq c_1 \left(\sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^2(D \cap B_{\rho,j}) + \beta_0 \left(\sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^1(D \cap \partial B_{\rho,j}) \right)^2 \right)^{3/2} \end{aligned}$$

and the right hand side goes to zero by (3.7) and (3.7). Therefore the volume change from (b) vanishes as $k \rightarrow \infty$. Finally, by (3.7) and (3.1.1), we have $\lim_k \sum_{\mathcal{E}_{k,j}} \mathcal{H}^2(N_D \triangle B_D) =$

0, so the area added in (c) vanishes as $k \rightarrow \infty$. Hence, we have

$$\mathcal{H}^2(\mathbf{S}_{k,j}) \leq \mathcal{H}^2(S'_{k,j}) + T_h(\mathbf{S}_{k,j}, S'_{k,j}) + \varepsilon_{k,j}$$

for all $S'_{k,j} \in \mathcal{D}(B_{\rho_1,j})[\mathbf{S}_{k,j}]$, where $\lim_k \varepsilon_{k,j} = 0$.

Let $D'_{k,j,i} \in \mathcal{D}(B_{\rho,j})[D_{k,j,i}]$ be mutually disjoint for $i = 1, \dots, n$. Let $\Omega_{k,j,i}$ be the open set in $B_{\rho_1,j}$ so that

$$\partial[\Omega_{k,j,i}] \lrcorner B_{\rho_1,j} = [D_{k,j,i}] \lrcorner B_{\rho_1,j}.$$

Let $\Omega'_{k,j,i}$ be the analogous open set for $D'_{k,j,i}$. We claim that

$$\sum_i \left(\mathcal{H}^2(D_{k,j,i}) - \int_{\Omega_{k,j,i}} h \, d\mathcal{H}^3 \right) \leq \sum_i \left(\mathcal{H}^2(D'_{k,j,i}) - \int_{\Omega'_{k,j,i}} h \, d\mathcal{H}^3 \right) + \varepsilon''_{k,j}$$

for some $\lim_k \varepsilon''_{k,j} \rightarrow 0$.

Consider the competitor $S'_{k,j} \in \mathcal{D}(B_{\rho_1,j})[\mathbf{S}_{k,j}]$ given by replacing each $D_{k,j,i}$ by $D'_{k,j,i}$. This competitor is well-defined (i.e. there are no self-intersections) because $(\mathbf{S}_{k,j} \setminus \bigcup_i D_{k,j,i}) \cap B_{\rho,j} = \emptyset$ by definition, and $D'_{k,j,i} \setminus \overline{B}_{\rho,j}$ is a subset of $D_{k,j,i}$. Hence, (3.7) follows directly from (3.7).

We conclude with some notation. Let $B_{k,j,i}$ be the open set in $B_{\rho_1,j}$ so that

$$\partial B_{k,j,i} \setminus D_{k,j,i} \subset \partial B_{\rho_1,j} \quad \text{and} \quad D_{\rho_1,\sigma_{k,j},j}^- \subset \partial B_{k,j,i}.$$

If $B_{k,j,i} \subset B_{k,j,i'}$ for some $i \neq i'$, then we say $D_{k,j,i} < D_{k,j,i'}$. If $B_{k,j,i} = \Omega_{k,j,i}$ (as defined above), then we say $D_{k,j,i}$ is oriented up; otherwise, it is oriented down. By selecting a subsequence and relabeling, we can assume that

$$D_{k,j,1} > D_{k,j,2} > \dots > D_{k,j,n},$$

and that $D_{k,j,i}$ is oriented up if and only if $D_{k',j,i}$ is oriented up for all $k' \neq k$. By the assumption that there is an open set Λ_k with $\partial\Lambda_k \cap U = \Sigma_k \cap U$, we conclude that $D_{k,j,i}$ is oriented up if and only if $D_{k,j,i-1}$ and $D_{k,j,i+1}$ are oriented down (i.e. the orientation alternates with respect to the order of the discs).

Parity for Stacked discs. Let Ω_k and Ω be defined as in Theorem 3.3.1 with respect to the set U . Let $x \in \text{spt}\|V\|$. We claim that

$$\Theta^2(\|V\|, x) \equiv 0 \pmod{2} \implies x \notin \partial^*\Omega,$$

and

$$\Theta^2(\|V\|, x) \equiv 1 \pmod{2} \implies x \in \overline{\partial^*\Omega}.$$

Let W be any neighbourhood of x . Taking j large enough, we have $B_{\rho_1,j} \subset W$. Let $\Omega_{k,j,i}$ be defined as above for the stacked discs $D_{k,j,i}$. Let $\Omega_{k,j}$ be the open subset of $B_{\rho_1,j}$ satisfying

$$\partial[\Omega_{k,j}] \llcorner B_{\rho_1,j} = \sum_{i=1}^n [D_{k,j,i}] \llcorner B_{\rho_1,j}.$$

Let $\Omega^{j,i}$ and Ω^j be the limits of $\Omega_{k,j,i}$ and $\Omega_{k,j}$ in k respectively, in the sense of Theorem 3.3.1.

We first show that $\mathcal{H}^3((\Omega^j \triangle \Omega) \cap B_{\rho,j}) = 0$. Indeed, we have

$$\begin{aligned} \mathcal{H}^3((\Omega^j \triangle \Omega) \cap B_{\rho,j}) &= \lim_k \mathcal{H}^3((\Omega_{k,j} \triangle \Omega_k) \cap B_{\rho,j}) \\ &\leq \lim_k \mathcal{H}^3(\beta_k) + \sum_{D \in \mathcal{E}_{k,j}} \mathcal{H}^3(\Lambda_D \cap B_{\rho,j}) = 0, \end{aligned}$$

by Lemma 3.5.2(3.5.2) and (3.7) (where β_k are the bubbles called “ B ” in Lemma 3.5.2). Hence, we have

$$\partial^*\Omega \cap B_{\rho,j} = \partial^*\Omega^j \cap B_{\rho,j},$$

so it suffices to work with Ω^j .

Let $\omega^j := (\mu_{r_j} \circ \phi_x^{-1})(\Omega^j)$. Let $v_{j,i} := (\mu_{r_j} \circ \phi_x^{-1})_{\#} V_{j,i}$. By (3.7), we have

$$\text{spt} \left\| \sum_i v_{j,i} \right\| \cap b_\rho \subset b_\rho \cap \{-\sigma_j \leq x_3 \leq \sigma_j\}.$$

Let

$$b_{\rho,j}^+ := b_\rho \cap \{x_3 > \sigma_j\} \quad \text{and} \quad b_{\rho,j}^- := b_\rho \cap \{x_3 < -\sigma_j\}.$$

Since $\overline{\partial^* \omega^j} \cap b_\rho \subset \text{spt} \left\| \sum_i v_{j,i} \right\| \cap b_\rho$, we have either $b_{\rho,j}^+ \subset \omega^j$ or $b_{\rho,j}^+ \cap \omega^j = \emptyset$ (and analogously for $b_{\rho,j}^-$). We see that $D_{k,j,1}$ is oriented down if and only if $b_{\rho,j}^+ \subset \omega^j$. Similarly, $D_{k,j,n}$ is oriented up if and only if $b_{\rho,j}^- \subset \omega^j$. Note that $b_{\rho,j}^\pm$ contain a definite portion of $b_2^\pm$ (the upper and lower half balls of radius 2 in $T_x M$).

If $n \equiv 1 \pmod{2}$, then ω^j converges in L^1 as $j \rightarrow \infty$ (without loss of generality) to the upper half space $H^+ \subset T_x M$. Hence, every sufficiently small neighbourhood W of x satisfies

$$0 < \mathcal{H}^3(\Omega \cap W) < \mathcal{H}^3(W),$$

so we have $x \in \overline{\partial^* \Omega}$.

If $n \equiv 0 \pmod{2}$, then ω^j converges in L^1 as $j \rightarrow \infty$ to $T_x M$ or $\emptyset$, so $x \notin \partial^* \Omega$ (i.e. we showed x is not in the essential boundary of Ω , which contains the reduced boundary (see [Mag12, Theorem 16.2]).

$C^{1,\alpha}$ Regularity. By Lemma 3.4.7(3.4.7) and (3.7), we have

$$\mathcal{H}^2(D_{k,j,i}) \leq \mathcal{H}^2(D'_{k,j,i}) + c|T(D_{k,j,i}, D_{k,j,i'})| + \varepsilon_{k,j}$$

for any $D'_{k,j,i} \in \mathcal{D}(B_{\rho,j})[D_{k,j,i}]$.

We observe that everything we have proved thus far only requires that the function h is measurable and satisfies $|h| \leq c$. We note that (3.7) implies that $D_{k,j,i}$ is a minimising sequence for the problem with prescribing function $-c + 2c\mathbf{1}_{\Omega_{j,i}}$. Hence, all of the above work applies to $V_{j,i}$. In particular, $V_{j,i}$ has c -bounded first variation

and positive integer density in its support. By (3.7), there is a neighbourhood W of x so that $\Theta^2(V_{j,i}, y) = 1$ for all $y \in W \cap \text{spt}\|V_{j,i}\|$.

By [All72, §8], we have $V_{j,i} \llcorner G(W, 2) = \mathbf{v}(N_{j,i}, 1)$ for a $C^{1,\alpha}$ surface $N_{j,i} \subset W$. By (3.7), we have $N_{j,i} \cap W = \partial^* \Omega^{j,i} \cap W$. Hence, $N_{j,i}$ is on one side of $N_{j,i'}$ for any i, i' , and each of these surfaces intersects tangentially at x . We conclude by (3.7).

Decomposition into 1-Stacks and 2-Stacks. Take x so that $h(x) \neq 0$. Suppose without loss of generality (by swapping orientations and taking j sufficiently large) that $h|_{B_{\rho_1,j}} > 0$. We now decompose the stacked disc minimisation problem (3.7) into smaller units, which we call 1-stacks and 2-stacks.

We first separate the 1-stacks. If $D_{k,j,1}$ is oriented up, then we show that

$$\mathcal{H}^2(D_{k,j,1}) - \int_{\Omega_{k,j,1}} h \, d\mathcal{H}^3 \leq \mathcal{H}^2(D'_{k,j,1}) - \int_{\Omega'_{k,j,1}} h \, d\mathcal{H}^3 + \varepsilon_{k,j}$$

for all $D'_{k,j,1} \in \mathcal{D}(B_{\rho,j})[D_{k,j,1}]$, where $\Omega'_{k,j,1}$ is defined as above. The analogous statement holds if $D_{k,j,n}$ is oriented down by vertical reflection. The result follows immediately from Lemma 3.4.8.

Now we separate the 2-stacks. If $D_{k,j,i'}$ is oriented down for some $1 \leq i' < n$, then we show that

$$\sum_{i=i'}^{i'+1} \left(\mathcal{H}^2(D_{k,j,i}) - \int_{\Omega_{k,j,i}} h \, d\mathcal{H}^3 \right) \leq \sum_{i=i'}^{i'+1} \left(\mathcal{H}^2(D'_{k,j,i}) - \int_{\Omega'_{k,j,i}} h \, d\mathcal{H}^3 \right) + \varepsilon'_k$$

for all mutually disjoint $D'_{k,j,i} \in \mathcal{D}(B_{\rho,j})[D_{k,j,i}]$ for $i = i', i' + 1$, where $\Omega'_{k,j,i}$ is defined as above. The result follows immediately from Lemma 3.4.9.

Regularity of 1-Stacks. Suppose (3.7) holds⁴. Recall that we showed $\Theta^2(V_{j,1}, y) = 1$ for all $y \in W \cap \text{spt}\|V_{j,1}\|$ and

$$\text{spt}\|V_{j,1}\| \cap W = \partial^* \Omega^{j,1} \cap W.$$

⁴We note that (3.7) holds at any point in V of density 1, as well as in the case $D_{k,j,1}$ is oriented up.

Hence, we have

$$\|V_{j,1}\|_{\perp W} = |D\mathbf{1}_{\Omega^{j,1}}|_{\perp W}.$$

Inserting this equality into Theorem 3.3.1(3.3.1), we conclude that

$$H_{V_{j,1}}|_W = h\nu_{\partial^*\Omega^{j,1}}|_W,$$

which is a $C^{1,\alpha}$ vector field. Applying the regularity theorem of Allard [All72, §8] iteratively, we conclude that $N_{j,1}$ is a smooth surface with prescribed mean curvature h in W with respect to the set $\Omega^{j,1}$. Stability follows from smooth embeddedness and (3.3.1).

Regularity of 2-Stacks. Suppose $D_{k,j,i'}$ is oriented down for $1 \leq i' < n$, so (3.7) holds. For ease of notation, let $i' = 1$.

We show the mean convexity of $V_{j,1}$. Let X be a smooth vector field with compact support in $B_{\rho,j}$, and let Φ_t be the flow of X . Without loss of generality, we suppose that

$$\left. \frac{d}{dt} \right|_{t=0} \|(\Phi_t)_\# V_1\|(B_{\rho,j}) < 0,$$

(i.e. either the derivative is zero, or we can swap $X \mapsto -X$). By Lemma 3.4.7(3.4.7) and (3.7), we have (using $h \geq 0$)

$$\mathcal{H}^2(D_{k,j,1} \cap B_{\rho,j}) \leq \mathcal{H}^2(\Phi_t(D_{k,j,1}) \cap B_{\rho,j}) + c\mathcal{H}^3(\Omega_{k,j,1} \setminus \Phi_t(\Omega_{k,j,1})) + \varepsilon_k''.$$

Following the proof of Theorem 3.3.1, we have

$$\|V_{j,1}\|(B_{\rho,j}) \leq \|(\Phi_t)_\# V_{j,1}\|(B_{\rho,j}) + c\mathcal{H}^3(\Omega^{j,1} \setminus \Phi_t(\Omega^{j,1})).$$

Hence, we have

$$0 > \frac{d}{dt} \Big|_{t=0} \|(\Phi_t)_\# V_{j,1}\|(B_{\rho,j}) \geq -c \frac{d}{dt} \Big|_{t=0^+} \mathcal{H}^3(\Omega^{j,1} \setminus \Phi_t(\Omega^{j,1})).$$

By [Mag12, Proposition 17.8], we have

$$\frac{d}{dt} \Big|_{t=0^+} \mathcal{H}^3(\Omega^{j,1} \setminus \Phi_t(\Omega^{j,1})) \leq \int_{\partial^* \Omega_1} (X \cdot \nu_{\partial^* \Omega^{j,1}})_- d\mathcal{H}^2.$$

Hence, we observe that the first variation cannot be negative if $X \cdot \nu_{\partial^* \Omega^{j,1}} \geq 0$, which proves the desired weak mean convexity. By vertical reflection, the same holds for $V_{j,2}$.

Suppose $\text{spt}\|V_{j,1}\| \cap W' = \text{spt}\|V_{j,2}\| \cap W'$ for some open set $W' \subset W$. As shown above, $\Theta^2(V_{j,1} + V_{j,2}, x) = 2$ for all $x \in \text{spt}\|V_{j,1} + V_{j,2}\| \cap W'$. By (3.7), we have $\partial^*(\Omega^{j,1} \cup \Omega^{j,2}) \cap W' = \emptyset$, so in fact $\overline{\partial^*(\Omega^{j,1} \cup \Omega^{j,2})} \cap W' = \emptyset$. Since we already showed $\text{spt}\|V_{j,i}\| \cap W' = \partial^* \Omega^{j,i} \cap W'$, we conclude that $\mathcal{H}^3((\Omega^{j,1} \cup \Omega^{j,2}) \cap W') = \mathcal{H}^3(W')$. By Theorem 3.3.1(3.3.1), $V_{j,1} + V_{j,2}$ is stationary in W' . Then by Allard's regularity theorem [All72, §8],

$$(V_{j,1} + V_{j,2}) \llcorner G(W', 2) \cap \mathbf{v}(N, 2),$$

where $N \subset W'$ is a smooth minimal surface. Hence, $N_{j,1} \cap W' = N_{j,2} \cap W' = N$. Stability follows from smooth embeddedness and Theorem 3.3.1(3.3.1).

Maximum Principle. First, suppose $\Theta^2(\|V\|, x) = 2n + 1$. Then (without loss of generality) $D_{k,j,1}$ is oriented up. By the regularity of 1-stacks, $V_{j,1}$ is the varifold associated to a smooth surface $N_{j,1}$ with mean curvature $h > 0$ pointing up. Suppose for contradiction that $n \geq 1$. Then $V_{j,2}$ is the varifold associated to a $C^{1,\alpha}$ surface $N_{j,2}$ with generalised mean curvature pointing down. Moreover, $N_{j,2}$ is below $N_{j,1}$ and intersects $N_{j,1}$ tangentially at x , which contradicts the strong maximum principle (see Lemma 3.8.2). Hence, if $D_{k,j,1}$ is oriented up, then $n = 1$ and V is the varifold of a

smooth surface of prescribed mean curvature h pointing up in a neighbourhood of x .

Second, suppose $\Theta^2(\|V\|, x) = 2n$ with $n \geq 2$. By the above paragraph, V consists of n 2-stacks intersecting tangentially and on one side at x . Then $N_{j,2}$ and $N_{j,2n-1}$ are $C^{1,\alpha}$ surfaces on one side of each other, intersecting tangentially at x , and with generalised mean curvature pointing away from each other. By the strong maximum principle (see Lemma 3.8.2 in the next section) and the regularity of 2-stacks, there is a neighbourhood W'' of x and a minimal surface $N \subset W''$ satisfying $N_{j,i} \cap W'' = N$ for all i . Hence, $\Theta^2(\|V\|, y) = 2n$ for all $y \in W'' \cap \text{spt}\|V\|$. $\square$

3.8 Maximum principle

A maximum principle is required to patch together the limits of decomposed stacked disc minimisation problems. Since we obtain lower regularity than in the minimal surface setting, we need a maximum principle in the setting of $C^{1,\alpha}$ surfaces.

We adopt the following definition from [ZZ19].

Definition 3.8.1. Let N_i ($i = 1, 2$) be connected embedded two-sided C^1 surfaces in a connected open subset $W \subset M$ with $\partial N_i \cap W = \emptyset$. Let ν_i be the unit normal to N_i . We say that N_2 *lies on one side of* N_1 if N_1 divides W into two connected components $W_1 \cup W_2 = W \setminus N_1$, where ν_1 points into W_1 and either:

- $N_2 \subset \overline{W_1}$, which we write as $N_1 \leq N_2$; or
- $N_2 \subset \overline{W_2}$, which we write as $N_1 \geq N_2$.

Lemma 3.8.2 (Maximum Principle). *Let N_1 and N_2 be $C^{1,\alpha}$ surfaces in a connected open set W so that N_1 lies on one side of N_2 , $N_1 \geq N_2$, $N_2 \geq N_1$, and N_1 and N_2 intersect tangentially at $x \in W$. If each N_i has nonnegative mean curvature in the weak sense with respect to the normal ν_i , then $N_1 = N_2$ and each is a smooth minimal surface.*

Proof. Since the unit normals point in opposite directions, being smooth and minimal follows if we show $N_1 = N_2$.

Suppose for contradiction $N_1 \neq N_2$. Without loss of generality, there is a sequence $x_j \rightarrow x$ with $x_j \in N_1 \setminus N_2$. Let $P \subset T_x M$ be the common tangent plane of N_1 and N_2 at x , and let $\nu = \nu_1(x)$ be a unit normal to P , so that $\nu_2(x) = -\nu$. Let $r_0 \in (0, 1)$ be sufficiently small so that $\phi_x^{-1}(N_1 \cap B_r^M(x))$ and $\phi_x^{-1}(N_2 \cap B_{r_0}^M(x))$ are $C^{1,\alpha}$ graphs over $B \subset P$, given by $C^{1,\alpha}$ functions $u_1 \geq u_2$ and let $\rho \in (0, r_0/2)$. Let B_r denote $B_r^{\mathbb{R}^2}(0)$.

We shall use the same setup as in the proof of Proposition 3.9.1. We let

- $\tilde{g}$ be the rescaled pullback metric $(\phi_x \circ \mu_\rho)^*g$, so that

$$\|\tilde{g} - g_{\text{Euc}}\|_{C^\infty(B_2 \times [-1,1])} \leq \delta(\rho),$$

where $\delta(\rho) \rightarrow 0$ as $\rho \rightarrow 0$;

- $\tilde{u}_i : B_2 \rightarrow \mathbb{R}$ be defined by $\tilde{u}_i(x) = \rho^{-1}u_i(\rho x)$ so that

$$\|\tilde{u}_i\|_{C^1(B_2)} \leq \delta(\rho);$$

- $A(x, z, p)$ be such that with the rescaled metric $\tilde{g}$ on $\mathbb{R}^3$, we have $\mathcal{H}^2(\text{graph } u|_{B_r}) = \int_{B_r} A(x, u, Du) \, dx$ for all C^1 functions $u : B_2 \rightarrow \mathbb{R}$ and $r \in (0, 2)$;
- Q be the mean curvature operator for graphs over B_2 with respect to $\tilde{g}$.

Then $Q(u) \leq 0$ (resp. ≥ 0) in the weak sense in B_1 is equivalent to (in our choice of sign convention)

$$\int_{B_1} (A_p(x, u, Du) \cdot D\phi + A_z(x, u, Du)\phi) \leq 0 \text{ (resp. } \geq 0)$$

for all $\phi \in W_0^{1,2}(B_1)$ with $\phi \geq 0$.

By assumption, we have $Q(\tilde{u}_1) \geq 0$ and $Q(\tilde{u}_2) \leq 0$ in the weak sense. If we let $v = \tilde{u}_1 - \tilde{u}_2 \geq 0$, then $v \in C^{1,\alpha}$ is a non-negative weak supersolution of a linear elliptic equation with continuous coefficients (depending on $\tilde{u}_1$ and $\tilde{u}_2$) as in [GT01, Theorem 10.7], i.e.

$$\int_{B_1} (a^{ij} D_j v + b^i v) D_i \phi - (-b^i D_i v + cv) \phi \geq 0$$

for all non-negative $\phi \in W_0^{1,2}(B_1)$, where $\tilde{u}_t = t\tilde{u}_1 + (1-t)\tilde{u}_2$ and

$$\begin{aligned} a^{ij} &:= \int_0^1 A_{p_i p_j}(x, \tilde{u}_t, D\tilde{u}_t) dt \\ b^i &:= \int_0^1 A_{p_i z}(x, \tilde{u}_t, D\tilde{u}_t) dt \\ c &:= - \int_0^1 A_{zz}(x, \tilde{u}_t, D\tilde{u}_t) dt. \end{aligned}$$

Since N_1 and N_2 intersect tangentially at x , $v(0) = 0$, and $\inf_{B_{1/4}} v = 0$. Hence, the weak Harnack inequality for supersolutions [GT01, Theorem 8.18] implies $\|v\|_{L^1(B_{1/2})} = 0$. As v is $C^{1,\alpha}$, this means $v|_{B_{1/2}} \equiv 0$, contradicting the choice of the sequence $x_j \rightarrow x$. $\square$

3.9 Improvement from free boundary problems

We now use (and generalise) some results from the theory of free boundary problems to improve our results on the regularity of the limit surfaces in Theorem 3.7.2.

For the remainder of this section, let V be as in Theorem 3.7.2, and $x \in \text{spt}\|V\|$. Also, let W_x be the open neighbourhood of x provided by the Theorem 3.7.2, and let $N_1, N_2, \dots, N_{n_x}$ be the corresponding $C^{1,\alpha}$ surfaces meeting tangentially at x .

The main result of this section is the following.

Proposition 3.9.1. *The surfaces $N_1, \dots, N_{n_x}$ are $C^{1,1}$ in the interior of W_x .*

Proof. By Theorem 3.7.2, $V_{\perp}G(W_x, 2) = \sum_{l=1}^{n_x} \mathbf{v}(N_l, 1)$, where N_i are distinct $C^{1,\alpha}$ hypersurfaces with c -bounded generalised mean curvature, such that each N_l is on one side of $N_{l'}$ intersecting tangentially at x . Let then $n := n_x$ and let $P \subset T_x M$ be the common tangent plane of $N_1, N_2, \dots, N_n$ at x .

We shall represent the surfaces N_l locally as graphs over their common tangent plane P , which we shall for the rest of this proof identify with $\mathbb{R}^2$. For the rest of this proof, let us set $B_r := B_r^{\mathbb{R}^2}(0)$.

First, let $r_0 \in (0, 1)$ be sufficiently small so that for $l = 1, \dots, n$

$$\phi_x^{-1}(N_l \cap B_{r_0}^M(x)) = \text{graph } u_l$$

where $u_l : B_{r_0} \rightarrow \mathbb{R}$ is a $C^{1,\alpha}$ function such that $u_l(0) = 0$ and $Du_l(0) = 0$. Since the surfaces lie on one side of each other, we have

$$u_1 \geq u_2 \geq \dots \geq u_n.$$

Let now $\rho \in (0, r_0/2)$ such that $|u_l| < \rho$ on $B_{2\rho}$ for all $l = 1, \dots, n$, and let us rescale the metric g on $T_x M$, so that we can work over the unit ball in $\mathbb{R}^2$ with respect to the rescaled metric. To this end, let $\tilde{g}$ be the pullback metric $\tilde{g} := (\phi_x \circ \mu_\rho)^* g$ (where $\mu_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the dilation by $t > 0$), which satisfies

$$\|\tilde{g} - g_{\text{Euc}}\|_{C^\infty(B_2 \times [-1, 1])} \leq \delta(\rho),$$

where $\delta(\rho) \rightarrow 0$ as $\rho \rightarrow 0$, since (M, g) is a smooth Riemannian manifold.

Finally, we define the rescaled functions $\tilde{u}_l : B_2 \rightarrow (-1, 1)$ by $\tilde{u}_l(y) = \rho^{-1} u_l(\rho y)$. Note that we also have

$$\|\tilde{u}_l\|_{C^{1,\alpha}(B_2)} \leq \delta(\rho) < \frac{1}{2}$$

for sufficiently small $\rho > 0$.

For a C^1 function $v : B_2 \rightarrow \mathbb{R}$, we can parametrise the graph of v by the standard coordinates $y = (y_1, y_2)$ on B_2 and let $A(y, z, p)$ be such that with respect to the rescaled metric $\tilde{g}$ on $\mathbb{R}^3$, we have $\mathcal{H}^2(\text{graph } v|_{B_r}) = \int_{B_r} A(y, v, Dv) \, dy$ for $0 < r \leq 2$.

Let $U = \phi_x(B_\rho \times (-\rho, \rho))$ and consider the discs $D_l = N_l \cap \bar{U} = \phi_x(\text{graph } u_l|_{B_\rho(0)})$. By the proof of Theorem 3.7.2, notice that, after identifying U with $B_\rho \times (-\rho, \rho)$ and rescaling, the n -tuple $(\tilde{u}_1, \dots, \tilde{u}_l)$ is a critical point of the functional

$$I(v_1, \dots, v_n) = \int_{B_1} \left(\sum_{l=1}^n A(y, v_l, Dv_l) + B_l(y, v_l) \right) dy \quad (3.9.1)$$

among all n -tuples $(v_1, \dots, v_n)$ such that $v_l|_{\partial B_1} = \tilde{u}_l|_{\partial B_1}$ and $v_1 \geq v_2 \geq \dots \geq v_n$, where

- $A(y, z, p)$ is defined as above;
- $B_l(y, z)$ is defined by

$$B_l(y, z) = (-1)^{l-1} \int_{-1}^z H(y, t) \, dt$$

where $H(y, t) = \tilde{h}(y, t) \tilde{\sigma}(y, t)$, $\tilde{\sigma}(y, t)$ is the volume density with respect to the rescaled pullback metric $\tilde{g}$, and $\tilde{h}(y, t) = h(\phi_x(\rho y, \rho t))$.

The $\tilde{u}_l$ satisfy the Euler-Lagrange equations for this functional in the weak sense, i.e.

$$Q(\tilde{u}_l) + (-1)^{l-1} H(y, \tilde{u}_l) = 0 \text{ in } \{\tilde{u}_{l-1} > \tilde{u}_l > \tilde{u}_{l+1}\}$$

and

$$Q(\tilde{u}_l) = Q(\tilde{u}_{l+1}) = 0 \text{ in } \{\tilde{u}_l = \tilde{u}_{l+1}\}$$

where Q be the mean curvature operator for graphs over B_1 with respect to the rescaled pullback metric $\tilde{g}$. By Theorem 3.7.2, if a point x_0 lies in $\text{spt } \|V\|$ and

$n_{x_0} > 1$ is odd, then $h(x_0) = 0$. Thus, we see that the functions $\tilde{u}_l$ satisfy

$$\sum_{l=1}^n Q(\tilde{u}_l) = \sum_{l=1}^n (-1)^l H(y, \tilde{u}_l) \quad (3.9.2)$$

in the weak sense.

We can now combine the techniques in [Sil05] and [SY19] to prove the desired $C^{1,1}$ -regularity.

First, we notice that, as in [Sil05, Proposition 4.1], the difference between consecutive sheets $w_l := \tilde{u}_l - \tilde{u}_{l+1}$ satisfies a linear elliptic equation with a right-hand side which is uniformly bounded in L^∞ . Hence, by the Harnack inequality in [GT01], if $0 \in \partial\{\tilde{u}_l = \tilde{u}_{l+1}\}$, then there is a constant C such that

$$w_l(y) \leq C|y|^2$$

in $B_{1/2}$, proving quadratic growth from contact points. We can then run the same induction argument as in [SY19, Proposition 2.8] to prove boundedness of the second derivatives, provided we show an L^∞ -bound on the second derivative of $\sum_l \tilde{u}_l$ (as in [SY19, Lemma 2.7]).

In order to prove this bound, we look at the equation satisfied by the sum $\sum_l \tilde{u}_l$, as in section 5 of [Sil05]. Note that, in local coordinates, the mean curvature operator Q applied to a function u takes the form

$$(D_{p_j} A^i)(y, u, Du) D_{ij} u + (D_z A^i)(y, u, Du) D_i u + (D_{y_i} A^i)(y, u, Du)$$

where $A^i := D_{p_i} A$. Let $a^{ij} := D_{p_j} A^i = D_{p_i p_j} A$. Then, we can rewrite (3.9.2) as

$$\sum_{l=1}^n a^{ij}(y, \tilde{u}_l, D\tilde{u}_l) D_{ij} \tilde{u}_l = R(y) \quad (3.9.3)$$

where $R(y) = \sum_{l=1}^n (-1)^l H(y, \tilde{u}_l) - (D_z A^i)(y, \tilde{u}_l, D\tilde{u}_l) D_i \tilde{u}_l - (D_{y_i} A^i)(y, \tilde{u}_l, D\tilde{u}_l) \in C^{0,\alpha}$. Let us also write L_l for the linear operator $L_l(w) := a^{ij}(y, \tilde{u}_l, D\tilde{u}_l) D_{ij} w$. We can rewrite (3.9.3) as

$$L_n \left(\sum_{l=1}^n \tilde{u}_l \right) = R(y) + \sum_{k=1}^{n-1} (L_{k+1} - L_k) \left(\sum_{l=1}^k \tilde{u}_l \right) \quad (3.9.4)$$

Note that $L_k = L_{k+1}$ on the set $\{\tilde{u}_k = \tilde{u}_{k+1}\}$, and on the open set $\{\tilde{u}_k > \tilde{u}_{k+1}\}$, the collections $\{\tilde{u}_1, \dots, \tilde{u}_k\}$ and $\{\tilde{u}_{k+1}, \dots, \tilde{u}_n\}$ satisfy multiple membrane problems with fewer sheets. We can then use an induction (on the maximum number of simultaneously touching membranes near a point, with the base case of 2 membranes being treated in [Sil05]) argument similar to the proof of [SY19, Proposition 2.8], together with the proofs of Lemma 4.2 and Lemma 4.4 in [Sil05] to conclude that the right-hand side in (3.9.4) is in $C^{0,\alpha}$. The standard Schauder estimates (see [GT01, chapter 6]) then imply that $\sum_l \tilde{u}_l$ is in $C^{2,\alpha}$ with $C^{2,\alpha}$ norm controlled by the $C^{0,\alpha}$ norm of the right-hand side, which in turn depends only on universal constants and on the $C^{1,\alpha}$ norms of the $\tilde{u}_l$. $\square$

As a byproduct of the proof of Proposition 3.9.1, we get the following estimate.

Corollary 3.9.2. *Let the n -tuple $(\tilde{u}_1, \dots, \tilde{u}_N) \subset C^{1,\alpha}(B_1)$ be a critical point of the functional (3.9.1), with*

$$\begin{aligned} \|\tilde{u}_l\|_{C^1(B_1)} &< 1 \quad \forall l = 1, \dots, n \\ D_{p_i p_j} A|_{B_1 \times (-1,1) \times B_1} \cdot \xi_i \xi_j &\geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^2 \\ \|A\|_{C^3(B_1 \times (-1,1) \times B_1)} &\leq \Lambda \end{aligned}$$

and $H \in C^{1,\alpha}(B_1 \times (-1,1))$, as in the proof of Proposition 3.9.1. Then for all $l = 1, \dots, n$, $\tilde{u}_l \in C_{\text{loc}}^{1,1}(B_1)$, and there is a constant C depending on $n, \alpha, \lambda, \Lambda$ such

that

$$\sum_{l=1}^n \|\tilde{u}_l\|_{C^{1,1}(B_{1/2})} \leq C \sum_{l=1}^n (\|\tilde{u}_l\|_{C^{1,\alpha}(B_1)} + \|H\|_{C^{1,\alpha}(B_1 \times (-1,1))}).$$

Moreover, if $h(x) \neq 0$, we can prove a more precise regularity result. By Theorem 3.7.2, if $h(x) \neq 0$, then $n_x \in \{1\} \cup 2\mathbb{N}$ and

- if $n_x = 1$, then $V \llcorner G(W_x, 2) = \mathbf{v}(N, 1)$, where $N \subset W_x$ is a smooth stable surface with prescribed mean curvature h ;
- if $n_y = 2n$ for all $y \in \text{spt}\|V\| \cap W'_x$ for an open set $W'_x \subset W_x$ containing x , then $V \llcorner G(W'_x, 2) = \mathbf{v}(N, 2n)$, where $N \subset W'_x$ is a smooth stable minimal surface.

Hence, in both cases, the support of V in a neighbourhood of x coincides with a smooth surface.

We now deal with the case where $n_x = 2n$, and there is a sequence $x_j \rightarrow x$ with $x_j \in \text{spt}\|V\|$ and $n_{x_j} \neq 2n$. By Theorem 3.7.2, then $n = 1$ and the surfaces $N_1, N_2 \subset W_x$ intersect tangentially at x . Let T denote the touching set $T = N_1 \cap N_2 \subset W_x$ and let Γ be its boundary $\partial T \cap W_x$ in W_x .

Proposition 3.9.3. *The boundary Γ of the touching set T is locally contained in a $C^{1,\alpha}$ curve and satisfies $\mathcal{H}^1(\Gamma) < +\infty$*

Proof. This follows directly from the results in section 4 of [Sil05]. Indeed, the difference $\tilde{u}_1 - \tilde{u}_2$ is a solution to an obstacle problem to which the results of Caffarelli ([Caf77], [Caf80] and [Caf98]) apply. $\square$

Chapter 4

Minimisation over isotopy classes

In this chapter we follow the arguments of [MSY82], [CDL03], and [DLP10], upgrading from the regularity of minimisers of discs to the regularity of minimisers over isotopy classes.

4.1 Minimisation problem

Let (M, g) be a smooth 3-dimensional Riemannian manifold. Let $h : M \rightarrow \mathbb{R}$ be a smooth real valued function, and let $c := \sup_M |h|$.

As before, by rescaling M , we can assume that the conditions of §3.1.1 hold with $\rho_0 = 1$.

Let $U \subset M$ be an open set with C^1 boundary.

Let $\mathcal{O}(U)$ be the set of open sets $\Omega \subset M$ with smooth boundary so that $\partial\Omega$ has transverse (or empty) intersection with U .

In this part, we consider the following question.

Question 4.1.1. *Consider a sequence $\{\Omega_k\}_{k \in \mathbb{N}} \subset \mathcal{O}(U)$ and $\Sigma_k := \partial\Omega_k \cap U$ satisfying $\sup_k \mathcal{H}^2(\Sigma_k) < \infty$, $\sup_k \text{genus}(\Sigma_k) < \infty$, and*

$$\mathcal{A}^h(\Omega_k) \leq \mathcal{A}^h(\phi(1, \Omega_k)) + \varepsilon_k$$

for all $\phi \in \mathcal{I}(U)$ with $\varepsilon_k \rightarrow 0$. Does (a subsequence of) $\Sigma_k := \partial\Omega_j \cap U$ converge to a regular limit with prescribed mean curvature h ?

We first observe that all of the conclusions of Theorem 3.3.1 hold, where $\mathbf{v}(\Sigma_k) \rightarrow V$ and $\Omega_k \cap U \rightarrow \Omega$, by following the same proof.

4.2 Gamma-reduction

The key insight of [MSY82] is the introduction of a procedure, called γ -reduction, to cut away small necks that either create genus or separate large components. This procedure enables the reduction of the full isotopy minimisation problem to the disc minimisation problem, solved in chapter 3 above.

4.2.1 Definition of gamma-reduction

Let $\delta > 0$ be the constant from Lemma 3.1.2, and let $0 < \gamma < \delta^2/9$. Let $U \subset M$ be an open set, and let $\Sigma \subset U$ be a smooth embedded oriented surface with $\partial\Sigma \subset \partial U$. Let $\mathcal{C}(\Sigma, U)$ be the set of smooth embedded oriented surfaces $\Sigma' \subset U$ with $\partial[\Sigma'] = \partial[\Sigma]$.

Definition 4.2.1. For $\Sigma_1, \Sigma_2 \in \mathcal{C}(\Sigma, U)$, we write

$$\Sigma_2 \stackrel{(\gamma, U)}{\ll} \Sigma_1$$

and say Σ_2 is a γ -reduction of Σ_1 if the following hold:

1. Σ_2 is obtained from Σ_1 by surgery in U ; namely,
 - $\overline{\Sigma_1 \setminus \Sigma_2} =: A$ is diffeomorphic to the standard closed annulus,
 - $\overline{\Sigma_2 \setminus \Sigma_1} =: D_1 \cup D_2$ is diffeomorphic to the disjoint union of two standard closed discs,

- there is an open set $Y \subset U$ homeomorphic to the standard 3-ball with $\partial Y = A \cup D_1 \cup D_2$ and $Y \cap (\Sigma_1 \cup \Sigma_2) = \emptyset$.

$$2. \mathcal{H}^2(A) + \mathcal{H}^2(D_1) + \mathcal{H}^2(D_2) < 2\gamma.$$

3. If Γ is the connected component of Σ_1 containing A , then for each component of $\Gamma \setminus A$ we have either

- it is a disc of area at least $\delta^2/2$,
- it is not simply connected.

Definition 4.2.2. For $\Sigma_1, \Sigma_2 \in \mathcal{C}(\Sigma, U)$, we write

$$\Sigma_2 \stackrel{(\gamma, U)}{<} \Sigma_1$$

and say Σ_2 is a *strong γ -reduction* of Σ_1 , if there is an isotopy $\psi \in \mathcal{I}(U)$ satisfying

1. $\Sigma_2 \stackrel{(\gamma, U)}{\ll} \psi(1, \Sigma_1)$,
2. $\mathcal{H}^2(\psi(1, \Sigma_1) \triangle \Sigma_1) < \gamma$.

We say a surface is *(γ, U) -irreducible* (resp. *strongly (γ, U) -irreducible*) if it admits no (γ, U) reduction (resp. strong (γ, U) -reduction).

4.2.2 Structure of gamma-irreducible surfaces

We adapt [MSY82, Theorem 2] to our setting, allowing us to decompose γ -irreducible surfaces in small balls into a disjoint union of discs.

Theorem 4.2.3. *Suppose $\mathcal{H}^3(U) \leq \eta/8$. Let $U' \subset\subset U$ be an open set so that $\overline{U'}$ is diffeomorphic to $\overline{\mathbb{B}}$. Suppose $\Omega \subset M$ satisfies*

•

$$\mathcal{A}^h(\Omega) - \inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^h(\phi(1, \Omega)) =: E(\Omega) \leq 3\gamma/16.$$

- $\Sigma := \partial\Omega \cap U$ is strongly (γ, U) -irreducible,
- Σ has transverse intersection with $\partial U'$.

Let $\Gamma_1, \dots, \Gamma_q$ be the components of $\Sigma \cap \partial U'$, and let F_j denote the isoperimetric region in $\partial U'$ with boundary Γ_j . Suppose further that

$$\sum_{j=1}^q \mathcal{H}^2(F_j) \leq \gamma/8.$$

Let Σ_0 denote the union of all closed components Λ of Σ contained in an open set $K_\Lambda \subset U$ diffeomorphic to $\mathbb{B}$ so that $\partial K_\Lambda \cap \Sigma = \emptyset$.

Then $\mathcal{H}^2(\Sigma_0) \leq 2E(\Omega)$, and there are disjoint closed discs $D_1, \dots, D_p$ satisfying

1. $D_j \subset \Sigma \setminus \Sigma_0$,

2. $\partial D_j \subset \partial U'$,

3.

$$\left(\bigcup_{j=1}^p D_j \right) \cap U' = (\Sigma \setminus \Sigma_0) \cap U',$$

4. there is an isotopy $\phi \in \mathcal{I}(U)$ such that $\phi(t, x) = x$ for all $(t, x) \in [0, 1] \times W$, where W is a neighbourhood of $(\Sigma \setminus \Sigma_0) \setminus \bigcup_{j=1}^p (D_j \setminus \partial D_j)$, so that

$$\bigcup_{j=1}^p (\phi(1, D_j) \setminus \partial D_j) \subset U'$$

5.

$$\sum_{j=1}^p (\mathcal{H}^2(D_j) - c\mathcal{H}^3(\Omega^j)) \leq \sum_{j=1}^q \mathcal{H}^2(F_j) + E(\Omega),$$

where $\Omega^j \subset U$ is the region bounded by $D_j \cup F_{\partial D_j}$, which exists by the existence of the isotopy in (4).

Remark 4.2.4. Let $K_1, \dots, K_s \subset U$ be open 3-balls with $\Sigma_0 \subset \bigcup_{j=1}^s K_j$ and $\partial K_j \cap \Sigma = \emptyset$.

For each $\varepsilon > 0$, there is an open set $\bigcup_{j=1}^s K_j \subset W \subset U$ and an isotopy $\phi \in \mathcal{I}(U)$ satisfying

- $W \cap (\Sigma \setminus \Sigma_0) = \emptyset$,
- $\phi(t, x) = x$ for all $(x, t) \in (M \setminus W) \times [0, 1]$,
- $\phi(t, W) \subset W$,
- $\mathcal{H}^2(\phi(1, \Sigma_0)) < \varepsilon$,
- the union of the regions bounded by each component of $\phi(1, \Sigma_0)$ (given by Lemma 3.1.2) has volume bounded by ε ,
- and each component of $\phi(1, \Sigma_0)$ has diameter less than ε .

Let $\tilde{\Omega}$ be the region given by deleting Σ_0 in Σ . Each component that we collapse decreases $\mathcal{A}^h$ by Proposition 3.5.1, and we have

$$\mathcal{H}^2(\Sigma_0) \leq 2E(\Omega)$$

and

$$\mathcal{A}^h(\tilde{\Omega}) - \inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^h(\phi(1, \tilde{\Omega})) \leq E(\Omega).$$

Hence, the general case of Theorem 4.2.3 follows from the special case where $\Sigma_0 = \emptyset$, which we henceforth assume in the proof below.

Remark 4.2.5. Since $\Omega^j \subset U$ and $\mathcal{H}^3(U) \leq \eta$, we have by Proposition 3.5.1 that

$$\mathcal{H}^3(\Omega^j) \leq c_1^{2/3} \eta^{1/3} (\mathcal{H}^2(D_j) + \mathcal{H}^2(F_{\partial D_j})).$$

By our choice of η , we have

$$c\mathcal{H}^3(\Omega^j) \leq \frac{1}{2}(\mathcal{H}^2(D_j) + \mathcal{H}^2(F_{\partial D_j})).$$

Then conclusion (5) of Theorem 4.2.3 implies

$$\sum_{j=1}^p \mathcal{H}^2(D_j) \leq 3 \sum_{j=1}^q \mathcal{H}^2(F_j) + 2E(\Omega).$$

Proof of Theorem 4.2.3. The proof follows exactly as in [MSY82, Theorem 2], with the additional step of keeping track of volumes.

We induct on q , where the case $q = 0$ is trivial.

Assume Ω satisfies

- $\Sigma_0 = \emptyset$,
- $\sum_{j=1}^q \mathcal{H}^2(F_j) \leq \gamma/8$,
- $E(\Omega) \leq 3\gamma/8 - \frac{3}{2} \sum_{j=1}^q \mathcal{H}^2(F_j)$,
- Σ is strongly $\tilde{\gamma}$ irreducible for $\tilde{\gamma} = \gamma/4 + 4 \sum_{j=1}^q \mathcal{H}^2(F_j) + \frac{4}{3}E(\Omega)$.

Assume as an inductive hypothesis that the conclusions of the theorem are true for $\hat{\Omega} = \phi(1, \Omega)$ for some $\phi \in \mathcal{I}(U)$ if it satisfies the above assumptions with $q - 1$ in place of q and $\hat{\Omega}$ in place of Ω .

Without loss of generality (by relabeling), we have $F_q \cap \Gamma_j = \emptyset$ for all $j \neq q$. Since Σ is strongly $\tilde{\gamma}$ -irreducible and $\mathcal{H}^2(F_q) < \tilde{\gamma}$, there is a disc $D \subset \Sigma$ so that $\partial D = \Gamma_q$ and $\mathcal{H}^2(D) < \delta^2/2$. Then $D \cup F_q$ is homeomorphic to $\mathbb{S}^2$ and $\mathcal{H}^2(D \cup F_q) < \delta^2/2 + \delta^2/2 = \delta^2$, so Lemma 3.1.2 implies that $D \cup F_q = \partial K$ for $K \subset U$ an open 3-ball. Let Λ be the component of Σ containing D . Since $\Sigma_0 = \emptyset$ by assumption, we have $\Lambda \setminus D \not\subset K$. Hence, we have $(\Sigma \setminus D) \cap K = \emptyset$ (because $\Sigma \setminus D$ does not intersect F_q).

Let $\Sigma_* = (\Sigma \setminus D) \cup F_q$ and Ω_* the corresponding region. As in [MSY82, Theorem 2], for every $\varepsilon > 0$, there is a continuous isotopy α taking Σ_* to a smooth surface $\hat{\Sigma}_*$ and Ω_* to $\hat{\Omega}_*$ satisfying

- $\hat{\Omega}_* = \phi(1, \Omega)$ for some $\phi \in \mathcal{I}(U)$,
- $\hat{\Sigma}_* \cap \partial U' = \bigcup_{j=1}^{q-1} \Gamma_j$,
- $\mathcal{H}^2(\hat{\Sigma}_* \triangle \Sigma) < \mathcal{H}^2(D) + \mathcal{H}^2(F_q) + \varepsilon$,
- $\mathcal{H}^3(\hat{\Omega}_* \triangle \Omega) < \mathcal{H}^3(K) + \varepsilon/(2c)$,
- $\mathcal{H}^2(\hat{\Sigma}_*) < \mathcal{H}^2(\Sigma) + \mathcal{H}^2(F_q) - \mathcal{H}^2(D) + \varepsilon/2$,
- $\mathcal{H}^2(\hat{\Sigma}_*) \geq \mathcal{H}^2(\Sigma_*)$.

Using $\mathcal{H}^3(U) \leq \eta/8$, Proposition 3.5.1 implies $c\mathcal{H}^3(K) \leq \frac{1}{4}(\mathcal{H}^2(F_q) + \mathcal{H}^2(D))$. We assume $\varepsilon \leq \mathcal{H}^2(F_q)/4$. Observe that since Ω and $\hat{\Omega}_*$ are in the same isotopy class, we have

$$\begin{aligned} E(\hat{\Omega}_*) &< E(\Omega) + c\mathcal{H}^3(K) + \mathcal{H}^2(F_q) - \mathcal{H}^2(D) + \varepsilon \\ &\leq E(\Omega) + \frac{3}{2}\mathcal{H}^2(F_q) - \frac{3}{4}\mathcal{H}^2(D). \end{aligned}$$

By [MSY82, Remark 3.11], we have that $\hat{\Sigma}_*$ is strongly γ^* -irreducible, where

$$\begin{aligned} \gamma^* &= \gamma/4 + 4 \sum_{j=1}^q \mathcal{H}^2(F_j) + \frac{4}{3}E(\Omega) - \mathcal{H}^2(D) - 2\mathcal{H}^2(F_q) \\ &= \gamma/4 + 4 \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + \frac{4}{3}E(\Omega) + 2\mathcal{H}^2(F_q) - \mathcal{H}^2(D) \\ &\geq \gamma/4 + 4 \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + \frac{4}{3}E(\hat{\Omega}_*). \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
E(\hat{\Omega}_*) &\leq E(\Omega) + \frac{3}{2}\mathcal{H}^2(F_q) \\
&\leq 3\gamma/8 - \frac{3}{2}\sum_{j=1}^q \mathcal{H}^2(F_j) + \frac{3}{2}\mathcal{H}^2(F_q) \\
&= 3\gamma/8 - \frac{3}{2}\sum_{j=1}^{q-1} \mathcal{H}^2(F_j).
\end{aligned}$$

Hence, $\hat{\Omega}_*$ satisfies the inductive hypothesis. Then there are pairwise disjoint discs $\tilde{\Delta}_1, \dots, \tilde{\Delta}_p$ satisfying

- $\tilde{\Delta}_j \subset \hat{\Sigma}_*$,

- $\partial\tilde{\Delta}_j \subset \partial U'$,

-

$$\left(\bigcup_{j=1}^p \tilde{\Delta}_j\right) \cap U' = \hat{\Sigma}_* \cap U',$$

- there is an isotopy $\tilde{\psi} \in \mathcal{I}(U)$ such that $\tilde{\psi}(t, x) = x$ for all $(t, x) \in [0, 1] \times \tilde{W}$, where $\tilde{W}$ is a neighbourhood of $\hat{\Sigma}_* \setminus \bigcup_{j=1}^p (\tilde{\Delta}_j \setminus \partial\tilde{\Delta}_j)$, so that

$$\bigcup_{j=1}^p (\tilde{\psi}(1, \tilde{\Delta}_j) \setminus \partial\tilde{\Delta}_j) \subset U',$$

-

$$\sum_{j=1}^p (\mathcal{H}^2(\tilde{\Delta}_j) - c\mathcal{H}^3(\tilde{\Omega}^j)) \leq \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + E(\hat{\Omega}_*).$$

Letting $\Delta_j := (\alpha^{-1})(1, \tilde{\Delta}_j)$ (i.e. the time-reversal of the smoothing isotopy α for Σ_*), and taking $\psi := \tilde{\psi} * \alpha$ (i.e. the concatenation of isotopies), we have that

- $\Delta_j \subset \Sigma_*$,

- $\partial\Delta_j \subset \partial U'$,

•

$$\left(\bigcup_{j=1}^p \Delta_j \right) \cap U' = \Sigma_* \cap U',$$

- there is an isotopy $\psi \in \mathcal{I}(U)$ such that $\psi(t, x) = x$ for all $(t, x) \in [0, 1] \times W'$, where W' is a neighbourhood of $\Sigma_* \setminus \bigcup_{j=1}^p (\Delta_j \setminus \partial \Delta_j)$, so that

$$\bigcup_{j=1}^p (\psi(1, \Delta_j) \setminus \partial \Delta_j) \subset U',$$

•

$$\sum_{j=1}^p (\mathcal{H}^2(\Delta_j) - c\mathcal{H}^3(\tilde{\Omega}^j)) \leq \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + E(\hat{\Omega}_*).$$

Let β be a continuous isotopy so that $\beta(t, K) \subset K$, $\beta|_{\{t\} \times \Sigma \setminus D} = \mathbf{1}_{\Sigma \setminus D}$, and $\beta(1, D) = F_q$. Consider the following two cases:

Case 1: $F_q \subset \bigcup_{j=1}^p \Delta_j$. Select the discs $D_1, \dots, D_p$ by taking $D_{j_0} = (\Delta_{j_0} \setminus F_q) \cup D$ for the unique j_0 so that $F_q \subset \Delta_{j_0}$, and $D_j = \Delta_j$ for $j \neq j_0$. Define an isotopy by $\tilde{\phi} = \psi * \beta$. By smoothing $\tilde{\phi}$, we obtain the desired isotopy ϕ . We compute

$$\begin{aligned} \sum_{j=1}^p (\mathcal{H}^2(D_j) - c\mathcal{H}^3(\Omega^j)) &\leq \sum_{j=1}^p (\mathcal{H}^2(\Delta_j) - c\mathcal{H}^3(\tilde{\Omega}^j)) + \mathcal{H}^2(D) - \mathcal{H}^2(F_q) - c\mathcal{H}^3(K) + \varepsilon \\ &\leq \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + E(\hat{\Omega}_*) + \mathcal{H}^2(D) - \mathcal{H}^2(F_q) - c\mathcal{H}^3(K) + \varepsilon \\ &\leq \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + E(\Omega) + 2\varepsilon \\ &\leq \sum_{j=1}^q \mathcal{H}^2(F_j) + E(\Omega) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired inequality holds.

Case 2: $F_q \not\subset \bigcup_{j=1}^p \Delta_j$. Define the discs $D_1, \dots, D_{p+1}$ by $D_j = \Delta_j$ for $j < p+1$ and $D_{p+1} = D$. Define an isotopy $\tilde{\phi} = \hat{\beta} * \psi * \beta$, where $\hat{\beta} \in \mathcal{I}(U)$ satisfies $\hat{\beta}(t, x) = x$ for all $(x, t) \in (\Sigma \setminus D) \times [0, 1]$ and $\hat{\beta}(1, F_q)$ is a disc $\tilde{D} \subset U'$ with $\partial \tilde{D} = \partial D$,

$\tilde{D} \cap \partial U' = \partial \tilde{D}$, and $\tilde{D} \cap \psi(t, \Sigma_*) = \Gamma_q$ for all $t \in [0, 1]$. We claim that in this case there is a neighbourhood W of $\partial D = \Gamma_q$ so that $W \cap D \subset U'$. Otherwise we would have W with $W \supset \partial D$ and $W \cap (\Sigma \setminus D) \subset U'$, which would imply $F_q \subset \bigcup_{j=1}^p \Delta_j$, a contradiction. By smoothing $\tilde{\phi}$, we obtain the required isotopy. We compute

$$\begin{aligned}
\sum_{j=1}^{p+1} (\mathcal{H}^2(D_j) - c\mathcal{H}^3(\Omega^j)) &\leq \sum_{j=1}^p (\mathcal{H}^2(\Delta_j) - c\mathcal{H}^3(\tilde{\Omega}^j)) + \mathcal{H}^2(D) - c\mathcal{H}^3(K) + \varepsilon \\
&\leq \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + E(\hat{\Omega}_*) + \mathcal{H}^2(D) - c\mathcal{H}^3(K) + \varepsilon \\
&\leq \sum_{j=1}^{q-1} \mathcal{H}^2(F_j) + E(\Omega) + \mathcal{H}^2(F_q) + 2\varepsilon \\
&= \sum_{j=1}^q \mathcal{H}^2(F_j) + E(\Omega) + 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired inequality holds. $\square$

4.2.3 Gamma-reduction of minimising sequence

We now modify the initial minimising sequence $\{\Sigma_k\}$ to a new sequence $\{\tilde{\Sigma}_k\}$ with $\lim \mathbf{v}(\tilde{\Sigma}_k) = \lim \mathbf{v}(\Sigma_k) = V$ so that each $\tilde{\Sigma}_k$ is strongly (γ, U) -irreducible for some γ .

For $q \in \mathbb{N}$ and $\gamma \in (0, \delta^2/9)$, let $k_q(\gamma)$ be the largest integer such that there exist $\Sigma_q^{(j)} \in \mathcal{C}(\Sigma_q, U)$ for $j = 1, \dots, k_q(\gamma)$ with

$$\Sigma_q^{(k_q(\gamma))} \stackrel{(\gamma, U)}{<} \dots \stackrel{(\gamma, U)}{<} \Sigma_q^{(2)} \stackrel{(\gamma, U)}{<} \Sigma_q^{(1)} = \Sigma_q.$$

We claim that $k_q(\gamma)$ is bounded independent of q and γ . Indeed, there is an upper bound on the topology of Σ_q and $\mathcal{H}^2(\Sigma_q)$ independent of q and W by assumption. Then the uniform bound on $k_q(\gamma)$ follows from [MSY82, (3.9)].

We also note that $k_q(\gamma) \leq k_q(\gamma')$ for $\gamma < \gamma'$, which follows by definition.

Let $l(\gamma)$ be the nonnegative integer defined by

$$l(\gamma) := \limsup_{q \rightarrow \infty} k_q(\gamma).$$

l is an nondecreasing integer-valued function of γ , so there is $\gamma_0 \in (0, \delta^2/9)$ so that

$$l(\gamma) = l(\gamma_0) \text{ for all } \gamma \in (0, \gamma_0].$$

For each $n \geq \gamma_0^{-1}$, there is a $q_n \geq n$ so that $k_{q_n}(1/n) = l(1/n) = l(\gamma_0)$. We set

$$\tilde{\Sigma}_n := \Sigma_{q_n}^{(k_{q_n}(1/n))}.$$

Hence, $\tilde{\Sigma}_n$ is strongly (γ_0, U) -irreducible for all sufficiently large n .

By [MSY82, (3.10)], we have

$$\mathcal{H}^2(\tilde{\Sigma}_n \Delta \Sigma_{q_n}) \leq C/n$$

for some C independent of n . Hence, $\lim \mathbf{v}(\tilde{\Sigma}_n) = \lim \mathbf{v}(\Sigma_k)$.

Let $\tilde{\Omega}_n \subset M$ be the open set corresponding to $\tilde{\Sigma}_n$ (which is obtained from Ω_{q_n} in the obvious way at each (γ, U) -reduction). Moreover, since Σ_{q_n} can be recovered from $\tilde{\Sigma}_n$ by deleting discs and adding thin tubes, we have

$$\mathcal{A}^h(\tilde{\Omega}_n) \leq \inf_{\phi \in \mathcal{I}(U)} \mathcal{A}^h(\phi(1, \tilde{\Omega}_n)) + \varepsilon'_n$$

where $\varepsilon'_n \rightarrow 0$ as $n \rightarrow \infty$.

4.3 Interior regularity

We prove the interior regularity part of Theorem 2.0.1. To clarify, the “sufficiently small” assumption on U is $\mathcal{H}^3(U) \leq \eta/8$, where η is defined in (3.5).

Proof of Theorem 2.0.1 Interior Regularity. By §4.2.3, it suffices to consider a minimising sequence $\{\Sigma_k\}$ so that Σ_k is strongly (γ, U) -irreducible. By Remark 4.2.4, we can suppose there are no closed components of Σ_k in U contained in an open 3-ball whose boundary does not intersect Σ_k .

Let $x_0 \in \text{spt}\|V\|$. Let $\rho_0(x_0) := d(x_0, \partial U)$. By the monotonicity of (3.3.1), there is a constant c_2 so that

$$\sigma^{-2}\|V\|(B_\sigma(x_0)) \leq c_2\rho^{-2}\|V\|(B_\rho(x_0))$$

for any $\sigma < \rho \leq \rho_0(x_0)$.

By the coarea formula,

$$\int_{\rho-\sigma}^{\rho} \mathcal{H}^1(\Sigma_k \cap \partial B_s(x_0)) \, ds \leq \mathcal{H}^2(\Sigma_k \cap (\overline{B}_\rho(x_0) \setminus B_{\rho-\sigma}(x_0)))$$

for almost every $\rho \in (0, \rho_0(x_0))$, $\sigma \in (0, \rho)$. Taking $\sigma = \rho/2$, (4.3) implies

$$\rho^{-1} \int_{\rho/2}^{\rho} \mathcal{H}^1(\Sigma_k \cap \partial B_s(x_0)) \, ds \leq C\rho$$

for all sufficiently large k , where C depends on μ , $\rho_0(x_0)$, and $\|V\|(U)$. Then there is a sequence $\rho_k \in (3\rho/4, \rho)$ so that Σ_k has transverse intersection with $\partial B_{\rho_k}(x_0)$ and

$$\mathcal{H}^1(\Sigma_k \cap \partial B_{\rho_k}(x_0)) \leq C\rho \leq C\eta_1\rho_0(x_0)$$

for all sufficiently large k , provided $\rho \leq \eta_1\rho_0(x_0)$ for some $\eta_1 \in (0, 1)$ to be determined. For η_1 sufficiently small and k sufficiently large, we can apply Theorem 4.2.3 to Σ_k with

$U' = B_{\rho_k}(x_0)$. Hence, there are discs $D_k^{(1)}, \dots, D_k^{(q_k)} \subset \Sigma_k$ and isotopies $\phi^{(k)} \in \mathcal{I}(U)$ so that for k sufficiently large, we have

- $\partial D_k^{(l)} \subset \partial B_{\rho_k}(x_0)$,
- $\Sigma_k \cap B_{\rho_k}(x_0) = (\bigcup_{l=1}^{q_k} D_k^{(l)}) \cap B_{\rho_k}(x_0)$,
- $\phi^{(k)}(1, D_k^{(l)}) \setminus \partial D_k^{(l)} \subset B_{\rho_k}(x_0)$,
- $\sum_{l=1}^{q_k} \mathcal{H}^2(D_k^{(l)}) \leq \tilde{C} \eta_1^2 \rho_0(x_0)^2$ (see (4.2.5)).

For η_1 sufficiently small and k sufficiently large, we can apply Lemma 3.5.2 to produce discs $\tilde{D}_k^{(l)}$ with $\partial \tilde{D}_k^{(l)} = \partial D_k^{(l)}$, $\tilde{D}_k^{(l)} \setminus \partial D_k^{(l)} \subset B_{\rho_k}(x_0)$, and

$$\mathcal{H}^2(\tilde{D}_k^{(l)}) \leq \mathcal{H}^2(D_k^{(l)}) + T_h(\tilde{D}_k^{(l)}, D_k^{(l)}).$$

By taking a subsequence, we can assume that $\rho_k \rightarrow \rho_\infty$. Take an arbitrary $\alpha > 0$ small. We can construct oriented smooth discs $\tilde{\Sigma}_k \in \mathcal{D}^*(B_{\rho_\infty - \alpha}(x_0))$ so that (for all k sufficiently large)

- $\tilde{\Sigma}_k \cap B_{\rho_\infty - \alpha}(x_0) = \bigcup_l \tilde{D}_k^{(l)} \cap B_{\rho_\infty - \alpha}(x_0)$,
- $\partial \tilde{\Sigma}_k \cap \overline{B_{\rho_\infty - \alpha}(x_0)} = \emptyset$,
- $\tilde{\Sigma}_k \setminus B_{\rho_\infty - \alpha}(x_0)$ is connected.

Indeed, we can inductively add bridges outside $B_{\rho_\infty - \alpha}(x_0)$ joining the discs $\tilde{D}_k^{(l)}$. Then $\tilde{\Sigma}_k$ satisfies the hypotheses of Theorem 3.7.2 with the set $B_{\rho_\infty - \alpha}(x_0)$. Hence, the interior regularity of Theorem 2.0.1 follows. $\square$

4.4 Boundary regularity

We prove the boundary regularity part of Theorem 2.0.1. We follow the proof of [DLP10].

4.4.1 Wedge property

Definition 4.4.1. Let $U = B_\rho(x) \subset M$ for $\rho \ll 1$ and let $\sigma \subset \partial U$ be a disjoint union of finitely many smooth Jordan curves. An open set $A \subset U$ *meets ∂U in σ transversally* if there is a positive angle θ_0 so that

- $\partial A \cap \partial U \subset \sigma$,
- for every $p \in \partial A \cap \partial U$, if we choose coordinates so that $T_p \partial U = \{z = 0\}$ and $T_p \sigma = \{y = z = 0\}$, then every point $q = (q_1, q_2, q_3) \in A$ satisfies $\frac{q_3}{q_2} \geq \tan(1/2 - \theta_0)$.

Lemma 4.4.2 (Wedge Property). *There is a $r_0 < 1$ so that the following holds for $\rho < r_0$. Let V and $\Sigma := \partial \Omega_0 \cap U$ as in Theorem 2.0.1. Then there exists an open set $A \subset U$ which meets ∂U in $\partial \Sigma$ transversally and satisfies $\text{spt}\|V\| \subset \overline{A}$.*

We use the following result, see [DLP10, Lemma 5.3].

Proposition 4.4.3. *If $\beta \subset \partial B_1(0) \subset \mathbb{R}^3$ is a disjoint union of finitely many C^2 Jordan curves, then the convex hull of β meets $\partial B_1(0)$ in β transversally.*

Proof of Lemma 4.4.2. Consider the rescaled exponential coordinates induced by the chart $f_{x,\rho} : \overline{B}_\rho(x) \rightarrow \overline{B}_1(0)$ given by $f_{x,\rho}(z) = (\exp_x^{-1}(z))/\rho$. We apply Proposition 4.4.3 and let B be the convex hull of $\beta = f(\partial \Sigma)$.

Fix $\xi \in \beta$ and let π_1 and π_2 be the halfplanes delimiting the wedge of B . We assume without loss of generality that $\pi_1 = \{x_3 \leq a\}$ for some $a \in (0, 1)$.

Let $C_t = (0, 0, -t)$, and let $S_t = B_2(C_t)$. Assuming ρ is sufficiently small, (3.3.1) implies $\|\delta(f_\# V)\| \leq \|f_\# V\|$. By [Whi10, Theorem 7], $\text{spt}\|V\|$ is contained in $\overline{S}_{t_0}$, where t_0 is such that $\partial S_{t_0} \cap \partial B_1(0) \subset \pi_1$. The same argument applies to π_2 and ξ is arbitrary, so the conclusion follows. $\square$

4.4.2 Tangent cones at the boundary

Let $x \in \partial U$. As in the previous section, consider the chart $f_{x,\rho} : B_\rho(x) \rightarrow B_1(0)$ given by $f_{x,\rho}(z) = (\exp_x^{-1}(z))/\rho$. Fix coordinates in $\mathbb{R}^3$ so that $f_{x,\rho}(U \cap B_\rho(x))$ converges to the half ball $B_1^+(0) = B_1(0) \cap \{x_1 > 0\}$ as $\rho \rightarrow 0$.

We first follow the ideas of [DLR18, Lemma 6.4].

Lemma 4.4.4. $\|V\|(\partial\Sigma) = 0$.

Proof. Let $V^r = V \llcorner G_2(\overline{U} \setminus \partial\Sigma)$ and $V^s = V - V^r$.

Let X be a compactly supported vector field on $\overline{U}$ that vanishes on $\partial\Sigma$. Let $\phi_\delta \in C_c^\infty(\overline{U} \setminus \partial\Sigma)$ satisfy

- $\phi_\delta = 1$ outside $B_{2\delta}(\partial\Sigma)$,
- $|\nabla \phi_\delta| \leq C\delta^{-1}$.

Since X vanishes on $\partial\Sigma$, we have $|X| \leq C\delta$ in $B_{2\delta}(\partial\Sigma)$. Hence, $|\nabla(\phi_\delta X)| \leq C$. Then by dominated convergence and the fact that V has bounded first variation in $\overline{U} \setminus \partial\Sigma$, we have

$$\begin{aligned} \delta V^r(X) &= \int_{\overline{U}} \operatorname{div}_\pi X(p) \, dV^r(p, \pi) \\ &= \lim_{\delta \rightarrow 0} \int_{\overline{U}} \operatorname{div}_\pi(\phi_\delta X)(p) \, dV^r(p, \pi) \\ &= \lim_{\delta \rightarrow 0} \int_{\overline{U}} \operatorname{div}_\pi(\phi_\delta X)(p) \, dV(p, \pi) \\ &= \lim_{\delta \rightarrow 0} \int_{\overline{U}} H_V \cdot (\phi_\delta X)(p) \, dV(p, \pi) \\ &= \lim_{\delta \rightarrow 0} \int_{\overline{U}} H_V \cdot X(p) \, dV(p, \pi) = \delta V(X). \end{aligned}$$

Hence, $\delta V^s = 0$.

Let W be a small tubular neighbourhood of $\partial\Sigma$. For each $p \in W \setminus \partial\Sigma$, let q be the unique nearest point in $\partial\Sigma$. Let $X(p)$ be the tangent vector at p to the minimising

geodesic joining p to q parametrised on $[0, 1]$. Set $X(p) = 0$ for $p \in \partial\Sigma$. If e_1, e_2, e_3 is a smooth orthonormal frame over $\partial\Sigma$ with e_1 tangent to $\partial\Sigma$, we have $\nabla_{e_1}X = 0$, $\nabla_{e_2}X = -e_2$, and $\nabla_{e_3}X = -e_3$ along $\partial\Sigma$. For any 2-plane π , we have

$$\operatorname{div}_\pi X(p) = \sum_{i=1}^2 g(\nabla_{f_i} X, f_i)$$

for $\{f_i\}$ an orthonormal basis for π . Hence, $\operatorname{div}_\pi X(p) \leq -1$. Then

$$0 = \delta V^s(X) = \int_{\bar{U}} \operatorname{div}_\pi X(p) \, dV^s(p, \pi) \leq -\|V^s\|(\bar{U}) = -\|V\|(\partial\Sigma),$$

so $\|V\|(\partial\Sigma) = 0$. □

By Lemma 4.4.4, we can apply Allard's boundary monotonicity formula [All75, 3.4(2)]. Then the varifolds $V_{x,\rho} := (f_{x,\rho})_\# V$ have uniformly bounded mass with respect to ρ . Hence, if we take a sequence $\rho_j \rightarrow \infty$, there is a subsequence (not relabeled) so that $V_{x,\rho_j} \rightarrow W$. We note by [Sim83, Theorem 42.7] that W is stationary in $B_1^+(0)$.

By Lemma 4.4.2, there is a positive angle θ_0 so that (after changing coordinates) $\|W\|$ is supported in $\{|x_2| \leq x_1 \tan(\theta_0)\}$. Then $\operatorname{spt}\|W\| \cap \{x_1 = 0\} = \{x_1 = x_2 = 0\} =: l$.

By the same argument as Lemma 4.4.4, we have $\|W\|(l) = 0$. Hence, by Allard's boundary monotonicity formula [All75, 3.4(2)], we have

$$\|W\|(B_\rho(0)) = \pi\theta(\|V\|, x)\rho^2$$

By Allard's reflection principle [All75, 3.2], we have $W' := W + r_\# W$ is stationary. By (4.4.2) and [All72, §5.1, Corollary 2], W' is a cone, which implies W is a cone.

By Allard's boundary regularity theorem (see [All75, §4]), it suffices to show that

any such W is a half-disc of the form

$$P_\theta := \{y_2 = y_1 \tan(\theta)\} \cap B_1^+(0)$$

for some $\theta \in (-\pi/2, \pi/2)$.

By the classification of one dimensional stationary integral varifolds from [AA76], we have

$$W = \sum_{i=1}^N k_i P_{\theta_i}$$

for $\theta_i \in [-\theta_0, \theta_0]$ and $k_i \in \mathbb{N}$. It therefore remains to show $N = 1$ and $k_1 = 1$.

4.4.3 Reduction to the area minimisation case

Consider a subsequence so that

$$S_n := f_{x_0, \rho_n}(\Sigma_{k_n} \cap B_{\rho_n}(x_0))$$

converges to W . We further suppose that $\rho_n^{-2} \varepsilon_{k_n} \rightarrow 0$. For any isotopy ϕ supported in $B_1^+(0)$, let $\tilde{\phi}$ be the isotopy of $B_{\rho_n}(x_0) \cap U$ given by conjugating with the chart f_{x_0, ρ_n} . We have

$$\begin{aligned} \mathcal{H}^2(\Sigma_{k_n} \cap B_{\rho_n}(x_0)) &= \int_{\Omega_{k_n} \cap B_{\rho_n}(x_0) \cap U} h \, d\mathcal{H}^3 \\ &\leq \mathcal{H}^2(\tilde{\phi}(1, \Sigma_{k_n} \cap B_{\rho_n}(x_0))) - \int_{\tilde{\phi}(1, \Omega_{k_n} \cap B_{\rho_n}(x_0) \cap U)} h \, d\mathcal{H}^3 + \varepsilon_{k_n}. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{H}^2(\Sigma_{k_n} \cap B_{\rho_n}(x_0)) &= \rho_n^2 \mathcal{H}^2(S_n) + o(\rho_n^2), \\ \mathcal{H}^2(\tilde{\phi}(1, \Sigma_{k_n} \cap B_{\rho_n}(x_0))) &= \rho_n^2 \mathcal{H}^2(\phi(1, S_n)) + o(\rho_n^2), \\ \mathcal{H}^3(B_{\rho_n}(x_0) \cap U) &= \rho_n^3 \mathcal{H}^3(B_1^+(0)) + o(\rho_n^3) = o(\rho_n^2), \end{aligned}$$

we have

$$\mathcal{H}^2(S_n) \leq \mathcal{H}^2(\phi(1, S_n)) + \varepsilon'_n,$$

where $\varepsilon'_n \rightarrow 0$. Hence, S_n is ε'_n -area minimising in its isotopy class, so the rest of the proof follows exactly as in [DLP10, §8]. Therefore, the boundary regularity of Theorem 2.0.1 follows.

Chapter 5

Estimates and compactness for minimisers

We develop a compactness theory to prove an analogue of the curvature estimates for stable surfaces that works in our lower regularity setting.

5.1 $C^{1,1}$ compactness

Let $U \subset M$ be a precompact open set with C^1 boundary.

Let Σ^2 be a compact orientable surface (possibly with boundary) and let $f : \Sigma \rightarrow M^3$ be 2-sided $C^{1,1}$ immersion such that $f(\Sigma) \subset U$ and $f(\partial\Sigma) \subset \partial U$. Let $\nu : \Sigma \rightarrow TM$ denote a (Lipschitz) choice of unit normal vector field for the immersion. We endow Σ with the pullback metric f^*g and for $p \in \Sigma$, we denote by $\mathcal{B}_r^\Sigma(p)$ the image $f(B_r^\Sigma(p)) \subset M$ of the ball of radius r centred at p in Σ .

Note that, since f is a $C^{1,1}$ immersion, for any $p \in \Sigma$ there is $0 < \rho(p) \leq \rho_0$ such that if Σ_p is the component of $f^{-1}(B_{\rho(p)}^M(f(p)))$ which contains p , then

- $\Sigma_p \cap \partial\Sigma = \emptyset$,
- f is injective on Σ_p ,

- and $\phi_{f(p)}^{-1}(f(\Sigma_p))$ can be represented as the graph of a $C^{1,1}$ function u_p over an open set $B_p \subset df_p(T_{f(p)}\Sigma) \subset T_{f(p)}M$ containing 0.

Then, for any $p \in \Sigma$ we can define the *pointwise Lipschitz constant* $\text{Lip}_p \nu$ of ν to be the following quantity

$$\text{Lip}_p \nu := \limsup_{B_p \ni \xi \rightarrow 0} \frac{|\nu_p(0) - \nu_p(\xi)|}{|\xi|}$$

where

- $\nu_p : B_p \rightarrow T_{f(p)}M$ is the unit normal to the graph of u_p in $T_{f(p)}M \cong \mathbb{R}^3$, given by

$$\nu_p = \frac{(-\nabla u_p, 1)}{\sqrt{1 + |\nabla u_p|^2}}$$

in geodesic normal coordinates,

- gradients and distances are taken with respect to the pullback metric on $\mathbb{R}^3$.

Remark 5.1.1. It is easy to see that for every $p \in \Sigma$ there is $0 < r(p) \leq \rho(p)$ such that for any $\xi, \eta \in \phi_x^{-1}(\mathcal{B}_{r(p)}^\Sigma(p))$ we have

$$|\nu_p(\xi) - \nu_p(\eta)| \leq 2 \text{Lip}_p \nu |\xi - \eta|.$$

To simplify the notation, we redefine $\rho(p)$ to be the same as $r(p)$.

We start by proving a simple technical fact, which plays the same role in the proof of compactness as [CM11, Lemma 2.4].

Lemma 5.1.2. *Let Σ^2 be a compact orientable surface (possibly with boundary) and let $f : \Sigma \rightarrow M^3$ be 2-sided $C^{1,1}$ immersion such that $f(\Sigma) \subset U$ and $f(\partial\Sigma) \subset \partial U$. Let $\nu : \Sigma \rightarrow TM$ be a (Lipschitz) choice of unit normal vector field for the immersion such that $\sup_{p \in \Sigma} \text{Lip}_p \nu \leq C_0$. Then, for any $K \subset\subset U$ there is $r_K = r_K(\rho_0, \text{dist}_M(K, \partial U), C_0) > 0$ such that if $x = f(p) \in f(\Sigma) \cap K$, then*

1. $\mathcal{B}_{2r_K}^\Sigma(p)$ can be written as the graph of a $C^{1,1}$ function u on an open neighbourhood of 0 in $T_x M$, such that $|\nabla u| \leq 1$ and $\text{Lip } \nabla u \leq 2\sqrt{2}C_0$,

2. the connected component of $B_{r_K}^M(x) \cap f(\Sigma)$ containing x is contained in $\mathcal{B}_{2r_K}^\Sigma(p)$.

Proof. Since $K \subset\subset U$, then $\delta_K := \text{dist}_M(K, \partial U) > 0$. Hence $\text{dist}_{f(\Sigma)}(f(\Sigma) \cap K, f(\partial\Sigma)) \geq \delta_K > 0$. Let $r = \min\{\rho_0/2, \delta_K/2\} > 0$ and let $p \in \Sigma$ be such that $x = f(p) \in f(\Sigma) \cap K$. Then $\mathcal{B}_r^\Sigma(p) \subset B_r^M(x) \subset\subset U$ stays a fixed positive distance away from ∂U .

Now, let $y = f(q) \in \mathcal{B}_r^\Sigma(p)$ and let $\tilde{\gamma} : [0, 1] \rightarrow \Sigma$ be a curve from p to q in Σ such that $\gamma = f \circ \tilde{\gamma}$ is a minimising constant speed geodesic from x to y in $f(\Sigma)$.

By compactness, we can pick $0 = t_0 < t_1 < \dots < t_N < t_{N+1} = 1$ so that the segment $\overline{x_j x_{j+1}} = \gamma([t_j, t_{j+1}]) \subset \mathcal{B}_r^\Sigma(p)$ (which is a minimising geodesic from $x_j = f(p_j)$ to $x_{j+1} = f(p_{j+1})$) lies in $f(\Sigma_{p_j})$ for $j = 0, \dots, N$. Hence, by (3.1.1) and Remark 5.1.1,

$$\begin{aligned} |d\phi_x^{-1}\nu(p_j) - d\phi_x^{-1}\nu(p_{j+1})| &\leq 4|d\phi_{x_j}^{-1}\nu(p_j) - d\phi_{x_j}^{-1}\nu(p_{j+1})| \\ &\leq 8(\text{Lip}_{p_j} \nu) d_\Sigma(x, y) |t_j - t_{j+1}|, \end{aligned}$$

and, using the uniform pointwise Lipschitz bound,

$$\begin{aligned} |d\phi_x^{-1}\nu(p) - d\phi_x^{-1}\nu(q)| &\leq \sum_{j=0}^N |d\phi_x^{-1}\nu(p_j) - d\phi_x^{-1}\nu(p_{j+1})| \\ &\leq 8C_0 d_{f(\Sigma)}(x, y) \sum_{j=0}^N |t_j - t_{j+1}| \\ &= 8C_0 d_{f(\Sigma)}(x, y) \\ &\leq 8C_0 r. \end{aligned}$$

In particular, this proves that for any $y = f(q) \in \mathcal{B}_r^\Sigma(p)$, we have $|d\phi_x^{-1}\nu(p) -$

$|d\phi_x^{-1}\nu(q)| \leq 8C_0r$. By taking $r \leq 2r_K := \min\{\rho_0/2, \delta_K/2, \frac{1}{16C_0}\}$, we have that

$$|d\phi_x^{-1}\nu(p) - d\phi_x^{-1}\nu(q)| \leq \frac{1}{2}$$

for all $y = f(q) \in \mathcal{B}_r^\Sigma(p)$, which implies that the whole of $\mathcal{B}_r^\Sigma(p)$ can be written as the graph of a $C^{1,1}$ function u over an open subset of $df_p(T_{f(p)}\Sigma) \subset T_{f(p)}M$.

Then it is immediate to see (as in [CM11, Lemma 2.4]) that u satisfies $|\nabla u| \leq 1$ and $\text{Lip } \nabla u \leq 2\sqrt{2}C_0$.

The second statement follows from the first as in [CM11, Lemma 2.4]. $\square$

The next lemma is essentially a geometric Arzelà-Ascoli theorem, and it follows indeed from the usual Arzelà-Ascoli compactness theorem. The argument is well-known and variations of it appear in several different sources and contexts (see for example [ACGL22, Theorem 11.8]). We shall now outline it for the sake of completeness.

Lemma 5.1.3. *Let $\{\Sigma_i\}_{i \in \mathbb{N}}$ be a sequence of compact orientable surfaces and let $f_i : \Sigma_i \rightarrow U$ be a sequence of 2-sided $C^{1,1}$ immersions with unit normal vector field ν_i in a precompact open set $U \subset M$ with $f(\partial\Sigma_i) \subset \partial U$. If the immersions have locally bounded area, i.e. for every $K \subset\subset U$ there is a finite c_K such that $\sup_i \mathcal{H}^2(f_i(\Sigma_i) \cap K) \leq c_K$, and*

$$C_0 := \sup_i \sup_{p \in \Sigma_i} \text{Lip}_p \nu_i < \infty,$$

then there is a subsequence (not relabeled) so that f_i converges in $C_{loc}^{1,\alpha}$ to a $C^{1,1}$ immersion $f : \Sigma \rightarrow U$ with $\sup_{p \in \Sigma} \text{Lip}_p \nu \leq C_0$.

Sketch of proof. Let $K \subset\subset U$. Note that, by Lemma 5.1.2, there is a positive constant r_K such that for any i , if $x \in f_i(\Sigma_i) \cap K$, then for all $r \leq r_K$, the component of $B_r^M(x) \cap f(\Sigma_i) \subset B_{2r}^{\Sigma_i}(x)$ which contains x is the graph of a $C^{1,1}$ function satisfying explicit $C^{1,1}$ -bounds depending on C_0 only.

We can cover the compact set $\overline{K}$ with finitely many balls $\{B_k = B_{r_K/2}^M(z_k)\}_{k=1}^N$.

For any i , if $f_i(\Sigma_i) \cap B_k \neq \emptyset$, then there is some $x_i \in f_i(\Sigma_i) \cap B_k$, and the component of $f_i(\Sigma_i) \cap B_k$ which contains x_i lies in $B_{r_K}^M(z_k)$. This is the graph of a function u_i such that $|\nabla u_i| \leq 1$, so, in particular, it has area bounded below by some uniform constant κr_K^2 . By the uniform upper bound on $\mathcal{H}^2(\Sigma_i)$, we conclude that for any $k = 1, \dots, N$ and $i \in \mathbb{N}$, if $B_k \cap f_i(\Sigma_i) \neq \emptyset$, then $f_i(\Sigma_i) \cap B_{r_K}^M(z_k)$ contains a union of a finite number $N_{k,i}$ of $C^{1,1}$ graphs, and $N_{k,i}$ is bounded above by a universal constant N_0 depending on K and the upper bounds on area and the Lipschitz constant.

Finally, for any $k \in \{1, \dots, N\}$ such that $f_i(\Sigma_i) \cap B_k \neq \emptyset$ for infinitely many i 's we can select points x_i in the intersections and the corresponding tangent planes $T_{x_i}f(\Sigma_i)$. Up to a subsequence, we may assume $x_i \rightarrow x \in B_k$ and $T_{x_i}f(\Sigma_i) \rightarrow P$, a two-dimensional subspace of $T_x M$. Note that the corresponding functions u_i are uniformly bounded and satisfy $|\nabla u_i| \leq 1$, $\text{Lip } \nabla u_i \leq 2\sqrt{2}C_0$, then, by Arzelà-Ascoli, up to a subsequence, they converge (in $C^{1,\alpha}$ for any $0 < \alpha < 1$) to some $C^{1,1}$ function u defined on $B_{r_K}^{T_x M} \cap P$ with the same $C^{1,1}$ bound.

We then repeat this argument for all $k = 1, \dots, N$ and take a diagonal subsequence. It is clear that the graphs we obtained match up on the overlaps between balls (because the surfaces $f_i(\Sigma_i)$ do), so we have produced our immersed limit surface restricted to $K \subset\subset U$.

Lastly, we take an exhaustion of U by compact sets, repeat this procedure inductively and check that the limit surfaces coincide on the overlaps. $\square$

5.2 Curvature estimates

We now use these compactness ideas to prove the curvature estimates.

For convenience, we denote by $\mathcal{P}$ the set of pairs (V, Ω) where V is a varifold and $\Omega \subset M$ is a set of finite perimeter with the property that there is a sequence $\Omega_j \subset M$

with smooth boundary satisfying

$$\mathbf{1}_{\Omega_j} \xrightarrow{L^1} \mathbf{1}_\Omega, \quad D\mathbf{1}_{\Omega_j} \xrightarrow{*} D\mathbf{1}_\Omega, \quad \mathbf{v}(\partial\Omega_j) \rightharpoonup V.$$

We let $\mathcal{P}_c$ denote the pairs $(V, \Omega) \in \mathcal{P}$ so that V has c -bounded first variation.

For $(V, \Omega) \in \mathcal{P}$, we define

$$\mathcal{A}^h(V, \Omega) := \|V\|(M) - \int_{\Omega} h \, d\mathcal{H}^3.$$

Definition 5.2.1. A pair $(V, \Omega) \in \mathcal{P}$ is a *weakly stable h -surface in U* if

- (V, Ω) has the regularity of Theorem 2.0.1 in U ,
- for any $\phi \in \mathcal{I}(U)$, we have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}^h(\phi(t, \cdot)_{\#} V, \phi(t, \Omega)) &= 0, \\ \left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{A}^h(\phi(t, \cdot)_{\#} V, \phi(t, \Omega)) &\geq 0. \end{aligned}$$

We check that weakly stable h -surfaces satisfy the usual stability inequality.

Proposition 5.2.2. *If (V, Ω) is a weakly stable h -surface in U with corresponding immersion $f : \Sigma \rightarrow U$, then for any $\psi \in C_c^\infty(\Sigma)$ which is the pullback by f of a smooth function on U , we have*

$$\int_{\Sigma} |\nabla^\Sigma \psi|^2 - (|A^\Sigma|^2 + \text{Ric}_M(\nu, \nu)) \psi^2 \, d\mathcal{H}^2 \geq \int_{\partial^* \Omega} (H^\Sigma h + \partial_\nu h - |H^\Sigma|^2) \psi^2 \, d\mathcal{H}^2.$$

Proof. This formula follows from the standard computation of the second variation of area and potential energy (see [BdCE88, Proposition 2.5]), combined with the observation that all the integrands are well defined measurable and integrable functions since the immersion is $C^{1,1}$. □

The next result is the $C^{1,1}$ analogue of the usual curvature estimates for stable minimal (or prescribed mean curvature) surfaces (see e.g. [ZZ20, Theorem 3.6]).

Theorem 5.2.3. *Let $U \subset M$ be an open set and $c > 0$. Let $(V, \Omega) \in \mathcal{P}_c$ be a weakly stable h -surface in U for a function with $\|h\|_{C^1} \leq c$ and $\|V\|(U) \leq m_0$. Let $f : \Sigma \rightarrow U$ be the $C^{1,1}$ immersion inducing the varifold V in U . Then there is a constant $C = C(U, c, m_0) > 0$ so that*

$$\sup_{p \in \Sigma} \min\{1, d_\Sigma(p, f^{-1}(\partial U))\} \operatorname{Lip}_p \nu \leq C.$$

Proof. We argue by contradiction, using a standard point-picking argument (cf. [Whi16, Lecture 3]).

Suppose there is a sequence of such immersions $f_i : \Sigma_i \rightarrow M$ so that

$$\sup_{p \in \Sigma_i} \min\{1, d_{\Sigma_i}(p, f_i^{-1}(\partial U))\} \operatorname{Lip}_p \nu_i = R_i \rightarrow \infty.$$

Let us denote the local Lipschitz constant $\operatorname{Lip}_p \nu_i$ by $l_i(p)$.

Now pick $p_i \in \Sigma_i$ so that

$$\min\{1, d_{\Sigma_i}(p_i, f_i^{-1}(\partial U))\} l_i(p_i) \geq \frac{1}{2} R_i. \quad (5.2.1)$$

We shall rescale the metric on U by $l_i(p_i) \rightarrow \infty$ and obtain a new metric $\tilde{g}_i = l_i(p_i)g$, so that, with respect to the new metric, $\tilde{l}_i(p_i) = 1$. By (5.2.1), the rescalings $(U, \tilde{g}_i)$ converge locally smoothly to $(\mathbb{R}^3, g_{\text{Euc}})$, and, under the rescaling,

$$d_{\tilde{\Sigma}_i}(p_i, f_i^{-1}(\partial U)) \geq \frac{1}{2} R_i.$$

Assume, without loss of generality that $f_i(p_i) \rightarrow 0 \in \mathbb{R}^3$. For $p, p_i \in \Sigma_i$, if (under the rescaling) $d_{\tilde{\Sigma}_i}(p, p_i) \leq \frac{1}{4} R_i$, then, since the quantity $d_{\Sigma_i}(\cdot, f_i^{-1}(\partial U)) l_i(\cdot)$ is scale

invariant, $\tilde{l}_i(p) \leq \frac{R_i}{R_i/2 - R_i/4} = 4$. Thus,

$$\sup_{p \in B_{R_i/4}^{\tilde{\Sigma}_i}(p_i)} \tilde{l}_i(p) \leq 4.$$

In any compact set, the mass bound m_0 and the mononicity formula give a uniform mass bound for the rescaled immersions. Recalling that $R_i \rightarrow \infty$, we conclude that, by Lemma 5.1.3, (up to a subsequence) the rescaled immersions converge in $C_{\text{loc}}^{1,\alpha}$ to a complete, 2-sided $C^{1,1}$ immersion $f : \Sigma \rightarrow \mathbb{R}^3$ with Lipschitz unit normal vector field ν such that $\sup_{\Sigma} \text{Lip}_p \nu \leq 4$.

Claim 1: f is a $C^{1,1}$ immersion with locally ordered sheets. Indeed, the $C_{\text{loc}}^{1,\alpha}$ graphical convergence preserves the local ordering of sheets of the immersions f_i .

Claim 2: The image of f is a smooth embedded minimal surface. Since each sheet of the rescaled immersion in a small ball has c/R_i bounded first variation, the lower semicontinuity of the first variation (see [Sim83, Theorem 42.7]) implies that each sheet of f in a small ball is stationary. By Allard's regularity theorem [All72], f is a smooth minimal immersion. Since the sheets are ordered locally, the strong maximum principle for minimal surfaces implies that each component of the image of f is a smoothly embedded minimal surface.

Claim 3: f is a stable minimal immersion. By claim 2, f is a covering of a smooth minimal embedding, so we can assume without loss of generality that f is an embedding. We consider the limit of (5.2.2), where we fix a function ψ on Σ with compact support and transplant it to Σ_i using the graphical identification. Since the rescalings satisfy $|H^{\Sigma_i}| \leq c/R_i$ and $|\nabla h_i| \leq c/R_i$, the right hand side of (5.2.2) vanishes in the limit. Since the convergence is $C_{\text{loc}}^{1,\alpha}$, every term on the right hand side converges to the corresponding term for Σ except the $|A^{\Sigma_i}|^2$ term. Since the ambient space converges locally smoothly to Euclidean $\mathbb{R}^3$, [Lan85, Theorem 4.3] implies that the term $\int_{\Sigma_i} |A^{\Sigma_i}|^2 u^2 d\mathcal{H}^2$ is lower semicontinuous (we have local weak convergence

in $W^{2,2}$ by the uniform $C^{1,1}$ bounds). Hence, Σ satisfies the stability inequality in $\mathbb{R}^3$.

By the stable Bernstein theorem in $\mathbb{R}^3$ due to [FCS80], [dCP79], and [Pog81], Σ is flat. By Corollary 3.9.2, Σ_i converges in $C_{\text{loc}}^{1,1}$ to the flat limit. However, we chose points so that $\text{Lip}_{p_i} \nu_i \geq 1$ where $p_i \rightarrow 0$, which yields a contradiction. $\square$

As usual, this estimate directly implies $C_{\text{loc}}^{1,\alpha}$ pre-compactness for weakly stable h -surfaces (cf. [ZZ20, Theorem 3.6]).

Chapter 6

Application to existence in the 3-sphere

6.1 Min-max setup

We set up an appropriate setting to carry out the min-max procedure of [CDL03] for the prescribed mean curvature functional.

Let $v > 0$, $\kappa > 0$, and $c > 0$ satisfy

$$\frac{8\pi}{(1 + \kappa)^2 + c^2/4} > 4\pi + vc.$$

Consider a smooth metric g on $\mathbb{S}^3$ so that

- $R_g \geq 6$,
- $\text{Vol}(\mathbb{S}^3, g) \leq v$,
- $(\mathbb{S}^3, g)$ isometrically embeds in $\mathbb{R}^N$ with $|A_{\mathbb{S}^3}| \leq 1 + \kappa$.

Note that the round metric satisfies these conditions with $v = 2\pi^2$, $\kappa = 0$, and $c = 0.547$. Moreover, there is an explicit C^2 graphical neighbourhood of the standard

embedding $\mathbb{S}^3 \subset \mathbb{R}^3 \subset \mathbb{R}^N$ and an explicit constant $c > 0$ for which these conditions are satisfied (by rescaling). Let $h : \mathbb{S}^3 \rightarrow \mathbb{R}$ be a smooth function with $|h| \leq c$. Moreover, we assume that h satisfies the conditions $(\dagger)$ or $(\ddagger)$ from [ZZ20]. By [ZZ20, Proposition 3.8 and Corollary 3.18], the set of functions satisfying these conditions is open and dense, all positive functions satisfy $(\ddagger)$, and any such function h has the unique continuation property for surfaces of prescribed mean curvature h .

Let g_0 be the round metric on $\mathbb{S}^3$. Let $x_0 \in \mathbb{S}^3$. Let $\{\Omega_t\}_{t \in [0,1]}$ be given by

$$\Omega_t := B_{\pi t}^{g_0}(x_0).$$

Note that we use the round metric in particular only to define the standard sweepout; everything else uses the general metric g . Let $\mathcal{M}$ be the set of smooth maps $\psi : [0, 1] \times [0, 1] \times M \rightarrow M$ so that

- $\psi(s, t, \cdot)$ is a diffeomorphism of $\mathbb{S}^3$,
- $\psi(s, 0, \cdot) = \text{id}$.

We define the min-max width $\omega_1(g, h)$ by

$$\omega_1(g, h) := \inf_{\psi \in \mathcal{M}} \sup_{t \in [0,1]} \mathcal{A}^h(\psi(1, t, \Omega_t)).$$

We say $\{\Omega_t^j = \psi_j(1, t, \Omega_t)\}_{j \in \mathbb{N}}$ for $\psi_j \in \mathcal{M}$ is a *minimising sequence* if

$$\lim_{j \rightarrow \infty} \sup_t \mathcal{A}^h(\Omega_t^j) = \omega_1(g, h).$$

Given a minimising sequence, we say $\{\Omega_{t_j}^j\}$ is a *min-max sequence* if

$$\lim_{j \rightarrow \infty} \mathcal{A}^h(\Omega_{t_j}^j) = \omega_1(g, h).$$

By the standard compactness theory (see [Mag12, Theorem 12.26] and [Sim83,

chapter 1, Theorem 4.4]), for any min-max sequence $\{\Omega_{t_j}^j\}$, there is a set $\Omega \subset \mathbb{S}^3$ and a varifold V satisfying

$$\mathbf{1}_{\Omega_{t_j}^j} \xrightarrow{L^1} \mathbf{1}_\Omega, \quad D\mathbf{1}_{\Omega_{t_j}^j} \xrightarrow{*} D\mathbf{1}_\Omega, \quad \mathbf{v}(\partial\Omega_{t_j}^j) \rightharpoonup V,$$

and

$$\|V\|(\mathbb{S}^3) - \int_\Omega h \, d\mathcal{H}^3 = \omega_1(g, h).$$

We show the following theorem.

Theorem 6.1.1. *Suppose $(\mathbb{S}^3, g)$ and h satisfy the constraints (6.1) and $\int_{\mathbb{S}^3} h \, d\mathcal{H}_g^3 \geq 0$. Then there is an open set $\Omega \subset \mathbb{S}^3$ satisfying*

- $\mathcal{A}^h(\Omega) = \omega_1(g, h)$,
- $\partial\Omega$ is a disjoint union of smooth embedded spheres with prescribed mean curvature h with respect to Ω .

Since reversing orientations corresponds to reversing the sign of h , Theorems 2.1.3 and 2.1.5 follow from Theorem 6.1.1.

6.2 Density estimates in the 3-sphere

We exhibit an upper and lower bound for the mass of a min-max varifold for the $\mathcal{A}^h$ functional with a point of density at least two. We show that these two bounds are contradictory in the case of our chosen setup.

We first show an upper bound, using the upper bound for c , the upper bound for the volume of $(\mathbb{S}^3, g)$, and the lower bound for the scalar curvature of g .

Proposition 6.2.1. *Suppose $(\mathbb{S}^3, g)$ satisfies the hypotheses of §6.1. Suppose $(V, \Omega) \in$*

$\mathcal{P}$ satisfies $\|V\|(\mathbb{S}^3) - \int_{\Omega} h \, d\mathcal{H}^3 = \omega_1(g, h)$. Then

$$\|V\|(\mathbb{S}^3) \leq 4\pi + vc.$$

Proof. By [MN12, Theorem 1.1], $R_g \geq 6$ implies that $\omega_1(g, 0) \leq 4\pi$. Let $h_+ := \min\{0, h\}$ and $h_- := \min\{0, -h\}$. We compute

$$\begin{aligned} \|V\|(\mathbb{S}^3) - \int_{\mathbb{S}^3} h_+ \, d\mathcal{H}^3 &\leq \|V\|(\mathbb{S}^3) - \int_{\Omega} h \, d\mathcal{H}^3 \\ &= \omega_1(g, h) \\ &\leq \omega_1(g, 0) + \int_{\mathbb{S}^3} h_- \, d\mathcal{H}^3 \\ &\leq 4\pi + \int_{\mathbb{S}^3} h_- \, d\mathcal{H}^3. \end{aligned}$$

Since $|h| \leq c$ and $\mathcal{H}^3(\mathbb{S}^3, g) = v$, the conclusion follows. $\square$

To prove the lower bound, we recall a monotonicity formula for the Willmore energy (modified for our setting).

Lemma 6.2.2 ([Sim93, (1.4)] and [Top98, §6]). *If V is an integer rectifiable 2-varifold in $\mathbb{R}^N$ with bounded first variation and finite mass, then*

$$\pi\Theta^2(V, x_0) \leq \frac{1}{16} \int_{\mathbb{R}^4} |H_V|^2 \, d\|V\|$$

for any $x_0 \in \mathbb{R}^N$.

Proof. We note that the proof given in the first part of the proof of Lemma 1 from [Top98] applies to varifolds (where our convention for mean curvature differs from Topping's by a factor of 2). The only point where the C^1 assumption is used is in showing that the rightmost term in [Top98, (21)] vanishes as $\sigma \rightarrow 0$. Instead, we use

the more trivial bound:

$$\begin{aligned} \int_{B_\sigma} (\sigma^{-2} - \rho^{-2}) X \cdot H_V \, d\|V\| &\leq ((\sigma^{-2} - \rho^{-2})\sigma c) \|V\|(B_\sigma) \\ &\leq c\sigma \frac{\|V\|(B_\sigma)}{\sigma^2} = O(\sigma) \end{aligned}$$

as $\sigma \rightarrow 0$ by the monotonicity formula and assumption of bounded support and finite mass. $\square$

We apply this monotonicity formula to obtain the mass lower bound.

Proposition 6.2.3. *Suppose $(\mathbb{S}^3, g)$ satisfies the hypotheses of §6.1. If V is a varifold in $(\mathbb{S}^3, g)$ with c -bounded first variation and $\Theta^2(V, x) \geq 2$ for some $x \in \mathbb{S}^3$, then*

$$\|V\|(\mathbb{S}^3) \geq \frac{8\pi}{(1 + \kappa)^2 + c^2/4}.$$

Proof. Using the isometric embedding of $(\mathbb{S}^3, g)$ into $\mathbb{R}^N$, we view V as a 2-varifold in $\mathbb{R}^N$. By the bound on the second fundamental form of $\mathbb{S}^3 \subset \mathbb{R}^N$, we have

$$|H_V^{\mathbb{R}^N}|^2 \leq 4(1 + \kappa)^2 + c^2.$$

By Lemma 6.2.2 and the assumption of a point of density at least 2, we have

$$2\pi \leq \frac{1}{16} \int_{\mathbb{R}^N} |H_V^{\mathbb{R}^N}|^2 \, d\|V\| \leq \frac{4(1 + \kappa)^2 + c^2}{16} \|V\|(\mathbb{S}^3),$$

which yields the desired inequality. $\square$

Combining the mass upper and lower bounds, we deduce the desired conclusion about the density of special varifolds appearing in the min-max construction.

Corollary 6.2.4. *Suppose $(\mathbb{S}^3, g)$ satisfies the hypotheses of §6.1. Suppose $(V, \Omega) \in \mathcal{P}$ satisfies $\|V\|(\mathbb{S}^3) - \int_\Omega h \, d\mathcal{H}^3 = \omega_1(g, h)$. Further suppose that the varifold V has*

integer density everywhere and c -bounded first variation in $(\mathbb{S}^3, g)$. Then $\Theta^2(V, x) = 1$ for all $x \in \text{spt}\|V\|$.

Proof. Suppose otherwise for contradiction. Then by Proposition 6.2.3 and Proposition 6.2.1, we must have

$$4\pi + vc \geq \frac{8\pi}{(1 + \kappa)^2 + c^2/4}.$$

This inequality contradicts (6.1). □

6.3 Min-max proof

We show that each step in [CDL03] and [DLP10] can be modified to work in our setting.

6.3.1 Nontrivial width

The fact that $\omega_1(g, h) > 0$ follows immediately from the isoperimetric inequality. By the assumption $\int_{\mathbb{S}^3} h \, d\mathcal{H}^3 \geq 0$, $\omega_1(g, h)$ cannot be achieved by a trivial solution (meaning a min-max sequence whose limit satisfies $V = 0$ and $\Omega \in \{\mathbb{S}^3, \emptyset\}$).

6.3.2 Pull-tight

The pull-tight argument of [ZZ19, §4] and [ZZ20, §5] applies to our setting, as the modification to an arbitrary minimising sequence uses ambient isotopies. Hence, there is a minimising sequence $\{\Omega_t^j\}$ with the property that every min-max sequence $\{\Omega_{t_j}^j\}$ satisfies $\mathbf{v}(\partial\Omega_{t_j}^j) \rightharpoonup V$, where V is a varifold with c -bounded first variation and bounded mass.

6.3.3 Almost-minimising

We say $\Omega \subset \mathbb{S}^3$ is ε -almost minimising in $U \subset \mathbb{S}^3$ if there is no isotopy $\phi \in \mathcal{I}(U)$ satisfying

$$\mathcal{A}^h(\phi(t, \Omega)) \leq \mathcal{A}^h(\Omega) + \varepsilon/8,$$

$$\mathcal{A}^h(\phi(1, \Omega)) \leq \mathcal{A}^h(\Omega) - \varepsilon.$$

We say a sequence $\{\Omega^j\}$ is almost minimising in U if Ω^j is ε_j almost minimising in U with $\varepsilon_j \rightarrow 0$.

With this definition, we see that the argument of [CDL03, §5] is general and does not depend on the specific choice of functional. Hence, there is a function $r : \mathbb{S}^3 \rightarrow \mathbb{R}^+$ and a min-max sequence $\{\Omega^j\}$ satisfying the conclusions of (2) so that $\{\Omega^j\}$ is almost minimising in any annulus centred at any $x \in \mathbb{S}^3$ with outer radius at most $r(x)$.

To allow application of our regularity theory, we assume (without loss of generality) that $\mathcal{H}^3(B_{r(x)}(x)) \leq \eta/8$ for all $x \in M$.

6.3.4 Regularity of weak replacements

Definition 6.3.1. Let $(V, \Omega) \in \mathcal{P}_c$. A pair $(V', \Omega') \in \mathcal{P}_c$ is a *weak replacement* for (V, Ω) in $U \subset \mathbb{S}^3$ if

- $V' \llcorner G(M \setminus \overline{U}, 2) = V \llcorner G(M \setminus \overline{U}, 2)$ and $\Omega \setminus \overline{U} = \Omega' \setminus \overline{U}$,
- $|||V||(\mathbb{S}^3) - |||V'||(\mathbb{S}^3)| \leq c\mathcal{H}^3(U)$,
- (V', Ω') is in the $C_{\text{loc}}^{1,\alpha}$ graphical closure of the set of weakly stable h -surfaces in U .

Lemma 6.3.2. *If there is a function $r : \mathbb{S}^3 \rightarrow \mathbb{R}$ so that $(V, \Omega) \in \mathcal{P}_c$ has a weak replacement in any annulus centred at any $x \in \mathbb{S}^3$ with outer radius at most $r(x)$, then V is integer rectifiable and any tangent cone to V at any $x \in \text{spt}\|V\|$ is an integer multiple of a plane.*

Proof. We observe that the same proof as [CDL03, Lemma 6.4] works in our setting.

Density lower bound. The monotonicity formula implies there is a constant c_2 so that

$$\frac{\|V\|(B_\sigma(x))}{\pi\sigma^2} \leq \frac{c_2\|V\|(B_\rho(x))}{\pi\rho^2}$$

for any varifold V with c -bounded first variation, $x \in \mathbb{S}^3$, and $\sigma < \rho \ll 1$. Let $r < r(x)$ sufficiently small. Let (V', Ω') be a replacement for (V, Ω) in $A(x, r, 2r)$. By the maximum principle of [Whi10], V' is not identically 0 on $A(x, r, 2r)$. Since $V' \llcorner G(A(x, r, 2r), 2)$ is the varifold of a $C^{1,1}$ immersion, there is a $y \in A(x, r, 2r)$ with $\Theta^2(V', y) \geq 1$. Then we have

$$\frac{\|V\|(B_{4r}(x))}{16\pi r^2} \geq \frac{\|V'\|(B_{4r}(x))}{16\pi r^2} - Cr \geq \frac{\|V'\|(B_{2r}(y))}{16\pi r^2} - Cr \geq \frac{c_2}{4} - Cr.$$

Hence, we have a uniform positive lower bound for the density by taking r sufficiently small. By [All72, Theorem 5.5], V is rectifiable.

Tangent cones. Let $x \in \text{spt}\|V\|$. Let C be a tangent cone to V at x , and let $\rho_n \rightarrow 0$ satisfy

$$(f_{x, \rho_n})_\# V \rightharpoonup C.$$

Let (V'_n, Ω'_n) be a replacement for (V, Ω) in $A(x, \rho_n/4, 3\rho_n/4)$. Let $W'_n = (f_{x, \rho_n})_\# V'_n$. Up to a subsequence (not relabeled), we have $W'_n \rightharpoonup C'$, where C' is a stationary varifold. By the definition of the replacement, we have

$$C' \llcorner G(B_1(0) \setminus A(0, 1/4, 3/4), 2) = C \llcorner G(B_1(0) \setminus A(0, 1/4, 3/4), 2),$$

$$\|C'\|(B_\rho(0)) = \|C\|(B_\rho(0)) \quad \forall \quad \rho \in (0, 1/4) \cup (3/4, 1).$$

Since C is a cone, (6.3.4) implies

$$\frac{\|C'\|(B_\sigma(0))}{\pi\sigma^2} = \frac{\|C'\|(B_\rho(0))}{\pi\rho^2}$$

for all $\rho \in (0, 1/4) \cup (3/4, 1)$. By stationarity and monotonicity, we conclude that C' is a cone. Since C and C' agree on $B_{1/4}(0)$, we have $C' = C$. By Theorem 5.2.3, the convergence in $A(0, 1/4, 3/4)$ is $C_{\text{loc}}^{1,\alpha}$ graphical. By the same argument as in the proof of Claims 1 and 2 in Theorem 5.2.3 (i.e. the mean curvature of each sheet vanishes in the limit, and the ordering of sheets is preserved so we can apply the strong maximum principle), C' is the varifold of a smooth embedded minimal surface with integer multiplicity on the annulus $A(0, 1/4, 3/4)$. Then C' is (an integer multiple of) a smooth embedded minimal cone in $\mathbb{R}^3$, and hence a plane with integer multiplicity. $\square$

6.3.5 Regularity of replacements

Definition 6.3.3. Let $(V, \Omega) \in \mathcal{P}_c$. A pair $(V', \Omega') \in \mathcal{P}_c$ is a *replacement* for (V, Ω) in $U \subset \mathbb{S}^3$ if

- $V' \llcorner G(M \setminus \overline{U}, 2) = V \llcorner G(M \setminus \overline{U}, 2)$,
- $|||V||(\mathbb{S}^3) - |||V'||(\mathbb{S}^3)| \leq c\mathcal{H}^3(U)$,
- $V' \llcorner G(U, 2)$ is an embedded smooth stable surface with prescribed mean curvature h with respect to Ω' .

Remark 6.3.4. Observe that any replacement is also a weak replacement.

Definition 6.3.5. The pair $(V, \Omega) \in \mathcal{P}_c$ has the *good replacement property* if

1. there is a positive function $r : U \rightarrow \mathbb{R}^+$ such that there is a replacement (V', Ω') for (V, Ω) in any annulus centred at x with outer radius at most $r(x)$ for all $x \in U$,

2. there is a positive function $r' : U \rightarrow \mathbb{R}^+$ such that there is a replacement (V'', Ω'') for the replacement (V', Ω') from (1) in any annulus centred at x with outer radius at most $r(x)$ and any annulus centred at y with outer radius at most $r'(y)$ for any $y \in U$,
3. there is a positive function $r'' : U \rightarrow \mathbb{R}^+$ such that there is a replacement (V''', Ω''') for the replacement (V'', Ω'') from (2) in any annulus centred at z with outer radius at most $r''(z)$ for any $z \in U$.

Lemma 6.3.6. *If $(V, \Omega) \in \mathcal{P}_c$ has the good replacement property, then V is an embedded smooth surface with prescribed mean curvature h with respect to Ω .*

Proof. Since unique continuation and curvature estimates hold, the proof is the same as the proof of [CDL03, Proposition 6.3]. The only exception is the argument showing $\Sigma \subset \text{spt}\|V\|$ on $B_\rho(x) \setminus \{x\}$, which in [CDL03] uses that $\|V\|(\mathbb{S}^3) = \|V'\|(\mathbb{S}^3)$. Instead, we follow the argument from [ZZ19, Theorem 6.1 Claim 5] and [ZZ20, Theorem 7.1 Claim 5], which applies to our setting. $\square$

6.3.6 Construction of replacements

The construction in [CDL03, §7] is general and applies to our setting (see also [ZZ20, §6]). We need only make two observations.

First, to go from the regularity of minimisers in isotopy classes to the regularity of minimisers in constrained isotopy classes, we need a squeezing lemma like [CDL03, Lemma 7.4]. We note that the same proof applies in our setting, as volume changes after squeezing will be arbitrarily small.

Second, we observe that the replacements constructed in this way are a priori only weak replacements. However, Lemma 6.3.2 and Corollary 6.2.4 imply that the weak replacements we construct are in fact strong replacements under the assumptions of Theorem 6.1.1.

6.3.7 Genus bound

We follow the argument of [DLP10].

The analogue of [DLP10, Proposition 3.2], i.e. regularity of the constrained isotopy minimisation problem, follows from the above interior and boundary regularity theorems, combined with the observation that the squeezing argument at the boundary from [DLP10, Lemma 6.1] works in our setting (after squeezing, volume changes will be arbitrarily small).

The proof of [DLP10, Proposition 2.1] in [DLP10, §4] follows the same argument combined with the following observations. First, the min-max limit V is a smooth embedded surface of prescribed mean curvature and satisfies unique continuation. Hence, when we solve local constrained minimisation problems and take limits using the compactness theory developed in chapter 5, the apriori $C^{1,1}$ local minimisers converge in $C_{\text{loc}}^{1,\alpha}$ to the smooth limit V . Since local minimisers are smooth where they have density 1, we can always take the local constrained minimisers to be smooth stable embedded surfaces of prescribed mean curvature. Second, the estimate [DLP10, Lemma 4.2] holds for smooth surfaces with bounded mean curvature, where the constant C depends on the mean curvature bound. The proof follows by keeping track of the nonvanishing left hand side in the first variation formula [DLP10, (A.2)].

Equipped with Simon's Lifting Lemma [DLP10, Proposition 2.1], the rest of the proof is identical.

Part II

Boundary behaviour of Allen–Cahn solutions

Chapter 7

The Allen–Cahn equation with Neumann boundary conditions and limit-interfaces

7.1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. For each small $\varepsilon > 0$, consider the *Allen–Cahn energy functional* arising from the van der Waals–Cahn–Hilliard theory of phase transition [CH58]:

$$E_\varepsilon(u) := \int_\Omega \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \tag{7.1.1}$$

where $u : \Omega \rightarrow \mathbb{R}$ belongs to the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$. Here, $W : \mathbb{R} \rightarrow \mathbb{R}$ is a nonnegative double-well potential with strict minima at ± 1 , where $W(\pm 1) = 0$. If $u \in H^1(\Omega)$ is a critical point of E_ε , then by standard elliptic

theory $u \in C^3(\overline{\Omega})$ and it satisfies the Euler–Lagrange equation:

$$\begin{cases} -\varepsilon^2 \Delta u + W'(u) = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \quad (7.1.2)$$

where ν is the outward (with respect to Ω) unit normal of $\partial\Omega$. In the literature, (7.1.2) is often known as the *Allen–Cahn equation*. This equation and its parabolic counterpart (see e.g. [RSK89]) play an important role in the theory of phase transitions.

There is an extensive literature on the semi-linear elliptic equation (7.1.2). Much of the development was centred around a celebrated conjecture of De Giorgi concerning entire (i.e. $\Omega = \mathbb{R}^n$) solutions. For past results and recent developments on this important problem, we refer the interested reader to the survey [CW18].

The Allen–Cahn equation is closely related to the theory of minimal hypersurfaces. This connection arises when one considers the asymptotic behaviour of solutions to the Allen–Cahn equation as $\varepsilon \rightarrow 0^+$. Heuristically, the nodal sets of solutions u_ε to the Allen–Cahn equation will converge in some sense to a sharp limit interface which is an embedded minimal hypersurface (up to—at worst—a small singular set). For minimising solutions, this was made precise by Modica [Mod87] and Kohn–Sternberg [KS89] using De Giorgi’s theory of Γ -convergence. Minimisation techniques were further used to show the existence of stable solutions of (7.1.2) in two-dimensional non-convex domains. Furthermore, the existence of unstable solutions in two-dimensions was later studied by Kowalczyk [Kow05]. Finally, Sternberg and Zumbrun in [SZ98] studied the connectivity properties of phase boundaries in strictly convex domains.

For the case of general critical points, the interior regularity theory of (7.1.2) has been developed in a series of papers by Padilla–Tonegawa [PT98], Hutchinson–Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa–Wickramasekera [TW12]. Building on this regularity theory, in his PhD thesis, Guaraco [Gua18] developed a min-

max theory for minimal hypersurfaces in closed Riemannian manifolds based on the Allen–Cahn equation. This approach has proved to be particularly successful when the ambient space is of dimension two and three, as seen, for example, in the work of Chodosh and Mantoulidis [Man21, CM23, CM20].

Although we have by now a rather satisfactory interior theory, much less is known about the boundary behaviour of solutions of (7.1.2) in the singular limit as $\varepsilon \rightarrow 0^+$. One major obstacle is represented by the phenomenon of boundary concentration. When Ω is the unit ball, Malchiodi, Ni and Wei [MNW07] constructed radially symmetric solutions u_ε to (7.1.2) whose nodal sets cluster towards $\partial\Omega$. Malchiodi and Wei [MW07] later removed the symmetry assumption and showed that such boundary clustering can occur for any mean-convex domain $\Omega \subset \mathbb{R}^n$. These results show that the limit-interface can collapse to the boundary even when Ω is *convex*.

Our goal in this part of this thesis and in subsequent forthcoming work is to fill in this gap in the literature and to generalise the Allen–Cahn min-max theory to the free boundary setting. More precisely, in this thesis we study the boundary behaviour of the limit-interface arising from general critical points of the Allen–Cahn energy functional E_ε in (7.1.1), i.e. solutions satisfying (7.1.2), subject only to a uniform energy bound. In particular, we do not assume that u_ε is *minimising* or *stable*, and we show that the limit-interface is *integer rectifiable* up to the boundary (when regarded as an $(n - 1)$ -varifold *in* $\mathbb{R}^n$). Moreover, we show that the limit varifold is *stationary with free boundary*, i.e. a critical point of the area functional with respect to variations which preserve the boundary $\partial\Omega$ as a set (we note that a different Dirichlet boundary value problem, related to Plateau’s problem, was recently studied by Guaraco and Lynch [GL23]) This provides the optimal regularity of the limit-interface under our assumptions, thus extending the interior regularity result of Hutchinson–Tonegawa [HT00] to the free boundary setting.

We now explain in more details what the key difficulties in extending the interior

regularity theory to the boundary are, and how we overcome these new difficulties. One of the main ingredients in the interior regularity theory for general critical points is the almost monotonicity formula obtained in [HT00, Proposition 3.4]. Using a maximum principle argument and a reflection process modelled on [GJ86], Tonegawa [Ton03, Theorem 3.7] proved a *global* monotonicity formula holding up to the boundary, under the assumption that Ω is *convex*. One of our main results in this chapter is a boundary monotonicity-type formula (Proposition 7.4.16), which differs from that of [Ton03] in two respects. First, our result does not require the convexity of $\partial\Omega$. Second, whereas the monotonicity formula in [Ton03] contains terms integrated on balls reflected across the boundary as in [GJ86], our monotonicity formula does not involve any reflection and is thus of simpler application. Note that a similar monotonicity formula for free boundary stationary varifolds was obtained by Martin Li in joint work with Guang and Zhou in [GLZ20]. We note that it would be possible—though we chose not to pursue it here—to use a reflection technique similar to the one adopted in [GJ86] and [Ton03], and obtain a global monotonicity formula involving the energy density on balls reflected across the boundary, resembling that of [Ton03].

Since we chose not to consider reflected balls, a technical difficulty we have to overcome is the fact that our monotonicity-type formula only applies to balls that are either disjoint from the boundary or centred exactly at the boundary. This makes some arguments in the boundary setting more delicate than the corresponding ones in the interior regularity theory. For example, in contrast to the interior regularity theory of [HT00], we can only establish a uniform density ratio upper bound up to the boundary, but not a uniform lower bound (c.f. Proposition 7.5.2). This is again related to the aforementioned boundary clustering phenomenon, which is a major obstacle in our setting. Nonetheless, as in the interior setting, we can prove the vanishing of the discrepancy measure (c.f. Theorem 7.5.4) which allows us to obtain control on the first variation of the limit interface, and to establish the integer-

rectifiability of the limit varifold, as in [HT00].

The chapter is organised as follows. In section 7.2, we introduce the notations and some preliminary results that will be used throughout the rest of the chapter. We then state our assumptions and the main result of this chapter (Theorem 7.3.3) in section 7.3. In section 7.4, we derive our key monotonicity-type formula and several of its consequences. Much of the work is then devoted to proving pointwise and integral estimates on the discrepancy function. Using our monotonicity formula, we then derive at the diffuse level (i.e. $\varepsilon > 0$) uniform bounds on the energy density ratio for balls that are either centred at a boundary point or completely disjoint from the boundary. In section 7.5, we establish bounds on the density ratio for the limit measure and prove the vanishing of discrepancy measure, which implies the rectifiability of the limit interface. The last section, section 7.6, shows further that the limit interface is in fact an integral varifold up to the boundary.

7.2 Notation and preliminaries

In this section, we introduce our notation and some preliminary concepts that will be used throughout the rest of this chapter.

7.2.1 Ambient space

Let $(\widetilde{M}, g)$ be a smooth complete n -dimensional Riemannian manifold *without boundary*. We use dist_g and ∇ to denote the intrinsic distance and Levi–Civita connection on $\widetilde{M}$ induced by g . For any $x \in \widetilde{M}$, we denote by $\exp_x$ the exponential map and $\text{inj}_g(x)$ the injectivity radius of $(\widetilde{M}, g)$ at x . We use $B_r(x)$ to denote the open geodesic ball of radius r centred at $x \in \widetilde{M}$. For a given subset $A \subset \widetilde{M}$, we denote by $\mathbf{1}_A$ its indicator function; by $\overline{A}$ and A° respectively the closure and interior of A with respect to the topology on $(\widetilde{M}, g)$. Moreover, we set $\text{inj}_g(A) := \inf_{x \in A} \text{inj}_g(x)$ and

$U_r(A) := \cup_{x \in A} B_r(x)$. For any linear transformations $A, B : T_x \widetilde{M} \rightarrow T_x \widetilde{M}$, we define the Frobenius (or Hilbert–Schmidt) inner product and norm by

$$\langle A, B \rangle_g := \sum_{i,j=1}^n A_{ij} B_{ij} \quad \text{and} \quad |A|_g := \sqrt{\langle A, A \rangle}$$

where A_{ij}, B_{ij} are matrix components of A, B under an orthonormal basis of $T_x \widetilde{M}$ with respect to g . Moreover, $\|A\|_g$ denotes the operator norm. We shall sometimes use $\cdot$ to denote the inner product g on the tangent spaces of $\widetilde{M}$ or omit the subscript g when there is no ambiguity. Vectors and co-vectors are identified via the metric g wherever necessary.

Throughout this chapter, we shall assume $(\widetilde{M}, g)$ has positive injectivity radius $\text{inj}_g(\widetilde{M}) \geq i_0 > 0$ and bounded sectional curvature, i.e. $|\sec_{(\widetilde{M}, g)}| \leq k_0$, for some constant $k_0 \geq 0$. The classical Hessian comparison theorem implies (see [CM11, Lemma 7.1]) that the distance function $r(\cdot) = \text{dist}_g(\cdot, x)$ from any $x \in \widetilde{M}$ satisfies

$$\left| \text{Hess } r(X, X) - \frac{1}{r} |X - (X \cdot \nabla r) \nabla r|^2 \right| \leq \sqrt{k_0} \quad (7.2.1)$$

provided that $|X| = 1$ and $r < \min\{i_0, \frac{1}{\sqrt{k_0}}\}$. Taking the trace of (7.2.1) along directions orthogonal to ∇r yields

$$\left| \Delta r - \frac{n-1}{r} \right| \leq (n-1) \sqrt{k_0}. \quad (7.2.2)$$

Moreover, the standard Bochner formula [CM11, Proposition 1.47] implies that for any smooth function f on subsets of $\widetilde{M}$,

$$\left| \frac{1}{2} \Delta |\nabla f|^2 - |\text{Hess } f|^2 - \nabla f \cdot \nabla(\Delta f) \right| \leq (n-1) k_0 |\nabla f|^2. \quad (7.2.3)$$

On the other hand, by the Rauch Comparison Theorem, there exists a constant $r_0 > 0$

depending only on n and k_0 such that for all $r < \min\{r_0, i_0\}$, we have

$$\frac{1}{2}\omega_n r^n \leq \text{Vol}(B_r(x)) \leq 2\omega_n r^n, \quad (7.2.4)$$

$$\frac{1}{2}n\omega_n r^{n-1} \leq \text{Area}(\partial B_r(x)) \leq 2n\omega_n r^{n-1}, \quad (7.2.5)$$

where ω_k is the volume of the k -dimensional unit ball in $\mathbb{R}^k$ with respect to the Euclidean metric.

Let $\mathbf{G}_x(n, k)$ denote the space of unoriented k -dimensional subspaces of $T_x \widetilde{M}$. The orthogonal complement of $S \in \mathbf{G}_x(n, k)$ is denoted by $S^\perp \in \mathbf{G}_x(n, n - k)$. For $a \in T_x \widetilde{M}$, we denote by $a \otimes a : T_x \widetilde{M} \rightarrow T_x \widetilde{M}$ the matrix with components $a_i a_j$ with respect to an orthonormal basis. Finally, for $S \in \mathbf{G}_x(n, k)$, we identify S with the corresponding orthogonal projection of $T_x \widetilde{M}$ onto S . In the case of $k = n - 1$, we also identify $S \in \mathbf{G}_x(n, n - 1)$ with the unit vector $\pm N \in T_x \widetilde{M}$ which is perpendicular to S . Note that we may express this relation by $S = I - N \otimes N$ where I is the identity map on $T_x \widetilde{M}$. The correspondence is a homeomorphism with respect to the naturally endowed topologies on $\mathbf{G}_x(n, n - 1)$ and $\mathbb{S}^{n-1}/\{\pm 1\}$.

7.2.2 Boundary geometry

Let $\Omega \subset \widetilde{M}$ be an open and connected domain with smooth non-empty boundary $\partial\Omega$, whose outward unit normal is denoted by ν . As in [Bre16], we define the local interior ball curvature at $x \in \partial\Omega$ by

$$\overline{Z}_\Omega(x) := \sup_{y \in \partial\Omega, 0 < \text{dist}_g(x, y) \leq \frac{1}{2} \text{inj}_g(\widetilde{M})} \left(-\frac{2 \exp_x^{-1}(y) \cdot \nu(x)}{\text{dist}_g(x, y)^2} \right).$$

The local exterior ball curvature $\underline{Z}_\Omega(x)$ can similarly be defined by considering the infimum instead. When $(\widetilde{M}, g)$ is Euclidean space, this agrees with the notion of (global) interior and exterior ball curvature as given in [ALM13] (c.f. [HL23]). By

letting y approach x , it is clear that $\underline{Z}_\Omega(x) \leq \lambda_1(x) \leq \dots \leq \lambda_{n-1}(x) \leq \overline{Z}_\Omega(x)$ where $\lambda_i(x)$ are the principal curvatures of $\partial\Omega$ at x with respect to the normal ν . Note that $\overline{Z}_\Omega = \underline{Z}_\Omega = 0$ when Ω is a half space in $\mathbb{R}^n$.

We will always assume that Ω has bounded ball curvatures, i.e. $\max\{|\overline{Z}_\Omega|, |\underline{Z}_\Omega|\} \leq \kappa_0$ for some constant $\kappa_0 \geq 0$. Therefore, the signed distance function $d(\cdot) = \text{dist}_g(\cdot, \partial\Omega)$, with $d > 0$ in Ω , is smooth in the tubular neighbourhood $U_{\frac{1}{2\kappa_0}}(\partial\Omega)$ and satisfies the following bounds¹:

$$|\nabla d| = 1, \quad \text{Hess } d(\nabla d, \cdot) = 0 \quad \text{and} \quad \|\text{Hess } d\| \leq C_1 \quad (7.2.6)$$

where $C_1 > 0$ is a constant depending only on n , k_0 and κ_0 . Moreover, for any $x \in \partial\Omega$, the radial distance function $r(\cdot) = \text{dist}_g(\cdot, x)$ restricted to $\partial\Omega$ satisfies

$$|r\nabla r \cdot \nu| \leq \frac{\kappa_0}{2} r^2 \quad (7.2.7)$$

provided that $r < \frac{1}{2} \text{inj}_g(\widetilde{M})$. This follows from the definition and the observation that $\exp_x^{-1} y$ is the parallel transport of $r\nabla r$ from y to x along a radial geodesic. It is also easily seen from the definition that there exists a constant $r_1 > 0$ depending only on n , k_0 and κ_0 such that for any $x \in \partial\Omega$, $r < \min\{r_1, i_0\}$, we have

$$\frac{1}{3} \omega_n r^n \leq \text{Vol}(B_r(x) \cap \Omega) \leq \frac{2}{3} \omega_n r^n, \quad (7.2.8)$$

$$\frac{1}{3} n \omega_n r^{n-1} \leq \text{Area}(\partial B_r(x) \cap \Omega) \leq \frac{2}{3} n \omega_n r^{n-1}.$$

7.2.3 Functions, vector fields and measures

Let $U \subset \widetilde{M}$ be an open subset. We use $C_c(U)$ to denote the space of real-valued continuous functions on U with compact support, and $L^p(U)$ to denote the space of

¹See [GT01, Appendix 14.6] for the calculations in $\mathbb{R}^n$. The Riemannian case follows from standard Jacobi field estimates.

functions on U which are in p -integrable with respect to the volume measure dv_g on $(\widetilde{M}, g)$. For any subset $A \subset \widetilde{M}$, we let $C^k(A)$ be the space of all functions defined on A which can be extended to a C^k function on an open subset $U \subset \widetilde{M}$ containing A . We will use $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$ to denote respectively the positive and negative part of a real-valued function f .

Let $\mathfrak{X}_c(U)$ be the space of smooth, compactly supported vector fields on U ; and $\mathfrak{X}_c^{tan}(U)$ be the subspace consisting of all $X \in \mathfrak{X}_c(U)$ such that $X(x) \in T_x \partial\Omega$ for all $x \in \partial\Omega \cap U$. Note that $\mathfrak{X}_c^{tan}(U) = \mathfrak{X}_c(U)$ if $\partial\Omega \cap U = \emptyset$. A function $u \in L^1(U)$ is said to be in $BV(U)$ (see [EG15], and also [MPPP07] in the Riemannian setting) if

$$\sup \left\{ \int_U u \operatorname{div}_g X \, dv_g : X \in \mathfrak{X}_c(U), |X| \leq 1 \right\} < +\infty$$

where div_g is the divergence operator with respect to g . For any $u \in BV(U)$, we use Du to denote its distributional derivative which is a vector-valued Radon measure on U . The corresponding variation measure is then denoted by $\|Du\|$. A subset $A \subset U$ is said to have finite perimeter in U if $\mathbf{1}_A \in BV(U)$. For a set A of finite perimeter we call $\|\partial A\| := \|D\mathbf{1}_A\|$ the perimeter measure and $\|\partial A\|(U) := \|D\mathbf{1}_A\|(U)$ the perimeter of A in U . We shall use $\partial^* A$ to denote the reduced boundary and $\nu_{\partial^* A}$ the generalised outward normal (c.f. [EG15, section 5.7]). These concepts can be defined locally, as in [EG15, Definition 5.2].

For any $0 \leq k \leq n$, the k -dimensional Hausdorff measure on $(\widetilde{M}, g)$ is denoted by $\mathcal{H}^k$. Note that $\mathcal{H}^n$ agrees with the volume measure dv_g on $(\widetilde{M}, g)$. Given a subset $A \subset U$ and a measure μ on U , the restriction of μ to A is denoted by $\mu \llcorner A$. For any Radon measure μ on U and $\phi \in C_c(U)$ we often write $\mu(\phi)$ for $\int \phi d\mu$. A sequence $\{\mu_i\}_{i \in \mathbb{N}}$ of Radon measures on U is said to converge (in the weak-* topology) to a Radon measure μ on U if $\mu_i(\phi) \rightarrow \mu(\phi)$ for all $\phi \in C_c(U)$. We will denote this weak convergence by $\mu_i \rightarrow \mu$. We omit $d\mu$ if μ is the volume measure dv_g . We write $\operatorname{spt} \mu$

for the support of μ , which is the smallest relatively closed subset $V \subset U$ such that $\mu(U \setminus V) = 0$. Thus, $x \in \text{spt } \mu$ if and only if $\mu(B_r(x)) > 0$ for all $r > 0$. For any $x \in U$, we denote the k -dimensional density of μ at x by

$$\Theta^k(\mu, x) := \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k}$$

provided that the limit exists. Clearly $\Theta^k(\mu, x) = 0$ for all $x \notin \text{spt } \mu$. Therefore, we often regard $\Theta^k(\mu, \cdot)$ as a function defined on $\text{spt } \mu$.

7.2.4 Varifolds

We recall some definitions from geometric measure theory and refer the readers to the standard references [All72, Sim83] for more details. A nice description of varifolds in Riemannian manifolds without using isometric embedding can be found in [Sch22]. We shall only consider codimension one varifolds in this chapter.

For any open set $U \subset \widetilde{M}$, let $G(U) := \cup_{p \in U} \mathbf{G}_p(n, n-1)$. A general varifold in U is a Radon measure on $G(U)$. We denote the set of all general varifolds in U by $\mathbf{V}_{n-1}(U)$, equipped with the weak topology. For $V \in \mathbf{V}_{n-1}(U)$, let $\|V\|$ be the weight measure of V , which is the Radon measure on U defined by

$$\|V\|(\phi) := \int_{G(U)} \phi(x) dV(x, S), \quad \forall \phi \in C_c(U).$$

We say $V \in \mathbf{V}_{n-1}(U)$ is rectifiable if there exists an $\mathcal{H}^{n-1}$ -measurable, countably $(n-1)$ -rectifiable subset $\Sigma \subset U$ and a locally $\mathcal{H}^{n-1}$ -integrable function Θ defined on Σ such that

$$V(\phi) = \int_{\Sigma} \phi(x, T_x \Sigma) \Theta(x) d\mathcal{H}^{n-1}(x) \quad \forall \phi \in C_c(G(U)),$$

where $T_x \Sigma \in \mathbf{G}_x(n, n-1)$ is the unique approximate tangent space of Σ at x which

exists $\mathcal{H}^{n-1}$ -a.e. on Σ . Note that $\Theta^{n-1}(\|V\|, x) = \Theta(x)$ for any such x . A rectifiable varifold V is uniquely determined by its weight $\|V\| = \Theta \mathcal{H}^{n-1} \llcorner \Sigma$ through the above formula. For this reason, we naturally say a Radon measure μ on U is rectifiable when one can associate a rectifiable varifold V such that $\|V\| = \mu$. If $\Theta(x) \in \mathbb{N}$ for $\mathcal{H}^{n-1}$ -a.e. $x \in \Sigma$, we say V is integer rectifiable or an integral varifold. The set of all integral varifolds in U is denoted by $\mathbf{IV}_{n-1}(U)$. We say that V is a unit-density varifold if $\Theta(x) = 1$ for $\|V\|$ -a.e. x .

For $V \in \mathbf{V}_{n-1}(U)$ let δV be the first variation of V given by

$$\delta V(X) := \int_{G(U)} \langle \nabla X(x), S \rangle_g dV(x, S) \quad \forall X \in \mathfrak{X}_c(U).$$

We shall denote the total variation of δV by $\|\delta V\|$. If $\|\delta V\|$ is a Radon measure on U (i.e. V has locally bounded first variation on U), we may apply the Radon-Nikodym theorem to δV with respect to $\|V\|$. Writing the singular part of $\|\delta V\|$ with respect to $\|V\|$ as σ_V , we have a $\|V\|$ -measurable vector field H , a $\|\delta V\|$ -measurable vector field η with $|\eta| = 1$ $\|\delta V\|$ -a.e., and a Borel set $Z \subset U$ such that $\|V\|(Z) = 0$ and

$$\delta V(X) = \int_U X(x) \cdot H(x) d\|V\|(x) + \int_Z X(x) \cdot \eta(x) d\sigma_V(x) \quad (7.2.9)$$

for all $X \in \mathfrak{X}_c(U)$. We call H the generalised mean curvature vector, η the generalised outward co-normal, σ_V the generalised boundary measure, and Z the generalised boundary of V .

To study the boundary behaviour, we shall also need the notion of *free-boundary varifolds* introduced in [Ede20, section 3] (see also [DM21, Definition 2.4]). Given a smoothly embedded hypersurface $S \subset U$ oriented by the normal ν_S , a varifold $V \in \mathbf{V}_{n-1}(U)$ is said to have free boundary in S if there exists a $\|V\|$ -measurable vector field H^T which is locally integrable with respect to $\|V\|$ and $H^T(x) \cdot \nu_S(x) = 0$

for $\|V\|$ -a.e. $x \in S$, satisfying

$$\delta V(X) = \int_U X(x) \cdot H^T(x) d\|V\|(x)$$

for all $X \in \mathfrak{X}_c(U)$ with $X \cdot \nu_S = 0$ on S . Note that we use the notation H^T to distinguish it from the generalised mean curvature vector H defined in (7.2.9). In fact, if H exists, then H^T is the tangential part of H along S (see [Ede20, Definition 3.0.3]). It was shown in [DM21, Corollary 4.7]² that any such free-boundary varifold with $H^T \in L^1_{loc}$ with respect to $\|V\|$ has locally bounded first variation in U . Moreover, by [Ede20, Proposition 3.2], an integral varifold $V \in \mathbf{IV}_{n-1}(U)$ has free boundary in S if and only if V has locally bounded first variation in U , and the generalised boundary measure σ_V is supported in S , and $\eta = \nu_S$ at σ_V -a.e. $x \in Z$. For the case at hand, we will mostly take $S = \partial\Omega$.

7.2.5 Double-well potential

Throughout this chapter, we fix a double-well potential $W : \mathbb{R} \rightarrow \mathbb{R}$ which is a non-negative smooth function satisfying the following properties:

- $W(\pm 1) = W'(\pm 1) = 0$,
- W has a unique local non-degenerate maximum at $\gamma \in (-1, 1)$,
- there exist some $\alpha \in (0, 1)$ and $\kappa \in (0, 1)$ such that $W''(t) \geq \kappa$ for all $|t| \geq \alpha$.

We also define the energy constant:

$$h_0 := \int_{-1}^1 \sqrt{2W(s)} ds.$$

²Although the statement was presented for varifolds in Euclidean space, the arguments there are local and hence the proof carries over to the Riemannian setting.

Remark 7.2.1. A common choice of the double-well potential is given by $W(s) = \frac{1}{4}(1 - s^2)^2$ with the unique local maxima at $\gamma = 0$. Note that our definition of the constant h_0 differs from [HT00, Gua18] (denoted by σ there) by a factor of 2 but agrees with that in [KT18]. Some of our results established in this chapter actually hold for more general potentials with multiple wells, such as those treated in [Ton03].

7.2.6 Constants and cutoffs

We shall use the convention that $C(\alpha, \beta, \gamma, \dots)$ denotes a positive constant (which may change from line to line) that depends only on the parameters $\alpha, \beta, \gamma, \dots$. Throughout this chapter, we fix a smooth cutoff function $\chi : [0, \infty) \rightarrow [0, \infty)$ such that

- $\chi(s) = s$ for $s \in [0, 1]$,
- $\chi(s) = 2$ for $s \geq 4$,
- $0 \leq \chi'(s) \leq 1$ and $0 \leq -\chi''(s) \leq 1/2$ for all s .

For any constant $a > 0$, we denote $\chi_a : [0, \infty) \rightarrow [0, \infty)$ to be the rescaled cutoff function $\chi_a(s) := a\chi(a^{-1}s)$. Note that $0 \leq \chi'_a \leq 1$ and $0 \leq -\chi''_a \leq a^{-1}/2$ everywhere.

7.3 Assumptions and main results

7.3.1 Assumptions

From now on, we will assume the following, unless we explicitly state otherwise. Let $(\widetilde{M}, g)$ be a smooth complete Riemannian manifold with

- injectivity radius lower bound $\text{inj}_g(\widetilde{M}) \geq i_0 > 0$ and
- sectional curvature bound $|\sec_{(\widetilde{M}, g)}| \leq k_0$;

and $\Omega \subset \widetilde{M}$ be an open domain with smooth non-empty boundary $\partial\Omega$ such that

- Ω has bounded ball curvatures $\max\{|\overline{Z}_\Omega|, |\underline{Z}_\Omega|\} \leq \kappa_0$.

The outward unit normal of $\partial\Omega$ is denoted by ν .

We also fix a bounded open set $U \subset \widetilde{M}$ with Lipschitz boundary. Suppose we have

- a sequence $\{\varepsilon_i\}_{i=1}^\infty$ of positive real numbers $\varepsilon_i \rightarrow 0$;
- a sequence $\{\lambda_i\}_{i=1}^\infty$ of real numbers such that $|\lambda_i| \leq \Lambda_0$ for all i ;
- a sequence $\{u_i\}_{i=1}^\infty$ of smooth functions on $\overline{\Omega} \cap U$ with $\|u_i\|_{L^\infty} \leq C_0$ and $E_{\varepsilon_i}(u_i) \leq E_0$ satisfying the (volume-constrained) Allen–Cahn equation

$$-\varepsilon_i \Delta u_i + \frac{W'(u_i)}{\varepsilon_i} = \lambda_i \quad \text{in } \Omega \cap U \quad (7.3.1)$$

together with the homogeneous Neumann boundary condition

$$\frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega \cap U. \quad (7.3.2)$$

Remark 7.3.1. We can relax the a-priori regularity of u_i to lie only inside certain Sobolev spaces. The interior regularity for weak solutions to (7.3.1) follows from standard elliptic PDE theory [GT01] while the boundary regularity along $\partial\Omega$ for weak solutions to (7.3.1) and (7.3.2) follows from [ADN59].

As in [HT00], after passing to a subsequence, the u_i converge a.e. and in L^1 to some $u \in BV(\Omega \cap U)$ such that $u = \pm 1$ a.e.. Hence, $\{u = 1\}$ and $\{u = -1\}$ are sets of finite perimeter in $U \cap \Omega$ and we write $M := \partial^* \{u = 1\}$ for the reduced boundary of $\{u = 1\}$ in Ω with outward-pointing unit normal ν_M (see [EG15, section 5.7]).

Moreover, we have

$$\|\partial\{u = 1\}\|(\Omega \cap U) = \frac{1}{2} \int_{\Omega \cap U} |Du| \leq \frac{E_0}{h_0}. \quad (7.3.3)$$

For each i , we define a Radon measure μ_i on U by

$$\mu_i := \frac{1}{h_0} \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \right) dv_g \llcorner \Omega \cap U, \quad (7.3.4)$$

and the associated varifold $V_i \in \mathbf{V}_{n-1}(U)$ by

$$V_i(\phi) := \int_{\{|\nabla u_i| \neq 0\}} \phi \left(x, I - \frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \right) d\mu_i \quad (7.3.5)$$

for $\phi \in C_c(G(U))$. Note that as measures on U , both μ_i and V_i are supported in $U \cap \overline{\Omega}$ and $\|V_i\| = \mu_i \llcorner \{|\nabla u_i| \neq 0\}$. Moreover, the uniform energy bound $E_{\varepsilon_i}(u_i) \leq E_0$ implies the uniform bound $\|V_i\|(U) \leq E_0/h_0$ and hence (after passing to a subsequence) the V_i converge as varifolds to some $V \in \mathbf{V}_{n-1}(U)$. The first variation of V_i is given by (see [PT98])

$$\delta V_i(X) = \int_{\{|\nabla u_i| \neq 0\}} \left\langle \nabla X, I - \frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \right\rangle_g d\mu_i \quad (7.3.6)$$

for every $X \in \mathfrak{X}_c(U)$.

7.3.2 Boundary behaviour

As in [KT18], we can also consider the behaviour of $u_i|_{\partial\Omega \cap U}$. First of all, we prove a useful lemma which gives a local bound for the Allen–Cahn energy restricted to the boundary. This can be viewed as a localised version of [KT18, Lemma 4.2].

Lemma 7.3.2. *For any $\tilde{U} \Subset U$, there exists a constant $C_2 > 0$ depending only on n ,*

$k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)$ such that

$$\begin{aligned} \int_{\partial\Omega \cap \tilde{U}} \frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} d\mathcal{H}^{n-1} \leq C_2 \int_{\Omega \cap U} \frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \\ + \Lambda_0 C_0 (C_2 dv_g(\Omega \cap U) + \mathcal{H}^{n-1}(\partial\Omega \cap U)). \end{aligned}$$

Proof. Let $X \in \mathfrak{X}_c(U)$. By the calculations in [KT18, Lemma 4.1], we have

$$\begin{aligned} h_0 \delta V_i(X) &= \int_{\Omega \cap \{|\nabla u_i| \neq 0\}} \nabla X \cdot \left(\frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \right) \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} - \frac{W(u_i)}{\varepsilon_i} \right) \\ &\quad - \int_{\Omega \cap \{|\nabla u_i| = 0\}} (\nabla X \cdot I) \frac{W(u_i)}{\varepsilon_i} + \lambda_i \int_{\Omega} u_i \text{div}_g X \\ &\quad + \int_{\partial\Omega} \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} - \lambda_i u_i \right) (X \cdot \nu) d\mathcal{H}^{n-1}. \end{aligned} \quad (7.3.7)$$

We choose a vector field $X = -\zeta \nabla(\chi_a \circ d)$ where $d(\cdot) := \text{dist}_g(\cdot, \partial\Omega)$, $a = (8\kappa_0)^{-1}$ and $\zeta \in C_c(U)$ is a smooth cutoff function such that $\zeta = 1$ on $\tilde{U}$; $0 \leq \zeta \leq 1$ and $|\nabla \zeta| \leq \frac{2}{\text{dist}_g(\tilde{U}, \partial U)}$. Note that $X = \zeta \nu$ along $\partial\Omega \cap U$ and $\|\nabla X\|_{L^\infty} \leq C(n, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U))$ by (7.2.6). Moreover, by (7.3.6), it is easily seen that

$$|h_0 \delta V_i(X)| \leq \|\nabla X\|_{L^\infty} \int_{\Omega \cap U} \frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i}.$$

By examining each of the terms on the right-hand side of (7.3.7), we then obtain the desired inequality, using the assumptions that $\|u_i\|_{L^\infty} \leq C_0$ and $|\lambda_i| \leq \Lambda_0$. $\square$

By the same arguments as in [HT00] (c.f. [KT18, Theorem 3.1]), using Lemma 7.3.2, after passing to a subsequence, $u_i \lfloor \partial\Omega \cap U$ converges a.e. and in L^1_{loc} with respect to $\mathcal{H}^{n-1}$ to some $\tilde{u} \in BV_{loc}(\partial\Omega \cap U)$ such that $\tilde{u} = \pm 1$ $\mathcal{H}^{n-1}$ -a.e.. Hence, $\{\tilde{u} = 1\}$ and $\{\tilde{u} = -1\}$ are sets of locally finite perimeter in $\partial\Omega \cap U$. However, we note that $\tilde{u}$ may not coincide with the trace of u on $\partial\Omega \cap U$ (see [KT18] and the discussions therein).

7.3.3 Main result

We are now ready to state the main result of this chapter.

Theorem 7.3.3. *Under the assumptions of section 7.3.1, after possibly passing to a subsequence, we have*

- $\lambda_i \rightarrow \lambda$,
- $u_i \rightarrow u$ a.e. and in $L^1(\Omega \cap U)$ with respect to dv_g ,
- $u_i \llcorner \partial\Omega \cap U \rightarrow \tilde{u}$ a.e. and in $L^1_{loc}(\partial\Omega \cap U)$ with respect to $\mathcal{H}^{n-1}$,
- $V_i \rightarrow V$ as varifolds in U ,

where $\lambda \in \mathbb{R}$, $u \in BV(\Omega \cap U)$, $\tilde{u} \in BV_{loc}(\partial\Omega \cap U)$ and $V \in \mathbf{V}_{n-1}(U)$. Moreover, the following statements are true:

(1) (Equipartition of energy) For each $\phi \in C_c(U)$,

$$\frac{1}{2}\|V\|(\phi) = \lim_{i \rightarrow \infty} \frac{1}{h_0} \int_{\bar{\Omega} \cap U} \frac{\varepsilon_i |\nabla u_i|^2}{2} \phi = \lim_{i \rightarrow \infty} \frac{1}{h_0} \int_{\bar{\Omega} \cap U} \frac{W(u_i)}{\varepsilon_i} \phi.$$

(2) (Uniform convergence away from limit interface) $M = \partial^*\{u = 1\} \subset \text{spt } \|V\| \cap \Omega$ and $u_i \rightarrow \pm 1$ locally uniformly on $(\Omega \cap U) \setminus \text{spt } \|V\|$ as $i \rightarrow \infty$. For each $b \in (0, 1)$, the subsets $\{|u_i| \leq 1 - b\}$ locally converge in the Hausdorff sense to $\text{spt } \|V\|$ inside $\Omega \cap U$ as $i \rightarrow \infty$.

(3) (Regularity of limit interface) $V \in \mathbf{IV}_{n-1}(U)$ with density

$$\Theta(x) = \begin{cases} \text{odd} & \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in M, \\ \text{even} & \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in (\text{spt } \|V\| \cap \Omega) \setminus M. \end{cases}$$

In other words, V is integer rectifiable up to the boundary $\partial\Omega$ even when $\|V\|(\partial\Omega) > 0$.

(4) (*Variational property of limit interface*) V has free boundary in $\partial\Omega \cap U$ with

$$H^T(x) = \begin{cases} -\frac{2\lambda}{h_0\Theta(x)}\nu_M(x) & \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in M, \\ 0 & \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \text{spt } \|V\| \setminus M. \end{cases}$$

Moreover, $\text{spt } \sigma_V \subset \partial\Omega \cap U$ and $\eta(x) = \nu(x)$ for σ_V -a.e. x on the generalised boundary of V .

Remark 7.3.4. When $U \subset \Omega$, Theorem 7.3.3 reduces to the interior regularity result in [HT00] (see also [Gua18, Man21] for the adaption to the Riemannian setting).

Remark 7.3.5. One key conclusion of Theorem 7.3.3 is that the limit interface V is integer-rectifiable *up to the boundary* $\partial\Omega$. To the best of the author's knowledge, in the existing literature, boundary behaviour is only partially treated and under additional restrictive assumptions. For example, the first three conclusions in Theorem 7.3.3 were obtained in [Ton03] in the case of convex Euclidean domains. In their recent work [KT18], Kagaya–Tonegawa considered the more general Allen–Cahn functional with boundary contact energy and studied the boundary behaviour of the limit interface *under the assumption* of vanishing discrepancy measure. In this thesis, we prove, in the case of zero boundary contact energy, the vanishing of the discrepancy measure without any convexity assumption on $\partial\Omega$, even when boundary concentration of $\|V\|$ occurs.

Remark 7.3.6. As in [Ton02, Ton05], one can consider variable chemical potentials lying in suitable Sobolev spaces instead of constants on the right hand side of (7.3.1).

Remark 7.3.7. By [DM21, Theorem 1.3], we can conclude that the $(n-2)$ -dimensional part of the generalised measure $\sigma_V \llcorner \{x | \Theta_*^{n-2}(\sigma_V, x) > 0\}$ is $(n-2)$ -rectifiable. It would be interesting to study more refined regularity and singular structure of the full generalised measure σ_V , which is highly non-trivial due to the lack of a suitable monotonicity formula (c.f. [Ton03, section 4]). We will address this in future work.

7.4 Monotonicity formula and consequences

Let $(\widetilde{M}, g)$ and $\Omega, U \subset \widetilde{M}$ be as given in section 7.3.1. Throughout this section, we assume $u \in C^\infty(\overline{\Omega} \cap U)$ is a solution to

$$-\varepsilon \Delta u + \frac{W'(u)}{\varepsilon} = \lambda \quad \text{in } \Omega \cap U, \quad (7.4.1)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \cap U, \quad (7.4.2)$$

where $\varepsilon \in (0, 1)$, $|\lambda| \leq \Lambda_0$, $\|u\|_{L^\infty} \leq C_0$ and $E_\varepsilon(u) \leq E_0$. We define the energy density $e_\varepsilon(u)$ and discrepancy function $\xi_\varepsilon(u)$ respectively by

$$e_\varepsilon(u) := \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \quad (7.4.3)$$

and

$$\xi_\varepsilon(u) := \frac{\varepsilon |\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}. \quad (7.4.4)$$

Our first goal is to prove an almost-monotonicity inequality (c.f. [Gua18, Appendix B]) of the form:

$$\frac{d}{d\rho} \left(e^{c\rho} \rho^{-(n-1)} \int_{B_\rho(x) \cap \Omega} e_\varepsilon(u) \right) \geq e^{c\rho} \rho^{-(n-1)} \int_{B_\rho(x) \cap \Omega} -(\xi_\varepsilon(u))^+$$

for some $c > 0$ and all sufficiently small ρ . Next, we establish up-to-the-boundary estimates, in both pointwise and integral forms, for u and $\xi_\varepsilon(u)$. Together with the almost-monotonicity inequality, these estimates allow us to control the energy density ratio

$$I_{\varepsilon, u}(r, x) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r(x) \cap \Omega} e_\varepsilon(u) \quad (7.4.5)$$

which will be instrumental in deriving the regularity of the limit interface in later sections.

7.4.1 Almost monotonicity formula

Recall that the monotonicity formula for minimal surfaces is obtained by applying the first variation formula for the area functional to (suitably truncated) radial test vector fields. In the Allen–Cahn setting, the almost monotonicity formula (which can be viewed as an ε -version of the monotonicity formula for minimal surfaces) can be obtained by similar arguments. The following *Pohozaev identity* plays the role of the first variation formula in this setting.

Lemma 7.4.1 (Pohozaev identity). *Let $u \in C^2(\overline{\Omega} \cap U)$ be a solution to the semilinear equation $\Delta u = F(u)$ in $\Omega \cap U$ where $F = G'$ for some smooth function $G : \mathbb{R} \rightarrow \mathbb{R}$. For any $X \in \mathfrak{X}_c(U)$, we have*

$$\begin{aligned} & \int_{\Omega \cap U} \left[\left(\frac{|\nabla u|^2}{2} + G(u) \right) \operatorname{div}_g X - \nabla X(\nabla u, \nabla u) \right] dv_g \\ &= \int_{\partial\Omega \cap U} \left[\left(\frac{|\nabla u|^2}{2} + G(u) \right) (X \cdot \nu) - (\nabla u \cdot X) \frac{\partial u}{\partial \nu} \right] d\mathcal{H}^{n-1}. \end{aligned} \quad (7.4.6)$$

Proof. Multiply both sides of the equation $\Delta u = F(u)$ by $X \cdot \nabla u$ and then integrate by parts twice (see [Xio18, Lemma 9] for a similar calculation). $\square$

For the Allen–Cahn equation (7.4.1), we choose $G(t) = \varepsilon^{-2} \widetilde{W}(t)$ where

$$\widetilde{W}(t) := W(t) - \varepsilon \lambda t + \varepsilon \Lambda_0 C_0.$$

Note that the role of the constant term $\varepsilon \Lambda_0 C_0$ is simply to ensure that $\widetilde{W}(u) \geq 0$ under our assumptions. For this choice of G and using the Neumann boundary condition (7.4.2), the Pohozaev identity (7.4.6) reads

$$\begin{aligned} & \int_{\Omega \cap U} \left[\left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) \operatorname{div}_g X - \varepsilon \nabla X(\nabla u, \nabla u) \right] dv_g \\ &= \int_{\partial\Omega \cap U} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{\widetilde{W}(u)}{\varepsilon} \right) (X \cdot \nu) d\mathcal{H}^{n-1}. \end{aligned} \quad (7.4.7)$$

We define $\tilde{e}_\varepsilon(u)$, $\tilde{\xi}_\varepsilon(u)$ and $\tilde{I}_{\varepsilon,u}(r, x)$ as in (7.4.3), (7.4.4) and (7.4.5) with W replaced by $\tilde{W}$. We will first establish the almost monotonicity formula for $\tilde{I}_{\varepsilon,u}$ from which the corresponding almost monotonicity formula for $I_{\varepsilon,u}$ follows.

For the purpose of making the test vector fields compactly supported, we let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be any smooth decreasing cutoff function such that $\varphi(t) = 1$ for $t \leq \frac{1}{2}$ and $\varphi(t) = 0$ for $t \geq 1$. For any $x \in \tilde{M}$ and $0 < \rho < \text{inj}_g(x)$, the function $\varphi\left(\frac{\text{dist}_g(x, \cdot)}{\rho}\right)$ is a smooth function which is compactly supported inside $B_\rho(x)$.

From now on, we fix $x \in U \cap \bar{\Omega}$ and denote $r(\cdot) := \text{dist}_g(x, \cdot)$. For any $0 < \rho < \min\{\text{dist}_g(x, \partial U), i_0\}$, we consider the test vector field $X \in \mathfrak{X}_c(U)$ defined by

$$X := \varphi\left(\frac{r}{\rho}\right) r \nabla r$$

which is a truncated radial vector field centred at x supported inside $B_\rho(x)$. When $\rho < \frac{1}{\sqrt{k_0}}$, a direct computation together with (7.2.1) and (7.2.2) gives

$$\left| \text{div}_g X(x) - \left(n\varphi + \frac{r}{\rho} \varphi' \right) \right| \leq (n-1) \sqrt{k_0} r \varphi$$

and

$$\left| \nabla X(\nabla u, \nabla u) - \left(|\nabla u|^2 \varphi + \frac{r}{\rho} (\nabla u \cdot \nabla r)^2 \varphi' \right) \right| \leq \sqrt{k_0} r |\nabla u|^2 \varphi$$

where φ and φ' are both evaluated at $\frac{r}{\rho}$. Observing the relation

$$\rho \frac{d}{d\rho} \left[\varphi\left(\frac{r}{\rho}\right) \right] = -\frac{r}{\rho} \varphi'\left(\frac{r}{\rho}\right).$$

and plugging the above estimates into the left hand side of the Pohozaev identity

(7.4.7), we obtain the inequality

$$\begin{aligned} & \frac{d}{d\rho} \left[\rho^{-(n-1)} \int_{\Omega \cap U} \tilde{e}_\varepsilon(u) \varphi \right] - \rho^{-(n-1)} \frac{d}{d\rho} \left[\int_{\Omega \cap U} \varepsilon (\nabla u \cdot \nabla r)^2 \varphi \right] + \rho^{-n} \int_{\Omega \cap U} \tilde{\xi}_\varepsilon(u) \varphi \\ & \geq - (n+1) \sqrt{k_0} \left[\rho^{-(n-1)} \int_{\Omega \cap U} \tilde{e}_\varepsilon(u) \varphi \right] - \rho^{-(n-1)} \int_{\partial \Omega \cap U} \tilde{e}_\varepsilon(u) |\nabla r \cdot \nu| \varphi \, d\mathcal{H}^{n-1}. \end{aligned} \quad (7.4.8)$$

Note that we have used the fact that $r \leq \rho$ in the support of φ and $\varepsilon |\nabla u|^2 \leq 2\tilde{e}_\varepsilon(u)$.

We now consider two separate cases:

- Case 1: $x \in \Omega \cap U$ and $B_\rho(x) \Subset \Omega \cap U$;
- Case 2: $x \in \partial \Omega \cap U$ $B_\rho(x) \Subset U$.

In the first case, since there is no boundary term because $B_\rho(x) \cap \partial \Omega = \emptyset$, we obtain the following interior almost-monotonicity inequality (c.f. [Gua18, Appendix B]), which reduces to [HT00, Lemma 3.1] when $k_0 = 0$.

Lemma 7.4.2 (Interior almost-monotonicity inequality). *For any $x \in \Omega \cap U$ and $0 < \rho < \min\{i_0, \frac{1}{\sqrt{k_0}}\}$ such that $B_\rho(x) \Subset \Omega \cap U$, we have*

$$\frac{d}{d\rho} \left[e^{(n+1)\sqrt{k_0}\rho} \rho^{-(n-1)} \int_{B_\rho(x)} \tilde{e}_\varepsilon(u) \right] \geq e^{(n+1)\sqrt{k_0}\rho} \rho^{-n} \int_{B_\rho(x)} -\tilde{\xi}_\varepsilon(u). \quad (7.4.9)$$

Proof. It follows directly from (7.4.8) by dropping the non-negative term and the boundary term, after letting $\varphi \rightarrow \mathbf{1}_{(-\infty, 1)}$. $\square$

For the second case, we need to handle the extra boundary term in (7.4.8). The following lemma allows us to rewrite the boundary term in terms of an interior integral (and its derivative).

Lemma 7.4.3. *For any $x \in \partial \Omega \cap U$ and $0 < \rho < \min\{i_0, \frac{1}{2\kappa_0}\}$ such that $B_\rho(x) \Subset U$, we have*

$$\int_{\partial \Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) \, d\mathcal{H}^{n-1} \leq (n+1)C_2 \int_{\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) + 6 \frac{d}{d\rho} \left[\int_{\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) \right].$$

Proof. Let φ be the radial cutoff function centred at x as defined before. For any $0 < \rho < \min\{\text{dist}_g(x, \partial U), \frac{1}{2\kappa_0}\}$, we can consider the test vector field $X \in \mathfrak{X}_c(U)$ defined by

$$X = -\varphi \left(\frac{r}{\rho} \right) \nabla d$$

where d is the signed distance function from $\partial\Omega$ as defined in section 7.2.2. Note that X is smooth and compactly supported in $B_\rho(x)$ with $X \cdot \nu = \varphi$ along $\partial\Omega$. By (7.2.6), we have the following estimates:

$$\text{div}_g X \leq (n-1)C_1\varphi - \frac{1}{\rho}\varphi'$$

and

$$-\varepsilon \nabla X(\nabla u, \nabla u) \leq \left(C_1\varphi - \frac{1}{\rho}\varphi' \right) \varepsilon |\nabla u|^2.$$

By using these bounds in the Pohozaev identity (7.4.7), and noting that $r \geq \frac{\rho}{2}$ in the support of φ' and $\varepsilon |\nabla u|^2 \leq 2\tilde{e}_\varepsilon(u)$, we obtain the desired inequality after letting $\varphi \rightarrow \mathbf{1}_{(-\infty, 1)}$. $\square$

We are now ready to establish the almost-monotonicity inequality centred at a boundary point $x \in \partial\Omega$.

Lemma 7.4.4 (Boundary almost-monotonicity inequality). *There exists a constant*

$$c_1 = c_1(n, k_0, \kappa_0) \geq 0$$

so that the following holds: for any $x \in \partial\Omega \cap U$ and $0 < \rho < \frac{1}{2} \min\{i_0, \frac{1}{\kappa_0}\}$ such that $B_\rho(x) \Subset U$, we have

$$\frac{d}{d\rho} \left[e^{c_1\rho} \rho^{-(n-1)} \int_{\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) \right] \geq \frac{e^{c_1\rho}}{1 + 3\kappa_0\rho} \rho^{-n} \int_{\Omega \cap B_\rho(x)} -\tilde{\xi}_\varepsilon(u). \quad (7.4.10)$$

Proof. First of all, note that by (7.2.7), we have the following estimate on the bound-

any term of (7.4.8) for any $\rho < \frac{1}{2}i_0$,

$$\rho^{-(n-1)} \int_{\partial\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) |\nabla r \cdot \nu| d\mathcal{H}^{n-1} \leq \frac{\kappa_0}{2} \rho^{-(n-2)} \int_{\partial\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) d\mathcal{H}^{n-1}.$$

On the other hand, Lemma 7.4.3 implies that when $\rho < \min\{i_0, \frac{1}{2\kappa_0}\}$, we have the inequality

$$\begin{aligned} 6\rho \frac{d}{d\rho} \left[\rho^{-(n-1)} \int_{\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) \right] &\geq \rho^{-(n-2)} \int_{\partial\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u) d\mathcal{H}^{n-1} \\ &\quad - [(n+1)C_1\rho + 6(n-1)]\rho^{-(n-1)} \int_{\Omega \cap B_\rho(x)} \tilde{e}_\varepsilon(u). \end{aligned}$$

Putting it back to (7.4.8), using $\rho\kappa_0 < \frac{1}{2}$, after letting $\varphi \rightarrow \mathbf{1}_{(-\infty, 1)}$ and rearranging terms, we get the desired inequality with $c_1 = \frac{n+1}{4}C_1 + 3(n-1)\kappa_0$. $\square$

Remark 7.4.5. As noted in section 7.1, it would be possible to mimic the approach of [GJ86] and [Ton03], by reflecting across the boundary $\partial\Omega$, and prove an almost-monotonicity inequality for the sum of the energy density ratio on a ball centred in $\bar{\Omega}$ and its reflection. We chose, for the sake of simplicity, to avoid reflection across the boundary entirely, and consider balls that are either disjoint from $\partial\Omega$, or centred at points lying on $\partial\Omega$, as in [GLZ20].

7.4.2 Auxiliary estimates

We first derive some a-priori estimates for solutions to (7.4.1) and (7.4.2). These estimates hold locally uniformly in space *up to the boundary* $\partial\Omega$ and uniformly in ε . We fix throughout this subsection an open subset $\tilde{U} \Subset U$.

We begin with the C^0 -estimate. Our proof uses a maximum principle argument similar to that in [HT00, Proposition 3.2], except that we have to include an extra barrier term in order to prevent the extrema from occurring along the boundary.

Lemma 7.4.6 (Uniform C^0 estimate). *There exist constants*

$$\varepsilon_2 = \varepsilon_2(\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_2 = c_2(\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) \geq 1$$

such that for all $0 < \varepsilon < \varepsilon_2$, we have

$$\|u\|_{C^0(\bar{\Omega} \cap \tilde{U})} \leq 1 + c_2 \varepsilon < 2$$

Proof. Let $x \in \bar{\Omega} \cap \tilde{U}$ and $\rho < \frac{1}{6} \min\{i_0, \frac{1}{\sqrt{k_0}}, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial U)\}$, and $r(\cdot) := \text{dist}_g(x, \cdot)$.

Inside the ball $B_{3\rho}(x) \Subset U$, we fix a smooth radial function $\zeta = \zeta(r)$ such that

- $\zeta = 1 + \frac{c_2 \varepsilon}{2}$ inside $B_\rho(x)$;
- $\zeta = 1 + C_0$ on the annulus $B_{3\rho}(x) \setminus B_{2\rho}(x)$; and
- inside the ball $B_{3\rho}(x)$, we have by (7.2.2)

$$1 + \frac{c_2 \varepsilon}{2} \leq \zeta \leq 1 + C_0, \quad |\nabla \zeta| \leq 2C_0 \rho^{-1}, \quad |\Delta \zeta| \leq C(n)C_0 \rho^{-2}$$

where c_2 will be chosen later and then ε is sufficiently small so that $c_2 \varepsilon < \min\{1, C_0\}$.

We first claim that $u \leq 1 + c_2 \varepsilon$ on $B_\rho(x) \cap \Omega$ for all sufficiently small ρ by a contradiction argument. Suppose $\sup_{B_\rho(x) \cap \Omega} u > 1 + c_2 \varepsilon$. We divide into two cases as before:

- Case 1: $x \in \Omega \cap \tilde{U}$ and $B_{3\rho}(x) \Subset \Omega \cap U$;
- Case 2: $x \in \partial\Omega \cap \tilde{U}$ and $B_{3\rho}(x) \Subset U$.

For the first case, we consider as in [HT00, Proposition 3.2] the auxiliary function $f := u - \zeta$ which is smooth inside $B_{3\rho}(x)$. One can verify

- $f \leq -1$ along the boundary $\partial B_{3\rho}(x)$, and
- $\sup_{B_{3\rho}(x)} f > \frac{c_2\varepsilon}{2} > 0$.

Therefore, $f(x_0) = \sup_{B_{3\rho}(x)} f > \frac{c_2\varepsilon}{2}$ at an interior maxima $x_0 \in B_{3\rho}(x)$. By the maximum principle, at x_0 we have

$$\begin{aligned}
0 &\geq \varepsilon \Delta f = \varepsilon^{-1} W'(u) - \lambda - \varepsilon \Delta \zeta \\
&= \varepsilon^{-1} W'(tu + (1-t)\zeta)|_{t=0}^{t=1} + \varepsilon^{-1} W'(\zeta) - \lambda - \varepsilon \Delta \zeta \\
&\geq \varepsilon^{-1} f \int_0^1 W''(tu + (1-t)\zeta) dt - \lambda - \varepsilon \Delta \zeta \\
&\geq \frac{c_2\kappa}{2} - \lambda - \varepsilon \Delta \zeta,
\end{aligned}$$

where we used that $u > \zeta \geq 1$ at x_0 and the properties of W in section 7.2.5. If $c_2 \geq 1$ is sufficiently large, depending only on $\Lambda_0, C_0, W, n, i_0, k_0, \text{dist}_g(\tilde{U}, \partial U)$ and $\text{dist}_g(x, \partial\Omega)$, we arrive at a contradiction.

For the second case, we consider another auxiliary function

$$f := u - \zeta + \frac{4C_0}{\rho} \chi_a \circ d$$

where $d(\cdot) = \text{dist}_g(\cdot, \partial\Omega)$ and χ is the cutoff function defined in section 7.2.6 with $a = \frac{\varepsilon\rho}{16C_0}$. One can again verify

- $f \leq -1 + \frac{\varepsilon}{2} < -\frac{1}{2}$ along the boundary $\partial B_{3\rho}(x) \cap \Omega$;
- $\sup_{B_{3\rho}(x) \cap \Omega} f > \frac{c_2\varepsilon}{2} > 0$; and
- $\frac{\partial f}{\partial \nu} \leq -2C_0\rho^{-1} < 0$ along $B_{3\rho}(x) \cap \partial\Omega$.

Hence, $\sup_{B_{3\rho}(x) \cap \Omega} f$ is again achieved at an interior maxima $x_0 \in B_{3\rho}(x) \cap \Omega$. After applying the maximum principle at x_0 , a similar calculation as before (note that we

still have $u \geq 1$ and $\zeta > 1$ by our choice of a) gives

$$\frac{c_2 \kappa}{2} \leq \lambda + \varepsilon \Delta \zeta + \frac{1}{2} \max_{t \in [1, 1+C_0]} W''(t) - \frac{4\varepsilon C_0}{\rho} \Delta(\chi_a \circ d).$$

By the definition of χ_a in section 7.2.6 and (7.2.6), we will again obtain a contradiction for c_2 sufficiently large, depending only on $\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0$ and $\text{dist}_g(\tilde{U}, \partial U)$. This proves the desired upper bound for u . The lower bound can be obtained similarly by considering the auxiliary function $f = u + \zeta$ for Case 1 and $f = u + \zeta - \frac{4C_0}{\rho} \chi_a \circ d$ for Case 2. $\square$

Using the C^0 -estimate in Lemma 7.4.6, one obtains boundary gradient estimates by standard PDE theory of semi-linear elliptic equations.

Lemma 7.4.7 (Uniform gradient estimates). *There exists a constant*

$$c_3 = c_3(\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) \geq 1$$

such that for all $0 < \varepsilon < \varepsilon_2$, we have the gradient estimate

$$\|\varepsilon \nabla u\|_{C^0(\bar{\Omega} \cap \tilde{U})} \leq c_3. \quad (7.4.11)$$

Proof. The interior gradient estimate can be found e.g. in [HL11, Proposition 2.19]. The boundary gradient estimates along $\partial\Omega$ follows e.g. from [Xu17, Theorem 1.1] (c.f. [XZ17] for the extension to the Riemannian setting). $\square$

We can actually obtain better estimates away from the transition region $\{|u| \leq \alpha\}$ which imply that $\frac{W(u)}{\varepsilon}$ is of order $\mathcal{O}(\varepsilon^{2\eta-1})$ for any $\eta \in (0, 1)$.

Lemma 7.4.8 (Refined uniform C^0 estimates away from the transition region). *For*

any $\eta \in (0, 1)$, there exist constants

$$\varepsilon_4 = \varepsilon_4(\eta, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_4 = c_4(\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) \geq 1$$

such that for all $0 < \varepsilon < \varepsilon_4$, $x \in \bar{\Omega} \cap \tilde{U}$ and $\varepsilon^{1-\eta} \leq r \leq \frac{1}{6} \min\{i_0, \frac{1}{\sqrt{k_0}}, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial U)\}$, we have

(i) if $u \geq \alpha$ on $B_{3r}(x) \cap \Omega$, then $1 - c_4\varepsilon^\eta \leq u \leq 1 + c_4\varepsilon$ on $B_r(x) \cap \Omega$;

(ii) if $u \leq -\alpha$ on $B_{3r}(x) \cap \Omega$, then $-1 - c_4\varepsilon \leq u \leq -1 + c_4\varepsilon^\eta$ on $B_r(x) \cap \Omega$.

Proof. We only prove (i) as the arguments are the same for (ii). Let $x \in \bar{\Omega} \cap \tilde{U}$ and $r < \frac{1}{6} \min\{i_0, \frac{1}{\sqrt{k_0}}, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial U)\}$. Suppose $u \geq \alpha$ on $B_{3r}(x) \cap \Omega$. By Lemma 7.4.6, we have $u \leq 1 + c_2\varepsilon$ provided that $\varepsilon < \varepsilon_2$. It remains to prove that $u \geq 1 - c_4\varepsilon^\eta$ on $B_r(x) \cap \Omega$ for some suitable choice of c_4 , an appropriate range of r and sufficiently small ε .

We will argue by contradiction as in the proof of Lemma 7.4.6. Suppose, on the contrary, that $\alpha \leq \inf_{B_r(x) \cap \Omega} u < 1 - c_4\varepsilon^\eta$. We fix a smooth radial function ζ such that

- $\zeta = 1 - \frac{c_4\kappa\varepsilon^\eta}{4C}$ inside $B_r(x)$;
- $\zeta = \alpha + \frac{c_4\kappa\varepsilon^\eta}{4C}$ on the annulus $B_{3r}(x) \setminus B_{2r}(x)$; and
- inside the ball $B_{3r}(x)$, we have by (7.2.2)

$$\alpha + \frac{c_4\kappa\varepsilon^\eta}{4C} \leq \zeta \leq 1 - \frac{c_4\kappa\varepsilon^\eta}{4C}, \quad |\nabla\zeta| \leq 2r^{-1}, \quad |\Delta\zeta| \leq C(n)r^{-2}$$

where $C = C(W) := \max\{1, \max_{s \in [\alpha, 2]} |W''(s)|\}$, c_4 will be chosen later and ε is sufficiently small that $c_4\varepsilon^\eta < \min\{1, \frac{2C(1-\alpha)}{\kappa}\}$.

We now consider the auxiliary function

$$f := u - \zeta + \frac{4}{r}\chi_a \circ d$$

where $d(\cdot) = \text{dist}_g(\cdot, \partial\Omega)$ and χ is the cutoff function defined in section 7.2.6 with $a = \frac{c_4\kappa r\varepsilon^\eta}{16C}$. One can verify (note that $C \geq \kappa$)

- $f \geq -\frac{3}{4}\frac{c_4\kappa\varepsilon^\eta}{C} \geq -\frac{3}{4}c_4\varepsilon^\eta$ along the boundary $\partial B_{3r}(x) \cap \Omega$;
- $\inf_{B_{3r}(x) \cap \Omega} f < -\frac{3}{4}c_4\varepsilon^\eta$; and
- $\frac{\partial f}{\partial \nu} \geq \frac{2}{r} > 0$ along $B_{3r}(x) \cap \partial\Omega$, if non-empty.

Hence, no matter whether $x \in \Omega$ or $x \in \partial\Omega$, $\inf_{B_{3r}(x) \cap \Omega} f$ is achieved at an interior minimum $x_0 \in B_{3r}(x) \cap \Omega$. After applying the maximum principle at x_0 , by our choice of a and a similar calculation as before (note that we still have $\alpha \leq u(x_0) < 2$ and $\alpha \leq \zeta(x_0) \leq 1$) gives

$$\frac{3c_4\kappa}{4}\varepsilon^{-1+\eta} \leq -\lambda - \varepsilon\Delta\zeta - \frac{4\varepsilon}{r}\Delta(\chi_a \circ d).$$

By the definition of χ_a in section 7.2.6 and (7.2.6), this implies

$$\frac{3\kappa}{4}c_4 \leq \Lambda_0\varepsilon^{1-\eta} + C(n)\varepsilon^{2-\eta}r^{-2} + 4(n-1)C_2\varepsilon^{2-\eta}r^{-1} + \frac{32C}{c_4\kappa}\varepsilon^{2-2\eta}r^{-2}.$$

For $r \geq \varepsilon^{1-\eta}$, this yields a contradiction when c_4 is sufficiently large, depending only on $\Lambda_0, W, n, i_0, k_0, \kappa_0$ and $\text{dist}_g(\tilde{U}, \partial U)$. $\square$

Remark 7.4.9. In case $B_{3r}(x) \Subset \Omega$, we do not need to include the term involving χ_a in the auxiliary function f and we have the stronger conclusion that $u \geq 1 - c_4\varepsilon$ (i.e. with $\eta = 1$), c.f. [HT00, Proposition 4.1]. With a different choice of auxiliary function ζ , we can also establish the same estimates with $\eta = 1$. However, unlike in

[HT00] (see the remark after Proposition 3.2), apparently one cannot obtain better estimates even when $\lambda = 0$ by iterating the arguments above.

7.4.3 Estimates for the discrepancy function

Note that in order for the almost-monotonicity formula in Lemma 7.4.2 and 7.4.4 to be effective, we need some control on the right hand side of the inequalities involving the discrepancy function ξ_ε defined in (7.4.4).

In this subsection, we obtain a crude pointwise upper bound for the discrepancy function. We want to point out that our bound is weaker than the interior bound obtained in [HT00, Proposition 3.3] but holds up to the boundary $\partial\Omega$. Note that we have chosen to work at the original scale, rather than rescaling by ε as in [HT00]. This does not affect our subsequent analysis, but certain expressions will change by appropriate powers of ε .

Proposition 7.4.10 (Pointwise upper bound on discrepancy function). *There exists a constant*

$$\varepsilon_5 = \varepsilon_5(\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

such that for all $0 < \varepsilon < \varepsilon_5$, we have

$$\sup_{\bar{\Omega} \cap \tilde{U}} \xi_\varepsilon(u) \leq \varepsilon^{-4/5}. \quad (7.4.12)$$

Proof. We consider as in [HT00] the following auxiliary function

$$\xi_G := \frac{\varepsilon}{2} |\nabla u|^2 - \varepsilon^{-1} W(u) - \varepsilon^{-1} G(u)$$

where $G \in C^\infty(\mathbb{R})$ is to be chosen later. By the Bochner formula (7.2.3) and a similar computation as in [HT00, Lemma 3.5], we have the following differential inequality

at points where $|\nabla u| > 0$:

$$\begin{aligned} & \varepsilon^2 \Delta \xi_G - \frac{2(W' + G') \nabla u}{|\nabla u|^2} \cdot \nabla \xi_G + 2(G'' + (n-1)k_0 \varepsilon^2) \xi_G \\ & \geq \varepsilon^{-1} (G')^2 + \varepsilon^{-1} G' W' - 2\varepsilon^{-1} (G'' + (n-1)k_0 \varepsilon^2) (W + G) + \lambda (W' + G') \end{aligned} \quad (7.4.13)$$

where the functions G and W and their derivatives are evaluated at u .

Let $\delta \in (0, 1)$ be fixed later and define (recall Lemma 7.4.6)

$$G = G_\delta(r) := \delta \left(1 + \int_{-1-c_2\varepsilon_2}^r \exp \left(- \int_{-1-c_2\varepsilon_2}^t \frac{|W'(s)| + \delta}{2(W(s) + \delta)} ds \right) dt \right).$$

Noting that $\frac{W}{W'}$ is a smooth function which vanishes at ± 1 , one easily verifies that the following holds on $r \in [-1 - c_2\varepsilon_2, 1 + c_2\varepsilon_2]$,

- $\delta \leq G \leq 5\delta$,
- $C^{-1}\delta^2 < G' \leq \delta$,
- $0 < -G'' = G' \frac{|W'(r)| + \delta}{2(W(r) + \delta)} \leq C$

for some constant $C = C(W) > 1$. By Lemma 7.4.6, we have (the functions G and W and their derivatives are evaluated at u)

$$G'W' - 2G''(W + G) = G' \left(W' + \frac{|W'| + \delta}{W + \delta} (W + G) \right) \geq \delta G' \quad (7.4.14)$$

provided that $\varepsilon < \varepsilon_2$.

Let $x \in \overline{\Omega} \cap \tilde{U}$ and $R = \frac{1}{6} \min\{i_0, \frac{1}{\sqrt{k_0}}, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial U), 1\}$. We fix a smooth radial cutoff function ζ centred at x such that

- $\zeta = 1$ on $B_{R/8}(x)$
- $\zeta = 0$ outside $B_{R/4}(x)$,
- $0 \leq \zeta \leq 1$, $|\nabla \zeta| \leq 8R^{-1}\zeta^{1/2}$, $|\Delta \zeta| \leq C(n)R^{-2}$

where $C(n) > 0$ is a constant depending only on n . Let $d(\cdot) := \text{dist}_g(\cdot, \partial\Omega)$ and χ be the cutoff function defined in section 7.2.6 with $a = (8\hat{C})^{-1}\varepsilon^{1/5}R$ where $\hat{C} > 0$ is a constant to be fixed later. Define a new auxiliary function

$$f := \zeta\xi_G + \varepsilon^{-1}R^{-1}\hat{C}\chi_a \circ d.$$

We claim that for sufficiently small ε , we have

$$\sup_{\Omega} f \leq \frac{1}{2}\varepsilon^{-4/5}. \quad (7.4.15)$$

If the supremum is non-positive, then we must have $\xi_G \leq 0$ and by our choice of G that

$$\xi_{\varepsilon} = \xi_G + \varepsilon^{-1}G(u) \leq 5\varepsilon^{-1}\delta$$

and thus $\xi_{\varepsilon}(x) \leq \varepsilon^{-4/5}$ provided that $\delta \leq \frac{1}{5}\varepsilon^{1/5}$. So, without loss of generality, we assume that $\sup_{\Omega} f > 0$ and is achieved at some point $x_0 \in B_{R/4}(x) \cap \overline{\Omega}$ (note that $f \leq \frac{1}{4}\varepsilon^{-4/5}$ outside $B_{R/4}(x)$ by our choice of a). First, we claim that $x_0 \notin \partial\Omega$. By the Neumann boundary condition (7.4.2) and [Ton03, Lemma 3.1], along $\partial\Omega$ we have

$$\frac{\partial f}{\partial \nu} = \frac{\partial \zeta}{\partial \nu}\xi_G - \varepsilon\zeta B(\nabla u, \nabla u) - \varepsilon^{-1}R^{-1}\hat{C}$$

where B is the second fundamental form of $\partial\Omega$ with respect to ν (i.e. $B(\tau, \tau) := \langle \nabla_{\tau}\nu, \tau \rangle$). Since κ_0 bounds the maximum absolute value of the principal curvatures of $\partial\Omega$, by Lemma 7.4.7 and our choice of ζ , we have $\frac{\partial f}{\partial \nu} < 0$ along $\partial\Omega$ by taking $\hat{C} := 10c_3^2$. Hence, we can assume without loss of generality that $x_0 \notin \partial\Omega$. Suppose (7.4.15) does not hold, then at an interior maximum x_0 we have by our choice of a that

$$\frac{1}{2}\varepsilon\zeta|\nabla u|^2 \geq \zeta\xi_G > \frac{1}{2}\varepsilon^{-4/5} - 2a\hat{C}\varepsilon^{-1}R^{-1} = \frac{1}{4}\varepsilon^{-4/5}.$$

From this and (7.4.11), we have at x_0 ,

$$\xi_G \leq \frac{c_3^2}{2} \varepsilon^{-1} \quad \text{and} \quad \zeta^{-1} \leq 2c_3^2 \varepsilon^{-1/5}. \quad (7.4.16)$$

Since x_0 is an interior maximum point of f , we have $\nabla f = 0$ and $\Delta f \leq 0$ at x_0 . The first order condition gives $\zeta \nabla \xi_G = -\xi_G \nabla \zeta - \varepsilon^{-1} R^{-1} \hat{C} \chi'_a \nabla d$, which, together with (7.4.16), implies

$$\zeta^{1/2} |\nabla \xi_G| \leq 4c_3^2 \varepsilon^{-1} R^{-1} + \sqrt{2} \hat{C} c_3 \varepsilon^{-11/10} R^{-1} \leq C(c_3) \varepsilon^{-11/10} R^{-1}.$$

On the other hand, the second order condition at x_0 says

$$\zeta \Delta \xi_G \leq -\xi_G \Delta \zeta - 2 \nabla \zeta \cdot \nabla \xi_G - \varepsilon^{-1} R^{-1} \hat{C} \chi'_a \Delta d - \varepsilon^{-1} R^{-1} \hat{C} \chi''_a |\nabla d|^2.$$

Using the first order condition, our choice of ζ , together with (7.2.6) and (7.4.16), we have

$$\begin{aligned} \Delta \xi_G &\leq \left(-\xi_G \zeta^{-1} \Delta \zeta + 2 \xi_G \zeta^{-2} |\nabla \zeta|^2 \right) + 2 \zeta^{-2} \varepsilon^{-1} R^{-1} \hat{C} \chi'_a (\nabla d \cdot \nabla \zeta) \\ &\quad - \varepsilon^{-1} R^{-1} \hat{C} \zeta^{-1} \chi'_a \Delta d - \varepsilon^{-1} R^{-1} \hat{C} \zeta^{-1} \chi''_a |\nabla d|^2 \\ &\leq C(n, c_3) \varepsilon^{-6/5} R^{-2} + C(c_3) \varepsilon^{-13/10} R^{-2} \\ &\quad + C(C_1, c_3, n) \varepsilon^{-6/5} R^{-1} + C(c_3) \varepsilon^{-7/5} R^{-2} \\ &\leq C(C_1, c_3, n) \varepsilon^{-7/5} R^{-2}. \end{aligned}$$

Combining the estimates above, for sufficiently small ε depending only on C_1 , n , $\hat{C}$ and c_3 , we have at x_0

$$(\text{LHS of (7.4.13)}) \leq C(C_1, c_3, n) \varepsilon^{3/5} R^{-2} + C(W, c_3) \varepsilon^{-1/5} R^{-1} + C(n, k_0, c_3) \varepsilon$$

On the other hand, by our choice of G , together with (7.4.14)

$$\begin{aligned}
(\text{RHS}) \text{ of (7.4.13)} &\geq \varepsilon^{-1}(G')^2 + \varepsilon^{-1}\delta G' - C(n, k_0, W)\varepsilon - C(W)\Lambda_0 \\
&\geq \varepsilon^{-1}(C^{-2}\delta^4 + C^{-1}\delta^3) - C(n, k_0, W)\varepsilon - C(W)\Lambda_0 \\
&\geq 2C^{-2}\varepsilon^{-1}\delta^4 - C(n, k_0, W)\varepsilon - C(W)\Lambda_0.
\end{aligned}$$

Combining these we arrive at the inequality

$$\delta^4 \leq C''\varepsilon^{4/5}$$

for some constant $C'' > 0$ depending only on $\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0$ and $\text{dist}_g(\tilde{U}, \partial U)$. Therefore, we arrive at a contradiction by taking $\delta = (2C'')^{1/4}\varepsilon^{1/5} < 1$ for sufficiently small ε , and (7.4.15) holds. Proposition 7.4.10 then follows immediately. $\square$

Remark 7.4.11. With a different choice of the auxiliary function G , resembling the one in [Kag19], it is possible to obtain a better estimate of order $\varepsilon^{-3/5}$ in Proposition 7.4.10.

7.4.4 Consequences of the almost monotonicity formula

We now combine the results obtained in the previous subsections to derive some consequences of the almost monotonicity formula (Lemma 7.4.2 and 7.4.4). We first establish a uniform lower bound of the energy density ratio (7.4.5) on balls of radius $\varepsilon \leq r \leq \mathcal{O}(\varepsilon^{4/5})$ centred at a boundary point in the transition region $\{|u| \leq 1 - b\}$.

Lemma 7.4.12 (Energy density ratio lower bound I). *For any $b \in (0, 1 - \alpha)$, there exist constants*

$$\varepsilon_6 = \varepsilon_6(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_6 = c_6(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0$$

such that for all $0 < \varepsilon < \varepsilon_4$, $x \in \partial\Omega \cap \tilde{U}$ with $|u(x)| \leq 1 - b$, and $\varepsilon \leq r \leq c_6 \varepsilon^{4/5}$, we have

$$I_{\varepsilon,u}(r, x) \geq c_6.$$

In fact, the same holds for $x \in \Omega \cap \tilde{U}$ provided furthermore that $r < \text{dist}_g(x, \partial\Omega)$.

Proof. We first assume $x \in \partial\Omega$. Since $|u(x)| \leq 1 - b$, we have $|u| \leq 1 - \frac{b}{2}$ on $B_{\frac{b\varepsilon}{2c_3}}(x) \cap \Omega$ by (7.4.11) provided that $\varepsilon < \varepsilon_2$. As a result, by (7.2.8) and (7.4.5) we have for sufficiently small ε that

$$\tilde{I}_{\varepsilon,u}(\varepsilon, x) \geq I_{\varepsilon,u}(\varepsilon, x) \geq \frac{1}{\omega_{n-1}\varepsilon^{n-1}} \int_{B_{\frac{b\varepsilon}{2c_3}}(p) \cap \Omega} \frac{W(u)}{\varepsilon} \geq 4c'_6 \quad (7.4.17)$$

where $c'_6 := \frac{1}{12} \frac{\omega_n}{\omega_{n-1}} \left(\frac{b}{2c_3} \right)^n \min_{|s| \leq 1 - \frac{b}{2}} W(s) > 0$.

Let $c''_6 := \frac{\omega_{n-1}}{\omega_n} c'_6$. Assuming $\varepsilon < \varepsilon_5$ is sufficiently small such that $c''_6 \varepsilon^{4/5} < \frac{1}{2} \min\{\text{dist}(\tilde{U}, \partial U), i_0, \frac{1}{\kappa_0}\}$, we can apply the boundary almost monotonicity formula (7.4.10) and the upper bound on discrepancy function (7.4.12) to see that for any $\varepsilon \leq r \leq c''_6 \varepsilon^{4/5}$ (note that $\tilde{\xi}_\varepsilon(u) \leq \xi_\varepsilon(u)$ by definition)

$$\begin{aligned} \tilde{I}_{\varepsilon,u}(r, x) &\geq e^{-c_1(r-\varepsilon)} \tilde{I}_{\varepsilon,u}(\varepsilon, x) - \frac{2}{3} \frac{\omega_n}{\omega_{n-1}} \varepsilon^{-4/5} (r - \varepsilon) \\ &\geq e^{-c_1 c'_6 \varepsilon^{4/5}} \tilde{I}_{\varepsilon,u}(\varepsilon, x) - \frac{2}{3} \frac{\omega_n}{\omega_{n-1}} c''_6 > 2c'_6 \end{aligned}$$

for ε sufficiently small such that (7.4.17) holds and $e^{-c_1 c'_6 \varepsilon^{4/5}} \geq \frac{3}{4}$. Setting $c_6 := \min\{c'_6, c''_6\}$, the asserted lower bound for $I_{\varepsilon,u}(r, x)$ follows from the estimate

$$\tilde{I}_{\varepsilon,u}(r, x) \leq I_{\varepsilon,u}(r, x) + \frac{4}{3} \frac{\omega_n}{\omega_{n-1}} \Lambda_0 C_0 r \quad (7.4.18)$$

which holds for all sufficiently small r depending only on k_0 and κ_0 . The case of interior points $x \in \Omega$ can be obtained similarly, using the interior monotonicity formula (7.4.9). $\square$

In order to prove a uniform lower bound of the energy density ratio on balls of radius $r \geq \mathcal{O}(\varepsilon^{4/5})$, the pointwise upper bound in Proposition 7.4.10 will not be sufficient for our purpose. We need a more refined integral upper bound for the discrepancy function. It is important to note that the centre x in the proposition below may or may not lie in $\partial\Omega$, and does not need to be inside the transition region $\{|u| \leq 1 - b\}$.

Proposition 7.4.13 (Integral upper bound for discrepancy function). *For any $b \in (0, 1 - \alpha)$, there exist constants*

$$r_2 = r_2(n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) \in (0, 1),$$

$$\varepsilon_7 = \varepsilon_7(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_7 = c_7(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 1$$

such that for any $0 < \varepsilon < \varepsilon_7$, $x \in \bar{\Omega} \cap \tilde{U}$ and $c_6 \varepsilon^{4/5} \leq r \leq r_2$, we have

$$\frac{1}{r^n} \int_{B_r(x) \cap \Omega} \xi_\varepsilon^+(u) \leq \frac{c_7}{r^{7/8}} (I_{\varepsilon, u}(r, x) + 1). \quad (7.4.19)$$

Proof. Let $b \in (0, 1 - \alpha)$. Fix $x \in \bar{\Omega} \cap \tilde{U}$ and $c_6 \varepsilon^{4/5} \leq r \leq r_2$ where r_2 is sufficiently small with $r_2 < \frac{1}{2}\{r_0, r_1, \text{dist}_g(x, \partial U)\}$. We divide the domain of integration $B_r(x) \cap \Omega$ into a union of three disjoint subsets as follow:

$$\mathcal{A} := \left((B_r(x) \cap \Omega) \setminus B_{r-2\varepsilon^{9/10}}(x) \right) \cup \left((B_r(x) \cap \Omega) \cap U_{2\varepsilon^{9/10}}(\partial\Omega) \right),$$

$$\mathcal{B} := \{y \in (B_r(x) \cap \Omega) \setminus \mathcal{A} : B_{\varepsilon^{9/10}}(y) \cap \{|u| \leq 1 - b\} \neq \emptyset\},$$

$$\mathcal{C} := \{y \in (B_r(x) \cap \Omega) \setminus \mathcal{A} : B_{\varepsilon^{9/10}}(y) \cap \{|u| \leq 1 - b\} = \emptyset\}.$$

Note that $r \geq c_6 \varepsilon^{4/5} \geq 2\varepsilon^{9/10}$ for all sufficiently small ε (depending on c_6). We will

estimate the (LHS) of (7.4.19) over each of the three subsets defined above.

We begin with the estimate on $\mathcal{A}$. By (7.2.5) and the bound on ball curvatures, for r_2 small enough (depending only on n , k_0 and κ_0) we have the following volume estimate

$$dv_g(\mathcal{A}) \leq C(n)r^{n-1}\varepsilon^{9/10}.$$

Combining this estimate with (7.4.12) and $r \geq c_6\varepsilon^{4/5}$, we have

$$\frac{1}{r^n} \int_{\mathcal{A}} \xi_{\varepsilon}^+(u) \leq C(n)\varepsilon^{1/10}r^{-1} \leq C(n)c_6^{-1/8}r^{-7/8}$$

which is clearly bounded by the (RHS) of (7.4.19) provided that $c_7 \geq C(n)c_6^{-1/8}$.

We now proceed with the estimate on $\mathcal{B}$. We first need to an estimate of the volume $dv_g(\mathcal{B})$ by considering a covering of $\mathcal{B}$ by balls $\{B_{\varepsilon^{9/10}}(z)\}$ with centres $z \in \{|u| \leq 1 - b\} \cap (\Omega \cap B_{r-\varepsilon^{9/10}}(x) \setminus \overline{U_{\varepsilon^{9/10}}(\partial\Omega)})$. By the Vitali covering lemma, we can extract a finite disjoint family of such balls $\{B_{\varepsilon^{9/10}}(z_i)\}_{i=1}^N$ so that $\mathcal{B} \subset \cup_{i=1}^N B_{5\varepsilon^{9/10}}(z_i)$. Hence, we have from (7.2.4) that the volume estimate

$$dv_g(\mathcal{B}) \leq N \cdot C(n)\varepsilon^{9n/10} \tag{7.4.20}$$

holds for all sufficiently small ε depending on r_0 . It remains to bound N from above. Applying Lemma 7.4.12 (with $x = z_i$), since $\varepsilon \leq \varepsilon^{9/10} \leq c_6\varepsilon^{4/5}$ and $\text{dist}(z_i, \partial\Omega) > \varepsilon^{9/10}$, we have $I_{\varepsilon,u}(\varepsilon^{9/10}, z_i) \geq c_6$. Summing over all i , noting that $\{B_{\varepsilon^{9/10}}(z_i)\}_{i=1}^N$ are disjoint balls contained in $B_r(x) \cap \Omega$, we have

$$N\omega_{n-1}c_6\varepsilon^{9(n-1)/10} \leq \int_{B_r(x) \cap \Omega} e_{\varepsilon}(u) = \omega_{n-1}r^{n-1}I_{\varepsilon,u}(r, x).$$

This gives an upper estimate on N . By substituting back into (7.4.20), we obtain the volume estimate

$$dv_g(\mathcal{B}) \leq C(n, c_6)\varepsilon^{9/10}r^{n-1}I_{\varepsilon,u}(r, x).$$

Using this estimate together with (7.4.12) and $r \geq c_6 \varepsilon^{4/5}$, we have

$$\frac{1}{r^n} \int_{\mathcal{B}} \xi_{\varepsilon}^{+}(u) \leq C(n, c_6) \varepsilon^{1/10} r^{-1} I_{\varepsilon, u}(r, x) \leq C(n, c_6) r^{-7/8} I_{\varepsilon, u}(r, x)$$

which is clearly bounded by the (RHS) of (7.4.19) with c_7 sufficiently large depending only on n and c_6 .

Finally, it remains to prove the estimate on $\mathcal{C}$. Consider the Lipschitz function

$$\phi(y) := \min \left\{ 1, \varepsilon^{-9/10} \text{dist}_g \left(y, (\tilde{U} \setminus (B_r(x) \cap \Omega)) \cup \{|u| \leq 1 - b\} \right) \right\}$$

which clearly satisfies

- $\phi = 0$ on $(\tilde{U} \setminus (B_r(x) \cap \Omega)) \cup \{|u| \leq 1 - b\}$,
- $\phi = 1$ on $\mathcal{C}$,
- $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq \varepsilon^{-9/10}$ everywhere.

Taking gradient on both sides of (7.4.1) and then multiplying by $\phi^2 \nabla u$, we have by Bochner formula (7.2.3) that

$$\varepsilon \phi^2 \left(\frac{1}{2} \Delta |\nabla u|^2 - |\text{Hess } u|^2 + (n-1)k_0 |\nabla u|^2 \right) \geq \varepsilon \phi^2 \langle \nabla \Delta u, \nabla u \rangle = \frac{W''(u)}{\varepsilon} \phi^2 |\nabla u|^2.$$

Integrating the above on $B_r(x) \cap \Omega$ and using integration by parts (note that $\phi = 0$ on $B_r(x) \cap \partial\Omega$), we obtain

$$\begin{aligned} & \int_{B_r(x) \cap \Omega} \left(\varepsilon \phi^2 |\text{Hess } u|^2 + \frac{W''(u)}{\varepsilon} \phi^2 |\nabla u|^2 \right) \\ & \leq -\varepsilon \int_{B_r(x) \cap \Omega} \phi \langle \nabla \phi, \nabla |\nabla u|^2 \rangle + C(n, k_0) \varepsilon \int_{B_r(x) \cap \Omega} \phi^2 |\nabla u|^2 \\ & = -2\varepsilon \int_{B_r(x) \cap \Omega} \phi \text{Hess } u(\nabla u, \nabla \phi) + C(n, k_0) \varepsilon \int_{B_r(x) \cap \Omega} \phi^2 |\nabla u|^2 \\ & \leq \frac{1}{2} \int_{B_r(x) \cap \Omega} \varepsilon \phi^2 |\text{Hess } u|^2 + 2 \int_{B_r(x) \cap \Omega} \varepsilon |\nabla u|^2 |\nabla \phi|^2 + C(n, k_0) \varepsilon \int_{B_r(x) \cap \Omega} \phi^2 |\nabla u|^2 \end{aligned}$$

where we have used Cauchy-Schwarz and Young's inequalities in the last inequality. Since $\text{spt } \phi \subset \{|u| > 1 - b\}$, we have $W''(u) \geq \kappa$ on $\text{spt } \phi$. Absorbing the first term on right to the left hand side and combining this with $|\nabla \phi| \leq \varepsilon^{-9/10}$, we obtain for ε sufficiently small (depending on n, k_0 and κ)

$$\frac{\kappa}{2\varepsilon} \int_{B_r(x) \cap \Omega} \phi^2 |\nabla u|^2 \leq 2\varepsilon^{-4/5} \int_{B_r(x) \cap \Omega} |\nabla u|^2.$$

Since $\phi = 1$ on $\mathcal{C}$ and $0 \leq \phi \leq 1$, together with $r \geq c_6 \varepsilon^{4/5}$, we have

$$\frac{1}{r^n} \int_{\mathcal{C}} \xi_{\varepsilon}^+(u) \leq \frac{4\omega_{n-1}}{\kappa} \varepsilon^{1/5} r^{-1} I_{\varepsilon,u}(r, x) \leq C(n, W, c_6) r^{-3/4} I_{\varepsilon,u}(r, x)$$

which is clearly bounded by the (RHS) of (7.4.19) with c_7 sufficiently large depending only on n, W and c_6 (note that $r^{-3/4} \leq r^{-7/8}$). $\square$

Using Proposition 7.4.13, we can extend Lemma 7.4.12 beyond the scale of $\mathcal{O}(\varepsilon^{4/5})$.

Proposition 7.4.14 (Energy density ratio lower bound II). *For any $b \in (0, 1 - \alpha)$, there exist constants*

$$r_3 = r_3(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) \in (0, 1),$$

$$\varepsilon_8 = \varepsilon_8(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_8 = c_8(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0$$

such that for all $0 < \varepsilon < \varepsilon_8$, $x \in \partial\Omega \cap \tilde{U}$ with $|u(x)| \leq 1 - b$, and $\varepsilon \leq r \leq r_3$, we have

$$I_{\varepsilon,u}(r, x) \geq c_8.$$

In fact, the same holds for $x \in \Omega \cap \tilde{U}$ provided furthermore that $r < \text{dist}_g(x, \partial\Omega)$.

Proof. By Lemma 7.4.12, the proposition holds for $\varepsilon \leq r \leq c_6 \varepsilon^{4/5}$ with $c_8 = c_6$ and

$\varepsilon_8 = \varepsilon_6$. The point is to use the almost monotonicity formula in Lemma 7.4.2 and 7.4.4 together with Proposition 7.4.13 to show that $I_{\varepsilon,u}(r, x)$ cannot further decrease too much as r increases beyond the scale $c_6\varepsilon^{4/5}$ to some definite radius r_3 to be determined.

We first deal with the case $x \in \partial\Omega$. By the boundary almost monotonicity formula Lemma 7.4.4 together with (7.4.19), for $\varepsilon < \varepsilon_7$ we obtain for $c_6\varepsilon^{4/5} \leq \rho \leq \frac{1}{2} \min\{r_2, i_0, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial\tilde{U})\}$,

$$\frac{d}{d\rho} \left(\omega_{n-1} e^{c_1\rho} \tilde{I}_{\varepsilon,u}(\rho, x) \right) \geq -\frac{c_7}{\rho^{7/8}} e^{c_1\rho} (\tilde{I}_{\varepsilon,u}(\rho, x) + 1). \quad (7.4.21)$$

We define r_3 as the supremum of all $r' \in \left[c_6\varepsilon^{4/5}, \frac{1}{2} \min\{r_2, i_0, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial\tilde{U})\} \right]$ such that

$$e^{c_1\rho} \tilde{I}_{\varepsilon,u}(\rho, x) \geq \frac{1}{2} c_6 e^{c_1 c_6 \varepsilon^{4/5}} \text{ for all } \rho \in [c_6\varepsilon^{4/5}, r'].$$

Note that the left-hand side is a continuous function of ρ . By Lemma 7.4.12, we have $r_3 > c_6\varepsilon^{4/5}$. If $r_3 = \frac{1}{2} \min\{r_2, i_0, \frac{1}{\kappa_0}, \text{dist}_g(x, \partial\tilde{U})\}$, then we are done by taking $\varepsilon_8 = \min\{\varepsilon_6, \varepsilon_7\}$ and $c_8 = \frac{1}{4} c_6 e^{-c_1 r_2}$ (we further reduce r_3 depending on k_0, κ_0, C_0 and Λ_0 according to the estimate (7.4.18)). Otherwise, by continuity we must have

$$e^{c_1 r_3} \tilde{I}_{\varepsilon,u}(r_3, x) = \frac{1}{2} c_6 e^{c_1 c_6 \varepsilon^{4/5}}. \quad (7.4.22)$$

By the definition of r_3 and (7.4.21), we have for $\rho \in [c_6\varepsilon^{4/5}, r_3]$

$$\frac{d}{d\rho} \log \left(e^{c_1\rho} \tilde{I}_{\varepsilon,u}(\rho, x) \right) \geq -C(n, c_6, c_7, r_2) \rho^{-7/8}.$$

Integrating ρ over $[c_6\varepsilon^{4/5}, r_3]$, using (7.4.22) and Lemma 7.4.12,

$$\log \frac{1}{2} \geq \log \left(\frac{e^{c_1 r_3} \tilde{I}_{\varepsilon,u}(r_3, x)}{e^{c_1 c_6 \varepsilon^{4/5}} \tilde{I}_{\varepsilon,u}(c_6 \varepsilon^{4/5}, x)} \right) \geq -8C(n, c_6, c_7, r_2) r_3^{1/8}.$$

This gives the desired positive lower bound for r_3 independent of ε and x . The proof for the interior case $x \in \Omega$ is similar where we use Lemma 7.4.2 instead. $\square$

We now proceed to establish an upper bound for the energy density ratio in terms of the uniform energy bound E_0 , within the same range of radii as in Proposition 7.4.14. Note that we do not require the centre point x to lie inside the transition region $\{|u| \leq 1 - b\}$.

Proposition 7.4.15 (Energy density ratio upper bound). *For any $b \in (0, 1 - \alpha)$, there exist constants*

$$\varepsilon_9 = \varepsilon_9(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_9 = c_9(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 1$$

such that for all $0 < \varepsilon < \varepsilon_9$, $x \in \partial\Omega \cap \tilde{U}$, and $\varepsilon \leq r \leq r_3$, we have

$$I_{\varepsilon, u}(r, x) \leq c_9(E_0 + 1).$$

In fact, the same holds for $x \in \Omega \cap \tilde{U}$ provided furthermore that $r < \text{dist}_g(x, \partial\Omega)$.

Proof. We only give the proof for $x \in \partial\Omega$, as the interior case is similar. We first treat the case when $c_6\varepsilon^{4/5} \leq r \leq r_3$. Let $\delta_0 := \max\{1, e^{c_1 r_3} \tilde{I}_{\varepsilon, u}(r_3, x)\} > 0$. Note that

$$\delta_0 \leq C(c_1, n) r_3^{-(n-1)} E_0 + C(c_1, n) \Lambda_0 C_0 \leq C(c_1, n, r_3, \Lambda_0, C_0)(E_0 + 1). \quad (7.4.23)$$

Note that $e^{c_1 r} \tilde{I}_{\varepsilon, u}(r, x) \leq \delta_0$ clearly implies the desired upper bound of $I_{\varepsilon, u}(r, x)$.

We can assume without loss of generality that $e^{c_1 \rho} \tilde{I}_{\varepsilon, u}(\rho, x) \geq \delta_0$ for all $\rho \in [r, r_3]$.

Integrating (7.4.21) over $[r, r_3]$, we obtain as in Proposition 7.4.14 that

$$\log \left(\frac{e^{c_1 r_3} \tilde{I}_{\varepsilon, u}(r_3, x)}{e^{c_1 r} \tilde{I}_{\varepsilon, u}(r, x)} \right) \geq -C(c_7, n, c_1, r_3).$$

This implies

$$\tilde{I}_{\varepsilon,u}(r, x) \leq C(c_7, n, c_1, r_3)\delta_0$$

which is the desired upper bound by (7.4.23).

It remains to treat the case $\varepsilon \leq r \leq c_6\varepsilon^{4/5}$. Integrating the almost monotonicity formula (7.4.10) over $[r, c_6\varepsilon^{4/5}]$ and using the pointwise bound (7.4.12), we obtain

$$e^{c_1 r} \tilde{I}_{\varepsilon,u}(r, x) \leq C(n, c_1, r_3, c_6) + e^{c_1 c_6 \varepsilon^{4/5}} \tilde{I}_{\varepsilon,u}(c_6 \varepsilon^{4/5}, x)$$

which yields the desired result as $\tilde{I}_{\varepsilon,u}(c_6 \varepsilon^{4/5}, x) \leq c_9(E_0 + 1)$. $\square$

We conclude this section with the following proposition which says that the energy density ratio is almost monotone modulo a term of order $r^{1/8}$ for the range of radii in $[c_6 \varepsilon^{4/5}, r_3]$.

Proposition 7.4.16 (Almost monotonicity of energy density ratio). *For any $b \in (0, 1 - \alpha)$, there exist constants*

$$\varepsilon_{10} = \varepsilon_{10}(b, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$c_{10} = c_{10}(b, E_0, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 1$$

such that for any $0 < \varepsilon < \varepsilon_{10}$, $x \in \partial\Omega \cap \tilde{U}$ and $c_6 \varepsilon^{4/5} \leq s \leq r \leq r_3$, we have

$$I_{\varepsilon,u}(r, x) \geq e^{-c_{10}(r-s)} I_{\varepsilon,u}(s, x) - c_{10} r^{1/8}; \quad (7.4.24)$$

In fact, the same holds for $x \in \Omega \cap \tilde{U}$ provided furthermore that $r < \text{dist}(x, \partial\Omega)$.

Proof. We again just provide the proof for the case $x \in \partial\Omega$. Using Proposition 7.4.15 and (7.4.21), for $\varepsilon < \min\{\varepsilon_7, \varepsilon_9\}$, we have for $\rho \in [c_6 \varepsilon^{4/5}, r_3]$ that

$$\frac{d}{d\rho} \left(e^{c_1 \rho} \tilde{I}_{\varepsilon,u}(\rho, x) \right) \geq -C(c_1, c_7, c_9, E_0) \rho^{-7/8}.$$

Integrating ρ over $[s, r]$ gives

$$\tilde{I}_{\varepsilon, u}(r, x) \geq e^{-c_1(r-s)} \tilde{I}_{\varepsilon, u}(s, x) - C(c_1, c_7, c_9, E_0) r^{1/8},$$

from which (7.4.24) easily follows by (7.4.18) (since $r \leq r^{1/8}$). $\square$

7.5 Density ratio bounds and rectifiability of the limit varifold

In this section, we assume u_i , ε_i and λ_i are sequences as in section 7.3.1. Recall that the associated Radon measures μ_i on U and the associated varifolds $V_i \in \mathbf{V}_{n-1}(U)$ are defined in (7.3.4) and (7.3.5) respectively. By the compactness of Radon measures, after possibly passing to a subsequence, there exists a Radon measure μ on U and a varifold $V \in \mathbf{V}_{n-1}(U)$ such that

$$\mu_i \rightarrow \mu \quad \text{and} \quad V_i \rightarrow V$$

in the sense of measures. Note that $\|V_i\| = \mu_i \llcorner \{|\nabla u_i| \neq 0\}$ from the definition and observe that the support (as measures on U) of both μ and $\|V\|$ is contained in $\overline{\Omega} \cap U$. The main result of this section is to show that V is rectifiable in U and that $\|V\| = \mu$. Note that the rectifiability of V in the interior of Ω was established in [HT00, Gua18]. Our main point here is to prove that the same continues to hold up to the boundary $\partial\Omega$.

7.5.1 Density ratio bound

We first begin with a lemma which says that any point in $\text{spt } \mu$ can be approximated by a sequence of points x_i in the transition region $\{|u| \leq \alpha\}$.

Lemma 7.5.1. *For each $x \in \text{spt } \mu$, there exists (after possibly passing to a subsequence) a sequence $x_i \in (\overline{\Omega} \cap U) \cap \{|u_i| \leq \alpha\}$ such that $x_i \rightarrow x$.*

Proof. We argue by contradiction as in the proof of Proposition 4.1 in [HT00]. Let $x \in \text{spt } \mu$. We first assume $x \in \partial\Omega$. Suppose, on the contrary, that there exists $\bar{r} > 0$ such that $B_{\bar{r}}(x) \Subset U$ and $|u_i| > \alpha$ on $B_{\bar{r}}(x) \cap \overline{\Omega}$ for all large i . By shrinking $\bar{r}$ such that $B_{\bar{r}}(x) \cap \overline{\Omega}$ is connected and passing to a subsequence, we can assume $u_i > \alpha$. By Lemma 7.4.8 (shrinking $\bar{r}$ further if necessary) with a fixed $\eta \in (\frac{1}{2}, 1)$, we have for all large i that $|u_i - 1| \leq c_4 \varepsilon_i^\eta$ and hence $\varepsilon_i^{-1}W(u_i) = \mathcal{O}(\varepsilon_i^{2\eta-1})$ on $B_{\bar{r}/3}(x) \cap \Omega$. The Bochner formula (7.2.3) then implies that for all large i (such that $(n-1)k_0 \leq \frac{\kappa}{2\varepsilon_i}$)

$$\Delta \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} \right) \geq \frac{\kappa}{2\varepsilon_i} |\nabla u_i|^2.$$

Multiplying the above by any nonnegative $\phi \in C_c^2(B_{\bar{r}}(x) \cap \overline{\Omega})$ and integrating by parts, we obtain

$$\begin{aligned} & \int_{B_{\bar{r}}(x) \cap \Omega} \phi \frac{\varepsilon_i |\nabla u_i|^2}{2} \\ & \leq \frac{\varepsilon_i^2}{\kappa} \int_{B_{\bar{r}}(x) \cap \Omega} (\Delta \phi) \frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{\varepsilon_i^2}{\kappa} \int_{B_{\bar{r}}(x) \cap \partial\Omega} \left(\varepsilon_i \phi B(\nabla u_i, \nabla u_i) - \frac{\partial \phi}{\partial \nu} \frac{\varepsilon_i |\nabla u_i|^2}{2} \right) d\mathcal{H}^{n-1} \\ & \leq \frac{E_0}{\kappa} \varepsilon_i^2 \|\phi\|_{C^2} + \frac{1+2\kappa_0}{\kappa} \varepsilon_i^2 \|\phi\|_{C^1} \int_{B_{\bar{r}}(x) \cap \partial\Omega} e_{\varepsilon_i}(u_i) d\mathcal{H}^{n-1} \end{aligned}$$

where B is the second fundamental form of $\partial\Omega$. By the proof of Lemma 7.3.2 together with the estimates $\varepsilon_i^{-1}W(u_i) = \mathcal{O}(\varepsilon_i^{2\eta-1})$ with $\eta > 1/2$ and letting $i \rightarrow \infty$, we have $\mu(B_{\bar{r}/3}(x)) = 0$, contradicting that $x \in \text{spt } \mu$. The case $x \notin \partial\Omega$ is simpler, and the proof is in [HT00, Proposition 4.1]. $\square$

As a key step towards proving the rectifiability of the limit varifold V , we now establish a locally uniform density ratio upper bound for the limit measure μ , which holds globally up to the boundary $\partial\Omega$. Note that compared to [HT00, Proposition 4.1]), we do not claim at this stage that there is a uniform lower bound for the

density ratio centred at a boundary point. This is due to the possible complication that the x_i 's in Lemma 7.5.1 could be a sequence of interior points approaching a boundary point $x \in \text{spt } \mu \cap \partial\Omega$. This poses new difficulties when we want to apply our monotonicity formulas, which hold for balls that are either centred at a boundary point (Lemma 7.4.4) or stay away from the boundary (Lemma 7.4.2) entirely.

Proposition 7.5.2 (Density ratio bounds for the limit measure). *There exist constants*

$$\theta_1 = \theta_1(\Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0,$$

$$\theta_2 = \theta_2(E_0, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U)) > 0$$

such that for any $x \in (\text{spt } \mu \cap \tilde{U}) \cap \partial\Omega$, we have for all $0 < r < \frac{r_3}{2}$,

$$\frac{\mu(B_r(x))}{\omega_{n-1}r^{n-1}} \leq \theta_2. \quad (7.5.1)$$

Moreover, for any $x \in (\text{spt } \mu \cap \tilde{U}) \cap \Omega$, we have for all $0 < r < \frac{1}{2} \min\{r_3, \text{dist}_g(x, \partial\Omega)\}$,

$$\theta_1 \leq \frac{\mu(B_r(x))}{\omega_{n-1}r^{n-1}} \leq \theta_2. \quad (7.5.2)$$

Here, r_3 is the constant appearing in Proposition 7.4.14 with $b = \frac{1-\alpha}{2}$.

Proof. Let $x \in (\text{spt } \mu \cap \tilde{U}) \cap \partial\Omega$. Fix any $0 < r < \frac{r_3}{2}$. Since $\mu_i \rightarrow \mu$ as measures, we will have for all large i

$$\frac{\mu(B_r(x))}{\omega_{n-1}r^{n-1}} \leq \frac{2\mu_i(B_r(x))}{\omega_{n-1}r^{n-1}} = \frac{2}{h_0} I_{\varepsilon_i, u_i}(r, x).$$

Hence, the desired assertion follows from Proposition 7.4.15 with $\theta_2 := \frac{2}{h_0} c_9(E_0 + 1)$.

For $x \in (\text{spt } \mu \cap \tilde{U}) \cap \Omega$. By Lemma 7.5.1, after passing to a subsequence, we have a sequence of interior points $x_i \in \Omega \cap \tilde{U}$ such that $|u_i(x_i)| \leq \alpha$ for all i and $x_i \rightarrow x$. Fix any $0 < r < \frac{1}{2} \min\{r_3, \text{dist}_g(x, \partial\Omega)\}$. The upper bound θ_2 is obtained exactly as

before. For the lower bound, we observe for all large i

$$\frac{\mu(B_r(x))}{\omega_{n-1}r^{n-1}} \geq \frac{\mu_i(B_r(x))}{2\omega_{n-1}r^{n-1}} \geq \frac{\mu_i(B_{r/2}(x_i))}{2\omega_{n-1}r^{n-1}} = \frac{1}{2^n h_0} I_{\varepsilon_i, u_i}(r/2, x_i).$$

Hence, the desired assertion follows from Proposition 7.4.14 with $\theta_1 := \frac{c_8}{2^n h_0}$. $\square$

Exactly the same argument as in the proof of Lemma 7.5.1 and Proposition 7.5.2 yields the following result.

Proposition 7.5.3 (Uniform convergence away from limit interface). *Either $u_i \rightarrow 1$ or $u_i \rightarrow -1$ locally uniformly on $(\Omega \cap U) \setminus \text{spt } \|V\|$. In particular, we have $M := \partial^* \{u = 1\} \subset \text{spt } \|V\| \cap (\Omega \cap U)$. Moreover, $\frac{\varepsilon_i |\nabla u_i|^2}{2}$ and $\frac{W(u_i)}{\varepsilon_i}$ both converges locally uniformly to 0 on $(\Omega \cap U) \setminus \text{spt } \|V\|$.*

7.5.2 Vanishing of discrepancy measure

Let $e_i := e_{\varepsilon_i}(u_i)$ and $\xi_i := \xi_{\varepsilon_i}(u_i)$ denote the energy density and discrepancy function as defined in (7.4.3) and (7.4.4). By compactness of Radon measures (on U), after possibly passing to a subsequence, we can assume that there exists a Radon measure $|\xi|$ on U such that

$$|\xi_i| dv_g \llcorner \Omega \cap U \rightarrow |\xi|$$

in the sense of measures. We call $|\xi|$ the discrepancy measure. Since $|\xi_i| \leq e_i$, we clearly have $\text{spt } |\xi| \subset \text{spt } \mu \subset \overline{\Omega} \cap U$.

Theorem 7.5.4 (Vanishing of discrepancy measure). *The discrepancy measure $|\xi|$ vanishes on U , i.e. $|\xi| = 0$. In other words, the discrepancy functions ξ_i converge to 0 in $L^1_{loc}(\overline{\Omega} \cap U)$.*

Proof. We first claim that for all $x \in \text{spt } |\xi| \cap \partial\Omega$, we have

$$\liminf_{r \rightarrow 0} \frac{|\xi|(B_r(x))}{\omega_{n-1}r^{n-1}} = 0. \quad (7.5.3)$$

We argue by contradiction as in [HT00, Proposition 4.3]. Suppose on the contrary that (7.5.3) does not hold, i.e. there exist $\delta_0 > 0$, $R_0 > 0$ and $x_0 \in \text{spt } |\xi| \cap \partial\Omega$ such that

$$|\xi|(B_r(x_0)) \geq \delta_0 \omega_{n-1} r^{n-1} \quad \text{for all } 0 < r \leq R_0. \quad (7.5.4)$$

By the density ratio upper bound (7.5.1), there exists $\theta_2 > 0$ such that

$$\frac{\mu(B_r(x_0))}{\omega_{n-1} r^{n-1}} \leq \theta_2 \quad \text{for all } 0 < r \leq \frac{r_3}{2}. \quad (7.5.5)$$

Fix $0 < t < \min\{R_0, \frac{r_3}{2}, r_1, r_2, \frac{1}{\kappa_0}\}$ to be chosen later. Since $|\xi_i| dv_g \llcorner \Omega \cap U \rightarrow |\xi|$ and $\mu_i \rightarrow \mu$, by (7.5.4) and (7.5.5), for any fixed $N \in \mathbb{N}$, there exists some $i > N$ such that

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r(x_0) \cap \Omega} |\xi_i| \geq \frac{\delta_0}{2} \quad \text{for all } r \in [t^2, t] \quad (7.5.6)$$

and

$$\frac{\mu_i(B_t(x_0))}{\omega_{n-1} t^{n-1}} \leq 2\theta_2. \quad (7.5.7)$$

Note that $|\xi_i| = \xi_i^+ + \xi_i^-$. By (7.5.6) and Proposition 7.4.13, for all i large enough (so that $\varepsilon_i < \varepsilon_7$ and $c_6 \varepsilon_i^{4/5} \leq t^2$), we have

$$\begin{aligned} \frac{1}{(1 + 3\kappa_0 \rho) \rho^n} \int_{B_\rho(x_0) \cap \Omega} -\xi_i &\geq \frac{1}{4\rho^n} \int_{B_\rho(x_0) \cap \Omega} |\xi_i| - \frac{2}{\rho^n} \int_{B_\rho(x_0) \cap \Omega} \xi_i^+ \\ &\geq \frac{\delta_0}{8} \rho^{-1} - \frac{2c_7}{\rho^{7/8}} (I_{\varepsilon_i, u_i}(\rho, x_0) + 1) \end{aligned}$$

for all $\rho \in [t^2, t]$. Since $x_0 \in \text{spt } \mu$, by Proposition 7.4.15, for all i large enough (so that $\varepsilon_i < \min\{\varepsilon_9, t^2\}$), we have

$$\frac{1}{(1 + 3\kappa_0 \rho) \rho^n} \int_{B_\rho(x_0) \cap \Omega} -\xi_i \geq \frac{\delta_0}{8} \rho^{-1} - C(c_7, c_9, E_0) \rho^{-7/8}$$

for all $\rho \in [t^2, t]$. By the almost monotonicity formula (7.4.10) (note that $\tilde{\xi}_i \leq \xi_i$),

$$\frac{d}{d\rho} \left(\omega_{n-1} e^{c_1 \rho} \tilde{I}_{\varepsilon_i, u_i}(\rho, x_0) \right) \geq \frac{\delta_0}{8} e^{c_1 \rho} \rho^{-1} - C(c_7, c_9, E_0) e^{c_1 \rho} \rho^{-7/8},$$

for all $\rho \in [t^2, t]$. Integrating over $[t^2, t]$ and using (7.4.18), (7.5.7), we obtain

$$C(n, c_1, \Lambda_0, C_0) \left(\frac{h_0}{\omega_{n-1}} 2\theta_2 + t \right) \geq \frac{\delta_0}{8} \log \frac{1}{t} - C(c_7, c_9, E_0, c_1) t^{1/8},$$

which gives a contradiction for t sufficiently small. Hence, we have proved (7.5.3).

By the differentiation theory for Radon measures (applied to $|\xi|$ and $d\mathcal{H}^{n-1} \llcorner \partial\Omega$), c.f. [Sim83, P.26], we conclude from (7.5.3) that $|\xi|(\partial\Omega) = 0$, i.e. $|\xi|$ vanishes on $\partial\Omega$. By a similar argument using the interior monotonicity formula (7.4.9), we can show that (7.5.3) also holds for interior points $x \in \text{spt } |\xi| \cap \Omega$. Combining this with the density lower bound in (7.5.2), we have

$$\liminf_{r \rightarrow 0} \frac{|\xi|(B_r(x))}{\mu(B_r(x))} = 0,$$

which implies $|\xi| = 0$ inside $\Omega \cap U$ as well. $\square$

7.5.3 Rectifiability of limit varifold

Let V_i be the varifold associated with u_i as in (7.3.5). After passing to a subsequence, we can also assume that there exist $u \in BV(\Omega \cap U)$, $\lambda \in \mathbb{R}$ and $V \in \mathbf{V}_{n-1}(U)$ such that

$$\lambda_i \rightarrow \lambda, \quad u_i \rightarrow u \text{ a.e.}, \quad V_i \rightarrow V \text{ in the varifold sense.}$$

Recall that $M := \partial^* \{u = 1\}$ is the reduced boundary of $\{u = 1\}$ in $\Omega \cap U$ with outward normal ν_M . Our goal in this subsection is to show that V is in fact rectifiable up to the boundary $\partial\Omega$.

Lemma 7.5.5 (First variation of limit varifold). *There exists a constant*

$$c_{11} = c_{11}(E_0, \Lambda_0, C_0, W, n, i_0, k_0, \kappa_0, \text{dist}_g(\tilde{U}, \partial U), dv_g(\Omega \cap U), \mathcal{H}^{n-1}(\partial\Omega \cap U)) > 1$$

such that for any $X \in \mathfrak{X}_c(U)$ with $\text{spt } X \subset \tilde{U}$, we have

$$\left| h_0 \delta V(X) + \lambda \int_{\Omega \cap U} X \cdot Du \right| \leq c_{11} \sup_{\partial\Omega \cap U} |X \cdot \nu|. \quad (7.5.8)$$

In particular, for any $X \in \mathfrak{X}_c^{\text{tan}}(U)$, we have

$$\delta V(X) = -\frac{2\lambda}{h_0} \int_M X \cdot \nu_M d\mathcal{H}^{n-1}. \quad (7.5.9)$$

Remark 7.5.6. If $\lambda = 0$, then (7.5.9) implies that V is a free boundary stationary varifold (c.f. [GLZ20, Definition 3.1]).

Proof. Let $X \in \mathfrak{X}_c(U)$ with $\text{spt } X \subset \tilde{U}$. Note that $\delta V_i(X) \rightarrow \delta V(X)$ as $i \rightarrow \infty$. On the other hand, by the calculations in [KT18, Lemma 4.1], we have

$$\begin{aligned} h_0 \delta V_i(X) &= \int_{\Omega \cap \{|\nabla u_i| \neq 0\}} \nabla X \cdot \left(\frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \right) \xi_i - \int_{\Omega \cap \{|\nabla u_i| = 0\}} (\nabla X \cdot I) \frac{W(u_i)}{\varepsilon_i} \\ &\quad + \lambda_i \int_{\Omega} u_i \text{div}_g X + \int_{\partial\Omega} \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} - \lambda_i u_i \right) (X \cdot \nu) d\mathcal{H}^{n-1}. \end{aligned}$$

Note that the first and second term of the (RHS) above are both bounded by $C(n)\|X\|_{C^1} \int_{\tilde{U} \cap \Omega} |\xi_i|$, which converges to 0 as $i \rightarrow \infty$ by Theorem 7.5.4. For the third term, since $\lambda_i \rightarrow \lambda$ and $u_i \rightarrow u \in BV(\Omega \cap U)$ a.e., by the Dominated Convergence Theorem, we have

$$\lim_{i \rightarrow \infty} \lambda_i \int_{\Omega} u_i \text{div}_g X = \lambda \int_{\Omega} u \text{div}_g X = -\lambda \int_{\Omega} X \cdot Du + \lambda \int_{\partial\Omega} Tu(X \cdot \nu) d\mathcal{H}^{n-1}$$

where $Tu \in L^1(\partial\Omega)$ is the trace of u on $\partial\Omega$ (c.f. [EG15, Theorem 5.6]). Moreover,

by the boundedness of the trace operator T , we have

$$\|Tu\|_{L^1(\partial\Omega\cap U)} \leq C(n, i_0, k_0, \kappa_0)\|u\|_{BV(\Omega\cap U)} \leq C(n, i_0, k_0, \kappa_0, E_0, W, dv_g(\Omega\cap U)).$$

Combining all the above and applying Lemma 7.3.2 for the last term gives (7.5.8). $\square$

Finally, we can show the main result of this subsection.

Theorem 7.5.7 (Rectifiability of limit varifold). *The limit varifold $V \in \mathbf{V}_{n-1}(U)$ satisfies $\|V\| = \mu$ and is rectifiable.*

Proof. Recall that $\|V_i\| = \mu_i \llcorner \{|\nabla u_i| \neq 0\}$ and since $h_0\mu_i \llcorner \{|\nabla u_i| = 0\} \leq |\xi_i|dv_g \llcorner \Omega \cap U$ which converges to 0 by Theorem 7.5.4, we have $\|V_i\| \rightarrow \mu$ and thus $\|V\| = \mu$. From (7.5.8), we know that V has locally bounded first variation in U . By Allard's Rectifiability Theorem [Sim83, Theorem 42.4], it remains to show that $\|V\| = \mu$ has a locally uniform density lower bound on $\text{spt } \|V\|$.

We claim that V has free boundary in $\partial\Omega \cap U$ as described in section 7.2.4. Recall $M = \partial^*\{u = 1\} \subset \Omega \cap U$. By the structure theorem for sets of finite perimeter (c.f. [EG15, Theorem 5.15], $\|\partial\{u = 1\}\| = \mathcal{H}^{n-1} \llcorner M$ which is a Radon measure on U by (7.3.3). Consider the Radon measures $\mu = \|V\|$ and $\sigma := \mathcal{H}^{n-1} \llcorner M$ on U , the density ratio lower bound (7.5.2) implies (c.f. [EG15, section 1.6]) that σ is absolutely continuous with respect to μ . Together with Lemma 7.5.5, the Radon–Nikodym Theorem implies that

$$\delta V(X) = -\frac{2\lambda}{h_0} \int_{\Omega \cap U} X \cdot \nu_M d\sigma = -\frac{2\lambda}{h_0} \int_{\Omega \cap U} (X \cdot \nu_M) D_\mu \sigma d\mu$$

for any $X \in \mathfrak{X}_c^{\text{tan}}(U)$. Therefore, V has free boundary in $\partial\Omega \cap U$ with $\|V\|$ -measurable vector field

$$H^T(x) := \begin{cases} -\frac{2\lambda}{h_0}(D_\mu \sigma)\nu_M & \text{for } x \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Note the the density lower bound in Proposition 7.5.2 implies that $H^T \in L^\infty$.

To finish the proof, note that $\|V\|$ has a locally uniform density lower bound on $\text{spt } \|V\| \cap \Omega \cap U$ by (7.5.2). As V has free boundary in $\partial\Omega \cap U$ with $H^T \in L^\infty$, the “reflected” density of V is upper semi-continuous (c.f. [GJ86, Corollary 3.2]) and hence $\|V\|$ has a locally uniform density lower bound up to the boundary $\partial\Omega$. This completes the proof. $\square$

We now give the proof of our main result, Theorem 7.3.3, modulo the integrality of the limit varifold V which will be presented in the next section.

Proof of Theorem 7.3.3. (1) follows directly from Theorem 7.5.4 and 7.5.7. (2) follows from Lemma 7.5.1, Proposition 7.5.2 and 7.5.3. The proof of (3) will be given in section 7.6. Finally, (4) follows from the proof of Theorem 7.5.7 and that $\Theta(x) = (D_\mu\sigma(x))^{-1}$. The rest then follows from [Ede20, Proposition 3.2]. $\square$

7.6 Integrality of the limit varifold

We will establish in this section that the limit varifold V is integral in U , i.e. $V \in \mathbf{IV}_{n-1}(U)$. Since V is rectifiable in U by Theorem 7.5.7, V has a unique approximate tangent space T at $\mathcal{H}^{n-1}$ -a.e. $x \in \text{spt } \|V\|$. In other words, for any such $x \in \text{spt } \|V\|$ and any sequence of positive real numbers $r_i \rightarrow 0$, we have

$$(\Phi_{r_i})_\# V \rightarrow \Theta(x)|T|$$

as varifolds³, where $(\Phi_{r_i})_\# V$ is the pushforward of the varifold V by the map $\Phi_{r_i} : (\widetilde{M}, g) \rightarrow (\widetilde{M}, r_i^{-2}g)$, $\Theta(x)$ is the $(n-1)$ -dimensional density of V at x and $|T|$ is the unit density varifold associated to the hyperplane $T \subset T_x \widetilde{M} \cong \mathbb{R}^n$.

³To be more precise, one can first locally isometrically embed $(\widetilde{M}, g)$ into some Euclidean space $\mathbb{R}^L$ and then do rescalings in $\mathbb{R}^L$ (c.f. [All72]). Alternatively, one can also do the rescalings intrinsically using the exponential map (c.f. [Sch22]). Both are equivalent and thus we will be using either formulation.

We shall assume $x \in \partial\Omega \cap U$, as the argument for interior points is similar and follows from a minor adaptation of [HT00] (at least in the Euclidean setting). Fix any sequence of positive $r_i \rightarrow 0$. We use $\tilde{g}_i := r_i^{-2}g$ to denote the rescaled metric on $\widetilde{M}$. Note that $(\widetilde{\Omega} \cap U, x, \tilde{g}_i) \rightarrow (\mathbb{R}_+^n, 0, g_E)$ smoothly in the sense of pointed Cheeger-Gromov convergence, where $\mathbb{R}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is the upper half-space equipped with the standard Euclidean metric g_E . Since $V_i \rightarrow V$, after passing to a subsequence, we can further assume as $i \rightarrow \infty$

$$(\Phi_{r_i})_{\#}V_i \rightarrow \Theta(x)|T| \quad \text{and} \quad \varepsilon_i r_i^{-1} \rightarrow 0.$$

Let $\tilde{V}_i = (\Phi_{r_i})_{\#}V_i$, $\tilde{\varepsilon}_i = \varepsilon_i r_i^{-1}$ and $\tilde{\lambda}_i = r_i \lambda_i$. Then u_i satisfies, with respect to the rescaled metric $\tilde{g}_i$,

$$-\tilde{\varepsilon}_i \Delta_{\tilde{g}_i} u_i + \frac{W'(u_i)}{\tilde{\varepsilon}_i} = \tilde{\lambda}_i \quad \text{in } \Omega \cap U, \quad (7.6.1)$$

$$\partial_\nu u_i = 0 \quad \text{on } \partial\Omega \cap U. \quad (7.6.2)$$

Note that we have $\|u_i\|_{L^\infty} \leq C_0$, $|\tilde{\lambda}_i| \leq r_i \Lambda_0 \rightarrow 0$. Moreover, by Proposition 7.4.15, for any fixed $R > 0$, we have for all sufficiently large i

$$(E_{\tilde{\varepsilon}_i} \llcorner (B_R(x), \tilde{g}_i))(u_i) = \omega_{n-1} R^{n-1} I_{\varepsilon_i, u_i}(r_i R, x) \leq \omega_{n-1} R^{n-1} c_9 (E_0 + 1)$$

for all sufficiently large i , so that all the results obtained in previous sections can be applied (with slight modifications for a convergent sequence of varying metrics), after observing that $\tilde{V}_i$ is the associated varifold of u_i with respect to the rescaled metric $\tilde{g}_i$ and $\tilde{V}_i \rightarrow \tilde{V} := \Theta(x)|T|$. Since $\tilde{\lambda}_i \rightarrow 0$, by Lemma 7.5.5, $\tilde{V}_i \rightarrow \tilde{V} := \Theta(x)|T|$ is a stationary varifold with free boundary in $\mathbb{R}_+^{n-1} := \partial\mathbb{R}_+^n$. As $\tilde{V}$ is supported in $\mathbb{R}_+^n$ and T is a hyperplane, we must have $T = \{x_n = 0\}$.

For simplicity, by a suitable choice of coordinates, we can assume without loss of generality that $\widetilde{\Omega} = \mathbb{R}_+^n$ is equipped with a metric $\tilde{g}_i$ converging locally smoothly to

the Euclidean metric g_E and $U = B_4(0)$ is the Euclidean ball of radius 4 centred at the origin. From now on, all balls and distances will be taken with respect to g_E . As in section 7.4, we denote $u_i, \tilde{\lambda}_i, \tilde{\varepsilon}_i, \tilde{g}_i$ by $u, \tilde{\lambda}, \tilde{\varepsilon}, \tilde{g}$. We use $T : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ to denote the orthogonal projection $T(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ with respect to g_E and $T^\perp : \mathbb{R}^n \rightarrow \mathbb{R}$ to denote the map $T^\perp(x_1, \dots, x_n) = x_n$. We also let $\nu = (\nu_1, \dots, \nu_n)$ be the unit normal to the level sets of u

$$\nu := \begin{cases} \frac{\nabla u}{|\nabla u|} & \text{when } |\nabla u| \neq 0, \\ 0 & \text{when } |\nabla u| = 0, \end{cases}$$

and

$$e_{\tilde{\varepsilon}} := \frac{\tilde{\varepsilon}|\nabla u|^2}{2} + \frac{W(u)}{\tilde{\varepsilon}}, \quad \xi_{\tilde{\varepsilon}} := \frac{\tilde{\varepsilon}|\nabla u|^2}{2} - \frac{W(u)}{\tilde{\varepsilon}}.$$

We first establish a proposition which shows that the smallness of the discrepancy and tilt excess imply the solution u is close in $\tilde{\varepsilon}$ -scale to the one-dimensional heteroclinic solution near the transition region which must be rather far away from the boundary.

Proposition 7.6.1 (Sheet separation). *For any given $s, b \in (0, 1)$, there exist constants*

$$\eta_1 = \eta_1(s, b, W) \in (0, 1),$$

$$L = L(s, b, W) > 1$$

such that the following holds: for any $0 < \eta < \eta_1$, suppose

(i) $y \in \mathbb{R}_+^n$ with $T(y) = 0$ and $|u(y)| \leq 1 - b$;

(ii) u satisfies (7.6.1) on $B_{4\tilde{\varepsilon}L}(y) \cap \mathbb{R}_+^n$ and (7.6.2) along $B_{4\tilde{\varepsilon}L}(y) \cap \partial\mathbb{R}_+^n$ with $\tilde{\varepsilon} \in (0, 1)$, $|\tilde{\lambda}| \leq \eta$, $\|\tilde{g} - g_E\|_{C^2(B_{4\tilde{\varepsilon}L}(y))} \leq \eta$ and $|\xi_{\tilde{\varepsilon}}| \leq \eta\tilde{\varepsilon}^{-1}$ on $B_{4\tilde{\varepsilon}L}(y) \cap \mathbb{R}_+^n$;

(iii)

$$\int_{B_{4\tilde{\varepsilon}L}(y) \cap \mathbb{R}_+^n} |\xi_{\tilde{\varepsilon}}| + (1 - (\nu_n)^2) \tilde{\varepsilon} |\nabla u|^2 \leq \eta (4\tilde{\varepsilon}L)^{n-1},$$

then $B_{3\tilde{\varepsilon}L}(y) \cap \partial\mathbb{R}_+^n = \emptyset$, $T^{-1}(0) \cap \{x \in B_{3\tilde{\varepsilon}L}(y) : u(x) = u(y)\} = \{y\}$ and

$$|I_{\tilde{\varepsilon},u}(\tilde{\varepsilon}L, y) - h_0| \leq s.$$

Proof. Exactly the same proof (the weaker hypothesis $|\xi_{\tilde{\varepsilon}}| \leq \eta \tilde{\varepsilon}^{-1}$ is sufficient as pointed out in [Ton02, Proposition 4.8]) of [HT00, Proposition 5.6] goes through using the boundary C^2 -estimates in [ADN59] and shows that u is in $\tilde{\varepsilon}$ -scale C^1 -close to the unique one-dimensional finite-energy heteroclinic solution $q(x_n)$ satisfying $q(|y|) = u(y)$. In particular, this implies $\frac{\partial u}{\partial x_n} \neq 0$ on $B_{3\tilde{\varepsilon}L}(y) \cap \mathbb{R}_+^n$ from which the assertion follows by (7.6.2). $\square$

The second proposition deals with decomposing the limit varifold V into stacked single-layered interfaces. Note that by Proposition 7.6.1, we only need to consider the case where the transition regions stay sufficiently away from the boundary and hence the interior arguments in [HT00, Ton02, Ton05] go through with trivial modifications.

Proposition 7.6.2 (Stacking argument). *For any given $R, \tilde{E}_0 > 0$, $s \in (0, 1)$, $N \in \mathbb{N}$, there exists a constant*

$$\eta_2 = \eta_2(R, C_0, \tilde{E}_0, s, N, n, W) > 0$$

such that the following holds: for any $0 < \eta < \eta_2$, suppose

- (i) $Y \subset \mathbb{R}_+^n$ has no more than $N + 1$ elements, $T(y) = 0$ for all $y \in Y$, $\text{diam } Y \leq \eta R$ and there exists $a > 0$ such that $|y - z| > 3a$ for all $y, z \in Y$; moreover, $U_{3a}(Y) \cap \partial\mathbb{R}_+^n = \emptyset$;

- (ii) u satisfies (7.6.1) on $U_R(Y) \cap \mathbb{R}_+^n$ and (7.6.2) along $U_R(Y) \cap \partial\mathbb{R}_+^n$ with $\|u\|_{L^\infty} \leq$

C_0 , $\tilde{\varepsilon} < \eta$, $\|\tilde{g} - g_E\|_{C^2(U_R(Y))} \leq \eta$ and $|\tilde{\lambda}| \leq \eta$; moreover, for each $y \in Y$, we have

$$\int_a^R \frac{d\tau}{\tau^n} \int_{B_\tau(y) \cap \mathbb{R}_+^n} \xi_\varepsilon^+ \leq \eta R;$$

(iii) for each $y \in Y$ and $r \in [a, R]$, we have

$$\int_{B_r(y) \cap \mathbb{R}_+^n} |\xi_\varepsilon| + (1 - (\nu_n)^2) \tilde{\varepsilon} |\nabla u|^2 \leq \eta r^{n-1},$$

$$\int_{B_r(y) \cap \mathbb{R}_+^n} \tilde{\varepsilon} |\nabla u|^2 \leq \tilde{E}_0 r^{n-1};$$

then we must have

$$\sum_{y \in Y} \frac{1}{\omega_{n-1} a^{n-1}} \int_{B_a(y)} e_{\tilde{\varepsilon}} \leq s + \frac{1+s}{\omega_{n-1} R^{n-1}} \int_{U_R(Y)} e_{\tilde{\varepsilon}}.$$

Proof. The proof of Proposition 7.6.2 requires a technical lemma similar to [HT00, Lemma 5.4] except that we have a weaker hypothesis on the upper bound of the discrepancy function (c.f. [Ton02, Proposition 4.7]). We then argue in exactly the same way by plugging in the suitably cut off radial vector field centred at a point in Y (c.f. the proof of [HT00, Lemma 5.4]), and control the error terms in the monotonicity formula as in Lemma 7.4.12 and Proposition 7.4.14. $\square$

The third proposition says that the energy is uniformly small in $\tilde{\varepsilon}$ outside the transition region (c.f. [HT00, Proposition 5.1] and [Ton05, Proposition 4.5]).

Proposition 7.6.3 (Uniform smallness of energy outside transition region). *Suppose that $\|\tilde{g} - g_E\|_{C^2(B_4(0) \cap \mathbb{R}_+^n)} \leq 1$ and $|\tilde{\lambda}| \leq 1$. Given $s > 0$, there exist positive constants*

$$\varepsilon_{12} = \varepsilon_{12}(s, C_0, n, W) > 0,$$

$$b = b(s, C_0, n, W, E_0) \in (0, 1)$$

such that for any $0 < \tilde{\varepsilon} < \epsilon_{12}$, we have

$$\int_{B_3(0) \cap \mathbb{R}_+^n \cap \{|u| \geq 1-b\}} \frac{W(u)}{\tilde{\varepsilon}} \leq s.$$

The proof of Proposition 7.6.3 requires two lemma as in [HT00]. Define

$$Z_\alpha := \left\{ x \in B_4(0) \cap \mathbb{R}_+^n : |u(x)| \leq \frac{1+\alpha}{2} \right\}.$$

Lemma 7.6.4. *Suppose that $\|\tilde{g} - g_E\|_{C^2(B_4(0) \cap \mathbb{R}_+^n)} \leq 1$ and $|\tilde{\lambda}| \leq 1$. There exist positive constants*

$$\varepsilon_{13} = \varepsilon_{13}(W) > 0,$$

$$c_{13} = c_{13}(n, W) > 0,$$

such that for any $0 < \tilde{\varepsilon} < \varepsilon_{13}$, $x \in B_3(0) \cap \mathbb{R}_+^n$ with $|u(x)| < 1 - \tilde{\varepsilon}^\beta$ for some β satisfying

$$\frac{1}{c_{13}|\log \tilde{\varepsilon}|} < \beta < \min \left\{ \frac{2}{3}, \frac{1}{c_{13}\tilde{\varepsilon}|\log \tilde{\varepsilon}|} \right\},$$

then we have

$$\text{dist}(x, Z_\alpha) \leq c_{13}\beta\tilde{\varepsilon}|\log \tilde{\varepsilon}|.$$

Proof. We argue exactly in the same way as in [HT00, Lemma 5.2], except that we have to suitably modify the function ψ . By a translation, we can assume without loss of generality $x = 0$, and denote $\tilde{\Omega} = \tilde{\varepsilon}^{-1}(\Omega - x)$. By the Fredholm alternative, for any $0 < R < \tilde{\varepsilon}^{-1}$, there exists a unique solution $\varphi = \varphi_R$ to the following mixed boundary value problem:

$$\begin{cases} \Delta \varphi = \frac{\kappa}{4} \varphi & \text{on } B_R(0) \cap \tilde{\Omega}, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{along } B_R(0) \cap \partial \tilde{\Omega}, \\ \varphi = 1 & \text{along } \partial B_R(0) \cap \tilde{\Omega}. \end{cases}$$

Note that φ_R is continuous on $\overline{B_R(0) \cap \tilde{\Omega}}$ up to the corner $\partial B_R(0) \cap \partial \tilde{\Omega}$. Moreover, $\varphi_R > 0$ by the strong maximum principle and unique continuation property (see e.g. [Lie13, section 1.8]). We can then define $\psi_R = (\varphi_R(0))^{-1} \varphi_R$ which satisfies

$$\begin{cases} \Delta \psi_R = \frac{\kappa}{4} \psi_R & \text{on } B_R(0) \cap \tilde{\Omega}, \\ \frac{\partial \psi_R}{\partial \nu} = 0 & \text{along } B_R(0) \cap \partial \tilde{\Omega}, \\ \psi_R(0) = 1. \end{cases}$$

Using the Harnack inequality and standard Schauder estimates [ADN59], there exists a constant $c_{13} > 0$ depending only on n and κ such that $\psi_R(x) > e^{|x|/c_{13}}$ for all $|x| \geq 1$. Using ψ_R in place of ψ , the rest of the proof follows from the arguments in [HT00, Lemma 5.2]. $\square$

Lemma 7.6.5. *Suppose that $\|\tilde{g} - g_E\|_{C^2(B_4(0) \cap \mathbb{R}_+^n)} \leq 1$ and $|\tilde{\lambda}| \leq 1$. There exist positive constants*

$$\varepsilon_{14} = \varepsilon_{14}(C_0, W, n) > 0,$$

$$c_{14} = c_{14}(C_0, W, n, E_0) > 0,$$

such that for any $0 < \tilde{\varepsilon} < \varepsilon_{14}$ and $\tilde{\varepsilon} < r < r_3$, we have

$$\mathcal{L}^n(\{x \in B_3(0) \cap \mathbb{R}_+^n : \text{dist}(x, Z_\alpha) < r\}) \leq c_{14}r.$$

Proof. The proof follows from the same covering arguments as in [HT00, Lemma 5.3], using Proposition 7.4.14 (with $b = \frac{1-\alpha}{2}$) and the arguments in the proof of Proposition 7.4.13. $\square$

Proof of Proposition 7.6.3. It follows exactly as in [HT00, Proposition 5.1] using Lemma 7.6.4 and 7.6.5. $\square$

Proof of Theorem 7.3.3 (continued). We now finish the proof of Theorem 7.3.3 by showing that $V \in \mathbf{IV}_{n-1}(U)$. We use the same notation as in the beginning of this

section with $x = 0 \in \partial\Omega \cap U$. Recall that

$$\tilde{V}_i := (\Phi_{r_i})_{\#} V_i \rightarrow \Theta_0 |T| \quad \text{and} \quad \tilde{\varepsilon}_i := \varepsilon_i r_i^{-1} \rightarrow 0. \quad (7.6.3)$$

Our goal here is to prove that $\Theta_0 \in \mathbb{N}$.

First of all, by Theorem 7.5.4 and (7.6.3), we have

$$\lim_{i \rightarrow \infty} \int_{B_4(0) \cap \mathbb{R}_+^n} |\xi_{\tilde{\varepsilon}_i}| = \lim_{i \rightarrow \infty} \int_{B_4(0) \cap \mathbb{R}_+^n} (1 - \nu_n^2) \tilde{\varepsilon}_i |\nabla u_i|^2 = 0. \quad (7.6.4)$$

Let $N \in \mathbb{N}$ be the smallest positive integer such that $N > \Theta_0$. Note that $\|\tilde{g}_i - g_E\|_{C^2(B_R(0) \cap \mathbb{R}_+^n)} \rightarrow 0$ and $|\tilde{\lambda}_i| \rightarrow 0$ as $i \rightarrow \infty$. Fix an arbitrary small $s > 0$. By Proposition 7.6.3 and (7.6.4), we can choose some $b > 0$ such that for all sufficiently large i , we have

$$\int_{B_3(0) \cap \mathbb{R}_+^n \cap \{|u_i| \geq 1-b\}} \left(\frac{\tilde{\varepsilon}_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\tilde{\varepsilon}_i} \right) \leq s.$$

With these choices of s and b , we choose $0 < \eta < \min\{\eta_1, \eta_2\}$ and $L > 1$ as in Proposition 7.6.1 and 7.6.2 (with $R = 1$ and $\tilde{E}_0 = c_9(E_0 + 1)$). Following [HT00], we define the good set G_i consisting of all $x \in B_2(0) \cap \mathbb{R}_+^n \cap \{|u_i| \leq 1 - b\}$ such that

$$\int_{B_r(x) \cap \mathbb{R}_+^n} |\xi_{\tilde{\varepsilon}_i}| + (1 - \nu_n^2) \tilde{\varepsilon}_i |\nabla u_i|^2 \leq \eta r^{n-1} \quad \text{for all } r \in [4\tilde{\varepsilon}_i L, 1].$$

By the Besicovitch covering theorem and (7.6.4), we have

$$\lim_{i \rightarrow \infty} \|\tilde{V}_i\|(B_2(0) \cap \mathbb{R}_+^n \cap \{|u_i| \leq 1 - b\} \setminus G_i) = 0$$

and

$$\lim_{i \rightarrow \infty} \mathcal{H}^{n-1}(T(B_2(0) \cap \mathbb{R}_+^n \cap \{|u_i| \leq 1 - b\} \setminus G_i)) = 0.$$

On the other hand, by (7.6.3) and Proposition 7.4.14, we obtain

$$\lim_{i \rightarrow \infty} \text{dist}(G_i, T) = 0. \quad (7.6.5)$$

Now, for any $x \in B_1(0) \cap \partial \mathbb{R}_+^n$ and $|t| \leq 1 - b$, we let

$$Y := T^{-1}(x) \cap G_i \cap \{u_i = t\} \quad \text{and} \quad a := L\tilde{\varepsilon}_i.$$

We would like to apply Proposition 7.6.2 to this Y for all sufficiently large i . Note that $\|\tilde{g}_i - g_E\| \rightarrow 0$ and $\tilde{\lambda}_i \rightarrow 0$ as $i \rightarrow \infty$. By Proposition 7.6.1 and (7.6.5), all assumptions in (i) of Proposition 7.6.2 are satisfied (except perhaps the cardinality of Y) for large i . Noting how the constants in section 7.4 scale with respect to the blow-up radii $r_i \rightarrow 0$ (especially Proposition 7.4.10 and 7.4.13), (ii) is also satisfied for large i . Condition (iii) is essentially the definition of G_i , together with Proposition 7.4.15. Note that Y cannot contain more than $N - 1$ elements since otherwise we will obtain by Proposition 7.6.1 and 7.6.2 that

$$N \leq (N + 1)s + (1 + s)(\Theta_0 + s)$$

which is a contradiction for s sufficiently small (depending only on N). The rest of the argument is the same as [HT00, Theorem 1], and we must have $\Theta_0 = N - 1 \in \mathbb{N}$. $\square$

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