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On the Manipulability of Tournament Rules by Throwing Matches

Sathyanarayanan, Sacheth

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On the Manipulability of Tournament Rules by Throwing Matches

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Master of Science in Engineering

Recommended for Acceptance
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Abstract

A Tournament Rule takes as input a set of $\binom{n}{2}$ pairwise match results on n teams and (possibly randomly) selects a ranking of teams, with a prize associated to each place in the ranking. Two teams may manipulate their match outcomes to try and improve their expected prize winnings. [10] and subsequent works seek to understand the minimum manipulability among all "reasonable" tournament rules (i.e. tournament rules so that if team x defeats every other team, it is ranked first with probability 1), when these two teams fix the match between them.

Our work instead considers the possibility that two teams may both fix the match between them, and additionally throw matches to outside teams (that is, they can intentionally lose a match to a non-colluding team that they could have won). Our main result establishes that no two teams can gain more than 1/3 in expected prize winnings in Nested Randomized King of the Hill (introduced in [4], similar to QuickSort), by together fixing their match and throwing matches to external teams. This is optimal, as any "reasonable" tournament admits the possibility of two teams gaining 1/3 in expected prize-winnings just by fixing the match between them.

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1 Introduction

In this paper, we examine the topic of manipulation in tournaments. A (round-robin) tournament comprises of a set of n teams that play all possible pairwise matches among each other. Organizers of the tournament use a *tournament rule* to rank the teams from 1 through n based on how good they are; we also assume that they award prizes to the teams as a function of their rank. We're most interested in minimally-manipulable tournament rules that satisfy a fairness condition called Condorcet consistence: if there exists a team that defeats all other teams (such a team is called a Condorcet winner), then this team must be ranked first with probability 1.

Depending on the construction of the tournament rule, participants may have an incentive to manipulate their matches. Such manipulation is undesirable for a variety of reasons — it is a breach of sportsmanship and violates the principle of fair play, it undermines the integrity of the tournament and the sport as a whole, and it demonstrates a lack of respect for the other competitors and the audience. Tournament rules that are not resistant to manipulation create an environment where colluders operate with an undeserved sense of security, ultimately compromising the legitimacy of the tournament. Though most forms of real-world manipulation in sports involve some incentive outside the scope of the tournament (i.e., betting or bribery), there have been several examples of teams manipulating matches so as to increase their chance of winning the tournament.

Prior works define manipulation, within this context, as a set S of colluding teams fixing the results of matches among themselves (with the goal of increasing their total expected prize), and bound how much a set of teams can gain in expected prize by such manipulation. We initiate the study of a more general notion of manipulation: we allow teams within S to deliberately lose matches to external teams that they would have otherwise won (called throwing), in addition to potentially fixing their own matches. This new definition of manipulation is more representative of the broader set of incentives at play and better captures real-world occurrences. For instance, NBA teams often intentionally throw matches with the goal of winning high draft picks, a process known as "tanking." In early 2023, the Dallas Mavericks were fined $\$750,000$ ([9]) for tanking. Another example of such manipulation is in the world of Badminton — eight players from China, South Korea and Indonesia were disqualified for intentionally throwing matches with the goal of facing easier opponents later ([1]) on during the 2012 London Olympics.

1.1 Prior Work

There has been an extensive study of tournament rules within the TCS community ([3], [2], [13], [12], [8], [7], [10], [11], [6], [5], [4]). In particular, there has been an effort to find minimally-manipulable tournament rules that satisfy the Condorcet Consistence fairness criteria.

All prior works define manipulation as a set S of colluding teams fixing the results of matches among themselves (with the goal of increasing their total expected prize). Most prior work also only deals with tournament rules that select a winner and not a ranking of all teams. The authors of [10], [11] and [6] demonstrated a variety of tournament rules and an associated b such that no two teams could fix the result of their match and increase the probability of one of them winning the tournament by more than b — a tournament rule that satisfies this property is said to be $2-$SNM-b. For instance, the Nested Randomized King of the Hill (similar to QuickSort, described below), Randomized Single Elimination Bracket (randomly sort the teams and run a single elimination bracket) and Randomized Death Match (repeatedly pick two remaining teams at random and eliminate the loser) tournament rules are all $2-$SNM-$1/3$ (this bound is tight and optimal over all Condorcet consistent tournament rules). There also exist a variety of other tournament rules that are $2-$SNM-$1/2$ or $2-$SNM-1.

Significantly less is known for the case where more than 2 teams collude. [10] showed that for any Condorcet consistent tournament rule, k teams can fix the results of matches among them and increase the probability of one of them winning by at least $(k-1)/(2k-1)$ i.e., every Condorcet consistent tournament rule is at least $k-$SNM$-(k-1)/(2k-1)$. [11] proved that no Condorcet consistent tournament rule is $k-$SNM$-1/2$ for all k and demonstrated a tournament rule that is $k-$SNM-$2/3$ for all k. In the specific case where $k=3$, [5] showed that the Randomized Death Match is $3-$SNM$-31/60$, and that the bound is tight.

The study of tournament rules that output a ranking instead of just a winner was initiated by [4], who extended the definition of manipulation to allow teams to fix the result of their match to increase their total expected winnings (assume that there exists a prize associated with each rank) and not just the probability of one of them being ranked first. They also show that the under the Nested Randomized King of the Hill (NRKotH) tournament rule, no two teams can fix the result of their match and increase their total expected prize by more than $1/3$.

1.2 Our Results

We initiate the study of manipulation by fixing as well as throwing matches. Our focus is on the Nested Randomized King of the Hill (NRKotH) tournament rule, which can be described as follows: (a) pick a team

u (called the pivot) uniformly at random from the set of all teams, (b) fix u's ranking to be below all teams that defeat u and above all teams that lose to u, and (c) recurse independently on the set of all teams that defeat u and on the set of all teams that lose to u.

Our main result bounds how much two teams can gain in expected prize by throwing matches to external teams and potentially fixing the result of their match. To be more specific:

Main Result (see Theorem 5.1) For any prize vector p, no two teams u and v can throw matches to external teams and potentially fix the result of their match to gain more than $1/3$ in expected prize under the NRKotH tournament rule. This bound is tight and optimal over all Condorcet consistent tournament rules.

We note that in addition to satisfying Condorcet consistence, NRKotH also satisfies two significantly stronger fairness criteria ([4]): "Cover consistence" and "Consistence under Expectation."

Within a tournament, a team u is said to "cover" another team v if u defeats v and every team that v defeats. A tournament rule is said to be cover consistent if u is ranked lower (better) than v whenever u covers v. A tournament rule is said to be consistent under expectation if its expected rank, over the randomness of the tournament rule, is 1 plus the number of teams it loses to.

2 Preliminaries

2.1 Setup

We introduce notation consistent with prior works ([2], [10], [11], [6], [4]) and introduce new terminology to reflect updated definitions of manipulation.

Definition 2.1 (Tournament)**.** *A (round robin) tournament T on n teams is a complete, directed graph on n vertices whose edges denote the outcome of a match between two teams. In particular, an edge from vertex x to vertex y means that team x defeats team y in T.*

Definition 2.2 (Tournament Rule)**.** *A tournament rule r is a function that maps tournaments T to a distribution over rankings σ (where $\sigma(x)$ denotes the ranking of x), where $r_\sigma(T) := \Pr[r(T) = \sigma]$ denotes the probability that ranking σ is output on tournament T under rule r. We use the notation $r_{x,j}(T) := \sum_{\sigma|\sigma(x)=j} r_\sigma(T)$ to denote the probability that team x is ranked j^{th}.*

Definition 2.3 (Prize Vector)**.** *A prize vector is a vector $\vec{p} \in \mathbb{R}^n$ such that $p(j) \geq p(j+1)$ for all j. The semantic meaning is that the team ranked j^{th} receives $p(j)$ in prize money. A top$-k$ prize vector is a prize vector such that first k elements are 1 while the last $n-k$ elements are 0.*

Like prior work, we are interested in tournament rules which satisfy basic notions of fairness. Prior work mostly considers rules which are *Condorcet-Consistent*: whenever a team is undefeated, that team wins (in our language, is ranked first) with probability one.

Definition 2.4 (Condorcet-Consistent)**.** *Team i is a Condorcet winner of a tournament T if i beats every other team. A tournament ranking rule r is Condorcet-consistent if for every tournament T with a Condorcet winner i, $r_{i,1}(T) = 1$ (whenever T has a Condorcet winner, that team is ranked first with probability 1).*

Finally, we are interested in how manipulable tournament ranking rules are. Throughout the next few definitions, we let S be a subset of the teams in the tournament and let $W_T(u)$ be the set of teams that u defeats in T.

Definition 2.5 (Match fixing)**.** *Two two teams x and y are said to fix their match in tournaments T and T' if x defeats y in T but y defeats x in T'.*

Definition 2.6 ($S-$adjacent tournaments)**.** *For any set S of colluding teams, two tournaments T and T' are said to be $S-$adjacent if two teams in S can potentially fix the result of their match and teams in S can potentially throw (deliberately lose) matches that they would have otherwise won. More formally, T and T' are $S-$adjacent if they are equivalent, except possibly for the results of matches between x and y where either (a) $x, y \in S$ or (b) $x \in S$ and $y \in W_T(x)$. Note that, trivially, T and T are $S-$adjacent for all S and all T.*

Definition 2.7 (Manipulating a Tournament)**.** *For a set S of colluding teams, tournament T, tournament rule r, and prize vector p, we define $\alpha_S^p(r, T)$ to be the maximum prize money that S can possibly gain in expectation by manipulating T to an $S-$adjacent T'. We define $\alpha_k^p(r)$ to be the maximum of $\alpha_S^{\vec{p}}(r, T)$ over all tournaments and all sets S satisfying $|S| = k$ and $\alpha_k(r) = \sup_p \alpha_k^{\vec{p}}(r)$ to be the supremum of $\alpha_k^{\vec{p}}(r)$ over all prize vectors in $[0,1]^n$. Finally, we define $\alpha_k = \inf_{Condorcet\ consistent\ r} \alpha_k^r$ to be the lowest bound on manipulation achievable over all Condorcet consistent tournament rules.*

Now, we introduce the tournament rule that we study — Nested Randomized King of the Hill. This tournament ranking rule is similar to the QuickSort sorting algorithm. We also present a different interpretation of this rule in Section 2.3.

Definition 2.8 (Nested Randomized King of the Hill)**.** *The tournament ranking rule Nested Randomized King of the Hill (NRKotH) proceeds as follows, when given as input a tournament T on n teams:*

1. *If $n = 0$, return an empty ordering. Else, continue.*
2. *Pick a team, u, uniformly at random. Call u the pivot.*
3. *Let B denote the teams that beat u, and L denote the teams that lose to u.*
4. *Run NRKotH on B and L, and call the outputs σ_B and σ_L respectively.*
5. *For all teams $b \in B$, set $\sigma(b) := \sigma_B(b)$.*
6. *Set $\sigma(u) := |B| + 1$.*
7. *For all teams $\ell \in L$, set $\sigma(\ell) := \sigma_L(\ell) + |B| + 1$.*
8. *Output σ.*

That is, NRKotH picks a uniformly random team, u. All teams that beat u are ranked above u, and all teams that lose to u are ranked below u. With each set of teams, NRKotH is called recursively to determine their relative ranking.

2.2 Theme and Notation

In this paper, we consider a set S of two colluding teams. We define T to be the "normal" tournament, where no manipulating occurs and T' to be manipulated tournament. In general, we use the $'$ (prime) notation to refer to properties of T' instead if T. In our main result, we assume that T and T' are $S-$adjacent, but in certain other results, we assume that T and T' satisfy some different condition.

We also define u and v be the two colluding teams (i.e., $S = \{u, v\}$) and assume that u defeats v in T. Some of our results (for example Lemmas 5.17 and Lemma 5.22 among others) apply to both u and v regardless of the result of the match between u and v. In such situations, and in any situation where we want to make a claim about both u and v, we refer to one of u and v as x and the other as y (so $S = \{u, v\} = \{x, y\}$).

We now introduce standardized notation to refer to any team based on the result of its match against u and v in T and T'. We also introduce specific labels for certain classes of teams.

Definition 2.9 (Notation for teams)**.** *Let S_1 and S_2 represent two subsets of $S = \{u, v\}$, the set of colluding teams. We define $X_{S_1}^{S_2}$ to be the set of all teams (apart from u and v) that (a) defeat teams in S_1 and lose to teams in $S \backslash S_1$ in T and (b) defeat teams in S_2 and lose to teams in $S \backslash S_2$ in T'. To be more formal,*

$$X_{S_1}^{S_2} = \{x \notin S | x \succ_T y_1, x \prec_T y_2, x \succ_{T'} y_3, x \prec_{T'} y_4 \text{ for all } y_1 \in S_1, y_2 \in S \backslash S_1, y_3 \in S_2, y_4 \in S \backslash S_2\},$$

where $a \succ_{\mathcal{T}} b$ means that team a defeats team b in tournament $\mathcal{T}$. For example, $X_u^{u,v}$ represents the set of teams that defeat u and lose to v in T but defeat both u and v in T'. Note that since u and v throw matches in T' that they would have won in T, we have that $S_1 \subseteq S_2$. This is because it is impossible for a team to defeat u or v in T but not in T'. The set of all teams that only u throws its match to, apart from (potentially) v is $X_u \cup X_{u,v}^v$; similarly, the set of all team that only v throws its match to is $X_v \cup X_{u,v}^u$. Finally, the set of all teams that both u and v throw their matches to is X_{uv}.

Definition 2.10 (Separator)**.** *Let T be a tournament on a set of teams including x, y and w. We see that w "separates" x and y if either (a) w defeats x and loses to y in T or (b) w defeats y and loses to x in T.*

Definition 2.11 (Strong Separator)**.** *We define a strong separator to be any team that either defeats u and loses to v in both T and T' or defeats v and loses to u in both T and T'. Note that a team is a strong separator if and only if it is in $X_u^u \cup X_v^v$.*

Definition 2.12 (Nearly Strong Separator)**.** *Let x be one of u and v and let y be the other. A nearly strong separator is a team that loses to x and defeats y in T and defeats both x and y in T', but x is a Condorcet loser in T'. Put differently, a nearly strong separator is a team ion X_y^{xy}, where x is a Condorcet loser in T'.*

2.3 An Alternate View of NRKotH

In this section we provide an alternate interpretation of NRKotH that will aid in our analysis. Before we do so, we define the notions of projections and groups:

Definition 2.13 (Projection of a tournament)**.** *Let T be a tournament defined on a set U of teams and let U' be a subset of U. We define the projection of T onto U' to be that tournament $\mathcal{T}$ defined on U' where t_1 defeats t_2 in $\mathcal{T}$ if and only if t_1 defeats t_2 in T for all teams $t_1, t_2 \in U'$.*

Definition 2.14 (Groups)**.** *A group is a set of teams on which we recursively call NRKotH. Initially, all teams are in the same group. During a recursive call to NRKotH where we select team p as the pivot, we replace the group containing all teams in this recursive call with (a) the group containing all teams that*

defeat the pivot, (b) the group containing just the pivot, and (c) the group containing all teams that lose to the pivot.

For instance, consider a tournament with five teams $t_1, t_2, \ldots, t_5$ where t_i defeats t_j when $i > j$. Initally, there is just one group $\{t_1, t_2, t_3, t_4, t_5\}$. Upon calling NRKotH on this group, if we select t_3 as the pivot, the groups become $\{t_1, t_2\}, \{t_3\}, \{t_4, t_5\}$. Instead, if we select t_5 as the pivot, the groups become $\{t_1, t_2, t_3, t_4\}, \{t_5\}$.

Note that $G(u)$ changes over time, as different pivots are selected. For instance, in the example above, initially $G(t_i) = \{t_1, t_2, t_3, t_4, t_5\}$ for all i. After t_3 is selected as the pivot, $G(t_1) = G(t_2) = \{t_1, t_2\}$, $G(t_3) = \{t_3\}$ and $G(t_4) = G(t_5) = \{t_4, t_5\}$.

Definition 2.15 ($\gamma-$ordered implementation of NRKotH)**.** *The $\gamma-$ordered implementation of NRKotH takes as input a permutation γ of all teams, which we refer to as the pivot order. It proceeds as follows on a tournament T on n teams: for $i = 1 : n$, process the projection of T onto the group $G(\gamma(i))$ with pivot $\gamma(i)$.*

The $\gamma-$ordered implementation of NRKotH has the effect of pre-deciding what the pivots are during each call to NRKotH. Since the order in which groups are processed is irrelevant, we have:

Remark 2.1. *If we select γ uniformly at random from all possible pivot orders, the ranking produced by NRKotH and NRKotH$_\gamma$ are identically distributed.*

Using the $\gamma-$ordered implementation, we can think of NRKotH as proceeding in n different rounds — during each round, we select team $\gamma(i)$ as the pivot within its group.

Definition 2.16 (Current group)**.** *We define $G_i(u)$ to be the group containing team u right after running round i of NRKotH. Note that $G_0(u)$ is the set of all teams in the tournament and $G_n(u) = \{u\}$ for all teams u.*

Definition 2.17 (Relevant matches)**.** *We define the relevant matches of u to be the set of all matches between u and $\gamma(i)$ if $G_{i-1}(u) = G_{i-1}(\gamma(i))$ for all $i = 1 : n$. In other words, the relevant matches of u is the set of all matches between u and a pivot in its group.*

The relevant matches are the only matches whose result we take into account while constructing the ranking — the results of all other matches are irrelevant. Note that the notion of the "number of all relevant matches" is equivalent to the notion of the "number of comparisons" in the QuickSort sorting algorithm.

Each of the groups can be totally ordered under $>$ such that for any two teams u and v, during any round if $G_i(u) > G_i(v)$ then u's rank will be lower (i.e. better) than v's rank. We can define $>$ by creating an

ordered list of the groups. Initially, all teams are in the same group. During every recursive call, we remove the group containing all teams in this recursive call from the ordered list with the groups in Definition 2.14 in exactly that order — that is, the group defined in (a) occurs before the group defined in (b), which occurs before the group defined in (c). If group G_1 occurs before G_2 in the ordered list, then $G_1 > G_2$.

Definition 2.18 (Highest and Lowest Groups)**.** *At any stage, the highest group G_h is defined to be that group such that $G_h > G$ for all other groups G. Similarly, the lowest group G_ℓ is defined to be that group such that $G > G_\ell$ for all other groups G. Note that G_h is the group containing all teams that have won all relevant matches so fat while G_ℓ is the group containing all teams that have lost all relevant matches so far.*

Since $G_n(u) = \{u\}$, the total order on the groups gives us the required ranking. For all analysis in this paper, we adopt the $\gamma-$ordered implementation of NRKotH, where we select a uniformly random pivot order γ. In some cases, we also use the principle of deferred decisions to delay fixing values of γ.

Definition 2.19 (Generating function)**.** *Consider a tournament T defined on $X \cup Y \cup \{t\}$ where all teams in X defeat t and all teams in Y lose to t. We can think of generating the pivot order γ as first fixing the order γ' of all teams in $X \cup Y$ and then inserting t at some position, say r_t. We now define the generating function $f_{\gamma'}(r_t)$ to be the rank assigned to team t if we run NRKotH on T with pivot order γ. Note that the prize awarded to t is now $p(f_{\gamma'}(r_t))$.*

Definition 2.20 (Relative Ordering)**.** *Consider some tournament T defined on a set of $n \geq 2$ teams, including t_1 and t_2, and let γ be some ordering of the teams in T. Consider running NRKotH on T and T', where we select the pivots in the order γ. We define the function $RO_T(t_1, t_2, \gamma)$ to be 1 if t_1 is ranked better than t_2 and 0 otherwise.*

Remark 2.2. *Under NRKotH, the probability that team t_1 is ranked better than team t_2 in a tournament T on $n \geq 2$ teams is*

$$\sum_{\gamma \in \Gamma} \frac{1}{n!} RO_T(t_1, t_2, \gamma),$$

where Γ is the set of all $n!$ orderings of all the teams. This follows directly from the fact that NRKotH uniformly selects the ordering of its pivots from all possible orderings.

2.4 Properties of NRKotH

Consider running NRKotH with pivot order γ on a tournament T with n teams. As we proceed through rounds of NRKotH, two teams x and y stay in the same group until the first i where either $\gamma(i) \in \{x, y\}$ or

$G_{i-1}(\gamma(i)) = G_{i-1}(x) = G_{i-1}(y)$ and $\gamma(i)$ separates x and y. Once this happens, $G_i(x) \neq G_i(y)$. In other words, x and y stay in the same group until either one of them is selected as the pivot or some team in the same group as them that separates them is selected as the pivot.

Definition 2.21 (Partitioner)**.** *For any two teams x and y, we define the partitioner of x and y to be the pivot that causes x and y to be in different pivots.*

Observation 2.1. *If $\gamma(i)$ is the partitioner of x and y, none of the pivots in rounds before i are x or y.*

Observation 2.2. *Consider a tournament T defined on $X \cup Y \cup \{t\}$ where all teams in X defeat t and all teams in Y lose to t. If t is selected as a pivot before all teams in Y, then its rank is at least $|X| + 1$. Similarly, if t is selected as a pivot before all teams in X, its rank is at most $|X| + 1$.*

Proof. We first prove that if t is selected as a pivot before all teams in Y, its rank is at least $|X| + 1$. Note that every pivot before t is in X. This means until t is selected as the pivot, it will also be in the lowest group (the group containing all teams that have lost all relevant matches so far). Thus, by the time t is selected as the pivot, all teams in X are either in the same group as u or in a higher group. When u is selected as the pivot, all teams in X that are in the same group will also be ranked lower than u. It follows that all teams in X will have a lower rank than t and the result follows. Very similarly, we can show that if t is selected as a pivot before all teams in X, all teams in Y will be ranked higher than it. □

We now state and prove an important property of NRKotH, called Monotonicity (introduced in [3]).

Lemma 2.1 (Monotonicity)**.** *If some team x throws matches, then, for any pivot order γ, its rank cannot decrease.*

Proof. Let t be some team that is initially ranked lower than x when running NRKotH with pivot order γ on T. It suffices to show that t is ranked lower than x when running NRKotH with pivot order γ on T' as well. We compare the outputs of the $\gamma-$ordered implementation of NRKotH when run on T and T'. As we proceed through the rounds of running NRKotH on T, x and t are in the same group until their partitioner — call this $\gamma(i)$ — is selected as the pivot. At this point, t moves to a higher group than x (since it is ranked lower than x).

Now consider iterating through γ until before we reach $\gamma(i)$ and analyzing the result of running NRKotH on T'. If the current pivot is in a different group than x and t, we move on to the next pivot — all the groups look exactly as they did in T. If the current pivot is in the same group as x and t but x does not throw to its match to this team, we move onto the next pivot — all the groups look exactly as they did in T. The only

case we are left with is where the current pivot loses to x in T but defeats x in T'. This causes x to move to a lower group in T' than in T while t stays in the same group (since the results of its match against the first pivot stays the same). At this point, we conclude the analysis and note that here x is ranked worse than t.

Thus, just before $\gamma(i)$ is selected as the pivot, either x is already in a lower group than t and we are done (this happens if x threw none of its relevant matches so far), or they are in the same group and all groups look exactly like they did in T. Now when $\gamma(i)$ is selected as the pivot, since x was pushed to a lower group than t in T, it must be pushed to a lower group than t in T' as well. This concludes our proof. □

Lemma 2.2 (Consider only top$-k$ prize vectors)**.** *Assume that for any two tournaments T and T', the maximum increase in the expected prize to $S = \{u, v\}$ by manipulating T to T' is at most $b \geq 0$ for all top$-k$ prize vectors. Then, the maximum increase in the expected prize to $S = \{u, v\}$ by manipulating T to T' is at most b for all prize vectors.*

Proof. We first note that the increase in prize by manipulating T to T' is a linear function of the prize vectors. To see why, consider some prize vector p. Assume that the probability of team t being ranked i^{th} in T is $r_t(i)$ and in T' is $r'_t(i)$. Then, the expected increase in prize to S is given by

$$\sum_{i=1}^{n} r'_u(i)p(i) + \sum_{i=1}^{n} r'_v(i)p(i) - \sum_{i=1}^{n} r_u(i)p(i) - \sum_{i=1}^{n} r_v(i)p(i) = \sum_{i=1}^{n} p(i)\left(r'_u(i) + r'_v(i) - r_u(i) - r_v(i)\right)$$

This is a linear combination of the entries of the prize vector since none of the r_t and r'_t depend on the prize vector. Let p_i denote the top$-i$ prize vector. For example, $p_1 = \langle 1, 0, 0, \ldots, 0\rangle$ and $p_2 = \langle 1, 1, 0, \ldots, 0\rangle$. Note that (almost) every prize vector can be express as a convex combination of the p_i vectors:

$$p = \sum_{i=1}^{n-1} \left(p(i) - p(i+1)\right) p_i + p(n)p_n.$$

The sum of the coefficients here is $p(1)$ and so, in the event that $p(1) = 1$, the value of function (that we are trying to maximize) on p can be expressed as a convex combination of the value of the function on the p_k. It follows that it suffices to maximize the function on the p_k. For the $p(1) < 1$ case, it suffices to show that if the value of the function at $p_c = \langle c, 0, 0 \ldots, 0\rangle$ is at most the value of the function at p_1 for any constant $c \leq 1$. To see why this is true, note that the value of the function at p_c is given by

$$c\left(r'_u(1) + r'_v(1) - r_u(1) - r_u(1)\right) < r'_u(1) + r'_v(1) - r_u(1) - r_u(1) \leq b,$$

when there is non-negative gain (if the gain is negative, the quantity on the left is at most $0 < b$ and we are done). Thus, the value of the function at p_c is at most p_c. We can express any vector p as a convex combination of the $p_{p(1)}, p_2, \ldots, p_n$ just like we did above to show that the value of the function at p is at most b. □

Since it suffices to consider only top$-k$ prize vectors, henceforth, whenever we use the term prize vector, we mean a top$-k$ prize vector $\langle 1, \ldots 1, 0, \ldots, 0\rangle$ for some k.

3 Why is this problem interesting?

At least at first glance, it is reasonable to expect that u throwing matches to external cannot help by much — it certainly cannot help itself. Moreover, perhaps it is the case that whenever u throws matches to external teams, S could gain more by just having u throw its match to v instead. Whenever u throws some matches, it does so to benefit v at the cost of hurting itself; at this point, it seems reasonable to expect that $\{u, v\}$ should not gain when v throws any matches — after all, doing so could nullify that help it received from u.

It turns out, however, that none of these are true. We provide examples of tournaments where (a) S can gain up to 1/3 by u simply throwing matches to external teams, (b) S cannot gain anything by u throwing its match to v, but can gain something by u throwing match to external teams, and (c) S gains something by just u throwing an external match, but it gains even more by both u and v throwing external matches simultaneously.

For all these examples, we assume that only the winner matters. That is, the prize vector is $\langle 1, 0, 0, \ldots\rangle$. Note that these examples can be trivially extended to a top$-k$ prize vector by adding in $k-1$ Condorcet winners (the results of the matches among these Condorcet winner is irrelevant).

(a) Here, we demonstrate an example where by u just throwing matches, $\{u, v\}$ can gain a lot (arbitrarily close to 1/3) in expected prize. Let the teams in the tournament (see Figure 1) be u, v, w and $\{y_i\}_{i=1}^n$. Assume that u, v and w form a cycle (i.e. u defeats v, v defeats w and w defeats u) and that all of u, v and w defeat all teams y_i. In this tournament, one of u, v or w will be the winner with probability 1/3 and so the expected prize to $\{u, v\}$ is 2/3. Now assume that u throws all its matches to the y_i. In the resulting tournament, one of u or v wins if the first pivot is any team apart from u. This happens with probability $(n+1)/n$ and so the expected prize to $\{u, v\}$ becomes $n/(n+1)$. This corresponds to a gain of 1/3 as $n \to \infty$. In this tournament, S could have gained the 1/3 just by u throwing its match

to v, for all n (this makes v a Condorcet winner). The next example shows that this is not always possible.

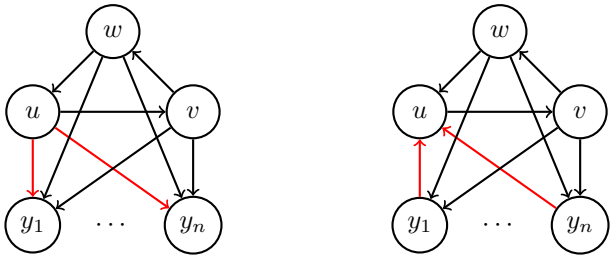


Figure 1: Left: $\{u, v\}$ gets expected prize 2/3. Right: u throws to all the y_i; $\{u, v\}$ gets expected prize $(n + 1)/n$.

(b) Here, we demonstrate an example where $\{u, v\}$ cannot gain anything by u and v fixing their match but can gain something by u throwing one match to an external team. Consider a tournament (see Figure 2) with teams u, v, x, y, and w. Assume that x defeats both u and v, y loses to both u and v, y defeats x, u defeats v and w is a Condorcet loser. In this tournament, for u or v to win, y must be selected as the pivot before u, v or x (w is a Condorcet loser and is irrelevant). Since this happens with probability 1/4, the expected prize awarded to S is 1/4. If u and v fix their match, the expected prize to S is still 1/4 (note that u and v are copies), which corresponds to a gain of 0. On the other hand, if u throws its match to w, the expected prize to $\{u, v\}$ is 4/15 (they get prize 1 if y is selected as the first pivot and 1/3 if w is selected as the first pivot). This corresponds to a gain of 1/60. Thus, $\{u, v\}$ gains nothing when u and v fix the result of their match, but they gain 1/60 when u throws its match to an external team. This shows that there are situations in which it is better to throw a match to an external team instead of fixing a match with a fellow colluder.

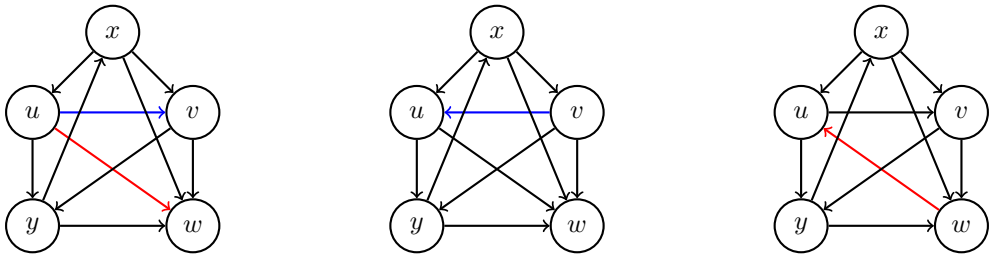


Figure 2: Left: $\{u, v\}$ gets expected prize 1/4. Middle: u and v fix their match; $\{u, v\}$ gets expected prize 1/4. Right: u throws to w; $\{u, v\}$ gets expected prize $4/15 > 1/4$.

(c) In this example, we have that $\{u, v\}$ increases their total expected prize by a little bit when u throws one match and by even more when v also throws one match (in addition to the match u throws).

Consider a tournament (see Figure 3) with teams u, v, x, y, w_u and w_v. Assume that x defeats u and v; y loses to u, v and x; u defeats v; w_v loses to u, v and y and defeats w_u and x; and w_u is a Condorcet loser. The expected prize to $\{u, v\}$ in this tournament is 1/5. If u throws its match to w_u, the expected prize increases by 1/120 to 5/24. If u throws its match to w_u and v throws its match to w_v, the expected prize increases by 1/45 (from the original) to 2/9.

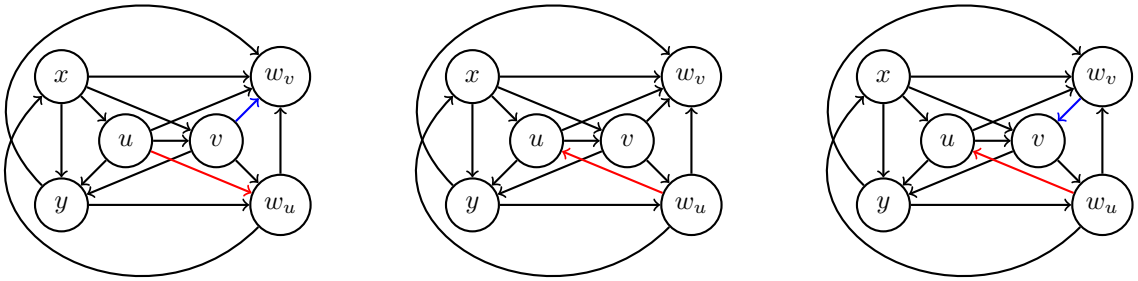


Figure 3: Left: $\{u, v\}$ gets expected prize 2/3. Middle: u throws to w_u; $\{u, v\}$ gets expected prize $5/24 > 1/5$. Right: u throws to w_u and v throws to w_v; $\{u, v\}$gets expected prize $2/9 > 5/24$..

4 Recursive Analysis

In order to analyze the outcome of NRKotH on a tournament, we employ a form of recursive analysis on the pivot order. In this section, we outline the form of such an analysis.

In the most general setting, we want to compare the outcome of running NRKotH on two tournaments T and T' defined on a set U of n teams that satisfy some boolean criteria $\mathcal{C}(T, T')$ (for instance, in our main result, we specify the criteria that T and T' are $S-$adjacent tournaments). More specifically, we want to bound the expected prize that S can gain by manipulating T to T'. To do so, we consider the $\gamma-$ordered implementation of NRKotH, where γ is a uniformly randomly chosen pivot order.

We partition the set of all teams into two disjoint sets: $\mathfrak{R}$, "recursive" teams and $\mathfrak{T}$, "terminating" teams. Intuitively, a team t is a "recursive" team if after selecting t as the pivot, we are left with two smaller versions of the same tournaments. More formally, $p \in \mathfrak{T}$ if there exists two tournaments $\mathcal{T}$ and $\mathcal{T}'$ that are defined on a proper subset of U such that $C(\mathcal{T}, \mathcal{T}')$ is satisfied and after selecting t as the pivot, the expected increase in prize to S from T to T' is at most the expected increase in prize to S from $\mathcal{T}$ to $\mathcal{T}'$. A "terminating" team is a team t such that when t is selected as a pivot, we already have a bound on the expected increase

in prize to S; we use the notation $b(t)$ to denote this bound.

We conduct the analysis as follows for any fixed pivot order γ: we start by considering T and T' and iterate through all pivots in the order $\gamma(1), \gamma(2), \ldots, \gamma(n)$. Assume that we are currently processing $\gamma(i)$. If $\gamma(i)$ is not a team in the tournaments we are considering, we ignore it and move on to $\gamma(i+1)$. If $\gamma(i) \in \mathfrak{R}$, we consider $\mathcal{T}$ and $\mathcal{T}'$ and move on to $\gamma(i+1)$. If $\gamma(i) \in \mathcal{T}$, we terminate our analysis and claim that that the upper bound for γ is the bound we already have for the terminating team $\gamma(i)$.

Note that the recursive teams do not matter in determining the maximum expected gain. We continually shrink the set of teams under consideration until we pick one team from $\mathfrak{T}$. Since γ is picked uniformly at random, each team in $\mathfrak{T}$ is equally likely to be the first one picked. If we assume that $\mathfrak{T} = \{t_1, t_2, \ldots, t_k\}$ then this tells us the expected gain to S from T to T' is at most $\sum_{i=i}^{k} b(t_i)/k$.

We now use this recursive analysis to re-prove a result from [4]: show that u and v cannot fix the result of their match and gain more than $1/3$ prize in expectation.

Theorem 4.1 (u and v fixing their match)**.** *Assume that T and T' are equivalent with the exception that u defeats v in T and loses to v in T'. Then, $\{u, v\}$ cannot gain more than $1/3$ prize in expectation by manipulating T to T' for any prize vector p.*

Proof. The main design challenge in this proof is deciding which teams to select as terminating teams and which teams to select as recursive teams. Obvious choices for terminating teams are v and u. Let's see why these work.

First consider v. When v is selected as the first pivot, its rank goes up by 1 since it now defeats u. However, u moves down from the group of teams that defeat v to the group of teams that lose to v. Intuitively, any increase in v's prize is offset by a decrease in u's prize. Let's be more formal with this. Assume that there are n_v teams (including u) that defeat v in T. Then, v's prize in T is $p(n_v + 1)$ and its prize in T' is $p(n_v)$. Since u defeats v in T, its prize is at least $p(n_v)$, and since it loses to v in T', its prize is at most $p(n_v + 1)$. Thus, the total prize to $\{u, v\}$ in T is at least $p(n_v + 1) + p(n_v)$ while the total prize to $\{u, v\}$ in T' is at most $p(n_v) + p(n_v + 1)$. Thus, the maximum increase from T to T' is at most 0, which tells us that v is $0-$terminating.

Now consider u. When u is selected as the first pivot, its rank goes up by 1, which could correspond to a decrease in its prize. However, v moves up from the group of teams that lose to u into the group of teams that defeat v, which could correspond to a prize increase of 1. Thus, in total, the prize to $\{u, v\}$ could increase by 1 (since u does not gain), which means that u is $1-$terminating. In the most general case, we cannot hope to do better since it is possible that v gets prize 0 in T, but it defeats all teams that lose to u,

and emerges a Condorcet winner in T'.

Another class of teams that are terminating are those that separate u and v in T. If such a team is selected, then u and v are immediately put in different groups and the result of their match does not matter. Thus, the prize awarded to $\{u, v\}$ in T and T' is exactly the same, which means that such teams are $0-$terminating.

The only teams left to consider are those that defeat/lose to both u and v. We claim that all such teams are recursive. Let's see why. First consider some team x that defeats both u and v and let L be the set of teams that lose to u and v in T. Similarly, let L be the set of teams that lose to x in T'. Since the results of all of x's matches are the same in T and T', we have that $L = L'$. By the definition of NRKotH, the expected prize awarded to $\{u, v\}$ in T is equal to the expected prize awarded to $\{u, v\}$ in $\mathcal{T}$, the projection of T onto L with prize vector $p_{-|L|}$, the last $|L|$ entries of p. Similarly, the expected prize awarded to $\{u, v\}$ in T' is equal to the expected prize awarded to S in $\mathcal{T}'$, the projection of T' onto L with prize vector $p_{-|L|}$. This increase is bounded by the increase from $\mathcal{T}$ to $\mathcal{T}'$, over all prize vectors (if it's there's a bound for all prize vectors, there's a bound for a specific vectors). We now note that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent with the exception that u defeats v in $\mathcal{T}$ but loses to v in $\mathcal{T}'$. Since we have reduced the problem to a smaller version of the same problem, x is recursive. Very similarly, we can show that any x that loses to both u and v is recursive by considering the projection of T/T' onto B/B', the set of teams that defeat x in T/T'.

In summary, we have that v is $0-$terminating, u is $1-$terminating, separators are $0-$terminating and all other teams are recursive. What happens if there are no separators? We have one $0-$terminating team and one $1-$terminating team, which gives us a bound of $1/2$ — not quite good enough! It turns out that if there are no separators, $\{u, v\}$ cannot gain anything in the first place. We omit a formal proof (presented in [4, Lemma 3.3]) but we note that intuitively this follows from the fact that NRKotH is anonymous: since there are no separators, u and v are copies and so the prize awarded to them is independent of the result of their match.

Henceforth, it suffices to consider the case where there is at least one separator, which means that we have at least two $0-$terminating teams and one $1-$terminating team while the rest are all recursive. This gives us the required bound of $1/3$. □

5 Main Result — u and v throw and potentially fix

In this section, we prove our main result. As noted before, we let u and v simultaneously throw matches to external teams in addition to potentially fixing the result of their own match, and we assume, without loss

of generality, that u defeats v in T.

Theorem 5.1. *In any tournament T, no two teams can throw matches to external teams, potentially fix the result of their own match and increase their total expected prize by more than $1/3$ under the NRKotH tournament rule for any prize vector all of whose elements are in $[0, 1]$. This bound is optimal over all Condorcet consistent tournament rules. More precisely, $\alpha_2(NRKotH) = \alpha_2 = 1/3$.*

Proof of Theorem 5.1:

We first note that the bound is tight since the $1/3$ bound is also tight in the event that u and v only fix the result of their match (from [4]). Additionally Example (b) in Section 3 shows that $\{u, v\}$ can gain arbitrarily close to $1/3$ by having u just throw matches. We also note that this bound is optimal over all Condorcet consistent tournament rules because the $1/3$ bound is also tight when u and v just fix the result of their match (from [4]).

The proof of this theorem is structurally similar to the proof of Theorem 4.1. Our goal is to find a set of terminating teams and recursive teams such that the bounds of the terminating teams average out to $1/3$. Let's assume that there are k_1 teams to which only u throws its match, k_2 teams to which only v throws its match, k_3 teams to which both u and v both throw their match, and k teams in total to whom a match is thrown. That is,

$$\begin{aligned} k_1 &= |X^u| + |X^{uv}_v|, \\ k_2 &= |X^v| + |X^{uv}_u|, \\ k_3 &= |X^{uv}|, \\ k &= k_1 + k_2 + k_3. \end{aligned}$$

As in the case where the colluders can only fix the result of their match, u and v are terminating teams. The bound for u is still, in general, 1 (Lemma 5.2) but we can do slightly better in some cases. In particular, if u and v do not fix their match (Lemma 5.6) or if they do fix their match but there are no strong separators (Lemma 5.4), we can prove a bound of $k_1/(k_1 + 1)$. This comes from the fact that there is no gain if v is selected as a pivot before any of the teams only u throws its match to.

The bound for v also depends on whether or not u and v fix the result of their match — if they do, v's bound is 0 (Lemma 5.3); otherwise, it's $k_2/(k_2 + 1)$ (Lemma 5.7). In addition, all strong separators (teams in X^v_v or X^u_u) are $0-$terminating (Lemma 5.9) and all teams in X^{uv}_{uv} and X are recursive (Lemma 5.11).

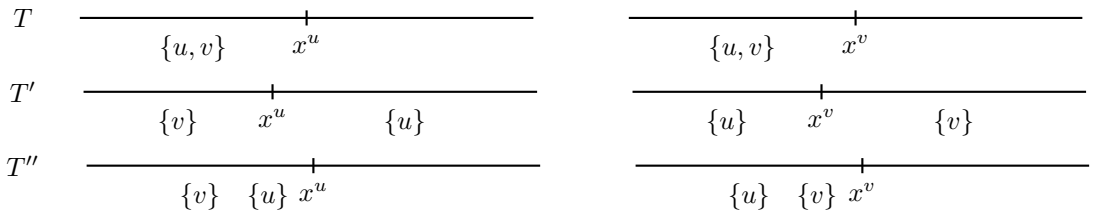


Figure 4: Left: selecting a team in $x^u \in X^u$ as the first pivot in T (top), the given tournament; T' (middle), the tournament where u throws its match to x^u; and T'' (bottom), the tournament where u defeats x^u but throws its match to all teams that defeat x^u. Right: selecting a team in $x^v \in X^v$ as the first pivot in T (top), the given tournament; T' (middle), the tournament where v throws its match to x^v; and T'' (bottom), the tournament where v defeats x^v but throws its match to all teams that defeat x^v.

Things get much more interesting when we look at teams to whom a match is thrown. First consider some team $x^u \in X^u$ to whom u throws its match. Recall that x^u loses its match to both u and v in T and defeats only u in T'. We can pictorially (see Figure 4) represent what happens when x^u is selected as the first pivot.

In Figure 4, we use a line to represent the ranking: teams to the left have a lower (better) rank than teams to the right. We use a tick mark to represent the current pivot. Teams to the left of the pivot are the teams that defeat it, while teams to the right of the pivot are the teams that lose to it. In T, both u and v defeat x^u and are positioned to its left. In T', since u throws its match to x^u, it moves to the right of x^u while v stays on to the left. We now define a new tournament T'' that is equivalent to T' with the exception that u defeats x^u, but loses to all teams that defeat x^u.

For any pivot order, the expected prize awarded to $\{u, v\}$ in T' is at most the expected prize awarded to $\{u, v\}$ in T''. To see why, note that v's rank stays exactly the same for any pivot order in T' and T''. Since u is a Condorcet loser among the teams that defeat x^u (by definition, it loses to all these teams in T'') inserting it into v's group will not affect v's rank; u will simply be ranked last among this group. On the other hand, we are decreasing u's rank — and potentially increasing its prize — by having it defeat x^u. Thus, once x^u is selected as the first pivot, we can bound the expected increase in prize to $\{u, v\}$ from T to T' by the expected increase in prize to S from T to T''.

Let B and B'' represent the set of teams that defeat x^u in T and T'' respectively. Note that $B = B'' = B' \cup \{u\}$. The expected prize awarded to $\{u, v\}$ in T and T'' equals the expected prize awarded to $\{u, v\}$ in $\mathcal{T}$ and $\mathcal{T}''$, the projections of T and T'' onto B and B'' respectively. However, note that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent with the exception that u throws all its matches and v potentially throws some matches. Thus, the problem of finding a bound when x^u is selected as the first pivot reduces to the problem of finding the

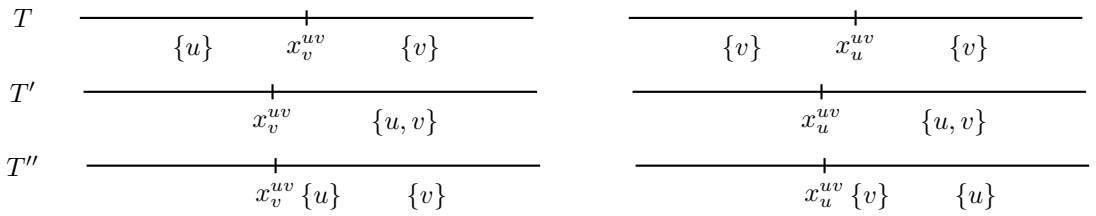


Figure 5: Left: selecting a team in $x_v^{uv} \in X_v^{uv}$ as the first pivot in T (top), the given tournament; T' (middle), the tournament where u throws its match to x_u^{uv}; and T'' (bottom), the tournament where u loses to x_u^{uv} but defeats all teams that lose to x_u^{uv}. Right: selecting a team in $x_u^{uv} \in X_u^{uv}$ as the first pivot in T (top), the given tournament; T' (middle), the tournament where v throws its match to x_u^{uv}; and T'' (bottom), the tournament where v loses to x_u^{uv} but defeats all teams that lose to x_u^{uv}.

maximum gain from $\mathcal{T}$ to $\mathcal{T}'$ when u throws all its matches and v throws some matches. This value is at most $1/4$ if v throws its match in T' to all teams that defeat u and lose to v in T, or no such teams exist (Lemma 5.17); otherwise, the bound is $1/3$ (Lemma 5.19).

Very similarly, we have that the problem of finding a bound when some team $x^v \in X^v$ is picked as the first pivot reduces to the problem of finding the maximum gain from $\mathcal{T}$ to $\mathcal{T}'$ when v throws all its matches and u throws some matches. This value is at most $1/4$ if u throws its match in T' to all teams that defeat v and lose to u in T, or no such teams exist (Lemma 5.17); otherwise, the bound is $11/40$ (Lemma 5.18). Note that we get an improvement over the previous case ($11/40$ instead of $1/3$), which comes from the fact that v loses to u in T.

We now consider what happens when $x_v^{uv} \in X_v^{uv}$ is selected as the first pivot (see Figure 5). Recall that x_v^{uv} defeats only v in T but defeats both u and v in T'. Define T' to be the tournament equivalent to T with the exception that u loses to x_v^{uv} but defeats every team that loses to it in T. For any pivot order, the prize awarded to $\{u, v\}$ in T'' is at most the expected prize awarded to $\{u, v\}$ in T. To see why, note that in going from T to T'', v's rank stays exactly the same for any pivot order in T' and T''. Inserting u, a Condorcet winner, into the group containing v does not affect v's rank since u is definitely going to be ranked first among the group of teams that defeat x_v^{uv}. On the other hand, we are increasing u's rank — and potentially decreasing its prize — by having it defeat x_v^{uv}. Thus, once x_v^{uv} is selected as the first pivot, we can bound the expected increase in prize to $\{u, v\}$ from T to T' by the expected increase in prize to S from T'' to T'.

Define L'' and L' to be the set of teams that lose to x_v^{uv} in T'' and T' respectively. By the definition of NRKotH, the expected prize awarded to $\{u, v\}$ in T'' and T' equals the expected prize awarded to $\{u, v\}$ in $\mathcal{T}''$ and $\mathcal{T}'$, the projections of T'' and T' onto L'' and L' respectively. The expected increase from T to T' is thus at most the expected increase from $\mathcal{T}''$ to $\mathcal{T}'$. We now note that $\mathcal{T}''$ and $\mathcal{T}'$ are equivalent

with the exception that u is a Condorcet winner in $\mathcal{T}''$, and that u and v throw some matches. It follows that bounding the increase from T to T' when x_v^{uv} is selected as the first pivot reduces to the problem of bounding the increase when u is a Condorcet winner and u and v throw some matches; the result of the match between u and v in $\mathcal{T}'$ is the same as the result of their match in T' — they may or may fix their match. Very similarly, we can show that bounding the increase when x_u^{uv} is selected as the first pivot reduces to the problem of bounding the increase when v is a Condorcet winner and u and v throw some matches. Again note that u and v may or may not fix the result of their match in T'. This is exactly the same as the case as when x_u^{uv} is selected as the first pivot since the result of the match between u and v is irrelevant (we rewrite T to make either u or v a Condorcet winner). What does make a difference, however, is the result of the match between u and v in T' (i.e., whether or not they fix their match).

Let α and β be the winner and loser of the match between u and v in T' respectively. Then, the bound when $x_\alpha^{\alpha\beta}$ is selected as the first pivot is $1/4$ if α defeats, in T, every team that defeats α and loses to β in T', or no such teams exist (Lemma 5.22); otherwise, the bound is $1/3$ (Lemma 5.24). The bound when $x_\beta^{\alpha\beta}$ is selected as the first pivot is $1/4$ if β defeats, in T, every team that defeats β and loses to α in T', or no such team exists (Lemma 5.22); otherwise, the bound is $11/40$ (Lemma 5.23). If the bounds we get selecting x_u^{uv} and x_v^{uv} look curiously similar to the bounds we get when x^u and x^v are selected as the first pivot, it is because these problems are, in some sense, equivalent — the Bridge Lemma (Lemma 5.21) establishes that the "Condorcet winner throws some" problem reduces to the "colluder throws all" problem.

The last class of teams to consider are those that both u and v throw their match to i.e. teams $x^{uv} \in X^{uv}$ — all such teams are $0-$terminating (Lemma 5.8).

We now have everything we need to prove our bound. We first note that strong separators are double-edged swords. On one hand, they provide an extra 0 in our analysis (since they are $0-$terminating); on the other hand, they defeat one of u/v and lose to the other in both T and T', which means that our bounds for picking teams in $X^u \cup X^v \cup X_v^{uv} \cup X_u^{uv}$ as the first pivot need to be $1/3$ or $11/40$ instead of $1/4$.

In particular, if $|X_v^v| = 0$, then teams in X_v^{uv} and X^v are $1/4-$terminating. This is because every team that defeats v and loses to u in T' must now be in X^v (since $|X_v^v| = 0$), but v defeats all such teams in T; this gives us the $1/4$. For X^v, every team that defeats v and loses to u in T must be in X_v^{uv} (since $|X_v^v| = 0$), but u loses to such teams in T'; this gives us the $1/4$ bound. Very similarly, if $|X_u^u| = 0$, we have that all teams in X_u^{uv} and X^u are $1/4-$terminating. Our bound for u is also 1 when there are strongly separating teams.

We now break our analysis up into two cases: Case 1, where u and v do not fix their match and Case 2,

where they do.

Case 1: u and v fix their match If there are no strong separators, from the reasoning above, every team in $X_u \cup X_v \cup X_v^{uv} \cup X_u^{uv}$ is $1/4-$terminating. Here's a quick summary of the bounds of the other terminating teams: u is $k_1/(k_1+1)-$terminating, v is $0-$terminating, all teams in X^{uv} are $0-$terminating, and all teams in $X_{uv}^{uv} \cup X$ are recursive. The bound now becomes

$$\frac{1}{k_1+k_2+k_3+2} \cdot \left(\frac{k_1}{k_1+1} + \frac{k_1+k_2}{4} \right) \leq \frac{1}{k+2} \cdot \left(\frac{k}{k+1} + \frac{k}{4} \right)$$

since the $x/(x+1)$ function is increasing over the non-negative integers. This function is always less than $1/3$ (Lemma A.1 in the Appendix) and the result follows.

If strong separators do exist, the bound for teams in $X_u \cup X_v \cup X_v^{uv} \cup X_u^{uv}$ is $1/3$ (it is $11/40$ for some of these teams, but this does not really matter). The bound for u is also 1 instead of $k_1/(k_1+1)$, while the bound for v remains 0. The strong separator also guarantees us at least 1 extra 0. As before, the bounds for all teams in X^{uv} is 0 and all teams in $X_{uv}^{uv} \cup X$ are recursive.

In total, we have one $1-$terminating team, at least two $0-$terminating teams and some number of recursive teams. This immediately gives us the $1/3$ bound.

Case $2: u$ and v do not fix their match Here, the bounds for u and v are $k_1/(k_1+1)$ and $k_2/(k_2+1)$. This by itself is not enough to get a $1/3$ bound since we have two nearly $1-$terminating teams and we at least need four $0-$terminating teams to offset this. This is not necessarily possible when there are fewer than 4 strong separators.

In order to address this, we examine whether or not u and v can have non-zero bounds simultaneously. It turns out that this is not always possible — in certain cases, there can be non-zero gain only when at most one of u or v is selected as the first pivot. To see why, let us understand what needs to happen for there to be some gain when u is the first pivot. Let $b(u)$ and $b(v)$ represent the gain when u and v are selected as the first pivot respectively. When u is the first pivot, since it throws some matches, its rank increases and its prize in T' is at most as much as its prize in T. Moreover, since u and v do not fix the result of their match, v is ranked worse than u in both T and T'. If u gets prize 0 in T, then both u and v get prize 0 in both T and T', which means $b(u) = 0$. This means that u must get prize 1 in T. If u gets prize 0 in T' then v also gets prize 0 in T'. which means $b(u) \leq 0$. This means that u must get prize 1 in T'. Note that u's rank in T' is $R_1 = |X_{uv}^{uv}| + |X_u^{uv}| + |X_v^{uv}| + |X^{uv}| + |X_u^u| + |X^u| + 1$, and so the prize corresponding to all ranks up

to (and including) this must be 1. If only the top R_1 teams get prize 1, then v gets prize 0 in T' when u is selected as the first pivot (since it gets ranked after u), which means that $b(u) \leq 0$. Thus, at least the top $R_1 + 1$ teams must get prize 1.

Now consider what happens when v is selected as the first pivot. Note that v's rank in T is $R_2 = |X_{uv}^{uv}| + |X_v^{uv}| + |X_v^v| + 2$, since u defeats v. Also, u's rank must be strictly less than this. Since the top R_1 teams definitely get prize 1, if $R_2 \leq R_1 + 1$, then u and v both get prize 1 when v is selected as the first pivot; this means that we must have $b(v) \leq 0$ since it is impossible for them to get more than prize 1 each in T'. Let's assume that $R_2 = R_1 + 2$. If the prize corresponding to position $R_2 = R_1 + 2$ is 0, then v gets prize 0 when it is selected as the pivot in T and T' (since its rank gets higher in T'), which implies that $b(v) \leq 0$. It follows that either one of $b(u)$ or $b(v)$ is at most 0 or $R_2 \geq R_1 + 3$.

First consider the case where at least one of $b(u)$ or $b(v)$ is at most 0. Without loss of generality, assume that $b(v) = 0$; we now have that $b(u) = k_1/(k_1 + 1)$. We can employ the same proof technique as in Case 1. In particular, if there are no strong separators, we have that all teams in $X^u \cup X^v \cup X_u^{uv} \cup X_v^{uv}$ are $1/4$ terminating, which gives us a bound of

$$\frac{1}{k_1 + k_2 + k_3 + 2} \cdot \left(\frac{k_1}{k_1 + 1} + \frac{k_1 + k_2}{4} \right) \leq \frac{1}{3}.$$

If there are strong separators, we have at least two 0s (from the strong separator and v) and all teams in $X^u \cup X^v \cup X_u^{uv} \cup X_v^{uv}$ are $1/3$ terminating. This gives us the $1/3$ bound.

Now, we assume that $R_2 \geq R_1 + 3$. Plugging in the values for R_1 and R_2, we get that

$$|X_v^v| \geq |X_u^u| + |X_u^{uv}| + |X^u| + |X^{uv}| + 2$$

If $|X_u^u| \geq 1$ then $|X_v^v| \geq 3$. Thus, there are at least 4 strong separators, all of which are $0-$terminating. All teams in $X^u \cup X^v \cup X_u^{uv} \cup X_v^{uv}$ are $1/3-$terminating, all teams in X^{uv} are $0-$terminating and all teams in $X_{uv}^{uv} \cup X$ are recursive. This gives a bound of $2/6 = 1/3$.

Henceforth, we can assume that $|X_u^u| = 0$, which means that all teams in $X^u \cup X_u^{uv}$ are $1/4-$terminating.. Since there exists teams in X_v^v, we need to use the $11/40$ instead of $1/4$ bound for teams in $X^v \cup X_v^{uv}$ (since we assume that v loses to u in T', we can use $11/40$ instead of $1/3$). Since $|X_v^v| \geq 2$, we have at least 2 strongly separating teams that are $0-$terminating. As before, we also have that all teams in X^{uv} are $0-$terminating and that all teams in $X_{uv}^{uv} \cup X$ are recursive. Since u and v are $k_1/(k_1 + 1)-$terminating and

$k_2/(k_2+1)-$terminating respectively, we have that our total bound (in the worst case) is at most

$$\frac{1}{k_1+k_2+k_3+4}\cdot\left(\frac{k_1}{k_1+1}+\frac{k_2}{k_2+1}+\frac{11k_1}{40}+\frac{11k_2}{40}\right)\leq\frac{1}{k_1+k_2+4}\cdot\left(\frac{k_1}{k_1+1}+\frac{k_2}{k_2+1}+\frac{11}{40}(k_1+k_2)\right)$$

This worst case happens when $|X^u|=|X^{uv}_u|=|X^{uv}|=0$ (which is why we only have 11/40 instead of 1/4 in our equation) and $R_2=R_1+3$. The value of this function is always less than 1/3 (Lemma A.2 in the Appendix), which concludes our proof.

5.1 Terminating Teams

In this section, we establish that u, v, all teams that both u and v throw to, and all strong separators are terminating.

5.1.1 Teams u and v are terminating

Lemma 5.2 (u is $1-$terminating when potentially fix)**.** *If u and v both throw matches to external teams and potentially fix their match, then u is $1-$terminating.*

Proof. Since u wins at most as many matches in T' as it does in T, when it is selected as the first pivot, its rank does not decrease, which means its prize does not increase. In order for $\{u,v\}$ to gain more than 1 in prize, v must gain more than 1. This is impossible, however, since each entry of the prize vector is between 0 and 1. □

Lemma 5.3 (v is $0-$terminating when fix)**.** *If u and v both throw matches to external teams and fix their match, then v is $0-$terminating.*

Proof. Consider picking v as the first pivot. Since v throws matches to external teams, it's rank does not decrease. Since u initially defeats v and later loses to v, its rank also does not decrease. Since the ranks of both u and v do not decrease, the prize to $\{u,v\}$ does not increase and it follows that v is $0-$terminating. □

Lemma 5.4 (u is $k_1/(k_1+1)-$terminating when fix + no strong separators)**.** *Assume that u and v both throw matches to external teams and fix their match. If there are no strong separators, u is $k_1/(k_1+1)-$terminating, where $k_1=|X^u\cup X^{uv}_v|$ is the number of teams to which only u throws its match.*

Proof. In T, since there are no strong separators, the only teams that defeat u are all teams in X^{uv}_{uv} and X^{uv}_u. In T', the teams that defeat u are all teams in X^{uv}_{uv}, X^{uv}_u, X^{uv}_v, X^u, X^{uv} and $\{v\}$. Of the teams that

defeat u in T', the only teams that lose to v are those in X^u. From Observation 2.2, if v is selected as a pivot before all teams in X^u, then its rank is at least $|X^{uv}_{uv}| + |X^{uv}_u| + |X^{uv}_v| + |X^{uv}| + 1$, which is at least u's rank in T. It follows that v's prize in T' is at most u's prize in T.

Very similarly, the teams that lose to u in T' are all teams in X and X^v while the teams that lose to u in T are all teams in X, X^u, X^v, X^{uv}, X^{uv}_v and $\{v\}$. Of the teams that lose to u in T, the only teams that defeat v in T are those in X^{uv}_v. From Observation 2.2, if v is selected as a pivot before all teams in X^{uv}_v, it will be ranked better than all teams that lose to it, or that u's prize in T' is at most v's prize in T.

It follows that if v is selected as a pivot before all teams in $X^u \cup X^{uv}_v$, the expected increase in prize to $\{u, v\}$ from T to T' will be 0. Since this happens with probability $1/(k_1 + 1)$, $\{u, v\}$ can gain only with probability $k_1/(k_1 + 1)$. Furthermore, this gain can be at most 1 since u does not gain anything (its rank does not decrease and so its prize does not increase). It follows that u is $k_1/(k_1 + 1)-$terminating. □

Before moving on to the case where u and v do not fix their match, we introduce a helpful lemma. This lemma applies to both u and v, regardless of the result of their match. To capture this, let x be one of u and v and let y be the other.

Lemma 5.5 (y before all w where x throws and does not fix)**.** *Assume that x throws matches to external teams but does not fix its match against y and that y does not throw any matches to external teams. Let W be the set of all teams that x throws its match to. Then, for any pivot order γ where y occurs before all teams in W, y's rank is higher in T' than in T.*

Proof. Let t be some team that is initially ranked lower than y when running NRKotH with pivot order γ on T. It suffices to show that t is ranked lower than y when running NRKotH with pivot order γ on T' as well. We compare the outputs of the $\gamma-$ordered implementation of NRKotH when run on T and T'. Let $\gamma(i)$ be the partitioner of y and t when running NRKotH on T. From Observation 2.1, none of the pivots before round i are y; since all teams in W occur after v in the pivot order, they cannot be in W either.

Since none of the pivots from $\gamma(1)$ to $\gamma(i-1)$ are in W, the results of all matches between these pivots and any team that is not in W stay the same (this is because one of these pivots could be x). In other words, the relevant matches — as well as their results — that do not involve x or teams in W stay the same in T and T'. The consequence of this is that after round j (for all $1 \leq j \leq i - i$), the groups after running NRKotH on T and T' are equal with the possible exception that teams in W move to a higher group. Note that if x is not selected as a pivot before $\gamma(i)$, all relevant matches will stay the same and all groups will be exactly the same. If it is selected, however, its ranking could go down, but the group it is in will stay the

same (it will be the singleton group containing itself).

If $t \notin W$, from the argument above, y and t are in the same group right before $\gamma(i)$ is selected as the pivot. If $t \in W$, either they are in the same group or t was pushed up to a higher group, in which case we are done since t is ranked lower than y in T'. Henceforth, assume that y and t are in the same group. Let this group be $G' := G'_{i-1}(v) = G'_{i-1}(t)$. As we showed above, this group is equal to $G := G_{i-1}(v) = G_{i-1}(t)$, the group containing y and t right before $\gamma(i)$ is selected as the pivot in T, modulo containing teams in W. In other words $G' \cup W = G \cup W$.

Thus, $\gamma(i)$ (which is not in W) is in G'. If $\gamma(i) \in \{y, t\}$ then y is still ranked worse than t since the result of the match between y and t stays the same. Otherwise, we must have that $\gamma(i)$ defeats y and loses to t. The results of these matches but also stay the same in T' since $\gamma(i) \notin W$. This is because the results of the matches between y and t or between $\gamma(i)$ and y and $\gamma(i)$ and t stays the same in T and T'. Since the relative ordering of y and t is decided by what happens when $\gamma(i)$ is selected as the pivot, and the results of the relevant matches stay the same, the relative ordering of y and t is the same in T and T'. □

Lemma 5.6 (u is $k_1/(k_1+1)-$terminating when no fix)**.** *Assume that u and v both throw matches to external teams and do not fix their match. Then, u is $k_1/(k_1+1)-$terminating, where $k_1 = |X^u \cup X^{uv}_v|$ is the number of teams to which only u throws its match.*

Proof. Consider picking u as the first pivot. Since u throws some matches, its rank cannot decrease and so its prize cannot increase. The only way to increase the prize to $\{u, v\}$ is to increase the prize to v. Now, v is in a group without u and from Lemma 2.1, it is disincentivized from throwing any matches. In other words, once u is selected as the first pivot, the gain when v throws matches is at most the gain when v does not throw any matches. Thus, it suffices to bound the total increase when u is picked as the first pivot and v does not throw any matches.

From Lemma 5.5, v can gain only if it does not occur before all teams in W, which happens with probability $k_1/(k_1 + 1)$. Furthermore, if there is any gain, it is at most 1 (u cannot gain, v can gain at most 1) It follows that u is $k_1/(k_1 + 1)-$terminating. □

Lemma 5.7 (v is $k_2/(k_2+1)-$terminating when no fix)**.** *Assume that u and v both throw matches to external teams and do not fix their match. Then, v is $k_2/(k_2+1)-$terminating, where $k_2 = |X^v \cup X^{uv}_u|$ is the number of teams to which only v throws its match.*

Proof. The proof of this is identical to the proof of Lemma 5.6 □

5.1.2 Teams that both u and v throw to are terminating

Lemma 5.8 (Teams in X^{uv} are 0−terminating)**.** *If u and v both throw matches to external teams and potentially fix the result of their match, all teams in X^{uv} are 0−terminating.*

Proof. Consider some team $x^{uv} \in X^{uv}$ and assume that exactly n_x teams defeat x^{uv} in T. The the prize awarded to S in T is at least $p(n_x - 1) + p(n_x)$ since both u and v defeat x^{uv} while the prize awarded to S in T' is at most $p(n_x) + p(n_x + 1)$, since both u and v lose to x^{uv}. This corresponds to an increase of at most 0, which concludes our proof. □

5.1.3 (Nearly) Strong Separators are terminating

Lemma 5.9 (Strong Separators are 0−terminating)**.** *If u and v both throw matches to external teams and potentially fix the result of their match, all Strong Separaotors are 0−terminating.*

Proof. Consider picking some Strongly Separating team t as the first pivot. Assume, without loss of generality, that t defeats u and loses to v in both T and T'. We define B (resp. B') and L (resp. L') to be the set of teams that defeat and lose to t in T (resp. T') respectively. Since none of t's matches changes from T to T', we have $B = B'$ and $L = L'$.

Once t is selected as the first pivot, u and v are separated into two different groups. From Lemma 2.1, both u and v are now disincentivized from throwing matches. In other words, once t is selected as the first pivot, the maximum gain from T to T' when u and v both throw matches is at most the maximum gain when none of u and v throw any matches. However, if none of u and v throw any matches then $T = T'$ (even if u and v do fix their match, this match is irrelevant), which means that the gain is 0. □

This lemma also extends to Nearly Strong Separators. This isn't directly used in the proof of the main result but becomes relevant for future results.

Lemma 5.10 (Nearly Strong Separators are 0−terminating)**.** *If u and v both throw matches to external teams and potentially fix the result of their match, all Nearly Strong Separators are 0−terminating.*

Proof. Let t be a Nearly Strong Separator. We know that t defeats one of u and v in T (call this team y) and loses to the other (call this team x). Furthermore, t defeats both x and y in T' and x is a Condorcet loser in T'. Let us fix some pivot order. Assume that t gets rank r_t in T. Since x defeats t in T, its rank is at most $r_t - 1$ in T. Since it is the Condorcet loser in T', its rank in T' is n.

The group containing y in T and T' is the same with the exception that we insert x, a Condorcet loser. This has the effect of decreasing y's rank in T' by 1, compared to its rank in T. Thus, if y has rank r_y in T, it has rank $r_y - 1$ in T'. The total prize awarded to S in T is at least $p(r_t - 1) + p(r_y)$ while the total prize awarded to S in T' is $p(n) + p(r_y - 1)$, which corresponds to an increase of $p(n) + p(r_y - 1) - p(r_t - 1) - p(r_y)$. Since y loses to t in T, we have that $r_y > r_t$, which implies that $p(r_y - 1) \leq p(r_t - 1)$, and so the total increase is at most 0. □

5.2 Recursive Teams

Lemma 5.11 (Teams in $X \cup X_{uv}^{uv}$ are recursive)**.** *If u and v both throw matches to external teams and potentially fix their match, then all teams in X and X_{uv}^{uv} are recursive. Here, we recurse on two smaller tournaments $\mathcal{T}$ and $\mathcal{T}'$ that are $\{u, v\}-$adjacent.*

Proof. Consider some team $x \in X$. Let B and L be the set of teams that defeat and lose to x in T. Similarly, let B' and L' be the set of teams that defeat and lose to x in T'. Since the results of all of x's matches are the same, we have $B = B'$ and $L = L'$.

We also define $\mathcal{T}$ and $\mathcal{T}'$ be the projections of T and T' onto B and $B' = B$. The maximum increase in prize to S from T to T' for any prize vector is at most the maximum increase in prize to S from $\mathcal{T}$ to $\mathcal{T}'$ over all prize vectors. However, note that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent with the exception that u and v throw some matches. It follows that $x \in X$ is recursive. Very similarly, we can show that any team $x_{uv}^{uv} \in X_{uv}^{uv}$ is recursive by considering the projections of T and T' onto the set of teams that lose to x_{uv}^{uv}. □

5.3 Colluder throws all

In this section, we consider the situation where one of u or v throws all its matches. Let $x \in \{u, v\}$ be the team that throws all its matches and let y be the other team. Though we make no assumption about the result of the match between x and y in T, since x throws all its matches, it must be the case that x loses to y in T'.

We first note that it x throws all its matches and y throws some matches, y is not incentivized to throw any more matches.

Lemma 5.12 (x throws all implies y does not throw)**.** *Assume that we have a bound b on the total increase in expected prize awarded to $\{x, y\}$ when x throws all its matches and y throws some matches. Then, the total increase in expected prize when x throws all its matches and y throws even more matches is also at*

most b.

Proof. Once x has thrown all its matches, it is going to be ranked last regardless of what y does. Since y throwing even more matches cannot help itself (Lemma 2.1), any bound on the total expected gain to $\{x, y\}$ when x throws all its matches and y throws some matches also holds when x throws all its matches and y throws even more matches. □

Now, we can tackle the case where x throws all its matches and y throws some matches. The *hard* case here is where x and y are copies in T'. This is because if there were some team separating x and y in T', such a team would be a (Nearly) Strong Separator and, consequently, be $0-$terminating. In the absence of such separators, we only have one $1-$terminating team, one $0-$terminating team and some number of recursive teams. This gives us a bound of $1/2$, which is pretty high, and Recursive Analysis does not get us very far.

Instead, we use a different approach — analyze the entire pivot order, instead of just the first pivot. Before we describer our approach, we generalize the notion of y being $0-$terminating and introduce an important claim.

Lemma 5.13 (y before x, where x throws all)**.** *If y occurs before x in the pivot order γ, then $\{x, y\}$ cannot gain in prize by x throwing all of its matches and y throwing some of its matches when running NRKotH with pivot order γ.*

Proof. From Lemma 5.12, we can assume that y does not throw any matches. Since x throws all its matches, it will be ranked last in T'. It is clear, therefore, that x cannot gain in prize. It now suffices to show that y also cannot gain if it occurs as a pivot before u.

Let t be some team that is initially ranked lower than y when running NRKotH with pivot order γ on T. It suffices to show that t is ranked lower than y when running NRKotH with pivot order γ on T' as well. We compare the outputs of the $\gamma-$ordered implementation of NRKotH when run on T and T'. Let $\gamma(i)$ be the partitioner of y and t when running NRKotH on T. From Observation 2.1, none of the pivots before round i are y; since y occurs before x in the pivot order, they cannot be x either.

Since none of the pivots from $\gamma(1)$ to $\gamma(i-1)$ are x, the results of all matches between these pivots and any team that is not x stay the same. In other words, the relevant matches — as well as their results — that do not involve x stay the same in T and T' (it is possible, however, that x's relevant matches are different in T and T'). The consequence of this is that after round j (for all $1 \leq j \leq i - i$), the groups after running NRKotH on T and T' are equal with the possible exception that x is removed from the group it is in when running NRKotH on T and inserted into a lower group when running NRKotH on T'.

Now consider $G' := G'_{i-1}(v) = G'_{i-1}(t)$, the group containing y and t right before $\gamma(i)$ is selected as the pivot when running NRKotH on T'. As we showed above, this group is equal to $G := G_{i-1}(v) = G_{i-1}(t)$ modulo containing x. In other words $G' \cup \{x\} = G \cup \{x\}$. Thus, $\gamma(i)$ is in G' and also partitions y and t in T' in the same way it did in T. This is because the results of the matches between y and t or between $\gamma(i)$ and y and $\gamma(i)$ and t stays the same in T and T'. As a result, t is ranked lower than y in T' as well. We now have that y does not gain anything, which concludes the proof. □

Claim 5.14. *Let U and V be two random variables uniformly selected from $[1:n]$, for $n > 1$, such that $U < V$. Then, for any function $g : [1:n] \to [0,1]$, we have $|\mathbb{E}[g(V-1)] - \mathbb{E}[g(U)]| \leq 1/2$.*

Proof. Since $U < V$, there are $\binom{n}{2}$ values for the ordered pair (U, V), each of which are equally likely. For a given value $U = i$, then there are $n - i$ possible values for V (the values in the series $i+1, i+2, \ldots, n$). Thus, the probability that $U = i$ (conditioned on $U < V$) is $(n-i)/\binom{n}{2}$. It follows that the expectation of $g(U)$ is $\mathbb{E}[g(U)] = \sum_{i=1}^{n-1} g(i)(n-i)/\binom{n}{2}$.

For a given value $V = i$, there are $i - 1$ possible values of U (the values in the series $1, 2, \ldots, i-1$). It follows that the expectation of $g(V-1)$ is $\mathbb{E}[g(V-1)] = \sum_{i=2}^{n} g(i-1)(i-1)/\binom{n}{2} = \sum_{i=1}^{n-1} g(i) \times i/\binom{n}{2}$. Given these values, we have

$$\mathbb{E}[g(V-1)] - \mathbb{E}[g(U)] = \sum_{i=1}^{n-1} g(i)\,(i-(n-i))\,/\binom{n}{2} = \sum_{i=1}^{n-1} g(i) \times (2i-n)\,/\binom{n}{2}.$$

We first show that $\mathbb{E}[g(V-1)] - \mathbb{E}[g(U)] \leq 1/2$. For $i \leq n/2$, we have $g(i) \times (2i-n) \leq 0$, and so we can zero out the corresponding terms. For $i > n/2$, $2i - n$ is positive and we can use the fact that $g(i) \leq 1$ to get

$$\mathbb{E}[g(V-1)] - \mathbb{E}[g(U)] \leq \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} (2i-n)/\binom{n}{2}$$

For even n, this sum equals $\frac{(n-2)}{2(n-1)} < 1/2$. For odd n, this equals $\frac{(n-1)}{2n} < 1/2$. We now show that $\mathbb{E}[g(U)] - \mathbb{E}[g(V-1)] \leq 1/2$. Contrary to the previous case, for $i \geq n/2$, we have $g(i) \times (n-2i) \leq 0$ and we can zero out the corresponding terms. For $i < n/2$, $n - 2i$ is positive and we can use the fact that $g(i) \leq 1$ to get

$$\mathbb{E}[g(U)] - \mathbb{E}[g(V-1)] \leq \sum_{i=1}^{\lfloor n/2 \rfloor} (n-2i)/\binom{n}{2} = \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} (2i-n)/\binom{n}{2} < 1/2$$

as we showed above. □

Lemma 5.15 (Copy throws all)**.** *Assume that x and y are copies in T (that is, none of the other teams separate x and y). If x throws all its matches and y throws some matches, then $\{x, y\}$ cannot gain more than $1/4$ in expected prize.*

Proof. From Lemma 5.12, we can assume that y does not throw any matches. Since x and y are copies in T, every team in this tournament (that is not x or y) must either defeat both x and y or lose to both x and y in T. Those teams that defeat x and y in T continue to defeat both of them in T'. On the other hand, the teams that lose to both x and y in T only lose to y in T' (since x throws its match to them). In other words, every team (that is not x or y) must lie in either X_{xy}^{xy} or X^x.

We define $\mathcal{T}$ to be the tournament where y is removed from T; in other words, $\mathcal{T}$ is the projection of T onto $X_{xy}^{xy} \cup X^x \cup \{x\}$. We now fix a permutation γ' of all teams excluding y and define $f_{\gamma'}$ to be the generating function of x in $\mathcal{T}$. If we insert x in γ' at position r_x, the rank of x will be $f_{\gamma'}(r_x)$.

In order to generate the pivot order γ on which to run NRKotH on T and T', we can first fix the order γ' of all teams in $X_{xy}^{xy} \cup X^x$ and then insert x and y such that they occupy positions r_x and r_y respectively. If we insert y before x, then from Lemma 5.13, S cannot gain anything by manipulating T to T'. Henceforth, we assume that x appears before y in γ; that is, $r_x < r_y$. All pivots in γ before x either both defeat or both lose to x and y in T — exactly how they treat x in $\mathcal{T}$. Thus, when running NRKotH on T, the rank of x becomes $f_{\gamma'}(r_x)$ if x defeats y or $f_{\gamma'}(r_x) + 1$ if y defeats x. Since y defeats all teams that x defeats and loses to all teams that defeat u, it gets ranked right next to x. It follows that the ranks awarded to x and y are $f_\gamma(r_x)$ and $f_\gamma(r_x) + 1$, or that the prize awarded to S is $p(f_{\gamma'}(r_x)) + p(f_{\gamma'}(r_x) + 1)$.

We now consider the prize awarded to S in T'. Consider the same orderings γ' and γ as above. Since x throws all its matches, its position in the pivot order γ does not *matter* — selecting x as a pivot only seals its position at the end and does not affect any of the other teams. The rank of y after running NRKotH on T' is now $f_{\gamma'}(r_y - 1)$. We subtract 1 to account for the fact that x appears before y; thus, the "effective" position of y, which is number of teams before it in γ that *matter* is $r_y - 1$. Also note that the rank of x is n (it loses all its matches and is ranked last). Thus, the total prize awarded to S in T' is $p(f_{\gamma'}(r_y - 1)) + p(n)$.

We now have that the maximum gain in expected prize by manipulating T to T' is $p(f_{\gamma'}(r_y - 1)) + p(n) - p(f_{\gamma'}(r_x)) - p(f_{\gamma'}(r_x) + 1) \leq p(f_{\gamma'}(r_y - 1)) - p(f_{\gamma'}(r_x))$. Setting $g = p \circ f_{\gamma'}$ and noticing that x and r_x as well as y and r_y are identically distributed, Claim 5.14 tells us that the expected prize awarded to S does not increase by more than $1/2$ from T to T' for any pivot order γ where x occurs before y. Since x occurs before y with probability $1/2$ over all possible pivot orders, the maximum gain in expected prize is $1/4$. □

We can now use the argument from the previous lemma as well as a recursive analysis to make two more claims. Before we do this, we demonstrate that teams in X_{xy}^{xy} and X^x are both recursive teams.

Lemma 5.16 (Teams in X_{xy}^{xy} and X^x are recursive)**.** *If x throws all its matches and y throws some matches, then all teams in X_{xy}^{xy} and X^x are recursive. here, we recurse onto two tournaments $\mathcal{T}$ and $\mathcal{T}'$ that are equivalent with the exception that x throws all its matches and y throws some matches.*

Proof. The fact that all teams in X_{xy}^{xy} are recursive follows from Lemma 5.11. Consider some team $x_{xy}^{xy} \in X_{xy}^{xy}$. Let B and L be the set of teams that defeat and lose to x^x in T. Similarly, let B' and L' be the set of teams that defeat and lose to x^x in T'. Note that $B = B'$ and $L = L'$. The maximum gain from T to T' is at most the maximum gain from $\mathcal{T}$ to $\mathcal{T}'$, where $\mathcal{T}$ and $\mathcal{T}'$ are the projections of T and T' onto L and $L' = L$. Now note that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent with the exception that x throws all its matches and y throws some matches. It follows that x_{xy}^{xy} is recursive.

Now consider some team $x^x \in X^x$. Let B and L be the set of teams that defeat and lose to x^x in T. Similarly, let B' and L' be the set of teams that defeat and lose to x^x in T'. Note that $B = B' \cup \{x\}$ and $L' = L \cup \{x\}$. Consider bringing x up from L' to B' such that it loses to all teams in B'. We are allowed to do this since this potentially increases the prize awarded to x in T' but does not affect y — x is a Condorcet loser and does not affect the rankings of any other teams in B'.

The expected increase in prize to S from T to T' is at most the expected increase in prize from $\mathcal{T}$ to $\mathcal{T}'$, where $\mathcal{T}$ and $\mathcal{T}$ are the projections of T and T' onto B and B' respectively. However, note that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent with the exception that x throws all its matches and y throws some matches. It follows that $x^x \in X^x$ is recursive. □

Lemma 5.17 (Special case of x throws all)**.** *Assume that x throws all its matches and that y throws some matches. If y throws its match against all teams that defeat x and lose to y (or no such teams exist), then $\{x, y\}$ cannot gain more than $1/4$ in expected prize.*

Proof. From Lemma 5.12, we can assume that y does not throw any more matches, apart from the ones it throws to teams that defeat x and lose to y. If x and y are copies in T, the result follows from Lemma 5.15. Thus, we can assume that there exists at least separator of x and y in T. We break our analysis down into two cases — Case 1 : there is exactly one separator of x and y in T and Case 2 : there are at least two separators of x and y in T.

Case 1: Exactly one separator. Let's call this separator w. From the constraints of the problem, we have that either (a) w defeats y and loses to x in both T and T' or (b) w defeats x and loses to y in T and defeats both x and y in T'. All other teams (apart from u, v and w) must be in X_{xy}^{xy} or X^x. Let $Z \subseteq X_{xy}^{xy} \cup X^x$ be the set of teams that separate y and w in T (note that $x, y, w \notin Z$).

Consider running NRKotH on a fixed pivot order γ of all teams. Define $\mathcal{I} = Z \cup \{x, y, w\}$ and let f be the first pivot in γ that also lies in $\mathcal{I}$. If $f \in Z$, this means that w will not be in the group containing y in both T and T' right after f is selected as the pivot. Let G be the group containing y in T right after f is selected as the pivot. This group G will comprise of x, y as well as potentially teams in $X_{xy}^{xy} \cup X^x$; this is because w has not yet been selected as a pivot and so x and y are copies with respect to all pivots selected so far in T. The group G' containing y in T' will be G or $G \backslash \{x\}$; this is because everything stays the same in T' with the exception that x could lose some matches to some pivots and end up in a worse group. In T', if x is not already in G', bring it up from the group it is in (the lowest group) and insert it in G' — we maintain the fact that x loses all its matches to teams in G'. We are allowed to do this since this operation increases the prize to x in T' without affecting the prize to y since x is a Condorcet loser and would have been ranked below y anyway. Now, the maximum expected increase in prize S can gain from T to T' is at most the maximum expected increase it can gain from the projection of T onto G to the projection of T' onto G'. Note that x and y are copies in G (since $w \notin G$) and x throws all its matches in G'. From Lemma 5.15, we know that this gain is at most $1/4$.

If $f = y$, we know that y occurs before x in the pivot order and from Lemma 5.13, the gain is at most 0.

Now assume that $f = w$. Since none of the pivots so far have been in Z, x, y and w must be in the same group G in T right before w is selected as the pivot. In T', y and w will also be in the same group G' since the results of their matches do not change and x will be in the lowest group right before f is selected as the pivot. As we argued above, G' is either G or $G \backslash \{x\}$. As we did in the case where $f \in Z$, in T' let us bring x up to group G' such that it loses all matches against teams in G'. This only increases x's prize and does not affect y's prize since x is a Condorcet loser and would have been ranked below y anyway. Let us see what happens right after f is selected as the pivot. One possibility is that w defeats y and loses to x in T and defeats both x and y in T'; here, it is a Nearly Strong Separator since x is a Condorcet loser in T' and the gain is 0 from Lemma 5.10. The other possibility is that it defeats x and loses to y in T and defeats both in T'. Here, the ranks of both u and v individually decreases and the total gain is 0.

Finally, consider the case where $f = u$. If w occurs before y in the rest of the pivot order, the gain is at most 1 (u definitely does not gain, v gains at most 1). Assume that y occurs before w. Consider fixing an

order of all pivots except x and y — call this γ'. Let $\mathcal{T}$ be the tournament T where y is removed. That is, $\mathcal{T}$ is the projection of T onto $X_{xy}^{xy} \cup X^x \cup \{x, w\}$. Let $f_{\gamma'}$ be the generating function of x in $\mathcal{T}$; that is, if x is inserted in position r_x, it has rank $f_{\gamma'}(r_x)$.

Now, we analyze tournament T. As described above, consider a pivot order γ' of all teams in $X_{xy}^{xy} \cup X^x \cup \{w\}$. Assume that w is at position r_w and suppose that we insert x and y at positions r_x and r_y such that $r_x < r_y < r_w$ (this guarantees that x appears before y, which appears before w in γ). Let's first assume that w defeats y and x in T and defeats both x and y in T'. Once x is selected as the pivot, x's rank in T will be almost identical to its rank in $\mathcal{T}$. In particular, if x defeats y, since x and y are copies until x is selected as the pivot, x's rank will be $f_{\gamma'}(r_x)$; on the other hand, if y defeats x, its rank increases by 1 to become $f_{\gamma'}(r_x) + 1$. Thus, the total prize awarded to S is at least $p(f_{\gamma'}(r_x) + 1) + p(n)$, since y gets prize at least $p(n)$. Now assume that w defeats x and loses to y in T and defeats both in T'. If x defeats y in T, the rank awarded to x in T is $f_{\gamma'}(r_x)$ while the rank awarded to y is $f_{\gamma'}(r_x) + 1$ since y defeats all teams that lose to u. If x loses to y, then the rank awarded to x is $f_{\gamma'}(r_x) + 1$ and the rank awarded to y is at most $f_{\gamma'}(r_x)$. Thus, the total prize awarded to S in T is at least $p(f_{\gamma'}(r_x)) + p(f_{\gamma'}(r_x) + 1) \geq p(f_{\gamma'}(r_x)) + p(n)$

In tournament T', the team x does not matter in the pivot order since it is a Condorcet Loser. If w defeats y and x in T and defeats both x and y in T', the rank awarded to y will be $f(r_y - 1) + 1$. We subtract 1 from r_y to account for the fact that $r_x < r_y$ (and so the effective number of teams ahead of y goes down by 1) and add 1 to the rank to account for the fact that x defeats w in T but y loses to w. The rank of x in T' will be n and so the total prize awarded to S in T' is $p(f(r_y - 1) + 1) + p(n)$. If w defeats x and loses to y in T and defeats both in T', the rank awarded to y is $f_{\gamma'}(r_y - 1)$, where we subtract 1 from r_y to account for the fact that $r_x < r_y$.

This corresponds to an increase of $p(f_{\gamma'}(r_y - 1) + 1) - p(f_{\gamma'}(r_x) + 1)$ or $p(f_{\gamma'}(r_y - 1)) - p(f_{\gamma'}(r_x))$, both of which are at most $1/2$ from Claim 5.14. Since the probability that y appears before w is $1/2$, we have the maximum expected increase when $f = u$ is $3/4$.

Since every element in $\mathcal{I}$ is equally likely to be selected as the first pivot, we have the maximum expected increase in prize is $1/(|Z| + 3) \cdot (|Z|/4 + 3/4) = 1/4$, which is the desired result.

Case 2: At least two separators. Here, x is $1-$terminating (it does not gain anything while y gains at most 1) and y is $0-$terminating; this is because y must occur before x in the pivot order and from Lemma 5.13, the gain is at most 0.

We also have that all separators are $0-$terminating. Consider some separator w. One possibility is that

w defeats y and loses to x in T and defeats both x and y in T'; here, it is a Nearly Strong Separator since x is a Condorcet loser in T' and the gain is 0 from Lemma 5.10. The other possibility is that it defeats x and loses to y in T and defeats both in T'. Here, the ranks of both u and v individually decrease and the total gain is 0. From Lemma 5.16, all teams in X_{uv}^{uv} and X^u are recursive.

To summarize, we have one $1-$terminating team, at least three $0-$terminating teams, and some number of recursive teams. This gives us the required bound of $1/4$. □

We now consider the most general case in this setting — there could be teams that defeat x and lose to y in both T and T'. Here, we obtain different bounds based on the result of the match between x and y : $11/40$ if x loses to y and $1/3$ otherwise. We now introduce two different lemmas to prove this. We also refer to the teams u and v instead of x and y since we know the result of the match in T.

Lemma 5.18 (v throws all)**.** *If v throws all its matches and u throws some matches, the expected prize that $\{u, v\}$ can gain by manipulating T to T' is at most* $11/40$.

Proof. From Lemma 5.12, we can assume that u does not throw any matches. From Lemmas 5.15 and 5.17, it suffices to consider the case where there exists at least one team that defeats v and loses to u in T. Every team in the tournament (that is not u or v) is in $X_{uv}^{uv}, X^v, X_u^{uv}$, or X_v^v and $|X_v^v| \geq 1$. We now split our analysis into two cases: Case 1 is where $|X_v^{uv}| = 0$ and $|X_u^u| = 1$ and Case 2, where $|X_u^{uv}| + |X_v^v| \geq 2$.

Case 1: ($|X_u^{uv}| = 0$ **and** $|X_v^v| = 1$) Let w be the team in X_v^v and let $Z \subseteq X_{uv}^{uv} \cup X^u$ be the set of teams that separate u and w in T (note that $u, v, w \notin Z$). If $|Z| = 0$, this means that w loses to all teams in X_{uv}^{uv} and defeats all teams in X^v in T and T'.

Consider running NRKotH on a fixed pivot order γ of all teams. Define $\mathcal{I} = Z \cup \{u, v, w\}$ and let f be the first pivot in γ that also lies in $\mathcal{I}$. If $f \in Z$, this means that w will not be in the group containing u in both T and T' right after f is selected as the pivot. Let G be the group containing u in T right after f is selected as the pivot. This group G will comprise of u, v as well as potentially teams in $X_{uv}^{uv} \cup X^v$; this is because w has not yet been selected as a pivot and so u and v are copies with respect to all pivots selected so far in T. The group G' containing u in T' will be G or $G \backslash \{v\}$; this is because everything stays the same in T' with the exception that v could lose some matches to some pivots and end up in a worse group. In T', if v is not already in G', bring it up from the group it is in (the lowest group) and insert it in G' — we maintain the fact that v loses all its matches to teams in G. We are allowed to do this since this operation increases the prize to v in T' without affecting the prize to u since v is a Condorcet loser and would have

been ranked below u anyway. Now, the maximum expected increase in prize S can gain from T to T' is at most the maximum expected increase it can gain from the projection of T onto G to the projection of T' onto G'. Note that u and v are copies in G (since $w \notin G$) and v throws all its matches in G'. From Lemma 5.15, we know that the gain is at most $1/4$.

If $f = u$, we know that v occurs before u in the pivot order and from Lemma 5.13, the gain is at most 0.

Now assume that $f = w$. Since none of the pivots so far have been in Z, u, v and w must be in the same group G in T right before w is selected as the pivot. In T', u and w will also be in the same group G' since the results of their matches do not change and v will be in the lowest group right before f is selected as the pivot. As we argued above, G' is either G or $G \backslash \{v\}$. As we did in the case where $f \in Z$, in T' let us bring v up to group G' such that it loses to all teams in G'. This only increases v's prize and does not affect u's prize since v is a Condorcet loser and would have been ranked below u anyway. Let us see what happens right after f is selected as the pivot. Since w defeats v and loses to u in both T and T', it is a Strong Separator, and from Lemma 5.9, the gain is at most 0.

Finally, consider the case where $f = v$. We claim that if the next pivot in $\mathcal{I}$ is u, the gain is at most $1/2$. After u, assume that the next pivot in $\mathcal{I}$ is some team $z \in \{w\} \cup Z$ (that is, the order of the pivots in $\mathcal{I}$ is v and then u and then z). Consider fixing an order of all pivots except u and v — call this γ'. Let $\mathcal{T}$ be the tournament T where u is removed. That is, $\mathcal{T}$ is the projection of T onto $X_{uv}^{uv} \cup X^v \cup \{v, w\}$. Let $f_{\gamma'}$ be the generating function of v in $\mathcal{T}$; that is, if v is inserted in position r_v, it has rank $f_{\gamma'}(r_v)$.

Now, we analyze tournament T. As described above, consider a pivot order γ' of all teams in $X_{uv}^{uv} \cup X^v \cup \{w\}$. Assume that z is at position r_z and suppose that we insert u and v at positions r_u and r_v such that $r_v < r_u < r_z$ (this guarantees that v appears before u, which appears before z in γ). Once v is selected as the pivot, it gets rank $f_{\gamma'}(r_v) + 1$, where we add 1 since u defeats v in T. Since u gets selected as a pivot before w as well as all separators of u and w, all relevant pivots (i.e. pivots that continue to stay in the same group as u) between v and u must defeat both u and w; thus, when u is selected as the pivot, it will be in the same group as w and will get rank $f_{\gamma'}(r_v) - 1$, since it will be ranked above w. Thus, the total prize awarded to S is $p(f_{\gamma'}(r_v) + 1) + p(f_{\gamma'}(r_v) - 1)$.

In tournament T', the team v does not matter in the pivot order since it is a Condorcet Loser and it will get prize $p(n)$. The prize awarded to u is $p(f_{\gamma'}(r_u - 1) - 1)$. We subtract 1 from r_u to account for the fact that $r_v < r_u$ (and so the effective number of teams ahead of u goes down by 1) and subtract 1 from $f_{\gamma'}(r_u - 1)$ to account for the fact that w defeats v but u defeats w in T'. The total prize to S is now $p(f_{\gamma'}(r_u - 1) - 1) + p(n)$. The total increase in prize from T to T' is $p(f_{\gamma'}(r_u - 1) - 1) - p(f_{\gamma'}(r_v) - 1)$. From

Claim 5.14, this is at most 1/2.

In the event that v is not the next pivot in $\mathcal{I}$ after u, the gain is at most 1 (v definitely does not gain, u gains at most 1). Since the probability that u occurs as the pivot right after v is $1/(|Z|+2)$, we have that the expected gain when u is the first pivot is $1/(2(|Z|+2)) + (|Z|+1)/(|Z|+2) = (|Z|+3/2)/(|Z|+2)$.

Since every element in $\mathcal{I}$ is equally likely to be selected as the first pivot, we have the maximum expected increase in prize is

$$\frac{1}{|Z|+3} \cdot \left(\frac{|Z|}{4} + \frac{|Z|+3/2}{|Z|+2} \right)$$

This function is maximized, over the positive integers, when $|Z| = 2$ or 3 and takes value 11/40 (Lemma A.3 in the Appendix), which gives us the desired result.

Case 2: ($|X_u^{uv}| + |X_v^v| > 1$) We first claim that v is 1−terminating; this is because v definitely does not gain anything, while u could gain up to 1. We also have that u is 0−terminating since if u is the first pivot, it must occur before v and there is no gain here from Lemma 5.13.

All teams in X_u^{uv} are Nearly Strong Separators and are 0−terminating from Lemma 5.10. All teams in X_v^v are Strong Separators and are 0−terminating from Lemma 5.9. From Lemma 5.16, all teams in X_{uv}^{uv} and X^v are recursive.

To summarize, we have one team that is 1−terminating, at least three teams that are 0−terminating and some number of recursive teams. This gives us the required bound of 11/40 (note that the bound is *not* 1/4 since you could recurse onto Case 1, where the bound is $11/40 > 1/4$). □

Lemma 5.19 (u throws all)**.** *Let T and T' be two tournaments that are equivalent with the exception that u throws all its matches and v throws some matches. The expected prize that S can gain by manipulating T to T' is at most* 1/3.

Proof. From Lemma 5.12, we can assume that v does not throw any matches. From Lemmas 5.15 and 5.17, it suffices to consider the case where there exists at least one team that defeats u and loses to v in T. Every team in the tournament (that is not in u or v) is in $X_{uv}^{uv}, X^u, X_v^{uv}$, or X_u^u and $|X_u^u| \geq 1$.

Note that u is 1−terminating from Lemma 5.2. We also have that v is 0−terminating since if v is the first pivot, it must occur before u and there is no gain here from Lemma 5.13.

All teams in X_v^{uv} are Nearly Strong Separators and are 0−terminating from Lemma 5.10. All teams in X_u^u are Strong Separators and are 0−terminating from Lemma 5.9. From Lemma 5.16, all teams in X_{uv}^{uv}

and X^u are recursive.

To summarize, we have one team that is 1−terminating, at least 2 teams that are 0−terminating, and some number of recursive teams. This gives us the required bound of 1/3. □

Remark 5.1. *This is the best we can prove. For instance, consider a tournament T with teams u, v, w and y where u defeats v, v defeats w and w defeats u; further assume that y is a Condorcet loser. It is easy to see that u, v and w win the tournament with equal probability. Under the $p = \langle 1, 0, 0, 0\rangle$ prize vector, u and v have expected prize $2/3$. If u throws all, v becomes a Condorcet winner and the expected prize to $\{u, v\}$ becomes 1, corresponding to a gain of $1/3$.*

5.4 Condorcet Winner throws some

In this section, we consider the situation where one of u or v is a Condorcet Winner and throws some of its matches. We prove this by showing that this problem basically reduces to the problem of one of u or v throwing all their matches. This way, we get the exact same bounds as in the previous section for corresponding situations.

Definition 5.1 (Flip of a tournament)**.** *The flip of a tournament T is defined to be that tournament T_f such that for any two teams x and y, x defeats y in T_f if and only if y defeats x in T_f.*

Definition 5.2 (Flip of a prize vector)**.** *The flip of a prize vector p is defined to be that prize vector p_f such that the i^{th} component of p_f is defined to be 1 minus the $(n-i+1)^{st}$ component of p.*

We now show an interesting property of flipping both the tournament and the prize vector:

Lemma 5.20 (Invariant)**.** *The expected prize awarded to some team t in a tournament T with prize vector p plus the expected prize awarded to t in T_f with prize vector p_f equals 1.*

Proof. Assume that there are n teams in the tournament and consider some pivot order γ. We claim that team t gets rank r when we run NRKotH on T with pivot order γ, then t will get rank $n-r+1$, when we run NRKotH on T_f with pivot order γ. We prove this claim by strong induction on n.

Base Case If $n = 1$, the claim is trivially true.

Inductive Step Assume the result is true for all tournaments with fewer than n teams and consider some tournament with n teams. Let B and L be the sets of teams that defeat and lose to $\gamma(i)$, the

first pivot in T. Similarly, let B' and L' be the sets of teams that defeat and lose to $\gamma(i)$ in T_f. By the definition of f, $B = L'$ and $L = B'$. The rank of $\gamma(i)$ in T is $|B| + 1$ and the rank of $\gamma(i)$ in T_f is $|B'| + 1 = |L| + 1 = n - |B| - 1 + 1 = n - |B| = n - (|B| + 1) + 1$. Thus, the result is true for $\gamma(i)$.

Now consider some team t in $B = L'$. By the definition of NRKotH, the rank of t in T is the rank of t in $\mathcal{T}$, the projection of T onto B. The rank of t in T_f is $|B'|+1$ plus the rank of t in $\mathcal{T}'$, the projection of T_f onto $L' = B$. Note that since we are projecting flipped tournaments onto the same set, we have that $\mathcal{T}' = f(\mathcal{T})$. Assume that the rank of t in $\mathcal{T}$ is r. From the Inductive Hypothesis, the rank of t in $\mathcal{T}'$ is $|B|-r+1$. We know have that the rank of t in T is r will the rank of t in T_f is $|B'|+1+|B|-r+1 = |L|+1+|B|-r+1 = n-r+1$. Thus, the claim is true for all $t \in B$. Very similarly, we can show that the claim is true for all $t \in L$.

Thus, we have shown, using strong induction, that if team t gets rank r when we run NRKotH on T with pivot order γ, then t will get rank $n - r + 1$, when we run NRKotH on T_f with pivot order γ.

Let Γ be the set of all possible pivot orders and let $r_T(\gamma)$ be the rank awarded to team t in tournament T when the pivot order is γ and the prize vector is p. The expected prize awarded to t in T_f is now given by

$$\frac{1}{n!}\sum_{\gamma\in\Gamma} p_f(r_{T_f}(\gamma)) = \frac{1}{n!}\sum_{\gamma\in\Gamma} p_f(n - r_T(\gamma) + 1) = \frac{1}{n!}\sum_{\gamma\in\Gamma}(1 - p(r_T(\gamma))) = 1 - \frac{1}{n!}\sum_{\gamma\in\Gamma} p(r_T(\gamma)),$$

which is 1 minus the expected prize awarded to t in T. □

We can now use this invariant to describe how to reduce problems of the form "Condorcet winner throws some matches" to problems of the form "colluder throws all matches."

Lemma 5.21 (Bridge Lemma)**.** *For any tournament T, let $E^p_{S;T}$ represent the total expected prize awarded to $S = \{u, v\}$ when running NRKotH on T. Assume that for all pairs of tournaments T and T' that satisfy some condition $C(T, T')$ and for all prize vectors, we have that $E^p_{T'} - E^p_T \leq b$. Then, for any two tournaments $\mathcal{T}$ and $\mathcal{T}$ such that $C(f(\mathcal{T}'), f(\mathcal{T}))$ is satisfied, we must have that $E^p_{\mathcal{T}'} - E^p_{\mathcal{T}} \leq b$.*

Proof. From Lemma 5.20 and the linearity of expectation, $E^p_{\mathcal{T}'} = 2 - E^{f(p)}_{f(\mathcal{T}')}$ and $E^p_{\mathcal{T}} = 2 - E^{f(p)}_{f(\mathcal{T})}$. Thus,

$$E^p_{\mathcal{T}'} - E^p_{\mathcal{T}} = E^{f(p)}_{f(\mathcal{T})} - E^{f(p)}_{f(\mathcal{T}')} \leq b,$$

where the last inequality comes from the fact that $C(f(\mathcal{T}'), f(\mathcal{T}))$ is satisfied. □

We can now use this Lemma as well as results from the previous section to prove new results about the case when a Condorcet winner throws some matches. As in the previous section, we call the colluding teams

x and y (instead of u and v) when the result applies to both u and v; that is, we do not want to assume the result of the match between the colluding teams in T.

Lemma 5.22 (Improved Condorcet winner throws some)**.** *Let T be a tournament in which x is a Condorcet winner and let T' be equivalent to T with the exception that x and y throw some matches and y defeats, in T, every team that defeats y and loses to x in T'. Then, the maximum expected prize that $S = \{x, y\}$ can gain by manipulating T to T' is at most* $1/4$.

Proof. This follows from Lemmas 5.21 and 5.17. □

Lemma 5.23 (Condorcet winner becomes u, v winner)**.** *Let T be a tournament in which x is a Condorcet winner and let T' be equivalent to T with the exception that x and y throw some matches. If x defeats y in T', the maximum expected prize that $S = \{x, y\}$ can gain by manipulating T to T' is at most* $11/40$.

Proof. This follows from Lemmas 5.21 and 5.18. □

Lemma 5.24 (Condorcet winner becomes u, v loser)**.** *Let T be a tournament in which x is a Condorcet winner and let T' be equivalent to T with the exception that x and y throw some matches. If y defeats x in T', the maximum expected prize that $S = \{x, y\}$ can gain by manipulating T to T' is at most* $1/3$.

Proof. This follows from Lemmas 5.21 and 5.19. □

6 Conclusion and Open Problems

We initiate the study of manipulation in tournament rules by colluding teams throwing matches to other external teams in addition to potentially fixing the result of their own match. Specifically, we show that no teams can throw matches to external teams and potentially fix the result of their own match to gain more than $1/3$ in expected prize under the Nested Randomized King of the Hill where all prizes lie in $[0, 1]$. This bound is tight and optimal over all Condorcet consistent tournament rules.

One possible direction for future work is examining the general case where more than 2 colluding teams fix the results of their matches and throw matches to external teams. It would also be interesting to see whether other tournament rules satisfy the same manipulability guarantees as NRKotH.

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A Appendix

Lemma A.1. *The function*

$$f(x) = \frac{1}{x+2} \cdot \left(\frac{x}{x+1} + \frac{x}{4} \right)$$

takes value at most $1/3$ *over the non-negative integers.*

Proof. We show that the global maximum of this function in $[0, \infty)$ is less than $1/3$. The first derivative is

$$f'(x) = \frac{-x^2 + 2x + 5}{2(x+1)^2(x+2)^2},$$

and this equals 0 when $x = 1 \pm \sqrt{6}$. The only critical point in $[0, \infty)$ is $x_0 = 1 + \sqrt{6}$ and the function takes value $f(x_0) = 11/4 - \sqrt{6} < 1/3$ at this point. The value of $f'(x)$ for x just less than x_0 is positive and for x just higher than x_0 is negative, which shows that x_0 is at a local maximum. It remains to check that $f(0) = 0 < f(x_0)$ and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{1}{x + 3 + 2/x} + \frac{1}{4 + 8/x} \right) = \frac{1}{4} < f(x_0).$$

This shows that the maximum value of f over all non-negative reals is $f(x_0) < 1/3$ and so the function takes values at most $1/3$ over the non-negative integers. □

Lemma A.2. *The function*

$$f(x, y) = \frac{1}{x + y + 4} \cdot \left(\frac{x}{x+1} + \frac{y}{y+1} + \frac{11}{40}(x + y) \right)$$

takes value at most $1/3$ *over the non-negative integers.*

Proof. We first note that for all non-negative x and y, we have that

$$\frac{x}{x+1} + \frac{y}{y+1} \leq \frac{2(x+y)}{x+y+2}.$$

Using simple algebra, this inequality reduces to $(x - y)^2 \geq 0$, which is always true. This trick lets us reduce the number of dimensions by 1. In particular, we now want to maximize the function

$$f(z) = \frac{1}{z+4} \cdot \left(\frac{2z}{z+2} + \frac{11z}{40} \right)$$

over non-negative integers z. We claim that the global maximum of this function in $[0, \infty)$ is less than $1/3$. The first derivative is

$$f'(z) = \frac{-9k^2 + 44k + 204}{10(k+2)^2(k+4)^2)}$$

and this equals 0 for $z = 22/9 \pm 4\sqrt{145}/9$. The only critical point in $[0, \infty)$ is $z_0 = 22/9 + 4\sqrt{145}/9$ and the function takes value $f(z_0) = 109/40 - \sqrt{29/5} < 1/3$ at this point. It remains to check that $f(0) = 0 < f(z_0)$ and

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \left(\frac{2}{z + 6 + 8/z} + \frac{11}{40(1 + 4/z)} \right) = \frac{11}{40} < f(x_0).$$

This shows that the maximum value of f over all non-negative reals is $f(z_0) < 1/3$ and so the function takes values at most $1/3$ over the non-negative integers. □

Lemma A.3. *The function*

$$f(x) = \frac{1}{x+3} \cdot \left(\frac{x}{4} + \frac{x + 3/2}{x+2} \right)$$

takes values at most $11/40$ *over the non-negative integers.*

Proof. Unfortunately, this function takes value higher than $11/40$ over the non-negative reals — for instance, $f(\sqrt{6}) = 3/\left(6 + 2\sqrt{6}\right) > 11/40$. However, we note that to the right of $x_0 = \sqrt{6}$, the first derivative is always negative. To see why, note that

$$f'(x) = \frac{-(x^2 - 6)}{4(x+2)^2(x+3)^2.}$$

Thus, for $x > \sqrt{6}$, $f'(x)$ is negative and the function is ever decreasing. Thus, the maximum of the function over the positive integers must be less than $\sqrt{6}$. We can compute $f(0) = 1/4 = 0.25$, $f(1) = 13/48 \approx 0.2708$ and $f(2) = f(3) = 11/40$ to get that the f takes value at most $11/40$ over the non-negative integers. □