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Counting rational points on low-degree del Pezzo surfaces with automorphic forms

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Princeton, NJ : Princeton University

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Counting rational points on low-degree del Pezzo surfaces with automorphic forms

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

We study Manin's conjecture for certain families of low-degree del Pezzo surfaces, i.e. asymptotics for the number of rational points of increasing height. In general, problems about rational points on del Pezzo surfaces are considered harder as the degree gets lower. In this thesis, we resolve Manin's conjecture for all Châtelet surfaces over $\mathbb{Q}$, which are del Pezzo surfaces of degree four, and establish the first asymptotic count for a set of rational points on a del Pezzo surface of degree one.

In Chapter 2, we prove the key analytic ingredient necessary in both arguments – a bound on the sums of the absolute values of the Hecke eigenvalues of cuspidal automorphic forms of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ along polynomial values. We use pieces of the Sato-Tate distribution to prove a logarithmic power savings over the trivial bound in the applicable cases; additionally we classify, when the polynomial is solvable, exactly when a logarithmic power savings occurs using analysis about the base change of the cuspidal representation.

In Chapter 3, we study Manin's conjecture for Châtelet surfaces – the proper smooth models of the affine surfaces $x^2 + \Delta y^2 = f(z)$, where $f(z)$ is a squarefree polynomial of degree 3 or 4. Previous works handle the case of $\Delta = 1$ (and extend to any $\Delta > 0$ such that $\mathbb{Q}(\sqrt{-\Delta})$ has class number one). We resolve the conjecture by viewing the point-count through an automorphic lens and connecting it to the sums of Chapter 2.

In Chapter 4, we study rational points on the singular del Pezzo surface of degree one defined by $y^2 = x^3 + AxQ(u,v)^2 + BQ(u,v)^3$, where $4A^3 - 27B^2 \neq 0$ and $Q(u,v)$ is a positive-definite quadratic form. The main algebraic tool used is a parameterization of the integral points on quadratic twists of elliptic curves by binary quartic forms; this parameterization was first observed by Mordell and uses syzygys of binary quartic forms. We then reduce the problem to correlation sums of binary quadratic and binary quartic forms, which are analyzed using the central analytic ingredients of Chapters 2 and 3. As a result, we establish an asymptotic count for rational points that are integral with respect to the singularity on these del Pezzo surfaces.

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To my father, who taught me to be curious.

To my mother, who taught me to be strong.

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Chapter 1

Introduction

1.1 Diophantine equations and naive heuristics

Let us begin with a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$. A fundamental question is whether this polynomial has an integral solution, i.e. does there exist $\mathbf{x} \in \mathbb{Z}^n$ such that $f(\mathbf{x}) = 0$? The study of integer solutions to polynomial equations dates back to Diophantus of Alexandria in 250 A.D. and has since pervaded every aspect of mathematics. To understand these solutions, it often helps to approach the problem using algebro-geometric framework. Let R denote a ring and define the variety V as the solution set of our polynomial:

$$V(R) = \{\mathbf{x} \in R^n : f(\mathbf{x}) = 0\}.$$

Some motivating questions in the study of Diophantine equations are:

- For a given ring R, is $V(R)$ non-empty?
- For a given ring R, is $V(R)$ finite?
- If $V(R)$ is infinite, can we define a height on $V(R)$ and derive asymptotics for the number of points of bounded height?

The focus of this thesis is to address the third problem for families of polynomials f that correspond to specific low degree del Pezzo surfaces.

To start, we consider a naive heuristic approach to this problem when $R = \mathbb{Z}$ and f is a fixed homogeneous form of degree d. Take the height on $V(\mathbb{Z})$ to be the ℓ^∞-norm, i.e.

$$H(\mathbf{x}) = \max_{i \in \{1,\ldots,n\}} |x_i|.$$

Our goal is to estimate the counting function:

$$N(B; V) = \#\{\mathbf{x} \in V(\mathbb{Z}) : H(\mathbf{x}) \leq B\} = \#\{\mathbf{x} \in \mathbb{Z}^n : H(\mathbf{x}) \leq B, f(\mathbf{x}) = 0\}.$$

Observe that for any vector $\mathbf{x}$ satisfying $H(\mathbf{x}) \leq B$, we know that $f(\mathbf{x}) \ll B^d$. Thus, the probability that a random value of $f(\mathbf{x})$ is zero should be $\asymp B^{-d}$. From this heuristic, we are lead to predict that for $n \geq d$,

$$N(B; V) \asymp B^{n-d}.$$

However, this heuristic is too naive to be true. Quickly, we realize that there are examples of polynomials f for which the prediction fails. Consider $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - 3x_3^2$: the only integral

solution to $f(x_1, x_2, x_3) = 0$ is given by $\mathbf{x} = \mathbf{0}$, whereas the heuristic suggests that $N(B; V) \asymp B$. This example exemplifies potential local obstructions – when local conditions (specifically a mod 4 condition) can prevent the existence of the expected number of integral points. Furthermore, we know that there can exist obstructions to rational points that go beyond local conditions, such as the Brauer-Manin obstructions; this heuristic is far too simple to see these types of obstructions!

Another way in which this heuristic can fail is when there are accumulating subvarieties. For example, consider the polynomial $f(x_1, x_2, x_3, x_4, x_5) = x_1^4 + x_2^4 - x_3^4 - x_4^4 - x_5^4$. Since there are five variables and f is of degree four, the heuristic predicts that $N(B; V) \asymp B$. However, the plane defined by $\{x_1 = x_3, x_2 = x_4, x_5 = 0\}$ always satisfies $f(x_1, x_2, x_3, x_4, x_5) = 0$ and contains $\gg B^2$ points of height $\leq B$. So, in this example the heuristic predicts too few points. These examples illustrate the need for a more sophisticated and subtle approach to counting solutions to Diophantine equations.

To introduce the more sophisticated prediction, we switch to counting rational points instead of integral points, i.e. $R = \mathbb{Q}$. When $V \subset \mathbb{P}^{n-1}$ is a projective variety defined by f, then we can consider the counting function

$$N(B; V) = \#\{[\mathbf{x}] = [x_1 : \ldots : x_n] \in \mathbb{P}^{n-1}(\mathbb{Q}) : H([\mathbf{x}]) \leq B, f(x_1, \ldots, x_n) = 0\},$$

where f is the defining (homogeneous) polynomial of V. Since every point $[\mathbf{x}] \in \mathbb{P}^{n-1}(\mathbb{Q})$ has a unique representative $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ with $\gcd(x_1, \ldots, x_n) = 1$, this counting function is equivalent to

$$N(B; V) = \#\{\mathbf{x} \in \mathbb{Z}^n : H(\mathbf{x}) \leq B, \gcd(x_1, \ldots, x_n) = 1, f(x_1, \ldots, x_n) = 0\}.$$

Around 1990, Manin [1, 21] conjectured the size of this rational point-counting function $N(B; V)$ for sufficiently nice polynomials f (specifically the polynomial whose zero sets are Fano varieties); in particular, he suggests that

$$N(B; V) \sim c_V B^a \log(B)^{b-1},$$

where the exponents a and b are given geometric interpretations and c_V is allowed to be zero. This prediction is consistent with results from the Hardy-Littlewood circle method (when n is exponentially larger than d). Manin's conjecture was verified by Franke, Manin, and Tschinkel [21] for generalized flag manifolds, employing techniques that connect their point-counting function with automorphic forms, specifically Eisenstein series, of certain semisimple algebraic groups. We will

state Manin's conjecture explicitly in §1.3 for del Pezzo surfaces; first, we introduce the setting of del Pezzo surfaces.

1.2 Del Pezzo surfaces and Fano varieties

Del Pezzo surfaces are a particularly nice class of surfaces to study these point-counting problems on – a motivating reason to study rational points on del Pezzo surfaces in particular is that it is expected that if a del Pezzo surface has rational points, then it should have a "rich" (and infinite) class of rational points.

Definition 1.2.1. A **del Pezzo surface** is a smooth, projective, geometrically integral surface S with an ample anticanonical sheaf; these are Fano varieties of dimension two.

Definition 1.2.2. Given a del Pezzo surface S, the **degree** d is the self-intersection number of its canonical divisor.

One feature of del Pezzo surface that gives an indication of the "richness" of their rational points is that every del Pezzo surface S (over $\overline{\mathbb{Q}}$) is either:

- $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and hence has degree 8,
- the blow-up of $\mathbb{P}^2$ at $9 - d$ points in general position.

Here the points on $\mathbb{P}^2$ are considered in general position there is no set of three points that are colinear and no set of six points lying on a conic. Implied in the above description is that $1 \leq d \leq 9$, although this fact can be shown in a separate manner (see [19, §8] for further characteristics of del Pezzo surfaces). Since, for $d \leq 7$, a del Pezzo surface of degree d is the blow-up at $9 - d$ points, questions about the rational points of del Pezzo surfaces are considered to be *harder for low degree del Pezzo surfaces*.

In Chapter 4, we will study a family of **singular del Pezzo surfaces**; that is, they contain isolated singularities that are rational double points. For these surfaces, there is a minimal desingularization $\pi : \tilde{S} \to S$ with $\tilde{S}$ a smooth del Pezzo surface and $-K_{\tilde{S}} = \pi^*(-K_S)$. Furthermore, the blow-ups in the desingularization process will produce (-2)-curves over the rational double points. These singularities and their respective exceptional divisors can be classified by their associated Dynkin diagrams (over $\overline{\mathbb{Q}}$) (see [19, §8] for more details on the classification of del Pezzo surfaces over $\overline{\mathbb{Q}}$ and see [4] for further discussion on the rational points on del Pezzo surfaces.) Although singular del Pezzo surfaces have similarities to smooth del Pezzo surfaces, often they have extra

structures to them that can be exploited when it comes to problems about their rational points.

Another way of viewing del Pezzo surfaces is that they are two-dimensional Fano varieties. Before beginning our discussion of Manin's conjecture, let us dive deeper into the world of projective varieties (following [28, Chapter II Section 1.10]) and explain why being Fano is important to our setting.

We return to our original (far too naive) heuristic for this motivation. The Kodaira classification of projective varieties separates varieties into three distinct types: varieties of general type, varieties of intermediate type, and Fano varieties. Consider a degree d hypersurface V in $\mathbb{P}^n$; the naive heuristic predicts that

$$N(B;V) \asymp B^{n+1-d}.$$

Not coincidentally, for this surface the anticanonical divisor is $\mathscr{O}(n+1-d)$ and can be used to classify the type of projective variety of the hypersurface. If V is a variety of general type, then its canonical sheaf is ample and $n+1-d<0$. Of course, we should not expect a negative exponent for $N(B;V)$, but the heuristic perhaps lends credence to Lang's conjecture regarding the finiteness of $V(\mathbb{Q})$ for varieties of general type. If V is a variety of intermediate type, then $n+1-d=0$; this case is the borderline case where it is unclear whether to expect for a given variety V whether $V(\mathbb{Q})$ is finite or infinite (for example, elliptic curves fall into this case). Finally, if V is a Fano variety then the anticanonical sheaf is ample and $n+1-d>0$. The naive heuristic guesses that $V(\mathbb{Q})$ contains many rational points; in some sense, the class of rational points is expected to be "rich", where "rich" means that $N(B;V) \gg B^{\alpha}$ for some $\alpha>0$.

When V is Fano and $V(\mathbb{Q})$ is nonempty, our intuition is that $N(B;V) \gg B^{\alpha}$ for some $\alpha>0$, which matches the naive heuristic. However, this prediction says nothing about the existence of local or more complicated obstructions to the existence of rational points – it is still possible for $V(\mathbb{Q})$ to be empty when V is a Fano variety! Nevertheless, the heuristic provides some intuition for why we choose to study del Pezzo surfaces, which are the Fano varieties of dimension two.

1.3 Manin's conjecture (for del Pezzo surfaces)

Let X now denote a Fano projective variety and we take $U \subset X$ to be a Zariski open set. Define the **growth rate** of U to be β_U satisfying that if $U(\mathbb{Q})$ is Zariski-dense then

$$B^{\beta_U-\epsilon} \ll_{\epsilon} N(B;U) = \#\{\mathbf{x} \in U(\mathbb{Q}) : H(\mathbf{x}) \leq B\} \ll_{\epsilon} B^{\beta_U+\epsilon},$$

where $H(\mathbf{x})$ denotes an anticanonical height on $X(\mathbb{Q})$. Manin's insight is to relate this quantity β_U to geometric properties of X. We point out that by counting points on this open subset $U \subset X$, we avoid the issue of accumulating subvarieties.

Let $\mathrm{Div}(X)$ denote the divisor group of X, $[K_X]$ denote the canonical divisor of X, and $[H] \in \mathrm{Div}(X)$ denote a divisor corresponding to a hyperplane section. Then we define

$$a(X) := \inf\{r \in \mathbb{Z} : r[H] + [K_X] \text{ is effective}\}.$$

Since we assumed X is a Fano, we know that $a(X) > 0$. Furthermore, when X is a del Pezzo surface, we have the exact calculation that $a(X) = 1$. In [1, Conjecture B], this quantity $a(X)$ is related to the growth rate:

Conjecture 1.3.1 (Manin's conjecture for the growth rate)**.** *Assume that X is a Fano variety and $X(\mathbb{Q})$ is Zariski-dense. Then there exists a Zariski open subset $U \subset X$ such that $\beta_U = a(X)$.*

Manin then goes beyond the growth rate of the open set $U \subset X$ and hypothesizes an asymptotic – he predicts that for a sufficiently large number field $K/\mathbb{Q}$ and assuming $X(K)$ is Zariski-dense,

$$N_K(B; U) \sim cB^{a(X)} \log(B)^{b(X)}$$

where $b(X)$ also has a geometric interpretation ([1, Conjecture C']). Manin originally stated his conjecture over arbitrarily large number fields but the conjectures have since been refined for rational points over $\mathbb{Q}$. One notable constraint of working with $\mathbb{Q}$ rather than a number field $K/\mathbb{Q}$ is potential obstructions to the existence of rational points – by taking a suitable finite extension, it can be guaranteed that $X(K) \neq \emptyset$.

In this thesis, however, we work with integral and rational points. So, let us state a version of Manin's conjecture over $\mathbb{Q}$ for del Pezzo surfaces after [4, Conjecture 2.3]:

Conjecture 1.3.2 (Manin's conjecture for del Pezzo surfaces)**.** *Let S be a del Pezzo surface of degree d and assume that $S(\mathbb{Q}) \neq \emptyset$. Then there exists a Zariski open subset $U \subset S$ such that*

$$N(B; U) = c_S B \log(B)^{\rho_S - 1}(1 + o(1)),$$

where ρ_S is the Picard rank of S.

Remark 1.3.3. Furthermore, this open set $U \subset S$ can be constructed for a given del Pezzo surface by

removing lines and thin sets from S, i.e. the accumulating subvarieties. Also the geometric Picard rank, i.e. the rank of the Picard group over $\overline{\mathbb{Q}}$, of S is given by $10 - d$.

Remark 1.3.4. A version of Manin's conjecture exists for singular del Pezzo surfaces by instead computing the growth rate and Picard rank of the minimal desingularization $\pi : \tilde{S} \to S$.

Remark 1.3.5. The leading constant c_S was given an interpretation by Peyre [32] in terms of adelic measures and other geometric information, and later by Salberger [33] in terms of universal torsors. It is of note that, as presented in [28, §7], Peyre's interpretation of the leading constant also has a connection to certain Artin L-functions and is given by

$$c_S = \alpha(S)\beta(S) \cdot \mathrm{Res}_{s=1} L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) \cdot \tau_H(S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}),$$

where $\alpha(S)$, $\beta(S)$ are computable constants defined in [28], $L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})$ is the Artin L-function of the Galois representation $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$, and τ_H denotes a Tamagawa measure. In particular, in this formulation, the power of $\log(B)$ in the asymptotic corresponds to the order of the pole of $L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})$ at $s = 1$. A heuristic for this power of $\log(B)$ for cubic surfaces is given in [4, Section 8.3] by relating the Picard rank to the pole of a certain Dirichlet series; this idea of determining the exponent of $\log(B)$ via the pole of certain L-functions will be repeated in Chapter 3 when studying Manin's conjecture for Châtelet surfaces.

We end this subsection by stating the main point-counting results of this thesis.

1.3.1 Châtelet surfaces

The first family of del Pezzo surfaces we study (in §3) will be Châtelet surfaces over $\mathbb{Q}$. Let $\Delta \in \mathbb{Q}^\times$ satisfy $\sqrt{-\Delta} \notin \mathbb{Q}$, and let $f(z) \in \mathbb{Q}[z]$ be a separable polynomial of degree 3 or 4. A **Châtelet surface** $X_{\Delta,f}$ over $\mathbb{Q}$ is defined as a proper smooth model of the affine surface:

$$x^2 + \Delta y^2 = f(z).$$

$X_{\Delta,f}$ will be a del Pezzo surface of *degree four*.

Châtelet surfaces were first studied by Châtelet in [7, 8], who was interested in parameterizing the rational points over number fields of these surfaces (see [9]). Later, these surfaces were the first examples of surfaces that are not homogeneous spaces for which the Hasse principle, weak approximation, and other interesting problems about rational points were resolved. For certain pairs $(\Delta, f(z))$, $X_{\Delta,f}$ may fail the Hasse principle due to a Brauer-Manin obstruction. In [11,

12], Colliot-Thélène, Sansuc, and Swinnerton-Dyer show that the **Brauer-Manin obstruction**, a cohomological obstruction that boils down to quadratic reciprocity in this scenario, is the only possible cause of why such a surface may fail the Hasse principle; these results hold not only over $\mathbb{Q}$, but over any number field.

For Châtelet surfaces, Manin's conjecture evaluates to the following prediction for counting rational points of $X_{\Delta,f}$:

Conjecture 1.3.6 (Manin's conjecture for Châtelet surfaces)**.** *Let $\Delta \in \mathbb{Q}^\times$ satisfy that $\sqrt{-\Delta} \notin \mathbb{Q}$ and let $f(z) \in \mathbb{Z}[z]$ be a separable polynomial of degree 3 or 4 that decomposes as a product of r irreducible polynomials $f = f_1 \dots f_r$. Let $X_{\Delta,f}/\mathbb{Q}$ be a Châtelet surface defined by $x^2 + \Delta y^2 = f(z)$. Assume that $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$. Then as $B \to \infty$,*

$$N(X_{\Delta,f}, B) \sim C_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1},$$

where $C_{\Delta,f} > 0$ and

$$\rho_{\Delta,f} = 2 + \#\{1 \leq i \leq r : \sqrt{-\Delta} \in \mathbb{Q}[z]/(f_i(z))\}.$$

Remark 1.3.7. This clean formulation of Manin's conjecture over $\mathbb{Q}$ is possible thanks to the fact that if $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$ then $X_{\Delta,f}(\mathbb{Q})$ is Zariski-dense. Indeed by [12, Proposition 9.8.(iii)], if $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$ then $X_{\Delta,f}$ is birational to a geometrically integral nonconical cubic surface in $\mathbb{P}^3$ with a pair of conjugate singular points. Then by [13, Theorem A] of Coray and Tsfasman, $X_{\Delta,f}$ must be unirational and thus $X_{\Delta,f}(\mathbb{Q})$ is Zariski-dense.

Moreover, the question of when $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$ is settled by the seminal result of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [12]; it demonstrates that $X_{\Delta,f}(\mathbb{Q}) = \emptyset$ if and only if there is either a Brauer-Manin obstruction or a local obstruction to the existence of rational points. Furthermore, as discussed in [12], a Brauer-Manin obstruction to the Hasse principle may occur only when $f(z)$ is the product of two irreducible quadratics.

Numerous instances of Manin's conjecture for Châtelet surfaces have been resolved. In [5], Browning established an upper bound of the correct order of magnitude for Châtelet surfaces with $\Delta > 0$. In [27], Iwaniec and Munshi derive a lower bound when $f(z)$ is an irreducible cubic polynomial and Δ can be either positive or negative. In their seminal result, de la Bretèche, Browning, and Peyre [15] introduced the use of the torsor method, among other tools, to prove Manin's conjecture for $x^2 + y^2 = f(z)$ when f decomposes as a product of distinct linear factors. Later, de la Bretèche and Browning [16] showed Manin's conjecture for the case that $\Delta = 1$ and $f(z)$ contains an irreducible

cubic factor. In [17], the asymptotic for Manin's conjecture is established for $x^2 + y^2 = f(z)$ where f is an irreducible quartic or a product of two irreducible quadratic factors over $\mathbb{Q}(i)$ by de la Bretèche and Tenenbaum; these factorization cases are often regarded as the trickiest cases. Finally, Destagnol [18] resolved Manin's conjecture for $\Delta = 1$ by proving the asymptotic for $x^2 + y^2 = f(z)$ where f is a product of two linearly independent linear factors and an irreducible quadratic factor, the remaining unknown factorization. In the above cases, the techniques are amenable to replacing $x^2 + y^2$ with any positive-definite quadratic form of class number one.

In this thesis, we resolve Manin's conjecture for **all** Châtelet surfaces over $\mathbb{Q}$. Specifically, we prove the following theorem (which will appear in Chapter 3 as Theorem 3.1.3):

Theorem 1.3.8. *Let $\Delta \in \mathbb{Z}$ be a squarefree integer satisfying that $\sqrt{-\Delta} \notin \mathbb{Q}$. Let $f(z) \in \mathbb{Z}[z]$ be a separable polynomial of degree 3 or 4. Let $X_{\Delta,f}/\mathbb{Q}$ be the corresponding Châtelet surface. Then as $B \to \infty$, there exists a constant $c_{\Delta,f}$ such that*

$$N(X_{\Delta,f}, B) = c_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1} + O_{\Delta,f}(B \log(B)^{\rho_{\Delta,f}-1-10^{-10}}).$$

Moreover, $c_{\Delta,f}$ is zero if and only if there is a Brauer-Manin obstruction or a local obstruction to rational points on $X_{\Delta,f}$.

In Theorem 1.3.8, we do not assume a priori that $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$. Our proof of the aforementioned asymptotic result gives an explicit, albeit complicated, expression for $c_{\Delta,f}$. Notably, $c_{\Delta,f}$ is closely related to certain auxiliary varieties that appear in the descent arguments of Colliot-Thélène, Coray, and Sansuc in [10]. Since our expression for $c_{\Delta,f}$ is complicated, we determine when it vanishes but do not pursue an adelic interpretation along the lines of Peyre's prediction [32].

1.3.2 del Pezzo surfaces of degree one

The second family of del Pezzo surfaces we study in Chapter 4 are elliptic surfaces that also happen to be singular del Pezzo surfaces of degree 1. Let $A, B \in \mathbb{Z}$ satisfy that $4A^3 - 27B^2 \neq 0$ and $Q(u, v)$ be a positive-definite quadratic form. We desire to count the set of rational points on the elliptic surface

$$S_{A,B,Q} := \{y^2 = x^3 + AxQ(u, v)^2 + BQ(u, v)^3\}.$$

These surfaces live in weighted projective space $\mathbb{P}(2, 3, 1, 1)$; thus, the rational points are the integral points $(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z})$ satisfying that $\gcd(x, u, v) = 1$, which implies that $\gcd(x, y, u, v) = 1$ for this surface.

Our methods, however, will limit the set of rational points for which we can derive asymptotics; we will count a specific set of rational points.

Definition 1.3.9. A rational point $(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z})$ is called **integral with respect to the singularity** if $\gcd(x, y, Q(u, v)) = 1$. From the structure of $S_{A,B,Q}$, the condition $\gcd(x, y, Q(u, v)) = 1$ is equivalent to $\gcd(x, Q(u, v)) = 1$.

To see that this gcd condition is connected to the singularity of $S_{A,B,Q}$, we recall that the singularity of $S_{A,B,Q}$ is given by $x = y = Q(u, v) = 0$. We must further restrict which integers A, B, and positive-definite quadratic forms $Q(u, v)$ we study; this assumption comes from the automorphic techniques in our method for counting points. Let $\mathcal{F}(A, B)$ denote the finite set of binary quartic forms with invariants $-4A$ and $-4B$, up to equivalence by $\mathrm{SL}_2(\mathbb{Z})$.

Definition 1.3.10. Let $A, B \in \mathbb{Z}$ and $Q(u, v)$ be a positive-quadratic form. The tuple $(A, B, Q(u, v))$ is called **disassociated** if the quadratic extension determined by $Q(u, v)$, denoted as K_Q is not contained in the splitting field of F, for all $F \in \mathcal{F}(A, B)$.

Our second main point-counting result is a partial progress towards Manin's conjecture for $S_{A,B,Q}$, and the first instance of an asymptotic statement for counting rational points on a *del Pezzo surface of degree one.*

Theorem 1.3.11. *Let $A, B \in \mathbb{Z}$ such that $4A^3 - 27B^2 \neq 0$ and $Q(u, v)$ be a positive-definite quadratic form. Assume that $(A, B, Q(u, v))$ is disassociated. Let $U_1 = \mathbb{P}(2, 3, 1, 1)\backslash\{y = 0\}$ and $U_2 = \mathbb{P}(2, 3, 1, 1)\backslash\{Q(u, v) = 0\}$. Define the point-counting function*

$$N(T) := \#\{(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z}) \cap U_1 \cap U_2 : h((x, y, u, v)) \leq T, \gcd(x, Q(u, v)) = 1\}.$$

Then the following asymptotic holds as $T \to \infty$:

$$N(T) = c_{A,B,Q}T + O_{A,B,Q}(T \log(T)^{-10^{-10}}).$$

Remark 1.3.12. The assumption that $(A, B, Q(u, v))$ is disassociated is stronger than necessary for our methods to produce as asymptotic for $N(T)$. Let $\mathcal{H}$ denote the Hilbert splitting field of K_Q; let K_F denote the splitting field of a binary form F. If we have that either $K_Q \not\subset K_F$ or $K_Q \subset K_F$ and $\mathcal{H} \cap K_F = K_Q$, then there exists an explicit exponent $0 \leq \beta_{A,B,Q} \leq 2$ such that

$$N(T) = c_{A,B,Q}T \log(T)^{\beta_{A,B,Q}} + O_{A,B,Q}(T \log(T)^{\beta_{A,B,Q}-10^{-10}}).$$

Moreover, for a given A, B and $Q(u,v)$, this exponent $\beta_{A,B,Q}$ is computable by combining the explicit parameterization given in §4.2 of the integral points and utilizing Mordell's fundamental domain for binary quartics with given invariants (see [30] for explicit bounds). In this subsection, we also give examples of computing when the leading constant $c_{A,B,Q}$ is nonzero, although computing this more generally is still unknown.

While asymptotics for Manin's conjecture have not been established for any del Pezzo surface of degree one, both upper and lower bounds have been explored. Munshi uses sieve methods to investigate in [31] singular surfaces of the form $\{y^2 = (x + \lambda Q(u,v))(x + \delta Q(u,v))(x + \overline{\delta} Q(u,v))\}$, where $\lambda \in \mathbb{Z}$, δ is a generator of an imaginary quadratic field, and $Q(u,v)$ is a positive-definite quadratic form. He establishes that the number of rational points of height up to T is bounded by $\ll T^{5/4+\epsilon}$ for any $\epsilon > 0$. In [29], Le Boudec improves this bound to $T^{1+\epsilon}$, for any $\epsilon > 0$ for surfaces of the form $\{y^2 = (x - e_1 Q(u,v))(x - e_2 Q(u,v))(x - e_3 Q(u,v))\}$, where $e_1, e_2, e_3 \in \mathbb{Z}$ are distinct and $Q(u,v)$ is a quadratic form.

For nonsingular del Pezzo surfaces of degree one, which are elliptic surfaces, one can use the bounds derived by Bhargava, Shankar, Taniguchi, Thorne, Tsimerman, and Zhao [2] and Helfgott and Venkatesh [25] for integral points on elliptic curves to establish nontrivial upper bounds of size $O(T^{2.87+\epsilon})$. On the other side, Frei, Loughran, and Sofos [22] established a lower bound of the correct order of magnitude for del Pezzo surfaces of degree one after passing to an extension of $\mathbb{Q}$ of degree at most 138240. Additionally, motivated by an observation of Heath-Brown in [24], Bonolis and Browning explain in [3] how to achieve an upper bound of $O_\epsilon(B^{2+\epsilon})$ using a suitable rank growth hypothesis for elliptic curves.

The new tool employed in establishing Theorem 1.3.11 is the use of syzygies. The integral points of $S_{A,B,Q}$ with respect to the singularity can be parameterized using a syzygy of Cayley [6] and Hermite [26] for binary quartic forms; this syzygy specifically connects binary quartics to the quadratic twists of elliptic curves. This crucial observation was initially made by Mordell [30] during his seminal investigation of the rational points of elliptic curves. Recently, Duke [20] provided the quantitative version of this relationship that we will utilize in §4.2. A loose summary of the parameterization is such: the solutions to the equation

$$\{(x,y,n) : y^2 = x^3 + Axn^2 + Bn^3, \gcd(x,n) = 1\}$$

are given by

$$\left\{(H_F(\mathbf{m}), T_F(\mathbf{m}), F(\mathbf{m})) : \mathbf{m} \in \mathbb{Z}^2, F \in \mathcal{F}(A, B)\right\},$$

where $H_F(m, n)$ (resp. $T_F(m, n)$) is a binary quartic (resp. sextic) that is formed with the coefficients of F. This parameterization will be further elucidated in §4.2. We note that this parameterization makes the set of integral points with respect to the singularity a thin set of type II (as defined by Serre [34, §9.7]); however, we expect that the set of rational points counted in Conjecture 1.3.6 to not be thin.

1.3.3 Correlation sums

In both Theorem 1.3.8 (when $\Delta > 0$) and Theorem 1.3.11, the main analytic ingredient an estimate of the following correlation sum for a positive-definite quadratic form $Q(u, v)$ and a squarefree binary cubic or quartic $F(x, y)$:

$$\sum_{x,y \leq X} r_Q(F(x, y)),$$

where $r_Q(n) = \#\{u, v \in \mathbb{Z} : Q(u, v) = n\}$. It is through this sum that we connect our problem to the theory of modular forms.

1.4 A connection to automorphic forms

We find ourselves often wanting to estimate the correlation sum:

$$\sum_{x,y \leq X} r_Q(F(x, y)). \tag{1.4.1}$$

To do so, we will connect the problem to the theory of modular forms through the representation number $r_Q(n)$ and the theta function for $Q(u, v)$.

Let K_Q denote the splitting field of the positive-definite quadratic form $Q(u, v)$, $\Delta(Q)$ denote the discriminant, and χ_Q denote the corresponding real quadratic character. Consider the θ-function defined by $Q(u, v)$:

$$\theta_Q(z) := \sum_n r_Q(n) e^{2\pi i n z}.$$

Since $Q(u, v)$ is a positive-definite binary form, $\theta_Q(z)$ will be holomorphic modular form of weight one for $\Gamma_0(4\Delta(Q), \chi_Q) \subset \mathrm{SL}_2(\mathbb{Z})$. Thus, it will have a *decomposition into Eisenstein and cuspidal*

components:

$$\theta_Q(z) = E_Q(z) + C_Q(z).$$

Let us denote the Fourier coefficient of $E_Q(z)$ (resp. $C_Q(z)$) as $\lambda_E(n)$ (resp. $\lambda_C(n)$). We correspondingly have a decomposition of the representation function:

$$r_Q(n) = \lambda_E(n) + \lambda_C(n).$$

Finally, we can decompose our correlation sum:

$$\sum_{x,y\leq X} r_Q(F(x,y)) = \sum_{x,y\leq X} \lambda_E(F(x,y)) + \sum_{x,y\leq X} \lambda_C(F(x,y)). \tag{1.4.2}$$

Our strategy is to prove an asymptotic for the Eisenstein component and derive upper bounds for the cuspidal component.

The Eisenstein coefficient has an explicit expression in terms of class group characters; let $\mathcal{C}$ denote the class group of K_Q and $[Q]$ the class corresponding to $Q(u,v)$, then we can write

$$\lambda_E(n) = \sum_{\substack{\psi\in\mathcal{C}\\ \mathrm{ord}(\psi)\leq 2}} \overline{\psi([Q])} \sum_{\substack{\mathfrak{a}\subset K_Q\\ N(\mathfrak{a})=n}} \psi(\mathfrak{a}).$$

Note that the above sum is over the genus characters of K_Q. More generally, Siegel [35, 36, 37] gives an interpretation of the Eisenstein coefficient for positive-definite quadratic forms in $n \geq 2$ variables in terms of local densities:

$$\lambda_E(n) = \prod_{p\in\mathscr{P}\cup\infty} \beta_{p,Q}(n),$$

where the local factors $\beta_{p,Q}$ are defined as

$$\beta_{\infty,Q}(n) = \lim_{\epsilon\to 0} \frac{\mathrm{vol}(\{(u,v)\in\mathbb{R}^2 : Q(u,v)\in[n-\epsilon, n+\epsilon]\})}{2\epsilon},$$

$$\beta_{p,Q}(n) = \lim_{k\to\infty} \frac{\#\{(u,v)\in(\mathbb{Z}/p^k\mathbb{Z})^2 : Q(u,v) = n \bmod p^k\}}{p^k}.$$

Given the explicit nature of these expressions for $\lambda_E(n)$, it is not surprising that the Eisenstein contribution, i.e.

$$\sum_{x,y\leq X} \lambda_E(F(x,y)),$$

can often be handled using more classical analytic number theory techniques – these techniques are

by no means trivial and can often carry their own difficulties and subtleties (as will be considered for much of Chapter 3). Nevertheless, the Eisenstein component often does not require an appeal to the theory of more general automorphic forms.

It is of note that there are known evaluations and asymptotics for

$$\sum_{x,y\leq X} \tau(F(x,y)),$$

where τ is the divisor function and $F(x,y)$ is a binary form of degree ≤ 4. In particular, Greaves [23] studied the cubic case and Daniel [14] proved the quartic case; the techniques of Daniel are crucial for our analysis of the Eisenstein contribution.

The novel feature of this thesis is the handling of the cuspidal component. Let π denote a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ and let $\lambda_\pi(n)$ denote the normalized Hecke eigenvalues. Observe that any cusp form $C(z)$ of $\Gamma_0(4\Delta,\chi)$ can be decomposed into Hecke eigenforms with corresponding cuspidal automorphic representations. In Chapter 2, we analyze when the sum

$$\sum_{x,y\leq X} |\lambda_\pi(F(x,y))|$$

exhibits logarithmic savings. In particular, we tie the problem to questions about the base change of π to the splitting field of $F(x,y)$.

Theorem 1.4.1. *Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ that satisfies the Ramanujan-Petersson conjecture. Let $P(x) \in \mathbb{Z}[x]$ be irreducible and solvable and K denote the splitting field of $P(x)$. Assume that the **base change** π_K to $GL_2(\mathbb{A}_K)$ is **cuspidal**, then the following bound holds:*

$$\sum_{n\leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X\log(X)^{-0.066}.$$

The above savings comes the Sato-Tate distribution of $\lambda_\pi(n)$. Additionally, the above result can be achieved for binary forms: if K_F denote the splitting field of $F(x,1)$ and π_{K_F} is cuspidal, then

$$\sum_{x,y\leq X} |\lambda_\pi(F(x,y))| \ll X^2\log(X)^{-c}.$$

On the other hand, the Eisenstein component, when nonzero, is expected to grow asymptotically to $X^2\log(X)^\beta$, where $\beta \geq 0$ depends on the factorization of $F(x,y)$ over K_Q. Thus, under the

assumption that the base change is cuspidal, the cuspidal contribution in the decomposition (1.4.2) of our correlation sum is part of the error term. In Chapter 2, we will further discuss the case when π_{K_F} is not cuspidal and the relevant structure with induced representations.

Remark 1.4.2. The analysis of the logarithmic power savings can be done more generally for non-negative multiplicative functions. For instance, fix $C > 1$ and $g(p^k) \leq C^k$ for each prime p. Let $P(x) \in \mathbb{Z}[x]$ be an irreducible polynomial. Then we call the following the "trivial bound":

$$\sum_{n \leq X} g(P(n)) \leq \sum_{n \leq X} C^{\Omega(P(n))} \ll X \log(X)^{C-1}.$$

After applying the sieve methods of §2.2, we see that

$$\sum_{n \leq X} g(P(n)) \ll X \exp\left(\sum_{p \leq X} \frac{(g(p)-1)}{p} \cdot \#\{a \in \mathbb{F}_p : P(a) = 0\} \right).$$

Let $\beta > 0$ and $\mathscr{P}_\beta$ be the set of primes satisfying that $\#\{a \in \mathbb{F}_p : P(a) = 0\} \neq 0$ and $g(p) \leq C - \beta$. Let us write the density of $\mathscr{P}_\beta$ as:

$$\lim_{X \to \infty} \frac{\#\{p \leq X : p \in \mathscr{P}_\beta\}}{\#\{p \leq X : p \in \mathscr{P}\}} =: \delta_\beta.$$

Then, if $\delta_\beta > 0$, we will have the savings over the "trivial bound" given by:

$$\sum_{n \leq X} g(P(n)) \ll X \log(X)^{C-1-\beta\delta_\beta}.$$

In Theorem 1.4.1, we benefit from being to analyze explicitly the automorphic structure behind $g(n)$ to achieve extra savings.

1.5 A structural note

This thesis consists of three chapters corresponding to each of the theorems stated above; each chapter is its own separate paper. Although there is cross-referencing between the three chapters, due to the interconnectedness of the material, each chapter is designed to be self-contained enough to be read independently of the other two.

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Chapter 2

Bounds on sums of Hecke eigenvalues along polynomial sequences

2.1 Introduction

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ that satisfies the Ramanujan-Petersson conjecture; let λ_π denote the normalized Hecke eigenvalues of π. We fix an irreducible nonconstant polynomial $P(x) \in \mathbb{Z}[x]$. In this paper, we focus on when the following sum exhibits savings over the trivial bound:

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)|. \tag{2.1.1}$$

In particular, we connect the above sum to the base change of π to the splitting field of $P(x)$, denoted as K. Let us assume that from now on $P(x)$ is a solvable polynomial.

Theorem 2.1.1. *Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ that satisfies the Ramanujan-Petersson conjecture. Let $P(x) \in \mathbb{Z}[x]$ be irreducible and solvable, and let K denote the splitting field of $P(x)$. Assume that the **base change** π_K to $GL_2(\mathbb{A}_K)$ is **cuspidal**, then the following bound holds:*

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X \log(X)^{-0.066}.$$

Observe that Theorem 2.1.1 applies to those π's corresponding to holomorphic cusp forms ([10, 11]). Our result provides savings over the trivial bound for (2.1.1), despite the presence of the absolute values around the Hecke eigenvalues. Here, the trivial bound is given by using the pointwise bound that $|\lambda_\pi(p)| \leq 2$; in other words, the correlation sum is $O(X \log(X))$. One may expect that there should be further cancellation, even potentially power-saving cancellation, on the sum

$$\sum_{n \leq X} \lambda_\pi(P(n)).$$

Indeed, when $P(x)$ is a linear polynomial, this stronger bound exists. Blomer [1] analyzes sums of the above form when $P(x)$ is quadratic and π corresponds to a holomorphic cusp form of weight $k \geq 4$ and determines when the sum exhibits power-saving cancellation; these sums were further studied in [30, 31] by Templier and Tsimerman. However, when $P(x)$ is of higher degree, it is not clear how to successfully approach these sums (as discussed by Sarnak in [26]). When π is no longer cuspidal, asymptotics are known for certain variants of this problem; Greaves [16] and Daniel [7] derive asymptotics for this expression when λ_π is replaced by the divisor function, which is the Fourier coefficient of an Eisenstein series, along binary cubic and quartic forms.

By instead considering (2.1.1) and attacking these larger sums with sieve methods, we gain flexibility in the polynomial arguments that can be studied; the caveat of this approach is that one

should expect a limit of a logarithmic power saving. Consider, for example, when $P(x) = x$: if π satisfies the Sato-Tate distribution (see [5, 29]), then the following asymptotic for the absolute values of the Hecke eigenvalues was computed in the concluding remarks of [12] by Elliot, Moreno, and Shahidi:

$$\sum_{n \leq X} |\lambda_\pi(n)| \sim \frac{cX}{\log(X)^{1-\frac{8}{3\pi}}}.$$

We note that while there can be nontrivial savings in the L^1-norm, Rankin [25], and independently Selberg [27], showed that the L^2-norm is always of magnitude $\asymp X$:

$$\sum_{n \leq X} |\lambda_\pi(n)|^2 = c_\pi X + O(X^{1-2/5})$$

for *all* cuspidal automorphic representations π. As a consequence of Cauchy-Schwartz, this implies that the L^1-norm is bounded $O(X)$; we will use L^q-norms for $q = 2, 4$, which can be analyzed via analytic properties of $L(s, \pi, \mathrm{sym}^q)$ to deduce the power of logarithm savings.

There have been many successful upper bounds for cases of (2.1.1): Elliot, Moreno, and Shahidi [12] considered the case of the Ramanujan τ-function and their method forms the basis for our proof. Holowinsky [17] studied the case of (2.1.1) for the polynomial $x(x + \ell)$, which appears in the application to holomorphic analogue of Quantum Unique Ergodicity (QUE); Chiriac and Yang [4] considered when π is defined by a non-CM holomorphic newform with trivial character along arbitrary polynomials.

We also briefly remark that sums of the form (2.1.1) have numerous applications in number theory. They played a critical role in the resolution of the holomorphic analogue of QUE by Holowinsky and Soundararajan [18, 19], i.e. mass equidistribution for Hecke eigenforms; notably the bounds achieved depended logarithmically on the coefficients of the polynomial argument $x(x + \ell)$ and hence summing over ℓ is well-controlled. These sums also appear in the recent conditional proof of the mixing conjecture by Blomer, Brumley, and Khayutin [2].

Remark 2.1.2. In Theorem 2.1.1, we focus on the case where $P(x)$ is an irreducible polynomial for simplicity. However, for many of the applications, we will wish for $P(x)$ to be a squarefree polynomial. The sieve used (Theorem 2.2.1) has been generalized to squarefree polynomials and from these generalizations we can deduce that the following holds:

Let $P(x) = P_1(x) \dots P_r(x)$ be a squarefree polynomial and $P_i(x)$ irreducible factors. Let K_i be the splitting field of $P_i(x)$. Assume that the base change π_{K_i} is cuspidal for all $i \in \{1, \dots, r\}$. Then

there exists a positive $c_r > 0$ such that

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X \log(X)^{-c_r}.$$

In §2.4, we establish the connection between (2.1.1) and the base change of π to the splitting field of $P(x)$, denoted as K. In doing so, the problem reduces to the linear case over number fields. After introducing a new argument exploiting the structure of dihedral representations, we establish the following general upper bound for cuspidal representations:

Theorem 2.1.3. *Let K be a number field and π an automorphic representation of $GL_2(\mathbb{A}_K)$ that is tempered at all finite places of K. Then there exists a positive constant $c_\pi > 0$ such that*

$$\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ N(\mathfrak{a}) \leq X}} |\lambda_\pi(\mathfrak{a})| \ll_\pi X \log(X)^{-c_\pi}$$

if and only if π is cuspidal. Moreover, if π is cuspidal, then there is a universal $c > 0.066$ such that $c_\pi \geq c$.

This upper bound is novel in its lack of assumptions on the moments of π, i.e. requirements on the sizes of

$$\sum_{n \leq X} |\lambda_\pi(n)|^k$$

for $k > 2$. In particular, we do not require $\mathrm{sym}^2(\pi)$ (or any higher symmetric power) to be a cuspidal representation. In the proof of Theorem 2.1.3, the second and fourth moments will be considered. Additionally, for π non-cuspidal, the proof provided shows that unless π is constructed with a quadratic Hecke character, there is a positive constant $\beta_\pi > 0$ such that

$$\sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ N(\mathfrak{a}) \leq X}} |\lambda_\pi(\mathfrak{a})| \asymp_\pi X \log(X)^{\beta_\pi}.$$

Finally, we hope to understand (2.1.1) when π_K is not a cuspidal representation. The work on the base change of GL_2 representations by Langlands [22] classifies exactly when this phenomenon occurs for $P(x)$ a solvable polynomial; we use this to establish lower bounds for the potential size of the sum.

Theorem 2.1.4. *Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ that satisfies the*

Ramanujan-Petersson conjecture. Let $P(x) \in \mathbb{Z}[x]$ be irreducible and K be the splitting field of $P(x)$ be solvable. Assume that the base change π_K is not cuspidal; hence, it has the form

$$\pi = Ind_{W_F}^{W_{\mathbb{Q}}} \theta$$

for some quadratic subfield $F \subset K$ and unitary Hecke character θ of F. If θ has finite order, let L_θ denote the ray class field of θ.

If θ is of infinite order or if $Gal(L_\theta/L_\theta \cap K) \not\subset \ker(\theta^2)$, then

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X \log(X)^{1/2}.$$

Otherwise, the following lower bound holds:

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \geq \sum_{\substack{n \leq X \\ P(n) \ squarefree}} \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ N(\mathfrak{a}) = |P(n)|}} 1.$$

Remark 2.1.5. We can see that a necessary requirement for π_K to be noncuspidal is that $F \subset K$. In this case, if instead we looked at the representation induced from the trivial character $\Psi = \mathrm{Ind}_{W_F}^{W_{\mathbb{Q}}} 1$, we have that

$$\sum_{n \leq X} |\lambda_\Psi(|P(n)|)| \asymp_{\Psi,P} X \log(X).$$

In comparison, if $F \not\subset K$, then the corresponding sum of the representation induced from the trivial character is of size $\asymp_{\Psi,P} X$. When $F \subset K$, this extra power of $\log(X)$ is a consequence of the pole at $s = 1$ of the Dirichlet series:

$$\sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(N(\mathfrak{a}))}{N(\mathfrak{a})^s},$$

where χ is the quadratic character associated to the quadratic field F.

Remark 2.1.6. We note that there are instances of squarefree polynomials $P(x) \in \mathbb{Z}[x]$ such that

$$\sum_{n \leq X} \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ N(\mathfrak{a}) = |P(n)|}} 1 = 0;$$

in particular, any Châtelet surface with a local or Brauer-Manin obstruction would produce such an example. However, it is expected that if $P(x)$ is irreducible that such Brauer-Manin obstructions do not occur (as was shown by Colliot-Theléne and Sansuc [6] conditionally on Schinzel's Hypothesis H).

Additionally, if $P(x)$ is irreducible and there are no local obstructions to solutions of $N(\mathfrak{a}) = P(n)$, then the following lower bound is expected to hold when $F \subset K$:

$$\sum_{\substack{n \leq X \\ P(n) \text{ squarefree}}} \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ N(\mathfrak{a}) = |P(n)|}} 1 \gg_{P,F} X \log(X).$$

In Proposition 3.8.1, the analog of the above lower bound is shown when $P(x)$ is replaced by an irreducible binary form of degree at most 4.

Finally, let us comment on a natural generalization of this problem, when $P(x)$ is replaced by an absolutely irreducible *multivariate* polynomial:

$$\sum_{x_i \leq X} |\lambda_\pi(|P(x_1, \dots, x_n|)|.$$

First, if $P(\mathbf{x})$ is a homogeneous binary form, then we achieve the same theorems as the single variable case (see the remarks after Theorem 2.2.1). In particular, if K is the splitting field of $P(x, 1)$, then if the base change π_K is cuspidal then

$$\sum_{x,y \leq X} |\lambda_\pi(|P(x, y)|)| \ll X^2 \log(X)^{-c}.$$

In a similar vein for $n \geq 3$ or nonhomogeneous polynomials with $n = 2$, after an appropriate sieve for nonnegative multiplicative functions along polynomial values, this problem reduces to questions about the Sato-Tate distribution of π and the number of local solutions of $P(\mathbf{x}) \bmod p$. Under the assumption of a wide range of conjectures (including those surrounding connections between the Hasse-Weil L-function of the variety $\{P(x_1, \dots, x_n) = 0\}$ and automorphic representations), one can achieve conditional logarithmic power savings over the trivial bound again by using second and fourth moment calculations.

2.1.1 Outline and notation

Let us first set some standard notation:

- p, $\mathfrak{p}$, or $\mathfrak{P}$ will denote primes or prime ideals,
- sums over p, $\mathfrak{p}$, or $\mathfrak{P}$ are sums over primes or prime ideals,
- $\varrho(p) = \#\{x \in \mathbb{F}_p : P(x) = 0\}$ will denote the local count of the polynomial $P(x)$,

- π, Ξ will denote GL_2 representations, whereas ψ, ξ, θ will denote GL_1 representations,
- Let $\Pi(G)$ denote the irreducible admissible automorphic representations of G. The base change map sends

$$\Pi(\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})) \to \Pi(\mathrm{GL}_2(\mathbb{A}_K))$$

and is analyzed for solvable extensions by Langlands in [22].

We end this section by outlining the rest of the paper. In §2.2, we establish the necessary background on sieves; in particular, we require a sieve for nonnegative multiplicative functions along polynomial values that was originally derived by Nair [23]:

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \ll_{\pi,P} X \exp\left(\sum_{p \leq X} \frac{\varrho(p)(|\lambda_\pi(p)| - 1)}{p}\right),$$

where $\varrho(p) = \#\{x \in \mathbb{F}_p : P(x) = 0\}$. We also discuss an elementary lower bound sieve in the case $P(x) = x$.

In §2.3, we establish Theorem 2.1.3. For the upper bound, we split into the case when π is a Galois representation and otherwise. When π is a Galois representation, the inner sum over primes can be dealt with directly via the Chebotarev density theorem (see §2.3.1). Otherwise, we relate $|\lambda_\pi(p)|$ to its second and fourth moments; in these computations, it will suffice to know that $\mathrm{sym}^2(\pi)$ is a cuspidal representation (see §2.3.2). We will use heavily the fact that for any cuspidal automorphic representation Ξ, the Rankin-Selberg L-function, $L(s, \Xi \times \overline{\Xi})$, has a pole of order *exactly one*. Finally, in §2.3.3, we compute directly the lower bound in the case when π is noncuspidal by using the isobaric decomposition $\pi = \psi_1 \boxplus \psi_2$, where ψ_1, ψ_2 are unitary Hecke characters (see [21] for a description of isobaric decomposition).

In §2.4, we will prove Theorem 2.1.1 and Theorem 2.1.4. First, we reduce Theorem 2.1.1 to Theorem 2.1.3 via base change; specifically, if K is the splitting field of $P(x)$, then all but finitely many primes p that split in K satisfy:

$$\lambda_\pi(p)\varrho(p) = \sum_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ N(\mathfrak{p}) = p}} \lambda_{\pi_K}(\mathfrak{p}).$$

Then in §2.4.2, we analyze when π_K is noncuspidal. The key idea is that if π_K is noncuspidal, then π must be induced by a Hecke character and we can instead look at the problem of base change on GL_1. Then if $\theta^2 \circ N_{K/F}$ is nontrivial, we obtain a saving over the trivial bound – this condition is

implied by the condition stated in Theorem 2.1.4.

In §2.5, we discuss the particular example of representations induced by class group characters of quadratic fields. In this case, K_θ is given by the Hilbert class field. This section is written for a future application to counting rational points on del Pezzo surfaces of degree one.

2.2 Sieve methods

2.2.1 Upper bound sieves

We begin by discussing the classical upper bound that will handle our linear case, i.e. Theorem 2.1.3. Let $f(n)$ be a nonnegative multiplicative function supported on squarefree numbers. Then if $f(n)$ satisfies the following condition for the sum over primes:

$$\sum_{w\leq p<X} \frac{f(p)\log(p)}{p} \leq a\log(X/w)+b,$$

for all w with $2 \leq w < X$ and $a \geq 0, b \geq 1$ constants, then one can derive that[1]

$$\sum_{n\leq X} f(n) \ll X \prod_{p\leq X}\left(1+\frac{f(p)-1}{p}\right).$$

We apply a sieve deduced by Nair [23] for nonnegative multiplicative functions along polynomial values – Nair also derived a version for short intervals that we will not need. It was later generalized by de la Bretéche and Browning [8] and de la Bretéche and Tenenbaum [9] for binary forms; by Browning and Sofos [3] for sums over prime ideals.

Theorem 2.2.1 (Nair, [23]). *Let $f(n)$ be a nonnegative multiplicative function such that there exist constants $A, B \geq 0$ such that for any natural number v, $f(p^v) \leq A^v$, and in general, $f(n) \leq Bn^\epsilon$. Then the following bound holds for any irreducible polynomial $P(x) \in \mathbb{Z}[x]$:*

$$\sum_{n\leq X} f(|P(n)|) \ll_{P,A,B} X\exp\left(\sum_{p\leq X}\frac{\varrho(p)(f(p)-1)}{p}\right),$$

where $\varrho(p) = \#\{a \bmod p : P(a) = 0\}$.

Furthermore, Nair and Tenenbaum [24] generalized the above result for functions in several variables. A consequence of their work is that if $P_1(x), \ldots, P_k(x) \in \mathbb{Z}[x]$ are distinct irreducible

[1] See [13, Theorem A.1] for further details

polynomials and $f_1, \dots, f_k$ are nonnegative multiplicative functions that are bounded on prime values, then

$$\sum_{n \leq X} f_1(|P_1(n)|) \dots f_k(|P_k(n)|) \ll_{P_1,\dots,P_k,f_1,\dots,f_k} X \exp\left(\sum_{i=1}^{k} \sum_{p_i \leq X} \frac{\varrho_{P_i}(p_i)(f_i(p_i)-1)}{p_i}\right),$$

where $\varrho_{P_i}(p) = \#\{a \bmod p : P_i(a) = 0\}$. A consequence of the work of de la Bretéche and Tenenbaum [9] for binary forms is that a similar bound holds for $g_1, \dots, g_k$ nonnegative multiplicative functions and $F_1, \dots, F_k$ distinct irreducible binary forms:

$$\sum_{x,y \leq X} \prod_{i=1}^{k} g_i(|F_i(x,y)|) \ll_{g_1,\dots,g_k,F_1,\dots,F_k} X^2 \exp\left(\sum_{i=1}^{k} \sum_{p_i \leq X} \frac{\varrho_{F_i}(p_i)(g_i(p_i)-1)}{p_i^2}\right),$$

where $\varrho_{F_i}(p) = \#\{(a,b) \bmod p : F(a,b) = 0\}$. For this paper, we will stick however to the case of one irreducible polynomial, $P(\mathbf{x})$, and the particular nonnegative multiplicative functions $|\lambda_\pi(n)|$.

Remark 2.2.2. We note that Chiriac and Yang [4] have derived a multivariate analog of the above sieve that can be applied when $P(\mathbf{x}) \in \mathbb{Z}[x_1, ..., x_n]$ is an absolutely irreducible polynomial.

2.2.2 Lower bound sieves

In Theorem 2.1.3, we also want to consider lower bounds of this sum. To do so, we use a corresponding lower bound sieve.

Theorem 2.2.3 ([13, Theorem A.4])**.** *Let $f(n)$ be a multiplicative nonnegative function and satisfy that:*

$$\sum_{y < p \leq x} \frac{f(p)\log(p)}{p} \leq a\log(x/y) + b,$$

$$\sum_{p \leq y} f(p)\log(p) \gg y.$$

Then $f(n)$ satisfies that

$$\sum_{n \leq X} f(n) \gg X \prod_{p \leq X} \left(1 + \frac{f(p)-1}{p}\right).$$

Remark 2.2.4. If $f(n)$ is bounded on prime values, then the first condition is always satisfied.

Corollary 2.2.5. *Let K be a number field and $g : I_K \to \mathbb{R}$ be a nonnegative multiplicative function*

on the ideals of K that is bounded on prime ideals. If

$$\sum_{\substack{\mathfrak{p}\subset\mathcal{O}_K\\ N(\mathfrak{p})\leq y}} g(\mathfrak{p})\log(N(\mathfrak{p})) \gg y,$$

then the following bound is satisfied:

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})\leq X}} g(\mathfrak{a}) \gg_K X \prod_{N(\mathfrak{p})\leq X}\left(1+\frac{g(\mathfrak{p})-1}{N(\mathfrak{p})}\right).$$

Proof. Apply Theorem 2.2.3 with the nonnegative multiplicative function

$$f(n)=\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K\\ N(\mathfrak{a})=n}} g(\mathfrak{a}).$$

□

Remark 2.2.6 (Non-existence of a lower bound sieve for polynomial values). One might expect, given Theorem 2.2.3, that a similar bound would hold for the sum:

$$\sum_{n\leq X} f(|P(n)|).$$

However, this can not happen; for instance, take $f(p)=\mathbf{1}_{p\equiv 3 \bmod 4}$ and $P(x)=x^2+1$. Then it is clear that

$$\sum_{n\leq X} f(|P(n)|)=0.$$

However, if we consider what the analogous lower bound to Theorem 2.2.3 would produce,

$$X\prod_{p\leq X}\left(1+\frac{\varrho(p)(f(p)-1)}{p}\right)=X\prod_{\substack{p\leq X\\ p\equiv 1 \bmod 4}}\left(1-\frac{2}{p}\right)\gg X\log(X)^{-1},$$

this causes immediate issues. More complicatedly, such a bound would immediately contradict the existence of a possible Brauer-Manin obstruction for Châtelet surfaces.

2.3 Linear polynomials

In this section, we prove Theorem 2.1.3. Recall that K is a number field and π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ that is tempered at all finite places of K. Consequently, we know that the normalized Hecke eigenvalues satisfy that

$$|\lambda_\pi(\mathfrak{p})| \leq 2,$$

for all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$.

To prove the theorem, we split the study into different cases. In the first two subsection, we prove that the upper bound holds whenever π is cuspidal. In the final subsection (in §2.3.3), we will show that when π is not cuspidal, a sufficient lower bound holds.

For the upper bound, we first apply Theorem 2.2.1, as the assumptions are satisfied:

$$\sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \ll_K X \exp\left(\sum_{N(\mathfrak{p})\leq X} \frac{|\lambda_\pi(\mathfrak{p})|-1}{N(\mathfrak{p})}\right).$$

In order to study the sum

$$S_\pi(X) := \sum_{N(\mathfrak{p})\leq X} \frac{|\lambda_\pi(\mathfrak{p})|-1}{N(\mathfrak{p})},$$

we relate $|\lambda_\pi(\mathfrak{p})|$ to the Hecke eigenvalues of $\mathrm{sym}^2(\pi)$, the automorphic representation on $\mathrm{GL}_3(\mathbb{A}_K)$. This causes us to need to break the proof into two cases: when π is dihedral and when π is non-dihedral.

2.3.1 Galois representations

First, we tackle the case when π is a Galois representation, and hence has finite image. There is a Galois extension L/K such that

$$\pi : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(\mathbb{C}).$$

Additionally, the Hecke eigenvalues are of the form

$$\lambda_\pi(\mathfrak{p}) = \mathrm{Tr}(\pi(\mathrm{Frob}_\mathfrak{p})) = \chi_\pi(\mathrm{Frob}_\mathfrak{p}),$$

where χ_π denotes the character of the representation π. In this case, we can see that

$$S_\pi(X) = \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)| \sum_{\substack{N(\mathfrak{p}) \leq X \\ \mathrm{Frob}_\mathfrak{p} = \sigma}} \frac{1}{N(\mathfrak{p})} - \log\log(X) + O_K(1).$$

Recall that χ_π is invariant under conjugation and hence we can apply the Chebotarev density theorem to determine that

$$S_\pi(X) = \left(\frac{1}{|\mathrm{Gal}(L/K)|} \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)| - 1 \right) \cdot \log\log(X) + O_K(1).$$

In order to establish Theorem 2.1.3 in this case, we need to prove that

$$\frac{1}{|\mathrm{Gal}(L/K)|} \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)| < 1.$$

To do so, we apply Cauchy-Schwartz:

$$\frac{1}{|\mathrm{Gal}(L/K)|} \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)| \leq \left(\frac{1}{|\mathrm{Gal}(L/K)|} \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)|^2 \right)^{1/2}.$$

Since π is an irreducible representation, we know that

$$\frac{1}{|\mathrm{Gal}(L/K)|} \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)|^2 = 1.$$

Observe that the Cauchy-Schwartz inequality is in fact a strict inequality if and only if $|\chi(\sigma)| = 1$ for every $\sigma \in \mathrm{Gal}(L/K)$. On the other hand, since π is cuspidal, it is a nontrivial irreducible representation and hence $\chi(1) = \dim(\pi) \geq 2$.

Consequently, we see that

$$S_\pi(X) \leq -|\mathrm{Gal}(L/K)|^{-1} \cdot \log\log(X) + O_K(1),$$

and thus we have that

$$\sum_{N(\mathfrak{a}) \leq X} |\lambda_\pi(\mathfrak{a})| \ll_K X \log(X)^{-|\mathrm{Gal}(L/K)|^{-1}}.$$

.

Remark 2.3.1. Furthermore, for a fixed Galois representation, we can understand completely the

exponent

$$\sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \sim c_\pi X \log(X)^{\beta_\pi}$$

where β_π is one fewer than the L^1-norm of the trace function of π.

Remark 2.3.2. One can instead look at cuspidal representations of the Weil group $W_{L/K}$, of which Galois representations are a subset, and derive a similar logarithmic savings over the trivial bound for the L^1-norm of π.

2.3.2 Using the second and fourth moment

Now, let us assume that π is any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ satisfying the Ramanujan-Petersson conjecture; we will use information about the L^2-norm and L^4-norm of π to derive nontrivial savings on the L^1-norm of π. We follow the proof of Elliott, Moreno, and Shahidi [12] regarding the Ramanujan τ-function. Similar proofs also appear in the work of Chiriac and Yang [4], and Holowinsky [17].

Let us define

$$\delta := \inf_{-1\leq y\leq 2} y^{-2}(1 + y/2 - (1 + y)^{1/2}) \approx 0.067.$$

We remark that the definition of δ is strictly positive and independent of π. By the definition of δ, the following relation holds for any prime ideal $\mathfrak{p}$:

$$\begin{aligned} |\lambda_\pi(\mathfrak{p})| - 1 &\leq \frac{1}{2}(\lambda_\pi(\mathfrak{p})\lambda_{\overline{\pi}}(\mathfrak{p}) - 1) - \delta\left(\lambda_\pi(\mathfrak{p})\lambda_{\overline{\pi}}(\mathfrak{p}) - 1\right)^2 \\ &\leq -\delta + \left(\frac{1}{2} + 2\delta\right)(\lambda_\pi(\mathfrak{p})\lambda_{\overline{\pi}}(\mathfrak{p}) - 1) - \delta\left(\lambda_\pi(\mathfrak{p})^2\lambda_{\overline{\pi}}(\mathfrak{p})^2 - 2\right). \end{aligned} \tag{2.3.1}$$

Let $\pi \boxtimes \pi'$ denote the Rankin-Selberg convolution of π and π'. From the above inequality, we know

$$S_\pi(X) \leq -\delta \log\log(X) + (1/2 + 2\delta) \cdot T_{\pi\boxtimes\overline{\pi}}(X) - \delta \cdot T_{\pi\boxtimes\pi\boxtimes\overline{\pi}\boxtimes\overline{\pi}}(X) + O_K(1),$$

where we define

$$T_{\pi\boxtimes\overline{\pi}}(X) = \sum_{N(\mathfrak{p})\leq X} \frac{\lambda_{\pi\boxtimes\overline{\pi}}(\mathfrak{p}) - 1}{N(\mathfrak{p})}, \qquad T_{\pi\boxtimes\pi\boxtimes\overline{\pi}\boxtimes\overline{\pi}}(X) = \sum_{N(\mathfrak{p})\leq X} \frac{\lambda_{\pi\boxtimes\pi\boxtimes\overline{\pi}\boxtimes\overline{\pi}}(\mathfrak{p}) - 2}{N(\mathfrak{p})}.$$

In other words, $T_{\pi\boxtimes\overline{\pi}}$ measures the second moment of π and $T_{\pi\boxtimes\pi\boxtimes\overline{\pi}\boxtimes\overline{\pi}}$ measures the fourth moment. At this point, we split into the case when π is *dihedral* and *non-dihedral*.

If π is a dihedral representation, then it must also be a Galois representation [14, 20] and thus §2.3.1 applies. Now if π is dihedral representation of $\mathrm{Gal}(L/K) \cong D_{2n}$, then §2.3.1 shows that

$$\sum_{N(\mathfrak{a}) \leq X} |\lambda_\pi(\mathfrak{a})| \ll X \log(X)^{-1/2n}.$$

However, this exponent will not be uniform in the choice of cuspidal representation π. We look further at the value of the exponent and the L^1-norm of the trace function in this case:

$$\frac{1}{|\mathrm{Gal}(L/K)|} \sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_\pi(\sigma)| - 1.$$

Since π is a two-dimensional irreducible representation of D_{2n}, we can calculate explicitly this expression as:

$$\sum_{\sigma \in \mathrm{Gal}(L/K)} |\chi_{\pi(\sigma)}| = 2 + \sum_{a=1}^{n-1} 2|\cos(2\pi a/n)|.$$

Define the function

$$g(n) = \frac{1}{n} \sum_{a=0}^{n-1} |\cos(2\pi a/n)|,$$

and its maximum value as

$$\delta' := \sup_{n \in \mathbb{Z}} g(n).$$

We claim that $\delta' \leq 4/5$. First, we note that we must have $n \geq 3$ for there to exist two-dimensional irreducible representations of D_{2n}. Additionally, for $n \geq 5$, we can see that

$$\left| g(n) - \int_0^1 |\cos(2\pi x)| dx \right| < 1/10.$$

Since we know that

$$\int_0^1 |\cos(2\pi x)| dx = \frac{2}{\pi} \approx 0.6366,$$

after computing the values of $g(3)$ and $g(4)$, we see that $\delta' < 4/5$. Hence, for any dihedral representation, we have a uniform exponent of logarithmic power savings given by:

$$\sum_{N(\mathfrak{a}) \leq X} |\lambda_\pi(\mathfrak{a})| \ll X \log(X)^{-1/5}.$$

So, it remains to analyze when π is not dihedral. We claim that in this case:

$$T_{\pi\boxtimes\overline{\pi}}(X) = O_\pi(1), T_{\pi\boxtimes\pi\boxtimes\overline{\pi}\boxtimes\overline{\pi}}(X) = O_\pi(1). \tag{2.3.2}$$

Before we prove the above claim, we make the observation that both sums converge if and only if the following limits exist:

$$\lim_{s\to 1^+} \zeta_K(s)^{-1} L(s, \pi \boxtimes \overline{\pi}), \tag{2.3.3}$$

$$\lim_{s\to 1^+} \zeta_K(s)^{-2} L(s, \pi \boxtimes \pi \boxtimes \overline{\pi} \boxtimes \overline{\pi}). \tag{2.3.4}$$

Let us start with the $T_{\pi\boxtimes\overline{\pi}}(X)$ and the L-function $L(s, \pi \boxtimes \overline{\pi})$. Since π is cuspidal, we know that $L(s, \pi \boxtimes \overline{\pi})$ has a *simple pole* at $s = 1$; consequently, the limit (2.3.3) exists and hence the first part of (2.3.2) is true. Another way of viewing this statement is that $L(s, \pi, \mathrm{Ad})$ is analytic at $s = 1$, where $\mathrm{Ad}(\pi)$ is the Gelbart-Jacquet lift of [14].

Furthermore, we can decompose $\pi \boxtimes \overline{\pi} = \Lambda^2(\pi) \boxplus \mathrm{sym}^2(\pi)$; here we use $\boxplus$ to denote the isobaric sum (see [21]). If π is non-dihedral, we know that $\mathrm{sym}^2(\pi)$ is cuspidal. Now we consider $T_{\pi\boxtimes\pi\boxtimes\overline{\pi}\boxtimes\overline{\pi}}(X)$ and (2.3.4). From our decomposition of $\pi \boxtimes \overline{\pi}$, we know that $L(s, \pi \boxtimes \pi \boxtimes \overline{\pi} \boxtimes \overline{\pi})$ has the same analytic behavior at $s = 1$ as the product $\zeta_K(s) L(s, \pi, \mathrm{sym}^2)^2 L(s, \mathrm{sym}^2(\pi) \boxtimes \mathrm{sym}^2(\overline{\pi}))$ – we know that $L(s, \pi, \mathrm{sym}^2)$ is entire at $s = 1$ since $\mathrm{sym}^2(\pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_K)$. Similarly, we know that since $\mathrm{sym}^2(\pi)$ is cuspidal, $L(s, \mathrm{sym}^2(\pi) \boxtimes \mathrm{sym}^2(\overline{\pi}))$ has a pole of order one at $s = 1$; here $\mathrm{sym}^2(\pi) \boxtimes \mathrm{sym}^2(\overline{\pi})$ is a $\mathrm{GL}_3 \times \mathrm{GL}_3$ Rankin-Selberg convolution. Hence, this L-function has a pole of *order two* at $s = 1$. Thus, the limit (2.3.4) exists and this establishes the second half of claim (2.3.2).

Hence, in the case when π is non-dihedral, we know that

$$\sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \ll_\pi X \log(X)^{-\delta}.$$

Now, taking $c = \min(\delta, \delta')$, this completes the cuspidal case of Theorem 2.1.3.

2.3.3 Lower bounds when π is not cuspidal

Let π be a non-cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_K)$, so π has the isobaric decomposition (see [21])

$$\pi = \psi_1 \boxplus \psi_2,$$

where ψ_1, ψ_2 are $\mathrm{GL}_1(\mathbb{A}_K)$-representations, i.e. quasi-characters. Since π is tempered at each finite place, we take ψ_1 and ψ_2 to be unitary.

Let us assume that $|\lambda_\pi(\mathfrak{p})|$ satisfies the assumptions of Theorem 2.2.3. This would then imply that

$$\sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \gg_K X \exp\left(\sum_{N(\mathfrak{p})\leq X} \frac{|\lambda_\pi(\mathfrak{p})|-1}{N(\mathfrak{p})}\right).$$

Now, since $\pi = \psi_1 \boxplus \psi_2$, we know that

$$\lambda_\pi(\mathfrak{p}) = \psi_1(\mathfrak{p}) + \psi_2(\mathfrak{p}).$$

Hence, we can estimate:

$$S_\pi(X) = \sum_{N(\mathfrak{p})\leq X} \frac{|\psi_1(\mathfrak{p})+\psi_2(\mathfrak{p})|-1}{N(\mathfrak{p})} = \sum_{N(\mathfrak{p})\leq X} \frac{|1+\psi_1^{-1}\psi_2(\mathfrak{p})|-1}{N(\mathfrak{p})}.$$

Define the character $\psi := \psi_1^{-1}\psi_2$ and define the angle $\theta_\mathfrak{p} \in [0,1]$ such that $\psi(\mathfrak{p}) = e(2\pi i\theta_\mathfrak{p})$. Thus, we have that

$$S_\pi(X) = \sum_{N(\mathfrak{p})\leq X} \frac{\sqrt{2+2\cos(2\pi\theta_\mathfrak{p})}-1}{N(\mathfrak{p})}.$$

We split our analysis into two cases: when $\psi = 1$ and when ψ is nontrivial. First, if $\psi = 1$, then $\theta_\mathfrak{p} = 0$ for all primes $\mathfrak{p}$. So, we have that

$$S_\pi(X) = \sum_{N(\mathfrak{p})\leq X} \frac{1}{N(\mathfrak{p})} = \log\log(X) + O_K(1),$$

and thus

$$\sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \gg_K X\log(X).$$

Now, if $\psi \neq 1$, then we know that for any $A > 0$:

$$\sum_{N(\mathfrak{p})\leq X} \cos(2\pi\theta_\mathfrak{p}) = O_\psi(X\log(X)^{-A}).$$

Hence, we have that

$$\lim_{X\to\infty} \mathbb{E}_{N(\mathfrak{p})\leq X} \sqrt{2+2\cos(2\pi\theta_\mathfrak{p})} \geq 1.$$

Consequently, we have that

$$S_\pi(X) \gg_\pi 1, \qquad \sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \gg_\pi X.$$

Thus, in order to complete the proof of Theorem 2.1.3, it suffices to prove that the multiplicative function $|\lambda_\pi(\mathfrak{p})|$ satisfies the assumptions of Theorem 2.2.3.

Finally, let us check that when $\pi = \psi_1 \boxplus \psi_2$

$$\sum_{N(\mathfrak{p})\leq y} |\lambda_\pi(\mathfrak{p})| \log(N(\mathfrak{p})) \gg y.$$

Again, let $\psi = \psi_1^{-1}\psi_2$ and $\psi(\mathfrak{p}) = e(2\pi i\theta_\mathfrak{p})$. Then we can write

$$\sum_{N(\mathfrak{p})\leq y} |\lambda_\pi(\mathfrak{p})| \log(N(\mathfrak{p})) = \sum_{N(\mathfrak{p})\leq y} |1+\psi(\mathfrak{p})| \log(N(\mathfrak{p})) = \sum_{N(\mathfrak{p})\leq y} \sqrt{2+2\cos(2\pi\theta_\mathfrak{p})} \cdot \log(N(\mathfrak{p})).$$

We already established that

$$\lim_{X\to\infty} \mathbb{E}_{N(\mathfrak{p})\leq X} \sqrt{2+2\cos(2\pi\theta_\mathfrak{p})} \geq 1.$$

Consequently:

$$\sum_{N(\mathfrak{p})\leq y} \sqrt{2+2\cos(2\pi\theta_\mathfrak{p})} \cdot \log(N(\mathfrak{p})) \gg y$$

as desired. Thus $|\lambda_\pi(\mathfrak{p})|$ satisfies the conditions of Theorem 2.2.3. □

Remark 2.3.3. We note that as a consequence of Theorem 2.2.1 that when π is noncuspidal,

$$\sum_{N(\mathfrak{a})\leq X} |\lambda_\pi(\mathfrak{a})| \asymp_\pi X \log(X)^{\beta_\pi},$$

where β_π is defined as

$$\beta_\pi := \lim_{X\to\infty} \frac{1}{\log\log(X)} \sum_{N(\mathfrak{p})\leq X} \frac{\sqrt{2+2\cos(2\pi\theta_\mathfrak{p})}}{N(\mathfrak{p})} - 1.$$

Furthermore, β_π is strictly positive unless ψ is a quadratic character.

2.4 Higher degree polynomials

In the following section, we prove Theorem 2.1.1 and Theorem 2.1.4. Before doing so, we will review base change for GL_1 and GL_2 automorphic representations; as this theory will be critical in both sections. For a more comprehensive discussion of base change, see [15, 22].

Let us recall the following notation:

- F is a fixed number field; in our applications it will always be $F = \mathbb{Q}$,
- $G = \mathrm{GL}_1$ or GL_2 defined over F,
- For a number field extension E/F, $G_{E/F}$ is the restriction of scalars of G over E, i.e. $G_{E/F} = G(\mathbb{A}_E)$,
- The Weil group of a number field F is given by a tuple $(W_F, \varphi, \{r_E\}_{E/F})$ such that W_F is a topological group with a continuous homomorphism:

$$\varphi : W_F \to \mathrm{Gal}(\overline{F}/F)$$

with dense image that satisfies:

  - For any finite extension E/F with $E \subset \overline{F}$, define $W_E = \varphi^{-1}(\mathrm{Gal}(\overline{F}/E))$. Since φ is continuous, W_E is open in W_F; additionally, it induces a bijection

$$W_F/W_E \xrightarrow{\sim} \mathrm{Gal}(\overline{F}/F)/\mathrm{Gal}(\overline{F}/E) \cong \mathrm{Gal}(E/F),$$

where the last isomorphism holds when E/F is Galois.

  - For each finite extension E/F, the map r_E gives an isomorphism of topological groups

$$r_E : \mathbb{A}_E^\times/E^\times \xrightarrow{\sim} W_E^{\mathrm{ab}},$$

where W_E^{ab} is the maximal abelian Hausdorff quotient of W_E. Furthermore, these maps r_E satisfy four conditions that are described explicitly in [28, (W1),(W2),(W3),(W4)].

We also set $\Pi(G)$ to denote the set of irreducible admissible automorphic representations of $G(\mathbb{A}_F)$. We aim to construct a map that describes base change:

$$\Pi(G) \to \Pi(G_{E/F})$$

$$\pi \mapsto \pi_{E/F}.$$

Let us assume that E/F is a solvable number field extension; hence there exists a chain of subfields

$$F = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \ldots \subsetneq E_{n-1} \subsetneq E_n = E$$

where E_{i+1}/E_i is a cylic extension of prime order. Base change on L-groups gives us a map

$$\Pi(G) \to \Pi(G_{E_1/F}) \to \ldots \to \Pi(G_{E_{n-1}/F}) \to \Pi(G_{E/F})$$

that is compatible with local data; when $G = \mathrm{GL}_2$, this map was constructed and proved by Langlands in [22].

The GL_1 base change problem

First, we review the case of GL_1-base change. Let θ be a one-dimensional representation of W_F, i.e. a quasicharacter of $\mathbb{A}_F^\times/F^\times$. Let E/F denote a cyclic extension. The base change lift

$$\Pi(\mathrm{GL}_1(\mathbb{A}_F)) \to \Pi(\mathrm{GL}_1(\mathbb{A}_E))$$

is given explicitly by the map

$$\theta \mapsto \theta \circ N_{E/F} =: \theta_E.$$

In particular, for an ideal $I \subset E$,

$$\lambda_{\theta_E}(I) = \theta(N_{E/F}(I)) = \lambda_{\theta_F}(N_{E/F}(I)).$$

Next, observe that $\widehat{\mathrm{Gal}(E/F)}$ are themselves quasi-characters that are trivial on $N_{E/F}(\mathbb{A}_E^\times/E^\times)$ and hence will be trivial after base change to E.

Proposition 2.4.1 ([15, Proposition 1])**.** *Let E/F be a cyclic extension. The lifting θ_E will be cuspidal if and only if there exists $\zeta \in \widehat{Gal(E/F)}$ such that $\theta = \zeta$.*

The GL_2 base change problem

Now, we discuss the more complicated case of GL_2-base change. The base change for global fields is discussed in [15, Theorem 2]. We recall the relevant parts of the theorem:

Theorem 2.4.2 ([15, Theorem 2 (a),(b)]). *Let $\pi \in \Pi(G)$ be a cuspidal representation. Then there exists a unique lifting $\pi_{E/F} \in \Pi(G_{E/F})$. Since E/F is solvable, denote the chain of cyclic extension:*

$$F = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_{n-1} \subsetneq E_n = E.$$

Then, $\pi_{E/F}$ is cuspidal if and only if there exists E_{i+1}/E_i a quadratic extension and θ a quasicharacter of $\mathbb{A}_{E_{i+1}}^{\times}/E_{i+1}^{\times}$ such that

$$\pi_{E_i} = Ind_{W_{E_{i+1}}}^{W_{E_i}} \theta.$$

In this case, $\pi_{E_{i+1}}$ is a principal series given by quasicharacter θ, i.e. $\pi(\theta, \theta^{-1})$.

We further make the observation that for π an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, the Fourier coefficients of $\pi_{E/F}$ at prime ideals $\mathfrak{P} \subset \mathcal{O}_E$ will be given by

$$\lambda_{\pi_{E/F}}(\mathfrak{P}) = \lambda_\pi(N_{E/F}(\mathfrak{P})). \tag{2.4.1}$$

This connection between the Hecke eigenvalues of π and $\pi_{E/F}$ will be the key relationship between the correlation sum of Theorem 2.1.1 and this automorphic picture.

2.4.1 Cuspidal base change

In this subsection, we handle the situation when the base change π_K is cuspidal.

Proof of Theorem 2.1.1. We will reduce Theorem 2.1.1 to Theorem 2.1.3, where K is the splitting field of $P(x)$. Again, let us start by applying the sieve (Theorem 2.2.1):

$$\sum_{n \leq X} |\lambda_\pi(|P(n)|)| \ll_P X \exp\left(\sum_{p \leq X} \frac{\varrho(p)(|\lambda_\pi(p)| - 1)}{p}\right).$$

We observe that since $\varrho(p) = \#\{x \bmod p : P(x) = 0\}$, at all but finitely many primes,

$$\varrho(p) = \#\{\mathfrak{p} \subset \mathcal{O}_K : N_{K/\mathbb{Q}}(\mathfrak{p}) = p\}.$$

Since $P(x)$ is irreducible, we have

$$\sum_{p \leq X} \frac{\varrho(p)|\lambda_\pi(p)|}{p} - \frac{\varrho(p)}{p} = \sum_{N(\mathfrak{p}) \leq X} \frac{|\lambda_\pi(N(\mathfrak{p}))|}{N(\mathfrak{p})} - \log\log(X) + O_K(1) = \sum_{N(\mathfrak{p}) \leq X} \frac{|\lambda_\pi(N(\mathfrak{p}))| - 1}{N(\mathfrak{p})} + O_K(1).$$

Now by (2.4.1), we know that this gives the relationship

$$\sum_{p\leq X}\frac{\varrho(p)(|\lambda_\pi(p)|-1)}{p}=\sum_{N(\mathfrak{p})\leq X}\frac{|\lambda_{\pi_K}(\mathfrak{p})|-1}{N(\mathfrak{p})}+O_K(1)=S_{\pi_K}(X)+O_K(1),$$

using the notation of §2.3. Since we have assumed that π_K is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$, by the proof of Theorem 2.1.3,

$$S_{\pi_K}(X)\leq -c\log\log(X)+O_{\pi,K}(1).$$

Thus, we establish the desired estimate

$$\sum_{n\leq X}|\lambda_\pi(|P(n)|)|\ll_{\pi,P} X\log(X)^{-c}.$$

□

2.4.2 Non-cuspidal base change

Let K again denote the splitting field of $P(x)$ and we assume that it is solvable. We assume that π_K is not cuspidal and want to determine the size of

$$\sum_{n\leq X}|\lambda_\pi(|P(n)|)|.$$

Proof of Theorem 2.1.4. First, we want to determine cases of automorphic representations $\pi\in\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ where π_K is not cuspidal and there still exists nontrivial savings in the above sum, via the sieve method of Theorem 2.2.1.

Since K is solvable, there exists a chain of subfields:

$$\mathbb{Q}=E_0\subsetneq E_1\subsetneq\ldots\subsetneq E_{n-1}\subsetneq E_n=K$$

with E_{i+1}/E_i cyclic extensions of prime order. Since π_K is not cuspidal, by Theorem 2.4.2, we know that there exists some index i such that $[E_{i+1}:E_i]=2$ and

$$\pi_{E_i}=\mathrm{Ind}_{W_{E_{i+1}}}^{W_{E_i}}\theta,$$

where θ is a quasi-character of $\mathbb{A}_{E_{i+1}}^{\times}/E_{i+1}^{\times}$. Furthermore, this fact tells us that

$$\pi = \mathrm{Ind}_{W_{E_{i+1}}}^{W_{\mathbb{Q}}} \theta.$$

Hence, since π is a two-dimensional representation, we must have that $E_i = \mathbb{Q}$ and E_{i+1} a quadratic extension, that we will hereby denote as F.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ sitting over p. Then as a consequence of the structure of π, we know that

$$\lambda_\pi(p) = \sum_{N_{F/\mathbb{Q}}(\mathfrak{p})=p} \theta(\mathfrak{p}).$$

Furthermore, let $\mathfrak{P} \subset \mathcal{O}_K$ be a prime ideal sitting above $\mathfrak{p}$. We also know that

$$\lambda_{\pi_K}(\mathfrak{P}) = \lambda_\pi(N_{K/\mathbb{Q}}(\mathfrak{P})) = \sum_{\substack{\mathfrak{p}\subset\mathcal{O}_F \\ N_{K/\mathbb{Q}}(\mathfrak{P})=N_{F/\mathbb{Q}}(\mathfrak{p})}} \theta(\mathfrak{p}).$$

Now, we recall that after an application of Theorem 2.2.1, we want to determine the size of

$$\begin{aligned} S_{\pi_K}(X) &:= \sum_{\substack{\mathfrak{P}\subset\mathcal{O}_K \\ N_{K/\mathbb{Q}}(\mathfrak{P})\le X}} \frac{|\lambda_{\pi_K}(\mathfrak{P})| - 1}{N_{K/\mathbb{Q}}(\mathfrak{P})} \\ &= \sum_{\substack{\mathfrak{P}\subset\mathcal{O}_K \\ N_{K/\mathbb{Q}}(\mathfrak{P})\le X}} N_{K/\mathbb{Q}}(\mathfrak{P})^{-1} \left(\left| \sum_{\substack{\mathfrak{p}\subset\mathcal{O}_F \\ N_{K/\mathbb{Q}}(\mathfrak{P})=N_{F/\mathbb{Q}}(\mathfrak{p})}} \theta(\mathfrak{p}) \right| - 1 \right) + O_{K,\pi}(1). \end{aligned}$$

Since $\lambda_{\pi_K}(\mathfrak{P})$ only depends on the value of $N_{K/\mathbb{Q}}(\mathfrak{P}) = p$, we group the summand by the prime ideals sitting above p. It is worth observing as well that since we sum over prime ideals, only those primes satisfying $N_{K/\mathbb{Q}}(\mathfrak{P})$ is prime will contribute to the growth of the summand; the other primes can be grouped into the $O_{K,\pi}(1)$ term. Thus, since

$$\sum_{N_{K/\mathbb{Q}}(\mathfrak{P})\le X} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{P})} = \sum_{p\le X} \frac{\#\{\mathfrak{P}\subset\mathcal{O}_K : N_{K/\mathbb{Q}}(\mathfrak{P}) = p\}}{p} + O_K(1),$$

we achieve that

$$S_{\pi_K}(X) = \sum_{p\le X} \frac{\#\{\mathfrak{P}\subset\mathcal{O}_K : N_{K/\mathbb{Q}}(\mathfrak{P}) = p\}}{p} \left(\left| \sum_{\substack{\mathfrak{p}\subset\mathcal{O}_F \\ N_{F/\mathbb{Q}}(\mathfrak{p})=p}} \theta(\mathfrak{p}) \right| - 1 \right) + O_K(1).$$

Since θ is a unitary character of F and $\lambda_\pi(p) \in \mathbb{R}$, we have a stronger replacement for (2.3.1):

$$\left| \sum_{N_{F/\mathbb{Q}}(\mathfrak{p})=p} \theta(\mathfrak{p}) \right| \leq \frac{3}{2} + \frac{1}{4} \left(\sum_{N_{F/\mathbb{Q}}(\mathfrak{p})=p} \theta^2(\mathfrak{p}) \right).$$

Furthermore, if p is a prime such that there does not exists $\mathfrak{p} \subset \mathcal{O}_F$ with $N_{F/\mathbb{Q}}(\mathfrak{p}) = p$, then $\lambda_\pi(p) = 0$. Thus, we have the upper bound

$$\begin{aligned} S_{\pi_K}(X) \leq \frac{1}{2} \sum_{\substack{p \leq X \\ \exists \mathfrak{p}: N_{F/\mathbb{Q}}(\mathfrak{p})=p}} & \frac{\#\{\mathfrak{P} \subset \mathcal{O}_K : N_{K/\mathbb{Q}}(\mathfrak{P}) = p\}}{p} \\ & - \sum_{\substack{p \leq X \\ \nexists \mathfrak{p}: N_{F/\mathbb{Q}}(\mathfrak{p})=p}} \frac{\#\{\mathfrak{P} \subset \mathcal{O}_K : N_{K/\mathbb{Q}}(\mathfrak{P}) = p\}}{p} \\ & + \frac{1}{4} \sum_{\substack{p \leq X \\ \exists \mathfrak{p}: N_{F/\mathbb{Q}}(\mathfrak{p})=p}} \frac{\#\{\mathfrak{P} \subset \mathcal{O}_K : N_{K/\mathbb{Q}}(\mathfrak{P}) = p\}}{p} \times \left(\sum_{\substack{\mathfrak{p} \subset \mathcal{O}_F \\ N_{F/\mathbb{Q}}(\mathfrak{p})=p}} \theta^2(\mathfrak{p}) \right). \end{aligned}$$

We make the following comments about these sums: (1) for the first sum, this is equivalent to counting

$$\frac{1}{2} \sum_{N_{F/\mathbb{Q}}(\mathfrak{p}) \leq X} \frac{\#\{\mathfrak{P} \subset \mathcal{O}_K : N_{K/F}(\mathfrak{P}) = \mathfrak{p}\}}{N_{F/\mathbb{Q}}(\mathfrak{p})} + O_K(1) = \frac{1}{2} \log\log(X) + O_K(1),$$

(2) the second sum is empty since $F \subset K$, (3) after counting the number of ideals with $N_{F/\mathbb{Q}}(\mathfrak{p}) = p$, the third sum becomes

$$\frac{1}{2} \sum_{N_{F/\mathbb{Q}}(\mathfrak{p}) \leq X} \frac{\#\{\mathfrak{P} \subset \mathcal{O}_K : N_{K/F}(\mathfrak{P}) = \mathfrak{p}\}}{N_{F/\mathbb{Q}}(\mathfrak{p})} \times \theta^2(\mathfrak{p}) + O_K(1) = \frac{1}{2} \sum_{N_{K/\mathbb{Q}}(\mathfrak{P}) \leq X} \frac{\theta^2(N_{K/F}(\mathfrak{P}))}{N_{K/\mathbb{Q}}(\mathfrak{P})} + O_K(1).$$

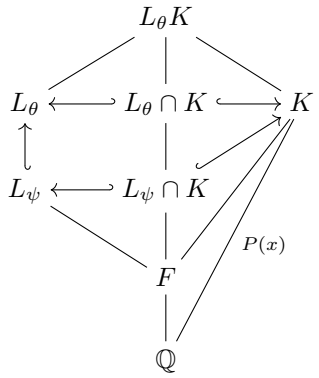
Thus, we know that

$$S_{\pi_K}(X) \leq \frac{1}{2} \log\log(X) + \frac{1}{2} \sum_{N_{K/\mathbb{Q}}(\mathfrak{P}) \leq X} \frac{\theta^2(N_{K/F}(\mathfrak{P}))}{N_{K/\mathbb{Q}}(\mathfrak{P})} + O_K(1),$$

the upper bound in the first part of Theorem 2.1.4 holds so long as $\theta^2 \circ N_{K/F}$ is not a trivial character; in other words, we need to understand the **$\mathbf{GL}_1$-base change** of θ^2 to the extension K/E_i.

Return to the GL_1-base change problem

By Proposition 2.4.1, we know that θ_K^2 is trivial, i.e. noncuspidal, if and only if $\theta^2 \in \widehat{G}$ for $G = \mathrm{Gal}(K/F)$. Let us denote $\psi = \theta^2$. We want to find another interpretation of our condition for when $\psi_K \equiv 1$ that will be more beneficial for our analysis in the second part of Theorem 2.1.4. Consider the following diagram, where L_ψ (resp. L_θ) denotes the ray class field of ψ (resp. θ). Since $\ker(\theta) \subset \ker(\psi)$, we can take $L_\psi \subset L_\theta$.



Let G_ψ denote the Galois group of $L_\psi \cap F$. We know that $\psi_K = 1$ if and only if $\psi \in \widehat{G}$; we claim this can occur if and only if $\psi \in \widehat{G_\psi}$, i.e. if and only if $\psi_{L_\psi \cap K} = 1$. This is clear since ψ is a character of $\mathrm{Gal}(L_\psi/F)$, and hence the interpretation of ψ as an element of $\widehat{G}$ must be as a character on G_ψ extended trivially to G. Now by definition of the ray class field of L_ψ, this in fact tells us that $L_\psi = L_\psi \cap K$, and hence we have that $L_\psi \subset K$.

Lemma 2.4.3. *Let ψ be a quasi-character of F. Then $\psi_K = 1$ if and only if $L_\psi \subset K$, where L_ψ denotes the kernel field of ψ.*

Now we recall that in our set up $\psi = \theta^2$ – we want to determine when $L_{\theta^2} \subset K$ in order to determine when $\theta_K^2 = 1$. Instead of analyzing the extension L_ψ, we can instead work with L_θ. We can view θ^2 as a character of the Galois group $\mathrm{Gal}(L_\theta/F)$ and see that $\theta_K^2 = 1$ if and only if $\theta_{L_\theta \cap K}^2 = 1$, and thus, if and only if $\theta^2 \in \widehat{G_\theta}$ (where $G_\theta = \mathrm{Gal}(L_\theta \cap K/F)$). In turn, this occurs if and only if $\mathrm{Gal}(L_\theta/L_\theta \cap K) \subset \ker(\theta^2)$, where $\ker(\theta^2)$ is considered as a subgroup of $\mathrm{Gal}(L_\theta/F)$. This proves the first part of Theorem 2.1.4.

Remark 2.4.4. We note that $\ker(\theta) = \{1\} \subset \mathrm{Gal}(L_\theta/F)$, but we can have that $\{1\} \subsetneq \ker(\theta^2)$. So, this condition does not necessarily imply that $L_\theta \subset K$.

It remains to prove the second part of the theorem when $\mathrm{Gal}(L_\theta/L_\theta \cap K) \subset \ker(\theta^2)$. We return

to our original sum:

$$\begin{aligned}\sum_{n\le X}|\lambda_\pi(|P(n)|)| &\ge \sum_{\substack{n\le X\\ P(n)\text{ squarefree}}}|\lambda_\pi(|P(n)|)|\\ &= \sum_{\substack{n\le X\\ P(n)\text{ squarefree}}}\left|\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N_{F/\mathbb{Q}}(\mathfrak{a})=|P(n)|}}\theta(\mathfrak{a})\right|\\ &= \sum_{\substack{n\le X\\ P(n)\text{ squarefree}}}\prod_{\substack{p|P(n)\\ \exists\mathfrak{p}\subset\mathcal{O}_F: N_{F/\mathbb{Q}}(\mathfrak{p})=p}}|1+\theta^2(\mathfrak{p})|.\end{aligned}$$

Above, $\mathfrak{p}$ denotes a choice of prime ideal of $\mathcal{O}_F$ such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = p$. Since $p \mid P(n)$, there exists a $\mathfrak{P} \subset \mathcal{O}_K$ such that $N_{K/\mathbb{Q}}(\mathfrak{P}) = p$ and that $N_{K/F}(\mathfrak{P}) = \mathfrak{p}$; let $\mathfrak{P}' = N_{K/L_\theta\cap K}(\mathfrak{P})$. Since $\mathrm{Gal}(L_\theta/L_\theta\cap K)\subset\ker(\theta^2)$ and hence $\theta^2_{L_\theta\cap K}=1$, we know that

$$\theta^2(\mathfrak{p}) = \theta^2(N_{L_\theta\cap K/F}(\mathfrak{P}')) = \theta^2_{L_\theta\cap K}(\mathfrak{P}') = 1.$$

Consequently,

$$\sum_{n\le X}|\lambda_\pi(|P(n)|)| \ge \sum_{\substack{n\le X\\ P(n)\text{ squarefree}}}\prod_{\substack{p|P(n)\\ \exists\mathfrak{p}: N_{F/\mathbb{Q}}(\mathfrak{p})=p}}2 = \sum_{\substack{n\le X\\ P(n)\text{ squarefree}}}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N_{F/\mathbb{Q}}(\mathfrak{a})=|P(n)|}}1.$$

This completes the proof.

□

2.5 Example: class group characters of quadratic fields

In this section, we demonstrate an application of Theorem 2.1.1 and Theorem 2.1.4 in the case when ξ is a class group character. Specifically, let F be a quadratic extension of $\mathbb{Q}$, ξ a class group character of F, and Ξ the representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with Hecke eigenvalues:

$$\lambda_\Xi(n) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N_{F/\mathbb{Q}}(\mathfrak{a})=n}}\xi(\mathfrak{a}).$$

Observe that $\Xi = \mathrm{Ind}_{W_F}^{W_\mathbb{Q}}\xi$. Again, let K denote the splitting field of $P(x)$ and assume that K is solvable.

Since ξ is a class group character, its kernel field is contained in H, the Hilbert class field; in

the argument above, we replace L_ξ with H (since $\mathrm{Gal}(H/H\cap K)\cong \mathrm{Gal}(L_\xi/L_\xi\cap K)$). First, let us determine when Ξ_K is cuspidal. Since Ξ is cuspidal, we know that ξ^2 is nontrivial; here we have used that class group character of order ≤ 2 by Gauss genus theory give rise to Eisenstein representations. Note that if $F\not\subset K$, then Ξ_K must be cuspidal by Theorem 2.4.2. In this case, Theorem 2.1.1 applies and we have that

$$\sum_{n\leq X}|\lambda_\Xi(|P(n)|)|\ll_{\Xi,P} X\log(X)^{-0.066}.$$

From now on, let us assume that $F\subset K$ and hence the trivial bound is given by

$$\sum_{n\leq X}|\lambda_\Xi(|P(n)|)|\ll_{\Xi,P} X\log(X).$$

Now, by Theorem 2.1.4, if $\mathrm{Gal}(H/H\cap K)\not\subset\ker(\xi^2)$, then we have the savings:

$$\sum_{n\leq X}|\lambda_\Xi(|P(n)|)|\ll_{\Xi,P} X\log(X)^{1/2}.$$

Otherwise, we have the corresponding lower bound (which is expected to have size $\gg X\log(X)$ unless there is a Brauer-Manin or local obstruction).

Corollary 2.5.1. *Let F be a quadratic field and ξ be a class group character of F. Let H be the Hilbert class field of F and assume that the splitting field K of $P(x)\in\mathbb{Z}[x]$ be solvable. If $F\not\subset K$, then the following holds:*

$$\sum_{n\leq X}\left|\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N(\mathfrak{a})=|P(n)|}}\xi(\mathfrak{a})\right|\ll_{\xi,P} X\log(X)^{-0.066}.$$

If $F\subset K$ and Gal$(H/H\cap K)\not\subset\ker(\xi^2)$, then

$$\sum_{n\leq X}\left|\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N(\mathfrak{a})=|P(n)|}}\xi(\mathfrak{a})\right|\ll_{\xi,P} X\log(X)^{1/2}.$$

Otherwise, we have that

$$\sum_{n\leq X}\left|\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N(\mathfrak{a})=|P(n)|}}\xi(\mathfrak{a})\right|\gg\sum_{\substack{n\leq X\\ P(n)\ squarefree}}\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F\\ N(\mathfrak{a})=|P(n)|}}1.$$

Remark 2.5.2. Since ξ^2 is nontrivial, we know that $\ker(\xi^2)$ is not the whole class group, i.e. $\ker(\xi^2)\subsetneq \mathrm{Gal}(H/F)$.

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Chapter 3

On Manin's conjecture for Châtelet surfaces

3.1 Introduction

Let $\Delta \in \mathbb{Q}^\times$ satisfy $\sqrt{-\Delta} \notin \mathbb{Q}$, and let $f(z) \in \mathbb{Q}[z]$ be a separable polynomial of degree 3 or 4. A Châtelet surface $X_{\Delta,f}$ over $\mathbb{Q}$ is defined as a proper smooth model of the affine surface:

$$x^2 + \Delta y^2 = f(z).$$

The purpose of this paper is to prove Manin's conjecture for all Châtelet surfaces, including when $\Delta < 0$.

Châtelet surfaces were first studied by Châtelet in [6, 7], who was interested in parameterizing the rational points over number fields of these surfaces (see [8]). Later, these surfaces were the first examples of surfaces that are not homogeneous spaces for which the Hasse principle, weak approximation, and other interesting problems about rational points were resolved. For certain pairs $(\Delta, f(z))$, $X_{\Delta,f}$ may fail the Hasse principle due to a Brauer-Manin obstruction. In [14, 15], Colliot-Thélène, Sansuc, and Swinnerton-Dyer show that the Brauer-Manin obstruction is the only possible cause of why such a surface may fail the Hasse principle; these results hold not only over $\mathbb{Q}$, but over any number field.

We first present Manin's conjecture, as applied to Châtelet surfaces. There exists a map $\psi : X_{\Delta,f} \to \mathbb{P}^4$, whose image is a singular del Pezzo surface of degree four (as described by Browning in [5]). This map then induces a height function on $X_{\Delta,f}$ by pulling back the natural height H on $\mathbb{P}^4$ via ψ. We define the point-counting function:

$$\tilde{N}(X_{\Delta,f}, B) = \#\{x \in X_{\Delta,f}(\mathbb{Q}) : H(x) \leq B\}.$$

Manin's conjecture, originally written as [1, Conjecture C'] by Batyrev and Manin and stated over $\mathbb{Q}$ explicitly by Browning as [4, Conjecture 2.3], predicts the behavior of $\tilde{N}(X_{\Delta,f}, B)$ as $B \to \infty$; additionally, in [43], Peyre gives an interpretation of the constant in terms of local densities and the Brauer group. In the following conjecture, the exponent was worked out in [5, Lemma 1] by Browning.

Conjecture 3.1.1 (Manin's conjecture for Châtelet surfaces)**.** *Let $\Delta \in \mathbb{Q}^\times$ satisfy that $\sqrt{-\Delta} \notin \mathbb{Q}$ and let $f(z) \in \mathbb{Z}[z]$ be a separable polynomial of degree 3 or 4 that decomposes as a product of r irreducible polynomials $f = f_1 \dots f_r$. Let $X_{\Delta,f}/\mathbb{Q}$ be a Châtelet surface defined by $x^2 + \Delta y^2 = f(z)$.*

Assume that $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$. *Then as* $B \to \infty$,

$$\tilde{N}(X_{\Delta,f}, B) \sim C_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1},$$

where $C_{\Delta,f} > 0$ *and*

$$\rho_{\Delta,f} = 2 + \#\{1 \leq i \leq r : \sqrt{-\Delta} \in \mathbb{Q}[z]/(f_i(z))\}.$$

Remark 3.1.2. This clean formulation of Manin's conjecture over $\mathbb{Q}$ is possible thanks to the fact that if $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$ then $X_{\Delta,f}(\mathbb{Q})$ is Zariski-dense. Indeed by [15, Proposition 9.8.(iii)], if $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$ then $X_{\Delta,f}$ is birational to a geometrically integral nonconical cubic surface in $\mathbb{P}^3$ with a pair of conjugate singular points. Then by [17, Theorem A] of Coray and Tsfasman, $X_{\Delta,f}$ must be unirational and thus $X_{\Delta,f}(\mathbb{Q})$ is Zariski-dense.

Moreover, the question of when $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$ is settled by the seminal result of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [15]; it demonstrates that $X_{\Delta,f}(\mathbb{Q}) = \emptyset$ if and only if there is either a Brauer-Manin obstruction or a local obstruction to the existence of rational points. Furthermore, as discussed in [15], a Brauer-Manin obstruction to the Hasse principle may occur only when $f(z)$ is the product of two irreducible quadratics. There is a finite and effective computation for determining when $X_{\Delta,f}(\mathbb{Q}) = \emptyset$; this computation uses auxiliary varieties that we will introduce in (3.1).

Numerous instances of Manin's conjecture for Châtelet surfaces have been resolved. In [5], Browning established an upper bound of the correct order of magnitude for Châtelet surfaces with $\Delta > 0$. In [38], Iwaniec and Munshi derive a lower bound when $f(z)$ is an irreducible cubic polynomial and Δ can be either positive or negative. In their seminal result, de la Bretèche, Browning, and Peyre [19] introduced the use of the torsor method, among other tools, to prove Manin's conjecture for $x^2 + y^2 = f(z)$ when f decomposes as a product of distinct linear factors. Later, de la Bretèche and Browning [21] showed Manin's conjecture for the case that $\Delta = 1$ and $f(z)$ contains an irreducible cubic factor. In [25], the asymptotic for Manin's conjecture is established for $x^2 + y^2 = f(z)$ where f is an irreducible quartic or a product of two irreducible quadratic factors over $\mathbb{Q}(i)$ by de la Bretèche and Tenenbaum; these factorization cases are often regarded as the trickiest cases. Finally, Destagnol [28] resolved Manin's conjecture for $\Delta = 1$ by proving the asymptotic for $x^2 + y^2 = f(z)$ where f is a product of two linearly independent linear factors and an irreducible quadratic factor, the remaining unknown factorization. In the above cases, the techniques are amenable to replacing $x^2 + y^2$ with any positive-definite quadratic form of class number one.

Châtelet surfaces belong to a wider class of surfaces; Châtelet surfaces are considered singular

del Pezzo surfaces of degree four. Manin's conjecture was established by Batyrev and Tschinkel [2] for all del Pezzo surfaces of degree ≥ 7 and nonsingular del Pezzo surfaces of degree 6, as these are toric varieties. Additionally, Manin's conjecture is known for many more examples of del Pezzo surfaces of lower degree, as is discussed by Browning in [4, Chapter 2].

Next, we introduce our main result. In [5], Browning shows that

$$\tilde{N}(X_{\Delta,f}, B) = \frac{1}{2} \cdot \#\Big\{((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v) \\ \gcd(x,y,t) = \gcd(u,v) = 1, \max(|tu^2|, |tv^2|, |x|, |y|) \leq B\Big\},$$

where $F(u,v)$ denotes the homogenization of $f(z)$. We modify this height function slightly to better suit our techniques. Define:

$$N(X_{\Delta,f}, B) := \frac{1}{2} \cdot \#\Big\{((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v) \\ \gcd(x,y,t) = \gcd(u,v) = 1, \max(|tu^2|, |tv^2|, \|x \pm y\sqrt{-\Delta}\| \leq \nu_{\Delta,f} B\Big\},$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{C}$ and

$$\nu_{\Delta,f} = \begin{cases} \left(\max_{\mathbf{y} \in [-1,1]^2} |F(\mathbf{y})|\right)^{1/2}, & \Delta > 0 \\ 1, & \Delta < 0. \end{cases}$$

This new height is equivalent up to a constant (depending on Δ and f) to the natural height on $X_{\Delta,f}$ induced by ψ.

Theorem 3.1.3. *Let $\Delta \in \mathbb{Z}$ be a squarefree integer satisfying that $\sqrt{-\Delta} \notin \mathbb{Q}$. Let $f(z) \in \mathbb{Z}[z]$ be a separable polynomial of degree 3 or 4. Let $X_{\Delta,f}/\mathbb{Q}$ be the corresponding Châtelet surface. Then as $B \to \infty$, there exists a constant $c_{\Delta,f}$ such that*

$$N(X_{\Delta,f}, B) = c_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1} + O_{\Delta,f}(B \log(B)^{\rho_{\Delta,f}-1-10^{-10}}).$$

Moreover, $c_{\Delta,f}$ is zero if and only if there is a Brauer-Manin obstruction or a local obstruction to rational points on $X_{\Delta,f}$.

Corollary 3.1.4. *Manin's conjecture holds for all Châtelet surfaces over $\mathbb{Q}$.*

In Theorem 3.1.3, we do not assume a priori that $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$. Our proof of the aforementioned asymptotic result gives an explicit, albeit complicated, expression for $c_{\Delta,f}$. Notably, $c_{\Delta,f}$ is closely related to auxiliary varieties; these varieties appear in the work of Colliot-Thélène, Coray, and Sansuc in [10] when $f(z)$ is the product of two irreducible quadratics. Since our expression for $c_{\Delta,f}$ is complicated, we determine when it vanishes but do not pursue an adelic interpretation along the lines of Peyre's prediction [43].

Let $f(z) = f_1(z)\dots f_r(z)$ be a decomposition into irreducible factors and let $\alpha_1, \dots, \alpha_r \in \mathbb{Q}^\times$ satisfy that $\alpha_1 \dots \alpha_r = 1$. We define the auxiliary variety:

$$X^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r} := \{(z,(x_i,y_i)_{i=1}^r) : x_i^2 + \Delta y_i^2 = \alpha_i f_i(z) \neq 0\}.$$

The relationship between $X^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r}$ and $X_{\Delta,f}$ will be expanded upon further in §3.12. For now, let us define the following condition:

(A) There exists a tuple $(\alpha_1,\dots,\alpha_r) \in (\mathbb{Q}^\times)^r$ satisfying $\alpha_1 \dots \alpha_r = 1$ such that

$$X^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r}(\mathbb{Q}_v) \neq \emptyset$$

for all places v of $\mathbb{Q}$.

We will show that our constant $c_{\Delta,f}$ is nonzero if and only if condition (A) holds (see Proposition 3.12.7). Additionally, if $f(z)$ is irreducible, then a consequence is that $c_{\Delta,f}$ is nonzero if and only if there are no local obstructions to rational points on $X_{\Delta,f}$.

In this way, Theorem 3.1.3 gives an *analytic proof of the existence of many rational points on* $X_{\Delta,f}$ *via direct counting* if condition (A) holds. Now, by the work of Colliot-Thélène, Coray, and Sansuc, and an application of Harari's "formal lemma" [32], one can interpret condition (A) in terms of the Brauer-Manin and local obstructions of $X_{\Delta,f}$. In particular, this allows us to conclude that if $c_{\Delta,f}$ is zero, then $X_{\Delta,f}(\mathbb{Q}) = \emptyset$ as well.

Finally, we highlight the new ingredients that allow us to push beyond $\Delta = 1$. In §2, we provide a detailed outline of our proof for general Châtelet surfaces. First, we decompose our count into contributions from Eisenstein series and cusp forms. Consider the following function for Δ a positive squarefree integer:

$$\theta(z) = \sum_{\substack{I \subset \mathcal{O}_{\mathbb{Q}(\sqrt{-\Delta})} \\ I \text{ is principal}}} \exp(2\pi i z \cdot N(I)).$$

This function is a modular form of weight one and has a decomposition into Eisenstein and cuspidal components. If $\Delta = 1$, then the fact that $x^2 + y^2$ has class number one implies that $\theta(z)$ is purely Eisenstein, i.e. the cuspidal part is zero. We will find that in general the cuspidal contribution is negligible in comparison with the Eisenstein contribution.

When $\Delta < 0$, i.e. the indefinite case, a similar picture forms with careful analysis; the main additional obstacle in this case is that $\mathbb{Q}(\sqrt{-\Delta})$ has an infinite unit group. To mitigate this difficulty, we restrict our point-count to mildly-short boxes and intervals of length scaled by $\ell = \log(B)^{-10^{-10}}$. This restriction allows us to estimate accurately the count

$$\#\{|x \pm y\sqrt{-\Delta}| \leq B : x^2 + \Delta y^2 = t^2 F(u,v)\}$$

and convert our sum into a question about principal ideals I with norm $t^2F(u,v)$. Of course, summing over short intervals and boxes complicates our estimates and sometimes adds an extra factor of $\ell^{-O(1)}$, where this exponent can be made explicit. Fortunately, the error terms achieved by previous results on Manin's conjecture, whose techniques we mimic, achieve savings of $\log(B)^{-10^{-5}}$ which will be more powerful than $\ell^{-O(1)}$. Once the sum is written in terms of principal ideals of $\mathbb{Q}(\sqrt{-\Delta})$, we can again decompose the count in terms of Eisenstein and cuspidal contributions as before; this is equivalent to decomposing the Maass cusp form into Eisenstein and cuspidal ingredients.

Second, we demonstrate that the cuspidal contribution is always negligible in comparison with the Eisenstein contribution. Let $C(z)$ denote the cuspidal part of $\theta(z)$ and let $\lambda_C(n)$ denote its Fourier coefficients. Our cuspidal contribution reduces to estimating sums of the form

$$\sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} \left|\lambda_C(t^2F(u,v))\right|,$$

and of the form

$$\sum_{t \leq B} \sum_{d|t} \mu(d) \sum_{|u|,|v| \leq (B/t)^{1/2}} \lambda_C\left(t^2F(u,v)/d^2\right).$$

In the first case, we study correlation sums of the absolute values of its Fourier coefficients $\lambda_C(n)$ along polynomial values; although we sacrifice cancellation in the sign of $\lambda_C(n)$, we still gain a saving of a power of log. In particular, we use a Nair-type sieve [41] to reduce the problem to computing the first few Sato-Tate moments of the corresponding L-functions (see Rankin's [44] work on the moments of the Ramanujan τ-function). We remark that this method was used by Holowinsky [35]

to study correlation sums of Fourier coefficients of cusp forms with quadratic polynomials. Heath-Brown [34] also used this technique to study the equidistribution of solutions to certain Châtelet surfaces and his work inspired our proof techniques. While this technique handles most cases, there are delicate issues concerning the cuspidality of the base change lift of $C(z)$ to the splitting field of $f(z)$; in these cases, the absolute values are too expensive of a sacrifice.

Fortunately, in the above exceptional cases, we can appeal to the more robust method of handling the second sum. We determine that since $C(z)$ is a cusp form, we can derive savings for the sum

$$\sum_{d\leq X}\mu(d)\sum_{t\leq X/d}\lambda_C(at^2).$$

Moreover, our cusp forms are induced from class group characters of $\mathbb{Q}(\sqrt{-\Delta})$. This extra structure allows us to track explicitly the dependency in the above bound on a. Thus, we can derive a polylogarithmic saving on

$$\sum_{|u|,|v|\leq B^{1/2}}\sum_{d\leq B/\max(|u|,|v|)^2}\mu(d)\sum_{t\leq B/d\max(|u|,|v|)^2}\lambda_C(t^2F(u,v)).$$

This saving is a general feature for cusp form coefficients. The inner sum is controlled by the poles of the symmetric square L-function of $C(z)$ (see [45] for a discussion of the poles); if there is no pole, then the desired cancellation follows from the inner sum. If $C(z)$ is dihedral and the symmetric square L-function has a pole, then an asymptotic for the inner sum is well-understood and the cancellation follows from the Möbius sum.

For the Eisenstein contribution, we follow the methods of [19, 21, 25, 28, 33] that resolved the $\Delta = 1$ case. The third new feature is that for more general Δ, due to Gauss genus theory (see [30]), the Eisenstein Fourier coefficients now contain an extra summation over genus characters. This extra complication can still be handled by the above techniques for the $\Delta = 1$ case and we achieve an explicit expression for the leading constant $c_{\Delta,f}$.

The fourth and last new ingredient is the analysis of the vanishing of the leading constant $c_{\Delta,f}$. This is resolved in §3.12 and involves the auxiliary varieties introduced in (3.1). In particular, our leading term brings us to a non-negative sum which is approximately of the form

$$\sum_{(\alpha_1,\ldots,\alpha_r)/\sim}\prod_p\mu_p(X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}_p)).$$

where $\mu_p(X^*_{\Delta,f_1,...,f_r,\alpha_1,...,\alpha_r}(\mathbb{Q}_p))$ is zero if and only if $X^*_{\Delta,f_1,...,f_r,\alpha_1,...,\alpha_r}(\mathbb{Q}_p) = \emptyset$. Here, the sum over $(\alpha_1, ..., \alpha_r)/\sim$ is over a sufficient and finite set of auxiliary varieties. Thus, we prove that $c_{\Delta,f} = 0$ if and only if there is a local obstruction for every auxiliary variety.

3.2 Outline

In this section, we outline our proof strategy; our goal is to highlight the new techniques and convey the essential ideas, without the technical detail provided in the rest of the paper. By leaving out some of the more notationally-heavy complications, we hope to provide a clear roadmap for the rest of the paper. We warn the reader that in this section some of the claims are vague (and will be made precise in later sections) and some are technically false (but morally true and the modifications needed to correct the statements will be given later).

Let us start with the case when $\Delta > 0$. If $\Delta = 1$, Theorem 3.1.3 is known; so assume that $\Delta > 1$ (for example, we can take $\Delta = 5$). The main challenge to overcome is when $\mathbb{Q}(\sqrt{-\Delta})$ has class number greater than one. From our definition (3.1), we see that:

$$N(X_{\Delta,f}, B) = \frac{1}{2} \sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} r_{x^2+\Delta y^2}(t^2 F(u, v)),$$

where $r_{x^2+\Delta y^2}(n) = \#\{(x, y) \in \mathbb{Z}^2 : x^2 + \Delta y^2 = n\}$. Consider the binary theta series

$$\theta(z) = \sum_n r_{x^2+\Delta y^2}(n) \exp(2\pi i n z) = \sum_{x,y} \exp(2\pi i z(x^2 + \Delta y^2),$$

which is a **modular form of weight one** in $M_1(\Gamma_0(-\Delta), \chi)$ (see [37, §22.3]), where χ denotes the real quadratic character for $\mathbb{Q}(\sqrt{-\Delta})$. As such, we can decompose $\theta(z)$ in terms of its Eisenstein and cuspidal contributions $\theta(z) = E(z) + C(z)$. Note that since $\theta(z)$ is a weight one modular form, the Fourier coefficients of the Eisenstein component and the cuspidal component are expected to be of the same magnitude.

The Fourier coefficients of the Eisenstein part can be written explicitly in terms of **genus characters**, i.e. the class group character of order ≤ 2. As such, we have a nice description of the Fourier coefficients of the Eisenstein part:

$$\lambda_E(n) \approx (1 \star \chi)(n) := \sum_{d|n} \chi(d).$$

For the cuspidal part, we can write $\lambda_C(n)$ as a fixed linear combination of Fourier coefficients $\lambda_\Psi(n)$, where Ψ is the automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ given by a class group character ψ of order ≥ 3. The critical fact that we can restrict to those ψ of order ≥ 3 follows from Gauss genus theory for binary quadratic forms, as discussed by Fomenko in [30]. We can describe these coefficients $\lambda_\Psi(n)$ explicitly:

$$\lambda_\Psi(n) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbb{Q}(\sqrt{-\Delta})} \\ N(\mathfrak{a})=n}} \psi(\mathfrak{a}).$$

From this decomposition of $r_{x^2+\Delta y^2}(n)$, we can see that Theorem 3.1.3 follows from gaining asymptotics on

$$\sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} (1 \star \chi)(t^2 F(u,v))$$

and sufficient upper bounds on

$$\sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} \lambda_\Psi(t^2 F(u,v)).$$

Note that we have left out the gcd-conditions here.

Before analyzing the two sums above, let us discuss what happens when $\Delta < 0$. There are two challenges involved in the indefinite case: again, we run into issues when $\mathbb{Q}(\sqrt{-\Delta})$ has class number greater than one; we also must handle the fact that there are infinitely many solutions to the equation $x^2 + \Delta y^2 = 1$. To handle the second issue, we restrict to counting in short boxes.

Let $\ell, L = w(B)$, a small value going slowly to zero as $B \to \infty$. We eventually take $w(B) = \log(B)^{-10^{-10}}$. Define the box

$$\mathcal{B}(\mathbf{x}_0, L) := \mathbf{x}_0 + [0, L]^2,$$

and the interval

$$\mathcal{I}(\theta, \ell) := \{\exp(2\pi i \theta') : \theta' \in [\theta, \theta + \ell]\}.$$

Then we consider the restricted counting function:

$$N(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),\mathcal{I}(\theta,\ell)) := \frac{1}{2}\cdot\#\left\{((x,y),(u,v),t)\in\mathbb{Z}^2\times\mathbb{Z}^2\times\mathbb{Z}_+ : \begin{array}{c} x^2+\Delta y^2 = t^2F(u,v), \\ (u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L), \\ \|x\pm y\sqrt{-\Delta}\|\le B, \\ \arg(x+y\sqrt{-\Delta})\in\mathcal{I}(\theta,\ell), \\ \gcd(x,y,t)=\gcd(u,v)=1 \end{array}\right\}.$$

In other words, we restrict (u,v) to a mildly shorter box $(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)$ and constrain the argument $\arg(x+y\sqrt{-\Delta})\in\mathcal{I}(\theta,\ell)$. Here we define the argument as

$$\arg(x+y\sqrt{-\Delta}) = \frac{|x+y\sqrt{-\Delta}|}{|x-y\sqrt{-\Delta}|} \bmod |\varepsilon|^2,$$

where ε is the fundamental unit to Pell's equation $x^2+\Delta y^2=1$. Our strategy is to produce an asymptotic for $N(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),\mathcal{I}(\theta,\ell))$; this asympotic would then recover an estimate for $N(X_{\Delta,f},B)$ by splitting the region $\{|x|,|y|\le B,|u|,|v|\le(B/t)^{1/2}\}$ into short boxes and arguments in $\mathcal{I}(\theta,\ell)$, and then summing over $\mathbf{x}_0$ and θ. This computation is done in §3.5.7.

By restricting to the shorter box and interval, we can make precise the count

$$\#\{|x\pm y\sqrt{-\Delta}|\le B : x^2+y^2=t^2F(u,v), \arg(x+y\sqrt{-\Delta})\in\mathcal{I}(\theta,\ell)\}.$$

In particular, we can find an explicit constant

$$\kappa(\mathbf{x}_0,\theta) = c_\Delta\cdot\max(-\log|F(\mathbf{x}_0)|,0)\cdot\frac{|\mathcal{I}(\theta,\ell)|}{4\pi}$$

satisfying that

$$\begin{aligned} N(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),\mathcal{I}(\theta,\ell)) &= (\kappa(\mathbf{x}_0,\theta)+O(w(B))) \\ &\times\sum_{t\le B}\sum_{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)} \#\{I\subset\mathbb{Q}(\sqrt{-\Delta}) : I \text{ a principal ideal and } N(I)=t^2F(u,v)\} \\ &+ E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),\mathcal{I}(\theta,\ell)), \end{aligned}$$

where $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta, \ell))$ is a linear combination of remainder terms of the form

$$\sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} |\lambda_\Xi(t^2 F(u,v))|$$

for Ξ a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ given by an unitary nontrivial Hecke Grössencharacter ξ on $\mathbb{Q}(\sqrt{-\Delta})$ of infinite order (see Definition 3.3.2).

For the main term, we again use a decomposition into Eisenstein and cuspidal components. In particular, we can express

$$\#\{I \subset \mathbb{Q}(\sqrt{-\Delta}) : I \text{ principal and } N(I) = n\} = \frac{1}{|C|} \sum_{\psi \in \hat{C}} \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_{\mathbb{Q}(\sqrt{-\Delta})} \\ N(\mathfrak{a}) = n}} \psi(\mathfrak{a}),$$

where C is the class group of $\mathbb{Q}(\sqrt{-\Delta})$. Again, from Gauss's genus theory, the Eisenstein contribution comes from the class group characters of order ≤ 2; the Fourier coefficient for the Eisenstein part is approximately

$$\lambda_E(n) \approx (1 \star \chi)(n).$$

Similarly, the cuspidal Fourier coefficient is a linear combination of sums λ_Ψ where Ψ is the representation given by ψ a class group character of order ≥ 3. This matches the process completed in the positive definite case.

In the end, Theorem 3.1.3 reduces to estimating the sum

$$\sum_{t \leq B} \sum_{(u,v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L)} (1 \star \chi)(t^2 F(u,v)),$$

and providing a suitable upper bound on the sums

$$\sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} \lambda_\Xi(t^2 F(u,v)),$$

$$\sum_{t \leq B} \sum_{|u|,|v| \leq (B/t)^{1/2}} |\lambda_\Xi(t^2 F(u,v))|.$$

where Ξ is the representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ induced by a Hecke character ξ. Additionally, (3.2) will only be used when ξ is a class group character of order ≥ 3 (and hence has finite image), whereas (3.2) only occurs when ξ is a Hecke Grössencharacter (which has infinite image). We remark that it is likely that with more careful analysis we could remove the absolute values in (3.2); however, we

achieve the necessary savings with the absolute values added.

Our method for estimating the Eisenstein contribution follows closely the previous approaches when $\Delta = 1$; indeed, when $\Delta = 1$, $\theta(z) = E(z)$ and so the whole sum is contained in the Eisenstein part. We particularly mimic the proofs of de la Bretèche and Tenenbaum in [25] and the proof of Heath Brown in [33]. However, we must consider when $(u, v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$ lies in a short box – fortunately, all of the techniques are either amenable to this issue or we are fortunate that the saving expected is far larger than the loss from summing over $|u|, |v| \leq (B/t)^{1/2}$ instead.

Using estimates on lattice points in convex regions, we establish a near-optimal **level of distribution** result for squarefree binary forms in §3.9. Write $F(x, y) = \prod_{i=1}^{r} F_i(x, y)$ as the decomposition into irreducible forms. We show that

$$\sum_{d_i \leq D_i} \left| \sum_{\substack{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ d_i | F_i(u,v)}} 1 - \frac{BL^2}{t} \cdot \frac{\#\{x \bmod d_1 \dots d_r : d_i \mid f_i(x)\}}{d_1 \dots d_r} \right| \ll D + \sqrt{\frac{BD}{t}} \exp(\sqrt{\log\log(B)}),$$

where $D = D_1 \dots D_r$. This result allows us to take $D \leq \frac{B}{t} \log(B/t)^{-\varepsilon}$ for any $\epsilon > 0$ and still obtain a reasonable bound. We remark that if $F(x, y)$ has linear factors then the shape of the above bound is slightly different. This level of distribution result is obtained in [25] for $F(x, y)$ irreducible or a product of two irreducible quadratics; we closely follow their techniques. Also, Marasingha [39] studies a general level of distribution result for squarefree binary forms and achieves a savings for $D \leq \frac{B}{t} \log(B/t)^{-A}$ for any large enough integer $A > 0$.

This step is the primary obstacle to studying $F(x, y)$ of higher degree; for $\deg(F) \leq 3$, the level of distribution is sufficient and for $\deg(F) = 4$, it is almost sufficient (up to a power of $\log(B)$). For $\deg(F) > 4$, the level of distribution would be a power of B away from necessary and this gap can not be covered by the argument as written below.

The other main input into our proof comes directly from de la Bretèche and Tenenbaum in [24] when $F(x, y)$ is irreducible or the product of two irreducible quadratics; we describe their idea in detail in §3.10 and discuss how to handle other factorizations of $F(x, y)$. Define the **twisted Hooley Δ-function**:

$$\Delta(n, \chi) := \sup_{\substack{D \in \mathbb{R} \\ 0 \leq v \leq 1}} \left| \sum_{\substack{d | n \\ d \in [D, e^v D]}} \chi(d) \right|.$$

Applying Nair's sieve and [24, Theorem 1.1], we achieve that for $D \geq \frac{B}{t}\log(B/t)^{-10^{-5}}$,

$$\sum_{\substack{|u|,|v|\leq (B/t)^{1/2}\\ \exists d_i \sim D_i : d_i | F_i(u,v)}} \Delta(F(u,v),\chi) \ll \frac{B}{t}\cdot \log(B/t)^{\rho_{\Delta,f}-2-10^{-5}}.$$

Note that since we take $L = w(B) = \log(B)^{-10^{-10}}$, this bound is still $\ll \frac{BL^2}{t}\log(B)^{\rho_{\Delta,f}-2-10^{-10}}$. This procedure handles the large moduli.

Together, these results allow us to re-express the Eisenstein contribution as:

$$BL^2 \sum_{t\leq B}\frac{1}{t}\sum_{(a_i) \bmod \Delta}\prod_{i=1}^{r}(1+\chi(a_i))\cdot \#\{\mathbf{x} \bmod \Delta : F_i(\mathbf{x}) \equiv a_i \bmod \Delta\} \\ \times \sum_{d_i \leq D_i}\frac{\chi(d_1\dots d_r)\cdot \#\{x \bmod d_1\dots d_r : d_i \mid f_i(x)\}}{d_1\dots d_r} \\ + O\left(\sum_{t\leq B}\frac{B}{t}\log(B/t)^{\rho_{\Delta,f}-2-10^{-5}}\right).$$

However, the above expression is an oversimplification – in particular, it assumes that $(1\star\chi)(n)$ is a totally multiplicative function, which it is not. Instead, we must substitute the identity that

$$(1\star\chi)(mn) = \sum_{c \mid \gcd(m,n)}\mu(c)\chi(c)(1\star\chi)(n/c)(1\star\chi)(m/c).$$

Consequently, we re-express our main term as:

$$\sum_{(u,v)\in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)}(1\star\chi)(F(u,v)) = \sum_{(c_1,\dots,c_r)}{}'\mu(c_1\cdots c_r)\chi(c_1\cdots c_r)\sum_{\substack{(u,v)\in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ c_i \mid F_i(u,v)}}\prod_{i=1}^{r}(1\star\chi)(F_i(u,v)/c_i).$$

Here the sum over $(c_1,\dots,c_r)$ is restricted to a finite set of possible vectors where the entries divide the pairwise resultants of the irreducible factors of $f(z)$.

Proceeding as before with our level of distribution result and bounds on the large moduli with the twisted Hooley Δ-function, our main term is actually of the form:

$$BL^2 \sum_{(c_1,\dots,c_r)}{}'\mu(c_1\dots c_r)\chi(c_1\dots c_r)\sum_{t\leq B}\frac{1}{t}\cdot\sum_{(a_i) \bmod \Delta}\prod_{i=1}^{r}(1+\chi(a_i))\#\{\mathbf{x} \bmod \Delta : F_i(\mathbf{x}) \equiv a_i \bmod \Delta\} \\ \times \sum_{d_i \leq D_i}\frac{\chi(d_1\dots d_r)\#\{x \bmod c_1\dots c_r d_1\dots d_r : c_i d_i \mid f_i(x)\}}{c_1\dots c_r d_1\dots d_r}.$$

Before analyzing this leading term further, let us discuss the cuspidal contribution.

First, we start with the finite image case. To study (3.2), we first change the order of summation:

$$\sum_{t\leq B}\sum_{|u|,|v|\leq (B/t)^{1/2}}\lambda_{\Xi}(t^2F(u,v)) = \sum_{|u|,|v|\leq B^{1/2}}\sum_{t\leq B/\max(|u|,|v|)^2}\lambda_{\Xi}(t^2F(u,v)).$$

At this point, we need to address a previous false claim about the function $\lambda_{\Xi}(n)$ in (3.2) – because we are counting those (x,y) such that $\gcd(x,y,t)=1$, we need to restrict our sum to ideals satisfying that $N(I)=t^2F(u,v)$ where $I=\mathfrak{a}\mathfrak{t}$, $\gcd(\mathfrak{t},\bar{\mathfrak{t}})=\gcd(\mathfrak{a},\bar{\mathfrak{t}})=1$. In other words, our sum (3.2) is of the form:

$$\sum_{|u|,|v|\leq B^{1/2}}\sum_{N(\mathfrak{a})=F(u,v)}\xi(\mathfrak{a})\sum_{\substack{N(\mathfrak{t})\leq B/\max(|u|,|v|)^2\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\xi(\mathfrak{t})^2.$$

Now, since ξ^2 is a nontrivial class group character, we achieve a power saving upper bound in the sum over ideals $\mathfrak{t}$. In particular, we claim that

$$\sum_{\substack{N(\mathfrak{t})\leq B/\max(|u|,|v|)^2\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\xi(\mathfrak{t})^2 \ll \prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)\cdot\left(\frac{B}{\max(|u|,|v|)^2}\right)^{5/6}.$$

The above is a consequence of the prime ideal theroem with nontrivial Hecke characters.

To complete our analysis of the finite image cuspidal contribution, we apply a generalization of Nair's sieve for non-negative multiplicative functions along polynomial values by de la Bretèche and Tenenbaum [23]. In particular, this sieve gives us that

$$\sum_{|u|,|v|\leq B^{1/2}}\sum_{N(\mathfrak{a})=F(u,v)}\prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)$$
$$\ll B\cdot\exp\left(\sum_{p\leq B^{1/2}}\frac{\#\{x \bmod p : f(x)=0 \bmod p\}}{p}\cdot\Big((1\star\chi)(1+p^{-2/3})-1\Big)\right) \ll B\log(B)^{\rho_{\Delta,f}-2}.$$

Using partial summation, we achieve that

$$\sum_{\substack{N(\mathfrak{t})\leq B/\max(|u|,|v|)^2\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\xi(\mathfrak{t})^2\sum_{|u|,|v|\leq (B/N(\mathfrak{t}))^{1/2}}\sum_{\substack{N(\mathfrak{a})=F(u,v)\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\xi(\mathfrak{a}) \ll B\log(B)^{\rho_{\Delta,f}-2}.$$

This is negligible compared to the main term of size $B\log(B)^{\rho_{\Delta,f}-1}$.

We can also view this sum more generally through the perspective of the Fourier coefficients of cusp forms. Let $C(z)$ be a weight one cusp form with central character χ; the gcd condition $\gcd(x,y,t)=1$ leads us to the sum

$$\sum_{t\leq B}\sum_{d|t}\mu(d)\sum_{|u|,|v|\leq (B/t)^{1/2}}\lambda_C\left(\frac{t^2F(u,v)}{d^2}\right)=\sum_{|u|,|v|\leq B^{1/2}}\sum_{d\leq B/\max(|u|,|v|)^2}\mu(d)\sum_{t\leq B/(d\max(|u|,|v|)^2)}\lambda_C(t^2F(u,v)).$$

This naturally brings us to analyzing the sum

$$\sum_{d\leq X}\mu(d)\sum_{n\leq X/d}\lambda_C(an^2)$$

and want to track the dependency on a above. Using Perron's formula, we can rewrite the above in terms of $L(s,\mathrm{sym}^2(C))$ and $\zeta(s)^{-1}$. Since $C(z)$ is a cusp form, $L(s,\mathrm{sym}^2(C))$ can have at most a pole of order one at $s=1$ (see [45] for a criteria for when there is a pole); thus, $L(s,\mathrm{sym}^2(C))\zeta(s)^{-1}$ converges at $s=1$. This allows us to derive a bound

$$\sum_{d\leq X}\mu(d)\sum_{n\leq X/d}\lambda_C(an^2)\ll \mathfrak{S}_a X\log(X)^{-10},$$

where $\mathfrak{S}_a$ is a finite product over the primes dividing a. When $C(z)$ is induced by a class group character, we have explicit bounds on $\mathfrak{S}_a$; in particular, the following holds for primes p:

$$\mathfrak{S}_p=\begin{cases}2+O(1/p^{2/3}), & \chi(p)=1,\\ O(1/p^{2/3}), & \chi(p)=-1.\end{cases}$$

Plugging the asymptotic above in and applying Nair's sieve as before, we achieve that the contribution from the cuspidal components with finite image is bounded by

$$\ll B\sum_{|u|,|v|\leq B^{1/2}}\frac{\mathfrak{S}_{F(u,v)}}{\max(|u|,|v|)^2}\log(B/\max(|u|,|v|)^2)^{-10}\ll B\log(B)^{\rho_{\Delta,f}-2}.$$

This is again negligible compared to the main term of size $B\log(B)^{\rho_{\Delta,f}-1}$.

In the Grössencharacter case, i.e. (3.2), we aim to acheive a small power of $\log(B)$ savings from the **absolute values of the Fourier coefficients of Ξ**. For this argument, we apply the same

version of Nair's sieve above with our non-negative multiplicative function now taken to be $|\lambda_\Xi(n)|$. Consequently, we have that

$$\sum_{|u|,|v|\leq (B/t)^{1/2}} |\lambda_\Xi(t^2F(u,v))| \ll \frac{B}{t}\cdot \prod_{p\ll B/t}\left(1+\frac{(|\lambda_\Xi(p)|-1)\cdot \#\{x \bmod p : f(x)\equiv 0 \bmod p\}}{p}\right).$$

To understand the above product, we study the sum

$$\sum_{\substack{\mathfrak{p}\subset \mathcal{O}_{\mathbb{Q}(\sqrt{-\Delta})}\\ N(\mathfrak{p})\ll B/t}} \frac{\#\{x \bmod N(\mathfrak{p}) : f(x)\equiv 0 \bmod N(\mathfrak{p})\}}{2N(\mathfrak{p})}\cdot |\xi(\mathfrak{p})+\xi(\mathfrak{p})^{-1}|.$$

One can show that when p splits in $\mathbb{Q}(\sqrt{-\Delta})$,

$$|\lambda_\Xi(p)|-1 \leq \frac{1}{2}+\frac{1}{4}\lambda_{\Xi^2}(p).$$

When p does not split in $\mathbb{Q}(\sqrt{-\Delta})$, by definition $|\lambda_\Xi(p)|-1=-1$. Using (3.2), we can reduce (3.2) to analyzing

$$\left(\frac{-1}{4}+\frac{3}{4}(\rho_{\Delta,f}-2)\right)\log\log(B/t)+O(1)+\sum_{\substack{\mathfrak{p}\subset \mathcal{O}_{\mathbb{Q}(\sqrt{-\Delta})}\\ N(\mathfrak{p})\ll B/t}} \frac{\xi^2(\mathfrak{p})\#\{x \bmod N(\mathfrak{p}) : f(x)\equiv 0 \bmod N(\mathfrak{p})\}}{N(\mathfrak{p})}.$$

It remains to understand the final sum. At this point, we recall that we only analyze this sum Ξ is induced by a Grössencharacter ξ, and hence ξ^2 is nontrivial. We must then lift the nontrivial Hecke character ξ^2 to K_f, the splitting field of $f(z)$ over $\mathbb{Q}(\sqrt{-\Delta})$, to understand the sum above; in other words, we need to understand the **base change** of ξ^2 to K_f. This new Hecke character $\tilde{\xi}=\xi^2\circ N_{K_f/\mathbb{Q}(\sqrt{-\Delta})}$ on K_f is necessarily nontrivial since ξ had infinite order; it is, however, not true that the base change of ξ to K_f is always nontrivial if we remove the assumption that ξ has infinite order.

The prime ideal theorem for Hecke Grössencharacters determines that

$$\sum_{\substack{\mathfrak{p}\subset \mathcal{O}_{\mathbb{Q}(\sqrt{-\Delta})}\\ N(\mathfrak{p})\ll B/t}} \frac{\xi^2(\mathfrak{p})\#\{x \bmod N(\mathfrak{p}) : f(x)\equiv 0 \bmod N(\mathfrak{p})\}}{N(\mathfrak{p})} = o(\log\log(B/t)).$$

Consequently, for such character ξ,

$$\sum_{t\leq B}\sum_{|u|,|v|\leq (B/t)^{1/2}} |\lambda_\Xi(t^2F(u,v))| \ll \sum_{t\leq B}\frac{B}{t}\cdot \log(B)^{\rho_{\Delta,f}-2-1/5} \ll B\log(B)^{\rho_{\Delta,f}-1-1/5}.$$

We remark that this approach was given in [34] by Heath-Brown and our approach is strongly inspired by his work. We also note that our savings here is large enough to overcome the fact that we have moved from summing over $(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$ to $|u|, |v| \leq (B/t)^{1/2}$ as long as $w(B) \leq \log(B)^{-1/49}$.

We would like to highlight that this approach to the Grössencharacter case can also be used to approach (3.2) in most cases. In particular, we only require that the base change of ξ^2 to K_f to be a nontrivial Hecke character to achieve a small polylogarithmic savings; this lift will be acceptable unless K_f is not independent from the Hilbert class field of $\mathbb{Q}(\sqrt{-\Delta})$.

Finally, we comment on how this technique can be applied to a wider class of cusp forms. For instance, let $C(z) \in S_1(\Gamma_0(-\Delta), \chi)$ be a cusp form and consider the sum

$$\sum_{|u|,|v|\leq X} |\lambda_C(t^2F(u,v))| \ll \frac{B}{t} \cdot \prod_{p \ll B/t} \left(1 + \frac{(|\lambda_C(p)| - 1) \cdot \#\{x \bmod p : f(x) \equiv 0 \bmod p\}}{p}\right).$$

We have again applied Nair's sieve here. We can replace (3.2) with Hecke relations:

$$|\lambda_C(p)| - 1 \leq -\frac{1}{18} + \frac{4}{9}\lambda_{\mathrm{sym}^2(C)}(p) - \frac{1}{18}\lambda_{\mathrm{sym}^4(C)}(p).$$

We can then proceed with the argument above under the assumption of certain analytic properties of the base change of $\mathrm{sym}^2(C)$ and $\mathrm{sym}^4(C)$ to K_f. In summary, the savings in this bound, despite the addition of absolute values around the Fourier coefficients, comes from the early Sato-Tate moments of $C(z)$ and its base change to K_f. This idea will be expanded upon in forthcoming work.

Finally, we return to the Eisenstein contribution, i.e. the main term and how we determine when $c_{\Delta,f}$ is zero. Let us define $\xi(s; f, \chi)$ to be the following Dirichlet series for $\Re(s) > 1$:

$$\xi(s; f, \chi) = \sum_{d_i} \frac{\chi(d_1 ... d_r)\#\{x \bmod d_1 ... d_r : d_i \mid f_i(x)\}}{(d_1 ... d_r)^s}.$$

For every irreducible factor $f_i(z)$ of $f(z)$, let us take $K_i = \mathbb{Q}[z]/f_i(z)$. Then we can see that

$$\xi(s; f, \chi) = A_{\Delta,f}(s) \cdot \prod_{i=1}^{r} L_{K_i}(s, \chi), \qquad A_{\Delta,f}(1) \neq 0$$

where $L_{K_i}(s, \chi)$ is the quadratic twist of the Dedekind zeta function $\zeta_{K_i}(s)$ by χ and $A_{\Delta,f}(s)$ converges for $\Re(s) > 3/4$. We will see that the order of the pole at $s = 1$ of $L_{K_i}(s, \chi)$ is one if

$\sqrt{-\Delta} \in K_i$ and zero otherwise. As a consequence, $\xi(s; f, \chi)$ has a pole of order $\rho_{\Delta,f} - 2$ at $s = 1$.

Since $\xi(s; f, \chi)$ is intimately tied to our sum over $d_i \leq D_i$ in our main term (3.2), we find that (3.2) can be re-written as

$$BL^2 \log(B) \sum_{(c_1,\dots,c_r)}{}' \mu(c_1 \dots c_r)\chi(c_1 \dots c_r) \sum_{(a_i) \bmod \Delta} \prod_{i=1}^{r} (1 + \chi(a_i)) \#\{\mathbf{x} \bmod \Delta : F_i(\mathbf{x}) \equiv a_i \bmod \Delta\}$$
$$\times \operatorname{Res}_{s=1} \xi(s; f, \chi) \log(B)^{\rho_{\Delta,f}-2} \times \prod_{p | c_1 \dots c_r} \mathfrak{S}_p,$$

where $\mathfrak{S}_p$ are local factors. It should be noted that $c_1 \dots c_r$ itself must divide the product of resultants $\prod_{i \neq j} \operatorname{Res}(f_i, f_j)$.

In §3.12, we will show that $\mathfrak{S}_p$ is zero if and only if $X^*_{\Delta,f_1,\dots,f_r,c_1,\dots,c_r}(\mathbb{Q}_p)$ is empty (see (3.1) for the definition of $X^*_{\Delta,f_1,\dots,f_r,c_1,\dots,c_r}$). Similarly, we will find that

$$\sum_{a_i \bmod \Delta} \prod_{i=1}^{r} (1 + \chi(a_i)) \cdot \#\{\mathbf{x} \bmod \Delta : F_i(\mathbf{x}) \equiv a_i \bmod \Delta\} = 0$$

if and only if for some $p \mid -\Delta$, $X^*_{\Delta,f_1,\dots,f_r,c_1,\dots,c_r}(\mathbb{Q}_p) = \emptyset$. We remark that the true (more complicated) computation involves a sum over genus characters as well. Moreover, we show that these are all of the possible local obstructions to $X^*_{\Delta,f_1,\dots,f_r,c_1,\dots,c_r}$ and that the finite set of $(c_1, \dots, c_r)$ in the restricted sum are the only ones necessary to consider.

Consequently, we have the following statement: *our leading constant $c_{\Delta,f}$ is zero if and only if for every vector $(c_1, \dots, c_r)$, there is a local obstruction to rational points on the torsor $X^*_{\Delta,f_1,\dots,f_r,c_1,\dots,c_r}$.* Finally, the results of Colliot-Thélène, Coray, and Sansuc [10, 12] and Harari's "formal lemma" [32] (see Theorem 3.12.4) allows us to interpret this result in terms of the Brauer-Manin obstruction; specifically, the leading constant $c_{\Delta,f} = 0$ if and only if there is either a Brauer-Manin obstruction or local obstruction to rational points on $X_{\Delta,f}$. We remark that in §3.12.1 we discuss in detail the case when $f(z)$ is irreducible and there is only one auxiliary variety – $X_{\Delta,f}$ itself; in this case, our leading constant is a product of local densities.

3.3 Background and notation

3.3.1 Notation

First, let us establish the notation that we will use for the remainder of the paper.

- f will always denote a single variable polynomial and F a binary form,

- $f(z) = f_1 ... f_r(z)$ is the decomposition of $f(z)$ into irreducible polynomials,
- $d \sim D$ if $d \in [D, 2D]$,
- $e(\alpha) = \exp(2\pi i \alpha)$,
- $e_p(\alpha) = \exp(2\pi i \alpha / p)$,
- $\log_k(x) = \log\log\log ... \log(x)$ where there are k-iterated logarithms,
- If $\mathbf{v} = (v_i)$ denotes a vector, then the lower-case unbolded letter v denotes the product of the entries of $\mathbf{v}$,
- $K = \mathbb{Q}(\sqrt{-\Delta})$ is the quadratic field extension generated by $x^2 + \Delta y^2$,
- $N(x + y\sqrt{-\Delta}) = x^2 + \Delta y^2$,
- $\mathcal{O}_K$ is the ring of integers of K,
- I,$\mathfrak{a}, \mathfrak{p}, \mathfrak{t}, \mathfrak{k}$ will denote ideals of $\mathcal{O}_K$,
- $\mathcal{U}$ will denote the ring of units inside of $\mathcal{O}_K$,
- C will denote the class group of K,
- $\hat{C}$ will denote the group of class group characters of K, which can also be viewed as Hecke characters of K,
- χ will denote the real quadratic character such that $\zeta_K(s) = \zeta(s)L(s, \chi)$, and we will denote the modulus of the character as $-\Delta$,
- We define the Δ-part of an integer n as:

$$p_\Delta(n) = \prod_{\substack{p \text{ prime} \\ p^e \| n \\ p | \Delta}} p^e.$$

  We also define the part of n coprime to Δ as

$$p_{\neg\Delta}(n) = n / p_\Delta(n) = \prod_{\substack{p \text{ prime} \\ p^e \| n \\ p \nmid \Delta}} p^e,$$

- K_{f_i} denotes splitting field of f_i over K and $K_f = \prod_{i=1}^r K_{f_i}$,

- $\mathfrak{P}$ will denote a prime ideal on $\mathcal{O}_{K_f}$,
- For a binary form $F(u,v)$, we denote

$$\varrho_F(n) := \#\{(u,v) \bmod n : F(u,v) \equiv 0 \bmod n\}.$$

We also consider the single-variable version of this local count:

$$\varrho_f(n) := \#\{x \bmod n : f(x) \equiv 0 \bmod n\}.$$

- We define the Dirichlet series for $\Re(s) > 1$:

$$\xi(s; f, \chi) = \prod_p \left(1 + \frac{\chi(p)\varrho_f(p)}{p^s}\right).$$

We also introduce some notation specific to the indefinite case, i.e. when $\Delta < 0$. In this case, K is a real quadratic field.

Definition 3.3.1. Let (x_0, y_0) denote the fundamental solution to Pell's equation $x^2 + \Delta y^2 = 1$. Then we write $\varepsilon = x_0 + y_0\sqrt{-\Delta}$ as the **fundamental unit**. Every element of $u \in \mathcal{U} \subset \mathcal{O}_K$ will satisfy that for some $k \in \mathbb{Z}$,

$$u = \varepsilon^k.$$

Next, let us fix $(x,y) \in \mathbb{Z}$. Then we define two new variables:

$$\alpha = x + y\sqrt{-\Delta},\ \overline{\alpha} = x - y\sqrt{-\Delta}.$$

Then we say that (x,y) (or α) is **minimal** if $x \geq 0$ and

$$1 \leq \left|\frac{\alpha}{\overline{\alpha}}\right| \leq |\varepsilon^2|.$$

We note that for any principal ideal I, there will be an unique minimal generator (x,y) such that $(x + y\sqrt{-\Delta}) = (\alpha) = I$. We denote the minimal element of $\mathcal{O}_K$ equivalent to (x,y) up to a unit as $(x,y) \bmod \varepsilon$. We may also denote this element as $\alpha \bmod \varepsilon$.

Finally, let us define our Hecke Grössencharacter Ψ that will detect the "argument" of α, as alluded to in the outline. This definition is inspired by the choice of Grössencharacter taken by

Heath-Brown in [34].

Definition 3.3.2. Define the Hecke character Ψ on principal ideals $I = (\alpha)$ to be:

$$\Psi(I) = \operatorname{sgn}(\alpha\overline{\alpha}) \exp\left(\pi i \frac{\log|\alpha| - \log|\overline{\alpha}|}{\log|\varepsilon|}\right).$$

Ψ can then be lifted trivially to the set of all ideals of $\mathcal{O}_K$.

Remark 3.3.3. Note that Ψ is indeed well-defined on ideals as the following holds:

$$\log|\varepsilon\alpha| - \log|\overline{\varepsilon\alpha}| = 2\log|\varepsilon| + \log|\alpha| - \log|\overline{\alpha}|.$$

Additionally, we know that $|\Psi(\mathfrak{a})| = 1$ for any ideal $\mathfrak{a} \subset \mathcal{O}_K$, so Ψ is a nontrivial unitary Hecke Grössencharacter.

3.3.2 Hooley's Δ-function

Next, we recall the definition and state our notation for the following arithmetic functions:

Definition 3.3.4. The divisor function will be denoted as

$$\tau(n) := \sum_{d|n} 1.$$

Definition 3.3.5. Hooley's Δ-function is defined as

$$\Delta(n) := \sup_{D\in\mathbb{R}, 0\le v\le 1} \sum_{\substack{d|n \\ d\in[D, e^v D]}} 1.$$

We note that the average of $\tau(n)$ satisfies

$$\sum_{n\le X} \tau(n) \sim X\log(X).$$

On the other hand, the average of the Δ-function has a small saving:

$$\sum_{n\le X} \Delta(n) \ll X\log(X)^{o(1)}.$$

This feature is particularly useful in our level of distribution result proved in §3.9, since we will need to be careful about tracking any and all powers of $\log(X)$. We will use the following case of a more

general theorem proved by de la Bretèche and Tenenbaum in [27].

Theorem 3.3.6 (de la Bretèche and Tenenbaum, [27, (1.7)]). *Let $f \in \mathbb{Z}[x]$ be an irreducible polynomial. Then the following bound holds:*

$$\sum_{n \leq X} \Delta(n) \varrho_f(n) \ll_f X \log\log(X)^{5/2}.$$

Next, we introduce the twisted Hooley Δ-function by the quadratic character χ.

Definition 3.3.7. The Hooley Δ-function twisted by χ is defined as

$$\Delta(n, \chi) := \sup_{\substack{D \in \mathbb{R} \\ 0 \leq v \leq 1}} \left| \sum_{\substack{d \mid n \\ d \in [D, e^v D]}} \chi(d) \right|.$$

Observe that $\Delta(n, \chi) \leq \Delta(n) \leq \tau(n)$ for all n.

Remark 3.3.8. We also note that the following inequalities hold:

$$\Delta(n) \geq \sup_{D \in \mathbb{R}} \left| \sum_{\substack{d \mid n \\ d \sim D}} 1 \right| =: \Delta_2(n),$$

$$\Delta(n, \chi) \geq \sup_{D \in \mathbb{R}} \left| \sum_{\substack{d \mid n \\ d \sim D}} \chi(d) \right| =: \Delta_2(n, \chi).$$

This falls from the fact that $2 \leq e^v$ for some $0 \leq v \leq 1$.

In [24], de la Bretèche and Tenenbaum showed that the average of $\Delta(n, \chi)^2$ is smaller than the average of the divisor function. Moreover, this bound also holds for correlations of $\Delta(n, \chi)^2$ with the number of local solutions of a polynomial.

Theorem 3.3.9 (de la Bretèche and Tenenbaum, [24, Theorem 1.1]). *For χ a nonprincipal quadratic character mod M and $f \in \mathbb{Z}[x]$ a polynomial irreducible over $\mathbb{Q}(\sqrt{-M})$. Then, the following bound holds for an explicit constant $c_{f,M} > 0$:*

$$\sum_{n \leq X} \Delta(n, \chi)^2 \varrho_f(n) \ll_{\chi, F} X \exp(c_{f,M} \sqrt{\log_2(X) \log_3(X)}).$$

3.3.3 A sieve on binary forms

We will often need a sieve for averages of non-negative multiplicative functions along polynomial values. The concept for this sieve was first introduced by Nair in [41] for the average of multiplicative functions along single-variable polynomials; later, this work was generalized by Nair and Tenenbaum in [42].

Theorem 3.3.10 (Nair [41]). *Let $f \in \mathbb{Z}[x]$ be a squarefree polynomial with no fixed prime divisors. Let g be a non-negative multiplicative function that is uniformly bounded on prime values. Then we have that*

$$\sum_{n \leq X} g(|f(n)|) \ll_g X \prod_{p \leq X} \left(1 - \frac{\varrho_f(p)}{p}\right) \exp\left(\sum_{p \leq X} \frac{g(p)\varrho_f(p)}{p}\right).$$

This sieve was first generalized to binary forms by de la Bretèche and Browning [20] for irreducible forms. The most general version for binary forms was established by de la Bretèche and Tenenbaum in [23], and this is the result that we shall use.

Define $M_k(A, B, \epsilon)$ to be a class of sub-multiplicative functions in k variables satisfying for $G \in M_k(A, B, \epsilon)$,

$$G(a_1, ..., a_k) \leq \min(A^{\Omega(a_1 \cdots a_k)}, B(a_1 ... a_k)^{\epsilon}).$$

We note that $\tau(n), \Delta(n), \Delta(n, \chi)$ all belong to $M_1(2, B(x) = x, \epsilon)$.

Theorem 3.3.11 (de la Bretèche and Tenenbaum, [23, Theorem 1.1]). *Assume $G \in M_k(A, B, \epsilon)$ for a fixed $A, B \geq 1$. Let $F(x, y) = \prod_{i=1}^{k} F_i(x, y)$ be a squarefree product of primitive binary forms. Fix $\alpha > 0$ and assume that $Y \geq X^{\alpha}$ and $x_0, y_0 \in [X, 2X]$. Then we have that*

$$\sum_{x,y \in [x_0, x_0+Y] \times [y_0, y_0+Y]} G(|F_1(x, y)|, ..., |F_k(x, y)|) \ll_{A,B,\alpha} Y^2 E(X + Y) \prod_{p \leq X+Y} \left(1 - \frac{\varrho_F(p)}{p^2}\right),$$

where we define

$$E(X + Y) = \sum_{s = s_1 ... s_k \leq X+Y} G(s_1, ..., s_k) \frac{\#\{x, y \bmod s : s_i \mid F_i(x, y) \forall i\}}{s^2}.$$

In particular, if $g(n)$ is a non-negative multiplicative function bounded by $\tau(n)$ and $F(x, y) = \prod_{i=1}^{k} F_i(x, y)$ is a squarefree primitive binary form, then we have that

$$\sum_{x,y \leq X} \prod_{i=1}^{k} g(F_i(x, y)) \ll X^2 \prod_{i=1}^{k} \prod_{p \leq X} \left(1 + \frac{\varrho_{F_i}(p)(g(p) - 1)}{p^2}\right).$$

Sometimes, we will wish to apply the following corollary of Theorem 3.3.11 where we specify certain divisors of $F(x,y)$:

Corollary 3.3.12. *Let $F(x,y) \in \mathbb{Z}[x,y]$ be an irreducible primitive binary form and $g \in M_1(A,B,\epsilon)$. We say that $g(t) = 0$ if $t \notin \mathbb{Z}$. Then we have that*

$$\sum_{x,y \le X} g(|F(x,y)/c|) \ll_{A,B} \frac{X^2 (A^{\deg(F)} \cdot \deg(F))^{\Omega(c)}}{c} \prod_{p \ll X} \left(1 + \frac{\varrho_F(p)(g(p)-1)}{p^2}\right).$$

Proof. We rearrange our sum as:

$$\sum_{x,y \le X} g(|F(x,y)/c|) = \sum_{\substack{\alpha \bmod c \\ c \mid F(\alpha)}} \sum_{x,y \ll X/c} g(|F_\alpha(x,y)/c|),$$

where we define $F_\alpha(x,y) = F(cx + \alpha_1, cy + \alpha_2)$. Since $c \mid F(\alpha)$, we have that $F_\alpha(x,y)/c$ is itself an irreducible binary form with integer coefficients. Moreover, $F_\alpha(x,y)/c = c^{e_\alpha} \tilde{F}_\alpha(x,y)$ for some primitive binary form $\tilde{F}_\alpha$ and power $e_\alpha \le \deg(F)$. So, we use submultiplicativity and apply Theorem 3.3.11 to the inside sums. Hence, we have the relation:

$$\sum_{x,y \ll X/c} g(|F_\alpha(x,y)/c|) \ll \frac{X^2 A^{e_\alpha \Omega(c)}}{c^2} \prod_{p \ll X} \left(1 + \frac{\varrho_{F_\alpha/c}(p)(g(p)-1)}{p^2}\right).$$

Since $g \in M_1(A,B,\epsilon)$, we can see the contribution from the primes $p \mid c$ is bounded by $A^{\omega(c)}$. So, let us assume from now on that $p \nmid c$. Now, we consider

$$\varrho_{F_\alpha/c}(p) = \#\{\mathbf{x} \bmod p : F_\alpha(x,y)/c \equiv 0 \bmod p\} = \#\{\mathbf{x} \bmod p : F(cx + \alpha_1, cy + \alpha_2) \equiv 0 \bmod p\}.$$

For $p \nmid c$, we know that c is invertible in $\mathbb{F}_p$ and thus we get that $\varrho_{F_\alpha/c}(p) = \varrho_F(p)$.

So, our original sum can be bounded by

$$\sum_{x,y \le X} g(|F(x,y)/c|) \ll X^2 \cdot \frac{A^{\deg(F)\cdot\Omega(c)}}{c^2} \cdot \prod_{p \ll X} \left(1 + \frac{\varrho_F(p)(g(p)-1)}{p^2}\right) \sum_{\substack{\alpha \bmod c \\ c \mid F(\alpha)}} 1.$$

Finally, we recall that $\varrho_F(c) \le c \deg(F)^{\omega(c)}$. So, we have our desired bound.

□

3.3.4 Upper bounds on fixed sizes of divisors

We introduce two lemmas that bound the contribution of large divisors of a polynomial. These results will be particularly useful in §3.10.

First, we write a result that bounds the number of integers with particularly large divisors; the following formulation was stated in [33] by Heath-Brown.

Lemma 3.3.13 (Heath-Brown, [33, Lemma 5.1]). *There exists an explicit constant $\eta > 0$ such that*

$$\#\{m \ll X : \exists d \sim D, d \mid m\} \ll \frac{X}{\log(D)^{\eta}}.$$

Next, we introduce a modification of the above lemma for binary forms that was proved by de la Bretèche and Tenenbaum in [25].

Lemma 3.3.14 (de la Bretèche and Tenenbaum, [25, (7.41)]). *Let $F(x, y)$ be a binary quartic. Then there exists an explicit constant $\eta > 0$ such that*

$$\#\{|x|, |y| \leq X : \exists d \sim D : d \mid F(x, y)\} \ll X^2 \cdot \left(\frac{X^2/D + D/X^2 + \log\log(X)}{\log(D)^{\eta}} \right).$$

We remark that in both of the cases above η is explicit and greater than $1/1000$.

3.3.5 Dirichlet series and Rankin-Selberg L-functions

In this subsection, we will discuss the connection between $\xi(s; f, \chi)$, and a quadratic twist of a Dedekind zeta function. First, we want to establish another similar Dirichlet series that will appear in our computations later. Recall from the notation section that for a vector $\mathbf{d} = (d_1, \ldots, d_r)$, we write the product of the entries as $d = d_1 \cdots d_r$. Let $\Re(s) > 1$ and define the following

$$\xi(s; F, \chi) = \sum_{d_i=1}^{\infty} \frac{\chi(d)\varrho_F(\mathbf{d})}{d^s},$$

where we define

$$\varrho_F(\mathbf{d}) = \#\{\mathbf{x} \in (\mathbb{Z}/d\mathbb{Z})^2 : d_i \mid F_i(\mathbf{x})\}/\sim$$

and that $\mathbf{x} \sim \mathbf{x}'$ if $\mathbf{x} \equiv \lambda \mathbf{x}' \bmod d$. For vectors $(p_1, \ldots, p_r)$ with prime entries p_i, we know that $\varrho_F((p_i)) \leq 4^r$; so, $\xi(s; F, \chi)$ converges for all $\Re(s) > 1$. We also note that we have an Euler product

for $\xi(s;F,\chi)$:

$$\prod_p \Bigg(1+\frac{\chi(p)}{p^s}\cdot(\varrho_F((p,1,\dots,1))+\dots+\varrho_F((1,\dots,p))) \\ +\frac{\chi(p)^2}{p^{2s}}\cdot(\varrho_F((p,p,1,\dots,1))+\dots+\varrho_F((1,\dots,p,p)))+\dots\Bigg).$$

This expansion allows us to see that

$$\xi(s;F,\chi)=S_{F,\chi}(s)\prod_p\left(1+\frac{\chi(p)}{p^s}\cdot(\varrho_F((p,1,\dots,1))+\dots+\varrho_F((1,\dots,p)))\right),$$

where $S_{F,\chi}(s)$ converges for $\Re(s)>1/2$.

We note that at all but finitely many primes p, we have that

$$\varrho_F((p,1,\dots,1))+\dots+\varrho_F((1,\dots,p))=\varrho_f(p)=\sum_{i=1}^r\varrho_{f_i}(p).$$

Here we recall from §3.3.1 that $\varrho_f(p)=\#\{x\in\mathbb{F}_p : f(x)=0 \bmod p\}$. Thus, we get that

$$\xi(s;F,\chi)=R_{f,\chi}(s)S_{F,\chi}(s)S_{f,\chi}(s)\prod_{i=1}^r\xi(s;f_i,\chi),$$

where we have that $R_{f,\chi}(s)$ a product over finitely many primes $p\mid F(1,0)$ and $S_{f,\chi}(s)$ converges for $\Re(s)>1/2$. Now, this tells us that $\xi(s;F,\chi)$ has a pole at $s=1$ if and only if $\xi(s;f,\chi)$ has a pole at $s=1$. The pole at $s=1$ of both of these functions will be determined by the pole of a quadratic twist of a Dedekind zeta function.

Define the field $L_{f_i}=\mathbb{Q}[z]/f_i(z)$. Let us consider the ζ-function of L_{f_i}:

$$\zeta_{L_{f_i}}(s)=\prod_{\mathfrak{P}\subset\mathcal{O}_{L_{f_i}}}\left(1+N(\mathfrak{P})^{-s}+N(\mathfrak{P})^{-2s}+\dots\right).$$

It is known that $\zeta_{L_{f_i}}(s)$ has a meromorphic continuation to $\mathbb{C}$, a functional equation, and a simple pole at $s=1$. Since $\deg(f_i)\le 4$, we know that the Galois group of L_{f_i} is solvable. Thus, by the resolution of Dedekind's conjecture for solvable groups by Uchida [47] and van der Waall [48], we have that

$$\zeta_{L_{f_i}}(s)=\zeta(s)\prod_{\pi_j}L(s,\pi_j),$$

where π_j are nontrivial irreducible representations and $L(s, \pi_j)$ are entire.

Next, we can view the Dirichlet L-function $L(s, \chi)$ as a L-function of a nontrivial irreducible representation, which we denote by χ as well. Then the quadratic twist of $\zeta_{L_{f_i}}(s)$ by χ is given by:

$$L(s; f_i \times \chi) = L(s, \chi) \prod_{\pi_j} L(s, \pi_j \times \chi).$$

We then have the following Dirichlet series and Euler product expansions for $\Re(s) > 1$:

$$L(s; f_i \times \chi) = \sum_{\mathfrak{A} \subset \mathcal{O}_{L_{f_i}}} \chi(N(\mathfrak{A}))N(\mathfrak{A})^{-s} = \prod_{\mathfrak{P}} \left(1 + \chi(N(\mathfrak{P}))N(\mathfrak{P})^{-s} + \chi(N(\mathfrak{P}))^2 N(\mathfrak{P})^{-2s} + \ldots\right).$$

Note that since $L_{f_i} = \mathbb{Q}[z]/f_i(z)$, the following relation is true:

$$\varrho_{f_i}(p) = \#\{\mathfrak{P} \subset \mathcal{O}_{L_{f_i}} : N(\mathfrak{P}) = p\}.$$

This tells us that

$$\xi(s; f_i, \chi) = S_{L_{f_i}, \chi}(s) L(s; f_i \times \chi),$$

where $S_{L_{f_i}, \chi}(s)$ converges for $\Re(s) > 1/2$. Additionally, we see that $S_{L_{f_i}, \chi}(1) \neq 0$. Thus, we get the following lemma:

Lemma 3.3.15. *The Dirichlet series $\xi(s; F, \chi)$ and $\xi(s; f, \chi)$ have meromorphic continuations to $\Re(s) > 1/2$. Additionally, assume that $L(s; f_i \times \chi)$ has a pole of order m_i at $s = 1$. Then the order of the pole at $s = 1$ of both $\xi(s; F, \chi)$ and $\xi(s; f, \chi)$ is given by $m_1 + \ldots + m_r$.*

Corollary 3.3.16. *The order of the pole of $\xi(s; F, \chi)$ and $\xi(s; f, \chi)$ at $s = 1$ is $\rho_{\Delta, f} - 2$, where $\rho_{\Delta, f}$ is defined in Conjecture 3.1.1.*

Proof. It suffices to show that the order of the pole at $s = 1$ of $L(s; f_i \times \chi)$ is given by

$$\begin{cases} 1, & \sqrt{-\Delta} \in L_{f_i} \\ 0, & \text{else.} \end{cases}$$

Since χ is nontrivial and π_j are nontrivial irreducible representations, $L(s; f_i \times \chi)$ has a pole if and only if $\pi_j = \chi$ for some j. This phenomenon occurs if and only if $\sqrt{-\Delta} \in \mathbb{Q}[z]/(f_i(z)) = L_{f_i}$. $\square$

3.3.6 Theta series and class group characters

In this subsection, we will discuss the connection between this problem and modular forms, as alluded to in §3.2. Note that if C is the class group of K, then

$$\mathbf{1}_{\mathfrak{a} \text{ is principal}} = \frac{1}{|C|} \sum_{\psi \in \hat{C}} \psi(\mathfrak{a}).$$

Consequently, we get that

$$\sum_{N(\mathfrak{a})=n} \mathbf{1}_{\mathfrak{a} \text{ is principal}} = \frac{1}{|C|} \sum_{\psi \in \hat{C}} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}).$$

On the other hand, consider the following function:

$$\theta(z) = \sum_{\substack{N(\mathfrak{a})=n \\ \mathfrak{a} \text{ is principal}}} e(z \cdot N(\mathfrak{a})).$$

As commented in [37, Equation (22.55)], for $\Delta > 0$, $\theta(z)$ will be a Mellin transform of an Epstein zeta function and will be a modular form of weight one in $M_1(\Gamma_0(-\Delta), \chi)$. In fact, if $\Delta > 0$, then this is a recognizable theta series:

$$\theta(z) = \sum_{x,y} e(z \cdot (x^2 + \Delta y^2)).$$

Since $\theta(z) \in M_1(\Gamma_0(-\Delta), \chi)$, we can decompose it into its Eisenstein and cuspidal part. We present the following explicit decomposition of the Fourier coefficients, as shown by Blomer in [3, Lemma 2.2] for $\Delta > 0$. This decomposition is also discussed by Fomenko in [30].

Lemma 3.3.17. *Let $\theta(z) = E(z) + C(z)$, where $E(z)$ is an Eisenstein series and $C(z)$ is a cusp form. Let $\lambda_E(n)$ (resp. $\lambda_C(n)$) denote the Fourier coefficients of $E(z)$ (resp. $C(z)$). Then we have that*

$$\lambda_E(n) = \frac{1}{|C|} \sum_{ord(\psi) \leq 2} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}),$$

$$\lambda_C(n) = \frac{1}{|C|} \sum_{ord(\psi) \geq 3} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}).$$

Additionally, Iwaniec discusses in [36, §12.5] how to express the Eisenstein term in even more detail when K is a quadratic field. Because we are working with *binary quadratic forms*, the Eisenstein component contains not only the trivial component, but the contribution from all of the genus

characters. Furthermore, we can describe these genus characters explicitly. Let $q_1 q_2 = -\Delta$ and $\gcd(q_1, q_2) = 1$. Then all of our genus characters are of the form:

$$\psi(\mathfrak{p}) = \chi_{q_1,q_2}(\mathfrak{p}) = \begin{cases} \chi_{q_1}(N(\mathfrak{p})), & \mathfrak{p} \nmid q_1, \\ \chi_{q_2}(N(\mathfrak{p})), & \mathfrak{p} \nmid q_2. \end{cases}$$

These correspond to a basis of Eisenstein series with Fourier coefficients given by

$$\varepsilon_{q_1,q_2}(n) = \sum_{d|n} \chi_{q_1}(d)\chi_{q_2}(n/d),$$

with χ_{q_1}, χ_{q_2} real quadratic characters satisfying $\chi_{q_1}\chi_{q_2} = \chi$. Note that the trivial character corresponds to $q_1 = 1$ and $q_2 = -\Delta$. Thus, we get that

$$\lambda_E(n) = \frac{1}{|C|} \sum_{q_1 q_2 = -\Delta} \varepsilon_{q_1,q_2}(n).$$

We will apply the above decompositions in §3.5.

3.4 Upper bounds

In this section, we will establish some upper bounds for $N(X_{\Delta,f}, B)$ that are necessary for our proof. First, we recall that an upper bound of the correct order of magnitude was established by Browning [5] for the positive-definite case.

Theorem 3.4.1 (Browning, [5])**.** *Assume that $\Delta > 0$ is a nonsquare integer. Then as $B \to \infty$, we have that*

$$N(X_{\Delta,f}, B) \ll B \log(B)^{\rho_{\Delta,f}-1}.$$

We will need a similar upper bound for this counting function on short boxes $N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1)$; more explicitly, we will need upper bounds on

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) := \frac{1}{2} \cdot \#\Big\{((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v),$$
$$t \le B/\exp(\sqrt{\log(B)}), (u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L), \|x + y\sqrt{-\Delta}\|, \|x - y\sqrt{-\Delta}\| \le \nu_{\Delta,f} B,$$
$$\gcd(x,y,t) = \gcd(u,v) = 1\Big\}.$$

In the above sum, we have already removed the large torsors – this process will occur in §3.5.1 before

our applications of the lemmas below.

Lemma 3.4.2. *Assume that $\Delta > 0$ is a nonsquare integer. If $L \gg \exp(-(\log(B))^{1/3})$, then we have that*

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) \ll L^2 B \log(B)^{\rho_{\Delta,f}-1}$$

Proof. We first expand the definition of $S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1)$:

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) \ll \sum_{t \leq B} \sum_{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)} \#\{x^2 + \Delta y^2 = t^2 F(u,v), \gcd(x,y,t) = 1\},$$

Our first step is to rewrite this inner count with a sum over ideals. Consider the principal ideal $I = (x + y\sqrt{-\Delta})$. Since $\gcd(x,y,t) = 1$ and $N(I) = t^2 F(u,v)$, we know that for any prime p dividing t, we must have that $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K and that (p) does not divide I. As a result, we can write

$$I = \mathfrak{t}^2 \mathfrak{a}$$

where $N(\mathfrak{t}) = t$, $N(\mathfrak{a}) = F(u,v)$, and $\gcd(\mathfrak{t}, \bar{\mathfrak{t}}) = 1$. Additionally, we have that $\gcd(t, I, \overline{I}) = 1$, which occurs if and only if $\gcd(\mathfrak{a}, \bar{\mathfrak{t}}) = 1$. Here we write that $\gcd(\mathfrak{a}, \mathfrak{b}) = 1$ if $\mathfrak{a}$ and $\mathfrak{b}$ do not share any common prime ideal factors. We point out that from now on we use the convention that $N(\mathfrak{t}) = t$.

So, we can bound the above sum by:

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) \ll \sum_{N(\mathfrak{t}) \leq B} \sum_{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)} \#\{\mathfrak{a} : N(\mathfrak{a}) = F(u,v)\}.$$

Next, we can bound this sum of ideals as follows:

$$\#\{\mathfrak{a} : N(\mathfrak{a}) = n\} \ll (1 \star \chi)(n),$$

and henceforth replace $\#\{\mathfrak{a} : N(\mathfrak{a}) = F(u,v)\}$ by $(1 \star \chi)(F(u,v))$ in the sum above.

Note that $(1 \star \chi)(n)$ is a non-negative multiplicative function that is bounded by $\tau(n)$. Applying Theorem 3.3.11, we get the following upper bound:

$$\begin{aligned} S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) &\ll \sum_{N(\mathfrak{t}) \leq B} \sum_{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)} (1 \star \chi)(F(u,v)) \\ &\ll \sum_{N(\mathfrak{t}) \leq B} \frac{1}{N(\mathfrak{t})} \cdot BL^2 \prod_{p \ll B/t} \left(1 + \frac{\varrho_f(p)\chi(p)}{p}\right). \end{aligned}$$

We recognize this product over primes as the factors in the Euler product of $\xi(s; f, \chi)$. Since $\xi(s; f, \chi)$ has a pole of order $\rho_{\Delta,f} - 2$ at $s = 1$ (Corollary 3.3.16), we achieve that

$$\begin{aligned} S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) &\ll L^2 B \sum_{N(\mathfrak{t}) \leq B} \frac{1}{t} \log(B/t)^{\rho_{\Delta,f}-2} \\ &\ll L^2 \cdot B \log(B)^{\rho_{\Delta,f}-1}. \end{aligned}$$

□

For the indefinite case, we can similarly achieve an optimal upper bound.

Lemma 3.4.3. *Assume that $\Delta < 0$ is an integer. If $L \gg \exp(-(\log(B))^{1/3})$, then we have that*

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) \ll \left(\left| \frac{\log|F(\mathbf{x}_0)|}{\log|\varepsilon|} \right| + O(1) \right) L^2 B \log(B)^{\rho_{\Delta,f}-1}.$$

Proof. We first expand the definition of $S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1)$:

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) \ll \sum_{t \leq B} \sum_{(u,v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L)} \sum_{\substack{x + y\sqrt{-\Delta} \bmod \varepsilon \\ x^2 + \Delta y^2 = t^2 F(u,v)}} \#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\}.$$

Consider the count of units

$$\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\}.$$

We can see that

$$|\varepsilon^k \alpha| \leq B \iff k \leq \frac{\log(B)}{\log|\varepsilon|} - \frac{\log|\alpha|}{\log|\varepsilon|}.$$

Since α is minimal, we have by definition that

$$1 \leq \frac{|\alpha|}{|\overline{\alpha}|} = \frac{|\alpha|^2}{\alpha\overline{\alpha}} \leq |\varepsilon|^2.$$

Because $\alpha\overline{\alpha} = t^2 F(u, v)$, we can see that

$$\frac{\log|\alpha|}{\log|\varepsilon|} = \frac{\log|t^2 F(u,v)|}{2\log|\varepsilon|} + O(1).$$

On the other hand, since $(u, v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L)$, we have that

$$\frac{\log|t^2 F(u,v)|}{2\log|\varepsilon|} = \frac{\log(B)}{\log|\varepsilon|} + \frac{\log|F(\mathbf{x}_0)|}{2\log|\varepsilon|} + O(L).$$

Thus, we have that

$$|\varepsilon^k \alpha| \leq B \iff k \leq \left| \frac{\log |F(\mathbf{x}_0)|}{2 \log |\varepsilon|} \right| + O(1).$$

A similar computation with $|\overline{\varepsilon}^k \overline{\alpha}|$ reveals an analogous lower bound. So, we have that

$$\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\} = \left| \frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} \right| + O(1).$$

Plugging this in above, we can see the following:

$$\begin{aligned} S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) &\ll \left(\frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} + O(1) \right) \sum_{t \leq B} \sum_{(u,v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L)} \#\{\alpha \bmod \varepsilon : \alpha \overline{\alpha} = t^2 F(u,v)\} \\ &\ll \left(\frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} + O(1) \right) \sum_{N(\mathfrak{t}) \leq B} \sum_{(u,v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L)} \#\{\mathfrak{a} \subset \mathcal{O}_K : N(\mathfrak{a}) = F(u,v)\}. \end{aligned}$$

Now, as in the positive-definite case, we can again use the following bound:

$$\#\{N(\mathfrak{a}) = n\} \ll (1 \star \chi)(n).$$

So, we can apply Theorem 3.3.11:

$$\begin{aligned} S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) &\ll \left(\frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} + O(1) \right) \sum_{N(\mathfrak{t}) \leq B} \sum_{(u,v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L)} (1 \star \chi)(t^2 F(u,v)) \\ &\ll \left(\frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} + O(1) \right) \sum_{N(\mathfrak{t}) \leq B} \frac{1}{N(\mathfrak{t})} \cdot BL^2 \prod_{p \ll B/t} \left(1 + \frac{\varrho_f(p) \chi(p)}{p} \right) \\ &\ll \left(\frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} + O(1) \right) L^2 B \sum_{N(\mathfrak{t}) \leq B} \frac{1}{N(\mathfrak{t})} \log(B/t)^{\rho_{\Delta,f} - 2} \\ &\ll \left(\frac{\log |F(\mathbf{x}_0)|}{\log |\varepsilon|} + O(1) \right) L^2 \cdot B \log(B)^{\rho_{\Delta,f} - 1}. \end{aligned}$$

Again, we have used that the product over primes is related to $\xi(s; f, \chi)$, which has a pole of order $\rho_{\Delta,f} - 2$ at $s = 1$ (Corollary 3.3.16).

□

Remark 3.4.4. Intuitively, we expect for a generic α that

$$\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\} = O(1).$$

However, we also expect that for certain α, this quantity can be as large as $\log(B)$. This discrepancy

is now hidden in the $\log|F(\mathbf{x}_0)|/\log|\varepsilon|$ term, as $\mathbf{x}_0$ ranges over values in $[-1,1]^2$. For most $\mathbf{x}_0$, $|F(\mathbf{x}_0)|$ is of approximately constant size, but for a small subset of $\mathbf{x}_0$, the value $|F(\mathbf{x}_0)|$ can be quite small.

3.5 Initial reductions

In §3.2, we discussed that a key ingredient in our proof is a restriction to short boxes and the use of Hecke Grössencharacters. In this section, we will demonstrate explicitly why these new tools are crucial for our method. First, we recall some notation. Let $\mathbf{x}_0 \in [-1,1]^2$ and L be length to be determined later. Recall that we denote:

$$\mathcal{B}(\mathbf{x}_0, L) := \mathbf{x}_0 + [0,L]^2.$$

3.5.1 Removal of large torsors

We will first remove the contribution from the large torsors – when $t \in [B/\exp(\sqrt{\log(B)}), B]$.

Lemma 3.5.1. *Assume that Δ is a squarefree integer and $\sqrt{-\Delta} \notin \mathbb{Q}$. Define the contribution from large t as:*

$$L(X_{\Delta,f}, B) := \frac{1}{2} \cdot \#\Big\{((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v),$$
$$|u|,|v| \in (B/t)^{1/2}, \|x + y\sqrt{-\Delta}\|, \|x - y\sqrt{-\Delta}\| \leq \nu_{\Delta,f} B,$$
$$t \in [B/\exp(\sqrt{\log(B)}), B], \gcd(x,y,t) = \gcd(u,v) = 1\Big\}.$$

Then we have the upper bound:

$$L(X_{\Delta,f}, B) \ll B\log(B)^{\rho_{\Delta,f} - 3/2}.$$

Proof. Fix a $t \in [B/\exp(\sqrt{\log(B)}), B]$ for now. First, assume that $\Delta > 0$; then, we have that

$$\#\{x^2 + \Delta y^2 = t^2 F(u,v) : |u|,|v| \leq (B/t)^{1/2}, \gcd(x,y,t) = \gcd(u,v) = 1\}$$
$$\ll \sum_{|u|,|v| \leq (B/t)^{1/2}} \#\{\mathfrak{a} : N(\mathfrak{a}) = t^2 F(u,v)\} \ll (1 \star \chi)(t) \sum_{|u|,|v| \leq (B/t)^{1/2}} \#\{\mathfrak{a} : N(\mathfrak{a}) = F(u,v)\}.$$

The above comes from the initial analysis of the sum in Lemma 3.4.2. Since $\#\{\mathfrak{a} : N(\mathfrak{a}) = n\} \ll$

$(1 \star \chi)(n)$ and this is a non-negative multiplicative function, we apply Theorem 3.3.11 to achieve

$$\#\{x^2 + \Delta y^2 = t^2 F(u,v) : |u|, |v| \leq (B/t)^{1/2}, \gcd(x,y,t) = \gcd(u,v) = 1\}$$
$$\ll \frac{B \cdot (1 \star \chi)(t)}{t} \prod_{p \ll B/t} \left(1 + \frac{\varrho_f(p)\chi(p)}{p}\right) \ll \frac{B \cdot (1 \star \chi)(t)}{t} \log(B/t)^{\rho_{\Delta,f} - 2}.$$

Next, consider when $\Delta < 0$. We claim that the same upper bound holds for a fixed large t. In Lemma 3.4.3, we saw that

$$\begin{aligned}
&\#\{x^2 + \Delta y^2 = t^2 F(u,v) : |u|, |v| \leq (B/t)^{1/2}, \gcd(x,y,t) = \gcd(u,v) = 1\} \\
&\qquad \ll \sum_{|u|,|v| \leq (B/t)^{1/2}} \sum_{\substack{\alpha = x + y\sqrt{-\Delta} \bmod \varepsilon \\ x^2 + \Delta y^2 = t^2 F(u,v)}} \#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\} \\
&\qquad \ll (1 \star \chi)(t) \sum_{|u|,|v| \leq (B/t)^{1/2}} \left(\frac{\log(B)}{\log|\varepsilon|} - \frac{\log|t^2 F(u,v)|}{2\log|\varepsilon|} + O(1)\right) \#\{N(\mathfrak{a}) = F(u,v)\} \\
&\ll \frac{B \cdot (1 \star \chi)(t)}{t} \log(B/t)^{\rho_{\Delta,f} - 2} + \sum_{|u|,|v| \leq (B/t)^{1/2}} \left(\frac{\log(B)}{\log|\varepsilon|} - \frac{\log|t^2 F(u,v)|}{2\log|\varepsilon|}\right) \#\{N(\mathfrak{a}) = F(u,v)\}.
\end{aligned}$$

Since $|u|, |v| \leq (B/t)^{1/2}$, we have that $t^2 F(u,v) = B^2 F(\mathbf{y})$ for some $\mathbf{y} \in [-1,1]^2$. So, we get that

$$\frac{\log(B)}{\log|\varepsilon|} - \frac{\log|t^2 F(u,v)|}{2\log|\varepsilon|} = -\frac{\log|F(\mathbf{y})|}{2\log|\varepsilon|}.$$

Assume that $1 \geq L_t \geq (t/B)^{1/3}$ and divide $[-1,1]^2$ into boxes of sidelength L_t. Then for $\mathbf{y} \in \mathcal{B}(\mathbf{x}_0, L_t)$, we have that

$$-\frac{\log|F(\mathbf{y})|}{2\log|\varepsilon|} = -\frac{\log|F(\mathbf{x}_0)|}{2\log|\varepsilon|} + O(L_t).$$

Then if we sum over $(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L_t)$:

$$\begin{aligned}
&\left(-\frac{\log|F(\mathbf{x}_0)|}{\log|\varepsilon|} + O(1)\right) \sum_{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L_t)} \#\{N(\mathfrak{a}) = F(u,v)\} \\
&\qquad \ll \left(-\frac{\log|F(\mathbf{x}_0)|}{\log|\varepsilon|} + O(1)\right) \cdot \frac{BL_t^2 \cdot (1 \star \chi)(t)}{t} \log(B/t)^{\rho_{\Delta,f} - 2}.
\end{aligned}$$

Here we have used that $\#\{N(\mathfrak{a}) = F(u,v)\} \ll (1 \star \chi)(F(u,v))$ and Theorem 3.3.11.

For our final step in the indefinite case, we sum over $\mathbf{x}_0 \in [-1,1]^2$ spaced L_t apart. We can see

that

$$\begin{aligned}
&\#\{x^2+\Delta y^2 = t^2F(u,v) : |u|,|v| \le (B/t)^{1/2}, \gcd(x,y,t) = \gcd(u,v) = 1\} \\
&\ll \frac{B\cdot(1\star\chi)(t)}{t}\log(B/t)^{\rho_{\Delta,f}-2} + \frac{B\cdot(1\star\chi)(t)}{t}\log(B/t)^{\rho_{\Delta,f}-2}\sum_{\mathbf{x}_0}\left(-\frac{\log|F(\mathbf{x}_0)|}{2\log|\varepsilon|}+O(L_t)\right)L_t^2 \\
&\ll \frac{B\cdot(1\star\chi)(t)}{t}\log(B/t)^{\rho_{\Delta,f}-2}\left(1+\int_{[-1,1]^2}-\frac{\log|F(\mathbf{y})|}{2\log|\varepsilon|}d\mathbf{y}+O(L_t)\right).
\end{aligned}$$

Since this integral converges and $L_t \in [(t/B)^{1/3}, 1]$, we derive that when $\Delta < 0$:

$$\begin{aligned}
&\#\{x^2+\Delta y^2 = t^2F(u,v) : |u|,|v| \le (B/t)^{1/2}, \gcd(x,y,t) = \gcd(u,v) = 1\} \\
&\ll \frac{B\cdot(1\star\chi)(t)}{t}\log(B/t)^{\rho_{\Delta,f}-2}.
\end{aligned}$$

Thus, this bound holds independent of the sign of Δ.

Finally, we sum over the values of t, achieving that

$$\begin{aligned}
&\sum_{t\in[B/\exp(\sqrt{\log(B)}),B]} \#\{x^2+\Delta y^2 = t^2F(u,v) : |u|,|v| \le (B/t)^{1/2}, \gcd(x,y,t) = \gcd(u,v) = 1\} \\
&\ll \sum_{t\in[B/\exp(\sqrt{\log(B)}),B]} \frac{B\cdot(1\star\chi)(t)}{t}\log(B/t)^{\rho_{\Delta,f}-2} \\
&\ll \sum_{N(\mathfrak{t})\in[B/\exp(\sqrt{\log(B)}),B]} \frac{B}{N(\mathfrak{t})} \ll B\log(B)^{\rho_{\Delta,f}-3/2}.
\end{aligned}$$

This completes the proof of this lemma. □

Comparing this to Theorem 3.1.3, it is clear that this is an error term; hence, it is safe to **assume from here on that** $t \le B/\exp(\sqrt{\log(B)})$**.**

3.5.2 Short boxes and intervals

From now on, we take $t \le B/\exp(\sqrt{\log(B)})$ and fix a short box and short interval. Let us recall for $\theta \in [0,1]$ and $\ell \le 1$ a length to be determined later, we defined the interval:

$$\mathcal{I}(\theta,\ell) := \{e(\theta') : \theta' \in [\theta, \theta+\ell]\} \subset S^1.$$

Let us define the following point-counting functions: if $\Delta > 0$, we write

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) := \frac{1}{2} \cdot \#\Big\{((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v),$$
$$(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L), \|x + y\sqrt{-\Delta}\|, \|x - y\sqrt{-\Delta}\| \leq B \cdot \max_{\mathbf{y} \in [-1,1]} |F(\mathbf{y})|^{1/2},$$
$$t \leq B/\exp(\sqrt{\log(B)}), \gcd(x,y,t) = \gcd(u,v) = 1\Big\}.$$

If $\Delta < 0$, we write

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta,\ell)) := \frac{1}{2} \cdot \#\Big\{((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v),$$
$$(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L), \Psi(x + y\sqrt{-\Delta}) \in \mathcal{I}(\theta,\ell), |x + y\sqrt{-\Delta}|, |x - y\sqrt{-\Delta}| \leq B,$$
$$t \leq B/\exp(\sqrt{\log(B)}), \gcd(x,y,t) = \gcd(u,v) = 1\Big\}.$$

We remark that restricting $\Psi(x + y\sqrt{-\Delta}) \in \mathcal{I}(\theta,\ell)$ allows us to control the argument of $\alpha = x + y\sqrt{-\Delta}$. Similarly, taking $(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$ for $L = o(1)$ gives us precision on the value of $t^2F(u,v)$.

In the next few sections, we will establish the following estimates on $N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), I(\theta,\ell))$.

Proposition 3.5.2. *Assume $\Delta > 0$ is a squarefree integer and $\sqrt{-\Delta} \notin \mathbb{Q}$, and let B be a sufficiently large number. Let $\mathbf{x}_0 \in [-1,1]^2$ and $L \gg \log(B)^{-10^{-10}}$. Then there exists a constant $c_{\Delta,f}$ such that as $B \to \infty$,*

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) = L^2 \cdot c_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-10}})).$$

Moreover, $c_{\Delta,f} = 0$ if and only if there is a Brauer-Manin obstruction or a local obstruction, i.e. $X_{\Delta,f}(\mathbb{Q}) = \emptyset$.

Proposition 3.5.3. *Assume $\Delta < 0$ is a squarefree integer and $\sqrt{-\Delta} \notin \mathbb{Q}$, and let B be a sufficiently large number. Let $\mathbf{x}_0 \in [-1,1]^2$, $\theta \in [0,1]$, and $\ell, L \gg \log(B)^{-10^{-10}}$. Then we have that there exists a constant $b_{\Delta,f}$ such that as $B \to \infty$,*

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta,\ell)) = \kappa(\mathbf{x}_0, \mathcal{I}(\theta,\ell)) \cdot b_{\Delta,f} \cdot L^2 B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-10}})),$$

where we define

$$\kappa(\mathbf{x}_0, \mathcal{I}(\theta,\ell)) = \max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|, 0) \cdot \frac{|\mathcal{I}(\theta,\ell)|}{4\pi}.$$

Moreover, $b_{\Delta,f}$ is zero if and only if there is a Brauer-Manin obstruction or a local obstruction, i.e. $X_{\Delta,f}(\mathbb{Q}) = \emptyset$.

Remark 3.5.4. We reiterate that in §3.12 we will show that these constants $c_{\Delta,f}$ and $b_{\Delta,f}$ are zero if and only if for every relevant vector $\mathbf{c}$, there is a local obstruction to rational points on the variety $X^*_{\Delta,f_1,\dots,f_r,\mathbf{c}}$ as defined in (3.1).

3.5.3 Decomposition of the positive-definite case

For this subsection, let us assume that $\Delta > 0$. In this case, while we write $N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1)$ in order to standardize notation, we do not genuinely need short boxes in this situation.

Note that in (3.5.2), the conditions that $\|x \pm y\sqrt{-\Delta}\| = \sqrt{x^2 + \Delta y^2} \leq B \cdot (\max_{\mathbf{y}\in[-1,1]} |F(\mathbf{y})|)^{1/2}$ holds trivially. Thus, we get that

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) = \frac{1}{2} \sum_{t \leq B/\exp(\sqrt{\log(B)})} \sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L) \\ \gcd(u,v)=1}} \#\{x^2+\Delta y^2 = t^2 F(u,v) : \gcd(x,y,t) = 1\}.$$

Our first step is to rewrite this inner sum with a sum over ideals; we repeat the argument discussed in §4 for this. Consider the principal ideal $I = (x + y\sqrt{-\Delta})$. Since $\gcd(x, y, t) = 1$ and $N(I) = t^2 F(u, v)$, we know that for any prime p dividing t, we must have that $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K and that (p) does not divide I. As a result, we can write

$$I = \mathfrak{t}^2 \mathfrak{a}$$

where $N(\mathfrak{t}) = t$, $N(\mathfrak{a}) = F(u, v)$, and $\gcd(\mathfrak{t}, \overline{\mathfrak{t}}) = 1$. Additionally, we have that $\gcd(t, I, \overline{I}) = 1$, which occurs if and only if $\gcd(\mathfrak{a}, \overline{\mathfrak{t}}) = 1$. Here we write that $\gcd(\mathfrak{a}, \mathfrak{b}) = 1$ if $\mathfrak{a}$ and $\mathfrak{b}$ do not share any common prime ideal factors.

Consequently, we can rewrite the sum as:

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) = \frac{|\mathcal{U}|}{2} \sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1}} \sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L) \\ \gcd(u,v)=1}} \sum_{\substack{N(\mathfrak{a})=F(u,v) \\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}} \mathbf{1}_{\mathfrak{t}^2\mathfrak{a} \text{ is principal}}.$$

Recall that the group of units $\mathcal{U}$ is finite in this case. From (3.3.6), we know that we can rewrite the

above sum with class group characters:

$$N(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),S^1)=\frac{|\mathcal{U}|}{2|C|}\sum_{\psi\in\hat{C}}\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2\sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sum_{\substack{N(\mathfrak{a})=F(u,v)\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{a}).$$

Following Lemma 3.3.17, we split the above sum into its Eisenstein and cuspidal parts. In particular, let us define for $\psi\in\hat{C}$ satisfying $\mathrm{ord}(\psi)\geq 3$:

$$E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);0,\psi)=\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2\sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sum_{\substack{N(\mathfrak{a})=F(u,v)\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{a}).$$

We can see that

$$\sum_{\mathrm{ord}(\psi)\geq 3}E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);0,\psi)$$

will be the contribution to $N(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),S^1)$ given by the cuspidal class group characters; this be shown in §3.6 to be an error term.

For the Eisenstein contribution, i.e. the genus characters as described in §3.3.6, we must make an appropriate modification to deal with the condition $\gcd(\mathfrak{a},\bar{\mathfrak{t}})=1$. For this we note that

$$\sum_{\substack{N(\mathfrak{a})=n\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\chi_{q_1,q_2}(\mathfrak{a})=\sum_{\mathfrak{k}|\bar{\mathfrak{t}}}\mu(\mathfrak{k})\sum_{N(\mathfrak{a})=n/N(\mathfrak{k})}\chi_{q_1,q_2}(\mathfrak{a}\mathfrak{k})=\sum_{\mathfrak{k}|\bar{\mathfrak{t}}}\mu(\mathfrak{k})\chi_{q_1,q_2}(\mathfrak{k})\sum_{d|n/N(\mathfrak{k})}\chi_{q_1}(d)\chi_{q_2}(n/d).$$

From now on, we will denote the norm of $\mathfrak{k}$ as $N(\mathfrak{k})=k$. Since $\mathfrak{k}\mid\mathfrak{t}$ and so $\gcd(\mathfrak{k},\Delta)=1$, it follows from the definition of $\chi_{q_1,q_2}(\mathfrak{a})$ given in [37, §12.5] that $\chi_{q_1,q_2}(\mathfrak{k})=1$. So, for a genus character parameterized by q_1,q_2 satisfying $q_1q_2=-\Delta$ and $\gcd(q_1,q_2)=1$, we define:

$$\begin{aligned}M_{q_1,q_2}(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L))&=\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\sum_{\mathfrak{k}|\bar{\mathfrak{t}}}\mu(\mathfrak{k})\sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\varepsilon_{q_1,q_2}(F(u,v)/k)\\ &=\sum_{\substack{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{k},\bar{\mathfrak{k}})=1}}\mu(\mathfrak{k})\sum_{\substack{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\sum_{\substack{(u,v)\in(B/kt)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\varepsilon_{q_1,q_2}(F(u,v)/k),\end{aligned}$$

where we write the Fourier coefficients of the corresponding Eisenstein series as

$$\varepsilon_{q_1,q_2}(n)=\sum_{d|n}\chi_{q_1}(d)\chi_{q_2}(n/d).$$

Remark 3.5.5. We note (and will expand in detail further in §3.8.2) that

$$\varepsilon_{q_1,q_2}(n) \approx (1 \star \chi)(n).$$

This function $(1 \star \chi)$ is the one studied by de la Bretèche and Tenenbaum in [25] to prove the case of $x^2 + y^2 = f(z)$ for f an irreducible binary quartic. As such, the analysis from the Eisenstein part will strongly match how previous authors have proved the asymptotic for $\Delta = 1$.

We can now write that:

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) = \frac{|\mathcal{U}|}{2|C|} \sum_{q_1 q_2 = -\Delta} M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) + \sum_{\substack{\psi \in \hat{C} \\ \mathrm{ord}(\psi) \geq 3}} O(E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)).$$

3.5.4 Decomposition of the indefinite case

For this subsection, assume that $\Delta < 0$. In this case, the group of units $\mathcal{U}$ is infinite and generated by the fundamental unit ε. We wish to count:

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta, \ell)) = \frac{1}{2} \sum_{t \leq B/\exp(\sqrt{\log(B)})} \sum_{\substack{(u,v) \in (B/t)^{1/2} \mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} \sum_{\substack{x + y\sqrt{-\Delta} \bmod \varepsilon \\ x^2 + \Delta y^2 = t^2 F(u,v) \\ \Psi(x + y\sqrt{-\Delta}) \in \mathcal{I}(\theta,\ell) \\ \gcd(x,y,t)=1}} \#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\}$$

Our goal is to express the above sum in terms of $M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L))$ and $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$ (and similar functions). To do so, we must first estimate this count $\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\}$ and remove the condition that $\Psi(x + y\sqrt{-\Delta}) \in \mathcal{I}(\theta, \ell)$.

Counting the number of units

To estimate the count $\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\}$, we apply bounds similar to the proofs of Lemma 3.4.3 and Lemma 3.5.1. Note that $|\varepsilon^k \alpha| \leq B$ if and only if

$$k \leq \frac{\log(B)}{\log|\varepsilon|} - \frac{\log|\alpha|}{\log|\varepsilon|}.$$

Since $\Psi(\alpha) \in \mathcal{I}(\theta, \ell)$ and α is minimal, we know that

$$\frac{\log|\alpha| - \log|\overline{\alpha}|}{\log|\varepsilon|} = \frac{2\log|\alpha|}{\log|\varepsilon|} - \frac{\log|\alpha \cdot \overline{\alpha}|}{\log|\varepsilon|} \in [\theta, \theta + \ell].$$

Recall that $\alpha \cdot \overline{\alpha} = x^2 + \Delta y^2 = t^2 F(u,v)$. Since $(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$, we know that $t^2 F(u,v) = B \cdot (F(\mathbf{x}_0) + O(L))$. So, we can replace this term with

$$\frac{\log|\alpha \cdot \overline{\alpha}|}{\log|\varepsilon|} = \frac{\log|t^2 F(u,v)|}{\log|\varepsilon|} = \frac{2\log(B)}{\log|\varepsilon|} + \frac{\log|F(\mathbf{x}_0)|}{\log|\varepsilon|} + O_\varepsilon(L).$$

Combining all of these factors together, we get that

$$k \leq -\frac{\theta}{2} - \frac{\log|F(\mathbf{x}_0)|}{2\log|\varepsilon|} + O(\ell + L).$$

A similar computation with $|\overline{\varepsilon}^k \overline{\alpha}| \leq B$ reveals that we also need

$$k \geq -\frac{\theta}{2} + \frac{\log|F(\mathbf{x}_0)|}{2\log|\varepsilon|} + O(\ell + L).$$

Thus, we get that

$$\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\overline{\varepsilon}^k \overline{\alpha}| \leq B\} = \max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|, 0) + O(\ell + L).$$

Remark 3.5.6. Since $\mathbf{x}_0 \in [-1,1]^2$, we expect $\log|F(\mathbf{x}_0)|/\log|\varepsilon|$ to often be negative.

Plugging (3.5.4) in, we have that

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta,\ell)) = \left(\frac{\max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|, 0)}{2} + O(\ell + L)\right) \times \sum_{t \leq B/\exp(\sqrt{\log(B)})} \sum_{\substack{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} \sum_{\substack{x + y\sqrt{-\Delta} \bmod \varepsilon \\ x^2 + \Delta y^2 = t^2 F(u,v) \\ \Psi(x+y\sqrt{-\Delta}) \in \mathcal{I}(\theta,\ell) \\ \gcd(x,y,t)=1}} 1.$$

Applying the Erdős-Turán inequality

In this subsection, we remove the condition that $\Psi(x+y\sqrt{-\Delta}) \in \mathcal{I}(\theta,\ell)$; we follow the proof structure set out by Heath-Brown in [34]. We recall the following application of the Erdős-Turán inequality [29], as written in [40, Theorem 1]:

Lemma 3.5.7 (Erdős-Turán, [29]). *Let* $\mathcal{A}_t(n) = \{x + y\sqrt{-\Delta} \bmod \varepsilon : x^2 + \Delta y^2 = n, \gcd(x,y,t) = 1\}$

and $\mathcal{I} \subset S^1$. Then we have that as $n \to \infty$ and for any $H \geq 1$,

$$\sum_{x+y\sqrt{-\Delta}\in\mathcal{A}} \mathbf{1}_{\Psi(x+y\sqrt{-\Delta})\in\mathcal{I}} - \frac{|\mathcal{I}|}{2\pi}\cdot \#\mathcal{A} \ll H^{-1}\#\mathcal{A} + \sum_{1\leq h\leq H}\left(\frac{1}{H+1}+\min(|\mathcal{I}|,(\pi h)^{-1})\right)\left|\sum_{\substack{x+y\sqrt{-\Delta}\in\mathcal{A}\\ \gcd(x,y,t)=1}}\Psi(x+y\sqrt{-\Delta})^h\right|.$$

Applying the above to (3.5.4), we achieve that:

$$N(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L),\mathcal{I}(\theta,\ell)) = \frac{|\mathcal{I}(\theta,\ell)|}{4\pi}\cdot(\max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|+O(\ell+L)) \times\left(M'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L))(1+O(\ell^{-1}H^{-1}))+O\left(\ell^{-1}\sum_{1\leq h\leq H}\frac{1}{h}A'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h)\right)\right),$$

where we define

$$M'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L)) := \sum_{t\leq B/\exp(\sqrt{\log(B)})}\sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sum_{\substack{x+y\sqrt{-\Delta}\bmod\varepsilon\\ x^2+\Delta y^2=t^2F(u,v)\\ \gcd(x,y,t)=1}}1,$$

$$A'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h) = \sum_{t\leq B/\exp(\sqrt{\log(B)})}\sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\left|\sum_{\substack{x+y\sqrt{-\Delta}\bmod\varepsilon\\ x^2+\Delta y^2=t^2F(u,v)\\ \gcd(x,y,t)=1}}\Psi(x+y\sqrt{-\Delta})^h\right|.$$

Let us take $H=\log(B)^5$. Lemma 3.4.3 tells us that

$$M'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L)) \ll L^2B\log(B)^{\rho_{\Delta,f}-1}.$$

Hence, we can see that

$$\ell^{-1}H^{-1}M'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L)) \ll \ell^{-1}L^2B\log(B)^{\rho_{\Delta,f}-6}$$

will more than a sufficient error term. So, it remains to better understand $M'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L))$ and $A'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h)$.

Remark 3.5.8. We would like to point out that there is an extra set of absolute values in the definition of $A'(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h)$ compared to $E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);0,\psi)$, despite them both being error terms that come from the cuspidal contribution. This leads us to requiring different types of analysis on

each sum.

Class group characters

We again apply the results of §3.3.6 and rewrite the above expressions in terms of class group characters. Consider the following sum for $0 \leq h \leq H$:

$$\sum_{\substack{x+y\sqrt{-\Delta} \bmod \varepsilon \\ x^2+\Delta y^2 = t^2 F(u,v) \\ \gcd(x,y,t)=1}} \Psi(x+y\sqrt{-\Delta})^h.$$

Since every principal ideal $I = (x + y\sqrt{-\Delta})$ with $N(I) = t^2F(u, v)$ will have a unique generator $x + y\sqrt{-\Delta}$ that is minimal, we can write the above sum as:

$$\sum_{\substack{N(I)=t^2F(u,v) \\ I \text{ is principal} \\ \gcd(x,y,t)=1}} \Psi(I)^h.$$

Next, we again note that since we require $\gcd(x, y, t) = 1$, we must have that for any prime p dividing t, the ideal $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits and we can not have (p) dividing I. Thus, we can write

$$I = \mathfrak{t}^2\mathfrak{a},$$

where $N(\mathfrak{t}) = t$, $N(\mathfrak{a}) = F(u, v)$, and $\gcd(\mathfrak{t}, \overline{\mathfrak{t}}) = \gcd(\mathfrak{a}, \overline{\mathfrak{t}}) = 1$. So, we can write the above sum as

$$\sum_{\substack{N(\mathfrak{t})=t \\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1}} \Psi(\mathfrak{t})^{2h} \sum_{\substack{N(\mathfrak{a})=F(u,v) \\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}} \Psi(\mathfrak{a})^h \cdot \mathbf{1}_{\mathfrak{t}^2\mathfrak{a} \text{ is principal}}.$$

Applying (3.3.6), the above sum is equivalent to:

$$\frac{1}{|C|} \sum_{\psi \in \hat{C}} \sum_{\substack{N(\mathfrak{t})=t \\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1}} \Psi(\mathfrak{t})^{2h}\psi(\mathfrak{t})^2 \sum_{\substack{N(\mathfrak{a})=F(u,v) \\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}} \Psi(\mathfrak{a})^h\psi(\mathfrak{a}).$$

From Lemma 3.3.17, we can decompose the contribution from the Eisenstein and cuspidal parts. Let us start with the main term given by $M'(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L))$ above; in particular, we have that

$$M'(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) = \frac{1}{|C|} \sum_{q_1q_2=-\Delta} M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) + \sum_{\substack{\psi \in \hat{C} \\ \text{ord}(\psi) \geq 3}} O(E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)),$$

In the above decomposition, $M(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L))$ and $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$ are defined in (3.5.3) and (3.5.3).

On the other hand, for $A'(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h)$, we add in the corresponding absolute value signs. In particular, define for $1 \leq h \leq H = \log(B)^5$ and $\psi \in \hat{C}$:

$$A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi) = \sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t}, \bar{\mathfrak{t}})=1}} \sum_{\substack{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} \sigma(F(u,v), \Psi^h \psi; \bar{\mathfrak{t}}),$$

where we have defined $\sigma(n, \psi; t)$ as

$$\sigma(n, \psi; \mathfrak{t}) := \left| \sum_{\substack{N(\mathfrak{a})=n \\ \gcd(\mathfrak{a}, \mathfrak{t})=1}} \psi(\mathfrak{a}) \right|.$$

Lemma 3.3.17 tells us that

$$A'(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h) \leq \sum_{\psi \in \hat{C}} A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi).$$

3.5.5 Unifying the positive-definite and indefinite notation

Let us summarize our findings. For $\Delta > 0$, we saw that

$$N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), S^1) = \frac{|\mathcal{U}|}{2|C|} \sum_{q_1 q_2 = \Delta'} M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) + \sum_{\substack{\psi \in \hat{C} \\ \mathrm{ord}(\psi) \geq 3}} O(E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)).$$

On the other hand, for $\Delta < 0$, we had the relation

$$\begin{aligned} N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta, \ell)) = \frac{|\mathcal{I}(\theta, \ell)|}{4\pi} \cdot (\max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|, 0) + O(\ell + L)) \\ \times \left(\frac{1}{|C|} \sum_{q_1 q_2 = \Delta'} M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) \right. \\ \left. + \frac{1}{|C|} \sum_{\substack{\psi \in \hat{C} \\ \mathrm{ord}(\psi) \geq 3}} O(E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)) + \ell^{-1} \sum_{1 \leq h \leq H} \frac{1}{h} \sum_{\psi \in \hat{C}} O(A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi)) \right). \end{aligned}$$

Thus, to prove Proposition 3.5.2 and Proposition 3.5.3, it will suffice to prove the following results for our cuspidal and Eisenstein parts.

Proposition 3.5.9 (The cuspidal contribution – the finite image case)**.** *Let $L \gg \log(B)^{-1/10}$. For*

ord$(\psi) \geq 3$, *we have that*

$$E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi) \ll_{\Delta,f} L^2 \cdot B \log(B)^{\rho_{\Delta,f}-1-\frac{1}{3}}.$$

Proposition 3.5.10 (The cuspidal contribution – the Grössencharacter case)**.** *Let* $L \gg \exp(-\log(B)^{1/3})$. *For* $1 \leq h \leq \log(B)^5$ *and* $\psi \in \hat{C}$, *we have that*

$$A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi) \ll_{B,\Delta,f} L^2 \cdot B \log(B)^{\rho_{\Delta,f}-1-1/5}.$$

Proposition 3.5.11 (The Eisenstein contribution)**.** *Let us assume that* $L \gg \log(B)^{-10^{-10}}$. *There exists a constant* $C_{q_1,q_2,f}$ *such that as* $B \to \infty$, *we have that*

$$M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) = C_{q_1,q_2,f} L^2 \cdot B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{\rho_{\Delta,f}-1-10^{-8}})).$$

Remark 3.5.12. We will further discuss $C_{q_1,q_2,f}$ in §3.11 and §3.12 – in particular, we relate $C_{q_1,q_2,f}$ to Brauer-Manin and local obstructions to the Hasse principle. However, it turns out that it is simpler to classify directly when $c_{\Delta,f}$ of Theorem 3.1.3 is zero, as it will become necessary to consider the sum over genus characters

$$\sum_{q_1 q_2 = -\Delta} C_{q_1,q_2,f}.$$

Remark 3.5.13. We can see that the above two propositions that our size constraints on the length of our short boxes L and short intervals ℓ come from the Eisenstein term. Additionally, we can see that the necessity to look at short boxes and intervals comes only from the case when $\Delta < 0$ and $x^2 + \Delta y^2$ is indefinite (as it was used in §3.5.3 to estimate $\#\{k \in \mathbb{Z} : |\varepsilon^k \alpha|, |\varepsilon^{-k}\overline{\alpha}| \leq B\}$). Our condition in Proposition 3.5.10 that $L \gg \exp(-\log(B)^{1/3})$ comes simply by the desire to have $(B/t)^{1/2}|\mathcal{B}(\mathbf{x}_0, L)| \gg 1$. The condition $L \gg \log(B)^{-1/10}$ in Proposition 3.5.9 allows us to compare $L^2 B$ and LB against the corresponding savings.

3.5.6 Interpretation with cusp forms

At this point, let us point out that if ξ is a Hecke character of K, then there is a corresponding automorphic representation Ξ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Moreover, the Fourier coefficients of Ξ will be given by

$$\lambda_\Xi(n) = \sum_{N(\mathfrak{a})=n} \xi(\mathfrak{a}).$$

Since any class group character is a Hecke character, and Ψ is by definition a Hecke character, we can interpret $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$ and $A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi)$ in terms of these cuspidal automorphic representations Ξ.

First, let Ξ be the automorphic representation for $\Psi^h\psi$ for $h \geq 1$. Then we will see that $A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi)$ is upper bounded by the sum:

$$\sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t}, \bar{\mathfrak{t}})=1}} \sum_{\substack{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} |\lambda_\Xi(F(u,v))|.$$

In Proposition 3.5.10, we will achieve our savings despite the absolute values placed around the Fourier coefficients – this savings will come from studying the first few Sato-Tate moments of Ξ.

Second, let Φ be the automorphic representation for the class group character ψ with $\mathrm{ord}(\psi) > 2$. Our error term $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$ can then be written as (approximately):

$$\sum_{t \leq B/\exp(\sqrt{\log(B)})} \sum_{d|t} \mu(d) \sum_{\substack{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} \lambda_\Phi\left(\frac{t^2 F(u,v)}{d^2}\right).$$

This viewpoint will be expanded on in §3.6.1. In Proposition 3.5.9, we achieve our saving by the sum over $\mathfrak{t}$ and d and optimal upper bounds on the contribution from the sum over (u, v). It is important in this argument that unlike $A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi)$, there are no absolute values.

3.5.7 Proof of the asymptotic for Theorem 3.1.3

In both cases, Lemma 3.5.1 allows us to reduce the problem to estimating:

$$\begin{aligned} S(X_{\Delta,f}, B) := \frac{1}{2} \cdot \#\Big\{ ((x,y),(u,v),t) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}_+ : x^2 + \Delta y^2 = t^2 F(u,v), \\ |u|, |v| \in (B/t)^{1/2}, \|x \pm y\sqrt{-\Delta}\| \leq \nu_{\Delta,f} B, \\ t \leq B/\exp(\sqrt{\log(B)}), \gcd(x,y,t) = \gcd(u,v) = 1 \Big\}. \end{aligned}$$

For this sum, we split into short intervals. We choose $L = \lceil \log(B)^{10^{-10}} \rceil^{-1}$. If $\Delta > 0$, take $\ell = 1$ and so $\mathcal{I}(\theta, \ell) = S^1$; if $\Delta < 0$, we let $\ell = \lceil \log(B)^{-10^{-10}} \rceil$. Let $\mathbf{x}_0$ and θ then range over the corresponding dividing of the box $[-1, 1]^2$ and interval $[0, 1]$. We know that

$$S(X_{\Delta,f}, B) = \sum_{\mathbf{x}_0} \sum_{\theta} N(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L), \mathcal{I}(\theta, \ell)).$$

If $\Delta > 0$, we apply Proposition 3.5.2:

$$\begin{aligned} S(X_{\Delta,f}, B) &= c_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-10}})) \cdot \sum_{\mathbf{x}_0} L^2 \\ &= 4 \cdot c_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-10}})). \end{aligned}$$

This completes the positive-definite case.

On the other hand, if $\Delta < 0$, we apply Proposition 3.5.3 to achieve:

$$\begin{aligned} S(X_{\Delta,f}, B) &= b_{\Delta,f} B \log(B)^{\varrho_{a,f}-1}(1 + O(\log(B)^{-10^{-7}})) \cdot \sum_{\mathbf{x}_0} \sum_{\theta} \kappa(\mathbf{x}_0, \mathcal{I}(\theta, \ell)) \\ &= b_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-7}})) \cdot \sum_{\mathbf{x}_0} \sum_{\theta} L^2 \cdot \frac{\ell}{2} \cdot \max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|, 0). \end{aligned}$$

Recall that $b_{\Delta,f} = 0$ if and only if there is a Brauer-Manin or local obstruction.

Finally, we can see that this sum can be approximated by the integral:

$$\begin{aligned} &\sum_{\mathbf{x}_0} \sum_{\theta} L^2 \cdot \frac{\ell}{2} \cdot \max(-\log|F(\mathbf{x}_0)|/\log|\varepsilon|, 0) \\ &\qquad = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1} \max(-\log|F(x,y)|/\log|\varepsilon|, 0) d\theta dx dy + O(\log(B)^{-10^{-10}}). \end{aligned}$$

Since this integral is a constant only depending on f and ε, we can write this expression as $d_{\Delta,f} + O(\log(B)^{-10^{-10}})$. Moreover, it is clear that $d_{\Delta,f} > 0$. Plugging this back into $S(X_{\Delta,f}, B)$, we have that

$$S(X_{\Delta,f}, B) = b_{\Delta,f} d_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-10}})).$$

Taking $c_{\Delta,f} = b_{\Delta,f} d_{\Delta,f}$ finishes the proof. □

3.5.8 Proof of Corollary 3.1.4

Note that we have restricted in Theorem 3.1.3 to when $\Delta \in \mathbb{Z}$ is squarefree and $f(z) \in \mathbb{Z}[z]$, whereas the general definition for Châtelet surfaces allows $\Delta \in \mathbb{Q}$ and $f(z) \in \mathbb{Q}[z]$. Consider the initial surface:

$$x^2 + \Delta_0 y^2 = f_0(z)$$

for $\Delta_0 \in \mathbb{Q}$ a nonsquare and $f_0(z) \in \mathbb{Q}[z]$. After clearing denominators, we can see that rational points on the above surface will be in direct bijection with the rational points on the surface:

$$(Qx)^2 + \Delta_1 y^2 = f_1(z)$$

for $Q, \Delta_1 \in \mathbb{Z}$ and $f_1(z) \in \mathbb{Z}[z]$. After modifying the height function accordingly (by a factor of Q), we can see that the problem of counting rational points on this surface is equivalent to counting rational points (with the modified height function) on the surface

$$x^2 + \Delta_1 y^2 = f_1(z)$$

for $\Delta_1 \in \mathbb{Z}$ and $f_1(z) \in \mathbb{Z}[z]$.

Next, if Δ_1 is not squarefree, we can write $\Delta_1 = \Delta\delta^2$. Hence, our surface can be written as

$$x_1^2 + \Delta(\delta y)^2 = f_1(z).$$

After modifying the height function by a factor of δ, we can see that counting rational points on the above surface is equivalent to counting rational points on

$$x^2 + \Delta y^2 = f(z),$$

where $f(z) = f_1(z) \in \mathbb{Z}[z]$. Thus, Theorem 3.1.3 indeed resolves Manin's conjecture for all Châtelet surfaces over $\mathbb{Q}$.

□

3.6 The cuspidal contribution – the finite image case

In this section, we prove Proposition 3.5.9. Let us recall the definition of $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$ from (3.5.3):

$$\sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \psi(\mathfrak{t})^2 \sum_{\substack{(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L) \\ \gcd(u,v)=1}} \sum_{\substack{N(\mathfrak{a})=F(u,v) \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}} \psi(\mathfrak{a}).$$

We also recall that ψ is a class group character of order ≥ 3. As described in the outline, we aim to change the order of summation and achieve savings from the sum over $\mathfrak{t}$; however, our situation is slightly complicated by the fact that we are summing over short-ish boxes $\mathcal{B}(\mathbf{x}_0, L)$.

Let $\epsilon(B)$ to be determined later. We say that $t \sim_\epsilon T$ if $t \in [T, T(1+\epsilon(B))]$. Consider the space:

$$D(\mathbf{x}_0, L) = \cup_{T_i} (B/T_i)^{1/2} \mathcal{B}(\mathbf{x}_0, L),$$

where T_i ranges over the "dyadic interals of difference $(1+\epsilon(B))$" up to $B/\exp(\sqrt{\log(B)})$. First, we claim that almost all of the points (u, v) counted in $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$ are in $D(\mathbf{x}_0, L)$.

Lemma 3.6.1. *Assume $\epsilon(B) < L$. Let (u, v) be in $(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$ for some t, but not contained in $D(\mathbf{x}_0, L)$. Then (u, v) is $O(\epsilon(B))$ close to the corners of the box $\mathcal{B}(\mathbf{x}_0, L)$. In other words, either*

$$(u, v) \in (B/t)^{1/2}(x_0, y_0 + L) + [0, O(\epsilon(B))]^2$$

or

$$(u, v) \in (B/t)^{1/2}((x_0 + L, y_0)) + [0, O(\epsilon(B))]^2.$$

Proof. Since $(u, v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$, we know that

$$\frac{t^{1/2}}{B^{1/2}}(u, v) \in \mathcal{B}(\mathbf{x}_0, L).$$

We also know that if $T_1 \leq t \leq T_2$, then

$$\frac{T_1^{1/2}}{B^{1/2}}(u, v) \notin \mathcal{B}(\mathbf{x}_0, L), \frac{T_2^{1/2}}{B^{1/2}}(u, v) \notin \mathcal{B}(\mathbf{x}_0, L).$$

Let us assume that $u \leq v$. Then we must have that $\frac{T_1^{1/2}}{B^{1/2}}u < x_0$ and $\frac{T_2^{1/2}}{B^{1/2}}v > y_0 + L$. Thus, we get that $\frac{t^{1/2}}{B^{1/2}}u - x_0 \leq O(\epsilon(B) \cdot \frac{t^{1/2}}{B^{1/2}}u)$. Similarly, we achieve that $y_0 + L - \frac{t^{1/2}}{B^{1/2}}v = O(\epsilon(B) \cdot \frac{t^{1/2}}{B^{1/2}}v)$. □

Next, we justify changing the order of summation for $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$.

Lemma 3.6.2. *Let us define*

$$D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi) = \sum_{T_i} \sum_{\substack{(u,v) \in (B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} \sum_{\mathfrak{a}: N(\mathfrak{a}) = F(u,v)} \psi(\mathfrak{a}) \sum_{\substack{\mathfrak{t}: N(\mathfrak{t}) \sim_\epsilon T_i \\ \gcd(\mathfrak{t}, \bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a}, \bar{\mathfrak{t}})=1}} \psi(\mathfrak{t})^2.$$

Here the sum over T_i is a sum over powers of $(1+\epsilon(B))$ up to $B/\exp(\sqrt{\log(B)})$. Assume that $\epsilon(B) = o(L)$. Then we can bound

$$|E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi) - D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi)| \ll L\epsilon(B)B\log(B)^{\rho_{\Delta,f}-1}$$

Proof. First, we compare $D(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)$ to an intermediate sum:

$$I(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)=\sum_{T_i}\sum_{\substack{(u,v)\in(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sum_{N(\mathfrak{a})=F(u,v)}\psi(\mathfrak{a})\sum_{\substack{N(\mathfrak{t})\sim_\epsilon T_i\\ (u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1\\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2.$$

Assume $(u,v)\in(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L)$, but $(u,v)\notin(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)$ for some $t\in[T_i,(1+\epsilon(B))T_i]$. Since $\epsilon(B)\le L$, we can isolate such points as satisfying that they are within $O(\epsilon(B))$ of the edges of $(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L)$. Thus, we have that (using the trivial bound $|\psi^2(\mathfrak{t})|\le 1$)

$$\begin{aligned}|D(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)-I(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)|&\ll\sum_{T_i}\epsilon(B)T_i\cdot\sum_{(u,v)\in(B/T_i)^{1/2}\partial\mathcal{B}(\mathbf{x}_0,L)+O(\epsilon(B))}(1\star\chi)(F(u,v))\\&\ll\epsilon(B)\sum_{T_i}T_i\cdot\frac{B}{T_i}\cdot L\epsilon(B)\cdot\log(B)^{\rho_{\Delta,f}-2}\\&\ll L\epsilon(B)^2B\log(B)^{\rho_{\Delta,f}-2}\cdot\frac{\log(B)}{\epsilon(B)}\ll L\epsilon(B)B\log(B)^{\rho_{\Delta,f}-1}.\end{aligned}$$

Here we have used Theorem 3.3.11 as before with the non-negative multiplicative function $(1\star\chi)(n)$. Also, observe that $(B/T_i)^{1/2}(\partial\mathcal{B}(\mathbf{x}_0,L)+O(\epsilon(B)))\supset(B/T_i)^{1/2}\partial\mathcal{B}(\mathbf{x}_0,L)+O(\epsilon(B))$ and the larger one has area $BT_i^{-1}L\epsilon(B)$. Here we have used that there are $\ll\log(B)/\epsilon(B)$ such values of T_i.

Next, we want to compare $I(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)$ and $E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);0,\psi)$. We can see that

$$I(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)=\sum_{\substack{(u,v)\in D(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sum_{N(\mathfrak{a})=F(u,v)}\psi(\mathfrak{a})\sum_{\substack{t\le B/\exp(\sqrt{\log(B)})\\ (u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)}}\sum_{\substack{N(\mathfrak{t})=t\\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1\\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2,$$

whereas we have that

$$E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);0,\psi)=\sum_{\substack{(u,v)\in\cup_t(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sum_{N(\mathfrak{a})=F(u,v)}\psi(\mathfrak{a})\sum_{\substack{t\le B/\exp(\sqrt{\log(B)})\\ (u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)}}\sum_{\substack{N(\mathfrak{t})=t\\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1\\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2.$$

Thus, the difference is summing over exactly those (u,v) not contained in $D(\mathbf{x}_0,L)$. Thus, we have

that

$$\begin{aligned}
&|I(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)-E(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);0,\psi)| \\
&\qquad\ll \sum_{\substack{N(\mathfrak{t})\le B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \sum_{\substack{(u,v)\in(B/t)^{1/2}((x_0,y_0+L)+O(\epsilon(B))) \\ \cup(B/t)^{1/2}((x_0+L,y_0)+O(\epsilon(B)))}} (1\star\chi)(F(u,v)) \\
&\qquad\ll \sum_{N(\mathfrak{t})\le B/\exp(\sqrt{\log(B)})} \epsilon(B)^2\cdot\frac{B}{t}\log(B/t)^{\rho_{\Delta,f}-2} \\
&\qquad\ll B\epsilon(B)^2\log(B/t)^{\rho_{\Delta,f}-1}.
\end{aligned}$$

Again, we have applied Theorem 3.3.11 on the sum over (u,v) with the non-negative multiplicative function $(1\star\chi)(n)$. □

Finally, we wish to bound $D(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)$. Now, we place the absolute values:

$$D(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);\psi)\le\sum_{T_i}\sum_{\substack{(u,v)\in(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L) \\ \gcd(u,v)=1}}\sum_{N(\mathfrak{a})=F(u,v)}\left|\sum_{\substack{N(\mathfrak{t})\sim_\epsilon T_i \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2\right|.$$

Lemma 3.6.3. *For $T\gg_K 1$, and $\psi^2\neq 1$, the following bound holds:*

$$\sum_{\substack{N(\mathfrak{t})\le T \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2\ll_K\prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)T^{5/6}.$$

Corollary 3.6.4. *Let $\epsilon(B)=o(1)$. Then we have that*

$$\sum_{\substack{N(\mathfrak{t})\sim_\epsilon T \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2\ll_K\prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)T^{5/6}.$$

The above corollary comes from noting that since $\epsilon(B)=o(1)$,

$$\begin{aligned}
&\sum_{\substack{N(\mathfrak{t})\sim_\epsilon T \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2=\sum_{\substack{N(\mathfrak{t})\le(1+\epsilon)T \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2-\sum_{\substack{N(\mathfrak{t})<T \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2 \\
&\ll_K\prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)((1+\epsilon)T)^{5/6}+\prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)T^{5/6}\ll_K\prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^{2/3}}\right)T^{5/6}.
\end{aligned}$$

Proof of Lemma 3.6.3. Define the Dirichlet series:

$$\xi_{\mathfrak{a}}(s,\psi^2) := \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_K \\ \gcd(\mathfrak{b},\overline{\mathfrak{b}})=1 \\ \gcd(\mathbf{a},\overline{\mathfrak{b}})=1}} \frac{\psi(\mathfrak{b})^2}{N(\mathfrak{b})^s}.$$

This series converges absolutely for $\Re(s) > 1$. We note that

$$\xi_{\mathfrak{a}}(s,\psi^2) = \prod_{\mathfrak{p}|\mathbf{a}} \left(1 - \frac{\psi(\mathfrak{p})^2}{N(\mathfrak{p})^s}\right) \cdot \xi(s,\psi^2),$$

where we define

$$\xi(s,\psi^2) := \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_K \\ \gcd(\mathfrak{b},\overline{\mathfrak{b}})=1}} \frac{\psi(\mathfrak{b})^2}{N(\mathfrak{b})^s}.$$

We can relate $\xi(s,\psi^2)$ to a Hecke L-function as such:

$$\xi(s,\psi^2) \cdot \sum_{\substack{I\subset\mathcal{O}_K \\ I \text{ principal}}} \frac{1}{N(I)^s} = \xi(s,\psi^2)\zeta(2s) = L_K(s,\psi^2) := \sum_{\mathfrak{b}\subset\mathcal{O}_K} \frac{\psi(\mathfrak{b})^2}{N(\mathfrak{b})^s},$$

where $L_K(s,\psi^2)$ is a Hecke L-function. Thus, we have that

$$\xi_{\mathfrak{a}}(s,\psi^2) = f_{\mathbf{a}}(s) L_K(s,\psi^2),$$

where

$$f_{\mathbf{a}}(s) = \zeta(2s)^{-1} \cdot \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{\psi(\mathfrak{p})^2}{N(\mathfrak{p})^s}\right).$$

Consequently, $\xi_{\mathbf{a}}(s,\psi^2)$ is analytic for $\Re(s) > 1/2$ (since $\psi^2 \neq 1$, $L_K(s,\psi^2)$ converges at $s = 1$).

Let us consider the following weighted sum:

$$S(T) = \sum_{\substack{N(\mathfrak{t})\leq T \\ \gcd(\mathfrak{t},\overline{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\overline{\mathfrak{t}})=1}} \psi(\mathfrak{t})^2 (T - N(\mathfrak{t})).$$

By Perron's formula, we have that for any $\sigma > 1$ the following holds:

$$S(T) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f_{\mathfrak{a}}(s)\zeta(2s)L_K(s,\psi^2) \frac{T^{1+s}}{s(s+1)} ds.$$

Take $1/2 < \alpha < 1$. Let us shift the contour above to the line $\Re(s) = \alpha$. Since $\psi^2 \neq 1$, we know that

$L(s,\psi^2)$ will not have a pole at $s=1$. Thus, we have that the above is

$$\begin{aligned} S(T) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f_{\mathbf{a}}(s)\zeta(2s)L_K(s,\psi^2)\frac{T^{1+s}}{s(s+1)}ds \\ &\ll_\alpha \prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^\alpha}\right)T^{1+\alpha}\int_{\alpha-i\infty}^{\alpha+i\infty}|L_K(s,\psi^2)|\frac{ds}{s(s+1)}. \end{aligned}$$

Now, for $\Re(s)=\alpha$, the convexity bound gives us that

$$|L_K(s,\psi^2)| \ll_K |s|^{\frac{1-\alpha}{2}+\epsilon}.$$

So, we have that the inner integral is bounded above by

$$\int_{-\infty}^{\infty}(\alpha+|t|)^{\frac{1-\alpha+\epsilon}{2}-2}dt<\infty,$$

which converges to a constant depending on α. Thus, we have that

$$S(T)\ll_\alpha \prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^\alpha}\right)T^{1+\alpha}.$$

Next, we use the following trick. Consider the difference

$$S(T+T')-S(T)=T'\sum_{\substack{N(\mathfrak{t})\le T\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1\\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}}\psi(\mathfrak{t})^2+O_K\left(T'^2\right)$$

where the error term comes from the ideals with $N(\mathfrak{t})\in(T,T+T']$ and the trivial bound $|\psi^2|\le 1$. Applying the above bound (3.6), we also have that

$$S(T+T')-S(T)\ll_\alpha \prod_{\mathfrak{p}|\mathfrak{a}}\left(1+\frac{1}{N(\mathfrak{p})^\alpha}\right)(T+T')^{1+\alpha}.$$

Let us take $\beta=\frac{1+\alpha}{2}$; since $1/2<\alpha<1$, we have that $3/4<\beta<1$. Then we can see that by

comparing the above bounds and taking $T' = T^{\beta}$,

$$\sum_{\substack{N(\mathfrak{t}) \leq T \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1}} \psi(\mathfrak{t})^2 \ll T' + \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 + \frac{1}{N(\mathfrak{p})^{\alpha}}\right) T^{1+\alpha} T'^{-1}$$

$$\ll \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 + \frac{1}{N(\mathfrak{p})^{\alpha}}\right) T^{\beta}.$$

This brings us to our final choices of α and β. Let us choose $\alpha = 2/3$ and $\beta = 5/6$. These provide us with the final bound.

□

Now, let us plug this into our estimates for $D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi)$. This gives us that:

$$D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi) \leq \sum_{T_i} T_i^{5/6} \sum_{\substack{(u,v) \in (B/T_i)^{1/2} \mathcal{B}(\mathbf{x}_0, L) \\ \gcd(u,v)=1}} (1 \star \chi)(F(u,v)) \cdot \prod_{p | F(u,v)} (1 + 1/p^{2/3}).$$

Applying Nair's sieve (Theorem 3.3.11) to the inner sum, we get that

$$\begin{aligned}
&\sum_{(u,v) \in (B/T_i)^{1/2} \mathcal{B}(\mathbf{x}_0, L)} (1 \star \chi)(F(u,v)) \prod_{p | F(u,v)} (1 + p^{-2/3}) \\
&\ll \frac{BL^2}{T_i} \exp\left(\sum_{p \ll B/T_i} \frac{\varrho_f(p)}{p} \cdot \Big((1 \star \chi)(p) \cdot (1 + p^{-2/3}) - 1 \Big) \right) \\
&\ll \frac{BL^2}{T_i} \exp\left((\rho_{\Delta,f} - 2) \log\log(B/T_i) + \sum_{p \ll B/T_i} \frac{2\varrho_f(p)}{p^{5/3}} \right) \\
&\ll \frac{BL^2}{T_i} \log(B)^{\rho_{\Delta,f} - 2}.
\end{aligned}$$

Thus, we have that

$$D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi) \ll L^2 B \log(B)^{\rho_{\Delta,f} - 2} \sum_{T_i} T_i^{-1/6}.$$

We note that this inner sum is a geometric series, since T_i is summed over powers of $(1 + \epsilon(B))$ up to $B/\exp(\sqrt{\log(B)})$:

$$\sum_{T_i} T_i^{-1/6} = \sum_{k=0}^{K} (1 + \epsilon(B))^{-k/6} < \frac{1}{1 - (1 + \epsilon(B))^{-1/6}},$$

where $K = (\log(B) - \sqrt{\log(B)})/\log(1+\epsilon(B))$. Now we know that

$$\frac{1}{1-(1+\epsilon(B))^{-1/6}} \ll \epsilon(B)^{-1}.$$

So, we get that

$$D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi) \leq \epsilon(B)^{-1} BL^2 \log(B)^{\rho_{\Delta,f}-2}.$$

Finally, we choose $\epsilon(B)$. Let us take $\epsilon(B) = \log(B)^{-1/2}$. So, we get that

$$D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi) \ll BL^2 \log(B)^{\rho_{\Delta,f}-3/2}.$$

On the other hand, by Lemma 2.2,

$$|E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi) - D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi)| \ll LB \log(B)^{\rho_{\Delta,f}-3/2}.$$

Hence,

$$E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi) \ll LB \log(B)^{\rho_{\Delta,f}-3/2} + BL^2 \log(B)^{\rho_{\Delta,f}-3/2}.$$

Since $L \gg \log(B)^{-10^{-1}}$, we can also safely say that

$$E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi) \ll L^2 B \log(B)^{\rho_{\Delta,f}-1-1/3}.$$

This concludes the proof of Proposition 3.5.9. □

3.6.1 Interpretation with cusp forms

Now, let us reframe the above proof using the cusp form interpretation of our sum $E(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); 0, \psi)$. We consider the expression after our change of variables (Lemma 3.6.2),

$$D(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); \psi) = \sum_{T_i} \sum_{\substack{(u,v)\in(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L) \\ \gcd(u,v)=1}} \sum_{\substack{N(\mathfrak{t}^2\mathfrak{a})=t^2F(u,v) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1 \\ \gcd(\mathfrak{a},\bar{\mathfrak{t}})=1 \\ N(\mathfrak{t})\sim_\epsilon T_i}} \psi(\mathfrak{t}^2\mathfrak{a}).$$

Let Ξ denote the cuspidal automorphic representation on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ given by ψ; since ψ is a class group character, the representation Ξ has finite image. By replacing the condition $\gcd(\mathfrak{t}, \bar{\mathfrak{t}}) = 1$ with

a Möbius sum over divisors of $N(\mathfrak{t})$, the above is equivalent to

$$\sum_{T_i} \sum_{\substack{(u,v)\in(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}} \sum_{t\sim_\epsilon T_i} \sum_{d|t} \mu(d)\lambda_\Xi\left(\frac{t^2F(u,v)}{d^2}\right)$$

$$= \sum_{T_i} \sum_{\substack{(u,v)\in(B/T_i)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}} \sum_{d\le T_i} \mu(d) \sum_{t\sim_\epsilon T_i/d} \lambda_\Xi(t^2F(u,v)).$$

We will return to this sum at the end of the section, but for now let us consider the sum

$$\sum_{d\le X} \mu(d) \sum_{t\le X/d} \lambda_\Xi(at^2).$$

Let $C(z)$ denote a holomorphic cusp form in $S_k(\Gamma_0(-\Delta),\chi)$ and $\lambda_C(n)$ denote its normalized Fourier coefficients. Let us take c to be a constant such that $|\lambda_C(p)|\le c$ for all primes p. We claim that the following bound holds, *independent of whether of not $C(z)$ is dihedral.*

Lemma 3.6.5. *Let $C(z)\in S_k(\Gamma_0(-\Delta),\chi)$ and assume that $|\lambda_C(p)|\le c$ for all primes c. Then we have the bound*

$$\sum_{d\le X} \mu(d) \sum_{t\le X/d} \lambda_C(at^2) \ll (c+1)^{\omega(a)} X\log(X)^{-40c}.$$

Remark 3.6.6. The exponent of $40c$ is fairly arbitrary, but we choose a number for concreteness that will suffice in future bounds.

Proof. We define the weighted sum:

$$S(X) := \sum_{d\le X} d\mu(d) \sum_{t\le X/d} \lambda_C(at^2)(X/d-t).$$

From Perron's formula, we know that for $\sigma>1$,

$$S(X) = \frac{1}{2\pi i}\sum_{d\le X} d\mu(d) \int_{\sigma-i\infty}^{\sigma+i\infty} \xi(s;a,C)\cdot\frac{(X/d)^{s+1}}{s(s+1)}ds,$$

where we define the Dirichlet series

$$\xi(s;a,C) := \sum_n \frac{\lambda_C(an^2)}{n^s}.$$

We now relate the above Dirichlet series to the L-function $L(s,\mathrm{sym}^2(C))$. Define α_p and β_p such

that

$$1 - \lambda_C(p)p^{-s} + \chi(p)p^{-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}).$$

Let us define

$$g_a(s) := \prod_{p^e \| a} \left(\lambda_C(p^e) + \lambda_C(p^{e+1})p^{-s} + ...\right)\left(1 - \alpha_p^2 p^{-s}\right)\left(1 - \beta_p^2 p^{-s}\right)\left(1 - \chi(p)p^{-s}\right).$$

Then we have the relation for $\Re(s) > 1$:

$$\xi(s; a, C) = g_a(s)L(2s, \chi^2)^{-1}L(s, \mathrm{sym}^2(C)).$$

In particular, $\xi(s; a, C)$ and $L(s, \mathrm{sym}^2(C))$ share the same order of the pole at $s = 1$. We refer the reader to Shimura's [45] classificaton of when $L(s, \mathrm{sym}^2(C))$ has a pole at $s = 1$; in our proof, it will turn out that there is cancellation independent of if $C(z)$ is dihedral or not.

Let us bring back the Möbius sum. We can see that

$$S(X) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \xi(s; a, C)\frac{X^{s+1}}{s(s+1)} \cdot \sum_{d \le X} \frac{\mu(d)}{d^s} ds.$$

Take $\sigma = 1 + \frac{1}{\log(X)} > 1$, we know that for any $A > 0$,

$$\sum_{d \le X} \frac{\mu(d)}{d^{s+1}} = \zeta(s)^{-1}\left(1 + O(\log(X)^{-A})\right).$$

Thus, we have that for any $A' > 0$,

$$S(X) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} g_a(s)L(2s, \chi^2)^{-1}L(s, \mathrm{sym}^2(C))\zeta(s)^{-1}\frac{X^{s+1}}{s(s+1)} ds + O\left(\sup_{\Re(s)=\sigma} |g_a(s)| \cdot X^2 \log(X)^{-A'}\right).$$

Next, we want to add cutoffs at $\sigma + iD$ and $\sigma - iD$ for D to be determined later. This gives us that the integral above is equal to

$$\frac{1}{2\pi i}\int_{\sigma - iD}^{\sigma + iD} \xi(s; a, C)\zeta(s)^{-1}\frac{X^{s+1}}{s(s+1)} ds + O\left(\sup_{\Re(s)=\sigma} |g_a(s)| \cdot \frac{X^2 \log(X)^2}{D}\right)$$

Finally, let us shift our line of integration to $\Re(s) = \alpha$ for some $\alpha > 1 - \log(D)^{-8}$; in other words, we stay within a zero free region for $\zeta(s)$. We note that if $L(s, \mathrm{sym}^2(C))$ has a pole at $s = 1$, then it is a simple pole and will be cancelled by the zero of $\zeta(s)^{-1}$. If $L(s, \mathrm{sym}^2(C))$ is holomorphic at

$s = 1$, then again $\xi(s; a, C)\zeta(s)^{-1}$ will be holomorphic at $s = 1$. Thus, we get the formula

$$S(X) = O\left(\sup_{\substack{\Im(s)=\pm D \\ \alpha \leq \Re(s) \leq \sigma}} |g_a(s)| \cdot \frac{X^2 \log(X)^2}{D}\right) + O\left(X^{2-\log(D)^{-8}} \int_{\alpha - iD}^{\alpha + iD} \frac{|\xi(s; a, C)||\zeta(s)|^{-1}}{|s||s+1|} ds\right).$$

Recall that the convexity bound [37, (5.8),(5.20)] gives us that

$$|\xi(s; a, C)| \ll |g_a(s)|(|s| + 1)^{1-\alpha}.$$

We also know from [46, (3.6.5)] that since $\Re(s) = \alpha > 1 - \log(D)^{-8}$, we have that

$$|\zeta(s)|^{-1} \ll \log(|s| + 3)^7.$$

Thus, in this region,

$$\int_{\alpha - iD}^{\alpha + iD} \frac{|\xi(s; a, C)||\zeta(s)|^{-1}}{|s||s+1|} ds \ll \sup_{\Re(s)=\alpha} |g_a(s)|.$$

Thus, we have that

$$S(X) \ll \sup_{\alpha \leq \Re(s) \leq \sigma} |g_a(s)| \cdot \left(\frac{X^2 \log(X)^2}{D} + X^{2-\log(D)^{-9}}\right).$$

Take $D = \log(X)^{101c}$ and $A' = 101c$. Consequently, we have that

$$S(X) \ll \sup_{\alpha \leq \Re(s) \leq \sigma} |g_a(s)| X^2 \log(X)^{-99c}.$$

Finally, we need to remove our weight. Let $Y < X$ be determined later and consider

$$S(X+Y) - S(X) = \sum_{d \leq X+Y} d\mu(d) \sum_{t \leq (X+Y)/d} \lambda_C(at^2)((X+Y)/d - t) - \sum_{d \leq X} d\mu(d) \sum_{t \leq X/d} \lambda_C(at^2)(X/d - t).$$

Rearranging the expression, we get that

$$\begin{aligned} S(X+Y) - S(X) = Y \sum_{d \leq X} \mu(d) \sum_{t \leq X/d} \lambda_C(at^2) \\ + \sum_{d \leq X} d\mu(d) \sum_{X/d < t \leq (X+Y)/d} \lambda_C(at^2)((X+Y)/d - t) \\ + \sum_{X < d \leq X+Y} d\mu(d) \sum_{t \leq (X+Y)/d} \lambda_C(at^2)((X+Y)/d - t). \end{aligned}$$

Since $Y < X$, the last sum only contains $t = 1$ and hence is bounded by

$$\ll |g_a(1)|X \cdot \left| \sum_{X \leq d \leq X+Y} \mu(d) \right|.$$

Consider the second sum. Note that since $t > X/d$, we have that $(X+Y)/d - t \leq Y/d$. So, the second sum is bounded by

$$Y \sum_{d \leq X} \sum_{X/d < t \leq (X+Y)/d} |\lambda_C(at^2)| \ll c^{\omega(a)} Y^2 \log(X)^c,$$

where c is the bound on the prime values of $|\lambda_C(p)|$.

Thus, we know that

$$\sum_{d \leq X} \mu(d) \sum_{t \leq X/d} \lambda_C(at^2) = Y^{-1}(S(X+Y) - S(X)) + O\left(|g_a(1)|XY^{-1} \left| \sum_{X \leq d \leq X+Y} \mu(d) \right| \right) + O\left(c^{\omega(a)} Y \log(X)^c \right).$$

Applying (3.6.1), we have that

$$\sum_{d \leq X} \mu(d) \sum_{t \leq X/d} \lambda_C(at^2) \ll \sup_{\alpha \leq \Re(s) \leq \sigma} |g_a(s)| \cdot Y^{-1} X^2 \log(X)^{-99c} + |g_a(1)| \cdot Y^{-1} X \left| \sum_{X \leq d \leq X+Y} \mu(d) \right| + c^{\omega(a)} Y \log(X)^c.$$

We take $Y = X \log(X)^{-42c}$. We also note that by definition of $g_a(s)$, we know that

$$\sup_{\alpha \leq \Re(s) \leq \sigma} |g_a(s)| \ll (c+1)^{\omega(a)}.$$

This gives us that

$$\sum_{d \leq X} \mu(d) \sum_{t \leq X/d} \lambda_C(at^2) \ll (c+1)^{\omega(a)} X \log(X)^{-40c}.$$

□

Finally, let us return to our original sum. Let us consider the non-short interval version for simplicity, but as in Lemma 3.6.2 and Lemma 3.6.3, the above can be modified for our slightly-short

intervals. We consider the sum:

$$\sum_{\substack{|u|,|v|\leq B^{1/2}\\ \gcd(u,v)=1}} \sum_{t\leq B/\max(|u|,|v|)^2} \sum_{d|t} \mu(d)\lambda_{\Xi}\left(\frac{t^2F(u,v)}{d^2}\right).$$

At this point, we use the fact that if Ξ is induced from a class group character, then $c=1$. Moreover, we can calculate that

$$\sup_{\alpha\leq\Re(s)<\sigma} |g_p(1)| \ll \begin{cases} 1+p^{-\alpha}, & \chi(p)=1,\\ 0, & \chi(p)=-1. \end{cases}$$

We note that we can assume that $\alpha > 1/2$ in the computations above, as this was the lower bound for the zero-free region. Thus, the bounds in the proof of Lemma 3.6.5 give us a more precise upper bound:

$$\ll \sum_{|u|,|v|\leq B^{1/2}} \frac{B\log(B/\max(|u|,|v|)^2)^{-40}}{\max(|u|,|v|)^2}\cdot(1\star\chi)(F(u,v))\cdot\prod_{\substack{p|F(u,v)\\ \chi(p)=1}}(1+p^{-2/3}).$$

Now, let us split $[0,B^{1/2}]$ into dyadic regions indexed by B_i. The diagonal contribution is clearly the largest above, so we have the upper bound

$$\sum_{B_i}\frac{B}{B_i^2}\log(B/B_i^2)^{-40}\sum_{|u|,|v|\sim B_i}(1\star\chi)(F(u,v))\prod_{\substack{p|F(u,v)\\ \chi(p)=1}}(1+p^{-2/3})$$
$$\ll B\log(B)^{\rho_{\Delta,f}-2}\sum_{B_i}\log(B/B_i^2)^{-40}\leq B\log(B)^{\rho_{\Delta,f}-2}\sum_{n=1}^{\log(B)}n^{-40}\ll B\log(B)^{\rho_{\Delta,f}-2}.$$

Compared to the main term of $\log(B)^{\rho_{\Delta,f}-1}$, this is negligible. We end this section by remarking that the process of changing the order of summation in Lemma 3.6.2 produces a weaker error term due to the extra factor of $\epsilon(B)$, but this process can be similarly done with this perspective.

3.7 The cuspidal contribution – the Grössen case

In this section, we will prove Proposition 3.5.10. Let us recall the definition of $A(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h,\psi)$ (as defined in 3.5.4):

$$A(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h,\psi)=\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}}\sum_{\substack{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(u,v)=1}}\sigma(F(u,v),\Psi^h\psi;\bar{\mathfrak{t}}),$$

where we have defined $\sigma(n, \psi; t)$ as

$$\sigma(n, \psi; \mathfrak{t}) := \left| \sum_{\substack{N(\mathfrak{a})=n \\ \gcd(\mathfrak{a},\mathfrak{t})=1}} \psi(\mathfrak{a}) \right|.$$

Recall also that ψ is any class group character and that $h \geq 1$. Moreover, Ψ (defined in Definition 3.3.2) is a Hecke Grössencharacter of infinite order.

Now, $\sigma(n; \Psi^h\psi, \bar{\mathfrak{t}})$ is a multiplicative, non-negative function, and bounded by $d(n)$. Thus, we know that by Theorem 3.3.11,

$$\begin{aligned} A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi) &\leq \sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \sum_{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)} \sigma(F(u,v); \Psi^h\psi, \bar{\mathfrak{t}}) \\ &\ll L^2 \sum_{N(\mathfrak{t})\leq B} \frac{B}{N(\mathfrak{t})} \prod_{i=1}^{r} \prod_{p \ll B/t} \left(1 + \frac{\varrho_{f_i}(p)(\sigma(p; \Psi^h\psi, \bar{\mathfrak{t}}) - 1)}{p}\right). \end{aligned}$$

Here we have used the observation that for all but finitely many primes, $\frac{\varrho_F(p)}{p^2} = \frac{\varrho_f(p)}{p} + O(p^{-2})$. From now on, let us fix an irreducible factor $g(z) = f_i(z)$; we study

$$\prod_{p \ll B/t} \left(1 + \frac{\varrho_g(p)(\sigma(p; \xi, \bar{\mathfrak{t}}) - 1)}{p}\right),$$

where we take $\xi = \Psi^h\psi$.

We note that $\sigma(p; \xi, \bar{\mathfrak{t}}) = 0$ if $\chi(p) = -1$, as in that case there are no prime ideals $\mathfrak{p}$ such that $N(\mathfrak{p}) = p$. Additionally, we know that t is product of primes that split in K, so we must have that $\gcd(t, p) = 1$ for any p such that $\chi(p) = -1$. Motivated by these choices, we can split the above product as:

$$\prod_{\substack{p \ll B/t \\ \chi(p)=-1}} \left(1 - \frac{\varrho_g(p)}{p}\right) \prod_{\substack{p \ll B/t \\ \chi(p)=1 \\ \gcd(p,t)=1}} \left(1 + \frac{\varrho_g(p)(\sigma(p; \xi, \bar{\mathfrak{t}}) - 1)}{p}\right) \prod_{\substack{p \ll B/t \\ p|t}} \left(1 + \frac{\varrho_g(p)(\sigma(p; \xi, \bar{\mathfrak{t}}) - 1)}{p}\right).$$

Consider the primes p such that $p \mid t$. In this case

$$\sigma(p; \xi, \bar{\mathfrak{t}}) = \left| \sum_{\substack{N(\mathfrak{p})=p \\ \gcd(\mathfrak{p},\bar{\mathfrak{t}})=1}} \xi(\mathfrak{p}) \right| = |\xi(\mathfrak{p})| = 1.$$

Here we have used that since $p \mid t$ and $\gcd(\mathfrak{t}, \bar{\mathfrak{t}}) = 1$, we must have that exactly one of the primes $\mathfrak{p}$ above p satisfies $\gcd(\mathfrak{p}, \bar{\mathfrak{t}}) = 1$. So, the contribution from these primes is given by:

$$\prod_{\substack{p \ll B/t \\ p \mid t}} \left(1 + \frac{\varrho_g(p)(\sigma(p; \xi, \bar{\mathfrak{t}}) - 1)}{p}\right) = \prod_{\substack{p \ll B/t \\ p \mid t}} 1 = 1.$$

Next, we consider the primes p satisfying that $\chi(p) = 1$ and $p \nmid t$. Since $\gcd(p, t) = 1$, we know that

$$\sigma(p; \xi, \bar{\mathfrak{t}}) = \sigma(p; \xi, 1) = |\xi(\mathfrak{p}) + \bar{\xi}(\mathfrak{p})| = 2|\cos(\theta_p)|,$$

where $\mathfrak{p}$ is a prime over p and $\theta_p \in [0, 2\pi]$ is an angle. By the following identity on cosine,

$$2|\cos(\theta)| \leq \frac{3}{2} + \frac{1}{2}\cos(2\theta),$$

we know that

$$\sigma(p; \xi, \bar{\mathfrak{t}}) \leq \frac{3}{2} + \frac{1}{4}(\xi(\mathfrak{p})^2 + \bar{\xi}(\mathfrak{p})^2).$$

Thus, we can rewrite the second product as

$$\prod_{\substack{p \ll B/t \\ \chi(p)=1 \\ \gcd(p,t)=1}} \left(1 + \frac{\varrho_g(p)(\sigma(p; \xi, \bar{\mathfrak{t}}) - 1)}{p}\right) \ll \prod_{\substack{p \ll B/t \\ \chi(p)=1}} \left(1 + \frac{3\varrho_g(p)}{2p} + \sum_{N(\mathfrak{p})=p} \frac{\varrho_g(p)\xi(\mathfrak{p})^2}{N(\mathfrak{p})} - \frac{\varrho_g(p)}{p}\right).$$

The above analysis tells us that (3.7) can be simplified to the following expression:

$$\prod_{p \ll B/t} \left(1 + \frac{\varrho_g(p)(\sigma(p; \xi, \bar{\mathfrak{t}}) - 1)}{p}\right) \ll \exp\left(\sum_{p \ll B/t} \frac{-\varrho_g(p)}{p} + \sum_{\substack{p \ll B/t \\ \chi(p)=1}} \frac{3\varrho_g(p)}{2p} + \sum_{\substack{p \ll B/t \\ N(\mathfrak{p})=p}} \frac{\varrho_g(N(\mathfrak{p}))\xi^2(\mathfrak{p})}{N(\mathfrak{p})}\right).$$

In the third summation above, we observe that for all but finitely many primes, the existence of a prime ideal $\mathfrak{p}$ satisfying that $N(\mathfrak{p}) = p$ implies that $\chi(p) = 1$.

Since $g(z)$ is irreducible, we have that

$$\sum_{p \ll B/t} \frac{\varrho_g(p)}{p} = \log\log(B/t) + O(1).$$

On the other hand,

$$\sum_{\substack{p \ll B/t \\ \chi(p)=1}} \frac{3\varrho_g(p)}{2p} = \frac{3}{4}\log\log(B/t) + \frac{3}{4} \cdot \mathbf{1}_{\sqrt{-\Delta} \in \mathbb{Q}[z]/g(z)} \cdot \log\log(B/t) + O(1).$$

The above follows from rewriting the above sum as

$$\sum_{p \ll B/t} \frac{3\varrho_g(p)}{4p} + \sum_{p \ll B/t} \frac{3\chi(p)\varrho_g(p)}{4p}$$

and using Corollary 3.3.16 to evaluate the second sum.

It remains to handle the final sum in (3.7). We observe that if $\sqrt{-\Delta} \notin \mathbb{Q}[z]/g(z)$, then

$$\varrho_g(p) = \#\{\mathfrak{P} \subset \mathcal{O}_{K_g} : N_{K_g/K}(\mathfrak{P}) = \mathfrak{p}\}$$

for any prime $\mathfrak{p}$ over p. Otherwise, $K \subset \mathbb{Q}[z]/g(z)$ and since $[K : \mathbb{Q}] = 2$, we have that

$$\varrho_g(p) = 2 \cdot \#\{\mathfrak{P} \subset \mathcal{O}_{K_g} : N_{K_g/K}(\mathfrak{P}) = \mathfrak{p}\}$$

for any prime $\mathfrak{p}$ over p. Therefore, we can rewrite this sum as

$$\sum_{N(\mathfrak{p}) \ll B/t} \frac{\varrho_g(N(\mathfrak{p}))\xi^2(\mathfrak{p})}{N(\mathfrak{p})} \ll \sum_{\substack{\mathfrak{P} \subset \mathcal{O}_{K_g} \\ N(\mathfrak{P}) \ll B/t}} \frac{\xi^2(N_{K_g/K}(\mathfrak{P}))}{N(\mathfrak{P})} + O(1).$$

Here the factor of $O(1)$ comes from a sum over primes $\mathfrak{P}$ such that $N_{K/\mathbb{Q}}(\mathfrak{P}) = p^e$ for $e > 1$.

We consider the character $\xi^2 \circ N_{K_g/K}$ on K_g; this will be a Hecke character. Since Ψ has infinite order, we can see that since $h \geq 1$, $\Psi^{2h}\psi^2 \circ N_{K_g/K}$ is a nontrivial Hecke character. We can also view $\xi^2 \circ N_{K_g/K}$ as the base change of ξ^2 to K_g. Since $\xi^2 \circ N_{K_g/K}$ is a nontrivial Hecke character on K_g, the prime ideal theorem with Grössencharacters ([37, Theorem 5.13]) states the following: for some constant $c > 0$,

$$\sum_{N(\mathfrak{a}) \leq X} \Lambda(\mathfrak{a})\Psi^h(\mathfrak{a})\psi(\mathfrak{a}) \ll X \exp\left(-c\frac{\log(X)}{\sqrt{\log(X)} + \log(h)}\right) \log(xh)^4.$$

Taking $X = B/t$ and $h \leq \log(B)^5$, we consequently have that

$$\sum_{\substack{\mathfrak{P} \subset \mathcal{O}_{K_g} \\ N(\mathfrak{P}) \ll B/t}} \frac{\xi^2(\mathfrak{P})}{N(\mathfrak{P})} = o(\log\log(B/t)).$$

Consequently, we have that

$$\prod_{p \ll B/t} \left(1 + \frac{\varrho_g(p)(\sigma(p;\xi,\bar{\mathfrak{t}}) - 1)}{p}\right) \ll \exp\left((-\frac{1}{4} + \frac{3}{4}(\varrho_{\Delta,g} - 2)) \cdot \log\log(X) + O(1) + o(\log\log(B/t)\right)$$
$$\ll \log(B/t)^{\varrho_{\Delta,g} - 2 - 1/5}.$$

Finally, we sum over ideals $\mathfrak{t}$:

$$A(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); h, \psi) \ll L^2 \sum_{\substack{N(\mathfrak{t}) \ll B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{B}{N(\mathfrak{t})} \cdot \log(B/t)^{\rho_{\Delta,f} - 2 - 1/5} \ll L^2 B \log(B)^{\rho_{\Delta,f} - 1 - 1/5}.$$

This completes the proof of Proposition 3.5.9. □

Example 3.7.1 (A counterexample when ξ has finite order). In this example, we show an example of when the bound of Proposition 3.5.10 fails for a character of finite order. Let $\Delta = 23$ and thus $K = \mathbb{Q}(\sqrt{-23})$. Take $f(z) = z^3 - z - 1$. Then $f(z)$ generates the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$, which has class group $\mathbb{Z}/3\mathbb{Z}$. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal such that $N(\mathfrak{p}) \mid F(u,v)$ for some $(u,v) \in (B/t)^{1/2}\mathcal{B}(\mathbf{x}_0, L)$. Consider $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K_f/K) \cong \mathbb{Z}/3\mathbb{Z}$. We note that

$$\#\{\mathfrak{P} \subset \mathcal{O}_{K_f} : N(\mathfrak{P}) = \mathfrak{p}\} = \mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}) = \begin{cases} 3, & \mathrm{Frob}_{\mathfrak{p}} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\mathfrak{p}$ splits in K_f if and only if $\mathfrak{p}$ is a principal ideal. However, if $N(\mathfrak{p}) \mid F(u,v)$, then we must be in the first case since then $\#\{\mathfrak{P} \subset \mathcal{O}_{K_f} : N(\mathfrak{P}) = \mathfrak{p}\} > 0$; in conclusion, for any prime $\mathfrak{p}$ such that $N(\mathfrak{p}) \mid F(u,v)$, $\mathfrak{p}$ is a principal ideal. Thus, we have that for *any class group character* ψ,

$$\sum_{N(\mathfrak{a}) = F(u,v)} \psi(\mathfrak{a}) = \sum_{N(\mathfrak{a}) = F(u,v)} 1.$$

In this case, we get that

$$\sum_{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)}\sum_{N(\mathfrak{a})=F(u,v)}\psi(\mathfrak{a})=\sum_{(u,v)\in(B/t)^{1/2}\mathcal{B}(\mathbf{x}_0,L)}\sum_{N(\mathfrak{a})=F(u,v)}1,$$

is exactly the same as the contribution from the Eisenstein part (when $\psi=1$, since if the class group is $\mathbb{Z}/3\mathbb{Z}$ there are no other genus characters).

Remark 3.7.2. We observe that in order for our argument above for Proposition 3.5.10 to achieve a polylogarithmic savings, it suffices for $\xi^2\circ N_{K_g/K}$ to be nontrivial for every irreducible factor $g(z)$. If ξ has infinite order, this is immediate. For most pairs $(\Delta,f(z))$, the above condition also holds for ξ of finite order. Let H be the Hilbert class field of K. The base change of a class group character ξ^2 to K_g will also be nontrivial if $K_g\cap H=K$.

Remark 3.7.3. As mentioned in §3.5.6, we can also interpret $A(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);h,\psi)$ as a correlation sum of Fourier coefficients of cusp forms along polynomial values. Let Ξ be a cuspdial automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. We can use the argument above to study

$$\sum_{n\leq X}\lambda_{\Xi}(p(n))$$

for $p(x)\in\mathbb{Z}[x]$ a polynomial. Again, our problem reduces to analytic properties of the base change of $\mathrm{sym}^2(\Xi)$; in particular, one hopes that it is cuspidal. This problem will be studied further in forthcoming work.

3.8 The Eisenstein contribution

For the next four sections, we will work towards the proof of Proposition 3.5.11. Recall from (3.5.3) that we define

$$M_{q_1,q_2}(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L))=\sum_{\substack{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{k},\overline{\mathfrak{k}})=1}}\mu(\mathfrak{k})\sum_{\substack{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{tk},\overline{\mathfrak{tk}})=1}}S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k),$$

where we define

$$S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k)=\sum_{\substack{(x,y)\in(B/kt)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(x,y)=1}}\varepsilon_{q_1,q_2}(F(x,y)/k).$$

We proceed to analyze $S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k)$ in this section and in §3.11 we resume the summation over $\mathfrak{k}$ and $\mathfrak{t}$. Let us record our final goal for $S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k)$:

Proposition 3.8.1. *Let $F(x, y)$ be a squarefree binary form of degree ≤ 4 and $L \gg \log(B)^{-10^{-10}}$. Then we have that for t and k satisfying $tk \leq B/\exp(\sqrt{\log(B)})$, the following holds:*

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k) = \frac{BL^2}{tk^2} \cdot C_{q_1,q_2,f,k}(B/tk) \cdot \left(1 + O\left(\log(B)^{-10^{-7}}\right)\right)$$
$$+O\left(\frac{B(8\deg(F))^{\omega(k)}}{tk^{3/2}} \cdot \log(B)^{\rho_{\Delta,f}-2-10^{-7}} + \frac{B^{3/4}}{(tk)^{3/4}} \cdot \left(L + (4\deg(F))^{\omega(k)}\nu_1(k)^{1/2}\right) \cdot \log(B)^{-10^{-7}}\right),$$

where $C_{q_1,q_2,f,k}(B/tk)$ is an expression that depends on the factorization of $F(x, y)$ and

$$\nu_1(k) = \sum_{\alpha \in \mathcal{U}(k)} \frac{1}{\lambda_1(\Lambda(\alpha))}.$$

Additionally, for a fixed value of k, there exists a constant $c_{q_1,q_2,f,k}$ such that

$$C_{q_1,q_2,f,k}(B/tk) = c_{q_1,q_2,f,k} \log(B/tk)^{\rho_{\Delta,f}-2}.$$

Remark 3.8.2. At this stage the proof is essentially a combination of the proofs provided by de la Bretèche and Tenenbaum in [25] (for irreducible quartics and products of two irreducible quadratics) and by Heath-Brown in [33] (for the product of four linear factors) for $\Delta = 1$. In essence, we have reduced ourselves to case when K has class number one.

First, we must apply a few necessary reductions to put the sum in a more malleable form.

3.8.1 Multiplicativity of ε_{q_1,q_2}

We recall from (3.5.3) that

$$\varepsilon_{q_1,q_2}(n) = \sum_{d|n} \chi_{q_1}(d)\chi_{q_2}(n/d).$$

This function is multiplicative, but not totally multiplicative. We would like to write:

$$\varepsilon_{q_1,q_2}(F(x, y)) \approx \prod_{i=1}^{n} \varepsilon_{q_1,q_2}(F_i(x, y)),$$

but the above relation is not quite true. Instead, we can use the following multiplicativity relation.

Lemma 3.8.3. *Let n and m be integers. Then*

$$\varepsilon_{q_1,q_2}(nm) = \sum_{c|\gcd(m,n)} \mu(c)\chi(c)\varepsilon_{q_1,q_2}(n/c)\varepsilon_{q_1,q_2}(m/c).$$

Proof. Since ε_{q_1,q_2} is multiplicative, it suffices to check this property on prime powers. Let p be a prime. So, we can assume that $\gcd(p,t) = 1$. Then expanding the definition, we have that

$$\varepsilon_{q_1,q_2}(p^{e+f}) = \sum_{i=0}^{e+f} \chi_{q_1}(p)^i \chi_{q_2}(p)^{e+f-i}.$$

On the other hand, we have that

$$\varepsilon_{q_1,q_2}(p^e)\varepsilon_{q_1,q_2}(p^f) = \sum_{i=0}^{e+f} \#\{k \le e, \ell \le f : k+\ell = i\} \cdot \chi_{q_1}(p)^i \chi_{q_2}(p)^{e+f-i}.$$

Now, because $\chi(p) = \chi_{q_1}(p)\chi_{q_2}(p)$, the following equality is true:

$$\varepsilon_{q_1,q_2}(p^e)\varepsilon_{q_1,q_2}(p^f) - \chi(p)\varepsilon_{q_1,q_2}(p^{e-1})\varepsilon_{q_1,q_2}(p^{f-1}) = \varepsilon_{q_1,q_2}(p^{e+f}).$$

□

Thus, we can write:

$$\varepsilon_{q_1,q_2}(F(x,y)/k) = \sum_{\mathbf{c}} \mu(\mathbf{c})\chi(\mathbf{c}) \prod_{i=1}^{r} \varepsilon_{q_1,q_2}(F_i(x,y)/k_i c_i),$$

where $k = k_1 ... k_r$ and $c_i k_i \mid F_i(x,y)$ for each i, and $\mathbf{c}$ is given by tuples of products of integers dividing $\gcd(F_i(x,y), F_j(x,y))$ for $i \neq j$. These breakdowns depend on the factorization of $F(x,y)$ (for instance, if $F(x,y)$ is irreducible then there is no need for the above sum). We write down these decompositions explicitly in §3.11.1.

Moreover, since we will take $\gcd(x,y) = 1$, we have that $\gcd(F_i(x,y), F_j(x,y)) = \mathrm{Res}(F_i, F_j)$, the resultant of F_i and F_j, for $i \neq j$. Note that $\mathrm{Res}(F_i, F_j)$ is a finite constant only depending on our polynomial $F(x,y)$. Furthermore, this restricts the sum above to those $\mathbf{c}$ satisfying that $c_i \mid \prod_{i \neq j} \mathrm{Res}(F_i, F_j)$. This fact also allows us to determine that k decomposes into finitely many combinations $k_1 \dots k_r$ where $k_i \mid F_i(x,y)$ – we choose one without loss of generality and write it as

$\mathbf{k} = (k_1, ..., k_r)$. Plugging the above into (3.8), we get that:

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k) = \sum_{\mathbf{c}} \mu(\mathbf{c})\chi(\mathbf{c})S(\mathbf{c}, \mathbf{k}),$$

where we define:

$$S(\mathbf{c}, \mathbf{k}) := \sum_{\substack{(x,y)\in(B/kt)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(x,y)=1}} \prod_{i=1}^{r} \varepsilon_{q_1,q_2}(F_i(x,y)/c_ik_i)$$

3.8.2 Interpretation ε_{q_1,q_2} as $(1 \star \chi)$

Next, we would like to make precise our claim that

$$\varepsilon_{q_1,q_2}(n) \approx (1 \star \chi)(n).$$

As we have alluded to before, these two functions are morally equivalent.

Recall that $\chi_{q_1}\chi_{q_2} = \chi$. Then we have the relation:

$$\begin{aligned}\varepsilon_{q_1,q_2}(n) = \sum_{d|n} \chi_{q_1}(n/d)\chi_{q_2}(d) &= \sum_{d|p_{\neg q_1}(n)} \chi_{q_1}(p_{\neg q_1}(n)/d)\chi_{q_2}(p_{q_1}(n)d)\\ &= \chi_{q_1}(p_{\neg q_1}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{d|p_{\neg q_1}(n)} \chi(d) = \chi_{q_1}(p_{\neg q_1}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{d|p_{\neg -\Delta}(n)} \chi(d)\\ &= \chi_{q_1}(p_{\neg q_1}(n))\chi_{q_2}(p_{q_1}(n))(1 \star \chi)(p_{\neg -\Delta}(n)).\end{aligned}$$

Here, we have use that $\chi(d) = 0$ if $\gcd(d, -\Delta) > 1$. Recall at this point we are using abusive notation that $-\Delta$ denotes the modulus of the quadratic character χ.

We can thus reexpress (3.8.1) as:

$$\begin{aligned}S(\mathbf{c}, \mathbf{k}) &= \sum_{\substack{(x,y)\in(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(x,y)=1\\ k_ic_i|F_i(x,y)}} \chi_{q_1}(p_{\neg q_1}(F(x,y)/kc))\chi_{q_2}(p_{q_1}(F(x,y)/kc)) \prod_{i=1}^{r}(1\star\chi)(p_{\neg -\Delta}(F_i(x,y)/k_ic_i))\\ &= \sum_{\substack{n\ll(B/tk)^2\\ p|n \implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{a_i \bmod -\Delta\\ \gcd(a_i,-\Delta)=1}} \prod_{i=1}^{r} \chi_{q_1}(a_i)\\ &\qquad \times \sum_{\substack{(x,y)\in(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ \gcd(x,y)=1\\ k_ic_i|F_i(x,y)\\ p_{-\Delta}(F(x,y)/kc)=n\\ p_{\neg -\Delta}(F_i(x,y)/k_ic_i)\equiv a_i \bmod -\Delta}} \prod_{i=1}^{r}(1 \star \chi)(p_{\neg -\Delta}(F_i(x,y)/k_ic_i)).\end{aligned}$$

Note here that from (3.8.1), we can assume that $\chi(c_i) \neq 0$ for all c_i and hence $\gcd(c_i, -\Delta) = 1$. Moreover, since $k = N(\mathfrak{k})$ where $\gcd(\mathfrak{k}, \bar{\mathfrak{k}}) = 1$, we must have that $\gcd(k, -\Delta) = 1$ as well.

Let us bound away the contribution from the large n.

Lemma 3.8.4. *Let $F(x, y)$ be a squarefree binary form of degree ≤ 4. Define for n such that $p \mid n \implies p \mid -\Delta$ and a vector of residues $\mathbf{a}$ mod $-\Delta$:*

$$U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) := \sum_{\substack{(x,y) \in (B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ k_i c_i \mid F_i(x,y) \\ p_{-\Delta}(F(x,y/c))=n \\ p_{\neg -\Delta}(F_i(x,y)/k_i c_i) \equiv a_i \bmod -\Delta}} \prod_{i=1}^{r} (1 \star \chi)(p_{\neg -\Delta}(F_i(x,y)/k_i c_i)).$$

Then we have that

$$U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) \ll_{\Delta, f} \frac{BL^2 \log(B/t)^{\rho_{\Delta,f} - 2} (8 \deg(F))^{\omega(k)}}{tk^2 n}.$$

Proof. Note that since $(1 \star \chi)(n)$ is a non-negative function,

$$U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) \leq \sum_{n = n_1 \dots n_r} \sum_{\substack{(x,y) \in (B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ n_i \mid F_i(x,y) \\ k \mid F(x,y)}} \prod_{i=1}^{r} (1 \star \chi)(F_i(x,y)/k_i c_i).$$

Here, we have used the fact that $(1 \star \chi)(d p_{\neg u}(n))) = (1 \star \chi)(p_{\neg u}(n))$ for any d satisfying that if $p \mid d$ then $p \mid -\Delta$. We apply Theorem 3.3.11, we have that the inner sum above is bounded by

$$\frac{BL^2}{tk} \cdot \log(B/tk)^{\rho_{\Delta,f} - 2} \cdot \prod_{i=1}^{r} \frac{(8 \deg(F))^{\omega(n_i k_i)}}{n_i k_i} \ll \frac{BL^2 \log(B/tk)^{\varrho_{\Delta,Q} - 2} (8 \deg(F))^{\omega(k)}}{tk^2 n}.$$

Here we have used Corollary 3.3.16 and that $\omega(n_i) \leq \omega(\Delta)$ for any value of n_i. □

Consequently, we can bound the contribution from $n > N$:

$$\sum_{n > N} \sum_{\mathbf{a} \bmod -\Delta} U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) \ll \frac{BL^2 \log(B/tk)^{\rho_{\Delta,f} - 2} (8 \deg(F))^{\omega(k)}}{tk^2} \cdot \sum_{\substack{n > N \\ p \mid n \implies p \mid -\Delta}} \frac{1}{n}.$$

Since the sum over n satisfying that $p \mid n \implies p \mid -\Delta$ is a geometric series, we can see that

$$\sum_{n > N} \sum_{\mathbf{a} \bmod -\Delta} U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) \ll \frac{BL^2 \log(B/tk)^{\rho_{\Delta,f} - 2} (8 \deg(F))^{\omega(k)}}{Ntk^2}.$$

For $N = \log(B/tk)^{10^{-7}}$, this error term is sufficient for Propositon 3.5.11. So, from now on we can

assume that $n \leq \log(B/tk)^{10^{-7}} \ll \log(B)^{10^{-7}}$ (since $tk \leq B/\exp(\sqrt{\log(B)})$).

Remark 3.8.5. Since we assume that $n \leq \log(B)^{10^{-7}}$ still grows as $B \to \infty$, we must be somewhat careful in tracking the dependency of future bounds on n. However, n is still quite small, so we can allow for any reasonable polynomial dependency on n.

Finally, let us define a new function for a fixed n and tuple $\mathbf{a}$ satisfying that $\gcd(a_i, -\Delta) = 1$ for all i:

$$S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) = \sum_{\substack{(x,y) \in (B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ \gcd(x,y)=1 \\ k_i c_i | F_i(x,y) \\ p_{q_i}(F(x,y)/c) = n_i \\ p_{\neg -\Delta}(F_i(x,y)/k_i c_i) \equiv a_i \bmod -\Delta}} \prod_{i=1}^{r} (1 \star \chi)(p_{\neg -\Delta}(F_i(x,y)/k_i c_i)).$$

3.8.3 Removing the gcd condition

Now, we will remove the condition that $\gcd(x, y) = 1$ using Möbius inversion. In particular, we know that

$$S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}) = \sum_{b \leq (B/tk)^{1/2}} \mu(b) \sum_{\substack{(x,y) \in (B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ b | x, y \\ k_i c_i | F_i(x,y) \\ p_{q_i}(F(x,y)/c) = n_i \\ p_{\neg -\Delta}(F_i(x,y)/k_i c_i) \equiv a_i \bmod -\Delta}} \prod_{i=1}^{r} (1 \star \chi)(p_{\neg -\Delta}(F_i(x,y)/k_i c_i)).$$

We can bound away the contribution given by large values of b:

Lemma 3.8.6. *Let $F(x, y)$ be a squarefree binary form of degree ≤ 4. Define for a fixed b:*

$$U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}; b) := \sum_{\substack{(x,y) \in (B/kt)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \\ b | x, y \\ k_i c_i | F_i(x,y) \\ p_{q_i}(F(x,y)/c) = n_i \\ p_{\neg -\Delta}(F_i(x,y)/k_i c_i) \equiv a_i \bmod -\Delta}} \prod_{i=1}^{r} (1 \star \chi)(p_{\neg -\Delta}(F_i(x,y)/k_i c_i)).$$

Then for any $B \geq X \gg 1$, we have that

$$\sum_{X \leq b \ll (B/kt)^{1/2}} \mu^2(b) U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}; b) \ll_{\Delta, F} \frac{BL^2 \log(B/kt)^{\rho_{\Delta,f} - 2} (8 \deg(F))^{\omega(k)}}{Xtk^2}.$$

Proof. Again, by the nonnegativity of $(1 \star \chi)(n)$, we have that

$$U(n, \mathbf{a}, \mathbf{c}, \mathbf{k}; b) \leq \sum_{(x,y) \in b^{-1}(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L)} \prod_{i=1}^{r} (1 \star \chi)(b^{\deg(F_i)} F_i(x,y)/k_i c_i).$$

Now we apply Lemma 3.8.3 to separate the contribution from $b^{\deg(F_i)}$ and $F_i(x,y)/k_ic_i$:

$$\sum_{e_i|b}\mu(e_i)\chi(e_i)\prod_{i=1}^{r}(1\star\chi)(b^{\deg(F_i)}/e_i)\sum_{\substack{(x,y)\in b^{-1}(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ e_i|F_i(x,y)}}\prod_{i=1}^{r}(1\star\chi)(F_i(x,y)/k_ic_ie_i).$$

After using Theorem 3.3.11, we have that the above is bounded by

$$\sum_{e_i|b}\mu^2(e_i)\prod_{i=1}^{r}(1\star\chi)(b^{\deg(F_i)}/e_i)\cdot\frac{BL^2\log(B/tk)^{\rho_{\Delta,f}-2}(8\deg(F))^{\omega(k)}}{b^2tk^2}\cdot\prod_{i=1}^{r}\frac{(8\deg(F))^{\omega(e_i)}}{e_i}.$$

Summing over $b\geq X$, we get the expression

$$\begin{aligned}\frac{BL^2\log(B/tk)^{\rho_{\Delta,f}-2}(8\deg(F))^{\omega(k)}}{tk^2}\sum_{b\geq X}\frac{\mu^2(b)}{b^2}\cdot\prod_{i=1}^{r}\sum_{e_i|b}\mu^2(e_i)(1\star\chi)(b^{\deg(F_i)}/e_i)\cdot\frac{(8\deg(F))^{\omega(e_i)}}{e_i}\\ \ll\frac{BL^2\log(B/tk)^{\rho_{\Delta,f}-2}\deg(F)^{\omega(k)}}{tk^2}\sum_{b\geq X}\mu^2(b)\cdot\frac{a_b}{b^2},\end{aligned}$$

where we define:

$$a_b:=\prod_{i=1}^{r}\prod_{p|b}\left(1+\frac{8\deg(F)\cdot(1\star\chi)(b^{\deg(F_i)}/p)}{p}\right).$$

This function a_b is multiplicative in b, so we relate this sum to a Dirichlet series with an Euler product expansion. If we consider the singular series:

$$\begin{aligned}\gamma(s):=\sum_{b=1}^{\infty}\mu^2(b)\cdot\frac{a_b}{b^s}=\prod_{p}\left(1+p^{-s}\prod_{i=1}^{r}\prod_{p|b}\left(1+\frac{8\deg(F)\cdot(1\star\chi)(b^{\deg(F_i)}/p)}{p}\right)\right)\\ =\prod_{p}\left(1+p^{-s}+O_{F,\chi}(p^{-1-s})\right).\end{aligned}$$

Thus, we can see that the singular series will converge at $s=2$ and in fact the convergence is at the same rate as the zeta function. We can now estimate that

$$\sum_{b\geq X}\mu^2(b)\cdot\frac{a_b}{b^2}\ll\frac{1}{X}.$$

Combining this with the earlier bounds, we get that:

$$\frac{BL^2\log(B/tk)^{\rho_{\Delta,f}-2}(8\deg(F))^{\omega(k)}}{tk^2}\sum_{b\geq X}\frac{\mu^2(b)}{b}\cdot\prod_{i=1}^{r}\sum_{e_i|b}\mu^2(e_i)(1\star\chi)(b^{\deg(F_i)}/e_i)\cdot\frac{(8\deg(F))^{\omega(e_i)}}{e_i}$$
$$\ll\frac{BL^2\log(B/tk)^{\rho_{\Delta,f}-2}(8\deg(F))^{\omega(k)}}{tk^2X}.$$

□

If we take $X=\log(B)^{10^{-7}}$, we will get a sufficient error term for Propostion 3.5.11. So from now on, we assume that $b\leq\log(B)^{10^{-7}}$. Again, we must track how all future error terms depend on b, since it does grow with B; fortunately, since b is small in comparison with B, we can allow any reasonable polynomial dependency on b.

We define a new function to be estimated for a fixed b:

$$S(n,\mathbf{a},\mathbf{c},\mathbf{k};b)=\sum_{\substack{(x,y)\in(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ b|x,y\\ k_ic_i|F_i(x,y)\\ p_{q_j}(F(x,y)/c)=n_j\\ p_{\neg-\Delta}(F_i(x,y)/k_ic_i)\equiv a_i \bmod -\Delta}}\prod_{i=1}^{r}(1\star\chi)(p_{\neg-\Delta}(F_i(x,y)/k_ic_i)).$$

3.8.4 Expanding the Dirichlet convolution

In our final reduction, we will expand the Dirichlet convolution $(1\star\chi)(n)$. We note that

$$\begin{aligned}(1\star\chi)(p_{\neg-\Delta}(F_i(x,y)/k_ic_i))&=\sum_{d_i|p_{\neg-\Delta}(F_i(x,y)/k_ic_i)}\chi(d_i)\\&=\sum_{\substack{d_i|p_{\neg-\Delta}(F_i(x,y)/k_ic_i)\\ d_i\leq\sqrt{p_{\neg-\Delta}(F_i/k_ic_i)}}}\chi(d_i)+\sum_{\substack{d_i|p_{\neg-\Delta}(F_i(x,y)/k_ic_i)\\ d_i\leq\sqrt{p_{\neg-\Delta}(F_i/k_ic_i)}}}\chi(a_i/d_i)\\&=(1+\chi(a_i))\sum_{\substack{d_i|p_{\neg-\Delta}(F_i(x,y)/k_ic_i)\\ d_i\leq\sqrt{p_{\neg-\Delta}(F_i/k_ic_i)}}}\chi(d_i).\end{aligned}$$

We remark that if $p_{\neg-\Delta}(F_i(x,y)/k_ic_i)$ is a square, then the second equality should be modified to be a sum over $d_i\leq\sqrt{p_{\neg-\Delta}}$ and $d_i<\sqrt{p_{\neg-\Delta}}$. However, the contribution from those (x,y) such that $p_{\neg-\Delta}(F_i(x,y)/k_ic_i)$ is a square can be shown to be negligible via the large sieve. From now on, we

ignore these terms. Let us define:

$$S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d}) = \sum_{\substack{(x,y)\in(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)\\ b|x,y\\ k_ic_id_i|F_i(x,y)\\ p_{q_j}(F(x,y)/c)=n_j\\ p_{\neg -\Delta}(F_i(x,y)/k_ic_i)\equiv a_i \bmod -\Delta}} 1.$$

Then we can rewrite (3.8.2) as:

$$S(n,\mathbf{a},\mathbf{c},\mathbf{k};b) = \prod_{i=1}^{r}(1+\chi(a_i)) \sum_{d_i\leq \max_{(x,y)\in(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)}\sqrt{|F_i(x,y)|/n_ik_ic_i}} \prod_{i=1}^{r}\chi(d_i)\cdot S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d}).$$

Our next goal is to establish either asymptotics on average or upper bounds for $S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d})$. To do so, we will have different treatments for various ranges of $\mathbf{d}$. For "small" $\mathbf{d}$ (the precise definition of this will be given in §3.11.1, as it depends on the factorization of $F(x,y)$), we use a level of distribution result established in §3.9 – this will provide our main term. For "large" $\mathbf{d}$, we will prove upper bounds in §3.10 using Nair's sieve that will go into our final error term.

Remark 3.8.7. Since there are many variables in the final expression we wish to estimate $S(n,\mathbf{a},\mathbf{c},\mathbf{k};b)$, let us remark now on their approximate "size" and what aspects need to be tracked throughout the following proof.

- n is a small constant ($n \ll \log(B)^{10^{-7}}$; it also satisfies that $p \mid n \implies p \mid -\Delta$ and hence $\sum n^{-e}$ for any $e > 0$ will converge to a constant only depending on $-\Delta$. Its importance appears in the computations for evaluating the main term.

- $\mathbf{a}$ appear with respect to congruences mod $n(-\Delta)$. This aspect is a nuisance when establishing the error term but is relevant for evaluating the main term.

- $\mathbf{c}$ has entries that must divide their respect resultants $\mathrm{Res}(F_i,F_j)$ for $i \neq j$. Since the pairwise resultants are fixed constants given a form F, the dependency on $\mathbf{c}$ is not needed to be tracked for the error term. However, tracking the dependence on $\mathbf{c}$ is crucial for evaluating the main term. When F is irreducible, it is not necessary to sum over $\mathbf{c}$ (i.e. $\mathbf{c} = 1$).

- $\mathbf{k}$ satisfies that $k \leq B/\exp(\sqrt{\log(B)})$ and its dependence must be tracked throughout the error term computations, as we later sum over k (recall that by construction, for each k there is a unique vector chosen as $\mathbf{k}$). It is not as relevant for the main term computation.

- b is a small constant ($b \leq \log(B)^{10^{-7}}$) and its dependence must be tracked throughout the error

term computations. However, it is not particularly relevant for the main term computations.

3.9 Level of distribution

In this section, we will prove a level of distribution result for squarefree binary forms of degree ≤ 4 alluded to in the paragraphs above. Let us write it out now in its full technical detail:

Proposition 3.9.1. *Let $F(x,y) = \prod_{i=1}^{r} F_i(x,y)$ be a squarefree binary form of degree ≤ 4, where $F_i(x,y)$ are irreducible factors of $F(x,y)$. For a fixed $n, \mathbf{a}, \mathbf{c}, \mathbf{k}, \mathbf{d} = (d_i)$ and b, we define the following local function:*

$$\varrho_F(\mathbf{d}; b, n, \mathbf{a}, \mathbf{c}, \mathbf{k}) := \#\{\mathbf{x} \bmod bcdkn(-\Delta) : k_i c_i d_i \mid F_i(x,y), \gcd(x,y,kcd) = 1,$$
$$b \mid \gcd(x,y), p_{-\Delta}(F(x,y)) = n, p_{\neg -\Delta}(F_i(x,y)/k_i c_i) \equiv a_i \bmod -\Delta\}/\sim,$$

where we say that $\mathbf{x} \sim \mathbf{x}'$ if $\mathbf{x} \equiv \lambda \mathbf{x}' \bmod kcd$ for some $\lambda \in \mathbb{Z}$. For a tuple $\mathbf{D} = (D_i)_{i=1}^{r}$, we define the error quantity as:

$$E(\mathbf{D}) = \sum_{d_i \sim D_i} \left| S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}) - \frac{\varrho_F(\mathbf{d}; b, n, \mathbf{a}, \mathbf{c}, \mathbf{k})}{cdb^2 n^2 (-\Delta)^2} \cdot \frac{BL^2}{tk^2} \right|.$$

Let $D = D_1 ... D_r$. If $F(x,y)$ has no linear factors, then we have that for an explicit constant $\gamma_r > 0$,

$$E(\mathbf{D}) \ll D + \frac{\tau(b^4)}{b} \cdot \sqrt{D} \cdot \frac{B^{1/2} L (16 \deg(F))^{\omega(k)}}{t^{1/2} k} \exp(\gamma_r \sqrt{\log_2(B) \log_3(B)}).$$

If $F(x,y)$ has a linear factor, then we have that

$$E(\mathbf{D}) \ll D + \frac{B^{1/2} L}{t^{1/2}} \cdot \frac{\tau(b^{\deg(F)})}{b} \cdot \left(\frac{(16 \deg(F))^{\omega(k)}}{k} \cdot \sqrt{D} \exp(\gamma_r \sqrt{\log_2(B) \log_3(B)}) \right)$$
$$+ \frac{B^{1/2} L}{t^{1/2}} \cdot \frac{\tau(b^{\deg(F)})}{b} \cdot \left(\frac{\log(kcD)}{k^{1/2}} \max_{i : \deg(F_i) = 1} D_i \right).$$

Remark 3.9.2. The constant γ_r is worked out explicitly in the work of Hall and Tenenbaum [31] on generalized Hooley's Δ-functions.

Our proof when $F(x,y)$ has no linear factors follows the proof of the level of distribution result written in de la Bretèche and Tenenbaum in [25]. In this case, the level of distribution is near optimal in the sense that we can take $D \leq \frac{BL^2}{tk} \log(B)^{-\epsilon}$ for any $\epsilon > 0$; in other words, we can let ϵ be very small, such as 10^{-7}. If $F(x,y)$ has linear factors, then we get a slightly worse level of

distribution result – we will need $D_i \leq \frac{BL^2}{tk}\log(B)^{-1-\epsilon}$ in the final term. Luckily, our results for large moduli are more flexible in this case. We also would like to point out the work of Marasingha [39] towards level of distribution results for arbitrary squarefree binary forms.

3.9.1 Lattices of $F(x,y)$

First, we will explore the distribution of lattices corresponding to the divisors of $F(x,y)$ and how they connect to our error term. We start by fixing a tuple $\mathbf{d} = (d_i)_{i=1}^r$. For a collection of squarefree binary forms $F_1(x,y),...,F_k(x,y)$, we define the following divisor lattices:

$$\Lambda(d_1,...,d_k) := \{(x,y) \in \mathbb{Z}^2 : d_i \mid F_i(x,y)\}.$$

$$\Lambda(d_1,...,d_k;b) := \{(x,y) \in \mathbb{Z}^2 : d_i \mid F_i(x,y), b \mid \gcd(x,y)\}.$$

It is noteworthy that these will be the union of lattices of determinant $d_1...d_n$; this observation was made by Daniel in [18].

On the other hand, let

$$\Lambda(n,\mathbf{a}) := \{\mathbf{x} \in \mathbb{Z}^2 : p_{-\Delta}(F(x,y)/c) = n, p_{\neg -\Delta}(F_i(x,y)/k_ic_i) \equiv a_i \bmod -\Delta\}.$$

Observe that the above is not a lattice, despite the notation, but is still defined by congruence conditions mod $n(-\Delta)$ in $\mathbb{Z}^2$ and will be the shift of a finite union of lattices with determinant $n^2(-\Delta)^2$. Then, we can see that we would like to count

$$\#(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L) \cap \Lambda(c_1k_1d_1,\ldots,c_rk_rd_r;b) \cap \Lambda(n,\mathbf{a}).$$

From now on, we use the shorthand notation:

$$\Lambda(\mathbf{ckd};b) := \Lambda(c_1k_1d_1,\ldots,c_rk_rd_r;b).$$

Next, we further decompose $\Lambda(\mathbf{ckd};b)$ into lattices corresponding to solutions of $F_i(x,y)$ mod $k_ic_id_i$. Let $\alpha = (\alpha_1,\alpha_2)$ satisfy that $F_i(\alpha) \equiv 0 \bmod k_ic_id_i$ for all i, $b \mid \alpha_1,\alpha_2$ and that $\gcd(\alpha_1,\alpha_2,kcd) = 1$. Then we define

$$\Lambda(\alpha;\mathbf{ckd},b) := \{\mathbf{x} \in \mathbb{Z}^2 : \mathbf{x} \equiv \lambda\alpha \bmod bcdk \text{ for some } \lambda \in \mathbb{Z}\}.$$

It follows that $\Lambda(\alpha; \mathbf{ckd}, b)$ is a lattice with determinant $kcdb^2$. Let us write

$$\mathcal{U}(\mathbf{ckd}; b) = \{\alpha \bmod kcdb : F_i(\alpha) \equiv 0 \bmod k_i c_i d_i, b \mid (\alpha_1, \alpha_2), \gcd(\alpha_1, \alpha_2, kcd) = 1\}/\sim,$$

where equivalence is defined as $\alpha \sim \alpha'$ if and only if $\alpha = \lambda\alpha' \bmod cd$ for $\lambda \neq 0$. Then we can see that

$$\#(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \cap \Lambda(\mathbf{ckd}; b) \cap \Lambda(n, \mathbf{a}) \\ = \sum_{\alpha \in \mathcal{U}(\mathbf{ckd}; b)} \#(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \cap \Lambda(\alpha; \mathbf{ckd}, b) \cap \Lambda(n, \mathbf{a}).$$

Recall from Proposition 3.9.1 that we defined

$$\varrho_F(\mathbf{d}; b, n, \mathbf{a}, \mathbf{c}, \mathbf{k}) = \#\{\mathbf{x} \bmod bcdkn(-\Delta) : k_i c_i d_i \mid F_i(x, y), \gcd(x, y, kcd) = 1, \\ b \mid \gcd(x, y), p_{-\Delta}(F(x, y)) = n, p_{\neg -\Delta}(F_i(x, y)/k_i c_i) \equiv a_i \bmod -\Delta\}/\sim,$$

where

$$\mathbf{x} \sim \mathbf{x}' \iff \mathbf{x} \equiv \lambda \mathbf{x}' \bmod kcd.$$

From standard lattice estimates, we have that

$$\left|\#(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0, L) \cap \Lambda(\mathbf{ckd}; b) \cap \Lambda(n, \mathbf{a}) - \frac{BL^2}{t} \cdot \frac{\varrho_F(\mathbf{d}; b, n, \mathbf{a}, \mathbf{c}, \mathbf{k})}{kcdb^2n^2(-\Delta)^2}\right| \\ \ll E(\mathbf{ckd}, b; n, \mathbf{a}) + O(1)$$

where we define

$$E(\mathbf{ckd}, b; n, \mathbf{a}) := \sum_{\alpha \in \mathcal{U}(\mathbf{ckd}; b)} \frac{B^{1/2}L}{(tk)^{1/2}\lambda_1(\Lambda(\alpha; \mathbf{ckd}, b) \cap \Lambda(n, \mathbf{a}))}.$$

In this context, we use $\lambda_1(\Lambda)$ to denote the first successive minima of the lattice. The above bound comes from applying the classical geometry of numbers estimate to each lattice.

3.9.2 Bounding the error term

Next, we sum this error term over when $d_i \sim D_i$:

$$\sum_{d_i \sim D_i} E(\mathbf{ckd}, b; n, \mathbf{a}) \ll \frac{B^{1/2}L}{(tk)^{1/2}} \sum_{d_i \sim D_i} \sum_{\alpha \in \mathcal{U}(\mathbf{ckd}; b)} \frac{1}{\lambda_1(\Lambda(\alpha; \mathbf{ckd}, b) \cap \Lambda(n, \mathbf{a}))}.$$

We now note that $\lambda_1(\Lambda(\alpha; \mathbf{ckd}, b) \cap \Lambda(n, \mathbf{a})) \geq \lambda_1(\Lambda(\alpha; \mathbf{ckd}, b))$. Let $\mathbf{v}$ be the shortest nontrivial vector in $\Lambda(\alpha; \mathbf{ckd}, b)$; then, $\mathbf{v}$ satisfies that $\gcd(v_1, v_2) = b$ and $k_i c_i d_i \mid F_i(\mathbf{v})$. Moreover, since $\det(\Lambda(\alpha; \mathbf{ckd}, b)) = cdkb^2$, we know that $\lambda_1(\Lambda) \leq b\sqrt{kcd}$. Thus, we can upper bound the above sum by

$$\begin{aligned}\sum_{d_i \sim D_i} E(\mathbf{ckd}, b; n, \mathbf{a}) &\ll \frac{B^{1/2}L}{(tk)^{1/2}} \sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ \gcd(v_1, v_2) = b}} \frac{1}{|\mathbf{v}|} \prod_{i=1}^{r} \#\{d_i \sim D_i : k_i c_i d_i \mid F_i(\mathbf{v})\} \\ &\ll \frac{B^{1/2}L}{(tk)^{1/2}} \left(\sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ \gcd(v_1, v_2) = b \\ F(\mathbf{v}) \neq 0}} \frac{1}{|\mathbf{v}|} \prod_{i=1}^{r} \Delta_2(F_i(\mathbf{v})/k_i c_i) + \sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ \gcd(v_1, v_2) = b \\ F(\mathbf{v}) = 0}} \frac{1}{|\mathbf{v}|} \cdot \sum_{i=1}^{r} D_i \mathbf{1}_{F_i(\mathbf{v})=0}. \right)\end{aligned}$$

Here we are using the notation that $\Delta_2(x) = 0$ if $x \notin \mathbb{Z}$ and $D = \prod_{i=1}^{r} D_i$. We also note that since F is a squarefree binary form, for each vector $\mathbf{v}$ satisfying $\gcd(v_1, v_2) = b$, there can be at most one form F_i such that $F_i(\mathbf{v}) = 0$. Moreover, that form must be linear.

For the first sum, let us recall from Remark 3.3.8, we can bound the expression by

$$\frac{\tau(b^{\deg(F)})}{b} \sum_{\substack{|\mathbf{v}| \leq \sqrt{kcD} \\ F(\mathbf{v}) \neq 0}} \frac{1}{|\mathbf{v}|} \prod_{i=1}^{r} \Delta(F_i(\mathbf{v})/k_i c_i).$$

Applying Corollary 3.3.12, and partial summation, we find that this is

$$\ll \frac{\tau(b^{\deg(F)})}{b} \cdot \sqrt{kcD} \cdot \frac{(16 \deg(F))^{\omega(ck)}}{ck} \cdot \prod_{i=1}^{r} \left(\sum_{s_i \ll \sqrt{kcD}} \frac{\varrho_{F_i}(s_i)(\Delta(s_i) - 1)}{s_i^2} \right).$$

Subsequently, applying Theorem 3.3.6 gives us that the contribution from those $\mathbf{v}$ with $F(\mathbf{v}) \neq 0$ is bounded by

$$\ll \frac{\tau(b^{\deg(F)})(16 \deg(F))^{\omega(ck)}}{b(ck)^{1/2}} \cdot \sqrt{D} \cdot \exp(\gamma_r \sqrt{\log_2(kcD) \log_3(kcD)}),$$

where $\gamma_r > 0$ is the constant depending on r as calculated explicitly in [31]. At this point, we recall that $\mathbf{c} = (c_i)$ has entries $c_i \mid \mathrm{Res}(F_i, F_j)$ for $i \neq j$. Since these resultants are fixed constants, the contribution from the factors depending on c above can be thrown into the implied constant.

Next, we handle the zeros of $F(\mathbf{v}) = 0$. First, we note that if $F_i(x, y)$ has degree ≥ 2, then the number of vectors $\mathbf{v}$ with $\gcd(v_1, v_2) = b$ such that $F_i(\mathbf{v}) = 0$ is zero – if such a solution existed, it would imply that the binary form $F_i(x, y)$ is reducible. Thus, for $F(x, y)$ squarefree with *no linear factors*, we have that

$$\sum_{d_i \sim D_i} E(\mathbf{ckd}, b; n, \mathbf{a}) \ll \frac{\tau(b^4)}{b} \cdot \sqrt{D} \cdot \frac{B^{1/2} L(16 \deg(F))^{\omega(k)}}{t^{1/2} k} \exp(\gamma_r \sqrt{\log_2(B) \log_3(B)}).$$

3.9.3 Linear factors

From now on, we assume that $F(x, y)$ has at least one linear factor. Since F is squarefree, we can see that

$$\sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ \gcd(v_1, v_2) = b \\ F(\mathbf{v}) = 0}} \frac{1}{|\mathbf{v}|} \sum_{i=1}^{r} D_i \mathbf{1}_{F_i(\mathbf{v})=0} \ll \sum_{\substack{i=1 \\ \deg(F_i)=1}}^{r} D_i \sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ \gcd(v_1, v_2) = b \\ F_i(\mathbf{v}) = 0}} \frac{1}{|\mathbf{v}|}.$$

Now, we consider for a linear factor $L(x, y)$, the sum

$$\sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ L(\mathbf{v})=0 \\ \gcd(v_1, v_2) = b}} \frac{1}{|\mathbf{v}|} \leq \frac{1}{b} \sum_{\substack{|\mathbf{v}| \leq \sqrt{kcD} \\ L(\mathbf{v})=0}} \frac{1}{|\mathbf{v}|} \ll_L \frac{\log(kcD)}{b}.$$

Thus, we get that the contribution from the linear factors is given by

$$\sum_{\substack{i=1 \\ \deg(F_i)=1}}^{r} D_i \sum_{\substack{|\mathbf{v}| \leq b\sqrt{kcD} \\ \gcd(v_1, v_2) = b \\ F_i(v) = 0}} \frac{1}{|\mathbf{v}|} \ll_F \frac{\log(kcD)}{b} \max_{i:\deg(F_i)=1} D_i.$$

Therefore, if $F(x, y)$ *has a linear factor*, we have that

$$\sum_{d_i \sim D_i} E(\mathbf{ckd}, b; n, \mathbf{a}) \ll \frac{B^{1/2} L}{t^{1/2}} \cdot \frac{\tau(b^{\deg(F)})}{b} \times \left(\frac{(16 \deg(F))^{\omega(k)}}{k} \sqrt{D} \exp(\gamma_r \sqrt{\log_2(B) \log_3(B)}) + \frac{\log(kcD)}{k^{1/2}} \max_{i:\deg(F_i)=1} D_i \right).$$

This completes the proof of Proposition 3.9.1. □

3.10 Large moduli and Hooley's Δ-function

In this section, we will discuss how to handle the sum for larger $\mathbf{d}$: let us write

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{D}) := \sum_{d_i \sim D_i} \chi(d) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}),$$

where $S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d})$ is defined in (3.8.4). If $F(x, y)$ has a linear factor $F_1(x, y)$, let us define

$$T_L(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; D_1) := \sum_{d_1 \sim D_1} \sum_{d_i \ll (B/tk)^{\deg(F_i)/4} (n_i k_i)^{-1/2}} \chi(d) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}).$$

We will provide an upper bound for these sums. This proof is based off of de la Bretèche and Tenenbaum's [25] and Heath Brown's [33] (for $F(x, y)$ with linear factors) treatment of the large moduli terms.

Proposition 3.10.1. *Let F be a squarefree binary form. Let $\mathbf{D} = (D_i)_{i=1}^r$ satisfy that $\log(D) \gg \log(B/kt)$. Then the following bound holds:*

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{D}) \ll \left(\frac{B}{b^2 t k^2} + \frac{B^{3/4} \nu_1(k)^{1/2}}{b t^{3/4} k^{5/4}} \right) \cdot (4 \deg(F))^{\omega(k)} \times \log(B/tk)^{\rho_{\Delta, f}/2 - 1 - 10^{-5}} \cdot \left(\frac{B}{tkD} + \frac{tkD}{B} + \log\log(B/tk) \right)^{1/2}.$$

Proposition 3.10.2. *Let F be a squarefree binary form with a linear factor $F_1(x, y)$. Then, the following bound holds:*

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; D_1) \ll \frac{B(16 \deg(F))^{\omega(k)}}{tk^2} \cdot \log(B/tk)^{\rho_{\Delta, f} - 2 - 10^{-5}}.$$

3.10.1 Relation to the twisted Hooley's Δ-function

We first observe that

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{D}) \leq \sum_{\substack{(x,y) \in (B/tk)^{1/2} \mathcal{B}(\mathbf{x}_0, L) \\ b | x, y \\ \exists d_i \sim D_i : k_i c_i d_i | F_i(x,y) \\ p_{-\Delta}(F(x,y)/c) = n \\ p_{\neg -\Delta}(F_i(x,y)/k_i c_i d_i) \equiv a_i \bmod -\Delta}} \prod_{i=1}^r \Delta(F_i(x, y)/k_i c_i, \chi).$$

Since $\Delta(n,\chi) \geq 0$, we can remove a few conditions and ask to bound:

$$\sum_{\substack{|x|,|y|\leq (B/tk)^{1/2} \\ b|x,y \\ \exists d_i \sim D_i : k_i c_i d_i | F_i(x,y)}} \prod_{i=1}^{r} \Delta(F_i(x,y)/k_i c_i, \chi).$$

After applying Cauchy-Schwarz, we achieve the upper bound of

$$T(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{D}) \leq \Big(\#\{|x|,|y| \leq (B/tk)^{1/2} : b \mid (x,y), \exists d_i \sim D_i, k_i c_i d_i \mid F_i(x,y)\}\Big)^{1/2} \times \left(\sum_{\substack{|x|,|y|\leq (B/tk)^{1/2} \\ b|x,y}} \prod_{i=1}^{r} \Delta(F_i(x,y)/k_i c_i, \chi)^2\right)^{1/2}.$$

Let us apply Lemma 3.3.14 to the first quantity:

$$\begin{aligned} &\#\{|x|,|y| \leq (B/tk)^{1/2} : b \mid (x,y), \exists d_i \sim D_i, k_i c_i d_i \mid F_i(x,y)\} \\ &\qquad \leq \#\{|x|,|y| \leq (B/tk)^{1/2} : b \mid (x,y), k \mid F(x,y), \exists d \sim D, d \mid F(x,y)\} \\ &\ll \left(\frac{\deg(F)^{\omega(k)}}{b^2 k} \cdot \frac{B}{tk} + \nu_1(k)\left(\frac{B}{tk}\right)^{1/2}\right) \cdot \left(\frac{B}{tkD} + \frac{tkD}{B} + \log\log(B/tk)\right) \log(B/tk)^{-10^{-4}}. \end{aligned}$$

We note that tracking through the proof of equation (7.41) in [25], one can add the lattice constraints that $b \mid x, y$ and achieve the upper bound mentioned with an additional factor of b^{-2} (recall that $b \leq \log(B)^{10^{-7}}$ is much smaller that $B/tk \geq \exp(\sqrt{\log(B)})$). Similarly, after tracking through the proof of (7.41) in [25] with the lattice condition that $k \mid F(x,y)$, we achieve the bound above; in §3.9, it was seen that this condition gives $\deg(F)^{\omega(k)}$ such lattices of determinant k.

Next, we apply Corollary 3.3.12 to the second sum:

$$\sum_{\substack{|x|,|y|\leq (B/tk)^{1/2} \\ b|x,y}} \prod_{i=1}^{r} \Delta(F_i(x,y)/k_i c_i, \chi)^2 \ll \frac{B}{tk\log(B/tk)} \cdot \frac{(16\deg(F))^{\omega(k)}}{b^2 k} \prod_{i=1}^{r} \left(\sum_{s_i \ll B/tk} \Delta(s_i,\chi)^2 \frac{\varrho_{F_i}(s_i)}{s_i^2}\right).$$

Here we have again restricted to the lattice defined by $b \mid (x,y)$; however, $b \leq \log(B)^{10^{-7}}$ is much smaller that B/tk. If $F_i(x,y)$ is irreducible over the quadratic field K, then by Theorem 3.3.9, we have that

$$\sum_{s_i \ll B/tk} \Delta(s_i,\chi)^2 \cdot \frac{\varrho_{F_i}(s_i)}{s_i^2} \ll \sum_{s_i \ll B/tk} \Delta(s_i,\chi)^2 \cdot \frac{\varrho_{f_i}(s_i)}{s_i} \ll \exp(C\sqrt{\log_2(B)\log_3(B)}),$$

for an explicit $C > 0$. Otherwise, we use a trivial bound:

$$\sum_{s_i \ll B/tk} \Delta(s_i, \chi)^2 \cdot \frac{\varrho_{F_i}(s_i)}{s_i^2} \ll \sum_{s_i \ll B/tk} \Delta(s_i)^2 \cdot \frac{\varrho_{F_i}(s_i)}{s_i^2} \ll \log(B/tk) \exp(C\sqrt{\log_2(B)\log_3(B)}).$$

Here we have used the results of de la Bretèche and Tenenbaum in [26, Theorem 1.1] for averages of higher powers of $\Delta(n)$ correlated with local counts of polynomials.

Combining these two bounds, we have that

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{D}) \leq \left(\frac{B}{b^2tk^2} + \frac{B^{3/4}\nu_1(k)^{1/2}}{bt^{3/4}k^{5/4}}\right) \cdot (4\deg(F))^{\omega(k)} \cdot \left(\frac{B}{tkD} + \frac{tkD}{B} + \log\log(B/tk)\right)^{1/2} \times \exp(C'\sqrt{\log_2(B)\log_3(B)}) \cdot \log(B/tk)^{\rho_{\Delta,f}/2-1-10^{-4}}.$$

This completes the proof of Proposition 3.10.1. □

3.10.2 Linear factors

Let us assume that $F(x, y)$ has a linear factor that we denote as $F_1(x, y) = L(x, y)$. Again, we enlarge our region the terms being summed are non-negative. In this case, we have that

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; D_1) \leq \sum_{\substack{|x|,|y| \leq (B/tk)^{1/2} \\ b|x,y}} \Delta(L(x, y)/k_1r_1c_1, \chi; D_1) \prod_{i=2}^{r} (1 \star \chi)(F_i(x, y)/k_ic_i, \chi),$$

where we define the localized twisted Hooley Δ function as

$$\Delta(n, \chi; D) := \left| \sum_{\substack{d|n \\ d \sim D}} \chi(d) \right|.$$

Next, we separate the sum by certain values of $L(x, y)$:

$$T(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; D_1) \leq \sum_{\substack{m \ll (B/tk)^{1/2}/k_1r_1c_1 \\ b|m \\ \exists d_1 \sim D_1, d_1|m}} \Delta(m, \chi; D_1) \sum_{\substack{|x|,|y| \leq (B/tk)^{1/2} \\ b|x,y \\ L(x,y) = mk_1r_1c_1}} \prod_{i=2}^{r} (1 \star \chi)(F_i(x, y)/k_ic_i, \chi).$$

For now let us fix a m. The restriction $L(x, y) = mk_1r_1c_1$ allows us to rewrite the polynomial

$F_i(x,y)$ as a single variable polynomial $F_{i,m}(x)$. In particular, we have that

$$\sum_{\substack{|x|,|y|\leq (B/tk)^{1/2}\\ b|x,y\\ L(x,y)=mk_1c_1}} \prod_{i=2}^{r}(1\star\chi)(F_i(x,y)/k_ic_i,\chi) \ll \sum_{\substack{x\ll (B/tk)^{1/2}\\ b|x}} \prod_{i=2}^{r}(1\star\chi)(F_{i,m}(x)/k_ic_i,\chi).$$

So, we apply Corollary 3.3.12 to the sum above:

$$\ll \frac{B^{1/2}}{b(tk)^{1/2}}\cdot\frac{(16\deg(F))^{\omega(k)}}{k}\prod_{i=1}^{r}\exp\left(\sum_{p_i\ll B/tk}\frac{\varrho_{F_{i,m}}(p_i)\chi(p_i)}{p_i}\right).$$

Now, we can see that if $\varrho'_{F_i}(p)$ is the original number of roots of $F_i(x,1) \bmod p$, then for p not dividing m, we have that

$$\varrho_{F_i,m}(p) = \varrho_{F_i}(p).$$

So, we write the above sum as

$$\sum_{i=2}^{r}\sum_{\substack{p_i\ll X\\ p_i\nmid m}}\frac{\varrho'_{F_i}(p_i)\chi(p_i)}{p_i} + \sum_{i=2}^{r}\sum_{p_i|m}\frac{\varrho_{F_{i,m}}(p_i)\chi(p_i)}{p_i}.$$

We consider

$$\exp\left(\sum_{i=2}^{r}\sum_{p|m}\frac{\varrho_{F_{i,m}}(p_i)\chi(p_i)}{p_i} - \sum_{p_i|m}\frac{\varrho'_{F_i}(p_i)\chi(p_i)}{p_i}\right) \ll \prod_{i=2}^{r}\prod_{p_i|m}\left(\frac{1-\varrho_{F_{i,m}}(p_i)/p_i}{1-\varrho'_{F_i}(p_i)/p_i}\right) \ll \prod_{i=2}^{r}\left(\frac{\sigma(m)}{m}\right)^{\deg(F_i)},$$

where we use the notation $\sigma(m) = \sum_{d|m} d$. Here, we have used [33, (4.6)]. Now since $m \ll B/tk$, the above is bounded by

$$\ll \prod_{i=2}^{r}\log\log(B/tk)^{\deg(F_i)} \ll \log\log(B/tk)^{\deg(F)-1}.$$

In summary, we have that

$$\sum_{\substack{x\ll (B/tk)^{1/2}\\ b|x}}\prod_{i=2}^{r}(1\star\chi)(F_{i,m}(x)/k_ic_i,\chi) \ll \frac{B^{1/2}\log\log(B/tk)^{\deg(F)-1}}{b(tk)^{1/2}}\cdot\frac{(8\deg(F))^{\omega(k)}}{k} \times \exp\left(\sum_{i=2}^{r}\sum_{p_i\ll B/tk}\frac{\varrho'_{F_i}(p_i)\chi(p_i)}{p_i}\right).$$

Since $L(x,y)$ was a linear factor of $F(x,y)$, we know that the Picard rank satisfies

$$\rho_{\Delta,F} = \rho_{\Delta,F_2\dots F_r}.$$

Thus, we get that

$$\sum_{\substack{x\ll(B/tk)^{1/2}\\ b|x}} \prod_{i=2}^{r}(1\star\chi)(F_{i,m}(x)/k_ic_i,\chi) \ll \frac{B^{1/2}\log(B/tk)^{\rho_{\Delta,F}-2}\log\log(B/tk)^{\deg(F)-1}}{b(tk)^{1/2}}\cdot\frac{(16\deg(F))^{\omega(k)}}{k}.$$

Here we have again used Corollary 3.3.16 that the order of the pole of $\xi(s;F,\chi)$ is given by $\rho_{\Delta,f}-2$.

Applying the above bound to the inner sum of (3.10.2), we see that

$$T(n,\mathbf{a},\mathbf{c},\mathbf{k},b;D_1) \leq \frac{B^{1/2}\log(B/tk)^{\rho_{\Delta,f}-2}\log\log(B/tk)^{\deg(F)-1}}{b(tk)^{1/2}}\cdot\frac{(16\deg(F))^{\omega(k)}}{k} \times \sum_{\substack{m\ll(B/tk)^{1/2}/k_1c_1\\ b|m\\ \exists d_1\sim D_1, d_1|m}} \Delta(m,\chi;D_1).$$

Applying the Cauchy-Schwarz inequality to the second sum, this final sum is bounded by:

$$\ll \#\{m\ll(B/tk)^{1/2}/k_1c_1 : b\mid m, \exists d_1\sim D_1, d_1\mid m\}^{1/2}\times\left(\sum_{\substack{m\ll(B/tk)^{1/2}/k_1c_1\\ b|m}}\Delta(m,\chi)^2\right)^{1/2}.$$

Now we apply Lemma 3.3.13 to the first sum, and Theorem 3.3.10 and Theorem 3.3.9 to the second to get the upper bound:

$$T(n,\mathbf{a},\mathbf{c},\mathbf{k},b;D_1) \leq \frac{B\log(B/t)^{\rho_{\Delta,f}-2-10^{-5}}}{tk}\cdot\frac{(16\deg(F))^{\omega(k)}}{k}.$$

This completes the proof of Proposition 3.10.2.

3.11 The asymptotics

In this section, we will finally prove Proposition 3.8.1 and establish the asymptotic in Proposition 3.5.11. To do so, we bring in ingredients from the preceding sections. Unfortunately, it is in this section where we are confronted with a notationally-challenging problem – while morally equivalent, the proofs for various factorizations of $F(x,y)$ must be treated slightly differently. As such, we have

to do casework on the factorization-type of $F(x,y)$ in §3.11.1.

We also highlight that the calculations in §3.11.2 are the jumping-off point for computing the leading constant in §3.12. In particular, the notation that is used in §3.12 is established in this section.

3.11.1 Proof of Proposition 3.8.1

Let us first recall the statement of Proposition 3.8.1:

Proposition. *Let $F(x,y)$ be a squarefree binary form of degree ≤ 4 and $L \gg \log(B)^{-10^{-10}}$. Then we have that for t and k satisfying $tk \leq B/\exp(\sqrt{\log(B)})$, the following holds:*

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k) = \frac{BL^2}{tk^2} \cdot C_{q_1,q_2,f,k}(B/tk) \cdot \left(1 + O\left(\log(B)^{-10^{-7}}\right)\right)$$
$$+O\left(\frac{B(8\deg(F))^{\omega(k)}}{tk^{3/2}} \cdot \log(B)^{\rho_{\Delta,f}-2-10^{-7}} + \frac{B^{3/4}}{(tk)^{3/4}} \cdot \left(L + (4\deg(F))^{\omega(k)}\nu_1(k)^{1/2}\right) \cdot \log(B)^{-10^{-7}}\right),$$

where $C_{q_1,q_2,f,k}(B/tk)$ is an expression that depends on the factorization of $F(x,y)$ and

$$\nu_1(k) = \sum_{\alpha \in \mathcal{U}(k)} \frac{1}{\lambda_1(\Lambda(\alpha))}.$$

Additionally, for a fixed value of k, there exists a constant $c_{q_1,q_2,f,k}$ such that

$$C_{q_1,q_2,f,k}(B/tk) = c_{q_1,q_2,f,k} \log(B/tk)^{\rho_{\Delta,f}-2}.$$

Remark 3.11.1. Note that since $tk \leq B/\exp(\sqrt{\log(B)})$, we have that $\log(B) \ll \log(B/tk) \ll \log(B)$. We use these interchangeably in this section when referring to error terms.

Before delving into the case work for this proof, let us outline the general strategy. After applying the reductions discussed in §3.8, the sum that we wish to analyze takes the form:

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k) \sim \sum_{c_{ij} | \mathrm{Res}(F_i, F_j)} \mu(c)\chi(c) \sum_{b \leq \log(B)^{10^{-7}}} \mu(b) \cdot \sum_{\substack{n \leq \log(B)^{10^{-7}} \\ p|n \implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))$$
$$\sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i, -\Delta)=1}} \prod_{i=1}^{r} \chi_{q_1}(a_i)(1+\chi(a_i)) \sum_{d_i \ll (B/tk)^{\deg(F_i)/2}} \prod_{i=1}^{r} \chi(d_i) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}).$$

We proceed by splitting up the sum over d_i into sums over "small" and "large" moduli – either $d \leq (B/tk)^{\deg(F)/4}\log(B)^{-\epsilon}$ or $d \geq (B/tk)^{\deg(F)/4}\log(B)^{-\epsilon}$. We remark that if $F(x,y)$ has linear

factors, the cutoff point for being considered a "small" or "large" modulus is shifted by $\log(B)$.

For "small" moduli, our level of distribution result (Proposition 3.9.1) enables us to achieve the main term:

$$\frac{BL^2}{tk^2}\cdot\sum_{c_{ij}|\mathrm{Res}(F_i,F_j)}\frac{\mu(c)\chi(c)}{c}\sum_{b\leq\log(B)^{10^{-7}}}\frac{\mu(b)}{b^2}\cdot\sum_{\substack{n\leq\log(B)^{10^{-7}}\\ p|n\implies p|-\Delta}}\frac{\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))}{n^2}$$
$$\sum_{\substack{\mathbf{a}\bmod -\Delta\\ \gcd(a_i,-\Delta)=1}}\frac{\prod_{i=1}^r\chi_{q_1}(a_i)(1+\chi(a_i))}{\Delta^2}\sum_{d_i\ll(B/tk)^{\deg(F_i)/2}}\prod_{i=1}^r\chi(d_i)\frac{\varrho_F(\mathbf{d};b,n,\mathbf{a},\mathbf{c},\mathbf{k})}{d}.$$

For "large" moduli, the analysis on the twisted Hooley Δ-function (Propostion 3.10.1 or Proposition 3.10.2) gives us that these terms contribute negligibly. For these terms, we establish a bound of the form:

$$\sum_{d\geq(B/tk)^{\deg(F)/2}\log(B)^{-\epsilon}}\chi(d)S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d})\ll\log(B)^{\rho_{\Delta,f}-2-10^{-5}}\cdot\left(\frac{B}{b^2tk^2}+\frac{B^{3/4}\nu_1(k)^{1/2}}{(tk)^{3/4}}\right).$$

Note that since $b,n\leq\log(B)^{10^{-7}}$ compared to our saving of $\log(B)^{-10^{-5}}$, we will get a sufficient error term here.

After careful analysis, we can rewrite this main term as

$$\frac{BL^2}{tk^2}\cdot\cdot\sum_{c_{ij}|\mathrm{Res}(F_i,F_j)}\mu(c)\chi(c)\cdot\bigtriangledown_{q_1,q_2}(\mathbf{c})\cdot V(\mathbf{c};B/tk),$$

where $\bigtriangledown_{q_1,q_2}(\mathbf{c})$ is a constant that detects local obstructions at primes $p\mid-\Delta$ for $X^*_{\Delta,f_1,\ldots,f_r,\mathbf{c}}$(formally defined in (Definition 3.11.3)) and $V(\mathbf{c};B/tk)$ is a sum related to the Dirichlet series $\xi(s;F,\chi)$ (defined in (3.3.5)). We determine that $V(\mathbf{c};B/tk)$ satisfies that

$$V(\mathbf{c};B/tk)\sim c_{q_1,q_2,f,k,\mathbf{c}}\log(B/tk)^{\rho_{\Delta,f}-2}.$$

Additionally, in §3.12, we will analyse the constant $c_{q_1,q_2,f,k,\mathbf{c}}$ further.

Now, we proceed with the full proof of Proposition 3.8.1. We start with the case work on the factorizaton of $F(x,y)$.

Case: $F(x,y)$ is irreducible

Since $F(x,y)$ is irreducible, we can skip the reductions in §3.8.1. Applying the reductions of §3.8.2, §3.8.3, and §3.8.4, we achieve:

$$\begin{aligned}
S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k) &= \sum_{\substack{n\ll\log(B)^{10^{-7}}\\ p|n\implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{a \bmod -\Delta\\ \gcd(a,-\Delta)=1}} \chi_{q_1}(a)S(n,a,1,k) \\
&\qquad + O\left(\frac{BL^2}{tk^2}\cdot\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right) \\
&= \sum_{b\le\log(B)^{10^{-7}}} \mu(b) \sum_{\substack{n\ll\log(B)^{10^{-7}}\\ p|n\implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{a \bmod -\Delta\\ \gcd(a,-\Delta)=1}} \chi_{q_1}(a)S(n,a,1,k;b) \\
&\qquad + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right) \\
&= \sum_{b\le\log(B)^{10^{-7}}} \mu(b) \sum_{\substack{n\ll\log(B)^{10^{-7}}\\ p|n\implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{a \bmod -\Delta\\ \gcd(a,-\Delta)=1}} \chi_{q_1}(a)(1+\chi(a)) \\
&\qquad \times \sum_{d\le\max\sqrt{|F(x,y)|/n_1n_2k}} \chi(d)S(n,a,1,k,b;d) + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right).
\end{aligned}$$

Above, the maximum is taken over $(x,y)\in(B/tk)^{1/2}\mathcal{B}(\mathbf{x}_0,L)$. For "small" moduli $d\le D := (B/tk)^{\deg(F)/4}(nk)^{-1/2}\log(B)^{-10^{-5}}$, we apply Proposition 3.9.1, our level of distribution result. From this, we can see that

$$\begin{aligned}
\sum_{d\le D}\chi(d)S(n,a,1,k,b;d) &= \frac{BL^2}{tk^2}\sum_{d\le D}\frac{\chi(d)\varrho_F(d;b,n,a,1,k)}{db^2n^2(-\Delta)^2} \\
&\qquad + O\left(\frac{B}{tk^{3/2}n^{1/2}}\log(B)^{-10^{-5}} + \frac{BL(8\deg(F))^{\omega(k)}}{tk^{7/4}n^{1/4}}\log(B)^{-10^{-6}}\right).
\end{aligned}$$

Remark 3.11.2. If $F(x,y)$ is an irreducible cubic form, the level of distribution error for any $D\le(B/tk)^{3/4}(nk)^{-1/2}$ is sufficient. So, in this case we do not need to separately handle these "large" moduli.

On the other hand, Proposition 3.10.1 for "large" moduli tells us that

$$\begin{aligned}
&\sum_{(B/tk)^{\deg(F)/4}(nk)^{-1/2}\gg d\ge D}\chi(d)S(n,a,1,k,b;d) \\
&\qquad\ll\left(\frac{B}{b^2tk^{3/2}}+\frac{B^{3/4}\nu_1(k)^{1/2}}{bt^{3/4}k^{3/4}}\right)\cdot(4\deg(F))^{\omega(k)}\cdot\log(B)^{\rho_{\Delta,f}/2-1-10^{-6}}.
\end{aligned}$$

Together, this gives us that

$$S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k) = \frac{BL^2}{tk^2} \cdot \sum_{b \leq \log(B)^{10^{-7}}} \mu(b) \sum_{\substack{n \ll \log(B)^{10^{-7}} \\ p|n \Longrightarrow p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))$$
$$\times \sum_{\substack{a \bmod -\Delta \\ \gcd(a,-\Delta)=1}} \chi_{q_1}(a)(1+\chi(a)) \sum_{d \leq D} \frac{\chi(d)\varrho_F(d; b, n, a, 1, k)}{db^2n^2(-\Delta)^2}$$
$$+ O\left((8\deg(F))^{\omega(k)} \cdot \left(\frac{B}{tk^{3/2}} + \frac{B^{3/4}\nu_1(k)^{1/2}}{t^{3/4}k^{3/4}}\right) \log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right).$$

Here we have used that $n, b \leq \log(B)^{10^{-7}}$ are small constants. Note that $\rho_{\Delta,f} \geq 2$ by definition and hence $\rho_{\Delta,f} - 2 \geq \rho_{\Delta,f}/2 - 1$.

Let us rework this main term. First, we can undo the Möbius inversion from §3.8.3 – we extend the sum over $b \leq \log(B)^{10^{-7}}$ to one over all b by again bounding away the contribution from large b with the Nair sieve (Theorem 3.3.11). We can also use multiplicativity of ϱ_F to write:

$$\varrho_F(d; n, a, 1, k) = \varrho_F(kd)\varrho_F(n, a),$$

where we have defined:

$$\varrho_F(kd) = \#\{\mathbf{x} \bmod kd : \gcd(x, y, kd) = 1, F(x, y) \equiv 0 \bmod kd\}/\sim$$

$$\varrho_F(n, a) = \#\{\mathbf{x} \bmod n(-\Delta) : \gcd(x, y, -\Delta) = 1, p_{-\Delta}(F(x, y)) = n, p_{\neg -\Delta}(F(x, y)) \equiv a \bmod -\Delta\}.$$

Note that we have used that $\gcd(k, -\Delta) = 1$ by construction and $\gcd(d, -\Delta) = 1$ for those d with nonzero contribution to the sum. Consequently, the main term can be rewritten as:

$$\frac{BL^2}{tk^2} \cdot \bigtriangledown_{q_1,q_2} \cdot V(k, D),$$

where we define:

$$\bigtriangledown_{q_1,q_2} := \sum_{p|n \Longrightarrow p|-\Delta} \frac{\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_2}(n))}{n^2} \sum_{\substack{a \bmod -\Delta \\ \gcd(a,-\Delta)=1}} \frac{\chi_{q_1}(a)(1+\chi(a))}{(-\Delta)^2} \cdot \varrho_F(n, a).$$

$$V(k, D) = \sum_{d \leq D} \frac{\chi(d)\varrho_F(kd)}{d}.$$

Both the terms $\bigtriangledown_{q_1,q_2}$ and $V(k,D)$ will be generalized in the more complicated cases below.

Case: $F(x,y)$ is the product of two irreducible factors

Write $F(x,y) = F_1(x,y)F_2(x,y)$ as the decomposition of $F(x,y)$ into irreducible factors. If one of the factors is linear, assume without loss of generality that it is F_1. Following the reductions of §3.8, we have that:

$$\begin{aligned}
S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k) &= \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c)S(\mathbf{c}=(c,c),\mathbf{k}) \\
&= \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c) \times \sum_{\substack{n\ll \log(B)^{10^{-7}} \\ p|n \implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i,-\Delta)=1}} \chi_{q_1}(a)S(n,\mathbf{a},\mathbf{c},\mathbf{k}) \\
&\qquad + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right) \\
&= \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c) \sum_{\substack{n\ll \log(B)^{10^{-7}} \\ p|n \implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i,-\Delta)=1}} \chi_{q_1}(a) \\
&\qquad \times \sum_{b\le \log(B)^{10^{-7}}} \mu(b)S(n,\mathbf{a},\mathbf{c},\mathbf{k};b) + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right) \\
&= \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c) \sum_{\substack{n\ll \log(B)^{10^{-7}} \\ p|n \implies p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i,-\Delta)=1}} \chi_{q_1}(a)(1+\chi(a_1))(1+\chi(a_2)) \\
&\qquad \times \sum_{b\le \log(B)^{10^{-7}}} \mu(b) \sum_{d_i \le \max \sqrt{|F_i(x,y)|/n_ik_i}} \chi(d)S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d}) \\
&\qquad + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right).
\end{aligned}$$

If F_1 and F_2 are both irreducible quadratics, we write that $\mathbf{d} \le \mathbf{D}$ if $\mathbf{d} = (d_1,d_2)$ satisfies that $d_1d_2 \le D := (B/tk)^{\deg(F)/4}(nk)^{-1/2}\log(B)^{-10^{-5}}$. If F_1 is linear, we define $D_1 := (B/tk)^{1/4}(n_1k_1)^{-1/2}\log(B)^{-1-10^{-5}}$. We write $\mathbf{d} \le \mathbf{D}$ when $d_1 \le D_1$ – for such vectors $\mathbf{d}$, we automatically have that $d \le (B/tk)^{\deg(F)/4}(nk)^{-1/2}\log(B)^{-1-10^{-5}}$. Proposition 3.9.1 tells us that for $\mathbf{d} \le \mathbf{D}$, we have that:

$$\begin{aligned}
\sum_{d_i \le D_i} \chi(d)S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d}) &= \frac{BL^2}{tk^2} \cdot \sum_{\mathbf{d}\le\mathbf{D}} \frac{\chi(d)\varrho_F(\mathbf{d};b,n,\mathbf{a},\mathbf{c},\mathbf{k})}{cdb^2n^2(-\Delta)^2} \\
+ O\left(\frac{B}{tk^{3/2}n^{1/2}}\log(B)^{-10^{-5}} + \frac{BL(8\deg(F))^{\omega(k)}}{tk^{7/4}n^{1/4}}\log(B)^{-10^{-6}} + \frac{B^{3/4}L}{(tk)^{3/4}(k_1n_1)^{1/2}}\log(B)^{-10^{-5}}\right).
\end{aligned}$$

Note that this last term does not exist when $F_1(x,y)$ and $F_2(x,y)$ are both irreducible quadratics.

Also we observe an abuse of notation above – here c denotes the product of the entries of the vector $\mathbf{c}$ compared to the entries of the vector earlier on this page.

Applying either Proposition 3.10.1 (if F_1, F_2 are both quadratic) or Proposition 3.10.2 (if F_1 or F_2 is linear), we achieve the upper bound:

$$\sum_{\mathbf{d}\not\leq\mathbf{D}} \chi(d)S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d}) \ll \frac{B(16\deg(F))^{\omega(k)}}{tk^2}\cdot\log(B/tk)^{\rho_{\Delta,f}-2-10^{-5}} + \left(\frac{B}{b^2tk^{3/2}}+\frac{B^{3/4}\nu_1(k)^{1/2}}{bt^{3/4}k^{3/4}}\right)\cdot(4\deg(F))^{\omega(k)}\cdot\log(B)^{\rho_{\Delta,f}/2-1-10^{-6}}.$$

Here we have used that $0 \leq \rho_{\Delta,f}/2 - 1 \leq \rho_{\Delta,f} - 2$.

Together, we get the estimate:

$$S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k) = \frac{BL^2}{tk^2}\cdot C_{q_1,q_2,f,k}(B/tk)\cdot(1+O(\log(B)^{-10^{-7}})) + O\left(\frac{B(16\deg(F))^{\omega(k)}}{tk^{3/2}}\log(B)^{\rho_{\Delta,f}-2-10^{-7}} + \frac{B^{3/4}\cdot((4\deg(F))^{\omega(k)}\nu_1(k)^{1/2}+L)}{(tk)^{3/4}}\cdot\log(B)^{-10^{-7}}\right),$$

where we define:

$$C_{q_1,q_2,f,k}(B/tk) := \sum_{c|\mathrm{Res}(F_1,F_2)}\frac{\mu(c)\chi(c)}{c^2}\sum_{p|n\implies p|\Delta}\frac{\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))}{n^2} \times \sum_{\substack{\mathbf{a}\bmod\Delta\\ \gcd(a_i,\Delta)=1}}\frac{\chi_{q_1}(a)(1+\chi(a_1))(1+\chi(a_2))}{\Delta^2}\sum_{\mathbf{d}\leq\mathbf{D}}\frac{\chi(d)\varrho_F(\mathbf{d};n,\mathbf{a},\mathbf{c},\mathbf{k})}{d}.$$

Now, we generalize the definitions of $\bigtriangledown_{q_1,q_2}$ and $V(k,D)$ from before (in (3.11.1), (3.11.1)).

Definition 3.11.3. We define the constant:

$$\bigtriangledown_{q_1,q_2}(\mathbf{c}) := \sum_{p|n\implies p|-\Delta}\frac{\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))}{n^2}\sum_{\substack{\mathbf{a}\bmod-\Delta\\ \gcd(a_i,-\Delta)=1}}\frac{\chi_{q_1}(a)}{(-\Delta)^2}\times\prod_{i=1}^{r}(1+\chi(a_i))\varrho_F(n,\mathbf{ca}).$$

Here the function $\varrho_f(n,\mathbf{a})$ denotes:

$$\varrho_F(n,\mathbf{a}) = \#\{\mathbf{x}\bmod n(-\Delta) : \gcd(x,y,-\Delta)=1, p_{-\Delta}(F(x,y))=n, p_{\neg-\Delta}(F_i(x,y))\equiv c_ia_i \bmod -\Delta\}.$$

This constant is independent of $\mathbf{k}$, and will detect when there is an obstruction to rational points, as will be explained in §3.12. We observe that it does contain a twist depending on the values of $\mathbf{c}$; these twists are nonexistence in the case when F is irreducible, as seen in (3.11.1).

We also define

$$V(\mathbf{c},\mathbf{k},D) := \sum_{\mathbf{d}\leq\mathbf{D}} \frac{\chi(d)\varrho_F(\mathbf{ckd})}{d},$$

where $\mathbf{d}\leq\mathbf{D}$ denotes summing over the appropriate range of "small" moduli and

$$\varrho_F(\mathbf{ckd}) := \#\{\mathbf{x} \bmod ckd : \gcd(x,y,ckd)=1, c_ik_id_i \mid F_i(x,y)\}/\sim .$$

Then, from the multiplicativity of ϱ_F, we see that

$$C_{q_1,q_2,f,k}(B/tk) = \sum_{c|\mathrm{Res}(F_1,F_2)} \frac{\mu(c)\chi(c)}{c^2}\cdot \bigtriangledown_{q_1,q_2}(\mathbf{c})\cdot V(\mathbf{c},\mathbf{k},D).$$

Case: $F(x,y)$ is a product of three irreducible factors

The proofs for the case of three and four irreducible factors are essentially equivalent to the proof when there are two irreducible factors; we write them for completeness.

Since $\deg(F(x,y))\leq 4$, we must have at least two linear factors. We denote these as $F_1(x,y)$ and $F_2(x,y)$. Applying the reductions of §3.8, we have the following estimates:

$$\begin{aligned}
S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k) &= \sum_{c_1c_2|\mathrm{Res}(F_1F_2,F_3)} \mu(c_1c_2)\chi(c_1c_2) \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c) S(\mathbf{c}=(cc_1,cc_2,c_1c_2),\mathbf{k}) \\
&= \sum_{c_1c_2|\mathrm{Res}(F_1F_2,F_3)} \mu(c_1c_2)\chi(c_1c_2) \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c) \sum_{\substack{n\ll\log(B)^{10^{-7}}\\ p|n\Longrightarrow p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \\
&\qquad \times \sum_{\substack{\mathbf{a} \bmod -\Delta\\ \gcd(a_i,-\Delta)=1}} \chi_{q_1}(a) S(n,\mathbf{a},\mathbf{c},\mathbf{k}) + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right) \\
&= \sum_{c_1c_2|\mathrm{Res}(F_1F_2,F_3)} \mu(c_1c_2)\chi(c_1c_2) \sum_{c|\mathrm{Res}(F_1,F_2)} \mu(c)\chi(c) \sum_{\substack{n\ll\log(B)^{10^{-7}}\\ p|n\Longrightarrow p|-\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \\
&\qquad \times \sum_{\substack{\mathbf{a} \bmod -\Delta\\ \gcd(a_i,-\Delta)=1}} \chi_{q_1}(a)(1+\chi(a_1))(1+\chi(a_2))(1+\chi(a_3)) \sum_{b\ll\log(B)^{10^{-7}}} \mu(b) \\
&\qquad\qquad \times \sum_{d_i\leq\sqrt{\max|F_i(x,y)|/(k_in_i)^{1/2}}} \chi(d) S(n,\mathbf{a},\mathbf{c},\mathbf{k},b;\mathbf{d}) \\
&\qquad\qquad\qquad + O\left(\frac{BL^2}{tk^2}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}\right).
\end{aligned}$$

Let $D_i := (B/tk)^{1/4}(n_ik_i)^{-1/2}\log(B)^{-1-10^{-5}}$ for each i such that F_i is linear. We write that $\mathbf{d}\leq\mathbf{D}$ if $d_i\leq D_i$ for each linear factor F_i. Otherwise, we write that $\mathbf{d}\not\leq\mathbf{D}$. Applying Proposition 3.9.1,

we get that

$$\sum_{d_i \leq D_i} \chi(d) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}) = \frac{BL^2}{tk^2} \cdot \sum_{\mathbf{d} \leq \mathbf{D}} \frac{\chi(d) \varrho_F(\mathbf{d}; b, n, \mathbf{a}, \mathbf{c}, \mathbf{k})}{cdb^2 n^2 (-\Delta)^2}$$
$$+ O\left(\frac{B}{tk^{3/2} n^{1/2}} \log(B)^{-10^{-5}} + \frac{BL(8 \deg(F))^{\omega(k)}}{tk^{7/4} n^{1/4}} \log(B)^{-10^{-6}} + \frac{B^{3/4} L}{(tk)^{3/4} (k_1 n_1)^{1/2}} \log(B)^{-10^{-5}} \right).$$

On the other hand, Proposition 3.10.2 implies that the following upper bound holds:

$$\sum_{\mathbf{d} \not\leq \mathbf{D}} \chi(d) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}) \ll \frac{B(16 \deg(F))^{\omega(k)}}{tk^{3/2} b^2} \log(B)^{\rho_{\Delta, f} - 2 - 10^{-5}}.$$

Together, we get the estimate that:

$$S(X_{\Delta, f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k) = \frac{BL^2}{tk^2} \cdot C_{q_1, q_2, f, k}(B) \cdot (1 + O(\log(B)^{-10^{-7}}))$$
$$+ O\left(\frac{B(16 \deg(F))^{\omega(k)}}{tk^{3/2}} \log(B)^{\rho_{\Delta, f} - 2 - 10^{-7}} + \frac{B^{3/4} L}{(tk)^{3/4}} \log(B)^{-10^{-7}} \right),$$

where we have that:

$$C_{q_1, q_2, f, k}(B/tk) := \sum_{c_1 c_2 | \mathrm{Res}(F_1 F_2, F_3)} \frac{\mu(c_1 c_2) \chi(c_1 c_2)}{c_1^2 c_2^2} \sum_{c | \mathrm{Res}(F_1, F_2)} \frac{\mu(c) \chi(c)}{c^2} \cdot \bigtriangledown_{q_1, q_2}(\mathbf{c}) \cdot V(\mathbf{c}, \mathbf{k}, D).$$

Case: $F(x,y)$ is the product of four irreducible factors

Let $F(x,y) = F_1F_2F_3F_4$, where all four are linear facts. Applying the reductions of §3.8, we have the following estimate for $S(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L); q_1, q_2, t, k)$:

$$\sum_{\substack{c \mid \mathrm{Res}(F_1F_2, F_3F_4) \\ c = c_1c_2 \\ c = c_3c_4}} \mu(c)\chi(c) \sum_{\substack{e \mid \mathrm{Res}(F_1, F_2) \\ f \mid \mathrm{Res}(F_3, F_4)}} \mu(e)\chi(e)\mu(f)\chi(f) S(\mathbf{c} = (c_1e, c_2e, c_3f, c_4f), \mathbf{k})$$

$$= \sum_{\substack{c \mid \mathrm{Res}(F_1F_2, F_3F_4) \\ c = c_1c_2 \\ c = c_3c_4}} \mu(c)\chi(c) \sum_{\substack{e \mid \mathrm{Res}(F_1, F_2) \\ f \mid \mathrm{Res}(F_3, F_4)}} \mu(e)\chi(e)\mu(f)\chi(f)$$

$$\times \sum_{\substack{n \ll \log(B)^{10^{-7}} \\ p \mid n \implies p \mid -\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i, -\Delta) = 1}} \chi_{q_1}(a) S(p_{q_1}(n), p_{q_2}(n), \mathbf{a}, \mathbf{c}, \mathbf{k})$$

$$+ O\left(\frac{BL^2}{tk^2} \log(B)^{\rho_{\Delta,f} - 2 - 10^{-7}}\right)$$

$$= \sum_{\substack{c \mid \mathrm{Res}(F_1F_2, F_3F_4) \\ c = c_1c_2 \\ c = c_3c_4}} \mu(c)\chi(c) \sum_{\substack{e \mid \mathrm{Res}(F_1, F_2) \\ f \mid \mathrm{Res}(F_3, F_4)}} \mu(e)\chi(e)\mu(f)\chi(f)$$

$$\times \sum_{\substack{n_j \ll \log(B)^{10^{-7}} \\ p \mid n \implies p \mid -\Delta}} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)) \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i, -\Delta) = 1}} \chi_{q_1}(a) \prod_{i=1}^{4} (1 + \chi(a_i)) \sum_{b \ll \log(B)^{10^{-7}}} \mu(b)$$

$$\times \sum_{d_i \leq \sqrt{\max |F_i| / (k_i n_i)^{1/2}}} \chi(d) S(p_{q_1}(n), p_{q_2}(n), \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d})$$

$$+ O\left(\frac{BL^2}{tk^2} \log(B)^{\rho_{\Delta,f} - 2 - 10^{-7}}\right).$$

Let $D_i := (B/tk)^{1/4}(n_ik_i)^{-1/2} \log(B)^{-1-10^{-5}}$ for each i. We write that $\mathbf{d} \leq \mathbf{D}$ if $d_i \leq D_i$ for each linear factor F_i. Otherwise, we write that $\mathbf{d} \not\leq \mathbf{D}$. Applying Proposition 3.9.1, we get that

$$\sum_{d_i \leq D_i} \chi(d) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}) = \frac{BL^2}{tk^2} \cdot \sum_{\mathbf{d} \leq \mathbf{D}} \frac{\chi(d) \varrho_F(\mathbf{d}; b, n, \mathbf{a}, \mathbf{c}, \mathbf{k})}{cdb^2n^2(-\Delta)^2}$$
$$+ O\left(\frac{B}{tk^{3/2}n^{1/2}} \log(B)^{-10^{-5}} + \frac{BL(8 \deg(F))^{\omega(k)}}{tk^{7/4}n^{1/4}} \log(B)^{-10^{-6}} + \frac{B^{3/4}L}{(tk)^{3/4}(k_1n_1)^{1/2}} \log(B)^{-10^{-5}}\right).$$

On the other hand, Proposition 3.10.2 implies that the following upper bound holds:

$$\sum_{\mathbf{d} \not\leq \mathbf{D}} \chi(d) S(n, \mathbf{a}, \mathbf{c}, \mathbf{k}, b; \mathbf{d}) \ll \frac{B(16 \deg(F))^{\omega(k)}}{tk^{3/2}b^2} \log(B)^{\rho_{\Delta,f} - 2 - 10^{-5}}.$$

Together, we get the estimate that:

$$S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,t,k)=\frac{BL^2}{tk^2}\cdot C_{q_1,q_2,f,k}(B)\cdot(1+O(\log(B)^{-10^{-7}}))$$
$$+O\left(\frac{B(16\deg(F))^{\omega(k)}}{tk^{3/2}}\log(B)^{\rho_{\Delta,f}-2-10^{-7}}+\frac{B^{3/4}L}{(tk)^{3/4}}\log(B)^{-10^{-7}}\right),$$

where we have that:

$$C_{q_1,q_2,f,k}(B/tk):=\sum_{\substack{c|\mathrm{Res}(F_1F_2,F_3F_4)\\ c=c_1c_2\\ c=c_3c_4}}\frac{\mu(c)\chi(c)}{c^2}\sum_{\substack{e|\mathrm{Res}(F_1,F_2)\\ f|\mathrm{Res}(F_3,F_4)}}\frac{\mu(e)\chi(e)\mu(f)\chi(f)}{e^2f^2}\cdot\bigtriangledown_{q_1,q_2}(\mathbf{c})\cdot V(\mathbf{c},\mathbf{k},D).$$

Completion of the proof of Proposition 3.8.1

Noting that $\bigtriangledown_{q_1,q_2}(\mathbf{c})$ is a finite constant, and since $F(x,y)$ is a squarefree form we know that the sum over $\mathbf{c}$ is given by a finite sum, we see that in order to establish Proposition 3.8.1 it suffices to prove that for any fixed $\mathbf{c}$ we have that:

$$V(\mathbf{c},\mathbf{k},D)=c_{q_1,q_2,f,k,\mathbf{c}}\log(B/tk)^{\rho_{\Delta,f}-2}(1+O(\log(B)^{-1/2})).$$

Let us define the Dirichlet series:

$$\xi(s;F,\chi,\mathbf{c},\mathbf{k})=\sum_{\mathbf{d}}\frac{\chi(d)\varrho_F(\mathbf{ckd})}{d^s}.$$

Since $\varrho_F(n)\leq\deg(F)^{\omega(n)}$ for all but finitely many values of n, we can see that the above Dirichlet series converges for $\Re(s)>1$. Moreover, for $\Re(s)>1$, we have the Euler product expansion:

$$\xi(s;F,\chi,\mathbf{c},\mathbf{k})=\mathfrak{S}'_{\mathbf{c},\mathbf{k}}(s)\prod_{p\nmid ck}\left(1+\frac{\chi(p)}{p^s}\cdot(\varrho_F((p,1,...,1))+\ldots+\varrho_F((1,...,p)))+\frac{\chi(p)^2}{p^{2s}}\cdot\ldots\right),$$

where we define:

$$\mathfrak{S}'_{\mathbf{c},\mathbf{k}}(s):=\prod_{\substack{p|ck\\ p^{e_i}\|c_ik_i}}\left(\varrho_F((p^{e_1},...,p^{e_r}))+\frac{\chi(p)}{p^s}\cdot(\varrho_F((p^{e_1+1},...,p^{e_r}))+\ldots+\varrho_F((p^{e_1},...,p^{e_r+1})))+\frac{\chi(p)^2}{p^{2s}}\cdot\ldots\right).$$

Note that this product is over a finite number of primes (depending on $\mathbf{c}$ and $\mathbf{k}$) and that the local factors at $p\nmid ck$ match those in the Euler product expansion of $\xi(s;F,\chi)$ (as defined in (3.3.5)).

Thus, we get that for $\Re(s) > 1$,

$$\xi(s; F, \chi, \mathbf{c}, \mathbf{k}) = \mathfrak{S}_{\mathbf{c},\mathbf{k}}(s) \cdot \xi(s; F, \chi),$$

where we take:

$$\mathfrak{S}_{\mathbf{c},\mathbf{k}}(s) := \mathfrak{S}'_{\mathbf{c},\mathbf{k}}(s) \cdot \prod_{p|ck} \left(1 + \frac{\chi(p)}{p^s} \cdot (\varrho_F((p, 1, ..., 1)) + \ldots + \varrho_F((1, ..., p))) + \frac{\chi(p)^2}{p^{2s}} \cdot \ldots\right)^{-1}.$$

By Lemma 3.3.15, we can see that $\xi(s; F, \chi, \mathbf{c}, \mathbf{k})$ will have a meromorphic continuation to $\Re(s) > 1/2$ and a pole of order $\rho_{\Delta,f} - 2$ at $s = 1$. Thus, we have that:

$$V(\mathbf{c}, \mathbf{k}, D) = \sum_{\mathbf{d} \leq \mathbf{D}} \frac{\chi(d)\varrho_F(\mathbf{ckd})}{d} = \mathfrak{S}_{\mathbf{c},\mathbf{k}}(1)\mathrm{Res}_{s=1}\xi(s; F, \chi) \cdot \log(D)^{\rho_{\Delta,f}-2}(1 + O(\log(D)^{-1})).$$

Recalling that $D = (B/tk)^{\deg(F)/4}k^{-1/2}\log(B)^{-\varepsilon}$, where $\varepsilon = 10^{-5}$ if there are no linear factors and $\varepsilon = -1 - 10^{-5}$ if F has linear factors, we establish the claim (3.11.1) for a fixed value of k and $\mathbf{c}$:

$$V(\mathbf{c}, \mathbf{k}, D) = c_{q_1,q_2,f,k,\mathbf{c}} \log(B/tk)^{\rho_{\Delta,f}-2}(1 + O(\log(B)^{-1/2})),$$

where $c_{q_1,q_2,f,k,\mathbf{c}} = \mathfrak{S}_{\mathbf{c},\mathbf{k}}(1) \cdot \mathrm{Res}_{s=1}\xi(s; F, \chi) \cdot (\deg(F)/4)^{\rho_{\Delta,f}-2}$. □

3.11.2 Proof of Proposition 3.5.11

Let us first recall the statement of Proposition 3.5.11.

Proposition. *Let us assume that $L \gg \log(B)^{-10^{-10}}$. There exists a constant $C_{q_1,q_2,f}$ such that as $B \to \infty$, we have that*

$$M_{q_1,q_2}(X_{\Delta,f}, \mathcal{B}(\mathbf{x}_0, L)) = C_{q_1,q_2,f}L^2 \cdot B\log(B)^{\rho_{\Delta,f}-1}(1 + O(\log(B)^{-10^{-8}})).$$

In this subsection, we will explain how to boost the estimate of Proposition 3.8.1 to Proposition 3.5.11 by summing over t and k. Additionally, we shall keep track of our leading constant to be plugged into the analysis of §3.12.

Proof. Recall that

$$M_{q_1,q_2}(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L)) = \sum_{\substack{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{k},\bar{\mathfrak{k}})=1}} \mu(\mathfrak{k}) \sum_{\substack{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} S(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L);q_1,q_2,N(\mathfrak{t}),N(\mathfrak{k})).$$

Applying Proposition 3.8.1, we can see that this is given by:

$$\begin{aligned}
M_{q_1,q_2}(X_{\Delta,f},\mathcal{B}(\mathbf{x}_0,L)) &= BL^2 \sum_{\substack{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{k},\bar{\mathfrak{k}})=1}} \frac{\mu(\mathfrak{k})}{N(\mathfrak{k})^2} \sum_{\substack{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} C_{q_1,q_2,f,N(\mathfrak{k})}(B/N(\mathfrak{tk})) \\
&\quad + O\left(BL^2\log(B)^{\rho_{\Delta,f}-2-10^{-7}} \sum_{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})} \frac{1}{N(\mathfrak{k})^2} \sum_{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})} \frac{1}{N(\mathfrak{t})} \right) \\
&\quad + O\left(B\log(B)^{\rho_{\Delta,f}-2-10^{-7}} \sum_{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})} \frac{(8\deg(F))^{\omega(k)}}{N(\mathfrak{k})^{3/2}} \sum_{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})} \frac{1}{t} \right) \\
&+ O\left(B^{3/4}\log(B)^{-10^{-7}} \sum_{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})} \frac{L+(4\deg(F))^{\omega(N(\mathfrak{k}))}\nu_1(N(\mathfrak{k}))^{1/2}}{N(\mathfrak{k})^{3/4}} \sum_{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})} \frac{1}{N(\mathfrak{t})^{3/4}} \right).
\end{aligned}$$

First, let us bound away the error terms. Since we know that

$$\sum_{N(\mathfrak{a})\leq X} \frac{1}{N(\mathfrak{a})} \ll_\Delta \log(X), \sum_{N(\mathfrak{a})\leq X} \frac{C^{\omega(N(\mathfrak{a}))}}{N(\mathfrak{a})^\alpha} \ll_\Delta 1$$

for any $\alpha > 1$ and any $C \in \mathbb{N}$, we can see that the first and second terms are both bounded by $B\log(B)^{\rho_{\Delta,f}-1-10^{-7}}$.

For the final error term, we see that:

$$\begin{aligned}
&\sum_{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})} \frac{L+(4\deg(F))^{\omega(N(\mathfrak{k}))}\nu_1(N(\mathfrak{k}))^{1/2}}{N(\mathfrak{k})^{3/4}} \sum_{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)})} \frac{1}{N(\mathfrak{t})^{3/4}} \\
&\quad \ll_\Delta \sum_{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)})} \frac{L+(4\deg(F))^{\omega(N(\mathfrak{k}))}\nu_1(N(\mathfrak{k}))^{1/2}}{N(\mathfrak{k})^{3/4}} \cdot \left(\frac{B}{N(\mathfrak{k})}\right)^{1/4} \\
&\quad \ll_\Delta B^{1/4}L\cdot\log(B) + B^{1/4} \sum_{k\leq B/\exp(\sqrt{\log(B)})} \frac{(4\deg(F))^{\omega(k)}\nu_1(k)^{1/2}}{k} \cdot \#\{\mathfrak{k} : N(\mathfrak{k}) = k\}.
\end{aligned}$$

Observe that $\#\{\mathfrak{k} : N(\mathfrak{t}) = k\} \ll 2^{\omega(k)}$. On the other hand, we have that

$$\sum_{k \leq B/\exp(\sqrt{\log(B)})} \frac{\nu_1(k)^{1/2}}{k} \cdot \#\{\mathfrak{k} : N(\mathfrak{k}) = k\} \leq \left(\sum_{k \leq B/\exp(\sqrt{\log(B)})} \frac{1}{k} \right)^{1/2} \left(\sum_{k \leq B/\exp(\sqrt{\log(B)})} \frac{\nu_1(k) 4^{\omega(k)}}{k} \right).$$

The first term above is bounded by $\log(B)^{1/2}$. Expanding out the second term, we have

$$\sum_{k \leq B/\exp(\sqrt{\log(B)})} \frac{\nu_1(k)(32\deg(F)^2)^{\omega(k)}}{k} \leq \sum_{k \leq B} \frac{(32\deg(F)^2)^{\omega(k)}}{k} \sum_{\alpha \in \mathcal{U}(k)} \frac{1}{\lambda_1(\Lambda(\alpha))}.$$

Since we know that

$$\sum_{k \leq B} \sum_{\alpha \in \mathcal{U}(k)} \frac{1}{\lambda_1(\Lambda(\alpha))} \ll \sum_{0 \neq |\mathbf{v}| \leq \sqrt{B}} \frac{\#\{k \leq B : k \mid F(\mathbf{v})\}}{|\mathbf{v}|} \ll \sqrt{B}\log(B),$$

by partial summation the second term is bounded absolutely. In summary, we have that

$$\sum_{k \leq B/\exp(\sqrt{\log(B)})} \frac{\nu_1(k)^{1/2}}{k} \cdot \#\{\mathfrak{k} : N(\mathfrak{k}) = k\} \ll \log(B)^{1/2}.$$

Thus, this final error term is again upper bounded by $B\log(B)^{\rho_{\Delta,f}-1-10^{-7}}$. For all of the error terms above, we recall that $L \gg \log(B)^{-10^{-10}}$.

It remains to analyze the main term by expanding the definition of $C_{q_1,q_2,f,N(\mathfrak{k})}$:

$$BL^2 \cdot \sum_{c_{ij} | \mathrm{Res}(F_i,F_j)} \frac{\mu(\mathbf{c})\chi(\mathbf{c})}{c} \cdot \bigtriangledown_{q_1,q_2}(\mathbf{c}) \sum_{\substack{N(\mathfrak{k}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{k},\bar{\mathfrak{k}})=1}} \frac{\mu(\mathfrak{k})}{N(\mathfrak{k})^2} \sum_{\substack{N(\mathfrak{t}) \leq B/k\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} V(\mathbf{c},\mathbf{k},D).$$

Here we use $\mu(\mathbf{c})\chi(\mathbf{c})$ to denote the appropriate product depending on the factorization of $F(x,y)$ (as written down explictly in §3.11.1). Let us expand out the definition of $V(\mathbf{c},\mathbf{k},D)$ and change

the order of summation (using $\mathfrak{t}' = \mathfrak{t}\mathfrak{k}$):

$$\sum_{\substack{N(\mathfrak{k})\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{k},\bar{\mathfrak{k}})=1}} \frac{\mu(\mathfrak{k})}{N(\mathfrak{k})^2} \sum_{\substack{N(\mathfrak{t})\leq B/k\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} V(\mathbf{c},\mathbf{k},D)$$
$$= \sum_{\substack{N(\mathfrak{t}')\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t}',\bar{\mathfrak{t}'})=1}} \frac{1}{N(\mathfrak{t}')} \sum_{\mathfrak{k}|\mathfrak{t}'} \frac{\mu(\mathfrak{k})}{N(\mathfrak{k})} \sum_{\mathbf{d}\leq (B/N(\mathfrak{t})')^{\deg(F)/4}} \frac{\chi(d)\varrho_F(\mathbf{ckd})}{d}.$$

Note that since $k = N(\mathfrak{k}) \mid t = N(\mathfrak{t})$, we know that if $p \mid k$ then $\chi(p) = 1$; hence, $\chi(k) = 1$. For a fixed $t = N(\mathfrak{t})$, we can see that the relevant Dirichlet series to look at is:

$$\xi(s;F,\chi,\mathbf{c},t) := \sum_{\mathbf{d}} \frac{\chi(kd)\varrho_F(\mathbf{ckd})}{d^s} \sum_{k|t} \frac{\mu(k)}{k^s} = \sum_{\gcd(d_i,t)=1} \frac{\chi(d)\varrho_F(\mathbf{cd})}{d^s}.$$

We want to establish a relationship between this Dirichlet series and $\xi(s;F,\chi)$, which was studied in §3.3.5. First, this Dirichlet series will converge for $\Re(s) > 1$ and has the Euler product expansion:

$$\xi(s;F,\chi,\mathbf{c},t) = \mathfrak{S}'_{\mathbf{c},t}(s) \cdot \prod_{p\nmid ct} \left(1 + \frac{\chi(p)}{p^s} \cdot (\varrho_F((p,1,...,1)) + \ldots + \varrho_F((1,...,p))) + \frac{\chi(p)^2}{p^{2s}} \cdot \ldots\right),$$

where we define:

$$\mathfrak{S}'_{\mathbf{c},t}(s) = \prod_{\substack{p^{e_i}\| c_i \\ p\nmid t}} \left(\varrho_F((p^{e_1},...,p^{e_r})) + \frac{\chi(p)}{p^s} \cdot (\varrho_F((p^{e_1+1},...,p^{e_r})) + \ldots + \varrho_F((p^{e_1},...,p^{e_r+1}))) + \frac{\chi(p)^2}{p^{2s}} \cdot \ldots\right).$$

Thus, we have that for $\Re(s) > 1$,

$$\xi(s;F,\chi,\mathbf{c},t) = \mathfrak{S}_{\mathbf{c},t}(s) \cdot \xi(s;F,\chi)$$

where we define:

$$\mathfrak{S}_{\mathbf{c},t}(s) := \mathfrak{S}'_{\mathbf{c},t}(s) \cdot \prod_{p|ct} \left(1 + \frac{\chi(p)}{p^s} \cdot (\varrho_F((p,1,...,1)) + \ldots + \varrho_F((1,...,p))) + \frac{\chi(p)^2}{p^{2s}} \cdot \ldots\right)^{-1}.$$

Let us define:

$$R(e_1,...,e_r) = \sum_{f_i\geq e_i} \frac{\chi(p^{f-e})}{p^{f-e}} \varrho_F(p^{f_1},...,p^{f_r}),$$

We can see that

$$\mathfrak{S}_{\mathbf{c},t}(s) = \prod_{p^{e_i}\|c_i} \frac{R(e_1,...,e_r)}{R(0,...,0)} \prod_{\substack{p|t \\ p\nmid c}} R(0,...,0)^{-1}.$$

By Lemma 3.3.15, $\xi(s;F,\chi,\mathbf{c},t)$ has a meromorphic continuation to $\Re(s) > 1/2$ and a pole of order $\rho_{\Delta,f}-2$ at $s=1$. So, we can write (3.11.2) as

$$BL^2 \cdot \sum_{c_{ij}|\mathrm{Res}(F_i,F_j)} \frac{\mu(\mathbf{c})\chi(\mathbf{c})}{c} \cdot \bigtriangledown_{q_1,q_2}(\mathbf{c}) \cdot \sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})}$$
$$\times d_{\Delta,f} \cdot \mathfrak{S}_{\mathbf{c},N(\mathfrak{t})}(1) \cdot \mathrm{Res}_{s=1}\xi(s;F,\chi) \cdot (\log(B/N(\mathfrak{t}))^{\rho_{\Delta,f}-2} + O(\log(B)^{-1/2})),$$

where $d_{\Delta,f} = (\deg(F)/4)^{\rho_{\Delta,f}-2}$. Alternatively, we can also conclude at this point that

$$C_{q_1,q_2,f} = \sum_{\mathbf{c}} \frac{\mu(\mathbf{c})\chi(\mathbf{c})}{c} \cdot \bigtriangledown_{q_1,q_2}(\mathbf{c}) \sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \cdot \mathrm{Res}_{s=1} \sum_{\gcd(d_i,t)=1} \frac{\chi(d)\varrho_F(\mathbf{cd})}{d^s},$$

The above expression is the one that will become relevant in §3.12.

However, to achieve Proposition 3.5.11, it remains to complete the sum over $\mathfrak{t}$. Next, consider the Dirichlet series:

$$\gamma(s;\mathbf{c}) := \sum_{\gcd(\mathfrak{t},\bar{\mathfrak{t}})=1} \frac{\mathfrak{S}_{\mathbf{c},N(\mathfrak{t})}(1)}{N(\mathfrak{t})^s} = \prod_{p^{e_i}\|c_i} \frac{R(e_1,...,e_r)}{R(0,...,0)} \cdot \sum_{\gcd(\mathfrak{t},\bar{\mathfrak{t}})=1} N(\mathfrak{t})^{-s} \prod_{\substack{p|t \\ p\nmid c}} R(0,...,0)^{-1}$$
$$= \cdot \prod_{p^{e_i}\|c_i} \frac{R(e_1,...,e_r)}{R(0,...,0)} \cdot \sum_t t^{-s} \prod_{\substack{p|t \\ p\nmid c}} R(0,...,0)^{-1} \cdot \#\{N(\mathfrak{t}) = t : \gcd(\mathfrak{t},\bar{\mathfrak{t}}) = 1\}.$$

While this expression looks complicated, in order to establish the analytic behavior at $s=1$, we only need to establish a few properties. First, since $\mathbf{c}$ is a fixed vector, the behavior at the finitely many primes $p \mid c$ will not change the analytic properties of the above function. Second, we can determine the order of the pole at $s=1$ by considering the linear terms:

$$\gamma(s;\mathbf{c}) = \mathfrak{S}_{\mathbf{c}}(s) \prod_{\chi(p)=1} \left(1 + 2p^{-s}\left(1 + \frac{1}{p}\cdot(\varrho_F((p,1,...,1)) + ... + \varrho_F((1,...,p))) + \dots\right)^{-1} + \dots\right)$$

Here we have used that if $p \mid t$ then $\chi(p) = 1$. So, we can see from the linear terms that $\gamma(s;\mathbf{c})$ has a pole of order one at $s=1$.

Consequently, we have that by partial summation

$$\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{\mathfrak{S}_{\mathbf{c},N(\mathfrak{t})}(1)}{N(\mathfrak{t})}\log(B/N(\mathfrak{t}))^{\rho_{\Delta,f}-2} = \sigma_{\mathbf{c}}\log(B)^{\rho_{\Delta,f}-1}(1+O(\log(B)^{-1/2})).$$

Here $\sigma_{\mathbf{c}}$ is a constant only depending on $\mathbf{c}$, F, and K. This concludes the proof of the proposition with

$$C_{q_1,q_2,f} = (d_{\Delta,f}\cdot \mathrm{Res}_{s=1}\xi(s;F,\chi)) \sum_{c_{ij}|\mathrm{Res}(F_i,F_j)} \frac{\mu(\mathbf{c})\chi(\mathbf{c})\sigma_{\mathbf{c}}}{c}\cdot \bigtriangledown_{q_1,q_2}(\mathbf{c}).$$

□

3.12 Analysis of the leading constant

Finally, we want to establish that our leading constant $c_{\Delta,f}$ is zero if and only if there is a local or Brauer-Manin obstruction to rational points on $X_{\Delta,f}$. We can see from the reductions of section §3.5 that it suffices to classify when

$$\sum_{q_1q_2=-\Delta} C_{q_1,q_2,f} = 0.$$

In particular, we want to deduce exactly when the following expression is zero:

$$\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \sum_{q_1q_2=-\Delta}\sum_{c_{ij}|\mathrm{Res}(F_i,F_j)} \frac{\mu(\mathbf{c})\chi(\mathbf{c})}{c}\cdot \bigtriangledown_{q_1,q_2}(\mathbf{c}) \times \mathrm{Res}_{s=1}\sum_{\substack{\mathbf{d}\\ \gcd(d_i,t)=1}} \frac{\chi(d)\varrho_F(\mathbf{cd})}{d^s}$$

Here we use $\mathrm{Res}_{s=1}$ to denote the first nonzero coefficient in the Laurent series expansion at $s=1$. We will begin our analysis gradually by starting with the combinatorially simplest case: when $f(z)$ is irreducible. In this case, it is known by [14] that $X_{\Delta,f}$ satisfies the Hasse principle; we aim for the stronger result that if there are no local obstructions to $X_{\Delta,f}$ then there are $\asymp B\log(B)^{\rho_{\Delta,f}-2}$ rational points of height at most B.

Second, we introduce the auxiliary varieties that are inherent to our analysis – this comes to the sum over c_{ij}. These varieties also appear in the finite and effective computation of [10] and [12] for determining if $X_{\Delta,f}(\mathbb{Q})=\emptyset$. In this subsection, we also establish that if $f(z)$ has a linear factor then there exists an auxiliary variety containing a rational point.

Finally, we will classify when $c_{\Delta,f}=0$ by handling the cases when $f(z)$ factorizes. This will include the case when $f(z)$ factorizes as a product of two quadratics – the only scenario when the

Hasse principle may fail.

3.12.1 The irreducible case

Let $f(z)$ be an irreducible polynomial. Then our expression (3.12) becomes:

$$\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \sum_{q_1 q_2 = -\Delta} \bigtriangledown_{q_1,q_2}(1) \cdot \mathrm{Res}_{s=1} \prod_{p \nmid t} \left(1 + \frac{\chi(p)\varrho_F(p)}{p^s} + \dots\right).$$

Since $\bigtriangledown_{q_1,q_2}(1)$ definitionally is independent of $\mathfrak{t}$, we can move this sum to the outside. In Lemma 3.12.12, we find that

$$\bigtriangledown(1) = \sum_{q_1 q_2 = -\Delta} \bigtriangledown_{q_1,q_2}(1) \neq 0$$

if and only if for all $p \mid -\Delta$, $X_{\Delta,f}(\mathbb{Q}_p) \neq \emptyset$. In other words, $\bigtriangledown(1)$ detects the local obstructions for primes dividing $-\Delta$.

Next, we consider

$$\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \cdot \mathrm{Res}_{s=1} \prod_{p \nmid t} \left(1 + \frac{\chi(p)\varrho_F(p)}{p} + \dots\right)$$

We claim that the above is zero if and only if there is a local obstruction to $X_{\Delta,f}$ at a prime $p \nmid -\Delta$.

Note that at all but finitely many primes p, $\varrho_F(p) = \varrho_f(p)$. Since f is irreducible, $\varrho_f(p)$ is also the number of ideals in $\mathbb{Q}[z]/f(z)$ of norm p. So, at all primes p not dividing the discriminant D_f of $f(z)$, the local factor matches the local factor of $L_{\mathbb{Q}[z]/f(z)}(s,\chi)$, where we recall that

$$L_{\mathbb{Q}[z]/f(z)}(s,\chi)\zeta_{\mathbb{Q}[z]/f(z)}(s) = \zeta_{K[z]/f(z)}(s).$$

Consequently, we know that for $\Re(s) > 1$

$$\prod_p \left(1 + \frac{\chi(p)\varrho_F(p)}{p^s} + \dots\right) = \prod_{p \mid D_f} \left(1 + \frac{\chi(p)\varrho_F(p)}{p^s} + \dots\right) \cdot c_f L_{\mathbb{Q}[z]/f(z)}(s,\chi),$$

where c_f is a nonzero constant (formed by the product of local factors of $L_{\mathbb{Q}[z]/f(z)}(s,\chi)$ for primes dividing D_f).

Thus, we have that (3.12.1) becomes

$$c_f \cdot \mathrm{Res}_{s=1} L_{\mathbb{Q}[z]/f(z)}(s,\chi) \prod_{p \mid D_f} \left(1 + \frac{\chi(p)\varrho_F(p)}{p} + \dots\right) \sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \prod_{p \mid t} \left(1 + \frac{\varrho_F(p)}{p} + \dots\right)^{-1}.$$

Here we have used that if $p \mid t$ then we must have that $\chi(p) = 1$. Let us note that

$$\mathrm{Res}_{s=1} L_{\mathbb{Q}[z]/f(z)}(s,\chi) \neq 0,$$

since if $K \not\subset \mathbb{Q}[z]/f(z)$ then the series converges and $L_{\mathbb{Q}[z]/f(z)}(1,\chi) > 0$. If $K \subset \mathbb{Q}[z]/f(z)$, then this is the residue of a simple pole and is nonzero.

Now, we wish to determine when these local factors $p \mid D_f$ can be zero.

Lemma 3.12.1. *We have that*

$$1 + \frac{\chi(p)\varrho_F(p)}{p} + \frac{\chi(p)^2\varrho_F(p^2)}{p^2} + \dots = 0$$

if and only if $\chi(p) = -1$, p^e divides the coefficients of f for some odd integer e, and $\varrho_F(p^{e+1}) = 0$. In this case, $X_{\Delta,f}(\mathbb{Q}_p) = \emptyset$.

Proof. First, if $\chi(p) \in \{0,1\}$, then the sum above is clearly strictly positive and therefore can not be zero. So, we can assume that $\chi(p) = -1$. Second, for all but finitely many primes p, we have that $\varrho_F(p^k) \leq 4$ for all powers k. Hence, for $p > 4$ and f not being identically zero in $\mathbb{F}_p$, the discriminant of f, the above is bounded below by a geometric series and also strictly positive and thus nonzero.

It remains to deal with small primes and those dividing the coefficients of f. Next, we note that if p divides the coefficients of f, then we know that both $\varrho_F(p) = p$ and that there must exist some power e where $p^e \| F$ (so, $\varrho(p^{e+1}) < p^{e+1}$). For $k \geq e+1$, we then have that $\varrho_F(p^k) \leq \varrho_F(p^{e+1})$. On the other hand,

$$\sum_{i=0}^{e} \frac{\chi(p^i)\varrho_F(p^i)}{p^i} = \sum_{i=0}^{e} (-1)^i \in \{0,1\}.$$

If $p > 4$ and $\varrho_F(p^{e+1}) > 0$ or e is even, then again we can bound the sum below by a geometric series and get a strictly positive value. Otherwise, if $\varrho_F(p^{e+1)}) = 0$, $p > 4$, and e is odd, then the local factor is zero. In this case, F must be of the form $p^e\tilde{F}$, where $\tilde{F}$ is an integral binary form satisfying $\varrho_{\tilde{F}}(p) = 0$.

Finally, we want to show that these conditions are equivalent to when $X_{\Delta,f}(\mathbb{Q}_p) = \emptyset$. Since

$\chi(p) = -1$, we can see that if e is odd then the equation

$$x^2 + \Delta y^2 = p^e \tilde{F}(u, v)$$

can have a solution if and only if we have that $p \mid \tilde{F}(u, v)$ for some nontrivial (u, v). In other words, $\varrho_{\tilde{F}}(p) > 0$. Hence, $X_{\Delta,f}(\mathbb{Q}_p) = \emptyset$ if and only if $p^e \| F$ for some odd integer e and $\varrho_F(p^{e+1}) = 0$.

Finally, we need to handle the small primes $p = 2, 3$. If $\chi(p) = -1$, $\varrho_F(p^{e+1}) = 0$, and e is odd, we again get that there can be a local obstruction as above – in these cases, we can see similarly see that this condition is equivalent to $X_{\Delta,f}(\mathbb{Q}_p) = \emptyset$. Otherwise, if e is even or if $\varrho_F(p^{e+1}) > 0$, we again bound the sum with a geometric series. However, we replace the inequality $\varrho_F(p^{e+1}) \leq 4p^e$ with $\varrho_F(p^{e+1}) \leq p^{e+1}$ to achieve the lower bound.

□

We must also discuss the sum over the ideals $\mathfrak{t}$.

Lemma 3.12.2.

$$\sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t}, \bar{\mathfrak{t}}) = 1}} \frac{1}{N(\mathfrak{t})} \prod_{p \mid t} \left(1 + \frac{\varrho_F(p)}{p} + \ldots\right)^{-1} = c \log(B)(1 + o(1)),$$

where $c \neq 0$.

Proof. Consider the Dirichlet series for $\Re(s) > 1$:

$$D(s) = \sum_{\mathfrak{t}} N(\mathfrak{t})^{-s} \cdot \prod_{p \mid t} \left(1 + \frac{\varrho_F(p)}{p} + \ldots\right)^{-1} = \prod_{\mathfrak{p}} \left(1 + \frac{1}{N(\mathfrak{p})^s} + O\left(\frac{1}{N(\mathfrak{p})^{1+s}} + \frac{1}{N(\mathfrak{p})^{2s}}\right)\right).$$

We can see that this series has a simple pole at $s = 1$, and thus

$$\sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t}, \bar{\mathfrak{t}}) = 1}} \frac{1}{N(\mathfrak{t})} \cdot \prod_{p \mid t} \left(1 + \frac{\chi(p)\varrho_F(p)}{p} + \ldots\right)^{-1} = \zeta(2)^{-1} \mathrm{Res}_{s=1} D(s) \cdot \log(B)(1 + o(1)).$$

In particular, we can see that the sum over $\mathfrak{t}$ is never zero (since the residue of the simple pole of $D(s)$ at $s = 1$ is nonzero). □

Returning to (3.12), we have that the sum is now of the form:

$$\bigtriangledown(1) \cdot \prod_{p \mid D_f} \left(1 + \frac{\chi(p)\varrho_F(p)}{p} + \ldots\right) \cdot \left(c_f \cdot \mathrm{Res}_{s=1} L_{\mathbb{Q}[z]/f(z)}(s, \chi)\right) \cdot \left(\zeta(2)^{-1} \mathrm{Res}_{s=1} D(s)\right) \cdot \log(B)(1 + o(1)).$$

We saw before that $\bigtriangledown(1) = 0$ if and only if for some $p \mid -\Delta$, $X_{\Delta,f}(\mathbb{Q}_p) = \emptyset$. Similarly, by Lemma 3.12.1 the product over primes dividing D_f is zero if and only if for $p \mid D_f$ with $\chi(p) = -1$, we have that $X_{\Delta,f}(\mathbb{Q}_p) = \emptyset$ (see Lemma 3.12.10). All of the other factors above have been shown to be nonzero. By Lemma 3.12.10, these are also all of the possible local obstructions to $X_{\Delta,f}$ when $f(z)$ is irreducible.

Thus, we have proved the following result.

Proposition 3.12.3. *Let $f(z)$ be irreducible. Then the leading constant $c_{\Delta,f}$ is zero if and only if there is a local obstruction for $X_{\Delta,f}$. Consequently, the Hasse principle holds for $X_{\Delta,f}$.*

3.12.2 Descent and auxiliary varieties

To analyze (3.12) when $f(z)$ factorizes, we must first introduce the auxiliary varieties $X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}$ (defined in (3.1)). Fortunately, these auxiliary varieties are well-studied and appear as crucial objects in Colliot-Thélène, Coray, and Sansuc's work on the Brauer-Manin obstruction and descent. Specifically, in a sequence of papers [11, 13], Colliot-Thélène and Sansuc established the method of descent. In our proof, we would like to use the fibrations discussed by Colliot-Thélène, Coray, and Sansuc in [10] when $f(z)$ factorizes as the product of two irreducible quadratics. It is of note that in the proof of [12, Proposition 3] it is established that a similar set of auxiliary surfaces are intimately tied to these questions; observe that in [12, Proposition 3], these varieties are denoted as W_g and take into account extra homogenization when $f(z)$ has an odd degree irreducible factor.

Let us recall from (3.1) the definition of our auxiliary varieties:

$$X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r} = \{(z,(x_i,y_i)_{i=1}^r : x_i^2 + \Delta y_i^2 = \alpha_i f_i(z) \neq 0\}.$$

In the next subsection, we describe how $X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}$ relates to $X_{\Delta,f}$ in many ways. Most importantly is the following consequence of Harari's "formal lemma" [32, Corollary 2.6.1] (see also [16, Theorem 13.4.3] and the example [16, p. 343]) and was demonstrated to the author by Colliot-Thélène in [9]. In [10] and [12], certain cases are also shown under certain degree parity conditions.

Theorem 3.12.4 (Colliot-Thélène [9])**.** *Assume that $X_{\Delta,f}(\mathbb{Q}_v) \neq \emptyset$ for every place v and there is no Brauer-Manin obstruction to the Hasse principle for $X_{\Delta,f}$. Then there exists an $(\alpha_1,\ldots,\alpha_r) \in (\mathbb{Q}^\times)^r$ satisfying $\alpha_1\ldots\alpha_r = 1$, $X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}_v) \neq \emptyset$ for all places v.*

Proof. By assumption, let $(x_v,y_v,z_v) \in X_{\Delta,f}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}(X)}$; we can furthermore assume that $(x_v,y_v,z_v) \in X^*_{\Delta,f}(\mathbb{A}_\mathbb{Q})$. Let $U_{\Delta,f}$ denote the smooth compactification of $X^*_{\Delta,f}$ and hence $U_{\Delta,f} \subset X_{\Delta,f}$. Then

let S denote the finite set of places of $\mathbb{Q}$ such that

$$S := \{v : \exists \gamma \in \mathrm{Br}(U_{\Delta,f}), \gamma(x_v, y_v, z_v) \neq 0\}.$$

Let $\beta_i = (-\Delta, f_i) \in \mathrm{Br}(U_{\Delta,f})$ denote an element of the Brauer group of $U_{\Delta,f}$, for each $i = 1, \ldots, r-1$. Then Harari's formal lemma [16, Theorem 13.4.3] gives an extension of $(x_v, y_v, z_v)_{v\in S}$ to an adelic point $(M_v)_{v\in\mathscr{P}\cup\{\infty\}}$ such that

$$\sum_{v\in\mathscr{P}\cup\{\infty\}} \beta_i(M_v) = 0, \forall i = 1, \ldots, r-1.$$

Note that Harari's formal lemma specifically applies to those $\beta_i \in \mathrm{Br}(U_{\Delta,f})\backslash\mathrm{Br}(X_{\Delta,f})$; however, this equality trivially holds for any element of $\mathrm{Br}(X_{\Delta,f})$ since $(x_v, y_v, z_v) \in X_{\Delta,f}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}(X)}$.

Next, we consider the exact sequence

$$1 \to \mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times) \to \oplus_{v\in\mathscr{P}\cup\{\infty\}}\mathbb{Q}_v^\times/N_{K_v/\mathbb{Q}_v}(K_v^\times) \xrightarrow{\gamma} \mathbb{Z}/2\mathbb{Z} \to 1.$$

Consider for $i \in \{1, \ldots, r-1\}$ the map

$$(M_v) \mapsto (f_i(z_v)) \in \oplus_{v\in\mathscr{P}\cup\{\infty\}}\mathbb{Q}_v^\times/N_{K_v/\mathbb{Q}_v}(K_v^\times).$$

Since we know that

$$\gamma((f_i(z_v))_v) = \sum_{v\in\mathscr{P}\cup\{\infty\}} \beta_i(M_v) = 0 \in \mathbb{Z}/2\mathbb{Z},$$

there must exist an element $\alpha_i \in \mathbb{Q}^\times$ such that

$$f_i(z_v) = \alpha_i N_{K/\mathbb{Q}}(x_i + y_i\sqrt{-\Delta}) \neq 0.$$

Furthermore, the above holds simultaneously for all $\beta_1, \ldots, \beta_{r-1}$, so we must have that the auxiliary variety

$$\begin{gathered} x_1^2 + \Delta y_1^2 = \alpha_1 f_1(z) \neq 0 \\ \vdots \\ x_{r-1}^2 + \Delta y_{r-1}^2 = \alpha_{r-1} f_{r-1}(z) \neq 0 \\ x_r^2 + \Delta y_r^2 = (\alpha_1 \cdots \alpha_{r-1})^{-1} f_r(z) \neq 0 \end{gathered}$$

contains an adelic point; in other words, for $\alpha_r = (\alpha_1 \cdots \alpha_{r-1})^{-1}$, we have that $(M_v)_{v\in\mathscr{P}\cup\{\infty\}} \in$

$X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{A}_{\mathbb{Q}})$. □

Remark 3.12.5. We note that for any point $(z,(x_i,y_i)_{i=1}^r) \in X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q})$, we can create a point in $X_{\Delta,f}(\mathbb{Q})$ given by

$$x+y\sqrt{-\Delta} = \prod_{i=1}^{r}(x_i+y_i\sqrt{-\Delta}),\ z=z.$$

Thus, if the family of varieties $X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}$ satisfies the Hasse principle, Theorem 3.12.4 gives that the Brauer-Manin obstruction is the only possible obstruction to the Hasse principle. In fact, this is exactly the approach Colliot-Thélène, Coray, and Sansuc [10] used to prove that when $f = f_1(z)f_2(z)$ is the product of two irreducible quadratics the only possible obstruction to the Hasse principle is the Brauer-Manin obstruction. Moreover, in [12, Proposition 3 and 4], Colliot-Thélène and Sansuc used this strategy with a variation of the auxiliary varieties to prove that under the assumption of Schinzel's Hypothesis H, the only possible obstruction to the Hasse principle for Châtelet surfaces is the Brauer-Manin obstruction.

However, in the resolution of the question of the Hasse principle for Châtelet surfaces, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [14][15], use a different parameterization of the rational points on $X_{\Delta,f}$ – this parameterization is given by torsors. This allows them to approach the hardest case, when $f(z)$ is irreducible, which is circuitous with the auxiliary varieties above. Fortunately for us, this case when $f(z)$ is irreducible is actually the clearest example for our argument, as demonstrated in the previous subsection.

Next, we make some more observations on connections between rational points on $X_{\Delta,f}$ and those on $X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}$.

Lemma 3.12.6. *Assume that $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$. Then there exists $(\alpha_1,\ldots,\alpha_r) \in (\mathbb{Q}^\times)^r$ such that*

$$X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}) \neq \emptyset.$$

Proof. First note that if $X_{\Delta,f}(\mathbb{Q}) \neq \emptyset$, then $X_{\Delta,f}$ must be unirational (see [15, Proposition 9.8.(iii)] and [17, Theorem A]) and hence we have that $X_{\Delta,f}(\mathbb{Q})$ is Zariski-dense; so, $X^*_{\Delta,f}(\mathbb{Q}) \neq \emptyset$. Assume that for some $(x_0,y_0,z_0) \in \mathbb{Q}$ we have that $x_0^2+\Delta y_0^2 = f(z_0) \neq 0$. Then set $\alpha_i = f_i(z_0) \neq 0$ for each i and consider

$$x_i^2+\Delta y_i^2 = \alpha_i f_i(z).$$

This variety clearly has a rational point at $z=z_0$ for each i. Thus, $X^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}) \neq \emptyset$. □

For the remainder of the section, we aim to prove the following proposition:

Proposition 3.12.7. *Let $\Delta \in \mathbb{Z}$ be a squarefree integer satisfying that $\sqrt{-\Delta} \notin \mathbb{Q}$. Let $f(z) \in \mathbb{Z}[z]$ be a separable polynomial of degree 3 or 4. Let $X_{\Delta,f}$ be the corresponding Châtelet surface. Then as $B \to \infty$,*

$$N(X_{\Delta,f}, B) = c_{\Delta,f} B \log(B)^{\rho_{\Delta,f}-1} + O(B \log(B)^{\rho_{\Delta,f}-1-10^{-10}})$$

with $c_{\Delta,f}$ nonzero if and only if there exists an $(\alpha_1, ..., \alpha_r) \in (\mathbb{Q}^\times)^r$ such that $\alpha_1 ... \alpha_r = 1$ and $X^*_{\Delta,f_1,...,f_r,\alpha_1,...,\alpha_r}(\mathbb{Q}_v) \neq \emptyset$ for all places v.

By Theorem 3.12.4, this tells us that $c_{\Delta,f} = 0$ if and only if there is either a local or Brauer-Manin obstruction to rational points on $X_{\Delta,f}$.

Finite classes of torsors

Now, we show that we actually only need to consider a finite set of values $(\alpha_1, ..., \alpha_r)$ to understand $X_{\Delta,f}$. In [12, Proposition 3], it is shown that one must only look at a finite set of values $(\alpha_1, ..., \alpha_r)$ that belong to the kernels of certain natural maps. This sentiment of restricting to a finite set of representatives $(\alpha_1, ..., \alpha_r)$ is repeated in the lemma below, but first we shall homogenize our varieties.

Let us fix a tuple $(\alpha_1, ..., \alpha_r) \in \mathbb{Q}^r$ satisfying $\alpha_1 ... \alpha_r = 1$. We observe that we can instead look at the integral points on the homogenization of these variety $X^*_{\Delta,f_1,...,f_r,\alpha_1,...,\alpha_r}$:

$$Y^*_{\Delta,f_1,...,f_r,\alpha_1,...,\alpha_r} := \{x_i^2 + \Delta y_i^2 = \alpha_i t_i^2 F_i(u,v) \neq 0, \gcd(x_i, y_i, t_i) = \gcd(u,v) = 1, \forall i\}.$$

A version of this reduction is described explicitly by de la Bretèche and Browning in the derivation of [22, Lemma 4.2] for $\Delta = 1$. Next, we note that if $\alpha_i = \alpha_i' \beta_i$, where $\beta_i \in \mathbb{Z}$ and satisfies that $\beta_i = b_i^2 \prod_{\chi(p_j)=1} p_j^{e_j}$, i.e. β_i is a product of a square and primes that split in K, then we have that

$$X_{\Delta,f_1,...,f_r,\alpha_1,...,\alpha_r} \cong X^*_{\Delta,f_1,...,f_r,\alpha_1',...,\alpha_r'}.$$

By the equivalence of these varieties, we can assume that $\alpha_i \in \mathbb{Z}$ are squarefree and a product of primes satisfying $\chi(p) = -1$. We can also assume that $f_1, ..., f_r$ all satisfy that the coefficients of f_i are at most divisible by a squarefree integer coprime to α_i. Moreover, we also know that the product of all of the α_i is a square.

Lemma 3.12.8. *Let $\alpha_1, ..., \alpha_r \in \mathbb{Z}$ be squarefree and assume that if $p \mid \alpha_i$ then $\chi(p) = -1$. Moreover, assume that $\alpha_1 ... \alpha_r$ is a square. If $Y^*_{\Delta, f_1, ..., f_r, \alpha_1, ..., \alpha_r}(\mathbb{Q}) \neq \emptyset$, then*

$$\alpha_i \mid \prod_{j \neq i} Res(F_i, F_j).$$

Proof. Let $((x_i, y_i, t_i), u, v)$ be a point on $Y^*_{\Delta, f_1, ..., f_r, \alpha_1, ..., \alpha_r}$. Since $p \mid \alpha_i$ implies that $\chi(p) = -1$, we must have that $\alpha_i \mid F_i(u, v)$ for each i in order for the equation

$$x_i^2 + \Delta y_i^2 = \alpha_i t_i^2 F_i(u, v) \neq 0$$

to be solvable. Let us assume that α_i has the following prime factorization:

$$\alpha_i = \prod_{\chi(p) = -1} p^{e_{p,i}},$$

with $e_{p,i} \in \{0, 1\}$. Since the product of all of the α_i must be a square, we must have that

$$e_{p,i} = \sum_{j \neq i} e_{p,i,j}$$

where $p^{e_{p,i,j}} \mid \gcd(\alpha_i, \alpha_j)$. Then we also have that $p^{e_{p,i,j}} \mid F_j(u, v)$ as well. Since $\gcd(u, v) = 1$, this implies that $p^{e_{p,i,j}} \mid \text{Res}(F_i, F_j)$. Since this holds for every pair i and j and all prime factors of α_i, α_j, we have that

$$\alpha_i \mid \prod_{j \neq i} \text{Res}(F_i, F_j).$$

□

Remark 3.12.9. We note that if there is some b_i that divides all of the coefficients of F_i, then $b_i \mid \text{Res}(F_i, F_j)$ for any $j \neq i$. This is analogous to tracking the primes that divide the coefficients of $f(z)$ when $f(z)$ is irreducible in the case described above.

From Lemma 3.12.8, we see that it suffices to consider a finite set of tuples $(\alpha_1, ..., \alpha_r)$ in order to prove Proposition 3.12.7. In particular, we can restrict ourselves to those $\alpha_1, ..., \alpha_r$ that are squarefree and divide the pairwise resultants. These line up with the tuples $\mathbf{c}$ that we sum over in (3.12).

Possible local obstructions

First, we want to determine for which primes p we can have the possibility of local obstructions for $Y^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r}$.

Lemma 3.12.10. *Fix $\alpha_1,\dots,\alpha_r \in \mathbb{Z}$ as in Lemma 1.4. Let ℓ be a finite prime. If $\ell \nmid -\Delta\prod_{i=1}^r \alpha_i$, then $Y^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r}(\mathbb{Q}_\ell) \neq \emptyset$. On the other hand, if $\ell \mid -\Delta$, then we have that $Y^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r}(\mathbb{Q}_\ell) \neq \emptyset$ if and only if there exists (u,v) with $\gcd(u,v)=1$ and*

$$\chi_\ell(\alpha_i F_i(u,v)\Delta^{e_i}/\ell^{2e_i}) = 1, \forall i = 1,\dots,r.$$

Here χ_ℓ denotes the quadratic character detecting squares mod ℓ and e_i is the exponent such that $\ell^e \| F_i(u,v)$. Finally, if $\ell \mid \prod_{i=1}^r \alpha_i$ and $\ell^{e_i}\|\alpha_i$, then $Y^*_{\Delta,f_1,\dots,f_r,\alpha_1,\dots,\alpha_r}(\mathbb{Q}_\ell) \neq \emptyset$ if and only if $\varrho_F((\ell^{e_i})) \neq 0$.

Remark 3.12.11. By our assumptions on α_i, we have that $e_i \in \{0,1\}$.

Proof. First, let us start with the equation

$$x^2 + \Delta y^2 = c.$$

For $\ell \nmid -c\Delta$, there will always be solutions over $\mathbb{Q}_\ell$ by the Hasse-Weil theorem and Hensel's Lemma; hence there is no chance of a local obstruction at these places. If $\ell \mid -\Delta$, then we consider

$$x^2 + \Delta y^2 \equiv x^2 \equiv c \bmod \ell.$$

If $\chi_\ell(c) = 1$, then there is no obstruction for $\mathbb{Q}_\ell$; if $\chi_\ell(c) = -1$ then there is an obstruction for $\mathbb{Q}_\ell$. On the other hand, if $\ell \mid c$, then we can not yet conclude anything. However, we get that x must also be a multiple of ℓ, and hence we get the equation

$$\ell^2 x'^2 + \Delta y^2 \equiv c \bmod \ell^2.$$

Dividing by ℓ, we can see that if $\chi_\ell(c\Delta/\ell) \in \{-1,1\}$, we can determine if there is an obstruction for $\mathbb{Q}_\ell$. Continuing this process, we get that if $\ell^e \| c$, then there is a solution in $\mathbb{Q}_\ell$ if and only if $\chi_\ell(c\Delta^e/\ell^e) = 1$. Lastly, assume $\ell \mid c$ and $\gcd(\ell,-\Delta) = 1$. Now, if $\chi(\ell) = -1$, we know that there can only be a solution to $x^2 + \Delta y^2 = c$ if $c = \ell^{2e}m$, for $\gcd(m,\ell) = 1$. Furthermore, $x^2 + \Delta y^2 = \ell^{2e}$ has a solution given by $x = \ell^e$. On the other hand, if $\chi(\ell) = 1$, $x^2 + \Delta y^2$ factorizes as a product

of linear factors mod ℓ. Thus, there is a nonzero solution to $x^2 + \Delta y^2 = c \bmod \ell$, and by Hensel's lemma this can be lifted to a solution in $\mathbb{Q}_\ell$.

Now, we return to the original variety $Y^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}$. First, we consider $\ell \nmid -\Delta \prod_{i=1}^r \alpha_i$. For these places, $x_i^2 + \Delta y_i^2 = \alpha_i F_i(u,v)$ has many solutions mod ℓ by Lang-Weil and by Hensel's Lemma, these solutions can be extended to elements of $\mathbb{Q}_\ell$. Thus, there will be no local obstructions at these places.

If $\ell \mid -\Delta$, then our conditions above give that $Y^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}_\ell) \neq \emptyset$ if and only if there exists (u,v) with $\gcd(u,v) = 1$ such that

$$\chi_\ell(\alpha_i F_i(u,v)\Delta^{e_i}/\ell^{2e_i}) = 1, \forall i.$$

Finally, let $\ell \mid \prod_{i=1}^r \alpha_i$. Note that by assumptions on the choice of $\{\alpha_i\}$, we can assume that if $\ell \mid \alpha_i$ then $\chi(\ell) = -1$. Let e_i be the power such that $\ell^{e_i} \| \alpha_i$. We can see that $Y^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}_\ell) \neq \emptyset$ gives us a point (u,v) such that $\ell^{e_i \bmod 2} \mid F_i(u,v)$ (as we need an even power of ℓ to divide $\alpha_i F_i(u,v)$). In other words, we must have that $\varrho_F((\ell^{e_i \bmod 2})) > 0$. This is in fact sufficient because from such a point with $\ell^{e_i \bmod 2} \mid F_i(u,v)$, we can build a solution in $Y^*_{\Delta,f_1,\ldots,f_r,\alpha_1,\ldots,\alpha_r}(\mathbb{Q}_\ell)$.

□

3.12.3 Computing the constant

Let us return to analyzing (3.12) in the general case when $f(z)$ factorizes; in this section, we prove Proposition 3.12.7. First, let us fix a t and rearrange the inner expression. Let us define the local factor at a prime p as:

$$L_p((p^{e_i})) = \sum_{i_1,\ldots,i_r} \frac{\chi(p^{i_1+\ldots+i_r})\varrho_F((p^{e_1+i_1},\ldots,p^{e_r+i_r}))}{p^{i_1+\ldots+i_r}}.$$

We can see that for a fixed t, we are looking at:

$$\mathrm{Res}_{s=1} \sum_{\substack{\mathbf{d} \\ (d_i,tR)=1}} \frac{\chi(d)\varrho_F(\mathbf{d})}{d^s} \times \sum_{c_{ij} \mid \mathrm{Res}(F_i,F_j)} \frac{\mu(\mathbf{c})\chi(\mathbf{c})}{c} \left(\sum_{q_1 q_2 = -\Delta} \bigtriangledown_{q_1,q_2}(\mathbf{c}) \right) \times \prod_{\substack{p \mid c \\ p \nmid t}} L_p(\mathbf{c}_p) \times \prod_{\substack{p \mid c \\ p \mid t}} \varrho_F(\mathbf{c}_p),$$

where we define $R = \prod_{i \neq j} \mathrm{Res}(F_i, F_j)$ and $\mathbf{c}_p$ the vector containing the p-parts of $\mathbf{c}$. It now behooves us to inspect this sum of $\bigtriangledown_{q_1,q_2}(\mathbf{c})$ over q_1 and q_2 and attempt to move it to the outside.

Lemma 3.12.12. *Let* $\mathbf{c}$ *be a fixed vector. Then*

$$\bigtriangledown(\mathbf{c}) := \sum_{q_1q_2=-\Delta} \bigtriangledown_{q_1,q_2}(\mathbf{c}) \geq 0.$$

Additionally, $\bigtriangledown(\mathbf{c}) = 0$ *if and only if there exists* $\ell \mid -\Delta$ *such that* $Y^*_{\Delta,f_1,\dots,f_r,\mathbf{c}}(\mathbb{Q}_\ell) = \emptyset$.

Proof. Let us expand out the definition of $\bigtriangledown_{q_1,q_2}(\mathbf{c})$:

$$\sum_{q_1q_2=-\Delta} \bigtriangledown_{q_1,q_2}(\mathbf{c}) = \sum_{q_1q_2=-\Delta} \sum_{p\mid n \implies p\mid\Delta} \frac{\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))}{n^2\Delta^2} \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i,-\Delta)=1}} \prod_{i=1}^{r}(1+\chi(a_i))\chi_{q_1}(a)\varrho_F(n,\mathbf{ca})$$

$$= \sum_{p\mid n \implies p\mid-\Delta} \frac{1}{n^2\Delta^2} \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i,-\Delta)=1}} \prod_{i=1}^{r}(1+\chi(a_i))\varrho_F(n,\mathbf{ca}) \sum_{q_1q_2=-\Delta} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(a).$$

Observe that if n satisfies that $p \mid n \implies p \mid -\Delta$, then we have that

$$\varepsilon_{q_1,q_2}(n) = \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n)).$$

In particular, if $p_{\neg-\Delta}(F_i(u,v)/c_i) \equiv a_i \bmod -\Delta$ and that $p_\Delta(F_i(u,v)) = n$, then we know that

$$\varepsilon_{q_1,q_2}(F(u,v)/c) = \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(a)(1\star\chi)(F(u,v)/cp_{-\Delta}(F(u,v))).$$

Thus, we can rewrite the above expression as:

$$\bigtriangledown(\mathbf{c}) = \sum_{p\mid n \implies p\mid-\Delta} \frac{1}{n^2\Delta^2} \sum_{\substack{\mathbf{a} \bmod -\Delta \\ \gcd(a_i,-\Delta)=1}} \prod_{i=1}^{r}(1+\chi(a_i)) \sum_{\substack{u,v \bmod -n\Delta \\ p_{-\Delta}(F(u,v))=n \\ p_{\neg-\Delta}(F_i(u,v)/c_i)\equiv a_i \bmod -\Delta \\ \gcd(u,v,\Delta)=1}} \sum_{q_1q_2=-\Delta} \frac{\varepsilon_{q_1,q_2}(F(u,v)/c)}{(1\star\chi)(F(u,v)/c)}.$$

We recall that

$$\sum_{q_1q_2=\Delta} \varepsilon_{q_1,q_2}(n) = r_{\mathcal{G}}(n) = \sum_{\substack{N(\mathfrak{a})=n \\ [\mathfrak{a}]\in\mathcal{G}}} 1 \geq 0,$$

where $\mathcal{G}$ is the principal genus. Moreover, $r_{\mathcal{G}}(n) = 0$ if $(1\star\chi)(n) = 0$, so the above expression is well-defined and non-negative. This completes the proof that $\bigtriangledown(\mathbf{c}) \geq 0$.

Let us determine when it is zero. We note that this is only relevant when $\chi(a_i) = 1$ for all i; for now, fix a value of $(u,v) \bmod -n\Delta$ that satisfies the restrictions in the above sum. In this case, the

above sum is zero if

$$\sum_{q_1 q_2 = -\Delta} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(a) = 0.$$

This indicates that for some q_1, q_2, we have that $\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(a) = -1$. We note that we must have that $\chi_{q_1}(a) = 1$ since $\chi(a) = 1$ and $q_1 \mid -\Delta$. Thus, either $\chi_{q_1}(p_{q_2}(n)) = -1$ or $\chi_{q_2}(p_{q_1}(n)) = -1$. Assume without loss of generality that it is $\chi_{q_1}(p_{q_2}(n)) = -1$. Let us consider the equation for our (u, v) chosen:

$$x^2 + \Delta y^2 = cF(u, v)/p_{q_1}(n) \bmod q_1.$$

Since $\chi_{q_1}(cF(u, v)/p_{q_1}(n)) = \chi_{q_1}(p_{q_2}(n)) = -1$, this equation is unsolvable; therefore, there must be some $\ell \mid q_1$ where there is an obstruction to this equation. This gives us that $\chi_\ell(cF(u, v)\Delta^e/\ell^{2e}) = -1$ for some $\ell \mid -\Delta$ and e the exponent such that $\ell^e \| F(u, v)$. However, by multiplicativity, we have that

$$\chi_\ell(cF(u, v)\Delta^e/\ell^{2e}) = \prod_{i=1}^{r} \chi_\ell(c_i F_i(u, v)\Delta^{e_i}/\ell^{2e_i}) = -1.$$

So, for the a_i and (u, v) chosen above, we have that for some i, we know that $\chi_\ell(c_i F_i(u, v)\Delta^{e_i}/\ell^{2e_i}) = -1$.

Now, we return to our expression. We can see that $\bigtriangledown(\mathbf{c}) = 0$ if and only if for every n satisfying $p \mid n \implies p \mid -\Delta$, and $\mathbf{a}$ such that $\chi(a_i) = 1 \forall i$, either

$$\varrho_F(n, \mathbf{ca}) = 0$$

or for any point $(u, v) \bmod -n\Delta$ as chosen above, we get that there exists some i and $\ell \mid -\Delta$ such that

$$\chi_\ell(c_i F_i(u, v)\Delta^{e_i}/\ell^{2e_i}) = -1.$$

We remark that this second condition is the local conditions set out by Lemma 3.12.10 for $Y^*_{\Delta, f_1, \ldots, f_r, \mathbf{c}}(\mathbb{Q}_\ell)$.

The first condition, on the other hand, would give us that for all tuples (a_i) such that $\chi(a_i) = 1 \forall i$, we have that $\varrho_F(n, \mathbf{ca}) = 0$. However, this simply tells us that there are no values of $(u, v) \bmod -n\Delta$ satisfying the local constraints set out. This immediately tells us that there can not be $\mathbb{Q}_{-\Delta}$-points on $\{x_i^2 + \Delta y_i^2 = c_i F_i(u, v)\}$. Thus the first condition is contained in the set of conditions for $\mathbb{Q}_\ell$ set out by Lemma 3.12.10 (in fact, it is a consequence of the relation $\prod_{\ell \mid -\Delta} \chi_\ell = \chi$). This completes the lemma. □

Lemma 3.12.13. *For any p_i such that $\chi(p_i) = 1$,*

$$\bigtriangledown(\mathbf{c}) = \bigtriangledown((p_1c_1, ..., p_rc_r)).$$

Proof. We note that by a change of variables,

$$\begin{aligned}
&\sum_{\substack{\mathbf{a} \bmod \Delta \\ \gcd(a_i,\Delta)=1}} \prod_{i=1}^{r}(1+\chi(a_i))\varrho_F(n,(a_ic_ip_i)) \times \sum_{q_1q_2=\Delta} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(a) \\
&\quad = \sum_{\substack{\mathbf{a} \bmod \Delta \\ \gcd(a_i,\Delta)=1}} \prod_{i=1}^{r}(1+\chi(a_ip_i))\varrho_F(n,\mathbf{ca}) \times \sum_{q_1q_2=\Delta} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(ap_1...p_r) \\
&\quad\quad = \sum_{\substack{\mathbf{a} \bmod \Delta \\ \gcd(a_i,\Delta)=1}} \prod_{i=1}^{r}(1+\chi(a_i))\varrho_F(n,\mathbf{ca}) \times \sum_{q_1q_2=\Delta} \chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\chi_{q_1}(a)
\end{aligned}$$

Here we use that $\chi(p_i) = 1$ for each i, and since $q_1 \mid -\Delta$, $\chi_{q_1}(p_i) = 1$ for each i as well. Thus, we have that $\bigtriangledown(\mathbf{c}) = \bigtriangledown((c_1p_1, ..., c_rp_r))$.

□

Having completed the above lemmas, we can see that (3.12) at a fixed t becomes

$$\left(\mathrm{Res}_{s=1} \sum_{\substack{\mathbf{d} \\ \gcd(d_i,tR)=1}} \frac{\chi(d)\varrho_F(\mathbf{d})}{d^s}\right) \times \left(\sum_{\substack{c_{ij}|\mathrm{Res}(F_i,F_j) \\ p|c \implies \chi(p)=-1 \\ \mu^2(\mathbf{c})=1}} \frac{\bigtriangledown(\mathbf{c})}{c} \cdot \prod_{p|c} L_p(\mathbf{c})\right) \times \left(\sum_{\substack{c'_{ij}|\mathrm{Res}(F_i,F_j) \\ p|c' \implies \chi(p)=1}} \frac{\mu(\mathbf{c}')}{c'} \prod_{\substack{p|c' \\ p\nmid t}} L_p(\mathbf{c}')\right).$$

We observe that the middle sum is independent of t and contains all of the local data of $\bigtriangledown(\mathbf{c})$. So, we can now safely bring back the summation over t. We have that (3.12) is equal to:

$$\begin{aligned}
&\left(\sum_{\substack{c_{ij}|\mathrm{Res}(F_i,F_j) \\ p|c \implies \chi(p)=-1 \\ \mu^2(\mathbf{c})=1}} \frac{\bigtriangledown(\mathbf{c})}{c} \cdot \prod_{p|c} L_p(\mathbf{c}_p)\right) \\
&\qquad \times \left(\sum_{\substack{N(\mathfrak{t})\le B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \sum_{\substack{c'_{ij}|\mathrm{Res}(F_i,F_j) \\ p|c' \implies \chi(p)=1}} \frac{\mu(\mathbf{c}')}{c'} \times \mathrm{Res}_{s=1} \sum_{\substack{\mathbf{d} \\ \gcd(d_i,t)=1}} \frac{\chi(d)\varrho_F(\mathbf{c}'\mathbf{d})}{d^s}\right).
\end{aligned}$$

We now have two goals – first, to show that the leftmost sum is zero if and only if there is a local obstruction to $Y^*_{\Delta,f_1,...,f_r,\mathbf{c}}$ for every $\mathbf{c}$, and second, to show that the rightmost expression is nonzero.

First, let us analyze further these local factors $L_p(\mathbf{c})$.

Lemma 3.12.14. *Let $p \mid c$ as above and $\chi(p) = -1$.*

(a) $L_p(1) > 0$,

(b) $L_p(\mathbf{c}) \geq 0$,

(c) $L_p(\mathbf{c}) = 0$ *if and only if* $Y^*_{\Delta,f_1,\ldots,f_r,\mathbf{c}}(\mathbb{Q}_p) = \emptyset$.

Proof. Let f_i be the largest exponent such that $p^{f_i} \mid F_i$. We remark that in our decomposition of $F(u,v)$ into irreducible components, we can assume that $f_i = 0$ for all but one i. Then we have that if $j_i \leq f_i$ for all i, $\varrho_F((p^{j_1},\ldots,p^{j_r})) = p^{j_1+\ldots+j_r}$. Otherwise, for any $j_i > f_i$, we have that $\varrho_F((p^{j_1},\ldots,p^{j_r})) \leq \varrho_F((p^{j_1},\ldots,p^{j_i-1},\ldots,p^{j_r})$. (Moreover, from Hensel's lemma, for sufficiently large j_i, this is an equality.) Thus, we either achieve a geometric series for $L_p(\mathbf{c})$ or a finite sum. These relations ensure us that for any $\mathbf{c}$, we have that

$$L_p(\mathbf{c}) \geq 0.$$

In particular, by considering the geometric series or finite sum $L_p(\mathbf{c})$, we get that $L_p(\mathbf{c}) = 0$ if and only if the leading term $\varrho_F(\mathbf{c}_p) = 0$, where $\mathbf{c}_p$ is the vector of p-factors of entries of $\mathbf{c}$. If $\mathbf{c} = 1$, this is false – as, $\varrho_F(1) = 1$. Hence, $L_p(1) > 0$. On the other hand, if $\varrho_F(\mathbf{c}_p) = 0$, we have by Lemma 3.12.10, this is exactly when $Y^*_{\Delta,f_1,\ldots,f_r,\mathbf{c}}(\mathbb{Q}_p) = \emptyset$. Thus, we achieve (c) and conclude the lemma. □

We consider the sum:

$$\sum_{\substack{c_{ij} \mid \mathrm{Res}(F_i,F_j) \\ p \mid c \implies \chi(p) = -1 \\ \mu^2(\mathbf{c}) = 1}} \frac{\bigtriangledown(\mathbf{c})}{c} \cdot \prod_{p \mid c} L_p(\mathbf{c}).$$

Since this is a sum of non-negative terms, we can see that the above can only be zero if for every $\mathbf{c}$ with $p \mid c \implies \chi(p) = -1$, we get that either $\bigtriangledown(\mathbf{c}) = 0$ or $L_p(\mathbf{c})$ for some $p \mid c$. However, this is exactly the conditions set out by Lemma 3.12.10 as the potential local obstructions to rational points on $Y^*_{\Delta,f_1,\ldots,f_r,\mathbf{c}}$. Thus, we get the following result.

Lemma 3.12.15.

$$\sum_{\substack{c_{ij} \mid Res(F_i,F_j) \\ p \mid c \implies \chi(p) = -1 \\ \mu^2(\mathbf{c}) = 1}} \frac{\bigtriangledown(\mathbf{c})}{c} \cdot \prod_{p \mid c} L_p(\mathbf{c}) = 0$$

if and only if for every such $\mathbf{c}$ *in the summand, there is a local obstruction to* $Y^*_{\Delta,f_1,\ldots,f_r,\mathbf{c}}$.

Finally, it remains to show that the following is nonzero:

$$\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \sum_{\substack{c'_{ij}|\mathrm{Res}(F_i,F_j)\\ p|c' \implies \chi(p)=1}} \frac{\mu(\mathbf{c}')}{c'} \times \mathrm{Res}_{s=1} \sum_{\substack{\mathbf{d}\\ \gcd(d_i,t)=1}} \frac{\chi(d)\varrho_F(\mathbf{c}'\mathbf{d})}{d^s}.$$

Let us rearrange this expression as

$$\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)}\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} \frac{1}{N(\mathfrak{t})} \prod_{\substack{p|R'\\ p\nmid t}} \left(\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p,1)}{c_p}\right) \prod_{\substack{p|R'\\ p|t}} \left(\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')\varrho_F(\mathbf{c}'_p)}{c_p}\right) \times \mathrm{Res}_{s=1} \prod_{p\nmid tR'} L_p(1,s),$$

where we define

$$L_p((p^{e_i}),s) = \sum_{i_1,\dots,i_r} \frac{\chi(p^{i_1+\dots+i_r})\varrho_F((p^{e_1+i_1},\dots,p^{e_r+i_r}))}{p^{(i_1+\dots+i_r)s}},$$

R' to be the part of $\prod_{i\neq j}\mathrm{Res}(F_i,F_j)$ made of primes with $\chi(p)=1$, and $\mathbf{c}'_p$ sums over the possible p-parts of the vectors $\mathbf{c}$ dividing these resultants. For example, if $r=2$, then the only vectors are $(1,1)$ and (p,p); if $r=3$ on the other hand, then $\mathbf{c}'_p$ can be $(1,1,1),(p,p,1),(1,p,p),(p,1,p)$, etc.

Now, the expression above can be rewritten as

$$\prod_{p|R'}\left(\sum_{\mathbf{c}'_p}\frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p}\right)\times\left(\mathrm{Res}_{s=1}\prod_{p\nmid R'}L_p(1,s)\right)$$
$$\times\sum_{\substack{N(\mathfrak{t})\leq B/\exp(\sqrt{\log(B)})\\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} N(\mathfrak{t})^{-1}\prod_{\substack{p|t\\ p\nmid R'}}L_p(1,s)^{-1}\prod_{\substack{p|t\\ p|R'}}\left(\sum_{\mathbf{c}'_p}\frac{\mu(\mathbf{c}')\varrho_F(\mathbf{c}'_p)}{c'_p}\right)\left(\sum_{\mathbf{c}'_p}\frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p}\right)^{-1}.$$

Let us first show that the first product is nonzero.

Lemma 3.12.16. *Let $p\mid R'$. Then*

$$\sum_{\mathbf{c}'_p}\frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p}\neq 0.$$

Proof. We first note that since $p\mid R'$, $\chi(p)=1$. Next, if $\mathbf{c}'_p=(p^{e_i})$, then

$$\frac{L_p(\mathbf{c}'_p)}{c'_p}=\sum_{i_1\geq e_1,\dots,i_r\geq e_r}\frac{\varrho_F((p^{i_1},\dots,p^{i_r}))}{p^{i_1+\dots+i_r}}.$$

So, if $r = 2$, we have that the above sum is

$$\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p} = \sum_{i_1 \geq 0, i_2 \geq 0} \frac{\varrho_F((p^{i_1}, p^{i_2}))}{p^{i_1+i_2}} - \sum_{i_1 \geq 1, i_2 \geq 1} \frac{\varrho_F((p^{i_1}, p^{i_2}))}{p^{i_1+i_2}} \geq 1.$$

This comes from the fact that $\varrho_F(1, 1) = 1$.

For $r = 3$ and $r = 4$, however, we must use a different argument. We note that in both cases we have multiple linear factors – these will produce geometric series with the multiplicative difference of $1/p$, and hence have denominators $p - 1$. We use this fact freely in the casework below.

If p divides three of the pairwise resultants, we have that

$$\begin{aligned}\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p} &= 1 + \sum_{\substack{i_1=0\\i_2=0\\i_3=1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} + \sum_{\substack{i_1=0\\i_2=1\\i_3=0}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} + \sum_{\substack{i_1=1\\i_2=0\\i_3=0}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} + \sum_{\substack{i_1\geq 1\\i_2\geq 1\\i_3\geq 1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} \\ &\quad - \sum_{\substack{i_1=1\\i_2\geq 1\\i_3\geq 1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} - \sum_{\substack{i_1\geq 1\\i_2=1\\i_3\geq 1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} - \sum_{\substack{i_1\geq 1\\i_2\geq 1\\i_3=1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} \\ &= 1 + \frac{2}{p-1} + \frac{\{0,1,2\}}{p-1} + \sum_{\substack{i_1\geq 1\\i_2\geq 1\\i_3\geq 1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} - \sum_{\substack{i_1=1\\i_2\geq 1\\i_3\geq 1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} - \sum_{\substack{i_1\geq 1\\i_2=1\\i_3\geq 1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} - \sum_{\substack{i_1\geq 1\\i_2\geq 1\\i_3=1}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}}.\end{aligned}$$

Here $\{0, 1, 2\}$ denotes the cases when F has a quadratic factor with 0 or 2 solutions, and when F has a linear factor and hence 1 solution. We note that the above is equal to

$$\begin{aligned}&1 + \frac{2}{p-1} + \frac{\{0,1,2\}}{p-1} - \frac{2\varrho(p,p,p)}{p^3} - \frac{\varrho(p,p,p^2)}{p^4} - \frac{\varrho(p,p^2,p)}{p^4} - \frac{\varrho(p^2,p,p)}{p^4} + \sum_{\substack{i_1\geq 2\\i_2\geq 2\\i_3\geq 2}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}} \\ &\qquad = 1 + \frac{2}{p-1} + \frac{\{0,1,2\}}{p-1} - \frac{2\cdot\{0,1,2\}}{p} - \frac{3\cdot\{0,1,2\}}{p^2} + \sum_{\substack{i_1\geq 2\\i_2\geq 2\\i_3\geq 2}} \frac{\varrho_F((p^{i_j}))}{p^{i_1+i_2+i_3}}.\end{aligned}$$

Now, we note that for $p \geq 2$,

$$1 + \frac{2}{p-1} + \frac{\{0,1,2\}}{p-1} - \frac{2\cdot\{0,1,2\}}{p} - \frac{3\cdot\{0,1,2\}}{p^2} > 0.$$

Thus, we have that

$$\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p} > 0$$

when p divides three of the pairwise resultants.

Finally, if F splits into four linear factors and if p divides all four of the pairwise resultants, we again can expand in terms of the indices i_1, i_2, i_3, i_4. Since every factor is linear, we know that

$$\varrho_F((p^{i_1}, ..., p^{i_4})) = \begin{cases} p^{-\max(i_1,...,i_4)}, & p^{i_j} \mid \mathrm{Res}(F_j, F_k) \forall k \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

By going through the combinatorics, we again get that for any $p \geq 2$

$$\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p} > 0.$$

□

It remains to show that the following is nonzero:

$$\sum_{\substack{N(\mathfrak{t}) \leq B/\exp(\sqrt{\log(B)}) \\ \gcd(\mathfrak{t},\bar{\mathfrak{t}})=1}} N(\mathfrak{t})^{-1} \prod_{\substack{p \mid t \\ p \nmid R'}} L_p(1,s)^{-1} \prod_{\substack{p \mid t \\ p \mid R'}} \left(\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')\varrho_F(\mathbf{c}'_p)}{c'_p} \right) \left(\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p} \right)^{-1}.$$

We note that the above sum is asymptotic to $A\log(B) + O(\sqrt{\log(B)})$, where A is the residue at $s = 1$ of the Dirichlet series

$$\prod_{\substack{\mathfrak{p} \\ p \nmid R'}} \left(1 + N(\mathfrak{p})^{-s} \left(L_p(1)^{-1}\right) + \ldots\right) \prod_{\substack{\mathfrak{p} \\ p \mid R'}} \left(1 + N(\mathfrak{p})^{-s} \left(\left(\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')\varrho_F(\mathbf{c}'_p)}{c'_p} \right) \left(\sum_{\mathbf{c}'_p} \frac{\mu(\mathbf{c}')L_p(\mathbf{c}'_p)}{c'_p} \right)^{-1} \right) + \ldots \right).$$

Since $L_p(1)^{-1} = 1 + O(1/p)$, the above series has a simple pole at $s = 1$; since A is the residue at $s = 1$, $A \neq 0$ as desired.

Thus, $c_{\Delta,f} = 0$ *if and only if for all of the* $\mathbf{c}$ *above from Lemma 3.12.8,* $Y^*_{\Delta,f_1,...,f_r,\mathbf{c}}(\mathbb{Q}_\ell) = \emptyset$ *for some place* ℓ. This completes the proof of Proposition 3.12.7, and thus Theorem 3.1.3. □

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Chapter 4

Integral points on a family of degree one del Pezzo surfaces

4.1 Introduction

In 2002, Swinnerton-Dyer [37] wrote that

> It is known that Del Pezzo surfaces of degree $d > 4$ satisfy the Hasse principle and weak approximation; indeed those of degree 5 or 7 necessarily contain rational points. Del Pezzo surfaces of degree 2 or 1 have no aesthetic merits and have attracted little attention; it seems sensible to ignore them until the problems coming from those of degrees 4 and 3 have been solved.

In this paper, we will attempt the non-sensible task of studying the rational and integral points on certain singular del Pezzo surfaces of degree one; specifically, our surfaces will be elliptic surfaces that parameterize certain families of quadratic twists of elliptic curves.

Let $A, B \in \mathbb{Z}$ be integers satisfying that $4A^3 - 27B^2 \neq 0$ and let $Q(u, v)$ be a positive-definite binary quadratic form. We will study the singular del Pezzo surfaces of degree one defined by the equation:

$$S_{A,B,Q} := \{y^2 = x^3 + AxQ(u, v)^2 + BQ(u, v)^3\}.$$

These surfaces live in weighted projective space $\mathbb{P}(2, 3, 1, 1)$; thus, the rational points are the integral points $(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z})$ satisfying that $\gcd(x, u, v) = 1$, which implies that $\gcd(x, y, u, v) = 1$ for this surface.

We define the height function on $S_{A,B,Q}$ as the restriction of the natural height on $\mathbb{P}(2, 3, 1, 1)$:

$$\tilde{h}((x, y, u, v)) = \max(|x|^{1/2}, |y|^{1/3}, |u|, |v|).$$

The following refinement of Manin's conjecture specialized for the surface $S_{A,B,Q}$ predicts the count for rational points on $S_{A,B,Q}$ over the fixed number field $\mathbb{Q}$; we present the version given in [7, Conjecture 2.3]:

Conjecture 4.1.1 (Batyrev-Manin [2, Conjecture C'])**.** *Define the open sets* $U_1 = \mathbb{P}(2, 3, 1, 1)\backslash\{y = 0\}$ *and* $U_2 = \mathbb{P}(2, 3, 1, 1)\backslash\{Q(u, v) = 0\}$. *Define the point-counting function*

$$\tilde{N}(T) := \#\{(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z}) \cap U_1 \cap U_2 : \tilde{h}((x, y, u, v)) \leq T, \gcd(x, u, v) = 1\}.$$

Assume that $S_{A,B,Q}(\mathbb{Q}) \neq \emptyset$. *Then there exists a constant* $C_{A,B,Q}$ *such that*

$$\tilde{N}(T) \sim C_{A,B,Q} T \log(T)^{\varrho_{A,B,Q}-1},$$

where ϱ is the rank of the Picard group over $\mathbb{Q}$ of the desingularization $\tilde{S}_{A,B,Q}$ of $S_{A,B,Q}$.

This constant $C_{A,B,Q}$, known as Peyre's constant [34], can be expressed in terms of a product of local densities, the Brauer group, and the volume of a certain polytope in $\mathrm{Pic}(\tilde{S}_{A,B,Q})_{\mathbb{R}}$; we will not focus on this constant in this paper. Note that $C_{A,B,Q}$ is not guaranteed to be nonzero in the above prediction.

The exponent in the prediction can be explicitly calculated as well for $S_{A,B,Q}$. Let $\Delta(Q)$ be the discriminant of $Q(u,v)$ and $K_Q = \mathbb{Q}(\sqrt{\Delta(Q)}) = \mathbb{Q}[x]/(Q(x,1))$. In the §4.5, we compute that the Picard rank $\varrho_{A,B,Q}$ satisfies that

$$\varrho_{A,B,Q} = \begin{cases} 5, & z^3 - A\Delta(Q)z + B\Delta(Q)^{3/2} \text{ splits completely over } K_Q, \\ 4, & z^3 - A\Delta(Q)z + B\Delta(Q)^{3/2} \text{ splits into a linear and quadratic factor over } K_Q, \\ 3, & \text{otherwise.} \end{cases}$$

Hence, for a generic A, B and $Q(u,v)$, the conjecture asserts

$$\tilde{N}(T) \asymp T\log(T)^2.$$

While asymptotics for Manin's conjecture have not been established for any del Pezzo surface of degree one, both upper and lower bounds have been explored. Munshi uses sieve methods to investigate in [32] singular surfaces of the form $\{y^2 = (x + \lambda Q(u,v))(x + \delta Q(u,v))(x + \overline{\delta} Q(u,v))\}$, where $\lambda \in \mathbb{Z}$, δ is a generator of an imaginary quadratic field, and $Q(u,v)$ is a positive-definite quadratic form. He establishes that the number of rational points of height up to T is bounded by $\ll T^{5/4+\epsilon}$ for any $\epsilon > 0$. In [29], Le Boudec improves this bound to $T^{1+\epsilon}$, for any $\epsilon > 0$ for surfaces of the form $\{y^2 = (x - e_1 Q(u,v))(x - e_2 Q(u,v))(x - e_3 Q(u,v))\}$, where $e_1, e_2, e_3 \in \mathbb{Z}$ are distinct and $Q(u,v)$ is a quadratic form.

For nonsingular del Pezzo surfaces of degree one, which are elliptic surfaces, one can use the bounds derived by Bhargava, Shankar, Taniguchi, Thorne, Tsimerman, and Zhao [4] and Helfgott and Venkatesh [26] for integral points on elliptic curves to establish nontrivial upper bounds of size $O(T^{2.87+\epsilon})$. On the other side, Frei, Loughran, and Sofos [23] established a lower bound of the correct order of magnitude for del Pezzo surfaces of degree one after passing to an extension of $\mathbb{Q}$ of degree at most 138240. Additionally, motivated by an observation of Heath-Brown in [25], Bonolis and Browning explain in [5] how to achieve an upper bound of $O_\epsilon(T^{2+\epsilon})$ using a suitable rank growth hypothesis for elliptic curves.

The lowest degree del Pezzo surface for which Manin's conjecture is known is a singular del Pezzo surface of degree two studied by Baier and Browning in 2013 [1]. For higher degrees, much more is known. In [11] and [12], de la Bretèche and Browning derive Manin asymptotics for certain del Pezzo surfaces of degree four. In [16], de la Bretèche, Browning, and Derenthal prove an asymptotic for $x_1x_2^2 + x_2x_0^2 + x_3^3 = 0$, a cubic surface; and in [15], de la Bretèche and Browning also prove Manin's conjecture for nonsingular quartic surfaces with conic fibrations. For more known cases of Manin's conjecture, especially for higher degrees, we refer to [6] and [7].

For our surface, we will count a specific subset of the rational points.

Definition 4.1.2. A rational point $(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z})$ is called **integral with respect to the singularity** if $\gcd(x, y, Q(u, v)) = 1$. From the structure of $S_{A,B,Q}$, the condition $\gcd(x, y, Q(u, v)) = 1$ is equivalent to $\gcd(x, Q(u, v)) = 1$. [1]

To see that this gcd condition is connected to the singularity of $S_{A,B,Q}$, we recall that the singularity of $S_{A,B,Q}$ is given by $x = y = Q(u, v) = 0$. By viewing the singularity as a boundary divisor, we can view these points in the framework developed by Derenthal and Wilsch [20] and by Chambert-Loir and Tschinkel [9] for integral points with respect to boundary divisors. In both of the aforementioned papers, and the later work of Santens [35], conjectures are formulated for counting integral points with respect to boundary divisors under certain geometric assumptions (for example, split log-Fano varieties) that $S_{A,B,Q}\backslash\{x = y = Q(u, v) = 0\}$ do not satisfy. Furthermore, it is both expected, and verified through our parameterization of the integral points, that the set of integral points with respect to the singularity to be contained in a thin set; since in many formulations of Manin's conjecture for integral points one excludes a thin set, it could be that this is the appropriate thin set to be excluded.

We must further restrict which integers A, B, and positive-definite quadratic forms $Q(u, v)$ we study; this assumption comes from the automorphic techniques in our method for counting points. Let $\mathcal{F}(A, B)$ denote the finite set of binary quartic forms with invariants $-4A$ and $-4B$, up to equivalence by $\mathrm{SL}_2(\mathbb{Z})$.

Definition 4.1.3. Let $A, B \in \mathbb{Z}$ and $Q(u, v)$ be a positive-quadratic form. The tuple $(A, B, Q(u, v))$ is called **disassociated** if the quadratic extension determined by $Q(u, v)$, denoted as K_Q is not contained in the splitting field of F, for all $F \in \mathcal{F}(A, B)$.

[1] Another interpretation of this gcd condition is that (x, y, u, v) is integral with respect to the singularity if and only if $(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z}/p^k\mathbb{Z})$ is nonsingular for every prime power p^k.

Finally, for simplicity, we define a slightly different height function:

$$h((x,y,u,v)) = \max(|x|^{1/2}, |y|^{1/3}, |Q(u,v)|^{1/2}).$$

This height function is within a scalar multiple of the natural height function $\tilde{h}((x,y,u,v))$ coming from weighted projective space.

Theorem 4.1.4. *Let $A, B \in \mathbb{Z}$ such that $4A^3 - 27B^2 \neq 0$ and $Q(u,v)$ be a positive-definite quadratic form. Assume that $(A, B, Q(u,v))$ is disassociated. Let $U_1 = \mathbb{P}(2,3,1,1)\backslash\{y = 0\}$ and $U_2 = \mathbb{P}(2,3,1,1)\backslash\{Q(u,v) = 0\}$. Define the point-counting function*

$$N(T) := \#\{(x,y,u,v) \in S_{A,B,Q}(\mathbb{Z}) \cap U_1 \cap U_2 : h((x,y,u,v)) \leq T, \gcd(x, Q(u,v)) = 1\}.$$

Then the following asymptotic holds as $T \to \infty$:

$$N(T) = c_{A,B,Q}T + O_{A,B,Q}(T\log(T)^{-10^{-10}}).$$

Remark 4.1.5. The assumption that $(A, B, Q(u,v))$ is disassociated is stronger than necessary for our methods to produce as asymptotic for $N(T)$. Let $\mathcal{H}$ denote the Hilbert splitting field of K_Q; let K_F denote the splitting field of a binary form F. If we have that either $K_Q \not\subset K_F$ or $K_Q \subset K_F$ and $\mathcal{H} \cap K_F = K_Q$, then there exists an explicit exponent $0 \leq \beta_{A,B,Q} \leq 2$ such that

$$N(T) = c_{A,B,Q}T\log(T)^{\beta_{A,B,Q}}1 + O_{A,B,Q}(T\log(T)^{\beta_{A,B,Q}-10^{-10}}).$$

Moreover, for a given A, B and $Q(u,v)$, this exponent $\beta_{A,B,Q}$ is computable by combining the explicit parameterization given in §4.2 of the integral points and utilizing Mordell's fundamental domain for binary quartics with given invariants (see [31] for explicit bounds).

Remark 4.1.6. In this paper, we do not give a criteria for when $c_{A,B,Q}$ is nonzero – in §4.4 we look at specific examples of surfaces and their respective leading constants. In all cases, lower bounds for Manin's conjecture are established. However, in §4.4.3, we also give an example of a surface when a Brauer-Manin obstruction completely obstructs integral points in one of the fibers – this phenomenon must be considered in any eventual evaluation of the constant $c_{A,B,Q}$.

The new tool employed in establishing Theorem 4.1.4 is the use of syzygies. The integral points of $S_{A,B,Q}$ with respect to the singularity can be parameterized using a syzygy of Cayley [8] and Hermite

[27] for binary quartic forms; this syzygy specifically connects binary quartics to the quadratic twists of elliptic curves. This crucial observation was initially made by Mordell [31] during his seminal investigation of the rational points of elliptic curves. Recently, Duke [22] provided the quantitative version of this relationship that we will utilize in §4.2. A loose summary of the parameterization is such: the solutions to the equation

$$\{(x, y, n) : y^2 = x^3 + Axn^2 + Bn^3, \gcd(x, n) = 1\}$$

are given by

$$\left\{(H_F(\mathbf{m}), T_F(\mathbf{m}), F(\mathbf{m})) : \mathbf{m} \in \mathbb{Z}^2, F \in \mathcal{F}(A, B)\right\},$$

where $H_F(m, n)$ (resp. $T_F(m, n)$) is a binary quartic (resp. sextic) that is formed with the coefficients of F. This parameterization will be further elucidated in §4.2. We note that this parameterization makes the set of integral points with respect to the singularity a thin set of type II (as defined by Serre [36, §9.7]); however, we expect that the set of rational points counted in Conjecture 4.1.1 to not be thin.

Our key analytic input will be an asymptotic for correlation sums between binary quartic and binary quadratic forms. First, let $Q(u, v)$ be a positive-definite integral quadratic form. We define

$$r_Q(n) := \#\{u, v \in \mathbb{Z} : Q(u, v) = n\}.$$

We want to study the correlations of $r_Q(n)$ with the number of solutions of a binary form. Based on the analysis of the similar correlation sum in Chapter 3, we will prove the following theorem:

Theorem 4.1.7. *Let $F(x, y)$ be a squarefree binary form of degree ≤ 4 and $Q(u, v)$ a positive-definite integral quadratic form. Assume that the splitting field of $F(x, y)$, denoted as K_F, and the Hilbert class field of K_Q are independent. Let $\mathcal{R} \subset \mathbb{R}^2$ be a fixed compact subset. Let $\alpha \bmod M$ be a fixed residue class in $(\mathbb{Z}/M\mathbb{Z})^2$. Then there exists a constant $c_{\alpha,F,Q}$ such that*

$$\sum_{\substack{x,y \in X\mathcal{R} \\ \gcd(x,y)=1 \\ (x,y) \equiv \alpha \bmod M}} r_Q(F(x, y)) = c_{\alpha,F,Q} X^2 |\mathcal{R}| \log(X)^{\beta_{F,Q}} + O_{\mathcal{R},F,Q,M}(X^2 (\log(X))^{\beta_{F,Q} - 10^{-10}}),$$

where we have that

$$\beta_{F,Q} = \begin{cases} 2, & F(x,y) = Q_1(x,y)Q_2(x,y) \text{ where } Q_1, Q_2 \text{ are quadratic and } K_{Q_1} = K_{Q_2} = K_Q, \\ 1, & F(x,y) = Q_1(x,y)Q_2(x,y) \text{ where } Q_1, Q_2 \text{ are quadratic and } K_{Q_1} = K_Q \neq K_{Q_2}, \\ 1, & F(x,y) \text{ is irreducible and } K_Q \subset K_F, \\ 0, & \text{otherwise.} \end{cases}.$$

Remark 4.1.8. Observe that for Theorem 4.1.4, by the assumption that $(A, B, Q(u,v))$ are disassociated, all of the relevant exponents $\beta_{F,Q}$ will be zero.

For $G(u,v) := u^2 + v^2$, there are many results in the literature covering such correlation sums. Heath-Brown [24], and later de la Bretèche and Browning [13], analyzed and proved asymptotics when $F(x,y)$ factors into four distinct linear forms L_1, L_2, L_3, L_4:

$$\sum_{x,y \leq X} r_G(L_1(x,y))r_G(L_2(x,y))r_G(L_3(x,y))r_G(L_4(x,y)) = c_{L_i,Q}X^2 + O(X^2 \log(X)^{-\delta}).$$

Note that using different methods, Matthiesen [30] also established the above estimate for $F(x,y)$ a product of arbitrarily many distinct linear factors. Later, in [14], de la Bretèche and Browning, a later Destagnol in [21], resolved the case when $F(x,y)$ splits into two linear factors and an irreducible quadratic. Finally, in [18], de la Bretèche and Tenenbaum solved the hardest case: when $F(x,y)$ is an irreducible binary quartic form. We remark that all of the techniques used in the aforementioned results work for any positive-definite quadratic form of class number one, and all but [30] build upon the seminar work of Daniel [10] for divisor functions along binary quartics. In Chapter 3, we followed through on this insight and gives a proof for the above correlation sums for any $Q(u,v) = u^2 + \Delta v^2$ where Δ is any squarefree integer.

In §4.3, we rework the tools from Chapter 3 to handle an arbitrary positive-definite form and establish Theorem 4.1.7. We remark that in combination with the ideas of Chapter 3 for $\Delta < 0$, we can also handle arbitrary indefinite binary quadratic forms (by adding in the use of argument-detecting Hecke Grossencharacters and restricting to short-ish intervalas). For the sake of clarity and simplicity of notation, we do not work out the indefinite case in this paper.

Finally, in §4.4 we go through examples of Theorem 4.1.4 and some cases when $(A, B, Q(u,v))$ are disassociated. In these examples, we also establish some lower bounds for Manin's conjecture

for the following surfaces:

$$y^2 = x^3 - x(u^2 + v^2)^2,$$

$$y^2 = x^3 - x(u^2 + 5v^2)^2,$$

$$y^2 = x^3 - \frac{97}{48}x(u^2 + v^2)^2 + \frac{955}{864}(u^2 + v^2)^3.$$

In the second example, we see an interesting complication regarding the geometric interpretation of the exponent of $\log(B)$. In the last example, we see a case where the Brauer-Manin obstruction can block the existence of integral points on one of the fibers given by the syzygy parameterization.

Remark 4.1.9. We do not generally produce points on our surfaces $S_{A,B,Q}$ since we do not give a criteria for when $c_{A,B,Q} > 0$. Mordell [31] and Duke [22] study the equation

$$y^2 = x^3 + Axn^2 + Bn^3;$$

it is clear that as one averages over n there should be rational points on this variety. However, introducing a quadratic form $Q(u, v)$ in place of n can bring about extra complications with the potential of a Brauer-Manin obstruction. To understand when there exists integral points with respect to the singularity, it would be necessary to first understand further the elements of $\mathcal{F}(A, B)$ (beyond the fact that $\mathcal{F}(A, B)$ is finite).

4.2 Syzygys of binary quartic forms

First, we recall some facts about binary quartics. Then, we will discuss the reduction of Theorem 4.1.4 to Theorem 4.1.7 via this parameterization given by syzygys of binary quartic forms.

4.2.1 Binary quartic forms and Mordell's parameterization

Let us write

$$F(x, y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Then the space of integral binary quartics $F(x, y)$ can be parameterized by the coefficient variables a, b, c, d, e. Thus, there is an action of $\mathrm{SL}_2(\mathbb{Z})$ on the space of integral binary quartics:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot F(x, y) = F(\alpha x + \beta y, \gamma x + \delta y).$$

This action then induces an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}[a,b,c,d,e]$, and we can ask for the generators of the ring of invariants:

$$\mathbb{Z}[a,b,c,d,e]^{\mathrm{SL}_2(\mathbb{Z})} = \mathbb{Z}[I,J].$$

Here, $I(a,b,c,d,e)$ and $J(a,b,c,d,e)$ are defined as follows:

$$I = ae - 4bd + 3c^2$$

$$J = ace + 2bcd - b^2e - d^2a - c^3.$$

For $F(x,y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$, we will also denote these expressions as $I(F)$ and $J(F)$.

Example 4.2.1. The discriminant polynomial $\Delta(F)$ of a binary quartic form $F(x,y)$ will also be invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$; so, $\Delta(F) \in \mathbb{Z}[a,b,c,d,e]^{\mathrm{SL}_2(\mathbb{Z})}$ should have an expression in terms of $I(F)$ and $J(F)$. This is indeed true and the formula is reminscient of the discriminant of an elliptic curve:

$$\Delta(F) = I(F)^3 - 27J(F)^2.$$

Next, in order to write down the syzygy between $I(F)$ and $J(F)$, we must define two covariant functions.

Definition 4.2.2. The **Hessian** of a binary quartic form $F(x,y)$ is the binary quartic form[2]:

$$H_F(x,y) = \frac{1}{44}\det\begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix}.$$

Given a binary quartic form $F(x,y)$, we call the following function the **Jacobian covariant polynomial**:

$$T_F(x,y) = \frac{-1}{8}\det\begin{pmatrix} F_x & F_y \\ H_{F,x} & H_{F,y} \end{pmatrix}.$$

This polynomial $T_F(x,y)$ will be a binary sextic form.

Lemma 4.2.3. *If $F(x,y)$ is a squarefree binary quartic form, then $F(x,y)$ and $H_F(x,y)$ will share no common factors over $\overline{\mathbb{Q}}$.*

Finally, we introduce our **fundamental syzygy**, which is due independently to Cayley [8] and

[2]Here, $F_x = \frac{\partial F(x,y)}{\partial x}$ and $F_{xy} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$.

Hermite [27], that will form the basis of Mordell's parameterization.

Theorem 4.2.4 (Cayley [8], Hermite [27]). *For $F(x,y)$ a binary quartic form, the following relation holds:*

$$T_F(x,y)^2 = -4H_F(x,y)^3 + I(F)\cdot H_F(x,y)F(x,y)^2 - J(F)\cdot F(x,y)^3.$$

In [31], Mordell noted that the syzygy above takes the same shape as the quadratic twists of an elliptic curve. In particular, by setting $y' = T_F(x,y)/2$, $x' = -H_F(x,y)$, and $n = F(x,y)$, we get the elliptic curve

$$y'^2 = x'^3 - \frac{I(F)}{4}x'n^2 - \frac{J(F)}{4}n^3.$$

From this observation, one can parameterize the solutions of the equation $y'^2 = x'^3 + Ax'n^2 + Bn^3$ using binary quartics. This parameterization first appears in a paper of Mordell [31]; we will introduce the quantitative form by Duke in [22].

Define $\Gamma_\infty \subset \mathrm{SL}_2(\mathbb{Z})$ to be the group that fixes the point at ∞:

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\}.$$

We also need the notion of an admissible binary quartic form.

Definition 4.2.5. A binary quartic form $F(x,y) = ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4$ is **admissible** if $\gcd(a,b) = 1$.

Finally, we define the space of binary quartic forms with invariants $-4A$ and $-4B$:

$$\mathscr{F}(A,B) := \{\text{admissible integral binary quartic forms } F(x,y) : I(F) = -4A, J(F) = -4B, \text{ and } F(1,0) > 0\}.$$

We note that this set is invariant under the action of Γ_∞. In fact, without the conditions of admissibility and that $F(1,0) > 0$, this space would be invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. In Theorem 1.7 of Bhargava and Shankar [3], the congruences necessary on A and B so that this set is nonempty are listed.

In the following result, the connection between the set $\mathscr{F}(A,B)$ and the integral points on the affine elliptic surface $y^2 = x^3 + Axn^2 + Bn^3$ (the surface consisting of the quadratic twists of an elliptic curve) is made explicit.

Theorem 4.2.6 (Duke, [22, Proposition 2])**.** *Let $A, B, \in \mathbb{Z}$ satisfy that $4A^3 - 27B^2 \neq 0$. Then consider the map*

$$\mathscr{F}(A,B)/\Gamma_\infty \to \{(x,y,n) \in \mathbb{Z}^2 \times \mathbb{Z}_+ : y^2 = x^3 + Axn^2 + Bn^3, \gcd(x,n) = 1\}$$

$$F \xrightarrow{\varphi} (-H_F(1,0), \frac{1}{2}T_F(1,0), F(1,0)).$$

Moreover, this map will be a bijection.

The above result gives us a nice expression for the following counting function for the integer points on the elliptic curve of height at most T:

$$v(n,T) := \#\left\{(x,y) \in \mathbb{Z}^2 : \begin{array}{c} y^2 = x^3 + Axn^2 + Bn^3, \\ \gcd(x,n) = 1, |x|^{1/2}, |y|^{1/3} \leq T, y \neq 0 \end{array}\right\}.$$

Corollary 4.2.7. *Let $A, B \in \mathbb{Z}^2$ satisfy that $4A^3 - 27B^2 \neq 0$. Then we have that*

$$v(n,T) = \sum_{F \in \mathscr{F}(A,B)/\Gamma_\infty} \mathbf{1}_{T_F(1,0)\neq 0} \mathbf{1}_{|H_F(1,0)|^{1/2}, |T_F(1,0)/2|^{1/3} \leq T}.$$

However, this sum is still not ideal for our application, as it sums over an infinite set $\mathscr{F}(A,B)/\Gamma_\infty$. Instead, we look to the the set of representatives under the action of $\mathrm{SL}_2(\mathbb{Z})$:

$$\mathcal{F}(A,B) := \{\text{integral binary quartic forms } F(x,y) : I(F) = -4A, J(F) = -4B\}/\mathrm{SL}_2(\mathbb{Z}).$$

Mordell [31] showed that the number of equivalence classes $\#\mathcal{F}(A,B)$ is finite. Morally, we want to use the following idea:

$$\mathcal{F}(A,B) \times \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty \approx \mathscr{F}(A,B)/\Gamma_\infty.$$

However, the above relation is not quite true; we need to make appropriate adjustments to handle our restriction to admissible forms and the condition that $F(1,0) > 0$.

Definition 4.2.8. Fix a binary quartic form F. A pair $(m_1, m_2) \in \mathbb{Z}^2$ is **admissible for** F if

$$\begin{pmatrix} m_1 & * \\ m_2 & * \end{pmatrix} \cdot F$$

is an admissible binary quartic form.

We note that in order for $(m_1, m_2) \in \mathbb{Z}^2$ to be admissible for a fixed binary quartic F, it must satisfy that $\gcd(m_1, m_2) = 1$. In fact, it is shown in [22, Proposition 4] that (m_1, m_2) are admissible for F *if and only if* the pair (m_1, m_2) satisfies

$$\gcd(m_1, m_2) = 1, \text{ and } \forall p \mid \Delta(F) \text{ prime}, (F(m_1, m_2), H_F(m_1, m_2)) \neq (0, 0) \bmod p.$$

We denote the $q = \prod_{p|\Delta(F)} p$. Let us denote by R_F the following congruence conditions:

$$R_F := \{\mathbf{m} \bmod q : \forall p \mid q \text{ prime}, (F(\mathbf{m}), H_F(\mathbf{m})) \neq (0, 0) \bmod p\}.$$

We can now make observation (4.2.1) rigorous, as was done in [22, Proposition 4]; we repeat Duke's proof in §4.2.3.

Lemma 4.2.9. *Let $A, B \in \mathbb{Z}^2$ satisfy that $4A^3 - 27B^2 \neq 0$. Define $Aut(F) \subset SL_2(\mathbb{Z})$ as the automorphism group of F. Then we have that*

$$v(n, T) = \sum_{F \in \mathcal{F}(A,B)} \frac{1}{\#Aut(F)} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^2 \\ \gcd(m_1, m_2) = 1 \\ \overline{\mathbf{m}} \in R_F \\ F(\mathbf{m}) = n}} \mathbf{1}_{T_F(\mathbf{m}) \neq 0} \mathbf{1}_{|H_F(\mathbf{m})|^{1/2}, |T_F(\mathbf{m})|^{1/3} \leq T}$$

Remark 4.2.10. Note that $\mathrm{Aut}(F)$ is finite for any element $F \in \mathcal{F}(A, B)$.

4.2.2 Reducing Theorem 4.1.4 to Theorem 4.1.7

Let $A, B \in \mathbb{Z}$ satisfy that $4A^3 - 27B^2 \neq 0$. Then we wish to count

$$N(T) = \#\{(x, y, u, v) \in S_{A,B,Q}(\mathbb{Z}) : h((x, y, u, v)) \leq T, \gcd(x, Q(u, v)) = 1, y \neq 0\}.$$

We can rewrite this in terms of the function $v(n, T)$ as

$$N(T) = \sum_{n \leq T^2} v(n, T) r_Q(n),$$

where $r_Q(n) = \#\{u, v \in \mathbb{Z} : Q(u, v) = n\}$. Plugging in Lemma 4.2.9, the expression above can be written as

$$\begin{aligned} N(T) &= \sum_{F \in \mathcal{F}(A,B)} \frac{1}{\#\mathrm{Aut}(F)} \sum_{n \leq T^2} r_Q(n) \sum_{\substack{\mathbf{m} \in \mathcal{A}_F \\ T_F(\mathbf{m}) \neq 0 \\ |H_F(\mathbf{m})|^{1/2}, |T_F(\mathbf{m})/2|^{1/3} \leq T}} \mathbf{1}_{F(\mathbf{m})=n} \\ &= \sum_{F \in \mathcal{F}(A,B)} \frac{1}{\#\mathrm{Aut}(F)} \sum_{\substack{\mathbf{m} \in \mathcal{A}_F \\ T_F(\mathbf{m}) \neq 0 \\ |H_F(\mathbf{m})|^{1/2}, |T_F(\mathbf{m})/2|^{1/3}, |F(\mathbf{m})|^{1/2} \leq T}} r_Q(F(\mathbf{m})). \end{aligned}$$

Next, we note that since $F \in \mathcal{F}(A, B)$, the discriminant $\Delta(F) = 16(4A^3 - 27B^2) \neq 0$. Thus, we automatically have that our sum is over squarefree binary quartic forms. Applying Lemma 4.2.3 and using that F, H_F, T_F are all homogeneous, we can see that the conditions

$$\{\mathbf{m} \in \mathbb{R}^2 : |H_F(\mathbf{m})|^{1/2}, |T_F(\mathbf{m})/2|^{1/3}, |F(\mathbf{m})|^{1/2} \leq T, T_F(\mathbf{m}) \neq 0\} =: T^{1/2}\mathcal{R}_F$$

where $\mathcal{R}_F$ is a compact region; Lemma 4.2.3 ensures that this region will be noncuspidal. Moreover, this region will have boundary satisfying that $|\partial T^{1/2}\mathcal{R}_F| \ll_F T^{1/2}$. We can rewrite the sum $N(T)$ as

$$N(T) = \sum_{F \in \mathcal{F}(A,B)} \frac{1}{\#\mathrm{Aut}(F)} \sum_{\substack{\mathbf{m} \in T^{1/2}\mathcal{R}_F \\ \gcd(m_1, m_2) = 1 \\ \mathbf{m} \in R_F}} r_Q(F(\mathbf{m})).$$

Here we recall R_F as the set of congruence conditions from (4.2.1). So, we apply Theorem 4.1.7 to each of these sums individually to achieve Theorem 4.1.4 – observe that since $(A, B, Q(u, v))$ are disassociated that $\beta_{F,Q} = 0$ for all $F \in \mathcal{F}(A, B)$. Note that $\#\mathcal{F}(A, B)$ and $\#R_F$ are both finite constants only dependent on $A, B \in \mathbb{Z}$. □

Remark 4.2.11. To count rational points instead of integral points with respect to the singularity using the method of syzygies described above, it seems that one would need to sum over binary quartic forms $F(x, y)$ with invariants ranging in a thin set

$$(I(F), J(F)) \in \{\lambda^2 A, \lambda^3 B : \lambda \in \mathbb{Z}\}.$$

However, the problem of tracking the dependence on F and estimating the main term seems to require precise understanding of the class number of binary quartics.

4.2.3 Proofs of Lemmas from §4.2.1

The proof of Lemma 4.2.3, which states that F and H_F share no common factors over $\overline{\mathbb{Q}}$, is computational.

Proof of Lemma 4.2.3. Assume that $F(x,y)$ and $H_F(x,y)$ share a common factor $(x+ry)$ over $\overline{\mathbb{Q}}$. We write that $F = (x+ry)C(x,y)$, where $C(x,y)$ is a cubic form. Then we can see that

$$F_x = C(x,y) + (x+ry)C_x(x,y),$$

$$F_y = rC(x,y) + (x+ry)C_y(x,y).$$

Thus, we can compute that

$$F_{xx} = 2C_x(x,y) + (x+ry)C_{xx}(x,y),$$

$$F_{xy} = C_y(x,y) + rC_x(x,y) + (x+ry)C_{xy}(x,y),$$

$$F_{yx} = rC_x(x,y) + C_y(x,y) + (x+ry)C_{yx}(x,y),$$

$$F_{yy} = 2rC_y(x,y) + (x+ry)C_{yy}(x,y).$$

Now, by the definition of $H_F(x,y)$, we can see that $(x+ry) \mid H_F(x,y)$ over $\mathbb{Q}$ if and only if

$$(x+ry) \mid 4rC_x(x,y)C_y(x,y) - (C_y(x,y) + rC_x(x,y))^2 = -(C_y(x,y) - rC_x(x,y))^2.$$

We expand out in terms of $C(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$:

$$\begin{aligned} C_y(x,y) - rC_x(x,y) &= (3dy^2 + 2cxy + bx^2) - r(3ax^2 + 2bxy + cy^2) \\ &= -3rax^2 + b(x^2 - 2rxy) + c(2xy - ry^2) + 3dy^2. \end{aligned}$$

Now, if $(x+ry) \mid C_y(x,y) - rC_x(x,y)$, then the above should be zero if we plug in $(r,-1)$. Hence, we get the following linear condition on a,b,c,d:

$$-3r^3a + 3r^2b - 3rc + 3d = 0.$$

We note that solutions to the above linear condition can be parameterized as $a' = a$, $b' = b - ar$,

$c' = c - b'r$, $d = c'r$, which is equivalent to $C(x,y) = (x+ry)(a'x^2 + b'xy + c'y^2)$. So, if $(x+ry)$ divides the Hessian $H_F(x,y)$, we must have that it divides $C(x,y)$ and hence it forces $F(x,y)$ to have a double root. Since, we have assumed $F(x,y)$ to be squarefree, this gives us a contradiction. $\square$

In Lemma 4.2.9, $v(n,T)$ is expressed in terms of the finite set of representative $\mathcal{F}(A,B)$ by following the proof of [22, Proposition 4]. We repeat it here for the purposes of self-containment, as the lemma is crucial for the proof of the main theorem.

Proof of Lemma 4.2.9. Recall that

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty = \left\{ \begin{pmatrix} m_1 & * \\ m_2 & * \end{pmatrix} : m_1, m_2 \in \mathbb{Z} \right\}.$$

Fix a binary quartic $F(x,y)$ with invariants $I(F) = -4A$ and $J(F) = -4B$. Define the set

$$\mathcal{A}_F := \{(m_1,m_2) \in \mathbb{Z}^2 : (m_1,m_2) \text{ is admissible for } F\} \subset \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty.$$

Then consider the map:

$$\varphi_F : \mathcal{A}_F \to \{(x,y,n) \in \mathbb{Z}^2 \times \mathbb{Z}_+ : y^2 = x^3 + Axn^2 + Bn^3, \gcd(x,n) = 1\},$$

$$\varphi_F(m_1,m_2) = (-H_F(m_1,m_2), T_F(m_1,m_2)/2, F(m_1,m_2)).$$

Since (m_1,m_2) are admissible for F, we know that the tuple $(-H_F(m_1,m_2), T_F(m_1,m_2), F(m_1,m_2))$ will lie in the set $\{(x,y,n) : y^2 = x^3 + Axn^2 + Bn^3, \gcd(x,n) = 1\}$.

Now, the kernel of this map will have size $\#\mathrm{Aut}(F)$ since φ_F is derived from the injection in Theorem 4.2.6; in particular,

$$\varphi_F = \varphi \circ \left(F \mid \begin{pmatrix} m_1 & * \\ m_2 & * \end{pmatrix} \right).$$

Finally, since the map φ from Theorem 4.2.6 was a bijection, the union of $\mathrm{Im}(\varphi_F)$, as F ranges in $\mathcal{F}(A,B)$, will cover all of $\{(x,y,n) : y^2 = x^3 + Axn^2 + Bn^3, \gcd(x,n) = 1\}$. Thus, we get that

$$v(n,T) = \sum_{F \in \mathcal{F}(A,B)} \frac{1}{\#\mathrm{Aut}(F)} \# \left\{ \mathbf{m} \in \mathcal{A}_F : \begin{array}{c} F(\mathbf{m}) = n, T_F(\mathbf{m}) \neq 0, \\ \max(|H_F(\mathbf{m})|^{1/2}, |T_F(\mathbf{m})/2|^{1/3}) \leq T \end{array} \right\}.$$

Plugging in the definition of the congruence conditions R_F and rearranging the sum, we achieve the

lemma. □

4.3 Correlations of binary quadratics and binary quartics

The goal of this section is to establish Theorem 4.1.7. This proof will follow closely the proof of Proposition 3.5.2 with $L = 1$, so we aim to highlight the differences in the two proofs. There are two key changes in this formulation. First, we let $Q(u, v)$ be *any positive-definite binary quadratic*, rather than of the form $x^2 + ay^2$. Second, we allow for a *congruence restriction* $(x, y) \equiv \alpha \bmod M$; however, we are allowing ourselves arbitrary dependency on M in the error term, which eases the difficulty of this constraint. We also point out that in this work we do not handle the indefinite case (although we expect this case to follow with the method presented in Chapter 3 for indefinite forms), nor do we inspect the leading constant.

There is also a noticeable difference in that we do not have the extra summation over the torsors indexed by t; this is due to counting integral points rather than rational points. Thus, the argument in §3.6 is not applicable for this correlation sum; correspondingly, the restrictions of $(A, B, Q(u, v))$ come from the gaps in the arguments of §3.7.

4.3.1 Notation

- K_Q is the quadratic extension of $\mathbb{Q}$ by the binary form $Q(u, v)$; this is an imaginary quadratic field.
- χ is the real quadratic field such that $\zeta_{K_Q}(s) = \zeta(s)L(s, \chi)$.
- Δ is the discriminant of $Q(u, v)$.
- $\mathfrak{a}, \mathfrak{p}$ are ideals of $\mathcal{O}_{K_Q}$.
- $\mathcal{C}$ denotes the class group of K_Q.
- $\mathcal{H}$ denotes the Hilbert class field of K_Q; then $\mathrm{Gal}(\mathcal{H}/K_Q) \cong \mathcal{C}$,
- K_F denotes the splitting field of a binary form $F(u, v)$.
- We define the q-part of an integer n as:

$$p_q(n) = \prod_{\substack{p \text{ prime} \\ p^e \| q \\ p | q}} p^e.$$

We also define the part of n coprime to q as

$$p_{\neg q}(n) = n/p_q(n) = \prod_{\substack{p \text{ prime} \\ p^e \| n \\ p \nmid q}} p^e,$$

4.3.2 Decomposition of representation number

Let us start by decomposing the representation number $r_Q(n)$ into Eisenstein and cuspidal components using class groups. Let $\mathcal{U}$ denote the class group of K_Q. Then we know that

$$r_Q(n) = |\mathcal{U}| \#\{\mathfrak{a} \subset \mathcal{O}_{K_Q} : N(\mathfrak{a}) = n, \mathfrak{a} \text{ lies in } [Q]\},$$

where $[Q]$ is the ideal class given by the quadratic form $Q(u, v)$. If $Q(u, v) = u^2 + av^2$, then $[Q] = 1$ is the principal class.

Using class group characters, we can reexpress the sum above as:

$$r_Q(n) = \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\psi \in \hat{\mathcal{C}}} \overline{\psi([Q])} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}).$$

As in §3.3.6, the Fourier coefficients of the Eisenstein and cuspidal components are respectively:

$$\lambda_E(n) := \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\substack{\psi \in \hat{\mathcal{C}} \\ \text{ord}(\psi) \leq 2}} \overline{\psi([Q])} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}) = \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\substack{q_1 q_2 = \Delta \\ \gcd(q_1, q_2)=1}} \psi_{q_1,q_2}([Q]) \sum_{d|n} \chi_{q_1}(d)\chi_{q_2}(n/d).$$

$$\lambda_C(n) := \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\substack{\psi \in \hat{\mathcal{C}} \\ \text{ord}(\psi) \geq 3}} \overline{\psi([Q])} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}).$$

Above, ψ_{q_1,q_2} denotes the genus character parameterized by the pair (q_1, q_2). So, we can rewrite the sum in Theorem 4.1.7 in terms of:

$$\mathcal{M}(X; \mathcal{R}, \alpha \bmod M; q_1, q_2) = \sum_{\substack{x,y \in X\mathcal{R} \\ \gcd(x,y)=1 \\ (x,y) \equiv \alpha \bmod M}} \sum_{d|F(x,y)} \chi_{q_1}(d)\chi_{q_2}(F(x,y)/d)$$

$$\mathcal{N}(X; \mathcal{R}, \alpha \bmod M; \psi) = \sum_{\substack{x,y \in X\mathcal{R} \\ \gcd(x,y)=1 \\ (x,y) \equiv \alpha \bmod M}} \sum_{N(\mathfrak{a})=F(x,y)} \psi(\mathfrak{a}).$$

As in Chapter 3, the contribution from the Eisenstein part (i.e. $\mathcal{M}(X; \mathcal{R}, \alpha \bmod M; q_1, q_2)$) will be

our main term, and the cuspidal contribution will form an error term.

4.3.3 The cuspidal contribution

For the cuspidal contribution, we directly apply Corollary 2.5.1, which is the consequence of a sieve argument in combination with analysis about the base change of GL_2 automorphic representations.

Proposition 4.3.1. *Let $\psi \in \hat{\mathcal{C}}$ and ord$(\psi) \geq 3$. If $K_Q \not\subset K_F$, then*

$$\mathcal{N}(X; \mathcal{R}, \alpha \bmod M; \psi) \ll_{F,Q,\mathcal{R}} X^2 \log(X)^{-1/25}.$$

If $K_Q \subset K_F$ and $K_F \cap \mathcal{H} = K_Q$, then

$$\mathcal{N}(X; \mathcal{R}, \alpha \bmod M; \psi) \ll_{F,Q,\mathcal{R}} X^2 \log(X)^{1/2}.$$

Proof. Note that

$$\mathcal{N}(X; \mathcal{R}, \alpha \bmod M; \psi) \leq \sum_{x,y \in X\mathcal{R}} \left| \sum_{N(\mathfrak{a})=F(x,y)} \psi(\mathfrak{a}) \right|.$$

Applying the sieve of [17] for squarefree binary forms instead of the sieve of Nair [33] for single variable polynomials, the above is an immediate consequence of Corollary 2.5.1. Observe that since $\deg(F) = 4$, we can evaluate that $c_{\xi,F} > 1/25$ in the argument of Theorem 2.1.1. □

Consequently, recalling the definition of $\lambda_C(n)$ (4.3.2), we know that

$$\sum_{\substack{(x,y) \in X\mathcal{R} \\ \gcd(x,y)=1 \\ (x,y) \equiv \alpha \bmod M}} \lambda_C(F(x,y)) \ll_{F,Q,\mathcal{R}} X^2 \log(X)^{\beta_{F,Q}-1/25}.$$

4.3.4 The Eisenstein contribution

Finally, we wish to derive an asymptotic for (4.3.2) in the same vein as Proposition 3.8.1 when $t = k = 1$; the main difference in our analysis is the tracking of the congruence condition $\alpha \bmod M$. However, since we allow arbitrary dependency on $\alpha \bmod M$, this does not bring any technical complications to the proof, only notational. We also point out that the argument below is *not* self-contained and will cite many lemmas in Chapter 3; a similar argument is also presented in [18] for analyzing these sums. Our goal in this subsection is to establish:

Proposition 4.3.2. *Let F be a squarefree binary quartic form. Let $\gcd(q_1, q_2) = 1$ and $q_1 q_2 = \Delta(Q)$. Then for any fixed convex region $\mathcal{R}$ and congruence condition $\alpha \bmod M$,*

$$\mathcal{M}(X; \mathcal{R}, \alpha \bmod M; q_1, q_2) = c_{\alpha \bmod M, q_1, q_2, F, Q} X^2 |\mathcal{R}| \log(X)^{\beta_{F,Q}} + O_{\mathcal{R}, M, F, Q}(X^2 \log(X)^{\beta_{F,Q} - 10^{-10}}),$$

where $\beta_{F,Q}$ is the exponent defined in Theorem 4.1.7.

Proof. First, we move the congruence condition that $(x, y) \equiv \alpha \bmod M$ into the quartic form; we do so by observing that

$$\sum_{\substack{(x,y) \in X\mathcal{R} \\ \gcd(x,y)=1 \\ (x,y) \equiv \alpha \bmod M}} \sum_{d | F(x,y)} \chi_{q_1}(d) \chi_{q_2}(F(x,y)/d)$$

$$= \sum_{\substack{(x',y') \in M^{-1}(X\mathcal{R} - \alpha) \\ \gcd(Mx' + \alpha_1, My' + \alpha_2) = 1}} \sum_{d | F(Mx' + \alpha_1, My' + \alpha_2)} \chi_{q_1}(d) \chi_{q_2}(F(Mx' + \alpha_1, My' + \alpha_2)/d).$$

Let us define $F_{\alpha,M}(x', y') := F(Mx' + \alpha_1, My' + \alpha_2)$; we know that $F_{\alpha,M}$ is a square binary quartic form. We also write $\mathcal{R}_{\alpha,M} := M^{-1}(X\mathcal{R} - \alpha)$, which will still be a convex region. Furthermore, $|\mathcal{R}_{\alpha,M}| = M^{-2} X |\mathcal{R}|$ and $\partial \mathcal{R}_{\alpha,M} \ll_{\mathcal{R}} M^{-1} X$. In summary,

$$\mathcal{M}(X; \mathcal{R}, \alpha \bmod M; q_1, q_2) = \sum_{\substack{\mathbf{x} \in \mathcal{R}_{\alpha,M} \\ \gcd(Mx_1 + \alpha_1, Mx_2 + \alpha_2) = 1}} \sum_{d | F_{\alpha,M}(\mathbf{x})} \chi_{q_1}(d) \chi_{q_2}(F_{\alpha,M}(\mathbf{x})/d).$$

Second, we use Lemma 3.8.3 to turn the sum over divisors $d \mid F_{\alpha,M}(\mathbf{x})$ into a sum over divisors of the irreducible factors of $F_{\alpha,M}(\mathbf{x})$. We repeat it below for ease of the reader:

Lemma. *Denote the Eisenstein coefficient as*

$$\varepsilon_{q_1,q_2}(n) = \sum_{d | n} \chi_{q_1}(d) \chi_{q_2}(n/d).$$

Let n and m be integers. Then

$$\varepsilon_{q_1,q_2}(nm) = \sum_{c | \gcd(m,n)} \mu(c) \chi(c) \varepsilon_{q_1,q_2}(n/c) \varepsilon_{q_1,q_2}(m/c).$$

By the reduction of §3.8.1, we have that

$$\mathcal{M}(X;\mathcal{R},\alpha \bmod M;q_1,q_2)=\sum_{\mathbf{c}}\mu(\mathbf{c})\chi(\mathbf{c})S(X;\mathcal{R},\alpha \bmod M;\mathbf{c}),$$

where the sum is over tuples $\mathbf{c}$ with $c_{ij} \mid \mathrm{Res}(F_i,F_j)$ for $F_1,...,F_r$ irreducible factors of $F_{\alpha,M}$ and

$$S(X;\mathcal{R},\alpha \bmod M;\mathbf{c})=\sum_{\substack{\mathbf{x}\in\mathcal{R}_{\alpha,M}\\ \gcd(Mx_1+\alpha_1,Mx_2+\alpha_2)=1\\ c_i|F_i(\mathbf{x})}}\prod_{i=1}^{r}\sum_{d_i|F_i(\mathbf{x})}\chi_{q_1}(d_i)\chi_{q_2}(F_i(\mathbf{x})/d_ic_i).$$

Third, we follow §3.8.2 and relate $\chi_{q_1}\star\chi_{q_2}$ with the Dirichlet convolution $1\star\chi$, where χ is the quadratic character generating K_Q. We obtain that

$$\begin{aligned}&S(X;\mathcal{R},\alpha \bmod M;\mathbf{c})\\ &\quad=\sum_{\substack{n\ll X\\ p|n\implies p|\Delta(Q)}}\chi_{q_1}(p_{q_2}(n))\chi_{q_2}(p_{q_1}(n))\sum_{\substack{\mathbf{a} \bmod \Delta(Q)\\ \gcd(a_i,\Delta(Q))=1}}\prod_{i=1}^{r}\chi_{q_1}(a_i)S(X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a}),\end{aligned}$$

where we define

$$S(X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a})=\sum_{\substack{\mathbf{x}\in\mathcal{R}_{\alpha,M}\\ \gcd(Mx_1+\alpha_1,Mx_2+\alpha_2)=1\\ c_i|F_i(\mathbf{x})\\ p_{\Delta(Q)}(F_{\alpha,M}(\mathbf{x}))=n\\ p_{\neg\Delta(Q)}(F_i(\mathbf{x})/c_i)\equiv a_i \bmod n\Delta(Q)}}\prod_{i=1}^{r}(1\star\chi)(p_{\neg\Delta(Q)}(F_i(\mathbf{x})/c_i)).$$

Next, we apply Möbius inversion to remove the gcd condition:

$$S(X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a})=\sum_{b\ll_{\mathcal{R}}X}\mu(b)S(X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a},b),$$

where we define

$$S(X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a},b)=\sum_{\substack{\mathbf{x}\in\mathcal{R}_{\alpha,M}\\ b|M\mathbf{x}+\alpha\\ c_i|F_i(\mathbf{x})\\ p_{\Delta(Q)}(F_{\alpha,M}(\mathbf{x}))=n\\ p_{\neg\Delta(Q)}(F_i(\mathbf{x})/c_i)\equiv a_i \bmod n\Delta(Q)}}\prod_{i=1}^{r}(1\star\chi)(p_{\neg\Delta(Q)}(F_i(\mathbf{x})/c_i))$$

By Lemma 3.8.4 and Lemma 3.8.6 with $t=k=1$, $L=1$, we can bound away the contribution from those $n,b\geq\log(X)^{-10^{-10}}$ to a suitable error term.

We now recall that in Chapter 3 to evaluate $S(X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a},b)$, we needed to establish two results: a level of distribution bound and an upper bound on the contribution of the large moduli. Define for a fixed set of divisors $\mathbf{d}=(d_i)_{i=1}^r$:

$$T(\mathbf{d};X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a},b)=\#\left\{\mathbf{x}\in\mathcal{R}_{\alpha,M}:\begin{array}{c} b\mid M\mathbf{x}+\alpha,\\ c_id_i\mid F_i(\mathbf{x}),\\ p_{\Delta(Q)}(F_{\alpha,M}(\mathbf{x}))=n,\\ p_{\neg\Delta(Q)}(F_i(\mathbf{x})/c_i)\equiv a_i \bmod n\Delta(Q)\end{array}\right\}.$$

First, we would like a nearly-sufficient level of distribution bound for our binary form $F_{\alpha,M}$. Since we allow arbitrary dependence on our congruence condition $\alpha \bmod M$, we can establish this with careful lattice estimates.

Lemma 4.3.3. *Let $F_{\alpha,M}=\prod_{i=1}^r F_i$ be a squarefree binary form of degree ≤ 4, where $F_i(x,y)$ are irreducible factors of $F(x,y)$. For a tuple $\mathbf{e}=(e_i)$, we denote the product of the entries as $e=e_1...e_r$. For a fixed $n,\mathbf{a},\mathbf{c},\mathbf{d}=(d_i)$ and b, we define the following local function:*

$$\begin{aligned}\varrho_F(\mathbf{d};\alpha \bmod M;\mathbf{c},n,\mathbf{a},b):=\#\{\mathbf{x} \bmod bcdn\Delta(Q):c_id_i\mid F_i(x,y),\\ b\mid \gcd(Mx_1+\alpha_1,Mx_2+\alpha_2),\gcd(Mx_1+\alpha_1,Mx_2+\alpha_2,cd\Delta(Q))=1,\\ p_{\Delta(Q)}(F_{\alpha,M}(x,y))=n,p_{\neg\Delta(Q)}(F_i(x,y)/c_i)\equiv a_i \bmod \Delta(Q)\}/\sim,\end{aligned}$$

where we say that $\mathbf{x}\sim\mathbf{x}'$ if $\mathbf{x}\equiv\lambda\mathbf{x}' \bmod cd$ for some $\lambda\in\mathbb{Z}$. For a tuple $\mathbf{D}=(D_i)_{i=1}^r$, we define the error quantity as:

$$E(\mathbf{D})=\sum_{d_i\sim D_i}\left|T(\mathbf{d};X;\mathcal{R},\alpha \bmod M;\mathbf{c},n,\mathbf{a},b)-|\mathcal{R}_{\alpha,M}|\cdot\frac{\varrho_F(\mathbf{d};\alpha \bmod M;\mathbf{c},n,\mathbf{a},b)}{cdb^2n^2\Delta(Q)^2}\right|.$$

If $F(x,y)$ has no linear factors, then we have that for an explicit constant $\gamma_r>0$,

$$E(\mathbf{D})\ll_{\mathcal{R},M,F,Q} D+\frac{\tau(b^4)}{b}\cdot\sqrt{D}\cdot X^{1/2}\exp(\gamma_r\sqrt{\log_2(X)\log_3(X)}).$$

If $F(x,y)$ has a linear factor, then we have that

$$\begin{aligned}E(\mathbf{D})\ll_{\mathcal{R},M,F,Q} D+X^{1/2}\cdot\frac{\tau(b^4)}{b}\cdot\left(\sqrt{D}\exp(\gamma_r\sqrt{\log_2(X)\log_3(X)})\right)\\ +X^{1/2}\cdot\frac{\tau(b^4)}{b}\cdot\left(\log(D)\cdot\max_{i:\deg(F_i)=1}D_i\right).\end{aligned}$$

Proof. Since $\mathcal{R}_{\alpha,M}$ is a convex region satisfying that $\partial\mathcal{R}_{\alpha,M} \ll_{\mathcal{R},M} X$, we apply the lattice point counting argument of Proposition 3.9.1, taking $L = t = k = 1$ and replacing the box $B^{1/2}\mathcal{B}(\mathbf{x}_0, L)$ with our region $\mathcal{R}_{\alpha,M}$. Finally, we recall that $c = O_F(1)$ and remove the dependency on c in the final bounds. □

In the above level of distribution, since $|\mathcal{R}_{\alpha,M}| = (X/M)^2|\mathcal{R}|$, $E(\mathbf{D})$ is sufficiently small for $D \leq X^2\log(X)^{-10^{-10}}$ if $F(x,y)$ has no linear factors (resp. $D_i \leq X^{1/2}\log(X)^{-1-10^{-10}}$ if $F(x,y)$ has a linear factor). So it remains to bound the contribution from the large moduli. For this, we cite two bounds from §3.10 with $L = t = k = 1$ and $B^{1/2} = X$.

Lemma 4.3.4 (Proposition 3.10.1)**.** *Let $F_{\alpha,M}$ be a squarefree binary form. Let $\mathbf{D} = (D_i)_{i=1}^r$ satisfy that $\log(D) \gg \log(X)$. Then the following bound holds:*

$$\sum_{d_i \sim D_i} T(\mathbf{d}; X; \mathcal{R}, \alpha \bmod M; \mathbf{c}, n, \mathbf{a}, b) \ll_{\mathcal{R},M,F,Q} \left(\frac{X^2}{b^2} + \frac{X^{3/2}}{b}\right) \cdot \log(X)^{\beta_{F,Q}-10^{-5}} \cdot \left(\frac{X^2}{D} + \frac{D}{X^2} + \log\log(X)\right)^{1/2}.$$

Here $\beta_{F,Q}$ is defined as in Theorem 4.1.7.

Proof. We make the remark that again replacing the box $B^{1/2}\mathcal{B}(\mathbf{x}_0, L)$ with the region $\mathcal{R}_{\alpha,M}$ brings no extra complications since $\mathcal{R}_{\alpha,M}$ can be contained in a box of side length $O_{\mathcal{R},M}(X)$. We also note that in the notation of Proposition 3.10.1, $\beta_{F,Q} = \varrho_{\Delta,F} - 2$ and that $\varrho_{\Delta,F}/2 - 1 \leq \varrho_{\Delta,F} - 2$. □

Lemma 4.3.5. *Proposition 3.10.2 Let $F_{\alpha,M}$ be a squarefree binary form with a linear factor, denoted as $F_1(x,y)$. Then, the following bound holds:*

$$\sum_{d_1 \sim D_1} \sum_{\substack{d_i \ll X^{\deg(F_i)/2} \\ i=2,\ldots,r}} T(\mathbf{d}; X; \mathcal{R}, \alpha \bmod M; \mathbf{c}, n, \mathbf{a}, b) \ll_{\mathcal{R},M,F,Q} X^2 \cdot \log(X)^{\beta_{F,Q}-10^{-5}}.$$

Proof. Since $\mathcal{R}_{\alpha,M}$ can be contained in a box of side length $O_{\mathcal{R},M}(X)$, the proof can proceed as in Proposition 3.10.2. □

Combining Lemmas 4.3.3, 4.3.4, and 4.3.5 as in §3.11, we achieve that

$$S(X; \mathcal{R}, \alpha \bmod M; \mathbf{c}, n, \mathbf{a}, b) = |\mathcal{R}_{\alpha,M}| \sum_{d_i \ll X^{\deg(F_i)/2}} \prod_{i=1}^{r} \chi(d_i) \cdot \frac{\varrho_F(\mathbf{d}; \alpha \bmod M; \mathbf{c}, n, \mathbf{a}, b)}{cdb^2n^2\Delta(Q)^2} + O_{\mathcal{R},M,F,Q}\left(X^2\log(X)^{\beta_{F,Q}-10^{-10}}\right).$$

Note that since we can separately bound the contribution from the large $b \geq \log(X)^{10^{-10}}$, we can undo our Möbius inversion and remove the dependency on b. Finally, it remains to consider the sum over d_i. Consider for a fixed $\mathbf{c}, n, \mathbf{a}$:

$$\xi(s; \alpha \bmod M; \mathbf{c}, n, \mathbf{a}) = \sum_{d_i} \prod_{i=1}^{r} \chi(d_i) \cdot \frac{\varrho_F(\mathbf{d}; \alpha \bmod M; \mathbf{c}, n, \mathbf{a})}{d^s},$$

where we take

$$\varrho_F(\mathbf{d}; \alpha \bmod M; \mathbf{c}, n, \mathbf{a}) = \#\{\mathbf{x} \bmod cdn\Delta(Q) : \gcd(Mx_1 + \alpha_1, Mx_2 + \alpha_2, cd\Delta(Q)) = 1,$$
$$c_i d_i \mid F_i(x,y), p_{\Delta(Q)}(F_{\alpha,M}(x,y)) = n, p_{\neg\Delta(Q)}(F_i(x,y)/c_i) \equiv a_i \bmod \Delta(Q)\}/\sim .$$

It is clear that since for p_i prime, $\varrho_F((p_i^{e_i}); \alpha \bmod M; \mathbf{c}, n, \mathbf{a})$ is bounded, the Dirichlet series $\xi(s; \alpha \bmod M, \mathbf{c}, n, \mathbf{a})$ converges absolutely for $\Re(s) > 1$. Furthermore it has an Euler product expansion and that

$$\xi(s; \alpha \bmod M; \mathbf{c}, n, \mathbf{a}) = R_{M,\mathbf{c},n,\mathbf{a}}(s) \prod_{i=1}^{r} \zeta_{K_{F_i}}(s, \chi),$$

where $R_{M,\mathbf{c},n,\mathbf{a}}(s)$ is a factor that converges for $\Re(s) > 3/4$, the field K_{F_i} is the splitting field of F_i, and $\zeta_{K_{F_i}}(s,\chi)$ is the twist of Dedekind zeta function of K_{F_i} by the quadratic character χ mod $\Delta(Q)$.

Thus, we can rewrite the expression as

$$S(X; \mathcal{R}, \alpha \bmod M; \mathbf{c}, n, \mathbf{a}) = |\mathcal{R}_{\alpha,M}| \frac{R_{M,\mathbf{c},n,\mathbf{a}}(1)}{cn^2\Delta(Q)^2} \cdot \prod_{i=1}^{r} \left(\mathrm{Res}_{s=1} \zeta_{K_{F_i}}(s,\chi) \cdot \log(X)^{\mathbf{1}_{K_Q \subset K_{F_i}}} \right)$$
$$+ O_{\mathcal{R},M,F,Q}\left(X^2 \log(X)^{\beta_{F,Q} - 10^{-10}} \right).$$

Here we have used $\mathrm{Res}_{s=1}\zeta_{K_{F_i}}(s,\chi)$ to denote the residue at $s=1$ of the L-function, or if $\zeta_{K_{F_i}}(s,\chi)$ is analytic at $s=1$ then its value. Recall at this point that $|\mathcal{R}_{\alpha,M}| = (X/M)^2|\mathcal{R}|$.

Finally, we observe the relation that

$$\beta_{F,Q} = \sum_{i=1}^{r} \mathbf{1}_{K_Q \subset K_{F_i}}.$$

In the expression for $S(X; \mathcal{R}, \alpha \bmod M; \mathbf{c}, n, \mathbf{a})$ above, the exponent of $\log(X)$ is $\beta_{F,Q}$ and additionally independent of $(\mathbf{c}, n, \mathbf{a})$. Thus, to establish the asymptotic of Proposition 4.3.2, it remains to

check that

$$\sum_{\substack{n \leq \log(X)^{10^{-10}} \\ p|n \implies p|\Delta(Q)}} \frac{R_{M,\mathbf{c},n,\mathbf{a}}(1)}{n} \ll_{M,\mathbf{c},\mathbf{a},F,Q} 1.$$

Here we have recalled that the sums over $\mathbf{c}$ and $\mathbf{a}$ are both finite (in the sense that neither grow with X). For this final fact, we note that for any choice of $n, \mathbf{a}$, the upper bound $\varrho_F(1; \alpha \bmod M; \mathbf{c}, n, \mathbf{a}) \ll_{F,Q,M} n$ holds. Hence, since the sum of n is over a geometric series, we know that the sum over n even converges. This completes the proof of Proposition 4.3.2

□

4.3.5 Proof of Theorem 4.1.7

Recalling that

$$r_Q(n) = \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\psi \in \hat{\mathcal{C}}} \overline{\psi([Q])} \sum_{N(\mathfrak{a})=n} \psi(\mathfrak{a}),$$

and applying Proposition 4.3.1 (for $\mathrm{ord}(\psi) \geq 3$) and Proposition 4.3.2 (for $\mathrm{ord}(\psi) \leq 2$), we establish that

$$\begin{aligned}
\sum_{\substack{x,y \in X\mathcal{R} \\ \gcd(x,y)=1 \\ (x,y) \equiv \alpha \bmod M}} r_Q(F(x,y)) &= \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\substack{q_1 q_2 = \Delta(Q) \\ \gcd(q_1,q_2)=1}} c_{\alpha \bmod M, q_1, q_2, F, Q} X^2 |\mathcal{R}| \log(X)^{\beta_{F,Q}} \\
&\quad + O_{\mathcal{R},M,F,Q}(X^2 \log(X)^{\beta_{F,Q} - 10^{-10}}) + \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\substack{\psi \in \hat{C} \\ \mathrm{ord}(\psi) \geq 3}} O_{F,Q,\mathcal{R}}(X^2 \log(X)^{\beta_{F,Q} - 1/25}) \\
&= c_{\alpha,F,Q} X^2 |\mathcal{R}| \log(X)^{\beta_{F,Q}} + O_{\mathcal{R},F,Q,M}(X^2 \log(X)^{\beta_{F,Q} - 10^{-10}}),
\end{aligned}$$

where we take the constant $c_{\alpha,F,Q}$ to be

$$c_{\alpha,F,Q} = \frac{|\mathcal{U}|}{|\mathcal{C}|} \sum_{\substack{q_1 q_2 = \Delta(Q) \\ \gcd(q_1,q_2)=1}} c_{\alpha \bmod M, q_1, q_2, F, Q}.$$

□

4.4 Examples and lower bounds

In this section, we compute Theorem 4.1.4 for certain values of A and B such that $4A^3 - 27B^2 \neq 0$. In particular, we check when $c_{A,B,Q} > 0$ (and consequently derive lower bounds towards Manin's

conjecture).

4.4.1 The surface $S_{-1,0,u^2+v^2}$

In this case, we look at the surface:

$$y^2 = x^3 - x(u^2+v^2)^2.$$

For this surface, we have that $I = 4$ and $J = 0$. As computed by Duke in [22, Example 1], there are two classes of binary quartics in $\mathcal{F}(1,0)$:

$$x^4 + 6x^2y^2 + y^4, \text{ and } x^4 + 4y^4.$$

First, we calculate the exponent $\beta_{-1,0,u^2+v^2}$. Note that $K_Q = \mathbb{Q}(i)$ since $Q(u,v) = u^2+v^2$; since K_Q has class number one, the Hilbert class field $\mathcal{H} = K_Q$ and the Proposition 4.3.1 can always be applied. Next, to calculate the exponents we treat each binary quartic separately:

- Consider $x^4 + 6x^2y^2 + y^4 = (x^2+3y^2)^2 - 8y^4$. The splitting field of this binary quartic form is given by
$$K_F = \mathbb{Q}(\sqrt{2\sqrt{2}-3}, \sqrt{-3-2\sqrt{2}}),$$
which does not contain K_Q. Thus, $\beta_{x^4+6x^2y^2+y^4,u^2+v^2} = 0$.

- Consider $x^4+4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$. The splitting field of this binary quartic form is given by
$$K_F = \mathbb{Q}(1+i, 1-i) = \mathbb{Q}(i) = K_Q.$$
Thus, $\beta_{x^4+4y^4,u^2+v^2} = 2$.

As a result, $\beta_{-1,0,u^2+v^2} = 1$ and we know that

$$N(T) = c_{-1,0,u^2+v^2} \cdot T\log(T)^2 + O(T\log(T)^{2-10^{-10}}).$$

Next, we hope to analyze $c_{-1,0,u^2+v^2}$ and show that it is strictly positive. Since u^2+v^2 is a norm form of $\mathbb{Q}(i)$, we can apply §3.12 to deduce that $c_{-1,0,u^2+v^2} = 0$ if and only if the Châtelet surface given by $\{x^2+y^2 = z^4+4\}$ has a local or Brauer-Manin obstruction. One can see that the points $u = 0, v = \pm 2, z = 0$ are rational points on the surface $u^2+v^2 = z^4+4$. So, $c_{x^4+4y^4,u^2+v^2} > 0$.

Consequently, in this case we deduce a lower bound towards Manin's conjecture. Specifically, let $\tilde{N}(T)$ be defined as in Conjecture 4.1.1; then we know that

$$\tilde{N}(T) \gg T\log(T)^2.$$

In comparison, Manin's conjecture predicts that since $\varrho_{-1,0,u^2+v^2} = 5$ that $\tilde{N}(T) \asymp T\log(T)^4$ in this case.

4.4.2 The surface $S_{-1,0,u^2+5v^2}$

In this case, we look at the surface

$$y^2 = x^3 - x(u^2+5v^2)^2.$$

Again, we have that $A = -1$ and $B = 0$; thus we take $I = 4$ and $J = 0$. Note that unlike in the previous case, $K_Q = \mathbb{Q}(\sqrt{-5})$ *does not have class number one.* However, from our computations of the splitting fields of $x^4 + 6x^2y^2 + y^4, x^4 + 4y^4$ above, we can see that $(-1, 0, u^2 + 5v^2)$ is a disassociated tuple. Thus, we can apply directly Theorem 4.1.4 and see that

$$N(T) = c_{-1,0,u^2+5v^2} \cdot T + O(T\log(T)^{-10^{-10}}).$$

Still, u^2+5v^2 is a norm form for the quadratic field $\mathbb{Q}(\sqrt{-5})$; thus, to determine if $c_{-1,0,u^2+5v^2} > 0$ it suffices to study two Châtelet surfaces:

$$x^2 + 5y^2 = z^4 + 6z^2 + 1, \qquad x^2 + 5y^2 = z^4 + 4.$$

In fact, if we can find a rational point on either Châtelet surface then we know that $c_{-1,0,u^2+5v^2} > 0$. We can easily see that $(\pm 1, 0, 1)$ (resp. $(0, \pm 1, \pm 1)$) is a rational point on the first (resp. second) Châtelet surface. Define $\tilde{N}(T)$ again as in Conjecture 4.1.1; we again deduce a lower bound towards Manin's conjecture:

$$\tilde{N}(T) \gg T.$$

In comparison, since $z^3 + 20z$ has two irreducible factors over K_Q, we have that $\varrho_{A,B,Q} = 4$. So, Manin's conjecture predicts in this case that $\tilde{N}(T) \asymp T\log(T)^3$.

4.4.3 The surface $S_{-97/48,955/864,u^2+v^2}$

Finally, we consider the surface

$$y^2 = x^3 - \frac{97}{48}x(u^2+v^2)^2 + \frac{955}{864}(u^2+v^2)^3.$$

In this case, we have that $A = -97/48$ and $B = 955/864$, which are not integers. Nevertheless, we can proceed to look at the integral solutions to this surface given by Mordell's parameterization. Using that $I = 97/12$ and $J = -955/216$ and the fundamental domain described by Mordell [31] in his finiteness theorem for the class number of binary quartics, we achieve that there are four classes in $\mathcal{F}(97/12, -955/216)$:

$$-x^4 + 5x^2y^2 - 6y^4, x^4 + 5x^2y^2 + 6y^4, -2x^4 + 5x^2y^2 - 3y^4, 2x^4 + 5x^2y^2 + 3y^4.$$

Note that all four of the above expression factor into a product of two irreducible quadratics. In particular,

$$-x^4 + 5x^2y^2 - 6y^4 = (-x^2 + 3y^2)(x^2 - 2y^2) \implies K_{-x^4+5x^2y^2-6y^4} = \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

$$x^4 + 5x^2y^2 + 6y^4 = (x^2 + 3y^2)(x^2 + 2y^2) \implies K_{x^4+5x^2y^2+6y^4} = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}).$$

$$-2x^4 + 5x^2y^2 - 3y^4 = (-x^2 + y^2)(2x^2 - 3y^2) \implies K_{-2x^4+5x^2y^2-3y^4} = \mathbb{Q}(\sqrt{3/2}).$$

$$2x^4 + 5x^2y^2 + 3y^4 = (x^2 + y^2)(2x^2 + 3y^2) \implies K_{2x^4+5x^2y^2+3y^4} = \mathbb{Q}(i, \sqrt{-3/2}).$$

From the above calculations, we can see that:

$$\beta_{-x^4+5x^2y^2-6y^4,u^2+v^2} = 0, \beta_{x^4+5x^2y^2+6y^4,u^2+v^2} = 0,$$

$$\beta_{-2x^4+5x^2y^2-3y^4,u^2+v^2} = 0, \beta_{2x^4+5x^2y^2+3y^4,u^2+v^2} = 1.$$

Thus, we get that $\beta_{-97/48,955/864,u^2+v^2} = 1$.

The reason we focus on this particular example is that it demonstrates an interesting phenomenon: there exists a Brauer-Manin obstruction to rational points in one of the fibers. Specifically, Iskovskikh's famous counter example to the Hasse principle for Châtelet surfaces [28] appears to force $c_{-x^4+5x^2y^2-6y^4,u^2+v^2} = 0$.

On the other hand, we can construct explicit points on the Châtelet surface

$$u^2 + v^2 = 3z^4 + 5z^2 + 2.$$

So, we must have that $c_{2x^4+5x^2y^2+3y^4,u^2+v^2} > 0$. Since we also calculated that $2x^4 + 5x^2y^2 + 3y^4$ is the only form for which $\beta_{F,Q} > 0$, we must have that on a particular thin subset of

$$y^2 = x^3 - \frac{97}{48}x(u^2 + v^2)^2 + \frac{955}{864}(u^2 + v^2)^3,$$

there are $\asymp T\log(T)$ integral points.

Remark 4.4.1. This example is a surface where there is a Brauer-Manin obstruction blocking the existence of integral points corresponding to one of the relevant binary quartic forms, but after ranging over all of the binary quartics in $\mathcal{F}(97/12, -955/216)$ there will still be integral points on the other fibers. It would be interesting to see if one can construct a surface $S_{A,B,Q}$ where the Brauer-Manin obstructions the existence of points on all fibers.

4.5 Computations of the Picard group

We want to compute the rank ϱ of the Picard group over $\mathbb{Q}$ of the desingularization of $S_{A,B,Q}$ and hence the exponent of $\log(T)$ in Conjecture 4.1.1. First, we consider the geometry of the surface $S_{A,B,Q}$ over $\overline{\mathbb{Q}}$. Let $\alpha, \overline{\alpha}$ be the two roots of the quadratic form $Q(x, 1) = 0$. We can see that there two singularities given by the points $[0, 0, \alpha, 1]$ and $[0, 0, \overline{\alpha}, 1]$ sitting in the weighted projective space $\mathbb{P}(2, 3, 1, 1)$. Additionally, we have that there are three lines given by

$$L_i = \{y = 0, x - \beta_i Q(u, v) = 0\}, i = 1, 2, 3,$$

where β_i are roots of the cubic $x^3 + Ax + B = 0$, and the two lines in $\mathbb{P}(2, 3, 1, 1)$:

$$E_+ = \{u - \alpha v = 0, y^2 = x^3\}, E_- = \{u - \overline{\alpha} v = 0, y^2 = x^3\}.$$

We exclude L_1, L_2, L_3, E_1, E_2 by restricting to the open set $U_1 \cap U_2$ as defined in Theorem 4.1.4.

Next, we will compute the minimal desingularization of $S_{A,B,Q}$ over $\overline{\mathbb{Q}}$. Then, we will find the rank of

$$\mathrm{Pic}_{\mathbb{Q}}(\tilde{S}) = \mathrm{Pic}_{\mathbb{Q}(\alpha)}(\tilde{S})^{\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})}.$$

4.5.1 Computations over $\overline{\mathbb{Q}}$

We denote the minimal desingularization of $S_{A,B,Q}$ as $\tilde{S}_{A,B,Q}$. To compute the desingularization, we first will blow up the singular point $[0,0,\alpha,1]$; we denote this blow up as S_α. We look at the image in three charts: $D(x)$, $D(y)$ and $D(u-\alpha)$. In $D(x)$, the defining equation for the blowup is given by

$$y_1^2 - x(1 + A(u_1x + \alpha - \overline{\alpha})^2u_1^2 + B(u_1x + \alpha - \overline{\alpha})^3u_1^3)) = 0,$$

where we have written $y = y_1x$ and $u - \alpha = u_1x$. So, the exceptional divisor is given by

$$\{y_1 = 0, x = 0\}$$

where u_1 is allowed to range over $\overline{\mathbb{Q}}$. We recall that $\alpha - \overline{\alpha} = \sqrt{\Delta(Q)}$. Let z_i denote the roots of the equation $1 + A\Delta(Q)z^2 + B(\sqrt{\Delta(Q)})^3z^3 = 0$. Then there are double points on the exceptional divisor at $y_1 = 0, x = 0$ and $u_1 = z_i$ for $i = 1, 2, 3$.

Next, we look at the chart $D(y)$; here, the defining equation is given by

$$y^2(1 + y(x_1^3 + A(u_1x + \alpha - \overline{\alpha})^2(x_1u_1^2) + B(u_1x + \alpha - \overline{\alpha})^3(u_1)^3) = 0,$$

where we have taken $x = x_1y$ and $u - \alpha = u_1y$. We can see that there are no additional singularities in this chart.

In our final chart, $D(u-\alpha)$, we get the defining equation

$$(u-\alpha)^2(y_1^2 - (u-\alpha)(x_1^3 + A(u-\overline{\alpha})^2x_1 + B(u-\overline{\alpha})^3) = 0.$$

The exceptional divisor is given by

$$\{y_1 = 0, u = \alpha, x_1 \in \overline{\mathbb{Q}}\}.$$

There are double points given by solutions λ_i to $\lambda^3 + A\Delta(Q)\lambda + B(\sqrt{\Delta(Q)})^3 = 0$.

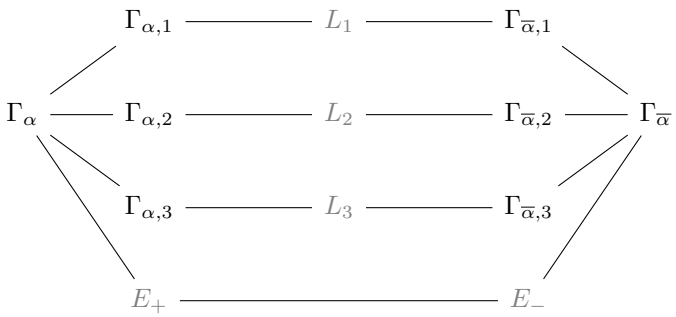
Thus, we can see that in our blow up S_α, there is an exception divisor $\mathbb{P}^1$, which we will denote as Γ_α. Additionally, there are three singularities corresponding to the roots z_i of $z^3 + A\Delta(Q)z + B(\sqrt{\Delta(Q)})^3$. We note that since we have assumed that $4A^3 - 27B^2 \neq 0$, these are three distinct roots.

So, we blow up S_α at each of these singularities corresponding to z_1, z_2, and z_3 to form S'_α. We

can see that this blowup will have three additional exceptional divisors $\Gamma_{\alpha,1}, \Gamma_{\alpha,2}, \Gamma_{\alpha,3}$ isomorphic to $\mathbb{P}^1$ that do not intersect each other and intersect Γ_α each at one point. Additionally, all four of these divisors will be (-2)-curves, i.e. $\Gamma_i \cdot \Gamma_i = -2$, since they are blow ups of double points.

Next, we can blow up S'_α at $[0, 0, \overline{\alpha}, 1]$ once to create $S'_{\overline{\alpha}}$. Analogously to the above computation, we will have an exceptional divisor $\Gamma_{\overline{\alpha}}$ which is isomorphic to $\mathbb{P}^1$, and three double points corresponding to roots of $z^3 + A\Delta(Q)z + B(-\sqrt{\Delta(Q)})^3 = 0$. We then have a final blow up at each of these singularities to create $S''_{\overline{\alpha}} = \tilde{S}_{A,B,Q}$. This process generates four new exceptional divisors, which we label $\Gamma_{\overline{\alpha}}, \Gamma_{\overline{\alpha},1}, \Gamma_{\overline{\alpha},2}, \Gamma_{\overline{\alpha},3}$. $\Gamma_{\overline{\alpha},1}, \Gamma_{\overline{\alpha},2}, \Gamma_{\overline{\alpha},3}$ will be isomorphic to $\mathbb{P}^1$, will not intersect each other, and will each intersect $\Gamma_{\overline{\alpha}}$ at one point. Again, these divisors will be (-2)-curves.

We can summarize the above findings by stating that $S_{A,B,Q}$ has 2 D_4 singularities. In particular, we get the following extended Dynkin diagram where the two D_4 singularities are pictured in black. The lines L_i pictured in dashed red will intersect both $\Gamma_{\alpha,i}$ and $\Gamma_{\overline{\alpha},i}$, and the lines E_+, E_- pictured in dash blue will intersect Γ_α and $\Gamma_{\overline{\alpha}}$ respectively.



We remark that from this Dynkin diagram, we can see that $\tilde{S}_{A,B,Q}$ is also a blowup of a Châtelet surface.

Since $S_{A,B,Q}$ has 2 D_4 singularities, we have that $\tilde{S}_{A,B,Q}$ is the minimal desingularization of S. Since $\tilde{S}_{A,B,Q}$ is the minimal desingularization of a del Pezzo surface of degree one, Derenthal [19] states that

$$\mathrm{rkPic}_{\overline{\mathbb{Q}}}(\tilde{S}_{A,B,Q}) = 9.$$

Let us denote by H the hyperplane section of $\tilde{S}_{A,B,Q}$. We claim that the above exceptional divisors, along with H, generate the entire Picard group over $\overline{\mathbb{Q}}$.

Lemma 4.5.1.

$$\mathrm{Pic}_{\overline{\mathbb{Q}}}(\tilde{S}) = \langle H, \Gamma_\alpha, \Gamma_{\alpha,1}, \Gamma_{\alpha,2}, \Gamma_{\alpha,3}, \Gamma_{\overline{\alpha}}, \Gamma_{\overline{\alpha},1}, \Gamma_{\overline{\alpha},2}, \Gamma_{\overline{\alpha},3} \rangle.$$

Proof. The hyperplane section will be indepedent of any of the exceptional divisors, so we will look

at the intersection numbers of the eight exceptional divisors. All of the six divisors coming from double points will have self-intersection -2, so:

$$\Gamma_{\beta,i} \cdot \Gamma_{\beta,i} = -2,$$

where we are using β to denote either α or $\overline{\alpha}$. Additionally, we have that

$$\Gamma_{\beta} \cdot \Gamma_{\beta,i} = 1$$

and

$$\Gamma_{\beta} \cdot \Gamma_{\beta} = -2.$$

Next, we note that we will have

$$\Gamma_{\alpha,i} \cdot \Gamma_{\overline{\alpha},j} = 0$$

for any $i, j \in \{\perp, 1, 2, 3\}$, since these are two separate singularities. Thus, we get the following intersection matrix:

$$\begin{array}{c} \begin{array}{cccccccc} \Gamma_{\alpha} & \Gamma_{\alpha,1} & \Gamma_{\alpha,2} & \Gamma_{\alpha,3} & \Gamma_{\overline{\alpha}} & \Gamma_{\overline{\alpha},1} & \Gamma_{\overline{\alpha},2} & \Gamma_{\overline{\alpha},3} \end{array} \\ \begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \end{array} \begin{array}{c} \Gamma_{\alpha} \\ \Gamma_{\alpha,1} \\ \Gamma_{\alpha,2} \\ \Gamma_{\alpha,3} \\ \Gamma_{\overline{\alpha}} \\ \Gamma_{\overline{\alpha},1} \\ \Gamma_{\overline{\alpha},2} \\ \Gamma_{\overline{\alpha},3} \end{array}$$

This matrix has a nonzero determinant, and so the exceptional divisors are linearly independent. Thus, along with H, we have that these exceptional divisors must generate $\mathrm{Pic}_{\overline{\mathbb{Q}}}(\tilde{S}_{A,B,Q})$.

□

Since we understand $\mathrm{Pic}_{\overline{\mathbb{Q}}}(\tilde{S})$, we aim to understand the Galois action on it. In particular, we want to compute

$$\mathrm{Pic}_{\mathbb{Q}}(\tilde{S}) = \mathrm{Pic}_{\overline{Q}}(\tilde{S})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = \mathrm{Pic}_{\mathbb{Q}(\alpha)}(\tilde{S})^{\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})}.$$

For considering the orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we note that it suffices to look at the action of $\mathrm{Gal}(K/\mathbb{Q})$, where K is a splitting field over $\mathbb{Q}(\alpha)$ of the polynomial $z^3 + A\Delta(Q)z + B(\sqrt{\Delta(Q)})^3$. It is clear that $[H]$ and $[\Gamma_\alpha + \Gamma_{\overline{\alpha}}]$ will be invariant under this action. Additionally, we have that $[\Gamma_{\alpha,1}] + [\Gamma_{\alpha,2}] + [\Gamma_{\alpha,3}] + [\Gamma_{\overline{\alpha},1}] + [\Gamma_{\overline{\alpha},2}] + [\Gamma_{\overline{\alpha},3}]$ will invariant under the Galois action. Hence, we can see that

$$\mathrm{rkPic}_{\mathbb{Q}}(\tilde{S}) \geq 3.$$

To determine the exact rank, we find that it is clearer to look at the picture over $\mathbb{Q}(\alpha)$.

4.5.2 Computation over $\mathbb{Q}(\alpha)$

Over $\mathbb{Q}(\alpha)$, we note that nothing changes in the first blowup to S_α. Then for the second blowup, rather than getting an exceptional divisor for each root of $P(z) = z^3 + A\Delta(Q)z + B(\sqrt{\Delta(Q)})^3$, we have a divisor corresponding to each irreducible factor of the polynomial over $\mathbb{Q}(\alpha)$ (as these will correspond to prime ideals). In particular, if $P(z)$ is irreducible, then we have the divisor $\Gamma_{\alpha,P(z)}$. If $P(z) = Q(z)L(z)$, where $Q(z)$ is an irreducible quadratic, then we have $\Gamma_{\alpha,Q(z)}$ and $\Gamma_{\alpha,L(z)}$. Finally, if $P(z) = (z - r_1)(z - r_2)(z - r_3)$, then we have a divisor corresponding to each root, $\Gamma_{\alpha,(z-r_1)}, \Gamma_{\alpha,(z-r_2)}, \Gamma_{\alpha,(z-r_3)}$. For the blowup over $[0, 0, \overline{\alpha}, 1]$, the same picture occurs except now we consider the factorization of $\overline{P(z)}$.

4.5.3 Computation over $\mathbb{Q}$

Finally, we want to consider the action of $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ on the divisors over $\mathbb{Q}(\alpha)$. We can see that

$$\Gamma_\alpha \mapsto \Gamma_{\overline{\alpha}}$$

$$\Gamma_{\alpha,P(z)} \mapsto \Gamma_{\overline{\alpha},\overline{P(z)}}.$$

So, if $P(z)$ is irreducible in $\mathbb{Q}(\alpha)$, we have that

$$\mathrm{Pic}_{\mathbb{Q}}(\tilde{S}) = \langle H, \Gamma_\alpha + \Gamma_{\overline{\alpha}}, \Gamma_{\alpha,P(z)} + \Gamma_{\overline{\alpha},\overline{P(z)}} \rangle$$

and hence has rank 3.

Otherwise, if $P(z) = Q(z)L(z)$ splits into an irreducible quadratic and a linear factor in $\mathbb{Q}(\alpha)$,

we have that the Picard group is:

$$\mathrm{Pic}_{\mathbb{Q}}(\tilde{S}) = \langle H, \Gamma_\alpha + \Gamma_{\overline{\alpha}}, \Gamma_{\alpha,Q(z)} + \Gamma_{\overline{\alpha},\overline{Q(z)}}, \Gamma_{\alpha,L(z)} + \Gamma_{\overline{\alpha},\overline{L(z)}} \rangle.$$

Thus, we get $\varrho = 4$ in this case.

Finally, if $P(z)$ splits completely, we have that the Picard group

$$\mathrm{Pic}_{\mathbb{Q}}(\tilde{S}) = \langle H, \Gamma_\alpha + \Gamma_{\overline{\alpha}}, \Gamma_{\alpha,z-r_1} + \Gamma_{\overline{\alpha},\overline{z-r_1}}, \Gamma_{\alpha,z-r_2} + \Gamma_{\overline{\alpha},\overline{z-r_2}}, \Gamma_{\alpha,z-r_3} + \Gamma_{\overline{\alpha},\overline{z-r_3}} \rangle.$$

This Picard group then has rank $\varrho = 5$.

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