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Polyfolds of Lagrangian Floer Theory in All Genera

Jemison, Michael

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POLYFOLDS OF LAGRANGIAN FLOER THEORY IN ALL GENERA

Michael Louis Jemison, Jr.

A DISSERTATION

PRESENTED TO THE FACULTY

OF PRINCETON UNIVERSITY

IN CANDIDACY FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

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Abstract

This dissertation describes a polyfold approach to the construction of an integration theory on moduli spaces of pseudo-holomorphic curves with Lagrangian boundary conditions. Polyfold theory was introduced by Hofer, Wysocki, and Zehnder in a series of articles. Polyfolds have been employed to study closed Gromov-Witten theory and Symplectic Field Theory. From its initial introduction, polyfold theory has seen generalizations which have allowed for a recyclable and modular construction of moduli spaces arising in symplectic geometry.

The present work sets the beginning of a geometric realization of the theory of Lagrangian Floer theory in all genera, generalizing the work of Fukaya, Oh, Ohta, and Ono. Similarly to the construction of Lagrangian Floer theory, construction of Lagrangian Floer theory in all genera separates into two pieces:

1. The construction of an integration theory on the moduli spaces of pseudo-holomorphic maps with Lagrangian boundary conditions

2. An ordered perturbation scheme on these moduli spaces which relates geometry (in particular integration) along the boundary of a given moduli space in terms of geometry of moduli spaces of lower order

The present article treats the first of these pieces. The second piece is left to subsequent work.

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1 Introduction

The present article describes a polyfold approach to the construction of an integration theory on moduli spaces of pseudo-holomorphic curves with Lagrangian boundary conditions. Polyfold theory was introduced by Hofer, Wysocki, and Zehnder in a series of articles ([HWZ4], [HWZ5], [HWZ6]). Polyfolds have been employed to study closed Gromov-Witten theory [HWZ1] and Symplectic Field Theory [FH6]. From its initial introduction, polyfold theory has seen generalizations [HWZ2], [FH7], [FH8] which have allowed for a recyclable and modular construction of moduli spaces arising in symplectic geometry.

The problem of constructing invariants from moduli-spaces of pseudo-holomorphic curves has also been approached by methods of finite-dimensional reduction (or Kuranishi methods), as introduced by Fukaya and Ono ([FO]). The case of pseudo-holomorphic curves of arbitrary genera with Lagrangian boundary conditions was discussed in ([L]) by the use of Kuranishi methods. The idea of finite dimensional reductions has been exploited in other works, in particular [FOOO], [MW0], [MW], [MW2], [P].

The present article sets the beginning of a geometric realization of the theory of Lagrangian Floer theory in all genera, as discussed in [CFL], generalizing the work of Fukaya, Oh, Ohta, and Ono ([FOOO]). Similarly to the construction of Lagrangian Floer theory, construction of Lagrangian Floer theory in all genera separates into two pieces:

1. The construction of an integration theory on the moduli spaces of pseudo-holomorphic maps with Lagrangian boundary conditions

2. An ordered perturbation scheme on these moduli spaces which relates geometry (in particular integration) along the boundary of a given moduli space in terms of geometry of moduli spaces of lower order

The present article treats the first of these pieces. The second piece is left to subsequent work.

1.1 Overview of Stable Maps with Boundary

In this article we will consider stable maps satisfying certain regularity conditions. Let

$$u : \mathfrak{O}(S, x) \to Q$$

denote a germ of a Riemann surface with boundary S around a point $x \in S$.

Definition 1.1.1. Let $m \geq 2$ be an integer $\delta > 0$. A germ of a continuous map $u : \mathfrak{O}(S, x) \to Q$ is of class (m, δ) near the point x, if for a smooth chart $\phi : U(u(x)) \to \mathbb{R}^{2n}$ mapping $u(x)$ to 0 and for a holomorphic polar coordinate $\sigma : [0, \infty) \times J \to S - \{x\}$ around x, the map

$$v(s, t) = \phi \circ u \circ \sigma(s, t)$$

where J may be either $I = [0, 1]$ or S^1, defined for s large, has weak partial derivatives up to order m and satisfies

$$||e^{\delta s} \frac{\partial}{\partial s^k} \frac{\partial}{\partial t^l} v||_{L^2} < \infty$$

for $k + l \leq m$. ■

Note that the definition is independent of the choice of smooth coordinates σ. A germ of a continuous map $u : \mathfrak{O}(S, x) \to Q$ is of class m near the point x if u is of class H^m_{loc}

near x.

Definition 1.1.2. A noded Riemann surface with boundary and marked points is a tuple $(S, j, M_i, M_b, D_i, D_b)$ in which (S, j) is an oriented smooth surface with boundary S and integrable complex structure j :

1. The subset $M_i \subset \text{int } S$ of S is the ordered set of interior marked points and M_b is the ordered subset of boundary marked points.

2. D_i is a finite collection of un-ordered pairs $\{x, y\}$ of points in int S so that $x \neq y$ and two pairs which intersect are identical.

3. D_b is a finite collection of un-ordered pairs $\{x', y'\}$ of points in ∂S so that $x' \neq y'$ and two pairs which intersect are identical.

All the sets M_i, M_b, D_i, D_b are mutually disjoint. D_i is the subset of interior nodal points, and D_b is set of boundary nodal points. We also require the boundary components of S to be ordered. ■

We let M denote the union $M_i \cup M_b$ and let D denote the union $D_i \cup D_b$. The Riemann surface S may consist of several smooth connected components C. The surface $(S, j, M_i, M_b, D_i, D_b)$ is called connected if the topological space $\overline{S}$, obtained by identifying the points x and y in the nodal pairs $\{x, y\} \in D_i \cup D_b$ is connected.

Definition 1.1.3. The double of a Riemann surface $(S, j, M_i, M_b, D_i, D_b)$ is defined by the tuple $(S^d, M_i \cup \sigma(M_i), M_b, D_i \cup \sigma(D_i), D_b)$ where S^d is the complex double of S and

$\sigma : S^d \to S^d$ is the natural anti-biholomorphism. For shorthand, we set

$$M_i^d = M_i \cup \sigma(M_i), \qquad M^d = (M_i^d, M_b) \tag{1.1.1}$$

$$D_i^d = D_i \cup \sigma(D_i), \qquad D^d = (D_i^d, D_b) \tag{1.1.2}$$

■

The arithmetic genus g_a of a connected Riemann surface with boundary $(S, j, M_i, M_b, D_i, D_b)$ is the integer g_a defined by

$$g_a = 1 + \#|D^d| + \sum_{C^d \text{ a connected component of } S^d} [g(C^d) - 1]$$

where $|D^d|$ is the set of nodal pairs and $g(C^d)$ is the genus of C^d. Note that the genus g_a gives the genus of the Riemann surface obtained by identifying the points in each nodal pair. In this article we will by shorthand refer to the marked points $M \cup |D|$ as special points, and we denote the special points contained in the domain component C by $\Sigma_C := C \cap (M \cup |D|)$.

Definition 1.1.4. Riemann surfaces $(S, j, M_i, M_b, D_i, D_b)$ and $(S', j', M_i', M_b', D_i', D_b')$ are equivalent if there exists a biholomorphic map $\phi : (S, j) \to (S', j')$ satisfying:

$$\phi(M_i) = M_i' \quad \phi(M_b) = M_b'$$

and

$$\phi(D_i) = D_i' \qquad \phi(D_b) = D_b'$$

In addition, we require that ϕ preserves the ordering of the marked points M and boundary components of S. ■

When ϕ is a self-equivalence of a Riemann surface $(S, j, M_i, M_b, D_i, D_b)$ we say that it's an automorphism, and the equivalence class of $(S, j, M_i, M_b, D_i, D_b)$ is denoted by $[(S, j, M_i, M_b, D_i, D_b)]$. We denote the automorphism group of a Riemann surface $(S, j, M_i, M_b, D_i, D_b)$ by $Aut(S, j, M, D)$.

Definition 1.1.5. We say $(S, j, M_i, M_b, D_i, D_b)$ is stable if its automorphism group is finite. ■

Stability is equivalent to the condition that the inequality

$$2g(C^d)+|\Sigma_{C^d}| \geq 3$$

holds for every connected component C of S (see Section 3.2 [L]). Now we consider the stability of maps:

Definition 1.1.6. Let Q be an even dimensional manfiold with $L \subset Q$ a totally real submanifold. Let $(\delta_0^m)_{m=0}^{\infty} \subset (0, 2\pi)$, and $(\delta_1^m)_{m=0}^{\infty} \subset (0, \pi)$ be strictly increasing sequences. The tuple $\alpha = (S, j, M, D, u)$ is a stable map of class m if the following holds:

1. $(S, j, M_i, M_b, D_i, D_b)$ is connected.

2. For some $m \geq 0$, the map $u : S \to Q$ is of class $(3 + m, \delta_0^m)$ at every interior nodal point and of class $(3+m, \delta_1^m)$ at every boundary nodal point and of class H_{loc}^{3+m} otherwise.

3. $u(x) = u(y)$ at every nodal pair $\{x, y\} \in D$.

4. $\int_C u^*\omega \geq 0$ for every component C of S.

5. If the underlying topological space of the component C of S is not stable, that is $2g(C^d)+|\Sigma_{C^d}| \leq 2$, then

$$\int_C u^*\omega > 0$$

6. $u(\partial S) \subset L$. ■

Definition 1.1.7. Two stable maps $\alpha = (S, j, M, D, u)$ and $\alpha' = (S', j', M', D', u')$ are called equivalent if there exists an isomorphism of Riemann surfaces

$$\phi : (S, j, M, D) \to (S', j', M', D')$$

such that

$$u' \circ \phi = u$$

An equivalence class of a curve α is denoted $[\alpha]$. ■

Definition 1.1.8. Let $(\delta_0^m)_{m=0}^\infty \subset (0, 2\pi)$, and $(\delta_1^m)_{m=0}^\infty \subset (0, \pi)$ be strictly increasing sequences. The collection of equivalence classes $[\alpha]$ where α has class $(3+m, \delta_0^m, \delta_1^m)$ will be referred to as the space of stable curves with boundary in L of class $(3 + m, \delta_0^m, \delta_1^m)$ and denoted by Z or $Z^{3+m,\delta_0^m,\delta_1^m}(Q, \omega)$. ■

In Section 4, we will establish a natural paracompact topology on Z:

Theorem 1.1.1. (compare Theorem 1.6 [HWZ1]) For a pair $(\delta_0^m, \delta_1^m) \in (0, 2\pi) \times (0, \pi)$ the space $Z^{3+m,\delta_0^m,\delta_1^m}(Q, \omega)$ has a natural second countable paracompact Hausdorff topology. ■

1.2 Brief Overview of Ep-Groupoids and Polyfolds

Recall that a groupoid is a small category in which all morphisms are invertible. Also recall that a Lie groupoid is a groupoid in which the object set and morphism set have

a smooth structure. An *ep*-groupoid is similar to a Lie groupoid, except the objects and morphisms are M-polyfolds, the associated maps are *sc*-smooth, and the source and target maps admit an additional properness property. The notions of M-polyfold and *sc*-smoothness are reviewed in Section 3.1.

Definition 1.2.1. Ep-Groupoid (compare Definition 2.28 [HWZ1]): An *ep*-groupoid $(T, \mathbf{T})$ is a groupoid T such that the object set T and morphism set $\mathbf{T}$ have M-polyfold structures. An *ep*-groupoid is equipped with the usual categorical maps:

a source (s):

$$s : \mathbf{T} \to T$$

a target (t):

$$t : \mathbf{T} \to T$$

an inverse (i):

$$i : \mathbf{T} \to \mathbf{T}$$

a composition map (m):

$$m : \mathbf{T}_s \times_t \mathbf{T} \to \mathbf{T}$$

a unit map (u):

$$u : T \to \mathbf{T}$$

These maps are required to be *sc*-smooth. We refer to these maps as structure maps. In addition, the following properties hold:

1. **(etale)** The source and target maps s and t are surjective local *sc*-diffeomorphisms.

2. **(proper)** For every point $x \in T$, there exists an open neighborhood $V(x)$ so that the map $t : s^{-1}(\overline{V(x)}) \to T$ is a proper mapping.

The orbit space of the *ep*-groupoid $(T, \mathbf{T})$ is the topological space obtained by quotienting T by the relations imposed by $\mathbf{T}$. We denote the orbit space by $|T|$. ■

In Section 7 we establish the following:

Theorem 1.2.1. Let $(\delta_0^m) \subset (0, 2\pi)$ and $(\delta_1^m) \subset (0, \pi)$ be strictly increasing sequences. Then the space $Z = Z^{3,\delta_0,\delta_1}(Q, \omega)$ possesses a natural polyfold structure such that the m^{th} level consists of equivalence classes of stable maps (S, j, M, D, u) such that u has class $(m + 3, \delta_0^m, \delta_1^m)$. ■

The polyfold Z has various open subsets of interest. Let $\overrightarrow{k} = (k_1, ..., k_h)$ be a tuple of integers $k_i \geq 0$ and let $\beta \in H_2(Q, L; \mathbb{Z})$ be a relative homology class. Denote by $Z_{g,h,M,\overrightarrow{k},\beta} \subset Z$ the subset consisting of equivalence classes $[\alpha] = [(S, j, M_i, M_b, D_i, D_b, u)]$ of stable maps satisfying:

1. If $b_1, \cdots, b_h$ are the ordered set of boundary components of S then $\#|b_i \cap M_b| = k_i$

2. The map $u : S \to Q$ represents the relative homology class $\beta \in H_2(Q, L)$

In particular, as an open subset, $Z_{g,h,M,\overrightarrow{k},\beta}$ inherits a naturally induced polyfold structure from Z.

1.3 Overview of Fredholm Theory

Fix a complex structure J on Q such that $\omega \circ (id \oplus J)$ is a Riemannian metric on Q. We define a set E consisting of equivalence classes of tuples

$$\widehat{\alpha} = (\alpha, \xi) = (S, j, M, D, u, \xi)$$

where $\alpha = (S, j, M, D)$ is a stable map and ξ is a continuous section of the bundle of anti-linear maps

$$\Omega^{0,1}((TS, j), u^*(TQ, J))$$

The section ξ further satisfies the condition that away from nodal and marked points ξ is of class H^2_{loc}. Near interior nodes ξ is of class $(2, \delta_0)$ and near boundary nodes ξ is of class $(2, \delta_1)$.

Two tuples (S, j, M, D, u, ξ) and $(S', j', M', D', u', \xi')$ are equivalent if there exists an equivalence

$$\phi : (S, j, M, D, u) \to (S', j', M', D', u')$$

of stable maps which satisfies

$$\xi' \circ T\phi = \xi$$

We denote the equivalence class of $\widehat{\alpha} = (S, j, M, D, u, \xi)$ as $[\widehat{\alpha}]$. In Section 4, we prove that E is a polyfold bundle over Z:

Theorem 1.3.1. The set E has a natural second countable paracompact Hausdorff topology and the natural projection $p : E \to Z$ is continuous. ■

The notion of a strong bundle ove an M-polyfold is recalled in Section 3.

Definition 1.3.1. Strong Bundles over Ep-groupoids (compare Definition 8.4

[HWZ3]): A strong bundle over the *ep*-groupoid $(T, \mathbf{T})$ is a pair (P, μ) in which $P : W \to T$ is a strong bundle over the object M-polyfold T and $\mu : \mathbf{T}_s \times_P W \to W$ a strong bundle map covering the target map $t : \mathbf{T} \to T$ such that the diagram

$$\begin{array}{ccc} \mathbf{T}_s \times_P W & \xrightarrow{\mu} & W \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle P} \\ \mathbf{T} & \xrightarrow{t} & T \end{array}$$

commutes, i.e.

$$P \circ \mu(\psi, w) = t(\psi)$$

In addition,

1. μ is a surjective local sc-diffeomorphism and linear on the fibers W_x

2. $\mu(1_x, w) = w$ for all $x \in T$ and $w \in W_x$

3. $\mu(\psi \circ \gamma, w) = \mu(\psi, \mu(\gamma, w))$ for all $\psi, \gamma \in \mathbf{T}$ and $w \in W$ satisfying

$$s(\gamma) = P(w) \quad \text{and} \quad t(\gamma) = s(\psi) = P(\mu(\gamma, w)) \quad \blacksquare$$

In Section 5, we consider the Cauchy-Riemann section $\overline{\partial}_J : Z \to E$:

$$\overline{\partial}_J[(S, j, M, D, u)] = [(S, j, M, D, u, \overline{\partial}_{J,j}(u))]$$

$$\overline{\partial}_{J,j}(u) = \frac{1}{2}[Tu + J(u) \circ Tu \circ j]$$

Suppose $(X, \mathbf{X})$ is an *ep*-groupoid which realizes Z as a polyfold. We will study the properties of the Cauchy-Riemann section lifted to the strong bundle $(W, \mathbf{W}) \to (X, \mathbf{X})$.

As we will discuss in Section 5, the verification of the Fredholm properties of the functorial lift $\overline{\partial}_J : (X, \mathbf{X}) \to (W, \mathbf{W})$ imply the applicability of the implicit function theorem. This in turn will allow for a theory of integration over moduli spaces of pseudo-holomorphic curves with Lagrangian boundary.

Theorem 1.3.2. The bundle $p : W \to Z$ is a strong polyfold bundle. The Cauchy-Riemann section $\overline{\partial}_J$ is an *sc*-Fredholm section. ■

Remark 1.3.1. Note that the compactness property of Theorem 1.3.2 is left to future work.

1.4 Integration over the Moduli Space of Stable Maps

Although it has been shown in the classical context that the Cauchy-Riemann operator on the space of stable maps from surfaces with boundary and Lagrangian boundary conditions is orientable when L is rel. spin ([Ge]), we do not demonstrate an orientability result in this article. Assuming the existence of an orientation for $\overline{\partial}_J$ and compactness for the solution set, the following is a consequence of Theorem 1.3.2:

Integration Let (Q, ω) be a closed symplectic manifold of dimension $2n$. Suppose $\overline{\partial}_J$ is orientable and $\overline{\partial}_J^{-1}(0)$ is a compact subset of $Z_{g,h,M,\overrightarrow{k},\beta}$. For a given homology class $\beta \in H_2(Q, L)$ and a tuple $(g, M, \overrightarrow{k})$ (where $M = (M_i, M_b)$) there exists a map:

$$\Psi_{g,h,M,\overrightarrow{k},\beta} : \Omega(L)^{\otimes \#|M_b|} \otimes \Omega(Q)^{\otimes \#|M_i|} \to \mathbb{R}$$

For a compatible complex structure J on Q and a generic sc^+ perturbation λ we have, a solution set consisting of a pair $(S_{g,h,M,\overrightarrow{k},\beta}(\overline{\partial}_J, \lambda), \nu)$ where

$$S_{g,h,M,\overrightarrow{k},\beta}(\overline{\partial}_J, \lambda) = \{z \in Z_{g,h,M,\overrightarrow{k},\beta} : \lambda(\overline{\partial}_J) > 0\}$$

and $\nu : Z_{g,h,M,\overrightarrow{k},\beta} \to \mathbb{Q}^+$ is given by

$$\nu(z) = \lambda(\overline{\partial}_J(z)) \in \mathbb{Q}^+$$

The pair $(S_{g,M,\overrightarrow{k},\beta}(\overline{\partial}_J, \lambda), \nu)$ is an oriented, compact, weighted, branched suborbifold of $Z_{g,h,M,\overrightarrow{k},\beta}$ with a well-defined integration theory. Ψ is given by the formula:

$$\Psi_{g,h,M,\overrightarrow{k},\beta}(\tilde{\omega}_1, \cdots, \tilde{\omega}_i, \cdots, \tilde{\omega}_{\#|M|}) = \int_{(S_{g,h,M,\overrightarrow{k},\beta}(\overline{\partial}_J,\lambda),\nu)} ev^*(\times_{i=1}^{\#|M|}\tilde{\omega}_i)$$

where $ev : Z_{g,h,M,\overrightarrow{k},\beta} \to L^{\#|M_b|} \times Q^{\#|M_i|}$ is the evaluation map. ■

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Part I

Construction of Ambient Polyfolds

In Part I, we will develop the ambient polyfolds suitable to conduct abstract perturbation theory. More specifically, Section 2 introduces Deligne-Mumford theory in the boundary setting. Section 3 defines procedures for gluing maps from a fixed Riemann surface into a symplectic manifold with Lagrangian boundary conditions. Finally, Section 4 establishes

a polyfold structure on the space of stable maps.

2 Deligne-Mumford Space of Curves with Boundary

In this section we discuss the results we will need to construct the polyfold structure on the space of stable maps with boundary. These are adaptations of the results presented in [HWZ3] for the closed case. Self-contained proofs for these results are given in Appendix. Alternatively, one can reduce the argumentation to the closed case.

2.1 Deligne-Mumford Spaces of Fixed Topological Type

In this section, we will describe orbifold charts on the Deligne-Mumford space of stable curves with boundary. Classically, this space is known to be a real analytic orbifold with boundary (see [DM], [KM], [Mu], and [RS2]). So in that sense, the material in this section is classical. We will however take an "ep-groupoid" point of view which will be important for considerations of the space of stable maps, to be discussed later.

Definition 2.1.1. The elements of the Deligne-Mumford space DM_{g,h,M_i,M_b} are equivalence classes of tuples (S, j, M, D) where:

1. S is a compact Riemann surface of genus g with boundary (see Definition 1.1.2) and ordered boundary components $b_1, \cdots, b_h$.

2. j is a complex structure on S.

3. $M = (M_i, M_b)$ where M_i is the set of ordered interior marked points and M_b is the set of ordered boundary marked points. We abbreviate the set of boundary marked

points on b_i by $\overrightarrow{m_{b_i}}$. The ordering of the points in $\overrightarrow{m_{b_i}}$ is chosen to agree with the orientation on b_i induced from the complex orientation on S.

4. (S, j, M, D) is a stable curve (see Definition 1.1.4).

5. $D = (D_i, D_b)$, where D_i of is the set of interior nodes and D_b is the set of boundary nodes.

6. Two tuples (S, j, M, D) and (S', j', M', D') are considered equivalent if there exists a biholomorphism $\phi : (S, j, M, D) \to (S', j', M', D')$ satisfying:

$$\phi(M_i) = M_i' \quad \phi(D) = D'$$

$$\phi : M \to M' \text{ is an order preserving bijection}$$

and

$$\phi(b_i) = b_i, \text{ for } 1 \leq i \leq h \ \blacksquare$$

We denote by $[(S, j, M, D)]$ the element of the Deligne-Mumford space represented by the stable curve (S, j, M, D). The Deligne-Mumford space is stratified by sets consisting of equivalence classes of surfaces of a fixed topological type. We formalize this notion in the definition below:

Definition 2.1.2. Let (S, j, M, D) be a Riemann surface with boundary. We denote the set of complex structures of S which induce the same orientation as j by $\mathfrak{J}_j$. $\blacksquare$

Definition 2.1.3. Topological Types (compare [HWZ3] Section 1.1.3) Let $\alpha = (S, j, M, D)$ represent the equivalence class of a stable connected element $[\alpha] \in DM_{g,h,M_i,M_b}$.

To α we associate an undirected graph T, which we refer to as the structure graph. We equip T with the following data:

1. To each of the connected components C of S, we associate a vertex V_C. We in turn equip the vertices with the following data:

1.a. A function $g : \{\text{Vertices of S}\} \to \mathbb{N}$, which is the value of the genus of the component corresponding to the vertex.

1.b. A list of cyclically ordered list $m_C = (M_i \cap C)$.

1.c. A list of cyclically ordered tuples $\overrightarrow{k}_C = ((\overrightarrow{M_b \cap b_1}), \cdots, (\overrightarrow{M_b \cap C}))$, where $b_1, \cdots, b_h$ are the boundary components of C.

2. We associate to T a set E_1 of type 1 edges: If components C and C' are connected by an interior node, there is a unique type 1 edge in E_1 between V_C and $V_{C'}$.

3. We also associate to T a set E_2 of type 2 edges: If components C and C' are connected by a boundary node, there is a unique type 2 edge in E_2 between V_C and $V_{C'}$.

An isomorphism of structure graphs is an isomorphism of undirected graphs which preserves the data associated to the edges and edge type. We denote by

$$[(T, g, (m_C)_{C \in S}, (\overrightarrow{k})_{C \in S}, V, E_1, E_2)]$$

the isomorphism class of the graph $(T, g, m_i, \overrightarrow{k}_C, V, E_1, E_2)$. ■

Consider the graph $\tau := (T, g, m_i, \overrightarrow{k}_C, V, E_1, E_2)$. Let DM_τ denote the collection of equivalence classes in DM_{g,h,M_i,M_b} with structure graphs isomorphic to τ. We have a natural topology on DM_τ :

Definition 2.1.4. Let $[\alpha] = [(S, j, M, D)] \in DM_\tau$ and let U be an open subset of $\mathfrak{J}_j$ containing j. Consider the subset of DM_τ, denoted by $V(\alpha, U)$, given by the set of equivalence classes

$$[(S, k, M, D)],\ k \in U, \text{ and the orientations induced by } k \text{ and } j \text{ agree}$$

where U is an open subset containing j in the space of complex structures on S. Let V_τ^1 denote the set of such subsets. ■

That V_τ^1 forms a topology follows analogously to the result in the case of closed Riemann surfaces proven in [HWZ3]:

Proposition 2.1.1. (compare Theorem 5.11 [HWZ3]) V_τ^1 is a basis for a Hausdorff topology on DM_τ.

Proof: First we show that V_τ^1 defines a topology. It is enough to show that given open subsets $V(\alpha, U)$, $V(\alpha', U')$ and an intersection $V(\alpha, U) \cap V(\alpha', U')$ and a point $[\alpha''] \in V(\alpha, U) \cap V(\alpha', U')$, there exists an open subset $V(\alpha'', U'')$ such that $V(\alpha'', U'') \subset V(\alpha, U) \cap V(\alpha', U')$. By (1) there exists morphisms

$$\phi : \alpha'' \to \alpha$$

$$\phi' : \alpha'' \to \alpha'$$

Observe that ϕ^*U and ϕ'^*U' are open C^∞-neighborhoods of j''. Set

$$U'' := \phi^*U \cap \phi'^*U'$$

then

$$V(\alpha'', U'') \subset V(\alpha, U) \cap V(\alpha', U')$$

Thus V^1_τ is a topology which we denote by $\mathfrak{T}_\tau$. Furthermore, by observing that the space of complex structures admits a metrizable topology (see Proposition 2.1.3) we see that $\mathfrak{T}_\tau$ admits a countable neighborhood basis.

Now we establish the Hausdorff property. Suppose we have two equivalence classes represented by curves:

$$\alpha = (S, j, M, D), \quad \alpha' = (S', j', D', M')$$

such that for any open neighborhoods V containing $[\alpha]$ and V' containing $[\alpha']$ we have $V \cap V' \neq \emptyset$. This stipulates the existence of sequences

$$j_k \to j \text{ and } j'_k \to j', \ \phi_k : (S, j_k, M, D) \to (S', j'_k, D', M')$$

such that the maps ϕ_k are biholomorphisms. The stability of the surfaces α and α' and Gromov-Compactness imply that a subsequence of the maps ϕ_k converge to a morphism

$$\phi : \alpha \to \alpha'$$

Thus $[\alpha] = [\alpha']$ and it follows $\mathfrak{T}_\tau$ is Hausdorff. ■

We will make use the notion of small disk structures to define a more convenient topology:

Definition 2.1.5. (Small Disk Structure) (compare Definition 2.3 [HWZ1]): A small disk structure associated to the Riemann surface $\alpha = (S, j, M, D)$, denoted by $\mathbf{D}$, assigns to every nodal point $x \in |D_i|$ ($x \in |D_b|$) a disk (half-disk) like neighborhood $D_x \subset S$ ($HD_x \subset S$) centered at x with smooth boundary. The disks and half-disks are mutually disjoint and disjoint from the set M. We will occasionally denote the half disks HD_x by D_x, for $x \in |D_b|$. Also the data of a small disk structure includes a choice of holomorphic coordinates, discussed in greater detail in Section 2.2. We further assume that the union

$$\mathbf{D} := \bigcup_{\{x,y\} \in D} D_x \cup D_y$$

is invariant under the action of $Aut(\alpha)$. ■

Definition 2.1.6. Consider a Riemann surface $\alpha = (S, j, M, D)$ with a small disk structure $\mathbf{D}$. We denote by $\mathfrak{J}_{D,j}$ the set consisting of complex structures which agree with j on $\mathbf{D}$:

$$\mathfrak{J}_{D,j} := \{k \in \mathfrak{J}_j \ : \ k|\mathbf{D} = j|\mathbf{D}\} \ \blacksquare$$

Definition 2.1.7. Let U be an open subset of $\mathfrak{J}_{D,j}$ which contains j. Given $[\alpha] = [(S, j, M, D)] \in DM_\tau$, consider the subset of DM_τ, denoted by $V(\alpha, U, \mathbf{D})$, given by the set of equivalence classes

$$\{[(S, k, M, D)] \in DM_\tau : \ k \in U\}$$

Let V_τ^2 denote the collection of such subsets $V(\alpha, U, \mathbf{D})$. ■

Proposition 2.1.2. (compare Theorem 5.12 [HWZ3]) V_τ^2 is a basis for a topology on DM_τ. The topologies induced by V_τ^2 and V_τ^1 agree.

Proof: Assume that $V(\alpha, U, \mathbf{D})$ is given and α' represents an element in $[\alpha'] \in V(\alpha, U, \mathbf{D})$. For definiteness, we set $\alpha = (S, j, M, D)$ and $\alpha' = (S, j', M, D)$. Note that:

$$j'|_{D_x} = j|_{D_x}, \text{ for all } x \in D$$

To prove the claim it is enough to find U' such that

$$V(\alpha', U') \subset V(\alpha, U, \mathbf{D})$$

Let $\{U'_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of open neighborhoods of j' defining a neighborhood basis for j'. Suppose for sake of contradiction, we have a sequence $j'_n \to j'$ such that (S, j'_n, M, D) does not represent an element in $V(\alpha, U, \mathbf{D})$ for any n. For every disk D_x of the small disk structure we fix a point on the boundary. Then there exists for every disk a unique biholomorphism

$$(D_x, j') \to (D_x, j'_n)$$

preserving the point and x. For each half-disk HD_x we find a unique biholomorphism

$$(HD_x, j') \to (HD_x, j'_n)$$

preserving x. Observe that these diffeomorphisms converge in C^{∞} since $j'_n \to j'$ in C^{∞}. Fixing such maps for every disk D_x and extending these maps smoothly to the entire surface S, we produce a sequence ϕ_k of diffeomorphisms preserving the points in $|D|$ and

converging in C^∞ to the identity so that $\phi_k^* j_k'$ coincides on $|\mathbf{D}|$ with j. Then:

$$[(S, \phi_k^* j_k', M, D)] = [(S, j_k', M, D)]$$

and $\phi_k^* j_k'$ belongs to U for k large. By construction $\phi_k^*|_{D_x} := j|_{D_x}$ for all $x \in D$. Thus we have the contradiction. ■

We will parametrize open subsets of DM_τ by open subsets of finite dimensional vector spaces. Towards this end, let $\alpha = (S, j, M, D)$ be a stable curve with boundary, and let $\Gamma_0(\alpha)$ denote the vector space of smooth sections η of the bundle $TS \to S$ satisfying:

$$\eta(D \cup M) = 0$$

$$\eta(l) \in T_l \partial S, \text{ for } l \in \partial S$$

Also denote by $\Omega^{0,1}(\alpha)$ the space of sections $\Gamma Hom(TS, TS)$ which are complex anti-linear.

Consider the classical linear Cauchy-Riemann operator $D : \Gamma_0(\alpha) \to \Omega^{0,1}(\alpha)$ given by

$$Df := \frac{\partial f}{\partial z} d\bar{z}$$

We will show that the quotient $H^1(\alpha) = \Omega^{0,1}(\alpha)/\mathrm{Im}\, D$ parametrizes deformations of the complex structure j.

The automorphism group G of α acts on $H^1(\alpha)$ via the natural representation (com-

pare Definition 2.9 [HWZ3])

$$G \times H^1(\alpha) \to H^1(\alpha), \quad (g, r) \to g * [r] := [(Tg)r(Tg)^{-1}]$$

Let E be a finite dimensional smooth vector space and consider a smooth family of complex structures $j : V \to \mathfrak{J}_j$ on S defined on a small open neighborhood V of the origin in E. Associated to j is a family of Riemann surfaces:

$$v \to \alpha_v = (S, j(v), M, D) \in DM_{g,h,M_i,M_b}$$

We call this family a deformation of α if $j(0) = j$. If V is a small open neighborhood of a finite dimensional vector space E, the differential $\delta v \to Dj(v)\delta v$ gives a linear map $E \to H^1(\alpha_v)$ which we write as

$$[Dj(v)]\delta v := [Dj(v)\delta v]$$

where $[Dj(v)\delta v]$ denotes the equivalence class of $Dj(v)\delta v$ in $H^1(\alpha_v)$. We refer to the linear map $[Dj(v)]$ as the Kodaira differential.

Definition 2.1.8. Good Families (compare Definition 1.4 [HWZ3]) Fix a connected and stable Riemann surface with boundary $\alpha = (S, j, M, D)$. Let E be a finite dimensional vector space and let $V \subset E_\alpha$ be an open neighborhood of the origin. Consider a smooth family $v \to j(v)$ of complex structures on S parametrized by V. Denote by $H^1(\alpha_v)$ the quotient

$$H^1(\alpha_v) := \Omega^{0,1}(\alpha_v)/\text{ Im } D(\Gamma_0(\alpha_v))$$

The family $v \to j(v)$ is effective if:

(1) $j(0) = j$

(2) $[Dj(g * v)(g * \delta v)] = [(Tg)(Dj(v)\delta v)(Tg)^{-1}]$

(3) $[j \cdot Dj(v)] : E_\alpha \to H^1(\alpha_v)$ is a real linear isomorphism for all $v \in V$.

We say E_α is a good vector space for α and that V is a good germ. ■

Proposition 2.1.3. (compare Lemma 3.8 [HWZ3]) The map

$$\{v \in \Omega^{0,1}(TS) : |v| < 1\} \to \mathfrak{J}_j$$

$$v \to j(v) := j(1 + v)(1 - v)^{-1}$$

is a homeomorphism onto $\mathfrak{J}_j$. The map is equivariant in the following sense:

$$j(g * v)(z) = Tg(g^{-1}(z)) \circ j(v)(g^{-1}(z)) \circ Tg^{-1}(z)$$

Proof: The argument follows identically to that presented in [HWZ3]. ■

In [HWZ3], the authors demonstrate the following result when S is closed:

Lemma 3.9 [HWZ3] The derivative $Dj(v)$ of the map $v \to j(v)$ is a linear map

$$Dj(v) : \Omega^{0,1}((TS, j), (TS, j)) \to \Omega^{0,1}((TS, j(v)), (TS, j(v))$$

It satisfies

$$j(v) \circ Dj(v)[\delta v] = Dj(v)[j \circ \delta v]$$

Thus $Dj(v)$ is complex linear. ■

The proof of the boundary version follows identically to that of Lemma 3.9 [HWZ3]:

Lemma 2.1.1. (compare Lemma 3.9 [HWZ3]) Let (S, j, M, D) be a Riemann surface with boundary. The derivative $Dj(v)$ of the map $v \to j(v)$ is a linear map

$$Dj(v) : \Omega^{0,1}((TS, j), (TS, j)) \to \Omega^{0,1}((TS, j(v)), (TS, j(v)))$$

We have

$$j(v) \circ Dj(v)[\delta v] = Dj(v)[j \circ \delta v]$$

$Dj(v)$ is complex linear. ■

We have the following from [HWZ3] in the closed case:

Theorem 3.10 [HWZ3] For every noded surface $\alpha = (S, j, M, D)$ there exists a good family $v \to j(v)$ which is complex. It is defined on a G-invariant open neighborhood V of $0 \in E$, where $E \subset \Omega^{0,1}(\alpha)$ is a complex G-invariant subspace with projection onto $H^1(\alpha)$ is an isomorphism. Explicitly,

$$j(v) = j \cdot (1 + v)(1 - v)^{-1},\ v \in V \subset E \quad ■$$

In the case of Riemann surfaces with boundary we have the following:

Proposition 2.1.4. (compare Theorem 3.10 [HWZ3]) For every Riemann surface with boundary $\alpha = (S, j, M, D)$ there exists a good family $v \to j(v)$. It is defined on a G-invariant open neighborhood V of $0 \in E$, where $E \subset \Omega^{0,1}(\alpha)$ is a complex G-invariant

subspace with projection onto $H^1(\alpha)$ is an isomorphism. Explicitly,

$$j(v) = j \cdot (1+v)(1-v)^{-1},\ v \in V \subset E$$

Proof: By construction we have:

$$j \cdot Dj(0)(\delta v) = -2(\delta v) \quad (1)$$

The operator $D : \Gamma_0(\alpha) \to \Omega^{0,1}(\alpha)$ is a Fredholm operator. So there exists a finite dimensional vector space $E \subset \Omega^{0,1}(\alpha)$ such that the projection onto the quotient $H^1(\alpha)$ is an isomorphism. We can assume E is G-invariant by employing an averaging argument as in [HWZ3] Theorem 3.10. From (1) above, it follows that $j(v)\cdot Dj(v)\cdot$ is an isomorphism for v sufficiently close to 0. ■

The following is a consequence of Riemann-Roch (see Appendix C [MS]):

Theorem 2.1.1. (compare Proposition 3.3, 3.4, 3.5 [HWZ3], Theorem C.1.10 [MS]) Suppose $[\alpha] = [(S, j, M_i, M_b, D_i, D_b)] \in DM_{g,h,M_i,M_b}$. The Cauchy-Riemann operator

$$D : \Gamma_0(\alpha) \to \Omega^{0,1}(\alpha)$$

$$f \to \frac{\partial f}{\partial z} d\overline{z}$$

is Fredholm and

$$\mathrm{ind}(D) := 3 \cdot \chi(\alpha) - 2\#M_i - \#M_b + 2\#D_i + \#D_b$$

$$= 6 - 6g - 3h - 2\#M_i - \#M_b + 2\#D_i + \#D_b \quad ■$$

Remark 2.1.1. Part (iii) of (see Theorem C.1.10 [MS]) implies that D is injective if

and only if α is stable. ■

2.2 Gluing Riemann Surfaces

[HWZ3] establishes the following in the case of closed Riemann surfaces:

Theorem 1.9 [HWZ3] Given a connected, closed stable nodal surface (S, j, M, D) and mutually disjoint closed disk-like neighborhoods D_x of the points x in $|D|$ with union such that

$$\bigcup_{x \in |D|} D_x$$

is invariant under the G-action, there exists a good complex family $v \to j(v) \in \mathfrak{J}_{D,j}$ such that $j(0) = j$. ■

From Theorem 1.9 [HWZ3] and a doubling argument we have:

Theorem 2.2.1. Given a connected stable nodal surface with boundary (S, j, M_i, M_b, D) and mutually disjoint closed disk like (half-disk like) neighborhoods D_x of the points $x \in |D_i|$ $(x \in |D_b|)$ such that

$$\bigcup_{x \in |D|} D_x$$

is invariant under the G-action, there exists a good complex family $v \to j(v) \in \mathfrak{J}_{D,j}$ such that $j(0) = j$. ■

To study nodal degeneration we introduce the notion of a gluing profile:

Definition 2.2.1. A gluing profile is a diffeomorphism

$$\psi : (0, 1] \to [0, \infty) \quad ■$$

Intuitively, the gluing profile will specify the rate of nodal degeneration specified by a given finite dimensional parametrization of the Deligne-Mumford space. As discussed in [HWZ3], the holomorphic Deligne-Mumford space is realized by the choice of profile $\phi_{log}(r) := \frac{-1}{2\pi}\log(r)$. The choice of profile we will need for polyfold theory, to ensure smoothness of reparametrization actions in Section 3, is the exponential gluing profile $\phi(r) = e^{\frac{1}{r}} - e$.

Definition 2.2.2. Given a Riemann surface with boundary $\alpha = (S, j, M, D)$ we define the vector space

$$N(D) := \mathbb{R}^{\#|D_b|} \oplus \mathbb{C}^{\#|D_i|}$$

We will draw the collection of gluing parameters from the partial quadrant:

$$B_D := [0, \frac{1}{2})^{\#|D_b|} \oplus B_{\frac{1}{2}}^{\#|D_i|}$$

where $B_{\frac{1}{2}}$ is the open ball of radius $\frac{1}{2}$ in $\mathbb{C}$. By $a = (a_{\{x,y\}\in D}) \in B_D$ or $\overline{a} \in B_D$ we denote a collection of gluing parameters for α. ■

First we examine the gluing of an interior node $\{x, y\}$. For each disk in the small disk structure we choose holomorphic polar coordinates

$$\psi_x : [0, \infty) \times S^1 \to D_x - \{x\}, \quad \psi_x(s, t) = \overline{\psi}_x(e^{-2\pi(s+it)})$$

where $\overline{\psi}_x : \{z \in \mathbb{C} : |z| \leq 1\} \to D_x$ is a biholomorphism satisfying $\overline{\psi}_x(0) = x$. For y we choose negative coordinates:

$$\psi_y : (-\infty, 0] \times S^1 \to D_y - \{y\}, \quad \psi_y(s, t) = \overline{\psi}_y(e^{2\pi(s+it)})$$

where $\overline{\psi}_y : \{z \in \mathbb{C} : |z| \leq 1\} \to D_y$ is a biholomorphism satisfying $\overline{\psi}_y(0) = y$.

We use these coordinates construct the glued neck $Z^{\{x,y\}}_{a_{\{x,y\}}}$:

Definition 2.2.3. Interior Neck $Z^{\{x,y\}}_{a_{\{x,y\}}}$ Suppose $a_{\{x,y\}} \neq 0$. Consider the subsets

$$D_{x,a_{\{x,y\}}} := \{z \in D_x : \ z = \psi_x(s,t), \ \ s \leq \phi(|a_{\{x,y\}}|)\}$$

and

$$D_{y,a_{\{x,y\}}} := \{z \in D_y : \ z = \psi_y(s,t), \ \ s \geq -\phi(|a_{\{x,y\}}|)\}$$

We take the quotient

$$Z^{\{x,y\}}_{a_{\{x,y\}}} := D_{x,a_{\{x,y\}}} \bigcup D_{y,a_{\{x,y\}}} / \sim$$

where $\sim$ is the relation which identifies

$$z = h_x(s,t)$$

with

$$z' = h_y(s',t')$$

if

$$s = s' + \phi(|a_{\{x,y\}}|) \ \text{ and } \ t = t' + \theta_{\{x,y\}} \mod 1$$

where $a_{\{x,y\}} = |a_{\{x,y\}}| e^{-2\pi i \theta_{\{x,y\}}}$. If $a_{\{x,y\}} = 0$ we define $Z^{\{x,y\}}_0 = D_x \cup D_y$ ■

Remark 2.2.1. Observe that the complex structure on $Z^{\{x,y\}}_{a_{\{x,y\}}}$ depends on the choice of complex coordinate charts. Indeed, there is an S^1 family of possible choices coming from the relative angle of rotation between the coordinate charts ψ_x and ψ_y. For a construction of $Z^{\{x,y\}}_{a_{\{x,y\}}}$ which captures this choice intrinsically. ■

For the boundary nodal pairs, we similarly choose positive polar coordinates and negative polar coordinates

$$\psi_x : [0,\infty) \times [0,1] \to HD_x - \{x\} \quad \text{and} \quad \psi_y : (-\infty, 0] \times [0,1] \to HD_y - \{y\}$$

The boundary neck is defined similarly:

Definition 2.2.4. Boundary Neck $Z^{\{x,y\}}_{a_{\{x,y\}}}$ Suppose $a_{\{x,y\}} \neq 0$. Consider the subsets

$$HD_{x,a_{\{x,y\}}} := \{z \in HD_x : z = \psi_x(s,t),\ s \leq \phi(|a_{\{x,y\}}|)\}$$

and

$$HD_{y,a_{\{x,y\}}} := \{z \in HD_y : z = \psi_y(s,t),\ s \geq -\phi(|a_{\{x,y\}}|)\}$$

We take the quotient

$$HZ^{\{x,y\}}_{a_{\{x,y\}}} := HD_{x,a_{\{x,y\}}} \bigcup HD_{y,a_{\{x,y\}}} / \sim$$

where $\sim$ is the relation which identifies

$$z = \psi_x(s,t)$$

with

$$z' = \psi_y(s',t')$$

if

$$s = s' + \phi(a_{\{x,y\}}) \ \text{ and } \ t = t'$$

If $a_{\{x,y\}} = 0$ we define $HZ^{\{x,y\}}_0 = HD_x \cup HD_y$. ■

Remark 2.2.2. We will also refer to $HZ^{\{x,y\}}_{a_{\{x,y\}}}$ as $Z^{\{x,y\}}_{a_{\{x,y\}}}$. ■

Remark 2.2.3. When $a_{\{x,y\}} \neq 0$, the neck has two global coordinate systems given by the positive coordinates inherited by D_x:

$$\{(s,t) : 0 \leq s \leq \phi(|a_{\{x,y\}}|)\}$$

and the negative coordinates inherited by D_y:

$$\{(s',t') : -\phi(|a_{\{x,y\}}|) \leq s' \leq 0\}$$

Given a point $x \in Z^{\{x,y\}}_{a_{\{x,y\}}}$ we will refer to the positive coordinate representative of x by $[(s,t)]$ and the negative coordinate representative by $[(s',t')]$. By

$$\psi_{a_{\{x,y\}}} : \{(s,t) : 0 \leq s \leq \phi(|a_{\{x,y\}}|)\} \to Z^{\{x,y\}}_{a_{\{x,y\}}}$$

we denote the positive coordinate chart on the glued neck. ■

Definition 2.2.5. The subset $Z^{\{x,y\}}_{a_{\{x,y\}}}(-h) \subset Z^{\{x,y\}}_{a_{\{x,y\}}}$ is defined by

$$Z^{\{x,y\}}_{a_{\{x,y\}}}(-h) = \{[s,t] \in Z^{\{x,y\}}_{a_{\{x,y\}}} : s \in [h, R-h]\} \quad ■$$

Definition 2.2.6. (Core of a Riemann Surface) Consider a real tuple $\overline{r} \in N(D)$. Then we define $S_{core,\overline{r}} \subset S$ to be the complement:

$$S_{core,\overline{r}} := S - \bigcup_{\{x,y\} \in D} \overline{\psi_x([r_{\{x,y\}}, \infty) \times J_{\{x,y\}})} \cup \overline{\psi_y((-\infty, -r_{\{x,y\}}] \times J_{\{x,y\}})}$$

where $J_{\{x,y\}} = [0,1]$ if $\{x,y\} \in D_b$ and $J_{\{x,y\}} = S^1$ if $\{x,y\} \in D_i$. When all entries of r are identical we will refer to $S_{core,\bar{r}}$ as $S_{core,r}$. We also abbreviate $S_{core,1}$ by S_{core}. ■

Definition 2.2.7. (Glued Surface) Given a Riemann surface with boundary (S, j, M, D), and a set of gluing parameters $(a_{\{x,y\}})_{\{x,y\}\in D} \in B_D$, we define the glued Riemann surface (S_a, j_a, M_a, D_a) as follows:

1. $S_a := S/\sim$ where $\sim$ identifies:

$$[h_x(s,t)] \sim [h_y(s',t')]$$

if $s - s' = \phi(|a_{\{x,y\}}|)$ and $t - t' = \log\frac{a_{\{x,y\}}}{-2\pi|a_{\{x,y\}}|} \mod 1$ for $\{x,y\} \in D_i$ and identifies

$$[h_x(s,t)] \sim [h_y(s,t)]$$

if $s - s' = \phi(|a_{\{x,y\}}|)$ for $\{x,y\} \in D_b$.

2. j_a is the complex structure naturally induced by j on S_a.

3. $(M_i)_a$ and $(M_b)_a$ are identified with M_i and M_b as subsets of S_{core}.

4. D_a is identified with the nodes in D for which $a_{\{x,y\}} = 0$. ■

Definition 2.2.8. (compare Definition 2.11 [HWZ1]): Let $a^l = (a^l_{\{x,y\}})_{\{x,y\}\in D}$ be a sequence of gluing parameters. A sequence of points $(z_l) \subset S_{a^l}$ is said to stay in finite distance to the core if there exsists a real number $h > 0$ satisfying $z_l \notin Z_{a^l}(-h)$ for l sufficiently large. ■

Definition 2.2.9. Let α be a Riemann surface with boundary equipped with a small disk structure $\mathbf{D}$. Let

$$\psi_x : \mathbb{R}^+ \times J \to D_x - \{x\}, \quad \psi_y : \mathbb{R}^- \times J \to D_y - \{y\}$$

be positive and negative holomorphic coordinates where $J = S^1$ if $\{x, y\} \in D_i$ and $J = I$ if $\{x, y\} \in D_b$. Then we say that

$$\bigcup_x \psi_x(\{0\} \times J) \subset S, \quad \bigcup_y \psi_y(\{0\} \times J) \subset S$$

are the left and right boundaries, respectively. We continue to denote the images of $\{0\} \times J$ under the coordinate charts ψ_x and ψ_y in the glued surfaces S_a as the left and right boundaries, respectively. ■

Lemma 2.2.1. Consider a biholomorphism of Riemann surfaces:

$$\psi : (S, j, D) \to (S', j', D')$$

Choose disk structures $\mathbf{D^1}$ for (S, j, D) and $\mathbf{D^2}$ for (S', j', D') such that

$$\psi(\mathbf{D^1}) = \mathbf{D^2}$$

Further fix holomorphic coordinates:

$$h_x : (\mathbb{R}^+ \times J_{\{x,y\}}, i) \to D_x, \quad h_y : (\mathbb{R}^- \times J_{\{x,y\}}, i) \to D_y$$

$$k_{\psi(x)} : (\mathbb{R}^+ \times J_{\{x,y\}}, i) \to D_{\psi(x)}, \quad k_{\psi(y)} : (\mathbb{R}^- \times J_{\{x,y\}}, i) \to D_{\psi(y)}$$

Then for gluing parameters $a_{\{x,y\}\in D}$ we have a family of biholomorphisms:

$$\psi_a : (S_a, j_a, D_a) \to (S'_{b(a)}, j'_{b(a)}, D'_{b(a)})$$

such that

$$b_{\{\psi(x),\psi(y)\}} := e^{-2\pi i(\theta_x - \theta_y)} a_{\{x,y\}}$$

where θ_x and θ_y are defined as follows

$$k^{-1}_{\psi(x)} \circ \psi \circ h_x(s,t) = (s, t+\theta_x), \quad k^{-1}_{\psi(y)} \circ \psi \circ h_y(s',t') = (s', t'+\theta_y)$$

Proof: The argument follows similarly to Proposition 2.11 [HWZ3]. We define the map $\psi_a : S_a \to S'_b$ piecewise on the core:

$$\psi_a := \psi \quad \text{on } S - \bigcup_{x\in|D|} \mathbf{D^1}$$

and on the neck:

$$\psi_a([z]_x^{a_{\{x,y\}}}) := [\psi(z)]_{\psi(x)}^{b_{\{\psi(x),\psi(y)\}}}$$

Here $[z]_x^{a_{\{x,y\}}}$ denotes the equivalence class of $z \in D_x$ in S_a and $[\psi(z)]_{\psi(x)}^{b_{\{\psi(x),\psi(y)\}}}$ denotes the equivalence class of $\psi(z) \in Z_{\psi(x)}^{b_{\{\psi(x),\psi(y)\}}}$. It suffices to show that if $h_x(s,t)$ and $h_y(s',t')$ are equivalent in $Z_{a_{\{x,y\}}}^{\{x,y\}}$, then $k_{\psi(x)} \circ \psi \circ h_x(s,t)$ and $k_{\psi(y)} \circ \psi \circ h_y(s',t')$ are equivalent in $Z_{b_{\{\psi(x),\psi(y)\}}}^{\{\psi(x),\psi(y)\}}$. The former equivalence requires

$$s - s' = R \,\text{and}\, t - t' = \theta, \text{ where } a_{\{x,y\}} = e^{-2\pi\theta}$$

while the latter equivalence requires

$$s - s' = R \text{ and } (t + \theta_x) - (t' + \theta_y) = \theta + \theta_x - \theta_y, \text{ where } a_{\{x,y\}} = e^{-2\pi\theta}$$

It is clear these conditions are the same. ■

Now we introduce the topology on DM_{g,h,M_i,M_b}.

Definition 2.2.10. Consider $[\alpha] \in DM_{g,h,M_i,M_b}$, and fix a representative (S, j, M, D) for α. Fix a small disk-structure $\mathbf{D}$ for α, a neighborhood U of j in $\mathfrak{J}_{D,j}$ and $\epsilon \in (0, 1)$. We define the set

$$V(\alpha, U, \mathbf{D}, \epsilon) := \{[S_a, k_a, M_a, D_a] \in DM_{g,h,M_i,M_b} \ : \ k \in U, \ a \in B_D, \ |a| < \epsilon\}$$

We denote the collection of such subsets by $\mathfrak{V}$. ■

Proposition 2.2.1. (compare Theorem 5.13 [HWZ3]) The collection of sets $\mathfrak{V}$ defines a basis for a second countable Hausdorff topology $\mathfrak{T}$ on DM_{g,h,M_i,M_b}.

Proof: The proof follows analogously to Theorem 5.13 [HWZ3]. ■

2.3 Good Uniformizing Families

For each interior nodal pair $\{x, y\} \in D_i$, there is a uniquely determined complex anti-linear map $\phi_{x,y} : T_yS \to T_xS$ such that if $\phi_{y,x}$ is its inverse, then the composition

$$T\overline{\psi}_y(0)^{-1} \circ \phi_{y,x} \circ T\overline{\psi}_x(0) : \mathbb{C} \to \mathbb{C}$$

is complex conjugation (Proposition 2.5 [HWZ3]). The composition

$$\sigma_{\{x,y\}}(g) := \phi_{x,y} \circ Tg(y)^{-1} \circ \phi_{g(x),g(y)} \circ Tg(x) : T_xS \to T_xS$$

is unitary (see Proposition 2.6 [HWZ3]). Thus $\sigma_{\{x,y\}}(g) \in S^1$. Consider the G-action

$$\sigma : G \to GL(N) \times GL(E)$$

given by

$$g * (a^0, a^1, e) = \sigma(g)(a^0, a^1, e) = (\sigma_N(g)a^0, a^1, g * e)$$

where $a^0 \in \mathbb{C}^{\#|D_i|}$, $a^1 \in \mathbb{R}^{\#|D_b|}$ and $\sigma_N(g)a^0 = b$ is given by

$$b_{\{g(x),g(y)\}} = \sigma_{\{x,y\}}(g) \cdot a^0_{\{x,y\}}$$

Now we use the above notions to define finite dimensional parametrizations of DM_{g,h,M_i,M_b} :

Definition 2.3.1. Good Uniformizing Family (compare Definition 2.12 [HWZ1]) Consider the stable Riemann surface with boundary $\alpha = (S, j, M, D)$ with automorphism group G, and fix a disk structure $\mathbf{D}$ for α. Let $V \subset E$ be a neighborhood of $0 \in E$ and consider a good deformation of j, $v \to j(v)$ parametrizd by V.

We define the family

$$(a, v) \to \alpha_{(a,v)}$$

$$(a, v) \to (S_a, j(v)_a, M_a, D_a)$$

of Riemann surfaces deformed by both gluing and variation of j. We restrict the parameters (a, v) to vary in some open neighborhood $(0, 0) \in O \subset N \times E$ which is invariant

under the action of G. The family $(a,v) \to \alpha_{(a,v)}$ is called a good uniformizing family with domain O if the following is satisfied:

(1) The set $U = \{[\alpha_{(a,v)}] : (a,v) \in O\}$ of equivalence classes of stable Riemann surfaces is open in the space DM_{g,h,M_i,M_b}.

(2) The map $p : O \to U$ defined by

$$p(a,v) = [\alpha_{(a,v)}]$$

induces a homeomorphism $[p] : O/G \to U$.

(3) If $\phi : \alpha_{(a,v)} \to \alpha_{(a',v')}$ is an isomorphism of noded, bordered Riemann surfaces where $(a,v) \in O$ and $(a',v') \in O$, then there exists an automorphism $g \in G$ satisfying $(a',v') = g * (a,v)$. Additionally, $\phi = g_a$.

(4)

$$[j \cdot Dj(v)] : E \to H^1(\alpha) \text{ is a real isomorphism for all } v \in V \quad \blacksquare$$

The main result on the parametrization of Deligne-Mumford spaces is:

Theorem 2.3.1. Existence of Good Uniformizing Families For each stable noded Riemann surface $\alpha = (S, j, M, D)$ there exists a good uniformizing family

$$(a,v) \to \alpha_{(a,v)}, \ \ (a,v) \in O$$

Proof: The proof is given in Section 8. $\blacksquare$

Theorem 2.3.1 follows from Theorem 2.19 [HWZ2] and a straightforward doubling argu-

ment. We will present another argument using lego decompositions in Section 8.

Good uniformizing families will form our orbifold charts for DM_{g,h,M_i,M_b}: Let O_λ be the parameter space parametrizing a good uniformizing family $(a,v) \to \alpha_{(a,v)}$. Let G_λ be the graph of the uniformizing family

$$G_\lambda := \{((a,v), \alpha_{(a,v)}) : (a,v) \in O_\lambda\}$$

Let Λ be a set of indexing graphs G_λ such that the images of the projections $p : G_\lambda \to DM_{g,h,M_i,M_b}$ cover DM_{g,h,M_i,M_b}. DM_{g,h,M_i,M_b} is the orbit space of a groupoid T_{g,h,M_i,M_b} with objects given by:

$$T_{g,h,M_i,M_b} := \bigsqcup_{\lambda \in \Lambda} G_\lambda$$

The morphism sets $M(G_\lambda, G_{\lambda'})$ are triples

$$\{((a,v), \phi, (a',v')) : (a,v) \in O_\lambda,\ (a',v') \in O_{\lambda'},\ \phi : \alpha_{a,v} \to \alpha'_{a',v'} \text{ is an equivalence}\}$$

The collection of all morphism sets is given by:

$$\mathbf{T}_{g,h,M_i,M_b} := \bigsqcup_{\lambda,\lambda' \in \Lambda} M(G_\lambda, G_{\lambda'})$$

Theorem 2.3.2. With respect to the exponential gluing profile, the object set T_{g,h,M_i,M_b} is a smooth manifold with boundary. The morphism sets $M(G_\lambda, G_{\lambda'})$ are also smooth manifolds with boundary. The structure maps are *sc*-smooth, and $(T, \mathbf{T})$ is an *ep*-groupoid. These charts realize DM_{g,h,M_i,M_b} as a smooth orbifold.

Proof: Theorem 2.3.1 is a consequence of Section 8. ■

The identification of the objects T by the morphisms $\mathbf{T}$ realizes DM_{g,h,M_i,M_b} as the quotient. We will make a similar identification of the space of stable maps Z as the orbit space of an *ep*-groupoid X.

From Theorem 2.1.1 and Proposition 2.1.4 , we have:

Corollary 2.3.1. $\dim DM_{g,h,M_i,M_b} = 3(2g+h-2) + 2\#M_i + \#M_b$ ■

2.4 The Universal Property of Uniformizing Families

We recall the discussion preceding Theorem 2.19 [HWZ1]: Let $\beta = (S', k, M', D')$ be a stable noded Riemann surface with boundary and small disk structure $\mathbf{D}'$. For each marked point $z \in M' := M'_i \cup M'_b$, choose an open neighborhood $V(z)$ such that the neighborhoods $V(z)$ are pairwise disjoint and such that they do not intersect $|D'|$ but *may* intersect $\mathbf{D}'$. If $M' = \{z_1, \cdots, z_{\#|M'|}\}$ we consider the space

$$V(M') := V(z_1) \times \cdots \times V(z_{\#|M'|})$$

Let $U_{D'}$ be a C^∞ open neighborhood of complex structures in $\mathfrak{J}_{D',k}$. Given the tuple

$$(b, k, \sigma) \in N_{D'} \times U_{D'} \times V(M'),$$

similarly to Definition 2.3.1, we associate the noded Riemann surface

$$\beta_{(b,k,\sigma)} := (S'_b, k'_b, M'(\sigma)_b, D'_b)$$

for $|b|$ sufficiently small, σ sufficiently close to σ_0 where $M'(\sigma_0) = M'$, and k' sufficiently close to k. For a family $w \to k(w)$ varying smoothly in a finite dimensional family, we

refer to the association

$$(b, w, \sigma) \to \beta_{(b,k(w),\sigma)} = (S'_b, k(w)_b, M(\sigma)_b, D_b)$$

as a smooth family.

Definition 2.4.1. Core-Continuity and Core-Smoothness (compare Section 2.1 [HWZ1]) Let notation be as in the preceding discussion. Consider a finite dimensional smooth manifold $U \subset U_{D'}$ parametrized by a finite dimensional germ $W \subset E'$. For each point $(b, w, \sigma) \in N_{D'} \oplus W \times V(M')$ we associate an isomorphism:

$$\phi(b, k(w), \sigma) : \beta_{(w,k(w),\sigma)} \to \alpha_{\mu(b,k(w),\sigma)}$$

Fix $(b_0, w_0, \sigma_0) \in N_{D'} \oplus W \times V(M')$, if $z \in S'_{b_0}$ is non-nodal then the image $\phi(b_0, k(w_0), \sigma_0)(z)$ is also non-nodal in S_{a_0}. If $z \in S'_{core}$, we identify a neighborhood of $z \in S'_{b_0}$ with a neighborhood of z considered as a point in S'. If $z \notin S_{core}$, there exists a nodal pair $\{x, y\} \in D$ such that z can be identified with a point in $D_x - \{x\}$ or $D_y - \{y\}$. Similarly, we can identify a neighborhood of the point $\phi(b_0, j(v_0), \sigma_0)(z)$ as a subset of S_{core} or $\mathbf{D}$.

Using these identifications, we can say that the family $(b, v, \sigma) \to \phi(b, k(w), \sigma)$ defines a family of diffeomorphisms between a germ of z and a germ of $\phi(b_0, k(w_0), \sigma_0)(z)$ for (b_0, w_0) sufficiently small. Core-continuity, resp. core-smoothness, means that all of these families of maps are continuous, resp. smooth, between the identified subsets of S' and S. ■

The following is proved in the Section 8 using the *sc*-implicit function theorem:

Theorem 2.4.1. (compare [HWZ1] Theorem 2.19) Fix a good family $(a, v) \to \alpha_{(a,v)}$

1. Consider the family $(b,k,\sigma) \to \beta_{(b,k,\sigma)}$ of noded Riemann surfaces with boundary, where $(b,k,\sigma) \in B_{\frac{1}{2}} \times U_{D'} \times V(M')$ and suppose that we have an isomorphism:

$$\psi : \beta_{(0,j',\sigma_0)} = (S', j', M'(\sigma_0), D') \to (S_{a_0}, j(a_0,v_0), M_{a_0}, D_{a_0}) = \alpha_{(a_0,v_0)}$$

for some parameter (a_0, v_0). Then there exists a uniquely determined core-continuous germ

$$(b,k,\sigma) \to \mu(b,k,\sigma) = (a(b,k,\sigma), w(b,k,\sigma))$$

defined for (b,k,σ) near $(0,j',\sigma_0)$ which satisfies $\mu(0,j',\sigma_0) = (a_0,v_0)$ and a core-continuous germ $(b,k,\sigma) \to \phi(b,k,\sigma)$ of isomorphisms

$$\phi(b,k,\sigma) : \beta_{(b,k,\sigma)} \to \alpha_{\mu(b,k,\sigma)}$$

between the noded Riemann surfaces with boundary satisfying $\phi(0,j',\sigma_0) = \psi$.

2. Also if $w \to k(w)$ is a smooth family of complex structures on S', with $k(0) = j'$ then the family $(b,w,\sigma) \to \phi(b,k(w),\sigma)$ of isomorphisms

$$\phi(b,k(w),\sigma) : \beta_{(b,k(w),\sigma)} \to \alpha_{\mu(b,k(w),\sigma)}$$

is core-smooth. ■

3 Gluing Maps

In Sections 3 and 4, we produce polyfold charts on the *ep*-groupoid of stable maps. These charts parametrize the neighborhood a stable map by combining the deformations of the

Riemann surface structure, described in Section 2, and continuous deformations of the map on a fixed domain. Section 3 is concerned with parametrizing the deformation of a map defined on a family of degenerating Riemann surfaces near the nodal degeneration on the domain.

3.1 Overview of M-Polyfolds

In Gromov-Witten theory we consider continuous families of reparametrizations of surfaces. However, the continuous reparametrization is never naively classically smooth. Using the notion of scaled smoothness, introduced by Hofer-Wysocki-Zehnder, we can extend the notion of smoothness to this situation in a manner which allows for the eventual construction of a Fredholm and integration theory on the moduli spaces of pseudo-holomorphic curves.

Definition 3.1.1. (compare Definition 1.1 [HWZ2]) An *sc*-structure on a Banach space E is a filtration by a sequence of subspaces

$$\cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_0 = E$$

where each subspace has a Banach space structure and the inclusion $E_{i+1} \to E_i$ is compact. Lastly, we require that the intersection $E_\infty := \bigcap_{i=0}^{\infty} E_i$ is dense in every E_i. ■

Given an *sc*-structure on E, we define a shifted *sc*-structure on E^j by the rule $(E^j)_i := E_{i+j}$.

We say a Banach space with an *sc*-structure is a scaled Banach space. We can extend a number of natural constructions on Banach spaces to scaled Banach spaces. Given E and F scaled Banach spaces, the direct sum $E \oplus F$ admits an *sc*-structure defined by

$(E \oplus F)_n := E_n \oplus F_n$.

Let E and F be scaled Banach spaces, then a linear operator $T : E \to F$ is an *sc*-operator if it preserves the filtration, i.e. $T(E_i) \subset F_i$ and $T : E_i \to F_i$ is continuous.

Proposition 3.1.1. Suppose $T : E \to F$ is an *sc*-linear map of *sc*-Hilbert spaces. Then $\ker T$ is an *sc*-Hilbert space.

Proof: $\ker T$ inherits an *sc*-filtration

$$\ker T_m := \ker T \cap E_m, \quad m \geq 0$$

from E. By hypothesis on E, the inclusions $\ker T_{m+1} \hookrightarrow \ker T_m$ are compact. It is enough to show that $\ker T \cap E_\infty$ is dense in each space $\ker T \cap E_m$. By hypothesis on E, given any $e \in E_m$, there exists a sequence $\{e_n\}_{n=0}^{\infty} \subset E_\infty$ which limits to e in E_m. $\ker T_m = \ker \mathrm{T}|_{E_m}$ is closed in every E_m, $m \geq 0$, so there exists an orthogonal splitting of Hilbert spaces (not necessarily of *sc*-Hilbert spaces)

$$E_m = \ker T_m \oplus C$$

for some subspace $C \subset E_m$ such that T is injective on C. Let e_n^1 and e_n^2 denote the projections of e_n onto $\ker T_m$ and C, respectively. Then $T(e_n) = T(e_n^1) + T(e_n^2) \to T(e) = 0$ as $n \to \infty$ implies that $T(e_n^2) \to 0$ in E_m. The injectivity of T implies that e_n^2 converges to 0 in E_m. In particular, this means that $e_n^1 \to e$ in E_m. Therefore, $\ker T_\infty$ is dense in E_m, for $m \geq 0$. ■

We say that a subset $C \subset E$ in an *sc*-Banach space is a partial quadrant if there exists an *sc*-linear isomorphism $T : E \to \mathbb{R}^n \oplus W$ such that $T(C) = [0, \infty)^n \oplus W$.

Now we come to the notion of *sc*-smoothness. Given *sc*-Banach spaces E and F, $U \subset E$ an open subset of a partial quadrant $C \subset E$, and $f : U \to F$ a map. If $f(U_i) \subset F_i$ and $f : U_i \to F_i$ is continuous for all $i \geq 0$, then we say the map f is *sc*-continuous, interchangeably we say f is sc^0. Given an open subset $U \subset E$, we define the tangent space of U to be $U^1 \oplus E$.

Definition 3.1.2. (compare Defintion 1.9 [HWZ2]) Consider scale Banach spaces E, F, a partial quadrant $C \subset E$ and an open subset $U \subset C$. A map $f : U \to F$ is sc^1 if the following holds:

1. f is sc^0

2. For each $x \in U_1$ there exists a bounded linear operator $Df(x) : E_0 \to F_0$ such that
$$\lim_{|h|_1 \to 0,\, x+h \in U_1} \frac{|f(x+h) - f(x) - Df(x)h|_0}{|h|_1} = 0$$

3. The map $Tf : TU \to TF$ given by $Tf(x,h) := (f(x), Df(x)h)$ is sc^0

The map $Tf : TU \to TF$ is the tangent map to f, if f is sc^1. We say f is sc^i if Tf is sc^{i-1}. We say f is sc-smooth or sc^∞ if f is sc^i for all $i \geq 0$. ■

Note that we don't require the map $(x,h) \to Tf(x,h)$ to be continuous in the operator norm. Also, observe condition (2) is weaker than the classical condition on the difference quotient. Further, observe that the differential of an sc^1 map can only be evaluated at points with regularity on level 1 or higher.

Theorem 3.1.1. Chain Rule (compare Theorem 1.1 [HWZ2]) Let $U \subset C \subset E$ and $V \subset D \subset F$ and $W \subset Q \subset G$ be relatively open subset of partial quadrants in

sc-Banach spaces and let $f : U \to V$ and $g : V \to W$ be sc^1-maps. Then the composition $g \circ f : U \to W$ is also sc^1- and

$$T(g \circ f) = (Tg) \circ (Tf) \qquad \blacksquare$$

We can generalize the notion of a smooth atlas to an atlas based on scale smooth retractions. Towards this end we have the following definition

Definition 3.1.3. (compare Definition 2.1 [HWZ2]) An sc-smooth model is a triplet (O, C, E), where E is an sc-Banach space, and $C \subset E$ is a partial quadrant in E, and $O = r(U)$ where $r : U \to U$ where $U \subset C$ is relatively open and $O \subset U$. $\blacksquare$

Definition 3.1.4. (compare Definitions 2.6 and 2.8 [HWZ2]) Let X be a topological space, then an M-polyfold chart around a point $x \in X$ is a 4-tuple $(U, \phi, (O, C, E))$ where (O, C, E) is an sc-smooth model, U an open neighborhood of x and $\phi : U \to O$ is a homeomorphism. We say that two charts $(U, \phi, (O, C, E))$ and $(U', \phi', (O', C', E'))$ are sc-smoothly comptaible provided $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ and its inverse are sc-smooth.

An M-polyfold is a Hausdorff paracompact topological space X, equipped with an equivalence class of sc-smooth atlases. $\blacksquare$

3.2 Interior Nodal Gluing

Recall the exponential gluing profile is given by:

$$\phi : (0, 1] \to [0, \infty), \qquad \phi(r) = e^{\frac{1}{r}} - e$$

Definition 3.2.1. Let $a \in B_{\frac{1}{2}}$. If $a = 0$, we define the set Z_0 to be the disjoint union

$$Z_0 := (\mathbb{R}^+ \times S^1) \bigsqcup (\mathbb{R}^- \times S^1)$$

If $a \neq 0$ we write a in polar form $a = |a| \cdot e^{2\pi i\theta}$. If $a \neq 0$, then

$$Z_a := [0, R] \times S^1 \bigsqcup [-R, 0] \times S^1 / (s - R, t - \theta) \sim (s', t')$$

where (s, t) gives the standard coordinates on $\mathbb{R}^+ \times S^1$ and (s', t') gives the standard coordinates on $\mathbb{R}^- \times S^1$, where $R := \phi(|a|)$. ■

Throughout this section, we fix an increasing sequence $(\delta_0^k)_{k=1}^{\infty} \subset (0, 2\pi)$.

Definition 3.2.2. If $a \neq 0$ we define the infinite cylinder C_a :

$$C_a := \mathbb{R}^+ \times S^1 \bigsqcup \mathbb{R}^- \times S^1 / \sim$$

where the equivalence relation $\sim$ identifies $(s, t) \in [0, R] \times S^1 \subset \mathbb{R}^+ \times S^1$ with points $(s', t') \in [-R, 0] \times S^1 \subset \mathbb{R}^- \times S^1$ which satisfy

$$s - s' = R \quad \text{and} \quad t - t' = \theta$$

For $a = 0$, set $C_0 = \emptyset$. ■

Definition 3.2.3. Let $\beta : \mathbb{R} \to [0, 1]$ be the smooth function uniquely specified by the conditions:

1. $\beta(-s) + \beta(s) = 1$

2. $\beta(s) = 1$ for all $s \leq -1$

3. $\beta'(s) < 0$ for $s \in (-1, 1)$ ■

For shorthand we use the notation $\mathbb{R}^{\pm} \times S^1$ to stand in for the disjoint union $\mathbb{R}^+ \times S^1 \sqcup \mathbb{R}^- \times S^1$.

Definition 3.2.4. Fix an increasing sequence $0 < \delta_0^0 < \cdots < \delta_0^k < \cdots < 2\pi$. For each $k \geq 0$, we define the Sobolev space:

$$H^{3+k,\delta_0^k}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n}) := \{u \in H^{3+k}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n}) : u \cdot e^{\delta_0^k |s|} \in H^{3+k}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})\}$$

The norm on $H^{3+k,\delta_0^k}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$ is defined by:

$$||u||^2_{k,\delta_0^k} := \sum_{|\alpha| \leq 3+k} \int_{\mathbb{R}^+ \times S^1} |D^\alpha u^+|^2 e^{2\delta_0^k |s|} ds\, dt + \int_{\mathbb{R}^- \times S^1} |D^\alpha u^-|^2 e^{2\delta_0^k v(s)}\, ds\, dt$$

where $v : \mathbb{R} \to \mathbb{R}$ is a smooth map which is equal to the absolute value function away from a compact neighborhood of 0. ■

Lemma 3.2.1. The sequence of Hilbert spaces

$$\cdots \longrightarrow H^{3+k,\delta_0^k}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n}) \longrightarrow \cdots \longrightarrow H^{3,\delta_0^0}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$$

defines the filtration for an *sc*-Hilbert space which we denote $H^{3,\delta_0^0}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$.

Proof: It is enough to show that the inclusions above are compact. Let $\{u_n\}_{n=0}^{\infty} \in H^{3+k,\delta_0^k}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$ be a bounded sequence. We will show this sequence is Cauchy in $H^{3+(k-1),\delta_0^{k-1}}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$. The proposition would be trivial if instead of infinite

cylinders, the domain spaces were cylinders of finite length. The intuition is that the exponential weights concentrate the mass of the sequence away from ∞ and thus reduce the situation to the case of finite length cylinders. Without loss of generality we may assume

$$||u_n|| = 1 \quad \text{for all } n \geq 0 \tag{3.2.1}$$

Claim Given $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\int_{[R_\epsilon,\infty)\times S^1 \sqcup (-\infty,-R_\epsilon]\times S^1} |D^\alpha {u_n}^{\pm}|^2 \cdot e^{2\delta_0^{k-1}|s|} ds\, dt < \epsilon \quad \text{for all } n$$

Proof of the Claim: Suppose for sake of contradiction there exists a subsequence u_{n_m} and a sequence $R_m \to \infty$ such that

$$\int_{[R_m,\infty)\times S^1 \sqcup (-\infty,-R_m]\times S^1} |D^\alpha u_{n_m}^{\pm}|^2 \cdot e^{2\delta_0^{k-1}|s|} ds\, dt > \epsilon \quad \text{for all } m \geq 0$$

Consider the inequalities:

$$\int_{[R_m,\infty)\times S^1 \sqcup (-\infty,-R_m]\times S^1} |D^\alpha u_{n_m}^{\pm}|^2 \cdot e^{2\delta_0^{k}|s|} ds\, dt$$

$$= \int_{[R_m,\infty)\times S^1 \sqcup (-\infty,-R_m]\times S^1} |D^\alpha u_{n_m}^{\pm}|^2 \cdot e^{2(\delta_0^{k}-\delta_0^{k-1})|s|} \cdot e^{2\delta_0^{k-1}|s|} ds\, dt$$

$$> \int_{[R_m,\infty)\times S^1 \sqcup (-\infty,-R_m]\times S^1} |D^\alpha u_{n_m}^{\pm}|^2 \cdot e^{2\delta_0^{k-1}|s|} \cdot e^{2(\delta_0^{k}-\delta_0^{k-1})R_m}\, ds\, dt$$

$$> e^{2(\delta_0^{k}-\delta_0^{k-1})R_m}\epsilon$$

However, taking $R_m \to \infty$ contradicts (3.2.1). ■

Given the claim, for each natural number $m \geq 0$, there exists an $R_m \in \mathbb{R}^+$ such that

$$\int_{[R_m,\infty)\times S^1 \sqcup (-\infty,-R_m]\times S^1} |D^\alpha u_n^\pm|^2 \cdot e^{2\delta_0^{k-1}|s|} ds\, dt < \frac{1}{m} \text{ for all } n \geq 0$$

For each m choose a cut-off function $\chi_m : \mathbb{R}^\pm \times S^1 \to [0,1]$ which is equal to 1 on $[0, R_m + 1]$ and equal to 0 on $[R_m + 2, \infty)$. Then for each m, consider the sequence $\{u_n^m\}_{n=0}^\infty := \{(\chi_m(s) \cdot u_n^+, \chi_m(-s) \cdot u_n^-)\}_{n=0}^\infty$. Set

$$H^{3+k}([0, \pm(R_m+2)]\times S^1, \mathbb{R}^{2n}) := H^{3+k}([0, R_m+2]\times S^1, \mathbb{R}^{2n}) \oplus H^{3+k}([-(R_m+2), 0]\times S^1, \mathbb{R}^{2n})$$

Since the inclusions

$$H^{3+k}([0, \pm(R_m + 2)] \times S^1, \mathbb{R}^{2n}) \hookrightarrow H^{2+k}([0, \pm(R_m + 2)] \times S^1, \mathbb{R}^{2n})$$

are compact, considering each map u_n^m as valued on $[0, \pm(R_m + 2)] \times S^1$, it follows that each sequence $\{u_n^m\}_{m=0}^\infty$ is Cauchy in $H^{2+k}([0, \pm(R_m + 2)] \times S^1, \mathbb{R}^{2n})$. Since the exponential weight is bounded on each compact subset, it follows that $\{u_n^m\}_{m=0}^\infty$ is Cauchy in $H^{2+k,\delta_{k-1}}(\mathbb{R}^\pm \times S^1, \mathbb{R}^{2n})$.

Given $\epsilon > 0$, choose $m \in \mathbb{N}$ such that $\frac{\epsilon}{3} < \frac{1}{m}$ and choose N is large enough such that for all $n, k \geq N$,

$$||u_n^m - u_k^m||_{2+k,\delta_{k-1}} < \frac{\epsilon}{3}$$

Observe:

$$||u_n - u_k||_{2+k,\delta_{k-1}} = ||u_n - u_n^m + u_n^m - u_k||_{2+k,\delta_{k-1}} \leq ||u_n - u_n^m||_{2+k,\delta_{k-1}}$$

$$+ ||u_n^m - u_k^m + u_k^m - u_k||_{2+k,\delta_{k-1}}$$

$$\leq ||u_n - u_n^m||_{2+k,\delta_{k-1}} + ||u_n^m - u_k^m||_{2+k,\delta_{k-1}} + ||u_k^m - u_k||_{2+k,\delta_{k-1}} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

The claim follows. ■

By a much simpler argument one can prove:

Lemma 3.2.2. The sequence of Hilbert spaces

$$\cdots \longrightarrow H^{3+k}(Z_a, \mathbb{R}^{2n}) \longrightarrow \cdots \longrightarrow H^3(Z_a, \mathbb{R}^{2n})$$

defines an *sc*-structure for $H^3(Z_a, \mathbb{R}^{2n})$. ■

Definition 3.2.5. We define the model space $E(2n)$ by

$$E(2n) := \mathbb{R}^{2n} \oplus H^{3,\delta_0^0}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$$

We label $(h^+, h^-) \in E(2n)$ as

$$h^+ = c + u^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}$$

$$h^- = c + u^- : \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}$$

where

$$u = (u^+, u^-) \in H^{3,\delta_0^0}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$$

and $c \in \mathbb{R}^{2n}$. The m^{th} level of $E(2n)$, denoted $E(2n)_m$, is defined by:

$$E(2n)_m := \mathbb{R}^{2n} \oplus H^{3+m,\delta_0^m}(\mathbb{R}^+ \times S^1) \oplus H^{3+m,\delta_0^m}(\mathbb{R}^- \times S^1)$$

The norm on $E(2n)_m$ is given by

$$||h||_{3,\delta_0^m} = |c|^2 + ||u^+||_{H^{3,\delta_0^m}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n})} + ||u^-||_{H^{3,\delta_0^m}(\mathbb{R}^- \times S^1, \mathbb{R}^{2n})} \quad \blacksquare$$

We define families of gluing and anti-gluing operators which extract the head and tail behavior, respectively, of pairs $(h^+, h^-) \in E(2n)$.

Definition 3.2.6. $\oplus$ **Gluing**: For $a \neq 0$ we define the gluing operator

$$\oplus_a : E(2n) \to H^3(Z_a, \mathbb{R}^{2n})$$

$$\oplus_a(h^+, h^-)([s,t]) = \beta(s - \frac{R}{2})h^+(s,t) + (1 - \beta(s - \frac{R}{2}))h^-(s - R, t - \theta)$$

where $R := \phi(|a|)$. For $a = 0$, we define $\oplus_0$ to be the identity. $\blacksquare$

Definition 3.2.7. For each integer $k \geq 0$ and $a \in B_{\frac{1}{2}} - \{0\}$, we define the Sobolev space:

$$H^{3+k,\delta_0^k}(C_a, \mathbb{R}^{2n}) := \{u \in H^{3+k}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) : u \cdot e^{\delta_0^k v(s - \frac{R}{2})} \in H^{3+k}(\mathbb{R} \times S^1, \mathbb{R}^{2n})\}$$

where $v : \mathbb{R} \to \mathbb{R}$ is a smooth map which is equal to the absolute value function away from a compact neighborhood of 0. The norm on $H^{3+k,\delta_0^k}(C_a, \mathbb{R}^{2n})$ is defined as:

$$||u||^2_{3+k,\delta_0^k} := \sum_{|\alpha| \leq 3+k} \int_{\mathbb{R} \times S^1} |D^\alpha u|^2 \cdot e^{2\delta_0^k v(s - \frac{R}{2})} ds\, dt \quad \blacksquare$$

Lemma 3.2.3. The sequence of Hilbert spaces

$$\cdots \longrightarrow H^{3+k,\delta_0^k}(C_a, \mathbb{R}^{2n}) \longrightarrow \cdots \longrightarrow H^{3,\delta_0^0}(C_a, \mathbb{R}^{2n})$$

defines an *sc*-structure on $H^{3,\delta_0^0}(C_a, \mathbb{R}^{2n})$.

Proof: The proof follows similarly to Lemma 3.2.1. ■

Definition 3.2.8. $\ominus$ **Gluing** For $a \neq 0$ define the minus gluing

$$\ominus_a : E(2n) \to H^{3,\delta_0^0}(C_a, \mathbb{R}^{2n})$$

$$\ominus_a(h^+, h^-)([s,t]) = -(1 - \beta(s - \frac{R}{2}))(h^+(s,t) - av_a(h^+, h^-))$$

$$+\beta(s - \frac{R}{2})(h^-(s - R, t - \theta) - av_a(h^+, h^-))$$

where

$$av_a(h^+, h^-) = \frac{1}{2}\int_{S^1} h^+(\frac{R}{2}, t) + h^-(-\frac{R}{2}, t)\, dt \quad ■$$

A linear algebra argument reveals the following:

Proposition 1.27 [HWZ0] For $a \in B_{\frac{1}{2}} - \{0\}$ we have an *sc*-direct sum decomposition

$$E(2n) = \ker(\ominus_a) \bigoplus \ker(\oplus_a) \quad ■$$

Let $\pi_a : E \to E$ be the projection onto $\ker(\ominus_a)$ along $\ker(\oplus_a)$ and consider the map

$$r : B_{\frac{1}{2}} \times E \to B_{\frac{1}{2}} \times E$$

defined by

$$r(a, h^+, h^-) = (a, \pi_a(h^+, h^-))$$

Theorem 1.28 [HWZ0] The map r is *sc*-smooth. ■

The image O of r is then, by construction, in bijective correspondence with the set

$$X(2n) := \{0\} \times E(2n) \bigsqcup_{0<|a|<\frac{1}{4}} \{a\} \times H^3(Z_a, \mathbb{R}^{2n})$$

via the map

$$\operatorname{Im} r \to X(2n)$$

$$(a, \pi_a(u^+, u^-)) \to (a, \oplus_a \pi_a(u^+, u^-))$$

Corollary 3.2.1. $X(2n)$ is an M-polyfold. ■

3.3 Interior Nodal Gluing in the Bundle of Anti-Linear Sections

3.3.1 Strong Bundles Over M-Polyfolds

We present the basic definitions relevant to strong bundles over M-polyfolds which appear in Section 2.5 of [HWZ2], for the convenience of the reader. We begin in local charts $U \subset C \subset E$ where U is a rel. open subset of a partial quadrant C in the *sc*-Banach space E. Given another sc-Banach space F, we define the product

$$U \lhd F$$

As a set $U \lhd F$ is the product $U \times F$. We equip $U \lhd F$ with a double filtration (m, k) for $0 \leq k \leq m + 1$ defined by

$$(U \lhd F)_{m,k} := U_m \oplus F_k$$

For $i = 0, 1$ we consider the *sc*-manifolds $(U \lhd F)[i]$ defined by the filtrations

$$(U \lhd F)[i]_m := U_m \oplus F_{m+i}, \quad m \geq 0$$

Definition 3.3.1. (compare Definition 2.22 [HWZ2]) A strong bundle map $\Phi : U \lhd F \to U' \lhd F'$ is a map which preserves the double filtration and has the form $\Phi(u, h) = (\phi(u), \Gamma(u, h))$ where Γ is linear in h, and for $i = 0, 1$ the maps

$$\Phi : U \lhd F[i] \to U' \lhd F'[i]$$

are *sc*-smooth. A strong bundle retraction is a strong bundle map $R : U \lhd F \to U \lhd F$ which satisfies $R \circ R = R$ of the form

$$R(u, h) = (r(u), \Gamma(u, h))$$

and $r : U \to U$ is an sc-smooth retraction. ■

Definition 3.3.2. (compare Definition 2.23 [HWZ2]) A local strong bundle retract is a tuple

$$(K, C \lhd F, E \lhd F)$$

where $K \subset C \lhd F$ is the image $K = R(U \lhd F)$ of the strong bundle retraction $R : U \lhd F \to U \lhd F$ of the form $R(u, h) = (r(u), \Gamma(u, h))$. We will abbreviate $(K, C \lhd F, E \lhd F)$ by $p : K \to O$, where p is the natural projection. ■

The retract K carries a double filtration $K_{m,k}$ for $m \geq 0$, and $0 \leq k \leq m + 1$:

$$K_{m,k} = K \cap (U_m \oplus F_k) = \{(u, h) \in U_m \oplus F_k : R(u, h) = (u, h)\}$$

$$= \{(u,h) \in O_m \oplus F_k : \Gamma(u,h) = h\}$$

Consider the spaces $K[i]$ by $K[i] = K_{0,i}$ for $i = 0,\ 1$, equipped with the filtrations

$$K[i]_m := K_{m,m+i} \text{ for all } m \geq 0$$

The projections $p : K[i] \to O$ define *sc*-smooth bundles for $i = 0, 1$. The definitions below formally state that a general strong bundle of M-polyfolds defined by a collection of local strong bundles with *sc*-smooth transition functions.

Definition 3.3.3. (compare Definition 2.25 [HWZ2]): A strong bundle chart for the bundle $P : Y \to X$ is a tuple

$$(\Phi, P^{-1}(V), (K, C \lhd F, E \lhd F))$$

such that $(K, C \lhd F, E \lhd F)$ is a strong bundle retract, say $p : K \to O$. In addition, $V \subset X$ is an open subset of X, homeomorphic to the retract O by a homeomorphism $\phi : V \to O$. In addition,

$$\Phi : P^{-1}(V) \to K$$

is a homeomorphism from $P^{-1}(V) \subset Y$ onto the retract K covering the homeomorphism $\phi : V \to O$, so that the diagram

$$\begin{array}{ccc} P^{-1}(V) & \xrightarrow{\Phi} & K \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{r} & O \end{array}$$

is commutative. The map Φ has the property that in the fibers $x \in V$, the map $\Phi : P^{-1}(x) \to p^{-1}(\phi(x))$ is a bounded linear isomorphism between Banach spaces. ■

Definition 3.3.4. (compare Definition 2.26/2.25 [HWZ2]) Two strong bundle charts $\Phi : P^{-1}(V) \to K$ and $\Phi' : P^{-1}(V') \to K'$ which satisfy $V \cap V' \neq \emptyset$ are compatible if the transition maps

$$\Phi' \circ \Phi^{-1}[i] : \Phi(P^{-1}(V \cap V'))[i] \to \Phi'(P^{-1}(V \cap V'))[i]$$

are sc-smooth diffeomorpisms for $i = 0, 1$. A strong bundle over the M-polyfold X is a continuous surjection $P : Y \to X$ from a paracompact Hausdorff space Y onto the M-polyfold X equipped with an equivalnce class of strong bundle atlases. ■

3.3.2 Local Model for Anti-Linear Sections Near the Node

Now we consider the model space for the fiber. We will utilize so called ″hat gluings″ defined as follows:

Definition 3.3.5. We define the *sc*-Hilbert space $F(2n) := H^{2+m,\delta_0^m}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$ with filtration given by

$$F_m(2n) := H^{2+m,\delta_0^m}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n}) \quad ■$$

Definition 3.3.6. For $a \neq 0$ we define the gluing operator

$$\widehat{\oplus}_a : F(2n) \to H^2(Z_a, \mathbb{R}^{2n})$$

$$\oplus_a(h^+, h^-)([s,t]) = \beta(s - \frac{R}{2})h^+(s,t) + (1 - \beta(s - \frac{R}{2}))h^-(s - R, t - \theta)$$

for $a = 0$ we define $\widehat{\oplus}_0$ to be the identity.

For $a \neq 0$ define the minus gluing

$$\widehat{\ominus}_a : F(2n) \to H^{2,\delta_0^0}(C_a, \mathbb{R}^{2n})$$

$$\widehat{\ominus}_a(h^+, h^-)([s,t]) = -(1-\beta(s-\frac{R}{2}))h^+(s,t) + \beta(s-\frac{R}{2})h^-(s-R, t-\theta) \quad \blacksquare$$

Note that the averaging term which is present in the definition of $\ominus_a$ is no longer present in the definition of $\widehat{\ominus}_a$. This is because the averaging term may add a nontrivial contribution to the asymptoptic limit of the map given $\pi_a(h^+, h^-)$, while sections in the fiber should decay exponentially to 0 at the node.

Proposition 1.27 [HWZ0] For $a \in B_{\frac{1}{2}} - \{0\}$, we have an *sc*-direct sum decomposition

$$F(2n) = \ker(\widehat{\ominus}_a) \bigoplus \ker(\widehat{\oplus}_a) \quad \blacksquare$$

Let $\widehat{\pi}_a : F(2n) \to F(2n)$ be the projection onto $\ker(\widehat{\ominus}_a)$ along $\ker(\widehat{\oplus}_a)$ and consider the map

$$\widehat{r} : B_{\frac{1}{2}} \times F(2n) \to B_{\frac{1}{2}} \times F(2n)$$

defined by

$$\widehat{r}(a, h^+, h^-) = (a, \widehat{\pi}_a(h^+, h^-))$$

Theorem 1.28 [HWZ0] The map $\widehat{r}$ is *sc*-smooth. $\blacksquare$

The image of $\widehat{r}$ is then, by construction, in bijective correspondence with the set

$$Y(2n) := \{0\} \times F(2n) \bigsqcup_{0<|a|<\frac{1}{2}} \{a\} \times H^2(Z_a, \mathbb{R}^{2n})$$

via the correspondence

$$\mathrm{Im}\,\widehat{r} \to Y(2n)$$

$$(a, \widehat{\pi}_a(u^+, u^-)) \to (a, \widehat{\oplus}_a \widehat{\pi}_a(u^+, u^-))$$

Corollary 3.3.1. $Y(2n)$ as an M-polyfold ■.

Consider the retraction

$$R : B_{\frac{1}{2}} \oplus E(2n) \lhd F(2n) \to B_{\frac{1}{2}} \oplus E(2n) \lhd F(2n)$$

given by

$$((a, (u^+, u^-)), (h^+, h^-)) \to ((a, \pi_a(u^+, u^-)), \widehat{\pi}_a(h^+, h^-))$$

By Theorems 1.28 and 1.27 [HWZ0] R is a $sc_{\lhd}$-smooth local strong bundle retraction. Let K denote the image of R and recall that we denote by O the image of r. Let $p : K \to O$ denote the local strong bundle.

Definition 3.3.7. (compare [HFWZ] Section 3.4) Define the set $W(2n)$ consisting of all pairs (u, h) where $u : Z_a \to \mathbb{R}^{2n} \in X(2n)$ and $h : Z_a \to \mathbb{R}^{2n} \in Y(2n)$. We define the bijection $(\oplus \times_a \widehat{\oplus}) : K \to W(2n)$ to be the map defined by

$$(a, (u^+, u^-), (h^+, h^-)) \to (\oplus_a(u^+, u^-), \widehat{\oplus}_a(h^+, h^-))$$

which is a bijection covering $\oplus : O \to X(2n)$. We have a local strong bundle chart:

$$\begin{array}{ccc} K & \xrightarrow{(\oplus \times_a \widehat{\oplus})} & W(2n) \\ \downarrow p & & \downarrow p \\ O & \xrightarrow{\oplus} & X(2n) \end{array} \qquad \blacksquare$$

3.4 Implantings into Manifolds

In this section, we will develop a gluing construction for maps valued in a symplectic manifold (Q, ω). The idea is simply to choose a coordinate chart around the nodal point in Q and perform the gluing in that chart.

Suppose S is a closed Riemann surface. Let D_x and D_y be small disks in the disk structure corresponding to an interior node $\{x, y\}$. Fix a map $u_0 : S \to Q$, and $\phi : \mathbf{R}(q) \to \mathbb{R}^{2n}$ be a chart around the point $u_0(x) = u_0(y) = q$ such that $\phi(q) = 0$. The open sets $\mathbf{R}_r(q) \subset \mathbf{R}(q)$ are the subsets $\phi^{-1}(B_r(0))$ where $B_r(0)$ is the open ball of radius r centered at 0. We assume that Q carries a Riemannian metric, g, over $\mathbf{R}_4(q)$ which is induced by pull-back of the Euclidean metric on $\mathbb{R}^{2n}$ by the chart map ϕ. Consider the exponential map exp under g. Then by hypothesis we have:

$$\phi(exp_p V(p)) = exp_{\phi(p)}(T\phi(p) \cdot V(p)) = \phi(p) + T\phi(p) \cdot V(p)$$

for $p \in \mathbf{R}_2(q)$ if $V(p) \in T_pQ$ satisfies $|V(p)|_{g,p} < 2$. Assume further that $u_0(D_x \cup D_y) \subset \mathbf{R}_1(q)$ and denote by H a C^0 neighborhood of the map u_0 which consists of continuous maps $v : S \to Q$ which satisfy the matching condition:

$$v(x) = v(y)$$

and

$$v(D_x \cup D_y) \subset \mathbf{R}_2(q)$$

Take a smooth map $u : S \to Q$ in H and consider the positive and negative holomorphic coordinates that come equipped with the small disk structure:

$$\psi_x : \mathbb{R}^+ \times S^1 \to D_x \quad \text{and} \quad \psi_y : \mathbb{R}^- \times S^1 \to D_y$$

Now consider the maps $u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^{2n}$ given by

$$u^+(s,t) = \phi \circ u \circ \psi_x(s,t)$$

and

$$u^-(s',t') = \phi \circ u \circ \psi_y(s',t')$$

Let $\oplus_a^0$ denote the gluing given in Definition 3.2.6, when the target is $\mathbb{R}^{2n}$. The gluing is defined by:

$$\phi(\oplus_a(u) \cdot \psi_a[s,t]) = \oplus_a^0(u^+, u^-)[s,t]$$

for $[s,t] \in Z_a$. The map $\oplus_a(u) : Z_a \to Q$ is equal to u near the boundaries of D_x and D_y so it extends to a map on all of S_a. Suppose η is a section of the pull-back bundle u^*TQ satisfying the matching condition

$$\eta(x) = \eta(y) \in T_pQ, \quad p = u_0(x) = u_0(y)$$

The glued section $\oplus_a(\eta)$ is a section of the pull-back $\oplus_a(u)^*TQ$ defined by

$$T\phi(\oplus_a(u)[s,t]) \cdot \oplus_a(\eta) \circ \psi_a[s,t] = \oplus_a^0(T\phi(u \circ \psi_x) \cdot \eta \circ \psi_x, T\phi(u \circ \psi_y) \cdot \eta \circ \psi_y)[s,t]$$

The stipulation that g restricted to $\mathbf{R}_4(q)$ is the pull-back of the Euclidean metric on $\mathbb{R}^{2n}$ implies the following:

Proposition 2.51 [HWZ1] If η is a section of the pull-back bundle of u^*TQ satisfying $\eta(x) = \eta(y)$ and $exp_u(\eta) \in H$, then

$$exp_{\oplus_a(u)}(\oplus_a(\eta)) = \oplus_a(exp_u(\eta)) \quad \blacksquare$$

Again suppose η is a section of the bundle u^*TQ satisfying $\eta(x) = \eta(y) \in T_pQ$, where $p = u(x) = u(y)$, then the implanted anti-gluing $\ominus_a(\eta) : C_a \to T_pQ$ is given by:

$$T\phi(p) \cdot \ominus_a(\eta)[s,t] = \ominus_a^0(T\phi(u \circ \psi_x) \cdot \eta \circ \psi_x, T\phi(u \circ \psi_y) \cdot \eta \circ \psi_y)[s,t]$$

3.4.1 Implanting Anti-Linear Sections

Consider a map $u : D_x \cup D_y \to \mathbf{R}_1(q)$. Anti-linear sections of the bundle u^*TQ for $u : D_x \cup D_y \to \mathbf{R}_1(q)$ can be represented by two vector fields $\xi^+ : D_x \to \mathbb{R}^{2n}$ and $\xi^- : D_y \to \mathbb{R}^{2n}$ defined by:

$$\xi^+(s,t) = T\phi(u \circ \psi_x(s,t)) \cdot \xi(\psi_x(s,t)) \circ \psi_x(s,t)\frac{\partial}{\partial s}$$

$$\xi^-(s',t') = T\phi(u \circ \psi_y(s',t')) \cdot \xi(\psi_y(s',t')) \circ \psi_y(s',t')\frac{\partial}{\partial s'}$$

for $(s,t) \in \mathbb{R}^+ \times S^1$ and $(s',t') \in \mathbb{R}^- \times S^1$. The hat gluing

$$\widehat{\oplus}_a(\xi) : (T_zZ_a, j_a(z)) \to (T_{\oplus_a u(z)}Q, J(\oplus_a(u)(z)))$$

is described by the vector field $\widehat{\oplus}_a^0(\xi^+, \xi^-)[s,t]$ defined by

$$T\phi(\oplus_a(u)\psi_a[s,t]) \cdot \widehat{\oplus}_a(\xi)(\psi_a[s,t]) \cdot T\psi_a\frac{\partial}{\partial s} := \widehat{\oplus}_a^0(\xi^+, \xi^-)[s,t]$$

The anti-gluing $\widehat{\ominus}_a(\xi)([s,t]) : (T_zC_a, j) \to (T_pQ, J(p))$ is given by:

$$T\phi(p) \cdot \widehat{\ominus}_a(\xi)(z) \cdot \frac{\partial}{\partial s} := \widehat{\ominus}_a^0(\xi^+, \xi^-)[s,t]$$

3.5 Boundary Nodal Gluing

For shorthand we use the notation $\mathbb{R}^{\pm} \times I$ to stand in for the disjoint union $\mathbb{R}^+ \times I \sqcup \mathbb{R}^- \times I$, where $I = [0,1] \subset \mathbb{R}$. Throughout this section, we fix an increasing sequence $(\delta_1^k)_{k=1}^{\infty} \subset (0, \pi)$.

Definition 3.5.1. Let $r \in [0, \frac{1}{2})$. We define the set Z_0 to be the disjoint union

$$Z_0 := (\mathbb{R}^+ \times I) \bigsqcup (\mathbb{R}^- \times I)$$

If $r \neq 0$, then we define

$$Z_r := [0,R] \times I \sqcup [-R, 0] \times I / (s - R, t) \sim (s', t')$$

where (s,t) gives the standard coordinates on $\mathbb{R}^+ \times I$ and (s',t') gives the standard coordinates on $\mathbb{R}^- \times I$, where $R := \phi(r)$. ■

Definition 3.5.2. If $r \neq 0$ we define the infinite cylinder C_r :

$$C_r := \mathbb{R}^+ \times I \sqcup \mathbb{R}^- \times I / \sim$$

where the equivalence relation $\sim$ identifies $(s,t) \in [0,R] \times I$ with points $(s',t') \in [-R, 0] \times I$ if

$$s = s' + R$$

For $r = 0$, set $C_0 = \emptyset$ ■

Definition 3.5.3. Fix an increasing sequence $0 < \delta_1^0 < \cdots < \delta_1^k < \cdots < \pi$. Let $H^{3+k,\delta_1^k}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$ denote the following weighted Sobolev space with real boundary conditions:

$$H^{3+k,\delta_1^k}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n}) := \{u \in H^{3+k}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n}) : u \cdot e^{\delta_1^k |s|} \in H^{3+k}(\mathbb{R}^{\pm} \times I, \mathbb{R}^{2n}),$$

$$u(\mathbb{R}^{\pm} \times \{0,1\}) \subset \mathbb{R}^n \times \{0\} \ \}$$

The norm on $H^{3+k,\delta_1^k}_{real}(\mathbb{R}^{\pm} \times I, \mathbb{R}^{2n})$ is defined as:

$$||u||^2_{k,\delta_1^k} := \sum_{|\alpha| \leq 3+k} \int_{\mathbb{R}^{\pm} \times [0,1]} |D^{\alpha} u|^2 e^{2\delta_1^k v(s)} \ ds \, dt$$

where $v : \mathbb{R} \to \mathbb{R}$ is a smooth map which is equal to the absolute value function away from a compact neighborhood of 0. ■

Lemma 3.5.1. The sequence of Hilbert spaces

$$\cdots \longrightarrow H^{3+k,\delta_1^k}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n}) \longrightarrow \cdots \longrightarrow H^{3,\delta_1^0}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$$

defines an *sc*-structure for $H^{3,\delta_1^0}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$.

Proof: The result follows similarly to Lemma 3.2.1. ■

Definition 3.5.4. For each integer $k \geq 0$ and $r \in (0, \frac{1}{2})$ we define the Sobolev space:

$$H^{3+k,\delta_1^k}(C_r, \mathbb{R}^{2n}) := \{u \in H^{3+k}_{real}(\mathbb{R} \times [0,1], \mathbb{R}^{2n}) : u \cdot e^{\delta_1^k |s - \frac{R}{2}|} \in H^{3+k}(\mathbb{R} \times [0,1], \mathbb{R}^{2n})\}$$

The norm on $H^{3+k,\delta_1^k}(C_r,\mathbb{R}^{2n})$ is defined as:

$$||u||^2_{3+k,\delta_1^k} := \sum_{|\alpha|\leq 3+k} \int_{\mathbb{R}\times[0,1]} |D^\alpha u|^2 \cdot e^{2\delta_1^k v(s-\frac{R}{2})} ds\, dt$$

where $v : \mathbb{R} \to \mathbb{R}$ is a smooth map which is equal to the absolute value function away from a compact neighborhood of 0. ■

Lemma 3.5.2. The sequence of Hilbert spaces

$$\cdots \longrightarrow H^{3+k,\delta_1^k}_{real}(C_r,\mathbb{R}^{2n}) \longrightarrow \cdots \longrightarrow H^{3,\delta_1^0}_{real}(C_r,\mathbb{R}^{2n})$$

defines the filtration for an *sc*-Hilbert space which we denote $H^{3,\delta_1^0}_{real}(C_r,\mathbb{R}^{2n})$.

Proof: The proof follows similarly to Lemma 3.2.1. ■

Definition 3.5.5. We define the Sobolev spaces

$$H^{3+m}_{real}(Z_r,\mathbb{R}^{2n}) := \{u \in H^3(Z_r,\mathbb{R}^{2n}) : u([0,\varphi(r)]\times\{0,1\}) \subset \mathbb{R}^n \times \{0\}\ \} \quad ■$$

We also have:

Lemma 3.5.3. The sequence of Hilbert spaces

$$\cdots \longrightarrow H^{3+k}(Z_r,\mathbb{R}^{2n}) \longrightarrow \cdots \longrightarrow H^3(Z_r,\mathbb{R}^{2n})$$

defines an *sc*-structure for $H^3(Z_r,\mathbb{R}^{2n})$. ■

Definition 3.5.6. We define the model space $E_{\text{real}}(2n)$ defined by

$$E_{\text{real}}(2n) := \mathbb{R}^n \times \{0\} \oplus H^{3,\delta_1^0}_{real}(\mathbb{R}^{\pm}\times[0,1],\mathbb{R}^{2n})$$

where we write $h \in E_{real}(2n)$ as $h = (h^+, h^-)$ with

$$h^+ = c + u^+ : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^{2n}$$

$$h^- = c + u^- : \mathbb{R}^- \times [0,1] \to \mathbb{R}^{2n}$$

$$u = (u^+, u^-) \in H^{3,\delta_1^0}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$$

The m^{th} level of $E_{real}(2n)$, denoted $E_{real}(2n)_m$, is given by

$$E_{real}(2n)_m := \mathbb{R}^n \times \{0\} \oplus H^{3+m,\delta_1^m}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$$

The norm on the m^{th} level is given by

$$||h||^2_{3,\delta_1^m} = |c|^2 + ||u^+||^2_{H^{3,\delta_1^m}_{real}\ (\mathbb{R}^+ \times [0,1], \mathbb{R}^{2n})} + ||u^+||^2_{H^{3,\delta_1^m}_{real}\ (\mathbb{R}^- \times [0,1], \mathbb{R}^{2n})} \quad \blacksquare$$

The maps u^+ and u^- trend towards 0, as $|s| \to \infty$. The constant $c \in \mathbb{R}^n \times \{0\}$ is the common asymptotic limit of h^+ and h^-.

Definition 3.5.7. For $r \neq 0$ we define the gluing operator

$$\oplus_r : E_{\text{real}} \to H^3_{real}(Z_r, \mathbb{R}^{2n})$$

$$\oplus_r (h^+, h^-)([s,t]) = \beta(s - \frac{R}{2}) h^+(s,t) + (1 - \beta(s - \frac{R}{2})) h^-(s - R, t - \theta)$$

For $r = 0$ we define $\oplus_0$ to be the identity.

For $r \neq 0$ define the minus gluing

$$\ominus_r : E_{\text{real}}(2n) \to H^{3,\delta_1^0}_{real}(C_r, \mathbb{R}^N)$$

$$\ominus_r(h^+,h^-)([s,t]) = -(1-\beta(s-\frac{R}{2}))(h^+(s,t)-av_r(h^+,h^-))$$

$$+\beta(s-\frac{R}{2})(h^-(s-R,t-\theta)-av_r(h^+,h^-))$$

where

$$av_r(h^+,h^-) = \frac{1}{2}\int_{[0,1]} \mathrm{Re}\, h^+(\frac{R}{2},t) + \mathrm{Re}\, h^-(-\frac{R}{2},t)\, dt \quad \blacksquare$$

Proposition 3.5.1. For every real number $r \in (0,\frac{1}{2})$ we have an *sc*-direct sum decomposition

$$E_{\mathrm{real}} = \ker(\ominus_r) \bigoplus \ker(\oplus_r)$$

Proof: The argument follows similarly to that of Theorem 1.27 [HWZ0]. $\blacksquare$

Definition 3.5.8. Let $\widetilde{\pi}_r : E_{\mathrm{real}}(2n) \to E_{\mathrm{real}}(2n)$ be the projection onto $\ker(\ominus_r)$ along $\ker(\oplus_r)$ and consider the map

$$q : [0,\frac{1}{2}) \times E_{\mathrm{real}}(2n) \to [0,\frac{1}{2}) \times E_{\mathrm{real}}(2n)$$

defined by

$$q(r,h^+,h^-) = (r,\widetilde{\pi}_r(h^+,h^-)) \quad \blacksquare$$

Lemma 3.5.4. The map q is *sc*-smooth. $\blacksquare$

The image of q is then, by construction, in bijective correspondence with the set

$$X_{\mathrm{real}}(2n) := \{0\} \times E_{\mathrm{real}}(2n) \bigsqcup_{r\in(0,\frac{1}{2})} \{r\} \times H^3_{\mathrm{real}}(Z_r,\mathbb{R}^{2n})$$

via the correspondence

$$\mathrm{Im}\, q \to X_{real}(2n)$$

$$(r, \widetilde{\pi}_r(u^+, u^-)) \to (r, \oplus_r \widetilde{\pi}_r(u^+, u^-))$$

Corollary 3.5.1. $X_{real}(2n)$ is an M-polyfold. ■

3.5.1 Boundary Nodal Gluing in the Fiber

Definition 3.5.9. We define the Sobolev space $H^{2+k,\delta_1^k}([\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$ by

$$H^{2+k,\delta_1^k}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n}) =: \{u \in H^{2+k}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n}) : u \cdot e^{2\delta_k |s|} \in H^{2+k}(\mathbb{R}^{\pm} \times I, \mathbb{R}^{2n})\}$$

The norm on H^{2+k,δ_1^k} is defined as:

$$||u||^2_{2+k,\delta_1^k} := \sum_{|\alpha| \leq 3+k} \int_{\mathbb{R}^{\pm} \times [0,1]} u \cdot e^{2\delta_1^k |s|} ds\, dt \quad ■$$

Definition 3.5.10. We define the model space $F_{\text{bdry}}(2n)$ by

$$F_{\text{bdry}}(2n) := H^{2,\delta_1^0}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$$

The filtration is given by

$$F_{\text{bdry}}(2n)_m := H^{2+m,\delta_1^m}(\mathbb{R}^{\pm} \times I, \mathbb{R}^{2n}) \quad ■$$

Proposition 3.5.2. $F_{\text{bdry}}(2n)$ is an *sc*-Hilbert space. ■

We also have hat versions of the boundary gluings defined analogously to $\oplus$, $\ominus$ see Definition 3.2.6 and 3.2.8.

Definition 3.5.11. For $r \in (0, \frac{1}{2})$, we define the gluing operator

$$\widehat{\oplus}_r : E_{\text{real}}(2n) \to H^3_{real}(Z_r, \mathbb{R}^{2n})$$

$$\widehat{\oplus}_r(h^+,h^-)([s,t]) = \oplus_r(h^+,h^-)$$

for $r=0$ we define $\widehat{\oplus}_0$ to be the identity.

For $r \in (0,\frac{1}{2})$ define the minus gluing

$$\widehat{\ominus}_r : F_{\text{bdry}} \to H^{2,\delta_1^0}(C_r,\mathbb{R}^{2n})$$

$$\widehat{\ominus}_r(h^+,h^-)([s,t]) = -(1-\beta(s-\frac{R}{2}))h^+(s,t) + \beta(s-\frac{R}{2})h^-(s-R,t) \quad \blacksquare$$

Proposition 3.5.3. For every real number $r \in (0,\frac{1}{2})$ we have an *sc*-direct sum decomposition

$$F_{\text{bdry}}(2n) = \ker(\ominus_r) \bigoplus \ker(\oplus_r) \quad \blacksquare$$

Let $\widehat{\widehat{\pi}}_r : F_{\text{bdry}}(2n) \to F_{\text{bdry}}(2n)$ be the projection onto $\ker(\ominus_r)$ along $\ker(\oplus_r)$ and consider the map

$$\widehat{q} : [0,\frac{1}{2}) \times F_{\text{bdry}}(2n) \to [0,\frac{1}{2}) \times F_{\text{bdry}}(2n)$$

defined by

$$\widehat{q}(r,h^+,h^-) = (r,\widehat{\widehat{\pi}}_r(h^+,h^-))$$

Proposition 3.5.4. The map $\widehat{q}$ is *sc*-smooth.

Proof See Appendix 3. $\blacksquare$

The image of $\widehat{q}$ is also, by construction, in bijective correspondence with the set

$$Y_{\text{bdry}}(2n) := \{0\} \times F_{\text{bdry}}(2n) \bigsqcup_{r\in(0,\frac{1}{2})} \{r\} \times H^2_{\text{bdry}}(Z_r,\mathbb{R}^{2n})$$

via the correspondence

$$\operatorname{Im} \widehat{q} \to Y_{\text{bdry}}(2n)$$

$$(a, \widehat{\widetilde{\pi}}_a(u^+, u^-)) \to (a, \widehat{\oplus}_a \widehat{\widetilde{\pi}}_a(u^+, u^-))$$

Corollary 3.5.2. $Y_{\text{bdry}}(2n)$ is an M-polyfold. ■

Consider the retraction

$$Q : [0, \frac{1}{2}) \oplus E_{real}(2n) \lhd F_{\text{bdry}}(2n) \to [0, \frac{1}{2}) \oplus E_{real}(2n) \lhd F_{\text{bdry}}(2n)$$

given by

$$((r, (u^+, u^-)), ((h^+, h^-)) \to ((r, \widetilde{\pi}_r(u^+, u^-)), \widehat{\widetilde{\pi}}_r(h^+, h^-))$$

Given Lemma 3.5.4 and Proposition 3.5.4, it follows that Q is a $sc_{\lhd}$-smooth retraction. Let K' denote the image of Q and denote by O' the image of q. Thus, we have a strong bundle $p : K' \to O'$.

Definition 3.5.12. Define the set $W_{\text{bdry}}(2n)$ consisting of all pairs (u, h) where $u : Z_r \to \mathbb{R}^{2n} \in X_{\text{real}}(2n)$ and $h : Z_r \to \mathbb{R}^{2n} \in Y_{\text{bdry}}(2n)$. We define the bijection $(\oplus \times_r \widehat{\oplus}) : K' \to W_{\text{bdry}}(2n)$ to be the map defined by

$$(r, (u^+, u^-), (h^+, h^-)) \to (\oplus_r(u^+, u^-), \widehat{\oplus}_r(h^+, h^-))$$

which covers $\oplus : O' \to X_{\text{real}}(2n)$. We have a local strong bundle isomorphism:

$$\begin{array}{ccc} K' & \xrightarrow{(\oplus \times_r \widehat{\oplus})} & W_{\text{bdry}}(2n) \\ \downarrow p & & \downarrow p \\ O' & \xrightarrow{\oplus} & X_{\text{real}}(2n) \end{array} \quad \blacksquare$$

Lastly, we note that we can define implantings for gluings at boundry nodes with La-

grangian boundary conditions in an analogous way to implantings for interior nodes. The precise notions will be introduced as needed.

3.6 Smoothness For Reparametrization Actions

In this section we follow Section 2.6 [HWZ1], adapting the hypotheses, statements, and proofs as appropriate to accomodate Lagrangian boundary conditions. Let HD_x, HD_y and $HD_{x'}$, $HD_{y'}$ denote half-disk like Riemann surfaces with smooth boundaries associated to boundary nodal pairs $\{x, y\}$ and $\{x', y'\}$, respectively. Consider the glued strips HZ_a and HZ'_b defined in Section 2 and, for $h \geq 0$ and a or b sufficiently small the substrips $HZ_a(-h)$, $HZ'_b(-h)$ see Definitions 2.2.3, 2.2.4, 2.2.5.

Given $\epsilon > 0$, let V be a germ around 0 in some Euclidean space and let $I_\epsilon := [0, \epsilon)$ be an interval in $\mathbb{R}$. We assume the following data are given:

1. A smooth map $(a, v) \to b(a, v) \in \mathbb{R}$ defined for $v \in V$ and $a \in I_\epsilon$ for ϵ sufficiently small, satisfying $b(0, v) = 0$ for $v \in V$.

2. For sufficiently large $h > 0$, a core-smooth family of holomorphic embeddings

$$\phi_{(a,v)} : Z_a(-h) \to Z'_{b(a,v)}$$

parametrized by $(a, v) \in I_\epsilon \oplus V$.

3. There exists $H > 0$ such that

$$Z'_{b(a,v)}(-H) \subset \phi_{(a,v)}(Z_a(-h))$$

for all $(a, v) \in I_\epsilon \oplus V$.

As usual we use cut-off functions β and β' to glue maps which are defined on finite cylinders Z_a and Z'_b, respectively. We equip the half-disks with the following choices of holomorphic coordinates:

$$\psi_x : \mathbb{R}^+ \times [0,1] \to HD_x, \qquad \psi_y : \mathbb{R}^- \times [0,1] \to HD_y$$

$$\psi'_{x'} : \mathbb{R}^+ \times [0,1] \to HD'_{x'}, \qquad \psi'_{y'} : \mathbb{R}^- \times [0,1] \to HD'_{y'}$$

We define an *sc*-Hilbert space E' as follows:

$$E' := \{(\xi^+, \xi^-) \in H^3(HD'_{x'}, \mathbb{R}^{2n}) \oplus H^3(HD'_{y'}, \mathbb{R}^{2n}) : (\xi^+ \circ \psi'_{x'}, \xi^- \circ \psi'_{y'}) \in H^{3,\delta_1^0}_{real}(\mathbb{R}^\pm \times I, \mathbb{R}^2)\}$$

We also define an *sc*-Hilbert space $E[-h]$ for $h \geq 0$:

$$E[-h] := \{(\eta^+, \eta^-) \in H^3(HD_x(-h), \mathbb{R}^{2n}) \oplus H^3(HD_y(-h), \mathbb{R}^{2n}) :$$

$$(\eta^+ \circ \psi_x, \eta^- \circ \psi_y) \in H^{3,\delta_1^0}_{real}([h, \infty) \times I, \mathbb{R}^2) \oplus H^{3,\delta_1^0}_{real}((-\infty, -h] \times I, \mathbb{R}^2) \}$$

Fix $h > 0$. We will abbreviate $E[-h]$ by E. Define the map:

$$\Phi : I_\epsilon \oplus V \oplus E' \to E$$

given by

$$(a, v, \xi) \to \eta$$

where η is defined by

$$\oplus_a(\eta) = \oplus'_{b(a,v)}(\xi) \circ \phi_{(a,v)} \quad \text{and} \quad \ominus_a(\eta) = 0$$

Theorem 3.6.1. (compare Theorem 2.57): Φ is *sc*-smooth.

Proof: We adapt the argument establishing the closed version of this result, Theorem 2.57 [HWZ1]. We make an adjustment to the family $\phi_{(a,v)}$ so that we may apply Theorem 8.8.3. The family $\phi_{(a,v)}$ is a core smooth family of holomorphic embeddings, so the images $\psi_x(\{0\} \times I)$, $\psi_y(\{0\} \times I)$ under the maps $\phi_{(a,v)}$ are smoothly varying curves in $Z'_{b(a,v)}$. By working in the local holomorphic coordinates, we can assume Z_a and $Z'_{b(a,v)}$ are both equipped with the standard complex structure. Assuming (a, v) is sufficiently close to $(0, v_0)$, there exists a constant h_0 such that these arcs do not intersect $Z_a(-h - h_0)$. Fix a family of core-smoothly varying isotopies $\iota_{a,v} : Z'_{b(a,v)} \to Z'_{b(a,v)}$ such that:

1. $\iota_{a,v} = \text{Identity on } \phi_{(a,v)}(Z_a(-h - h_0 - 1))$

2. $\iota_{a,v}(\phi_{(a,v)}(Z_a) = Z'_{b(a,v)}(-H)$

Consider the family of diffeomorphisms $\psi_{(a,v)} := \iota_{a,v} \circ \phi_{(a,v)}$. Observe that

$$\psi_{(a,v)} = \phi_{(a,v)} \text{ on } Z_a(-h - h_0 - 1)$$

By construction, for $j_{(a,v)} := \psi^*_{(a,v)} i$ we have a family of biholomorphisms:

$$\psi_{(a,v)} : (Z_a, j_{(a,v)}) \to (Z_{b(a,v)}(-H), i)$$

By construction $j_{(a,v)} = i$ on $Z_a(-h - h_0 - 1)$. Replace $Z_{b(a,v)}(-H)$ with $Z_{b'(a,v)}$ where $b'(a,v)$ is defined by the rule:

$$\phi(b(a,v)) - 2H = \phi(b'(a,v))$$

Both tubes are equipped with the standard complex structure and are biholomorphic via the identity map.

In view of Lemma 4.4 [HWZ0], the smoothness of $b'(a,v)$ follows from that of $b(a,v)$. Since $\psi_{(a,v)} = \phi_{(a,v)}$ away from the core, Theorem 8.8.3 implies that we may replace $\phi_{(a,v)}$ with $\psi_{(a,v)}$ and study the map:

$$\Phi : I_\delta \oplus V \oplus E' \to E,\ (a,v,\xi) \to \eta$$

where ξ is the unique solution of the equations, for $a \neq 0$:

$$\oplus_a(\eta) = \oplus'_{b'(a,v)}(\xi) \circ \psi_{(a,v)} \text{ and } \ominus_a(\eta) = 0$$

and for $a = 0$ Φ is defined by

$$\Phi(0,v,\xi) = (\xi^+ \circ \psi^+_{(0,v)}, \xi^- \circ \psi^-_{(0,v)})$$

Consider $\xi^\pm$ where $\xi^\pm = u^\pm + c$ where c is the asymptotic constant and $u^\pm \in H^{3+m,\delta_1^m}(\mathbb{R}^\pm \times I, \mathbb{R}^{2n})$. A linear algebra calculation reveals:

$$\eta^+ = c + (1 - \frac{\beta_a}{\gamma_a}) \cdot \text{Re} \int_0^1 \oplus_{b(a,v)}(u)(\psi_{(a,v)}(\frac{R}{2}, t))\, dt + \frac{\beta_a}{\gamma_a} \cdot \beta_{b(a,v)}(\psi_{(a,v)}) \cdot u^+(\psi_{(a,v)})$$

$$+\frac{\beta_a}{\gamma_a}\cdot(1-\beta_{b(a,v)}(\psi_{(a,v)}))\cdot u^-(\cdot - R(b(a,v)),\cdot)(\psi_{(a,v)})$$

There is a similar formula for η^-. The proof of smoothness of the map $(a,v,\xi)\to\eta^-$ follows similarly to the proof of the smoothness of $(a,v,\xi)\to\eta^+$. Thus we present only the latter. Consider the following maps:

(1) $I_\delta\oplus V\oplus H^{3,\delta_1^0}_{c,real}(R\times I,\mathbb{R}^{2n})\to\mathbb{R}^n\times\{0\}$

$$(a,v,\xi)\to c$$

(2) $I_\delta\oplus V\oplus H^{3,\delta_1^0}_{c,real}(\mathbb{R}^+\times I,\mathbb{R}^{2n})\oplus H^{3,\delta_1^0}_{c,real}(\mathbb{R}^-\times I,\mathbb{R}^{2n})\to H^{3,\delta_1^0}_{c,real}(\mathbb{R}^+\times I,\mathbb{R}^{2n})$

$$(a,v,u)\to(1-\frac{\beta_a}{\gamma_a})\cdot\mathrm{Re}\int_0^1\oplus_{b(a,v)}(u)(\psi_{(a,v)}(\frac{R}{2},t))\,dt$$

(3) $I_\delta\oplus V\oplus H^{3,\delta_1^0}_{c,real}(\mathbb{R}^+\times I,\mathbb{R}^{2n})\to H^{3,\delta_1^0}_{c,real}(\mathbb{R}^+\times I,\mathbb{R}^{2n})$

$$(a,v,u^+)\to\frac{\beta_a}{\gamma_a}\cdot\beta_{b(a,v)}(\psi_{(a,v)})\cdot u^+(\psi_{(a,v)})$$

(4) $I_\delta\oplus V\oplus H^{3,\delta_1^0}_{c,real}(\mathbb{R}^-\times I,\mathbb{R}^{2n})\to H^{3,\delta_1^0}_{c,real}(\mathbb{R}^+\times I,\mathbb{R}^{2n})$

$$(a,v,u^-)\to\frac{\beta_a}{\gamma_a}\cdot(1-\beta_{b(a,v)}(\psi_{(a,v)}))\cdot u^-(\cdot - R(b(a,v)),\cdot)(\psi_{(a,v)})$$

Map (1) is clearly *sc*-smooth. The other maps are clearly smooth away from $a=0$. Thus we will focus our analysis near $(0,v_0)$.

Consider the space $D^{m,\epsilon}$ of diffeomorphisms $u : \mathbb{R}^+ \times I \to \mathbb{R}^+ \times I$ having the form

$$u(s,t) = (s,t) + d + r(s,t)$$

where $d = (d, 0) \in \mathbb{R}^2$ and the map r has weak derivatives up to order m which weighted by $e^{\epsilon s}$ belong to $L^2(\mathbb{R}^+ \times I, \mathbb{R}^2)$.

Recall by Theorem 8.8.3 that given $\Delta > 1$, there exists a map

$$\widetilde{\psi} : I_\delta \oplus V \to \bigcap_{m \geq 3, 0 < \epsilon < \pi} D^{m,\epsilon},$$

defined in a neighborhood of $(0, v_0) \in I_\delta \oplus V$ such that:

For every $m \geq 3$ and every $0 < \epsilon < \pi$, the map $(a, v) \to \widetilde{\psi}_{(a,v)} \in D^{m,\epsilon}$ is smooth and satisfies

$$\widetilde{\psi}_{(a,v)}(s,t) = \psi_{(a,v)}(s,t) \text{ for } (s,t) \in [0, \frac{R(a)}{2} + \Delta] \times I$$

where $R(a) := e^{\frac{1}{|a|}} - e$. Thus we have:

$$\widetilde{\psi}_{(a,v)}(s,t) = (s,t) + d(a,v) + r(a,v)(s,t)$$

where the maps

$$I_\delta \oplus V \to \mathbb{R}^2, \ (a,v) \to d(a,v)$$

$$r : I_\delta \oplus V \to H^{m,\epsilon}_{real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}), \ (a,v) \to r(a,v), \ m \geq 3, \ 0 < \epsilon < \pi$$

are smooth.

Fix $\Delta > 1$. We may replace $\psi_{(a,v)}$ with $\widetilde{\psi}_{(a,v)}$ in the definition of the maps $(2)-(4)$ since the support of β_a is contained in $[0, \frac{R(a)}{2} + \Delta]$. We make this substitution, but retain the original notation throughout the remainder of the proof.

Proposition 3.6.1. (compare [HWZ1] Theorem 5.1): The composition

$$\widetilde{\Psi} : I_\delta \oplus V \oplus H^{3,\delta_1^0}_{real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \to H^{3,\delta_1^0}_{real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n})$$

defined by

$$\widetilde{\Psi}(a, v, u) = u \circ \psi_{(a,v)}$$

is well-defined and *sc*-smooth.

Proof of Proposition 3.6.1: The argument follows similarly to that of Theorem 5.1 [HWZ1]. Alternatively, one may use a doubling argument. ■

That each of the maps $(2)-(4)$ are *sc*-smooth follows from straightforward applications of Proposition 11.0.2 Theorem 2.8 [HWZ0] and Proposition 3.6.1. ■

Set $E := E[0]$. Suppose that (a_k) and (b'_k) are small gluing parameters converging to 0 and $\phi_k : Z_{a_k} \to Z'_{b'_k}$ is a sequence of holomorphic embeddings having the properties:

1. $\sup_{z \in Z_{a_k} - |D_{a_k}|,\, k \geq 1} ||T\phi_k(z)|| < \infty$

2. There exists a sequence ξ_k in E' converging on level 0 to $\xi_0 = (\xi^+, \xi^-) \in E'$ and define maps $\eta_k \in E$ as the unique solutions of the two equations:

$$\oplus_{a_k}(\eta_k) = (\oplus'_{b'_k}(\xi_k)) \circ \phi_k \quad \text{and} \quad \ominus_{a_k}(\eta_k) = 0$$

Proposition 3.6.2. (compare Proposition 2.59 [HWZ1]): Under the above assumptions, the sequence (ϕ_k) converges in C^∞_{loc} in finite distance to the left and right boundary to holomorphic embeddings $\phi^+ : HD_x \to HD'_{x'}$ and $\phi^- : HD_y \to HD'_{y'}$ satisfying $\phi^+(x) = x'$ and $\phi^-(y) = y'$. The sequence $(\eta_k) \subset E$ converges on level 0 to the map $\eta_0 = (\eta^+, \eta^-) \in E$ given by

$$\eta^+ = \xi^+ \circ \phi^+ \quad \text{and} \quad \eta^- = \xi^- \circ \phi^-$$

Proof: The maps ϕ_k converge in finite distance to the core, so by careful choice of isotopies with support near the core, we may assume $\phi_k = \phi_{k+N}$ in a neighborhood of the core for all $N \geq 0$. Further, as in Theorem 3.6.1, we may equip Z_{a_k} with a complex structure j_k such that

$$j_k = i \text{ on } Z_a(-h - h_0 - 1)$$

and

$$\phi_k : (Z_{a_k}, j_k) \to (Z_{b_k}, i)$$

We can express η_k^+ as follows:

$$\eta_k^+ = c + (1 - \frac{\beta_a}{\gamma_a}) \cdot \text{Re} \int_{[0,1]} \oplus'_{b'_k}(\xi_k)\phi_k(\frac{R(a_k)}{2}, t)\, dt + \frac{\beta_a}{\gamma_a} \cdot \beta'_{b'_k}(\phi_k) \cdot \xi_k^+(\phi_k)$$

$$+\frac{\beta_a}{\gamma_a} \cdot (1 - \beta'_{b'_k}(\phi_k)) \cdot \xi_k^-(\cdot - R(b'_k), 0)(\phi_k)$$

By Theorem 8.8.3, we may extend the maps ϕ_k to maps

$$\widetilde{\phi}_k : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^+ \times [0,1]$$

$$\widetilde{\phi}_k(s,t) = (s,t) + d(k) + r(k)(s,t)$$

such that

$$\widetilde{\phi}_k(s,t) = \phi_k(s,t) \text{ for } (s,t) \in [0, \frac{R(a_k)}{2} + \Delta] \times [0,1]$$

By Condition (1) and Gromov-Compactness, the maps ϕ_k converge in C^∞_{loc} to a pair of holomorphic maps:

$$(\phi^+, \phi^-) : (\mathbb{R}^+ \times [0,1], \mathbb{R}^- \times [0,1]) \to (\mathbb{R}^+ \times [0,1], \mathbb{R}^- \times [0,1])$$

This means that the maps $\widetilde{\phi}_k$ converge to ϕ in C^∞. We can re-write η^+_k as follows:

$$\eta^+_k = c + (1 - \frac{\beta_a}{\gamma_a}) \cdot \text{Re} \int_{[0,1]} \oplus'_{b(k)}(\xi_k)(\widetilde{\phi}_k(\frac{R(a_k)}{2}, t)dt + \frac{\beta_a}{\gamma_a} \cdot \beta'_{b(k)}(\widetilde{\phi}_k) \cdot \xi^+_k(\widetilde{\phi}_k)$$

$$+\frac{\beta_a}{\gamma_a} \cdot (1 - \beta'_{b(k)}(\widetilde{\phi}_k)) \cdot \xi^-_k(\widetilde{\phi}_k - (R(b(k)), 0))$$

Since $\xi_k \to \xi$ on level 0, it follows that η^+_k converges to a map η^+ and we have

$$\eta^+ = \xi^+ \circ \phi$$

Arguing similarly, one may establish the claim for η^-. ■

4 Polyfold Structure on the Space of Stable Maps with Lagrangian Boundary

Using the deformations studied in Sections 2 and 3, Section 4 constructs a compatible collection of local uniformizers on the space of stable maps.

4.1 Construction of Uniformizers

We will construct local charts on the polyfold $Z_{g,h,\vec{k},m}$ by constructing open subsets of *sc*-Hilbert retracts which locally parametrize the space of stable maps. The idea is that in a neighborhood of a stable map $(S, j, M_i, M_b, D_i, D_b, u)$ we may parametrize the smooth part of the curve by a choice of complex structure, the nodal degeneration by gluing parameters, and the map component by an appropriate subspace of $\Gamma(u^*TQ)$. This section follows the structure of [HWZ1] Section 3 and many of the definitions, propositions, and arguments in this section are similar to those appearing in [HWZ1] Section 3. We restrict ourselves to the situation where there is at least one marked point on each boundary component. This is enough for the construction of Lagrangian Floer theory in arbitrary genera as discussed in [CFL], for example. First we recall the definition of a stable map:

Definition 4.1.1. We fix two sequences $0 < \delta_0^0 < \cdots < \delta_0^m < \cdots < 2\pi$, $0 < \delta_1^0 < \cdots < \delta_1^m < \cdots < \pi$. Given a Riemann surface (S, j, M, D) which need not be stable, a stable map is a tuple

$$(S, j, M, D, u)$$

where $u : S \to Q$ is of class $(3, \delta_0^0, \delta_1^0)$ and

1. $\int_C u^*\omega \geq 0$ for every component C of S

2. If a domain component C of S has genus $g(C)$ and $\#\Sigma_C$ special points, and satisfies $2 \cdot g(C^d) + \#\Sigma_{C^d} \leq 2$, i.e. C is not stable, then

$$\int_C u^*\omega > 0 \quad \blacksquare$$

We also recall the notion of equivalence between stable maps:

Definition 4.1.2. Two stable maps $\alpha = (S, j, M, D, u)$ and $\alpha' = (S', j', M', D', u')$ are called equivalent if there exists an isomorphism $\phi : (S, j, M, D) \to (S', j', M', D')$ between the noded Riemann surfaces satisfying:

1. $\phi : (S, j) \to (S', j')$ is a biholomorphism

2. $\phi(M) = M'$ and ϕ preserves the order on the marked points

3. $\phi(D) = D'$

4. $u' \circ \phi = u$

We denote the set of stable maps by $H^{3,\delta_0^0,\delta_1^0}(Q)$ and the set of equivalence classes by $Z^{3,\delta_0^0,\delta_1^0}(Q)$. ■

Definition 4.1.3. Fix a Riemann surface with boundary S. Define

$$H^{3,\delta_0^0,\delta_1^0}(S, u^*TQ, u|^*_{\partial S}TL)$$

to be the set of sections η of u^*TQ which decay at the rate δ_0^0 at interior nodes and the rate δ_1^0 at boundary nodes. In addition, we impose the condition that $\eta|\partial S$ is a section of u^*TL. ■

The definition of stability for maps does not require the components of the underlying Riemann surface to be stable as long as $\int_C u^*\omega > 0$. It will be important to develop

a stabilization procedure for unstable underlying curves. We adapt Definition 3.1 from [HWZ1] to the case of Riemann surfaces with boundary:

Definition 4.1.4. Let S be a Riemann surface with boundary. A finite set Σ of points in S is a *stabilization* of $\alpha = (S, j, M, D, u)$ with automorphism group G if it satisfies the following:

(1) Σ is contained in the interior of C.

(2) Σ does not intersect $M \cup |D|$.

(3) Σ is preserved by G, i.e. $g(\Sigma) = \Sigma$ for all $g \in G$.

(4) If we consider $M_i \cup \Sigma$ as unordered marked points, the Riemann surface $(S, j, M_i \cup \Sigma, M_b, D)$ is stable.

(5) If $u(z) = u(z')$ for points $z, z' \in \Sigma$, then there exists $g \in G$ such that $g(z) = z'$.

(6) The image $u(\Sigma)$ does not intersect $u(|D| \cup M)$.

(7) At the points $z \in \Sigma$, the differential $Tu(z)$ is injective. At these points $u^*\omega(z)$ is non-degenerate and determines on T_zC the same orientation as the almost complex structure j. ■

Observe that by definition the automorphism group G^* of the nodal Riemann surface $(S, j, M \cup \Sigma, D)$ is finite and contains the automorphism group G of $\alpha = (S, j, M, D)$.

Lemma 4.1.1. (compare Lemma 3.2 [HWZ1]) Every stable map $\alpha = (S, j, M, D, u)$

possesses a stabilization Σ.

Proof: The proof follows similarly to that of Lemma 3.2 [HWZ1]. ■

Let $\alpha = (S, j, M, D, u)$ be a stable map equipped with the stabilization Σ. Label the images of the points $|D|$ and Σ under the map u as follows:

$$u(|D_i|) = \{w_1, \cdots, w_k\}$$

$$u(\Sigma) = \{w_{k+1}, \cdots, w_{k+j}\}$$

$$u(|D_b|) = \{w_{k+j+1}, \cdots w_{k+j+l}\}$$

Note that $\#u(|D_i|) + \#u(|D_b|)$ may be less than $k+l$ due to the non-injectivity of u. By definition we have $u(|D|) \cap u(\Sigma) = \emptyset$. For $i \in \{1, \cdots, k+j+l\}$ we fix a neighborhood $\mathbf{R}(w_i) \subset Q$ of w_i and a smooth bijective chart:

$$\phi_i : (\mathbf{R}(w_i), w_i) \to (\mathbb{R}^{2n}, 0)$$

We choose these maps so that the closures of the images of the ϕ_i for $i \in \{1, \cdots, k+j+l\}$ are mutually disjoint. Denote by $\mathbf{R}_r(w_i)$ the preimage of the r-ball $B_r(0)$ around 0 under the chart map ϕ_i. Next we fix a small disk/half disk structure $\mathbf{D}$, and in addition, closed disks D_z with smooth boundaries centered at the points $z \in \Sigma$ so that all disks are mutually disjoint and so that the following holds:

(*) For every $z \in |D| \cup \Sigma$ the image under u of the disks D_z is contained in $\mathbf{R}_1(w_i)$, where $w_i = u(z)$. Moreover the union $\cup_{z \in \Sigma} D_z$ is invariant under G^*.

Choose a Riemannian metric g on the manifold Q, which restricts to the pull-back of the standard metric on $\mathbb{R}^{2n}$ under $\phi_i|_{\mathbf{R}_4(w_i)}$. Consider the stable noded Riemann surface $\alpha = (S, j, M \cup \Sigma, D)$ with finite automorphism group G^*. Also consider a good family $v \to j(v)$, parametrized by a germ V in a vector space E_α, so that for a suitable G^* invariant open neighborhood O of $(0,0) \in N_D \oplus E$ we have the good uniformizing family

$$(a, v) \to (S_a, j(a, v), (M \cup \Sigma)_a, D_a)$$

We choose a C^0-open neighborhood of the map u, denoted by $U(u)$, in the space of C^0 maps from S to Q which is chosen small enough that images of the disks D_z under the maps in $U(u)$ land in $\mathbf{R}_2(w_i)$ for $w_i = u(z)$. Given this setup, consider the family of stable maps:

$$(a, v, u') \to (S_a, j(a, v), M_a, D_a, \oplus_a(u'))$$

where (a, v) belongs to the G^* invariant open neighborhood O of $(0,0) \in N \oplus E$ and where the continuous map $u' : S \to Q$ belongs to $U(u)$.

For ever $z \in \Sigma$, we define subdisks $SD_z(-h)$ which are contained in the interior of D_z. We further stipulate that $SD_z(-h)$ has smooth boundary so that the union is invariant under the action of G. We also assume the union of sub-disks centered at points in the stabilization are invariant under G^* and are concentric, i.e.

$$SD_z = D_z(-h) = \{\xi \in D_z : \xi = z \text{ or } \xi = \psi_z(s, t) \text{ for } s \geq h\} \tag{4.1.1}$$

for some $h > 0$, where ψ_z are positive holomorphic polar coordinates on D_z centered at

z. We also fix concentric subdisks/half subdisks

$$SD_x = D_x(-h) = \{\xi \in D_x : \xi = x \text{ or } \xi = \psi_x(s,t) \text{ for } s \geq h\}$$

$$SD_y = D_y(-h) = \{\xi \in D_y : \xi = y \text{ or } \xi = \psi_y(s,t) \text{ for } s \leq -h\}$$

$$SHD_{x'} = HD_{x'}(-h) = \{\xi \in D_{x'} : \xi = x' \text{ or } \xi = \psi_{x'}(s,t) \text{ for } s \geq h\}$$

$$SHD_{y'} = HD_{y'}(-h) = \{\xi \in D_{y'} : \xi = y' \text{ or } \xi = \psi_{y'}(s,t) \text{ for } s \leq -h\}$$

having the same invariance property under the action of G^*. For the gluing parameter $a = (a_{\{x,y\}})_{\{x,y\}\in D}$ consider the union

$$Z_a := \bigcup_{\{x,y\}\in D} Z^{\{x,y\}}_{a_{\{x,y\}}}$$

We also consider the union:

$$Z_a(-h) = \bigcup_{\{x,y\}\in D} Z^{\{x,y\}}_{a_{\{x,y\}}}(-h)$$

We have the following bordered analog of Proposition 3.3 from [HWZ1] which describes the behavior of morphisms near the neck:

Proposition 4.1.1. Fix a stable map $\alpha = (S, j, M, D, u)$. Then given $h > 0$, there exists a G^*-invariant open neighborhood $\mathfrak{O}' \subset \mathfrak{O}$ of $(0,0) \in N \oplus E$ and a G-invariant C^0- neighborhood $U'(u) \subset U(u)$ of the map u so that we have following:

If $u', u'' \in U'(u) \cap H^{3,\delta_0^0,\delta_1^0}(S,Q)$ are maps from S to Q and (a,v), $(b,w) \in \mathfrak{O}'$ and if

$$\phi : (S_a, j(a,v), M_a, D_a, \oplus_a(u')) \to (S_b, j(b,w), M_b, D_b, \oplus_b(u''))$$

is an isomorphism then there exists an automorphism g in the automorphism group G of the original stable map $\alpha = (S, j, M, D, u)$ with the property that

$$\phi(Z^{\{x,y\}}_{a_{\{x,y\}}}(-h)) \subset Z^{\{g(x),g(y)\}}_{b_{\{g(x),g(y)\}}} \tag{4.1.2}$$

at all nodal pairs $\{x, y\} \in D$. In addition, if $z \in \Sigma$ then

$$\phi(D_z(-h)) \subset D_{g(z)} \tag{4.1.3}$$

Proof: The intuition is that when (a, v), (b, w) are sufficiently small, any morphism

$$\phi : (S_a, j(a, v), M_a, D_a, \oplus_a u) \to (S_b, j(b, w), M_b, D_b, \oplus_b u')$$

must be close to automorphism $g \in Aut(\alpha)$. Conditions (4.1.2) and (4.1.3) then holds for ϕ because (4.1.2) and (4.1.3) are manifestly true for g and are both open conditions.

The proof is identical to [HWZ1] Proposition 3.3, except we substitute [HWZ1] Proposition 5.2 with the following version proved in Appendix 1:

Lemma 4.1.2. (compare Proposition 3.3 [HWZ1]) Consider the bordered stable map (S, j, M, D, u) and two sequences $u_k, u'_k \in H^{3,\delta^0_0,\delta^0_1}(S, Q)$ both converging to u in C^0. Let $\mathfrak{O} \subset N(D) \times E$ be an open subset. Assume that $(a_k, v_k) \to (0, 0) \in \mathfrak{O}$ and $(b_k, w_k) \to (0, 0) \in \mathfrak{O}$ and that

$$\phi_k : (S_{a_k}, j(a_k, v_k), M_{a_k}, D_{a_k}, \oplus_{a_k}(u_k)) \to (S_{b_k}, j(b_k, w_k), M_{b_k}, D_{b_k}, \oplus_{b_k}(u'_k))$$

is a sequence of isomorphisms. Then there exists a subsequence of ϕ_k which converges in C^∞_{loc} away from the nodes to an automorphism ϕ_0 of (S, j, M, D, u). ■

Suppose for the sake of contradiction, that we have two sequences $u'_k, u''_k \to u$ of maps in $U'(u) \cap H^{3,\delta_0^0,\delta_0^1}(S,Q)$ converging in C^0, sequences (a_k, v_k) and (b_k, w_k) which both converge to $(0,0)$, and a sequence of isomorphisms

$$\phi_k : (S_{a_k}, j(a_k, v_k), M_{a_k}, D_{a_k}, \oplus_{a_k}(u'_k)) \to (S_{b_k}, j(b_k, w_k), M_{b_k}, D_{b_k}, \oplus_{b_k}(u''_k))$$

which violate (4.1.2) and (4.1.3). By the Lemma, the maps ϕ_k converge to an isomorphism $\phi_0 : (S, j, M, D, u) \to (S, j, M, D, u)$. Therefore $\phi_0 = g \in G$ and by the invariance properties of subdisks, we have

$$\phi_0(D_z(-h)) = D_{g(z)}(-h) \subset D_{g(z)}$$

for $z \in \Sigma$ and

$$\phi_0(D_x(-h) \cup D_x(-h)) \subset D_{g(x)} \cup D_{g(y)}$$

for the nodal pairs $\{x, y\}$. After passing to a subsequence, it must hold that for k sufficiently large

$$\phi_k(D_z(-h)) \subset D_{g(z)}$$

and

$$\phi_k(D_x(-h) \cup D_y(-h)) \subset D_{g(x)} \cup D_{g(y)},$$

Thus we've reached a contradiction. ■

There is a similar proposition we will also need (compare with Proposition 3.5 [HWZ1]) which is also proved using the bordered analogue of Lemma 3.4 [HWZ1]:

Lemma 4.1.3. Given $h > 0$, there exists $H > 0$ and a G^*-invariant open neighborhood $\mathfrak{O}'' \subset \mathfrak{O}'$ of 0 and a C^0-neighborhood $U''(u) \subset U(u)$ of the map u so that the following

is satisfied: If $u', u'' \in U''(u)$, $(a,v),(b,w) \in \mathfrak{O}''$ and if

$$\phi : (S_a, j(a,v), M_a, D_a, \oplus_a(u')) \to (S_b, j(b,w), M_b, D_b, \oplus_b(u''))$$

is an isomorphism, then there exists an automorphism $g \in G$ satisfying

$$\phi(D_z(-H)) \subset D_{g(z)}(-h)$$

for all $z \in \Sigma$, and

$$\phi(Z^{\{x,y\}}_{a_{\{x,y\}}}(-H)) \subset Z^{\{g(x),g(y)\}}_{b_{\{g(x),g(y)\}}}(-h)$$

for every nodal pair $\{x,y\} \in D$. ■

The notion of good data, which summarizes the choices described above, will be important for constructing uniformizing families. Many of the conditions are similar to those appearing Definition 3.6 [HWZ1]

Definition 4.1.5. (compare Definition 3.6 [HWZ1]) Let $\alpha = (S, j, M, D, u)$ be a stable map and let G be the automorphism group of α. Good data centered at α consist of the following data:

(1) A stabilization Σ and small disk structure $\mathbf{D}$ associated to α and a good uniformizing family:

$$(a,v) \to (S_a, j(a,v), (M \cup \Sigma)_a, D_a), \quad (a,v) \in O$$

We fix disks D_z around every point in $z \in \Sigma$ which are mutually disjoint and invariant under G^*, where $G^* = Aut(S, j, M \cup \Sigma, D)$.

(2) Consider the set $W = u(|D| \cup \Sigma)$. We fix smooth charts around every point $w \in W$:

$$\phi_w : (\mathbf{R}(w), w) \to (\mathbb{R}^{2n}, 0)$$

so that the closures of $\mathbf{R}(w)$ are mutually disjoint. We set $\mathbf{R}_r(w) = \phi^{-1}(B_r(0))$ for $r > 0$. We stipulate that for every $z \in |D| \cup \Sigma$ the image of $u(D_z)$ of the disk D_z is contained in $\mathbf{R}_1(u(z))$. In addition, we impose the condition

$$\phi_w(L \cap \mathbf{R}(w)) \subset \mathbb{R}^n \times \{0\}, \quad \text{for all } w \in W$$

(3) The exponential map $exp : TQ \to Q$ on the manifold Q is defined with respect to a Riemannian metric g such that

$$(\phi_w^{-1})^* g | B_4(0)$$

is the Euclidean metric. Fix an open neighborhood of the zero-section in TQ, denoted by $\widetilde{\mathfrak{O}}$, which is fiberwise convex. We further suppose that $\widetilde{\mathfrak{O}}$ is contained in $S_{\varepsilon(\alpha)}T(Q)$, where $\varepsilon(\alpha) > 0$ is chosen so small that $exp : \widetilde{\mathfrak{O}}_q := \widetilde{\mathfrak{O}} \cap T_qQ \to Q$ is an embedding for every $q \in Q$. Further restrictions on the size of $\varepsilon(\alpha)$ will be fixed later.

(4) Choose for every point $w \in u(\Sigma)$ a $2n-2$ dimensional submanifold $M_w \subset Q$ such that $\phi_w(M_w) \subset \mathbb{R}^{2n}$. Furthermore, if $H_w := T_wM_w$ then

$$T_wQ = H_w \oplus \text{Im } Tu(z)$$

Note that if $u(z) = u(z')$ for $z, z' \in \Sigma$ then associate the same manifold M_w to both.

(5) We choose g such that $L \subset Q$ is a totally geodesic submanifold. We also choose

an open G-invariant neighborhood U of the zero-section $0 \in H_c^{3,\delta_0,\delta_1}(S, u^*TQ, u|^*_{\partial S}TL)$ so that every section $\eta \in U$ has its image in $u^*\widetilde{\mathfrak{O}}$.

(6) Concentric subdisks SD_z, as defined above (see Line 4.1.1), of D_z.

(7) The restriction $u|D_z$ is an embedding which intersects $M_{u(z)}$ uniquely at the point $u(z)$.

(8) We choose U sufficiently small such that if $\eta \in U$, then $f := exp_u(\eta) : S \to Q$ satisfies $f(D_z) \subset \mathbf{R}_2(u(z))$ for every $z \in \Sigma$. In addition, the restriction $f|D_z$ is an embedding which intersects $M_{u(z)}$ transversally at a single point $f(z')$, for some $z' \in SD_z$.

(9) If there exists an isomorphism

$$\phi : \alpha_{a,v,\eta} \to \alpha_{a',v',\eta'}$$

for $(a, v, \eta),\ (a', v', \eta') \in O \oplus U$ where

$$\alpha_{(a,v,\eta)} := (S_a, j(a, v), M_a, D_a, \oplus_a(exp_u(\eta))),$$

$$\alpha_{(a',v',\eta')} := (S_{a'}, j(a', v'), M_{a'}, D_{a'}, \oplus_{a'}(exp_u(\eta')))$$

then there exists an automorphism $g \in Aut(\alpha)$ such that $\phi(SD_z) \subset D_{g(z)}$ for all $z \in \Sigma$.

(10) If there exists an isomorphism

$$\phi : (S_a, j(a, v), (M \cup \Sigma)_a, D_a) \to (S_{a'}, j(a', v'), (M \cup \Sigma)_{a'}, D_{a'})$$

which also satisfies the conditions of (9), then $\phi = g_a$ for some $g \in Aut(\alpha)$ and $(a', v') = g * (a, v)$. ■

Remark 4.1.1. With regard to Condition (3) in Definition 4.1.5, we need to ensure that we can find a metric on Q which is locally Euclidean near the nodes and such that L is totally geodesic. We can construct such a metric as follows: Let $\chi : Q \to [0, 1]$ be a bump function such that

$$\chi = 1 \text{ on } \bigcup_{w \in D_b} \mathbf{R}_5(w)$$

and

$$\chi = 0 \text{ on the complement of } \bigcup_{w \in D_b} \mathbf{R}_6(w)$$

Let $g_{\text{Euclidean}}$ denote the Euclidean metric on $\mathbb{R}^{2n}$. Set $g'' := \sum_w \phi_w^* g_{\text{Euclidean}}$ and consider the metric $g' = (1 - \chi) \cdot h' + \chi \cdot g''$, where h' is any metric satisfying all the conditions of $'$good data$'$ except that L need not geodesic for h'. There exists a tubular neighborhood of $T \subset Q$ of L where the $'$fiber$'$ over a point $l \in L$ is contained in the set

$$\{exp'(v) : v \in N_l(L)\}$$

where $N(L)$ is the normal bundle of L in TQ with respect to the metric g' and exp' is exponentiation with respect to g'. Denote by $A : T \to T$ the map which acts as the antipodal map in the fibers. Then consider the metric $h = \frac{1}{2} \cdot (g' + A^* g')$. By construction L is totally geodesic for with respect to h. Since $g_{\text{Euclidean}}$ is invariant under the map

$$\phi_w \circ A \circ \phi_w^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \ (x, v) \to (x, -v)$$

it follows that $h|\mathbf{R}_5(w) = g''|\mathbf{R}_5(w)$. Let β be a bump function supported in a neighborhood of L contained in T with compact closure. Then we set $g := (1 - \beta)h' + \beta h$.

■

Remark 4.1.2. Consider good data centered at a smooth stable map $\alpha = (S, j, M, D, u)$. For $z \in \Sigma$ denote by $H_{u(z)}$ the plane $H_{u(z)} = T_{u(z)}M_{u(z)} \subset T_{u(z)}Q$ in the tangent space, which we refer to as the linear constraint associated with the point $z \in \Sigma$. We note that points $z, z' \in \Sigma$ satisfying $u(z) = u(z')$ are associated to the same constraint, $H_{u(z)}$. ■

Proposition 4.1.2. (compare Proposition 3.7 [HWZ1]) There exists good data centered at a stable map α representing $[\alpha] \in Z$.

Proof: The argument is similar to the closed case discussed in [HWZ1]. We include the argument for completeness. The existence of a good uniformizing family is given by Theorem 2.3.1. So up to the final property, (10), the proposition holds from Proposition 4.1.1 and Lemma 4.1.3. Assume the hypothesis of (10) holds, then we conclude that $\phi = g_a^*$ for an automorphism $g^* \in G^*$. However, by Proposition 4.1.1 if we assume the neighborhod U of the zero-section of u^*TQ in (5) and the neighborhood O of $(0, 0)$ in (1) are sufficiently small, then ϕ is necessarily C^∞ close on the core to an automorphism $g \in G$. Since $G \subset G^*$ is a finite group, it follows that $\phi = g \in G$. ■

Because our specifications of good data are stable under sufficiently small C^1 perturbations, without loss of generality we can assume our charts are centered at smooth families. More precisely, for any choice of good data centered at an H^3 map u we can find map $u' = exp_u\eta$, where η is sufficiently small, such that the original choice of stabilization, small disk structure, etc defines a choice of good data centered at u'.

Lemma 4.1.4. (compare Theorem 3.8 [HWZ1]) For every stable map $\alpha = (S, j, M, D, u)$ representing an element in Z there exists good data centered at a stable map $\alpha' = (S, j, M, D, u')$ where $u = exp_{u'}(\eta)$ and η is an arbitrarily small section in the open

neighborhood U of the zero-section $H_c^{3,\delta_0^0,\delta_1^0}((u')^*TQ, (u'|_{\partial S})^*TL)$. In addition, α and α' have the same automorphism group. ■

Define the *sc*-subspace of finite co-dimension, E_u, by:

$$E_u := \{\eta \in H_c^{3,\delta_0^0,\delta_1^0}(u^*TQ, u|_{\partial S}^*TL) : \eta(z) \in H_{u(z)} \text{ for } z \in \Sigma\}$$

Set $q_{\{x,y\}} = u(x) = u(y)$. For every nodal pair $\{x, y\} \in D$, we can implant the family of retractions originally defined in Section 2:

$$r_{\{x,y\}} : B_{\frac{1}{2}} \times T_{q_{\{x,y\}}}L \oplus H^{3,\delta_1^0}\Gamma(u|_{HD_x}^*TQ, u^*|_{\partial HD_x}TL))$$

$$\oplus H^{3,\delta_1^0}\Gamma(u|_{HD_y}^*TQ, u|_{\partial HD_y}^*TL)$$

$$\to B_{\frac{1}{2}} \times T_{q_{\{x,y\}}}L \oplus H^{3,\delta_1^0}\Gamma(u|_{HD_x}^*TQ, u|_{\partial HD_x}^*TL)) \oplus H^{3,\delta_1^0}\Gamma(u|_{HD_y}^*TQ, u|_{\partial HD_y}^*TL)$$

and for interior nodal pairs

$$q_{\{x,y\}} : B_{\frac{1}{2}} \times T_{q_{\{x,y\}}}Q \oplus H^{3,\delta_0^0}\Gamma(u|_{D_x}^*TQ) \oplus H^{3,\delta_0^0}\Gamma(u|_{D_y}^*TQ)$$

$$\to B_{\frac{1}{2}} \times T_{q_{\{x,y\}}}Q \oplus H^{3,\delta_0^0}\Gamma(u|_{D_x}^*TQ) \oplus H^{3,\delta_0^0}\Gamma(u|_{D_y}^*TQ)$$

The retraction is the identity near the left and right boundaries of D_x, D_y, and $HD_{x'}$, $HD_{y'}$, respectively, and so we may extend it to a retraction on E_u. We denote the extension by $r_{\{x,y\}}$ and observe that the retraction takes the form:

$$r_{\{x,y\}}(a, \eta) = (a, \pi_{a_{\{x,y\}}}^{\{x,y\}}(\eta))$$

where $\pi_{a_{\{x,y\}}}^{\{x,y\}}$ is the projection onto the kernel of $\ominus_{a_{\{x,y\}}}$. The projections obtained for vari-

ous nodal pairs $\{x, y\} \in D$ commute with each other. Denote by π_a for $a = (a_{\{x,y\}})_{\{x,y\}\in D}$ the product of all of the nodal projections.

We define the local retract K as follows:

$$K = \{(a, v, \eta) \in B_D \times E_u : (a, v) \in O \text{ and } \pi_a(\eta) = \eta\}$$

where O is specified in Definition 4.1.5. We shall abbreviate by $\mathfrak{O}$ the open subset of K given by:

$$\mathfrak{O} := \{(a, v, \eta) \in K : (a, v) \in O, \eta \in E_u \text{ and } \eta \in U \cap E_u\}$$

where U is the open neighborhood of the zero-section $0 \in H_c^{3,\delta_0^0,\delta_1^0}(S, u^*TQ, u|^*_{\partial S}TL)$ considered in property (5) of Definition (Good Data).

Definition 4.1.6. (compare with Definition 3.9 [HWZ1]) Assume we have good data, see Definition 4.1.5, centered at a stable map $\alpha = (S, j, M, D, u)$ with automorphism group G, and consider the good uniformizing family:

$$(a, v) \to \alpha_{(a,v)} := (S_a, j(a, v), (M \cup \Sigma)_a, D_a) \quad \text{with } (a, v) \in O$$

Then the associated family of stable maps

$$(a, v, \eta) \to \alpha_{(a,v,\eta)} := (S_a, j(a, v), M_a, D_a, \oplus_a(exp_u(\eta)))$$

for $(a, v, \eta) \in \mathfrak{O}$ is called a good uniformizing family of stable maps centered at the smooth stable map $\alpha = (S, j, M, D, u)$. ■

From Lemma 4.1.4, we have:

Proposition 4.1.3. (compare Proposition 3.10 [HWZ1]) Given $[\alpha'] \in Z$, there

exists a good uniformizing family

$$(a, v, \eta) \to \alpha_{(a,v,\eta)}, \quad (a, v, \eta) \in \mathfrak{O}$$

such that $\alpha_{(0,0,0)}$ is smooth and $[\alpha'] = [\alpha_{(a_0,v_0,\eta_0)}]$ for some $(a_0, v_0, \eta_0) \in \mathfrak{O}$. ■

Definition 4.1.7. (compare Definition 3.11 [HWZ1]) The action of $G = Aut(S, j, M, D, u)$ on $\mathfrak{O}$ is given by

$$(a', v', \eta') = g * (a, v, \eta)$$

Here $g \in G$, and $(a', v') = g * (a, v)$ and $\eta' \circ g = \eta$. ■

Proposition 4.1.4. (compare Proposition 3.12 [HWZ1]) Consider a good family $(a, v, \eta) \to \alpha_{(a,v,\eta)}$, $(a, v, \eta) \in \mathfrak{O}$. If

$$[\alpha_{(a,v,\eta)}] = [\alpha_{(a',v',\eta')}]$$

for (a, v, η), (a', v', η') in $\mathfrak{O}$, then there exists an automorphism $g \in G$ such that

$$(a', v', \eta') = g * (a, v, \eta)$$

Furthermore, every isomorphism between $\alpha_{(a,v,\eta)}$ and $\alpha_{(a',v',\eta')}$ is obtained by gluing an automorphism in G:

$$g_a : \alpha_{(a,v,\eta)} \to \alpha_{(a',v',\eta')}$$

Proof: By hypothesis, we've assumed the existence of an isomorphism

$$\phi : \alpha_{(a,v,\eta)} = (S_a, j(a, v), M_a, D_a, \oplus_a(exp_u(\eta))) \to$$

$$\alpha_{(a',v',\eta')} = (S_{a'}, j(a', v'), M_{a'}, D_{a'}, \oplus_{a'}(exp_u(\eta')))$$

with the property that

$$\oplus_a(exp_u(\eta)) = \oplus_{a'}(exp_u(\eta')) \circ \phi$$

Any point $z \in \Sigma$ is in the core of S_a, since z is not contained in $|D|$. On the core of S_a, $\oplus_a(exp_u(\eta)(z)) = exp_{u(z)}(\eta(z))$ for $z \in S_a$, and in particular this is true for $z \in \Sigma$. And by construction of the uniformizing family, $\eta(z) \in H_{u(z)} = T_{u(z)}M_{u(z)} \subset T_{u(z)}Q$ for $z \in \Sigma$.

By definition of good data, $exp_u\eta|_{D_z}$ is an embedding, transverse to the manifold $M_{u(z)}$, and intersects $M_{u(z)}$ precisely in the point z. By definition of good data, there exists an automorphism $g \in G$ such that

$$\phi(SD_z) \subset D_{g(z)}$$

By definition of the automorphism group, we have $u(g(z)) = u(z)$. Therefore,

$$exp_u(\eta') \circ \phi(z) = exp_u(\eta)(z) \in M_{u(z)} = M_{u(g(z))}$$

By hypothesis, $exp_u(\eta')|D_{g(z)}$ is an embedding and intersects $M_{u(g(z))}$ in exactly the point $g(z)$. Since $\phi(z) \in D_{g(z)}$, we must have

$$\phi(z) = g(z)$$

by Condition (7) of Definition 4.1.5. Σ is G invariant, so we can conclude that Σ is also ϕ invariant. So ϕ is an isomorphism of bordered stable nodal Riemann surfaces:

$$(S_a, j(a, v), (M \cup \Sigma)_a, D_a) \to (S_{a'}, j(a', v'), (M \cup \Sigma)_{a'}, D_{a'})$$

Now, by definition of good data we have an automorphism $g \in G$ such that $(a', v') = g * (a, v)$ and $\phi = g_a$. On the core this means $\phi = g_a = g$. Since $u \circ g = u$, we have

$$exp_u(\eta') \circ g = exp_u(\eta' \circ g) = exp_u(\eta)$$

Thus on the core we have $\eta' \circ g = \eta$. The isomorphism g_a takes $Z^{\{x,y\}}_{a_{\{x,y\}}}$ to $Z^{\{g(x),g(y)\}}_{a_{\{(g(x),g(y))\}}}$ for all nodal pairs $\{x, y\} \in D$ via the assignment:

$$g_a[z]_x^{a_{\{x,y\}}} = [g(z)]_{g(x)}^{a_{\{(g(x),g(y))\}}}$$

Using the hypotheses $u \circ g = u$, $\ominus_a \eta = 0$, we have the identities:

$$u_{a'} \circ g_a = u_a$$

$$\oplus_a(\eta' \circ g) = \eta'_{a'} \circ g_a$$

$$\ominus_a(\eta' \circ g) = 0$$

Then we have the chain of equalities:

$$exp_{u_a}\eta_a = exp_{u'_a}\eta'_{a'} \circ g_a = exp_{u_{a'} \circ g_a}\eta'_{a'} \circ g_a = exp_{u_a}(\eta' \circ g)_a \tag{4.1.4}$$

Since $\pi_a \eta = \eta$ and $\pi_a(\eta' \circ g) = \eta' \circ g$, Equation (4.1.4) implies $\eta = \eta' \circ g$. The claim follows. ■

The following theorem describes the extension of an isomorphism between stable maps in different uniformizing families to a germ of morphisms between stable maps parametrized by different uniformizing families.

Theorem 4.1.1. (compare Theorem 3.13 [HWZ1]) Consider two good uniformizing

families of stable maps $q \to \alpha_q$ parametrized by $q \in \mathfrak{O}$ and $q' \to \alpha'_{q'}$ parametrized by $q' \in \mathfrak{O}'$. We assume that for two points $q_0 \in \mathfrak{O}$ and $q'_0 \in \mathfrak{O}'$ there exists an isomorphism

$$\phi_0 : \alpha_{q_0} \to \alpha'_{q'_0}$$

between the associated stable maps. Then there exists a unique local germ of an *sc*-diffeomorphism

$$f : (\mathfrak{O}, q_0) \to (\mathfrak{O}', q'_0) \qquad q \to q' = f(q)$$

between the parameter spaces which satisfy $f(q_0) = q'_0$ and there exists a core-smooth germ of a family $q \to \phi_q$ of isomorphisms

$$\phi_q : \alpha_q \to \alpha'_{f(q)}$$

satisfying $\phi_{q_0} = \phi_0$.

Proof: Consider $q_0 = (a_0, v_0, \eta_0) \in \mathfrak{O}$ and $q'_0 = (a'_0, v'_0, \eta'_0) \in \mathfrak{O}'$. Suppose $\alpha_{q_0} = (S_{a_0}, j(a_0, v_0), M_{a_0},\ D_{a_0}, \oplus_{a_0} u)$ and $\alpha'_{q'_0} = (S'_{a'_0}, j'(a'_0, v'_0), M'_{a'_0}, D'_{a'_0}, \oplus_{a'_0} u')$. Further consider the subset $\Sigma' \subset S'$ of stabilizing points. By definition we have

$$\Sigma' \cap (M' \cup |D'|) = \emptyset$$

and

$$u'(\Sigma') \cap u'(M' \cup |D'|) = \emptyset$$

We will denote the preimage of Σ' under $\phi_0 : S_{a_0} \to S'_{a'_0}$ by

$$\Theta_0 = \phi_0^{-1}(\Sigma') \subset S_{a_0}$$

The subset Θ_0 need not be contained in the core of S_{a_0}, though by definition of an isomorphism of stable surfaces, it is disjoint from $M_{a_0} \cup |D_{a_0}|$. Let Θ be a smooth deformation of Θ_0. We have the following, which follows from Theorem 2.4.1:

Lemma 4.1.5. Let $U \subset S^{\#|\Sigma|}$ be an open subset containing Θ_0. There exists a germ of a smooth map

$$B_D \times V \times U \to B_{D'} \times V'$$

$$(a, v, \Theta) \to (a'(a, v, \Theta), v'(a, v, \Theta))$$

which satisfies

$$(a'(a_0, v_0, \Theta_0), v'(a_0, v_0, \Theta_0)) = (a'_0, v'_0)$$

such that there exists an associated core-smooth germ of isomorphisms

$$\phi_{(a,v,\Theta)} : (S_a, j(a, v), M_a \cup \Theta, D_a) \to (S_{a'}, j'(a', v'), (M' \cup \Sigma')_{a'}, D_{a'}),$$

where $a' = a'(a, v, \Theta)$ and $v' = v'(a, v, \Theta)$, and $\phi_{(a_0,v_0,\Theta_0)} = \phi_0$ ■

The diffeomorphism $\phi_0 : S_{a_0} \to S'_{a'_0}$ has the property:

$$\oplus_{a_0} exp_u(\eta_0) = \oplus_{a'_0}(exp'_{u'}(\eta'_0)) \circ \phi_0$$

If $z \in \Theta_0$, we have $\phi_0(z) = z' \in \Sigma'$, so

$$\oplus_{a_0} exp_u(\eta_0)(z) = \oplus_{a'_0} exp'_{u'}(\eta'_0)(z') \in M'_{u'(z')}$$

For (a, v, η) close to (a_0, v_0, η_0) on level 0 (in particular, the associated stable maps are C^1 close), it follows by Theorem 2.54 in [HWZ1] that $\oplus_a exp_u(\eta) : S_a \to Q$ intersects

$M'_{u'(z')} \subset Q$ transversally at unique points $z(a,v,\eta) \in S_a$ so

$$\oplus_a exp_u(\eta)(z(a,v,\eta)) \in M'_{u'(z')}$$

and the map

$$(a,v,\eta) \to z(a,v,\eta) \in S_a$$

is core-smooth. By the chain rule, near (a_0, v_0, η_0) an sc-smooth germ

$$q = (a,v,\eta) \to (a,v,z(a,v,\eta)) \to (a'(a,v,z(a,v,\eta)), v'(a,v,z(a,v,\eta)))$$

of functions and a core-smooth family of morphisms

$$\phi_q := \phi_{(a,v,\Theta_{(a,v,\eta)})} : (S_a, j(a,v), M_a \cup \Theta_q, D_a) \to (S_{a'(q)}, j'(a'(q), v'(q)), (M' \cup \Sigma')_{a'(q)}, D'_{a'(q)})$$

By construction we have

$$\phi_q(z(q)) = z' \in \Sigma'$$

for some $z' \in \Sigma'$. By C^1 closeness the maps

$$\oplus_a exp_u(\eta) \circ \phi^{-1}_{(a,v,\eta)} : S'_{a'(a,v,\eta)} \to Q$$

are still transverse to $M'_{u'(z')}$ at points $z' \in \Sigma'$. To complete the proof of the proposition, it is enough to prove the following:

Lemma 4.1.6. For $q = (a,v,\eta)$ sufficiently close to $q_0 = (a_0, v_0, \eta_0)$ there exists a section

$\eta'(q)$ of the bundle u'^*TQ near the section η'_0 uniquely defined by the properties:

$$\oplus_{a'(q)} exp_{u'}(\eta'(q)) = \oplus_a exp_u(\eta) \circ \phi_q^{-1} \tag{4.1.5}$$

$$\eta'(q) \text{ satisfies the linear constraints at the points of the stabilization} \tag{4.1.6}$$

$$(a'(q), v'(q), \eta'(q)) \in \mathfrak{O}' \tag{4.1.7}$$

In addition,

$$q \to \eta'(q)$$

is an sc-smooth map.

Proof: The argument follows similarly to Lemma 3.17 in [HWZ1], if one substitutes Theorem 2.57 in [HWZ1] for Theorem 3.6.1. ■ ■

4.2 Compatibility of Uniformizers

Consider two graphs of good uniformizing families:

$$q \to \alpha_q$$

$$\mathcal{G} = \{(q, \alpha_q) : q \in \mathfrak{O}\}$$

and

$$q' \to \alpha'_{q'}$$

$$\mathcal{G}' = \{(q', \alpha'_{q'}) : q' \in \mathfrak{O}'\}$$

We define the morphism set $M(\mathcal{G}, \mathcal{G}')$ between objects $\mathcal{G}$ and $\mathcal{G}'$ as follows

$$M(\mathcal{G}, \mathcal{G}') := \{((q, \alpha_q), \phi, (q', \alpha'_{q'})) : (q, \alpha_q) \in \mathcal{G}, (q', \alpha'_{q'}) \in \mathcal{G}', \phi : \alpha_q \to \alpha'_{q'} \text{is an equivalence}\}$$

Consider a subset $\widehat{\mathfrak{O}} \subset \mathfrak{O}$ which is the domain of definition of a bijective map $f : \widehat{\mathfrak{O}} \to \mathfrak{O}'$ of the form specified in Theorem 4.1.1. We define the subset $M(\mathcal{G}, \mathcal{G}')_{\widehat{\mathfrak{O}}} \subset M(\mathcal{G}, \mathcal{G}')$ as follows:

$$M(\mathcal{G}, \mathcal{G}')_{\widehat{\mathfrak{O}}} := \{((q, \alpha_q), \phi, (q', \alpha'_{q'})) : (q, \alpha_q) \in \widehat{\mathfrak{O}}\}$$

Let $\mathfrak{M}$ denote the collection of subsets of $M(\mathcal{G}, \mathcal{G}')$ of the form $M(\mathcal{G}, \mathcal{G}')_{\widehat{\mathfrak{O}}}$.

Proposition 4.2.1. $\mathfrak{M}$ is a basis for a Hausdorff topology on $M(\mathcal{G}, \mathcal{G}')$.

Proof: The proof follows analogously to [HWZ1] Lemma 3.18, substituting Lemma 3.12 [HWZ1] with Proposition 4.1.4. ■

Recall that the topology on $\mathcal{G}$ is induced from the projection map $\pi : \mathcal{G} \to \mathfrak{O}$. It is easy to see that the source s and target t maps are continuous with respect to this topology. Theorem 4.1.1 implies that given a morphism $\alpha_q \to \alpha'_{q'}$ there exists an injective map

$$\Gamma : \mathfrak{G} \to M(\mathcal{G}, \mathcal{G}'), \quad \Gamma(q, \alpha_q) = ((q, \alpha_q), \phi_q, (q'(q), \alpha'_{q'(q)}))$$

where $q'(q) = f(q)$ where $f : \mathfrak{O} \to \mathfrak{O}'$ is a local *sc*-diffeomorphism. By definition, we have

$$s \circ \Gamma = \mathrm{Id} \quad \text{and} \quad t \circ \Gamma(q, \alpha_q) = (q'(q), \alpha'_{q'(q)})$$

By Theorem 4.1.1 this means that $t \circ \Gamma$ is a local *sc*-diffeomorphism. It follows that Γ and s are local homeomorphisms and that s induces an *sc*-structure on the morphism sets $M(\mathcal{G}, \mathcal{G}')$. In particular, $\Gamma^{-1} = s$ and $t = (t \circ \Gamma) \circ \Gamma^{-1}$ are also a local *sc*-diffeomorphisms.

Thus we have:

Proposition 4.2.2. (compare Proposition 3.19 [HWZ1]): If $\mathcal{G}$ and $\mathcal{G}'$ are equipped with their natural M-polyfold structures, then the set $M(\mathcal{G}, \mathcal{G}')$ has a natural M-polyfold structure for which the source map s and the target map t are local sc-diffeomorphisms. ■

Proposition 4.2.3. (compare Proposition 3.20) [HWZ1]): Consider $\mathcal{G}$, $\mathcal{G}'$ and $M(\mathcal{G}, \mathcal{G}')$ equipped with the M-polyfold structures discussed above. Then:

1. The unit map

$$u : \mathcal{G} \to M(\mathcal{G}, \mathcal{G}), \quad u(q, \alpha_q) = ((q, \alpha_q), \mathrm{Id}, (q, \alpha_q))$$

is sc-smooth.

2. The inversion map

$$i : M(\mathcal{G}, \mathcal{G}') \to M(\mathcal{G}', \mathcal{G}), \quad i((q, \alpha_q), \phi, (q', \alpha'_{q'})) = ((q', \alpha'_{q'}, \phi^{-1}, (q, \alpha_q))$$

is sc-smooth.

Proof: In local charts, the source map s is sc-smooth and defines the smooth structure on the morphism sets. Thus the equation $s \circ u = \mathrm{Id}$ implies u is sc-smooth. Locally, $i = t^{-1} \circ s$, so sc-smoothness follows by the chain rule for sc-smooth maps. ■

Lastly, consider the multiplication map

$$m : M(\mathcal{G}', \mathcal{G}'')_s \times_t M(\mathcal{G}, \mathcal{G}') \to M(\mathcal{G}, \mathcal{G}'')$$

defined by

$$m(((q', \alpha'_{q'}), \psi, (q'', \alpha''_{q''})), ((q, \alpha_q), \phi, (q', \alpha'_{q'}))) = ((q', \alpha'_{q'}, \psi \circ \phi, (q'', \alpha''_{q''}))$$

Proposition 4.2.4. (compare Proposition [HWZ1] 3.21): The multiplication map is an *sc*-smooth map.

Proof: The argument follows identically as [HWZ1] Proposition 3.21 given the substitution of [HWZ1] Theorem 3.13 for Theorem 4.1.1. ■

4.3 Compactness Properties of $M(\mathcal{G}, \mathcal{G}')$

We will need the following compactness result, which is the open version of Theorem 3.22 in [HWZ1]:

Proposition 4.3.1. (compare Theorem 3.22 [HWZ1]) Let $\mathcal{G} = \{(q, \alpha_q) : q \in \mathfrak{O}\}$ be a graph where $q \to \alpha_q$ is a good uniformizing family of bordered stable maps. Let $\pi : \mathcal{G} \to \mathfrak{O}$ be the projection diffeomorphism. Then for every point $q \in \mathfrak{O}$ there is an open neighborhood $U^{\mathcal{G}}(q) \subset \mathfrak{O}$ such that if $\mathcal{G}$ and $\mathcal{G}'$ are two graphs and $q' \in \mathfrak{O}'$, then every sequence $\Phi_k \in M(\mathcal{G}, \mathcal{G}')$ of morphisms satisfying

$$\pi \circ s(\Phi_k) \in U^{\mathcal{G}}(q)$$

and

$$\pi' \circ t(\Phi_k) \to q'$$

possesses a subsequence converging in $M(\mathcal{G}, \mathcal{G}')$.

Remark 4.3.1. The above theorem will be used to show that the polyfold $Z_{g,h,\vec{k},m}$ is

Hausdorff. Indeed, this theorem reflects that objects in the orbit space cannot be too close without being equal. ■

The following discussion closely follows the material in [HWZ1] Section 3.3. Fix an M-polyfold chart $\mathcal{G} = \{(q, \alpha_q) : q \in \mathfrak{O}\}$ and a point $q_0 \in \mathfrak{O}$. We will construct a special neighhborhood $U^{\mathcal{G}}(q_0)$ for which we will verify the proposition. Suppose the chart $\mathcal{G}$ is centered at a stable map $\alpha = (S, j, M, D, u)$ and the good uniformizing family defining $\mathcal{G}$ is given by

$$q \to \alpha_q, \quad q = (a, v, \eta) \in \mathfrak{O}$$

$$(0, 0, 0) \to \alpha_0 := \alpha$$

Definition 4.3.1. We fix a family of metrics g_a, parameterized by $a \in B_D$, on each of the punctured surfaces

$$S_a - \bigcup_{a_{\{x,y\}}=0} \{x, y\}$$

satisfying:

1. If $a_{\{x,y\}} \in D_i$, and $a_{\{x,y\}} \neq 0$, the cylinder $Z^{\{x,y\}}_{a_{\{x,y\}}}$ is isometric to the standard cylinder $[0, R_{\{x,y\}}] \times S^1$ equipped with the product metric.

2. If $a_{\{x,y\}} \in D_i$, and $a_{\{x,y\}} = 0$ the metrics on the punctured disks $D_x - \{x\}$ and $D_y - \{y\}$ are isometric to the half cylinders $\mathbb{R}^+ \times S^1$ and $\mathbb{R}^- \times S^1$ equipped with the product metric, respectively.

3. If $a_{\{x,y\}} \in D_b$, and $a_{\{x,y\}} \neq 0$, the strips $HZ^{\{x,y\}}_{a_{\{x,y\}}}$ are isometric to the standard strip $[0, R_{\{x,y\}}] \times [0, 1]$ equipped with the product metric.

4. If $a_{\{x,y\}} \in D_b$, and $a_{\{x,y\}} = 0$, the metrics on the punctured strips $HD_x - \{x\}$ and $HD_y - \{y\}$ are isometric to the strips $\mathbb{R}^+ \times [0,1]$ and $\mathbb{R}^- \times [0,1]$ equipped with the product metric, respectively.

5. On the core of S_a, which can be canonically identified for all a, we take the metrics to be the same. ■

Now suppose $q_0 = (a_0, v_0, \eta_0) \in \mathfrak{O}$. We will construct $U^{\mathcal{G}}(q_0)$. Take a small open neighborhood P of $(a_0, v_0) \in O$, which satisfies $cl(P) \subset O$. Choose a Riemannian metric g satisfying conditions Good Data (Definition 4.1.5) on the symplectic manifold (Q, ω). Note that η_0 converges asymptotically to 0 at the nodes. Thus with respect to the metric g_{a_0}, the differential Tu_0 of $u_0 := \oplus_{a_0} exp_u(\eta_0) : S_{a_0} - |D_{a_0}| \to Q$ is uniformly bounded. For any map $w \in H^{3,\delta_0^0,\delta_1^0}(S_a, Q)$ we introduce the notation:

$$||Tw|| = \sup_{|h|=1,\, z \in S_a - |D_a|} ||Tw(z)h||_Q$$

where the sup is taken over all $z \in S_a - |D_a|$ and $h \in S_1(T_z S_a)$, where $S_1(T_z S_a)$ is the sphere of radius 1 in $T_z S_a$. Lastly, choose a small open neighborhood U_0 of η_0 in $H_c^{3,\delta_0^0,\delta_1^0}(u^*TQ, u|_{\partial S}^* TL)$ satisfying conditions (1) and (2) below:

$$(cl(P) \oplus cl(U_0)) \cap K \subset \mathfrak{O} \tag{4.3.1}$$

There exists a constant $C > 0$ such that $||T \oplus_a (exp_u(\eta)))|| \leq C$ for every $(a, v, \eta) \in P \oplus U_0$. (4.3.2)

We define $U^{\mathcal{G}}(q_0)$ by

$$U^{\mathcal{G}}(q_0) = (P \oplus U_0) \cap K$$

Proof of Proposition 4.3.1: Consider the M-polyfold

$$\mathcal{G}' := \{(q', \alpha'_{q'} : q' = (a', v', \eta') \in \mathfrak{O}'\}$$

where $q' \to \alpha'_{q'}$ is a good family centered at the stable map (S', j', M', D', u'). By G' we denote the automorphism group of (S', j', M', D', u'). Set $q'_0 := (a'_0, v'_0, \eta'_0) \in \mathfrak{O}'$. Restricting to a subsequence, we assume that the sequence

$$q_k := \pi \circ s(\Phi_k) := (a_k, v_k, \eta_k)$$

has the property that

$$(a_k, v_k) \to (a, v) \in cl(P)$$

By assumption we have the converging sequence:

$$q'_k := \pi' \circ t(\Phi_k) = (a'_k, v'_k, \eta'_k) \to q'_0 = (a'_0, v'_0, \eta'_0)$$

Associated with the sequence $\Phi_k \in M(\mathcal{G}, \mathcal{G}')$ we have the sequence

$$\phi_k : (S_{a_k}, j(a_k, v_k), M_{a_k}, D_{a_k}) \to (S'_{a'_k}, j'(a'_k, v'_k), M'_{a'_k}, D'_{a'_k})$$

of biholomorphisms satisfying

$$\oplus'_{a'_k}(exp'_{u'}(\eta'_k))) \circ \phi_k = \oplus_{a_k}(exp_u(\eta_k))$$

Lemma 4.3.1.

$$sup_{z \in S_{a_k} - |D_{a_k}|, k \geq 1} ||T\phi_k(z)|| < \infty$$

Proof: The proof Lemma 4.3.1 is in Appendix 1. ■

By Ascoli-Arezla, a subsequence of ϕ_k converges in C^∞_{loc} in finite distance to the core. By Proposition 3.6.2 we have convergence $\phi_0 = \lim_{k\to\infty}\phi_k$ in C^∞_{loc}. By hypothesis, we have $\eta'_k \to \eta'_0$. Thus by Theorem 3.6.1, we have that η_k converges in finite distance to the core and in the neck by Proposition 3.6.2. We set $\widehat{\eta}_0 := \lim_{k\to\infty}\eta_k$.

Set $\widehat{q}_0 := (a_0, v_0, \widehat{\eta}_0)$ and $\Phi_0 := ((\widehat{q}_0, \alpha_{q_0}, \phi_0, (q'_0, \alpha'_{q'_0})) \in M(\mathcal{G}, \mathcal{G}')$. It remains to show that $\lim_{k\to\infty}\Phi_k = \Phi_0$ in $M(\mathcal{G}, \mathcal{G}')$. It is enough to show that for any set $\mathfrak{V}$ in the basis of the topology for $M(\mathcal{G}, \mathcal{G}')$ containing Φ_0 that $\Phi_k \in \mathfrak{V}$ for k sufficiently large. By construction of the topology on $M(\mathcal{G}, \mathcal{G}')$ we may assume

$$\mathfrak{V} := \{((q, \alpha_q), \psi_q, (f(q), \alpha'_{f(q)})) : q \in \widehat{\mathfrak{O}} \subset \mathfrak{O}\}$$

where $\widehat{\mathfrak{O}} \subset \mathfrak{O}$ is a small open neighborhood of $\widehat{q}_0 \in \mathfrak{O}$ and f is a map as in Theorem 4.1.1. We may assume that $f(\widehat{q}_0) = q'_0$ and $\psi_{q_0} = \phi_0$.

Since $\psi_{q_k} \circ \phi_k^{-1} : \alpha'_{q'_k} \to \alpha'_{f(q_k)}$ is an isomorphism of stable maps, there exists by Proposition 4.1.4, an isomorphism $g' \in G'$ such that

$$f(q_k) = g' * q'_k \quad \text{and} \quad \psi_{q_k} \circ \phi_k^{-1} = (g')_{a'_k}$$

The sequence $\psi_{q_k} \circ \phi_k^{-1}$ converges in C^∞ to the identity map on $S'_{a'_0}$. In particular, we have $g' = \text{Identity} \in G'$, and $(g')_{a'_k} = \text{Identity on } S'_{a'_k}$. Thus $\psi_{q_k} = \phi_k$. This means $\Phi_k = ((q_k, \alpha_{q_k}), \phi_k, (q'_k, \alpha'_{q'_k}))$. ■

4.4 The Topology on Z

The space $\mathfrak{O}$ is equipped with a finite group action by *sc*-diffeomorphisms:

$$G \times \mathfrak{O} \to \mathfrak{O}, \quad (g, q) \to g * q.$$

The quotient $\mathfrak{O}/G$ inherits the second countability, paracompactness, and Hausdorff properties from $\mathfrak{O}$. Additionally, the projection map $\pi : \mathfrak{O} \to \mathfrak{O}/G$ is an open map. By Theorem 2.3.1 the map $\mathfrak{O}/G \to Z$ given by

$$[a, v, \eta] \to [S_a, j(a, v), M_a, D_a, exp_{\oplus_a u} \oplus_a (\eta)]$$

into the space of stable maps is injective. This inspires the following definition of the topology on Z:

Definition 4.4.1. We denote by $\mathfrak{N}$ the collection of subsets of Z consisting of sets U such that for all $z \in U$ there exists a good uniformizer $q \to \alpha_q$, parametrized by $\mathfrak{O}$, satisfying $z = [\alpha_{q_0}]$ for some $q_0 \in \mathfrak{O}$, and an open neighborhood $V(q_0) \subset \mathfrak{O}$ such that

$$\{[\alpha_q] \in Z : q \in V(q_0)\} \subset U \quad \blacksquare$$

We will follow the proof of Theorem 3.27 [HWZ1] in the proof of the following:

Theorem 4.4.1. (compare Theorem 3.27 [HWZ1]) The collection $\mathfrak{N}$ of subsets of the space Z of stable curves is a second countable, paracompact, Hausdorff topology which also is completely regular. In particular, the topology is metrizable. The non-singular elements in Z form a dense, open subset and Z is separable. $\blacksquare$

Lemma 4.4.1. (compare Lemma 3.28 [HWZ1]) The collection $\mathfrak{N}$ constitutes a topology on Z.

Proof: By construction $\emptyset \in \mathfrak{N}$ and $Z \in \mathfrak{N}$. In addition, it follows that $\mathfrak{N}$ is closed under union.

It is enough to show that if $U_1, U_2 \in \mathfrak{N}$, then $U := U_1 \cap U_2 \in \mathfrak{N}$. Suppose $z \in U$, then by construction we have good uniformizers $q \to \alpha_q$, parametrized by $\mathfrak{O}$, and $q' \to \alpha'_{q'}$, parametrized by $\mathfrak{O}'$ such that for $q_0 \in \mathfrak{O}$, $q'_0 \in \mathfrak{O}'$, we have

$$z = [\alpha_{q_0}] = [\alpha'_{q'_0}] \tag{4.4.1}$$

By construction we also have open subsets $V(q_0) \subset \mathfrak{O}$ and $V'(q'_0) \subset \mathfrak{O}'$ such that

$$\{[\alpha_q] : q \in V(q_0)\} \subset U_1 \quad \text{and} \quad \{[\alpha'_{q'}] : q' \in V'(q'_0)\} \subset U_2$$

By Equation (4.4.1), there exists an isomorphism $\phi_0 : \alpha_{q_0} \to \alpha'_{q'_0}$. Theorem 4.1.1, there exists a germ of *sc*-diffeomorphisms $f : (\mathfrak{O}, q_0) \to (\mathfrak{O}', q'_0)$ satisfying $f(q_0) = q'_0$ and a core smooth family of biholomorphisms

$$\phi_q : \alpha_q \to \alpha'_{f(q)}$$

such that $\phi_0 = \phi_{q_0}$. Consider a small open subset $V_1(q_0) \subset \mathfrak{O}$ containing q_0. It follows that

$$\{[\alpha_q] : q \in V_1(q_0)\}$$

is open and that

$$\{[\alpha_q] : q \in V_1(q_0)\} = \{[\alpha'_{q'}] : q' \in V'_1(q'_0) := f(V_1(q_0))\} \subset U_1 \cap U_2 = U \qquad \blacksquare$$

Lemma 4.4.2. (compare Lemma 3.29 [HWZ1]) Every point in the topological space $(Z, \mathfrak{N})$ has a countable basis of open neighborhoods.

Proof: The proof follows similarly to Lemma 3.30, substituting [HWZ1] Theorem 3.13 for Theorem 4.1.1. ■

Lemma 4.4.3. (compare Lemma 3.30 [HWZ1]) The topology $\mathfrak{N}$ is Hausdorff.

Proof: The proof follows similarly to Lemma 3.30, substituting Theorem [HWZ1] 3.22 for Proposition 4.1.1. ■

Lemma 4.4.4. (compare Lemma 3.31 [HWZ1]) The topology $\mathfrak{N}$ is completely regular.

Proof: The proof follows similarly to that of Lemma 3.31 [HWZ1], substituting Theorem 3.22 for Proposition 4.1.1. ■

Consider triples of the form (S, M, D) where S is an open Riemann surface, M is a finite set of ordered boundary marked points and un-ordered interior marked points, and a finite set of nodal pairs D, such that $|D| \cap M = \emptyset$. We say (S, M, D) and (S', M', D') are diffeomorphic if there exists a smooth orientation preserving diffeomorphism $\phi : S \to S'$ such that

$$\phi(M) = M', \qquad \phi(D) = D;$$

and preserves the ordering of the boundary components and boundary marked points. The diffeomorphism class of (S, M, D), denoted by $[[S, M, D]]$ consists of the triples diffeomorphic to (S, M, D). By Δ we denote the collection of all such diffeomorphism

types. Consider the map

$$\text{type} : Z \to \Delta \ : \ [(S, j, M, D, u)] \to [[S, M, D]]$$

Suppose $d \in \Delta$. Then by Z_d we denote the subset of Z consisting of stable maps with an underlying Riemann surface of type d. Z_d inherits a Hausdorff topology induced from $(Z, \mathfrak{N})$. Every element $z \in Z_d$ admits a representative of the form (S, j, M, D, u) where:

1. j is a smooth almost complex structure on S which induces the orientation on S

2. $u \in H_c^{3,\delta_0^0,\delta_1^0}(S, Q, L)$, i.e. $u \in H^3_{loc}$, obeys Lagrangian boundary conditions and decays at the rate $(3, \delta)$ at the nodes.

$H_c^{3,\delta_0^0,\delta_1^0}(S, Q, L)$ is a smooth Hilbert manifold. There exists an embedding $f : (Q, L) \to (\mathbb{R}^{2N}, \mathbb{R}^N)$, for some N sufficiently large, which induces an embedding:

$$F : H_c^{3,\delta_0^0,\delta_1^0}(S, Q, L) \to H_c^{3,\delta_0^0,\delta_1^0}(S, \mathbb{R}^{2N}, \mathbb{R}^N)$$

given by

$$u \to f \circ u$$

The Hilbert space topology on $H_c^{3,\delta_0^0,\delta_1^0}(S, \mathbb{R}^{2N}, \mathbb{R}^N)$ induces a topology on $H_c^{3,\delta_0^0,\delta_1^0}(S, Q, L)$ via the map F. We denote the induced topology on $H_c^{3,\delta_0^0,\delta_1^0}(S, Q, L)$ by τ. As discussed earlier, the space of almost complex structures on S is a complete metric space which we denote by $\mathfrak{J}$. By $H_d \subset H_c^{3,\delta_0^0,\delta_1^0}(S, Q, L)$ we denote the open subset consisting of all stable maps (S, j, M, D, u). H_d inherits a metric from τ by restriction, denoted τ_d. We set the

map

$$\Phi_d : J_d \times H_d \to Z_d$$

given by

$$(j, u) \to [(S, j, M, D, u)]$$

Lemma 4.4.5. (compare Lemma 3.32 [HWZ1]) For every type $d \in \Delta$ the canonical map $\Phi_d : J_d \times H_d \to Z_d$ is continuous, open, and surjective.

Proof: The argument follows similarly to [HWZ1] Lemma 3.32 with the modification that we subsitute [HWZ1] Theorem 2.19 for Theorem 2.4.1. ■

Lemma 4.4.6. (compare Lemma 3.33 [HWZ1]) For every type $d \in \Delta$, the Hausdorff topological space $(Z_d, \mathfrak{N}|Z_d)$ admits a countable basis.

Proof: Using Lemma 4.4.5, the result follows arguing as in [HWZ1] Lemma 3.33. ■

Lemma 4.4.7. (compare Lemma 3.34 [HWZ1]) The topology $\mathfrak{N}$ is secound countable.

Proof: The claim follows by arguing identically as in [HWZ1] Lemma 3.34. ■

Proof of Theorem 4.4.1 (compare Theorem 3.27 [HWZ1]) By Lemma 4.4.1, Lemma 4.4.4, and Lemma 4.4.7 and Urysohn's metrization theorem it follows that Z is metrizable and therefore paracompact.

Denote the set of un-noded elements in Z by int Z. It follows by the construction of basis elements by uniformizers that int Z is open and dense. The fact that each metric

space $J_d \times H_d$ is separable and the maps $\Phi_d : J_d \times H_d \to Z_d$ are open, continuous surjections, it follows that each Z_d is separable. Since the set Δ is countable, it follows that Z is separable. ■

4.5 The Polyfold Structure on the Space Z

Now we are able to construct the polyfold structure on the topological space Z of stable curves. By Proposition 4.1.3 for any $z \in Z$, we may find a collection of good uniformizing families centered at smooth maps which cover Z. Denote by Λ a family parametrizing such a collection of good uniformizing families, individually denoted $\mathcal{G}_\lambda$. In Section 7, we will specify the size of $\mathfrak{O}_\lambda$ by controlling the size of ε, which appears Section 4.1. We set $\mathcal{V}_\lambda = p(\mathfrak{O}_\lambda)$ cover Z, and without loss of generality, we assume that $\mathcal{V}_\lambda$ is locally finite.

We denote by X the disjoint union of the graphs

$$\mathcal{G}_\lambda = \{(q_\lambda, \alpha_{q_\lambda}) : q_\lambda \in \mathfrak{O}_\lambda\}$$

such that

$$X = \bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda$$

We refer to X as the object set of the *ep*-groupoid. The morphism set $\mathbf{X}$ is the disjoint union

$$\mathbf{X} = \bigcup_{\lambda, \lambda' \in \Lambda} M(\mathcal{G}_\lambda, \mathcal{G}_{\lambda'})$$

Theorem 4.5.1. (compare Theorem 3.35 [HWZ1]) The M-polyfold X is the object set and the M-polyfold $\mathbf{X}$ is the morphism set of a small category. The source and target maps $s, t : \mathbf{X} \to X$ are surjective local *sc*-diffeomorphisms. The structure maps $i : \mathbf{X} \to \mathbf{X}$, $u : X \to \mathbf{X}$, and $m : \mathbf{X}_s \times_t \mathbf{X} \to \mathbf{X}$ are *sc*-smooth. In addition, X satisfies

the properness assumption, and X is an *ep*-groupoid.

Proof: The only thing to check is properness. This follows from the argument of Theorem 3.35 in [HWZ1] substituting Proposition 4.3.1 on compactness for Theorem 3.22 [HWZ1]. ■.

The orbit space $|X|$ of the *ep*-groupoid X admits a homeomorphism $|p| : |X| \to Z$ induced from the canonical map $p : X \to Z$. Thus the pair $(X, |p|)$ defines a polyfold structure on Z. In summary,

Theorem 4.5.2. (compare Theorem 3.36 [HWZ1]): The M-polyfold groupoid X is an *ep*-groupoid and the pair $(X, |p|)$ defines a polyfold structure on Z. ■

Any two such *ep*-groupoids constructed as above are equivalent (see Definition 10.1 [HWZ2] and the discussion preceding Theorem 3.37 [HWZ1]).

Theorem 4.5.3. (compare Theorem 3.37 [HWZ1]) Given strictly increasing sequences $(\delta_0^i)_{i\geq 0} \subset (0, 2\pi)$ and $(\delta_1^i)_{i\geq 0} \subset (0, \pi)$, the second countable topological space Z of stable curves possesses a natural equivalence class of polyfold structures. ■

4.6 The Polyfold Structure of the Bundle $E \to Z$

Over each polyfold chart $\mathcal{G}_\lambda$ we will construct a strong M-polyfold bundle $W_\lambda \to \mathcal{G}_\lambda$. Our strong bundle will be the disjoint union:

$$W := \bigcup_{\lambda \in \Lambda} W_\lambda$$

Definition 4.6.1. Consider a good family $q \to \alpha_q$, parameterized by a set $\mathfrak{O}_\lambda$ and

centered at a smooth stable map $\alpha = (S, j, M, D, u)$. Recall from Section 4.1,

$$\mathfrak{O}_\lambda = \{(a, v, \eta) \in B_D \times E_u \ : \ (a, v) \in O \text{ and } \pi_a(\eta) = \eta, \ \eta \in U \cap E_u\}$$

Set

$$\text{Fill}\, \mathfrak{O}_\lambda = \{(a, v, \eta) \in B_D \times E_u \ : \ (a, v) \in O, \ \eta \in U \cap E_u\}$$

Let W_λ be the *sc*-Hilbert space of maps $z \to \xi(z)$, $z \in S - |D|$ where $\xi(z) : (T_z S, j) \to (T_{u(z)} Q, J(u(z)))$ is a complex anti-linear map. Moreover, the map $z \to \xi(z)$ is of Sobolev class $(2, \delta_0^0, \delta_1^0)$. The level k of the *sc*-structure is given by anti-linear maps of regularity $(2 + k, \delta_0^k, \delta_1^k)$. Consider the strong bundle retraction diagram:

$$\begin{array}{ccc} \text{Fill}\, \mathfrak{O}_\lambda \lhd W_\lambda & \xrightarrow{r \times \rho} & \text{Fill}\, \mathfrak{O}_\lambda \lhd W_\lambda \\ \downarrow & & \downarrow \\ \text{Fill}\, \mathfrak{O}_\lambda & \xrightarrow{\quad r \quad} & \text{Fill}\, \mathfrak{O}_\lambda \end{array}$$

where r is given by implanting the retractions $r_{\{x,y\}}$. We define the splicing core:

$$K_\lambda^R := \{(a, v, \eta, \xi) \in \text{Fill}\, \mathfrak{O}_\lambda \lhd W_\lambda : \ r_a(\eta) = \eta, \ \rho_a(\xi) = \xi\}$$

where ρ is defined similarly as r, by implanting local retractions $\widehat{r}_{\{x,y\}}$, $\widehat{q}_{\{x,y\}}$ on the space of anti-linear sections. We have a local strong local bundle:

$$K_\lambda^R \to \mathfrak{O}_\lambda$$

with projection given by $(a, v, \eta, \xi) \to (a, v, \eta)$. The automorphism group G of (S, j, M, D, u) acts on K_λ^R by

$$g * (a, v, \eta, \xi) = (g * a, g * v, \eta \circ g^{-1}, \xi \circ Tg^{-1}) \text{ for } g \in G \ \blacksquare$$

Remark 4.6.1. We denote the nodal parameters fixed in Section 2 for the family $\mathfrak{O}_\lambda$ by B_λ and we denote the good germ of complex deformations by V_λ. ■

For $v \in V_\lambda$, consider the complex linear bundle isomorphism

$$\delta(v) : (TS, j(v)) \to (TS, j)$$

given by

$$\delta(v)h = \frac{1}{2}(id - j \circ j(v))h$$

We have $\delta(v) \circ j(v) = j \circ \delta(v)$ and by construction $\delta(v)$ depends smoothly on $v \in V_\lambda$. Also, since $j(v) = j$, on the disks/half-disks defined by the small disk/half-disk structure we have

$$\delta(v)h = h \text{ for all } h \in T_xS, \text{ such that } x \in \bigsqcup_{z \in |D|} D_z$$

For varying topological families S_a we consider the complex linear map

$$\delta(a, v) : (TS_a, j(a, v)) \to (TS_a, j(a, 0))$$

given by

$$\delta(a, v)h = \frac{1}{2}(id - j(a, 0) \circ j(a, v))h$$

For any complex anti-linear map

$$\xi(z) : (T_zS, j) \to (T_{u(z)}Q, J(u(z)))$$

we have a glued section along the glued map $\oplus_a u$ defined by implanting the gluing $\widehat{\oplus}_a$:

$$\widehat{\oplus}_a(\xi)(z) : (T_zS_a, j(a, 0)) \to (T_{\oplus_a u(z)}Q, J(\oplus_a u(z)))$$

The composition $\xi \circ \delta(v)$ is a complex anti-linear map

$$(T_zS, j(v)) \to (T_{u(z)}Q, J(u(z)))$$

and satisfies the identity

$$\widehat{\oplus}_a(\xi \circ \delta(v)) = \widehat{\oplus}_a(\xi) \circ \delta(a, v)$$

Recall the exponential map $\widetilde{\mathfrak{O}} \stackrel{exp}{\to} Q$ defined in an open neighborhood of zero in $H_c^{3,\delta_0^0,\delta_1^0}(u^*TQ, u|_{\partial S}^*TL)$, where $\widetilde{\mathfrak{O}}$ was defined in the definition of 'Good Data.' Let $P_{xy} : T_xQ \to T_yQ$ be parallel transport with respect to the Levi-Civita connection associated to the metric g along the minimal geodesic joining x and y for x and y sufficiently close. Consider the map:

$$\Gamma : (T_xQ, J(x)) \to (T_yQ, J(y))$$

given by

$$v \to \frac{1}{2} \cdot [P_{xy} - J(y) \circ P_{xy} \circ J(x)] \cdot v$$

We assume $\mathfrak{O}_\lambda$ is chosen so small that $\Gamma(exp_p(\eta), p) : (T_pQ, J(p)) \to (T_{exp_p(\eta)}Q, J(exp_p(\eta)))$ is well-defined for all $(a, v, \eta) \in \mathfrak{O}_\lambda$. As a result of this definition, we have that Γ is the identity in the chosen coordinates near the image of the nodes under u, since g is the pull-back of Euclidean metric in chosen coordinates near these points.

Given $(a, v, \eta, \xi) \in K^R$, we define the complex anti-linear map

$$\Xi(a, v, \eta, \xi)(z) : (T_zS_a, j(a, v)) \to (T_{\oplus_a exp_u(\eta)(z)}Q, J(\oplus_a exp_u(\eta)(z)))$$

at points $z \in S_a - D_a$ by

$$\Xi(a, v, \eta, \xi)(z) = \Gamma[\oplus_a exp_u(\eta)(z), \oplus_a u(z)] \circ \widehat{\oplus}_a(\xi)(z) \circ \delta(a, v)(z)$$

Given $g \in G$ we have:

$$\Xi(a, v, \eta, \xi) \circ Tg_a^{-1} = \Xi(g * a, g * v, \eta \circ g^{-1}, \xi \circ Tg^{-1})$$

Recall the good uniformizing family mentioned above, $q \to \alpha_q$, which we write explicitly by

$$(a, v, \eta) \to (S_a, j(a, v), M_a, D_a, \oplus_a exp_u(\eta)), \qquad (a, v, \eta) = q \in \mathfrak{O}$$

We define the graph of the uniformizing family of anti-linear sections $\widehat{q} \to \widehat{\alpha}_{\widehat{q}}$ by

$$(a, v, \eta, \xi) \to (S_a, j(a, v), M_a, D_a, \oplus_a exp_u(\eta), \Xi(a, v, \eta, \xi))$$

where $(a, v, \eta, \xi) = \widehat{q} \in K^R$. For each $g \in G$, we have an isomorphism

$$\widehat{\alpha}_{\widehat{q}} \to \widehat{\alpha}_{g * \widehat{q}}$$

which is compatible with the action of g on $\mathcal{G}_\lambda$ and we have the identity:

$$\Xi(g * (a, v, \eta, \xi)) \circ Tg_a = \Xi(a, v, \eta, \xi)$$

Denote by W_λ the graph of the lifted family $\widehat{q} \to \widehat{\alpha}_{\widehat{q}}$. The collection of all lifted families of anti-linear sections for each $\lambda \in \Lambda$ produces a strong polyfold bundle

$$W = \bigcup_{\lambda \in \Lambda} W_\lambda \xrightarrow{\pi} X = \bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda$$

We say (S, j, M, D, u, ξ) is equivalent to $(S', j', M', D', u', \xi')$ if there exists an ismorphism $\phi : (S, j, M, D, u) \to (S', j', M', D', u')$ between the maps in the base spaces satisfying compatibility at the fiber

$$\xi' \circ T\phi = \xi$$

We denote by E the set of equivalence classes. By $P : W \to E$ we denote the quotient map sending a tuple to its equivalence class. We have a commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\Gamma} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{p} & Z \end{array}$$

Proposition 4.6.1. (Compare Proposition 3.39 [HWZ1]) : There exists a strong bundle map $\mu : \mathbf{X}_s \times_{\tilde{\mu}} W \to W$ which covers the target map $t : \mathbf{X} \to X$ of the *ep*-groupoid X satisfying

$$t \circ \pi_1(\Phi, w) = p \circ \mu(\Phi, w)$$

for all $(\Phi, w) \in \mathbf{X}_s \times_p W$, so that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{X}_s \times_p W & \xrightarrow{\mu} & W \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle p} \\ \mathbf{X} & \xrightarrow{t} & X \end{array}$$

Also μ has the following additional properties:

(1) μ is a surjective local *sc*-diffeomorphism and linear in the fibers W_x.

(2) $\mu(1_x, w_x) = w_x$ for all $x \in X$ and $w_x \in W_x$.

(3) $\mu(\Phi \circ \Psi, w) = \mu(\Phi, \mu(\Psi, w))$ for all morphisms $\Phi, \Psi \in \mathbf{X}$ and $w \in W$ satisfying $s(\Psi) = p(w)$ and $t(\Psi) = s(\Phi) = p(\mu(\Psi, w))$.

Proof: To define the map $\mu : \mathbf{X}_s \times_p W \to W$, we make the following notational shorthands:

$$q = (a, v, \eta), \qquad \widehat{q} = (a, v, \eta, \xi)$$

and

$$\alpha_q = (S_a, j(a, v), M_a, D_a, \oplus_a exp_u(\eta)), \quad \widehat{\alpha}_{\widehat{q}} = (S_a, j(a, v), M_a, D_a, \oplus_a exp_u(\eta), \Xi(a, v, \eta, \xi))$$

If $\Phi \in \mathbf{X}$ gives the morphism

$$\Phi := ((q, \alpha_q), \phi, (q', \alpha'_{q'})) : (q, \alpha_q) \xrightarrow{\phi} (q', \alpha'_{q'})$$

then the morphism

$$\phi : (S_a, j(a, v), M_a, D_a) \to (S'_{a'}, j'(a', v'), M'_{a'}, D'_{a'})$$

satisfies

$$\oplus'_{a'} exp'_{u'}(\eta') \circ \phi = \oplus_a exp_u(\eta)$$

If $w = (\widehat{q}, \widehat{\alpha}_{\widehat{q}}) \in W$, then $p(w) = (q, \alpha_q) = s(\Phi)$. Therefore, $(\Phi, w) \in \mathbf{X}_s \times_p W$ and we write

$$\mu(\Phi, (\widehat{q}, \widehat{\alpha}_{\widehat{q}})) = (\widehat{q}', \widehat{\alpha}'_{\widehat{q}'})$$

where $q' = (a', v', \eta')$ and $\widehat{q}' = (a', v', \eta', \xi') = (q', \xi')$. ξ' is defined as the uniquely specified

solution of the equations

$$\Xi(a,v,\eta,\xi) = \Xi'(a',v',\eta',\xi') \circ T\phi$$

$$\widehat{\ominus}'_{a'}(\xi') = 0$$

The rest of the argumentation follows formally as in Proposition 3.39 [HWZ1]. ■

The functoriality properties imply that the natural map $\Omega : W \to E$ given by

$$(a,v,\eta,\xi,\widehat{\alpha}_{(a,v,\eta,\xi)}) \to [\widehat{\alpha}_{(a,v,\eta,\xi)}]$$

is a homeomorphism on orbit spaces $|\Omega| : |W| \to E$. We have a commutative diagram:

$$\begin{array}{ccc} |W| & \xrightarrow{|\Omega|} & E \\ \downarrow & & \downarrow \\ |X| & \xrightarrow{|p|} & Z \end{array}$$

It follows that $(p : W \to X, |\Omega|, |p|)$ realizes $E \to Z$ as a strong polyfold bundle.

Part II

Pre-Fredholm Theory

In Part II we develop the Fredholm estimates necessary to establish Theorem 1.3.2. Section 5 introduces general results on polyfold Fredholm theory as well as general analytic results for the linearized Cauchy-Riemann operator. Section 6 produces an equivalent polyfold structure on the ep-groupoid of stable maps to that which was originally established in Section 4 which is more convenient for verification of the Fredholm estimates.

Verification of these estimates takes place in Section 7.

5 Fredholm Sections and Integration

5.1 Filling the Cauchy-Riemann Operator

In this section we recall some of the basic definitions and results from the Fredholm theory of polyfolds. For a fuller treatment see [HWZ2].

Definition 5.1.1. (compare Definition 3.1 [HWZ2]) An *sc*-germ of neighborhoods around $0 \in C$, denoted by $O(C,0)$ is a decreasing sequence

$$U_0 \supset U_1 \supset U_2 \supset \cdots$$

where U_m is a relatively open neighborhood of 0 in $C \cap E_m$. The tangent of $O(C,0)$ denoted by $TO(C,0)$ is the decreasing sequence

$$U_1 \oplus E_0 \supset U_2 \oplus E_1 \supset \cdots \qquad \blacksquare$$

Definition 5.1.2. (compare Definition 3.3 [HWZ2]) An *sc*-smooth section germ (f, x_0) of the strong bundle $P : Y \to X$ is a continuous section $f : V \to P^{-1}(V)$ defined on a (sufficiently small) open neighborhood V of the smooth point x_0, for which the following holds. There exists a strong bundle chart $(\Phi, P^{-1}(V), (K, C \lhd F, E \lhd F))$

$$\begin{array}{ccc} P^{-1}(V) & \xrightarrow{\Phi} & K \\ \downarrow P & & \downarrow P \\ (V, x_0) & \xrightarrow{\phi} & (O, 0) \end{array}$$

satisfying $\phi(x_0) = 0 \in O$ in which the principal part $\widehat{\mathbf{f}}$ of the local continuous section

$\widehat{\mathbf{f}} = \Phi \circ f \circ \phi^{-1} : O \to K$ has the property that the composition

$$\widehat{\mathbf{f}} \circ r : O(C, 0) \to F$$

is an sc-smooth germ. ■

So called sc^+-operators serve as compact perturbations in the polyfold framework:

Definition 5.1.3. (compare Definition 1.3 [HWZ2]) A sc-operator $S : E \to F$ between sc-Banach spaces is called an sc^+-operator, if$S(E_m) \subset F_{m+1}$ and $S : E \to F^1$ is an sc-operator. ■

Throughout this discussion on pre-Fredholm theory, we will concern ourselves with 'filled sections' of the Cauchy-Riemann section. Filling a section extends its domain to an open subset of an sc-Hilbert space. The filling is constructed in such a way that its solution set and Fredholm index agree with that of the original section. This replacement makes the analysis more tractable.

Definition 5.1.4. (compare Definition 3.1.4 [HWZ2]) Consider a local strong bundle $K \to O$. We suppose K is realized by a strong bundle retraction $R : U \lhd F \to U \lhd F$ of the form

$$R(u, f) = (r(u), \rho(r(u))f)$$

A smooth section germ $(f, 0)$ of the bundle $K \to O$ possesses a filling if there exists an sc^∞ section germ $(g, 0)$ of the bundle $U \lhd F \to U$ extending f and satisfying the following:

(1) $f(x) = g(x)$ for $x \in O$ close to 0.

(2) If $g(y) = \rho(r(y)) \cdot g(y)$ for a point $y \in U$ near 0, then $y \in O$.

(3) The linearization of the map

$$y \to [1 - \rho(r(y))] \cdot g(y)$$

at the point 0, restricted to $\ker(Dr(0))$ defines a topological linear isomorphism

$$\ker(Dr(0)) \to \ker(\rho(0)) \quad \blacksquare$$

Remark 5.1.1. The definition of a filling has several useful consequences, as demonstrated, for example, in [HWZ2] Pages 94-95. Here we briefly review the highlights. Observe that we have two splittings:

$$F = \rho(0)F \oplus (1 - \rho(0))F, \quad E = Dr(0)E \oplus (1 - Dr(0))E$$

With respect to these splittings we have the following relationship between the differentials of g and f at 0:

$$Dg(0) \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} Tf(0) & \rho(0) \cdot Dg(0)(1 - Dr(0)) \\ 0 & (1 - \rho(0)) \cdot Dg(0)(1 - Dr(0)) \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Combined with property (3) of Definition 5.1.1, this implies:

$$\text{kernel } Dg(0) = (\text{kernel } Tf(0)) \oplus \{0\}$$

Additionally, by Proposition 3.1.5 [HWZ2]:

(1) The operator $Tf(0) : Dr(0)E \to \rho(0)F$ is surjective if and only if the operator $Dg(0) : E \to F$ is surjective.

(2) $Tf(0)$ is a Fredholm operator (in the classical sense) if and only if $Dg(0)$ is a Fredholm operator and $\operatorname{ind} Tf(0) = \operatorname{ind} Dg(0)$. ■

Definition 5.1.5. Consider the section $\overline{\partial}_J^{\lambda} : \mathfrak{O}_\lambda \to W_\lambda$ given as follows

$$\Gamma(exp_u\eta(z), z) \circ \widehat{\oplus}_a \overline{\partial}_J^{\lambda}(a, v, \eta) \circ \delta(a, v)(z) = \overline{\partial}_{J,j(v)} exp_u\eta(z) \tag{5.1.1}$$

$$\widehat{\ominus}_{a_{\{x,y\}}} \overline{\partial}_J^{\lambda}(a, v, \eta) = 0 \quad \blacksquare \tag{5.1.2}$$

By construction we have commutativity of the following diagram:

$$\begin{array}{ccc} W_\lambda & \xrightarrow{P} & W \\ {\scriptstyle \overline{\partial}_J^{\lambda}}\uparrow & & \uparrow{\scriptstyle \overline{\partial}_J} \\ \mathfrak{G}_\lambda & \xrightarrow{p} & Z \end{array}$$

Consider a nodal point $\{x, y\} \in D_i$ and its image under u, $u(x) = u(y) := q_{\{x,y\}}$. Consider the Cauchy-Riemann section $\overline{\partial}_{q_{\{x,y\}}}$ defined as follows:

$$\overline{\partial}_{q_{\{x,y\}}} : H_c^{3,\delta_0}(C_a, T_{q_{\{x,y\}}}Q) \to H^{2,\delta_0}(C_a, T_{q_{\{x,y\}}}Q)$$

$$\eta \to \frac{\partial}{\partial s}\eta + J(q_{\{x,y\}})\frac{\partial}{\partial t}\eta$$

We define $\overline{\partial}_{q_{\{x,y\}}}$ similarly if $\{x, y\} \in D_b$:

$$\overline{\partial}_{q_{\{x,y\}}} : H_{c,real}^{3,\delta_0}(C_a, T_{q_{\{x,y\}}}Q) \to H_{\text{bdry}}^{2,\delta_0}(C_a, T_{q_{\{x,y\}}}Q)$$

$$\eta \to \frac{\partial}{\partial s}\eta + J(q_{\{x,y\}})\frac{\partial}{\partial t}\eta$$

The following are key analytic estimates:

Proposition 5.1.1. The linear Cauchy-Riemann operator on the infinite cylinder

$$\overline{\partial}_{q_{\{x,y\}}} : H_c^{3,\delta_0}(C_a, T_{q_{\{x,y\}}}Q) \to H^{2,\delta_0}(C_a, T_{q_{\{x,y\}}}Q) \tag{5.1.3}$$

is an isomorphism for $\delta_0 \in (0, 2\pi)$ and

$$\frac{1}{C_m} \cdot \left|\xi\right|_m \leq \left|\overline{\partial}_{q_{\{x,y\}}}\xi\right|_{m-1} \leq C_m \cdot \left|\xi\right|_m, \quad \xi \in H_c^{3+m,\delta_0^m}(C_a, T_{q_{\{x,y\}}}) \tag{5.1.4}$$

Similarly, the linear Cauchy-Riemann operator on the infinite strip with real boundary conditions

$$\overline{\partial}_{q_{\{x,y\}}} : H_{c,\,real}^{3,\delta_1}(C_a, T_{q_{\{x,y\}}}Q) \to H_{\text{bdry}}^{2,\delta_1}(C_a, T_{q_{\{x,y\}}}Q) \tag{5.1.5}$$

is an isomorphism for $\delta_1 \in (0, \pi)$ and

$$\frac{1}{C_m} \cdot \left|\xi\right|_m \leq \left|\overline{\partial}_{q_{\{x,y\}}}\xi\right|_{m-1} \leq C_m \cdot \left|\xi\right|_m, \quad \xi \in H_{c,real}^{3+m,\delta_1^m}(C_a, T_{q_{\{x,y\}}}) \quad \blacksquare \tag{5.1.6}$$

Remark 5.1.2. That (5.1.3) is an isomorphism follows from [LM]. That (5.1.4) is an isomorphism is due to Floer [F]. Alternatively, one may prove (5.1.4) is an isomorphism by a spectral flow argument [RS1]. ■

Recall the M-polyfolds Fill $\mathfrak{O}_\lambda$ and Fill W_λ established in Definition 4.6.1. Consider the section Fill $\overline{\partial}_J^\lambda$: Fill $\mathfrak{O}_\lambda \to$ Fill W_λ given by

$$\Gamma(exp_u\eta(z), z) \circ \widehat{\oplus}_a \text{Fill}\,\overline{\partial}_J^\lambda(a, v, \eta)(z) \circ \delta(a, v)(z) = \overline{\partial}_{J,j(v)} exp_u\eta(z) \tag{5.1.7}$$

$$\widehat{\ominus}_{a_{\{x,y\}}} \text{Fill}\, \overline{\partial}_J^{\lambda}(a,v,\eta) = \overline{\partial}_{q_{\{x,y\}}} \ominus_{a_{\{x,y\}}} (\eta_x, \eta_y) \tag{5.1.8}$$

Proposition 5.1.2. Fill $\overline{\partial}_J^{\lambda}$ is a filling for $\overline{\partial}_J^{\lambda}$.

Proof: Recall the retraction in the base $r : \text{Fill}\, \mathfrak{O}_\lambda \to \text{Fill}\, \mathfrak{O}_\lambda$ and the retraction in the fiber $\rho : \text{Fill}\, W_\lambda \to \text{Fill}\, W_\lambda$ are given by implanting retractions $r_{\{x,y\}}$, $q_{\{x,y\}}$ and $\widehat{r}_{\{x,y\}}$, $\widehat{q}_{\{x,y\}}$ respectively for $\{x,y\} \in D$. Smoothness considerations follow from the discussion in Section 7.

Condition (1) in Definition 5.1.3 follows from inspection of the system (5.1.1) – (5.1.2).

Condition (2) in Definition 5.1.3 follows from Proposition 5.1.1.

Condition (3) in Definition 5.1.3 follows from the fact that $Dr_{\{x,y\}}(0) = \text{Identity}$, $Dq_{\{x,y\}}(0) = \text{Identity}$ and $\widehat{r}_{\{x,y\}}(0) = \text{Identity}$, $\widehat{q}_{\{x,y\}}(0) = \text{Identity}$. ■

5.2 Fredholm Theory

Below we will discuss the *sc*-Fredholm properties which ensure the applicability of the *sc*-Implicit function theorem. The classical implicit function theorem is proved by a contraction argument. The *sc*-Fredholm property builds a contraction property into the definition:

Definition 5.2.1. (compare Definition 3.6 [HWZ2]) Let W be an *sc*-Banach space and $C = [0,\infty)^k \oplus \mathbb{R}^{n-k} \oplus W$ a partial quadrant. Then a basic germ is an *sc*-smooth germ

$$f : O(C,0) \to \mathbb{R}^N \oplus W$$

satisfying $f(0) = 0$ and having the following property: If $P : \mathbb{R}^N \oplus W \to W$ denotes the projection, the germ $P \circ f : O(C, 0) \to W$ has the form

$$P \circ f(a, w) = w - B(a, w)$$

for $(a, w) \in [0, \infty)^k \oplus \mathbb{R}^{n-k} \oplus W$ where $B(0, 0) = 0$. Moreover, for every $\epsilon > 0$ and every integer $m \geq 0$, the estimate

$$|B(a, w) - B(a, w')|_m \leq \epsilon \cdot |w - w'|_m$$

holds, if (a, w) and (a, w') are close enough to $(0, 0)$ on level m. ■

Definition 5.2.2. (compare Definition 3.7 [HWZ2]) Let f be an *sc*-smooth section of the strong bundle $P : Y \to X$ over the M-polyfold X, and let $x_0 \in X$ be a smooth point. Then (f, x_0) is an *sc*-Fredholm germ provided it possesses a filled version

$$(g, 0) : O(U, 0) \to U \lhd F$$

with the property that there exists a local sc^+ section $s : U \to U \lhd F$ satisfying $s(0) = g(0)$ and such that $(g - s, 0) : O(U, 0) \to U \lhd F$ is conjugated to a basic germ. ■

Definition 5.2.3. (compare Definition 3.8 [HWZ2]) A section f of the strong bundle $P : Y \to X$ over the M-polyfold X is an *sc*-Fredholm section, if it satisfies the following:

1. f is *sc*-smooth.

2. f is regularizing.

3. The germ (f, x) is *sc*-Fredholm for every $x \in X_\infty$. ■

Definition 5.2.4. (compare Definition 12.1 [HWZ2]) Let f be an *sc*-smooth section functor of a strong bundle over an *ep*-groupoid $(P : W \to X, \mu)$. f is an *sc*-Fredholm section functor if f is *sc*-Fredholm over the object M-polyfold X. ■

In the case that our polyfold has boundary, we also require that our section lies in good position to the boundary.

Definition 5.2.5. (compare Definition 3.9 [HWZ2]) Let $C \subset E$ be a partial quadrant in the sc-Banach space E and $N \subset E$ be a finite dimensional sc-subspace of E. The subspace N is in good position to the partial quadrant C, if the interior of $N \cap C$ in N is non-empty, and if N possesses a sc-complement P, so that

$$E = N \oplus P$$

having the following property: There exists $\epsilon > 0$, such that for pairs $(n, p) \in N \times P$ satisfying $|p|_0 \leq \epsilon |n|_0$ the statements $n \in C$ and $n + p \in C$ are equivalent. We call such an sc-complement P a good complement. ■

An application of the *sc*-implicit function theorem results in the following:

Proposition 5.2.1. (compare Theorem 3.5 and Corollary 3.6 [HWZ2]) Let f be an *sc*-Fredholm section of the strong bundle $P : Y \to X$ over the M-polyfold X and suppose that $x \in X$ satisfies $f(x) = 0$ and the following:

(a) The linearization $f'(x) : T_x X \to Y_x$ is surjective.

(b) The kernel N of $f'(x)$ is in good position to the boundary of X

Then there exists an open neighborhood U of x such that the following holds:

There exists a good complement Y of N in T_xX so that $K \oplus Y = T_xS$ and there exists a C^1 map $\sigma : V \to Y_1$ defined on a relatively open neighborhood V of 0 in $N \cap C_xX$ with the following properties:

1. $\sigma : O(N \cap C_xX, 0) \to (Y, 0)$

2. $\sigma(0) = 0$ and $D\sigma(0) = 0$.

3. $S(f,U) := \{x \in U \ : \ f(x) = 0\} = \{v + \sigma(v) \ : \ v \in V\}$ after perhaps replacing U by a smaller open neighborhood.

4. The map σ induces a tame sub M-polyfold structure on $S(f,U) \subset X$.

5. The tame sub-M-polyfold $S(f,u)$ admits a uniquely determined sc-smooth equivalent structure as a smooth manifold with boundary and corners.

6. For every $z \in S(f,U)$ the linearization $f'(z) : T_zX \to Y_z$ is surjective and $\ker(f'(z)) = T_zS(f,U)$. ■

The ultimate goal of studying the Fredholm properties of the Cauchy-Riemann section $\overline{\partial}_J : Z \to W$, is to develop an integration theory over the solution set $\overline{\partial}_J^{-1}(0)$ of pseudo-holomorphic curves. However, as discussed in the introduction, the Cauchy-Riemann

section need not be transverse to the zero-section. To rectify this, we apply the general machinery of polyfolds which establishes conditions under which we may perturb an *sc*-Fredholm section by multi-valued functors to ensure transversality. Furthermore, general polyfold theory establishes conditions under which the perturbed solution set may be integrated over.

This means that to establish an integration theory, it is enough to check to check *sc*-Fredholm condition, compactness, and orientability properties. The *sc*-Fredholm condition is verified in Section 7. The remaining conditions are left to subsequent work. Below we give an overview of the theory of integration on polyfolds.

Consider a general strong bundle of *ep*-groupoids $(P : W \to X, \mu)$ and an *sc*-Fredholm section $f : X \to W$. Perturbations of f are achieved via sc^+-multisection functors:

Definition 5.2.6. (compare Definition 13.4 [HWZ2]) Let $(P : W \to X, \mu)$ be a strong bundle over the *ep*-groupoid X. Then an sc^+-multisection is a functor

$$\Lambda : W \to \mathbb{Q}^+$$

such that we have the following local representation:

Around every object $x_0 \in X$, there exists an open neighborhood $U(x_0)$ on which the isotropy group G_{x_0} acts by its natural representation, and finitely many sc^+-sections $s_1, \cdots, s_k : U(x_0) \to W$ with associated positive rational numbers $\sigma_1, \cdots, \sigma_k$ such that

1. $\sum_{i=1}^{k} \sigma_i = 1$

2. $\Lambda(w) = \sum_{i\in\{1,\cdots,k:\, w=s_i(P(w))\}} \sigma_i$ for all $w \in W|U(x_0)$, where the empty sum is set to 0. ■

Functoriality implies that a multi-section Λ is invariant under equivalences in **W**. In particular, Λ factors through a map $|\Lambda| : |W| \to \mathbb{Q}^+$.

Now we describe the effect of perturbation by a multi-section on the linearization of an *sc*-Fredholm section f:

Definition 5.2.7. (compare Section 15.1 [HWZ2]) Let $(P : W \to X, \mu)$ be a strong bundle over an *ep*-groupoid. Denote by $Gr_F(x)$ the set consisting of all formal sums L of the form

$$L := \sum \sigma_L \cdot L$$

where all σ_L are non-negative and only a finite number are nonzero, and $L : T_xX \to W_x$ is an *sc*-Fredholm operator. We set

$$Gr_F(X) := \bigsqcup_{x\in X_\infty} Gr_F(x)$$

Let $\pi : Gr_F(X) \to X_\infty$ denote the natural projection. The set $Gr_F(X)$ admits a category structure:

A morphism between objects $L : T_xX \to W_x$ and $\phi_*L : T_yX \to W_y$ is realized by a morphism $\phi \in Mor(x, y)$ such that

$$\phi_*L \circ T\phi(\eta) = \mu(\phi, L(\eta)) \;\; \eta \in T_xX, \;\; s(\phi) = x, \;\; y = t(\phi) \quad ■$$

Theorem 5.2.1. (compare Theorem 15.1 [HWZ2]) Given P, f, Λ as in Definition

5.2.7, there exists a section functor

$$T_{(f,\Lambda)} : X_\infty \to Gr_F(X)$$

such that

1. $T_{(f,\Lambda)}(x) = 0$ for $x \in X_\infty - \text{supp}(\Lambda \circ f)$.

2. For $x \in \text{supp}(\Lambda \circ f)$ and any local sc^+ section structure $(s_i)_{i\in I}$, $(\sigma_i)_{i\in I}$ describing Λ locally near x we have:

$$T_{(f,\Lambda)}(y) = \sum_{\{i\in I : f(y)=s_i(y)\}} \sigma_i \cdot (f - s_i)'(y) \text{ for smooth } y \text{ near } x \ \blacksquare$$

In the strong *ep*-groupoid bundle $W \to X$ over an *ep*-groupoid of stable maps, we can measure the size of smooth sections $f : X \to W$ by taking advantage of the natural Hilbert space structure in the fibers. In a general strong bundle over an *ep*-groupoid we have the notion of an auxiliary norm:

Definition 5.2.8. (compare Definition 12.2 [HWZ2]) Let $(P : W \to X, \mu)$ be a strong bundle over an *ep*-groupoid. Consider the topological space:

$$W_{0,1} := \{h \in W \ : \ h \text{ has bi-regularity } (0,1) \ \}$$

and by $P : W_{0,1} \to X$ the projection onto the object space X. An auxiliary norm N is a continuous map $N : W_{0,1} \to [0, \infty)$ satisfying:

1. The restriction of N to each fiber $P^{-1}(x)$ is a complete norm.

2. If (h_k) is a sequence in $W_{0,1}$ such that $(P(h_k))$ converges in X to some x, and $N(h_k) \to 0$ then $h_k \to 0_x$ in $W_{0,1}$.

3. If $h = \mu(\phi, k)$ for $\phi \in \mathbf{X}$ and $k \in W_{0,1}$ satisfying $s(\phi) = P(k)$, then

$$N(h) = N(k) \quad \blacksquare$$

Definition 5.2.9. (compare Definition 13.5 and 13.6 [HWZ2]) Let $(P : W \to X, \mu)$ be a strong bundle and let Λ be an sc^+ multi-section. Also let N be an auxiliary norm. The support of Λ is the subset of $\text{supp}(\Lambda)$ of W defined by

$$\text{supp}(\Lambda) = \{w \in W \ : \ \Lambda(w) > 0\}$$

Also, we define for $x \in X$,

$$N(\Lambda)(x) := \max\{N(w) \ : \ w \in \text{supp}(\Lambda) \text{ satisfying } P(w) = x\} \quad \blacksquare$$

We want to use multi-sections to perturb *sc*-Fredholm sections into position such that their linearizations are surjective. This will guarantee that solutions in the interior of polyfold locally admit a geometric structure suitable for integration (branched, weighted orbifold structure). However, to ensure that points at the boundary admit a reasonable geometric structure we need the following stronger condition:

Definition 5.2.10. (compare Definition 15.6 [HWZ2]) Let f be an *sc*-Fredholm section of $(P : W \to X, \mu)$ and that $\Lambda : W \to \mathbb{Q}^+$ is an sc^+-multisection. The pair (f, Λ) is said to be in general position if for every $x \in \text{supp}(\Lambda \circ f)$ the linearization $T_{(f,\Lambda)}(x)$

restricted to the tangent space of the boundary at x is surjective. ■

The space of multi-sections is broad enough to perturb a general *sc*-Fredholm section into transverse position with good boundary geometry:

Theorem 5.2.2. (compare Theorem 15.4 [HWZ2]) Assume $(P : W \to X, \mu)$ is a strong bundle over a tame *ep*-groupoid admitting *sc*-smooth bump functions, and f a *sc*-Fredholm section functor with compact solution set $|S_f|$. Let N be an auxiliary norm and U be a saturated subset of X containing S_f so that (N, U) controls compactness. Given any continuous functor $h : X \to (0, 1]$, there exists an sc^+ multisection functor $\Lambda : W \to \mathbb{Q}^+$ with the following properties:

1. $N(\Lambda)(x) < h(x)$ for all $x \in X$.

2. The domain support of Λ is contained in U.

3. $T_{(f,\Lambda)}$ is surjective on $\text{supp}(\Lambda \circ f)$.

4. For $x \in \text{supp}(\Lambda \circ f)$ the kernels of the operators from $T_{(f,\Lambda)}(x)$ are in general position to ∂X.

In particular $\Lambda \circ f : X \to \mathbb{Q}^+$ is a compact, tame branched ep^+ subgroupoid. Also, the pertubation Λ can be taken to be structurable. ■

Here a branched ep^+ subgroupoid is defined as follows:

Definition 5.2.11. (compare Definition 9.1 [HWZ2]) Let X be an *ep*-groupoid. A branched ep^+-subgroupoid is a functor $\Theta : X \to \mathbb{Q}^+$ which satisfies the following:

1. If $\Theta(x) > 0$ then $x \in X_\infty$.

In addition, for every $x \in \text{supp}(\Theta)$ the following holds:

2. There exists an open neighborhood $U(x)$ and a finite family of submanifolds $M_i \subset U(x)$, $i \in I$, with $x \in M_i$ such that the inclusion $M_i \to U(x)$ is proper.

3. There exists for every $i \in I$ a positive rational number $\sigma_i > 0$

4. With the data in (ii) and (iii) we have an equality:

$$\Theta(y) = \sum_{\{i \in I \,:\, y \in M_i\}} \sigma_i \text{ for } y \in U(x)$$

Θ is called n-dimensional if for every point $x \in \text{supp}(\Theta)$ the manifolds M_i are n-dimensional. ■

See Definitions 9.6 and 9.8 [HWZ2] for the notions of tangent space and orientation in the context of branched ep^+ subgroupoids. The integration theory is provided by the following result:

Theorem 5.2.3. (compare Theorem 9.2 [HWZ2]) Set $S :=|\text{supp}\,\Theta|$. Let X be an ep-groupoid and $\widehat{\Theta} = (\Theta, \widehat{T}_\Theta) : X \to \mathbb{Q}^+$ be an oriented, tame, compact, branched ep^+-subgroupoid of dimension n. There exists a linear map:

$$\Phi_{\widehat{\Theta}} : \Omega^n_\infty(X) \to M(S, L(S)) : \ \omega \to \mu_\omega$$

where $M(S, L(S))$ is the vector space of finite signed measures on the Lebesgue σ-algebra

on S and $\Omega^n_\infty(X)$ is the space of de Rham n-forms on X. ■

The above considerations have natural generalizations to the context of polyfolds, which are equivalence classes of *ep*-groupoids.

Given the *sc*-Fredholm property, in order to produce an integration theory for $\overline{\partial}_J$, it remains to show that:

1. $\overline{\partial}_J^{-1}(0) \subset Z_{g,h,\vec{k},m,\beta}$ is compact and

2. $\overline{\partial}_J$ may be equipped with an orientation.

These points will not be discussed in detail in this article. We observe that although point 1 is closely related to Gromov Compactness, it must be checked that the induced topology on $\overline{\partial}_J^{-1}(0)$ agrees with the topology which appears in Gromov Compactness arguments. In the closed case Point 2 is considered in [HWZ1], in the polyfold context. Point 2 is also discussed in [Ge], in the context of Kuranishi structures.

6 A Lego Piece Construction on Spaces of Stable Maps with Lagrangian Boundary Conditions

In Section 4, we constructed local uniformizing charts on the space of stable maps. The purpose of this section is to construct alternative charts which allow for an easier verification of the *sc*-Fredholm property for the Cauchy-Riemann section

$$\overline{\partial} : Z \to E$$

$$[(S, j, M, D, u)] \to [(S, j, M, D, u, \frac{1}{2} \cdot [Tu + J \circ Tu \circ j])]$$

We will construct charts from fibered products of pieces which separately characterize nodal and classical behavior.

Remark 6.0.1. Let $\mathcal{G}_\lambda$ be a chart for the *ep*-groupoid $(X, \mathbf{X})$. Suppose $\mathcal{G}_\lambda$ is centered at the point (S, j, M, D, u). Recall the coordinate charts $\phi_{\{x,y\}} : (\mathbf{R}(w), w) \to \mathbb{R}^{2n}$ fixed in Section 4 for every $\{x, y\} \in D_i \cup D_b$, where $w = u(x) = u(y)$. We impose the additional property:

$$T\phi_{\{x,y\}} \circ J \circ T\phi^{-1}_{\{x,y\}}(0) = i,$$

where

$$i := \begin{bmatrix} 0 & \text{Identity}_{n\times n} \\ -\text{Identity}_{n\times n} & 0 \end{bmatrix} \in Gl(2n)$$

We recall that $\mathcal{G}_\lambda$ is equipped with the data of a small disk structure such that for every $(a, v, \eta) \in \mathfrak{O}_\lambda$ we have

$$exp_{\oplus_a(u)} \oplus_a (\eta)(D_x \cup D_y) \subset U(u(x))$$

We also assume the exponential map is induced by a metric which near every node is the pull-back of the Euclidean metric on $\mathbb{R}^{2n}$. We may occasionally shorten $\phi_{\{x,y\}}$ to ϕ_x or ϕ_y in what follows. ■

Suppose $\{x, y\} \in D_i$, and fix positive biholomorphic coordinates $\psi_x : (\mathbb{R}^+ \times S^1, i) \to (D_x, j|_{D_x})$. We set $D'_x = \psi_x([1, \infty) \times S^1)$.

Also, choose negative biholomorphic coordinates $\psi_y : (\mathbb{R}^- \times S^1, i) \to (D_y, j|_{D_y})$. We set $D'_y = \psi_y((-\infty, -1] \times S^1)$.

Denote by N_x the collar $\overline{D_x - D'_x} \subset S$, and by N_y the collar $\overline{D_y - D'_y} \subset S$. Observe that for $a \in B_{\frac{1}{2}}$ sufficiently small,

$$\oplus_a(u^+, u^-)|_{N_x \cup N_y} = u^+ \cup u^-|_{N_x \cup N_y}$$

Now suppose $\{x, y\} \in D_b$. Choose positive holomorphic coordinates $\psi_x : (\mathbb{R}^+ \times [0,1], i) \to HD_x$ and set $HD'_x := \psi_x([1,\infty) \times [0,1])$. Set $HN_x = \overline{HD_x - HD'_x}$.

Also, choose negative biholomorphic coordinates $\psi_y : (\mathbb{R}^- \times [0,1], i) \to HD_y$ and set $HD'_y := \psi_y((-\infty, -1] \times [0,1])$. We define $HN_y = \overline{HD_y - HD'_y}$. We define the following subsets of S:

$$\mathbf{N_x} = \bigcup_{\{x,y\} \in D_i} N_x, \quad \mathbf{N_y} = \bigcup_{\{x,y\} \in D_i} N_y$$

$$\mathbf{HN_{x'}} = \bigcup_{\{x',y'\} \in D_b} HN_{x'}, \quad \mathbf{HN_{y'}} = \bigcup_{\{x',y'\} \in D_b} HN_{y'}$$

We also define the set

$$\mathfrak{O}^{core}_\lambda := \{(v, \eta) \in V_\lambda \times H^3\Gamma(u|^*_{S_{core}} TQ) \ : \exists\, (a, v, \widetilde{\eta}) \in \mathfrak{O}_\lambda \text{ such that } \widetilde{\eta}|_{S_{core}} = \eta\}$$

Also consider the *sc*-Hilbert space:

$$\mathcal{O}^{core}_\lambda := \{\eta \in H^3\Gamma(u|^*_{S_{core}} TQ) \ : \ \eta(z) \in H_{u(z)} \text{ for } z \in \Sigma\}$$

It follows from the definition of $\mathfrak{O}^{core}_\lambda$ that $\mathfrak{O}^{core}_\lambda$ is an open subset of $V_\lambda \times \mathcal{O}^{core}_\lambda$. We introduce the following shorthands:

$$E_\lambda := \prod_{\{x,y\} \in D_i} B_{\frac{1}{2}} \times E(2n) \times \prod_{\{x',y'\} \in D_b} [0, 1/2) \times E_{\text{real}}(2n)$$

$$E^*_\lambda := \prod_{\{x,y\}\in D_i} \mathbb{C}\times E(2n) \times \prod_{\{x',y'\}\in D_b} \mathbb{R}\times E_{\text{real}}(2n)$$

$$X_\lambda := \prod_{\{x,y\}\in D_i} \{x,y\}\times X(2n) \times \prod_{\{x',y'\}\in D_b} \{x',y'\}\times X_{\text{real}}(2n)$$

$$F_\lambda := \prod_{\{x,y\}\in D_i} B_{\frac{1}{2}}\times F(2n) \times \prod_{\{x',y'\}\in D_b} \mathbb{R}^+\times F_{\text{bdry}}(2n)$$

$$Y_\lambda := \prod_{\{x,y\}\in D_i} \{x,y\}\times Y(2n) \times \prod_{\{x',y'\}\in D_b} \{x',y'\}\times Y_{\text{bdry}}(2n)$$

(Recall $E(2n)$, $E_{\text{real}}(2n)$, $X(2n)$, $X_{\text{bdry}}(2n)$, $F(2n)$, $F_{\text{bdry}}(2n)$, $Y(2n)$, $Y_{\text{bdry}}(2n)$ were defined in Section 3.) We denote the points of E_λ by tuples:

$$(a_{\{x,y\}}, \eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})_{\{x,y\}\in D}$$

or tuples

$$((a_{\{x,y\}}, \eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})_{\{x,y\}\in D_i}, (a_{\{x',y'\}}, \eta^+_{\{x',y'\}}, \eta^-_{\{x',y'\}})_{\{x',y'\}\in D_b})$$

or tuples

$$(\overline{a}, \overline{\eta}^+, \overline{\eta}^-)$$

where $(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}}) \in E(2n)$, $(\eta^+_{\{x',y'\}}, \eta^-_{\{x',y'\}}) \in E_{\text{real}}(2n)$. We employ a similar convention for labelling the points in X_λ, F_λ, Y_λ. As discussed in Section 3, we will identify $X(2n)$ and $X_{\text{bdry}}(2n)$ with the images of the retractions

$$r : B_{\frac{1}{2}}\times E(2n) \to B_{\frac{1}{2}}\times E(2n)$$

and

$$q : [0,\frac{1}{2})\times E_{\text{real}}(2n) \to [0,\frac{1}{2})\times E_{\text{real}}(2n)$$

respectively, and similarly for $F(2n)$ and $F_{\text{bdry}}(2n)$. We can collect the appropriate copies

of r and q to define a retraction $(r,q)_\lambda : E_\lambda \to E_\lambda$, such that $\mathrm{Im}\,(r,q)_\lambda$ is canonically identified with X_λ. Similarly, we identify Y_λ as a retract of F_λ.

We define the following *sc*-Hilbert spaces:

$$H := \prod_{\{x,y\}\in D_i} H^3(N_x,\mathbb{R}^{2n}) \times H^3(N_y,\mathbb{R}^{2n}) \times \prod_{\{x',y'\}\in D_b} H^3(HN_{x'},\mathbb{R}^{2n}) \times H^3(HN_{y'},\mathbb{R}^{2n})$$

$$H_i := \prod_{\{x,y\}\in D_i} H^3(N_x,\mathbb{R}^{2n}) \times H^3(N_y,\mathbb{R}^{2n})$$

$$H_b := \prod_{\{x',y'\}\in D_b} H^3(HN_{x'},\mathbb{R}^{2n}) \times H^3(HN_{y'},\mathbb{R}^{2n})$$

Consider the fibered product

$$\widetilde{\mathfrak{O}}_\lambda := \mathfrak{O}_\lambda^{core}{}_m \times_{f_i\times f_b} X_\lambda$$

where

$$f_i : X_\lambda \to H_i$$

is given by

$$f_i(t_x,t_y)_{\{x,y\}\in D_i} = (t_x\circ\psi_x^{-1}|_{N_x}, t_y\circ\psi_y^{-1}|_{N_y})_{\{x,y\}\in D_i}$$

and where

$$f_b : X_\lambda \to H_b$$

is given by

$$f_b(t_{x'},t_{y'})_{\{x',y'\}\in D_b} = (t_{x'}\circ\psi_{x'}^{-1}|_{HN_{x'}}, t_{y'}\circ\psi_{y'}^{-1}|_{HN_{y'}})_{\{x',y'\}\in D_b}$$

The map

$$m : \mathfrak{O}_\lambda^{core} \to H_i \times H_b$$

is given as follows

$$(v, \eta) \to ((T\phi_x \circ \eta|_{N_x}, T\phi_y \circ \eta|_{N_y})_{\{x,y\}\in D_i}, (T\phi_{x'} \circ \eta|_{HN_{x'}}, T\phi_{y'} \circ \eta|_{HN_{y'}})_{\{x',y'\}\in D_b})$$

Proposition 6.0.1. $\widetilde{\mathfrak{O}}_\lambda$ is an M-polyfold.

Proof: Consider the fibered product

$$\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda := {\mathfrak{O}_\lambda^{core}}_m \times_{f_i\times f_b} E_\lambda$$

where

$$f_i \times f_b : E_\lambda \to H$$

is given by

$$(f_i(\{t_x\}), f_b(\{t_{x'}\})) = ((t_x \circ \psi_x^{-1}|_{N_x}, t_y \circ \psi_y^{-1}|_{N_y})_{\{x,y\}\in D_i},$$

$$(t_{x'} \circ \psi_{x'}^{-1}|_{HN_{x'}}, t_{y'} \circ \psi_{y'}^{-1}|_{HN_{y'}})_{\{x',y'\}\in D_b})$$

We claim that $\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda$ is an open subset of a partial quadrant in an sc-Hilbert space: We can extend the map m to $V_\lambda \times \mathfrak{O}_\lambda^{core}$ in the obvious way. We can also extend $f_i \times f_b$ to E_λ^* in the obvious way. The maps m and $f_i \times f_b$ extended to E_λ^* and $\mathcal{O}_\lambda^{core}$ are linear sc^0 maps. The fibered product $(\mathcal{O}_\lambda^{core})_m \times_{f_i\times f_b} E_\lambda^*$ is the kernel of the map:

$$m \times 0 - 0 \times f_i \times f_b \ : \ \mathcal{O}_\lambda^{core} \times E_\lambda^* \to \ H$$

Therefore, $(\mathcal{O}_\lambda^{core})_m \times_{f_i\times f_b} E_\lambda^*$ is an sc-Hilbert space and $(\mathcal{O}_\lambda^{core})_m \times_{f_i\times f_b} E_\lambda^{**}$ is a partial

quadrant in $(\mathcal{O}_\lambda^{core})_m \times_{f_i \times f_b} E_\lambda^*$, and by construction, $\text{Fill}\widetilde{\mathfrak{O}}_\lambda$ is open in $(\mathcal{O}_\lambda^{core})_m \times_{f_i \times f_b} E_\lambda^{**}$. Finally, consider the retraction:

$$R := \text{Identity}_{\mathcal{O}_\lambda^{core}} \times (r,q)_\lambda \ : \ (\mathcal{O}_\lambda^{core})_m \times_{f_i \times f_b} E_\lambda^{**} \to (\mathcal{O}_\lambda^{core})_m \times_{f_i \times f_b} E_\lambda^{**}$$

By construction R is *sc*-smooth and a retraction such that $R(\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda) = \widetilde{\mathfrak{O}}_\lambda$. ■

Remark 6.0.2. Given $z := (v, \eta, a, \overline{\eta}^+, \overline{\eta}^-) \in \text{Fill}\,\widetilde{\mathfrak{O}}_\lambda$ there exists a unique section $\eta'(z) \in \Gamma(u^*TQ)$ such that $(a, v, \eta') \in \text{Fill}\,\mathfrak{O}_\lambda$ satisfies

$$\eta'(z)|_{S_{core}} = \eta$$

and

$$T\phi_x \circ \eta'(z)|_{D_x} = \eta^+_{\{x,y\}} \circ \psi_x^{-1}, \quad T\phi_y \circ \eta'(z)|_{D_y} = \eta^-_{\{x,y\}} \circ \psi_y^{-1}$$

$$T\phi_{x'} \circ \eta'(z)|_{HD_{x'}} = \eta^+_{\{x',y'\}} \circ \psi_{x'}^{-1}, \quad T\phi_{y'} \circ \eta'(z)|_{HD_{y'}} = \eta^-_{\{x',y'\}} \circ \psi_{y'}^{-1}$$

Restricting to an open subset of $\text{Fill}\,\mathfrak{O}_\lambda$ is necessary, we have a map $g_\lambda : \text{Fill}\,\mathfrak{O}_\lambda \to \text{Fill}\,\widetilde{\mathfrak{O}}_\lambda$ defined by

$$g_\lambda(a, v, \eta) = ((v, \eta|_{S_{core}}), (a, (T\phi_x \circ \eta \circ \psi_x, T\phi_y \circ \eta \circ \psi_y)_{\{x,y\} \in D}))$$

and an inverse map $f_\lambda : \text{Fill}\,\widetilde{\mathfrak{O}}_\lambda \to \text{Fill}\,\mathfrak{O}_\lambda$ defined by

$$f_\lambda(v, \eta, a, \overline{\eta}^+, \overline{\eta}^-) = (a, v, \eta'(v, \eta, a, \overline{\eta}^+, \overline{\eta}^-))$$

From the proof of Proposition 6.0.1, it follows that the tangent bundle of $\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda$ is

$$\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda \oplus ((\mathcal{O}_\lambda^{core})_m \times_{f_i \times f_b} E_\lambda^*)^1 \quad ■$$

For simplicity, we restrict Fill $\widetilde{\mathfrak{O}}_\lambda$ and Fill $\mathfrak{O}_\lambda$ to open subsets on which f_λ and g_λ are bijections.

Proposition 6.0.2. The map f_λ is an *sc*-diffeomorphism.

Proof: The topology on Fill $\widetilde{\mathfrak{O}}_\lambda$ is induced by the norm:

$$||(v,\eta,a,\overline{\eta}^+,\overline{\eta}^-)||_{\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda} := ||a||_{\widehat{B}_\lambda} + ||v||_{V_\lambda} + ||\eta||_{H^3\Gamma(u^*|_{S_{core}}TQ)} + ||(\overline{\eta}^+,\overline{\eta}^-)||_{E_\lambda}$$

That f_λ is an open embedding follows from the inequalities:

$$||f_\lambda(z)||_{\text{Fill}\,\mathfrak{O}_\lambda} \leq ||z||_{\text{Fill}\,\widetilde{\mathfrak{O}}_\lambda} \leq 2\cdot||f_\lambda(z)||_{\text{Fill}\,\mathfrak{O}_\lambda} \quad \blacksquare$$

Corollary 6.0.1. The map $f_\lambda : \widetilde{\mathfrak{O}}_\lambda \to \mathfrak{O}_\lambda$ is an open embedding. $\blacksquare$

Definition 6.0.1. We define the set

$$\widetilde{\mathcal{G}}_\lambda := \{((v,\eta,a,\overline{\eta}^+,\overline{\eta}^-)), (S_a, j(a,v), M_a, D_a, exp_{\oplus_a u} \oplus_a \eta'(v,\eta,a,\overline{\eta}^+,\overline{\eta}^-)) :$$

$$(a,\overline{\eta}^+,\overline{\eta}^-) \in X_\lambda,\ \ (a,v,\eta'(v,\eta,(a,\overline{\eta}^+,\overline{\eta}^-)) \in \mathfrak{O}_\lambda\} \quad \blacksquare$$

We define an *ep*-groupoid $(\widetilde{X}, \widetilde{\mathbf{X}})$ as follows:

Definition 6.0.2. The object set is given by

$$\widetilde{X} := \bigcup_{\lambda\in\Lambda} \widetilde{\mathcal{G}}_\lambda$$

We define the morphism sets $M(\widetilde{\mathcal{G}}_\lambda, \widetilde{\mathcal{G}}_{\lambda'})$ to be the set of tuples:

$$\{((v,\eta,a,\overline{\eta}^+,\overline{\eta}^-),\phi,(v',\eta',a',\overline{\eta'}^+,\overline{\eta'}^-)) : (f_\lambda(v,\eta,a,\overline{\eta}^+,\overline{\eta}^-),\phi,f_{\lambda'}(v',\eta',a',\overline{\eta'}^+,\overline{\eta'}^-)) \in M(\mathcal{G}_\lambda,\mathcal{G}_{\lambda'})\}$$

By $\widetilde{\mathbf{X}}$ we denote the union $\bigcup_{\lambda,\lambda'} M(\widetilde{\mathcal{G}}_\lambda, \widetilde{\mathcal{G}}_{\lambda'})$. We have a bijection:

$$\mathbf{f}_{\lambda,\lambda'} : M(\widetilde{\mathcal{G}}_\lambda, \widetilde{\mathcal{G}}_{\lambda'}) \to M(\mathcal{G}_\lambda, \mathcal{G}_{\lambda'})$$

$$((u,v),\phi,(u',v')) \to (f_\lambda(v,\eta,a,\overline{\eta}^+,\overline{\eta}^-),\phi,f_{\lambda'}(v',\eta',a',\overline{\eta'}^+,\overline{\eta'}^-))$$

Lastly, we define the *sc*-smooth diffeomorphisms

$$f := \bigcup_\lambda f_\lambda : \widetilde{X} \to X$$

and

$$\mathbf{f} := \bigcup_{\lambda,\lambda'} \mathbf{f}_{\lambda,\lambda'} : \widetilde{\mathbf{X}} \to \mathbf{X} \quad \blacksquare$$

Proposition 6.0.3. $(\widetilde{X}, \widetilde{\mathbf{X}})$ is an *ep*-groupoid. Furthermore, $(\widetilde{X}, \widetilde{\mathbf{X}})$ is equivalent to $(X, \mathbf{X})$.

Proof: From the above discussion, it follows that $\widetilde{X}$ and $\widetilde{\mathbf{X}}$ are M-polyfolds. We use the maps f and $\mathbf{f}$ to reduce the verification of the following *ep*-groupoid properties to Theorem 4.5.1:

1. The source and target maps $\widetilde{s}, \widetilde{t} : \widetilde{\mathbf{X}} \to \widetilde{X}$ are local diffeomorphisms.

Proof: Recall the sc^∞ embeddings $f : \widetilde{X} \to X$ and $\mathbf{f} : \widetilde{\mathbf{X}} \to \mathbf{X}$. Then we factor $\widetilde{s}$ as

$$\widetilde{s} := f^{-1} \circ s \circ \mathbf{f}$$

and $\widetilde{t}$ as

$$\widetilde{t} := f^{-1} \circ t \circ \mathbf{f} \quad \blacksquare$$

2. The unit map $\widetilde{u}$ and the inverse map $\widetilde{\iota}$ are sc^{∞}.

Proof: We factor $\widetilde{u}$ as

$$\widetilde{u} := \mathbf{f}^{-1} \circ u \circ f$$

and ι as

$$\widetilde{\iota} := \mathbf{f}^{-1} \circ \iota \circ \mathbf{f} \qquad \blacksquare$$

3. Every point $\widetilde{x} \in \widetilde{X}$ possesses an open neighborhood $V(\widetilde{x})$ of $\widetilde{x}$ in $\widetilde{X}$ such that the map

$$\widetilde{t} : \widetilde{s}^{-1}(cl_{\widetilde{X}}(V(\widetilde{x}))) \to \widetilde{X}$$

is proper.

Proof: By Theorem 4.5.1, for any point $x \in X$, there exists an open neighborhood $V(x)$ of x in X such that the map

$$t : s^{-1}(cl_X(V(x))) \to X \tag{6.0.1}$$

is proper. Set $x := f(\widetilde{x})$. Then there exists $V(x)$ such that Equation (6.0.1) defines a proper map. Then set $V(\widetilde{x}) := f^{-1}(V(x))$. Observe that we have the equality of sets

$$\widetilde{s}^{-1}(cl_{\widetilde{X}}(V(\widetilde{x}))) = \mathbf{f}^{-1} \circ s^{-1} \circ f(cl_{\widetilde{X}}(f^{-1}V(x))) = \mathbf{f}^{-1}(s^{-1}(cl_X(V(x))))$$

Thus we can factor $\widetilde{t}$ as follows:

$$\widetilde{t}|\widetilde{s}^{-1}(cl_{\widetilde{X}}(V(\widetilde{x}))) := f^{-1} \circ t \circ \mathbf{f}|\mathbf{f}^{-1}(s^{-1}(cl_X(V(x))))$$

Since f, $\mathbf{f}$ are embeddings, and t is proper on $cl_X(V(x))$, the claim follows. ■

4. By Claim 1., the fibered product $\widetilde{\mathbf{X}}_{\widetilde{s}} \times_{\widetilde{t}} \widetilde{\mathbf{X}}$ admits a natural M-polyfold structure. The multiplication map $\widetilde{m} : \widetilde{\mathbf{X}}_{\widetilde{s}} \times_{\widetilde{t}} \widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}}$ is sc^∞.

Proof: By Theorem 4.5.1, the multiplication map

$$m : \mathbf{X}_s \times_t \mathbf{X} \to \mathbf{X}$$

is sc^∞. Then we can factor $\widetilde{m}$ as follows:

$$\widetilde{m} := \mathbf{f}^{-1} \circ m \circ (\mathbf{f} \times \mathbf{f}) | \widetilde{\mathbf{X}}_{\widetilde{s}} \times_{\widetilde{t}} \widetilde{\mathbf{X}}$$

It follows that $\widetilde{m}$ is sc^∞. ■

We have established that $(\widetilde{X}, \widetilde{\mathbf{X}})$ is an *ep*-groupoid. Consider the *sc*-smooth functor:

$$F : (\widetilde{X}, \widetilde{\mathbf{X}}) \to (X, \mathbf{X})$$

given by f on objects and $\mathbf{f}$ on morphisms. It is enough to check the following properties:

1. F is faithful and full.

Proof: These properties follow by Remark 6.0.2. ■

2. The induced map $|F| : |\widetilde{X}| \to |X|$ between orbit spaces is a homeomorphism.

Proof: It is enough to show $|F|$ is open. Recall that the topology on $|X|$ is generated by subsets of the form

$$U_\lambda := \{[q] \ : \ q \in \mathfrak{O}_\lambda\} \tag{6.0.2}$$

and the topology on $|\widetilde{X}|$ is generated by subsets of the form

$$\widetilde{U}_\lambda := \{[q'] \ : \ q' \in \widetilde{\mathfrak{O}}_\lambda\} \tag{6.0.3}$$

(Note that the brackets [ ] used in lines (6.0.2) and (6.0.3) denote different equivalnce relations.) By construction $|F|$ takes $\widetilde{U}_\lambda$ bijectively onto U_λ. ■

3. F is a local *sc*-diffeomorphism on objects.

Proof: This follows from Remark 6.0.2. ■ ■

Remark 6.0.3. In the proof of Proposition 6.0.3, we introduced a functor

$$F : \widetilde{X} \to X$$

acting by f on objects and $\mathbf{f}$ on morphisms. Let $\widetilde{p} : \widetilde{X} \to Z$ be the quotient map. Note that $|\widetilde{p}|$ factors as $|\widetilde{p}| = p \circ |F|$. ■

Consider the bundle $W_\lambda \to \mathcal{G}_\lambda$ defined in Section 4. Consider the *sc*-Hilbert space

$$\mathcal{F}_\lambda^{core} := H^2\Omega^{0,1}((TS|_{S_{core}}, j), (u^*|_{S_{core}}TQ, J))$$

We define the local strong bundle:

$$\begin{array}{c} W_\lambda^{core} := \mathfrak{O}_\lambda^{core} \lhd \mathcal{F}_\lambda^{core} \\ \Big\downarrow \\ \mathfrak{O}_\lambda^{core} \end{array}$$

Recall the strong bundles defined in Section 3: $P : W(2n) \to X(2n)$ and $Q : W_{\text{bdry}}(2n) \to X_{\text{real}}(2n)$.

Define the product strong bundle $W_\lambda^{neck} \to X_\lambda$ as follows:

$$\begin{array}{c} W_\lambda^{neck} := \prod_{\{x,y\} \in D_i} \{x, y\} \times W(2n) \times \prod_{\{x',y'\} \in D_b} \{x', y'\} \times W_{\text{bdry}}(2n) \\ \Big\downarrow \prod_{D_i} P \times \prod_{D_b} Q \\ X_\lambda \end{array}$$

We also define the strong bundle $\text{Fill}\, W_\lambda^{neck} \to E_\lambda$ by

$$\begin{array}{c} \text{Fill}\, W_\lambda^{neck} := E_\lambda \lhd F_\lambda \\ \Big\downarrow \prod_{D_i} P \times \prod_{D_b} Q \\ E_\lambda \end{array}$$

We define the set $\widetilde{F}_\lambda$ to be the subset of $\mathcal{F}_\lambda^{core} \times F_\lambda$ consisting of the tuples

$$(\xi, \overline{\xi}^+, \overline{\xi}^-)$$

such that

$$T\phi_x(u) \circ \xi \cdot T\psi_x \cdot \frac{\partial}{\partial s}|_{[0,1]\times S^1} = \xi^+_{\{x,y\}}|_{[0,1]\times S^1}$$

$$T\phi_y(u) \circ \xi \cdot T\psi_y \cdot \frac{\partial}{\partial s}|_{[-1,0]\times S^1} = \xi^-_{\{x,y\}}|_{[-1,0]\times S^1}$$

and for $\{x', y'\} \in D_b$

$$T\phi_{x'}(u) \circ \xi \cdot T\psi_{x'} \cdot \frac{\partial}{\partial s}|_{[0,1]\times[0,1]} = \xi^+_{\{x',y'\}}|_{[0,1]\times[0,1]}$$

$$T\phi_{y'}(u) \circ \xi \cdot T\psi_{y'} \cdot \frac{\partial}{\partial s}|_{[-1,0]\times[0,1]} = \xi^-_{\{x',y'\}}|_{[-1,0]\times[0,1]}$$

Proposition 6.0.4. $\widetilde{F}_\lambda$ is an *sc*-Hilbert space.

Proof: We define the *sc*-Hilbert space:

$$Z := \prod_{\{x,y\}\in D_i} H^3([0,1]\times S^1, \mathbb{R}^{2n}) \times H^3([-1,0]\times S^1, \mathbb{R}^{2n})$$

$$\times \prod_{\{x',y'\}\in D_b} H^3([0,1]\times[0,1], \mathbb{R}^{2n}) \times H^3([-1,0]\times[0,1], \mathbb{R}^{2n})$$

The *sc*-structure on Z is the obvious one. Consider the map:

$$\upsilon : \mathcal{F}^{core}_\lambda \times F_\lambda \to Z$$

$$(\xi, \xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})_{\{x,y\}\in D}) \to (T\phi_x(u)\circ\xi\cdot T\psi_x \cdot \frac{\partial}{\partial s} - \xi^+_{\{x,y\}}, T\phi_y(u)\circ\xi\cdot T\psi_y \cdot \frac{\partial}{\partial s} - \xi^-_{\{x,y\}})_{\{x,y\}\in D}$$

Then $\ker \upsilon$ is an *sc*-Hilbert space by Proposition 3.1.1 and by construction $\ker \upsilon = \widetilde{F}_\lambda$.

■

We define the set $\operatorname{Fill} \widetilde{W}_\lambda$ to be the subset of $W^{core}_\lambda \times \operatorname{Fill} W^{neck}_\lambda$ consisting of tuples

$$((v, \eta, a, \overline{\eta}^+, \overline{\eta}^-), (\xi, \overline{\xi}^+, \overline{\xi}^-)) \in \operatorname{Fill} \widetilde{W}_\lambda$$

which satisfy

$$(\overline{\xi}^+, \overline{\xi}^-) \in Y_\lambda$$

We also define the local strong bundle $\operatorname{Fill} W_\lambda := \operatorname{Fill} \mathfrak{O} \lhd F_\lambda$.

Proposition 6.0.5. $\operatorname{Fill} \widetilde{W}_\lambda \to \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda$ is a strong bundle, and $\operatorname{Fill} W_\lambda \to \operatorname{Fill} \mathfrak{O}_\lambda$ is isomorphic to $\operatorname{Fill} \widetilde{W}_\lambda \to \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda$ as strong bundles.

Proof: $\operatorname{Fill} \widetilde{W}_\lambda$ is isomorphic to the local strong bundle $\operatorname{Fill} W_\lambda \to \operatorname{Fill} \mathfrak{O}_\lambda$ via the strong bundle isomorphism

$$\begin{array}{ccc} \operatorname{Fill} W_\lambda & \xrightarrow{\widehat{f}_\lambda} & \operatorname{Fill} \widetilde{W}_\lambda \\ \downarrow & & \downarrow \\ \operatorname{Fill} \mathfrak{O}_\lambda & \xrightarrow{f_\lambda} & \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda \end{array}$$

where map $\widehat{f}_\lambda$ is defined as follows:

$$\widehat{f}_\lambda(a, v, \eta, \xi, \widehat{\alpha}_{a,v,\eta,\xi}) = (f_\lambda(a, v, \eta), \xi', (\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})_{\{x,y\}\in D})$$

where: ξ' is defined by

$$\xi|_{S_{core}} = \xi'$$

and $(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})$ is defined by

$$(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = (T\phi_x \circ \xi \cdot T\psi_x \frac{\partial}{\partial s}, T\phi_y \circ \xi \cdot T\psi_y \frac{\partial}{\partial s}) \quad \blacksquare$$

Remark 6.0.4. Consider the retraction

$$\widehat{R} : W^{core}_\lambda \times \operatorname{Fill} W^{neck}_\lambda \to W^{core}_\lambda \times \operatorname{Fill} W^{neck}_\lambda$$

given by

$$((v, \eta, a, \overline{\eta}^+, \overline{\eta}^-), (\xi, \overline{\xi}^+, \overline{\xi}^-)) \to ((v, \eta, r(a, \overline{\eta}^+, \overline{\eta}^-)), (\xi, \widehat{r}(a, \overline{\xi}^+, \overline{\xi}^-)))$$

The retraction $\widehat{R}$ produces a local strong bundle chart for $\widetilde{W_\lambda} := \operatorname{Im} \widehat{R}$:

$$\begin{array}{ccc} \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda \lhd \widetilde{F}_\lambda & \xrightarrow{\widehat{R}} & \widetilde{W_\lambda} \subset \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda \lhd \widetilde{F}_\lambda \\ \downarrow & & \downarrow \\ \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda & \xrightarrow{r} & \widetilde{\mathfrak{O}}_\lambda \subset \operatorname{Fill} \widetilde{\mathfrak{O}}_\lambda \end{array}$$

By construction, $\widetilde{W_\lambda}$ is isomorphic to the strong bundle $W_\lambda \to \mathfrak{O}_\lambda$ via the strong bundle isomorphism

$$\begin{array}{ccc} W_\lambda & \xrightarrow{\widehat{f_\lambda}} & \widetilde{W_\lambda} \\ \downarrow & & \downarrow \\ \mathfrak{O}_\lambda & \xrightarrow{f_\lambda} & \widetilde{\mathfrak{O}}_\lambda \end{array} \quad \blacksquare$$

In summary we have:

Proposition 6.0.6. $\widetilde{W_\lambda} \to \widetilde{\mathfrak{O}}_\lambda$ is a strong bundle of M-polyfolds. Moreover, $W_\lambda \to \mathfrak{O}_\lambda$ is isomorphic to $\widetilde{W_\lambda} \to \widetilde{\mathfrak{O}}_\lambda$ as strong bundles. ■

By $\widetilde{W}$ we denote the strong bundle of M-polyfolds:

$$\widetilde{W} := \bigcup_{\lambda \in \Lambda} \widetilde{W_\lambda} \xrightarrow{\widetilde{\pi}} \widetilde{X} = \bigcup_{\lambda \in \Lambda} \widetilde{\mathcal{G}_\lambda}$$

Proposition 6.0.7. There exists a strong bundle map $\widetilde{\mu} : \widetilde{\mathbf{W}} := \widetilde{\mathbf{X}}_{\widetilde{s}} \times_p \widetilde{W} \to \widetilde{W}$ which covers the target map $\widetilde{t} : \widetilde{\mathbf{X}} \to \widetilde{X}$ of the *ep*-groupoid $(\widetilde{X}, \widetilde{\mathbf{X}})$ satisfying:

$$\widetilde{t} \circ \pi_1(\Phi, w) = \widetilde{\pi} \circ \widetilde{\mu}(\Phi, w)$$

for all $(\Phi, w) \in \widetilde{\mathbf{X}}_{\widetilde{s}} \times_{\widetilde{\pi}} \widetilde{W}$, so that the diagram

$$\begin{array}{ccc} \widetilde{\mathbf{X}}_{\widetilde{s}} \times_p \widetilde{W} & \xrightarrow{\widetilde{\mu}} & \widetilde{W} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \widetilde{\pi}} \\ \widetilde{\mathbf{X}} & \xrightarrow{\widetilde{t}} & \widetilde{X} \end{array}$$

commutes. Moreover, $\widetilde{\mu}$ has the following additional properties:

1. $\widetilde{\mu}$ is a surjective local *sc*-diffeomeomorphism and linear in the fibers $\widetilde{W}_x$

2. $\widetilde{\mu}(1_x, w_x) = w_x$ for all $x \in \widetilde{X}$ and $w_x \in \widetilde{W}_x$.

3. $\widetilde{\mu}(\Phi \circ \Psi, w) = \widetilde{\mu}(\Phi, \widetilde{\mu}(\Psi, w))$ for all morphisms $\Phi, \Psi \in \widetilde{\mathbf{X}}$ and $w \in \widetilde{W}$ satisfying $\widetilde{s}(\Psi) = \widetilde{\pi}(w)$ and $\widetilde{t}(\Psi) = \widetilde{s}(\Phi) = \widetilde{\pi}(\widetilde{\mu}(\Psi, w))$.

Proof: Define $\widetilde{\mu}$ as follows:

$$\widetilde{\mu}(\Phi, w) := \widehat{f}_\lambda^{-1} \cdot \mu(\mathbf{f}_\lambda(\Phi), \widehat{f}_\lambda(w)) \tag{6.0.4}$$

for $\Phi \in M(\mathcal{G}_\lambda, \mathcal{G}_{\lambda'})$ and $w \in \widetilde{W}_\lambda$ such that $s(\Phi) = w$. The claim follows from Proposition 4.6.1 and a straightforward diagram chase. ■

Definition 6.0.3. (compare [HWZ2] Definition 8.5) Let $(P : W \to X, \mu)$ and $(P' : W' \to X', \mu')$ be two strong bundles over *ep*-groupoids. A strong bundle map or strong bundle functor between strong bundles over *ep*-groupoids consists of a strong bundle map $\Phi : W \to W'$ covering the *sc*-smooth functor $\phi : X \to X'$ between *ep*-

groupoids having the property that

$$\Phi(\mu(\gamma, w)) = \mu'(\phi(\gamma), \Phi(w)),$$

for all $\gamma \in \mathbf{X}$ and $w \in W$ satisfying $s(\gamma) = P(w)$. ■

Proposition 6.0.8. $W \to X$ and $\widetilde{W} \to \widetilde{X}$ are isomorphic as strong bundles over *ep*-groupoids.

Proof: The claim follows by Equation (6.0.4). ■

We define the section

$$\overline{\partial}_{core,\lambda} : \mathfrak{O}^{core}_{\lambda} \to W^{core}_{\lambda}$$

by

$$\Gamma(exp_u \eta(z), z) \circ \overline{\partial}_{core,\lambda}(v, \eta) \circ \delta(v) = \overline{\partial}_{J,j(v)} exp_u \eta(z)$$

We also define the linearized Cauchy-Riemann operators as follows:

1. If $a_{\{x,y\}}$ is an interior node $\overline{\partial}_0$ is the linear Cauchy-Riemann operator

$$\frac{\partial}{\partial s} + \widehat{J}(0)\frac{\partial}{\partial t} : H^{3,\delta_0}_c(C_a, \mathbb{R}^{2n}) \to H^{2,\delta_0}(C_a, \mathbb{R}^{2n})$$

where C_a and

$$\widehat{J} := T\phi_x \circ J \circ T\phi_x^{-1}$$

2. If $a_{\{x,y\}}$ is a boundary node, then $\overline{\partial}_0$ is the linear Cauchy-Riemann operator

$$\frac{\partial}{\partial s} + \widehat{J}(0)\frac{\partial}{\partial t} : H^{3,\delta_0}_{c,real}(C_a, \mathbb{R}^{2n}) \to H^{2,\delta_0}_{\text{bdry}}(C_a, \mathbb{R}^{2n})$$

We also define the section

$$\text{Fill}\,\overline{\partial}_{neck,\lambda} : E_\lambda \to F_\lambda$$

by

$$(\overline{a}, \overline{\eta}^+, \overline{\eta}^-) \to (\overline{a}, (\overline{\xi}^+, \overline{\xi}^-)$$

where

$$\widehat{\oplus}_a(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = \frac{\partial}{\partial s}(u_a + \eta_a) + \widehat{J}(u_a + \eta_a)\frac{\partial}{\partial t}(u_a + \eta_a) \tag{6.0.5}$$

$$\widehat{\ominus}_a(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = \overline{\partial}_0(\ominus_a(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})) \tag{6.0.6}$$

where $u_a := \oplus_a(\phi \circ u \circ \psi_x, \phi \circ u \circ \psi_y)$ and $\eta_a := \oplus_a(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})$. We also define the section

$$\overline{\partial}_{neck,\lambda} : X_\lambda \to Y_\lambda$$

by

$$(\overline{a}, \overline{\eta}^+, \overline{\eta}^-) \to (\overline{a}, \overline{\xi}^+, \overline{\xi}^-)$$

where

$$\widehat{\oplus}_a(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = \frac{\partial}{\partial s}(u_a + \eta_a) + \widehat{J}(u_a + \eta_a)\frac{\partial}{\partial t}(u_a + \eta_a) \tag{6.0.7}$$

$$\widehat{\ominus}_a(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = 0 \tag{6.0.8}$$

We define the fibered product Cauchy-Riemann section Fill $\overline{\partial}_{J,\lambda}$ as follows

$$\text{Fill}\,\overline{\partial}_{J,\lambda} : \text{Fill}\,\widetilde{\mathfrak{O}}_\lambda \to \widetilde{F}_\lambda$$

$$\text{Fill}\ \overline{\partial}_{J,\lambda} := \overline{\partial}_{core,\lambda} \times \text{Fill}\,\overline{\partial}_{neck,\lambda}$$

Recall the Cauchy-Riemann section $\overline{\partial}_J : Z \to E$

$$\overline{\partial}_J([S, j, M_i, M_b, u]) = [S, j, M_i, M_b, u, \overline{\partial}_{J,j} u]$$

The commutativity of the diagrams below imply Corollary 6.0.2:

$$\begin{array}{ccc} W_\lambda & \xrightarrow{|\Omega|} & W \\ {\scriptstyle \overline{\partial}_J}\uparrow & & \uparrow{\scriptstyle \overline{\partial}_J} \\ \mathfrak{O}_\lambda & \xrightarrow{|p|} & Z \end{array} \qquad \begin{array}{ccc} W_\lambda & \xrightarrow{\widehat{f}_\lambda} & \widetilde{W}_\lambda \\ {\scriptstyle \overline{\partial}_J}\uparrow & & \uparrow{\scriptstyle \overline{\partial}_J} \\ \mathfrak{O}_\lambda & \xrightarrow{f_\lambda} & \widetilde{\mathfrak{O}}_\lambda \end{array}$$

Corollary 6.0.2. Let $\alpha_\lambda : \widetilde{\mathfrak{O}}_\lambda \to Z$ be the map given by $p \circ f_\lambda^{-1} : \mathfrak{O}_\lambda \to Z$. Let $\widehat{\alpha}_\lambda : \widetilde{\mathfrak{O}}_\lambda \to W$ be the chart given by $\Omega \circ \widehat{f}_\lambda^{-1}$. Then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{W}_\lambda & \xrightarrow{\widehat{\alpha}_\lambda} & W \\ {\scriptstyle \overline{\partial}_J}\uparrow & & \uparrow{\scriptstyle \overline{\partial}_J} \\ \widetilde{\mathfrak{O}}_\lambda & \xrightarrow{\alpha_\lambda} & Z \end{array} \quad \blacksquare$$

As a result of the corollary and the fact that the charts α_λ cover, it is enough to check that the section $\overline{\partial}_J : \widetilde{\mathfrak{O}}_\lambda \to \widetilde{W}_\lambda$ is *sc*-Fredholm for every $\lambda \in \Lambda$ in order to verify that $\overline{\partial}_J : Z \to W$ is a *sc*-Fredholm section. Now we verify that Fill $\widetilde{\mathfrak{O}}_\lambda$ is a filling for $\overline{\partial}_J$.

Proposition 6.0.9. The section Fill $\overline{\partial}_{J,\lambda}$: Fill $\widetilde{\mathfrak{O}}_\lambda \to \widetilde{F}_\lambda$ is a filling for the section $\overline{\partial}_J : \widetilde{\mathfrak{O}}_\lambda \to \widetilde{W}_\lambda$.

Proof: Recall the retractions r and $\widehat{R}$ from Remark 6.0.4. Condition 1 follows from observation of the systems (6.0.5) – (6.0.6) and (6.0.7) – (6.0.8).

Now we consider Condition 2. Note that if

$$\widehat{r} \circ \overline{\partial}_J(a, v, \eta, \overline{\eta}^+, \overline{\eta}^-) = \overline{\partial}_J(a, v, \eta, \overline{\eta}^+, \overline{\eta}^-)$$

then

$$\overline{\partial}_0(\ominus_a(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})) = 0$$

By Proposition 5.1.1, it follows that $\ominus_a(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}}) = 0$.

Condition 3 is trivial because it follows by Remark 11.0.1 $\ker Dr(0) = \ker \widehat{r}(0) = \{0\}$. ■

7 Pre-Fredholm Theory for Lagrangian Boundary Conditions

In this section we verify the *sc*-Fredholm estimates for the Cauchy-Riemann operator.

We will use the following theorem to check *sc*-smoothness:

Theorem 7.0.1. (compare Theorem 3.7.1 [HWZ2]) Let U be a relatively open neighbhorhood of 0 in the partial quadrant C of the *sc*-Hilbert space E, and let $\mathbf{f} : U \to F$ be an sc^∞ map into the *sc*-Hilbert space F, which satisfies the following:

(1) At every smooth point $x \in U$, the linearization $D\mathbf{f}(x) : E \to F$ is an *sc*-Fredholm operator and the index does not depend on x.

(2) There is an *sc*-splitting $E = B \oplus X$ in which B is a finite-dimensional subspace of E containing the kernel of $D\mathbf{f}(0)$, and $X \subset C$ such that the following holds for $b \in B \cap U$

small enough: If $(b_j) \subset B \cap U$ is a sequence converging to b and if $(\eta_j) \subset X$ is a sequence which is bounded on level m and satisfying

$$D\mathbf{f}(b_j)\eta_j = y_j + z_j$$

where $y_j \to 0$ in F_m and where the sequence (z_j) is bounded in F_{m+1}, then the sequence (η_j) possesses a convergent subsequence in X_m.

(3) For every $m \geq 0$ and $\epsilon > 0$, the estimate

$$||[D_2\mathbf{f}(b,0) - D_2\mathbf{f}(b,x)]h||_m \leq \epsilon \cdot ||h||_m$$

holds for all $h \in E_m$ and for $b \in B$ small and for $x \in X_{m+1}$ sufficiently small on level m.

Then f is *sc*-Fredholm near 0. ■

Remark 7.0.1. Going forward we will refer to hypothesis (2) as robustness and the bound in hypothesis (3) as the quadratic estimate. ■

We will choose a subtle but powerful coordinate change in order to simplify the verification of the hypotheses of Theorem 7.0.1. The idea is encapsulated in the following:

Proposition 7.0.1. (compare Proposition 1.13 [FH7]) Let X be an M-polyfold, and let Y be a set equipped with a map $g : X \to Y$ which is surjective. Suppose there exists a map $f : Y \to X$ such that $f \circ g : X \to X$ is *sc*-smooth and $g \circ f = Id_Y$. Then Y has a natural M-polyfold structure only depending on g but not on the choice of f provided there exists a map f with the above properties. For this M-polyfold structure, the following holds:

(1) $g : X \to Y$ is *sc*-smooth.

(2) For every $h : Y \to X$ for which $g \circ h = Id_Y$ and $h \circ g$ is sc-smooth the map $h : Y \to h(Y)$ is an sc-diffeomorphism onto a sub-M-polyfold of X. ■

This proposition asserts that any *sc*-smooth section of an M-polyfold retraction gives another choice of coordinates on the M-polyfold. In particular, the set

$$X = \{0\} \times E \bigcup_{a \in B_{\frac{1}{2}}} \{a\} \times H^3(Z_a, \mathbb{R}^{2n})$$

inherits a polyfold structure induced as the image of the sc-smooth retraction $B_{\frac{1}{2}} \times E \xrightarrow{r} B_{\frac{1}{2}} \times E$. One can use the inverse of the $\boxdot$ gluing map to produce section of the map

$$B_{\frac{1}{2}} \times E \to X$$

$$(a, e) \to \oplus_a e$$

Another section, which we will use in the next section, is described below:

Proposition 7.0.2. We define a map

$$\mathfrak{f} : X \to B_{\frac{1}{2}} \times E$$

by

$$\mathfrak{f}(a, u) = (a, \mathfrak{f}^+(a, u), \mathfrak{f}^-(a, u))$$

where:

$$\mathfrak{f}^+(a, u)(s, t) = \beta(s - \frac{R}{2} - 2)u([s, t])$$

$$\mathfrak{f}^-(a,u)(s',t') = \beta(-s' - \frac{R}{2} - 2)u([s',t']')$$

The map $\mathfrak{f}$ satisfies:

I 1. $\oplus \circ \mathfrak{f}(u) = u$ for $u \in X$ by construction.

2. $\mathfrak{f}$ is sc-smooth.

3. For each level m there exists a constant C_m independent of $|a| < \frac{1}{2}$ such that

$$||\mathfrak{f}(\oplus_a(h^+,h^-))||_m \leq C_m||(h^+,h^-)||_m \text{ for all } (h^+,h^-) \in E_m \quad \blacksquare$$

7.1 Pre-Fredholm Estimates at the Node

The verification of the *sc*-Fredholm properties, smoothness, regularizing, robustness and the quadratic estimate for interior nodal behavior follows from the considerations of [HWZ1]. The estimates on the core are classical. The most novel part of the verification of the *sc*-Fredholm properties, the analysis of nodal boundary behavior, is presented in this section. The estimates for cylindrical necks are established in [HWZ1]. Recall the following model spaces:

$$E_{real}(2n) := \mathbb{R}^n \times \{0\} \oplus H^{3,\delta_1^0}_{real}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$$

and

$$F_{\text{bdry}}(2n) := H^{2,\delta_1^0}(\mathbb{R}^{\pm} \times [0,1], \mathbb{R}^{2n})$$

Consider the map $H : B_{\frac{1}{2}} \times E_{real}(2n) \to F_{\text{bdry}}(2n)$, $H(a, h^+, h^-) = (\xi^+, \xi^-)$ defined by:

$$\widehat{\oplus}_a(\xi^+, \xi^-) = \overline{\partial}_J(u_a + h_a) \tag{7.1.1}$$

$$\widehat{\ominus}_a(\xi^+, \xi^-) = \overline{\partial}_0(\ominus_a(h^+, h^-)) \tag{7.1.2}$$

Note it is sufficient to consider the case when (u^+, u^-) is smooth since we can cover Z with charts centered at smooth maps and their images under the $\oplus$ gluing. We also assume that

$$\lim_{s\to\infty} u^+(s,t) = \lim_{s\to-\infty} u^-(s,t) = 0$$

Below we will substitute h_a for $\oplus_a(h^+, h^-)$ for shorthand.

Proposition 7.1.1. The map H is a filling for the Cauchy-Riemann equation. Moreover, H is *sc*-smooth.

Proof: The pair (ξ^+, ξ^-) is defined by the set of equations (7.1.1) and (7.1.2). These equations are equivalent to the following matrix equation:

$$\begin{bmatrix} \xi^+ \\ \xi^- \end{bmatrix} = \frac{1}{\beta_R(s)^2 + (1-\beta_R(s))^2} \begin{bmatrix} \beta_R & -(1-\beta_R) \\ (1-\beta_R) & \beta_R \end{bmatrix} \cdot \begin{bmatrix} \overline{\partial}_J(u_a + h_a) \\ \overline{\partial}_0(\ominus_a(h^+, h^-)) \end{bmatrix}$$

$$= \frac{1}{\beta_R(s)^2 + (1-\beta_R(s))^2} \begin{bmatrix} \beta_R\overline{\partial}_J(u_a + h_a) - (1-\beta_R)\overline{\partial}_0(\ominus_a(h^+, h^-)) \\ (1-\beta_R)\overline{\partial}_J(u_a + h_a) + \beta_R\overline{\partial}_0(\ominus_a(h^+, h^-)) \end{bmatrix}$$

It follows from Appendix 3 that the functions

$$(a, h^+, h^-) \to \frac{(1-\beta_R)}{\beta_R(s)^2 + (1-\beta_R(s))^2} \cdot \overline{\partial}_0\{\ominus_a(h^+, h^-)\}$$

$$(a, h^+, h^-) \to \frac{\beta_R}{\beta_R(s)^2 + (1-\beta_R(s))^2} \cdot \overline{\partial}_0\{\ominus_a(h^+, h^-)\}$$

are *sc*-smooth. Observe the following equalities:

$$\frac{\beta_R}{\beta_R(s)^2 + (1-\beta_R(s))^2} \cdot \overline{\partial}_J(u_a + h_a) = \frac{\beta_R}{\beta_R(s)^2 + (1-\beta_R(s))^2} \cdot \overline{\partial}_J\{\mathfrak{f}^+ \circ (u_a + h_a)\} \quad (7.1.3)$$

$$\frac{(1-\beta_R)}{\beta_R(s)^2 + (1-\beta_R(s))^2} \cdot \overline{\partial}_J(u_a + h_a) = \frac{(1-\beta_R)}{\beta_R(s)^2 + (1-\beta_R(s))^2} \cdot \overline{\partial}_J\{\mathfrak{f}^- \circ (u_a + h_a)\} \quad (7.1.4)$$

From Proposition 7.0.2 and Proposition 11.0.2, it follows that the right hand sides of (7.1.3) and (7.1.4) are *sc*-smooth functions of (a, h^+, h^-). Thus, so are the left hand sides. It follows that the section (η^+, η^-) is *sc*-smooth as well. ■

Lemma 7.1.1. *H is regularizing away from the left and right boundaries. More precisely, if $H(a, h^+, h^-) = (\xi^+, \xi^-) \in (F_{\text{bdry}})_m$, then*

$$(h^+, h^-) \in H^{3+m,\delta_m}((\epsilon, \infty) \times I \times \mathbb{R}^{2n}), H^{3+m,\delta_m}((-\infty, -\epsilon) \times I, \mathbb{R}^{2n})$$

for all $\epsilon > 0$.

Proof: Suppose $(\xi^+, \xi^-) \in (E_{real})_m$. Observe that by Proposition 5.1.1:

$$\frac{1}{C_m} \cdot |\ominus_a (h^+, h^-)|_{C_a, E_m} \leq |\widehat{\ominus}_a(\xi^+, \xi^-)|_{C_a, F_m} \leq C_m \cdot |\ominus_a (h^+, h^-)|_{C_a, E_m} \quad (7.1.5)$$

In addition, we have

$$\overline{\partial}_J h_a = \widehat{\oplus}_a(\xi^+, \xi^-) - \overline{\partial}_J u_a \quad (7.1.6)$$

Since u_a is smooth, $\overline{\partial}_J h_a \in H^{2+m}(Z_a)$. By classical non-linear elliptic estimates (see [A]), it follows that $\oplus_a(h^+, h^-) \in H^{3+m}_{loc}$ away from the left and right boundaries. Consider a family of bump functions $\gamma_\epsilon : \mathbb{R}^+ \to [0, 1]$, for $\epsilon > 0$, which satisfy:

1. $\gamma_\epsilon|[0, \frac{\epsilon}{2}] = 0$

2. $\gamma_\epsilon|[\epsilon, \infty] = 1$

3. $\frac{\partial}{\partial s}\gamma_\epsilon(s) \geq 0$

We suppose $\epsilon > 0$ is given so small that

$$\ominus_a(\gamma_\epsilon h^+, \gamma_\epsilon(-s)h^-) = \ominus_a(h^+, h^-)$$

Set $\gamma := \gamma_\epsilon$. Note that

$$\overline{\partial}_J(\oplus_a(\gamma h^+, \gamma(-s)h^-)) = \overline{\partial}_J \beta\gamma h^+ + \overline{\partial}_J(1-\beta)\gamma(-s)h^-$$

$$= \oplus_a(\gamma' \cdot h^+, \gamma'(-s) \cdot h^-) + \gamma_0 \cdot \overline{\partial}_J \oplus_a (h^+, h^-) \tag{7.1.7}$$

where $\gamma_0 : [0, R] \times I \to [0, 1]$ is defined by

$$\gamma_0 = \gamma \text{ on } [0, \epsilon] \times I$$

and

$$\gamma_0 = \gamma(-s + R) \text{ on } [R - \epsilon, R] \times I$$

and

$$\gamma_0 = 1 \text{ on } [\epsilon, R - \epsilon] \times I$$

From (7.1.6) and (7.1.7) we have

$$\overline{\partial}_J \oplus_a (\gamma h^+, \gamma(-s)h^-) \in H^{2+m}(Z_a) \tag{7.1.8}$$

From classical non-linear elliptic estimates [A], it follows that $\oplus_a(\gamma h^+, \gamma(-s)h^-) \in H^{3+m}(Z_a)$, since $\oplus_a(\gamma h^+, \gamma(-s)h^-)$ vanishes near the boundary. By Proposition 12.0.1 and (7.1.5) and (7.1.8) it follows that $(\gamma h^+, \gamma(-s)h^-) \in E_m^{real}$. ■

Proposition 7.1.2. Given $\varepsilon > 0$, on every level m we have the inequality

$$||(DH_a(h^+, h^-) - DH_a(0,0))(\eta^+, \eta^-)||_m \leq \varepsilon ||(\eta^+, \eta^-)||_m$$

for all $(\eta^+, \eta^-) \in E_m$ all $|a| \leq \frac{1}{2}$, and all $(h^+, h^-) \in E_m$ sufficiently small on level m.

Proof : We closely follow the argument of Proposition 7.7 in [FH1]. Recall the pair of equations which define H:

$$\overline{\partial}_J(u_a + h_a) = \widehat{\oplus}_a(H_a(h^+, h^-))$$

$$\overline{\partial}_0(\ominus_a(h^+, h^-)) = \widehat{\ominus}_a(H_a(h^+, h^-))$$

From the second equation we have:

$$\overline{\partial}_0(\ominus_a(\eta^+, \eta^-)) = \widehat{\ominus}_a(DH_a(h^+, h^-)(\eta^+, \eta^-))$$

The left hand side is independent of (h^+, h^-), so for all $(\eta^+, \eta^-) \in E_m$ and all $(h^+, h^-) \in E_m$, we have

$$\widehat{\ominus}_a(DH_a(h^+, h^-) - DH_a(0,0))(\eta^+, \eta^-)) = 0$$

Consider H_a^+ and H_a^- defined by $H_a = (H_a^+, H_a^-)$. From the equations:

$$\widehat{\oplus}_a(DH_a(h^+,h^-) - DH_a(0,0))(\eta^+,\eta^-) = \widehat{\oplus}_a(D\overline{\partial}_J(u_a+h_a) - D\overline{\partial}_J(u_a))(\eta^+,\eta^-) \quad (7.1.9)$$

$$\widehat{\ominus}_a(DH_a(h^+,h^-) - DH_a(0,0))(\eta^+,\eta^-) = 0 \quad (7.1.10)$$

we have

$$(DH_a^+(h^+,h^-) - DH_a^+(0,0))(\eta^+,\eta^-)$$

$$= \tau(s-\frac{R}{2}) \cdot (D\overline{\partial}_J(\mathfrak{f}^+ \cdot \oplus_a(u^+ + h^+, u^- + h^-) - D\overline{\partial}_J(\mathfrak{f}^+ \cdot \oplus_a(u^+,u^-))(\eta^+,\eta^-) \quad (7.1.11)$$

where τ is defined by

$$\tau(s) := \frac{\beta(s)}{(\beta(s))^2 + (1-\beta(s))^2}$$

Observe that by Theorem 2.26 [HWZ0] and (7.1.9) – (7.1.10), it suffices to show that (7.1.11) is bounded in order to establish the following claim:

Claim Given $\varepsilon > 0$, on every level m we have the inequality

$$||(DH_a^+(h^+,h^-) - DH_a^+(0,0))(\eta^+,\eta^-)||_m \leq \varepsilon ||(\eta^+,\eta^-)||_m$$

for all $(\eta^+,\eta^-) \in E_m$ all $|a| \leq \frac{1}{2}$, and all $(h^+,h^-) \in E_m$ sufficiently small on level m.

The map τ is bounded, therefore by (7.1.11), it is enough to show that the operator norm of

$$D\overline{\partial}_J(\mathfrak{f}^+ \circ \oplus_a(u_0^+ + h^+, u_0^- + h^-)) - D\overline{\partial}_J(\mathfrak{f}^+ \circ \oplus_a(u_0^+, u_0^-))$$

is small for all $|a| \leq \frac{1}{2}$ for all (h^+,h^-) sufficiently small on level m.

Since $\overline{\partial}_J : E_{real} \to F_{\text{bdry}}$ is level-wise smooth, its derivative is smooth and therefore locally Lipschitz. The set $\Sigma := \{\mathfrak{f}^+ \circ \oplus_a(u_0^+, u_0^-) : a \in [0, \frac{1}{2})\}$ is compact on every level. So for every level m we can find an open neighborhood $U_m \subset E_m$ of Σ and a constant D_m such that

$$||D\overline{\partial}_J(w) - D\overline{\partial}_J(w')||_{L(E_m,F_m)} \leq D_m \cdot ||w - w'||_m,\ w, w' \in U_m \tag{7.1.12}$$

Taking advantage of the compactness of Σ, we find for every level m a number $\sigma > 0$ such that $\Sigma + B_m(0, \sigma) \subset U_m$. For every level m we have a constant C_m such that

$$||\mathfrak{f}^+ \circ \oplus(a, u^+ + h^+, u^- + h^-) - \mathfrak{f}^+ \circ \oplus(a, u^+, u^-)||_m \leq C_m \cdot ||(h^+, h^-)||_m < C_m\sigma \tag{7.1.13}$$

for all $a \in [0, \frac{1}{2})$ and $(h^+, h^-) \in B_m(0, \sigma)$. Combining (7.1.12) and (7.1.13), the result follows.

Arguing similarly for DH_a^- establishes the result. ■

Fix a level $m \geq 0$. Consider the family of operators $\Gamma_a(k^+, k^-) : E_{real} \to F_{\text{bdry}}$ parametrized by $(k^+, k^-) \in (E_{real})_m$ defined by:

$$D\overline{\partial}_J(\oplus_a(u^+ + k^+, u^- + k^-))(\oplus_a(\eta^+, \eta^-)) = \widehat{\oplus}_a(\Gamma_a(k^+, k^-)(\eta^+, \eta^-)) \tag{7.1.14}$$

$$\overline{\partial}_0(\ominus_a(\eta^+, \eta^-)) = \widehat{\ominus}_a(\Gamma_a(k^+, k^-)(\eta^+, \eta^-)) \tag{7.1.15}$$

The operators in $\Gamma_a(k^+, k^-)$ are equal to the operators $DH_a(k^+, k^-)$. We partially follow Proposition 7.8 of [FH1]:

Proposition 7.1.3. (Robustness) Let $\{a_j\}_{j=0}^{\infty}$, $a_j \in [0, \frac{1}{2})$, be a sequence of real numbers which converge to $a \in [0, \frac{1}{2})$. Consider a sequence of maps $(x_j^+, x_j^-) \subset F_{\text{bdry}}$ which

converges to 0 on level m. Also suppose $(y_j^+, y_j^-) \subset F_{\text{bdry}}$ is a sequence of maps which is bounded on level $m+1$ and $(\eta_j^+, \eta_j^-) \subset E_{real}$ is a sequence bounded on level m. Suppose a sequence $(k_j^+, k_j^-) \subset E_{real}$ converges on level m to (k^+, k^-). There exists $\varepsilon > 0$ sufficiently small such that if $|(u^+ - k^+, u^- - k^-)|_m \leq \varepsilon$, and

$$\Gamma_{a_j}(k_j^+, k_j^-)(\eta_j^+, \eta_j^-) = (x_j^+, x_j^-) + (y_j^+, y_j^-) \tag{7.1.16}$$

then (η_j) has a subsequence converging on level m on slightly restricted strips.

Proof: Let $u_j := \oplus_{a_j}(u^+, u^-)$, $e_j := \oplus_{a_j}(\eta_j^+, \eta_j^-)$, and $k_j := \oplus_{a_j}(k_j^+, k_j^-)$. Using these abbreviations we can express (7.1.14) – (7.1.15), using (7.1.16), as follows:

$$\frac{\partial e_j}{\partial s} + J(u_j + k_j)\frac{\partial e_j}{\partial t} + (DJ(u_j + k_j)e_j)\frac{\partial (u_j + k_j)}{\partial t} = \widehat{\oplus}_{a_j}(x_j^+, x_j^-) + \widehat{\oplus}_{a_j}(y_j^+, y_j^-) \tag{7.1.17}$$

$$\overline{\partial}_0(\ominus_{a_j}(\eta_j^+, \eta_j^-)) = \widehat{\ominus}_{a_j}(x_j^+, x_j^-) + \widehat{\ominus}_{a_j}(y_j^+, y_j^-) \tag{7.1.18}$$

Take a cut-off function $\tau : \mathbb{R} \to [0, 1]$ satisfying $\tau = 0$ for $s \leq 1$ and $\tau = 1$ for $s \geq 2$. In addition, let $\gamma_{2r}^+(s) = \tau(s - 2r)$ and $\gamma_{2r}^-(s') = \tau(-s' - 2r)$. Replace (η_j^+, η_j^-) with $(\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-)$. Further, define $\overline{e}_{j,r} := \oplus_{a_j}(\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-)$. Then (7.1.17) can be re-expressed:

$$\begin{aligned} &\frac{\partial \overline{e}_{j,r}}{\partial s} + J(u_j + k_j)\frac{\partial \overline{e}_{j,r}}{\partial t} + (DJ(u_j + k_j)\overline{e}_j)\frac{\partial (u_j + k_j)}{\partial t} \\ &\quad = z_{j,r} + \widehat{\oplus}_{a_j}(\gamma_{2r}^+ x_j^+, \gamma_{2r}^- x_j^-) + \widehat{\oplus}_{a_j}(\gamma_{2r}^+ y_j^+, \gamma_{2r}^- y_j^-) \end{aligned} \tag{7.1.19}$$

where

$$z_{j,r} = \frac{\partial \gamma_{2r}^+}{\partial s}(s)\eta_j^+(s,t) - \frac{\partial \gamma_{2r}^-}{\partial s}\eta_j^-(s - R_j, t - \theta_j)$$

After replacing u_j with $\overline{u}_{j,r} = \tau(s - r)\tau(-s + R_j - r)u_j(s,t)$ and k_j with $\overline{k}_{j,r} = \tau(s -$

$r)\tau(-s+R_j-r)k_j(s,t)$, (7.1.19) becomes:

$$\frac{\partial \overline{e}_{j,r}}{\partial s} + J(\overline{u}_{j,r}+\overline{k}_{j,r})\frac{\partial \overline{e}}{\partial t} + (DJ(\overline{u}_{j,r}+\overline{k}_{j,r})\overline{e}_{j,r})\frac{\partial(\overline{u}_{j,r}+\overline{k}_{j,r})}{\partial t}$$

$$= z_{j,r} + \widehat{\oplus}_{a_j}(\gamma_{2r}^{+}x_j^{+}, \gamma_{2r}^{-}x_j^{-}) + \widehat{\oplus}_{a_j}(\gamma_{2r}^{+}y_j^{+}, \gamma_{2r}^{-}y_j^{-}) \tag{7.1.20}$$

and equation (7.1.18) becomes:

$$\overline{\partial}_0(\ominus_{a_j}(\eta_j^{+},\eta_j^{-})) = \widehat{\ominus}_{a_j}(\gamma_{2r}^{+}x_j^{+}, \gamma_{2r}^{-}x_j^{-}) + \widehat{\ominus}_{a_j}(\gamma_{2r}^{+}y_j^{+}, \gamma_{2r}^{-}y_j^{-}) \tag{7.1.21}$$

We have the following critical analytic estimates, which follow from the considerations in Proposition 5.1.1:

Proposition 7.1.4. Suppose $(\delta_1^m)_{m=0}^{\infty} \subset (0,\pi)$. The Cauchy-Riemann operator

$$\overline{\partial}_0 : H_{c,real}^{3,\delta_1^m}(C_a,\mathbb{R}^{2n}) \to H_{\text{bdry}}^{2,\delta_1^m}(C_a,\mathbb{R}^{2n})$$

is an *sc*-isomorphism, and for every $m \geq 0$ there exists a constant $C_m > 0$, which is independent of a, such that

$$\frac{1}{C_m}\big|\xi\big|_{H_{c,\,real}^{3+m,\delta_m}} \leq \big|\overline{\partial}_0\xi\big|_{H_{\text{bdry}}^{2+m,\delta_m}} \leq C_m\big|\xi\big|_{H_{c,\,real}^{3+m,\delta_m}}$$

for all $\xi \in H_{c,\,real}^{3+m,\delta_m}(C_a,\mathbb{R}^{2n})$. ■

Consider the space $H_{c,real}^{3+m,\delta_1^m}(C_a,\mathbb{R}^{2n})'$ which as a set agrees with $H_{c,real}^{3+m,\delta_1^m}(C_a,\mathbb{R}^{2n})$, however we equip $H_{c,real}^{3+m,\delta_1^m}(C_a,\mathbb{R}^{2n})'$ with a new norm given by

$$||u||'_{3+m,\delta_1^m} := \Big(\sum_{|\alpha|\leq m+3}\int_{\mathbb{R}\times[0,1]} |D^{\alpha}u|^2 \cdot e^{2\delta_1^m|s|}dsdt\Big)^{\frac{1}{2}}$$

Also consider the space $H^{2+m,\delta_1^m}_{\text{bdry}}(C_a,\mathbb{R}^{2n})'$ which as a set agrees with $H^{2+m,\delta_1^m}_{\text{bdry}}(C_a,\mathbb{R}^{2n})$, however we equip $H^{2+m,\delta_1^m}_{c,real}(C_a,\mathbb{R}^{2n})'$ with a new norm:

$$\big|\big|u\big|\big|'_{2+m,\delta_1^m} := \Big(\sum_{|\alpha|\leq m+2}\int_{\mathbb{R}\times[0,1]}|D^\alpha u|^2\cdot e^{2\delta_1^m|s|}dsdt\Big)^{\frac{1}{2}}$$

Proposition 7.1.5. For $\delta_1^m\in(0,\pi)$, the Cauchy-Riemann operator

$$\overline{\partial}_0 : H^{3,\delta_1^m}_{c,real}(C_a,\mathbb{R}^{2n})'\to H^{2,\delta_1^m}_{\text{bdry}}(C_a,\mathbb{R}^{2n})' \tag{7.1.22}$$

is an *sc*-isomorphism, and for every $m\geq 0$ there exists a constant $C_m>0$, which is independent of a, such that

$$\frac{1}{C_m}\big|\xi\big|_{H^{3+m,\delta_1^m}_{c,real}(C_a,\mathbb{R}^{2n})'}\leq\big|\overline{\partial}_0\xi\big|_{H^{2+m,\delta_1^m}_{\text{bdry}}(C_a,\mathbb{R}^{2n})'}\leq C_m\big|\xi\big|_{H^{3+m,\delta_1^m}_{c,real}(C_a,\mathbb{R}^{2n})'} \tag{7.1.23}$$

for all $\xi\in H^{3+m,\delta_1^m}_{c,real}(C_a,\mathbb{R}^{2n})'$.

Proof: As sets of maps, we have

$$H^{3,\delta_1^m}_{c,\,real}(C_a,\mathbb{R}^{2n}) = H^{3,\delta_1^m}_{c,\,real}(C_a,\mathbb{R}^{2n})'$$

$$H^{2,\delta_1^m}_{\text{bdry}}(C_a,\mathbb{R}^{2n}) = H^{2,\delta_1^m}_{\text{bdry}}(C_a,\mathbb{R}^{2n})'$$

Thus, isomorphism (7.1.22) follows from Proposition 7.1.4, however must verify equation (7.1.23). Consider the map

$$\text{Shift}_{\frac{R}{2}} : H^{3,\delta_0}_{c,\,real}(C_a,\mathbb{R}^{2n})\to H^{3,\delta_0}_{c,\,real}(C_a,\mathbb{R}^{2n})'$$

$$u(\cdot)\to u(\cdot+\frac{R}{2})$$

Observe:

$$\big|\big|u\big|\big|_{3+m,\delta_1^m} := \Big(\sum_{|\alpha|\leq 3+m} \int_{\mathbb{R}\times[0,1]} \Big| D^\alpha u(s,t) \Big|^2 e^{2\delta_1^m \left| s-\frac{R}{2} \right|} ds\, dt \Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{|\alpha|\leq 3+m} \int_{\mathbb{R}\times[0,1]} \Big| D^\alpha (u(s+\frac{R}{2},t)) \Big|^2 e^{2\delta_1^m |s|} ds\, dt \Big)^{\frac{1}{2}}$$

$$= \Big|\Big| \mathrm{Shift}_{\frac{R}{2}} u \Big|\Big|'_{3+m,\delta_1^m}$$

We have a similar calculation for $H^{2,\delta_0}_{\mathrm{bdry}}(C_a,\mathbb{R}^{2n})$ and $H^{2,\delta_0}_{\mathrm{bdry}}(C_a,\mathbb{R}^{2n})'$. The inequalities (7.1.23) follow from the fact that $\mathrm{Shift}_{\frac{R}{2}}$ is an isometry and the identity

$$\overline{\partial}_0 \circ \mathrm{Shift}_{\frac{R}{2}} = \mathrm{Shift}_{\frac{R}{2}} \circ \overline{\partial}_0 \quad \blacksquare$$

Consider the function $\alpha_{j,r} := \tau(s-r+2)\cdot\tau(-s+R_j-r+2)$. Note that α_j is identically equal to 1 on the support of $\tau(s-r)\cdot\tau(-s+R_j-r)$.

Proposition 7.1.6. (compare Proposition 7.9 in [FHWZ])

(1) Given $\widetilde{\epsilon}>0$, there exists $r>0$ such that for all large j the following holds:

$$\Big|\Big| \alpha_{j,r}\cdot (J(\overline{u}_{j,r}+\overline{k}_{j,r})-i)\cdot \frac{\partial}{\partial t} h \Big|\Big|'_{m+2} < \widetilde{\epsilon}\, \big|\big|h\big|\big|'_{m+3} \text{ for } h\in (E_{real})_m$$

(2) The sequence $(v_j^+, v_j^-)\in F_{m+1}$ defined by

$$\widehat{\ominus}_{a_j}(v_j^+,v_j^-)=0, \quad \widehat{\oplus}_{a_j}(v_j^+,v_j^-) = -(DJ(\overline{u}_{j,r}+\overline{k}_{j,r})\overline{e}_{j,r})\frac{\partial(\overline{u}_{j,r}+\overline{k}_{j,r})}{\partial t}$$

is bounded in F_{m+1}.

(3) The sequence $(w_j^+, w_j^-) \in F_{m+1}$ defined by

$$\widehat{\ominus}_{a_j}(w_j^+, w_j^-) = 0, \qquad \widehat{\oplus}_{a_j}(w_j^+, w_j^-) = z_j$$

is bounded on level $m+1$.

Proof: The sequence (k_j^+, k_j^-) limits to some $(k^+, k^-) \in (E_{real})_m$. We will make use of the abbreviations:

$$(m_j^+, m_j^-) := (k_j^+ - k_{j,\infty}, k_j^- - k_{j,\infty}), \quad \overline{m}_j := \tau(s-r)\cdot\tau(-s+R_j-r) \oplus_{a_j} (m_j^+, m_j^-)$$

and

$$(v^+, v^-) := (k^+ - k_\infty, k^- - k_\infty), \quad \overline{v}_j := \tau(s-r)\cdot\tau(-s+R_j-r) \oplus_{a_j} (v^+, v^-)$$

Set $\Sigma_k := [r-2, R_j - r + 2] \times I$. Given $\epsilon > 0$, first we will show for r, j and $\frac{1}{\varepsilon}$ sufficiently large we have:

$$\left| D^\beta (J(\overline{u}_{j,r} + \overline{k}_{j,r}) - J(k_\infty))(z) \right| < \epsilon \tag{7.1.24}$$

for indices β satisfying $|\beta| \leq 2 + m$ and $z \in \Sigma_k$. Observe:

$$\left| \overline{u}_{j,r} + \overline{k}_{j,r} - k_\infty \right|_{H^{3+m}(\Sigma_k, \mathbb{R}^{2n})}$$

$$= \left| \overline{u}_{j,r} + \overline{m}_{j,r} + \tau(s-r)\cdot\tau(-s+R_j-r)k_{j,\infty} - k_\infty \right|_{H^{3+m}(\Sigma_k, \mathbb{R}^{2n})}$$

$$\leq \left| \overline{u}_{j,r} + \overline{m}_{j,r} \right|_{H^{3+m}(\Sigma_k, \mathbb{R}^{2n})} + \left| \tau(s-r)\cdot\tau(-s+R_j-r)k_{j,\infty} - k_\infty \right|_{H^{3+m}(\Sigma_k, \mathbb{R}^{2n})}$$

$$\leq \left| \overline{u}_{j,r} + \overline{v}_{j,r} \right|_{H^{3+m}(\Sigma_k, \mathbb{R}^{2n})} + \left| \overline{m}_{j,r} - \overline{v}_{j,r} \right|_{H^{3+m}(\Sigma_k, \mathbb{R}^{2n})}$$

$$+\Big|\tau(s-r)\cdot\tau(-s+R_j-r)k_{j,\,\infty}-k_\infty\Big|_{H^{3+m}(\Sigma_k,\mathbb{R}^{2n})}$$

$$=I(r,j)+II(r,j)+III(r,j)$$

For any $(\zeta^+,\zeta^-)\in(E_{real})_m$, since τ is bounded we have the inequality

$$\Big(\Big|\Big|\tau(s-r)\cdot\tau(-s+R_j-r)\oplus_{a_j}(\zeta^+,\zeta^-)\Big|\Big|'_{3+m,\delta_1^m}\Big)^2$$

$$\leq C\Big(\sum_{|\alpha|\leq 3+m}\int_{[r,\infty)\times[0,1]}|D^\alpha\zeta^+|^2\cdot e^{2\delta_1^m\cdot|s|}ds\,dt+\sum_{|\alpha|\leq 3+m}\int_{(-\infty,-r]\times[0,1]}|D^\alpha\zeta^-|^2\cdot e^{2\delta_1^m\cdot|s|}ds\,dt\Big) \tag{7.1.25}$$

Given $\epsilon'>0$, we can choose $r>0$ sufficiently large such that the right hand side of inequality (7.1.25) is less than ϵ' for any fixed (ζ^+,ζ^-). In particular, this means we can choose $I(r,j)<\epsilon'$.

Since $k_j\to k$ on level m, it follows that for j sufficiently large we have $II(r,j)<\epsilon'$.

As $j\to\infty$,

$$III(r,j)\to|\tau(s-r)\cdot\tau(-s+R_j-r)k_\infty-k_\infty|_{H^{3+m}(\Sigma_k,\mathbb{R}^{2n})}$$

Observe that

$$|\tau(s-r)\cdot\tau(-s+R_j-r)k_\infty-k_\infty|_{H^{3+m}(\Sigma_k,\mathbb{R}^{2n})}\leq 4\cdot|k_\infty|$$

By the Sobolev embedding theorem, it follows that we can choose $\varepsilon>0$ sufficiently small such that $III(r,j)\leq\epsilon'$.

By the Sobolev embedding theorem, for any $\xi \in H^{3+m}(\Sigma_k, \mathbb{R}^{2n})$, we have the following inequality for fixed r:

$$|\xi|_{C^{m+1}(\Sigma_k,\mathbb{R}^{2n})} \leq C|\xi|_{H^{3+m}(\Sigma_k,\mathbb{R}^{2n})}$$

Having made such a choice of r for $\epsilon' > 0$, for $|\beta| \leq 2+m$, we have the following by the mean-value theorem and the smoothness of J:

$$\left|D^{\beta}\Big(J(\overline{u}_{j,r}+\overline{k}_{j,r})-J(k_{\infty})\Big)\right|_{C^0(\Sigma_k)} \leq M \cdot \left|\overline{u}_{j,r}+\overline{k}_{j,r}-k_{\infty}\right|_{C^{m+1}(\Sigma_k)} \leq 3M \cdot C \cdot \epsilon' \quad (7.1.26)$$

We choose $\epsilon' < \frac{\epsilon}{3M \cdot C}$. Given this choice of ϵ', (7.1.26) implies (7.1.24). Observe that the squared norm

$$\left(\left|\left|(J(\overline{u}_{j,r}+(\overline{k}_{j,r})-J(k_{\infty})\Big)\frac{\partial}{\partial t}h\right|\right|'_{\Sigma_k}\right)^2$$

is bounded above by integrals of the following type up to a multiplicative constant:

$$\int_{\Sigma_k}\left|D^{\beta}\Big(J(\overline{u}_{j,r}+(\overline{k}_{j,r}-J(k_{\infty})\Big)\right|^2 \cdot \left|D^{\gamma}\left(\frac{\partial}{\partial t}u\right)\right|^2 \cdot e^{2\delta_1^m|s|}ds\,dt$$

where $|\beta|+|\gamma| \leq 2+m$. Since the functions α_j are bounded, we can choose $r > 0$, j and $\frac{1}{\varepsilon}$ sufficiently large such that

$$\left|\left|\alpha_j \cdot (J(\overline{u}_j+\overline{k}_j)-J(k_{\infty}))\cdot\frac{\partial}{\partial t}h\right|\right|'_{m+2} \leq \frac{1}{2}\cdot\widetilde{\epsilon}\left|\left|h\right|\right|'_{m+3} \text{ for } h \in (E_{real})_m \qquad (7.1.27)$$

Choosing ε sufficiently small, we may force the operator norm of $J(k_\infty)-i$ small enough such that we have

$$||(J(k_{\infty})-i)\cdot\frac{\partial}{\partial t}h||'_{\Sigma_k} \leq \frac{1}{2}\cdot\widetilde{\epsilon}\,||h||'_{m+3} \qquad (7.1.28)$$

Since the difference $J(\overline{u}_j+\overline{k}_j)-i$ is supported on Σ_k, Claim (1) follows by combining

(7.1.27) and (7.1.28).

Claims (2) and (3) follow from the estimates on the $\boxdot$ gluing see Appendix 4. ■

Proposition 7.1.6 parts (2)-(3) allows for re-expression of (7.1.20)-(7.1.21) as follows:

$$\frac{\partial \overline{e}_j}{\partial s} + J(\overline{u}_j + \overline{k}_j)\frac{\partial \overline{e}_j}{\partial t} = \widehat{\oplus}_{a_j}(q_j^+, q_j^-) + \widehat{\oplus}_{a_j}(p_j^+, p_j^-) \tag{7.1.29}$$

$$\overline{\partial}_0(\ominus_{a_j}(\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-)) = \widehat{\ominus}_{a_j}(q_j^+, q_j^-) + \widehat{\ominus}_{a_j}(p_j^+, p_j^-) \tag{7.1.30}$$

where the sequence (q_j^+, q_j^-) converges to 0 in F_m and the sequence (p_j^+, p_j^-) is bounded in F_{m+1}. We fix the abbreviations

$$\overline{\partial}_j \cdot := \frac{\partial}{\partial s}\cdot + J(\overline{u}_j + \overline{k}_j)\frac{\partial}{\partial t}\cdot = \overline{\partial}_0 \cdot + (J(\overline{u}_j + \overline{k}_j) - i)\frac{\partial}{\partial t}\cdot$$

$$\overline{\partial}'_j \cdot := \overline{\partial}_0 \cdot + \alpha_j \cdot \Big(J(\overline{u}_j + \overline{k}_j) - i\Big)\frac{\partial}{\partial t}\cdot$$

Observe that

$$\overline{\partial}_j f = \overline{\partial}'_j f \tag{7.1.31}$$

for maps $f \in H^3(\mathbb{R}\times[0,1], \mathbb{R}^{2n})$ supported on $\Sigma'_k := [r, R_j - r]\times I$. In particular, the equality (7.1.31) holds for $\overline{e}_j$. We choose $\widetilde{\epsilon} < \frac{1}{2C_m}$, where C_m is the constant appearing in Proposition 5.1.1. Then by Proposition 7.1.6 (i), we have

$$\frac{1}{2C_m}|\xi|'_m \leq |\overline{\partial}'_j\xi|'_{m-1} \leq 2C_m|\xi|'_m, \quad \text{for all } \xi \in E_m^{real} \tag{7.1.32}$$

for j sufficiently large. In particular $\overline{\partial}'_j$ is an isomorphism. Now consider the system:

$$\overline{\partial}'_j \mathfrak{f}^+ \circ \overline{e}_j = \zeta_j^+ + \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-) + \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-) \tag{7.1.33}$$

$$\overline{\partial}_j' \mathfrak{f}^- \circ \overline{e}_j = \zeta_j^- + \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-) + \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-) \tag{7.1.34}$$

$$\overline{\partial}_0 \ominus_{a_j} (\gamma_{2r}^+ \eta_j^+, \gamma_{2r} \eta_j^-) = \widehat{\ominus}_{a_j}(q_j^+, q_j^-) + \widehat{\ominus}_{a_j}(p_j^+, p_j^-) \tag{7.1.35}$$

where

$$\zeta_j^+ := \frac{\partial \beta(s - \frac{R}{2} - 2)}{\partial s} \overline{e}_j, \qquad \zeta_j^- := -\frac{\partial \beta(s - \frac{R}{2} - 2)}{\partial s} \overline{e}_j$$

Note that the sequence $\zeta_j^{\pm}$ is bounded in F_{m+1} by Proposition 11.0.2. Also observe that by construction, the first two equations are valued on $\mathbb{R}^+ \times [0,1]$ and $\mathbb{R}^- \times [0,1]$, respectively. We re-write the system (7.1.33)-(7.1.35) as:

$$\overline{\partial}_j' \mathfrak{f}^+ \circ \overline{e}_j = \widetilde{q}_j^+ + \widetilde{p}_j^+ \tag{7.1.36}$$

$$\overline{\partial}_j' \mathfrak{f}^- \circ \overline{e}_j = \widetilde{q}_j^- + \widetilde{p}_j^- \tag{7.1.37}$$

$$\overline{\partial}_0 \ominus_{a_j} (\gamma_{2r}^+ \eta_j^+, \gamma_{2r}^- \eta_j^-) = \widehat{\ominus}_{a_j}(q_j^+, q_j^-) + \widehat{\ominus}_{a_j}(p_j^+, p_j^-) \tag{7.1.38}$$

where

$$\widetilde{q}_j^{\pm} := \zeta_j^{\pm} + \mathfrak{f}^{\pm} \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-), \quad \text{and} \quad \widetilde{p}_j^{\pm} := \mathfrak{f}^{\pm} \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-)$$

The terms $(\widetilde{q}_j^+, \widetilde{q}_j^-)$, $(\widetilde{p}_j^+, \widetilde{p}_j^-)$ vanish near the left and right boundaries. Therefore, we can consider (7.1.36) and (7.1.37) as equations on $\mathbb{R} \times [0,1]$.

Now consider the pair of solutions $(\xi_j^+, \xi_j^-) \in H^{3+m,\delta_1^m}(\mathbb{R} \times [0,1], \mathbb{R}^2) \oplus H^{3+m,\delta_1^m}(\mathbb{R} \times [0,1], \mathbb{R}^2)$ satisfying:

$$\overline{\partial}_j' \xi_j^+ = \widetilde{p}_j^+$$

$$\overline{\partial}_j' \xi_j^- = \widetilde{p}_j^-$$

By Equation (7.1.32), we have the chain of inequalities:

$$|(p_j^+, p_j^-)|_{F_m} \geq C'_m |(\mathfrak{f}^+ \circ \oplus_{a_j}(p_j^+, p_j^-), \mathfrak{f}^- \circ \oplus_{a_j}(p_j^+, p_j^-))|_{F_m}$$

$$= C'_m |(\overline{\partial}_j \xi_j^+, \overline{\partial}_j \xi_j^-)|_{F_m} \geq C'_m \cdot C_m (|\xi_j^+|'_m + |\xi_j^-|'_m)$$

where C'_m depends only on m. In particular, $|\xi_j^{\pm}|'_m \to 0$. Now consider the pair of solutions $(\mu_j^+, \mu_j^-) \in H^{3+(m+1),\delta_m}(\mathbb{R} \times [0,1], \mathbb{R}^2) \oplus H^{3+(m+1),\delta_m}(\mathbb{R} \times [0,1], \mathbb{R}^2)$ which satisfy

$$\overline{\partial}'_j \mu_j^+ = \widetilde{q}_j^+$$

$$\overline{\partial}'_j \mu_j^- = \widetilde{q}_j^-$$

Then by Equation (7.1.32), we have the inequality

$$|(\widetilde{q}_j^+, \widetilde{q}_j^-)|_{F_{m+1}} = |(\overline{\partial}'_j \mu_j^+, \overline{\partial}'_j \mu_j^-)|_{F_{m+1}} \geq C_{m+1} \cdot (|\mu_j^+|'_{m+1} + |\mu_j^-|'_{m+1})$$

Thus the sequence $|\mu_j^+|'_{m+1} + |\mu_j^-|'_{m+1}$ is bounded on level $m+1$. Since $\overline{\partial}'_j$ is an isomorphism we have the equalities (of functions on half-strips)

$$\xi_j^+ + \mu_j^+ = \mathfrak{f}^+ \circ \overline{e}_j$$

$$\xi_j^- + \mu_j^- = \mathfrak{f}^- \circ \overline{e}_j$$

We make the replacement:

$$(\xi_j^+, \xi_j^-) \to (\gamma_{2r-2}^+(s) \cdot \xi_j^+, \gamma_{2r-2}^-(s') \cdot \xi_j^-)$$

$$(\mu_j^+, \mu_j^-) \to (\gamma_{2r-2}^+(s) \cdot \mu_j^+, \gamma_{2r-2}^-(s') \cdot \mu_j^-)$$

Thus we have $(\xi_j^+, \xi_j^-) \in E_{m+1}$ and $(\mu_j^+, \mu_j^-) \in E_m$ and

$$\oplus_{a_j}(\xi_j^+ + \mu_j^+, \xi_j^- + \mu_j^-) = \oplus_{a_j}(\mathfrak{f}^+ \circ \overline{e}_j, \mathfrak{f}^- \circ \overline{e}_j) = \overline{e}_j$$

Now consider the solutions $\alpha_j \in H_c^{3+m,\delta_m}(C_a, \mathbb{R}^{2n})$, guaranteed to exist by Proposition 5.1.1 defined as follows:

$$\overline{\partial}_0 \alpha_j = \widehat{\ominus}_{a_j}(p_j^+, p_j^-)$$

By Proposition 12.0.1, there exists a unique pair $(\alpha_j^+, \alpha_j^-) \in E_m$ such that

$$\boxdot_{a_j}(\alpha_j^+, \alpha_j^-) = (0, \alpha_j)$$

By Equation (7.1.32) we have

$$C \cdot C' |(p_j^+, p_j^-)|_{F_m} \geq C |(1 - \widehat{\pi}_{a_j})(p_j^+, p_j^-)|_{F_m} \geq | \boxdot_{a_j} \{1 - \widehat{\pi}_{a_j}(p_j^+, p_j^-)\}|_{\widehat{G}_m} = |(0, \overline{\partial}_0 \alpha_j)|_{\widehat{G}_m}$$

$$\geq C'' |(0, \alpha_j)|_{G_m} = C'' | \boxdot (\alpha_j^+, \alpha_j^-)|_{G_m} \geq \frac{C''}{C'''} |(\alpha_j^+, \alpha_j^-)|_{E_m} \tag{7.1.39}$$

where C, C', and C'' depend only on m. In particular, we have $|(\alpha_j^+, \alpha_j^-)|_m \to 0$ on level m. For shorthand, we set $\widetilde{\alpha}_j := (\alpha_j^+, \alpha_j^-)$. By Proposition 12.0.1, there exists a unique pair $(\beta_j^+, \beta_j^-) \in E_{m+1}$ such that

$$\boxdot_{a_j}(\beta_j^+, \beta_j^-) = (0, \beta_j)$$

For shorthand we set $\widetilde{\beta}_j := (\beta_j^+, \beta_j^-)$. Similarly to Equation (7.1.39), we have an inequality:

$$|(q_j^+, q_j^-)|_{F_{m+1}} \geq C_{m+1} \cdot |(\beta_j^+, \beta_j^-)|_{E_{m+1}}$$

where C_{m+1} depends only on $m+1$. Since $\overline{\partial}_0$ is an isomorphism we have the equality:

$$\alpha_j + \beta_j = \widehat{\ominus}_{a_j}(\gamma_{2r}^+ \eta_j^+, \gamma_{2r}^- \eta_j^-)$$

We introduce the new sequences:

$$\widetilde{\xi}_j := \pi_{a_j}(\xi_j)$$

$$\widetilde{\mu}_j := \pi_{a_j}(\mu_j)$$

Then we have:

$$\oplus_{a_j}(\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j) = \overline{e}_j = \oplus_{a_j}(\gamma_{2r}^+ \mu_j^+, \gamma_{2r}^- \mu_j^-)$$

$$\ominus_{a_j}(\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j) = \ominus_{a_j}(\gamma_{2r}^+ \mu_j^+, \gamma_{2r}^- \mu_j^-)$$

By Proposition 12.0.1, we have:

$$\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j = (\gamma_{2r}^+ \eta_j^+, \gamma_{2r}^- \eta_j^-)$$

It follows that there exists a subsequence such that $\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j$ converges on level m. The claim follows. ■

Remark 7.1.1. Arguing similarly, we can verify that the Cauchy-Riemann operator on cylindrical necks is *sc*-smooth, satisfies the quadratic estimate, is regularizing away from the left and right boundaries and satisfies robustness away from the left and right boundaries. ■

Remark 7.1.2. Observe that we can control the size of domain on which the sequence η_j converges by adjusting the value of r which appeared in Equation (7.1.19). ■

7.2 Classical Estimates

Recall the definition of $\mathfrak{O}_\lambda^{core}$:

$$\mathfrak{O}_\lambda^{core} := \{(v,\eta) \in V_\lambda \times H^3\Gamma(u|^*_{S_{core}}TQ) \ : \exists\, (a,v,\widetilde{\eta}) \in \mathfrak{O}_\lambda \text{ such that } \widetilde{\eta}|_{S_{core}} = \eta\}$$

and of $\mathcal{O}_\lambda^{core}$:

$$\mathcal{O}_\lambda^{core} := \{\eta \in H^3\Gamma(u|^*_{S_{core}}TQ) \ : \ \eta(z) \in H_{u(z)} \text{ for } z \in \Sigma\}$$

Also, recall the section $\overline{\partial}_{core}$ over $\mathfrak{O}_\lambda^{core}$:

$$\overline{\partial}_{core} : \mathfrak{O}_\lambda^{core} \to W_\lambda^{core}$$

$$\overline{\partial}_{core}(v,\eta) := \Theta(u, exp_u(\eta)) \circ \overline{\partial}_{J,j(v)}(exp_u(\eta)) \circ \tau(v)$$

where

$$\Theta(z,z') = \Gamma(z,z')^{-1}, \quad \text{and} \quad \tau(v) = \delta(v)^{-1}$$

Proposition 7.2.1. The map $\overline{\partial}_{core}$ is *sc*-smooth and regularizing away from the left and right boundaries.

Proof: The claim with regards to smoothness follows by the smoothness of exponentiation ([E]). The regularizing property follows by classical elliptic theory (see [A]). ■

By the mean value theorem we have:

Proposition 7.2.2. Quadratic Estimate in the Classical Case with Boundary

(compare Proposition 4.25 [HWZ2]) Fix a level $m \geq 0$ and $\epsilon > 0$. Then

$$||[D_2\overline{\partial}_{core}(v,\eta) - D_2\overline{\partial}_{core}(v,0)]h||_{(\mathcal{F}^{core}_\lambda)_m} \leq \epsilon \cdot ||h||_{(\mathcal{O}^{core}_\lambda)_m}$$

for all $h \in (\mathcal{O}^{core}_\lambda)_m$ and for $(v,\eta) \in (V_\lambda \times \mathfrak{O}^{core}_\lambda)_m$ sufficiently close to $(0,0) \in (V_\lambda \times \mathfrak{O}^{core}_\lambda)_m$ and $\eta \in (\mathcal{F}^{core}_\lambda)_m$. ■

Proposition 7.2.3. Robustness in the Classical case with Boundary (compare Proposition 4.23 [HWZ2]) Let K be a finite-dimensional *sc*-subspace of $\mathcal{O}^{core}_\lambda$ and consider $(v_0,\eta_0) \in (\mathfrak{O}^{core}_\lambda)_\infty$ with $\eta_0 \in K$. We consider a sequence $(v_j,\eta_j) \in (\mathfrak{O}_\lambda)^{core}_\infty$ with $k_j \in K$ and converging to (v_0,η_0). If $(h_j) \in (\mathcal{O}^{core}_\lambda)_m$ is a bounded sequence satisfying

$$D_2\overline{\partial}_{core}(v_j,\eta_j)h_j = y_j + z_j \tag{7.2.1}$$

where $(y_j) \subset (\mathcal{F}^{core}_\lambda)_m$ converges to 0 and where the sequence (z_j) is bounded in $(\mathcal{F}^{core}_\lambda)_{m+1}$, then h_j converges in $(\mathcal{O}^{core}_\lambda)_m$ away from a small neighborhood of the neck.

Proof: The argument closely follows that of Proposition 4.23 [HWZ2]. Following the argument of Proposition 4.23 [HWZ2] we may derive the following expression:

$$D_2\overline{\partial}_{core}(v_j,\eta_j)h_j = (\nabla_{(T\exp_u(\eta_j)h_j,0)}\Theta)[\exp_u(\eta_j),u] \circ \Gamma(\exp_u(\eta_j),u) \circ \overline{\partial}_{J,j(v_j)}(v_j,\eta_j)$$

$$+ \ \Theta(\exp_u(\eta_j),u) \circ (\nabla_{h_j}\overline{\partial}_{J,j(v_j)})(\exp_u(\eta_j)) \circ \tau(v) \tag{7.2.2}$$

Consider the sequence of linear operators

$$L_j : E_m \to F_m$$

given by

$$h \to (\nabla_{(T\exp_u(\eta_j)h,0)}\Theta)[\exp_u(\eta_j), u] \circ \Gamma(\exp_u(\eta_j), u) \circ \overline{\partial}_{J,j(v_j)}(v_j, \eta_j)$$

These operators are sc^+ so we can re-write (7.2.1) as

$$(\nabla_{h_j}\overline{\partial}_{J,j(v_j)})(\exp_u(\eta_j)) = \Gamma(\exp_u(\eta_j), u)(y_j + z_j^*)\tau(v_j)(z) \tag{7.2.3}$$

where $y_j \to 0$ in H^{m+2} and $z_j^* := z_j - L_j(h_j)$ is bounded on H^{m+3}. Now we limit our attention to a small neighborhood of a point $z_0 \in \partial S$ away from the neck. (The case of z_0 in the interior is handled in Proposition 4.23 [HWZ2].) Choose a half-disk neighborhood $\mathbb{HD}$ of z_0 so small that the images $\exp_u(\eta_j)(\mathbb{HD})$ are contained in a fixed open neighborhood, $U(u(z_0))$, of $u(z_0)$. We fix a chart $\psi : U(u(z_0)) \subset Q \to \mathbb{R}^{2n}$ taking $u(z_0)$ to 0. We make the following substitutions and abbreviations:

$$\text{we denote } T\psi(\exp_u(\eta_j))h_j(z) \text{ by } h_j^*(z)$$

$$\text{we denote } T\psi(\exp_u(\eta_j)(z)) \circ \Gamma(\exp_u(\eta_j), u)y_j \circ \tau(v_j)(z) \text{ by } y_j^*(z)$$

$$\text{we denote } T\psi(\exp_u(\eta_j)(z)) \circ \Gamma(\exp_u(\eta_j), u)z_j^* \circ \tau(v_j)(z) \text{ by } z_j^{**}(z)$$

$$\text{we denote } T\psi \circ J(\psi^{-1}(z)) \circ T\psi^{-1}(z) \text{ by } J^*(z)$$

In local coordinates, (7.2.3) is given by

$$Dh_j^*(z) + J^*(z) \circ Dh_j^*(z) \circ j(v_j) + A(z)(h_j^*(z), \cdot) = y_j^* + z_j^{**} \tag{7.2.4}$$

where A is a compact bilinear operator. Fix an open neighborhood H of the point 1 in the disk, D, of radius 1 in $\mathbb{C}$ centered at the origin. Choose a smooth family of

biholomorphisms $\phi_v : (H, i) \to (\mathbb{HD}, j(v))$ with the property

$$\phi_v(1) = z_0$$

parametrized for v ranging in a compact neighborhood, C, of 0, restricting H and $\mathbb{HD}$ if necessary. We assume that ϕ_v factors as a composition:

$$(H, i) \xrightarrow{\phi|_D} (H', i) \xrightarrow{\widetilde{\phi}_v} (\mathbb{HD}, j(v))$$

where $\phi : D \to \mathbb{H}^+$ is a biholomorphism from the disk to the upper half-plane and $H' \subset \mathbb{H}^+$ is a neighborhood of 0. We also stipulate that $\phi(1) = 0$ and $\widetilde{\phi}_v(0) = z_0$ for all $v \in C$. We make the following substitutions and abbreviations:

$$\text{we denote } h_j^* \circ \phi_{v_j}(x) \text{ by } h_j^{**}(x)$$

$$\text{we denote } y_j^* \circ T\phi_{v_j}(x) \text{ by } y_j^{**}(x)$$

$$\text{we denote } z_j^{**} \circ T\phi_{v_j}(x) \text{ by } z_j^{***}(x)$$

$$\text{we denote } J^*(\phi_{v_j}(x)) \text{ by } J_j(x)$$

$$\text{we denote } A(\phi_{v_j}(x))(\cdot, \cdot) \text{ by } A_j(x)(\cdot, \cdot)$$

The above substitutions and (7.2.4) imply the equality:

$$Dh_j^{**}(z) + J_j(z) \circ Dh_j^{**}(z) \circ i + A_j(z)(h_j^{**}(z), \cdot) = y_j^{**}(z) + z_j^{***}(z) \tag{7.2.5}$$

Also, choose a smooth monotonic map $\kappa'_\epsilon : \mathbb{R}^+ \to [0, 1]$ for every $1 > \epsilon > 0$ satisfying the

following properties:

$$\kappa'_\epsilon(x) := x \quad \text{for } 0 \leq x \leq 1 - \frac{2\epsilon}{3}$$

$$\kappa'_\epsilon(x) := 1 \quad \text{for } x \geq 1$$

We define the map

$$\kappa_\epsilon : \mathbb{H}^+ \to \mathbb{H}^+$$

given in polar coordinates by

$$(r, \theta) \to (\kappa'_\epsilon(r), \theta)$$

Choose a smooth cut-off $k_\epsilon : D \to [0, 1]$ satisfying the following properties:

$$k_\epsilon(x) = 1 \ \text{ for } 0 \leq x \leq 1 - \epsilon, \ \ k_\epsilon(x) = 0 \ \text{ for } 1 - \frac{2\epsilon}{3} \leq x \leq 1$$

For $\epsilon > 0$ sufficiently small, we define the sequence of complex structures J_j^ϵ on the trivial bundle $D \times \mathbb{R}^{2n} \to D$ on the disk as follows:

$$J_j^\epsilon(x) := J(\widetilde{\phi}_{v_j} \circ k_\epsilon \cdot \phi(x)), \quad x \in D$$

We also consider the sequence of smooth maps:

$$A_j^\epsilon : D \to L(\mathbb{R}^{2n}, \mathbb{R}^{2n})$$

defined by

$$A_j^\epsilon(z) := A_j(z)(k_\epsilon(z) \cdot h_j^{**}(z), \cdot)$$

Consider the alterations in (7.2.5) which result from the substitutions

$$h_j^{**} \to k_\epsilon \cdot h_j^{**}, \quad J_j \to J_j^\epsilon, \quad A_j \to A_j^\epsilon :$$

$$D(k_\epsilon(z)\cdot h_j^{**}(z)) + J_j^\epsilon(z)\circ D(k_\epsilon(z)\cdot h_j^{**}(z))\circ i + A_j^\epsilon(z)(k_\epsilon(z)\cdot h_j^{**}(z),\cdot) =$$

$$Dk_\epsilon(z)\cdot h_j^{**}(z)+k_\epsilon(z)\cdot Dh_j^{**}(z)+J_j^\epsilon(z)\circ Dk_\epsilon(z)\cdot h_j^{**}(z)+k_\epsilon(z)\cdot Dh_j^{**}(z)\circ i+A_j^\epsilon(z)(k_\epsilon(z)\cdot h_j^{**}(z),\cdot)$$

$$= k_\epsilon(z)\cdot(Dh_j^{**}(z)+J_j^\epsilon(z)\circ h_j^{**}(z)\circ i+A_j^\epsilon(h_j^{**}(z),\cdot))+Dk_\epsilon(z)\cdot h_j^{**}(z)+J_j^\epsilon(z)\circ Dk_\epsilon(z)\cdot h_j^{**}(z)\circ i$$

$$= k_\epsilon\cdot(y_j^{**}+z_j^{***})+x_j \tag{7.2.6}$$

where

$$x_j = Dk_\epsilon(z)\cdot h_j^{**}(z) + J_j^\epsilon(z)\circ Dk_\epsilon(z)\cdot h_j^{**}(z)\circ i$$

is bounded on H^{m+3} so we can absorb it into the z_j^{***} term. We can re-write (7.2.6) as

$$D(k_\epsilon\cdot h_j(z)) + J_j^\epsilon(z)\circ D(k_\epsilon\cdot h_j(z))\circ i + A_j^\epsilon(z)(k_\epsilon\cdot h_j^{**}(z),\cdot) = y_j^{***}+z_j^{****} \tag{7.2.7}$$

where

$$y_j^{***} = k_\epsilon\cdot y_j^{**},\quad z_j^{****} = k_\epsilon\cdot z_j^{***}$$

Then y_j^{***} converges to 0 in H^{m+2} and z_j^{****} is bounded in H^{m+3}. Both terms y_j^{***} and z_j^{****} extend to D. J_j^ϵ and A_j^ϵ extend to D as does $h_j^{***} := k_\epsilon\cdot h_j^{**}$. Thus we can consider (7.2.7) as an equation on D. Now we can simplify (7.2.7) to

$$Dh_j^{***}(z) + J_j^\epsilon(z)\circ Dh_j^{***}(z)\circ i + A_j^\epsilon(z)(h_j^{***}(z),\cdot) = y_j^{***}+z_j^{****} \tag{7.2.8}$$

Consider the further abbreviations

$$\text{we will denote } y_j^{***}\cdot\frac{\partial}{\partial s} \text{ by } y_j'$$

$$\text{we will denote } z_j^{****}\cdot\frac{\partial}{\partial s} \text{ by } z_j'$$

Evaluating both sides of (7.2.8) at $\frac{\partial}{\partial s}$ we have

$$\frac{\partial}{\partial s}h_j^{***} + J_j^\epsilon \circ \frac{\partial}{\partial t}h_j^{***} + A_j^\epsilon(z)(h_j^{***}, T\phi_{v_j} \cdot \frac{\partial}{\partial s}) = y_j' + z_j' \tag{7.2.9}$$

The maps h_j^{***}, y_j', z_j' are compactly supported and h_j^{***}, y_j', $z_j' \in H^{m+3}(D, \mathbb{R}^{2n})$. Fix a point $z \in D$, where all the maps h_j^{***} vanish. Observe that the complex structures J_j^ϵ converge to a complex structure J_0^ϵ (since the are smoothly parametrized by a germ of a vector space) in C^∞. The families of linear maps

$$A_j^\epsilon(z)(h_j^{***}, T\phi_{v_j} \cdot \frac{\partial}{\partial s}) : D \to L(\mathbb{R}^{2n}, \mathbb{R}^{2n})$$

converge to a family

$$A_0^\epsilon(z)(h_j^{**}, T\phi_0 \cdot \frac{\partial}{\partial s}) : D \to L(\mathbb{R}^{2n}, \mathbb{R}^{2n})$$

as $v_j \to 0$. As in [HWZ2] Proposition 5.4, consider the subspace $W \subset H^{m+3}(D, \mathbb{R}^{2n})$ consisting of maps $u \in H^{m+3}(D, \mathbb{R}^{2n})$ satisfying

$$u(z) = 0$$

$$u(\partial D) \in \mathbb{R}^n \times \{0\}$$

The linear operator

$$L : W \to H^{m+2}(D, \mathbb{R}^{2n}), \quad u \to \frac{\partial}{\partial s}u + J_0^\epsilon \frac{\partial}{\partial t}u + A_0^\epsilon$$

is a Fredholm operator of index 0 with spectrum a countable discrete set of points with

no finite limit points. This is also true for the linear operators

$$L_j : W \to H^{m+2}(D, \mathbb{R}^{2n}), \quad u \to \frac{\partial}{\partial s} + J_j^\epsilon \frac{\partial}{\partial t} + A_j^\epsilon u$$

for large j. Since $J_j^\epsilon \to J_0^\epsilon$ and $A_j^\epsilon \to A_0^\epsilon$, this means that there exists an $\alpha > 0$ such that for j sufficiently large the operators $L_j + \alpha$ are linear isomorphisms. We also have a fixed constant C independent of j such that $||(L_j + \alpha)^{-1}|| < C$ for j sufficiently large. For j sufficiently large, consider the unique $w_j \in W$ satisfying

$$(L_j + \alpha)w_j = y_j'$$

Since $y_j' \to 0$ in $H^{m+2}(D, \mathbb{R}^{2n})$ we have

$$||w_j||_{H^{m+3}(D,\mathbb{R}^{2n})} = ||(L_j + \alpha)^{-1} y_j'||_{H^{m+3}(D,\mathbb{R}^{2n})} \leq C||y_j'||_{H^{m+2}(D,\mathbb{R}^{2n})}$$

which means that $w_j \to 0$ in $H^{m+3}(D, \mathbb{R}^{2n})$. Consider the sequence $e_j := h_j^{***} - w_j$ and observe that the sequence e_j is bounded in W. We have:

$$L_j(e_j) = z_j' + \alpha w_j =: z_j''$$

Observe that z_j'' is bounded in $H^{m+3}(D, \mathbb{R}^{2n})$. By classical elliptic regularity with boundary conditions, it follows that e_j is bounded in $H^{m+4}(D, \mathbb{R}^{2n})$ which implies e_j converges in $H^{3+m}(D, \mathbb{R}^{2n})$. In particular, this means h_j^{***} converges in a neighborhood of $1 \in D$. Unwinding the trail of substitutions, the claim follows. ■

7.3 Calculation of the Fredholm Index

We set the following abbreviation:

$$\mathcal{E}_\lambda := (\mathfrak{O}_\lambda^{core})_m \times_{(f_i,f_b)} E_\lambda^*$$

Proposition 7.3.1. Consider a chart $\mathfrak{O}$ parametrizing a smooth family of stable maps centered at a map $u : S \to Q$. For a smooth point $(a, v, \eta) \in \mathfrak{O}$ sufficiently close to $(0, 0, 0)$, $D\overline{\partial}_J(a_0, v_0, \eta_0)$ is sc-Fredholm. Furthermore, for (a, v, η) sufficiently small, the Fredholm index of $D\overline{\partial}_J(a, v, \eta)\cdot$ is constant and given by the Riemann-Roch formula (see [MS]):

$$\text{ind}(D\overline{\partial}_J(a_0, v_0, \eta_0)\cdot) = \mu+(n-3)(2-2g-h)+2\#M_i+\#M_b = \mu+n\chi(S)+\dim DM_{g,h,M_i,M_b}$$

where $\mu = \mu(u^*TQ, u^*|_{\partial S}TL)$ is the Maslov index and S has genus g, h boundary components, $\#M_i$ interior marked points and boundary marked points described by M_b.

Lemma 7.3.1. The operator $D_3\overline{\partial}_J(0, 0, 0) : E_u \to W_\lambda$ is sc-Fredholm.

Proof: By direct calculation (see [HWZ1] Page 190) we have:

$$(\nabla_{(h,0)}\Gamma)[u, u] \circ \mathbf{f}(0, 0, \eta_0) + D_3\mathbf{f}(0, 0, 0) \cdot h = \nabla_h\overline{\partial}_{J,j}u$$

where ∇ is the Levi-Civita connection. Since $(\nabla_{(\cdot,0)}\Gamma)[u, u]\circ(\mathbf{f}(0, 0, \eta_0)$ is sc^+, it is enough to show that $\nabla_{(\cdot)}\overline{\partial}_{J,j}u$ is sc-Fredholm. By definition:

$$\nabla_h\overline{\partial}_{J,j}u := \frac{1}{2}[\nabla h + J(u) \circ \nabla h \circ j \cdot + N(h, Tu\cdot) + J(u) \circ N(h, Tu \circ j\cdot)]$$

where N is torsion of ∇. In particular, we observe that $\nabla_{(\cdot)}\overline{\partial}_{J,j}u$ is a real linear Cauchy-Riemann operator.

In local coordinates near the positive node $\nabla_h\overline{\partial}_{J,j}u$ admits the form:

$$\frac{\partial}{\partial s}h + \widehat{J}(u^+)\frac{\partial}{\partial t}h + [D\widehat{J}(u^+)\cdot h]\cdot\frac{\partial}{\partial t}u^+ \tag{7.3.1}$$

Near the negative node, the expression of $\nabla_h\overline{\partial}_{J,j}u$ admits an analogous form.

Proposition 7.3.2. We have the inequality:

$$|\xi|_{m,\delta_0^m,\delta_1^m} \leq |\nabla_\xi\overline{\partial}_{J,j}u|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} + |\xi|_{H^{m-1}(S_{core},Q)}$$

for all $\xi \in H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(S,T^*Q)$. In particular, $\nabla_{(\cdot)}\overline{\partial}_{J,j}u$ is semi-Fredholm.

Proof: The argument loosely follows Lemma 4.13 [W]. Consider a family of smooth cut-off functions $\beta_r : S \to [0,1]$, parametrized by $r \in [1,\infty)$, such that $\beta_r = 1$ on $S_{core,r}$ and 0 on the complement of $S_{core,r+1}$, considering $S_{core,r}$ and $S_{core,r+1}$ as subsets of S. We have for all $r \in [1,\infty)$,

$$|\xi|_{m,\delta_0^m,\delta_1^m} |\beta_r\cdot\xi + (1-\beta_r)\cdot\xi|_{m,\delta_0^m,\delta_1^m} = |\beta_r\cdot\xi|_{m,\delta_0^m,\delta_1^m} + |(1-\beta_r)\xi|_{m,\delta_0^m,\delta_1^m}$$

By classical elliptic theory (see [A])

$$|\beta_r\xi|_{m,\delta_0^m,\delta_1^m} \leq C_r(|\nabla_{\beta_r\xi}\overline{\partial}_{J,j}u|_{H^{m-1}(S_{core},Q)} + |\beta_r\xi|_{H^{m-1}(S_{core},Q)})$$

$$\leq C_r(|\beta_r\nabla_\xi\overline{\partial}_{J,j}u|_{H^{m-1}(S_{core},Q)} + |(\overline{\partial}\beta_r)\xi|_{H^{m-1}(S_{core},Q)} + |\beta_r\xi|_{H^{m-1}(S_{core},Q)})$$

$$\leq C_r'(|\nabla_\xi \overline{\partial}_{J,j} u|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} + |\xi|_{H^{m-1}(S_{core},Q)}) \tag{7.3.2}$$

where C_r, C_r' depend only on r and m. Note that we use the real linearity of $\nabla_{(\cdot)}\overline{\partial}_{J,j}u$ to establish the second inequality. For every $\{x, y\} \in D$, we define the linear operators

$$\overline{\partial}_{J,x} : H^{3,\delta_0}_{real}(\mathbb{R}\times[0,1],\mathbb{R}^{2n}) \to H^{3,\delta_0}_{bdry}(\mathbb{R}\times[0,1],\mathbb{R}^{2n})$$

$$h^+ \to \frac{\partial}{\partial s}h^+ + \widehat{J}(u^+)\frac{\partial}{\partial t}h^+ + [D\widehat{J}(u^+)h^+]\frac{\partial}{\partial t}u^+$$

and

$$\overline{\partial}_{J,y} : H^{3,\delta_0}_{real}(\mathbb{R}\times[0,1],\mathbb{R}^{2n}) \to H^{3,\delta_0}_{bdry}(\mathbb{R}\times[0,1],\mathbb{R}^{2n})$$

$$h^- \to \frac{\partial}{\partial s}h^- + \widehat{J}(u^-)\frac{\partial}{\partial t}h^- + [D\widehat{J}(u^-)h^-]\frac{\partial}{\partial t}u^-$$

We can re-write $\overline{\partial}_{J,x}$ as follows:

$$\overline{\partial}_{J,x}h^+ = \frac{\partial}{\partial s}h^+ + \widehat{J}(u^+(\infty))\frac{\partial}{\partial t}h^+ + (\widehat{J}(u^+) - \widehat{J}(u^+(\infty)))\frac{\partial}{\partial t}h^+ + [D\widehat{J}(u^+)h^+]\frac{\partial}{\partial t}u^+$$

where $u^+(\infty) = \lim_{s\to\infty}u^+(s,t)$. Given $\epsilon > 0$, there exists $r > 0$ sufficiently large such that if h^+ is supported in $[r,\infty)$ then, arguing as in Lemma 4.19 [HWZ1], we have that

$$|(\widehat{J}(u^+) - \widehat{J}(u^+(\infty)))\frac{\partial}{\partial t}h^+|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} \leq \epsilon|h^+|_{m,\delta_0^m,\delta_1^m}$$

$$|[D\widehat{J}(u^+)h^+]\frac{\partial}{\partial t}u^+|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} \leq \epsilon|h^+|_{m,\delta_0^m,\delta_1^m}$$

Appealing to Proposition 5.1.1, for maps $h^+ \in H^{3,\delta_0}_{real}(\mathbb{R}^+\times[0,1],\mathbb{R}^{2n})$ supported on $[r,\infty)$ for r sufficiently large, we have:

$$|\overline{\partial}_{J,x}h^+|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} \geq C_m|h^+|_{m,\delta_0^m,\delta_1^m} \tag{7.3.3}$$

for some constant $C_m > 0$ depending only on m. Similarly, for $\overline{\partial}_{J,y}$ we have:

$$|\overline{\partial}_{J,y} h^-|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} \geq C_m |h^-|_{m,\delta_0^m,\delta_1^m} \tag{7.3.4}$$

where $h^- \in H^{3,\delta_0}(\mathbb{R}^- \times [0,1], \mathbb{R}^{2n})$ is supported on $(-\infty, -r]$ for some r chosen sufficiently large. Using (7.3.3) $-$ (7.3.4) and the observation that the coordinate change maps are smooth (hence bounded in C^∞), it follows that there exists a constant C'_m independent of ξ such that

$$|\nabla_\xi \overline{\partial}_{J,j} u|_{m-1,\delta_0^{m-1},\delta_1^{m-1}} \geq C'_m |(1-\beta_r)\xi|_{m,\delta_0^m,\delta_1^m} \tag{7.3.5}$$

Combining (7.3.2) and (7.3.5), the claim follows. ■

By (7.3.1), and applying estimates similar to Proposition 5.1.1 it follows that $\nabla_{(\cdot)} \overline{\partial}_{J,j} u$ is regularizing.

Consider the restriction of $\nabla_\xi \overline{\partial}_{J,j} u$ to a smooth connected component C of S. Then $\nabla_{(\cdot)} \overline{\partial}_{J,j} u$ extends to a real Cauchy-Riemann operator on non-decaying spaces of sections

$$\nabla_{(\cdot)} \overline{\partial}_{J,j} u | C : H^3(C, u^*TQ, u^*|_{\partial C} TL) \to H^2(C, \Omega^{0,1} T^*C \otimes u^*TQ)$$

Following [MS] Appendix C.1 we consider the subspace:

$$H^3_L := \{\xi \in H^3(C, u^*TQ) : \xi(\partial C) \subset u^*TL\}$$

We consider the adjoint $\nabla_{(\cdot)} \overline{\partial}_{J,j} u^*$ of $\nabla_{(\cdot)} \overline{\partial}_{J,j} u$,

$$\nabla_{(\cdot)} \overline{\partial}_{J,j} u^* : H^3_L \to H^2(C, u^*TQ)$$

The adjoint of any real linear Cauchy-Riemann operator D,

$$D = \overline{\partial}^{\nabla_0} + A,$$

where A is of order 0 and $\overline{\partial}^{\nabla_0}$ is the complex linear Cauchy-Riemann operator induced by a Hermitian connection ∇_0, is given by the following expression:

$$D^* = - \star J\partial^{\nabla_0} + A^*$$

where $\star$ denoes the Hodge star and

$$\partial^{\nabla_0} := \frac{1}{2}(\nabla_0\xi - J\nabla_0 \circ j)$$

In local holomorphic coordinates $s + it \in \mathbb{H}$, if $\eta = \xi ds - J\xi dt$, then

$$- \star J\partial^{\nabla}\eta = \frac{1}{\lambda^2}(-\nabla_{0s}\xi + i\nabla_{0t}\xi) \tag{7.3.6}$$

where the volume form is given by $\lambda^2 ds \wedge dt$. Using formula (7.3.6) we can directly calculate

$$< D^*\eta, \xi > \lambda^2 ds \wedge dt - < \eta, D\xi > \lambda^2 ds \wedge dt = d < \eta, J\xi > \tag{7.3.7}$$

for all $\xi \in \Gamma^\infty(C, u^*TQ)$ and $\eta \in \Omega^{0,1,\infty}(C, u^*TQ)$ (see [MS] Appendix C for details). (7.3.7) implies:

$$\int_C < D^*\eta, \xi > - < \eta, D\xi >= 0 \tag{7.3.8}$$

since $J \cdot TL = TL^\perp$. Consider the *sc*-Hilbert space:

$$H_L^{3,\delta_0^0,\delta_1^0} := \{\zeta \in \bigoplus_{C \text{ is a component in } S} H_L^3(C) : \zeta \text{ decays exponentially at the nodes } D,$$

$$\zeta(x) = \zeta(y), \text{ for } \{x, y\} \in D\}$$

From now on, we consider $\nabla_{(\cdot)}\overline{\partial}_{J,j}u^*$ as a real linear Cauchy-Riemann operator on $H_L^{3,\delta_0^0,\delta_1^0}$. On the positive neck $\nabla_{(\cdot)}\overline{\partial}_{J,j}u^*$ is given by:

$$[D\widehat{J}(u^+)^* \cdot \xi^+] \cdot \frac{\partial}{\partial t}u^+ - \frac{\partial}{\partial s}\xi^+ + \widehat{J}(u^+)^*\frac{\partial}{\partial t}\xi^+$$

It follows from Proposition 5.1.1 that the operators

$$-\partial_s + i\partial_t : H_c^{3,\delta_0^0}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to H^{2,\delta_0^0}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

$$-\partial_s + i\partial_t : H_{real}^{3,\delta_1^0}(\mathbb{R} \times [0,1], \mathbb{R}^{2n}) \to H^{2,\delta_1^0}(\mathbb{R} \times [0,1], \mathbb{R}^{2n})$$

are *sc*-linear isomorphisms. Thus we can argue as in Lemma 7.1.1 to demonstrate that $\nabla_{(\cdot)}\overline{\partial}_{J,j}u^*$ is regularizing. Arguing as in Proposition 7.3.2, it follows that $\nabla_{(\cdot)}\overline{\partial}_{J,j}u^*$ is semi-Fredholm. Below we set $D := \nabla_{(\cdot)}\overline{\partial}_{J,j}u$ and $D^* := \nabla_{(\cdot)}\overline{\partial}_{J,j}u$.

Lemma 7.3.2. (compare [MS] Theorem C.2.3)

(a) If $\eta \in H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(S, \Omega^{0,1}T^*S \otimes u^*TQ)$ and $\xi \in H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(S, u^*TQ)$ satisfy

$$\int_S < \eta, D\zeta >= \int_S < \xi, \zeta >$$

for every $\zeta \in H_L^{m,\delta_0^m,\delta_1^m}(S, u^*TQ)$, then $\eta \in H_L^{m,\delta_0^m,\delta_1^m}(S, \Omega^{0,1}T^*S \otimes u^*TQ)$ and $D^*\eta = \xi$.

(b) If $\xi \in H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(S, u^*TQ)$ and $\eta \in H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(S, \Omega^{0,1}T^*S \otimes u^*TQ)$ satisfy

$$\int_S < D^*\zeta, \xi >= \int_S < \zeta, \eta >$$

for every $\zeta \in H_L^{m,\delta_0^m,\delta_1^m}(S, \Omega^{0,1}T^*S \otimes u^*TQ)$, then $\xi \in H_L^{m,\delta_0^m,\delta_1^m}(S, u^*TQ)$ and $D\xi = \eta$.

Proof: (a) Without the exponential decay condition, the result is proven in [MS] Theorem C.2.3. However, the facts $\eta \in H^m$ and $D^*\eta = \xi$ are local and can be proven in local coordinates. Thus, the claim follows by classical elliptic estimates as in [MS] Theorem C.2.3. (b) Follows similarly. ■

Lemma 7.3.3. (compare [MS] Theorem C.2.3)

(a) $\text{Im } D \oplus \ker D^* = H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}\Omega^{0,1}((TS,j),(TS,j))$

(b) $H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(TS, u^*TQ) = \text{Im } D^* \ \oplus \ \ker D$.

Proof: (a) First we show the intersection $\text{Im } D \cap \ker D^*$ is trivial. Suppose η is in the intersection. Then $D^*\eta = 0$, which means that $\eta \in \text{Im } D^{\perp}$ by (7.3.8). But since $\eta \in \text{Im } D$, this means $\eta = 0$.

Observe that $\text{Im } D + \ker D^*$ is closed by Proposition 7.3.2. Suppose for sake of contradiction that $\text{Im } D + \ker D^* \neq H^{2,\delta_0^0,\delta_1^0}\Omega^{0,1}((TS,j),(TS,j))$. Then there exists $\alpha \in H^{2,\delta_0^0,\delta_1^0}\Omega^{0,1}((TS,j),(TS,j))$ such that

$$< D\eta + \lambda, \alpha >= 0 \quad \text{for all} \quad \eta \in H_L^{3,\delta_0^0,\delta_1^0}\Gamma(TS),\ \lambda \in \ker D^*$$

Then $\alpha \in \text{Im } D^{\perp}$, which implies $D^*\alpha = 0$ by (7.3.8). However, $\alpha \in \ker(D^*)^{\perp}$ as well, which means $\alpha = 0$. This establishes (a). (b) follows similarly. ■

Now it follows that D and D^* are Fredholm. The splittings in Lemma 7.3.3 and the

regularizing property, imply that D is sc-Fredholm. ■

Proposition 7.3.3. The index of $D\overline{\partial}_J(a,v,\eta)\cdot$ is independent of (a,v,η) for (a,v,η) sufficiently close to $(0,0,0)$.

Proof: Throughout the argument we assume that there is a single nodal parameter. It is immaterial to the argument whether the node has boundary or not. We will assume the single node is cylindrical. Recall by Proposition 6.0.9 that the section

$$\text{Fill}\,\overline{\partial}_J : \text{Fill}\,\widetilde{\mathfrak{S}}_\lambda \to \widetilde{F}_\lambda$$

is a filling for the section $\overline{\partial}_J : \widetilde{\mathfrak{S}}_\lambda \to \widetilde{W}_\lambda$ and that the section Fill $\overline{\partial}_J$ splits as

$$\text{Fill}\,\overline{\partial}_J := \overline{\partial}_{J,\,core} \times \text{Fill}\,\overline{\partial}_{J,\,neck}$$

Also, recall that the section Fill $\overline{\partial}_{J,\,neck}$ is defined as follows

$$\text{Fill}\,\overline{\partial}_{J,\,neck}(a,h^+,h^-) : B_D \times E(2n) \to F(2n)$$

$$(a,h^+,h^-) \to (\xi^+,\xi^-)$$

We set the abbreviation Fill $\overline{\partial}_{J,neck}(a,h^+,h^-) := G(a,h^+,h^-)$. Recall that G is defined by the system of equations:

$$\widehat{\oplus}_a(G(a,h^+,h^-)) = \Big[\frac{\partial}{\partial s}(u_a+h_a) + \widehat{J}(u_a+h_a)\frac{\partial}{\partial t}(u_a+h_a)\Big] \tag{7.3.9}$$

$$\widehat{\ominus}_a(G(a,h^+,h^-)) = \overline{\partial}_0 \ominus_a (h^+,h^-) \tag{7.3.10}$$

(Note that we suppressed the factor of $\frac{1}{2}$. We similarly suppress the factor of $\frac{1}{2}$ in $\overline{\partial}_0$ ap-

pearing the definition of $\overline{\partial}_0$ for sake of simplicity.) Consider the sc^+-section $G_0(a,h^+,h^-) = G(a,0,0)$. G_0 satisfies the following system of equations:

$$\widehat{\oplus}_a(G_0(a,h^+,h^-)) = \Big[\frac{\partial}{\partial s}u_a + \widehat{J}(u_a)\frac{\partial}{\partial t}u_a\Big]$$

$$\widehat{\ominus}_a(G_0(a,h^+,h^-)) = 0$$

We have the following by direct computation:

$$\widehat{\ominus}_a(\overline{\partial}_0 h^+, \overline{\partial}_0 h^-) =$$

$$\widehat{\ominus}_a\Big(\frac{\partial}{\partial s}h^+ + \widehat{J}(u^+{+}h^+)\frac{\partial}{\partial t}(u^+{+}h^+) - \widehat{J}(u^+)\frac{\partial}{\partial t}u^+, \frac{\partial}{\partial s}h^- + \widehat{J}(u^-{+}h^-)\frac{\partial}{\partial t}(u^-{+}h^-) - \widehat{J}(u^-)\frac{\partial}{\partial t}u^-\Big)$$
$$+\widehat{\ominus}_a\Big(i\frac{\partial}{\partial t}h^+ - \Big(\widehat{J}(u^+{+}h^+)\frac{\partial}{\partial t}(u^+{+}h^+) - \widehat{J}(u^+)\frac{\partial}{\partial t}u^+\Big), i\frac{\partial}{\partial t}h^- - \Big(\widehat{J}(u^-{+}h^-)\frac{\partial}{\partial t}(u^-{+}h^-) - \widehat{J}(u^-)\frac{\partial}{\partial t}u^-\Big)\Big)$$

By θ we denote:

$$-\widehat{\ominus}_a\Big(i\frac{\partial}{\partial t}h^+ - \Big(\widehat{J}(u^+{+}h^+)\frac{\partial}{\partial t}(u^+{+}h^+) - \widehat{J}(u^+)\frac{\partial}{\partial t}u^+\Big), i\frac{\partial}{\partial t}h^- - \Big(\widehat{J}(u^-{+}h^-)\frac{\partial}{\partial t}(u^-{+}h^-) - \widehat{J}(u^-)\frac{\partial}{\partial t}u^-\Big)\Big)$$

$$\in H^{3,\delta_0^0}(C_a, \mathbb{R}^{2n})$$

By direct calculation we have:

$$\begin{bmatrix}\widehat{\oplus}_a(G - G_0(a,h^+,h^-))\\ \widehat{\ominus}_a(G - G_0(a,h^+,h^-))\end{bmatrix} =$$

$$\begin{bmatrix}\widehat{\oplus}_a(\frac{\partial}{\partial s}h^+ + \widehat{J}(u^+{+}h^+)\frac{\partial}{\partial t}(u^+{+}h^+) - \widehat{J}(u^+)\frac{\partial}{\partial t}u^+, \frac{\partial}{\partial s}h^- + \widehat{J}(u^-{+}h^-)\frac{\partial}{\partial t}(u^-{+}h^-) - \widehat{J}(u^-)\frac{\partial}{\partial t}u^-)\\ \widehat{\ominus}_a(\frac{\partial}{\partial s}h^+ + \widehat{J}(u^+{+}h^+)\frac{\partial}{\partial t}(u^+{+}h^+) - \widehat{J}(u^+)\frac{\partial}{\partial t}u^+, \frac{\partial}{\partial s}h^- + \widehat{J}(u^-{+}h^-)\frac{\partial}{\partial t}(u^-{+}h^-) - \widehat{J}(u^-)\frac{\partial}{\partial t}u^-)\end{bmatrix}$$

$$+\begin{bmatrix}0\\ \theta\end{bmatrix}+\begin{bmatrix}\beta_a'(h^+-h^-)+A+B\\ \beta_a'(h^++h^--2av_a(h^+,h^-))\end{bmatrix} \tag{7.3.11}$$

where

$$A=\Big[\widehat{J}(u_a+h_a)\frac{\partial}{\partial t}(u_a+h_a)-\widehat{\oplus}_a(\widehat{J}(u^++h^+)\frac{\partial}{\partial t}(u^++h^+),\widehat{J}(u^-+h^-)\frac{\partial}{\partial t}(u^-+h^-))\Big]$$

and

$$B=-\Big[\widehat{J}(u_a)\frac{\partial}{\partial t}u_a-\widehat{\oplus}_a(\widehat{J}(u^+)\frac{\partial}{\partial t}u^+,\widehat{J}(u^-)\frac{\partial}{\partial t}u^-)\Big]$$

By further calculation we have:

$$\begin{bmatrix}(G-G_0(a,h^+,h^-))^+\\ (G-G_0(a,h^+,h^-))^-\end{bmatrix}=\begin{bmatrix}\frac{\partial}{\partial s}h^++\widehat{J}(u^++h^+)\frac{\partial}{\partial t}(u^++h^+)-\widehat{J}(u^+)\frac{\partial}{\partial t}u^+\\ \frac{\partial}{\partial s}h^-+\widehat{J}(u^-+h^-)\frac{\partial}{\partial t}(u^-+h^-)-\widehat{J}(u^-)\frac{\partial}{\partial t}u^-\end{bmatrix}$$

$$+\frac{1}{\gamma_a}\begin{bmatrix}\beta_a & \beta_a-1\\ 1-\beta_a & \beta_a\end{bmatrix}\cdot\begin{bmatrix}0\\ \theta\end{bmatrix}+\frac{1}{\gamma_a}\begin{bmatrix}\beta_a & \beta_a-1\\ 1-\beta_a & \beta_a\end{bmatrix}\cdot\begin{bmatrix}\beta_a'(h^+-h^-)+B+A\\ \beta_a'(h^++h^--2av_a(h^+,h^-))\end{bmatrix}$$

Consider the section

$$\Delta: B\times E\to F$$

Given by

$$\Delta(a,h^+,h^-)=\frac{1}{\gamma_a}\begin{bmatrix}\beta_a & \beta_a-1\\ 1-\beta_a & \beta_a\end{bmatrix}\cdot\begin{bmatrix}0\\ \theta\end{bmatrix}$$

More explicitly,

$$\Delta(a,h^+,h^-):=\frac{1}{\gamma_a}\begin{bmatrix}(\beta_a-1)\Big\{(\beta_a-1)\Big[\Big(\widehat{J}(u^++h^+)-i\Big)\frac{\partial}{\partial t}h^+\Big]+\beta_a\Big[\Big(\widehat{J}(u^-+h^-)-i\Big)\frac{\partial}{\partial t}h^-\Big]\Big\}\\ \beta_a\Big\{(\beta_a-1)\Big[\Big(\widehat{J}(u^++h^+)-i\Big)\frac{\partial}{\partial t}h^+\Big]+\beta_a\Big[\Big(\widehat{J}(u^-+h^-)-i\Big)\frac{\partial}{\partial t}h^-\Big]\Big\}\end{bmatrix}$$

$$+\frac{1}{\gamma_a}\begin{bmatrix}(\beta_a-1)\Big\{(\beta_a-1)\Big[\Big(\widehat{J}(u^++h^+)-\widehat{J}(u^+)\Big)\frac{\partial}{\partial t}u^+\Big]+\beta_a\Big[\Big(\widehat{J}(u^-+h^-)-\widehat{J}(u^-)\Big)\frac{\partial}{\partial t}u^-\Big]\Big\}\\ \beta_a\Big\{(\beta_a-1)\Big[\Big(\widehat{J}(u^++h^+)-\widehat{J}(u^+)\Big)\frac{\partial}{\partial t}u^+\Big]+\beta_a\Big[\Big(\widehat{J}(u^-+h^-)-\widehat{J}(u^-)\Big)\frac{\partial}{\partial t}u^-\Big]\Big\}\end{bmatrix}$$

$$:= I(a,h^+,h^-)+II(a,h^+,h^-)$$

Given $\epsilon>0$, using Remark 11.0.1 and employing a similar argument to Proposition 7.1.6 (1), choosing a and (h^+,h^-) sufficiently small we can ensure:

$$||DI(a,h^+,h^-)\cdot(\delta a,\delta\eta^+,\delta\eta^-)||_{F_{m+1}}\leq\epsilon\cdot(|\delta a|+||(\delta\eta^+,\delta\eta^-)||_{E_{m+1}})$$

We note that II is sc^+. When a is sufficiently small, the support of $\Delta(a,h^+,h^-)$ is contained in $[\frac{R}{2}-1,\infty)\cup(-\infty,-\frac{R}{2}+1]$. Since Δ is supported away from the left and right boundaries, the operator Δ extends by 0 to a section on the fibered product:

$$0\ \times\Delta:\mathfrak{O}^{core}_{\lambda\ m}\times_{f_i}E_\lambda\to\widetilde{F}_\lambda$$

Now consider the section

$$\Delta':B\times E\to F$$

$$\Delta'(a,h^+,h^-):=\frac{1}{\gamma_a}\begin{bmatrix}\beta_a & \beta_a-1\\ 1-\beta_a & \beta_a\end{bmatrix}\cdot\begin{bmatrix}A\\0\end{bmatrix}$$

$$=\frac{1}{2\gamma_a}\begin{bmatrix}\beta_a([\widehat{J}(u_a+h_a)\frac{\partial}{\partial t}(u_a+h_a)-\widehat{\oplus}_a(\widehat{J}(u^++h^+)\frac{\partial}{\partial t}(u^++h^+),\widehat{J}(u^-+h^-)\frac{\partial}{\partial t}(u^-+h^-))])\\ (1-\beta_a)([\widehat{J}(u_a+h_a)\frac{\partial}{\partial t}(u_a+h_a)-\widehat{\oplus}_a(\widehat{J}(u^++h^+)\frac{\partial}{\partial t}(u^++h^+),\widehat{J}(u^-+h^-)\frac{\partial}{\partial t}(u^-+h^-))])\end{bmatrix}$$

Observe that the support of $\Delta'(a,h^+,h^-)(\eta^+,\eta^-)$ is contained in $[\frac{R}{2}-1,\frac{R}{2}+1]\times[-\frac{R}{2}-1,-\frac{R}{2}+1]$. Given $\epsilon>0$, for (a,h^+,h^-) sufficiently small we have:

$$|D\Delta'(a,h^+,h^-)(\delta a,\delta\eta^+,\delta\eta^-)|_{(\widetilde{F}_\lambda)_m}\leq\epsilon\cdot|_{E_m}\cdot|(\delta a,\delta\eta^+,\delta\eta^-)|_{(\mathcal{E}_\lambda)_m}$$

Extension by 0 defines an operator on the fibered product

$$0 \times \Delta' : \mathfrak{O}^{core}_{\lambda}{}_{m} \times_{f_i} E_\lambda \to \widetilde{F}_\lambda$$

Consider the section:

$$\Delta'' : B \times E \to F$$

$$\Delta''(a, h^+, h^-) := \frac{1}{\gamma_a} \begin{bmatrix} \beta_a & \beta_a - 1 \\ 1 - \beta_a & \beta_a \end{bmatrix} \cdot \begin{bmatrix} \beta_a'(h^+ - h^-) \\ \beta_a'(h^+ + h^- - 2av_a(h^+, h^-)) \end{bmatrix}$$

By direct calculation, it follows that the operator $D\Delta''$ is sc^+ and is supported away from the left and right boundaries. Extension by 0 produces an operator on the fibered product

$$0 \times \Delta'' : \mathfrak{O}^{core}_{\lambda}{}_{m} \times_{f_i} E_\lambda \to \widetilde{F}_\lambda$$

Also consider the section:

$$\Delta''' : B \times E \to F$$

$$\Delta'''(a, h^+, h^-) := \frac{1}{\gamma_a} \begin{bmatrix} \beta_a & \beta_a - 1 \\ 1 - \beta_a & \beta_a \end{bmatrix} \cdot \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Observe that B is smooth and Δ''' is sc^+ and supported away from the left and right boundaries. Thus B extends by 0 to an sc^+ section on the fibered product

$$0 \times B : \mathfrak{O}^{core}_{\lambda}{}_{m} \times_{f_i} E_\lambda \to \widetilde{F}_\lambda$$

By construction we have:

$$G - G_0(0, h^+, h^-) = \begin{bmatrix} \frac{\partial}{\partial s} h^+ + \widehat{J}(u^+ + h^+) \frac{\partial}{\partial t}(u^+ + h^+) - \widehat{J}(u^+) \frac{\partial}{\partial t} u^+ \\ \frac{\partial}{\partial s} h^- + \widehat{J}(u^- + h^-) \frac{\partial}{\partial t}(u^- + h^-) - \widehat{J}(u^-) \frac{\partial}{\partial t} u^- \end{bmatrix}$$

Furthermore, by Remark 11.0.1

$$D(G - G_0)(0, h^+, h^-)(\delta a, \delta h^+, \delta h^-) = D_3(G)(0, h^+, h^-)(\delta h^+, \delta h^-)$$

In particular, the differential of $G - G_0$ at 0 is independent of a. By (7.3.11), we have:

$$(G - G_0)(a, h^+, h^-) := D(G - G_0)(0, h^+, h^-)(\delta a, \delta h^+, \delta h^-)$$

$$+(\Delta + \Delta' + \Delta'' + \Delta''')(a, h^+, h^-)(\delta a, \delta h^+, \delta h^-)$$

I.e., $G - G_0(a, h^+, h^-)$ differs from $G - G_0(0, h^+, h^-)$ by a sum of operators which are perturbation terms or sc^+. By direct calculation, it follows that

$$D_{1,2}\overline{\partial}_{core}(v, \eta) \cdot (\delta v, \delta \eta) = A(\delta v) + D_2\overline{\partial}_{core}(v, \eta) \cdot (\delta \eta)$$

where A is a compact operator such that $A(\delta v)|_{\mathbf{D} \cap S_{core}} = 0$ since the variation δv of complex structures on S_{core} vanishes on the small disks $\mathbf{D}$. Thus, A extends by 0 to an sc^+ operator on the fibered product:

$$A \times 0 : \mathfrak{O}_\lambda^{core}{}_m \times_{f_i} E_\lambda \to \widetilde{F}_\lambda$$

From the above, we have:

$$D\overline{\partial}_{core} \times \text{Fill}\, \overline{\partial}_{neck}(v, \eta, a, h^+, h^-)(\delta v, \delta \eta, \delta a, \delta h^+, \delta h^-)$$

$$= D_{2,4}\overline{\partial}_{core} \times \text{Fill}\, \overline{\partial}_{neck}(v, \eta, 0, h^+, h^-)(\delta \eta, \delta h^+, \delta h^-) + A(\delta v)$$

$$+ D(0 \times \Delta + \Delta' + \Delta'' + \Delta''')(a, h^+, h^-)(\delta a, \delta h^+, \delta h^-) \qquad (7.3.12)$$

It follows (7.3.12), that $D(\overline{\partial}_{core}\times\text{Fill}\overline{\partial}_{neck})(v,\eta,0,h^+,h^-)$ differs from $D(\overline{\partial}_{core}\times\text{Fill}\overline{\partial}_{neck})(v,\eta,a,h^+,h^-)$ by operators which are either compact or perturbation terms. It follows by Lemma 7.3.1 that $D\text{Fill}\overline{\partial}_{core}\times\overline{\partial}_{neck}(v,\eta,0,h^+,h^-)$ is *sc*-Fredholm, and in particular has finite Fredholm index. Thus

$$\text{ind}D\overline{\partial}_{core}\times\text{Fill}\,\overline{\partial}_{neck}(v,\eta,0,h^+,h^-)=\text{ind}D\overline{\partial}_{core}\times\text{Fill}\,\overline{\partial}_{neck}(v,\eta,a,h^+,h^-)$$

for all (v,η,a) sufficiently small. ■

Proposition 7.3.4. Let $\mathfrak{O}_\lambda$ be a chart parametrizing a uniformizing family, as constructed in Section 6. $D\overline{\partial}_J(a,v,\eta)\cdot$ is *sc*-Fredholm for all $(a,v,\eta)\in(\widetilde{\mathfrak{O}}_\lambda)_\infty$.

Proof: It is enough to show that

$$D\overline{\partial}_{J,\lambda}(a,v,g_\lambda(a,v,\eta)):V_\lambda\oplus\mathcal{E}_\lambda\to\widetilde{F}_\lambda$$

is *sc*-smooth. Consider a chart

$$\widetilde{\mathfrak{O}}_{\lambda'}\to Z$$

$$q'\to[q']$$

where $[0]=[x_0]\in Z$. It follows from Lemma 7.3.1 that the operator

$$D\overline{\partial}_{J,\lambda'}(0,0,0,0,0):V_{\lambda'}\oplus\mathcal{E}_{\lambda'}\to\widetilde{F}_{\lambda'}$$

is *sc*-Fredholm. By Theorem 4.1.1 there exists a germ $\mathfrak{O}'\subset\widetilde{\mathfrak{O}}_{\lambda'}$ and a map:

$$f:\mathfrak{O}'\to\widetilde{\mathfrak{O}}_\lambda$$

which parametrizes a family of morphisms

$$\phi_{f(q')} : \alpha_{q'} \to \alpha'_{f(q')}$$

such that

$$f(0) = (v_0, \eta_0, a_0, \overline{\eta}_0^+, \overline{\eta}_0^-)$$

and

$$\phi_{f(0)} = \text{Identity}$$

The family of morphisms $\phi_{f(\cdot)}$ induces a natural fiberwise isomorphism

$$\widehat{F} : \widetilde{F}_{\lambda'} | \, \mathfrak{O}' \to \widetilde{F}_{\lambda}$$

$$(a, v, \eta, \xi) \to \widetilde{\mu}(\phi_{f(a,v,\eta)}, \xi)$$

By naturality (Proposition 6.0.7) we have:

$$\begin{array}{ccc} \widetilde{F}_{\lambda'} | \, \mathfrak{O}' & \xrightarrow{\widehat{F}} & \widetilde{F}_{\lambda} \\ \downarrow & & \downarrow \\ \mathfrak{O}' & \xrightarrow{f} & \widetilde{\mathfrak{O}}_{\lambda} \end{array}$$

By construction, the Cauchy-Riemann operators in local coordinates satisfy the commutativity condition:

$$\begin{array}{ccc} T\widetilde{F}_{\lambda'} | \, \mathfrak{O}' & \xrightarrow{T\widehat{F}} & T\widetilde{F}_{\lambda} \\ {\scriptstyle D\overline{\partial}_{J,\lambda'}}\uparrow & & {\scriptstyle D\overline{\partial}_{J,\lambda}}\uparrow \\ T\,\mathfrak{O}' & \xrightarrow{Tf} & T\widetilde{\mathfrak{O}}_{\lambda} \end{array}$$

$D\overline{\partial}_{J,\lambda'}(0,0,0)$ is *sc*-Fredholm by Lemma 7.3.1, so it follows that $D\overline{\partial}_{J,\lambda}(a_0, v_0, \eta_0)$ is *sc*-Fredholm. ■

Proof of Proposition 7.3.1 First assume $\mathfrak{O}_\lambda$ is centered at a stable map for which the underlying Riemann surface is stable. It remains to verify the index formula:

$$\mathrm{ind}(D\overline{\partial}_J(a_0,v_0,\eta_0)\cdot) = n\chi(S_{a_0}) + \dim DM_{g,h,M_i,M_b} + \mu(u^*TQ, j, u^*|_{\partial S}TL) \qquad (7.3.13)$$

By Proposition 7.3.3, we know that the index of $D\overline{\partial}_J(a,v,\eta)$ is constant for tuples (a,v,η) close to (a_0,v_0,η_0). In particular, it is enough to prove the index formula for triples (a,v,η) such that all coordinates of a are non-zero. From the proof of Proposition 7.3.4, we can assume that the set of nodes is empty. In particular, we can assume $\mathcal{G}_\lambda$ is centered at a tuple $((0,0),(S,j(0),M,u))$, where $\alpha = (S,j,M)$ is a smooth Riemann surface. It then follows from classical index calculations that

$$\mathrm{ind}(D_2\mathrm{Fill}\,\overline{\partial}_J(0,0))\cdot = n\chi(S) + \mu((u^*TQ,j),(u^*|_{\partial S}TL))$$

Here the first coordinate is valued in V_λ and the second in E_u. By the proof of Proposition 7.3.3, we know that the index of the full differential ind $D\mathrm{Fill}\,\overline{\partial}_J(0,0)$ is independent of v. Thus

$$\begin{aligned}\mathrm{ind}(D\overline{\partial}_J(0,0)\cdot) &= \mathrm{ind}(D_2\mathrm{Fill}\,\overline{\partial}_J(0,0)\cdot) + \dim V_\lambda \\ &= n\chi(S) + \mu((u^*TQ,j),u^*|_{\partial STL}) + \dim V_\lambda \qquad (7.3.14)\end{aligned}$$

The claim follows since when α is non-singular, $\dim V_\lambda = \dim DM_{g,h,M_i,M_b}$.

If the underlying Riemann surface is unstable, then nontrivial stabilization conditions are imposed on E_u. These conditions decrease the index in (7.3.14) by $2\#\Sigma$. However, observe that

$$\dim DM_{g,h,M_i\cup\Sigma,M_b} = \dim DM_{g,h,M_i,M_b} + 2\#\Sigma$$

Thus the contribution to the index due to Σ is null, and (7.3.13) holds. ■

Theorem 7.3.1. The Cauchy-Riemann section $\overline{\partial}_J : \widetilde{\mathfrak{O}}_\lambda \to \widetilde{W}_\lambda$ is *sc*-Fredholm.

Proof: Smoothness of $\overline{\partial}_J$ follows from Proposition 7.2.1, Proposition 7.1.1, and Lemma 7.1.1. It remains to verify hypotheses $(1)-(3)$ in Theorem 7.0.1 near 0. Hypothesis (1) is satisfied by Proposition 7.3.1. Hypothesis (2) follows from Proposition 7.2.3, Proposition 7.1.3, and Remark 7.1.3. Hypothesis (3) follows from Proposition 7.2.2 and Proposition 7.1.2. ■

Theorem 1.3.2 from the introduction follows from Theorem 7.3.1 and Proposition 4.6.1.

Part III

Appendices

8 Appendix 1 : Lego Pieces and Deligne-Mumford Spaces

In this section we provide the analytical details for the construction of the Deligne-Mumford space for curves without boundary, in the polyfold framework. The corresponding analysis in the case of closed curves was taken up in [HWZ3], however in this article we approach the problem by using lego pieces. Our approach is similar to the approach taken in Section 2 of this article, in which we establish the *sc*-Fredholm property for the Cauchy-Riemann operator on the space of stable maps. The novel difficulty is that one must establish Fredholm properties on a family of maps with both degenerating domains and *targets*.

First, we develop the ambient space in which to apply the implicit function theorem. In this section we work on a fixed Riemann surface (S, j, M, D) equipped with two disk structures $\mathbf{D^1}$ and $\mathbf{D^2}$. Given $a \in B_D$, we denote the surface (S, j, M, D) glued with respect to $\mathbf{D^1}$ by (S_a, j_a, M_a, D_a) and the surface S glued with respect to $\mathbf{D^2}$ by $(\widehat{S}_a, \widehat{j}_a, \widehat{M}_a, \widehat{D}_a)$.

Definition 8.0.0. Let S and S' be smooth Riemann surfaces. We say a continuous map $u : S \to S'$ is an H^{m+3} diffeomorphism, for $m \geq 0$, if u maintains H^{m+3} regularity and admits an H^{m+3} inverse. ■

Definition 8.0.1. We define the set $\mathfrak{S}$ of morphisms:

$$\mathfrak{S} := \Big\{ u : (S_a, j_a, M_a, D_a) \to (\widehat{S}_b, \widehat{j}_b, \widehat{M}_b, \widehat{D}_b) :$$

$$u \text{ is an } H^3 \text{ diffeomorphism, and no coordinate of } a \text{ or } b \text{ is } 0 \Big\} \blacksquare$$

We begin by establishing an M-polyfold structure on $\mathfrak{S}$. Let $\mathfrak{S}_{ab}$ denote the subset of $\mathfrak{S}$ consisting of maps from S_a to $\widehat{S}_b$. Choose N sufficiently large and an embedding $\phi : \widehat{S}_b \to \mathbb{R}^N$, for b sufficiently small. We equip $\mathfrak{S}_{ab}$ with the M-polyfold structure it inherits as a subspace of $H^3(S_a, \mathbb{R}^N)$, which is induced by retraction along a tubular neighborhood.

Proposition 8.0.1. (compare Theorem 2.6 [HWZ0]) Let $\phi_v : S_a \to S_a$ be a family of diffeomorphisms parametrized by $v \in V$, an open neighborhood of the origin $0 \in \mathbb{R}^n$. The reparametrization map defined by

$$\widetilde{\Phi} : V \times \mathfrak{S}_{ab} \to \mathfrak{S}_{ab}, \quad (v, u) \to u \circ \phi_v$$

is *sc*-smooth.

Proof: Let E denote the *sc*-Hilbert space $E := H^3(S_a, \mathbb{R}^N)$, for N sufficiently large. The m^{th} level of E is given by $H^{3+m}(S_a, \mathbb{R}^N)$. We closely follow Theorem 2.6 [HWZ0]. We proceed by induction. The inductive hypothesis is:

(S_k) The map

$$\Phi : V \oplus E \to E$$

given by

$$(v, u) \to u \circ \phi_v$$

is sc^k and for every projection $\pi_j : T^k E \to E^j$ onto a factor of $T^k E$, the composition $\pi_j \circ T^k \Phi$ is a finite linear combination of maps of the form

$$V \oplus E_m \oplus (\mathbb{R}^n)^p \to E^j$$

$$(v, h, a_1, \cdots, a_p) \to \Phi(v, D^\alpha h) \cdot f(v, a_1, \cdots, a_p) \tag{8.0.1}$$

where $f : V \oplus (\mathbb{R}^n)^p \times S_a \to \mathbb{R}$ is a smooth function which is linear in the variables a_i. In addition, $|\alpha| \leq m - j$ and $p \leq k$.

(S_0) First we show that Φ is continuous. Consider a fixed point $(v_0, u_0) \in V \oplus E_m$, and for given $r > 0$ we choose a smooth map $w_0 : S_a \to \mathbb{R}^N$ which satisfies $|u_0 - w_0| \leq r$. We have the estimate:

$$|\Phi(v,u)-\Phi(v_0,u_0)|_m \leq |\Phi(v,u)-\Phi(v,w_0)|_m + |\Phi(v,w_0)-\Phi(v_0,w_0)|_m + |\Phi(v_0,w_0)-\Phi(v_0,u_0)|_m$$

$$:= I + II + III$$

The proof that I, II, III, all go to 0 as $v \to v_0$, $u \to u_0$, and $r \to 0$ follows identically as in Theorem 1.26 [HWZ0].

(S_1) The candidate for the linearization $D\Phi(v,u) : V \oplus E \to E$ at the point $(v,u) \in V \oplus E_1$ is:

$$D\Phi(v,u)(a,h) = \Phi(v,h) + \Phi(v,Du) \cdot D\phi_v \cdot a, \quad (a,h) \in V \oplus E_1 \tag{8.0.2}$$

The function $V \times S_a \to TS_a$ defined by

$$(v,x) \to D\phi_v(x) \cdot a$$

is smooth so it follows that the map:

$$T(V \oplus E) \to TE, \quad (v,u,a,h) \to (\Phi(v,u), D\Phi(v,u)(a,h))$$

is continuous where $D\Phi(v,u)(a,h)$ is defined by (8.0.2).

We will show that the right-hand side of (8.0.2) actually defines the linearization of Φ. With $(v,u,a,h) \in T(V \oplus E)$, consider:

$$\Phi(v+a,u+h) - \Phi(v,u) - D\Phi(v,u)(a,h)$$

$$= \Phi(v+a,h) - \Phi(v,h) + \int_0^1 [\Phi(v+\tau a, Du) D\phi_v \cdot a - \Phi(v,Du) \cdot D\phi_v \cdot a] d\tau$$

$$= I + II$$

First consider the term I. Φ is linear with respect to the second variable so we have:

$$\frac{1}{|a|+|h|_1}\cdot|\Phi(v+a,h)-\Phi(v,h)|_0=\frac{|h|_1}{|a|+|h|_1}\cdot|\Phi(v+a,\frac{h}{|h|_1})-\Phi(v,\frac{h}{|h|_1})|_0$$

for $h\neq 0$. Since the inclusion $E_1\to E_0$ is compact, by restricting to a subsequence we may assume that $\frac{h}{|h|_1}$ converges in E_0 as $h\to 0$. Φ is continuous, so we have:

$$\frac{1}{|a|+|h|_1}\cdot|\Phi(v+a,h)-\Phi(v,h)|_0\to 0$$

as $|a|+|h|_1\to 0$. Now we consider II. For $a\neq 0$,

$$\frac{1}{|a|+|h|_1}\cdot|\int_0^1[\Phi(v+\tau a,Du)\cdot D\phi_v\cdot a-\Phi(v,Du)\cdot D\phi_v\cdot a]d\tau|_0$$

$$\leq\frac{|a|}{|a|+|h|_1}\cdot\int_0^1|\Phi(v+\tau a,Du)D\phi_{v+\tau a}\cdot\frac{a}{|a|}-\Phi(v,Du)D\phi_v\cdot\frac{a}{|a|}|_0 d\tau$$

We note that Φ is sc^0 and $(v,x)\to D\phi_v(x)$ is smooth. Therefore, the above expression limits to 0 as $|a|+|h|_1\to 0$. This establishes (8.0.2).

We conclude the argument by assuming that the claim S_k holds and showing S_{k+1} holds. By induction, the composition $\pi\circ T^k\Phi$ is the linear combination of maps of the form (8.0.1). Consider a map of the form appearing in (8.0.1):

$$\Psi: V\oplus E_m\oplus(\mathbb{R}^n)^p\to E^j,\quad (v,h,a)\to\Phi(v,D^\alpha h)\cdot f(v,a)$$

By arguing similarly to the verification of (S_1), it follows that Ψ is sc^1. It is enough show that the compositions $\pi\circ T\Psi$ of the tangent map $T\Psi: T(V\oplus E_m\oplus(\mathbb{R}^n)^p)\to T(E^j)$ with projections onto factors of $T(E^j)$ are linear combinations of maps of the form (8.0.1).

The tangent map is given by

$$T\Psi(v,h,a,\delta v,\delta h,\delta a) = (\Psi(v,h,a), D\Psi(v,h,a)(\delta v,\delta h,\delta a))$$

where $(v,h,a) \in V \oplus E_m \oplus (\mathbb{R}^n)^p$ and $(\delta v,\delta h,\delta a) \in \mathbb{R}^n \oplus E_{m+1} \oplus (\mathbb{R}^n)^p$. By the induction hypothesis, the projection onto the first factor has the required form. Using the chain rule and the explicit form of the linearization of Φ described in (8.0.2) it follows that $D\Psi$ is a linear combination of maps of the required form as well. This completes the induction.

First suppose $\widehat{S}_b$ is closed. $\widehat{S}_b$ admits an embedding $\phi : \widehat{S}_b \hookrightarrow \mathbb{R}^N$ for N sufficiently large. There exists a tubular neighborhood $U \subset \mathbb{R}^N$, with a projection $\pi : U \to \phi(\widehat{S}_b)$. The subset $H^3(S_a, U)$ is an open subset of $H^3(S_a, \mathbb{R}^N)$ and so the map

$$\Phi|_{V \times H^3(S_a,U)} : V \times H^3(S_a,U) \to H^3(S_a,U)$$

is *sc*-smooth by the preceding argument. Consider the *sc*-smooth map

$$\widehat{\pi} : V \times H^3(S_a,U) \to V \times H^3(S_a, \phi(\widehat{S}_b))$$

$$(v,u) \to (v, \pi \circ u)$$

and the *sc*-smooth maps

$$\iota : V \times \mathfrak{S}_{ab} \to V \times H^3(S_a, \phi(\widehat{S}_b))$$

$$(v,u) \to (v, \phi \circ u)$$

$$\widehat{\phi} : V \times H^3(S_a, \phi(\widehat{S}_b)) \to V \times H^3(S_a, \widehat{S}_b)$$

$$(v, u) \to (v, \phi^{-1} \circ u)$$

The composition

$$\widehat{\phi} \circ \widehat{\pi} \circ \Phi \circ \iota : V \times \mathfrak{S}_{ab} \to V \times \mathfrak{S}_{ab}$$

is *sc*-smooth. Since $\widetilde{\Phi} := \widehat{\phi} \circ \widehat{\pi} \circ \Phi \circ \iota$, the claim follows when S_a is closed.

Now suppose S_a has boundary. Consider the space:

$$\mathfrak{S}'_{ab} := \{u : S_a \to \widehat{S}_b \ : \ u \text{ is an } H^3 \text{ embedding, and no coordinate of } a \text{ or } b \text{ is } 0\}$$

Note that maps in $\mathfrak{S}'_{ab}$ need not respect the boundary conditions. The argument for the closed case above shows that the map

$$\widetilde{\Phi} : V \times \mathfrak{S}'_{ab} \to \mathfrak{S}'_{ab}$$

$$(v, u) \to u \circ \phi_v$$

is *sc*-smooth. The restriction $\widetilde{\Phi}|V \times \mathfrak{S}_{ab}$ is therefore *sc*-smooth. ■

Lemma 8.0.1. (compare Lemma 3.10 [HWZ1]): Let $w_0 : (S_{a_0}, M_{a_0}, D_{a_0}) \to (\widehat{S}_{b_0}, \widehat{M}_{b_0}, \widehat{D}_{b_0}) \in \mathfrak{S}$ be an H^3 diffeomorphism and let $U(w_0)$ be an open subset containing w_0 in the space of H^3-diffeomorphisms $\mathrm{Diff}((S_{a_0}, M_{a_0}, D_{a_0}), (\widehat{S}_{b_0}, \widehat{M}_{b_0}, \widehat{D}_{b_0}))$. (We assume $\mathrm{Diff}((S_{a_0}, M_{a_0}, D_{a_0}), (\widehat{S}_{b_0}, \widehat{M}_{b_0}, \widehat{D}_{b_0}))$ is equipped with the topology induced by the exponential map.)

Let $\phi_a : S_a \to S_{a_0}$, $\psi_b : \widehat{S}_{b_0} \to \widehat{S}_b$ be core-smooth families of H^3 diffeomorphisms of triples parametrized by open subsets $\{a \in \mathbb{C} \ : |a - a_0| < \epsilon_0\}$, $\{b \in \mathbb{C} \ : |b - b_0| < \epsilon_0\}$, where $\epsilon_0 > 0$. Further, we require $\phi_{a_0} = Id$ and $\psi_{b_0} = Id$. Consider the set of maps $U(a_0, b_0, w_0, U(w_0), \epsilon_0)$ consisting of maps $u \in \mathfrak{S}$ such that $u = \psi_b \circ w \circ \phi_a$ for some

$w \in U(w_0)$, a such that $|a - a_0| < \epsilon_0$, and b such that $|b - b_0| < \epsilon_0$.

Then the subsets of the form $U(a_0, b_0, w_0, U(w_0), \epsilon_0)$ ranging over the choices of $(a_0, b_0, w_0, U(w_0), \{\phi_a\}, \{$ generate a topology for $\mathfrak{S}$.

Proof: Let $U_0 := U(a_0, b_0, w_0, U(w_0), \epsilon_0)$ be the set of diffeomorphisms consisting of points (a, b, u) satisfying $|a - a_0| < \epsilon_0$, $|b - b_0| < \epsilon_0$ and $u = \psi_b^0 \circ w \circ \phi_a^0$ for a diffeomorphism $w \in U(w_0)$ and maps:

$$\phi_a^0 : S_a \to S_{a_0}, \quad \psi_b^0 : \widehat{S}_{b_0} \to \widehat{S}_b$$

Similarly define $U_1 = U(a_1, b_1, w_1, U(w_1), \epsilon_1)$ according to the data $\epsilon_1, a_1, b_1, w_1, U(w_1), \{\phi_a^1\}, \{\psi_b^1\}$. If

$$(a_2, b_2, w_2) \in U_0 \cap U_1$$

then

$$w_2 = \psi_{b_2}^0 \circ v_0 \circ \phi_{a_2}^0 = \psi_{b_2}^1 \circ v_1 \circ \phi_{a_2}^1$$

for diffeomorphisms $v_0 \in U(w_0)$ and $v_1 \in U(w_1)$. By Proposition 8.0.1, we can find neighborhoods $V(v_0) \subset U(w_0)$, $V(v_1) \subset U(w_1)$ and δ_2 sufficiently small such that:

$$\psi_b^0 \circ V(v_0) \circ \phi_a^0 \subset \psi_b^1 \circ V(v_1) \circ \phi_a^1 \tag{8.0.3}$$

for all $|a - a_2| < \epsilon_2$ and $|b - b_2| < \epsilon_2$. Consider the smooth families of diffeomorphisms defined by

$$\widetilde{\phi}_a : (\phi_{a_2}^0)^{-1} \circ \phi_a^0 : S_a \to S_{a_2}$$

$$\widetilde{\psi}_a = \psi_b^0 \circ (\psi_{b_2}^0)^{-1} : \widehat{S}_{b_2} \to \widehat{S}_b$$

for $|a-a_2|$, $|b-b_2|<\epsilon_2$ satisfying $\widetilde{\phi}_{a_2}=Id$ on S_{a_2} and $\widetilde{\psi}_{b_2}=Id$ on $\widehat{S}_{b_2}$. Consider also the H^3 neighborhood of $w_2: S_{a_2}\to \widehat{S}_{b_2}$ in $\mathrm{Diff}(S_{a_2},\widehat{S}_{b_2})$ given by

$$U(w_2):=\psi^0_{b_2}\circ V(v_0)\circ\phi^0_{a_2}$$

We also introduce the set

$$U_2=U(a_2,b_2,w_2,U(w_2),\epsilon_2)$$

$$=\{(a,b,w):|a-a_2|<\epsilon_2,|b-b_2|<\epsilon_2 \text{ and } w=\widetilde{\psi}^0_b\circ u\circ\widetilde{\phi}^0_a \text{ for some } u\in U(w_2)\}$$

If $(a,b,w)\in U_2$ then $w=\widetilde{\psi}^0_b\circ u\circ\widetilde{\psi}^0_a$ and $u\in U(w_2)$. By definition there exists $v\in V(v_0)$ such that

$$u=\psi^0_{b_2}\circ v\circ\phi^0_{a_2}$$

Therefore,

$$w=\widetilde{\psi}^0_b\circ\psi^0_{b_2}\circ v\circ\phi^0_{a_2}\circ\widetilde{\phi}^0_a=\psi^0_b\circ v\circ\phi^0_a$$

By (8.0.3), it follows that we have:

$$w=\psi^1_b\circ v'\circ\phi^1_a$$

for $v\in V(v_1)\subset U(v_1)$ so $(a,b,w)\in U_0\cap U_1$. It follows that $U_2\subset U_0\cap U_1$. ■

Proposition 8.0.2. (compare [HWZ0] Theorem 1.42) With respect to the topology defined in Lemma 8.0.1, $\mathfrak{S}$ is second countable and paracompact. It is also an M-polyfold.

Proof: First suppose S is closed. Let $(a_0,b_0,w_0)\in\mathfrak{S}$. Without loss of generality, we may assume w_0 is smooth. Consider an open neighborhood O of the zero-section in $H^3\Gamma(w_0^*T\widehat{S}_{b_0},\widehat{M}_{b_0},\widehat{D}_{b_0})$. Let B_{a_0,δ_0}, B_{b_0,δ_0} be the open disks of radius δ_0 centered at a_0,

b_0, respectively. For δ_0, O sufficiently small, the map

$$exp : B_{a_0,\delta_0} \times B_{b_0,\delta_0} \times O \to \mathfrak{S}$$

provides the inverse chart for $\mathfrak{S}$. Explicitly, this map is given by

$$(a, b, \eta) \to (a, b, \psi_b \circ exp_{w_0}(\eta) \circ \phi_a), \quad (a, b, \eta) \in O$$

We can assume each of our charts are centered at smooth diffeomorphisms; thus we assume w_0 is smooth. Let another inverse chart be given by

$$(a', b', \eta') \to (a', b', \psi'_{b'} \circ exp_{w'_0}\eta' \circ \phi'_{a'})$$

Where the images of the charts intersect we have:

$$\psi_b \circ exp_{w_0}\eta \circ \phi_a = \psi'_b \circ exp_{w'_0}\eta' \circ \phi'_a \tag{8.0.4}$$

Consider the transition function

$$(a, b, \eta) \to (a, b, \eta')$$

By (8.4), η' is given by

$$\eta'(a, b, \eta) = exp^{-1}_{w'_0}\{(\psi'^{-1}_b \circ \psi_b \circ exp_{w_0}\eta \circ \phi_a \circ \phi'^{-1}_a)\}$$

By the smoothness of the exponential map on sections ([E] Lemma 4.1) and of reparametrization (Proposition 8.0.1) it follows that the transitions are *sc*-smooth.

When S has boundary we may maintain the same considerations as above except in place of O we consider an open neighborhood of the zero-section

$$O' \subset \Gamma(w_0^* T\widehat{S}_{b_0}, w_0|^*_{\partial S_{a_0}} T\widehat{S}_{b_0}, \widehat{M}_{b_0}, \widehat{D}_{b_0}) \quad \blacksquare$$

Now we introduce nodal behavior:

Definition 8.0.2. We define the *sc*-Hilbert space E_{cyl} to be the set of pairs (h^+, h^-)

$$h^+ \in H_c^{3,\delta_0^0}((\frac{1}{4}, \infty) \times S^1, \mathbb{R}^2), \quad h^- \in H_c^{3,\delta_0^0}((-\infty, -\frac{1}{4}) \times S^1, \mathbb{R}^2)$$

for which there exist constants $h_\infty^\pm \in \mathbb{R}^2$ such that

$$h^+ - h_\infty^+ \in H^{3,\delta_0^0}((\frac{1}{4}, \infty) \times S^1, \mathbb{R}^2), \quad h^- - h_\infty^- \in H^{3,\delta_0^0}((-\infty, -\frac{1}{4}) \times S^1, \mathbb{R}^2)$$

We define the *sc*-Hilbert space E_{strip} to be the set of pairs (h^+, h^-)

$$h^+ \in H_{c,\,real}^{3,\delta_1^0}((\frac{1}{4}, \infty) \times I, \mathbb{R}^2), \quad h^- \in H_{c,\,real}^{3,\delta_1^0}((-\infty, -\frac{1}{4}) \times I, \mathbb{R}^2)$$

for which there exist constants $h_\infty^\pm \in \mathbb{R} \times \{0\}$ such that

$$h^+ - h_\infty^+ \in H_{real}^{3,\delta_1^0}((\frac{1}{4}, \infty) \times I, \mathbb{R}^2), \quad h^- - h_\infty^- \in H_{real}^{3,\delta_1^0}((-\infty, -\frac{1}{4}) \times I, \mathbb{R}^2)$$

Lastly, we set

$$E_{neck} := \prod_{\{x,y\} \in D_i} E_{cyl} \times \prod_{\{x',y'\} \in D_b} E_{strip} \quad \blacksquare$$

Definition 8.0.3. $\mathfrak{D}^{3,\delta_0^0,\delta_1^0}_{neck}$ consists of tuples of maps

$$\left((u^+_{\{x,y\}},u^-_{\{x,y\}})_{\{x,y\}\in D_i},(u^+_{\{x',y'\}},u^-_{\{x',y'\}})_{\{x',y'\}\in D_b}\right)$$

where

$$u^+_{\{x,y\}}:(\frac{1}{4},\infty)\times S^1\to[0,\infty)\times S^1,\quad u^-_{\{x,y\}}:(-\infty,-\frac{1}{4})\times S^1\to(-\infty,0]\times S^1$$

$$u^+_{\{x',y'\}}:(\frac{1}{4},\infty)\times[0,1]\to[0,\infty)\times[0,1],\quad u^-_{\{x',y'\}}:(-\infty,-\frac{1}{4})\times[0,1]\to(-\infty,0]\times[0,1]$$

are H^3 embeddings.

Each of the maps in the tuple $\left((u^+_{\{x,y\}},u^-_{\{x,y\}})_{\{x,y\}\in D_i},(u^+_{\{x',y'\}},u^-_{\{x',y'\}})_{\{x',y'\}\in D_b}\right)$ are located in small neighborhood $\mathfrak{O}$ of the natural inclusion, the precise size of which is to be fixed later. However, $\mathfrak{O}$ is chosen small enough such that

$$u^+_{\{x,y\}}(s,t)=(s,t)+h^+_{\{x,y\}}(s,t),\quad u^-_{\{x,y\}}(s',t')=(s',t')+h^-_{\{x,y\}}(s',t')$$

$$u^+_{\{x',y'\}}(s,t)=(s,t)+h^+_{\{x',y'\}}(s,t),\quad u^-_{\{x',y'\}}(s',t')=(s',t')+h^-_{\{x',y'\}}(s',t')$$

for some tuple

$$\left((h^+_{\{x,y\}},h^-_{\{x,y\}})_{\{x,y\}\in D_i},(h^+_{\{x',y'\}},h^-_{\{x',y'\}})_{\{x',y'\}\in D_b}\right)\in E_{neck}\quad\blacksquare$$

Proposition 8.0.3. $\mathfrak{D}^{3,\delta_0^0,\delta_1^0}_{neck}$ is an M-polyfold. ■

Definition 8.0.4. (compare Section 4.1.3 [HWZ3]) Choose coordinates for every

disk $D_z^1 \in \mathbf{D^1}$ and $D_z^2 \in \mathbf{D^2}$ for every $z \in |D|$:

$$h_x : \mathbb{R}^+ \times J_{\{x,y\}} \to D_x^1, \quad h_y : \mathbb{R}^- \times J_{\{x,y\}} \to D_y^1$$

$$k_x : \mathbb{R}^+ \times J_{\{x,y\}} \to D_x^2, \quad k_y : \mathbb{R}^- \times J_{\{x,y\}} \to D_y^2$$

where $J_{\{x,y\}} = S^1$ if $\{x, y\} \in D_i$ and $J_{\{x,y\}} = [0, 1]$ if $\{x, y\} \in D_b$. Denote by $\mathfrak{D}^{3,\delta_0^0,\delta_1^0}$ the set consisting of maps $u : S \to S$ satisfying the following properties:

1. (a) u lies in a small neighborhood $\mathfrak{O}$ of the identity, the precise size of which is to be fixed later.

1. (b) In particular, $\mathfrak{O}$ is chosen small enough such that:

$$u(\mathbf{D}^1(-h)) \subset \mathbf{D^2}$$

for all $u \in \mathfrak{O}$ for some fixed $h > 0$. For concreteness we assume $h := \frac{1}{4}$.

2. $u(p) = p$ for $m \in M$, and $u(z) = z$ for $z \in |D|$.

3. In local coordinates

$$u^+_{\{x,y\}} := k_x^{-1} \circ u \circ h_x, \quad u^-_{\{x,y\}} = k_y^{-1} \circ u \circ h_y$$

we have:

$$(u^+_{\{x,y\}}, u^-_{\{x,y\}})_{\{x,y\} \in D} \in \mathfrak{D}^{3,\delta_0^0,\delta_1^0}_{neck} \quad \blacksquare$$

Proposition 8.0.4. $\mathfrak{D}^{3,\delta_0^0,\delta_1^0}$ is an M-polyfold.

Proof: Define the set $\mathfrak{D}$ to be the set of maps $u : S \to S$ satisfying Criteria 1 (a), 2, 3, and 4 in Definition 8.0.4. We equip $\mathfrak{D}$ with the $H^{3,\delta_0^0,\delta_1^0}$ topology. The set $\mathfrak{D}^{3,\delta_0^0,\delta_1^0}$ is an open subset of the space of maps $\mathfrak{D}$. Thus to establish the claim, it suffices to show that $\mathfrak{D}$ is an M-polyfold.

Set $(Tk_x^{-1} \cdot \eta \circ h_x, Tk_y^{-1} \cdot \eta \circ h_y) = (\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})$. Observe that $\mathfrak{D}$ near the identity is modelled on the *sc*-Hilbert space $\mathfrak{T}$, where $\mathfrak{T}$ is defined as follows:

$$\mathfrak{T} := \Big\{ \eta \in H^3_{loc}\Gamma(TS, T\partial S) \ : \ \big(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}}\big)_{\{x,y\}\in D} \in \mathfrak{D}^{3,\delta_0^0,\delta_1^0}_{neck}$$

$$\eta(p) = 0, \ p \in M \cup |D|, \ ||\eta||_{3,\delta_0^0,\delta_1^0} < \infty \Big\} \tag{8.0.5}$$

Consider the open subsets of $\mathfrak{T}$ defined by

$$\mathfrak{T}(\epsilon) := \Big\{ \eta \in H^3_{loc}\Gamma(TS, T\partial S) \ : \ \big(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}}\big)_{\{x,y\}\in D} \in \mathfrak{D}^{3,\delta_0^0,\delta_1^0}_{neck}$$

$$\eta(p) = 0, \ p \in M \cup |D|, \ ||\eta||_{3,\delta_0^0,\delta_1^0} < \epsilon \Big\}$$

Fix $\epsilon > 0$ sufficiently small so the exponential map

$$\exp : \mathfrak{T} \to \mathfrak{D}$$

$$\eta \to exp_{\mathrm{Id}}\, \eta$$

is a diffeomorphism. The exponential map provides the global chart on $\mathfrak{D}$. ■

8.1 Gluing Maps Between Riemann Surfaces

Consider $(\overline{u}^+, \overline{u}^-) := \left(u^+_{\{x,y\}}, u^-_{\{x,y\}}\right)_{\{x,y\}\in D} \in \mathfrak{D}_{neck}$. For $a \in B_D$, we define the tuple of glued maps:

$$\boxplus_a(\overline{u}^+, \overline{u}^-) := \left(\boxplus_{a_{\{x,y\}}} (u^+_{\{x,y\}}, u^-_{\{x,y\}})\right)_{\{x,y\}\in D}$$

where each map

$$\boxplus_{a_{\{x,y\}}}(u^+_{\{x,y\}}, u^-_{\{x,y\}}) : (\frac{1}{4}, R_{\{x,y\}} - \frac{1}{4}) \times J_{\{x,y\}} \to [0, R'_{\{x,y\}}] \times J_{\{x,y\}},$$

$$J_{\{x,y\}} = S^1 \text{ if } \{x,y\} \in D_i,\ J_{\{x,y\}} = I \text{ if } \{x,y\} \in D_b$$

is defined by the formula

$$\boxplus_{a_{\{x,y\}}}(u^+_{\{x,y\}}, u^-_{\{x,y\}})[(s,t)] := \beta_{R_{\{x,y\}}}(s)u^+_{\{x,y\}}(s,t)+(1-\beta_{R_{\{x,y\}}}(s))u^-_{\{x,y\}}(s'+R'_{\{x,y\}}, t'+\theta'_{\{x,y\}}) \tag{8.1.1}$$

and

$$s' = s - R_{\{x,y\}}$$

$$t' = t - \theta_{\{x,y\}}$$

$$R'_{\{x,y\}} = R_{\{x,y\}} + d^+_{\{x,y\}} - d^-_{\{x,y\}}$$

$$\theta'_{\{x,y\}} = \theta_{\{x,y\}} + \theta^+_{\{x,y\}} - \theta^-_{\{x,y\}}$$

Re-arranging equation (8.1.1):

$$\boxplus_{a_{\{x,y\}}}\left(u^+_{\{x,y\}}, u^-_{\{x,y\}}\right)(s,t) = (s,t)+(d^+_{\{x,y\}}, \theta^+_{\{x,y\}})+\beta_{R_{\{x,y\}}}(s)r^+(s,t)+(1-\beta_{R_{\{x,y\}}}(s))r^-_{\{x,y\}}(s',t')$$

We also define the anti-glued map

$$\ominus_{a_{\{x,y\}}}(u^+_{\{x,y\}}, u^-_{\{x,y\}}) : \mathbb{R} \times J_{\{x,y\}} \to \mathbb{R} \times J_{\{x,y\}}$$

$$\ominus_{a_{\{x,y\}}}(u^+_{\{x,y\}}, u^-_{\{x,y\}})(s,t) := -[1 - \beta_{R_{\{x,y\}}}(s)]r^+_{\{x,y\}}(s,t) + \beta_{R_{\{x,y\}}}(s)r^-_{\{x,y\}}(s',t')$$

Choosing $R_{\{x,y\}}$ large enough and the norms of $r^+_{\{x,y\}}$ and $r^-_{\{x,y\}}$ sufficiently small, we can ensure that

$$\boxplus_a(u^+, u^-)([\frac{1}{4}, R - \frac{1}{4}] \times S^1) \subset [0, R'] \times S^1$$

and that $\boxplus_a(u^+, u^-)$ is an embedding.

Now consider a globally defined glued map $\boxplus_a(u) : S_a \to \widehat{S}_{b(a,u)}$ defined by

$$k^{-1}_{b_{\{x,y\}}} \circ \boxplus_a(u) \circ h_{a_{\{x,y\}}}(s,t) := \boxplus_{a_{\{x,y\}}}(u^+_{\{x,y\}}, u^-_{\{x,y\}})(s,t)$$

if $(s,t) \in [\frac{1}{4}, R_{\{x,y\}} - \frac{1}{4}] \times J_{\{x,y\}}$ and

$$\boxplus_a(u)(z) := u(z) \text{ if } z \in S_{core}$$

The gluing parameters $b = b(a,u) = (b_{\{x,y\}})_{\{x,y\} \in D}$ are determined as follows:

If

$$a_{\{x,y\}} = |a_{\{x,y\}}|e^{-2\pi i\theta} \neq 0$$

then

$$b_{\{x,y\}} = |b_{\{x,y\}}|e^{-2\pi i\theta'}$$

where

$$\phi(|b_{\{x,y\}}|) = \phi(|a_{\{x,y\}}|) + d^+_{\{x,y\}}(u) - d^-_{\{x,y\}}(u)$$

for

$$\theta'_{\{x,y\}} = \theta_{\{x,y\}} + \theta^+_{\{x,y\}}(u) - \theta^-_{\{x,y\}}(u)$$

When $a = 0$, we set $b = 0$. In summary,

Theorem 8.1.1. (compare Proposition 4.3 [HWZ3]) Given a map $u \in \mathfrak{D}^{3,\delta^0_0,\delta^0_1}$, there exists a canonical glued diffeomorphism

$$\boxplus_a(u) : S_a \to \widehat{S}_{b(a,u)}$$

for a small enough. The gluing parameters $b(a, u)$ are determined by the formulas above. Lastly, we define the set $\overline{\mathfrak{S}}$ as follows:

$$\overline{\mathfrak{S}} := \mathfrak{S} \bigsqcup \mathfrak{D}^{3,\delta^0_0,\delta^0_1} \quad \blacksquare$$

8.2 M-Polyfold Structure on $\overline{\mathfrak{S}}$

Definition 8.2.1. We define the *sc*-Hilbert space E^0_{cyl} to be the set of pairs (h^+, h^-) :

$$h^+ \in H^{3,\delta_0}((\frac{1}{4}, \infty) \times S^1, \mathbb{R}^2), \quad h^- \in H^{3,\delta_0}((-\infty, -\frac{1}{4}) \times S^1, \mathbb{R}^2)$$

We also define the map

$$\rho : B_{\frac{1}{2}} \times E_{cyl} \to B_{\frac{1}{2}} \times E_{cyl}$$

$$(a, v^+, v^-) \to (a, \rho_a(v^+, v^-))$$

by the formula

$$\rho_a(v^+, v^-) = (v^+_\infty, v^-_\infty) + \pi_a(v^+ - v^+_\infty, v^- - v^-_\infty)$$

where π_a is the projection defined by:

$$\oplus_a \pi_a(r^+, r^-) = (r^+, r^-)$$

$$\ominus_a \pi_a(r^+, r^-) = 0 \quad \blacksquare$$

Definition 8.2.2. We define E^0_{strip} to be the *sc*-subspace of pairs E_{strip} consisting of pairs (v^+, v^-) such that $v^+_\infty = v^-_\infty$. Also, we define the map

$$\widetilde{\rho} : [0, \frac{1}{2}) \times E_{strip} \to [0, \frac{1}{2}) \times E_{strip}$$

$$(a, v^+, v^-) \to (a, \widetilde{\rho}_a(v^+, v^-))$$

by the formula

$$\widetilde{\rho}_a(v^+, v^-) = (v^+_\infty, v^-_\infty) + \widetilde{\pi}_a(v^+ - v^+_\infty, v^- - v^-_\infty)$$

where $\widetilde{\pi}_a$ is the projection defined by:

$$\oplus_a \widetilde{\pi}_a(r^+, r^-) = (r^+, r^-)$$

$$\ominus_a \widetilde{\pi}_a(r^+, r^-) = 0 \quad \blacksquare$$

Proposition 8.2.1. (compare Lemma 3.1, Theorem 3.2, Lemma 3.3 [HWZ1]) The map ρ is an *sc*-smooth retraction. In addition, the projection ρ_a is the projection onto the kernel of the map

$$P_a : E_{cyl} \to H_c^{3,\delta^0_0}(C_a)$$

given by

$$(h^+, h^-) \xrightarrow{P_a} \ominus_a(h^+ - h^+_\infty, h^- - h^-_\infty)$$

along the kernel of the map

$$P^+_a : E_{cyl} \to \mathbb{R}^2 \oplus H^3(Z_a)$$

given by

$$(h^+, h^-) \xrightarrow{P^+_a} (h^+_\infty - h^-_\infty, \oplus_a(h^+, h^- + h^+_\infty - h^-_\infty))$$

Proof: The *sc*-smoothness of ρ follows from Section 1. Now we will show ρ_a is a retraction. The argument closely follows Lemma 3.1 [HWZ1]. Suppose $(h^+, h^-) \in E_{cyl}$ and write $h^\pm = h^\pm_\infty + r^\pm$ and set $\pi_a(r^+, r^-) = (\eta^+, \eta^-)$. Then

$$\rho_a(h^+, h^-) = (h^+_\infty, h^-_\infty) + \pi_a(r^+, r^-) = (h^+_\infty, h^-_\infty) + (\eta^+, \eta^-)$$

We have the formula for the projection π_a:

$$\begin{bmatrix} \eta^+(s,t) \\ \eta^-(s',t') \end{bmatrix} = \frac{1}{\gamma_a} \begin{bmatrix} \beta_a & -(1-\beta_a) \\ 1-\beta_a & \beta_a \end{bmatrix} \cdot \begin{bmatrix} \beta_a r^+ + (1-\beta_a) r^- \\ (2\beta_a - 1) a v_a(r^+, r^-) \end{bmatrix}$$

From this formula we can deduce

$$(\eta^+, \eta^-) = (c, c) + (\xi^+, \xi^-)$$

where $c = a v_a(r^+, r^-)$ is the asymptotic limit of η^+ and η^-. Then we have

$$\rho_a(h^+, h^-) = (h^+_\infty + c, h^-_\infty + c) + (\xi^+, \xi^-)$$

Applying ρ_a to both sides:

$$\rho_a \circ \rho_a(h^+, h^-) = (h^+_\infty + c, h^-_\infty + c) + \pi_a(\xi^+, \xi^-)$$

$$= (h^+_\infty, h^-_\infty) + \pi_a(r^+, r^-) = \rho_a(h^+, h^-)$$

Part 2 follows similarly to Lemma 3.3 [HWZ1]. Also see the Proposition 8.2.2. ■

We also have a boundary version:

Proposition 8.2.2. (compare Lemma 3.3 [HWZ1]) The map $\widetilde{\rho}$ is an *sc*-smooth retraction. In addition, the projection $\widetilde{\rho}_a$ is the projection onto the kernel of the map

$$\widetilde{P}_a : E_{strip} \to H^{3,\delta_1^0}_{c,\,real}(C_a)$$

given by

$$(h^+, h^-) \xrightarrow{\widetilde{P}_a} \ominus_a(h^+ - h^+_\infty, h^- - h^-_\infty)$$

along the kernel of the map

$$\widetilde{P}^+_a : E_{strip} \to \mathbb{R} \oplus H^3_{real}(Z_a)$$

given by

$$(h^+, h^-) \xrightarrow{\widetilde{P}^+_a} (h^+_\infty - h^-_\infty, \oplus_a(h^+, h^- + h^+_\infty - h^-_\infty))$$

Proof: The *sc*-smoothness of $\widetilde{\rho}$ follows from Section 1. Now we will show $\widetilde{\rho}_a$ is a retraction. Suppose $(h^+, h^-) \in E_{strip}$ and write $h^\pm = h^\pm_\infty + r^\pm$ and set $\widetilde{\pi}_a(r^+, r^-) = (\eta^+, \eta^-)$. Then

$$\widetilde{\rho}_a(h^+, h^-) = (h^+_\infty, h^-_\infty) + \widetilde{\pi}_a(r^+, r^-) = (h^+_\infty, h^-_\infty) + (\eta^+, \eta^-)$$

We have the formula for the projection $\widetilde{\pi}_a$:

$$\begin{bmatrix} \eta^+(s,t) \\ \eta^-(s',t') \end{bmatrix} = \frac{1}{\gamma_a} \begin{bmatrix} \beta_a & -(1-\beta_a) \\ 1-\beta_a & \beta_a \end{bmatrix} \cdot \begin{bmatrix} \beta_a r^+ + (1-\beta_a) r^- \\ (2\beta_a - 1) a v_{a,\, real}(r^+, r^-) \end{bmatrix} \tag{8.2.1}$$

From this formula we can deduce

$$(\eta^+, \eta^-) = (c, c) + (\xi^+, \xi^-)$$

where $c = a v_{a,\, real}(r^+, r^-)$ is the asymptotic limit of η^+ and η^-. Then we have:

$$\widetilde{\rho}_a(h^+, h^-) = (h^+_\infty + c, h^-_\infty + c) + (\xi^+, \xi^-)$$

Applying $\widetilde{\rho}_a$ to both sides:

$$\widetilde{\rho}_a \circ \widetilde{\rho}_a(h^+, h^-) = (h^+_\infty + c, h^-_\infty + c) + \widetilde{\pi}_a(\xi^+, \xi^-)$$

$$= (h^+_\infty, h^-_\infty) + \widetilde{\pi}_a(r^+, r^-) = \widetilde{\rho}_a(h^+, h^-)$$

Now we establish Part 2. We first show that

$$E_{strip} = \ker P^+_a \bigoplus \ker P^-_a \tag{8.2.2}$$

Consider the linear map

$$P : E_{strip} \to \mathbb{R} \oplus H^3_{real}(Z_a) \oplus H^{3, \delta^0_1}_{c,\, real}(C_a)$$

$$(h^+, h^-) \to (h^+_\infty - h^-_\infty, \oplus_a(h^+, h^- + h^+_\infty - h^-_\infty), \ominus_a(h^+ - h^+_\infty, h^- - h^-_\infty)$$

Observe that 8.2.2 is equivalent to showing that P is bijective. Suppose $(h^+, h^-) \in \ker P$. Then $h^+_\infty - h^-_\infty = 0$ and

$$\oplus_a(h^+, h^-) = 0, \qquad \ominus_a(h^+, h^-) = 0$$

By Proposition 12.0.1 we have $h^+ = 0$ and $h^- = 0$, and injectivity follows. Now suppose $(\xi, \widehat{u}, \widehat{v}) \in \mathbb{R} \oplus H^3_{real}(Z_a) \oplus H^{3,\delta^0_1}_{c,\,real}(C_a)$. By Proposition 12.0.1 there is a unique pair $(u^+, u^-) \in E^0_{strip}$ solving the system

$$\oplus_a(u^+, u^-) = \widehat{u}, \qquad \ominus_a(u^+, u^-) = \widehat{v}$$

We have a decmposition $u^\pm = c + r^\pm$ where $(r^+, r^-) \in H^{3,\delta^1_0}_{real}(\mathbb{R}^\pm \times I)$ such that

$$\ominus_a(u^+, u^-) = \ominus_a(r^+, r^-) = \widehat{v}$$

We define the maps (h^+, h^-) by

$$h^+ = u^+, \qquad h^- = -\xi + u^-$$

By direct calculation:

$$h^+_\infty - h^-_\infty = \xi$$

$$\oplus_a(h^+, h^- + h^+_\infty - h^-_\infty) = \oplus_a(u^+, u^-) = \widehat{u}$$

and

$$\ominus_a(h^+ - h^+_\infty, h^- - h^-_\infty) = \ominus_a(r^+, r^-) = \widehat{v}$$

This establishes surjectivity and 8.2.2.

Given $(h^+, h^-) \in E_{strip}$ where $h^\pm = h^\pm_\infty + r^\pm$ we have a decomposition:

$$(h^+, h^-) = (w^+, w^-) + (v^+, v^-)$$

such that

$$(w^+, w^-) := (h^+_\infty, h^-_\infty) + \widetilde{\pi}_a(r^+, r^-), \qquad (v^+, v^-) := (\mathrm{Id} - \widetilde{\pi}_a)(r^+, r^-)$$

We will show $(w^+, w^-) \in \ker \widetilde{P}^-_a$ and $(v^+, v^-) \in \ker \widetilde{P}^+_a$. The projection $\mathrm{Id} - \widetilde{\pi}_a$ maps onto $\ker \oplus_a$ so $\oplus_a(v^+, v^-) = 0$. Using 8.2.1, it follows that v^+ and v^- share the asymptotic limit $c = -av_{a,\,real}(r^+, r^-)$. By direct calculation:

$$\widetilde{P}^+_a(v^+, v^-) = (v^+_\infty - v^-_\infty, \oplus_a(v^+, v^- + v^+_\infty - v^-_\infty)$$

$$= (0, \oplus_a(v^+, v^-)) = (0, 0)$$

Thus $(v^+, v^-) \in \ker \widetilde{P}^+_a$. Now observe $w^\pm_\infty = h^\pm_\infty + c$. By direct calculation:

$$\widetilde{\pi}_a(r^+ - c, r^- - c) = (\eta^+ - c, \eta^- - c), \quad \text{where } \widetilde{\pi}_a(r^+, r^-) = (\eta^+, \eta^-)$$

Using this we calculate:

$$\widetilde{P}^-_a(w^+, w^-) = \ominus_a(w^+ - w^+_\infty, w^- - w^-_\infty) = \ominus_a(w^+ - h^+_\infty - c, w^- - h^-_\infty - c)$$

$$= \ominus_a(\eta^+ - c, \eta^- - c) = \ominus_a \circ \widetilde{\pi}_a(r^+ - c, r^- - c) = 0$$

Thus $(w^+, w^-) \in \ker \widetilde{P}^-_a$. ■

Definition 8.2.3. We define the E_D to be the following *sc*-Hilbert space

$$E_D := \prod_{\{x,y\}\in D_i} \{x,y\} \times E_{cyl} \times \prod_{\{x',y'\}\in D_b} \{x',y'\} \times E_{strip}$$

We also define the map

$$\rho : B_D \times E_D \to B_D \times E_D$$

$$((a_{\{x,y\}}, u^+_{\{x,y\}}, u^-_{\{x,y\}})_{\{x,y\}\in D_i}, (a_{\{x',y'\}}, u^+_{\{x',y'\}}, u^-_{\{x',y'\}})_{\{x',y'\}\in D_b}) \to$$

$$((a_{\{x,y\}}, \rho(a_{\{x,y\}}, (u^+_{\{x,y\}}, u^-_{\{x,y\}})), (a_{\{x',y'\}}, \widetilde{\rho}(a_{\{x',y'\}}, (u^+_{\{x',y'\}}, u^-_{\{x',y'\}}))_{\{x',y'\}\in D_b})$$

We also define the *sc*-Hilbert space

$$E^0_D := \prod_{\{x,y\}\in D_i} \{x,y\} \times E^0_{cyl} \times \prod_{\{x',y'\}\in D_b} \{x',y'\} \times E^0_{strip} \quad \blacksquare$$

8.3 Lego Pieces

Remark 8.3.1. There exist a pair of real numbers $r > 0$ and $s > 0$ such that the identity map $Id : S \to S$ sends $S_{core,r'}$ into $\widehat{S}_{core,s'}$ for all $r' \geq r$ and $s' \geq s$. Without loss of generality, we assume $r = \frac{1}{2}$ and $s = 1$. $\blacksquare$

Definition 8.3.1. We define the unions of annuli:

$$A := \bigsqcup_{\{x,y\}\in D} h_x^{-1}((\frac{1}{4}, \frac{1}{2}] \times J_{\{x,y\}}) \cap D_x \cup h_y^{-1}([-\frac{1}{2}, -\frac{1}{4}) \times J_{\{x,y\}}) \cap D_y$$

$$B := \bigsqcup_{\{x,y\}\in D} h_x^{-1}([0,1] \times J_{\{x,y\}}) \cap D_x \cup h_y^{-1}([-1,0] \times J_{\{x,y\}}) \cap D_y$$

Let $\mathcal{S}_{core}$ denote the subset of $H^3(S_{core,\frac{1}{2}}, S_{core,1})$ consisting of a fixed open neighborhood in the space of H^3 morphisms of the natural inclusion $S_{core,\frac{1}{2}} \to \widehat{S}_{core,1}$. We choose $\mathcal{S}_{core}$

small enough such that

$$u(A) \subset \text{int } B$$

for all $u \in S_{core}$. ■

Here we specify the size of $\mathcal{S}_{core}$ with more precision:

Corollary 8.3.1. $\mathcal{S}_{core}$ is an M-polyfold modelled on the following *sc*-Hilbert space:

$$\mathcal{V} := H^3\Gamma(TS_{core,\frac{1}{2}}, T\partial S_{core,\frac{1}{2}}, M, D)$$

Given $\epsilon > 0$, we consider the following open subset:

$$\mathcal{V}(\epsilon) := \left\{ \eta \in \mathcal{V} : ||\eta||_3 < \frac{\epsilon}{3} \right\}$$

We fix ϵ sufficiently small as to guarantee

$$exp_{\text{Identity}}\eta(A) \subset B, \quad \text{Im } exp_{\text{Identity}}\eta \subset \widehat{S}_{core,1}$$

for all $\eta \in \mathcal{V}(\epsilon)$. The chart on $\mathcal{S}_{core}$ is given by:

$$exp : \mathcal{V}(\epsilon) \to \mathcal{S}_{core}$$

$$\eta \to exp_{Identity}\eta \quad ■$$

Definition 8.3.2. We define the open subset E_D^ϵ of E_D by

$$E_D^\epsilon := \left\{ (a, h^+_{\{x,y\}}, h^-_{\{x,y\}})_{\{x,y\}\in D} \in E_D \ : \ \left|(h^+_{\{x,y\}}, h^-_{\{x,y\}})_{\{x,y\}\in D}\right|_{E_D} < \frac{\epsilon}{3} \right\}$$

Definition 8.3.3. We define the annuli:

$$A_x := A \cap D_x, \quad A_y := A \cap D_y$$

for $\{x, y\} \in D$. We also define the *sc*-Hilbert space

$$H := \prod_{\{x,y\} \in D_i} H^3(A_x, \mathbb{R}^2) \times H^3(A_y, \mathbb{R}^2) \times \prod_{\{x',y'\} \in D_b} H^3(A_{x'}, \mathbb{R}^2) \times H^3(A_{y'}, \mathbb{R}^2)$$

Consider the map:

$$T : \mathcal{V} \times E_D \to H$$

$$(\eta, \psi^+_{\{x,y\}}, \psi^-_{\{x,y\}})_{\{x,y\} \in D} \to (Tk_x^{-1} \cdot \eta - \psi^+_{\{x,y\}} \circ h_x^{-1}|_{A_x}, Tk_y^{-1} \cdot \eta - \psi^-_{\{x,y\}} \circ h_y^{-1}|_{A_y})_{\{x,y\} \in D}$$

We define the *sc*-Hilbert space

$$\mathcal{T}'_{fiber} := \ker T$$

We define the open subset of $\mathcal{T}'_{fiber}$:

$$\mathcal{T}_{fiber}(\epsilon) := \ker T \cap \mathcal{V}(\epsilon) \times E_D^\epsilon \quad \blacksquare$$

Since $\epsilon > 0$ is fixed, we will refer to $\mathcal{T}_{fiber}(\epsilon)$ as $\mathcal{T}_{fiber}$.

Proposition 8.3.1. Consider the morphism:

$$\mathfrak{glue} : \mathcal{T}_{fiber} \to \mathfrak{T}$$

$$(\alpha, \beta) \to \mathfrak{glue}(\alpha, \beta)$$

(see line (8.0.5) for the definition of $\mathfrak{T}$) defined on the core by

$$\mathfrak{glue}(\alpha,\beta)|_{S_{core,\frac{1}{2}}} := \alpha$$

and on the neck by

$$\mathfrak{glue}(\alpha,\beta)|_{h_x((\frac{1}{4},\infty)\times J_{\{x,y\}})} := Tk_x \circ \beta \circ h_x^{-1}, \quad \mathfrak{glue}(\alpha,\beta)|_{h_y((-\infty,-\frac{1}{4})\times J_{\{x,y\}})} := Tk_y \circ \beta \circ h_y^{-1}$$

The map $\mathfrak{glue}$ is *sc*-smooth and a homeomorphism onto its image, which we denote by $\mathcal{T}(S)$ or $\mathcal{T}$. The inverse of $\mathfrak{glue}$, which we denote by $\mathfrak{separate}$, is given as follows:

$$\mathfrak{separate} : \mathcal{T} \to \mathcal{T}_{fiber}$$

$$v \to (v|_{S_{core,\frac{1}{2}}}, (Tk_x^{-1} \circ v \circ h_x, Tk_y^{-1} \circ v \circ h_y)_{\{x,y\}\in|D|})$$

Proof: Consider the *sc*-linear embedding:

$$\iota : H^3\Gamma(TS, T\partial S, M, D) \to \mathcal{V} \times E_D$$

given as follows:

$$\eta \to (\eta|_{S_{core,1}}, (Tk_x^{-1} \circ \eta \circ h_x, Tk_y^{-1} \circ \eta \circ h_y)_{\{x,y\}\in|D|})$$

Observe that the image of ι is contained in the kernel of the map

$$T : \mathcal{V} \times E_D \to H$$

Now consider the map

$$\mathfrak{glue} : \mathcal{T}_{fiber} \to H^3\Gamma((S, TS, T\partial S), M, D)$$

is given by

$$(\alpha, \beta) \to \mathfrak{glue}(\alpha, \beta)$$

where $\mathfrak{glue}(\alpha, \beta)$ defined on the core by

$$\mathfrak{glue}(\alpha, \beta)|_{S_{core,\frac{1}{2}}} := \alpha$$

and on the neck by

$$\mathfrak{glue}(\alpha, \beta)|_{h_x^{-1}((\frac{1}{4},\infty)\times J_{\{x,y\}})} := Tk_x^{-1} \circ \beta \circ h_x, \quad \mathfrak{glue}(\alpha, \beta)|_{h_y^{-1}((-\infty,-\frac{1}{4})\times J_{\{x,y\}})} := Tk_y^{-1} \circ \beta \circ h_y$$

Since ι is smooth it follows that the restriction of ι to $\mathcal{T}$ is smooth. We set:

$$\mathfrak{separate} := \iota|_{\mathcal{T}} \quad \blacksquare$$

We define the local *sc*-retracts:

$$X_{cyl} := \{(a, u^+, u^-) \in B_{\frac{1}{2}} \times E_{cyl} : \rho(a, u^+, u^-) = (a, u^+, u^-)\}$$

$$X_{strip} := \{(a, u^+, u^-) \in [0, \frac{1}{2}) \times E_{strip} : \widetilde{\rho}(a, u^+, u^-) = (a, u^+, u^-)\}$$

$$X_{neck} := \{(\overline{a}, (\overline{u}^+, \overline{u}^-)) \in B_D \times E_D \ : \ \rho(\overline{a}, (\overline{u}^+, \overline{u}^-)) = (\overline{a}, (\overline{u}^+, \overline{u}^-))\}$$

Consider the *sc*-smooth retraction:

$$\text{Identity} \times \rho : B_D \times \mathcal{T}_{fiber} \to B_D \times \mathcal{T}_{fiber}$$

$$(\overline{a}, \eta, , \overline{\eta}^+, \overline{\eta}^-) \longrightarrow (\overline{a}, \eta, \rho(\overline{a}, \overline{\eta}^+, \overline{\eta}^-))$$

By construction, Im (Identity $\times \rho$) $\subset \mathcal{T}_{fiber} \times X_{neck}$ is an M-polyfold.

Consider the map $R : B \times \mathcal{T} \to B \times \mathcal{T}$ defined on the core as follows:

$$R(a, \eta)|_{S_{core, \frac{1}{2}}} := \eta$$

and on the cylindrical neck as follows:

$$Tk_x^{-1} \cdot R(a, \eta) \circ h_x|_{[\frac{1}{4}, \infty) \times S^1} = \rho_a^+(Tk_x^{-1} \cdot \eta \circ h_x|_{[\frac{1}{4}, \infty) \times S^1}, Tk_y^{-1} \cdot \eta \circ h_y|_{(-\infty, -\frac{1}{4}] \times S^1})$$

$$Tk_y^{-1} \cdot R(a, \eta) \circ h_y|_{(-\infty, -\frac{1}{4}] \times S^1}) = \rho_a^-(Tk_x^{-1} \cdot \eta \circ h_x|_{[\frac{1}{4}, \infty) \times S^1}, Tk_y^{-1} \circ u \circ h_y|_{(-\infty, -\frac{1}{4}] \times S^1})$$

and on the strip like neck as follows:

$$Tk_{x'}^{-1} \cdot R(a, \eta) \circ h_{x'}|_{[\frac{1}{4}, \infty) \times I} = \widetilde{\rho}_a^+(Tk_{x'}^{-1} \cdot u \circ h_{x'}|_{[\frac{1}{4}, \infty) \times I}, Tk_{y'}^{-1} \cdot \eta \circ h_{y'}|_{(-\infty, -\frac{1}{4}] \times I})$$

$$Tk_{y'}^{-1} \cdot R(a, \eta) \circ h_{y'}|_{(-\infty, -\frac{1}{4}] \times I}) = \widetilde{\rho}_a^-(Tk_{x'}^{-1} \cdot \eta \circ h_{x'}|_{[\frac{1}{4}, \infty) \times I}, Tk_{y'}^{-1} \circ \eta \circ h_{y'}|_{(-\infty, -\frac{1}{4}] \times I})$$

By construction we have a commutative diagram:

$$\begin{array}{ccc} B_D \times \mathcal{T}_{fiber} & \xrightarrow{\mathfrak{glue}} & B_D \times \mathcal{T} \\ \downarrow{\scriptstyle \text{Identity} \times \rho} & & \downarrow{\scriptstyle R} \\ B_D \times \mathcal{T}_{fiber} & \xrightarrow{\mathfrak{glue}} & B_D \times \mathcal{T} \end{array}$$

As a direct corollary of the above diagram and Proposition 8.3.1 and the *sc*-smoothness of ρ, we have:

Corollary 8.3.2. The map $R : B_D \times \mathcal{T} \to B_D \times \mathcal{T}$ is an *sc*-smooth retraction. In particular, $R(B \times \mathcal{T})$ is an M-polyfold. ■

8.4 Construction of Chart maps

Definition 8.4.1. Let $\widehat{U}$ be an open neighborhood of $0 \in \mathcal{T}$. If $\widehat{U}$ is chosen sufficiently small, there exists a constant $\sigma_0 > 0$ such that $\boxplus_a(exp_{\text{Identity}}\eta) : S_a \to \widehat{S}_b$ is a diffeomorphism for $\eta \in \widehat{U}$ for $0 < |a| < \sigma_0$ and $b = b(a, exp_{\text{Identity}}\eta)$. When $a = 0$ we set $\boxplus_0(exp_{\text{Identity}}\eta) = exp_{\text{Identity}}\eta$ and $b(0, u^+, u^-) = 0$.

For $0 < \sigma < \sigma_0$, we define the set U_σ :

$$U_\sigma = \Big\{ \big(a, b(a, exp_{\text{Identity}}\eta\big), \boxplus_a(exp_{\text{Identity}}\eta)) \ : \ 0 \leq |a| < \sigma, \eta \in \widehat{U} \Big\} \tag{8.4.1}$$

When $\sigma = 0$, we set $U_0 = \widehat{U}$. ■

Proposition 8.4.1. (compare [HWZ3] Propositions 3.13 and 3.24) Let U_σ be as defined in (8.4.1). The set $\mathfrak{S} \cap U_\sigma$ is open in $\mathfrak{S}$. Given two open sets U_σ and V_σ of the form (8.4.1) and a point $(0, 0, exp_{\text{Identity}}\eta_0) \in \overline{\mathfrak{S}}$ which belongs to $U_\sigma \cap V_\sigma$, there exists a third set W of the form (8.4.1) centered at $(0, 0, exp_{\text{Identity}}\eta)$ such that

$$(0, 0, exp_{\text{Identity}}\eta) \in W \subset U \cap V$$

In addition, near the identity $\overline{\mathfrak{S}}$ admits a unique topology such that the open sets of $\mathfrak{S}$ and those of the form (8.4.1) are a basis.

Proof: Consider the set $U_\sigma \subset \overline{\mathfrak{S}}$

$$U_\sigma = \{(a, b(a, exp_{\text{Identity}}\eta), \boxplus_a(exp_{\text{Identity}}\eta)) : 0 \leq |a| < \sigma, \eta \in \widehat{U}\} \tag{8.4.2}$$

To show $U_\sigma \cap \mathfrak{S}$ is open, it is enough to show for any point

$$(a_0, b_0, w_0) = (a_0, b(a_0, exp_{\text{Identity}}\eta_0), \boxplus_{a_0}(exp_{\text{Identity}}\eta_0)) \in U_\sigma \cap \mathfrak{S}$$

that U_σ contains an open set $\mathfrak{O}$ of the form

$$\mathfrak{O} = \{(a, b, \psi_b \circ w \circ \phi_a) \in \mathfrak{S} : |a - a_0| < \epsilon, |b - b_0| < \epsilon \text{ and } w \in U(w_0)\}$$

where $\epsilon > 0$ and $U(w_0)$ is a small open H^3-neighborhood of $w_0 : S_{a_0} \to \widehat{S}_{b_0}$ in Diff$((S_{a_0}, M_{a_0}, D_{a_0}), (\widehat{S}_{b_0}, \widehat{M}$

It is enough to show that for any point $(a, b, \psi_b \circ w \circ \phi_a) \in \mathfrak{O}$ we have a solution $\xi \in \mathcal{T}$ such that

$$\boxplus_a (exp_{\text{Identity}}\xi) = \psi_b \circ w \circ \phi_a, \quad b := b(a, exp_{\text{Identity}}\xi) \tag{8.4.3}$$

where $(a, b(a, exp_{\text{Identity}}\xi), \boxplus_a(exp_{\text{Identity}}\xi)) \in U_\sigma$. By definition, on the neck we have:

$$[k_b^{-1} \circ \boxplus_a(exp_{\text{Identity}}\eta) \circ h_a(s, t)]$$

$$= [(s, t) + \oplus_a(Tk_x^{-1} \cdot \eta \circ h_x, Tk_y^{-1} \cdot \eta \circ h_y + (Tk_x^{-1} \cdot \eta \circ h_x)_\infty - (Tk_y^{-1} \cdot \eta \circ h_y)_\infty))(s, t)] \tag{8.4.4}$$

We can re-write (8.4.4) as:

$$[\oplus_a(Tk_x^{-1} \cdot \eta \circ h_x, Tk_y^{-1} \cdot \eta \circ h_y + (Tk_x^{-1} \cdot \eta \circ h_x)_\infty - (Tk_y^{-1} \cdot \eta \circ h_y)_\infty))(s, t)]$$

$$= [k_b^{-1} \circ \psi_b \circ u \circ \phi_a \circ h_a(s,t) - (s,t)]$$

It is enough to solve for $(\overline{q}^+, \overline{q}^-) \in E_D^0$ satisfying:

$$\oplus_{a_{\{x,y\}}}(q^+_{\{x,y\}}, q^-_{\{x,y\}})(s,t) = k^{-1}_{b_{\{x,y\}}} \circ \psi_{b_{\{x,y\}}} \circ u \circ \phi_{a_{\{x,y\}}} \circ h_{a_{\{x,y\}}}(s,t) - (s,t)$$

as the solution $(Tk_x^{-1} \cdot \eta \circ h_x, Tk_y^{-1} \cdot \eta \circ h_y)_{\{x,y\} \in D} \in \widehat{E}_D$ can be recovered from $(q^+_{\{x,y\}}, q^-_{\{x,y\}})$ via the relation

$$(Tk_x^{-1} \cdot \eta \circ h_x, Tk_y^{-1} \cdot \eta \circ h_y) = (q^+_{\{x,y\}}, q^-_{\{x,y\}} + (Tk_y^{-1} \cdot \eta \circ h_y)_\infty - (Tk_x^{-1} \cdot \eta \circ h_x)_\infty)$$

Setting

$$g_{\{x,y\}}(s,t) = k^{-1}_{b_{\{x,y\}}} \circ exp_{\text{Identity}} \eta \circ h_{a_{\{x,y\}}}(s,t) - (s,t)$$

we observe that the pair of equations below admit a unique solution:

$$\oplus_{a_{\{x,y\}}}(q^+_{\{x,y\}}, q^-_{\{x,y\}}) = g_{\{x,y\}}$$

$$\ominus_{a_{\{x,y\}}}(q^+_{\{x,y\}}, q^-_{\{x,y\}}) = 0$$

When $\{x,y\} \in D_i$ the solution is given by:

$$q^+_{\{x,y\}}(s,t) = \int_{[0,1]} g_{\{x,y\}}(\frac{R}{2}, t)dt + \frac{\beta_a(s)}{\gamma_a(s)} \cdot (g_{\{x,y\}}(s,t) - \int_{[0,1]} g_{\{x,y\}}(\frac{R}{2}, t)dt)$$

$$q^-_{\{x,y\}}(s',t') = \int_{[0,1]} g_{\{x,y\}}(\frac{R}{2}, t)dt + \frac{1-\beta_a(s)}{\gamma_a(s)} \cdot (g_{\{x,y\}}(s,t) - \int_{[0,1]} g_{\{x,y\}}(\frac{R}{2}, t)dt)$$

and when $\{x,y\} \in D_b$

$$q^+_{\{x,y\}}(s,t) = \int_{[0,1]} \text{Re}\, g_{\{x,y\}}(\frac{R}{2}, t)dt + \frac{\beta_a(s)}{\gamma_a(s)} \cdot (g_{\{x,y\}}(s,t) - \int_{[0,1]} \text{Re}\, g_{\{x,y\}}(\frac{R}{2}, t)dt)$$

$$q^{-}_{\{x,y\}}(s',t') = \int_{[0,1]} \text{Re}\, g_{\{x,y\}}(\frac{R}{2},t)dt + \frac{1-\beta_a(s)}{\gamma_a(s)} \cdot (g_{\{x,y\}}(s,t) - \int_{[0,1]} \text{Re}\, g_{\{x,y\}}(\frac{R}{2},t)dt)$$

On $S_{core,\frac{1}{4}}$, ξ is determined by

$$\xi|_{S_{core,\frac{1}{4}}} = exp^{-1}_{\text{Identity}} \psi_b \circ w \circ \phi_a$$

Observe that if $a = a_0$, $b = b_0$ and $\eta = \eta_0$, then $g = 0$ and so $Tk_x^{-1} \cdot \eta \circ h_x = 0$ and $Tk_y^{-1} \cdot \eta \circ h_y = 0$. It is easy to see that $(a_{\{x,y\}}, Tk_x^{-1} \cdot \eta \circ h_x, Tk_y^{-1} \cdot \eta \circ h_y)$ depends continuously on $(b_{\{x,y\}}, g_{\{x,y\}})$. Thus, if ϵ is sufficiently small and $U(\eta_0)$ is sufficiently small, then $\mathfrak{O} \subset U_\sigma$.

Consider now an intersection of two sets U_{σ_1}, U_{σ_2} and a point $(0,0,exp_{\text{Identity}}\eta_0) \in \overline{\mathfrak{S}} \in U_{\sigma_1} \cap U_{\sigma_2}$. Explicitly, we have:

$$U_{\sigma_1} := \{(a, b(a,\eta), \boxplus_a exp_{\text{Identity}}\eta) \ : \ 0 \leq |a| < \sigma_1, \ \eta \in U(\eta_1)\}$$

$$U_{\sigma_2} := \{(a, b(a,\eta), \boxplus_a exp_{\text{Identity}}\eta) \ : \ 0 \leq |a| < \sigma_2, \ \eta \in U(\eta_2)\}$$

By construction, $\eta_0 \in U(\eta_1) \cap U(\eta_2)$. If we set $\sigma_0 < \min(\sigma_1, \sigma_2)$, then

$$U_{\sigma_0} := \{(a, b(a,\eta), \boxplus_a exp_{\text{Identity}}\eta) \ : \ 0 \leq |a| < \sigma_0, \ \eta \in U(\eta_0)\}$$

satisfies

$$(0,0,exp_{\text{Identity}}\eta_0) \in U_{\sigma_0} \in U_{\sigma_1} \cap U_{\sigma_2}$$

and $U_{\sigma_0} \subset U_{\sigma_1} \cap U_{\sigma_2}$. ■

Inspection of the proof Proposition 8.4.1 reveals that the map $\boxplus \circ \mathfrak{glue}$ is a bijection on

Im (Identity) $\times \rho$. In particular, we have a chart map:

$$\phi : \{(a,\nu) \in B_D \times \mathcal{T}_{fiber} \ : \ \rho(a,\nu) = (a,\nu)\, , |a| < \sigma\} \to \overline{\mathfrak{S}}$$

given by

$$(a,\nu) \to \boxplus_a(\mathfrak{glue}(\nu))$$

8.5 Construction of the Strong Bundle

Definition 8.5.1. We define the set $\overline{\mathfrak{E}}$ of anti-linear sections to be the set consisting of pairs (u,ξ) where $u : S_a \to \widehat{S}_b \in \overline{\mathfrak{S}}$ and $\xi \in H^{2,\delta_0^0,\delta_1^0}\Omega^{0,1}(TS_a, u^*T\widehat{S}_b, M, D)$. ■

Let V and W denote two open neighborhoods of 0 in finite dimensional vector spaces E and E', respectively, and let

$$j : V \to \mathfrak{J}_{\mathbf{D^1},j}$$

and

$$k : W \to \mathfrak{J}_{\mathbf{D^2},j}$$

denote smooth parametrizations of complex structures on S such that $k(0) = j = j(0)$. Further suppose that $v \to j(v)$ is a good family.

Definition 8.5.2. We define the *sc*-Hilbert space F_{cyl} as follows:

$$F_{cyl} := H^{2,\delta_0^0}((\frac{1}{4},\infty) \times S^1, \mathbb{R}^2) \oplus H^{2,\delta_0^0}((-\infty,-\frac{1}{4}) \times S^1, \mathbb{R}^2)$$

The *sc*-filtration is the obvious one.

We define the retraction

$$\widehat{\pi} : B_{\frac{1}{2}} \times F_{cyl} \to B_{\frac{1}{2}} \times F_{cyl}$$

given by

$$(a, \eta^+, \eta^-) \to (a, \widehat{\pi}_a(\eta^+, \eta^-))$$

where $\widehat{\pi}_a(\eta^+, \eta^-)$ is defined by

$$\widehat{\oplus}_a \widehat{\pi}_a(\eta^+, \eta^-) = \widehat{\oplus}_a(\eta^+, \eta^-)$$

$$\widehat{\ominus}_a \widehat{\pi}_a(\eta^+, \eta^-) = 0 \quad \blacksquare$$

Definition 8.5.3. We define the space of maps F_{strip} as follows:

$$F_{strip} := H^{2,\delta_1^0}_{\text{bdry}}((\frac{1}{4}, \infty) \times I, \mathbb{R}^2) \oplus H^{2,\delta_1^0}_{\text{bdry}}((-\infty, -\frac{1}{4}) \times I, \mathbb{R}^2)$$

Also, we define the retraction

$$\widehat{\widetilde{\pi}} : [0, \frac{1}{2}) \times F_{strip} \to [0, \frac{1}{2}) \times F_{strip}$$

given by

$$(a, \eta^+, \eta^-) \to (a, \widehat{\widetilde{\pi}}_a(\eta^+, \eta^-))$$

where $\widehat{\widetilde{\pi}}_a(\eta^+, \eta^-)$ is defined by

$$\widehat{\oplus}_a \widehat{\widetilde{\pi}}_a(\eta^+, \eta^-) = \widehat{\oplus}_a(\eta^+, \eta^-)$$

$$\widehat{\ominus}_a \widehat{\widetilde{\pi}}_a(\eta^+, \eta^-) = 0 \qquad \blacksquare$$

Definition 8.5.4. We define the set of maps:

$$F_D := \prod_{\{x,y\} \in D_i} \{x, y\} \times F_{cyl} \times \prod_{\{x',y'\} \in D_b} \{x', y'\} \times F_{strip}$$

We denote points of F_D by $(a_{\{x,y\}}, \eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})_{\{x,y\}\in D}$ or by $(\overline{a}, \overline{\eta}^+, \overline{\eta}^-)$. In addition, we define the product retraction:

$$\widehat{\pi} : B_D \times F_D \to B_D \times F_D$$

by

$$\widehat{\pi}((a_{\{x,y\}}, \eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})_{\{x,y\}\in D_i}, (a_{\{x',y'\}}, \eta^+_{\{x',y'\}}, \eta^-_{\{x',y'\}})_{\{x',y'\}\in D_b})$$

$$= ((a_{\{x,y\}}, \widehat{\pi}_{a_{\{x,y\}}}(\eta^+_{\{x,y\}}, \eta^-_{\{x,y\}})_{\{x,y\}\in D_i}, (a_{\{x',y'\}}, \widehat{\widehat{\pi}}_{a_{\{x',y'\}}}(\eta^+_{\{x',y'\}}, \eta^-_{\{x',y'\}})_{\{x',y'\}\in D_b})) \quad \blacksquare$$

Consider the subset:

$$K_{cyl} := \{(a, \eta^+, \eta^-) \in B_{\frac{1}{2}} \times F_{cyl} : \widehat{\pi}(a, u^+, u^-) = (a, u^+, u^-)\}$$

$$K_{strip} := \{(a, \eta^+, \eta^-) \in B_{\frac{1}{2}} \times F_{strip} : \widehat{\widehat{\pi}}(a, u^+, u^-) = (a, u^+, u^-)\}$$

$$K := \{(\overline{a}, (\overline{\eta}^+, \overline{\eta}^-)) \in B_D \times F_D \ : \ \widehat{\pi}(\overline{a}, (\overline{\eta}^+, \overline{\eta}^-)) = (\overline{a}, (\overline{\eta}^+, \overline{\eta}^-))\}$$

Also consider the map of strong bundles:

$$\rho \times \widehat{\pi} : B_D \times E_D \lhd F_D \to B_D \times E_D \lhd F_D$$

given by:

$$(\overline{a}, (\overline{u}^+, \overline{u}^-), (\overline{\eta}^+, \overline{\eta}^-)) = (a, \rho(a, (\overline{u}^+, \overline{u}^-)), \widehat{\pi}(\overline{a}, (\overline{\eta}^+, \overline{\eta}^-)))$$

Observe that the diagram below provides a local strong bundle chart for the local strong

bundle retract Im $(\rho \times \widehat{\pi}) \to X$

$$\begin{array}{ccc} B_D \times E_D \lhd F_D & \xrightarrow{\rho \times \widehat{\pi}} & \rho \times \widehat{\pi}(B_D \times E_D \lhd F_D) \\ \downarrow & & \downarrow p \\ B_D \times E_D & \xrightarrow{\rho} & \rho(B_D \times E_D) = X \end{array}$$

Consider two families of complex linear isomorphisms (compare Section 4.3 [HWZ3]). The first of which is

$$I_2(w, z, z') : (T_z S, j) \to (T_{z'} S, k(w))$$

which is defined for w close to 0 and z' sufficiently close to z. The second is

$$I_1(v, z) : (T_z S, j(v)) \to (T_z S, j)$$

which is defined for all $z \in S$. The family I_2 is defined as follows:

Let $P_{z'z}$ be parallel transport along the minimal geodesic from z and z', then we set

$$I_2(w, z, z') := \frac{1}{2} \cdot [P_{z'z} - k(w)(z') \circ P_{z'z} \circ j(z)]$$

Observe that I_2 satisfies

$$I_2(0, z, z) = \text{Identity}$$

We define the family I_1 by

$$I_1(v, z) := \frac{1}{2}[\text{Id} - j(z) \circ j(v)(z)]$$

Observe that I_1 satisfies

$$I_1(v, z) := \text{Identity if } z \in \mathbf{D}$$

$$I_1(0,z) := \text{Identity}$$

We choose the parallel transport via a connection which is the product connection on the neck. In particular, in local coordinates I_2 is the identity.

Definition 8.5.5. Consider the *sc*-Hilbert space:

$$H' := \prod_{\{x,y\}\in D_i} H^2(A_x,\mathbb{R}^2)\times H^2(A_y,\mathbb{R}^2)\times \prod_{\{x',y'\}\in D_b} H^2(A_{x'},\mathbb{R}^2)\times H^2(A_{y'},\mathbb{R}^2)$$

Consider the map of *sc*-Hilbert spaces:

$$X : H^2\Omega^{0,1}((TS|_{core,\frac{1}{2}},j),(TS|_{core,\frac{1}{2}},j))\times F_D \to H'$$

$$\left(\xi,(\overline{a},\xi^+_{\{x,y\}},\xi^-_{\{x,y\}})_{\{x,y\}\in D}\right)\to \left(Tk_x^{-1}\cdot\xi\circ Th_x\cdot\frac{\partial}{\partial s}-\xi^+_{\{x,y\}},Tk_y^{-1}\cdot\xi\circ Th_y\cdot\frac{\partial}{\partial s}-\xi^-_{\{x,y\}}\right)_{\{x,y\}\in D}$$

Then $F_{fiber} := \ker X$ is an *sc*-Hilbert space. Set the abbreviation:

$$T_{fiber} := V\times W\times \rho(B_D\times \mathcal{T}_{fiber})$$

The bundle $V\times W\times\rho\times\widehat{\pi}(B_D\times\mathcal{T}_{fiber}\lhd F_{fiber})\to T_{fiber}$ is a local *sc*-strong bundle retract as realized by the strong bundle retraction diagram:

$$\begin{array}{ccc} V\times W\times B_D\times\mathcal{T}_{fiber}\lhd F_{fiber} & \xrightarrow{\text{Identity}\times\rho\times\widehat{\pi}} & V\times W\times B_D\times\mathcal{T}_{fiber}\lhd F_{fiber} \\ \downarrow & & \downarrow \\ V\times W\times B_D\times\mathcal{T}_{fiber} & \xrightarrow{\text{Identity}\times\rho} & V\times W\times B_D\times\mathcal{T}_{fiber} \end{array}$$

■

Proposition 8.5.1. The diagram below is a local strong bundle chart:

$$\begin{array}{ccc} V \times W \times \rho \times \widehat{\pi}(B_D \times \mathfrak{T}_{fiber} \lhd F_{fiber}) & \xrightarrow{\Theta} & \overline{\mathfrak{E}} \\ \downarrow & & \downarrow p \\ V \times W \times T_{fiber} & \xrightarrow{\boxplus \circ \mathfrak{glue}} & V \times W \times \overline{\mathfrak{S}} \end{array}$$

where the map

$$\Theta : V \times W \times T_{fiber} \lhd K_{fiber} \to \overline{\mathfrak{E}}$$

is given by

$$((v,w,\eta),(a,(\overline{\eta}^+,\overline{\eta}^-)),(\xi,(\overline{\xi}^+,\overline{\xi}^-))) \to (v,w,\boxplus(a,\mathfrak{glue}(\eta,\overline{\eta}^+,\overline{\eta}^-)),\theta)$$

where $\theta \in \Omega^{0,1}(TS_a, \boxplus(a,\mathfrak{glue}(\eta,\overline{\eta}^+,\overline{\eta}^-))^*TS_b)$ is defined by

$$\theta|_{S_{core,\frac{1}{2}}} := I_2(w,z,exp_{\text{Identity}}\eta(z)) \circ \xi \circ I_1(v,z)$$

and in local (positive) coordinates by

$$Tk^{-1}_{b_{\{x,y\}}} \circ \theta \circ Th_{a_{\{x,y\}}} \cdot \frac{\partial}{\partial s}|_{Z_{a_{\{x,y\}}}}(s,t) := \oplus_{a_{\{x,y\}}}(\xi^+_{\{x,y\}},\xi^-_{\{x,y\}}))(s,t) \text{ for } \{x,y\} \in D$$

Proof: Commutativity of the diagram is clear by construction. It is enough to show that Θ is a fiberwise bijection. Towards this end, suppose that $u : S_a \to S_b \in \text{Im } \mathfrak{glue} \circ exp(T_{fiber})$. Below we show that we can uniquely solve for any $\theta \in H^{3+m,\delta_0^m,\delta_1^m}(TS_a, u^*TS_b)$.

By hypothesis, there exist maps $(\overline{h}^+,\overline{h}^-) \in E_D$ and $\eta \in H^3\Omega^{0,1}((TS_{core,\frac{1}{2}},j),(TS_{core,\frac{1}{2}},j))$ such that

$$\mathfrak{glue}(exp_{Identity}\eta, \boxplus(a,((s,t)+h^+_{\{x,y\}},(s',t')+h^-_{\{x,y\}}))) = u$$

Over $TS_{core,\frac{1}{2}}$, we set

$$\xi = I_2(w, exp_{\text{Identity}}\eta(z), z) \circ \theta \circ I_1(v, z)^{-1}$$

Consider the anti-linear section θ in coordinates:

$$Tk^{-1}_{b_{\{x,y\}}} \circ \theta \circ Th_{a_{\{x,y\}}} \in H^{\delta_0^m, \delta_1^m}\Omega^{0,1}((TZ_{a_{x,y}}, i), ((h^{-1}_{b_{\{x,y\}}} \circ u \circ h_{a_{\{x,y\}}})^* TZ_{b_{\{x,y\}}}, i))$$

We set the abbreviation $\theta_{\{x,y\}} := Tk^{-1}_{b_{\{x,y\}}} \circ \theta \circ Th_{a_{\{x,y\}}} \cdot \frac{\partial}{\partial s}$ and identify the tangent spaces of $TZ_{a_{\{x,y\}}}$ and $TZ_{b_{\{x,y\}}}$ with $\mathbb{R}^2$. Consider the system of equations

$$\widehat{\oplus}_{a_{\{x,y\}}}(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})(s,t) = \theta_{\{x,y\}} \cdot \frac{\partial}{\partial s}$$

$$\widehat{\ominus}_{a_{\{x,y\}}}(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = 0$$

The unique is given by

$$\xi^+_{\{x,y\}} = \frac{\beta_{a_{\{x,y\}}}}{\gamma_{a_{\{x,y\}}}} \cdot \theta_{a_{\{x,y\}}} \cdot \frac{\partial}{\partial s} \tag{8.5.1}$$

$$\xi^-_{\{x,y\}} = \frac{1 - \beta_{a_{\{x,y\}}}}{\gamma_{a_{\{x,y\}}}} \cdot \theta_{a_{\{x,y\}}} \cdot \frac{\partial}{\partial s} \tag{8.5.2}$$

The (8.5.1)-(8.5.2) implies that $\beta = (\overline{a}, \xi, \overline{\xi}^+, \overline{\xi}^-) \in K_{fiber}$, and by construction β is mapped onto θ by Θ. ■

Corollary 8.5.1. $p : \overline{\mathfrak{E}} \to V \times W \times \overline{\mathfrak{S}} | \text{Im} \boxplus \circ \mathfrak{glue}$ is a strong bundle. ■

8.6 Cauchy-Riemann Section

Consider the Cauchy-Riemann section

$$\overline{\partial} : V \times W \times \overline{\mathfrak{S}} \to \overline{\mathfrak{E}}$$

$$(v, a, b, u) \to \overline{\partial}_{k(b,w),j(a,v)} u := \frac{1}{2} \cdot (Tu + k(b, w) \circ Tu \circ j(a, v)))$$

Our goal is to find a change of coordinates such that in a germ near the identity $S_0 := S \to \widehat{S}_0 := S$, $\overline{\partial}$ is expressed as an sc-Fredholm operator over a fixed domain. Consider the section

$$\overline{\partial}_{cyl} : X_{\text{cyl}} \to K_{\text{cyl}}$$

given by:

$$(a, (h^+, h^-)) \to \xi$$

where ξ is uniquely defined by the pair of equations

$$\widehat{\oplus}_a \xi = \frac{1}{2}(T \boxplus_a ((s, t) + h^+, (s', t') + h^-) + i \cdot \boxplus_a((s, t) + h^+, (s', t') + h^-) \cdot i)\frac{\partial}{\partial s} \quad (8.6.1)$$

$$\widehat{\ominus}_a \xi = 0 \quad (8.6.2)$$

Also consider the filling of $\overline{\partial}_{cyl}$:

$$\text{Fill}\, \overline{\partial}_{cyl} : B \times E_{cyl} \to F_{cyl}$$

given by

$$(a, h^+, h^-) \to \xi$$

where ξ is uniquely defined by the pair of equations

$$\widehat{\oplus}_a\xi = \frac{1}{2}(T \boxplus_a ((s,t)+h^+,(s',t')+h^-) + i\cdot \boxplus_a((s,t)+h^+,(s',t')+h^-)\cdot i)\frac{\partial}{\partial s} \quad (8.6.3)$$

$$\widehat{\ominus}_a\xi = \frac{1}{2}(T \ominus_a (h^+ - h^+_\infty, h^- - h^-_\infty) + i\cdot T\ominus_a (h^+ - h^+_\infty, h^- - h^-_\infty)\cdot i)\frac{\partial}{\partial\, s} \quad (8.6.4)$$

Set $h^+ - h^+_\infty = r^+$ and $h^- - h^-_\infty = r^-$. Then by direct calculation and the fact that (s,t) is holomorphic, we can simplify the above equations to the system:

$$\widehat{\oplus}_a\xi(z) = \overline{\partial}_0 r_a \quad (8.6.5)$$

$$\widehat{\ominus}_a\xi = \overline{\partial}_0 \ominus_a (r^+, r^-) \quad (8.6.6)$$

where $r_a := \oplus_a(r^+, r^-)$

Proposition 8.6.1. (compare Section 3.7 [FH1]) Fill $\overline{\partial}_{cyl}$ is *sc*-smooth.

Proof: The result follows similarly to Proposition 7.1.1. ■

Proposition 8.6.2. Fill $\overline{\partial}_{cyl}$ is regularizing away from the left and right boundaries.

Proof: The result follows nearly identically to Lemma 7.1.1. ■

Proposition 8.6.3. (compare Lemma 3.29) Fill $\overline{\partial}_{cyl}$ is a filling for $\overline{\partial}_{cyl}$.

Proof: We need to verify the 3 properties listed in Definition 5.1.1. With regards to the first property, note that for $(h^+,h^-)\in X_{cyl}$, we have $\rho_a(h^+,h^-) = (h^+,h^-)$ which is equivalent to

$$\ominus_a(h^+ - h^+_\infty, h^- - h^-_\infty) = 0$$

by Proposition 8.2.1. Thus the systems (8.6.1) – (8.6.2) and (8.6.3)-(8.6.4) coincide for $(h^+, h^-) \in X_{cyl}$.

With regards to the second property, suppose that

$$(a, \xi^+(a, h^+, h^-), \xi^-(a, h^+, h^-)) = \widehat{\pi}(a, \xi^+(a, h^+, h^-), \xi^-(a, h^+, h^-)),$$

for a point (a, h^+, h^-) close to 0. Then

$$\widehat{\ominus}_a(\xi^+, \xi^-) = 0 = \overline{\partial}_0 \ominus_a (h^+ - h^+_\infty, h^- - h^-_\infty)$$

Since $\overline{\partial}_0$ is an isomorphism, this means $\ominus_a(h^+ - h^+_\infty, h^- - h^-_\infty) = 0$. By Proposition 8.2.1 this is equivalent to $\rho_a(h^+, h^-) = 0$, so $(a, h^+, h^-) \in X_{cyl}$.

The third property follows because $D\rho(0) = \text{Identity}$ and $\widehat{\pi}_0 = \text{Identity}$, see Appendix 4.
■

For shorthand, we will refer to the filled section by $H(a, h^+, h^-) := (\xi^+, \xi^-)$. We begin by verifying the quadratic estimate:

Lemma 8.6.1. Given $\epsilon > 0$, the following holds for every level m:

$$||(DH_a(h^+, h^-) - DH_a(0,0)(\eta^+, \eta^-)||_m \leq \epsilon ||(\eta^+, \eta^-)||_m$$

for all $(\eta^+, \eta^-) \in (E_{cyl})_m$, all $a \in B_{\frac{1}{2}}$, and all $(h^+, h^-) \in (E_{cyl})_m$ sufficiently small on level m.

Proof: The claim follows from the observation that $DH_a(h^+, h^-) = DH_a(0,0)$ for all

$(h^+, h^-) \in (E_{cyl})_m$. ■

We introduce the shorthand $DH_a(k_j^+, k_j^-) := \Gamma_a(k_j^+, k_j^-)$. The operators $\Gamma_a(k_j^+, k_j^-)$ satisfy:

$$\overline{\partial}_0(\oplus_a(\eta^+ - \eta_\infty^+, \eta^- - \eta_\infty^-)) = \widehat{\oplus}_a(\Gamma_a(k_j^+, k_j^-)(\eta^+ - \eta_\infty^+, \eta^- - \eta_\infty^-)) \tag{8.6.7}$$

$$\overline{\partial}_0(\ominus_a(\eta^+ - \eta_\infty^+, \eta^- - \eta_\infty^-)) = \widehat{\ominus}_a(\Gamma_a(k_j^+, k_j^-)(\eta^+ - \eta_\infty^+, \eta^- - \eta_\infty^-)) \tag{8.6.8}$$

where $(\eta^+, \eta^-) \in (E_{cyl})_m$. The argument below follows similarly to Proposition 7.1.3, but is much simpler due to the fact that the complex structure on the target is eveywhere standard.

Proposition 8.6.4. Let $\{a_j\}_{j=0}^\infty$, $a_j \in B_{\frac{1}{2}}$, be a sequence of real numbers which converge to $a \in B_{\frac{1}{2}}$. Consider a sequence of maps $(x_j^+, x_j^-) \subset F_{\text{cyl}}$ which converges to 0 on level m. Also suppose $(y_j^+, y_j^-) \subset F_{\text{cyl}}$ is a sequence of maps which is bounded on level $m+1$ and $(\eta_j^+, \eta_j^-) \subset E_{cyl}$ is a sequence bounded on level m. Suppose a sequence $(k_j^+, k_j^-) \subset E_{cyl}$ converges on level m to (k^+, k^-). If we have the relation:

$$\Gamma_{a_j}(k_j^+, k_j^-)(\eta_j^+, \eta_j^-) = (x_j^+, x_j^-) + (y_j^+, y_j^-) \tag{8.6.9}$$

then (η_j) has a subsequence converging on level m on slightly restricted cylinders.

Proof: It is easy to see that the asymptoptic limits of any subsequence of (η_j) has itself a convergent subsequence. Thus it is enough to assume that $(\eta_j)_\infty = 0$ for all j. Let $u_j := \oplus_{a_j}(u^+, u^-)$ and $e_j := \oplus_{a_j}(\eta_j^+, \eta_j^-)$. Using these abbreviations we can express (8.6.9) as follows:

$$\frac{\partial e_j}{\partial s} + i\frac{\partial e_j}{\partial t} = \widehat{\oplus}_{a_j}(x_j^+, x_j^-) + \widehat{\oplus}_{a_j}(y_j^+, y_j^-) \tag{8.6.10}$$

$$\overline{\partial}_0(\ominus_a(\eta_j^+, \eta_j^-)) = \widehat{\ominus}_{a_j}(x_j^+, x_j^-) + \widehat{\ominus}_{a_j}(y_j^+, y_j^-) \tag{8.6.11}$$

Take a cut-off function $\tau : \mathbb{R} \to [0,1]$ satisfying $\tau = 0$ for $s \leq 1$ and $\tau = 1$ for $s \geq 2$. In addition, let $\gamma^+_{2r}(s) = \tau(s - 2r)$ and $\gamma^-_{2r}(s') = \tau(-s' - 2r)$. Replace (η^+_j, η^-_j) with $(\gamma^+_{2r}\eta^+_j, \gamma^-_{2r}\eta^-_j)$. Further, define $\overline{e}_j := \oplus_{a_j}(\gamma^+_{2r}\eta^+_j, \gamma^-_{2r}\eta^-_j)$. The first equation can be re-expressed:

$$\frac{\partial \overline{e}_j}{\partial s} + i\frac{\partial \overline{e}_j}{\partial t} = z_j + \widehat{\oplus}_{a_j}(\gamma^+_{2r}x^+_j, \gamma^-_{2r}x^-_j) + \widehat{\oplus}_{a_j}(\gamma^+_{2r}y^+_j, \gamma^-_{2r}y^-_j)$$

where

$$z_j = \frac{\partial \gamma^+_{2r}}{\partial s}(s)\eta^+_j(s,t) + \frac{\partial \gamma^-_{2r}}{\partial s}\eta^-_j(s - R_j, t - \theta_j)$$

The second equation becomes:

$$\overline{\partial}_0(\ominus_{a_j}(\eta^+_j, \eta^-_j)) = \widehat{\ominus}_{a_j}(\gamma^+_{2r}x^+_j, \gamma^-_{2r}x^-_j) + \widehat{\ominus}_{a_j}(\gamma^+_{2r}y^+_j, \gamma^-_{2r}y^-_j)$$

Using Proposition 7.1.6 part (3) we can re-express the system of equations (8.6.10) – (8.6.11) as:

$$\frac{\partial \overline{e}_j}{\partial s} + i\frac{\partial \overline{e}_j}{\partial t} = \widehat{\oplus}_{a_j}(q^+_j, q^-_j) + \widehat{\oplus}_{a_j}(p^+_j, p^-_j) \tag{8.6.12}$$

$$\overline{\partial}_0(\ominus_{a_j}(\gamma^+_{2r}\eta^+_j, \gamma_{2r}\eta^-_j)) = \widehat{\ominus}_{a_j}(q^+_j, q^-_j) + \widehat{\ominus}_{a_j}(p^+_j, p^-_j) \tag{8.6.13}$$

where the sequence (q^+_j, q^-_j) converges to 0 in F_m and the sequence (p^+_j, p^-_j) is bounded in F_{m+1}.

Now consider the system:

$$\overline{\partial}_0 \mathfrak{f}^+ \circ \overline{e}_j = k^+_j + \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(q^+_j, q^-_j) + \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(p^+_j, p^-_j) \tag{8.6.14}$$

$$\overline{\partial}_0 \mathfrak{f}^- \circ \overline{e}_j = k^-_j + \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(q^+_j, q^-_j) + \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(p^+_j, p^-_j) \tag{8.6.15}$$

$$\overline{\partial}_0 \ominus_{a_j} (\gamma^+_{2r}\eta^+_j, \gamma_{2r}\eta^-_j) = \widehat{\ominus}_{a_j}(q^+_j, q^-_j) + \widehat{\ominus}_{a_j}(p^+_j, p^-_j) \tag{8.6.16}$$

where

$$k_j^+ := \frac{\partial\beta(s-\frac{R}{2}-2)}{\partial s}\overline{e}_j, \qquad k_j^- := -\frac{\partial\beta(s-\frac{R}{2}-2)}{\partial s}\overline{e}_j$$

Note that $k_j^{\pm}$ is bounded in $E_m = F_{m+1}$ and by Proposition we can absorb it into the $\widehat{\oplus}_{a_j}(q_j^+, q_j^-)$ term. So we can re-write (8.6.14)-(8.6.16) as:

$$\overline{\partial}_0 \mathfrak{f}^+ \circ \overline{e}_j = \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-) + \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-) \tag{8.6.17}$$

$$\overline{\partial}_0 \mathfrak{f}^- \circ \overline{e}_j = \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-) + \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-) \tag{8.6.18}$$

$$\overline{\partial}_0 \ominus_{a_j} (\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-) = \widehat{\ominus}_{a_j}(q_j^+, q_j^-) + \widehat{\ominus}_{a_j}(p_j^+, p_j^-) \tag{8.6.19}$$

Consider $\mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-)$ and $\mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-)$ as equations on $\mathbb{R}\times S^1$. Now consider the pair of solutions $(\xi_j^+, \xi_j^-) \in H^{3+m,\delta_m}(\mathbb{R}\times S^1, \mathbb{R}^2) \oplus H^{3+m,\delta_m}(\mathbb{R}\times S^1, \mathbb{R}^2)$ satisfying:

$$\overline{\partial}_0 \xi_j^+ = \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-)$$

$$\overline{\partial}_0 \xi_j^- = \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(p_j^+, p_j^-)$$

By Proposition 5.1.1 and Proposition 7.0.2, we have the chain of inequalities:

$$|(p_j^+, p_j^-)|_{F_m} \geq C|(\mathfrak{f}^+ \circ \oplus_{a_j}(p_j^+, p_j^-), \mathfrak{f}^- \circ \oplus_{a_j}(p_j^+, p_j^-))|_{F_m}$$

$$= C|(\overline{\partial}_0\xi_j^+, \overline{\partial}_0\xi_j^-)|_{F_m} \geq C\cdot C'|\xi_j^+|'_m + |\xi_j^-|'_m$$

where C, C' depend only on m. In particular, $|\xi_j^{\pm}|'_m \to 0$. Now consider the pair of solutions $(\mu_j^+, \mu_j^-) \in H^{3+(m+1),\delta_m}(\mathbb{R}\times S^1, \mathbb{R}^2) \oplus H^{3+(m+1),\delta_m}(\mathbb{R}\times S^1, \mathbb{R}^2)$ which satisfy

$$\overline{\partial}_0 \mu_j^+ = \mathfrak{f}^+ \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-)$$

$$\overline{\partial}_0 \mu_j^- = \mathfrak{f}^- \circ \widehat{\oplus}_{a_j}(q_j^+, q_j^-)$$

Then by Proposition 5.1.1 and Proposition 7.0.2, we have the inequality

$$|(q_j^+, q_j^-)|_{F_{m+1}} \geq C_{m+1}|(\mathfrak{f}^+ \circ \oplus_{a_j}(q_j^+, q_j^-), \mathfrak{f}^- \circ \oplus_{a_j}(q_j^+, q_j^-))|_{F_{m+1}}$$

$$= C_{m+1}|(\overline{\partial}_0 \mu_j^+, \overline{\partial}_0 \mu_j^-)|_{F_{m+1}} \geq C_{m+1} C'_{m+1}(|\mu_j^+|'_{m+1} + |\mu_j^-|'_{m+1})$$

Thus the sequence $(|\mu_j^+|'_{m+1} + |\mu_j^-|'_{m+1})$ is bounded on level $m+1$. Since $\overline{\partial}_0$ is an isomorphism, we have that

$$\xi_j^+ + \mu_j^+ = \mathfrak{f}^+ \circ \overline{e}_j$$

$$\xi_j^- + \mu_j^- = \mathfrak{f}^- \circ \overline{e}_j$$

We make the replacement:

$$(\xi_j^+, \xi_j^-) \to (\gamma_{2r-2}^+(s) \cdot \xi_j^+, \gamma_{2r-2}^-(s') \cdot \xi_j^-)$$

$$(\mu_j^-, \mu_j^-) \to (\gamma_{2r-2}^+(s) \cdot \mu_j^+, \gamma_{2r-2}^-(s') \cdot \mu_j^-)$$

Then we can assume $(\xi_j^+, \xi_j^-) \in E_m$ and $(\mu_j^+, \mu_j^-) \in E_{m+1}$. This means that

$$\oplus_{a_j}(\xi_j^+ + \mu_j^+, \xi_j^- + \mu_j^-) = \oplus_{a_j}(\mathfrak{f}^+ \circ \overline{e}_j, \mathfrak{f}^- \circ \overline{e}_j) = \overline{e}_j$$

Now consider the solutions $\alpha_j \in H_c^{3+m,\delta_m}(C_a, \mathbb{R}^2)$, guaranteed to exist by Proposition 5.1.1

$$\overline{\partial}_0 \alpha_j = \widehat{\ominus}_{a_j}(p_j^+, p_j^-)$$

By Theorem 1.27 [HWZ0], there exists a unique pair $(\alpha_j^+, \alpha_j^-) \in E_m$ such that

$$\boxdot_{a_j}(\alpha_j^+, \alpha_j^-) = (0, \alpha_j)$$

By Proposition 5.1.1 we have

$$C \cdot C'|(p_j^+, p_j^-)|_{F_m} \geq C|(1 - \widehat{\pi}_{a_j})(p_j^+, p_j^-)|_{F_m} \geq |\boxdot_{a_j} \{1 - \widehat{\pi}_{a_j}(p_j^+, p_j^-)\}|_{\widehat{G}_m} = |(0, \overline{\partial}_0 \alpha_j)|_{\widehat{G}_m}$$

$$\geq C''|(0, \alpha_j)|_{G_m} = C''|\boxdot(\alpha_j^+, \alpha_j^-)|_{G_m} \geq \frac{C''}{C'''}|(\alpha_j^+, \alpha_j^-)|_{E_m} \tag{8.6.20}$$

In particular, we have $|(\alpha_j^+, \alpha_j^-)|_m \to 0$ on level m. For shorthand, we set $\widetilde{\alpha}_j := (\alpha_j^+, \alpha_j^-)$. Now consider the solutions β_j, guaranteed to exist by Theorem 1.27 [HWZ0]

$$\overline{\partial}_0 \beta_j = \widehat{\ominus}_{a_j}(q_j^+, q_j^-)$$

Since $\overline{\partial}_0$ is an isomorphism we have the equality:

$$\alpha_j + \beta_j = \widehat{\ominus}_{a_j}(\gamma_{2r}^+ \eta_j^+, \gamma_{2r}^- \eta_j^-)$$

By Theorem 1.27 [HWZ0], there exists a unique pair $(\beta_j^+, \beta_j^-) \in E_{m+1}$ such that

$$\boxdot_{a_j}(\beta_j^+, \beta_j^-) = (0, \beta_j)$$

For shorthand we set $\widetilde{\beta}_j := (\beta_j^+, \beta_j^-)$. By a similar sequence of inequalities as in (8.6.20), it follows that $\widetilde{\beta}_j$ is bounded in E_{m+1}. We introduce the new sequences:

$$\widetilde{\xi}_j := \pi_{a_j}(\xi_j)$$

$$\widetilde{\mu}_j := \pi_{a_j}(\mu_j)$$

Then we have:

$$\oplus_{a_j}(\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j) = \overline{e}_j := \oplus_{a_j}(\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-)$$

$$\ominus_{a_j}(\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j) = \ominus_{a_j}(\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-)$$

By Theorem 1.27 [HWZ0], we have:

$$\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j = (\gamma_{2r}^+\eta_j^+, \gamma_{2r}^-\eta_j^-)$$

From the above, there exists a subsequence such that $\widetilde{\xi}_j + \widetilde{\mu}_j + \widetilde{\alpha}_j + \widetilde{\beta}_j$ converges on level m. The claim follows. ■

Now consider the section $\overline{\partial}_{strip} : X_{strip} \to K_{strip}$ given by:

$$(a, (h^+, h^-)) \to \xi$$

where ξ is uniquely defined by the pair of equations

$$\widehat{\oplus}_a\xi = \frac{1}{2}(T \boxplus_a ((s,t) + h^+, (s',t') + h^-) + i \cdot \boxplus_a((s,t) + h^+, (s',t') + h^-) \cdot i)\frac{\partial}{\partial s} \quad (8.6.21)$$

$$\widehat{\ominus}_a\xi = 0 \quad (8.6.22)$$

Also consider the proposed filling of $\overline{\partial}_{strip}$,

$$\text{Fill}\,\overline{\partial}_{strip} : [0, \frac{1}{2}) \times E_{strip} \to F_{strip}$$

given by

$$(a, h^+, h^-) \to \xi$$

where ξ is uniquely defined by the pair of equations

$$\widehat{\oplus}_a\xi = \frac{1}{2}(T \boxplus_a ((s,t)+h^+,(s',t')+h^-) + i\cdot \boxplus_a((s,t)+h^+,(s',t')+h^-)\cdot i)\frac{\partial}{\partial s} \quad (8.6.23)$$

$$\widehat{\ominus}_a\xi = \frac{1}{2}(T \ominus_a (h^+-h^+_\infty, h^- - h^-_\infty) + i\cdot T\ominus_a (h^+-h^+_\infty, h^- - h^-_\infty)\cdot i)\frac{\partial}{\partial\, s} \quad (8.6.24)$$

Set $h^+ - h^+_\infty = r^+$ and $h^- - h^-_\infty = r^-$. Then by direct calculation and the fact that (s,t) is holomorphic, we can simplify the above equations to the system:

$$\widehat{\oplus}_a\xi(z) = \overline{\partial}_0 r_a$$

$$\widehat{\ominus}_a\xi = \overline{\partial}_0 \ominus_a (r^+, r^-)$$

We have the following results, arguing similarly to Propositions (8.6.1)-(8.6.3) and Lemma 8.6.1:

Proposition 8.6.5. Fill $\overline{\partial}_{strip}$ is *sc*-smooth. ■

Proposition 8.6.6. Fill $\overline{\partial}_{strip}$ is regularizing away from the left and right boundaries. ■

Proposition 8.6.7. Fill $\overline{\partial}_{strip}$ is a filling for $\overline{\partial}_{strip}$. ■

Lemma 8.6.2. Given $\epsilon > 0$, the following holds for every level m:

$$||(DH_a(h^+,h^-) - DH_a(0,0))(\eta^+,\eta^-)||_m \leq \epsilon||(\eta^+,\eta^-)||_m$$

for all $(\eta^+,\eta^-) \in E_m$, all $a \in B_{\frac{1}{2}}$, and all $(h^+,h^-) \in E_m$ sufficiently small on level m. ■

We introduce the shorthand $DH_a(k_j^+,k_j^-) := \Gamma_a(k_j^+,k_j^-)$. Arguing similarly to Proposition 8.6.4 we have:

Proposition 8.6.8. Let $\{a_j\}_{j=0}^{\infty}$, $a_j \in [0, \frac{1}{2})$, be a sequence of real numbers which converge to $a \in [0, \frac{1}{2})$. Consider a sequence of maps $(x_j^+, x_j^-) \subset F_{\text{bdry}}$ which converges to 0 on level m. Also suppose $(y_j^+, y_j^-) \subset F_{\text{bdry}}$ is a sequence of maps which is bounded on level $m+1$ and $(\eta_j^+, \eta_j^-) \subset E_{bdry}$ is a sequence bounded on level m. Suppose a sequence $(k_j^+, k_j^-) \subset E_{bdry}$ converges on level m to (k^+, k^-). If we have the relation:

$$\Gamma_{a_j}(k_j^+, k_j^-)(\eta_j^+, \eta_j^-) = (x_j^+, x_j^-) + (y_j^+, y_j^-) \tag{8.6.25}$$

then (η_j) has a subsequence converging on level m on slightly restricted strips. ■

Now consider $\overline{\partial}_{neck} : X_{neck} \to K_{neck}$ given by

$$(a_{\{x,y\}}, h^+_{\{x,y\}}, h^-_{\{x,y\}})_{\{x,y\}\in D} \to (a_{\{x,y\}}, h^+_{\{x,y\}}, h^-_{\{x,y\}}, \xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})_{\{x,y\}\in D}$$

where $(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})$ is defined by the pair of equations:

$$\widehat{\oplus}_{a_{\{x,y\}}}\xi_{\{x,y\}} \cdot \frac{\partial}{\partial s} = \frac{1}{2} \cdot (T \boxplus_{a_{\{x,y\}}} ((s,t) + h^+_{\{x,y\}}, (s',t') + h^-_{\{x,y\}})$$

$$+i \cdot T \boxplus_{a_{\{x,y\}}} ((s,t) + h^+_{\{x,y\}}, (s',t') + h^-_{\{x,y\}}) \cdot i) \cdot \frac{\partial}{\partial s}$$

$$\widehat{\ominus}_{a_{\{x,y\}}}\xi = 0$$

We have the following by Propositions (8.6.1)-(8.6.8) and Lemmas (8.6.1)-(8.6.2):

Corollary 8.6.1. The filling of $\overline{\partial}_{neck}$, Fill $\overline{\partial}_{neck} : B_D \times E_{neck} \to F_{neck}$ is given by

$$(a_{\{x,y\}}, h^+_{\{x,y\}}, h^-_{\{x,y\}})_{\{x,y\}\in D} \to (\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})_{\{x,y\}\in D}$$

where $\xi^{\pm}_{\{x,y\}}$ is uniquely defined by the pair of equations:

$$\widehat{\oplus}_{a_{\{x,y\}}}\xi^{\pm}_{\{x,y\}}(z) = \frac{1}{2}(T \boxplus_{a_{\{x,y\}}} ((s,t) + h^{+}_{\{x,y\}}, (s',t') + h^{-}_{\{x,y\}})$$

$$+i \cdot T \boxplus_{a_{\{x,y\}}} ((s,t) + h^{+}_{\{x,y\}}, (s',t') + h^{-}_{\{x,y\}}) \cdot i) \cdot \frac{\partial}{\partial s}$$

$$\widehat{\ominus}_{a_{\{x,y\}}}\xi^{\pm}_{\{x,y\}}(z) = \frac{1}{2}(T \ominus_{a_{\{x,y\}}} (h^{+}_{\{x,y\}} - h^{+}_{\{x,y\},\infty}, h^{-}_{\{x,y\}} - h^{-}_{\{x,y\},\infty})$$

$$+i \cdot T \boxplus_{a_{\{x,y\}}} (h^{+}_{\{x,y\}} - h^{+}_{\{x,y\},\infty}, h^{-}_{\{x,y\}} - h^{-}_{\{x,y\},\infty}) \cdot i) \cdot \frac{\partial}{\partial s} \quad \blacksquare$$

Proposition 8.6.9. Fill$\overline{\partial}_{neck}$ is smooth, regularizing away from left and right boundaries, satisfies the quadratic estimate, and satisfies robustness away from the left and right boundaries. ■

Now consider the section

$$\overline{\partial}_{core} : V \times W \times \mathfrak{V} \to H^2\Omega^{0,1}((TS|_{S_{core,\frac{1}{2}}}, j), (TS|_{S_{core,\frac{1}{2}}}, j))$$

given by

$$(v,w,\eta) \to \frac{1}{2} \cdot I_2(w, exp_{\text{Identity}}\eta(z), z) \cdot \Big\{ Texp_{\text{Identity}}\eta + k(w) \cdot Texp_{\text{Identity}}\eta \cdot j(v) \Big\} \cdot I_1(v,z)^{-1}$$

Proposition 8.6.10. $\overline{\partial}_{core}$ is smooth, and regularizing away from the left and right boundaries.

Proof: Smoothness follows since J is smooth, I_2 is smooth, and I_1 is smooth. The regularizing property follows as a result of classical elliptic theory. ■

Proposition 8.6.11. $\overline{\partial}_{core}$ satisfies the quadratic estimate and the robustness property.

Proof: The quadratic estimate property follows from the mean value theorem as in Proposition 7.2.2. The robustness property follows similarly to Proposition 7.2.3. ■

Consider the fiber product section

$$\overline{\partial}_{core} \times \overline{\partial}_{neck} : V \times W \times T_{fiber} \to K_{fiber}$$

and its proposed filling, the fiber product of $\overline{\partial}_{core}$ and the filling for $\overline{\partial}_{neck}$:

$$\overline{\partial}_{core} \times \text{Fill}\, \overline{\partial}_{neck} : B \times V \times W \times \mathcal{T}_{fiber} \to F_{fiber}$$

The following is a consequence of Corollary 8.6.1:

Lemma 8.6.3. $\overline{\partial}_{core} \times \text{Fill}\, \overline{\partial}_{neck}$ is a filling for $\overline{\partial}_{core} \times \overline{\partial}_{neck}$. ■

Definition 8.6.1. Let U denote an open subset of the space of complex structures in $\mathfrak{J}_{\mathbf{D}^2,j}$ containing j. Let V be a neighborhood of 0 in a finite dimensional vector space parametrizing a good family of complex structues in $\mathfrak{J}_{\mathbf{D}^1,j}$. Recall $\mathfrak{T}(\varepsilon)$ from the proof of Proposition 8.0.4. For fixed $\varepsilon > 0$ sufficiently small we set:

$$\mathcal{T}(S) := \mathfrak{T}(\varepsilon)$$

$$B^{m,\delta_0^m,\delta_1^m}(S) := U \times V \times \mathcal{T}(S)$$

$$E^{m-1,\delta_0^{m-1},\delta_1^{m-1}}(S) := B^{m,\delta_0^m,\delta_1^m}(S) \times H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}\Omega^{0,1}((TS,j),(TS,j)) \quad ■$$

Proposition 8.6.12. (compare Lemma 10.1 [HWZ3]) For every pair (δ_0^m, δ_1^m), $0 <$

$\delta_0^m < 2\pi$, $0 < \delta_1^m < \pi$, and $m \geq 3$, the map $T : B^{m,\delta_0^m,\delta_1^m}(S) \to E^{m-1,\delta_0^m,\delta_1^m}(S)$,

$$(v, k, \eta(\cdot)) \to (v, k, \eta(\cdot), I_2(k, exp_{\text{Identity}}\eta(\cdot), \cdot) \circ \overline{\partial}_{k,j(v)} exp_{\text{Identity}}\eta(\cdot) \circ I_1(v, \cdot)^{-1})$$

defines a smooth Fredholm section. The linearization of the map

$$(v, \eta) \to T(v, j, \eta)$$

at the solution $(0, 0)$ is an isomorphism.

Proof: We follow the proof of [HWZ3] Lemma 10.1. We introduce the following abbreviations:

$$L := D_3T(0, j, 0) : \mathfrak{T} \to H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}$$

$$S := D_1T(0, j, 0) : E \to H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}$$

$$F := D_{1,3}T(0, j, 0) : \mathfrak{T} \times E \to H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}$$

which is defined by $F(e, \eta) = Se + L\eta$. Here L acts as

$$f \to \frac{\partial f}{\partial \overline{z}} d\overline{z}$$

and S acts as

$$e \to j \cdot Dj(0)e = -2e$$

For each $z \in |D_i|$, choose holomorphic coordinates:

$$\tau_{z,1} : B_1 \to D_z^1, \quad \tau_{z,2} : B_1 \to D_z^2$$

For each $z \in |D_b|$, choose holomorphic coordinates:

$$\kappa_{z,1} : B_1 \cap \mathbb{H} \to D_z^1, \qquad \kappa_{z,2} : B_1 \cap \mathbb{H} \to D_z^2$$

Consider the maps

$$\gamma : \mathbb{R}^+ \times S^1 \to B_1, \quad \gamma(s,t) = e^{-2\pi(s+it)}$$

$$\delta : \mathbb{R}^+ \times [0,1] \to B_1 \cap \mathbb{H}, \quad \delta(s,t) = e^{-\pi(s+it)}$$

We also define the following compositions:

$$\tau_{z,1}^{polar} := \tau_{z,1} \circ \gamma, \quad \tau_{z,2}^{polar} := \tau_{z,2} \circ \gamma$$

$$\kappa_{z,1}^{polar} := \kappa_{z,1} \circ \delta, \quad \kappa_{z,2}^{polar} := \kappa_{z,2} \circ \delta$$

The section $\eta : S \to TS$ is given in local holomorphic coordinates by:

$$x \to \widehat{\eta}(x) = T\tau_{z,2}^{-1}(\tau_{z,1}(x))\eta(\tau_{z,1}(x)), \quad x \in B_1$$

$$x \to \widehat{\eta}(x) = T\kappa_{z,2}^{-1}(\kappa_{z,1}(x))\eta(\kappa_{z,1}(x)), \quad x \in B_1 \cap \mathbb{H}$$

and in polar coordinates:

$$(s,t) \to \widetilde{\eta}(s,t) = T{\tau_{z,2}^{polar}}^{-1}(\tau_{z,1}^{polar}(s,t))\eta(\tau_{z,1}^{polar}(s,t)), \quad (s,t) \in \mathbb{R}^+ \times S^1$$

$$(s,t) \to \widetilde{\eta}(s,t) = T{\kappa_{z,2}^{polar}}^{-1}(\kappa_{z,1}^{polar}(s,t))\eta(\kappa_{z,1}^{polar}(s,t)), \quad (s,t) \in \mathbb{R}^+ \times [0,1]$$

Consider the transition map:

$$\alpha_1(s,t) = (\tau_{z,2}^{polar})^{-1} \circ \tau_{z,1}^{polar} = (s,t) + (r_0, r_1)(s,t) + c$$

where $c = (c_0, c_1) \in \mathbb{R}^+ \times S^1$, $(r_0, r_1) : \mathbb{R} \times S^1 \to \mathbb{R}^2$ is periodic in t. Also consider the transition map

$$\alpha_2(s,t) = (\kappa_{z,2}^{polar})^{-1} \circ \kappa_{z,1}^{polar} = (s,t) + (w(s,t), 0) + b$$

where $b = (b_0, 0) \in \mathbb{R} \times [0,1]$ and $w : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^+$ is smooth. It is important to note that r_0 decays like $e^{-2\pi s}$ and w decays like $e^{-\pi s}$. By construction we have the relations:

$$\widetilde{\eta}(s,t) = T\gamma^{-1}(\gamma \circ \alpha_1(s,t))\widehat{\eta}(\gamma(s,t)) \tag{8.6.26}$$

$$\widetilde{\eta}(s,t) = T\delta^{-1}(\delta \circ \alpha_2(s,t))\widehat{\eta}(\delta(s,t)) \tag{8.6.27}$$

where

$$T\gamma^{-1}(\gamma \circ \alpha_1(s,t)) = \frac{-1}{2\pi} e^{2\pi[(s+r_0(s,t)+c_0)+i(t+r_1(s,t)+c_1)]} \tag{8.6.28}$$

$$T\delta^{-1}(\delta \circ \alpha_2(s,t)) = \frac{-1}{\pi} e^{\pi[(s+w(s,t)+b)+it]} \tag{8.6.29}$$

Suppose $(e, \eta) \in E \times \mathfrak{T}$ is in the kernel of F. Then since $\eta \in \mathfrak{T}$, in local coordinates near $z \in |D|$ we have:

$$\widetilde{\eta}(s,t) = h(s,t) + d$$

where $h : \mathbb{R}^+ \times S^1 \to \mathbb{R}^2$ and $d \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. It follows from (8.6.26) and (8.6.28) that $\widehat{\eta}$ converges to 0 as $z \to 0$. By Riemann's removable singularity theorem, $\widehat{\eta}$ extends by 0 to a holomorphic section satisfying $\widehat{\eta}(0) = 0$. Thus, η extends over every node $z \in |D|$ and satisfies $\eta(0) = 0$.

In local coordinates near $x \in |HD|$ we have:

$$\widetilde{\eta}(s,t) = g(s,t) + c$$

where $g : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^2$ and $c \in \mathbb{R} \times \{0\}$. By (8.6.27) and (8.6.29), it follows that $\widehat{\eta}$ converges to 0 as $z \to 0$. By Riemann's removable singularity theorem, $\widehat{\eta}$ extends by 0 to a holomorphic satisfying $\widehat{\eta}(0) = 0$. Thus, η extends over every node $z \in |HD|$ and satisfies $\eta(0) = 0$. This means that $\eta \in H^{3+m}\Gamma_0(S, M, D)$. In summary, we have $F(e, \eta) = 0$ and we know that $E \oplus \text{Im } L(\Gamma_0(TS, M, D) = H^{2+m}\Omega^{0,1}(TS, j)$, by hypothesis on E. Thus, we conclude that $(e, \eta) = 0$.

Now we consider the surjectivity of F. Towards this end, consider a section $\xi \in H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}$ compactly supported outside of the points $|D| \cup |HD|$. By Definition 2.1.8 there exists a pair $(e, \eta) \in E \times H^{3+m}\Gamma_0(TS, M, D)$ such that

$$j \cdot Dj(0)e + L(\eta) = \xi$$

Now we show $\eta \in H^{m,\delta_0^m,\delta_0^m}$. Towards this end, we observe that near the points in $|D| \cup |HD|$, ξ is 0 so η is holomorphic near these double points. From the discussion on injectivity, we can use this to show $\widehat{\eta}(0) = 0$. Near $z \in |D|$, we have

$$\widehat{\eta}(x) = \sum_{k \geq 1} a_k x^k = x[a_1 + g(x)]$$

where $g(x)$ is a holomorphic function $g(x) = \sum_{k \geq 2} a_k x^{k-1}$. Set

$$\beta_1(s,t) = 2\pi e^{2\pi[r_0(s,t) + ir_1(s,t)]}$$

Then in polar coordinates we have:

$$\widetilde{\eta}(s,t) = \frac{-1}{2\pi} e^{-2\pi[c_0 + ic_1]}[a_1 + [\beta_1(s,t) - 1]a_1 + \beta_1(s,t)g(e^{-2\pi(s+it)})]$$

From the decay property of r_0, it follows that

$$|\partial^\alpha[\beta_1(s,t)-1]| \geq C\cdot e^{2-\pi s}| \quad |\partial^\alpha \beta_1(s,t) g(e^{-2\pi(s+it)})| \leq Ce^{-2\pi s}$$

for $|\alpha| \leq m$ and some fixed C sufficiently large. In particular this means that $\widetilde{\eta}$ decomposes as $\widetilde{\eta} = h(s,t)+d$ where $h \in H^{m,\delta_0^m}$ and $d \in \mathbb{R}\times S^1$. Thus, it follows in a neighborhood of x, $\widetilde{\eta} \in H^{m,\delta_0^m}$. Similarly, near $x \in |HD|$, we have:

$$\widehat{\eta}(x) = \sum_{k\geq 1} b_k x^k = x[b_1 + f(x)]$$

where $f(x)$ is the holomorphic function $f(x) = \sum_{k\geq 2} b_k x^{k-1}$. Set

$$\beta_2(s,t) = e^{-\pi b}[b_1 + [\beta_2(s,t)-1] + \beta_2(s,t) f(e^{-\pi(s+it)})]$$

On $\mathbb{R}^+\times[0,1]$ we have:

$$\widetilde{\eta}(s,t) = \pi[b_1 + f(e^{-\pi(s+it)}]$$

From the decay property of w it follows that

$$|\partial^\alpha[\beta_2(s,t)-1]| \leq C\cdot e^{-\pi s}, \quad |\partial^\alpha[\beta_2(s,t) f(e^{-\pi(s+it)})]| \leq C\cdot e^{-\pi s}$$

for $|\alpha| \leq m$. In particular this means that $\widetilde{\eta}$ decomposes as $\widetilde{\eta} = k(s,t)+c$ where $k \in H^{m,\delta_1^m}_{real}$ and $c \in \mathbb{R}$. Thus in a neighborhood of x, $\widetilde{\eta} \in H^{m,\delta_1^m}_{c,\,real}$. We see that $\eta \in \Gamma^{m,\delta_0^m,\delta_1^m}$ and $F(e,\eta) = \xi$. Observe F is semi-Fredholm, following Proposition 7.3.2. In particular, the image of F is closed. The surjectivity of F follows from the fact that the set of sections ξ having compact support outside of $|D|\cup|HD|$ is dense in $H^{m-1,\delta_0^{m-1},\delta_1^{m-1}}$. ■

We have the following by the classical Banach space implicit function theorem:

Corollary 8.6.2. There exists an open subset $K \subset \mathfrak{J}_{\mathbf{D}^2,j}$ and a uniquely defined continuous map

$$g : K \to V \times B^{3,\delta_0^0,\delta_1^0}(S)$$

$$k \to (g_1(k), g_2(k))$$

which satisfies:

$$\overline{\partial}_{k,g_1(k)} exp_{\text{Identity}} g_2(k) = 0 \quad \blacksquare$$

Proposition 8.6.13. The section $\overline{\partial}_{core} \times \overline{\partial}_{neck}$ is *sc*-Fredholm.

Proof: Smoothness of $\overline{\partial}_{core} \times \overline{\partial}_{neck}$ follows from Proposition Propositions 8.6.9 and 8.6.10. It suffices to check the conditions of Theorem 7.0.1. To verify Condition (1) we can proceed as in Section 7.3, with Proposition 8.6.12 taking the place of Lemma 7.3.1 and directly calculate that the Fredholm index is constant for parameters near 0 as in Proposition 7.3.3.

Condition (2) follows from Proposition 8.6.9 and Proposition 8.6.11.

Condition (3) follows from Proposition 8.6.9 and Proposition 8.6.11. $\blacksquare$

Proposition 8.6.14. The diagram below is commutative:

$$\begin{array}{ccc} V \times W \times \text{Im } (\text{Id } \times \rho \times \widehat{\pi}) & \xrightarrow{\Theta} & \overline{\mathfrak{E}} \\ \uparrow{\scriptstyle \overline{\partial}_{core} \times \overline{\partial}_{neck}} & & \uparrow{\scriptstyle \overline{\partial}} \\ V \times W \times T_{fiber} & \xrightarrow{\boxplus \text{oglue}} & V \times W \times \overline{\mathfrak{S}} \end{array}$$

Proof: First we consider $\overline{\partial} \circ \boxplus \circ \mathfrak{glue}$:

$$(v, w, a, \eta, (h^+, h^-)) \xrightarrow{\mathfrak{glue}} (v, w, \boxplus(a, \mathfrak{glue}(\eta, h^+, h^-)) \xrightarrow{\boxplus} (v, w, \boxplus \circ \mathfrak{glue}(a, \eta, h^+, h^-)))$$

$$\xrightarrow{\overline{\partial}} (v, w, \boxplus(a, \mathfrak{glue}(\eta, h^+, h^-)), \overline{\partial}_{k(w),j(v)} \boxplus (a, \mathfrak{glue}(\eta, h^+, h^-)))$$

Going the other way around the diagram, we consider the principal part of $\Theta \circ \overline{\partial}_{core} \times \overline{\partial}_{neck}$:

$$(v, w, a, \eta, (h^+, h^-)) \overset{\overline{\partial}_{core} \times \overline{\partial}_{neck}}{\longrightarrow}$$

$$(I_2(w, exp_{\text{Identity}}\eta(z), z) \circ \overline{\partial}_{k(w),j(v)} exp_{\text{Identity}}\eta \circ I_1(v, z)^{-1}, \xi^+, \xi^-) \xrightarrow{\Theta} \theta$$

where $(\overline{\xi}^+, \overline{\xi}^-)$ is defined by:

$$\bigoplus_{a_{\{x,y\}}} (\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = \frac{1}{2} \cdot (T\boxplus_a((s,t)+h^+, (s',t')+h^-) + i \circ T\boxplus_a((s,t)+h^+, (s',t')+h^-) \circ i) \cdot \frac{\partial}{\partial s}$$

$$\widehat{\ominus}_a(\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}}) = 0$$

and θ is defined on the core by:

$$\theta|_{S_{core,\frac{1}{2}}} := I_2(w, z, exp_{\text{Identity}}\eta(z))$$

$$\circ I_2(w, exp_{\text{Identity}}\eta(z), z) \circ \overline{\partial}_{k(w),j(v)} exp_{\text{Identity}}\eta(z) \circ I_1(v, z)^{-1} \circ I_1(v, z)$$

$$= \overline{\partial}_{k(w),j(v)} exp_{\text{Identity}}\eta(z)$$

and on the neck (in positive coordinates) by

$$Tk^{-1}_{b_{\{x,y\}}} \circ \theta \circ Th_{a_{\{x,y\}}} \cdot \frac{\partial}{\partial s}|_{Z_{a_{\{x,y\}}}}(s,t) := \bigoplus_{a_{\{x,y\}}} (\xi^+_{\{x,y\}}, \xi^-_{\{x,y\}})(s,t)$$

$$= \frac{1}{2} \cdot (T \boxplus_a ((s,t) + h^+, (s',t') + h^-) + i \circ T \boxplus_a ((s,t) + h^+, (s',t') + h^-) \circ i) \cdot \frac{\partial}{\partial s}$$

$$= Tk^{-1}_{b_{\{x,y\}}} \circ \overline{\partial}_{k(b,w),j(a,v)} \boxplus (a, \mathfrak{glue}(\eta, h^+, h^-)) \circ Th_{a_{\{x,y\}}}$$

The claim follows. ■

Corollary 8.6.3. The section $\overline{\partial} : V \times W \times \overline{\mathfrak{S}}|\text{Im } \boxplus \circ \mathfrak{glue} \to \overline{\mathfrak{E}}$ is *sc*-Fredholm. ■

By Propositions 8.6.12, 13.0.1 (which is proved in Appendix 6), and Corollary 8.6.3:

Corollary 8.6.4. Suppose $j : V \to \mathfrak{J}_{\mathbf{D}^1,j}$ is a germ around the origin parametrizing a good family such that $j(0) = j$ and $k : W \to \mathfrak{J}_{\mathbf{D}^2,j}$ is a finite-dimensional smooth family such that $k(0) = j$. Then there exists a unique *sc*-smooth map

$$F : \mathfrak{O} \subset V \times B_D \to V \times W \times \overline{\mathfrak{S}}$$

$$F(a,v) = (f(a,v), v, a, b(a,v), \phi_{(a,v)})$$

$$\phi_{(a,v)} : (S_a, k(a,v), M_a, D_a) \to (\widehat{S}_{b(a,v)}, j(b(a,v), f(a,v)), M_{b(a,v)}, D_{b(a,v)})$$

defined on a small open neighborhood $\mathfrak{O}$ of $(0,0)$ and satisfying $F(0,0) = (0,0,0,0,\text{Identity})$ and

$$\overline{\partial}_{j(b(a,v),f(a,v)),k(a,v)}\phi_{(a,v)} = 0 \quad ■$$

8.7 Corollaries of the *sc*-Fredholm Property

We finally present the proof of Theorem 2.4.1:

Theorem 2.4.1 (compare Theorem 2.19 [HWZ1]) Fix a good family $(a,v) \to \alpha_{(a,v)}$

1. Consider the family $(b,k,\sigma) \to \beta_{(b,k,\sigma)}$ of noded Riemann surfaces with boundary, where $(b,k,\sigma) \in B_{\frac{1}{2}} \times U_{D'} \times V(M')$ and suppose that we have an isomorphism:

$$\psi : \beta_{(0,j',\sigma_0)} = (S', j', M'(\sigma_0), D') \to (S_{a_0}, j(a_0, v_0), M_{a_0}, D_{a_0}) = \alpha_{(a_0,v_0)}$$

for some parameter (a_0, v_0). Then there exists a uniquely determined core-continuous germ

$$(b,k,\sigma) \to \mu(b,k,\sigma) = (a(b,k,\sigma), w(b,k,\sigma))$$

defined for (b,k,σ) near $(0,j',\sigma_0)$ which satisfies $\mu(0,j',\sigma_0) = (a_0, v_0)$ and a core-continuous germ $(b,k,\sigma) \to \phi(b,k,\sigma)$ of isomorphisms

$$\phi(b,k,\sigma) : \beta_{(b,k,\sigma)} \to \alpha_{\mu(b,k,\sigma)}$$

between the noded Riemann surfaces with boundary satisfying $\phi(0,j',\sigma_0) = \psi$.

2. Also if $w \to k(w)$ is a smooth family of complex structures on S', with $k(0) = j'$ then the family $(b,w,\sigma) \to \phi(b,k(w),\sigma)$ of isomorphisms

$$\phi(b,k(w),\sigma) : \beta_{(b,k(w),\sigma)} \to \alpha_{\mu(b,k(w),\sigma)}$$

is core-smooth.

Proof: We begin with Claim 1. We will first assume $M(\sigma) = M(\sigma_0)$ for all $\sigma \in V(M)$ and treat the general case afterwards. Consider the map:

$$\psi : (S', j', M'(\sigma_0), D') \to (S, j, M, D)$$

Choose a finite dimensional good family $w \to k(w)$ of complex structures on S' such that $k(0) = \psi^* j = j'$ and satisfies $k(w)|\psi^{-1}D = j'|D$. Further, consider the family of glued surfaces

$$(b, w) \to (S'_b, j'(b, w), M'(\sigma_0)_b, D'_b), \quad \text{for } b \in N(D'),\ w \in W$$

By Corollary 8.6.4 we have a uniquely defined *sc*-smooth map

$$F : \mathfrak{O} \subset B_D \times W \to V \times W \times \overline{\mathfrak{O}}$$

$$F(a, w) = (f(a, w), w, a, b(a, w), \phi_{(a,w)})$$

defined on a small open neighborhood $\mathfrak{O}$ of the origin and satisfying $F(0, 0) = (0, 0, 0, 0, \text{Identity})$ and

$$\overline{\partial}_{k(b(a,w),w),j(a,f(a,w))}\phi_{(a,w)} = 0$$

where

$$\phi_{(a,w)} : (S'_a, j'(a, w), M'(\sigma_0), D'_a) \to (\widehat{S}'_{b(a,w)}, k(b(a, w), f(a, w)), M'(\sigma_0), \psi^{-1}D_{b(a,w)})$$

For $(b, w) \in N(D') \times W$ we have biholomorphisms

$$\psi_b : (\widehat{S}'_b, j'(b, w), M'(\sigma_0)_b, \phi^{-1}D_b) \to (S_b, j(b, w), M(\sigma_0)_b, D_b)$$

We set $\phi(c, k, \sigma_0) := \psi_{b(c,w)} \circ \phi_{c,w}$. For fixed σ_0, the continuity and uniqueness properties follow from Corollary 8.6.4.

Now consider a general triplet $(a, v, \sigma) \in B_{D'} \times U_{|D'|} \times V(M)$. Choose a smooth family

of isotopies

$$\phi_{M(\sigma)\to M(\sigma_0)} : (S', M(\sigma_0)) \to (S', M(\sigma))$$

for $\sigma \in V(M')$ such that $\phi_{M(\sigma)\to M(\sigma_0)}(M(\sigma_0)) = M(\sigma)$. Consider the open subset $\mathfrak{O}' \subset U_{|D'|} \times V(M)$ consisting of pairs (k, σ) such that $\phi^*_{M(\sigma)\to M(\sigma_0)} k \in \mathfrak{O}$ and $\phi_{M(\sigma)\to M(\sigma_0)}$ is supported away from $\mathbf{D}'$. By construction, we have a biholomorphism:

$$\phi^{-1}_{M(\sigma)\to M(\sigma_0)} : (S', k, M(\sigma), D') \to (S', \phi^*_{M(\sigma)\to M(\sigma_0)} k, M(\sigma_0), D')$$

Finally set $\phi(b, k, \sigma) := \phi(b, k, \sigma_0) \circ \phi^{-1}_{M(\sigma)\to M(\sigma_0)}$.

Claim 2 follows similarly. ■

Corollaries 8.7.1 and 8.7.2 below follow from Theorem 2.4.1:

Corollary 8.7.1. Let O be as in Definition 2.3.1. Consider the map

$$\Psi : \mathfrak{O} \to DM_{g,h,M_i,M_b}$$

given by:

$$(a, v) \to [(S_a, j(a, v), M_a, D_a)]$$

Then Ψ is an open map. ■

Corollary 8.7.2. (compare Theorem 3.15 [HWZ3]) Let $\alpha = (S, j, M, D)$ be a stable noded surface with automorphism group G and let $v \to j(v)$, $v \in V \subset E$ be a good family. Then there exists:

a. A C^∞ open neighborhood U of $\mathfrak{J}_{\mathbf{D},j}$, and

b. An open G-invariant neighborhood V' of $0 \in E$, with $\overline{V'} \subset V$

c. A small open subset $B_D(\epsilon) \subset B_D$

such that:

1. For every $g \in G$ there exists a smooth map

$$\Phi_g : B_D(\epsilon) \times U \to V' \times \overline{\mathfrak{S}}$$

$$(a,k) \to (\Phi_g^1(a,k), \Phi_g^2(a,k))$$

determined by the requirement

$$\Phi_g(0,j) = (0,g) \quad \text{and} \quad \overline{\partial}_{j(\Phi_g^1(a,k)),k} \Phi_g^2(a,k) = 0$$

2. For every solution (k,v,u) of

$$\overline{\partial}_{j(v),k} u = 0$$

with $(a,k) \in B_D(\epsilon) \times U$, $v \in V$, and $u \in \overline{\mathfrak{S}}$ there exists a unique $g \in G$ such that

$$\Phi_g(a,k) = (v,u) \quad \blacksquare$$

Theorem 2.3.1 Existence of Good Uniformizing Families For each stable noded Riemann surface $\alpha = (S, j, M, D)$ there exists a good uniformizing family

$$(a,v) \to \alpha_{(a,v)}, \qquad (a,v) \in O$$

Proof: Let V parametrize a good family. Its existence is guaranteed by Proposition 2.1.4. Let $B_D(\epsilon) \subset B_D$ and $V' \subset V$ be sufficiently small open neighborhoods of B_D and V, respectively. Consider the associated family

$$(a, v) \to \alpha_{(a,v)}, \quad (a, v) \in B_D(\epsilon) \times V'$$

Property 4 holds by construction. The verification of properties $1 - 3$ is given below:

Proof Property 1 (compare Lemma 3.17 [HWZ3]) Recall that the topology on DM_{g,h,M_i,M_b} is generated by subsets of the form:

$$V(\alpha, U, \mathbf{D}, \epsilon) := \{[S_a, k_a, M_a, D_a] \mid k \in U,\ k = j \text{ on } |\mathbf{D}|,\ |a| < \epsilon\}$$

Thus to show

$$U = \{[\alpha_{(a,v)}] \ : \ (a, v) \in O\}$$

is open, it is enough to show the existence of a C^∞ neighborhood $U \subset \mathfrak{J}_{\mathbf{D},j}$ of $j(v_0)$ such that for every pair $(a, k) \in B_D(\epsilon) \times U$ there exists a pair $(a', v) \in B_D \times V'$ such that

$$[S_{a'}, j(a', v), M_{a'}, D_{a'}] = [S_a, k_a, M_a, D_a]$$

The existence of (a', v) follows directly from Theorem 2.4.1. ■

Proof of Property 2 The argument establishing Property 1 shows that $[p]$ is an open map. Surjectivity follows from Theorem 2.4.1. Thus it is enough to show that $[p]$ is injective. This follows from Property 3.

Proof of Property 3 By Corollary 8.7.2, there is a unique morphism $\alpha_{(a,v)} \to \alpha_{(a',v')}$ in a neighborhood of $g : \alpha_{(0,0)} \to \alpha_{(0,0)}$. By Lemma 2.2.1 the unique morphism is g_a. ■ ■ ■

Theorem 2.3.2 With respect to the exponential gluing profile, the object set T_{g,h,M_i,M_b} is a smooth manifold with boundary. The morphism sets $M(G_\lambda, G_{\lambda'})$ are also smooth manifolds with boundary. The structure maps are *sc*-smooth, and $(T, \mathbf{T})$ is an *ep*-groupoid. These charts realize DM_{g,h,M_i,M_b} as a smooth orbifold.

Proof of Theorem 2.3.2 Recall the morphism sets $M(G_\lambda, G_{\lambda'})$:

$$M(G_\lambda, G_{\lambda'}) := \{((a,v), \phi, (a',v')) : (a,v) \in O_\lambda, (a',v') \in O_{\lambda'}, \phi : S_{a,v} \to S'_{a',v'} \text{ is an equivalence } \}$$

By Corollary 8.6.4, we know that the source and target maps:

$$s : M(G_\lambda, G_{\lambda'}) \to O_\lambda \quad ((a,v), \phi, (a',v')) \to (a,v)$$

$$t : M(G_\lambda, G_{\lambda'}) \to O_{\lambda'} \quad ((a,v), \phi, (a',v')) \to (a',v')$$

are smoothly compatible local bijections. Thus, $M(G_\lambda, G_{\lambda'})$ is a manifold with boundary, with smooth structure inherited from the source and target maps. The verification of the smoothness of the inverse, unit, and multiplication maps follow in a straightforward manner. The properness property follows from Gromov-Compactness. ■

8.8 Technical Results on Families of Holomorphic Maps

We summarize the set up of Theorem 1.46 [HWZ0], detailed in [HWZ0] pages 42-45: Denote by Γ the collection of pairs (a,b) of complex numbers such that $a \cdot b \neq 0$ and $|a|, |b| < \frac{1}{2}$. Denote by X the set consisting of all tuples (a,b,w) such that $(a,b) \in \Gamma$ and

$w : Z_a \to Z_b$ is an H^3 diffeomorphism of cylinders satisfying $w(p_a^{\pm}) = p_b^{\pm}$. Here the points $p_a^{\pm}$ are specified by the rule that $p_a^+ = [(0,0)]$ and $p_a^- = [(0',0')]$, and $p_b^{\pm}$ are defined similarly.

Proposition 8.8.1. (compare Proposition 1.42 [HWZ0]) The space X admits a natural second countable paracompact topology. X also admits a natural M-polyfold structure where the m^{th} level consists of H^{3+m} diffeomorphisms. ■

Consider the set of maps D_{cyl}^{3,δ_0^0} which consists of the pairs (u^+, u^-) of C^1 diffeomorphisms $u^{\pm} : \mathbb{R}^{\pm} \times S^1 \to \mathbb{R}^{\pm} \times S^1$ such that:

1. The maps $u^{\pm}$ admit H^3_{loc} regularity

2. There exist constants $(d^{\pm}, \theta^{\pm}) \in \mathbb{R} \times S^1$ and a map $r^{\pm} \in H^{3,\delta_0^0}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^2)$ such that

$$u^{\pm}(s,t) = (s + d^{\pm}, t + \theta^{\pm}) + r^{\pm}(s,t)$$

3. $u^{\pm}(0,0) = (0,0)$

Consider also the completion:

$$\overline{X} := X \bigsqcup D_{cyl}^{3,\delta_0^0}$$

Theorem 8.8.1. (compare Theorem 1.43 [HWZ0]) $\overline{X}$ admits a natural paracompact topology, extending that of X. ■

Consider the circle S_2^1 of radius 2. We can carry out the argument for Theorem 1.46 [HWZ0] adapted to the situation in which $\mathbb{R} \times S^1$, $\mathbb{R}^{\pm} \times S_2^1$, and $[0,R] \times S^1$ are replaced with $\mathbb{R} \times S_2^1$, $\mathbb{R}^{\pm} \times S_2^1$, and $[0,R] \times S_2^1$, respectively. The following result is a slight modification of Theorem 1.46 [HWZ0]:

Theorem 8.8.2. (compare Theorem 1.46 [HWZ0]) Consider a family of complex structures $(\mathbb{R}^+ \times S^1, j(v)(s,t))$, $v \in V$ such that $j(v)(s,t) = i$ for $s \geq s_0$, for some $s_0 > 0$. Let $(v_0, 0) \in V \oplus \mathbb{C}$ be fixed and let

$$(a, v) \to b(a, v) \quad \text{and} \quad (a, v) \to w(a, v)$$

be a family of holomorphic maps defined on some small open neighborhood $O(v_0, 0) \subset V \oplus \mathbb{C}$, containing $(v_0, 0)$, such that

$$w(a, v) : (Z_a, j(a, v), p_a^+, p_a^-) \to (Z_{b(a,v)}, i, p_{b(a,v)}^+, p_{b(a,v)}^-)$$

is a biholomorphic map between cylinders if $a \neq 0$.

Then given a constant $\Delta > 0$, on a possibly restricted neighborhood $O(v_0, 0)$, there exists a map

$$(a, v) \to \widetilde{w}(a, v) \in \bigcap_{m \geq 3, 0 < \epsilon < \pi} D^{m,\epsilon}$$

such that for every $m \geq 3$ and every $0 < \epsilon < \pi$ the map

$$(a, v) \to \widetilde{w}(a, v) \in D^{m,\epsilon}$$

is smooth and satisfies

$$\widetilde{w}(a, v)(s, t) = w(a, v)(s, t)$$

for all $(s, t) \in [0, \frac{1}{2}\phi(|a|) + \Delta] \times S_2^1$. In addition,

$$\widetilde{w}(a, v)(s, t) = (s + d_{(a,v)}, t + t_{(a,v)})$$

for $s \geq s(a)$, that is, for large s where large depends on the gluing parameter a. ■

Note that ϵ in the statement of the modified version of Theorem 1.46 [HWZ0], is constrained $0 < \epsilon < \pi$ since the diameter of the cylindrical ends has been adjusted from 1 to 2.

Theorem 8.8.3. (compare Theorem 1.46 [HWZ0]) Consider a family of complex structures $(\mathbb{R}^+ \times [0,1], j(v)(s,t))$, $v \in V$ such that $j(v)(s,t) = i$ for $s \geq s_0$, for some $s_0 > 0$. Let $(v_0, 0) \in V \oplus \mathbb{R}^+$ be fixed and let

$$(v,a) \to b(v,a) \text{ and } (v,a) \to w(v,a)$$

be a family of maps defined on some small open neighborhood $O(v_0, 0)$ and where

$$w(v,a) : (HZ_a, j(a,v)) \to (HZ_{b(v,a)}, i)$$

is the associated biholomorphic map between the strips if $a \neq 0$.

Then given a constant $\Delta > 0$, on a possibly restricted neighborhood $O(v_0, 0)$, there exists a map

$$(a,v) \to \widetilde{w}(a,v) \in \bigcap_{m \geq 3, 0 < \epsilon < \pi} D^{m,\epsilon}$$

such that for every $m \geq 3$ and every $0 < \epsilon < \pi$ the map

$$(a,v) \to \widetilde{w}(a,v) \in D^{m,\epsilon}$$

is smooth and satisfies

$$\widetilde{w}(a,v)(s,t) = w(a,v)(s,t)$$

for all $(s,t) \in [0, \frac{1}{2}\phi(|a|) + \Delta] \times I$. In addition,

$$\widetilde{w}(a,v)(s,t) = (s + d_{(a,v)}, t)$$

for $s \geq s(a)$, that is for large s where large depends on the gluing parameter a.

Proof of the Theorem: Consider the complex doubles of the maps $w(a,v)$

$$w(a,v)^D : (Z_a^D, j(a,v)^D) \to (Z^D_{b(a,v)}, i)$$

defined between cylinders of radius 2. By construction $w(a,v)(p_a^{\pm}) = p_b^{\pm}$. The claim follows from Theorem 8.8.2 ■

9 Appendix 2: Bubbling-Off Analysis

Lemma 9.0.1. (compare Lemma 3.4 [HWZ1]) We consider the bordered stable map (S, j, M, D, u) and two sequences of maps, $u_k, u_k' \in H^{3,\delta_0,\delta_1}(S, Q)$, from the noded Riemann surface with boundary S into the symplectic manifold Q converging to u in C^0. We assume that $(a_k, v_k) \to (0,0) \in \mathfrak{O}$ and $(b_k, w_k) \to (0,0) \in \mathfrak{O}$ and that

$$\phi_k : (S_{a_k}, j(a_k, v_k), M_{a_k}, D_{a_k}, \oplus_{a_k}(u_k)) \to (S_{b_k}, j(b_k, w_k), M_{b_k}, D_{b_k}, \oplus_{b_k}(u_k'))$$

is a sequence of isomorphisms. Then there is a subsequence of ϕ_k which converges in C^∞_{loc} away from the nodes to an automorphism ϕ_0 of (S, j, M, D, u).

Proof Assume for sake of contradiction that there exists a subset $z_k \in S_k$ satisfying

$$R_k = |\nabla \phi_k(z_k)| \to \infty$$

First we consider the case that the sequence z_k stays at a finite distance to the core of the sequence S_{a_k}. There are 3 cases to consider:

(1) The sequence of images $(\phi_k(z_k))$ is contained in a strip-like neck N_k' of S_{b_k} associated with a nodal pair $\{x', y'\} \in D_b$ and $dist(\phi_k(z_k), \partial N_k') \to \infty$.

(2) The sequence of images $(\phi_k(z_k))$ is contained in the cylindrical neck N_k of S_{b_k} associated with the nodal pair $\{x, y\} \in D_i$ and $dist(\phi_k(z_k), \partial N_k) \to \infty$.

(3) The sequence of images $(\phi_k(z_k))$ lies at a finite distance to the core region of the sequence S_{b_k}.

Let's consider case (3) first. Since z_k stays at a finite distance to the core, there exists k' sufficiently large such that we can assume $z_k \in S_{a_{k_0}}$ for some fixed k_0, for all $k \geq k'$. For simplicity, will refer to $S_{a_{k_0}}$ by S. If z_k converges to a point in the interior, the analysis in Lemma 3.4 [HWZ1] is sufficient to derive a contradiction. Let's suppose that z_k converges to a point $z^* \in \partial S$. By Hofer's Lemma [H] there exist sequences (ϵ_k) and (z_k) satisfying:

$$\epsilon_k \to 0, \qquad R_k \epsilon_k \to \infty$$

$$|\nabla \phi_k(z)| \leq 2R_k \text{ if } |z - z_k| \leq \epsilon_k$$

Choose a coordinate chart $\psi : B_1 \cap \mathbb{H}^+ \to S$, satisfying $\psi(0) = z^*$. We identify the points

$z_k \in S$ with their preimages under the coordinate chart ψ and work in local coordinates induced by ψ. There are two subcases to consider:

(3a)

$$\text{Im } z_k R_k \to \infty$$

(3b)

$$\text{Im } z_k R_k \to l \in [0, \infty)$$

Let's consider (3*b*) first. Consider the reparametrizations

$$\widetilde{\phi}_k(z) = \phi_k(\text{Re } z_k + \frac{z}{R_k})$$

defined for $|z| \leq \epsilon_k R_k$. Observe that $\widetilde{\phi}_k$ satisfies

$$|\nabla \tilde{\phi}_k(\text{Im } z_k \cdot R_k)| = 1 \tag{9.0.1}$$

$$|\nabla \tilde{\phi}_k(z)| \leq 2 \tag{9.0.2}$$

for all $z \in B_{\epsilon_k R_k} \cap \mathbb{H}^+$. Let $K \subset \mathbb{H}^+$ be a compact subset of $\mathbb{H}^+$. For k sufficiently large, K is contained in $B_{\epsilon_k R_k}(0) \cap \mathbb{H}^+$. Since $\widetilde{\phi}_k(0) = \phi_k(\text{Re } z_k) \to \phi_k(z_k) = z^*$ as $k \to \infty$, and we have uniform C^k bounds for every $k \geq 1$ ([A]). By the Ascoli-Arzela theorem, there exists a subsequence converging to a map $\phi : \mathbb{H}^+ \to S$ in C^∞_{loc}. The map ϕ is holomorphic and non-constant by (9.0.1). By removal of singularities, ϕ extends to a map $\phi : \mathbb{H}^+ \cup \{\infty\} \cong B_1 \to S$.

Note also that $\phi(\partial B_1) \subset \partial S$. By Schwarz reflection, we can double ϕ to a map $\phi^d : S^2 \to S^d$, where S^d is the double of S. The image of the map ϕ^d lands in a connected,

compact component C^d of S^d. Non-constant holomorphic maps are open, and the image of ϕ^d is closed (since the domain is compact) so the image of ϕ^d is all of C^d. By Riemann-Hurwitz, this means C^d is a sphere. Therefore, the original map ϕ itself surjects onto a disk component. The maps ϕ_k map special points to special points. The image of the map ϕ^d can contain two marked points if z^* is a boundary marked point and only one if z^* is not a boundary marked point. In either case, this implies that C is not a stable component of S.

Thus, the stability hypothesis implies that $\int_C u^*\omega > 0$. This means that we may find two points, p and q in B such that $\phi(p)$ and $\phi(q)$ are distinct and satisfy $dist(u'(\phi(p)), u'(\phi(q))) > 0$. By assumption, the maps ϕ_k satisfy $\oplus_{b_k}(u'_k) \circ \phi_k = \oplus_{a_k}(u_k)$. At finite distance to the core, for sufficiently large k, $\oplus_{b_k}(u'_k) = u'_k$ and similiarly for (u_k) so we have $u'_k \circ \phi_k = u_k$ at finite distance to the core, for sufficiently large k. We set the reparametrization maps $\tau_k : B_{\epsilon_k R_k} \cap \mathbb{H}^+ \to B_1$, given by $\tau_k(z) = \text{Re } z_k + \frac{z}{R_k}$. We consider the calculation:

$$0 < \text{dist}(u' \circ \phi(p), u' \circ \phi(q))$$

$$= \lim_k \text{dist}(u'_k \circ \phi_k \circ \tau_k(p), u'_k \circ \phi_k \circ \tau_k(q))$$

$$= \lim_k \text{dist}(u_k \circ \tau_k(p), u_k \circ \tau_k(q))$$

$$= \text{dist}(u(\text{Re } z_k), u(\text{Re } z_k)) = 0$$

This is, of course, a contradiction.

Now we examine case $(3a)$. Again, we employ Hofer's Lemma to find a sequence ϵ_k

and a replacement sequence for the (z_k) which satisfies:

$$\epsilon_k \to 0, \qquad R_k \epsilon_k \to \infty$$

$$|\nabla \phi_k(z)| \leq 2R_k \text{ if } |z - z_k| \leq \epsilon_k$$

Consider the reparametrized maps

$$\widetilde{\phi}_k(z) = \phi_k(z_k + \frac{z}{R_k})$$

which we consider to be defined on

$$\Omega_k := \{z \in \mathbb{C} : \text{Im } z \geq - \text{ Im } z_k R_k\}$$

Observe

$$|\nabla \widetilde{\phi}_k(z)| = \frac{1}{|R_k|} |\nabla \phi_k(z_k + \frac{z}{R_k})|$$

So we have

$$|\nabla \widetilde{\phi}_k(0)| = 1$$

and

$$|\nabla \widetilde{\phi}_k(z)| \leq 2$$

for $z \in B_{\epsilon_k R_k}(0) \cap \Omega_k$ and

$$\widetilde{\phi}_k(0) = \phi_k(z_k) \to z^{**}$$

as $k \to \infty$. For any compact subset $K \subset \mathbb{C}$, for some k' sufficiently large, we have $K \subset \Omega_k$, for all $k \geq k'$. We also have C^0 and C^1 bounds, as before. Thus we have C^∞_{loc} bounds, and by Ascoli-Arzela we have a C^∞_{loc} convergent subsequence converging to

a map $\widetilde{\phi} : \mathbb{C} \to S$. By removal of singularities, we have a map $\phi : S^2 \to S$. By a similar but simpler argument using Riemann-Hurwitz and so on, we derive a contradiction as in case $(3b)$.

Now we move to case (1). We recall that this means that we suppose that:

The sequence of images $(\phi_k(z_k))$ is contained in the strip-like neck N_k' of S_{a_k} associated with the nodal pair $\{x', y'\} \in D$ and $\text{dist}(\phi_k(z_k), \partial N_k') \to \infty$.

First we suppose $z \in \partial S$. We have the following subcases:

(1a)

$$\text{Im } z_k R_k \to \infty$$

(1b)

$$\text{Im } z_k R_k \to l \in [0, \infty)$$

Consider case $(1a)$. Note that since $(\phi_k(z_k), \partial N_k') \to \infty$, we can identify the target of ϕ_k with $\mathbb{R} \times [0, 1]$ for k sufficiently large. Thus we can represent the functions ϕ_k as a tuple

$$\phi_k(z) = (a_k(z), \theta_k(z)) \in \mathbb{R} \times [0, 1]$$

Assume we've made the appropriate replacements coming from Hofer's Lemma. We define the reparametrizations as follows:

$$\widetilde{\phi}_k(z) = (a_k(z_k + \frac{z}{R_k}) - a_k(z_k), \theta_k(z_k + \frac{z}{R_k}))$$

We take the domain of $\widetilde{\phi}_k$ to be

$$\Omega_k := \{z \in B_{\epsilon_k R_k} : \text{Im } z \geq -z_k R_k\}$$

The reparametrizations $\widetilde{\phi}_k$ satisfy

$$|\nabla \widetilde{\phi}_k(0)| = 1$$

and

$$|\nabla \widetilde{\phi}_k(z)| \leq 2$$

if $|z| \leq \epsilon_k R_k$. We have $\tilde{\phi}_k(0) = (0, \theta_k(z_k))$ which converges on a subsequence since $[0,1]$ is compact. This fact and the gradient bounds imply we have C^∞_{loc} bounds and by the Arzela-Ascoli theorem a C^∞_{loc} convergent subsequence, converging to a limit $\widetilde{\phi} : \mathbb{C} \to \mathbb{R} \times [0,1]$ satisfying

$$|\nabla \widetilde{\phi}(0)| = 1$$

By removal of singularities we can extend $\widetilde{\phi} : S^2 \to \mathbb{R}^1 \times [0,1] \cong \mathbb{H}^+ \subset \mathbb{C}$. However, there are no non-constant holomorphic maps into $\mathbb{C}$, so we have reached a contradiction.

Now let's consider case $(1b)$. Consider the reparametrizations:

$$\widetilde{\phi}_k(z) = \phi_k(\text{Re } z_k + \frac{z}{R_k}) - \phi_k(\text{Re } z_k)$$

with domain

$$\Omega_k := \{z \in B_{\epsilon_k R_k}(\text{Im } z_k R_k) \cap \mathbb{H}^+\}$$

These reparametrizations satisfy:

$$|\nabla \widetilde{\phi}_k(0)| = 1$$

$$|\nabla \widetilde{\phi}_k(z)| \leq 2$$

for $z \in B_{\epsilon_k R_k}(\text{Im } z_k R_k) \cap \mathbb{H}^+$. By the standard arguments, these conditions imply the existence of a subsequence which converges in C^∞_{loc} to $\widetilde{\phi} : \mathbb{H}^+ \to \mathbb{R}^1 \times [0,1] \cong \mathbb{H}^+$ satisfying

$$|\nabla \widetilde{\phi}_k(0)| = 1$$

In particular, $\widetilde{\phi}$ is non-constant and respects the real boundary conditions. Doubling $\widetilde{\phi}$, we get a map $\widetilde{\phi}^d : \mathbb{C} \to \mathbb{C}$. By removal of singularities, we can extend $\widetilde{\phi}^d : S^2 \to \mathbb{C}$. However, this contradicts the non-existence of non-constant holomorphic maps from S^2 to $\mathbb{C}$.

The analysis in the case z_k converges to a point in $\text{int}\, S$, follows similarly to the case $(1a)$ but simpler.

Case (2) follows similarly to case 1, but is simpler.

We have shown that z_k cannot stay in finite distance to the core. Therefore, we have

$$\text{dist}(z_k, \partial N_k) \to \infty$$

Again there are several cases to consider depending on the limiting behavior of the sequence $\phi(z_k)$. The analysis ruling each of these cases out follows similarly to what was

presented above. It follows that bubbling does not occur in the sequence ϕ_k, and by Gromov-Convergence the claim follows. ■

Lemma 9.0.2. A morphism $\Phi_k \in M(\mathcal{G}, \mathcal{G}')$ corresponds to a morphism

$$\phi_k : (S_{a_k}, j(a_k, v_k), M_{a_k}, D_{a_k}) \to (S'_{a'_k}, j'(a'_k, v'_k), M'_{a'_k}, D'_{a'_k})$$

Given the hypothesis of the theorem we have:

$$sup_{z \in S_{a_k} - |D_{a_k}|, k \geq 1} ||T\phi_k(z)|| < \infty$$

Proof: Suppose for sake of contradiction that there is a subsequence of points $c_k \in S_{a_k} - |D_{a_k}|$ such that

$$||T\phi_k(c_k)|| \to \infty$$

We consider three cases:

(1) The sequence of images $(\phi_k(z_k))$ is contained in the **boundary neck** N'_k of S'_k associated with the nodal pair $\{x', y'\} \in D_b$ and $dist(\phi_k(z_k), \partial N'_k) \to \infty$.

(2) The sequence of images $(\phi_k(z_k))$ is contained in the **interior neck** N'_k of S'_k associated with the nodal pair $\{x', y'\} \in D_i$ and $dist(\phi_k(z_k), \partial N'_k) \to \infty$.

(3) The sequence of images $(\phi_k(z_k))$ lies at a finite distance to the core region of S'_k.

Consider Case (3) first. There are two subcases:

3a. The points c_k also lie in finite distance to the core.

3b. The points c_k do not lie in finite distance to the core.

Consider case (3a) first. If this case holds, observe that c_k converges to the boundary or to an interior point. Suppose c_k converges to a point on the boundary. Then there are two cases to consider:

(3a.1)

$$\text{Im } c_k R_k \to \infty$$

(3a.2)

$$\text{Im } c_k R_k \to l \in [0, \infty)$$

As in Lemma 9.0.1, bubbling off analysis of Case (3a.1) results in a non-constant injective holomorphic map $\psi : S^2 \to S'_{a'_k}$ which has as its image a spherical component S of $S'_{a'_0}$. The component S contains at most one interior marked point from $M'_{a'_0}$ and no nodal points. Thus S is an unstable component, and must contain a stabilization point $z'_0 \in \Sigma'$. The following discussion occurs in finite distance to the core so we may canonically identify the maps in question below as defined on fixed surfaces which are independent of k. By definition of stabilization point, there exists a disk-like neighborhood $D'_{a'_0}$ centered at z'_0 with smooth boundary $\partial D'_{a'_0}$ such that the maps $u'_k : D'_{a'_0} \to Q$ are embeddings. By hypothesis we know $u'_k \to u'_0$ and $||T\phi_k(c_k)|| \to \infty$. Set $z_k^{1'} := \phi_k(z_k^1)$ and $z_k^{2'} := \phi_k(z_k^2)$. Then we can find two sequences (z_k^1) and (z_k^2) on the same component of S_{a_k}, with the properties:

$$(*) \quad d_{a_k}(z_k^1, z_k^2) \to 0 \text{ as } k \to \infty$$

$$(**) \quad z_k^{1'} = z'_0 \text{ and } z_k^{2'} \in \partial D'_{a'_0}$$

The maps u'_k are embeddings when restricted to the embedded disks $D'_{a'_0} = D'_{a'_k}$. Thus by $(**)$, the distances $d_Q(u'_k(z'_0), u'_k(z_k^{2'}))$ are bounded away from 0 as k approaches ∞. However, by $(*)$ assumption 2.2 in Section 1.5, the distances $d_Q(u_k(z_k^1), u_k(z_k^2))$ approach 0 as $k \to \infty$. But of course this is impossible since $u'_k \circ \phi_k = u_k$ implies

$$d_Q(u'_k(z'_0), u'_k(z_k^{2'}) = d_Q(u_k(z_k^1, u_k(z_k^2))$$

When we are in case (3a.2), we have a disk bubbling off. A similar analysis in this implies that it's image must be unstable, must contain a stabilization point, and we derive a contradiction similarly to case (3a.1).

Now consider case 3b. In this case, the c_k do not lie in finite distance to the core. Fix $k \geq 0$ sufficiently large. Then a bubbling off analysis produces either a map $\psi : S^2 \to S_{a_k}$ or a $\phi : \mathbb{H}^+ \to S_{a_k}$ depending on whether c_k limits to an interior point or boundary point. Furthermore, if c_k limits to a boundary point, the nature of the bubbling depends on the rate at which Im $z_k R_k$ grows. I.e,

$$\text{Im } c_k R_k \to \infty \quad \text{implies sphere bubbling}$$

$$\text{Im } c_k R_k \to l \in [0, \infty) \quad \text{implies disk bubbling}$$

We reach a contradiction in all cases, arguing similarly as in $(3a)$.

Now consider Case 1. Further, suppose the points c_k lie in finite distance to the core. Then bubbling off analysis results in, depending on the limiting behavior of c_k, we get

one of two injective, non-constant maps

$$\psi : \mathbb{C} \to \mathbb{R} \times [0,1] \cong \mathbb{H}^+ \hookrightarrow \mathbb{C}$$

where $\mathbb{H}^+$ is the upper-half plane or a map

$$\phi : \mathbb{H}^+ \to \mathbb{R} \times [0,1] \cong \mathbb{H}^+$$

The map ψ leads to a contradiction since by removal of singularities we can extend ψ to a map a non-constant map $\psi : S^2 \to \mathbb{C}$, which is impossible. The map ϕ leads to a contradiction since we can double it to a map $\phi^d : \mathbb{C} \to \mathbb{C}$. The map ϕ^d admits an extension to S^2, which again leads to a contradiction.

Now suppose c_k does not lie in finite distance to the core. If c_k belongs to an interior neck for sufficiently large k, bubbling off analysis results in a non-constant holomorphic map $\mathbb{C} \to \mathbb{R} \times [0,1] \cong \mathbb{H}^+$. However, this is impossible. Thus we may assume c_k belongs to a boundary neck for all k. Then the bubbling off analysis results in either a non-constant injective holomorphic $\mathbb{C} \to \mathbb{H}^+$ or a non-constant holomorphic map $\mathbb{H}^+ \to \mathbb{H}^+$. These lead to contradictions as before.

Case 2 follows similarly to [HWZ1]. ■

10 Appendix 3: Extension Operator

We establish properties of an extension operator for functions on $\mathbb{R}^+ \times I$:

Lemma 10.0.1. Let S^1_2 denote the circle of radius 2 with principal coordinates $\theta \in [0,2)$.

There exists an *sc*-linear extension operator $E : H^{3,\delta}(\mathbb{R}^+ \times [0,1], \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}^+ \times S^1_2, \mathbb{R}^N)$ such that $E(u)(s,\theta)|_{\mathbb{R}^+ \times I} = u(s,\theta)$ for $\theta \in [0,1]$, is *sc*-linear.

The following lemma is necessary:

Lemma 10.0.2. There exists a sequence of real numbers $\{a_k\}_{k=0}^{\infty}$ such that for every nonnegative integer n we have

$$\sum_{k=0}^{\infty} 2^{nk} a_k = (-1)^n \tag{10.0.1}$$

and

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| < \infty \tag{10.0.2}$$

Proof: See [Ad] Lemma 4.27. ■

We define the constant $K(n)$ for each natural number n by

$$K(n) := \sum_{k=0}^{\infty} 2^{nk} |a_k| < \infty$$

Proof of Lemma 10.0.1: We modify the proof in Lemma 4.28 [Ad] to include exponential weights. Let $\{a_k\}_{k=0}^{\infty}$ be the sequence from Lemma 10.0.2, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(t) = 1$ on $-\frac{1}{8} \leq t \leq \frac{1}{8}$ and $f(t) = 0$ if $|t| \geq \frac{1}{4}$.

Fix $\delta > 0$. First we show how to extend $\phi \in H^{3+m,\delta}(\mathbb{R}^+ \times [0,1], \mathbb{R}^N)$ to a map $E'\phi \in H^{3+m,\delta}(\mathbb{R}^+ \times [-\frac{1}{2}, 1], \mathbb{R}^N)$ defined by

$$E'\phi(s,t) = \phi(s,t) \text{ if } (s,t) \in \mathbb{R}^+ \times [0,1] \tag{10.0.3}$$

$$E'\phi(s,t) = \sum_{k=0}^{\infty} a_k f(-2^k t)\phi(s,-2^k t) \text{ if } (s,t) \in \mathbb{R}^+ \times [-\frac{1}{2}, 0] \tag{10.0.4}$$

The conditions on the support of f ensure that $E'\phi$ is a well-defined everywhere and a priori smooth away from points in $\mathbb{R}^+ \times \{0\}$. By Lemma 10.0.2 we have:

$$\sum_{k=0}^{\infty} a_k = 1$$

Therefore, expressions 10.0.3 and 10.0.4 agree along their domain of intersection. We will use Line (10.0.2) to verify that the higher derivatives calculated according to the expressions 10.0.3 and 10.0.4 agree along $\mathbb{R}^+ \times \{0\}$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N} \times \mathbb{N}$ and let D^α denote the derivative $(\frac{\partial}{\partial s})^{\alpha_1} \cdot (\frac{\partial}{\partial t})^{\alpha_2}$. Assume ϕ is smooth. By direct calculation, we have for $(s,t) \in \mathbb{R}^+ \times [-\frac{1}{2}, 0]$:

$$D^\alpha E'\phi(s,t) = \sum_{k=0}^{\infty} a_k \sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} (-2^k)^{\alpha_2} f^{(\alpha_2 - j)}(-2^k t) D^{(0,j)} D^{(\alpha_1,0)} \phi(s, -2^k t) = \sum_{k=0}^{\infty} \psi_{k,\alpha}(s,t)$$

where

$$\psi_{k,\alpha}(s,t) = a_k \sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} (-2^k)^{\alpha_2} f^{(\alpha_2 - j)}(-2^k t) D^{(0,j)} D^{(\alpha_1,0)} \phi(s, -2^k t)$$

Note that since f is constant near 0, for t in a sufficiently small open neighborhoodo of 0 (the quantification of 'sufficiently small' depends on k) we have

$$\psi_{k,\alpha}(s,t) = a_k (-2^k)^{\alpha_2} D^{(0,\alpha_2)} D^{(\alpha_1,0)} \phi(s, -2^k t)$$

Adding all $\psi_{k,\alpha}(s,t)$ as k ranges 0 to ∞ we have by Lemma 10.0.2:

$$D^\alpha E'\phi(s,0) = \sum_{k=0}^{\infty} \psi_{k,\alpha}(s,0) = \sum_{k=0}^{\infty} a_k (-2^k)^{\alpha_2} D^{(0,\alpha_2)} D^{(\alpha_1,0)} \phi(s,0) = D^\alpha \phi(s,0)$$

Thus $D^\alpha E\phi|_{\mathbb{R}^+ \times \{0\}} = D^\alpha \phi|_{\mathbb{R}^+ \times \{0\}}$ so the extension is smooth.

Now we show that E is sc^0. If $|\alpha| \leq 3+m$ then

$$|\psi_{k,\alpha}(s,t)|^2 e^{2\delta|s|} \leq K(m,f)^2 |a_k|^2 2^{2(3+m)k} \sum_{|\beta|\leq 3+m} |D^\beta \phi(s,-2^k t)|^2 e^{2\delta|s|}$$

where

$$K(m,f) = 2^{m+4} \cdot |f|_{C^{3+m}}$$

Observe:

$$|\psi_k|_{H^{3+m,\delta}(\mathbb{R}^+ \times [-1,0],\mathbb{R}^N)} \leq \sum_{(k,\alpha):|\alpha|\leq 3+m} |\psi_{k,\alpha}|_{H^{3+m,\delta}(\mathbb{R}^+ \times [-\frac{1}{4*2^k},0],\mathbb{R}^N)}$$

$$\leq K(m,f)|a_k|2^{(3+m)k} \{ \sum_{|\beta|\leq 3+m} \int_{\mathbb{R}^+ \times [-\frac{1}{4*2^k},0]} |D^\beta \phi(s,-2^k t)|^2 e^{2\delta|s|} ds\, dt \}^{\frac{1}{2}}$$

$$\leq K(m,f)|a_k|2^{(3+m)k} \{ 2^{-k} \sum_{|\beta|\leq 3+m} \int_{\mathbb{R}^+ \times [-\frac{1}{4},0]} |D^\beta \phi(s,-y_k)|^2 e^{2\delta|s|} ds\, dy_k \}^{\frac{1}{2}}$$

Here $y_k := 2^k t$. We have the inequality:

$$\sum_{k=0}^{\infty} \sqrt{2}^{-k} \{ \sum_{|\beta|\leq 3+m} \int_{\mathbb{R}\times [-\frac{1}{4},0]} |D^\beta \phi(s,-y_k)|^2 e^{2\delta|s|} ds\, dy_k \}^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}-1} \cdot |\phi|_{H^{3+m,\delta}(\mathbb{R}^+ \times [0,1],\mathbb{R}^N)}$$

Thus, we have:

$$\sum_{k=0}^{\infty} |\psi_k|_{H^{3+m,\delta}(\mathbb{R}^+ \times [0,1],\mathbb{R}^N)} \leq \frac{K(m,f)\cdot K(3+m)}{\sqrt{2}-1} \cdot |\phi|_{H^{3+m,\delta}(\mathbb{R}^+ \times [0,1],\mathbb{R}^N)}$$

This implies:

$$||E'\phi||_{H^{3+m,\delta}(\mathbb{R}^+ \times [0,1],\mathbb{R}^N)} \leq K(m) ||\phi||_{H^{3+m,\delta}(\mathbb{R}^+ \times [0,1],\mathbb{R}^N)}$$

for $K(m)$ dependent only on m (and f which is fixed). In the preceding analysis we supposed ϕ was smooth, however the result extends to arbitrary maps $\psi \in H^{3+m,\delta}(\mathbb{R}^+ \times$

$[0,1],\mathbb{R}^N)$ by a density argument. Similarly, we can extend $E'\phi$ across the boundary $\mathbb{R}^+ \times \{1\}$ to produce a map $E\phi \in H^{3+m,\delta}(\mathbb{R}^+ \times (-\frac{1}{2}, 1+\frac{1}{2}), \mathbb{R}^N)$. We consider $E\phi$ as a map valued on $\mathbb{R}^+ \times [0,2]/\sim := \mathbb{R}^+ \times S^1_2$ by extension by 0. ■

Corollary 10.0.1. Let S^1_2 denote the circle of radius 2 with principal coordinates $\theta \in [0,2)$. There exists an *sc*-linear extension operator $E : H^{3,\delta}(\mathbb{R}^- \times [0,1], \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}^- \times S^1_2, \mathbb{R}^N)$ such that $E(u)(s,\theta)|_{\mathbb{R}^- \times I} = u(s,\theta)$ for $\theta \in [0,1]$, is *sc*-linear. ■

The following extension results are straightforward:

Proposition 10.0.1. (a) There exists an *sc*-linear extension operator

$$E_1 : H^{3,\delta}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times S^1, \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R} \times S^1, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

(b) There exists an *sc*-linear extension operator

$$E_2 : H^{3,\delta}(\mathbb{R}^+ \times [0,1], \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times [0,1], \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R} \times [0,1], \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R} \times [0,1], \mathbb{R}^N) \blacksquare$$

We also have restriction maps:

Proposition 10.0.2. There exists an *sc*-linear retraction operator $R : H^{3,\delta}(\mathbb{R}^+ \times S^1_2, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times S^1_2, \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}^+ \times [0,1], \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times [0,1], \mathbb{R}^N)$ such that $R \circ E =$ Identity. ■

Consider the *sc*-linear mutually inverse reparameterization operators:

$$\mathfrak{reparam} : H^{3,\delta}(\mathbb{R}^+ \times S^1_2, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times S^1_2, \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times S^1, \mathbb{R}^N)$$

$$(u^+(\cdot,\cdot), u^-(\cdot,\cdot)) \to (u^+(2\cdot,\cdot), u^-(2\cdot,\cdot))$$

$$\mathfrak{reparam}^{-1} : H^{3,\delta}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times S^1, \mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}^+ \times S^1_2, \mathbb{R}^N) \oplus H^{3,\delta}(\mathbb{R}^- \times S^1_2, \mathbb{R}^N)$$

$$(u^+(\cdot,\cdot),u^-(\cdot,\cdot)) \to (u^+(\frac{1}{2}\cdot,\cdot),u^-(\frac{1}{2}\cdot,\cdot))$$

We define the *sc*-linear operators:

$$\widehat{E} := \mathfrak{reparam} \circ E$$

$$\widehat{R} := R \circ \mathfrak{reparam}^{-1}$$

Lastly, we can consider similar extension and restriction operators on the infinite strip $\mathbb{R}\times[0,1]$ and infinite cylinder $\mathbb{R}\times S^1$, respectively, which are defined similarly to $\widehat{E}$ and $\widehat{R}$:

$$\widehat{E}' : H^{3,\delta}(\mathbb{R}\times[0,1],\mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}\times S^1,\mathbb{R}^N)$$

$$\widehat{R}' : H^{3,\delta}(\mathbb{R}\times S^1,\mathbb{R}^N) \to H^{3,\delta}(\mathbb{R}\times[0,1],\mathbb{R}^N)$$

11 Appendix 4: Basic Analytical Proposition

In this section we discuss the boundary analogue of the following result:

Theorem 1.28 [HWZ0], Theorem 1.29, Lemma 2.14 If $\pi_a : E \to E$ denotes the linear projection of E onto $\ker \ominus_a$ along $\ker \oplus_a$, then the map

$$r : B_{\frac{1}{2}} \oplus E \to B_{\frac{1}{2}} \oplus E$$

$$(a,u^+,u^-) \to (a,\pi_a(u^+,u^-))$$

is an *sc*-smooth retraction and $Dr(0) = \text{Identity}$. In addition, the map

$$\widehat{r} : B_{\frac{1}{2}} \oplus F \to B_{\frac{1}{2}} \oplus F$$

$$(a, u^+, u^-) \to (a, \widehat{\pi}_a(u^+, u^-))$$

is an *sc*-smooth retraction and $D\widehat{r}(0) = \text{Identity}$. ■

Consider the spaces

$$L := L^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

$$F := H^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

$$E := H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

The key technical lemma in proving Theorems 1.28 and 1.29 in [HWZ0] is the following:

Proposition 11.0.1. (compare [HWZ0] Proposition 2.8) The following four maps

$$\Gamma_i : B_{\frac{1}{2}} \oplus G \to G, \quad i = 1, \cdots, 4$$

where $G = L$, $G = F$ or $G = E$ are *sc*-smooth.

(1) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval so that $f_1(+\infty) = 0$. Define

$$\Gamma_1(a, h)(s, t) = f_1(s - \frac{R}{2}) \cdot h(s, t)$$

if $a \neq 0$ and $\Gamma_1(0, h) = f(-\infty)h$ if $a = 0$.

(2) Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth function. Define

$$\Gamma_2(a, h)(s, t) = f_2(s - \frac{R}{2})h(s - R, t - \theta)$$

if $a \neq 0$ and $\Gamma_2(0,h) = 0$ if $a = 0$.

(3) Let $f_3 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval and satisfying $f_3(\infty) = 0$. Define

$$\Gamma_3(a,h)(s',t') = f_3(-s - \frac{R}{2})h(s',t')$$

if $a \neq 0$ and $\Gamma_3(0,h) = f_3(-\infty)h$ if $a = 0$.

(4) Let $f_4 : \mathbb{R} \to \mathbb{R}$ be a smooth function of compact support and define

$$\Gamma_4(a,h)(s',t') = f_4(-s' - \frac{R}{2})h(s'+R, t'+\theta)$$

if $a \neq 0$ and $\Gamma_4(0,h) = 0$ if $a = 0$. ■

Consider the spaces:

$$L' := L^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

$$F' := H^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

$$E' := H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

We need the following open version of [HWZ0] Proposition 2.8:

Proposition 11.0.2. (compare [HWZ0] Proposition 2.8) The following four maps

$$\Gamma_i' : B_{\frac{1}{2}} \oplus G' \to G', \quad i = 1, \cdots, 4$$

where $G' = L'$, $G' = F'$ or $G' = E'$ are *sc*-smooth.

(1) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval so that $f_1(+\infty) = 0$. Define

$$\Gamma_1'(a,h)(s,t) = f_1(s - \frac{R}{2}) \cdot h(s,t)$$

if $a \neq 0$ and $\Gamma_1'(0,h) = f(-\infty)h$ if $a = 0$.

(2) Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth function. Define

$$\Gamma_2'(a,h)(s,t) = f_2(s - \frac{R}{2})h(s - R, t - \theta)$$

if $a \neq 0$ and $\Gamma_2'(0,h) = 0$ if $a = 0$.

(3) Let $f_3 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval and satisfying $f_3(\infty) = 0$. Define

$$\Gamma_3'(a,h)(s',t') = f_3(-s - \frac{R}{2})h(s',t')$$

if $a \neq 0$ and $\Gamma_3'(0,h) = f_3(-\infty)h$ if $a = 0$.

(4) Let $f_4 : \mathbb{R} \to \mathbb{R}$ be a smooth function of compact support and define

$$\Gamma_4'(a,h)(s',t') = f_4(-s' - \frac{R}{2})h(s' + R, t' + \theta)$$

if $a \neq 0$ and $\Gamma_4'(0,h) = 0$ if $a = 0$.

Proof: Consider the commutative diagrams for $i = 1, 2, 3, 4$:

$$\begin{array}{ccc} B \times G' & \xrightarrow{\Gamma_i'} & G' \\ \downarrow{\scriptstyle \text{Identity} \times \widehat{E}} & & \uparrow{\scriptstyle \widehat{R}} \\ B \times G & \xrightarrow{\Gamma_i} & G \end{array}$$

Since all maps in the diagram besides Γ_i' are known to be *sc*-smooth by Proposition 11.0.1, it follows that Γ_i' is *sc*-smooth. ■

The following bound is established in [HWZ0] and is used in the proof of [HWZ0] Proposition 2.8:

[HWZ0] Lemma 2.12 Consider the map $a \to R(a) := e^{\frac{1}{|a|}} - e$, where $a = |a| \cdot e^{-2\pi i\theta}$. For every multi-index $\alpha = (\alpha_1, \alpha_2)$, there exists a constant C such that

$$|D^\alpha R(a)| \leq C \cdot R(a) \cdot [ln(R(a))]^{2|\alpha|}$$

for $0 < |a| < \frac{1}{2}$. ■

Proposition 11.0.3. (compare 1.28 [HWZ0]) Consider the map

$$\Phi_{real} : B_{\frac{1}{2}} \oplus H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \oplus H^{2,\delta_1}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n}) \to H^{2,\delta_1}_{real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \oplus H^{2,\delta_1}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n})$$

given by

$$(a, \xi^+, \xi^-) \to \widetilde{\pi}(a, \xi^+, \xi^-)$$

Φ_{real} is smooth since it is the restriction of Φ' to the space of maps with real boundary conditions. It also follows that $D\Phi_{real}(0) = \text{Identity}$.

Proof: Consider the equation:

$$\widetilde{\pi}_a(\xi^+, \xi^-) = (\eta^+, \eta^-)$$

Then by direct calculation:

$$\eta^+(s,t) = \frac{1}{2}\cdot(1-\frac{\beta_a}{\gamma_a}(s))\cdot([\xi^+]_{R,\,real}+[\xi^-]_{-R,\,real})+\frac{\beta_a^2}{\gamma_a}(s)\cdot\xi^+(s,t)+\frac{\beta_a(1-\beta_a)}{\gamma_a}(s)\cdot\xi^-(s-R,t)$$

where

$$[\xi^+]_{R,real} = \int_{[0,1]} \operatorname{Re}\,\xi^+(\frac{R}{2},t)\,dt, \quad \text{and} \quad [\xi^-]_{-R,real} = \int_{[0,1]} \operatorname{Re}\,\xi^-(-\frac{R}{2},t)\,dt$$

We prove the *sc*-smoothness of $(a, \xi^+, \xi^-) \to \eta^+$. The *sc*-smoothness of $(a, \xi^+, \xi^-) \to \eta^-$ follows similarly. Set $\xi^\pm = c + r^\pm$ then:

$$\eta^+(s,t) = c + \frac{1}{2}(1 - \frac{\beta_a}{\gamma_a}(s)) \cdot ([r^+]_{R,real} + [r^-]_{-R,real})$$

$$+\frac{\beta_a^2}{\gamma_a}(s)r^+(s,t) + \frac{\beta_a(1-\beta_a)}{\gamma_a}(s)r^-(s-R,t)$$

Thus it suffices to show the smoothness of the following maps:

1. $\Phi_1 : H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$

$$\xi^+ \to c$$

2. $\Phi_2 : B_{\frac{1}{2}} \times H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$

$$(a, r^+) \to [r^+]_{R,real}$$

3. $\Phi_3 : B_{\frac{1}{2}} \oplus (H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \times H^{3,\delta_0}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n})) \to H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n})$

$$(a, r^+, r^-) \to (1 - \frac{\beta_a}{\gamma_a}) \cdot ([r^+]_{R,real} + [r^-]_{-R,real})$$

4. $\Phi_4 : B_{\frac{1}{2}} \oplus H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \to H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n})$

$$(a, r^+) \to \frac{\beta_a^2}{\gamma_a} \cdot r^+$$

5. $\Phi_5 : B_{\frac{1}{2}} \oplus H^{3,\delta_0}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n}) \to H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n})$

$$(a, r^-) \to \frac{\beta_a(1 - \beta_a)}{\gamma_a} r^-(\cdot - R, \cdot)$$

Lemma 11.0.1. (compare [HWZ0] Lemma 2.18) The map $\Phi_1 : H^{3,\delta_0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$

$$\xi^+ \to c$$

is *sc*-smooth.

Proof: The map $\xi^+ \to c$ is *sc*-linear, thus *sc*-smooth. ■

Lemma 11.0.2. (compare [HWZ0] Lemmas 2.19, 2.20) The map $\Phi_2 : B_{\frac{1}{2}} \oplus H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \to \mathbb{R}^{2n}$ defined by

$$\Phi_2(a, h) = \int_{[0,1]} \mathrm{Re}\, h(\frac{R}{2}, t)\, dt = [h]_{R,real}$$

for $a \neq 0$ and $\Phi(0, h) = 0$ for $a = 0$ is *sc*-smooth.

In addition, the map $\Phi_3 : B_{\frac{1}{2}} \oplus (H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \times H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n})) \to H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times$

$I, \mathbb{R}^{2n})$ defined by

$$\Phi_3(a, r^+, r^-) = (1 - \frac{\beta_a}{\gamma_a}) \cdot ([r^+]_{R,real} + [r^-]_{-R,real})$$

for $a \neq 0$ and $\Psi(0, r^+, r^-) = 0$ for $a = 0$ is sc-smooth.

Proof: The argument follows identically to Lemma 2.19 [HWZ0]. The key estimate used in the argument is

$$|[\partial_s^j(v)]_R| \leq Ce^{-\delta_{m+1}\frac{R}{2}}|v|_{m+1} \tag{11.0.1}$$

for $j \leq m$, which follows from the Sobolev embedding

$$|e^{\delta_{m+1}\cdot}\partial_s^j(v)|_{C^0((\frac{R}{2}-1,\frac{R}{2}+1)\times S^1)} \leq C|e^{\delta_{m+1}\cdot}\partial_s^j v|_{H^{m+3}((\frac{R}{2}-1,\frac{R}{2}+1)\times S^1}$$

where C depends only on m. This type of Sobolev estimate continues to hold when S^1 is replaced by $[0, 1]$, so we derive the bound:

$$|[\partial_s^j v]_{R,real}| \leq C'e^{-\delta_{m+1}\frac{R}{2}}|v|_{m+1} \tag{11.0.2}$$

Using (11.0.2) instead of (11.0.1), the rest of the argument goes through as in Lemma 2.19 [HWZ0].

The key analytic bound with regards to verifying the smoothness of Φ_3 is again (11.0.2). The claim follows by substituting (11.0.1) for (11.0.2) and arguing as in Lemma 2.20 [HWZ0]. ■

Lemma 11.0.3. (compare Lemma 2.21 [HWZ0]) The map $\Phi_4 : B_{\frac{1}{2}} \oplus H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times$

$I, \mathbb{R}^{2n}) \to H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n})$, defined by

$$\Phi_4(a,r) = \frac{\beta_a^2}{\gamma_a} \cdot r$$

if $a \neq 0$ and $\Phi_4(0,r) = r$ if $a = 0$ is *sc*-smooth.

Proof: Consider the map $\Phi : B_{\frac{1}{2}} \oplus H_c^{3,\delta_0^0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}) \to H_c^{3,\delta_0^0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n})$, defined by

$$\Phi(a,r) = \frac{\beta_a^2}{\gamma_a} \cdot r$$

if $a \neq 0$ and $\Phi(0,r) = r$ if $a = 0$. Consider the commutative diagram:

$$\begin{array}{ccc} B \times G' & \xrightarrow{\Phi_4} & G' \\ \downarrow \widehat{E} & & \widehat{R} \uparrow \\ B \times G & \xrightarrow{\Phi} & G \end{array}$$

Φ is *sc*-smooth by Lemma 2.21 [HWZ0], and E and R are *sc*-smooth by Propositions 10.0.1 and 10.0.2. The claim follows. ■

Lemma 11.0.4. (compare Lemma 2.22 [HWZ0]) The map $\Phi_5 : B_{\frac{1}{2}} \oplus H^{3,\delta_0^0}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n}) \to H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n})$, defined by

$$\Phi_5(a,r) = \frac{\beta_a(1-\beta_a)}{\gamma_a} \cdot r(\cdot - R, \cdot)$$

if $a \neq 0$ and by $\Phi_5(0,r) = 0$ if $a = 0$ is *sc*-smooth.

Proof: Consider the map $\Phi : B_{\frac{1}{2}} \oplus H_c^{3,\delta_0^0}(\mathbb{R}^- \times S^1, \mathbb{R}^{2n}) \to H_c^{3,\delta_0^0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n})$, defined by

$$\Phi(a,r) = \frac{\beta_a(1-\beta_a)}{\gamma_a} \cdot r(\cdot - R, \cdot)$$

if $a \neq 0$ and by $\Phi(0,r) = 0$ is $a = 0$. Consider the commutative diagram:

$$\begin{array}{ccc} B \times H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^- \times I, \mathbb{R}^{2n}) & \xrightarrow{\Phi_5} & H^{3,\delta_1^0}_{c,\,real}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \\ \big\downarrow \widehat{E} & & \widehat{R} \big\uparrow \\ B \times H^{3,\delta_0^0}_{c}(\mathbb{R}^- \times S^1, \mathbb{R}^{2n}) & \xrightarrow{\Phi} & H^{3,\delta_0^0}_{c}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}) \end{array}$$

Φ is *sc*-smooth by Lemma 2.22 [HWZ0], and E and R are *sc*-smooth by Propositions 10.0.1 and 10.0.2. The claim follows. ■

Remark 11.0.1. Observe that:

1. $D\Phi_1(0)\xi^+ = c$, where $\xi^\pm = c + r^\pm$.

2. $D\Phi_2(0)r^+ = 0$, by the proof of Lemma 11.0.2 and Lemma 2.19 [HWZ0].

3. $D\Phi_3(0)r^+ = 0$, by the proof of Lemma 11.0.2 and Lemma 2.20 [HWZ0].

4. $D\Phi_4(0)r^+ = r^+$ by the proof of Lemma 11.0.3 and the proof of Lemma 2.21 [HWZ0].

5. $D\Phi_5(0)r^- = 0$ by the proof of Lemma 11.0.4 and the proof of Lemma 2.22 [HWZ0].

Thus, if we set $\Phi'(a, \xi^+, \xi^-) = \eta^+$, then $D\Phi'(0)\xi^+ = \xi^+$. The same holds for the map $(a, \xi^+, \xi^-) \to \eta^-$. In particular $D\widetilde{\pi}(0) = \text{Identity}$. ■

Remark 11.0.2. The discussion above suffices to prove the smoothness of

$$\widehat{\Phi} : B_{\frac{1}{2}} \oplus H^{2,\delta_1}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \oplus H^{2,\delta_1}(\mathbb{R}^- \times I, \mathbb{R}^{2n}) \to H^{2,\delta_1}(\mathbb{R}^+ \times I, \mathbb{R}^{2n}) \oplus H^{2,\delta_1}(\mathbb{R}^- \times I, \mathbb{R}^{2n})$$

given by

$$(a, \xi^+, \xi^-) \to \widehat{\widehat{\pi}}(a, \xi^+, \xi^-)$$

and to establish that $D\widehat{\Phi}(0) = \text{Identity}$. ■

12 Appendix 5: ⊡ Gluing

We briefly recall the definition of ⊡ in the closed case, following [HWZ0] Section 2.5. For every $a \in B_{\frac{1}{2}}$ we define a space G^a as follows:

$$G^0 = E \oplus \{0\}$$

When $a \neq 0$,

$$G^a = H^3(Z_a) \oplus H_c^{3,\delta_0}(C_a)$$

For shorthand, we set $Q^a = H^3(Z_a)$ and $P^a = H_c^{3,\delta_0}(C_a)$. The *sc*-structure filtrations are the obvious ones. By linear algebra, it follows that the map

$$\boxdot_a : (\oplus_a, \ominus_a) : E \to G^a$$

is an *sc*-linear isomorphism (see Theorem 1.27 [HWZ0]).

Definition 12.0.1. Given a pair $(k, \delta) \in \mathbb{N}_0 \times \mathbb{R}$ and a map $q : Z_a \to \mathbb{R}^{2n}$ we define the norm $||q||_{k,\delta}$ by

$$||q||_{k,\delta}^2 := \sum_{|\alpha| \leq k} \int_{[0,R]\times S^1} |D^\alpha q([s,t])|^2 \cdot e^{2\delta|s-\frac{R}{2}|}\, ds\, dt$$

We also define the averaging term:

$$[q]_a = \int_{S^1} q(\frac{R}{2}, t)\, dt$$

When Z_a has boundary we define:

$$||q||^2_{k,\delta} := \sum_{|\alpha|\leq k} \int_{[0,R]\times[0,1]} |D^\alpha q([s,t])|^2 \cdot e^{2\delta|s-\frac{R}{2}|}\, ds\, dt$$

and the averaging term:

$$[q]_{a,real} = \int_{[0,1]} \mathrm{Re}\, q(\frac{R}{2}, t)\, dt \qquad \blacksquare$$

Definition 12.0.2. Given a pair $(m,\delta) \in \mathbb{N}_0 \times \mathbb{R}$ and a map $p : C_a \to \mathbb{R}^{2n}$ with vanishing asymptotic constants we define:

$$||p||^2_{m,\delta} := \sum_{|\alpha|\leq m} \int_{\mathbb{R}\times S^1} |(D^\alpha p)[s,t])|^2 \cdot e^{2\delta_m|s-\frac{R}{2}|}\, ds\, dt$$

In the case C_a has boundary we set:

$$||p||^2_{m,\delta} := \sum_{|\alpha|\leq m} \int_{\mathbb{R}\times[0,1]} |(D^\alpha p)[s,t])|^2 \cdot e^{2\delta_m|s-\frac{R}{2}|}\, ds\, dt \qquad \blacksquare$$

Definition 12.0.3. Given a pair $(q,p) \in G^a$ we define the norm:

$$|(q,p)|_{G^a_m} := |[q]_a - p_\infty|^2 + e^{\delta_m R} \cdot (|q - [q]_a + p_\infty|^2_{3+m-\delta_m} + |\widehat{p}|^2_{3+m,\delta_m})$$

where $p_\infty = \lim_{s\to\infty} p(s,t)$ and $\widehat{p}([s,t]) = p([s,t]) - (1-2\beta_a)\cdot p_\infty$. $\blacksquare$

Remark 12.0.1. If $\boxdot(\eta^+,\eta^-)=(q,p)$, then direct calculation reveals:

$$|(q,p)|^2_{G^a_m} = |c|^2 + e^{\delta_m R}\cdot(|\widehat{\oplus}_a(r^+,r^-)|^2_{3+m,-\delta_m} + |\widehat{\ominus}_a(r^+,r^-)|^2_{3+m,\delta_m})$$

Here $[q]_a - p_\infty = c$, $\eta^\pm = c + r^\pm$. ■

[HWZ0] Theorem 2.23 For every level m there exists a constant $C_m > 0$ independent of $|a| < \frac{1}{2}$ so that the total gluing map $\boxdot_a : E \to G^a$ defined by

$$(\eta^+,\eta^-) \to \boxdot_a(\eta^+,\eta^-) := (\oplus_a(\eta^+,\eta^-), \ominus_a(\eta^+,\eta^-))$$

is an *sc*-isomorphism and satisfies the estimate

$$\frac{1}{C_m}\cdot|\eta|_{E_m} \leq |\boxdot_a(\eta)|_{G^a_m} \leq C_m\cdot|\eta|_{E_m} \quad \blacksquare$$

We establish the open analogue of Theorem 2.23 [HWZ0]:

Proposition 12.0.1. For every level m there exists a constant $C'_m > 0$ independent of $a \in [0, \frac{1}{2})$ so that the total gluing map $\boxdot_a : E_{real} \to G^a$ defined by

$$(\eta^+,\eta^-) \to \boxdot_a(\eta^+,\eta^-) := (\oplus_a(\eta^+,\eta^-), \ominus_a(\eta^+,\eta^-))$$

is an *sc*-isomorphism and satisfies the estimate

$$\frac{1}{C'_m}\cdot|\eta|_{(E_{real})_m} \leq |\boxdot_a(\eta)|_{G^a_m} \leq C'_m\cdot|\eta|_{(E_{real})_m} \tag{12.0.1}$$

Proof: That $\boxplus_a$ is an isomorphism follows by linear algebra. We consider the verification

of (12.0.1): Recall the extension operators

$$\widehat{E} : E_{real} \to E$$

$$\widehat{E}' : H^{3,\delta}_{c,\,real}(\mathbb{R} \times [0,1], \mathbb{R}^{2n}) \to H^{3,\delta}_{c}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

considered in Section 10. We have the chain of inequalities:

$$\frac{1}{K_m}|\widehat{E}\eta|_{(E_{real})_m} \leq |\eta|_{E_m} \leq K_m|\widehat{E}\eta|_{(E_{real})_m}$$

Suppose for sake of contradiction that for any $K > 0$ there exists an $\eta \in E_m$ with $|\eta|_{E_m} = 1$ such that

$$|\boxdot_a(\eta)|_{(E_{real})_m} \geq K \cdot |\eta|_{E_m}$$

Observe:

$$K_m|\boxdot_a \widehat{E}\eta|_{E_m} \geq |\boxdot_a \eta|_{E_{real\,m}} \geq K|\eta| \geq \frac{K}{K_m}|E\eta|_{E_m}$$

Taking $K \to \infty$, we contradict [HWZ0] Theorem 2.23. Thus there exists a $C'_m > 0$ such that

$$|\boxdot \eta|_{G^a_m} \leq C'_m|\eta|_{E_m}$$

for all $\eta \in E_m$ for which $|\eta|_{E_m} = 1$. Now suppose for sake of contradiction that for any $K > 0$ there exists an $\eta \in E_m$ with $|\eta|_{E_m} = 1$ such that

$$|\eta|_{E_m} \geq K|\boxdot_a(\eta)|_{G^a_m}$$

Observe:

$$K_m|\widehat{E}\eta|_{E_m} \geq |\eta|_{E_{real\,m}} \geq K|\boxdot_a(\eta)|_{G^a_m} \geq \frac{K}{K_m}|\boxdot_a \widehat{E}\eta|_{G^a_m}$$

Again taking $K \to \infty$ we reach a contradiction. ■

13 Appendix 6: Good Position

The following result is utilized in Appendix 1, as a necessary ingredient for Corollary 8.6.4.

Proposition 13.0.1. The kernel N of $D\text{Fill}\,\overline{\partial}_{core} \times \overline{\partial}_{neck}(0)$ is in good position to the boundary of $T_0(V \times W \times B_D \times \mathcal{T}_{fiber})$.

Proof: The good position condition requires the existence of a subspace

$$P \subset T_0(V \times W \times B_D \times \mathcal{T}_{fiber}) = E \times E' \times N(D) \times \mathcal{T}^*_{fiber}$$

such that

$$E \times E' \times N(D) \times \mathcal{T}_{fiber} = N \oplus P$$

and the existence of $\epsilon > 0$ such that for $n \in N$, $p \in P$, $|p| \leq \epsilon |n|$,

$$n + p \in C_0(E \times E' \times N(D) \times \mathcal{T}_{fiber}) \text{ if and only if } n \in C_0(E \times E' \times N(D) \times \mathcal{T}_{fiber})$$

Observe:

$$C_0(E \times E' \times N(D) \times \mathcal{T}_{fiber}) = E \times E' \times \mathbb{R}^{+\#|D_b|} \times \mathbb{C}^{\#|D_i|} \times \mathcal{T}_{fiber}$$

Now we analyze the kernel N. For the sake of simplicity assume $\#|D_b| = 1$ and $\#|D_i| = 0$.

The general case will be clear from what follows. Recall that the operator $\overline{\partial}_{core} \times \overline{\partial}_{neck}$ is given by

$$\overline{\partial}_{core} \times \overline{\partial}_{neck}(v, w, a, \eta, h^+, h^-) = (I_2(w, exp_{\text{Identity}}\eta, z) \circ \overline{\partial}_{k(w),j(v)} exp_{\text{Identity}}\eta \circ I_1(v, z), (\xi^+, \xi^-))$$

where (ξ^+, ξ^-) is specified by:

$$\oplus_a(\xi^+, \xi^-) = \frac{1}{2}(T \boxplus_a ((s,t) + h^+, (s',t') + h^-) + i \circ T \boxplus_a ((s,t) + h^+, (s',t') + h^-) \circ i) \cdot \frac{\partial}{\partial s}$$

$$\widehat{\ominus}_a(\xi^+, \xi^-) = \frac{1}{2}(T \ominus_a (h^+ - h^+_\infty, h^- - h^-_\infty) + i \circ T \ominus_a (h^+ - h^+_\infty, h^- - h^-_\infty) \circ i) \cdot \frac{\partial}{\partial s}$$

The pair (ξ^+, ξ^-) is explicitly defined by:

$$\begin{bmatrix} \xi^+ \\ \xi^- \end{bmatrix} = \begin{bmatrix} \overline{\partial}_0 h^+ \\ \overline{\partial}_0 h^- \end{bmatrix} + \frac{\beta'_a}{\gamma_a} \cdot \begin{bmatrix} \beta_a & \beta_a - 1 \\ 1 - \beta_a & \beta_a \end{bmatrix} \cdot \begin{bmatrix} h^+ - h^- - h^+_\infty + h^-_\infty \\ h^+ + h^- - 2a v_a(h^+, h^-) \end{bmatrix}$$

when $a > 0$ and, using Proposition 11.0.2

$$\begin{bmatrix} \xi^+ \\ \xi^- \end{bmatrix} = \begin{bmatrix} \overline{\partial}_0 h^+ \\ \overline{\partial}_0 h^- \end{bmatrix}$$

when $a = 0$. From the Remark 11.0.1, it follows that the differential of $\overline{\partial}_{neck}$ is independent of a at (a, h^+, h^-):

$$\mathrm{D}\,\overline{\partial}_{neck}(0,0,0) \cdot (\delta a, (\delta h^+, \delta h^-)) = \mathrm{D}_1\,\overline{\partial}_{neck}(0,0,0)(\delta a, (0,0))$$

$$+\mathrm{D}_2 \overline{\partial}_{neck}(0,0,0)(0, (\delta h^+, \delta h^-))$$

$$= (0) + \begin{bmatrix} \overline{\partial}_0 \delta h^+ \\ \overline{\partial}_0 \delta h^- \end{bmatrix}$$

In particular, for any $(\delta v, \delta w, \delta a, \delta \eta, \delta h^+, \delta h^-) \in N$, we also have $(\delta v, \delta w, \delta a', \delta \eta, \delta h^+, \delta h^-) \in N$, for every $\delta a' \in \mathbb{R}^{\#|D_b|} \times \mathbb{C}^{\#|D_i|}$. This means that the good position condition is satisfied for any *sc*-complement P of N and $\epsilon > 0$ for which P projects trivially onto $N(D)$. ■

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