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Fibrations in abelian varieties associated to Enriques surfaces

Sacca, Giulia

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Fibrations in abelian varieties associated to Enriques surfaces

Giulia Saccà

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

Let T be a general Enriques surface and let $f : S \to T$ be its universal cover. Consider a smooth curve $C \subset T$ of genus $g \geq 2$, set $D := f^{-1}(C)$, and let $\mathcal{C} \to |C|$ and $\mathcal{D} \to f^*|C| \subset |D|$ be the universal family and the restriction of the universal family respectively. We construct the relative Prym variety $P = \mathrm{Prym}(\mathcal{D}, \mathcal{C})$ of $\mathcal{D}$ over $\mathcal{C}$ and show that it is a (possibly singular) symplectic variety of dimension $2g - 2$. There is a morphism $P \to |C|$, which is a Lagrangian fibration and whose smooth fibers are $(g - 1)$-dimensional Prym varieties. We also prove that the smooth locus of P is simply connected. For any non zero integer $\chi = d - g + 1$ we consider the degree d relative compactified Jacobian $N = \mathrm{Jac}^d_A(|C|) \to |C|$, with respect to a polarization A on T. If χ is such that the Mukai vector $(0, [D], 2\chi)$ is primitive in $H^*(S, Z)$, and A if is general, we prove that N is smooth. Moreover, under some technical assumption that can be verified for low values of g and that are expected to be true in general, we show that $\pi_1(N) \cong \mathbb{Z}/(2)$, that $\omega_N \cong \mathcal{O}_N$, and that $h^{p,0}(N) = 0$ for $p \neq 0, 2g - 1$.

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A Nonna Ida,

To Olivia

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Introduction

Moduli spaces of sheaves on K3 surfaces are among the most studied objects in algebraic geometry. Part of their interesting features is that they inherit the structure coming from the K3 surface itself. For example, Mukai proved in [47] that the symplectic structure on the surface induces a holomorphic symplectic structure on the smooth locus of the moduli space. When smooth and projective, these moduli spaces provide examples of irreducible compact hyperkähler manifolds [9], [29], [47], [55].

The geometry of irreducible compact hyperkähler manifolds is rich, and very few examples are known. One of the reasons why hyperkähler manifolds have attracted much attention is that they are, together with Calabi-Yau manifolds and complex tori, the building blocks of Kähler manifolds with trivial first Chern class [9]. Up to deformation equivalence, the only known examples of compact irreducible hyperkähler manifolds are the following: in every even complex dimension we have smooth projective moduli spaces of sheaves on K3 surfaces and (fibers of the Albanese map of) smooth projective moduli spaces of sheaves on abelian surfaces [9], [29], [47], [55]; in dimension 10 and 6 we find O'Grady's exceptional examples, which are constructed as a symplectic resolution of the singularities of some singular moduli spaces of sheaves on a K3 surface [56] and on an abelian surface [57]. We should also remark that in [32], Kaledin, Lehn and Sorger proved that no new examples can be obtained by resolving singular moduli spaces. Part of this thesis has grown out of the attempt of finding new examples of irreducible holomorphic symplectic manifolds [6].

Given a compact irreducible holomorphic symplectic manifold M, a Lagrangian fibration on M is a proper morphism $p : M \to B$ whose general fiber is a Lagrangian submanifold of M. By a theorem of Matsushita [44], if $p : M \to B$ is any proper surjective morphism with connected fibers, and $\dim M > \dim B$, any irreducible component of a fiber of p is a Lagrangian subvariety of M. In particular, p is equidimensional. In all the known deformation classes there are examples of Lagrangian fibrations.

Given a linear system $|L|$ on a K3 surface S and an integer d, the degree d relative compactified Jacobian $\pi : J^d(|L|) \to |L|$ of $|L|$ is a moduli space of sheaves on S, and one can arrange things so that it is smooth and projective. In this case, it is an irreducible projective holomorphic symplectic manifold, called the Beauville-Mukai integrable system, and π is a Lagrangian fibration. Based on Sawon's work [64], it seems reasonable to expect that given a family $\mathcal{C} \to \mathbb{P}^g$ of genus g curves, if the relative compactified Jacobian $J(\mathcal{C}) \to \mathbb{P}^g$ is an irreducible holomorphic symplectic manifold, then it is a Beauville-Mukai integrable system. Because of this, to look for new examples it is natural to try to produce examples of integrable systems whose fibers are not Jacobian of curves but, for instance, *Prym varieties* or higher weight intermediate Jacobians. In [42] the authors construct a 4-dimensional singular holomorphic symplectic manifold admitting a Lagrangian fibration whose fibers are 2-dimensional Prym varieties, and which does not admit a symplectic resolution.

Generalizing [42], in [6] we give a systematic way of constructing Lagrangian fibrations of any even complex dimension, whose fibers are Prym varieties which, in most cases of the construction, are not Jacobians of curves. Let $f : S \to T$ be a K3 surface covering an Enriques surface. Let C be a smooth curve on T and set $D = f^{-1}(C)$. Let $\mathcal{C} \to |C|$ and $\mathcal{D} \to f^*|C| \subset |D|$ be the universal family and the restriction of the universal family respectively. We construct a (possibly singular) projective variety $\mathrm{Prym}(\mathcal{D}/\mathcal{C})$ that is equipped with a morphism

$$\pi : P = \mathrm{Prym}(\mathcal{D}/\mathcal{C}) \to |C| = \mathbb{P}^{g-1}, \tag{1}$$

whose fiber over a smooth curve C is the Prym variety $\mathrm{Prym}(D/C)$ of $D = f^{-1}(C)$ over C. The main results are the following.

Theorem 1 ([6])**.** *The smooth locus of the relative Prym variety carries a holomorphic symplectic structure. With respect to this symplectic structure the Prym variety of a double cover $D \to C$ is a Lagrangian subvariety and the map (1) is a completely integrable system.*

Theorem 2 ([6])**.** *The relative Prym variety* $\mathrm{Prym}(\mathcal{D}/\mathcal{C})$ *admits a symplectic resolution if and only if $|C|$ is a hyperelliptic linear system. When a symplectic resolution exists, it is deformation equivalent to a moduli space of sheaves on a K3 surface.*

Theorem 3 ([6])**.** *Let $\widetilde{P}$ be any normalization of the relative Prym variety P. Then $\widetilde{P}$ is simply connected.*

Theorem 1 provides, in every large enough even dimension, new examples of Lagrangian fibrations in principally polarized abelian varieties that are not Jacobians of curves.

On the other hand, not much work has been done on moduli spaces of sheaves on Enriques surfaces, even though it is natural to expect that their geometry is tightly related to that of moduli spaces of sheaves on the covering K3 surface. In [34] Kim studies the moduli space of stable vector bundles on an Enriques surface and shows that it admits a double cover onto a Lagrangian subvariety of some moduli space of sheaves on the covering K3 surface. This cover is unramified away from the singularities. In [35] he analyzes in more detail the case of rank two vector bundles. Hauzer's paper [28] improves his results, and also deals with other higher rank cases. Recently, and independently from this work, Zowislok [75] considers the pullback morphism of pure sheaves under finite unramified quotients of K3 or abelian surfaces and as in [35] relates the construction to Lagrangian subvarieties of hyperkähler manifolds. Lastly, in [58], Oguiso and Schröer prove that the Hilbert scheme of n points on a given Enriques surface T has the property that the canonical bundle is not trivial, but twice the canonical bundle is trivial.

The second part of this thesis studies the geometry of moduli spaces of pure sheaves of dimension one on a general Enriques surface T. If we consider pure dimension one sheaves whose first Chern class is linearly equivalent to a given curve C, these moduli spaces may be described as the relative compactified Jacobian of a linear system $|C|$. As in [34] and [75] we relate them to a Lagrangian subvariety of some moduli space of sheaves on the covering K3 surface S. The main results are the following

Theorem 4 ([62]). *Let T be a general Enriques surface and let S be the covering K3. Let $C \subset T$ be a curve of genus $g \geq 2$, let χ be a non zero integer and consider the Mukai vectors $w = (0, [C], \chi) \in H^*(T, \mathbb{Z})_{tf}$ and $v = (0, [f^{-1}(C)], 2\chi) \in H^*(S, \mathbb{Z})$. If A is a general polarization on T and v is primitive, then the moduli space $M_{w,A}(T)$ of A-semistable sheaves on T with Mukai vector w is a smooth projective variety of dimension $2g-1$, which admits an étale double cover onto a smooth Lagrangian subvariety of $M_{v,f^*A}(S)$.*

Using the same notation as above, and assuming some technical condition on $|C|$ (concerning the dimension of the fibers of the relative compactified Jacobian –see Assumption 4.1.5) that is verified for low values of g and is excepted to hold in general, we have also

Theorem 5 ([62]). *The canonical bundle of $M_{w,A}(T)$ is trivial, and the hodge numbers $h^{p,0}(M_{w,A}(T))$ vanish for $p \neq 0, 2g-1$. Moreover,*

$$\pi_1(M_{w,A}(T)) \cong \mathbb{Z}/(2), \quad \text{and if } \ g \geq 3, \ h^2(M_{w,A}(T)) = 11,$$

It is interesting to observe the different behavior of the canonical bundle of these odd dimensional moduli spaces with that of the Hilbert scheme of points on T. Over the open subset $U \subset |C|$ of smooth curves, we can identify the universal cover $\widetilde{M}_{w,A}(T)$ of $M_{w,A}(T)$ with the Stein factorization of the relative Norm map

$$M_{v,f^*A}(S)\big|_U \to M_{w,A}(T)\big|_U.$$

Theorem 6 ([62]). *If $g = 2$, then $M_{w,A}(T)$ and its universal cover $\widetilde{M}_{w,A}(T)$ are Calabi-Yau threefolds with Hodge numbers $h^{2,1} = h^{1,1} = 10$.*

The outline of the thesis is as follows

Chapter 1. We start by briefly recalling the notions of pure sheaves on a smooth projective variety, of stability of such sheaves, and the basic properties of their moduli spaces. We focus on pure dimension one sheaves on surfaces, giving criteria for stability and describing in detail examples that will be relevant in the following chapters. The sections on the behavior of stability, with respect to a given polarization, under the operation taking the dual or the pullback via cyclic covers are the foundations for the following chapters.

Chapter 2. Let $f : X \to Y$ be an étale double cover of smooth projective surfaces, let $C \subset Y$ be a curve of genus $g \geq 2$ and set $D = f^{-1}(C)$. Let $\{C\}$ and $\{D\}$ be the components of the Hilbert scheme containing C and D respectively, and consider the induced double cover $f : \mathcal{D} \to \mathcal{C}$, where $\mathcal{C} \to \{C\}$ and $\mathcal{D} \to f^*\{C\} \subset \{D\}$ are the universal family and the restriction of the universal family respectively. We construct the relative Prym variety $\mathrm{Prym}(\mathcal{D}, \mathcal{C})$ as the fixed locus of an involution τ on some (possibly singular) moduli space M of sheaves on X. The singularities come from the fact that, in order to ensure that the involution respects stability, we have to allow strictly semistable sheaves. We then describe in full detail the fibers of the natural morphism $\mathrm{Prym}(\mathcal{D}, \mathcal{C}) \to \{C\}$ over points corresponding to a number of classes of mildly singular curves.

Chapter 3. After collecting some results about linear systems and curves on K3 and Enriques surfaces, we specialize the results of Chapter 2 in the case of a K3 to Enriques cover $f : S \to T$. We prove Theorem 1 showing that in this case the involution $\tau : M \to M$, whose fixed locus is the relative Prym variety P, is a symplectic involution on a singular moduli space of sheaves on S. Using the Kuranishi map we then analyze the local structure of M and of P at the most general singular points. We prove that at these points P is locally isomorphic to its tangent cone. This shows that P is $\mathbb{Q}$-factorial and that it is terminal if and only if the linear system $|C|$ is numerically m-connected with $m \geq 2$ (non-hyperelliptic case). We then consider the case when the linear system

is hyperelliptic, and we show that the relative Prym variety admits a symplectic resolution. This resolution is birational and hence, by [29], deformation equivalent to a moduli space of sheaves on another K3 surface that can be explicitly described. The two-dimensional Prym variety is an elliptic K3 surface, of which we describe the singular fibers using some of the results of Chapter 2. Finally, using a theorem of Leibman [41] (as used also in [42]), the theory of Picard-Lefschetz, and the fact that M has a resolution that is an irreducible holomorphic symplectic manifold, we show that the smooth locus of P is simply connected.

Chapter 4. We use the results of Chapter 1 on cyclic covers to show that, under the assumptions of Theorem 4, the pullback morphism $\Phi : M_{w,A}(T) \to M_{v,H}(S)$ is an étale double cover of a Lagrangian subvariety $Y_{v,H} \subset M_{v,H}(S)$. To compute the invariants of $M_{w,A}(T)$ we assume that the dimension of the fibers of the natural support morphism $\nu : M_{w,A}(T) \to |C|$ does not jump too much (cf. Assumption 4.1.5). The computation of $\pi_1(M_{w,A}(T))$ uses the same techniques used for computing the fundamental group of P. As for the canonical bundle, the main step consists in computing the push-forward of the relative dualizing sheaf of ν. Via relative Serre duality, this is done by computing the higher direct images of the structure sheaf of $M_{w,A}(T)$ and comparing them with the higher direct images of the structure sheaf of $\mathcal{C} \to |C|$. This comparison relies on the theory of degeneration of Hodge bundles ([76], [37]). In the case of the second Betti number of $M_{w,A}(T)$ we also use the universal family of curves, whose second cohomology group is easily calculated. A crucial step is to show that there is an isomorphism of mixed Hodges structures $H^2(M_{w,A}(T)_U) \cong H^2(\mathcal{C}_U)$, which requires showing that the monodromy representation of $\pi_1(U)$ on the first cohomogy of a smooth fiber is irreducible. We show this fact using some of the results proved for the computation of $\pi_1(M_{w,A}(T))$. We end the chapter with a section containing a detailed description of the three dimensional case.

Chapter 1

Pure sheaves and their moduli

1.1 Pure sheaves of dimension one on surfaces

1.1.1 First definitions

Following [30] and [39] we define the notion of pure sheaves, the notion of stability for such sheaves, and we consider moduli spaces of pure sheaves of dimension one on surfaces and their first properties. We give criteria for the stability of pure sheaves of dimension one on a smooth projective surface and we describe in detail a few examples of moduli of sheaves supported on non integral curves. Some of these examples will be of great importance in Chapter 3.

Let X be a smooth projective variety of dimension n. Associated to any coherent sheaf F on X, is the ideal sheaf

$$\mathcal{I}_F := \ker[\mathcal{O}_X \to \mathcal{E}nd(F)],$$

This ideal sheaf is the kernel of the natural morphism $\mathcal{O}_X \to \mathcal{E}nd(F)$, which defines the $\mathcal{O}_X$-module structure on F, and defines a scheme structure on the closed subset

$$\{x \in X \ | F_x \neq 0\}.$$

Definition 1.1.1. *The support of F*

$$\mathrm{Supp}(F) \subset X,$$

is the subscheme of X associated to the ideal sheaf $\mathcal{I}_F$ defined above.

The dimension (resp. codimension) of the sheaf F is defined to be the dimension (reps. codimension) of $\mathrm{Supp}(F)$.

Definition 1.1.2. *A sheaf F on X is said to be pure of dimension d if*

$$\dim \mathrm{Supp}(F') = d, \quad \text{for every subsheaf} \quad 0 \subsetneq F' \subset F.$$

Clearly, being pure is a local notation. If locally on an affine open subset $U := \mathrm{Spec}\, A \subset X$ the sheaf F is defined by an A-module M, then F is pure of dimension d on U if and only if all the associated primes of M are of dimension d. For sheaves on non integral schemes or sheaves for which $\mathrm{Supp}(F) \subsetneq X$, being pure is the correct generalization of being torsion free.

Definition 1.1.3. *[30] Let F be a d-dimensional sheaf on X. For any $0 \leq i < d$, we define the sub sheaf*

$$T_i(F) \subset F,$$

as the maximal sub sheaf of F of dimension $\leq i$.

Hence, a sheaf F is pure of dimension d if and only if $T_0(F) = \cdots = T_{d-1}(F) = 0$.

We will now focus our attention on pure sheaves of codimension one on surfaces. In what follows, X will be a smooth projective surface. It turns out that on a smooth projective surface, pure sheaves of dimension one may have better properties than some torsion free sheaves. For example it is well known that the ideal sheaf $\mathcal{I}_x$ of a point x on a surface X surfaces is not reflexive since

$$\mathcal{H}om(\mathcal{H}om(\mathcal{I}_x, \mathcal{O}_X), \mathcal{O}_X) \cong \mathcal{O}_X.$$

On the other hand, if F is a pure dimension one sheaf on X, then $\mathcal{H}om(F, \mathcal{O}_X) = 0$. However, setting $\Gamma = \mathrm{Supp}(F)$ we get

$$\mathcal{E}xt^1_X(F, \omega_X) \cong \mathcal{H}om_\Gamma(F, \omega_\Gamma).$$

One can see this by tensoring the short exact sequence,

$$0 \to \mathcal{O}_X \xrightarrow{\gamma} \mathcal{O}_X(\Gamma) \to \mathcal{O}_\Gamma(\Gamma),$$

by ω_X, applying $\mathcal{H}om(F, \cdot)$, and using the fact that γ is locally defined by multiplication by a local

equation of Γ. Following the notation of [30] we define,

$$F^D := \mathcal{E}xt^1_X(F, \omega_X). \tag{1.1}$$

Proposition 1.1.4 ([30])**.** *Let X be a smooth projective surface, and let F be a pure dimension one sheaf on X.*

1. *F is reflexive, i.e., $F^{DD} \cong F$;*
2. *F has homological dimension one, i.e., the least length of a locally free resolution of F is one.*

In particular, for $i > 1$, $\mathcal{E}xt^i(F, G) = 0$ for any sheaf G on X.

Let F be a pure sheaf of dimension one on X, and let,

$$0 \to L_1 \xrightarrow{a} L_0 \to F \to 0,$$

be a length one locally free resolution of F, whose existence is guaranteed by the previous proposition. Following [39] we define the *determinantal support* as the subscheme $\mathrm{supp}(F)$ defined by the equation

$$\mathrm{supp}(F) := (\det a = 0) \subset X. \tag{1.2}$$

One can verify that the subscheme defined above is independent of the locally free resolution. Moreover, it defines the first Chern class of F, in the sense that

$$c_1(F) = [\det a = 0],$$

in $H^2(X, \mathbb{Z})$. The following examples show that the two notions of support defined above need not agree.

Example 1.1.5. Let $\Gamma \subset X$ be an integral curve, let $\Gamma' \subset X$ be the non reduced scheme defined by the ideal sheaf $\mathcal{O}_X(-2\Gamma)$. If E is a vector bundle on Γ, then,

$$\mathrm{Supp}(E) = \Gamma, \quad \text{and } \mathrm{supp}(E) = \Gamma'.$$

On the other hand, if $\mathrm{Supp}(\mathcal{O}_{\Gamma'}) = \mathrm{supp}(\mathcal{O}_{\Gamma'}) = \Gamma'$.

The definition of scheme theoretical support given in 1.1.2 is very natural, but has the disadvantage of not behaving well in families. On the other hand, this is not the case for the determinantal

support defined in (1.2):

Lemma 1.1.6. *Let $\mathcal{F}$ be a flat family of pure sheaves of dimension one on X parametrized by a scheme S. There is a subscheme $\Gamma \subset X \times S$, flat over S, such that for every $s \in S$, $\Gamma_s = \mathrm{supp}(\mathcal{F}_s)$.*

Proof. Let $\mathcal{L}_0 \to \mathcal{F} \to 0$ be a surjective morphism from a locally free $\mathcal{O}_{X\times S}$ module $\mathcal{L}_0$, and let $A : \mathcal{L}_1 \to \mathcal{L}_0$ be its kernel. Since F is flat, for every $s \in S$ the restriction of A to s induces a resolution,

$$0 \to \mathcal{L}_{1|X_s} \xrightarrow{A_s} \mathcal{L}_{0|X_s} \to \mathcal{F}_s \to 0. \tag{1.3}$$

Notice that since for every $s \in S$, $\mathcal{L}_{0|X_s}$ is locally free and $\mathrm{hd}(\mathcal{F}_s) = 1$, the sheaf $\mathcal{L}_{1|X_s}$ is also locally free. Set $\Gamma = (\det A = 0)$. There is an exact sequence,

$$0 \to \det \mathcal{L}_1 \to \det \mathcal{L}_0 \to \mathcal{O}_\Gamma \otimes \det \mathcal{L}_0 \to 0,$$

and since $\det \mathcal{L}_0$ is flat over S, $\mathcal{O}_\Gamma$ is flat if and only if for every $s \in S$ the restriction $\det \mathcal{L}_{1|X_s} \to \det \mathcal{L}_{0|X_s}$ is injective. This, however, follows directly from (1.3). □

For later use, we now calculate some $\mathcal{E}xt$-groups of certain pure sheaves of dimension one on X.

Lemma 1.1.7. *Let F and G be pure dimension one sheaves on X. Assume that F and G are locally free on their support and that $\Gamma_1 = \mathrm{Supp}(F)$ and $\Gamma_2 = \mathrm{Supp}(G)$ and have no common components. Then,*

$$H^0(X, \mathcal{E}xt^1(F, G)) = \mathbb{C}^{\mathrm{rk}(F)\,\mathrm{rk}(G)\ \Gamma_1\cdot\Gamma_2}.$$

Proof. First of all, notice that since F and G are pure of dimension one and since the intersection of their support is zero dimensional,

$$\mathcal{H}om(F, G) = 0.$$

Since F and G are locally free and the question is local, we can assume that $F = \mathcal{O}_{\Gamma_1}$ and $G = \mathcal{O}_{\Gamma_2}$. Consider the standard resolution of $\mathcal{O}_{\Gamma_2}$,

$$0 \to \mathcal{O}_X(-\Gamma_2) \to \mathcal{O}_X \to \mathcal{O}_{\Gamma_2} \to 0. \tag{1.4}$$

Applying the functor $\mathcal{H}om(\mathcal{O}_{\Gamma_1}, \cdot)$ we get

$$0 \to \mathcal{E}xt^1(\mathcal{O}_{\Gamma_1}, \mathcal{O}_X(-\Gamma_2)) \to \mathcal{E}xt^1(\mathcal{O}_{\Gamma_1}, \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{O}_{\Gamma_1}, \mathcal{O}_{\Gamma_2}) \to 0. \tag{1.5}$$

which is just the exact sequence (1.4) tensored by $\mathcal{E}xt^1(\mathcal{O}_{\Gamma_1}, \mathcal{O}_X) \cong \mathcal{O}_{\Gamma_1}(\Gamma_1)$. Hence, we are reduced to calculate H^0 of the skyscraper sheaf,

$$\mathcal{O}_{\Gamma_1}(\Gamma_1) \otimes \mathcal{O}_{\Gamma_2} \cong \mathcal{O}_{\Gamma_1} \otimes \mathcal{O}_{\Gamma_2}.$$

From (1.5) it also follows that the higher Tor-modules $\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{\Gamma_2,x}, \mathcal{O}_{\Gamma_2,x})$ vanish so that using Serre's intersection formula [65] we may conclude that

$$H^0(\mathcal{O}_{\Gamma_1} \otimes \mathcal{O}_{\Gamma_2}) \cong \mathbb{C}^{\Gamma_1 \cdot \Gamma_2}.$$

□

Corollary 1.1.8. *Let F and G be two pure dimension one sheaves on X, with support equal to Γ_1 and Γ_2 respectively. Suppose that Γ_1 and Γ_2 are reduced and meet transversally in smooth points. Then*

$$\mathrm{Ext}^1(F, G) = \mathbb{C}^{\mathrm{rk}(F)\,\mathrm{rk}(G)\ \Gamma_1 \cdot \Gamma_2}, \quad \mathrm{Ext}^2(F, G) = 0.$$

Proof. Since $\dim \mathrm{Supp}(\mathcal{E}xt^i(F, G)) = 0$ for every i, the local to global spectral sequence degenerates, so that

$$\mathrm{Ext}^1(F, G) = H^0(X, \mathcal{E}xt^1(F, G))$$

and the first statement follows from Lemma 1.1.7, while the the second statement follows from Serre duality and the fact that $\mathrm{Hom}(F, G) = 0$. □

1.1.2 Moduli spaces of pure sheaves of dimension one on surfaces

In order to construct a moduli space of sheaves with some fixed invariants on a smooth projective variety X, one needs to restrict to bounded families. Boundedness is usually achieved by fixing a polarization H on X and defining some notion of stability. For torsion free sheaves one can consider μ-stability or Gieseker-stability. Clearly, one cannot define slope-stability for torsion sheaves since their rank is equal to zero. However, one can still consider, as in the case of Gieseker-stability, the *reduced Hilbert polynomial*, and use this to define stability.

Fix a polarization H. Let F be a sheaf on X and let

$$P(F, m) = P_H(F, m) = \chi(F(m)) = \sum a_i(F) \frac{m^i}{i!}, \quad a_i(F) \in \mathbb{Z},$$

be its Hilbert polynomial. It is well known (cf. for eg. [26], Thm 7.5) that

$$\deg P(F,m) = \dim \mathrm{Supp}(F).$$

Let F be a pure sheaf of dimension d on X, the leading coefficient

$$a_d(F),$$

is always positive, and is called the multiplicity of F. Following [30] we define the rank of F as

$$\mathrm{rk}(F) := \frac{a_d(F)}{a_d(\mathcal{O}_\Gamma)}.$$

Clearly, this is just the usual rank if Γ is integral. The reduced Hilbert polynomial is, by definition, the polynomial with rational coefficients given by

$$p(F,m) = p_H(F,m) = \frac{P(F,m)}{a_d(F)}.$$

Definition 1.1.9. *Let F be d-dimensional sheaf on X, and let H be a polarization on X. The sheaf F is (semi)-stable with respect to H if it is pure and if for every proper non-trivial sub-sheaf $0 \subsetneq F' \subsetneq F$ we have*[1]

$$p(F',m)(\leq)p(F,m), \quad \textit{for } m \gg 0.$$

Equivalently, F is (semi)stable w.r.t. H if for every proper non trivial quotient $F \to F'' \to 0$, with F'' pure of dimension d,

$$p(F,m)(\leq)p(F'',m), \quad \textit{for } m \gg 0.$$

A sheaf F is called H-polystable if it is isomorphic to a direct sum of H-stable sheaves with reduced Hilbert polynomial equal to $p(F,m)$.

Any H-semistable sheaf F admits a filtration, called the *Jordan-Hölder* filtration,

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = F,$$

that has the property that the quotients $\mathrm{gr}_i(F) = F_i/F_{i-1}$ are H-stable with reduced Hilbert

[1] We follow the Notation 1.2.5 of [30]. If F is a sheaf that is (semi)-stable with respect to H, we will sometimes say that it is H-(semi)stable.

polynomial equal to $p_H(F,m)$. Such a filtration is not necessarily unique, but the graded sheaf

$$\mathrm{gr}_H(F) \cong \oplus_i \mathrm{gr}_i(F), \tag{1.6}$$

is unique. Two pure sheaves F and F' are said to be $\mathcal{S}$-equivalent with respect to H if

$$\mathrm{gr}_H(F) = \mathrm{gr}_H(F').$$

We now come to the definition of Mukai vector.

Definition 1.1.10. *Let F be a sheaf on X. The Mukai vector of F is*

$$v(F) = \mathrm{ch}(F)\sqrt{\mathrm{td}(X)} \in H^{2*}(X,\mathbb{Z})_{tf},$$

where $H^{2*}(X,\mathbb{Z})_{tf}$ denotes the torsion free part of the even cohomology.

We denote by $v_i \in H^{2i}(X,\mathbb{Z})_{tf}$ the i-th component of the Mukai vector.

Example 1.1.11. For example, if $X = S$ is a K3 surface, then

$$v(F) = (r, c_1(F), \frac{c_1^2}{2}(F) - c_2(F) + \mathrm{rk}(F)).$$

Example 1.1.12. If F is pure of dimension one on a smooth projective surface X, then

$$v(F) = (0, c_1(F), ch_2(F)) = (0, c_1(F), \chi(F) - \frac{1}{2}c_1(X)c_1(F)),$$

where $ch_2(F)$ is the degree two part of the Chern character of F.

In order to fix the numerical invariants of some class of sheaves, we will fix a Mukai vector $v \in H^{2*}(X,\mathbb{Z})_{tf}$. Given a polarization H, we will denote by

$$M_{v,H}(X),$$

the moduli space of H-(semi)stable sheaves with Mukai vector equal to v. The closed points of $M_{v,H}(X)$ parametrize $\mathcal{S}$-equivalence classes of H-semi stable sheaves, i.e., isomorphism classes of H-polystable sheaves with Mukai vector v.

Now let us specialize to the case where X is a smooth projective surface.

If F is pure of dimension one by the Hirzebruch-Riemann-Roch Theorem,

$$P(F,m) = (c_1(F)\cdot H)\ m + \chi(F),$$

so that F is H-(semi)stable if and only if for every proper non trivial sub sheaf $F' \subset F$ we have

$$\frac{\chi(F')}{c_1(F')\cdot H}(\leq)\frac{\chi(F)}{c_1(F)\cdot H}.$$

The analogous condition holds for quotient sheaves. By abuse of notation, we set (cf. [39])

$$\mu_H(F) := \frac{\chi(F)}{c_1(F)\cdot H}.$$

Let H be a polarization on X, and consider F a pure sheaf of dimension one that is H-stable. Set $v = v(F)$, and consider the moduli space $M_{v,H}(X)$ of H-semistable sheaves on X with Mukai vector v. Let $\mathcal{H}_v$ be the Hilbert scheme of subschemes of X whose class in cohomology equals the second component $v_1 \in H^2(X,\mathbb{Z})_{tf}$ of the Mukai vector. Following Le Potier [39] and using Lemma 1.1.6 we can define the *support morphism*,

$$\begin{aligned} \pi : M_{v,H}(X) &\longrightarrow \mathcal{H}_v, \\ F &\mapsto \mathrm{supp}(F) \end{aligned} \tag{1.7}$$

associating to a pure sheaf of dimension one, its determinantal support. If $\Gamma \subset X$ is a curve with $[\Gamma] = v_1$, we denote by

$$\{\Gamma\} \subset \mathcal{H}_v,$$

the irreducible component of $\mathcal{H}_v$ containing Γ, and by

$$M_{v,H}(X,\{\Gamma\}),$$

the pre-image under π of $\{\Gamma\}$. Similarly, we denote by

$$M_{v,H}(X,|\Gamma|),$$

the pre-image under π of $|\Gamma| \subset \{\Gamma\}$. If there is no risk of confusion, we simply write

$$M_{v,H}, \quad \text{resp. } M_{v,H}(\{\Gamma\}), M_{v,H}(|\Gamma|),$$

for $M_{v,H}(X)$ (resp. $M_{v,H}(X, \{\Gamma\})$, $M_{v,H}(X, |\Gamma|)$).

We can also define the *determinant morphism*

$$\begin{aligned} \det : M_{v,H} &\longrightarrow \mathrm{Pic}(X), \\ F &\longmapsto \det(F) \end{aligned} \tag{1.8}$$

Clearly

$$\det^{-1}(\mathcal{O}(\Gamma)) = M_{v,H}(X, |\Gamma|) \subset M_{v,H}(X).$$

In view of the following Lemma, we have a control on the fibers of π, at least over integral curves.

Lemma 1.1.13. *Let F be a pure dimension one sheaf on X, let H be a polarization and set $\Gamma = \mathrm{Supp}(F)$. If Γ is integral, then H-(semi)stability of F does not depend on H, and it is equivalent to the usual slope stability[2] of F viewed as a sheaf on Γ.*

Proof. Set $r = \mathrm{rk}(F)$ so that $c_1(F) = r[\Gamma]$, and let $F \to F''$ be a proper non trivial quotient. Then $\mathrm{Supp}(F'')$ is also equal to Γ and if we set $s = \mathrm{rk}(F'')$, then $c_1(F'') = s[\Gamma]$, and the Lemma follows. □

Corollary 1.1.14. *Let F be a pure sheaf of dimension one on X, suppose that $\Gamma = \mathrm{Supp}(F)$ is integral and that F, considered as a sheaf of Γ has rank one. Then F is stable w.r.t. any polarization H on X.*

In particular, if Γ is an integral curve of genus g with $[\Gamma] \in \mathcal{H}_v$, then

$$\pi^{-1}([\Gamma]) \cong \overline{\mathrm{Jac}}^d(\Gamma),$$

is the compactified Jacobian of rank one torsion free sheaves on Γ with fixed Euler characteristic $\chi = \chi(v) = d - g + 1$ determined by the last component $v_2 \in H^4(X, \mathbb{Q})$ of the Mukai vector.

Because of this, we will sometimes refer to the moduli space $M_{v,H}(\{\Gamma\})$ (or $M_{v,H}(|\Gamma|)$) as the degree

[2]The slope of a sheaf F on an integral curve Γ is the rational number $\frac{\deg F}{\mathrm{rk}(F)}$, where $\deg F := \chi(F) - \chi(\Gamma)$.

d relative compactified Jacobian of the curves in $\{\Gamma\}$ (or in $|\Gamma|$).

Theorem 1.1.15 ([2]). *Let Γ be a reduced and irreducible curve with locally planar singularities. For every $d \in \mathbb{Z}$, the compactified Jacobian $\overline{\mathrm{Jac}}^d(\Gamma)$ is irreducible and of dimension g.*

Notice that if $U \subset \mathcal{H}_v$ denotes the locus of smooth curves, then

$$\pi^{-1}(U) \to U,$$

is the relative Jacobian variety of the universal family of curves over U.

More generally, the fiber over a curve Γ is just the Simpson moduli space of $H\big|_\Gamma$-stable sheaves on Γ with given Hilbert polynomial determined by v.

Suppose that $v = (0, [\Gamma], v_2)$, where Γ is a curve of genus g and $v_2 = -g + 1 - \frac{1}{2}c_1(X) \cdot \Gamma$. Since the structure sheaf of any integral curve is H-stable, the assignment

$$\Gamma \mapsto \mathcal{O}_\Gamma,$$

defines, on any component $W \subset \mathcal{H}_v$ containing an integral curve, a rational section

$$s : W \dashrightarrow M_{v,H}, \tag{1.9}$$

of the support morphism (1.7).

As we will see, if $\Gamma = \mathrm{Supp}(F)$ is not integral the situation can be more complicated and in this case the polarization H will play a crucial role. Before stating the next propositions, dealing with the case when Γ is reduced but not irreducible, let us fix some notation. By a *sub curve* we mean an effective divisor

$$\Gamma' \subset \Gamma,$$

such that $\Gamma - \Gamma'$ is also effective. For a sub curve $\Gamma' \subset \Gamma$ we let,

$$\overline{\Gamma}' := \Gamma - \Gamma' \subset \Gamma,$$

be the complementary sub-curve.

Consider a sub curve $\Gamma' \subset \Gamma$. Since the restriction $F_{|\Gamma'} := F \otimes \mathcal{O}_{\Gamma'}$ is not necessarily pure, we

set

$$F_{\Gamma'} := F_{|\Gamma'}/T_0(F_{|\Gamma'}). \tag{1.10}$$

so that $F_{\Gamma'}$ is pure of dimension one. The sheaf $F_{\Gamma'}$ is the maximal quotient of F that is supported on Γ'. If

$$\Gamma = \sum \Gamma_i,$$

is the decomposition of Γ into irreducible components, then we set

$$F_i := F_{\Gamma_i}. \tag{1.11}$$

If

$$m : \coprod \Gamma_i \to \Gamma,$$

is the partial normalization that separates the irreducible components, the purity of F shows that F fits into a short exact sequence,

$$0 \to F \to \oplus F_i \to Q \to 0, \tag{1.12}$$

where Q is some skyscraper sheaf supported on $\cup_{i<j}\Gamma_i \cap \Gamma_j$.

The following proposition is well known (cf. for eg. Alexeev [1])

Proposition 1.1.16. *Let F be a pure dimension one sheaf on X, let H be a polarization, suppose that $\Gamma = \mathrm{Supp}(F)$ is reduced and that $\mathrm{rk}(F) = 1$. Then F is H-(semi)stable if and only if for every sub curve $\Gamma' \subset \Gamma$*

$$\mu_H(F)(\leq)\mu_H(F_{\Gamma'}).$$

Proof. Let $a : F \to G$ be a quotient of F, with G pure of dimension one. Then $\Gamma' = \mathrm{Supp}(G)$ is a subcurve of Γ, and a factors via the quotient morphism $F \to F_{\Gamma'}$. Since Γ' is reduced, $F_{\Gamma'} \to G$ is a surjective morphism of sheaves of rank one. It follows that $F_{\Gamma'} \cong G$ it is an isomorphism and the proposition is proved. □

In other words, to check stability it is enough to compare the slope of F with the slope of the maximal quotients of F that are supported on the sub curves of Γ.

Example 1.1.17. Consider a genus g curve $\Gamma \subset X$ that union of two integral curves Γ_1 and Γ_2 of genus respectively equal to g_1 and g_2. Set $\delta = \Gamma_1 \cdot \Gamma_2$, so that we can write $g = g_1 + g_2 + \delta - 1$. Fix

an integer d and set $\chi = d - g + 1$. The group $\mathrm{Jac}^d(\Gamma)$ of line bundles L on Γ with $\chi(L) = \chi$ is a disjoint union

$$\mathrm{Jac}^d(\Gamma) = \coprod_{d=d_1+d_2} \mathrm{Jac}^{d_1,d_2}(\Gamma), \tag{1.13}$$

where

$$\mathrm{Jac}^{d_1,d_2}(\Gamma) = \{L \in \mathrm{Jac}^d(\Gamma) |\ \deg(L_i) = d_i,\ \text{for } i = 1, 2\}.$$

Each of these components is isomorphic to the identity component $\mathrm{Jac}^{0,0}(\Gamma)$, which fits into the following exact sequence,

$$0 \to G \to \mathrm{Jac}^{0,0}(\Gamma) \xrightarrow{m^*} \mathrm{Jac}^0(\Gamma_1) \times \mathrm{Jac}^0(\Gamma_2) \to 0. \tag{1.14}$$

Here $m : \tilde{\Gamma} := \Gamma_1 \coprod \Gamma_2 \to \Gamma$ is the morphism that separates the two components, and the kernel of m^* can be expressed as (cf. EGA IV 21.5.8 [25])

$$G = (\prod_{x \in \Gamma} (m_* \mathcal{O}^*_{\tilde{\Gamma}})_x / \mathcal{O}_{\Gamma,x}) \operatorname{Im} H^0(\tilde{\Gamma}, \mathcal{O}^*_{\tilde{\Gamma}}). \tag{1.15}$$

In this case the skyscraper sheaf Q of (1.12) has Euler characteristic $\chi(Q) = \delta$. As mentioned above, to have boundedness we need to fix a polarization H and consider only the sheaves that are (semi)stable w.r.t. H. We let

$$\mathrm{Jac}^d_H(\Gamma)$$

be the set of H-stable[3] degree d line bundles on Γ. Set

$$k_1 = H \cdot \Gamma_1,\ \ k_2 = H \cdot \Gamma_2 \quad \text{and } \ k = H \cdot \Gamma = k_1 + k_2. \tag{1.16}$$

Using Proposition 1.1.16 we see that L is H-(semi)stable if and only if

$$\begin{cases} \dfrac{\chi}{k} (\leq) \dfrac{\chi_1}{k_1}, \\ \dfrac{\chi}{k} (\leq) \dfrac{\chi_2}{k_2}. \end{cases}$$

[3] In this example we are only considering line bundles that are stable. Indeed, as we see in Remark 1.1.18, line bundles that are strictly semi-stable are $\mathcal{S}$-equivalent to sheaves that are not locally free, so that in order to consider strictly semi-stable sheaves we need to compactify the generalized Jacobian and consider rank one torsion free sheaves as well.

Equivalently, since $\chi + \delta = \chi_1 + \chi_2$, L is H-(semi)stable if and only if

$$\frac{k_1}{k}\chi(\leq)\chi_1(\leq)\frac{k_1}{k}\chi + \delta. \tag{1.17}$$

From the above formula we can immediately deduce the following facts:

1. Strictly semi-stable sheaves exist if and only if $\frac{k_1}{k}\chi \in \mathbb{Z}$;
2. if $\frac{k_1}{k}\chi \in \mathbb{Z}$, then $\mathrm{Jac}^d_H(\Gamma)$ is the union of $\delta - 1$ g-dimensional components;
3. if $\frac{k_1}{k}\chi \notin \mathbb{Z}$, then $\mathrm{Jac}^d_H(\Gamma)$ is the union of δ components;
4. which among the components of (1.13) lie in $\mathrm{Jac}^d_H(\Gamma)$ depends on the rational number $\frac{k_1}{k}$, i.e., they depend on H.

Notice that this shows that the structure sheaf of Γ is not necessarily H-stable, i.e., that the identity component $\mathrm{Jac}^0(\Gamma)$ does not necessarily appear in $\mathrm{Jac}^0_H(\Gamma)$. Moreover, we see that if $\chi = 0$ there are strictly semistable sheaves for every polarization.

The example above shows that the moduli space $M_{v,H}(X)$ may depend strongly on H.

Remark 1.1.18. We should also remark that given a strictly H-semistable line bundle F on Γ, the graded sheaf $\mathrm{gr}_H(F)$ (cf. 1.6) is necessarily the direct sum of a sheaf supported on Γ_1 and of a sheaf supported on Γ_2. Hence, we see that the line bundle F is $\mathcal{S}$-equivalent to a non locally free sheaf.

If Γ is reduced, we will set

$$\pi^{-1}([\Gamma]) \cong \overline{\mathrm{Jac}}^d_H(\Gamma).$$

Theorem 1.1.19 ([14])**.** *Let $\Gamma \subset X$ be a connected reduced curve of genus g, and let d be any integer. Then,*

$$\dim \overline{\mathrm{Jac}}^d_H(\Gamma) = g.$$

Proof. This is Theorem 3.2 in [14], and the fact that the locus of H-polystable sheaves (that are not simple) is a product of compactified Jacobians of curves of lower genus, and hence has dimension strictly less than g. □

Corollary 1.1.20. *Let Γ be an integral curve and let $W \subset \{\Gamma\}$ be the locus of reduced curves. The restriction,*

$$M_{v,H}(\{\Gamma\})_{|W} \to W,$$

is flat.

Other problems may arise when dealing with non reduced curves and, in general, there can exist higher dimensional components of these moduli spaces.

Example 1.1.21. Consider a smooth curve Γ' of genus $\gamma' \geq 2$, and denote by Γ a non reduced scheme obtained by considering a double structure on Γ'. We denote by γ the genus of Γ. It was shown by Chen and Kass in [15], that all the components of the Simpson moduli space have dimension γ, except possibly a $(4\gamma'-3)$-dimensional component, which exists when $4\gamma'-3 \geq \gamma$ and parametrizes rank 2 semi-stable sheaves on Γ'. Let us now suppose that Γ and Γ' are contained in a smooth surface X, so that the scheme structure defining Γ is the one induced by the ideal sheaf $\mathcal{O}(-2\Gamma')$. By the adjunction formula,

$$\gamma = 4\gamma' - 3 - \deg \omega_{X\,|\Gamma'}.$$

It follows that the dimension of the Simpson moduli space does not jump if,

$$\deg \omega_{X\,|\Gamma'} \leq 0. \tag{1.18}$$

In particular, as soon as the canonical bundle of X is numerically trivial (K3, abelian, Enriques and bi-elliptic surfaces) then (1.18) is satisfied for any curve contained in X.

Something stronger is in fact true in the case of linear systems on K3 or abelian surfaces. Indeed, if we let $v = (0, D, \chi)$ be any primitive Mukai vector and if we choose a v-generic polarization H, it follows by [43] that the relative compactified Jacobian $M_{v,H}$ is flat over $|D|$ (cf. Remark 3.2.3). The proof, however, relies on the existence of a symplectic structure on these moduli spaces and cannot therefore be applied to moduli spaces of sheaves on other surfaces.

As we showed above, the answer to the following question should depend on the numerical properties of the canonical bundle of the surface.

Problem 1.1.22. *When is the relative compactified Jacobian of a linear system on a smooth projective surface flat over the base?*

We think that this should indeed be the case if the canonical bundle of the surface is numerically trivial.

Following [74], [73], and [59] we make the following definitions.

Definition 1.1.23. *Let $v \in H^{2*}(X,\mathbb{Z})_{tf}$ be a Mukai vector. A polarization H on X is called v-generic if for every H-semistable sheaf F with $v(F) = v$ and for every sub sheaf $F' \subset F$ with $p_H(F,m) = p_H(F',m)$, we have*

$$\mathbb{Z}v(F') = \mathbb{Z}v.$$

An immediate consequence of this definition is the following Lemma

Lemma 1.1.24. *Suppose that the Mukai vector $v \in H^{2*}(X,\mathbb{Z})_{tf}$ is primitive. If H is a v-generic polarization, then every H-semistable sheaf is H-stable.*

Proof. Indeed, let F be an H-semi stable sheaf on X with Mukai vector v, and let $F' \subset F$ be a sub sheaf with $p_H(F,m) = p_H(F',m)$. Since H is v-generic and v is primitive, $v(F') = v(F)$ which in turn forces $F' = F$. □

Proposition 1.1.25 (Section 1.4, [74])**.** *Let $v = (0,[\Gamma],v_2)$ be the Mukai vector of a pure dimension one sheaf F on X with $\chi = \chi(F) \neq 0$ and $\Gamma^2 > 0$. The set of polarizations $H \in \mathrm{Amp}(X)$ that are not v-generic is the union of finitely many linear subspaces in $\mathrm{Amp}(X)$, called the walls of v.*

Example 1.1.26. Let $\Gamma \subset X$ be a curve as in Example 1.1.17, and let H be a polarization such that $\frac{k_1}{k}\chi \in \mathbb{Z}$. If we set $v = (0,[\Gamma],\chi + \Gamma\cdot\omega_X/2)$, then H is not v-generic. Indeed, if F_1 is a rank one torsion free sheaf with $\chi(F_1) = \frac{k_1}{k}\chi$ and if F_2 is a rank one torsion free sheaf with $\chi(F_2) = \frac{k_2}{k}\chi$. Then

$$F = F_1 \oplus F_2,$$

is an H-polystable sheaf.

Definition 1.1.27. *Denote by $\mathcal{W}_v$ the union of all the walls of v. A connected component of $\mathrm{Amp}(X) \setminus \mathcal{W}_v$ is called a chamber.*

Let $H_0 \in \mathcal{W}_v$ be a non v-generic polarization. A chamber $\mathcal{C}$ is said to be adjacent to H_0, if $H_0 \in \overline{\mathcal{C}}$.

Lemma 1.1.28 ([75])**.** *Let $v \in H^{2*}(X,\mathbb{Z})_{tf}$ be the Mukai vector of a pure sheaf of dimension one on X. Let H_0 and H be polarizations on X such that H_0 is not v-generic, and H lies in a chamber adjacent to H_0. There is a projective birational morphism*

$$r : M_{v,H} \to M_{v,H_0}.$$

that is an isomorphism on the open subset of sheaves with integral support.

Proof. The existence of the morphism follows form Lemma 4.1 of [75], and the fact that it is an isomorphism on the open subset of sheaves with integral support follows from Corollary 1.1.14. □

Example 1.1.29. Let $\Gamma \subset X$ be an integral curve. Suppose that $[\Gamma] = [\Gamma_1 + \Gamma_2]$ in $\mathrm{Pic}(X)$ and such that, moreover, any sub curve of a reducible member of $|\Gamma|$ is linearly equivalent to Γ_1 or to Γ_2. Fix a non zero integer χ and set $v = (0, [\Gamma], \chi + \Gamma \cdot \omega_X/2)$. The equations in $\mathrm{Amp}(X)$ of the walls of v are of the form,

$$(\chi\Gamma_1 - \chi_1\Gamma) \cdot x = 0,$$

where χ_1 ranges in the finite set of integers defined by inequalities (1.17). Consider a wall $W_m \subset \mathcal{W}_v$ and choose a polarization $H_0 \in W_m$. Let H^+ (resp. H^-) be an ample divisor such that $(\chi\Gamma_1 - \chi_1\Gamma) \cdot H^+ > 0$ (resp. < 0), and such that it lies on a chamber adjacent to W_m. By Lemma 1.1.28 there are proper morphism $h^\pm : M_{v,H^\pm} \to M_{v,H_0}$ and a commutative diagram

$$\begin{array}{ccc} M_{v,H^+} & \dashleftarrow\dashrightarrow & M_{v,H^-} \\ & \searrow \quad \swarrow & \\ & M_{v,H_0} & \end{array}$$

1.2 Local structure of the moduli spaces

In this section, we will briefly recall a few results on the local structure of moduli spaces, that will be needed in the rest of this thesis.

1.2.1 First order deformations

Let X be a smooth projective surface, polarized by an ample line bundle H. Let F be an H-stable sheaf, and let M be the irreducible component of the moduli space of H-(semi)stable sheaves on X that contains the point $[F]$. The first result (Corollary 4.5.2 of [30]) is that the Zariski tangent space of M at $[F]$ is given by

$$T_{[F]}M = \mathrm{Ext}^1(F, F). \tag{1.19}$$

Recall that for any sheaf F on X and for any $i \geq 0$, there is a natural morphism

$$\mathrm{tr} : \mathrm{Ext}^i(F, F) \to H^i(X, \mathcal{O}_X), \tag{1.20}$$

called the trace morphism. The name comes from the fact that if F is locally free, then (1.20) is just the morphism

$$\mathrm{Ext}^i(F,F) = H^i(X, \mathcal{E}nd(F,F)) \to H^i(X, \mathcal{O}_X),$$

induced by taking the trace of an endomorphism of F

$$\mathrm{tr} : \mathcal{E}nd(F,F) \to \mathcal{O}_X.$$

If F is not locally free, then one uses a locally free resolution of F to define (1.20). For every $i \geq 0$ we set

$$\mathrm{Ext}^i(F,F)_0 = \ker[\mathrm{tr} : \mathrm{Ext}^i(F,F) \to H^i(X, \mathcal{O}_X)].$$

For $e \in \mathrm{Ext}^i(F,F)$ and $f \in \mathrm{Ext}^j(F,F)$, we let

$$e \cup f \in \mathrm{Ext}^{i+j}(F,F),$$

be the Yoneda product, which satisfies graded commutativity

$$\mathrm{tr}(e \cup f) = (-1)^{ij}\,\mathrm{tr}(f \cup e).$$

Both the trace maps and the Yoneda product are compatible with the natural conjugation action of $\mathrm{Aut}(F)$. By a result of Artamkin (Corollary in [7], or Theorem 4.5.3 in [30]) we know that the obstruction theory for deforming on X a stable F sheaf, has values in the kernel

$$\mathrm{Ext}^2(F,F)_0,$$

of the trace map. It follows that in order to show that a moduli space M is smooth at a point F, it is sufficient to show the vanishing of $\mathrm{Ext}^2(F,F)_0$.

Theorem 1.2.1 ([47])**.** *Let X be a surface with $\omega_X = \mathcal{O}_X$ (i.e. a K3 or an abelian surface) and let H be a polarization on X. Let F be a pure sheaf on X, and set $v = v(F)$. If F is H-stable, the moduli space $M_{v,H}$ is smooth at the point $[F]$.*

Proof. We said above that the obstructions to deforming F lie in $\mathrm{Ext}^2(F,F)_0$. However, if F is simple and $\omega_X = \mathcal{O}_X$ we know that $\mathrm{Ext}^2(F,F) = \mathrm{Hom}(F,F)$ is one dimensional and that the trace

morphism

$$\mathrm{Ext}^2(F,F) \to H^2(X,\mathcal{O}_X) \cong \mathbb{C},$$

is an isomorphism. Hence the obstruction vanish and $M_{v,H}$ is smooth at F. □

Corollary 1.2.2. *Let X be a surface with $\omega_X = \mathcal{O}_X$, let F be a pure sheaf on X whose Mukai vector $v = v(F)$ is primitive. Let H_0 be a non v-generic polarization, and let H be a v-generic polarization lying in a chamber that is adjacent to H_0. There is a projective birational morphism*

$$r : M_{v,H} \to M_{v,H_0},$$

that is a resolution of the singularities of M_{v,H_0}.

Proof. By Lemma 1.1.28, r is projective and birational. Since v is primitive and H is v-generic, the strictly semistable locus of $M_{v,H}$ is empty and hence by Mukai's theorem this moduli space is smooth and projective. □

Another example of smooth moduli space is the following

Theorem 1.2.3. *([39]) Let X be a smooth projective surface polarized by line bundle H, and let $\Gamma \subset X$ be a curve with*

$$\omega_X \cdot \Gamma < 0.$$

Let F be a pure sheaf of dimension one on X with $c_1(F) = [\Gamma]$ and set $v = v(F)$. If F is H-stable, then $M_{v,H}$ is smooth at F.

Proof. By Serre duality $\mathrm{Ext}^2(F,F)^\vee = \mathrm{Hom}(F, F \otimes \omega_X)$. Suppose there is a non zero morphism $F \to F \otimes \omega_X$ and let $I \subset F \otimes \omega_X$ be the image. Using the stability of F we have

$$\mu_H(F) \leq \mu_H(I) = \mu_H(I \otimes \omega_X^{-1}) + \frac{\omega_X \cdot \Gamma}{c_1(I) \cdot H} \leq \mu_H(F) + \frac{\omega_X \cdot \Gamma}{c_1(I) \cdot H}$$

and this gives a contradiction since $\omega_X \cdot \Gamma < 0$. □

Let F be a pure sheaf of dimension one on X, with $\mathrm{supp}(F) = \Gamma$ and $\chi(F) = \chi$ and $v(F) = v$. Suppose that F is H-stable. The tangent map of the determinant morphism $\det : M_{v,H} \to \mathrm{Pic}^0(X)$ at the point $[F]$ is given by the first trace morphism

$$\mathrm{Ext}^1(F,F) \to H^1(X,\mathcal{O}_X), \tag{1.21}$$

so that the tangent space to the fiber $M_{v,H}(|\Gamma|)$ over $\mathcal{O}(\Gamma)$ in $[F]$ is equal to $\mathrm{Ext}^1(F,F)_0$.

1.2.2 Kuranishi maps

In this subsection we will recall a few important results about the Kuranishi map, which is the main technical tool to study local structure of a moduli space of sheaves. A good reference on this subject is the appendix of [40].

Let X be a smooth projective surface with polarization H, and let F be an H-semistable sheaf. Set $v = v(F)$ and $M = M_{v,H}$. The local structure of M at $[F]$, is governed by the deformation theory of the unique polystable sheaf in the $\mathcal{S}$-equivalence class of F. For this reason, we will assume for the rest of the section that F is H-polystable. Set,

$$G = \mathbb{P}\,\mathrm{Aut}(F).$$

The Kuranishi map is a formal map

$$\kappa : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^2(F,F)_0$$

with zero differential

$$\kappa = \kappa_2 + \kappa_3 + \cdots .$$

that has the following properties

a) κ is equivariant under the natural action of G on $\mathrm{Ext}^1(F,F)$ and on $\mathrm{Ext}^2(F,F)_0$.

b) $\kappa^{-1}(0)/G$ is (isomorphic to) a formal neighborhood of $[F] \in M$,

c) The quadratic part κ_2 of κ, which is called the *moment map*, is given by the cup product

$$\begin{aligned} \kappa_2 : \ \mathrm{Ext}^1(F,F) &\longrightarrow \mathrm{Ext}^2(F,F)_0 \\ e &\mapsto \kappa_2(e) = \frac{1}{2} e \cup e \end{aligned} \tag{1.22}$$

d) $\kappa^{-1}(0)$ is locally isomorphic to a base for a Kuranishi family for F.

A pointed scheme (B,x) together with the sheaf $\mathcal{F}$ is a *Kuranishi family* for F if for any flat deformation $\mathcal{G} \to A \times X$ of F parametrized by an pointed scheme $(A,0)$, and any identification

$\phi : F \to \mathcal{G}_0$, there is a morphism $a : A \to B$ and an isomorphism $(a \times id)^*\mathcal{F} \cong \mathcal{G}$ inducing ϕ on the central fiber.

We should remark that only the tangent map $(da)_0$ is uniquely determined, and but the morphism $a : A \to B$ itself.

1.3 Stability, duality and finite pullbacks

It is well known that in the case of vector bundles, stability is preserved under standard operation such as tensoring by a line bundle or taking the dual. This is not necessarily the case for pure dimension one sheaves. In this section we tackle the question of when stability is preserved under such operations, or under pulling back by a cyclic cover. These results will be used in Section 2.1 and Section 4.1.

Before starting, we should stress the fact that in order to check the stability of a sheaf, one needs to examine the restriction of the sheaf itself to the sub curves of its support. In particular, the *multi degree* of a sheaf F, i.e., the collection of the degrees of the restriction F_{Γ_i} to each component plays an important role in the stability, with respect to a given polarization H, of the sheaf itself.

The next few lemmas will be of use to check whether stability is preserved under the operation of taking the dual.

Lemma 1.3.1. *Let $\Gamma = \Gamma_1 + \Gamma_2$ be the union of two curves that have no common components. For $i \neq j$, there is an exact sequence*

$$0 \to \mathcal{O}_{\Gamma_j}(-\Gamma_i) \to \mathcal{O}_\Gamma \to \mathcal{O}_{\Gamma_i} \to 0. \tag{1.23}$$

Proof. Tensoring $0 \to \mathcal{O}_X(-\Gamma_i) \to \mathcal{O}_X \to \mathcal{O}_{\Gamma_i} \to 0$ by $\mathcal{O}_\Gamma$ gives a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_\Gamma \to \mathcal{O}_{\Gamma_i} \to 0.$$

Since $\mathrm{Supp}(\mathcal{K}) = \Gamma_j$, the surjection $\mathcal{O}_X(-\Gamma_i) \to \mathcal{K}$ factors via $\mathcal{O}_X(-\Gamma_i) \to \mathcal{O}_{\Gamma_j}(-\Gamma_i)$. The proof follows by noticing that since $\mathcal{K} \to \mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is a surjective and generically injective morphism of pure sheaves it is an isomorphism. □

Let F be a pure sheaf of dimension one on a smooth surface X, and let $\Gamma = \Gamma_1 + \Gamma_2$ be a decomposition

of the support of F in two curves that have no common components. For $i, j = 1, 2$ and $i \neq j$ we set

$$F^{\Gamma_j} = \ker[F_{|\Gamma_i} \to F_{\Gamma_i}], \tag{1.24}$$

When no confusion is possible we will use the shorthand notation

$$F_i = F_{\Gamma_i}\,, \qquad F_j = F^{\Gamma_j} \tag{1.25}$$

Lemma 1.3.2. *Let F be a pure sheaf of dimension one on a smooth surface X, and let $\Gamma = \Gamma_1 + \Gamma_2$ be a decomposition of the support of F in two reduced curves that have no common components. Assume that Γ_1 and Γ_2 meet transversally in smooth points. Let Λ_F be the subset of $\Gamma_1 \cap \Gamma_2$ where F is locally free and set*

$$\Delta_F = \sum_{p \in \Lambda_F} p. \tag{1.26}$$

Then, for $j = 1, 2$

$$F^{\Gamma_j} \cong F_{\Gamma_j} \otimes \mathcal{O}_{\Gamma_j}(-\Delta_F),$$

Proof. For $i, j = 1, 2$, $i \neq j$, we may consider the short exact sequence

$$0 \to \mathcal{O}_{\Gamma_j}(-\Gamma_i) \to \mathcal{O}_\Gamma \to \mathcal{O}_{\Gamma_i} \to 0, \tag{1.27}$$

of the previous Lemma.

Tensoring by F we get the following exact sequence

$$\mathrm{Tor}^1_{\mathcal{O}_\Gamma}(F, \mathcal{O}_{\Gamma_i}) \to F_{|\Gamma_j}(-\Gamma_i) \xrightarrow{t} F \to F_{|\Gamma_i} \to 0,$$

where the sheaf $\mathrm{Tor}^1_{\mathcal{O}_\Gamma}(F, \mathcal{O}_{\Gamma_i})$ is supported on

$$T_F := \Gamma_1 \cap \Gamma_2 \setminus \Lambda_F.$$

The map t factors through a generically injective map $F_{\Gamma_j}(-\Gamma_i) \xrightarrow{s} F$. Since by definition $F_{\Gamma_j}(-\Gamma_i)$ is pure, s is actually injective and hence the following sequence

$$0 \to F_{\Gamma_j}(-\Gamma_i) \xrightarrow{s} F \to F_{|\Gamma_i} \to 0,$$

is exact. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{\Gamma_j}(-\Gamma_i) & \longrightarrow & F & \longrightarrow & F_{|\Gamma_i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \| & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \ker(\alpha) & \longrightarrow & F & \xrightarrow{\alpha} & F_{\Gamma_i} & \longrightarrow & 0 \end{array}$$

where $\alpha : F \to F_{\Gamma_i}$ is the composition of the restriction $F \to F_{|\Gamma_i}$ with the natural morphism β. For $p \in T_F$ the sheaf F is not locally free at p and $F_p \cong \mathcal{O}_{\Gamma_1,p} \oplus \mathcal{O}_{\Gamma_2,p}$. Thus, for $p \in T_F$,

$$F_{|\Gamma_i,p} \cong F_{\Gamma_i,p} \oplus \mathbb{C}_p,$$

so that

$$\ker(\beta) \cong \oplus_{p \in T_F} \mathbb{C}_p.$$

Since γ is injective it follows that $\mathrm{Coker}(\gamma) \cong \oplus_{p \in T_F} \mathbb{C}_p$, and thus

$$\ker(\alpha) \cong F_{\Gamma_j}(-\Gamma_i) \left(\sum_{p \in T_F} p \right) = F_{\Gamma_j}(-\Delta_F).$$

□

If F is a pure sheaf of dimension one on X, in (1.1) we defined the dual as

$$F^D = \mathcal{E}xt^1(F, \omega_X).$$

More generally, for any line bundle N on X we can set

$$j_N(F) := \mathcal{E}xt^1(F, N). \tag{1.28}$$

By Proposition 1.1.4,

$$j_N^2(F) \cong F.$$

For example, setting $\Gamma = \mathrm{Supp}(F)$, we have

$$F^\vee = j_{-\Gamma}(F) = \mathcal{E}xt^1(F, \mathcal{O}(-\Gamma)).$$

More generally, $j_N(F) = F^\vee \otimes \mathcal{O}(\Gamma) \otimes N$. In order to study the problem of wether stability is

preserved by taking the dual, it is convenient to consider the slight more general situation of wether stability is preserved under the map $F \mapsto j_N(F)$. To this aim, we start by computing the numerical invariants of $j_N(F)$ for a pure sheaf F of dimension one.

Lemma 1.3.3. *Let F be a pure sheaf of dimension one on X, and let N be a line bundle on X. Then,*

$$c_1(F) = c_1(j_N(F)).$$

Proof. Consider, a locally free resolution of F

$$0 \to L_1 \xrightarrow{a} L_0 \to F \to 0,$$

so that $c_1(F)$ is the class of the curve defined by the equation $\det a = 0$. Dualizing we get: $0 \to L_1^\vee \xrightarrow{a^\vee} L_0^\vee \to \mathcal{E}xt^1_S(F, \mathcal{O}_S) \to 0$. Since $(\det a = 0)$ and $(\det a^\vee = 0)$ define the same subscheme of S, the Lemma follows. □

As for the Hilbert polynomial with respect to a given polarization H, choosing $m \gg 0$, we can write

$$\begin{aligned} P_{j_N(F)}(m, H) =& \chi(\mathcal{E}xt^1_S(F, \mathcal{O}(N + mH)) = \dim H^0(\mathcal{E}xt^1_X(F, \mathcal{O}(N + mH)) \\ &= \dim \mathrm{Ext}^1(F \otimes \mathcal{O}(-N - mH), \mathcal{O}_X) \\ &= \dim H^1(F \otimes \mathcal{O}(-N) \otimes \mathcal{O}(-mH) \otimes \omega_X) \\ &= -\chi(F \otimes \mathcal{O}(-N) \otimes \mathcal{O}(-mH) \otimes \omega_X) \\ &= -P_{F \otimes N^\vee \otimes \omega_X}(-m, H). \end{aligned} \tag{1.29}$$

In particular,

$$\chi(j_N(F)) = -\chi(F \otimes N^\vee) = -\chi(F) + N \cdot c_1(F) - \omega_X \cdot c_1(F), \tag{1.30}$$

and

$$\mu_H(j_N(F)) = -\frac{\chi(F)}{c_1(F) \cdot H} + \frac{N \cdot c_1(F) - \omega_X \cdot c_1(F)}{H \cdot c_1(F)} = -\mu_H(F) + \frac{N \cdot c_1(F) - \omega_X \cdot c_1(F)}{H \cdot c_1(F)}. \tag{1.31}$$

Lemma 1.3.4. *Let H be a line bundle on X. For any $k \in Z$, set $N = H^k \otimes \omega_X$. Then, j_N respects H-(semi)stability.*

Proof. If $H = N \otimes \omega_X$, equation (1.31) becomes

$$\mu_H(j_N(F)) = -\mu_H(F) + k. \tag{1.32}$$

Let G be a quotient of $j_N(F)$ and let $\mathcal{K}$ be its kernel. Applying $\mathcal{H}om(\cdot, \mathcal{O}_X(N))$ to the exact sequence

$$0 \to \mathcal{K} \to \mathcal{E}xt^1(F, \mathcal{O}_X(N)) \to G \to 0.$$

using the fact that $\mathcal{K}$ is pure of dimension one, and that $j_N^2 = id$, we obtain an injection

$$0 \to \mathcal{E}xt^1(G, \mathcal{O}_X(N)) \to F.$$

Since F is H-(semi)stable, $\mu_H(j_H(G))(\leq)\mu_H(F)$ and the proof follows using (1.32). □

In general, j_N does not respect H stability. In order to decide when this is the case, it is useful to be able to describe the restriction of $j_N(F)$ to the sub curves of $\mathrm{Supp}(F)$.

Lemma 1.3.5. *Let N be a line bundle on X and consider a decomposition*

$$\Gamma = \Gamma_1 + \Gamma_2,$$

where Γ_1 and Γ_2 are two reduced curves having no common components and meeting transversally. Then, using the notation defined in (1.11) and in (1.26), we have

$$(j_N(F))_j = j_N(F_j \otimes \mathcal{O}(-\Delta_F)), \;\; j = 1, 2,$$

Proof. Let $i \neq j$, and consider the short exact sequence

$$0 \to F^j \to F \to F_i \to 0. \tag{1.33}$$

From Lemma 1.3.2 it follows that

$$F^j \cong F_j \otimes \mathcal{O}(-\Delta_F).$$

Applying $j_N(\cdot)$, since the lower and higher ext groups vanish for pure dimension one sheaves on a surface, we get

$$0 \to j_N(F_i) \to j_N(F) \xrightarrow{a} j_N(F_j \otimes \mathcal{O}(-\Delta_F)) \to 0.$$

Moreover, since $j_N(F_j \otimes \mathcal{O}(-\Delta_F))$ is torsion free and supported on Γ_j, the morphism a factors

through a surjective morphism

$$(j_N(F))_j \to j_N(F_j \otimes \mathcal{O}(-\Delta_F)).$$

However, this morphism is also injective because it is a generic isomorphism, and the Lemma is proved. □

Remark 1.3.6. *When F is locally free, the above lemma follows immediately from the fact that restriction commutes with taking the dual. Indeed, in this case $j_N(F_j \otimes \mathcal{O}(-\Delta_F)) = \mathcal{E}xt^1(F_j, -\Gamma_j) \otimes \mathcal{O}(\Gamma) \otimes N = F_j^\vee \otimes \mathcal{O}(\Gamma) \otimes N = (F^\vee \otimes \mathcal{O}(\Gamma) \otimes N)_j$.*

The following example shows that duality does not necessarily preserve stability.

Example 1.3.7. Let Γ_1 and Γ_2 be two smooth elliptic curves in X meeting in 2 points. Let Γ be the genus 3 curve that is the union of Γ_1 and Γ_2. A degree zero line bundle F is stable if and only if

$$-2\frac{k_1}{k} < d_1 < -2\frac{k_1}{k} + 2.$$

Let us choose $H = H^+$ so that $-2 < -2\frac{k_1}{k} < -1$. Then d_1 has to be equal to -1 (in which case $d_2 = 1$) or to 0 (in which case $d_2 = 0$). If instead we choose $H = H^-$ so that $-1 < -2\frac{k_1}{k} < 0$, then the multi-degree of H-stable line bundles are $(0, 0)$ and $(1, -1)$. It follows that the dual of a H^+-stable line bundle is always (H^-)-stable. Finally, if we let H_0 be an ample line bundle with $-2\frac{k_1}{k} = -1$, the only possible multi-degree is $(0, 0)$. For simplicity let us assume that the decomposition of Γ in $\mathrm{Pic}(X)$ is like in Example 1.1.29. With this notation, there is a commutative diagram

$$\begin{array}{ccc} M_{v,H^+} & \overset{j_{-\Gamma}}{\hookleftarrow\hookrightarrow} & M_{v,H^-} \\ {\scriptstyle h^+}\downarrow & & \downarrow{\scriptstyle h^-} \\ M_{v,H_0} & \overset{j_{-\Gamma}}{\hookleftarrow\hookrightarrow} & M_{v,H_0} \end{array}$$

The vertical morphism h^+ contracts the degree $(-1, 1)$ component of $\overline{\mathrm{Jac}}^0_{H^+}(\Gamma_1 + \Gamma_2)$ (recall that this component is a $\mathbb{P}^1$-bundle over $\mathrm{Jac}^{-1}(\Gamma_1) \times \mathrm{Jac}^1(\Gamma_2)$) to the singular locus of $\overline{\mathrm{Jac}}^0_{H_0}(\Gamma_1 + \Gamma_2)$, which is isomorphic to $\mathrm{Jac}^{-1}(\Gamma_1) \times \mathrm{Jac}^{-1}(\Gamma_2)$. Indeed, any line bundle with multi-degree equal to $(-1, 1)$ is strictly H_0-semistable, and $\mathcal{S}$-equivalent to a sheaf in $\mathrm{Jac}^{-1}(\Gamma_1) \times \mathrm{Jac}^{-1}(\Gamma_2)$.

More generally, we have

Example 1.3.8. Let H and N be line bundles on X, with H ample. Consider a genus g curve $\Gamma = \Gamma_1 + \Gamma_2$ that is the union of two integral curves Γ_1 and Γ_2 of genus equal to g_1 and g_2 respectively

that meet transversally in δ points. Let F be a pure sheaf of rank one on Γ, and as usual set $F_i = F_{\Gamma_i}$, $i = 1, 2$. Set

$$\chi = \chi(F), \quad \chi_i = \chi(F_i), \text{ for } i = 1, 2 \quad \text{and } q = \chi(Q),$$

where Q is the cokernel of the natural injection $F \to F_1 \oplus F_2$ so that

$$\Delta_F := \operatorname{supp}(Q),$$

is the locus of $\Gamma_1 \cap \Gamma_2$ where F is locally free. As usual, set

$$k_1 = H \cdot \Gamma_1, \ \ k_2 = H \cdot \Gamma_2 \quad \text{and } k = H \cdot \Gamma = k_1 + k_2.$$

As in Example 1.1.17 we know that F is H-stable if and only if

$$\frac{\chi}{k} < \frac{\chi_1}{k_1} < \frac{\chi}{k} + \frac{q}{k_1}. \tag{1.34}$$

Now, using Lemma 1.3.5 and formula (1.30), setting $n = \Gamma \cdot N$ and $n_i = \Gamma_i \cdot N$, for $i = 1, 2$, we see that $j_N(F)$ is H-stable if and only if,

$$-\frac{\chi}{k} + \frac{n}{k} - \frac{\omega_X \cdot \Gamma}{k} < -\frac{\chi_1}{k_1} + \frac{n_1}{k_1} - \frac{\omega_X \cdot \Gamma_1}{k} + \frac{q}{k_1} < -\frac{\chi}{k} + \frac{n}{k} - \frac{\omega_X \cdot \Gamma}{k} + \frac{q}{k_1}.$$

The following Lemma ensures that taking the dual works well in families.

Lemma 1.3.9. *Let $\mathcal{F}$ be a flat family of pure sheaves of dimension one on X parametrized by a scheme B. If $p : X \times S \to X$ denotes the natural projection, then $\mathcal{E}xt^1(\mathcal{F}, p^*\mathcal{O}_X)$ is a flat family of pure dimension one sheaves on X parametrized by X, and for every $s \in S$, there is an isomorphism $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{X\times S})_s \cong \mathcal{E}xt^1(\mathcal{F}_s, \mathcal{O}_S)$.*

Proof. The lemma follows from [3] in the following way. First, notice that for every $b \in B$ we have $\mathcal{E}xt^i_X(\mathcal{F}_b, \mathcal{O}_X) = 0$, for $i = 0, 2$. Then point (ii) of Theorem (1.10) in loc. cit. implies that $\mathcal{E}xt^1(\mathcal{F}, p^*\mathcal{O}_X)$ is flat and that the base change map $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{X\times S})_b \to \mathcal{E}xt^1(\mathcal{F}_s, \mathcal{O}_X)$ is an isomorphism. □

Let H be a polarization on X and let H be an H-stable sheaf. Set $v = v(F) = (0, [\Gamma], v_2)$ and choose

a line bundle N. The above Lemma guarantees that, setting

$$w := v(j_N(F)) = (0, [\Gamma], ch_2(j_N(F))),$$

we can define a birational involution

$$\begin{aligned} M_{v,H} &\dashrightarrow M_{w,H}, \\ F &\longmapsto j_N(F). \end{aligned} \tag{1.35}$$

that is compatible with the fibration structures $M_{v,H} \to \mathcal{H}_{[\Gamma]}$ and $M_{w,H} \to \mathcal{H}_{[\Gamma]}$. For example, we could choose N so that

$$\chi(F) = \frac{N \cdot \Gamma - \omega_X \cdot \Gamma}{2},$$

in order to have $v = w$ (cf. equation (1.30)), and get a birational involution

$$j_N : M_{v,H} \dashrightarrow M_{v,H},$$

which is defined on an open set containing the locus of sheaves with integral support. Notice that if $N = H \otimes \omega_X^\vee$ then this morphism is regular and it induces automorphisms of the fibers of the support morphism.

We should stress the fact that when $v = (0, [\Gamma], v_2)$, with $v_2 = -g(\Gamma) + 1 - \frac{1}{2}c_1(X) \cdot \Gamma$ and $N = -\Gamma$, then

$$j = j_{-\Gamma} : M_{v,H}\big|_U \to M_{v,H}\big|_U,$$

where $U \subset \{\Gamma\}$ is the open subset of smooth curves, is just the relative version of "-1" on the degree zero Jacobians.

Example 1.3.10. Consider a genus g curve $\Gamma = \Gamma_1 + \Gamma_2$ as in Example 1.1.17, and suppose that $\delta = \Gamma_1 \cdot \Gamma_2 = 2\nu$, for some $\nu \geq 2$, and fix an integer $\chi = d - g + 1$ with $d \neq 0$. Let N be a line bundle on X such that

$$\chi = \frac{N \cdot \Gamma - \omega_X \cdot \Gamma}{2}.$$

Assume that

$$H = N \otimes \omega_X^\vee,$$

is ample (or that it is anti-ample in which case we can set $H = N^\vee \otimes \omega_X$) and that

$$\frac{k_1}{k}\chi = \frac{1}{2}H \cdot \Gamma_1 \in \mathbb{Z},$$

so that H is not v-generic (for example, this is the case if $\chi = -g+1$ and $N = \Gamma$). Then j_N acts on $\overline{\mathrm{Jac}}^d_H(\Gamma)$, which by (2) of Example 1.1.17 has $\delta - 1 = 2\nu - 1$ irreducible components. Indeed, these components are the connected components of $\mathrm{Jac}^d_H(\Gamma)$: in our setting (1.17) becomes

$$\frac{k_1}{2} < \chi_1 < \frac{k_1}{2} + 2\nu,$$

and hence we see that the possible Euler characteristics of F_1 are given by

$$\chi_1 = \frac{k_1}{2} + \epsilon, \quad \text{with } \epsilon = 1, \ldots, 2\nu - 1.$$

In other words, the irreducible components of $\mathrm{Jac}^d_H(\Gamma)$ are parametrized by the pairs (χ_1, χ_2),

$$(\frac{k_1}{2}+1, \chi + 2\nu - \frac{k_1}{2} - 1), (\frac{k_1}{2}+2, \chi + 2\nu - \frac{k_1}{2} - 2), \ldots, (\frac{k_1}{2} + 2\nu - 1, \chi + 2\nu - \frac{k_1}{2} - 2\nu + 1). \quad (1.36)$$

By construction, j_N preserves H-stability. We wish to study the action of the involution

$$j_N : \overline{\mathrm{Jac}}^d_H(\Gamma) \to \overline{\mathrm{Jac}}^d_H(\Gamma).$$

on the irreducible components of $\mathrm{Jac}^d_H(\Gamma)$. Using Lemma 1.3.5 and the fact that F is locally free, we see that $(j_N(F))_1 = j_N(F_1 \otimes \mathcal{O}(-\Gamma_2)) = j_N(F_1) \otimes \mathcal{O}(\Gamma_2)$. Using (1.30) we see the action of j_N on the components of $\mathrm{Jac}^d_H(\Gamma)$ is given by

$$\chi_1 \mapsto -\chi_1 + k_1 + 2\nu,$$

i.e.,

$$(\frac{k_1}{2} + \epsilon, \chi + 2\nu - \frac{k_1}{2} - \epsilon) \mapsto (\frac{k_1}{2} + 2\nu - \epsilon, \chi - \frac{k_1}{2} - \epsilon).$$

In particular, we see that only the middle component (i.e., $\epsilon = \nu$) is preserved by j_N.

1.4 Stability and pullbacks via finite covers

In this subsection we will deal with how stability behaves under cyclic covers. Most of these results are adaptations to the pure dimension one case of the corresponding results for vector bundles of [69] or of [34]. It is surprising that in some sense, under pullback, stable pure dimension one sheaves behave better than stable vector bundles. Let

$$f : Y \to X,$$

be a cyclic cover of order r, and let η be the line bundle inducing it, so that

$$f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \eta \oplus \cdots \oplus \eta^{r-1}.$$

The first lemma guarantees that semi-stability is preserved if we choose an appropriate polarization.

Lemma 1.4.1. *Let F be a pure dimension one sheaf on X. Let A be an ample line bundle on X. If F is A-semistable, then f^*F is ${}^*A$-semistable on Y.*

Proof. First notice that since η is numerically trivial, tensoring a sheaf by η does not change the numerical invariants of the sheaf itself. In particular, the operation of tensoring by η preserves not only the slope, but also stability and semi-stability with respect to any line bundle.
Let $G \subset f^*F$ be a sub sheaf. Using the projection formula for f, we see that f_*G is a sub sheaf of the A-semi stable sheaf $F \oplus (F \otimes \eta) \oplus \cdots \oplus (F \otimes \eta^{r-1})$. Clearly, $\chi(G) = \chi(f_*G)$. Moreover, since G is pure of dimension one, $f_*c_1(G) = c_1(f_*G)$. Set $H = f^*A$, then $c_1(G) \cdot H = f_*c_1(G) \cdot A$, so that $\mu_H(G) = \mu_A(f_*G)$. Since $\mu_A(F) = \mu_{H'}(F \oplus (F \otimes \eta) \oplus \cdots \oplus (F \otimes \eta^{r-1}))$, the lemma is proved. □

Lemma 1.4.2. *Let E and F be two non isomorphic pure dimension one sheaves on S with the same slope. Suppose they are both A-stable and that $f^*E \cong f^*F$. Then,*

$$F \cong E \otimes \eta^j, \quad \text{for some } j = 1, \ldots, r-1.$$

Proof. If $f^*E \cong f^*F$, then we also have an isomorphism $E \otimes f_*\mathcal{O}_Y = f_*f^*E \cong f_*f^*F = F \otimes f_*\mathcal{O}_Y$. Since all maps from E to F are trivial, it follows that there is a $j = 1, \ldots, r-1$ such that the composition $E \to E \otimes f_*\mathcal{O}_Y \to F \otimes \eta^j$ is non zero. But E and $F \otimes \eta^j$ are stable of the same slope,

and hence

$$E \cong F \otimes \eta^j$$

□

It is then natural to ask wether stability, and not just semistability, is also preserved.

Lemma 1.4.3. *[69] Let F be a sheaf on X. If $F \cong F \otimes \eta^j$ for some $j \neq 0$, then f^*F is not simple. In particular, it cannot be stable.*

Proof. We have

$$\begin{aligned} \mathrm{Hom}(f^*F, f^*E) = \mathrm{Hom}(F, f_*f^*F) = \\ = \mathrm{Hom}(E, E) \oplus \mathrm{Hom}(E, E \otimes \eta) \oplus \cdots \oplus \mathrm{Hom}(E, E \otimes \eta^{r-1}) \cong \end{aligned} \tag{1.37}$$

Thus $\mathrm{Hom}(E, E) \oplus \mathrm{Hom}(E, E \otimes \eta^j) \subseteq \mathrm{Hom}(f^*E, f^*E)$, and the Lemma is proved. □

From now on we assume that $f : X \to Y$ is an étale double cover.

Lemma 1.4.4. *Let $f : X \to Y$ be an étale double cover, let $\iota : X \to X$ be the involution inducing the cover, and let H be a polarization on X. Let $D \subset X$ be a ι^*-invariant curve, χ a non zero integer and set $v = (0, [D], \chi - \omega_X \cdot D/2)$. There is a birational involution*

$$\begin{aligned} \iota^* : M_{v,H}(\{D\}) \dashrightarrow M_{v,H}(\{D\}), \\ F \longmapsto \iota^*F, \end{aligned}$$

that is compatible with the support morphism $\pi : M_{v,H}(\{D\}) \to \{D\}$ and the natural ι^*-action on $\{D\}$. If H is ι^*-invariant, then ι^* is a regular morphism.

Proof. It is clear that if F is an H-semistable pure sheaf with $v(F) = v$, then $\iota^*(F)$ is ι^*H-semistable with $v(\iota^*F) = v$. □

Corollary 1.4.5. *Assume that H is ι^*-invariant, and set*

$$M = M_{v,H}(\{D\}).$$

Let Z be any component of $\mathrm{Fix}(\iota^*) \subset M$. Then $Z \cap (M \setminus \mathrm{Sing}(M))$ is smooth.

Lemma 1.4.6. *Let B be a scheme, and let $\mathcal{F}$ on $Y \times B$ be a flat family of pure dimension one sheaves on Y. Then $f_B^*\mathcal{F}$ is a flat family of pure dimension one sheaves on X.*

Proof. Since f is a flat morphism, we only need to prove that for any pure dimension one sheaf F on Y, the pullback f^*F is pure of dimension one. Let $Q \subset f^*F$ be a sub sheaf with zero-dimensional support. Since $f_*Q \subset F \oplus F \otimes \omega_Y$, it follows that f_*Q, and thus Q, is the zero sheaf. □

As expected, the fixed locus of ι^*, if non empty, is related to some moduli space of sheaves on Y. This Lemma, together with Lemma 1.4.1 above, guarantees that we can indeed define a pullback morphism.

Let A be an ample line bundle on Y and let G be a pure sheaf of dimension one on Y. Let

$$w = (0, C, \chi - \frac{(\omega_Y \cdot C)}{2}),$$

be the Mukai vector of G, suppose that $\chi \neq 0$, and set

$$N = M_{w,A}(Y, \{C\}).$$

Set $D = f^{-1}(C)$,

$$H = f^*A, \qquad v = v(f^*G) = (0, D, 2\chi - (\omega_Y \cdot C).$$

and

$$M = M_{v,H}(X, \{D\}).$$

With this notation we can define the pullback morphism

$$\begin{gathered} \Phi : N = M_{w,A}(Y, \{C\}) \longrightarrow M = M_{v,H}(X, \{D\}, \\ [F] \longmapsto [f^*F] \end{gathered} \tag{1.38}$$

which factors through the fixed locus,

$$\Phi : N \to \mathrm{Fix}(\iota^*) \subset M,$$

and commutes with the support morphisms $N \to \{C\}$ and $M \to \{D\}$. If there is a sheaf G in N such that $G \ncong G \otimes \eta$, then Φ is generically $2:1$ and ramifies on the closed locus of sheaves such that $G \cong G \otimes \eta$.

Recall that since η is numerically trivial, tensoring by η preserve stability. It follows that if there is

a sheaf G in N such that $G \ncong G \otimes \eta$, then the assignment

$$G \mapsto G \otimes \eta,$$

defines an involution

$$\epsilon : N \to N, \tag{1.39}$$

such that

$$\Phi(N) \cong N/\langle \epsilon \rangle.$$

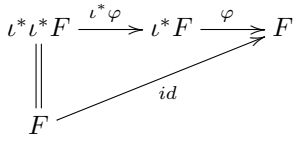
To determine the image of Φ we need the following

Lemma 1.4.7. *[34] Let F be a pure sheaf of dimension one on X and assume that it is ι^*-invariant. If F is simple, then*

$$F = f^*(G),$$

for some sheaf G on Y.

Proof. Since Y is a quotient of X by a $\mathbb{Z}/(2)$ action, the descent data translates into the existence of a morphism $\varphi : \iota^* F \to F$, such that the following diagram is commutative



Since F is simple this can always be achieved by multiplying any isomorphism $\iota^* F \to F$ by a suitable scalar. □

Chapter 2

Relative Prym varieties

2.1 The relative Prym variety

Given an étale double cover $f : X \to Y$ of smooth projective surfaces, consider curves $C \subset Y$ and $D = f^{-1}(D) \subset X$, and let $f : \mathcal{D} \to \mathcal{C}$ be the induced cover between two families of deformations of C and of D. In this section we generalize the notion of Prym variety of $D \to C$ and we construct the relative Prym variety of $\mathcal{D} \to \mathcal{C}$. We then describe in detail some example of Prym varieties for singular curves. The case where $X \to Y$ is a K3 to Enriques double cover will be fully studied in Chapter 3.

2.1.1 First definitions

We recall the classical definition of Prym variety. Let C be a smooth genus $g \geq 2$ curve, and let

$$f : D \to C,$$

be an étale double cover and let $\iota : D \to D$ be the covering involution. Then D is a smooth curve of genus $h = 2g - 1$. The involution ι^* acts on the Jacobian variety $\mathrm{Jac}(D)$, and the *Prym variety of D over C* is defined by

$$\mathrm{Prym}(D/C) := \mathrm{Fix}^\circ(-\iota^*) = \ker(1 + \iota^*)^\circ \subset \mathrm{Jac}^0(D), \tag{2.1}$$

where the superscript $^\circ$ stands for the identity component. Following Mumford [49], one can also define the Prym variety as the identity component of the kernel of norm map,

$$\begin{aligned} \mathrm{Nm} : \mathrm{Jac}^0(D) &\longrightarrow \mathrm{Jac}^0(D), \\ L = \mathcal{O}_D\left(\sum a_i x_i\right) &\longmapsto \det(f_* L) = \mathcal{O}_C\left(\sum a_i f(x_i)\right) \end{aligned} \tag{2.2}$$

or, equivalently, as the image of $(1-\iota^*) : \mathrm{Jac}^0(D) \to \mathrm{Jac}^0(C)$. From this it also follows that

$$\mathrm{Prym}(D/C) = H^1(D, \mathcal{O}_D)^- / H^1(D, \mathbb{Z})^-. \tag{2.3}$$

In [48] Mumford shows, among other things, that the decomposition in connected components of $\ker(\mathrm{Nm})$ is given by

$$\ker(\mathrm{Nm}) = (1-\iota^*)(\mathrm{Jac}^0(D)) \coprod (1-\iota^*)(\mathrm{Jac}^1(D)). \tag{2.4}$$

One can show (cf. [49] §6) that

$$f^*(\mathrm{Nm}(L)) = (1+\iota^*)L,$$

and since $\ker(\mathrm{Nm})$ and $\ker f^*$ both have two connected components, $\ker(1+\iota^*)$ has four connected components. In other words,

$$\ker(1+\iota^*) \cong \mathrm{Nm}^{-1}(0) \coprod \mathrm{Nm}^{-1}(\eta), \tag{2.5}$$

where $\eta \in \mathrm{Pic}^0(C)$ is the 2-torsion line bundle defining the double cover $D \to C$. Notice that in formula (2.4) and for any $d \in \mathbb{Z}$ we could put $\mathrm{Jac}^{2d}(D)$ and $\mathrm{Jac}^{2d+1}(D)$ in place of $\mathrm{Jac}^0(D)$ and of $\mathrm{Jac}^1(D)$, respectively. The Prym variety is a $(g-1)$-dimensional principally polarized abelian variety.

Our next aim is to give a relative version of this construction, whenever the curves C and D move in a family. The context in which we would like to carrie this out is the following. Consider an étale double cover of smooth projective surface

$$f : X \to Y,$$

let $C \subset X$ be a smooth curve of genus $g \geq 2$, set $D = f^{-1}(C)$ and $h = g(D) = 2g-1$. We let,

$$\iota : X \to X,$$

be the involution associated to the cover. Via the pullback f^*, we view $f^*\{C\}$ as a subvariety of $\{D\}$ and by abuse of notation we will write

$$\{C\} \subset \{D\}.$$

Let $\mathcal{C} \to \{C\}$ and $\mathcal{D} \to f^*\{C\} \cong \{C\}$ be the two universal families, and consider the relative cover

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & \mathcal{C} \\ & \searrow \quad \swarrow & \\ & \{C\} & \end{array}$$

Let F be a rank one torsion free sheaf on D with $\chi(F) = \chi = -h+1$, and set

$$v = v(F) = (0, [D], \chi + \frac{1}{2}\omega_X \cdot D).$$

Finally set

$$H_D := \mathcal{O}_X(D) \otimes \omega_X,$$

and assume that it is ample[1]. By Lemmas 1.3.4, 1.3.9, and 1.4.4 and formula (1.30), the assignment

$$F \longmapsto \tau_{-D}(F) := \mathcal{E}xt^1(\iota^* F, \mathcal{O}(-D))$$

defines a regular involution

$$\tau_{-D} : M_{v,H_D}(\{D\}) \longrightarrow M_{v,H_D}(\{D\}), \tag{2.6}$$

that commutes with the support morphism. We define the relative Prym variety of $\mathcal{D}$ over $\mathcal{C}$ as

$$\mathrm{Prym}_{v,H_D}(\mathcal{D}, \mathcal{C}) := \mathrm{Fix}^\circ(\tau) \subset M_{v,H_D}, \tag{2.7}$$

where the superscript $\circ$ indicates that we are considering the irreducible component of $\mathrm{Fix}^\circ(\tau)$ that

[1] If $\mathcal{O}_X(D) \otimes \omega_X$ is anti-ample, then we can set $H_D := (\mathcal{O}_X(D) \otimes \omega_X)^\vee$ and everything works in the same way.

contains the image of the restriction of the rational section (1.9) to $\{C\}$. We will also use the notation

$$P_{v,H_D} = P_{v,H_D}(\{C\}) = \mathrm{Prym}_{v,H_D}(\mathcal{D},\mathcal{C}).$$

Summing up, we have

Proposition 2.1.1. *The relative Prym variety $P_{v,H_D}(\{C\})$ is an irreducible variety of dimension*

$$\dim\{C\} + g - 1.$$

which comes with a surjective morphism,

$$\eta : P_{v,H_D}(\{C\}) \to \{C\},$$

whose general fiber is a principally polarized abelian variety of dimension $g-1$. The smooth locus of P_{v,H_D} contains the open locus $P_{v,H_D} \cap M^{reg}_{v,H_D}(\{C\})$, where $M^{reg}_{v,H_D}(\{C\})$ denotes the smooth locus of the moduli space $M_{v,H_D}(\{C\})$.

Proof. The morphism η is just the restriction of $\pi : M_{v,H_D} \to \{D\}$ to P_{v,H_D}. It is clear that this restriction factors through the inclusion $\{C\} \subset \{D\}$. Moreover, it is surjective since it is proper and its image contains the dense open subset of smooth curves. It is clear that if $[C] \in \{C\}$ is a smooth curve the fiber $\eta^{-1}(C)$ is the Prym variety (2.1). Lastly, the statement about the smooth locus is a consequence of the well known fact that the fixed locus of an automorphism of finite order on a manifold is smooth. □

Notice that since H_D is not v-generic, the strictly semi-stable locus of the moduli space M_{v,H_D} will in general be non empty. If X is a K3 or an abelian surface or if $D \cdot \omega_X < 0$ then M^{reg}_{v,H_D} contains (and will in general be equal to) the stable locus.

Clearly, one can consider the linear systems $|C|$ and $|D|$ instead of the Hilbert schemes and define, in the obvious way, the relative Prym variety

$$M_{v,H_D}(|C|) \supset P_{v,H_D}(|C|) \to |C|.$$

2.2 Prym varieties of singular curves

In this section we define and study the Prym varieties for an étale double cover

$$f : D \to C,$$

in a few cases where C is a *reduced* but mildly singular curve of genus g. These results will be used in Chapter 3 to describe the fibers of the relative Prym variety. Prym varieties of double covers of singular curves have also been considered in [27]. Set $h = g(D) = 2g - 1$. To fix notation, let $m : \widetilde{C} \to C$ and $n : \widetilde{D} \to D$ be the respective normalizations. Let $\widetilde{f} : \widetilde{D} \to \widetilde{C}$ be the induced double cover and let,

$$\widetilde{\iota} : \widetilde{D} \to \widetilde{D},$$

be the corresponding involution. With this notation there is a commutative diagram

$$\begin{array}{ccc} \widetilde{D} & \xrightarrow{n} & D \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{C} & \xrightarrow{m} & C \end{array}$$

If D is irreducible, then stability does not depend on the choice of a polarization and $\overline{\mathrm{Jac}}^0(D)$ is irreducible. In particular, the locus $\mathrm{Jac}^0(D)$ of line bundles is a group, and hence "taking the dual" is a well defined operation. On the other hand, if D is not irreducible, we need to use a polarization H in order to select a finite number of components of $\mathrm{Jac}^0(D)$, the one parametrizing the stable line bundles. As we saw in Examples 1.3.7 and 1.3.8, the dual of an H-stable line bundle is not necessarily H-stable so that, $\mathrm{Jac}^0_H(D)$ is not necessarily a group. However, we also saw examples where H-stability is preserved and $\mathrm{Jac}^0_H(D)$ is indeed a group (for e.g. in Example 1.3.7 when $H = H_0$).

Let us then consider a polarization H on D that is ι^*-invariant and such that $\mathrm{Jac}^0_H(D)$ is a group. Under these assumptions, the involution

$$\begin{aligned} \tau : \mathrm{Jac}^0_H(D) &\longrightarrow \mathrm{Jac}^0_H(D), \\ F &\longmapsto \iota^* F^\vee \end{aligned}$$

is well defined, and we can consider its fixed locus $\mathrm{Fix}(\tau) \subset \mathrm{Jac}^0_H(D)$. Clearly,

$$\mathrm{Fix}(\tau) = \ker(1+\iota^*) \subset \mathrm{Jac}^0_H(D).$$

The Prym variety of D over C is defined by

$$\mathrm{Prym}_H(D/C) := \overline{\ker(1+\iota^*)}^\circ \subset \overline{\mathrm{Jac}}^0_H(D).$$

In other words, we consider the locus of τ-invariant H-stable line bundles, we close it in the compactified Jacobian of H-semi stable rank one torsion free sheaves and then we take the connected component containing the identity. Notice that since there might be multiple components, it is important that first we take the closure and then the connected component containing the identity.

If the $f : X \to Y$ is a double cover of smooth projective surfaces, and if $C \subset Y$ and $D \subset X$, with $H = \omega_X \otimes \mathcal{O}(D)$ ample, then the Prym variety defined above is just the fiber over $[C] \in \{C\}$ of $P_{v,H_D} \to \{C\}$.

2.2.1 Irreducible nodal case

Here we consider the case where C is an irreducible curve with δ nodes $c_1, \ldots, c_\delta$ and no other singularities. Under this assumption, let

$$p_1, \iota(p_1), \ldots, p_\delta, \iota(p_\delta), \;\; i = 1, \ldots, \delta,$$

be the nodes of D, and finally for $i = 1, \ldots, \delta$ set

$$\{x_i, y_i\} = n^{-1}(p_i), \quad \text{so that} \quad \{\widetilde{\iota} x_i, \widetilde{\iota} y_i\} = n^{-1}(\iota p_i).$$

Recall that for irreducible curves the polarization is irrelevant. With this notation we have the following proposition

Proposition 2.2.1. *Let C be as above and suppose that $D = f^{-1}(C)$ is irreducible. Then $\overline{\ker(1+\iota^*)} \subset \overline{\mathrm{Jac}}^0(D)$ has 4 connected components. Each of these components is isomorphic to a rank δ degeneration of a principally polarized abelian variety.*

Proof. To prove the Proposition, we first consider the intersection

$$\mathrm{Fix}(\tau) \cap \mathrm{Jac}^0(D) = \ker(1+\iota^*)$$

and show that it has four connected components. Then we prove that these components stay disconnected even after passing to the closure in $\overline{\mathrm{Jac}}^0(D)$. For the first step we argue as follows. Since D and C are irreducible and nodal there are natural short exact sequences,

$$0 \to \prod_{i=1}^{\delta} \mathbb{C}^*_{p_i} \times \mathbb{C}^*_{\iota p_i} \to \mathrm{Jac}^0(D) \stackrel{n^*}{\to} \mathrm{Jac}^0(\widetilde{D}) \to 0,$$
$$0 \to \prod_{i=1}^{\delta} \mathbb{C}^*_{c_i} \to \mathrm{Jac}^0(C) \stackrel{m^*}{\to} \mathrm{Jac}^0(\widetilde{C}) \to 0.$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{i=1}^{\delta} \mathbb{C}^* & \longrightarrow & \ker(1+\iota^*) & \xrightarrow{\beta} & \ker(1+\widetilde{\iota}^*) & & \\
 & & \downarrow{\scriptstyle \zeta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i=1}^{\delta} \mathbb{C}^*_{p_i} \times \mathbb{C}^*_{\iota p_i} & \longrightarrow & \mathrm{Jac}^0(D) & \xrightarrow{n^*} & \mathrm{Jac}^0(\widetilde{D}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle (1+\iota^*)} & & \downarrow{\scriptstyle 1+\widetilde{\iota}^*} & & \\
0 & \longrightarrow & \ker(\alpha) & \longrightarrow & (1+\iota^*)\,\mathrm{Jac}^0(D) & \xrightarrow{\alpha} & (1+\widetilde{\iota}^*)\,\mathrm{Jac}^0(\widetilde{D}) & \longrightarrow & 0
\end{array} \tag{2.8}$$

Since $\prod_{i=1}^{\delta} \mathbb{C}^*$ is connected and since $\ker(1+\widetilde{\iota}^*)$ has four connected components (recall that $\widetilde{D}$ and $\widetilde{C}$ are smooth), to prove that $\ker(1+\iota^*)$ has four connected components it is sufficient to prove that β is surjective, i.e., that γ is surjective. By Lemma 1.4.7, it follows that every ι^*-invariant line bundle on D is the pull-back via f^* of a line bundle on C. It follows that

$$(1+\iota^*)\,\mathrm{Jac}^0(D) \subset f^*\,\mathrm{Jac}^0(C),$$

so that we can consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\alpha) & \longrightarrow & (1+\iota^*)\,\mathrm{Jac}^0(D) & \xrightarrow{\alpha} & (1+\widetilde{\iota}^*)\,\mathrm{Jac}^0(\widetilde{D}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \prod_{i=1}^{\delta} \mathbb{C}^* & \longrightarrow & f^*\,\mathrm{Jac}^0(C) & \longrightarrow & \widetilde{f}^*\,\mathrm{Jac}^0(\widetilde{C}) & \longrightarrow & 0.
\end{array}$$

Here, the fact that the kernel of the surjection $f^*\,\mathrm{Jac}^0(C) \to \widetilde{f}^*\,\mathrm{Jac}^0(\widetilde{C})$ is equal to $\prod_i \mathbb{C}^*$ follows

from the fact that the kernel of $\mathrm{Jac}^0(C) \to f^* \mathrm{Jac}^0(C)$ and of $\mathrm{Jac}^0(\widetilde{C}) \to \widetilde{f}^* \mathrm{Jac}^0(\widetilde{C})$ are both equal to $\mathbb{Z}/(2)$. Notice, however, that $\mathrm{Im}\,\gamma \cong \prod_{i=1}^{\delta} \mathbb{C}^*$, so that the series of inclusions

$$\prod_{i=1}^{\delta} \mathbb{C}^* \subset \ker(\alpha) \subset \prod_{i=1}^{\delta} \mathbb{C}^*,$$

is in fact a series of equalities. Thus γ is surjective and, as a consequence, β too is surjective.

To finish the proof, we just need to show that when we take the closure of $\ker(1+\iota)$ in the compactified Jacobian $\overline{\mathrm{Jac}}^0(D)$, the number of connected components does not change. It is well known (cf. for example [54]) that, in order to compactify $\mathrm{Jac}^0(D)$, one first compactifies the $(\prod_{i=1}^{\delta} \mathbb{C}^*_{p_i} \times \mathbb{C}^*_{\iota p_i})$-bundle over $\mathrm{Jac}^0(D)$ to a $(\prod_{i=1}^{\delta} \mathbb{P}^1 \times \mathbb{P}^1) = (\prod_{s \in \mathrm{Sing}(D)} \mathbb{P}^1_s)$-bundle over $\mathrm{Jac}^0(D)$. In loc. cit. Oda and Seshadri show that this bundle is the normalization of $\overline{\mathrm{Jac}}^0(D)$. Let us denote by Z this normalization. In order to obtain $\overline{\mathrm{Jac}}^0(D)$ we identify, following Oda and Seshadri [54], the boundary components of Z in the following way: for any 2δ-uple,

$$\Xi = \{\varepsilon_1, \varepsilon_1', \ldots, \varepsilon_\delta, \varepsilon_\delta'\}, \quad \text{with} \;\; \varepsilon_i, \varepsilon_i' \in \{1, -1\}, \;\; i = 1, \ldots, \delta$$

we consider the section

$$s_\Xi : \mathrm{Jac}^0(\widetilde{D}) \to Z,$$

defined by taking the 0-section of the j-th component of $\mathbb{P}^1_{p_1} \times \mathbb{P}^1_{\iota p_1} \times \cdots \mathbb{P}^1_{p_\delta} \times \mathbb{P}^1_{\iota p_\delta}$ if the j-th component of Ξ is equal to 1, and taking the ∞-section if it is equal to -1. The sections corresponding to Ξ and to $-\Xi$ (which are both identified to $\mathrm{Jac}^0(\widetilde{D})$) are then glued under a twist by the degree zero line bundle,

$$L_\Xi := \mathcal{O}_{\widetilde{D}} \left(\sum_i \varepsilon_i (x_i - y_i) + \sum_i \varepsilon_i' \, \widetilde{\iota}\,(x_i - y_i) \right).$$

Now observe that the map ζ in diagram (2.8) sends $(\lambda_1, \ldots, \lambda_\delta)$ to $(\lambda_1, \frac{1}{\lambda_1} \ldots, \lambda_\delta, \frac{1}{\lambda_\delta})$. Sending the λ_i's to zero, or infinity, it follows that the closure of $\ker(1 + \iota^*)$ in the normalization of $\overline{\mathrm{Jac}}^0(D)$ has four connected components, each of which intersects the image of the section s_Ξ if and only if

$$\varepsilon_i = -\varepsilon_i', \quad \text{for } i = 1, \ldots, \delta.$$

In this case then, by (2.4), L_Ξ belongs to the identity component of $\ker(1 + \widetilde{\iota}^*)$. It follows that tensoring by L_Ξ preserves each connected component of $\ker(1 + \widetilde{\iota}^*) \subset \mathrm{Jac}^0(\widetilde{D})$, and hence we may conclude that the fixed locus of τ on $\overline{\mathrm{Jac}}^0(D)$ has four connected components. $\square$

For later use we highlight the following Corollary.

Corollary 2.2.2. *Let C be an irreducible curve with one node and no other singularity, and let $D = f^{-1}(C)$ be the corresponding integral curve with two nodes. Then* $\mathrm{Prym}(D/C)$ *is a rank one degeneration of an abelian variety.*

2.2.2 Two reducible cases

The following example shows that the situation when C and D are not irreducible is slightly different. In Section 3.4.1 we will give an alternative, and more hands on, proof of the main result of this subsection in the case the curve C is the union of two smooth elliptic curves meeting in one point.

Let us consider the case where the curves $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$ are union of two smooth components intersecting transversally in δ (resp. 2δ) points. For simplicity we assume $\delta = 1$ so that $C_1 \cdot C_2 = 1$. Set

$$D_1 \cap D_2 = \{p, \iota p\},$$

and let $\{p_1, \iota p_1\}$ and $\{p_2, \iota p_2\}$ be the pair $\{p, \iota p\}$ viewed on D_1 and D_2 respectively. To fix ideas, we assume that the curve D is contained in a surface and we consider the case where the polarization is equal to[2]

$$H_D = \mathcal{O}(D) \otimes \omega_X.$$

so that it is not v-generic and since $\delta = 1$, by Example 1.1.17, $\mathrm{Jac}^0_{H_D}(D)$ is connected and hence a group. Also, for $i = 1, 2$ let

$$\iota_i : D_i \to D_i,$$

be the involution corresponding to the double cover $D_i \to C_i$, and let η_i be the line bundle defining the cover itself. In this case diagram (2.8) becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \ker(1+\iota^*) & \xrightarrow{\beta} & \ker(1+\widetilde{\iota}^*) & & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{Jac}^0(D) & \xrightarrow{n^*} & \mathrm{Jac}^0(\widetilde{D}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma=0} & & \downarrow{\scriptstyle (1+\iota^*)} & & \downarrow{\scriptstyle 1+\widetilde{\iota}^*} & & \\
0 & \longrightarrow & \ker(\alpha) & \longrightarrow & (1+\iota^*)\,\mathrm{Jac}^0(D) & \xrightarrow{\alpha} & (1+\widetilde{\iota}^*)\,\mathrm{Jac}^0(\widetilde{D}) & \longrightarrow & 0
\end{array}$$

[2]Without assuming that D is contained in a surface, we could just choose a ι^*-invariant polarization on D with multi degree $(k_1, k_2) = (2h_1 - 2 + 2\delta, 2h_2 - 2 + 2\delta)$, where for $i = 1, 2$ we have set $h_i = g(D_i)$.

and gives an exact sequence

$$0 \to \mathbb{C}^* \to \ker(1+\iota^*) \xrightarrow{\beta} \ker(1+\widetilde{\iota}^*) \to \ker(\alpha) \to 0,$$

so that to determine the number of connected components of $\ker(1+\iota^*) \subset \mathrm{Jac}^0(D)$ we need to compute $\ker(\alpha)$. Since in the case at hand, $\mathrm{Jac}^0(\widetilde{D}) \cong \mathrm{Jac}^0(D_1) \times \mathrm{Jac}^0(D_2)$, the fixed locus $\ker(1+\widetilde{\iota}^*)$ has 16 connected components. We claim that, $\ker(1+\iota^*)$ has 8 connected components. By what we said, this is equivalent to showing that $\ker(\alpha) \cong \mathbb{Z}/(2)$. To this aim, consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\alpha) & \longrightarrow & (1+\iota^*)\,\mathrm{Jac}^0(D) & \xrightarrow{\alpha} & (1+\widetilde{\iota}^*)\,\mathrm{Jac}^0(\widetilde{D}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \ker(\rho) \cong \mathbb{Z}/(2) & \longrightarrow & f^*\,\mathrm{Jac}^0(C) & \xrightarrow{\rho} & \widetilde{f}^*\,\mathrm{Jac}^0(\widetilde{C}) & \longrightarrow & 0 \\
 & & & & \uparrow\uparrow & & \uparrow\uparrow & & \\
 & & 0 & \longrightarrow & \mathrm{Jac}^0(C) & \xrightarrow{\sim} & \mathrm{Jac}^0(\widetilde{C}) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & \mathbb{Z}/(2)\times\mathbb{Z}/(2) & \longrightarrow & \mathbb{Z}/(2).
\end{array}$$

Suppose for a moment that $\ker(\alpha) = (0)$, then α would be an isomorphism and α^{-1} would give a section of the non trivial covering $f^*\,\mathrm{Jac}^0(C) \to f^*\,\mathrm{Jac}^0(\widetilde{C})$. This, however, is absurd and hence

$$\ker(\alpha) = \ker(\rho) \cong \mathbb{Z}/(2),$$

and the claim is proved.

Finally we observe that the closure in $\overline{\mathrm{Jac}}^0_{H_D}(D)$ of $\ker(1+\iota^*)$ is the union of 4 connected components, each of which is the union of two irreducible components. What we will show is that the closure every connected component of $\ker(1+\iota^*)$ intersects the closure of exactly one other connected component. Indeed, $\overline{\mathrm{Jac}}^0_{H_D}(D)$ is obtained from the $\mathbb{C}^*$-bundle over $\mathrm{Jac}^0(\widetilde{D}) = \mathrm{Jac}^0(D_1) \times \mathrm{Jac}^0(D_2)$ by first compactifying to a $\mathbb{P}^1$-bundle and then glueing the 0 and ∞ sections, (both identified with $\mathrm{Jac}^0(\widetilde{D})$), via the twist by line bundle $L := \mathcal{O}_{D_1}(p_1 - \iota p_1) \otimes \mathcal{O}_{D_2}(p_2 - \iota p_2)$. However, by (2.4), the class of $L_1 := \mathcal{O}_{D_1}(p_1 - \iota p_1)$ does not lie in the the identity component of $\ker(1+\iota_1^*)$, even though it lies $\mathrm{Nm}^{-1}(0) \subset \mathrm{Jac}^0(D_1)$. It follows that tensoring by L_1 preserves the two fibers $\mathrm{Nm}^{-1}(0)$ and $\mathrm{Nm}^{-1}(\eta_1)$ (cf. (2.5)), while interchanging the two components of each fiber. The analogous

statement holds for $L_2 := \mathcal{O}_{D_2}(p_2 - \iota p_2)$. We may conclude that tensoring by $L^{\otimes 2}$ preserves every component of $\ker(1+\widetilde{\iota}^*) = \ker(1+\iota_1^*) \times \ker(1+\iota_2^*)$. The action of tensoring by L thus divides these components in pairs, and within each pair the zero and infinity sections of the two components are identified.

A final remark can be made about the points at infinity of $\mathrm{Fix}(\tau)$. When the parameter λ of the $\mathbb{C}^*$-bundle $\ker(1+\iota^*)$ goes to zero or infinity the corresponding line bundle tends to a torsion free sheaf which is $\mathcal{S}$-equivalent to a polystable sheaf of the form $F_1 \oplus F_2$ where, for each $i = 1, 2$, the sheaf F_i is a stable sheaf supported on D_i (cf. Remark 1.1.18)

The previous discussion can be repeated almost word by word in the case in which the two curves C_1 and C_2 meet transversally in $\delta \geq 1$ points. The main difference in the case $\delta \geq 2$ is that $\overline{\mathrm{Jac}}^0_{H_D}(D)$ is not irreducible any more. However, as shown in Example 1.3.10 we see that among the $2\delta - 1$ components of $\mathrm{Jac}^0_{H_D}(D)$ only one them, the identity component, contains $\ker(1+\iota^*)$.

Proposition 2.2.3. *Let $C = C_1 \cup C_2$ (resp. $f^{-1}(C) = D = D_1 \cup D_2$) be the union of two smooth components intersecting transversally in δ (resp. 2δ) points. Then $\overline{\ker(1+\iota^*)} \subset \overline{\mathrm{Jac}}^0_{H_D}(D)$ has 4 connected components, each consisting in two irreducible components meeting at the boundary of $\overline{\mathrm{Jac}}^0_{H_D}(D)$. Moreover each connected component of $\mathrm{Fix}(\tau)$ contains points of the type $[F_1 \oplus F_2]$ where for each $i = 1, 2$ the sheaf F_i is a stable sheaf supported on D_i.*

Using Proposition 2.2.1, one can see that the analogous statement holds also if we assume that the components of C and of D are nodal and irreducible.

The last case we are going to describe is the case where C is irreducible, but D it not.

Proposition 2.2.4. *Let C be an irreducible curve with one node, let $\tilde{C}$ be the normalization of C, and let $f : D \to C$ be an étale double cover such that D is the union of two smooth components D_1 and D_2 that meet transversally in two points and that are interchanged by the involution (and are isomorphic to $\tilde{C}$). Then $\ker(1+\iota^*) \subset \overline{\mathrm{Jac}}^0_{H_D}(D)$ has two connected components both of which are isomorphic to the $\mathrm{Jac}^0(\tilde{C})$. Moreover, if H is a generic polarization (in the sense that $\frac{k_1}{k}\chi \notin \mathbb{Z}$) then $\ker(1+\iota^*) \subset \mathrm{Jac}^0_H(D)$ has 4 connected components each of which is isomorphic to $\mathrm{Jac}^0(\tilde{C})$.*

Proof. Under our assumptions diagram (2.8) becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & \ker(1+\iota^*) & \xrightarrow{\beta} & \mathrm{Jac}^0(D_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{Jac}^0(D) & \xrightarrow{n^*} & \mathrm{Jac}^0(D_1)\times\mathrm{Jac}^0(D_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle (1+\iota^*)} & & \downarrow{\scriptstyle 1+\tilde{\iota}^*} & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & (1+\iota^*)\,\mathrm{Jac}^0(D) & \xrightarrow{\alpha} & \mathrm{Jac}^0(D_1) & \longrightarrow & 0
\end{array}$$

So we see that $\ker(1+\iota^*)$ is equal to two copies of $\mathrm{Jac}^0(D) \cong \mathrm{Jac}^0(\tilde{C})$. Moreover, if we consider a v-generic polarization H, then we should repeat the above diagram for the number of irreducible components of $\overline{\mathrm{Jac}}^0_H(D)$ and we know from Example 1.1.17 that this number is equal to 2. □

The same result holds in the case that the components of D are not smooth (thus they are equal to only a partial normalization of C) but still intersect transversally in smooth points. In this case the components of the Prym varieties are equal to the compactified Jacobian of this partial normalization. Analougsly, we can formulate the proposition in the case where D_1 and D_2 intersect in more than two points.

A corollary of the study of the Prym varieties of these singular curves is the following computation of the number of maximal dimensional components of the fixed locus of τ.

Proposition 2.2.5. *If the linear systems $|C|$ and $|D|$ are sufficiently positive, for e.g. if the locus $Z \subset |C|$ of curves whose double cover $D \to C$ is not as in the statement of Proposition 2.2.1 or of Proposition 2.2.3 is of codimension ≥ 2, then $\mathrm{Fix}(\tau) \subset M_{v,H_D}(|C|)$ has four irreducible components that dominate $|C|$.*

Proof. Let $U \subset |C|$ be the set of curves C whose cover $D \to C$ is such that $\mathrm{Fix}(\tau) \subset \overline{\mathrm{Jac}}^0_{H_D}(D)$ has four connected components. The assumption of the proposition implies that U is simply connected. Now let

$$P_{v,H_D} \to W \xrightarrow{b} |C|,$$

be the Stein factorization of η, so that $W \to |C|$ is finite of degree 4. Since $b^{-1}(U) \subset W$ is a disconnected union of 4 components it follows that W is the union of four distinct irreducible components. □

2.2.3 Remarks on choices of the polarization

There a few possible generalizations, which we now describe, for the definition of the relative Prym variety we gave in (2.7). First of all, we can consider a non zero integer $\chi = d-h+1$ and the Mukai vector

$$v = v(F) = (0, [D], \chi + \frac{1}{2}\omega_X \cdot D).$$

Let N be a ι^*-invariant line bundle on X such that

$$\chi = \frac{N \cdot \Gamma - \omega_X \cdot \Gamma}{2},$$

and let H be an ample divisor. We define a birational involution

$$\begin{aligned} \tau_N : M_{v,H} &\dashrightarrow M_{v,H}, \\ F &\longmapsto \tau_N(F) := \mathcal{E}xt^1(\iota^* F, N), \end{aligned} \tag{2.9}$$

and the fixed locus $\overline{\mathrm{Fix}(\tau_N)} \subset M_{v,H}$ of τ_N as the closure of fixed locus of the restriction of τ_N on a dense open subset were it is biregular. The natural morphism $\mathrm{Fix}(\tau_N) \to |C|$ is generically $4:1$, but if $\chi \neq -h+1$ or if $\mathcal{O}_D(N) \neq \mathcal{O}_D(D)$ there is *a priori* no natural choice of an irreducible component.

However, if $\chi = -h+1$ and $N = \mathcal{O}(-D)$, even if H-stability is not preserved by τ_{-D}, we can set

$$P_{v,H} = \mathrm{Prym}_{v,H_D}(\mathcal{D}, \mathcal{C}) = \overline{\mathrm{Fix}^\circ(\tau_{-D})},$$

where by $\overline{\mathrm{Fix}^\circ(\tau_{-D})}$ we mean that we are considering the fixed locus the restriction of τ_N to an open subset where it is biregular, and then taking the closure of the irreducible component that contains the (closure of) the image of the rational section (1.9).

The advantage of choosing H_D as polarization is that we know that τ_{-D} is regular. This fact guarantees a better control on the local structure of $P_{v,H}$. On the other hand, by choosing the polarization H to be v-generic we may achieve a nicer structure of the moduli space (cf. for e.g. Theorem 1.2.1 and Theorem 1.2.3) while loosing, in general, a control on wether τ_{-D} is regular or not. Indeed, in the next two propositions we are going to see that as soon as the linear system $|C|$ has a member that breaks into the union of two components meeting transversally, the options of a v-generic polarization and of having τ_{-D} respecting H-stability are mutually exclusive. We should

point out that there in Section 3.4 we will see and example of when τ_{-D} does not respect stability, but the *a priori* birational map it defines is in fact a regular morphism.

Proposition 2.2.6. *Let $f : X \to Y$ be an étale double cover of two smooth projective surface. Let $C \subset Y$ be an irreducible curve and set $D = f^{-1}C$. Let χ be a non zero integer and set $v = (0, [D], \chi + \frac{1}{2}\omega_X \cdot D)$. Suppose that there is a member of $\{C\}$ that is the union of two integral components C_1 and C_2 that intersect transversally and whose inverse images $D_1 = f^{-1}(C_1)$ and $D_2 = f^{-1}(C_2)$ are also integral. Let N be a ι^*-invariant line bundle such that*

$$\chi = \frac{N \cdot D - \omega_X \cdot D}{2},$$

(so that τ_N, defined as in 2.9, preserves v and is involutive). If H is a v-generic polarization on X, then τ_N does not preserves H-stability.

Proof. Let H be a v-generic polarization and fix a ι^*-invariant line bundle N such that

$$\chi = \frac{N \cdot D - \omega_X \cdot D}{2},$$

so that τ_N preserves the Mukai vector and is involutive. Set $\delta = D_1 \cdot D_2$ and for $i = 1, 2$ set $h_i = g(D_i)$, and $k_i = H \cdot D_i$. As we saw in Example 1.1.17, the fact that H is v-generic translates in the fact that

$$\frac{\chi}{k} k_1 \notin \mathbb{Z}.$$

and the H-stability criterion for a line bundle F on $D_1 + D_2$ in terms of $\chi_i = \chi(F_i)$ is

$$\frac{k_1}{k}\chi < \chi_1 < \frac{k_1}{k}\chi + \delta.$$

Setting, for $i = 1, 2$, $m_i = N \cdot D_i - \omega_X \cdot D_i$ and $m = m_1 + m_2$, and using our assumption on N, we see that the line bundle $\tau_N(F)$ is H-stable if and only if

$$\chi\frac{k_1}{k} < -\chi_1 + \delta + m_1 < \chi\frac{k_1}{k} + \delta.$$

Let a be the round down of $\chi\frac{k_1}{k}$, so that we can write $m = a + s$, with $0 < s < 1$. If $\tau_N(F)$ is H-stable for every H-stable line bundle F, then

$$a + 1 \leq -\chi_1 + m_1 + \delta \leq a + \delta,$$

has to hold for every χ_1 that satisfies

$$a+1 \leq \chi_1 \leq a+\delta,$$

It follows that asking τ_N to preserve H-stability, forces

$$m_1 = 2a+1.$$

This leads us to a contradiction, since $m_1 = N \cdot D_1 - \omega_X \cdot D_1$ has to be even as it is the intersection number of two ι^*-invariant line bundles $N \otimes \omega_X^\vee$ and $\mathcal{O}_X(D)$. □

Conversely, we have the following

Proposition 2.2.7. *Consider the setting of Proposition 2.2.6. Let H be a polarization of X and suppose that there exist a ι^*-invariant line bundle N such that τ_N preserves v and H-stability. Then H is not v-generic and there exists a H-polystable sheaf with Mukai vector v of the form $F_1 \oplus F_2$ where for $i = 1, 2$, $\mathrm{supp}(F_i) = D_i$ and F_i is H-stable.*

Proof. Indeed, using the notation of the proof of Proposition 2.2.6, one can see that if τ_N preserves H-stability, then

$$\frac{m_1}{2} = \frac{k_1}{k}\chi, \quad \text{and} \quad \frac{m_2}{2} = \frac{k_2}{k}\chi,$$

are integers. It follows that H is not v-generic and that, moreover, we can find H-stable sheaves F_1 and F_2 supported on D_1 and D_2 respectively and such that $\chi(F_i) = \frac{\chi k_i}{k} = \frac{m_i}{2}$. □

Notice that for the statement of the Proposition to be true, we do not really need to require that τ_N is a regular morphism, but only that it is regular on an open subset of $M_{v,H}$ that intersects $\pi^{-1}[D_1 + D_2]$.

Remark 2.2.8. *Let $v = (0, D, \chi)$ and $v' = (0, D, \chi')$ be two Mukai vectors, and let N and N' be two line bundle such that $N \cdot D = \chi$ and $N' \cdot D = \chi'$. Assume that $H = N \otimes \omega_X$ and $H = N' \otimes \omega_X$ are ample, and consider the moduli spaces*

$$M = M_{v,H} \quad \text{and } M' = M_{v',H'},$$

with the involutions $\tau = \tau_N : M \to M$ and $\tau' = \tau_{N'} : M' \to M'$. Suppose that $N \otimes N'^\vee$ is divisible by two in $\mathrm{Pic}(X)$ and let L be such that $L^{\otimes 2} = N \otimes N'^\vee$. It is not hard to check that there is a

commutive diagram,

$$\begin{array}{ccc} M & \xrightarrow{\tau} & M \\ {\scriptstyle \otimes L}\downarrow & & \downarrow{\scriptstyle \otimes L} \\ M' & \xrightarrow{\tau'} & M' \end{array}$$

In particular, tensoring by L *defines an isomorphism of* $\mathrm{Fix}(\tau)$ *with* $\mathrm{Fix}(\tau')$*. Even if* $N \otimes N'^{\vee}$ *is not divisible by two in* $\mathrm{Pic}(X)$*, one can use the same argument to show that* M *and* M' *(and* $\mathrm{Fix}(\tau)$ *and* $\mathrm{Fix}(\tau')$*) are fiberwise isomorphic.*

Chapter 3

The relative Prym variety of a linear system on an Enriques surface

3.1 Linear systems on Enriques and K3 surfaces

In this section we collect a number of results, mostly from [17] and from [16], about curves and linear systems on Enriques and K3 surface and formulate them in view of our needs in the rest of this Chapter and in Chapter 4.

A smooth projective surface T is called an Enriques surface if it is smooth, if $H^1(T, \mathcal{O}_T) = 0$ and if the canonical bundle ω_T is a non trivial 2-torsion element of $\mathrm{Pic}(T)$, i.e. if $\omega_T \neq \mathcal{O}_T$ but $\omega_T^{\otimes 2} = \mathcal{O}_T$.

For the rest of this thesis T will denote an Enriques surface, and

$$f : S \to T, \tag{3.1}$$

will denote the universal cover of T. It is well known that S is a K3 surface. The double cover f is

just the cyclic double cover associated to ω_T. We denote by

$$\iota : S \to S, \tag{3.2}$$

the covering involution. Since there are no holomorphic two forms on T, the involution ι acts as -1 on the space $H^0(S, \omega_S)$ of global sections of the canonical bundle of S, i.e., ι is an anti-symplectic involution.

By a result of Namikawa (Proposition (2.3) of [50]) the ι^*-invariant of $\mathrm{NS}(S)$ is equal to $f^*(\mathrm{NS}(T))$. Moreover, since the pullback

$$f^* : \mathrm{NS}(T) \to \mathrm{NS}(S),$$

is injective, we deduce that $f^* \mathrm{NS}(T)$ is a primitive rank 10 sub-lattice of $\mathrm{NS}(S)$. In particular, the Picard number of S is greater or equal than 10. It is well known that[1]

$$\mathrm{NS}(T) = E_8(-1) \oplus U, \quad \text{and that} \quad H^2(S, \mathbb{Z}) = E_8(-2)^{\oplus 2} \oplus U^{\oplus 3},$$

where E_8 is the positive definite lattice associated to the Dynkin diagram of type E_8, and U is the rank two unimodular hyperbolic lattice. Moreover, if T is general in moduli (Proposition (5.6) of [50]), then

$$\mathrm{NS}(S) \cong f^* \mathrm{NS}(T). \tag{3.3}$$

From now on, when we say that T *is general*, we mean that (3.3) holds. If T is general, then

$$\mathrm{NS}(S) \cong \mathrm{NS}(T)(2)\,.$$

Consequently, the square of any class in $\mathrm{NS}(S)$ is divisible by 4. In particular, S and T do not contain any algebraic -2 classes. Since the arithmetic genus of an irreducible curve Γ in T or in S, is

$$g(\Gamma) = \frac{\Gamma^2}{2} + 1,$$

this means if T is general neither surface contains a smooth rational curve. A surface that does not contain any smooth rational curve is called *unodal*.

[1] Given a lattice Λ and a non-zero integer m, we denote by $\Lambda(m)$ the lattice obtained from Λ by multiplying its non-degenerate integral bilinear form by m.

For any sheaf F on T, we set

$$F' := F \otimes \omega_T. \tag{3.4}$$

Notice that $f^*F \cong f^*F'$. If $C \subset T$ is a curve with $C^2 \geq 0$, we usually denote by C' a section of $\mathcal{O}(C)'$.

Let $C \subset T$ be an irreducible curve of genus g. If $C^2 > 0$, it follows from the Hodge index Theorem that the induced double cover

$$D := f^{-1}(C) \to C,$$

is non trivial and that the line

$$\eta_C := \omega_{T|C},$$

is also non trivial. Notice that in this case

$$h := g(D) = 2g - 1.$$

Before recalling the results we will need on the geometry of the linear systems $|C|$ and $|D|$ we mention a few important facts on the topology of a covering morphism of smooth curves. Let us then consider an étale double cover of smooth curves,

$$f : D \to C,$$

which induces, at the level of the Picard groups a short exact sequence,

$$1 \to \mathbb{Z}/(2) \to \mathrm{Jac}(C) \xrightarrow{f^*} \mathrm{Fix}(\iota_t^*) \subset \mathrm{Jac}(D). \tag{3.5}$$

For every $d \in \mathbb{Z}$ one can show as in Lemma 1.4.7 that the fixed locus $\mathrm{Fix}(\iota^*) \subset \mathrm{Jac}^{2d}(D)$ is exactly the image $f^*(\mathrm{Jac}_d(C))$. For later use, we highlight the following

Remark 3.1.1. In [49] (Section 2, (vi)) it is shown that

$$\mathrm{Fix}(\iota^*) = \mathrm{Im}[(1 + \iota^*) : \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(D)].$$

Now, since

$$\mathrm{Pic}^0(D) = H^1(D, \mathcal{O})/H^1(D, \mathbb{Z}), \quad \mathrm{Pic}^0(C) = H^1(C, \mathcal{O})/H^1(C, \mathbb{Z}),$$

and

$$(1+\iota^*)H^1(D,\mathcal{O}) = H^1(D,\mathcal{O})^+,$$

it follows that

$$\begin{aligned} \mathrm{Fix}(\iota^*) &= f^*(H^1(C,\mathcal{O}))/f^*(H^1(C,\mathcal{O})) \cap H^1(D,\mathbb{Z}) \\ &= H^1(D,\mathcal{O})^+/H^1(D,\mathbb{Z}) \cap H^1(D,\mathcal{O})^+, \end{aligned} \tag{3.6}$$

where $H^1(D,\mathcal{O})^+$ denotes the ι-invariant subspace of $H^1(D,\mathcal{O})$.

The following Lemma will be used in Section 4.1.

Lemma 3.1.2. *Both inclusions*

$$(1+\iota^*)(H^1(D,\mathbb{Z})) \subset f^*H^1(C,\mathbb{Z}), \quad f^*H^1(C,\mathbb{Z}) \subset H^1(D,\mathbb{Z})^+ \tag{3.7}$$

are index 2 *sub lattices.*

Proof. From the short exact sequence (3.5) and Remark 3.1.1, it immediately follows that

$$f^*H^1(C,\mathbb{Z}) \subset H^1(D,\mathbb{Z})^+$$

is an index two sub lattice.

For the first inclusion, we argue as follows, identifying using Poincaré duality $H^1(C,\mathbb{Z})$ with $H_1(C,\mathbb{Z})$. The cover $f : D \to C$ is induced by an index two sub lattice $K \subset H_1(C,\mathbb{Z})$. Let α in $H_1(C,\mathbb{Z})$ be a primitive element that is not sent to zero under the natural projection onto $H_1(C,\mathbb{Z})/K = \mathbb{Z}/(2)$. Setting $\alpha_1 := \alpha$, it is possible to complete α to a symplectic basis $\{\alpha_i, \beta_i\}$, such that the elements α_i, for every $i \neq 1$, and β_i, for every i, lie in K. By construction,

$$f^*K \subset (1+\iota^*)H^1(D,\mathbb{Z}).$$

This is because the double cover is trivial when restricted to any cycle γ representing a class in K, so that

$$f^*\gamma = x + \iota x,$$

with x and ιx disjoint cycles. On the other hand, since $\tilde{\alpha} := f^{-1}(\alpha)$ is connected and $\iota(\tilde{\alpha}) = \tilde{\alpha}$, we have

$$2\tilde{\alpha} = (1+\iota)\tilde{\alpha}.$$

To conclude, it is sufficient to notice that $\tilde{\alpha}$ is not in $(1+\iota)H^1(D,\mathbb{Z})$ since we can find a $\tilde{\beta}$ in $H^1(D,\mathbb{Z})$ such that $\iota\tilde{\beta}=\tilde{\beta}$, and $(\tilde{\alpha},\tilde{\beta})=1$.[2] □

Corollary 3.1.3. $H^1(D,\mathbb{Z})^+/(1+\iota^*)H^1(D,\mathbb{Z}) \cong \mathbb{Z}/(2)\times\mathbb{Z}/(2)$.

Let us now get back to study the geometry of the linear systems $|C|$ and $|D|$.

The two ι^*-invariant subspaces of $|D|$ are

$$f^*|C|, \quad \text{and} \quad f^*|C'|.$$

In the sequel, we will drop the symbol f^* and consider $|C|$ and $|C'|$ as subspaces of $|D|$. The following Lemma will be used in Section 2.1.

Lemma 3.1.4. *Let T be a general Enriques surface, and let $C\subset T$ be an irreducible curve. If C is not divisible by 2 in* $\mathrm{NS}(T)$*, then D is irreducible.*

Proof. Since $D\to C$ has degree two and C is irreducible, D can have at most 2 components D_1 and D_2, each mapping birationally onto C. Since the two components are interchanged by the involution, and ι^* acts as the identity on $\mathrm{NS}(S)$, it follows that $D\sim D_1+\iota^*D_1\sim 2D_1$ contradicting the fact that C (and thus D) is not divisible by 2 in $\mathrm{NS}(T)$. □

Without the assumption that $\rho(S)=10$, the above Lemma does not necessarily hold.

If L is a line bundle on $X=T,S$, and $L^2\geq 0$, then

$$h^0(X,L)+h^2(X,L)=\frac{L^2}{2}+\chi(\mathcal{O}_X)+h^1(X,L),$$

and since $\chi(\mathcal{O}_T)=1$ and $\chi(\mathcal{O}_S)=2$, it follows that either L or $L^{-1}\otimes\omega_X$ is effective.

Suppose $X=S$. Since $H^1(X,\mathcal{O}_X)=0$, if $L=\mathcal{O}_X(D)$ is effective then $H^1(X,L)=\ker[H^1(D,\mathcal{O}_D(D))\to H^2(\mathcal{O}_S)]$, so that if D is connected then $H^1(X,L)=0$ and

$$\dim|D|=\frac{L^2}{2}+1=h,$$

If $L^2=0$ then the dimension of the linear system $|L|$ depends on the divisibility of the class of L in $\mathrm{NS}(S)$. Indeed, we can write $L=\mathcal{O}(kE)$ with $k\geq 1$ and $E\subset S$ a primitive elliptic curve. With

[2] We can choose for $\tilde{\beta}$ one of the connected components of $f^{-1}(\beta_0)$.

this notation

$$h^0(S, \mathcal{O}_S(kE)) = k + 1.$$

In other words,

$$|kE| = \mathrm{Sym}^k |E| \cong \mathbb{P}^k.$$

If $X = T$ and $L = \mathcal{O}_T(C)$ for some irreducible curve $C \subset T$, then $H^1(T, L) = H^1(C, \mathcal{O}_C(C)) = H^0(C, \eta_C)$. In particular, this means that if $L^2 > 0$ then

$$\dim |C| = \frac{L^2}{2} = g(C) - 1.$$

Now suppose that the class of L is primitive in $\mathrm{NS}(T)$ and that $L^2 = 0$. We can write $L = \mathcal{O}(e)$ where $e \subset T$ a curve of arithmetic genus one. If we set $E = f^{-1}(e)$ the class of the E is primitive in $\mathrm{NS}(S)$, it is in particular connected. Hence $\eta_e := \mathcal{O}_e(e)$ is non trivial and it follows that

$$H^1(T, L) = H^1(T, \mathcal{O}_T(e)) = 0,$$

and that $h^0(T, L) = 1$. The two invariant subspaces of the pencil $|E| = \mathbb{P}^1$ are the two points corresponding to the curves,

$$E := f^{-1}(e), \quad \text{and} \quad E' := f^{-1}(e').$$

On the other hand, if $L = \mathcal{O}(2e)$ and C is a section of $H^0(T, \mathcal{O}(2e))$, $f^{-1}(C)$ is disconnected and $\eta_C = \mathcal{O}_C$. Hence $h^1(T, L) = 1$ and

$$|2e| \cong \mathbb{P}^1,$$

is an elliptic pencil. This pencil has exactly two half fibers e and e' which, by Kodaira's classification of the singular fibers of an elliptic fibration, are either smooth, nodal rational curves, or a closed chain of smooth rational curves. By *primitive elliptic curve* we mean a primitive curve of the types above, i.e. a curve of arithmetic genus one that is either smooth, or is an irreducible nodal rational curve, or is a closed chain of smooth rational curves. We should point out that if T is general, then a primitive elliptic curve is necessarily smooth. Indeed, we know that it has to be irreducible. Suppose that there existed a irreducible nodal rational curve on a a general Enriques surface T. Its double cover in S would then have to be reducible, contradicting Lemma 3.1.4.

Notice that

$$\omega_T = \mathcal{O}_T(e - e').$$

If L is isotropic but not primitive, then

$$|ke| = \mathrm{Sym}^{\lfloor \frac{k}{2} \rfloor} |2e| \times |e| \cong \mathbb{P}^{\lfloor \frac{k}{2} \rfloor}.$$

We should remark that since $\mathrm{rk}\, E_8(-1) \oplus U \geq 5$, it always contained isotropic vectors and hence every Enriques surface contains a genus one pencil.

Now let Γ_1 and Γ_2 be two effective classes on T, or on S. By the Hodge index theorem it follows that, if $\Gamma_1^2, \Gamma_2^2 \geq 0$, then $\Gamma_1 \cdot \Gamma_2 \geq 0$. Moreover, $\Gamma_1 \cdot \Gamma_2 = 0$, if and only if, $\mathbb{Z}\Gamma_1 = \mathbb{Z}\Gamma_2$ in $\mathrm{NS}(T)$ (or in $\mathrm{NS}(S)$) and $\Gamma_1^2 = \Gamma_2^2 = 0$. By the Nakai-Moishezon-Kleiman criterion and the Hodge index theorem, it follows that if S and T are unodal, then

$$\mathrm{Amp}(T) = \mathcal{Q}_T^+, \quad \text{and} \quad \mathrm{Amp}(S) = \mathcal{Q}_S^+,$$

where $\mathcal{Q}_T^+$ (and $\mathcal{Q}_S^+$) is a connected component of the cone of classes with positive self intersection in $\mathrm{NS}(T)$ (and $\mathrm{NS}(S)$) respectively. Moreover, for both surfaces the cone of effective curves is equal to the closures of Q_T^+ and Q_S^+ in $\mathrm{NS}(T)$ and $\mathrm{NS}(S)$ respectively, which in turn equal the nef cone.

Proposition 3.1.5 ([17])**.** *Let L be a line bundle on T with $L^2 > 0$. If L is nef, then $|L|$ has no fixed component unless $L = 2e + R$, where e is a primitive curve of arithmetic genus 1 and R is a smooth rational curve with $e \cdot R = 1$. Moreover, if L has no fixed component then the general member is irreducible.*

Proof. This is just Proposition 3.1.6 and Corollary 3.1.2 of [17]. □

Proposition 3.1.6 (Reducibility Lemma,[17])**.** *Let L be an effective line bundle on T. Then L is linearly equivalent to an effective combination of curves of arithmetic genus one and zero.*

Proof. [17] □

Before continuing, we make two definitions which will play an important role in Chapter 3.

Definition 3.1.7 ([12])**.** *Let m be a positive integer. An effective divisor $C \subset T$ is said to be m-connected, if for every decomposition $C = C_1 + C_2$ into the sum of two effective divisors we have $C_1 \cdot C_2 \geq m$. A linear system $|L|$ is said to be m-connected if all its members are m-connected.*

Definition 3.1.8 ([63]). *A linear system $|L|$ on a K3 or an Enriques surface is said to be hyperelliptic if $L^2 = 2$ or if the associated morphism φ_L is of degree 2 onto a rational normal scroll of degree $n-1$ in $\mathbb{P}^n$.*

We now proceed to state a characterization of hyperelliptic linear systems on K3 or Enriques surfaces. First of all we recall the following proposition (cf. Proposition 4.5.1 of [17]).

Proposition 3.1.9 ([17]). *Let T be an Enriques surface, and let $C \subset T$ be an irreducible curve with $C^2 \geq 2$. The following are equivalent,*

1. *$|C|$ is a hyperelliptic curve;*
2. *$|C|$ has base points;*
3. *There exists a primitive elliptic curve e such that $C \cdot e = 1$.*

By Corollary 4.5.1 of [17] it follows that the general member of a hyperelliptic linear system is a smooth hyperelliptic curve.

Proposition 3.1.10. *Let S be a K3 surface and let $D \subset S$ be an irreducible curve, with $D^2 \geq 4$. The following are equivalent,*

1. *$|D|$ is a hyperelliptic;*
2. *the general member of $|D|$ is a smooth hyperelliptic curve;*
3. *there exists an elliptic pencil $|E_1|$ such that $C \cdot E_1 = 2$.*

Suppose, moreover, that S is unodal. If one of the above conditions is satisfied, then there exist an integer $n \geq 1$, and a primitive elliptic curve E_2 such that

$$D = nE_1 + E_2, \ \text{ with } E_1 \cdot E_2 = 2.$$

Moreover, the morphism

$$\varphi_D : S \to R \subset \mathbb{P}^{2n+1}, \tag{3.8}$$

is of degree two and maps S onto a rational normal scroll R of degree $2n$ in $\mathbb{P}^{2n+1}$, and R is isomorphic to a quadric surface whose two rulings are the images under φ_D of the elliptic pencils $|E_1|$ and $|E_2|$.

Proof. The equivalence of the three items above follows from Theorem 5.2 and Corollary 5.8 of Saint-Donat's paper [63]. As for the second part of the proposition, it follows from Proposition 5.7, from

Section 5.9 in loc.cit. and from the fact that the images in R of E_1 and E_2 are two lines. Indeed, the two points of the intersection $E_1 \cap E_2$ map to the same point under φ_D, since $\mathcal{O}(D)_{|E_1} \cong \mathcal{O}(E_2)_{|E_1}$. Thus the image ℓ of E_1 in $\mathbb{P}^{2n+1}$ has degree one and the map $E_1 \to \ell$ has 4 ramification points. Since the ramification curve meets D in $2h + 2 = 4n + 4$ points, it meets E_2 in 4 points. Hence, the image σ of E_2 in R is smooth and rational. Since $\ell \cdot \sigma = 1$, and $\ell^2 = \sigma^2 = 0$ the proposition is proved. □

Thus, if $|C|$ is hyperelliptic, so is $|D|$. In particular, if T is general, the class of any hyperelliptic curve C on T is of the form

$$ne_1 + e_2,$$

with $n \geq 1$ and $e_1 \cdot e_2 = 1$. And hence we see that a hyperelliptic linear system is 1-connected. It turns out that also the converse holds

Proposition 3.1.11. *Let T be a general Enriques surface and let L be an effective line bundle on T, with $L^2 > 0$. Then $|L|$ contains a member that is the union of two smooth curves meeting transversally in $\nu \geq 1$ points. Moreover, if $\nu = 1$ then $|L|$ is hyperelliptic.*

Proof. Before starting the proof, we should remark that the general member of a linear system L' with positive self intersection on a general T is smooth. Indeed, since L' is ample Proposition 3.1.5 guarantees that L has no fixed component. Hence, either $|L'|$ is base point free so that its general member is smooth by Bertini, or else $|L'|$ there are two simple base points, in which it is is hyperelliptic and the general member is smooth by the comment after Proposition 3.1.9.

By Proposition 3.1.6 we now that there is a member C in $|L|$ such that $C = C_1 + C_2$.

Suppose that $C_1 \cdot C_2 = 1$. By the Hodge Index theorem $1 = (C_1 \cdot C_2)^2 \geq C_1^2 C_2^2$, forcing on of the two curves, say C_1, to be equal to e_1 for some primitive elliptic curve. Since $C_1 \cdot C = C_1 \cdot C_2$, we may conclude that C is hyperelliptic.

We are left to prove that we can choose the two curves C_1 and C_2 so that they are smooth and that they intersect transversally. In order to achieve this it is sufficient to show that we can choose C_1 and C_2 to be either primitive elliptic curves, or to have positive self intersection number.

Suppose that $C_1^2 = C_2^2 = 0$ so that there exist primitive elliptic curves e_1 and e_2 and positive integers s and t such that $C_1 = se_1$ and $C_2 = se_2$. If $s = t = 1$ we are done, if not, we can set $C_1' = e_1 + e_2$ and $C_2' = (s-1)e_1 + (t-1)e_2$. If $C_1^2 = 0$ but $C_2^2 > 0$ then as before we can set $C_1 = se_1$ and if $s > 1$ we can set $C_1' = e_1$ and $C_2' = C_2 + (s-1)e_1$.

□

Corollary 3.1.12. *Let T be a general Enriques surface and let L be a line bundle on T. The linear system $|L|$ is hyperelliptic if and only if it is 1-connected.*

Corollary 3.1.13. *Let T be a general Enriques surface, and let $C \subset T$ be an irreducible curve. Set $D = f^{-1}(C)$. Then, there is a member of $f^*|C| \subset |D|$ that is a union of two smooth ι-invariant curves D_1 and D_2 meeting in 2ν points. Moreover, we have $\nu \geq 2$, unless $|D|$ is hyperelliptic, in which case $\nu = 1$.*

Proof. Keeping the notation in the proof of the previous proposition, it is enough to check that the curves $D_1 = f^{-1}(C_1)$ and $D_2 = f^{-1}(C_2)$ are connected. This, however, follows from the fact that, for $i = 1, 2$, either $C_i^2 > 0$ or C_i is a primitive elliptic curve. □

In Section 3.5 and Section 4.2 we will need the following proposition.

Proposition 3.1.14. *Let X be a general Enriques surface or a K3 surface covering a general Enriques surface, and let L be a line bundle on X with $L^2 > 0$. Then $|L|$ has reducible members in codimension one if and only if $|L|$ is hyperelliptic.*

Proof. We will prove the proposition when X is an Enriques surface, the proof for K3 surfaces is analogous. Since the irregularity of the surface is zero, any component of the discriminant parameterizing reducible curve is birational to a product of linear systems. Consider a reducible member $C_1 + C_2$, and set $\nu = C_1 \cdot C_2$. For $i = 1, 2$, letting g_i be the arithmetic genus of C_i, we have

$$g = g_1 + g_2 + \nu - 1.$$

First assume that if, for some $i = 1, 2$, the genus g_i is equal to one the corresponding curve C_i is a primitive elliptic curve. Then $\dim |C_i| = g_i - 1$, $\dim |C| = \dim |C_1| + \dim |C_2| + \nu$, and $\operatorname{codim}(|C_1| \times |C_2|, |C|) = \nu$. From Corollary 3.1.12 it follows that $\nu \geq 1$ and that $\nu = 1$ if and only if C is hyperelliptic.

Next, consider the case $C_1 \in |se_1| = \mathbb{P}^{\lfloor \frac{s}{2} \rfloor}$ for some primitive elliptic curve e_1 and some integer $s \geq 2$. Then $\dim |C| = g_2 - 1 + \nu$. Thus, if $g_2 \geq 2$, we have $\dim |C_1| \times |C_2| = \lfloor \frac{s}{2} \rfloor + g_2 - 1$, while

$\dim |C_1| \times |C_2| = \lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor$, if $C_2 = te_2$ with $t \geq 1$. It follows that

$$\operatorname{codim}(|C_1| \times |C_2|, |C|) = \begin{cases} \nu - \lfloor \frac{s}{2} \rfloor, & \text{if } g_2 \geq 2 \\ \nu - \lfloor \frac{s}{2} \rfloor - \lfloor \frac{t}{2} \rfloor, & \text{if } g_2 = 1. \end{cases}$$

In the first case, since $\nu = s\nu'$, with $\nu' \geq 1$ we are done, unless $s = 2$ and $\nu' = 1$. However if $s = 2$ and $\nu' = 1$, the curve C_2 is hyperelliptic of the form $\nu e_1 + e_2$, with $e_1 \cdot e_2 = 1$, and hence $L = \mathcal{O}_T((\nu + s)e_1 + e_2)$ is hyperelliptic.

As for the second case, we can set $\nu = st\nu'$ and thus we are done unless $\nu' = 1$, $s = 2$ and $t = 1$. This means that $C_2 = e_2$, with $e_1 \cdot e_2 = 1$ and, again, $L = 2e_1 + e_2$ is hyperelliptic. □

Next, we study the discriminant locus of a hyperelliptic linear system on an Enriques surface.

Proposition 3.1.15. *Let T be an Enriques surface, and let $|C| = |ne_1 + e_2|$ be a genus $g = n + 1 \geq 3$ hyperelliptic linear system on T. The discriminant locus $\Delta \subset |C|$ is the union of four irreducible components $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. The first two are hyperplanes and the general point parametrizes curves that are the union of two smooth curves meeting transversally in one point, the third component, whose general point parametrizes curves that are union of two smooth components meeting transversally in two points, is of degree $n - 1$, and the forth is of degree $4n + 12$ and generically it parametrizes singular but irreducible curves. Moreover, the general curve parametrized by this component has only one node.*

Proof. First of all, one can show with a classical Euler characteristic count shows that the degree of Δ is equal to $6(n + 2)$.

It is clear that the two hyperplanes,

$$\Delta_1 := [e_1] \times |(n-1)e_1 + e_2|, \quad \text{and} \quad \Delta_2 := [e_1'] \times |(n-1)e_1 + e_2'|, \tag{3.9}$$

constitute two components of the discriminant locus, and also that they parametrizes curves of the form $e_1 \cup \Gamma$ with $\Gamma \in |(n-1)e_1 + e_2|$ (resp. $e_1' \cup \Gamma$ with $\Gamma \in |(n-1)e_1 + e_2'|$). Secondly, the natural map

$$\phi : \mathbb{P}^1 \times \mathbb{P}^{n-2} = |2e_1| \times |(n-2)e_1 \times e_2| \to \Delta \subset \mathbb{P}^n.$$

is just the composition of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-2} \to \mathbb{P}^{2n-3}$ with the surjection $\mathbb{P}^{2n-3} \to \mathbb{P}^n$

induced by the natural map

$$H^0(T, \mathcal{O}(2e_1)) \otimes H^0(T, \mathcal{O}((n-2)e_1 + e_2)) \to H^0(T, \mathcal{O}(ne_1 + e_2)).$$

It follows that,

$$\mathbb{P}^1 \times \mathbb{P}^{n-2} \to \phi(\mathbb{P}^1 \times \mathbb{P}^{n-2}), \tag{3.10}$$

is finite and generically one to one, and that $\Delta_3 := \phi(\mathbb{P}^1 \times \mathbb{P}^{n-2})$ is a degree $n-1$ component of Δ whose general point parametrizes curves as in the statement of the theorem. Notice that the above map (3.10) is just the normalization of Δ_3.

We just need to prove that the remaining part Δ_4 of the discriminant is irreducible. By definition the rational map $\phi_{|C|}$ associated to the linear system maps T generically $2:1$ onto a degree $n-1$ rational surface $R \subset \mathbb{P}^n$, and contracts the curves e_1 and e_1'. A curve in $|C|$ is singular in the following three cases. If it covers a singular (hence reducible) hyperplane section of R, if it covers a smooth curve that is tangent to the ramification curve, or else if its image contains one of the two points P and Q to which either e_1 or e_1' is contracted. The ramification curve is described in Theorem 4.5.2 of [17]. It consists of the union of two lines ℓ_1 and ℓ_2 belonging the ruling that is the image of $|2e_1|$ and of an irreducible curve $B \subset R$. The irreducible curve B has two tacnodes in P and Q and is otherwise non singular (we are in the un-nodal case). The preimages, under $\phi_{|C|}$, of the lines through P and Q are the curves in Δ_1 and Δ_2, respectively. The set of hyperplane sections of R passing through ℓ_1 or ℓ_2 form a set of codimension 2, whereas the set of hyperplane sections of R that are tangent to B form an irreducible divisor $\Delta_4 \subset \mathbb{P}^n$. Moreover, the general hyperplane in this component is tangent to B in just one point and thus corresponds to curves in $|C|$ with only one node and no other singularity. □

3.2 The relative Prym variety associated to an Enriques surface

In this section we will analyze in more detail the relative Prym variety associated to a linear system on an Enriques surface.

We start with a definition.

Definition 3.2.1. *A compact Kähler manifold M is a hyperkähler manifold, or holomorphic sym-*

plectic manifold, if it is simply connected and if it admits a homolorphic symplectic form. If, moreover, the symplectic form is unique up to scalar M is said to be irreducible.

Let X be a K3 surface, let $v \in H^*(X, \mathbb{Z})$ be a Mukai vector, and let H be a polarization. Assume that v is such that the moduli space of H-semistable bundles $M = M_{v,H}(X)$ is not empty. In the fundamental paper [47], Mukai proved the following theorem

Theorem 3.2.2 (Mukai)**.** *The stable locus of M admits a symplectic form.*

We won't prove the theorem, but we will describe the symplectic form on the tangent space at a $[F] \in M$ corresponding to a stable sheaf F. Recall that the tangent space of M at F is naturally isomorphic to $\mathrm{Ext}^1(F, F)$. The symplectic form at the point $[F]$ is given by the following composition

$$\begin{array}{rccc} \sigma : \mathrm{Ext}^1(F,F) \times \mathrm{Ext}^1(F,F) & \xrightarrow{\cup} & \mathrm{Ext}^2(F,F) & \xrightarrow{\mathrm{tr}} H^2(S, \mathcal{O}_S) \cong \mathbb{C}, \\ (e, f) & \longmapsto & e \cup f & \longmapsto \quad \mathrm{tr}(e \cup f) \end{array} \tag{3.11}$$

where the identification $H^2(S, \mathcal{O}_S) = H^2(S, \omega_S) \cong \mathbb{C}$ is Serre dual to the isomorphism

$$H^0(S, \omega_S) = \mathbb{C}\sigma_S \cong \mathbb{C}, \tag{3.12}$$

defined by the choice of a, unique up to scalar, symplectic form $\sigma \in H^0(S, \omega_S)$.

Let us now fix some notation. The arbitrary étale double cover $f : X \to Y$ of smooth projective surfaces of Chapter 2 will be substituted with an étale double cover

$$f : S \to T,$$

where T is a general Enriques surface, and S is a K3 surface. We let $C \subset T$ be an irreducible curve of genus $g \geq 2$, and we set $D = f^{-1}(C)$ and $h = g(D) = 2g - 1$. Finally, we choose a non zero integer $\chi = d - h + 1$, set

$$v = (0, [D], \chi).$$

and for any polarization H on S, we consider the moduli space $M_{v,H} = M_{v,H}(S)$. By Theorem 1.2.1, Lemma 1.1.24, and Theorem 3.2.2 and we know that if v is primitive and H is v-generic, then $M_{v,H}$ is a projective hyperkähler.

We should point out that using the Hilbert-Chow morphism, Beauville has shown in [9] that the Hilbert scheme of points on a K3 surface admits a symplectic form. He also shows that this form is

unique up to scalar and that the Hilbert scheme is simply connected, i.e. that the Hilbert scheme of points on a K3 surface is an irreducible holomorphic symplectic manifold. Moreover, by work of [29] and [55], it turns out that any moduli space of sheaves on a K3 surface, if smooth and projective, is an irreducible holomorphic symplectic manifold. In particular, $M_{v,H}$ is simply connected and the symplectic form is unique up to a scalar.

Let us consider the support morphism

$$\pi : M_{v,H} \to |D|,$$

let $V \subset |D|$ be the smooth locus, and let $t \in V$ be a point and let F be a degree d line bundle on D_t. In [11] Beauville noticed that since the Yoneda product is compatible with the base change morphism

$$\mathrm{Ext}^1_{D_t}(F,F) \hookrightarrow \mathrm{Ext}^1_S(F,F),$$

the restriction of the pairing (3.11) to $\mathrm{Ext}^1_{D_t}(F,F)$ vanishes, and since the above injection is identified with the tangent map $T_{[F]}\,\mathrm{Jac}^d(D_t) \to T_{[F]}M_{v,H}$, it follows that $T_{[F]}\,\mathrm{Jac}^d(D_t) \subset T_{[F]}M_{v,H}$ is a Lagrangian subspace.

Remark 3.2.3. By a theorem of Matsushita [43], if H is v-generic then the support morphism $M_{v,H} \to |D|$ is actually equidimensional.

Lemma 3.2.4. *Let X be a projective K3 surface, let $D \subset X$ be an irreducible curve of genus $h \geq 2$, and consider the Mukai vector $v = (0, [D], \chi)$, where $\chi = -h+1$. Suppose that $|D|$ contains a member that is the union of two integral curves D_1 and D_2 with $D_1 \cdot D_2 \in 2\mathbb{Z}$, then D is not v-generic.*

Proof. As in Example 1.1.17, it is sufficient to notice that since $D^2 = 2h-2$ and $D_1 \cdot D_2$ is even,

$$(D_1 \cdot D)\frac{\chi}{D^2} = -\frac{D_1 \cdot D}{2},$$

is an integer. □

Keeping the notation of the Lemma, we should notice that even if $M_{v,D}$ is not smooth, the support morphism

$$\pi : M_{v,D} \to |D|,$$

is still equidimensional and that its restriction to the smooth locus $M^{reg}_{v,D} \to |D|$ where a symplectic form is defined is a Lagrangian fibration. Indeed, it is sufficient to consider a resolution $M_{v,H} \to$

$M_{v,D}$ as in Corollary 1.2.2 and notice that the support morphism are compatible.

The following propositions shows the fact that in every $\mathcal{S}$-equivalence class of sheaves there is a unique polystable sheaf, can be used to describe the singular locus of a moduli space.

Proposition 3.2.5. *Let S be a K3 surface, let $v = (0, D, \chi) \in H^*(S, \mathbb{Z})$ be a Mukai vector and let H any polarization. Set $M = M_{v,H}(S)$, and let $\Sigma \subset M$ be the singular locus of M. There is a stratification $\{\Sigma_\alpha\}$ of Σ in smooth locally closed subsets such that each Σ_α has a symplectic form.*

Proof. By Theorem 1.2.1, we know that Σ is contained in the strictly semi-stable locus. Hence if $[F] \in M$ is a singular point, we know that the sheaf F is H-semistable. Let $v_1 \cdots, v_s$ be points in the integral cohomology $H^*(S, \mathbb{Z})$ and let $n_1 \cdots, n_s$ be positive integers such that

$$\sum n_i v_i = v.$$

We say that a H-semistable sheaf F is of type $(n_1 v_1, \ldots, n_s v_s)$ if it is $\mathcal{S}$-equivalent to a sheaf

$$\oplus_{i=1}^{s} F_i^{n_i},$$

such that for $i = 1, \cdots s$, the sheaf F_i is H-stable with $v_i = v(F_i)$. We set

$$\Sigma_{(n_1 v_1, \ldots, n_s v_s)} = \{F \in M \,|\, F \text{ is of type } \sum n_i v_i\}.$$

The stratum $\Sigma_{(n_1 v_1, \ldots, n_s v_s)}$ is a locally closed subset of M, which is isomorphic to an open subset of the product,

$$M_{v_1,H} \times \cdots \times M_{v_s,H}$$

If non empty, this open subset is smooth, because we are assuming that all the F_i's are smooth. Since the smooth locus of all the $M_{v_i,H}$ has a symplectic form, it follows that the startum inherits a symplectic form. Moreover, it is clear that the union of the $\Sigma_{(n_1 v_1, \ldots, n_s v_s)}$ on all the possible decompositions

$$v = \sum n_i v_i,$$

is equal to Σ and also that there is only a finite number of such decompositions for which $\Sigma_{(n_1 v_1, \ldots, n_s v_s)}$ is non empty. □

Notice also that if for $i = 1, \ldots, s$ we set $D_i = c_1(F_i)$, the strata $\Sigma_{(n_1 v_1, \ldots, n_s v_s)}$ admits a morphism

to $|D_1| \times \cdots \times |D_s|$ that is a Lagrangian fibration.

Let us now come to the relative Prym variety.

Set $\chi = -h + 1$ and

$$v = (0, [D], -h + 1).$$

Setting $N = -D$ in (2.9) gives a birational involution

$$\begin{aligned} \tau = \tau_{-D} : M_{v,D} &\longrightarrow M_{v,D} \\ [F] &\longmapsto [\tau_{-D}(F)] \end{aligned} \tag{3.13}$$

that is the composition of the following two birational involutions:

$$\begin{aligned} \iota^* : M_{v,H} &\dashrightarrow M_{v,H}, \\ F &\longmapsto \iota^* F, \end{aligned}$$

of Lemma 1.4.4 and

$$\begin{aligned} M_{v,H} &\dashrightarrow M_{w,H}, \\ F &\longmapsto j_{-D}(F) = \mathcal{E}xt^1(F, \mathcal{O}(-D)). \end{aligned}$$

defined in (1.35). Notice that on a smooth fiber, j_{-D} is just the action by "-1". Recall that we have defined the relative Prym variety

$$P_{v,H} = \overline{\mathrm{Fix}^\circ(\tau)},$$

as the irreducible component of $\overline{\mathrm{Fix}(\tau)}$ that contains the image of the rational zero section.

We start with a Lemma

Lemma 3.2.6. *Set $M = M_{v,H}$. The involution $\iota^* : M \dashrightarrow M$ is anti-symplectic, i.e. if σ denotes the symplectic form on the smooth locus of M, then $\iota^*\sigma = -\sigma$.*
Moreover, the fibration $\pi : M \to |D|$ is equivariant with respect to the involution ι^* defined above.

Proof. Let F be an H-stable sheaf in M and consider two elements $e, f \in \mathrm{Ext}^1(F, F)$. We need to prove that

$$\sigma(\iota^* e, \iota^* f) = -\sigma(e, f).$$

By functoriality of the cup product and of the trace map, the following diagram is commutative,

$$\begin{array}{ccccc} \mathrm{Ext}^1(F,F)\times\mathrm{Ext}^1(F,F) & \xrightarrow{\cup} & \mathrm{Ext}^2(F,F) & \xrightarrow{\mathrm{tr}} & H^2(S,\mathcal{O}_S) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathrm{Ext}^1(\iota^*F,\iota^*F)\times\mathrm{Ext}^1(\iota^*F,\iota^*F) & \xrightarrow{\cup} & \mathrm{Ext}^2(\iota^*F,\iota^*F) & \xrightarrow{\mathrm{tr}} & H^2(S,\mathcal{O}_S). \end{array} \tag{3.14}$$

Hence, in order to prove the Lemma, we just need to prove that the identification $H^2(S,\mathcal{O}_S)\cong\mathbb{C}$ changes sign if we compose it with ι^*. This follows from the fact that identification $H^0(S,\omega_S)=\mathbb{C}\sigma\cong\mathbb{C}$ changes sign once we compose it with ι^*.

As for the second statement, it is a consequence of the very definitions of ι^* and of π. □

Proposition 3.2.7. *The birational involution (3.13) $\tau=\tau_{-D}$ is symplectic, i.e., if we let σ denote the smooth locus of $M_{v,H}$ we have $\tau^*\sigma=\sigma$.*

Proof. To prove that τ_{-D} is symplectic it is enough to prove that j_{-D} is anti-symplectic. We argue as follows. Set $M=M_{v,H}$, let $D_0\in|D|$ be a smooth curve, and set $J=\mathrm{Jac}^0(D_0)=\pi^{-1}(D_0)$. Since π is a Lagrangian fibration, the isomorphism $T_M\cong\Omega^1_M$ induced by the symplectic form σ, yields an isormophism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_J & \longrightarrow & T_{M|J} & \longrightarrow & \mathcal{N}_{J/M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & \mathcal{N}^\vee_{J/M} & \longrightarrow & \Omega^1_{M|J} & \longrightarrow & \Omega^1_J & \longrightarrow & 0. \end{array}$$

In particular for a point $x\in J$ we have the isomorphism

$$T_{D_0}|D|\cong\mathcal{N}_{J/M,x}\cong(T_xJ)^\vee. \tag{3.15}$$

Since the second isomorphism in (3.15) is given by σ, and since j^*_{-D} acts as the identity on $T_{D_0}|D|$ and as -1 on T_xJ, we conclude that $j^*_{-D}(\sigma)=-\sigma$. □

We let

$$M^{reg}_{v,H}\subset M_{v,H}, \tag{3.16}$$

be the largest open subset set of the smooth locus of $M_{v,H}$ on which τ is biregular.

With this notation we have the following

Corollary 3.2.8. *The smooth locus of* $\overline{\mathrm{Fix}(\tau)}$ *carries a holomorphic* 2*-form that is symplectic (i.e. non degenerate) on an open subset containing*

$$\overline{\mathrm{Fix}(\tau)} \cap M_{v,H}^{reg}.$$

Proposition 3.2.9. *The dimension of the relative Prym variety* $P_{v,H}$ *is equal to* $(2g-2)$*. The smooth locus of* $P_{v,H}$ *carries a holomorphic two form that is symplectic on a dense open subset. With respect to this symplectic form the natural morphism*

$$P_{v,H} \to |C|,$$

has a natural structure of Lagrangian fibration.

Proof. Since τ respects the fibration $\pi : M_{v,H} \to |D|$, there is a commutative diagram

$$\begin{array}{ccc} P_{v,H} & \longrightarrow & M_{v,H} \\ {\scriptstyle\eta}\downarrow & & \downarrow \\ |C| & \longrightarrow & |D| \end{array} \tag{3.17}$$

where, as usual, we identify $|C|$ with $f^*|C| \subset |D|$. The general fiber of this fibration is a principally polarized abelian variety of dimension $g-1$ since given any smooth curve C in $|C|$, with double cover $D = f^{-1}(C) \to C$, we have

$$\eta^{-1}(C) \cong \mathrm{Prym}(D/C).$$

Since $\dim|C| = g-1$ it follows that $\dim P_{v,H} = 2g-2$. Moreover, the general fibers of π are Lagrangian, and since the symplectic form defined on an open subset of $P_{v,H}$ is induced by that on $M_{v,H}$, the general fibers of ν are also Lagrangian. □

As we saw in Lemma 3.2.4, D is not v-generic. In particular, if we choose D as polarization the involution τ is biregular, but $M_{v,D}$ is not smooth. On the other hand, we should remark that if H lies in a chamber that is adjacent to a wall containing D then (cf. Corollary 1.2.2) there is a natural morphism

$$M_{v,H} \to M_{v,D},$$

that is compatible with the symplectic involutions τ on the two moduli spaces. It follows that there

is a projective morphism

$$P_{v,H} \to P_{v,D},$$

which is birational since it is an isomorphism on a dense open subset. Since $M_{v,H}$ has in general better singularities than $M_{v,D}$ (and is indeed smooth if v is primitive), it is natural to expect that also $P_{v,H}$ has better singularities than $P_{v,D}$. We will see in Example 3.3.7 and in Section 3.4 that in some cases $P_{v,H} \to P_{v,D}$ is a resolution of the singularities.

Lemma 3.2.10. *The fixed locus* $\overline{\mathrm{Fix}(\tau)}$ *has* 4 *irreducible components of dimension* $(2g-2)$. *All other components have smaller dimension and are contained in the strictly semistable locus of* $M_{v,H}$.

Proof. Indeed, if $Z \subset \overline{\mathrm{Fix}(\tau)}$ is a component that intersects the stable locus then it also intersects $M_{v,H}^{reg}$. It follows that there is dense open subset of Z that has a symplectic form, and since the fibers of the induced morphism $Z \to |C|$ are isotropic, their dimension is bounded by $\dim Z/2$. It follows that $\dim Z = 2g-2$ if and only if $Z \to |C|$ is dominant which happens if and only if Z intersect the Jacobian $\mathrm{Jac}^0(D_0)$ of a smooth curve D_0 covering a smooth curve C_0. If this is the case, then $Z \cap \mathrm{Jac}^0(D_0)$ is isomorphic to $\mathrm{Prym}(D_0, C_0)$ and hence Z has to be one of the components that map surjectively onto $|C|$. The fact that there are 4 of such components follows from Proposition 3.1.14 which implies that $|C|$ is sufficiently positive in the sense of Proposition 2.2.5.
On the other hand, if Z is contained in the strictly semistable locus then $Z \to |C|$ is not dominant. Moreover, by Proposition 3.2.5 the strata of the singular locus of $M_{v,H}$ admit a the symplectic form which in turn induces a symplectic form on a dense open subset of Z. Just as before we can bound the dimension of Z using the fact that the general fiber of $Z \to |C|$ is isotropic and that $\dim \pi(Z) < g-1$. □

As a corollary of the proof of this lemma we see that the components of maximal dimension all maps surjectively onto $|C|$.

The same argument as in the proof of the above Lemma, allows us to bound the dimension of the singular locus of the $(2g-2)$-dimensional components of $\overline{\mathrm{Fix}(\tau)}$.

Proposition 3.2.11. *Let* P *be a* $(2g-2)$*-dimensional component of* $\overline{\mathrm{Fix}(\tau)}$, *and let* $\mathrm{Sing}(P) \subset P$ *be its singular locus. Then* $\mathrm{codim}(\mathrm{Sing}(P), P) \geq 2$ *and equality holds if and only if* $|C|$ *is hyperelliptic.*

Proof. Let $\Sigma \subset P$ be an irreducible component of $\mathrm{Sing}(P)$. Then, $\pi(\Sigma)$ is contained in the closed set $R \subset |C|$ parametrizing reducible curves and by Proposition 3.1.14 we know that $\mathrm{codim}(R, |C|) = 1$ if and only if $|C|$ is hyperelliptic. By Proposition 3.2.5, we know that there is an open subset $U \subset \Sigma$

that carries a symplectic form. In particular, Σ cannot be of codimension one. Since the general fiber of $U \to R$ is isotropic, it follows using the same argument as in the proof of the Lemma 3.2.10 that $\operatorname{codim}(\operatorname{Sing}(P), P) \geq 2 \operatorname{codim}(R, |C|)$. In the hyperelliptic case we can apply Proposition 2.2.3 to the curves of Δ_1 or Δ_2 (notation as in Proposition 3.1.15) in order to produce a codimension 2 component of $\operatorname{Sing}(P)$. □

In Sections 3.3 and 3.4 we will analyze the singularities of the relative Prym varieties, and answer the natural question of whether there are any that admit a symplectic resolution.

Remark 3.2.12. As in Subsection 2.2.3 we can put ourselves in the following slightly more general setup: fix a non zero integer χ, set $v = (0, [D], \chi)$, and consider any ι^* invariant line bundle N on S with

$$\chi = 2N \cdot D.$$

For any polarization H on S we can consider the birational involution

$$\begin{aligned} \tau_N : M_{v,H} &\dashrightarrow M_{v,H}, \\ F &\longmapsto \tau_N(F) := \mathcal{E}xt^1(\iota^* F, N). \end{aligned}$$

Then just as in the previous case, one can show that τ_N is symplectic, that $\overline{\operatorname{Fix}(\tau_N)}$ has 4 maximal dimensional irreducible components whose singular locus is as in Proposition 3.2.11. Hence, every maximal dimensional component maps surjectively onto $|C|$ and has a dense open subset that carries a symplectic form with respect to which the support morphism is a lagrangian fibration.

3.3 Local structure of M and P

In this section we study, by studying the corresponding singularity of the moduli space the local structure of the relative Prym variety at the simplest singularities that can occur. This detailed analysis will reward us by allowing us to characterize the situations in which the relative Prym variety admits a symplectic resolution. We first identify the tangent cones of the moduli space and of the relative Prym variety at these singular points, then we show that the relative Prym variety is actually locally isomorphic to its tangent cone and from this we easily see in which situations the singularity of P admits or not a symplectic resolution.

As usual, we let T be a general Enriques surface, we let $f : S \to T$ be its universal cover, and we let

$C \subset T$ be a smooth curve of genus $g \geq 2$. We set $D = f^{-1}(C)$ so that $h = g(D) = 2g - 1$. We fix a non zero integer χ, consider a Mukai vector $v = (0, [D], \chi)$ and a ι^*-invariant line bundle N such that

$$2\chi = N \cdot D.$$

Finally we choose a polarization H such that

$$\tau_N : M_{v,H} \to M_{v,H},$$

is a morphism (for e.g. $H = \mathcal{O}(D)$, since on an unodal K3 surface any curve with positive self intersection is ample). Set

$$M = M_{v,H}, \quad \text{and} \quad \tau = \tau_N.$$

By

$$P \subset M,$$

we will denote a maximal i.e., $(2g-2)$-dimensiona,l irreducible component of $\mathrm{Fix}(\tau)$.

The singularities that we are going to study are those given by the simplest polystable sheaves that can occur, those that are the direct sum of two non isomorphic stable sheaves.

Proposition 3.3.1. *Let P be a $(2g-2)$-dimensional irreducible component of $\mathrm{Fix}(\tau_N) \subset M_{v,H}$. Then P contains an H-polystable sheaf $F = F_1 \oplus F_2$ such that F_1 and F_2 are H-stable, τ-invariant and such that $D_1 = \mathrm{supp}(F_1)$ and $D_2 = \mathrm{supp}(F_2)$ are reduced and have no common component.*

Proof. By Proposition 3.1.11 there is a member of $|C|$ that is the union of two smooth curves C_1 and C_2 meeting transversally[3]. Set $D_1 = f^{-1}(C_1)$ and $D_2 = f^{-1}(C_2)$. Having observed this, we can apply Proposition 2.2.7 and conclude that there is an H-polystable sheaf that is the direct sum of two H-stable sheaves F_1 and F_2 supported on D_1 and on D_2 respectively. To show that P contains sheaves of this type it is now sufficient to recall that $P \to |C|$ is surjective, to invoke Proposition 2.2.3 and to notice that, by Remark 2.2.8, $\overline{\ker(1+\iota^*)}$ is isomorphic to $\mathrm{Fix}(\tau_N) \cap \pi^{-1}([D_1 + D_2])$. □

Let

$$F = F_1 \oplus F_2,$$

be a sheaf as in the statement of the above proposition. We now proceed with the analysis of the local

[3]We don't really need that C_1 and C_2 are smooth, but just that they are reduced, and that there is an H-stable sheaf on C_1 (resp. C_2) with the same H-slope as F.

structure of M and of P at $[F]$. We start by fixing some notation and describing the setting.

Set

$$v_1 = v(F_1), \quad v_2 = v(F_2),$$

and, using the notation of the proof of Proposition 3.3.1, set

$$g_1 = g(C_1),\ \ g_2 = g(C_2),\ \ h_1 = g(D_1) = 2g_1 - 1,\ \ h_2 = g(D_2) = 2g_2 - 1, \quad \text{and } \nu = C_1 \cdot C_2.$$

Recall that the sheaf F belongs to a stratum of the singular locus of M that is isomorphic to an open subset of the product

$$M_1 \times M_2,$$

where for $i = 1, 2$ we have set $M_i := M_{v_i,H}$. Set

$$G = \mathbb{P}\operatorname{Aut}(F) \cong \mathbb{C}^*.$$

By Serre duality the bilinear form

$$\begin{aligned} \mu : \operatorname{Ext}^1(F_2, F_1) \times \operatorname{Ext}^1(F_1, F_2) &\longrightarrow \mathbb{C} \\ (f, f') &\longmapsto \mu(f, f') = \operatorname{tr}(f \cup f') \end{aligned} \tag{3.18}$$

is non degenerate, and we can identify $\operatorname{Ext}^1(F_2, F_1)$ with the dual of $\operatorname{Ext}^1(F_1, F_2)$. To make notation less heavy we write

$$\begin{aligned} U_1 &= \operatorname{Ext}^1(F_1, F_1), \quad U_2 = \operatorname{Ext}^1(F_2, F_2), \\ W &= \operatorname{Ext}^1(F_1, F_2), \quad W^\vee = \operatorname{Ext}^1(F_2, F_1), \end{aligned} \tag{3.19}$$

so that

$$\operatorname{Ext}^1(F, F) = U_1 \oplus U_2 \oplus W^\vee \oplus W.$$

Notice that,

$$U_1 \oplus U_2 = T_{[F_1 \oplus F_2]}(M_1 \times M_2) = T_{F_1} M_1 \oplus T_{F_2} M_2,$$

is the tangent space to the singular locus of M. The action of $\mathbb{C}^*$ on $\operatorname{Ext}^1(F, F)$ is trivial on the first two summands, and we may set up things so that on W, $\mathbb{C}^*$ acts by multiplication by $\lambda \in \mathbb{C}^*$

(and hence on $W^\vee$ by multiplication by λ^{-1}). The obstruction space

$$\mathrm{Ext}^2(F,F)_0$$

is one dimensional and, using the trace, we can identify it with $\mathrm{Ext}^2(F_1,F_1)\cong\mathbb{C}$. Using the notation established in subsection 1.2.2, it follows that

$$\kappa^{-1}(0)\subset \mathrm{Ext}^1(F,F),$$

is a hypersurface. Notice that, up to the constant factor $\frac{1}{2}$, the moment map (1.22) is given by

$$\begin{aligned} \mathrm{Ext}^1(F,F)=U_1\oplus U_2\oplus W^\vee\oplus W &\longrightarrow \mathrm{Ext}^2(F,F)_0=\mathbb{C}\\ e=(a,b,f,f') &\longmapsto \kappa_2(e)=\mu(f,f')\,. \end{aligned} \tag{3.20}$$

We mentioned in Subsection 1.2.2 that, locally around $[F]$, M is isomorphic to

$$\kappa^{-1}(0)/\mathbb{C}^*.$$

Let us now use this fact to identify the tangent cone $C(M)$ of M at F. First notice that the natural map

$$\begin{aligned} W^\vee\oplus W=\mathrm{Hom}(W,\mathbb{C})\oplus\mathrm{Hom}(\mathbb{C},W) &\longrightarrow \mathrm{End}(W)\\ (i,j) &\longmapsto ji \end{aligned}$$

factors through the action of $\mathbb{C}^*$ and exhibits the quotient of $\mathrm{Hom}(W,\mathbb{C})\oplus\mathrm{Hom}(\mathbb{C},W)$ by $\mathbb{C}^*$ as the group $\mathrm{End}_1(W)$ of endomorphisms of W of rank ≤ 1. Moreover,

$$\begin{aligned} \mathrm{Ext}^1(F,F)/\mathbb{C}^* &\cong \left(U_1\oplus U_2\oplus\mathrm{Hom}(W,\mathbb{C})\oplus\mathrm{Hom}(\mathbb{C},W)\right)/\mathbb{C}^*\\ &\cong U_1\times U_2\times\mathrm{End}_1(W). \end{aligned} \tag{3.21}$$

Recall that $\mathrm{End}_1(W)$ is isomorphic to the affine cone over the Segre embedding

$$s(\mathbb{P}W^\vee\times\mathbb{P}W)\subset\mathbb{P}(W^\vee\otimes W)$$

Let $C(M)$ be the tangent cone of M at the point $[F]$. Then $C(M)\subset Q/\mathbb{C}^*$, where

$$Q=\kappa_2^{-1}(0)=\{e\in\mathrm{Ext}^1(F,F)\mid e\cup e=0\}.$$

is the tangent cone in 0 of $\kappa^{-1}(0)$.

By choosing appropriate coordinates on $\mathrm{Ext}^1(F,F)$ and looking at $\mathbb{C}^*$-invariant polynomial functions on this vector space, one may deduce that the equation $k_2 = 0$ of $Q/\mathbb{C}^* \subset \mathrm{Ext}^1(F,F)/\mathbb{C}^*$ is *linear* in the invariant coordinates and is of the form,

$$A = 0\,, \quad \text{where} \quad A \in W \otimes W^\vee = \mathrm{End}(W)^\vee\,. \tag{3.22}$$

In particular, $Q/\mathbb{C}^*$ is irreducible and of the same dimension as $C(M)$ so that,

$$C(M) = Q/\mathbb{C}^* = U_1 \times U_2 \times \mathrm{End}_{1,A}(W)$$

where

$$\mathrm{End}_{1,A}(W) = \{a \in \mathrm{End}_1(W) \,|\, A(a) = 0\}.$$

Let us now come to the local study of P at $[F]$. Fixing an isomorphism

$$\phi : F \cong \tau(F)\,.$$

induces, by universality, an automorphism of the base B of a Kuranishi family of F

$$\tau_\phi : B \to B,$$

whose first order term is uniquely defined. Since τ_ϕ is not unique, it is not necessarily an involution, but it is so at the infinitesimal level and

$$d\tau_\phi = d\tau = \tau_* : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^1(F,F)\,. \tag{3.23}$$

Sometimes, and when no confusion is possible, we will write τ instead of τ_* to indicate the homomorphism (3.23).

Remark 3.3.2. If F is a stable sheaf, the action of τ_* on $\mathrm{Ext}^2(F,F)$ is equal to -1. To see this it suffices to prove that $\tau_*(e \cup f) = -e \cup f$ for $e, f \in \mathrm{Ext}^1(F,F)$. Interpreting the cup product in terms of composition of short exact sequences, we see that $\tau_*(e \cup f) = \tau_*(f) \cup \tau_*(e)$. The fact that τ is symplectic tells us that: $tr(e \cup f) = tr(\tau_*(e) \cup \tau_*(f)) = tr(\tau_*(f \cup e))$. Since $\tau_*(f \cup e) = \lambda f \cup e$, $\lambda \in \mathbb{C}^*$, and $tr(e \cup f) = -tr(f \cup e)$, we get $\lambda = -1$.

Our aim is to describe the tangent cone

$$C(P) = C_{[F]}(P),$$

of P at $[F]$. In order to do so, we consider the action, induced by (3.23), of τ on $(\mathrm{Ext}^1(F,F)/\mathbb{C}^*$. We have a series of inclusions

$$C(P) = C(M^\tau) \subseteq C(M)^\tau \subseteq (\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau, \tag{3.24}$$

where the superscript τ indicates that we are taking the fixed locus of τ.

Proposition 3.3.3. *Let P be a component of $\mathrm{Fix}(\tau)$ of maximal dimensional, and let $F = F_1 \oplus F_2 \in P$ be a polystable sheaf that is the direct sum of two τ-invariant stable sheaves supported on two smooth curves D_1 and D_2 of genus $h_1 = 2g_1 - 1$ and $h_2 = 2g_2 - 1$ respectively that intersect transversally in 2ν points. Let $C(P)$ be the tangent cone of P at $[F]$. Then*

$$C(P) = \mathbb{C}^{2g_1+2g_2-4} \times \mathcal{V}_2(\mathrm{Ext}^1(F_1,F_2))$$

where $\mathcal{V}_2(\mathrm{Ext}^1(F_1,F_2))$ is the affine cone over the degree two Veronese embedding of $\mathbb{P}(\mathrm{Ext}^1(F_1,F_2))$.

Proof. Since the action of τ on $(\mathrm{Ext}^1(F,F)/\mathbb{C}^*$ preserves the product structure (3.21) we have

$$(\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau \cong U_1^\tau \times U_2^\tau \times \mathrm{End}_1(W)^\tau.$$

Moreover, since τ_* interchanges the two summands W and $W^\vee$, the bilinear form on W defined by

$$q(v,w) = \tau^{-1}(v)(w)$$

is non degenerate and hence we can identify $\mathrm{End}_1(W)^\tau$ with the subspace $\mathrm{End}_1(W)^s$ of rank one endomorphisms that are symmetric with respect to q. Notice that this is compatible with the fact that, by construction and the fact that since $\tau(e \cup f) = \tau(f) \cup \tau(e)$ the linear form $A : \mathrm{End}(W) \to \mathbb{C}$ vanishes on the space of symmetric endomorphisms $\mathrm{End}(W)^s$. Since $\mathrm{End}_1(W)^s$ is isomorphic to the affine cone over the degree two Veronese embedding

$$\mathbb{P}W \hookrightarrow \mathbb{P}S^2W$$

of $\mathbb{P}W$, we have

$$\dim \mathrm{End}_1(W)^s = \dim W = \dim \mathrm{Ext}^1(F_1, F_2),$$

which, by Corollary 1.1.8 is equal to 2ν. We claim that

$$\dim U_1^\tau \times U_2^\tau = 2g_1 + 2g_2 - 4.$$

Assuming the claim for a moment, we have

$$\dim(\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau = 2g_1 + 2g_2 - 4 + 2\nu = 2(g-1) = \dim C(P),$$

and since $(\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau$ is irreducible, we may conclude that

$$C(P) = (\mathrm{Ext}^1(F,F)/\mathbb{C}^*)^\tau.$$

To prove the claim, recall that F is a fixed point of the induced action of τ on $M_1 \times M_2$. Since this action preserves the product structure, it follows that U_1^τ and U_2^τ are the tangent spaces to the fixed loci of τ on M_1 and M_2 respectively. Let $P_1 \subset M_1$ and $P_2 \subset M_2$ be the components of the fixed loci of τ in M_1 and M_2 containing F_1 and F_2 respectively. Since F_1 and F_2 are stable, they are smooth points of P_1 and P_2, and since $\dim P_1 = 2(g_1 - 2)$ and $\dim P_2 = 2(g_2 - 2)$ the claim follows. □

The next proposition tells us that locally around F, P is actually isomorphic to its tangent cone.

Proposition 3.3.4. *Let $[F] \in P \subset M$ be a polystable sheaf as in the statement of Proposition 3.3.1. Setting $D_1 = \mathrm{supp}(F_1)$ and $D_2 = \mathrm{supp}(F_2)$, suppose that $2\nu := D_1 \cdot D_2 \geq 3$. Then, locally around $[F]$, the relative Prym variety P is isomorphic to its tangent cone $C_{[F]}(P)$.*

Proof. Let Z denote the irreducible component of the singular locus of M containing $[F]$. Then a neighbourhood $\mathcal{Z}$ of $[F_1 \oplus F_2]$ in Z is isomorphic to $\mathcal{M}_1 \times \mathcal{M}_2$ where, for $i = 1, 2$, $\mathcal{M}_i \subset M_i$ is a suitable analytic neighborhood of $[F_i]$. As in Proposition 4.2 of [40], one can show that the Kuranishi map vanishes identically on $\mathrm{Ext}^1(F_1, F_1) \oplus \mathrm{Ext}^1(F_2, F_2)$ so that under the identification $(\mathcal{M}, [F]) \cong (\kappa^{-1}(0)//G,\ 0)$, the pointed space $(\mathcal{Z}, [F])$ is identified with $(\mathrm{Ext}^1(F_1, F_1) \oplus \mathrm{Ext}^1(F_2, F_2), 0)$. Consider the natural projection

$$\mathrm{Ext}^1(F,F) \to \mathrm{Ext}^1(F_1, F_1) \oplus \mathrm{Ext}^1(F_2, F_2).$$

Since this projection is G-equivariant, and the action of G on $k^{-1}(0)$ is induced by the linear action on $\mathrm{Ext}^1(F,F)$, also the restriction

$$\kappa^{-1}(0) \to \mathrm{Ext}^1(F_1,F_1) \oplus \mathrm{Ext}^1(F_2,F_2),$$

is G-equivariant. It follows that there is an induced morphism (recall that the G-action on the pure part of $\mathrm{Ext}^1(F,F)$ is trivial),

$$p : (\mathcal{M},[F]) \to (\mathcal{Z},[F]) \cong (\mathrm{Ext}^1(F_1,F_1) \oplus \mathrm{Ext}^1(F_2,F_2),0).$$

The inclusion $(\mathcal{Z},[F]) \subset (\mathcal{M},[F])$ is a section of p, and that the fiber of p over $[F]$ is identified with $(\kappa^{-1}(0) \cap \mathrm{Ext}^1(F_1,F_2) \oplus \mathrm{Ext}^1(F_1,F_2))//G$.

Now let Σ denote the irreducible component of the singular locus of P containing $[F]$. Then locally around $[F]$, Σ is isomorphic to $\mathcal{P}_1 \times \mathcal{P}_2 \subset \mathcal{M}_1 \times \mathcal{M}_2$ where, for $i = 1, 2$, $\mathcal{P}_i$ is a suitable neighborhood of $[F_i]$ in the relative Prym variety $P_i \subset M_i$. From Proposition 3.3.3 we know that the tangent cone of P in $[F]$ is locally isomorphic to

$$\mathcal{P}_1 \times \mathcal{P}_2 \times \mathrm{End}_1^s(W). \tag{3.25}$$

The projection $\mathcal{M} \to \mathcal{M}_1 \times \mathcal{M}_2$ induces a morphism $q : \mathcal{P} \to \mathcal{M}_1 \times \mathcal{M}_2$, and since the morphism induced by p at the level of tangent cones is equivariant with respect to the induced action of τ, the image of q in $\mathcal{M}_1 \times \mathcal{M}_2$ has the same dimension as, and therefore is equal to, $\mathcal{P}_1 \times \mathcal{P}_2$. Thus there is a morphism,

$$q : \mathcal{P} \to \mathcal{P}_1 \times \mathcal{P}_2.$$

At the level of tangent cones, q is just the projection onto the first two factors in (3.25). In particular, the fibration is flat and is thus a deformation of the central fiber $q^{-1}(0)$. Since the tangent cone to $q^{-1}(0)$ is isomorphic to $\mathrm{End}_1^s(W)$, we can apply the theorem of Grauert (cf. [24] and [13], Theorem 4.4) and conclude that $q^{-1}(0) \cong \mathrm{End}_1^s(W)$. Moreover, since $\mathrm{End}_1^s(W)$ is isomorphic to the cone over the degree two Veronese embedding of $\mathbb{P}W$, it is rigid as soon as $2\nu = \dim W \geq 3$. It follows, as in the previous case, that locally around $[F]$,

$$\mathcal{P} \cong \mathcal{P}_1 \times \mathcal{P}_2 \times \mathrm{End}_1^s(W).$$

□

We wrap up the results of the section in following Theorem:

Theorem 3.3.5. *Let T be a general Enriques surface and $f : S \to T$ its universal covering. Let $|C|$ be a non-hyperelliptic linear system of genus $g \geq 2$ on T. Set $D = f^{-1}(C)$, fix a non zero integer χ, and set $v = (0, [D], \chi - h + 1)$ with $h = 2g + 1 = g(D)$. Let N be a ι^*-invariant line bundle on S such that $2\chi = D \cdot N$, choose H so that $\tau_N : M_{v,H} \to M_{v,H}$ is a regular morphism, and let P be a component of $\mathrm{Fix}(\tau_N)$ of dimension $2g - 2$. Then P does not admit a symplectic resolution.*

Proof. By Proposition 3.3.1 we know that P contains an H-polystable sheaf $F = F_1 \oplus F_2$ such that F_1 and F_2 are H-stable, τ-invariant and such that $D_1 = \mathrm{supp}(F_1)$ and $D_2 = \mathrm{supp}(F_2)$ are reduced and have no common component. By Proposition 3.3.4 locally around $[F]$, the P is isomorphic to its tangent cone $C_{[F]}(P)$. Finally, using the notation of the last Proposition, we know from Proposition 3.3.3 that P is locally isomorphic to

$$P_1 \times \mathcal{P}_2 \times \mathrm{End}_1^s(W).$$

On the other hand, $\mathrm{End}_1^s(W)$ is isomorphic to the quotient of W by $\mathbb{Z}/2\mathbb{Z}$ acting by multiplication by -1 and thus is $\mathbb{Q}$-factorial. What we need is contained in Example 1.5 (ii) of [61]. Following the notation of Reid, we have

$$n = 2\nu, \text{ and } k = 2,$$

so that $b = \nu$ and $a = 1$. Also notice that the blow up $\widehat{\mathrm{End}}_1^s(W)$ of $\mathrm{End}_1^s(W)$ at the origin is smooth. From Reid's computation, it follows that the discrepancy of the exceptional divisor of the blow up is $\nu - 1$. As $\mathrm{End}_1^s(W)$ is $\mathbb{Q}$-factorial, any resolution factors via the blow up and therefore its discrepancy is bounded from below by the discrepancy of the blow up. In conclusion, the singularity is terminal if $\nu > 1$. Moreover, when $\nu = 1$ the singularity is canonical but not terminal.

□

Remark 3.3.6. It was proved by Kaledin [31] and by Namikawa [51], that if $w : X \to Y$ is a birational projective morphism from a smooth projective symplectic n-fold X to a normal variety Y, then w is a semi-small map, i.e. if Y_i denotes the set of points $y \in Y$ such that $\dim w^{-1}(y) = i$, then $\dim Y_i \leq n - 2i$. This, in particular, implies that if Y is $\mathbb{Q}$-factorial then $\mathrm{codim}\,\mathrm{Sing}(Y) \leq 2$. From this observation it follows that if we knew by other means that P (notation as above) were

normal and $\mathbb{Q}$-factorial at a general singular point, then we could conclude again that P has no symplectic resolution.

We finish this section with the following example of a situation where τ_H does not extend to a regular involution.

Example 3.3.7. Let $e, e' \subset T$ be two primitive elliptic curves on a general Enriques surface with $e \cdot e' = 2$ (for the existence of such a pair of curves see for example [16]). Set,

$$C = e + e',\ E = f^{-1}(e),\ E' = f^{-1}(e), \quad \text{and} \quad D = E + E'.$$

Since $g(C) = 3$ and $h = g(D) = 5$, we have $|C| \cong \mathbb{P}^2$ and $|D| = \mathbb{P}^5$. It is known (cf. [71]) that the only reducible curves in $|C|$ are the union of two primitive elliptic curves meeting in two points. Set $v = (0, [D], -4)$ and let H be a v-generic polarization that is adjacent to the wall containing D. Since the only reducible curves in $|C|$ are union of two primitive elliptic curves meeting in two points, the singularities of $P_{v,D}$ correspond to polystable sheaves of the form

$$F_1 \oplus F_2, \tag{3.26}$$

with $\dim \operatorname{Ext}^1(F_1, F_2) = 4$ and, for $i = 1, 2$, $\tau(F_i) = F_i$. It follows that, locally around $[F]$, $P_{v,D}$ is isomorphic to the cone over the degree two Veronese embedding of $\mathbb{P}^3$. We already noticed that the natural morphism

$$r : M_{v,H} \to M_{v,D}$$

is a resolution of the singularities of $M_{v,D}$. This map induces a birational projective morphism

$$s : P_{v,H} \to P_{v,D},$$

which we claim is a resolution of the singularities of $P_{v,D}$. Indeed, if $F \in M_{v,D}$ be a sheaf as in 3.26, then the fiber

$$r^{-1}([F]) \cong \mathbb{P}^3$$

is canonically identified with the set of H-stable sheaves whose Jordan-Hölder graded pieces with respect to D are isomorphic to F, and is hence equal to $\mathbb{P}\operatorname{Ext}^1(F_1, F_2)$ or $\mathbb{P}\operatorname{Ext}^1(F_2, F_1)$ depending

on which chamber H belongs to. One can check that

$$s^{-1}([F]) \cong \mathbb{P}^3.$$

It follows that s is just the blow up of $P_{v,D}$ in $[F]$ and that $P_{v,H}$ is smooth. Since s is not crepant, it follows that the holomorphic two form on $P_{v,H}$ is degenerate along the exceptional divisors. In particular, the birational the involution $\tau_{-D} : M_{v,H} \dashrightarrow M_{v,H}$ is not a regular morphism.

3.4 The hyperelliptic case

In this section, we will prove that the degree zero relative Prym variety of a hyperelliptic linear system is birational to a hyperkähler manifold. Depending on the choices of the polarization, in some case it turns out that it is actually a smooth compact hyperkähler manifold and in others we show that it admits a symplectic resolution. The hyperkähler manifolds arising from degree zero relative Prym varieties are all of $\mathrm{K3}^{[n]}$ type.

As usual, we let $f : S \to T$ be a general K3-Enriques cover, and we let $|C| = |ne_1 + e_2|$ be a hyperelliptic linear system on T of genus $g = n + 1 \geq 2$ (cf. Section 3.1 for the notation and the definitions). If we set $D = f^{-1}(C)$, then $|D|$ is a hyperelliptic linear system of genus $h = 2g - 1 = 2n + 1$. Set

$$v = (0, [D], -h + 1),$$

choose a polarization H on S, and consider the birational symplectic involution

$$\tau = \tau_{-D} : M_{v,H} \dashrightarrow M_{v,H},$$

defined in Section 2.1. Finally set,

$$M = M_{v,H}\,, \quad \text{and} \quad P = P_{v,H}.$$

The first result we want to prove is that, in the hyperelliptic case, the Prym involution τ on M comes from an actual involution on S.

Proposition 3.4.1. *Let $|C| = |ne_1 + e_2|$ be a hyperelliptic linear system on T and let $|D| = |f^*(C)|$.*

Set $v = (0, [D], -h+1)$, $h = g(D) = 2n+1$. Then there exists a symplectic involution

$$k : S \to S,$$

such that, for any ι^*-invariant polarization H, the birational involution

$$\tau : M \dashrightarrow M$$

defined in Section 2.1 concides with the birational involution

$$\begin{aligned} k^* : M &\dashrightarrow M \\ F &\mapsto k^*F. \end{aligned}$$

The proof of this proposition consists in defining the involution k. The linear system $|C|$ has two simple base points (Prop 4.5.1 of [17]) and defines a degree two map of T onto a degree $n-1$ surface in $\mathbb{P}^n$. The two base points are

$$e_1 \cap e_2', \text{ and } e_1' \cap e_2, \text{ if } n \text{ is odd}, \tag{3.27}$$

or

$$e_1 \cap e_2, \text{ and } e_1' \cap e_2', \text{ if } n \text{ is even}. \tag{3.28}$$

Following the notation in the Section 3.1 we have

$$E_i = f^{-1}(e_i), \text{ and } E_i' = f^{-1}(e_i'), \quad i = 1, 2.$$

Set $h = 2n+1$. The genus h linear system $|D| = |nE_1 + E_2|$ is also hyperelliptic, with the g^1_2 cut out by the elliptic pencil $|E_1|$. Consider the rational map

$$\varphi = \varphi_D : S \to R \dashrightarrow \mathbb{P}^h$$

associated to the linear system $|D|$. It is a degree two rational map onto a rational normal scroll of degree $h-1$. Let us denote by

$$\ell : S \to S,$$

the anti-symplectic involution defined by φ. Notice that any curve in $|D|$ is ℓ-invariant, and that ℓ induces the hyperelliptic involution.

The following result is well known.

Lemma 3.4.2. *Let D be a smooth hyperelliptic curve, and let $\ell : D \to D$ be the hyperelliptic involution. Then ℓ^* acts as $-id$ on* $\mathrm{Pic}^0(D)$.

Proof. Consider a degree m effective divisor $\mathcal{O}(\sum d_i)$ on D. Then $\mathcal{O}(\sum d_i)+j^*\mathcal{O}(\sum d_i) \sim mL$, where L is the hyperelliptic pencil. Since any degree zero line bundle can be written as the difference of two effective divisors of the same degree, the lemma follows. □

Finally, observe that the sub-linear system

$$W = f^*|C| \subset |D| \tag{3.29}$$

has 4 simple base points, that are the inverse image of (3.27) or of (3.28), By the same reasoning, the sub-linear system $W' = f^*|C'| \subset |D|$ has 4 simple base points that are the inverse image of (3.28) or of (3.27). Denote by $\{w_1, \ldots, w_4\}$ the base points of W and by $\{w'_1, \ldots, w'_4\}$ the base points of W'.

Lemma 3.4.3. *The two involutions ℓ and ι commute and their composition*

$$k = \iota \circ \ell, \tag{3.30}$$

is a symplectic involution with eight fixed points:

$$\{w_1, \ldots, w_4, w'_1, \ldots, w'_4\}. \tag{3.31}$$

Proof. It is sufficient to prove that for any smooth ι-invariant curve Γ in $|D|$, the $k_{|\Gamma}$ is an involution on Γ. But this is clear, since, $\ell^* = -id$ on $\mathrm{Pic}^0(\Gamma)$ and hence $(k^2)^*$ is the identity on $\mathrm{Pic}^0(\Gamma)$.

Since k is the composition of two anti-symplectic involutions, it is symplectic. As such, k has eight fixed points, which we readily describe. For simplicity, let us focus our attention on the two points $\{p, q\} = E'_1 \cap E_2$. By construction, we know that $\iota(p) = q$. Notice, however, that we can choose a smooth curve D in the linear system $|nE_1 + E_2|$ passing through those two points so that

$D \cap E_1 = \{p, q\}$. Since $|E_1|$ induces the g^1_2 on D, it follows that $\ell(p) = q$ and thus

$$k(p) = p, \text{ and } \ k(q) = q.$$

We can now argue in the same way for the remaining points of (3.31). □

This Lemma and the fact that $\ell^* = -1$ on $\mathrm{Pic}^0(D)$ for a smooth D, show that $\tau = k^*$ on a dense open subset of M, proving Proposition 3.4.1.

Remark 3.4.4. *If we choose $H = D$, so that τ is a morphism, then so is k^*. In fact, in this case, the linear system $|D|$ is k^*-invariant, and hence pulling back via k respects D-stability.*

Recall from Section 2.1, Proposition 2.2.6 that it is not possible to choose H such that $M = M_{v,H}$ is smooth and the map $F \mapsto \mathcal{E}xt^1(F, \mathcal{O}(-D))$ respects stability. It is therefore natural to ask wether there are choices of H such that $M_{v,H}(S)$ is smooth and, at the same time, k^* is, in fact, a regular morphism. Equivalently, we ask whether there exists a k^*-invariant ample class which is also v-generic. This would mean that the *a priori* only birational map τ can, in fact, be extended to a regular morphism.

To describe the v-generic polarization, we need to identify the walls.

Lemma 3.4.5. *The equations defining the walls relative to the Mukai vector $v = (0, D, \chi)$, with $D = nE_1 + E_2$, are of the form*

$$\frac{sE_1 \cdot x + \epsilon E_2 \cdot x}{nE_1 \cdot x + E_2 \cdot x}\ \chi = m, \tag{3.32}$$

with m ranging in a finite sets of integers, $\epsilon \in \{0, 1\}$ and $s = 0, \ldots, n-1$.

Proof. The finiteness of the number of walls follows from [74]. Each sub-curve of $|D|$ belongs to a linear subsystem of type $|sE_1 + \epsilon E_2|$, with s ranging from 0 to n, and $\epsilon \in \{0, 1\}$. Up to passing to a residual series, we may assume that $\epsilon = 1$. Clearly, equations (3.32) are satisfied by strictly semi-stable sheaves F with $[c_1(F)] = [sE_1 + E_2]$ and $\chi(F) = m$. The above equations are also sufficient for the existence of strictly semistable sheaves. Indeed, consider a smooth curve $\Gamma \in |sE_1 + E_2|$ and a smooth curve $\overline{\Gamma} \in |E_1|$. If equation (3.32) holds for some ample H with $[H] = x$ and some integer m, then we can choose a torsion free H-stable sheaf F_Γ on Γ with Euler characteristic equal to m, and an H-semi-stable sheaf $F_{\overline{\Gamma}}$ of rank $(n - s)$ on $\overline{\Gamma}$ such that $\chi(F_{\overline{\Gamma}}) = \chi - m$. The sheaf $F = F_\Gamma \oplus F_{\overline{\Gamma}}$ is then strictly H-semistable. Since the above are all the possible sub curves of $|D|$, there are no other walls.

□

We now turn to the question regarding the k^*-invariance of the polarization. With this question in mind, we describe the action of k^* on $\mathrm{NS}(S)$. Notice that since the involutions ι, ℓ and k commute, the induced actions on the Néron-Severi group are compatible. For a lattice Λ and an involution ϵ acting on Λ, we denote by Λ^{ε} and by $\Lambda^{-\varepsilon}$ the invariant and anti-invariant sub-lattices, respectively. Note that Λ^{ε} and $\Lambda^{-\varepsilon}$ are primitive in Λ.

Recall that $\mathrm{NS}(T) \cong U \oplus E_8(-1)$ and that $f^* \mathrm{NS}(T)$ is a primitive sub-lattice of $H^2(S,\mathbb{Z})$. By Proposition 3.1.10, it follows that $\mathrm{NS}(R) \cong U$ and that $\varphi^* \mathrm{NS}(R) = \langle E_1, E_2 \rangle$ is also primitive in $H^2(S,\mathbb{Z})$. For simplicity we will indicate by $\mathrm{NS}(T)$ and by $\mathrm{NS}(R)$ their respective pull-backs in $\mathrm{NS}(S)$.

Lemma 3.4.6. $\mathrm{NS}(S)^{k^*} = \langle E_1, E_2 \rangle \oplus (\mathrm{NS}(T)^{\perp})^{k^*}$, *so that if T is a general Enriques surface then* $\mathrm{NS}(S)^{k^*} = \langle E_1, E_2 \rangle$.

Proof. Since $\mathrm{NS}(S)^{\ell^*} = \mathrm{NS}(R) = \langle E_1, E_2 \rangle \subset \mathrm{NS}(T)$, we have $\langle E_1, E_2 \rangle = \mathrm{NS}(T)^{\ell^*} \cong \mathrm{NS}(S)^{\ell^*}$. Moreover, by [52], [46], [70]

$$H^2(S,\mathbb{Z})^{k^*} \cong U^{\oplus 3} \oplus E_8(-2), \text{ and } H^2(S,\mathbb{Z})^{-k^*} \cong E_8(-2).$$

Since k^* preserves the $(2,0)$ part of the Hodge decomposition we have

$$H^2(S,\mathbb{Z})^{-k^*} \subset H^{1,1}(S) \cap H^2(S,\mathbb{Z}) = \mathrm{NS}(S)\,.$$

In particular, since $\langle E_1, E_2 \rangle = \mathrm{NS}(T)^{\ell^*} = \mathrm{NS}(T)^{k^*}$, we have

$$\mathrm{NS}(S)^{k^*} = \langle E_1, E_2 \rangle \oplus (\mathrm{NS}(T)^{\perp})^{k^*}.$$

It follows that if the K3 surface satisfies $\mathrm{NS}(S) = \mathrm{NS}(T)$, then $\mathrm{NS}(S)^{k^*}$ is spanned by the classes of E_1 and E_2. This certainly happens if T is general (but if the Picard number of S is strictly greater than 10, then there are k^* invariant classes that do not come from T). □

Corollary 3.4.7. *Let T be a general Enriques surface. Let $\mathcal{W} \subset \mathrm{Amp}(S)$ be the union of all the walls. Then*

$$\mathrm{Amp}(S) \cap \mathrm{NS}(S)^{k^*} \setminus \mathcal{W} \cap \mathrm{NS}(S)^{k^*},$$

is non empty.

Proof. Since there are finitely many walls, it is sufficient to check that none of the equations (3.32)

vanish identically on $\mathrm{NS}(S)^{k^*} = \langle E_1, E_2 \rangle$. To this aim, consider $H \in \mathrm{Amp}(S) \cap \langle E_1, E_2 \rangle$. Then $H = aE_1 + bE_2$ for some positive integers a and b, so that the restrictions of the equations to $\mathrm{Amp}(S) \cap \mathrm{NS}(R)$ are

$$\frac{bk + \epsilon a}{bn + a}\chi = m, \tag{3.33}$$

and these are not identically zero. □

Proposition 3.4.8. *Keeping the notation of Theorem 3.4.1, we can always find an ample divisor H which is k^*-invariant and v-generic. In other words, there exists a polarization H such that $M_{v,H}$ is smooth and such that*

$$k^* : M_{v,H} \longrightarrow M_{v,H},$$

is a regular involution. In particular, if $H \in \mathrm{NS}(R) \setminus \mathcal{W} \cap \mathrm{NS}(R)$, the a priori *birational involution $\tau = \tau_{-D} : M_{v,H} \longrightarrow M_{v,H}$, extends to a regular morphism and the relative Prym variety $\mathrm{Prym}_{v,H}(\mathcal{D}/\mathcal{C})$ is smooth.*

Proof. It is sufficient to consider $H \in \mathrm{Amp}(S) \cap \mathrm{NS}(S)^{k^*} \setminus \mathcal{W} \cap \mathrm{NS}(S)^{k^*}$ which is non empty by the above Corollary. □

Since changing the polarization H does not change the birational class of the relative Prym variety and the above proposition ensures the existence of smooth Prym varieties, we can sum up the results we obtained so far in the following

Theorem 3.4.9. *Let T be an Enriques surface and $f : S \to T$ the universal covering. Let $|C| = |ne_1 + e_2|$ be a hyperelliptic linear system on T and let $|D| = |f^*(C)|$. Set $v = (0, [D], -h + 1)$, $h = g(D)$. For any polarization H on S, the relative Prym variety $P_{v,H} = \overline{\mathrm{Fix}^0(\tau)} = \overline{\mathrm{Fix}^0(k^*)}$, where $\tau = \tau_{-D}$ is birational to a projective hyperkähler manifold.*

Corollary 3.4.10. *The singular symplectic variety $P_{v,D}$ admits a symplectic resolution.*

Proof. First, recall that D is not v-generic. In fact, since the genus h of D is odd, $\chi = -h + 1$ is even, and hence D lies on the wall with equation

$$\frac{E_2 \cdot x}{nE_1 \cdot x + E_2 \cdot x}\ \chi = 2.$$

It is possible to choose an ample H as in Proposition 3.4.8 and such that, moreover, it lies in a v-chamber adjacent to the wall (or the intersection of walls) where D lies. Under these assumptions,

there is a natural projective birational morphism

$$\varepsilon : M_{v,H} \to M_{v,D},$$

which is a resolution of the singularities of $M_{v,D}$. Since ε commutes with both $k^* : M_{v,H} \to M_{v,H}$ and $\tau : M_{v,D} \to M_{v,D}$, there is an induced proper morphism

$$\varepsilon : P_{v,H} \to P_{v,D}.$$

which is still birational. Since $P_{v,H}$ is smooth and symplectic, it is a symplectic resolution of the singularities of $P_{v,D}$. □

The natural question is now to determine the deformation class of these smooth relative Pryms. As one may expect they are birational, and thus by [29] deformation equivalent, to moduli spaces of sheaves on the minimal resolution of the quotient of S by k.

To fix notation, let

$$\rho : S \to \overline{S} := S/\langle k \rangle,$$

be the quotient morphism. Then $\overline{S}$ is a singular K3 surface with 8 rational double points and we let

$$\eta : \widehat{S} \to \overline{S},$$

be its minimal resolution. It is well known that $\widehat{S}$ is a K3. Observe that a divisor in $|D|$ is k-invariant if and only if it is ι-invariant. In particular, any D in W is k-invariant so that if we set $\overline{D} = D/k$ there is an obvious bijection $W \cong |\overline{D}|$. The general curve D in W does not contain the points $\{w_1', \ldots, w_4'\}$ so that for $D \in W$, the double cover $D \to \overline{D}$ ramifies only in $\{w_1, \ldots, w_4\}$ and

$$g(\overline{D}) = \frac{h-1}{2} = g - 1.$$

Moreover, if $D \in W$ is smooth, then so is $\overline{D}$. In this case, the proper transform

$$\widehat{D} := \eta_*^{-1}(\overline{D}) \subset \widehat{S},$$

is also smooth and isomorphic to $\overline{D}$. Consider now the following commutative diagram,

$$\begin{array}{ccc} \widehat{k}\circlearrowright Z & \xrightarrow{\widehat{\rho}} & \widehat{S} \\ \downarrow \widehat{\eta} & & \downarrow \eta \\ k\circlearrowright S & \xrightarrow[\rho]{} & \overline{S} = S/k\,, \end{array}$$

where $Z \to S$ is the blow up of the surface at the 8 fixed points of the involution, and $Z \to \widehat{S}$ is the double cover ramified along the eight exceptional curves of η. We denote by

$$\widehat{k} : Z \to Z$$

the lift of k to Z so that $\widehat{k}$ is an involution on Z whose fixed locus is the union of the exceptional divisors. Let $R_1, \ldots, R_4$ be the exceptional divisors mapping to $\rho(w_1), \ldots, \rho(w_4)$, and $R'_1, \ldots, R'_4$ the exceptional divisors mapping to $\rho(w'_1), \ldots, \rho(w'_4)$. Since the general curve in $|D|$ does not pass through $w'_1, \ldots, w'_4$ and is smooth,

$$\widehat{D} \cdot R_i = 1, \tag{3.34}$$

and $\widehat{D} \cdot R'_i = 0$.

The general curve Γ in $\widehat{\rho}^*|\widehat{D}|$ is a smooth double cover of a curve in $|\widehat{D}|$, and, via $\widehat{\eta}$, maps isomorphically to its image in S. Indeed, $\widehat{\eta}$ induces an isomorphism

$$|\Gamma| \supset \widehat{\rho}^*|\widehat{D}| \cong W \subset |D|. \tag{3.35}$$

Theorem 3.4.11. *Set $w = (0, \widehat{D}, -g+2)$, and let H and $\widehat{H}$ be two ample line bundles on S and $\widehat{S}$ respectively. There is a rational map*

$$\begin{aligned} \psi : M_{w,\widehat{H}}(\widehat{S}\,) &\dashrightarrow M_{v,H}(S), \\ F &\mapsto \widehat{\eta}_*\widehat{\rho}^*F. \end{aligned} \tag{3.36}$$

defined on the open set of $M_{w,\widehat{H}}(\widehat{S}\,)$ parametrizing sheaves supported on irreducible curves. This map factors via the inclusion $P_{v,H} \subset M_{v,H}(S)$, and the induced map

$$\phi : M_{w,\widehat{H}}(\widehat{S}\,) \dashrightarrow P_{v,H}, \tag{3.37}$$

is birational.

Proof. For our purposes, it is enough to restrict our attention to the open subset of $M_{w,\widehat{H}}(\widehat{S})$ parametrizing sheaves with smooth support. Let $\mathcal{F}$ a family of pure sheaves of dimension one on S with Mukai vector w, supported on smooth curves, and parametrized by a scheme B. Then $\widehat{\eta}_{B*}\widehat{\rho}_B^*\mathcal{F}$ is a flat family of H-stable sheaves. Clearly, $\widehat{\rho}_B^*\mathcal{F}$ is flat over B and by formula (3.34) the support Γ_b of $\widehat{\rho}^*\mathcal{F}_b$ is smooth. If $\mathcal{E}$ is any flat family of pure dimension one sheaves on Z, with support on smooth curves belonging to $|\Gamma|$, parametrized by B, then $\widehat{\eta}_{B*}\mathcal{E}$ is a flat family of H-stable sheaves with support in W. This defines the rational map (3.36). Since $\widehat{k}$ is a lift of k, the two involutions coincide where $\widehat{\eta}$ is an isomorphism. Let $[F]$ be a point in $M_{w,\widehat{H}}(\widehat{S})$, since $\widehat{\rho}^*F$ is $\widehat{k}^*$-invariant, it follows that

$$\widehat{\eta}_*\widehat{\rho}^*F \in \mathrm{Fix}(k^*).$$

Hence ψ factors through the inclusion $P_{v,H} \subset M_{v,H}(S)$. The last assertion to prove is that the induced map $\phi : M_{w,\widetilde{H}}(\widetilde{S}) \dashrightarrow P_{v,H}$ is birational. As above, we assume that $\mathrm{supp}(F)$ is smooth. First, we show that ϕ is a local isomorphism at $[F]$. We claim that the induced map on tangent space

$$d\psi : \mathrm{Ext}^1_{\widetilde{S}}(F,F) \to \mathrm{Ext}^1_Z(\widehat{\rho}^*F, \widehat{\rho}^*F) \cong \mathrm{Ext}^1_S(\widehat{\eta}_*\widehat{\rho}^*F, \widehat{\eta}_*\widehat{\rho}^*F). \tag{3.38}$$

is injective. In fact, given a non trivial extension $0 \to F \to G \to F \to 0$, we can pull it back to Z to obtain a short exact sequence

$$0 \to \widehat{\rho}^*F \to \widehat{\rho}^*G \to \widehat{\rho}^*F \to 0. \tag{3.39}$$

If this sequence were split, the same would be true for

$$0 \to \widehat{\rho}_*\widehat{\rho}^*F \to \widehat{\rho}_*\widehat{\rho}^*G \to \widehat{\rho}_*\widehat{\rho}^*F \to 0.$$

However, this sequence is the direct sum of (3.39) and of

$$0 \to \widehat{\rho}^*F \otimes L \to \widehat{\rho}^*G \otimes L \to \widehat{\rho}^*F \otimes L \to 0,$$

where[4]

$$L := \frac{1}{2}\mathcal{O}_{\widetilde{S}}(-\sum R_i - \sum R_i').$$

Since these two exact sequence are non split by assumption, we get a contradiction. Hence, the

[4]Recall that $\sum R_i + \sum R_i'$ is divisible by two in $\mathrm{NS}(\widetilde{S})$.

induced map (3.38) is injective and ϕ is a local isomorphism. To end the proof of the theorem, we just need to prove that the degree of ϕ is one. It is enough to prove that if F_1 and F_2 are two sheaves on $\widehat{S}$ with Mukai vector w, then

$$\widehat{\rho}^* F_1 \cong \widehat{\rho}^* F_2, \quad \text{if and only if} \quad F_1 \cong F_2.$$

This follows from the projection formula. In fact, if $\widehat{\rho}^* F_1 \cong \widehat{\rho}^* F_2$, then

$$F_1 \oplus (F_1 \otimes L) \cong \widehat{\rho}_* \widehat{\rho}^* F_1 \cong \widehat{\rho}_* \widehat{\rho}^* F_2 \cong F_2 \oplus (F_2 \otimes L).$$

Since, for $i = 1, 2$, both F_i and $F_i \otimes L$ are stable and since $\deg F_i \otimes L \neq \deg F_j$, $i, j = 1, 2$ we must have an isomorphism $F_1 \cong F_2$.

□

Corollary 3.4.12. *Let $D = f^*C$ and $v = (0, [D], -h+1)$. If $|C|$ is a hyperelliptic linear system, and H is v-generic and k^*-invariant, the symplectic variety $P_{v,H}$ is an irreducible holomorphic symplectic manifold of K3 $^{[n]}$-type.*

Corollary 3.4.13. *Let $D = f^*C$, $v = (0, [D], -h + 1)$ and let H be a non v-generic polarization. If $|C|$ is a hyperelliptic linear system, then any resolution of $P_{v,H}$ is an irreducible holomorphic symplectic manifold that is of of K3 $^{[n]}$-type.*

3.4.1 The surface case

Let us now describe in more detail the relative Prym variety associated to a genus 2 linear system $|C| = |e_1 + e_2|$ on an Enriques surface T. Recall that $|C| \cong \mathbb{P}^1$, and that the linear system has two simple base points. If we assume the pair (T, C) to be general (cf. [62]) the linear system $|C| = \mathbb{P}^1$ has exactly 16 irreducible curves with one single node, and 2 reducible curves $e_1 + e_2$ and $e_1' + e_2'$ that are the union of two smooth elliptic curves meeting transversally in one point.

As usual we look at the 2-sheeted K3 cover $f : S \to T$ and we set $|D| = |f^*C|$, and $v = (0, [D], -2)$. We also choose a v-generic polarization H. We consider the involution $\tau = \tau_{-D} : M_{v,H} \to M_{v,H}$ and our goal is to describe $P_{v,H}$. More precisely, we will prove the following theorem.

Theorem 3.4.14. *The relative Prym variety $P_{v,H}$ is an elliptic K3 surface whose singular fibers consist in 16 irreducible curves with one node, and two fibers of type I_4 (i.e. a closed chain of 4 rational curves $R_1, \ldots, R_4$ with $R_i \cdot R_{1+1} = 1$, $i = 1, \ldots, 3$ and $R_4 \cdot R_1 = 1$).*

Proof. First of all, one can check that the set of v-generic polarizations is the union of two chambers $\mathcal{C}^+ = \{x \cdot E_1 > x \cdot E_2\}$, $\mathcal{C}^- = \{x \cdot E_1 < x \cdot E_2\}$. Since the isomorphism class of $P_{v,H}$ does not change as H varies within a chamber, and since both chambers have non zero intersection with the $\mathrm{NS}(S)^{k^*}$, we can assume that the v-generic polarization H is also k^*-invariant. Under this assumption, the symplectic involution k^* is regular and hence $P_{v,H} = \mathrm{Fix}^\circ(k^*)$ is an elliptic K3 surface (cf. Proposition 3.4.8). The fact that the fibration $P_{v,H} \to |C| \cong \mathbb{P}^1$ has 16 irreducible nodal fibers follows from the fact that there are exactly 16 curves in $|C|$ with one single node, from the fact that the induced double cover of these curves is irreducible, and from Corollary 2.2.2. What is left to prove is that the two fibers of $\nu_H : P_{v,H} \to |C|$ over the curve classes $[e_1 + e_2]$ and $[e_1' + e_2']$ consist each in a closed chain of four rational curves (as in the statement of the theorem). The proof of this fact involves three steps. First we show that the fibers Γ_1 and Γ_2 of $\nu_D : P_{v,D} \to |C|$ over $[e_1 + e_2]$ and $[e_1' + e_2']$ consist each in the union of two rational curves meeting transversely in two points. Secondly, we show that for each one of these two fibers the points where the two rational curves meet are singular points for the surface $P_{v,D}$. Finally we show that under the resolution

$$r : P_{v,H} \to P_{v,D}, \tag{3.40}$$

the preimage of both Γ_1 and Γ_2 consists in a closed chain of four rational curves. Once we prove the first and the second statement, the last one follows from the analysis carried out in Section 1.2.2, in particular from the fact that the singularities of $P_{v,D}$ at the points in question are of type A_1.

We will prove the steps above for the fiber over $[e_1 + e_2] \in |C|$, and for $H \in \mathcal{C}^+$. For the first statement, we look at the Lagrangian fibration $\pi_D : M_{v,D} \to |D|$ and we consider the restriction of the regular involution $\tau : M_{v,D} \to M_{v,D}$ to the fiber $\pi_D^{-1}([E_1 + E_2])$, which is just $\overline{\mathrm{Jac}}^0_D(E_1 + E_2)$, i.e. the moduli space of D-semi stable, rank one, degree zero torsion free sheaves . By the choice of polarization, this moduli space is irreducible (cf. Example 1.1.17) and the open locus of D-stable sheaves is the degree-$(0,0)$ generalized Jacobian, i.e. the $\mathbb{C}^*$-bundle over $\mathrm{Jac}^0(E_1) \times \mathrm{Jac}^0(E_2)$ defined by the exact sequence

$$0 \to \mathbb{C}^* \to \mathrm{Jac}^{(0,0)}(E_1 + E_2) \stackrel{n^*}{\to} \mathrm{Jac}^0(E_1) \times \mathrm{Jac}^0(E_2) \to 0, \tag{3.41}$$

where $n : E_1 \coprod E_2 \to E_1 + E_2$ is the normalization morphism separating the two components. To compactify the locus of D-stable sheaves one needs to add the set of $\mathcal{S}$-equivalence classes of D-semi stable sheaves, i.e., the set of isomorphism classes of D-polystable sheaves. This is exactly the set

of isomorphism classes of sheaves of the form $F = F_1 \oplus F_2$ with $\mathrm{supp}(F_i) = E_i$ and $\mu_D(F_i) = -1/2$, $i = 1, 2$, which this is just a copy of $\mathrm{Jac}^{-1}(E_1) \times \mathrm{Jac}^{-1}(E_2)$. From another viewpoint, the complete moduli space is obtained by compactifying the $\mathbb{C}^*$-bundle to a $\mathbb{P}^1$-bundle and then glueing the 0 and ∞ sections by twisting with the 2-torsion point $\mathcal{O}_{E_1}(x_1 - \iota x_1) \otimes \mathcal{O}_{E_2}(x_2 - \iota x_2)$, where x_1 and x_2 denote points on E_1 and E_2 (respectively) that are identified via n (cf. [54]). We need the following proposition.

Proposition 3.4.15. *With the assumptions made at the beginning of 3.4.1, the fixed locus of the restriction of τ to the fiber*

$$\pi_D^{-1}([E_1 + E_2]) \cong \overline{\mathrm{Jac}}{}^0_D(E_1 + E_2)$$

is the union of 4 one-dimensional components and of 8 isolated points. The one-dimensional components are the union of two rational curves meeting transversally in two points (an analogous statement holds for $\pi_D^{-1}([E'_1 + E'_2])$).

Assuming this proposition we conclude the proof of Theorem 3.4.14. Notice that the sheaves in $\mathrm{Sing}(\overline{\mathrm{Jac}}{}^0_D(E_1+E_2)) \cong \mathrm{Jac}^{-1}(E_1) \times \mathrm{Jac}^{-1}(E_2)$ correspond to $\mathcal{S}$-equivalence classes of D-semi stable sheaves and are singular points of $M_{v,D}$. Since for sheaf $[F = F_1 \oplus F_2] \in \mathrm{Jac}^{-1}(E_1) \times \mathrm{Jac}^{-1}(E_2)$, we have $\dim \mathrm{Ext}^1(F_1, F_2) = E_1 \cdot E_2 = 2$, from the proof of Theorem 3.3.5 we see that the singularity of $P_{v,D}$ at each of these points is a surface singularity of type A_1.

To end the proof of the theorem, we just need to prove that the resolution $P_{v,H} \to P_{v,D}$ is minimal. To see this, first observe that $\overline{\mathrm{Jac}}{}^0_H(E_1 + E_2)$ has two irreducible components, which are the closure of the generalized Jacobians of multi-degree $(0, 0)$ and $(1, -1)$ respectively (cf. [54]). The natural morphism (which is just the restriction of the natural morphism $M_{v,H} \to M_{v,D}$ to this fiber),

$$r : \overline{\mathrm{Jac}}{}^0_H(E_1 + E_2) \to \overline{\mathrm{Jac}}{}^0_D(E_1 + E_2),$$

contracts the $(1, -1)$-component to the singular locus, because the sheaves of multi-degree $(1, -1)$ are D-semi-stable and are $\mathcal{S}$-equivalent to a D-polystable sheaf of multi-degree $(-1, -1)$. Hence, for any point $(L_1, L_2) \in \mathrm{Jac}^{-1}(E_1) \times \mathrm{Jac}^{-1}(E_2)$, the fiber $r^{-1}((L_1, L_2))$ is just a fiber of the $\mathbb{C}^*$-bundle $\mathrm{Jac}^{(1,-1)}(E_1 + E_2) \to \mathrm{Jac}^1(E_1) \times \mathrm{Jac}^{-1}(E_2)$, so that the exceptional divisor of $P_{v,H} \to P_{v,D}$ over a τ-invariant polystable point $L_1 \oplus L_2$ is just the $\mathbb{P}^1$-compactification of $\mathbb{C}^* \cong r^{-1}((L_1, L_2))$. □

We now turn to the proof of Proposition 3.4.15.

Proof. First, we analyze the restriction of the fixed locus of τ to the open subset $\mathrm{Jac}^{(0,0)}(E_1+E_2)$. The action of τ on $\mathrm{Jac}^0(E_1)\times\mathrm{Jac}^0(E_2)$ preserves the fibration structure of (3.41), and the induced action on the base of that fibration is given by

$$\mathrm{Jac}^0(E_1)\times\mathrm{Jac}^0(E_2)\to\mathrm{Jac}^0(E_1)\times\mathrm{Jac}^0(E_2).$$
$$(F_1,F_2)\mapsto(\iota^*F_1^\vee,\iota^*F_2^\vee)$$

By abuse of notation we denote this involution again by the symbol τ. The action of ι^* on $\mathrm{Jac}^0(E_1)\times\mathrm{Jac}^0(E_2)$ is trivial, so that $\tau\equiv -id$ and hence the fixed locus of τ on $\mathrm{Jac}^0(E_1)\times\mathrm{Jac}^0(E_2)$ is exactly equal to the locus of 2-torsion points. In particular, if L is a τ-invariant, degree zero, line bundle on E_1+E_2 its restrictions L_1 and L_2 to E_1 and E_2, respectively, are 2-torsion points of the corresponding Jacobians, i.e., $n^*\,\mathrm{Fix}(\tau)\subset\mathrm{Jac}^0(E_1)[2]\times\mathrm{Jac}^0(E_2)[2]$. Set

$$E_1\cap E_2=\{x,\iota x\},\ \ E_1\cap E_2'=\{p,\iota p\},\quad\text{and}\quad E_1\cap E_2'=\{q,\iota q\}.$$

To each one of the points above, we append a subscript $i=1$ or 2 if we consider it as a point of E_1 or E_2. The sets of two-torsion points is given as follows

$$\mathrm{Jac}^0(E_1)[2]=\{\mathcal{O}_{E_1},\mathcal{O}_{E_1}(x_1-p_1),\mathcal{O}_{E_1}(\iota x_1-p_1),\mathcal{O}_{E_1}(\iota x_1-x_1)\}$$

$$\mathrm{Jac}^0(E_2)[2]=\{\mathcal{O}_{E_2},\mathcal{O}_{E_2}(x_2-q_2),\mathcal{O}_{E_2}(\iota x_2-q_2),\mathcal{O}_{E_2}(\iota x_2-x_2)\}$$

We look at the fibration (3.41). For each of the 16 two-torsion points of $\mathrm{Jac}^0(E_1)[2]\times\mathrm{Jac}^0(E_2)[2]$ we want to find which points in the corresponding fiber is fixed by τ. We start by recalling that a degree zero line bundle L on E_1+E_2 fits into an exact sequence

$$0\to L\to L_1\oplus L_2\to\mathbb{C}_x\oplus\mathbb{C}_{\iota x}\to 0,$$

which induces homomorphisms

$$(L_1\oplus L_2)_{|x}\xrightarrow{(a,b)}\mathbb{C}_x,\quad (L_1\oplus L_2)_{|\iota x}\xrightarrow{(a',b')}\mathbb{C}_{\iota x},\tag{3.42}$$

where a,b,a',b' are all isomorphisms. We then get identifications

$$(L_1)_{|x_1}\xrightarrow{ab^{-1}}(L_2)_{|x_2},\quad (L_1)_{|\iota x_1}\xrightarrow{a'b'^{-1}}(L_2)_{|\iota x_2},\tag{3.43}$$

Conversely, given L_1 and L_2 and identifications like in (3.43) we get a line bundle whose restriction to E_1 and E_2 are equal to L_1 and L_2, respectively. In other words, the fiber of n^* over (L_1, L_2) can be identified with $\mathbb{C}^* \cong \{(ab^{-1}, a'b'^{-1}) \in \mathbb{C}^* \times \mathbb{C}^*\}/\sim$, where $\sim$ denotes equivalence under the action of $\mathbb{C}^* = \mathbb{P}(\mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2))$.

Given a τ-invariant point $(L_1, L_2) \in \mathrm{Jac}^0(E_1)[2] \times \mathrm{Jac}^0(E_2)[2]$, we wish to determine the action of τ on the fiber $\mathbb{C}^* = (n^*)^{-1}(L_1, L_2)$. To this aim, choose isomorphisms

$$F_1 : L_1 \xrightarrow{\sim} \tau(L_1), \quad F_2 : L_2 \xrightarrow{\sim} \tau(L_2).$$

These choices are well defined up to the action of the automorphism groups of the line bundles and will not be relevant for our purposes. Given L with $n^*(L) = (L_1, L_2)$, the line bundle $\tau(L)$ fits into an exact sequence,

$$0 \to \tau(L) \to L_1 \oplus L_2 \overset{(F_1,F_2)}{\cong} \tau(L_1) \oplus \tau(L_2) \xrightarrow{e} \mathbb{C}_x \oplus \mathbb{C}_{\iota x} \to 0.$$

Here the epimorphism e is induced by

$$\begin{aligned}
(L_1)_{|x_1} &\xrightarrow{F_1(x_1)} (\tau L_1)_{|x_1} \xrightarrow{1/a'} \mathbb{C}_x \\
(L_2)_{|x_2} &\xrightarrow{F_2(x_2)} (\tau L_2)_{|x_2} \xrightarrow{1/b'} \mathbb{C}_x \\
(L_1)_{|\iota x_1} &\xrightarrow{F_1(x_1)} (\tau L_1)_{|\iota x_1} \xrightarrow{1/a} \mathbb{C}_{\iota x} \\
(L_2)_{|\iota x_2} &\xrightarrow{F_2(x_2)} (\tau L_2)_{|\iota x_2} \xrightarrow{1/b} \mathbb{C}_{\iota x}.
\end{aligned}$$

where we are viewing F_1 and F_2 as non zero sections of the one dimensional spaces

$$V_1 := \mathrm{Hom}(L_1, \tau(L_1)) \quad \text{and} \quad V_2 := \mathrm{Hom}(L_2, \tau(L_2))$$

respectively. In other words, and with a slight abuse of notation, the action of τ on

$$\mathbb{C}^* \cong \left\{ \left(\frac{a}{b}, \frac{a'}{b'} \right) \in \mathbb{C}^* \times \mathbb{C}^* \right\} / \sim$$

is given by,

$$\left(\frac{a}{b}, \frac{a'}{b'} \right) \mapsto \left(\frac{F_1(x_1)}{a'} \frac{b'}{F_2(x_2)}, \frac{F_1(\iota x_1)}{a} \frac{b}{F_2(\iota x_2)} \right),$$

so that $\tau(L) \cong L$ if and only if there exists $k \in \mathbb{C}^*$ such that

$$\begin{cases} k\dfrac{a}{b} = \dfrac{F_1(x_1)}{F_2(x_2)}\dfrac{b'}{a'} \\ k\dfrac{a'}{b'} = \dfrac{F_1(\iota x_1)}{F_2(\iota x_2)}\dfrac{b}{a}, \end{cases} \quad \text{i.e., if and only if} \quad \frac{F_1(x_1)}{F_2(x_2)} = \frac{F_1(\iota x_1)}{F_2(\iota x_2)}.$$

We may then conclude that L is τ-invariant if and only if ι acts on V_1 and V_2 with the same scalar. When $L_1 = \mathcal{O}_{E_1}$ or $L_1 = \mathcal{O}_{E_1}(\iota x_1 - x_1)$, ι acts on V_1 as $+1$. When $L_1 = \mathcal{O}_{E_1}(x_1 - p_1)$ or $L_1 = \mathcal{O}_{E_1}(\iota x_1 - p_1)$, ι acts on V_1 as -1. The analogous statement holds for L_2 and V_2.

We may thus conclude that the fixed locus of τ over the 8 points

$$(\mathcal{O}_{E_1}, \mathcal{O}_{E_2}),\ (\mathcal{O}_{E_1}, \mathcal{O}_{E_2}(\iota x_2 - x_2)),\ (\mathcal{O}_{E_1}(\iota x_1 - x_1), \mathcal{O}_{E_2}),\ (\mathcal{O}_{E_1}(\iota x_1 - x_1), \mathcal{O}_{E_2}(\iota x_2 - x_2))$$
$$(\mathcal{O}_{E_1}(x_1 - p_1), \mathcal{O}_{E_2}(x_2 - q_2)),\ (\mathcal{O}_{E_1}(x_1 - p_1), \mathcal{O}_{E_2}(\iota x_2 - q_2)),\ (\mathcal{O}_{E_1}(\iota x_1 - p_1), \mathcal{O}_{E_2}(x_2 - q_2)),$$
$$(\mathcal{O}_{E_1}(\iota x_1 - p_1), \mathcal{O}_{E_2}(\iota x_2 - q_2))$$

is equal to the whole fiber, whereas in the other cases it is empty. Since the compactification $\overline{\mathrm{Jac}}^0_D(E_1 + E_2)$ is the result of identifying the 0 and ∞-sections of the $\mathbb{C}^*$-bundle with a twist by $\mathcal{O}_{E_1}(x_1 - \iota x_1) \otimes \mathcal{O}_{E_2}(x_2 - \iota x_2)$, it follows that the closure of these 8 copies of $\mathbb{C}^*$ in the complete moduli space is the union of 4 connected components each of which is the union of two $\mathbb{P}^1$'s meeting transversally in two points belonging to the singular locus of $\overline{\mathrm{Jac}}^0_D(E_1 + E_2)$.

To conclude the proof of the Proposition, we remark that τ preserves the singular locus of $\overline{\mathrm{Jac}}^0_D(E_1 + E_2)$ which is isomorphic to $\mathrm{Jac}^{-1}(E_1) \times \mathrm{Jac}^{-1}(E_2)$. Here τ acts in the usual way, and it has 16 isolated fixed points; we have just seen that 8 of these points lie in a one-dimensional component, so we are left with 8 isolated points. □

A consequence of the proposition we just proved is the following.

Corollary 3.4.16. *With the assumptions made at the beginning of §3.4.1, there are four* 2*-dimensional components of* $\mathrm{Fix}(\tau) \subset M_{v,H}$*, which are pairwise isomorphic.*

Proof. Consider the fibration $\nu : P_{v,H} \to |C|$. From Proposition 2.2.1 that for every irreducible curve C, the restriction $\mathrm{Fix}(\tau)_{|\nu^{-1}[C]}$ has exactly four 2-dimensional connected components. However, using the above Proposition 3.4.15 we may conclude that the closure of these four 2-dimensional components cannot come together over $[e_1 + e_2]$ and $[e_1' + e_2']$. Letting $V \subset |C|$ be the locus of irreducible curves, we conclude that the Stein factorization of $\mathrm{Fix}(\tau) = \overline{\mathrm{Fix}(\tau)_{|\nu^{-1}(V)}} \to |C|$ is a 4 to

1 unramified cover of $|C| = \mathbb{P}^1$ and hence is trivial. Just as in the proof of the previous proposition we can see that each of these components is an elliptic K3 surface.

Recall that $|C|$ has two simple base points, and that $f^*|C| \subset |D|$ has 4 simple base points. Let $\mathcal{D} \to \mathbb{P}^1$ be the universal family of curves, let $s : \mathbb{P}^1 \to \mathcal{D}$ be one of the sections defined by the base points of $f^*|C|$, and let us fix smooth curves $D \to C$. Then tensoring by $\mathcal{O}_{\mathcal{D}}(s - \iota s)$ is a birational automorphism of $\mathrm{Fix}(\tau) \subset M_{v,H}$, that interchanges the two components intersecting $\mathrm{Nm}^{-1}(0) \subset \mathrm{Jac}^0(D)$ and the two components intersecting $\mathrm{Nm}^{-1}(\omega_{T|C}) \subset \mathrm{Jac}^0(D)$. The proof is concluded remarking that a birational automorphism of a K3 surface is, in fact, a regular automorphism. □

3.5 Fundamental group of the relative Prym variety

This section is devoted to showing that any resolution of the singularities of the degree zero relative Prym variety constructed in Section 3.2 is simply connected. We do this by showing that this is indeed the case for the smooth locus of the relative Prym variety. We will assume that $|C|$ is non hyperelliptic, since otherwise the result follows readily from Corollary 3.4.13.

The set-up is, as usual, the following. We let $C \subset T$ be a smooth, primitive, curve of genus $g \geq 2$ on a general Enriques surface T, and we set $D = f^{-1}(C)$. Considering the Mukai vector $v = (0, [D], -h + 1)$, with $h = g(D) = 2g - 1$, we set

$$M = M_{v,H}\,, \qquad P = P_{v,H} \tag{3.44}$$

The main result of this section is

Theorem 3.5.1. *Let $|C|$ be a non-hyperelliptic system, on a general Enriques surface T. Suppose that the general pencil $\mathbb{P}^1 \subset |C|$ contains only reduced and nodal curves. Let P be as in (3.44) the degree zero relative Prym variety associated to $|C|$. Then any resolution $\widetilde{P}$ of P is simply connected.*

Consider the support morphism

$$\pi : J = M \longrightarrow |D|. \tag{3.45}$$

and look at its restriction to P

$$\eta : P \to |C| \cong \mathbb{P}^{g-1}$$

Let U' be the locus of irreducible curves in $|C|$, $Z = |C| \smallsetminus U'$, and set

$$P' = \eta^{-1}(U).$$

We claim that to prove the theorem it is enough to prove the following Proposition.

Proposition 3.5.2. $\pi_1(P') = \{1\}$.

Indeed, let $\gamma : \widetilde{P} \to P$ be any resolution of singularities. Since P' is contained in the smooth locus of P, by [23] 0.7.B (cf. also [38] Proposition 2.10), the natural morphism $\pi_1(P') \to \pi_1(\widetilde{P})$ is surjective and the Proposition above would imply that

$$\pi_1(\widetilde{P}) = \{1\}.$$

The main ingredients in the proof are the following theorem of Leibman, as used in [42], and the theory of Picard-Lefschetz (cf. [5], Chapter XI).

Theorem 3.5.3 (Leibman, [41])**.** *Let $p\colon E \to B$ be a surjective morphism of smooth connected manifolds. Assume that p has a section s. Let $W \subset B$ be a union of locally closed submanifolds of real codimension at least two. Set $U = B \setminus W$ and $E_U = p^{-1}(U)$ and assume that $E_U \to U$ is a locally trivial fibration with fiber F. Consider the exact sequence*

$$1 \to \pi_1(F) \xrightarrow{j_*} \pi_1(E_U) \underset{p_*}{\overset{s_*}{\leftrightarrows}} \pi_1(U) \to 1. \tag{3.46}$$

Set $H = \ker(\pi_1(U) \to \pi_1(B))$ and view it, via s_*, as a subgroup of $\pi_1(E_U)$. Let $R = [\pi_1(F), H]$ be the commutator subgroup of $\pi_1(F)$ and H in $\pi_1(E_U)$. Then there is a split exact sequence

$$1 \to R \to \pi_1(F) \to \pi_1(E) \overset{s_*}{\leftrightarrows} \pi_1(B) \to 1. \tag{3.47}$$

Notice that, by definition, R is generated by elements of type

$$c^{-1}\tilde{\lambda}^{-1}c\tilde{\lambda}, \tag{3.48}$$

where $c \in \pi_1(F)$ and $\tilde{\lambda} = s_*(\lambda)$ is a lifting of $\lambda \in \pi_1(U)$ to $\pi_1(E_U)$.

Let

$$V, V' \subset |D|,$$

be the open loci of smooth and irreducible curves respectively, and let

$$\Sigma \subset |D|,$$

be the discriminant locus of the family. We will first apply Leibman's theorem to

$$M' := M_{V'} \to V',$$

equipped with the restriction of the rational section (1.9) to V' where it is regular. Notice that

$$\pi_1(M_{V'}) = \{1\}.$$

Indeed, let $r : M_H \to M$ be any resolution of the singularities of M as in Corollory 1.2.2. By Remark 3.2.3 and Proposition 3.1.14 it follows, identifying as we may $M_{V'} \subset M$ with its preimage in M_H, that $\operatorname{codim}(M_{V'}, M_H) \geq 2$.

Then, we will apply the same theorem to

$$P' \to U',$$

and, using the simply connectedness of $M_{V'}$, we will be able to deduce Theorem 3.5.1. Let us fix a base point $o \in U$ and denote by M_o and P_o the fibers of ν and π over o. Since $\pi_1(V')$ and $\pi_1(U')$ are trivial, the exact sequence (3.47) for $M_{V'} \to V'$ and $P' \to U'$ becomes respectively

$$\begin{aligned} 1 \to & R \to \pi_1(M_o) \to \pi_1(M) \to 1, \quad \text{and} \\ 1 \to & R' \to \pi_1(P_o) \to \pi_1(P') \to 1, \end{aligned}$$

where

$$R := [\pi_1(M_o), \pi_1(V)], \quad \text{and} \quad R' := [\pi_1(P_o), \pi_1(U)].$$

Notice that since $\pi_1(M') = 1$,

$$\pi_1(M_o) \cong R.$$

The strategy to prove the theorem is to use R to calculate R' and show that the inclusion $R' \subset \pi_1(P_o)$ is in fact an equality.

Before starting we need some preparation and some notation.

Remark 3.5.4. Let $\mathcal{X} \to \mathbb{P}^1$ is a Lefschetz pencil of curves, with discriminant locus $x_1, \ldots, x_n$. Generators for $\pi_1(\mathbb{P}^1 \setminus \{x_i\})$ can be given by choosing a point $o \in \mathbb{P}^1 \setminus \{x_i\}$, and joining x with the boundry of small disjoints discs $\Delta_1, \ldots, \Delta_n$ centered around $x_1, \ldots, x_n$. The family of curves parametrized by one of these disks is a family of smooth curves acquiring a simple node. Let α_λ be the vanishing cycle of this family. It is a classical result that the Picard-Lefschetz homomorphism of the family (i.e., the monodromy operator associated to the generator of the fundamental group of the disc) is given by (see, for instance, [5, Section X.9])

$$PL_\lambda(c) = c + (c \cdot \alpha_\lambda)\alpha_\lambda. \tag{3.49}$$

Recall that by a theorem of Zariski, the fundamental group of the complement of a hypersurface in a projective space is generated by the fundamental group of the complement of the intersection of the hypersurface with a general line.

Lemma 3.5.5. *Suppose that $E \to \mathbb{P}^r$ is the family of degree zero relative Jacobian varieties of a family $\mathcal{X} \to \mathbb{P}^r$ of curves. Suppose that $E \to \mathbb{P}^r$ has a rational section defined outside a closed subset $Z \subset |C|$ of codimension 2. Suppose furthermore that* $\operatorname{codim}(E_Z, E) \geq 2$*, and that $\mathbb{P}^r$ contains a Lefshetz pencil. Let R be as in Theorem 3.5.3. Then R is generated by vanishing cycles.*

Proof. Consider a general line $\ell \subset \mathbb{P}^r$ and use this to define a set of generators for the fundamental group of the smooth locus of the family, and hence a set of vanishing cycles. By unravelling the definition of Picard-Lefschetz transformation and of the commutator R of the theorem, one can check (cf. [6]) that, passing as we may to the additive notation, we have

$$c^{-1}\tilde{\lambda}^{-1}c\tilde{\lambda} = -c + PL_\lambda(c). \tag{3.50}$$

It follows that R is generated by

$$-c + c + (c, \alpha_\lambda)\alpha_\lambda = (c, \alpha_\lambda)\alpha_\lambda,$$

with α_λ vanishing cycle. □

Notice that in the above Lemma we could consider instead of the family over $\mathbb{P}^r$, its restriction to a simply connected open subset of $\mathbb{P}^r$.

We should also point out that we don't really need that $E \to \mathbb{P}^r$ contains a Lefschetz pencil, but

just that the fibers of a general pencil $\mathbb{P}^1 \subset \mathbb{P}^r$ are reduced and nodal. In this more general setting, the vanishing cycles will not necessarily be simple closed loops, but positive linear combinations of them.

Proposition 3.5.6. *Consider a point $o \in V \subset |D|$, and let D_o be the corresponding curve. Recall that $M_o \cong H^1(D_o, \mathcal{O})/H^1(D_o, \mathbb{Z})$. Under the isomorphism $\pi_1(M_o) \cong H_1(M_o, \mathbb{Z}) \cong H^1(D_o, \mathbb{Z})$, the first cohomology group of M_o is generated by vanishing cycles.*

Proof. Applying Theorem 3.5.3 to $M' \to V'$ yields

$$1 \to R \to \pi_1(M_o) \to \pi_1(E) \cong 1,$$

so that $\pi_1(M_t) \cong R$. To conclude the proof it is sufficient to use Lemma 3.5.5 and conclude that R is generated by vanishing cycles. □

Now we apply Theorem 3.5.3 to $P' \to U'$ and consider the following commutative diagram of abelian groups,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \longrightarrow & \pi_1(M_o) & \longrightarrow & 1 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & R' & \longrightarrow & \pi_1(P_o) & \longrightarrow & \pi_1(P) & \longrightarrow & 1, \end{array} \tag{3.51}$$

where the first two vertical maps are injections.

Since $R = [\pi_1(M_o), \pi_1(V)]$ and $R' = [\pi_1(P_o), \pi_1(U)]$, the inclusion $R' \subset R$ is determined by the inclusion

$$\pi_1(P_o) \subset \pi_1(M_o), \tag{3.52}$$

and by the pushforward

$$j_* \colon \pi_1(U) \to \pi_1(V), \tag{3.53}$$

induced by the inclusion $j : U \hookrightarrow V$. By (2.3), the inclusion (3.52) is just

$$H^1(D_o, \mathbb{Z})^- = (1 - \iota)H^1(D_o, \mathbb{Z}) \subset H^1(D_o, \mathbb{Z}), \tag{3.54}$$

i.e. $\pi_1(P_o)$ is just the ι-antinvariant sublattice of $\pi_1(M_o)$.

As for $j_* \colon \pi_1(U) \to \pi_1(V)$, we will choose a set of generators for $\pi_1(V)$ and for $\pi_1(U)$ that is

convenient to our purposes. Set

$$\delta = \deg \Delta, \quad \text{so that } \deg \Sigma = 2\delta.$$

Consider a general line $\ell \subset |C|$ and let $t_1, \ldots, t_\delta$ be the points of the intersection $\Delta_\ell = \ell \cap \Delta$. Since we are assuming that ℓ is a Lefschetz pencil, the curve D_{t_i} has two simple nodes and the two nodes are interchanged by the involution. In other words, ℓ meets Σ in its double locus. More specifically, we have the following

Lemma 3.5.7. *At each point t_i of Δ_ℓ such that D_{t_i} is irreducible, there are two branches of Σ that meet transversally. Deforming the curve along either branch results in smoothing out one of the two nodes of D_{t_i}. Moreover, the two branches are interchanged by the involution.*

Proof. Recall that by Lemma 3.1.4, the curve D_{t_i} is irreducible if and only if C_{t_i} is irreducible. Observe that it is enough to check that one can smooth out only one of the two nodes of D_{t_i} at a time. Indeed, since ι interchanges the two nodes the involution has to send a family that smoothes out one of the two nodes to a family that smoothes out the other node. It follows that if we can smooth out one node at a time, there have to be two distinct components of the discriminant meeting in t_i. The Lemma follows by noticing that since we are assuming that the surfaces are undoal, and the linear systems are not hyperelliptic, $|D|$ is actually very ample (see [63] Thm 6.1). In particular, the discriminant locus Σ is irreducible, and the general point parametrizes curves with just one node. Hence, we can deform D_{t_i} so that only one of the two nodes is smoothed out. □

Keeping the notation as above, one can show that even if the curve D_{t_i} is not irreducible but the general point of the component $\Sigma' \subset \Sigma$ containing D has fewer nodes than D_{t_i}, then there are two branches of Σ meeting in t_i and that the two branches are interchanged by the involution.

For simplicity, we assume that $\ell \subset |C|$ is a Lefschetz pencil. Choose a set of generators $\lambda_1, \ldots, \lambda_\delta$ of $\pi_1(\ell \setminus \{t_1, \ldots, t_\delta\})$ as in Remark 3.5.4 and let $\alpha_1, \ldots, \alpha_\delta$ be the corresponding vanishing cycles for $\mathcal{C}\big|_\ell$.

Then consider a small deformation ℓ' of ℓ in $|D|$, and assume that it is transverse to the discriminant Σ. Notice that $\ell' \subset |D|$ is a Lefschetz pencil, and that by the above lemma we can set,

$$\ell' \setminus \Sigma \cap \ell' = \{s_1, s_1', \ldots, s_\delta, s_\delta'\},$$

where s_i and s_i' are the intersections of ℓ' with the two branches of Σ that meet in t_i. as in Remark 3.5.4 , we can choose, for every i, cycles η_i and η_i' in V' so that the set

$$\{\eta_i, \eta_i'\}, \tag{3.55}$$

generates $\pi_1(\ell' \setminus \Sigma_\ell)$, and thus $\pi_1(V)$. Moreover, we can set things up in such a way that $\iota_* \eta_i = \eta_i'$ and that,

$$j_* \lambda_i = \eta_i \circ \eta_i'. \tag{3.56}$$

For more details we refer to [6].

Moreover, if for $i = 1, \ldots, \delta$, we let β_i and β_i' be the vanishing cycles associated to η_i and η_i', then

$$\iota(\beta_i) = \beta_i', \quad \text{and} \quad f^*(\alpha_i) = \beta_i + \beta_i'. \tag{3.57}$$

By Lemma 3.5.5, R is generated by elements of the form $(c, \beta_i)\beta_i$ and $(c, \beta_i')\beta_i'$, where c ranges in a set of generators of $H^1(D, \mathbb{Z})$.

The following Lemma, together with (4.7) ends the proof of the theorem.

Lemma 3.5.8. $R' \cong H^1(D_o, \mathbb{Z})^- \cong (1 - \iota)H^1(D_o, \mathbb{Z})$.

Proof. Using (3.50), (3.52) and (3.53) we can see that R' is generated by elements of the form,

$$c^{-1} PL_{\lambda_i \cdot \iota\lambda_i}(c).$$

where c is in $H^1(D, \mathbb{Z})^-$.

On the other hand, since β_i and $\iota\beta_i$ have trivial intersection number,

$$c^{-1} PL_{\lambda_i \cdot \iota\lambda_i}(c) = c^{-1} PL_\lambda PL_{\iota\lambda}(c).$$

and hence R' is generated by elements of the form

$$\begin{aligned} c^{-1} PL_\lambda PL_{\iota\lambda}(c) &= (c, \alpha_\lambda)\alpha_\lambda + (c, \iota\alpha_\lambda)\iota\alpha_\lambda + (c, \alpha_\lambda)(\alpha_\lambda, \iota\alpha_\lambda)\iota\alpha \\ &= (c, \alpha_\lambda)(\alpha_\lambda - \iota\alpha_\lambda). \end{aligned} \tag{3.58}$$

It is now sufficient to find, for each i, a $c_i \in H^1(D,\mathbb{Z})^-$ such that

$$(c_i, \alpha_i) = 1.$$

To this aim, notice that both α_i and $\iota\alpha_i$ are vanishing cycles and by construction there is a curve $C_i \in |C|$, having exactly two nodes, resulting from the vanishing of α_i and $\iota\alpha_i$, and no other singularities than the two nodes. But C_i is irreducible and therefore $C \setminus \{\alpha_i, \iota\alpha_i\}$ is connected and c_i can be readily constructed. □

Chapter 4

The relative compactified Jacobian of a linear system on an Enriques surface

4.1 Smoothness and first properties

In this Chapter we study the relative compactified Jacobian of a linear system on a general Enriques surface. We will first look at the smoothness properties, then we will compute some topological invariants and then show that the canonical bundle is trivial.

We start by applying the results of Section 1.4 to the following setting.

Let T be a general Enriques surface and let $C \subset T$ be an irreducible curve of genus $g \geq 2$. As usual $f : S \to T$ will be the universal cover of T, and D will denote the preimage of C in S. Let χ be a non zero integer and set

$$w = (0, [C], \chi), \quad \text{and} \quad v = (0, [D], 2\chi).$$

If A denotes a polarization on T, we set $H = f^*A$. Finally, set

$$N = M_{w,A}(T, |C|), \quad \text{and} \quad M = M_{v,H}(S, |D|). \tag{4.1}$$

and consider the pullback morphism (1.38)

$$\Phi : N \to \mathrm{Fix}(\iota^*) \subset M.$$

Let

$$Y = Y_{v,H} \subset \mathrm{Fix}(\iota^*), \tag{4.2}$$

be the connected component of $\mathrm{Fix}(\iota^*)$ containing $\mathrm{Im}\,\Phi$.

From now on let us assume that v is a primitive vector in $H^*(S, \mathbb{Z})$. This can be achieved for example if C (and thus D) is primitive, or at least not divisible by 2, in $\mathrm{NS}(T)$ (resp. $\mathrm{NS}(S)$).

Proposition 4.1.1. *Let v be primitive and choose A such that $H = f^*A$ is v-generic. Then Y and N are smooth.*

Proof. The smoothness of Y follows from Theorem 1.2.1, and Corollary 1.4.5. As for the smoothness of N it follows from the fact that if F is an A-stable sheaf on T with Mukai vector w, then

$$\mathrm{Ext}^2(F, F) = 0,$$

so that the obstruction space is trivial. Indeed, by Serre duality it is sufficient to show that $\mathrm{Hom}(F, F \otimes \omega_T) = 0$. This vanishing follows from from Lemma 1.4.3, since the assumptions of the Proposition imply that f^*G is H-stable, and thus simple, for every A-semistable sheaf G on T. This proves that stable locus of N is smooth. Moreover, the fact that the strictly semistable locus of M is empty implies that the strictly semistable locus of N is also empty and hence we are done. □

Proposition 4.1.2. *Let v be primitive and choose A such that $H = f^*A$ is v-generic. Then,*

$$\Phi : N \to Y,$$

is étale.

Proof. By Lemma 1.4.7 $\Phi(N) = Y$, as one can check that if $F = f^*G$ is H-semistable, then G is A-semistable. The fact that it is unramified follows directly from the proof of the above proposition, where we showed that if G is an A-stable sheaf corresponding to a point in N, then $G \ncong G \otimes \omega_T$. □

We should point out that one could prove that Φ is unramified directly showing that the tangent

map is injective. Indeed, the natural morphism

$$d\Phi : \mathrm{Ext}^1_T(G,G) \to \mathrm{Ext}^1_S(f^*G, f^*G),$$

is injective. This follows again from the fact that G is not isomorphic to $G \otimes \omega_T$.[1]

Lastly we have

Corollary 4.1.3. *Every component of* $\mathrm{Fix}(\iota^*)$ *is Lagrangian.*

Proof. By the choice of w we know that $\mathrm{Fix}(\iota^*)$ is non empty. By Lemma 1.4.7 every component is the image, under a finite pullback map, of some moduli space of A-stable sheaves on T with Mukai vector $w = (0, [C], \frac{\chi}{2})$. By Proposition 4.1.1 this moduli space is smooth of dimension equal to $\dim \mathrm{Ext}^1_T(G,G)$. Since for every $\dim \mathrm{Ext}^1_T(G,G) = -\chi(G,G) + 1 = C^2 + 1 = \dim M/2$, we now just need to show that every component is isotropic. This follows immediately by Lemma 3.2.6 which proves that the involution ι^* is anti-sympectic. □

By abuse of notation, let us denote by

$$\pi : Y \to |C|,$$

the restriction of the support morphism $M \to |D|$, and by

$$\nu : N \to |C|,$$

the support morphism for the moduli space of sheaves on T. There is a commutative diagram,

$$\begin{array}{ccc} N & \xrightarrow{\Phi} & Y \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & |C| & \end{array} \tag{4.3}$$

which shows that the double cover Φ restricts to a double cover between the fibers of π and of ν. For later use, let us define the following torsion line bundle on Y, by setting

$$\Phi_* \mathcal{O}_N \cong \mathcal{O}_Y \oplus L. \tag{4.4}$$

[1] As usual, set $G' = G \otimes \omega_T$, and let $0 \to G \to X \to G \to 0$ be a non trivial extension of G by G. Notice that since G is stable and not isomorphic to G', we can deduce that $\mathrm{Hom}(X, G') = 0$. If the pullback $0 \to f^*G \to f^*X \to f^*G \to 0$ splits, so does the sequence $0 \to f_*f^*G \to f_*f^*X \to f_*f^*G \to 0$. However, since $\mathrm{Hom}(X, G') = \mathrm{Hom}(X', G) = 0$, it follows that any morphism $X \oplus X' \to G \oplus G'$ is diagonal. Hence, any morphism splitting $0 \to f_*f^*G \to f_*f^*X \to f_*f^*G \to 0$ induces a splitting of $0 \to G \to X \to G \to 0$ as well.

Notice that $L^{\otimes 2} \cong \mathcal{O}_Y$, and that L generates the kernel of $\Phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(N)$.

Let

$$U, W \subset |C|,$$

be the open loci of smooth and reduced curves respectively. For any $t \in U$, the double cover $N_t \to Y_t$ is induced by the sequence

$$1 \to \mathbb{Z}/(2) \to \mathrm{Jac}(C) \xrightarrow{f^*} \mathrm{Fix}(\iota_t) \subset \mathrm{Jac}(D),$$

and by Theorem 1.1.19, the restriction

$$N_W \to W,$$

of ν to W is equidimensional and thus flat. However (cf. Example 1.1.21 and Problem 1.1.22), $N \to |C|$ might not be flat in general (see Assumption 4.1.5 below).

We sum up the results obtained so far in the following theorem

Theorem 4.1.4. *Let C be an irreducible curve of genus $g \geq 2$ on an Enriques surface T. Let χ be a non zero integer, set $w = (0, [C], \chi)$ and $v = (0, [D], 2\chi)$, where $D = f^{-1}(C)$. Assume that v is primitive, let A be an ample line bundle on T such that $H = f^*A$ is v-generic. The moduli space*

$$N_{w,A},$$

is a smooth projective variety of dimension $2g-1$ that admits a double cover to a smooth Lagrangian subvariety

$$Y_{v,H} \subset M_{v,H}.$$

Let

$$\Delta \subset |C|,$$

be the discriminant locus, and set $N_\Delta = \nu^{-1}(\Delta)$.

For the main theorems of Section 4.4.4 and Section 4.2, we need the following assumption

Assumption 4.1.5. *Let $W \subset |C|$ be the open locus of reduced curves, and let $N_W \to W$ be the restriction of ν to W. The linear system $|C|$ is such that*

$$\mathrm{codim}(N \setminus N_W, N) \geq 2.$$

This assumption is certainly satisfied if $N \to |C|$ is flat. Since $\operatorname{codim}(W, |C|) \geq 2$ this assumption is equivalent to asking that there are no irreducible components of N_Δ which map to codimension ≥ 2 subsets of Δ.

As mentioned in Section 1.3, after Problem 1.1.22, we expect that under the assumptions of the Theorem above, the support morphism

$$N \to |C|,$$

is indeed flat.

Notice that in some cases of low genus, where the curves of the linear system do not degenerate too much, one can show that the relative compactified Jacobian is equidimensional. Examples of linear systems that satisfy equidimensionality (and thus the above assumption) are

$$\begin{aligned} &|e_1 + e_2|, \text{ with } e_1 \cdot e_2 = 1, && g(C) = 2, \ \dim N = 3, \\ &|e + f| \text{ with } e \cdot f = 2, && g(C) = 3, \ \dim N = 5, \\ &|e_1 + e_2 + e_3| \text{ with } e_i \cdot e_j = 1 \text{ for } i \neq j, && g(C) = 4, \ \dim N = 7. \end{aligned} \tag{4.5}$$

where e_1, e_2, e_3, e and f are primitive elliptic curves. These linear systems do not contain non-reduced curves and hence, by Theorem 1.1.19, $N \to |C|$ is flat.

Lemma 4.1.6. *Let T be a general Enriques surface and let $|C|$ be a linear system on T. Suppose that $|C|$ satisfies Assumption 4.1.5. If $|C|$ is not hyperelliptic, then for every irreducible component Δ_i of the discriminant locus, the divisor N_{Δ_i} is irreducible. If $|C|$ is hyperelliptic of genus $g \geq 3$ then, using the notation of Proposition 3.1.15, for $i = 1, \ldots, 3$, N_{Δ_i} is irreducible, whereas N_{Δ_4} is the union of two irreducible components.*

Proof. Under the generality assumption on T, if $|C|$ is not hyperelliptic by Proposition 3.1.14 the general point of every component of the discriminant locus parameterizes irreducible curves. By Theorem 1.1.15 it follows that the general fiber of N_{Δ_i} is irreducible and, since Assumption 4.1.5 implies that every component of N_Δ maps to a component of Δ, that N_{Δ_i} itself is irreducible. To show that N_{Δ_i} is reduced, we can use the rational section if $d = 0$, or we can restrict to a general pencil since the relative compactified Jacobian of a general pencil is independent of the degree. If $|C|$ is hyperelliptic, then the statement is clear for Δ_1, Δ_2 where the general fiber of N is actually smooth and for Δ_3. For Δ_4 we use the fact that there is a simply connected open subset of Δ_4 that parametrizes curves that are the union of two smooth components meet transversally in 2 points

and then Example 1.1.17. □

Remark 4.1.7. *If we let k denote the number of irreducible components of Δ, the above Lemma shows that in the non hyperelliptic case, k is also the number of irreducible components of N_Δ, whereas in the hyperelliptic case, the number of irreducible components of N_Δ is equal to $k+1$. Notice also that if $|C|$ is very ample, then $k = 1$.*

4.2 The fundamental group of the relative compactified Jacobian of a linear system on an Enriques surface

This section is devoted to calculating the fundamental group of the relative compactified Jacobian variety N constructed in Section 4.1. For simplicity will do it in the case of degree zero Jacobian varieties, even though the arguments work whenever there is a rational section, defined on an open subset of $|C|$ whose complement has codimension ≥ 2.

First we will prove the main theorem of this section assuming that $|C|$ is a non-hyperelliptic linear system with the property that the generic line $\mathbb{P}^1 \subset |C|$ has only reduced and nodal members. Then we will show how to adapt the proof to the hyperelliptic case. In both cases, we will need Assumption 4.1.5. Notice that the linear systems described in (4.5) can be chosen general in moduli, in order to ensure the existence of Lefschetz pencils.

Let v be the Mukai vector $(0, D, \chi)$, where $D = f^{-1}(C)$ and $\chi = -h + 1$. Let $\eta : N \to |C|$ and $\pi : Y \to |C|$ be as in (4.1) and (4.2).

We first compute the fundamental group of Y, using the strategy and the techniques of Section 3.5, and adapting the proof to the present case. Recall that since the degree is zero, $\pi : Y \to |C|$ admits a rational section which is defined on an open subset containing the open subset, whose complement has codimension two, of reduced curves (this is clear if $|C|$ is not hyperelliptic since the locus of integral curve has codimension ≥ 2; if $|C|$ is hyperelliptic, one can check that the structure sheaf of the curve parametrized by the general point of every component of the discriminant is always stable).

Then we study the geometry of the double cover $N \to Y$ and prove some results on the induced covering map of some singular fibers. These results will be used to calculate in Section 4.3 the second Betti numbers of N and of Y.

The first result is

Theorem 4.2.1. *Let $|C|$ be a genus $g \geq 2$ linear system on a general Enriques surface T and suppose that it satisfies Assumption 4.1.5 and that the general pencil contains only nodal curves. Let Y be as in (4.2). Then,*

$$\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Proof. We use the notation of Section 3.5, and we apply Theorem 3.5.3 to the morphism

$$\pi : Y' = Y_{U'} \to U'.$$

First of all, notice that by Theorem 1.1.19 π is flat. Moreover, Assumption 4.1.5 tells us that there is an isomorphism

$$\pi_1(Y) = \pi_1(Y').$$

As in Section 3.5 we get a commutative diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \longrightarrow & \pi_1(M_o) & \longrightarrow & 1 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & R'' & \longrightarrow & \pi_1(Y_o) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1, \end{array} \tag{4.6}$$

where the first two vertical maps are injections. To prove the theorem, it is sufficient to show that

$$\pi_1(Y_o)/R'' \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2).$$

Since $R = [\pi_1(M_o, \pi_1(V)]$ and $R'' = [\pi_1(Y_o), \pi_1(U)]$, the inclusion $R'' \subset R$ is determined by the inclusion

$$\pi_1(Y_o) \subset \pi_1(M_o),$$

and by the pushforward

$$j_* \colon \pi_1(U) \to \pi_1(V),$$

induced by the inclusion $j : U \hookrightarrow V$. Recall that by (3.6) the inclusion (3.52) is

$$H^1(D_o, \mathcal{O})^+ \cap H^1(D_o, \mathbb{Z}) = H^1(D_o, \mathbb{Z})^+ \subset H^1(D_o, \mathbb{Z}), \tag{4.7}$$

i.e. that $\pi_1(Y_o)$ is just the ι-equivariant sublattice of $\pi_1(M_o)$.

Now assume that $|C|$ is not hyperelliptic, so that the members of a general pencil are irreducible. In this case, we can choose a set of generators of $\pi_1(U)$ and of $\pi_1(V)$ as in Section 3.5, and see that R'' is generated by elements of the form

$$\begin{aligned} c^{-1}PL_\lambda PL_{\iota\lambda}(c) &= (c,\alpha_i)\alpha_i + (c,\iota\alpha_i)\iota\alpha_i + (c,\alpha_i)(\alpha_i,\iota\alpha_i)\iota\alpha \\ &= (c,\alpha_i)(\alpha_i+\iota\alpha_i). \end{aligned}$$

As in Lemma 3.5.8 one can show that for every i there exist a $c_i \in H^1(D_o,\mathbb{Z})^+$ such that $(c,\alpha_i)=1$, and hence we can conclude that

$$R'' \cong (1+\iota)H^1(D_o,\mathbb{Z}).$$

Since $\pi_1(Y_o) = H^1(D_o,\mathbb{Z})^+$ using Corollary 3.1.3 we see that

$$\pi_1(Y') = \pi_1(Y_o)/R'' \cong \mathbb{Z}/(2)\times\mathbb{Z}/(2).$$

Now let us assume that $|C|$ is hyperelliptic. From Proposition 3.1.15 we know that the discriminant locus is the union of four components $\Delta_1,\dots,\Delta_4$. Let $\ell \subset |C|$ be a general pencil, set

$$t_1,\dots,t_\delta = \ell \cap \Delta,$$

and let λ_i be a simple closed loop around t_i. By deforming ℓ in $|D|$ in order to make it transversal to the discriminant locus Σ of $|D|$, we can describe, for every i, the image

$$j_*(\lambda_i) \in \pi_1(V),$$

and see how the monodromy representation along $j_*(\lambda_i)$ acts on the invariant cycles $H^1(D_o,\mathbb{Z})^+$. There are four cases:

1. If $t_i \in \Delta_4$, then C_{t_i} and D_{t_i} are irreducible and the analogue of formula (3.56) holds and hence $c^{-1}PL_{\lambda_i}PL_{\iota\lambda_i}(c) = (c,\alpha_i)(\alpha_i+\iota\alpha_i)$, where α_i is a vanishing cycle for D_o as in the non hyperelliptic case.

   Moreover, as in Lemma 3.5.8 we can show that there exist a $c_i \in H^1(D_o,\mathbb{Z})^+$ such that $(c,\alpha_i)=1$;

2. If $t_i \in \Delta_2 \cup \Delta_3$, then C_{t_i} is the union of a smooth elliptic curve and a smooth hyperellptic curve meeting in one point and D_{t_i} is the union of a smooth elliptic curve and a smooth hyperellptic

curve meeting in two points.

Let α_i be the simple closed loop in C_o whose vanishing generates the node of C_{t_i}. Since $\alpha_i = 0$ in $H^1(C_o, \mathbb{Z})$, $f^*(\alpha_i) = \epsilon_i + \iota\epsilon_i = 0$, where ϵ_i is the class of a simple closed loop in D_o. Since the component of $\Delta(|D|)$ containing D_{t_i} is birational to $|E_1| \times |(n-1)E_1 + E_2|$ and hence it parametrizes curves that are the union of a smooth elliptic curve and a smooth hyperellptic curve meeting in two points, there is only one branch of $\Delta(|D|)$ in t_i and we can write

$$j_*\lambda_i = \delta_i,$$

where the local monodromy along δ_i is $c \mapsto c + 2(c, \epsilon_i)\epsilon_i$. Since ϵ_i is anti-invariant, it follows that if c is invariant we have,

$$c^{-1}PL_{\delta_i}(c) = 2(c, \epsilon_i)\epsilon_i = 0.$$

This is in accordance with the fact that Y_{t_i} is smooth and so there is no monodromy around t_i;

3. If $t_i \in \Delta_4$, then C_{t_i} is the union of a smooth elliptic curve and a smooth hyperellptic curve meeting in two points and D_{t_i} is the union of a smooth hyperellptic curve Γ and two disjoint elliptic curves E and $E' = \iota(E)$, each meeting Γ in two points. Let α_1 and α_2 be the two simple closed loops in C_o that vanish as we degenerate the curve to C_{t_i}. We choose the orientation so that $\alpha_1 + \alpha_2 = 0$, and for $j = 1, 2$, we set $f^*(\alpha_j) = \beta_j + \iota\beta_j$. Up to relabeling, we can assume that the vanishing of β_1 and β_2 generate the nodes of $E \cap \Gamma$, and that the vanishing of $\iota\beta_1$ and $\iota\beta_2$ generate the nodes of $\iota(E) \cap \Gamma$. Consider the point t_i in $|D|$. There are two components of $\Sigma = \Delta(|D|)$ that meet in t_i: one that smoothes out the nodes of $E \cap \Gamma$ and one that smoothes out the nodes of $\iota(E) \cap \Gamma$. It follows that we can write

$$j_*\lambda_j = \xi_j \circ \iota\xi_j$$

where ξ_j and $\iota\xi_j$ each go around the two local branches of Σ. The local monodromy around η_j is defined by

$$c \mapsto c + 2(c, \beta_1)\beta_1 = c + 2(c, \beta_2)\beta_2,$$

and analogously for $\iota\eta_j$. Hence, if c is invariant, we have

$$c^{-1}PL_{\xi_j}PL_{\iota\xi_j}(c) = 2(c,\beta_1)(\beta_1+\iota\beta_1) = (c,\beta_1)(1+\iota)2\beta_1.$$

Moreover, we can find a $c \in H^1(D_o,\mathbb{Z})^+$ such that $(c,\beta_1) = 1$.

Notice that the union of the $\{\eta_i, \iota\eta_i\}$ as in point (1), the $\{\delta_i\}$ as in point (2), and the $\{\xi_i, \iota\xi\}$ as in point (3) generate $\pi_1(V)$. By our description of the local monodromy along this set of generators of $\pi_1(V)$, it follows that

$$R := [\pi_1(M_o), \pi_1(V)] = H^1(D_o,\mathbb{Z}),$$

is generated by the cycles of the form

$$\alpha_i, \iota\alpha_i, 2\epsilon_i, 2\beta_1,$$

and hence that

$$R'' := [\pi_1(Y_o), \pi_1(U)] = (1+\iota)H^1(D_o,\mathbb{Z}),$$

as desired. □

Let us now come to the double cover

$$\Phi : N \to Y,$$

and recall that we have denoted by L the 2-torsion line bundle on Y associated to this cover.

Proposition 4.2.2. *The double dover $N \to Y$ is induced by the index two subgroup*

$$f^*H^1(C,\mathbb{Z})/(1+\iota)H^1(D,\mathbb{Z}) \subset \pi_1(Y) = H^1(D,\mathbb{Z})^+/(1+\iota)H^1(D,\mathbb{Z}. \tag{4.8}$$

Proof. Let $u : J \to Y$ be the double cover induce by (4.8), and let L' be the associated two-torsion line bundle. We will prove that L and L' are isomorphic. First of all, we prove that for every $t \in U$, the induced coverings $N_t \to Y_t$ and $J_t \to Y_t$ are isomorphic, so that we find an isomorphism

$$L_t \cong L'_t, \qquad \forall t \in U.$$

Indeed, $N_t \to Y_t$ is induced by the surjection of the exact sequence

$$0 \to H^1(C_t,\mathbb{Z}) \xrightarrow{f^*} H^1(D_t,\mathbb{Z})^+ \to \mathbb{Z}/(2) \to 0.$$

whereas, by definition, $J_t \to Y_t$ is induced by the composition of $\pi_1(Y_t) \to \pi_1(Y)$ with the projection $\pi_1(Y) \to \pi_1(Y)/(f^*H^1(C,\mathbb{Z})/(1+\iota)H^1(D,\mathbb{Z}))$. But this is exactly

$$H^1(D_t,\mathbb{Z})^+ \to H^1(D_t,\mathbb{Z})^*/f^*H^1(C,\mathbb{Z}).$$

Now let $L_{U'}$ and $L'_{U'}$ be the restrictions of the two line bundles to the locus of sheaves with irreducible support $Y_{U'}$. Since $L_{U'}$ and $L'_{U'}$ agree on every fiber, we can write

$$L_{U'} \otimes L'_{U'} \cong \pi^*\mathcal{I},$$

for some $\mathcal{I}$ in $\mathrm{Pic}(U') = \mathbb{Z}$. However, since the rational section $s : |C| \dashrightarrow Y$ is defined on U' and since U' is simply connected, the restrictions $L_{|s(U')}$ and $L'_{|s(U')}$ to $s(U')$ are trivial, so that $s^*(L_{U'} \otimes L'_{U'}) = \mathcal{O}_{U'}$ and hence $\mathcal{I} \cong \mathcal{O}_{U'}$. Using Assumption 4.1.5 we may conclude that

$$L \cong L'.$$

□

As consequence of this proposition, we find that

$$\pi_1(N) \cong f^*H^1(C,\mathbb{Z})/(1+\iota)H_1(D,\mathbb{Z}) \cong H_1(C,\mathbb{Z})/f_*H_1(D,\mathbb{Z}). \tag{4.9}$$

Now let

$$\Psi : \widetilde{N} \to N,$$

be the universal covering space.

Using the so called Norm map, we can get a geometric interpretation of the covering $\widetilde{N}_U \to N_U$, at least in the case of degree zero Jacobians.Let $[C_t] \in |C|$ be a smooth curve, and set $D_t = f^{-1}(C_t)$. The Norm map (cf. (2.2)) is a morphism of abelian varieties $\mathrm{Norm} : \mathrm{Pic}^0(D_t) \to \mathrm{Pic}^0(C_t)$ defined by

$$\mathrm{Norm}_t : L \longmapsto \det(f_*L) \otimes \det(f_*\mathcal{O}_D)^\vee.$$

Let M_U be the restriction of $M = M_{v,H}(S)$ to U. We can consider a relative norm map

$$\mathrm{Norm}_U : M_U \longrightarrow N_U,$$

which is surjective. Over each curve $[C_t] \in U$ the kernel of Norm_t has two connected components (cf. [49] and (2.4)). It follows that Norm_t does not have connected fibers, and we can consider the Stein factorization,

$$M_U \to Q_U \to N_U.$$

We claim that the double cover $Q_U \to N_U$ is isomorphic to the double cover $\widetilde{N}_U \to N_U$. Indeed, the covering $Q_t \to N_t$ corresponds to the inclusion $f_* H_1(D_t, \mathbb{Z}) \subset H_1(C_t, \mathbb{Z})$, so that we can conclude with an argument just like that in the proof of Proposition 4.8.

We conclude this section by describing the induced covering space of the general singular fiber.

Proposition 4.2.3. *Let $C_0 \in |C|$ be an irreducible curve with one single node and no other singularity, and let N_0 be its compactified Jacobian. Then,*

$$\widetilde{N}_0 := \Psi^{-1}(N_0),$$

is irreducible.

Proof. Recall from [53] that if $0 \in |C|$ is a point corresponding to an irreducible curve with a single node, then the Jacobian N_0 of C_0 is l.c.i and has the following properties. The normalization $\widetilde{N}_0$ of N_0 is smooth and isomorphic to a $\mathbb{P}^1$-bundle over the Jacobian of the normalization $\widetilde{C}_0$ of C_0. This $\mathbb{P}^1$-bundle is obtained by compactifiying a natural $\mathbb{C}^*$-bundle on $\mathrm{Jac}^0(\widetilde{C})$ and the normalization morphism

$$n : \widetilde{N}_0 \to N_0,$$

identifies, up to a twist on the base, the zero and infinity sections of the $\mathbb{P}^1$-bundle. We claim that in order to prove that $\widetilde{N}_0$ is irreducible it is sufficient to prove that the covering

$$\Psi^{-1}(\mathrm{Sing}(N_0)) \to \mathrm{Sing}(N_0),$$

is non trivial. Let us assume this. Since $\mathrm{Sing}(\widetilde{N}_0) = \Psi^{-1}(\mathrm{Sing}(N_0))$, it then follows that $\mathrm{Sing}(\widetilde{N}_0)$ is connected and isogenous to $\mathrm{Jac}(\widetilde{C}_0)$. To prove that, under this hypothesis, $\widetilde{N}_0$ is irreducible we proceed by contradiction and suppose that $\widetilde{N}_0$ is the union $Z_1 \cup Z_2$ of two irreducible components. We then have $Z_1 \cap Z_2 \cong \mathrm{Sing}(\widetilde{N}_0)$. Moreover, since the morphism $Z_1 \cup Z_2 = \widetilde{N}_0 \to N_0$ is étale, the two branches meeting in $\mathrm{Sing}(\widetilde{N}_0)$ are smooth. Hence both Z_1 and Z_2 are smooth and thus the induced finite birational morphism $Z_i \to N_0$, $i = 1, 2$, factors through the normalization of N_0 via

a lifting $Z_i \to \widetilde{N}_0$. Under this last morphism the image of the connected set $Z_1 \cap Z_2$ should be the disjoint union of two sections of the $\mathbb{P}^1$-bundle $\widetilde{N}_0$, which is absurd.

Let us now prove that the induced covering of $\mathrm{Sing}(N_0) \cong \mathrm{Jac}(\widetilde{C}_0) \subset N_0$ is non trivial. There is an Abel-Jacobi morphism

$$i_0 : C_0 \to N_0,$$

that this an embedding and is such that the node of C_0 is exactly the intersection of C_0 with the singular locus of N_0 (cf. for eg. [22]). Moreover, one can see that $n^{-1}(C_0) \subset \widetilde{N}_0$ is the normalization $\widetilde{C}_0$ of C_0.

Notice that the image of $\pi_1(\mathrm{Sing}(N_0))$ in $\pi_1(N_0)$ coincides with

$$n_*(\pi_1(\widetilde{N}_0)) \subset \pi_1(N_0).$$

Notice that there are isomorphisms $H_1(\widetilde{N}_0, \mathbb{Z}) \cong H_1(\mathrm{Jac}(\widetilde{C}_0, \mathbb{Z}), \mathbb{Z}) \cong H_1(\widetilde{C}_0, \mathbb{Z})$ that are compatible with the normalization morphisms $\widetilde{N}_0 \to N_0$ and $\widetilde{C}_0 \to C_0$ and with the inclusion $C_0 \to N_0$. It follows that to prove that the covering of $\mathrm{Sing}(N_0)$ is non trivial it is sufficient to prove that the composition

$$H_1(\widetilde{C}_0, \mathbb{Z}) \xrightarrow{n_*} H_1(C_0, \mathbb{Z}) \xrightarrow{i_*} H_1(N, \mathbb{Z}) \cong \mathbb{Z}/(2),$$

is non trivial, where i is the embedding $C_0 \to N$.

Choose a base point $t \in U$ in a neighborhood of 0, and let C_t and D_t be the corresponding curves. We have seen (cf. (4.9)) that

$$H_1(N, \mathbb{Z}) \cong H_1(C_t, \mathbb{Z})/f_* H_1(D_t, \mathbb{Z}),$$

where the isomorphism is induced by the embeddings $C_t \subset N_t \subset N$.

First, we claim that i_* can be identified with the surjection

$$p : H_1(C_0, \mathbb{Z}) \to H_1(C_0, \mathbb{Z})/f_*(D_0, \mathbb{Z}).$$

In fact, if t is close enough to 0, we can consider retractions $r : C_t \to C_0$ and $s : D_t \to D_0$. These retractions induce the following commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & H_1(D_t,\mathbb{Z}) & \xrightarrow{s_*} & H_1(D_0,\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\mathbb{Z} & \longrightarrow & H_1(C_t,\mathbb{Z}) & \xrightarrow{r_*} & H_1(C_0,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_{t*}} & \swarrow{\scriptstyle i^*} & \downarrow{\scriptstyle p} & & \\
 & & H_1(N,\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}/(2) & \longrightarrow & 0
\end{array} \tag{4.10}$$

where the kernels of s_* and of r_* are generated by the vanishing cycles of D_0 and C_0 respectively. Thus p and i^* are identified and we have proved the claim.

The last thing that we need to prove is that

$$H_1(\widetilde{C}_0,\mathbb{Z}) \xrightarrow{n_*} H_1(C_0,\mathbb{Z}) \to H_1(C_0,\mathbb{Z})/f_*H_1(D_0,\mathbb{Z}) \tag{4.11}$$

is non zero. Let $m : \widetilde{D}_0 \to D_0$ be the normalization of D_0, and let $\widetilde{f} : \widetilde{D}_0 \to \widetilde{C}_0$ be the induced double cover. Since D_0 and $\widetilde{D}_0$ are irreducible, it is not hard to see that there is a commutative diagram

$$\begin{array}{ccccc}
H_1(\widetilde{D}_0,\mathbb{Z}) & \xrightarrow{\widetilde{f}_*} & H_1(\widetilde{C}_0,\mathbb{Z}) & \longrightarrow & \mathbb{Z}/(2) \\
\downarrow{\scriptstyle m_*} & & \downarrow{\scriptstyle n_*} & & \| \\
H_1(D_0,\mathbb{Z}) & \xrightarrow{f_*} & H_1(C_0,\mathbb{Z}) & \longrightarrow & \mathbb{Z}/(2).
\end{array}$$

Since $\operatorname{Coker} f_* \cong \operatorname{Coker} \widetilde{f}_*$, it follows that $n_*H_1(\widetilde{C}_0,\mathbb{Z})$ is not contained in $f_*H_1(D_0,\mathbb{Z})$, and thus (4.11) is non zero, and the proof is complete. □

Corollary 4.2.4. *Set $\xi = \nu \circ \Psi$. Then*

$$\xi : \widetilde{N} \to |C|,$$

is a fibration in abelian varieties and the restriction $\widetilde{N}_{\Delta_i}$ of $\widetilde{N}$ to the each component Δ_i of the discriminant locus parametrizing irreducible curves is irreducible and isomorphic to the pullback $\xi^*\Delta_i$.

4.3 The second Betti number

This section is devoted to computing the second Betti number of the relative compactified Jacobian N, constructed in Section 4.1 of a linear system $|C|$ on a general Enriques surface. The strategy is

to compare this cohomogy group with that of the universal family of curves over the linear system and, in fact, we will prove that the two groups have the same rank. In the whole section, unless otherwise stated, cohomology should be understood with complex coefficients.

Let $C \subset T$ be a smooth curve of genus $g \geq 2$. We will assume that $|C|$ satisfies Assumption 4.1.5.

Let χ be a non zero integer, consider the Mukai vector

$$w = (0, [C], \chi),$$

assume that $v = (0, D, 2\chi)$ is primitive, and let A be an ample line bundle on T as in the statement of Proposition 4.1.1. As in Section 4.1, we set

$$N = N_{w,A}, \quad \text{and } Y = \Phi(N) \subset M = M_{v,H},$$

where $H = f^*A$. Finally we let

$$\xi : \widetilde{N} \to N,$$

be the universal cover of N. We will prove the following theorem

Theorem 4.3.1. *Let $|C|$ be a base point free linear system on a general Enriques surface T, and let N be as above. Then*

$$h^2(N) = 11.$$

The same techniques will show the surprising fact that

$$h^2(N) = h^2(\widetilde{N}) = H^2(Y).$$

We will see that this is a consequence of the fact that the covering maps

$$\widetilde{N} \to N, \quad \text{and } N \to Y,$$

induce non trivial double covers of the fibers of $\nu : N \to |C|$ and of $\pi : Y \to |C|$ (cf. diagram (4.3), Proposition 4.2.2, and (4.9) and the discussion thereafter), and of Corollary 4.2.4.

Let $\mathcal{C} \subset T \times |C|$ be the universal family of curves. Since the linear system $|C|$ is base point free, the

first projection

$$q\colon \mathcal{C} \to T \tag{4.12}$$

is a fibration in $(g-2)$-dimensional projective spaces, the Leray spectral sequence degenerates and there is an isomorphism

$$H^2(\mathcal{C}) \cong H^2(T) \oplus H^2(\mathbb{P}^{g-2}) \cong \mathbb{C}^{11}. \tag{4.13}$$

In the same way we see that

$$H^1(\mathcal{C}) = 0.$$

As usual, let

$$U \subset |C|,$$

be the locus of smooth curves, and let $\mathcal{C}_U$ and N_U be the restrictions of ν and of q to U. As in Remark 4.1.7, we let k be the number of irreducible components of Δ and of N_Δ. We start with the following Lemma

Lemma 4.3.2. $H^1(\mathcal{C}_U) \cong H^1(N_U) \cong H^1(U) = \mathbb{C}^{k-1}$.

Proof. The last equality is just Proposition 1.3 of [21] and the fact that Δ has k irreducible components. Since $q : \mathcal{C}_U \to U$ is a smooth morphism, we have

$$H^1(\mathcal{C}_U) \cong H^0(U, R^1q_*\mathbb{C}) \oplus H^1(U),$$

and we just need to show that $H^0(U, R^1q_*\mathbb{C}) = 0$. This follows from the invariant cycle theorem [20], which in our setting asserts that

$$H^0(U, R^1q_*\mathbb{C}) = H^1(C_t)^{\text{inv}} = \text{Im}[H^1(\mathcal{C}) \to H^1(\mathcal{C}_U)],$$

where $H^1(C_t)^{\text{inv}}$ denotes the monodromy invariants of the first cohomology of a smooth curve C_t. Since $H^1(\mathcal{C}) = 0$, we are done. To finish the proof we just need to notice that there is a natural isomorphism of local systems

$$R^1q_*\mathbb{C}_{\mathcal{C}}\big|_U = R^1\nu_*\mathbb{C}_N\big|_U$$

□

Let us now consider the long exact sequences in cohomology for the pairs (N, N_U) and $(\mathcal{C}, \mathcal{C}_U)$. We

have the following Lemma

Lemma 4.3.3. *If $|C|$ is not hyperelliptic, there are exact sequences*

$$\begin{aligned} 0 \to \mathbb{C} \to H^2(N) \to H^2(N_U), \\ 0 \to \mathbb{C} \to H^2(\mathcal{C}) \to H^2(\mathcal{C}_U), \end{aligned} \tag{4.14}$$

and the inclusions $\mathbb{C} \subset H^2(N)$ and $\mathbb{C} \subset H^2(\mathcal{C})$ are identified with the pullbacks

$$\nu^* : H^2(\mathbb{P}^{g-1}) \to H^2(N), \text{ and } q^* : H^2(\mathbb{P}^{g-1}) \to H^2(\mathcal{C}).$$

Proof. By the previous Lemma, and the fact that $H^1(N) = H^1(\mathcal{C}) = 0$, we just have to show that $H^2(N, N_U)$ and $H^2(\mathcal{C}, \mathcal{C}_U)$ are both of dimension k. By Poincaré-Lefschetz duality (cf. [66] Chapter 6, Section §2 Thm 17 or also [60] Thm B.28),

$$H_i(N, N_U) \cong H^{2n-i}(N_\Delta). \tag{4.15}$$

Let $S(N_\Delta)$ be the singular locus of N_Δ. Taking $i = 2$, since the real codimension of $S(N_\Delta)$ in N_Δ is greater or equal to 2, we have

$$H^{2n-2}(N_\Delta) \cong H^{2n-2}(N_\Delta, S(N_\Delta)) \cong H_0(N_\Delta \setminus S(N_\Delta)) \cong \mathbb{C}^k, \tag{4.16}$$

where the second to last isomorphism is again given by Poincaré-Lefschetz duality[2] and the last isomorphism holds because by assumption N_Δ has k irreducible components. The same argument applies to show that $H^2(\mathcal{C}, \mathcal{C}_U) \cong \mathbb{C}^k$. □

Remark 4.3.4. *The Lemma above holds also for (Y, Y_U) and for $(\widetilde{N}, \widetilde{N}_U)$. Indeed, we only have to check that Y_Δ and $\widetilde{N}_\Delta$ have k irreducible components. For Y_Δ this is clear, and for $\widetilde{N}_\Delta$ this follows from Corollary 4.2.4 above and the fact that $|C|$ is not hyperelliptic.*

Recall that the cohomology groups of a smooth quasi-projective variety are endowed with a canonical mixed Hodge structure (MHS) [20]. In order to prove the Theorem, we claim that it is sufficient to prove that there is an isomorphism of mixed Hodge structures (MHS for short)

$$H^2(N_U) \cong H^2(\mathcal{C}_U). \tag{4.17}$$

[2]cit. Spanier, Thm 19 plus the fact the the pair $(N_\Delta, S(N_\Delta))$ is taut in N.

Indeed, if this is the case, there is also an isomorphism

$$W_2(H^2(N_U)) \cong W_2(H^2(\mathcal{C}_U),$$

where $W_2(H^2(N_U))$ and $W_2(H^2(\mathcal{C}_U)$ denote the lowest weight pieces of the MHSs. However, by Cor 3.2.17 of [20], we know that

$$W_2(H^2(N_U)) = \mathrm{Im}[H^2(N) \to H^2(N_U)],$$
$$W_2(H^2(\mathcal{C}_U) = \mathrm{Im}[H^2(\mathcal{C}) \to H^2(\mathcal{C}_U)].$$

Hence, if we assume (4.17), then (4.14) proves the theorem. Before proving (4.17) we need to prove the following proposition.

Proposition 4.3.5. *Let $|C|$ be a fixed component free linear system on a general Enriques surface T, and let $t \in U$ be a point. Suppose that the general pencil of $|C|$ contains only nodal curves. Then monodromy representation*

$$\rho : \pi_1(U) \to \mathrm{Aut}(H^1(C_t, \mathbb{C})), \tag{4.18}$$

is irreducible.

Proof. Let N^0 be the degree zero relative compactified Jacobian of $|C|$. Using Lemma 3.5.5 we know that the kernel R of the natural morphism

$$\pi_1(N_t) = H^1(C_t) \to \pi_1(N^0) \to 1,$$

is generated by vanishing cycles. Since $\pi_1(N^0) \cong \mathbb{Z}/(2)$, we may conclude that $H^1(C_t, \mathbb{C})$ is generated by vanish cycles. Let

$$\{\alpha_i\},$$

be the set of vanishing cycles associated to a set of generators $\{\lambda_i\}$ of $\pi_1(U)$ as in Remark 3.5.4. For simplicity assume that $\mathcal{C}\big|_\ell \to \ell$ is a Lefschetz pencil.

If the discriminant locus $\Delta \subset |C|$ is irreducible, Theorem 3.4 in [72] shows that the restriction of the monodromy representation to the subspace generated by the vanishing cycles is irreducible. Hence, if Δ is irreducible we are done. If this is not the case, to prove that there are no invariant subspaces,

we argue by contradiction. Suppose, therefore, that there is a non trivial invariant subspace

$$F \subset H_1(C_t, \mathbb{C}). \tag{4.19}$$

First of all, we check that F is a symplectic subspace of $H_1(C_t, \mathbb{C})$ equipped with the intersection pairing $(\cdot, \cdot)$. Since the $\{\alpha_i\}$ generate $H_1(C_t, \mathbb{C})$ and the intersection pairing is non degenerate, for every non zero $\beta \in F$ there exist an i such that $(\beta, \alpha_i) \neq 0$. Since F is a ρ-invariant subspace, it follows that $PL_{\lambda_i}(\beta) = -\beta + (\beta, \alpha_i)\alpha_i$ lies in F, and thus that

$$\alpha_i \in F, \tag{4.20}$$

i.e. F is a symplectic subspace. Since the monodromy action preserves the symplectic pairing the orthogonal complement $F^\perp$ is also invariant and, by the same argument as above, symplectic. Set

$$2n = \dim F, \quad 2m = \dim F^\perp.$$

The above argument also shows that every vanishing cycle α_i lies either in F or in $F^\perp$. In particular, we can decompose

$$R = R_F \oplus R_{F^\perp},$$

where R_F and $R_{F^\perp}$ are the sublattices generated by the vanishing cycles that lie in F and $F^\perp$ respectively. Since $R \subset H^1(C_t, \mathbb{Z})$ has index two and $(\cdot, \cdot)$ is unimodular, it follows that the determinant of the intersection matrix for R is 4. Since R_F and $R_{F^\perp}$ are orthogonal with respect to the symplectic pairing we can, up to switching F and $F^\perp$, assume that

$$R_F \subset H^1(C_t, \mathbb{Z}),$$

is primitive. Set $F_\mathbb{Z} = R_F$, $E_\mathbb{Z} = (F_\mathbb{Z})^\perp$ and $E = F^\perp$. By Corollary (1.2.6) of [8] we have

$$H^1(C_t, \mathbb{Z}) = F_\mathbb{Z} \oplus E_\mathbb{Z}. \tag{4.21}$$

Consider now the following commutative diagram,

$$\begin{array}{ccc} H^1(C_t,\mathbb{Z}) & \xrightarrow{\ j\ } & H^1(C_t,\mathcal{O}_{C_t}) \\ \downarrow & & \downarrow{\scriptstyle D}\ \sim \\ H^1_{dR}(C_t,\mathbb{C}) & \xrightarrow[\ p\ ]{} & H^{0,1}_{\overline{\partial}}(C_t) \end{array}$$

where the left hand side vertical arrow is the composition of the base change inclusion $H^1(C_t,\mathbb{Z}) \subset H^1(C_t,\mathbb{C})$, and the De Rham isomorphism. The top horizontal arrow is given by the exponential sequence, the right hand side vertical arrow is the Dolbeaut isomorphism, and the bottom arrow is the projection onto the Dolbeaut group. The two spaces $F' := D^{-1}p(F)$ and $E' := D^{-1}p(E)$ contain the lattices $j(F_{\mathbb{Z}})$ and $j(E_{\mathbb{Z}})$, of rank respectively equal to $2n$ and $2m$ and are thus of dimension equal to n and m respectively. It follows that

$$F_t := F'/j(F_{\mathbb{Z}}), \quad \text{and } E_t := E'/j(E_{\mathbb{Z}}),$$

are two smooth abelian varieties of dimensions equal to n and m respectively, that are principally polarized since $F_{\mathbb{Z}}$ and $E_{\mathbb{Z}}$ are unimodular. We can thus write

$$\mathrm{Jac}^0(C_t) \cong F_t \times E_t. \tag{4.22}$$

However, since the intersection product on $H^1(C_t,\mathbb{Z})$ can be viewed via the isomorphism $\wedge^2 H^1(C_t,\mathbb{Z}) \cong H^2(\mathrm{Jac}^0(C_t),\mathbb{Z})$ as the theta divisor of $\mathrm{Jac}^0(C_t)$, it follows that (4.22) is actually a decomposition as principally polarized abelian varieties. We have thus reached a contradiction since the Jacobian variety of a smooth curve is irreducible as principally polarized abelian variety. □

Proposition 4.3.6. *There are isomorphism of MHS*

$$H^2(N_U,\mathbb{C}) \cong H^2(\mathcal{C}_U,\mathbb{C}). \tag{4.23}$$

Proof. By Theorem 5.1 of [4], there is a functorial MHS on the cohomology of local systems that arise from families of smooth projective varieties. These MHS are compatible with the MHS on the cohomology of the total space of the family, in the sense (cf. point (3) of Theorem 5.1 of cit.) that they are subquotients of the canonical MHS on the cohomology of the total space.

On the other hand, consider a smooth projective morphism of smooth quasi-projective manifolds

$j : Y \to Z$. By the decomposition theorem (see for e.g.. Theorem 5.2.2 of [18]), it follows that

$$j_*\mathbb{Q}_Y \cong \bigoplus_{i\geq 0} R^i j_*\mathbb{Q}_Y[-i].$$

Taking cohomologies on both sides, it follows that the cohomology of Y is actually split, i.e. it is equal as MHS to the *direct sum* of the MHS on the cohomologies of the local systems. Applying these facts to

$$N_U \to U, \quad \text{and to } \mathcal{C}_U \to U,$$

it follows that it is sufficient to prove that for every $i, j \geq 0$ with $i + j = 2$, there are isomorphism of MHS

$$H^i(U, R^j\nu_*\mathbb{C}_N) \cong H^i(U, R^j q_*\mathbb{C}_{\mathcal{C}}). \tag{4.24}$$

Since clearly,

$$R^1\nu_*\mathbb{C}_{N_U} \cong R^1 p_*\mathbb{C}_{\mathcal{C}_U},$$

we only need to prove (4.24) for $(i, j) = (0, 2)$ and we will do this by showing that they are both one dimensional. For $t \in U$, it is clear that the monodromy acts trivially on the top cohomology $H^2(C_t)$ of the smooth curve C_t, and hence the space $H^0(U, R^2 p_*\mathbb{C}_{\mathcal{C}})$ is one dimensional. As for $H^0(U, R^2\nu_*\mathbb{C}_N)$ the situation is slightly more involved. Notice that

$$H^0(U, R^2\nu_*\mathbb{C}_N) \cong H^2(N_t, \mathbb{C})^{\text{inv}} \cong (\wedge^2 H^1(C_t, \mathbb{C}))^{\text{inv}}. \tag{4.25}$$

Now let

$$\rho : \pi_1(U) \to \text{Aut}(H^1(C_t, \mathbb{C}))$$

be the monodromy representation. Since ρ preserves the symplectic pairing $(\cdot, \cdot)$, the isomorphism $w : H^1(C_t, \mathbb{C}) \cong H_1(C_t, \mathbb{C})$ induced by $(\cdot, \cdot)$ is ρ-equivariant. By composing with w, it follows that any ρ-invariant element of $\wedge^2 H^1(C_t, \mathbb{C})$ can be thought of as a ρ-invariant morphism

$$\varphi : H^1(C_t, \mathbb{C}) \to H^1(C_t, \mathbb{C}). \tag{4.26}$$

By Proposition 4.3.5 above, ρ is an irreducible representation. It follows then by Schur's Lemma that

$$\varphi = \lambda id, \tag{4.27}$$

for some λ in $\mathbb{C}$, and hence that

$$(\wedge^2 H^1(C_t, \mathbb{C}))^{\text{inv}} \cong \mathbb{C}. \tag{4.28}$$

□

Notice that the invariants are actually defined over $\mathbb{Z}$. In fact, the theta divisor of $\text{Jac}(C_t)$ is always invariant under the monodromy, since it can be identified with the isomorphism $w : H^1(C_t, \mathbb{Z}) \cong H_1(C_t, \mathbb{Z})$. Since the group $H^2(N_t, \mathbb{Z})^{\text{inv}}$ is one dimensional, it is generated by the theta divisor of C_t.

Remark 4.3.7. *The morphisms $\widetilde{N} \to N$ and $N \to Y$ induce non trivial covers of the fibers of ν and of π, and hence for every i there are isomorphisms of local systems*

$$R^i\xi_*\mathbb{C}_{\widetilde{N}}\big|_U \cong R^i\nu_*\mathbb{C}_N\big|_U \cong R^i\pi_*\mathbb{C}_Y\big|_U.$$

Corollary 4.3.8. *If $|C|$ is not hyperelliptic, $h^2(\widetilde{N}) = h^2(N) = h^2(Y)$.*

Lastly, we consider the hyperelliptic case

Theorem 4.3.9. *Let $|C|$ be a hyperelliptic linear system of genus $g \geq 3$. Then*

$$h^1(N) = 11$$

Proof. First of all, we compute $H^2(\mathcal{C})$. Since $|C|$ has two simple base points, the projection $\mathcal{C} \to T$ factor via the blow up $\widetilde{T} \to T$ of the surface at the two base points. The induced morphism $\mathcal{C} \to \widetilde{T}$ is a fibration in $(g-2)$-dimensional projective spaces, and hence $H^2(\mathcal{C}) = \mathbb{C}^{13}$. By Proposition 3.1.15 Δ has 4 irreducible components so that

$$H^1(\mathcal{C}_U) \cong H^1(N_U) \cong H^1(U) = \mathbb{C}^3.$$

As in Lemma 4.3.3 we compute

$$h^2(\widetilde{N}, \widetilde{N}_U) = h^2(Y, Y_U) = h^2(N, N_U) = \#\text{ of irreducible components of } \cup_i N_{\Delta_i} = 5,$$
$$h^2(\mathcal{C}, \mathcal{C}_U) = \#\text{ of irreducible components of } \cup_i \mathcal{C}_{\Delta_i} = 7,$$

so that (4.14) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}^2 & \longrightarrow & H^2(N) & \xrightarrow{r} & H^2(N_U) \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & \mathbb{C}^4 & \longrightarrow & H^2(\mathcal{C}) & \xrightarrow{s} & H^2(\mathcal{C}_U). \end{array}$$

The second vertical arrow is the pullback via the Abel-Jacobi map $j : \mathcal{C}_U \subset N_U$. Since the linear system has base points, the morphism $\mathcal{C}_U \to U$ admits a section and the relative compactified Jacobian of $\mathcal{C}_U$ is independent of the degree. Hence we can assume that the Abel-Jacobi map sends the family of curves into the degree zero relative Jacobian. By Proposition 4.3.6 j^* is an isomorphism and the theorem is proved. □

4.4 The canonical bundle

To prove the main Theorem of this section, we are going to use Assumption 4.1.5 and also that the general line $\mathbb{P}^1 \subset |C|$ is such that all the singular members are reduced and nodal so that the local monodromy is unipotent.

Before starting, let us emphasize that

Lemma 4.4.1. *The canonical bundle of Y is trivial if and only if the canonical bundle of N is trivial.*

Proof. The only if part is clear, since $\Phi : N \to Y$ is étale, so we only need to address the if part. Recall the definition of L from formula (4.4). If $\omega_N \cong \mathcal{O}_N$, then $\Phi^*\omega_Y \cong \mathcal{O}_N$, implying either that ω_Y is trivial, or that it is isomorphic to L. To conclude the proof, we make the following two claims. Consider a point $t \in U$ and denote as usual by Y_t the fiber over t. The first claim is that L_{Y_t} is not trivial, which follows immediately from (3.5). Whilst the second claim is that $(\omega_Y)_{|Y_t}$ is trivial, which follows immediately from the fact that ω_{Y_t} and $N_{Y_t|Y}$ are trivial. Hence, ω_Y cannot be isomorphic to L, and the Lemma is proved. □

Remark 4.4.2. The same argument applies to show that the canonical bundle of the universal cover $\widetilde{N}$ is trivial if and only if the canonical bundle of N is trivial. In fact, the only thing that we use here is that on each fiber the double cover $Y_t \to N_t$ is non trivial, which is a consequence of formula (4.9).

Remark 4.4.3. The fact that the universal cover of N induces a non trivial cover of each fiber, establishes a difference with Enriques surfaces: any Enriques surface T admits an elliptic fibration

$T \to \mathbb{P}^1$ which has exactly two multiple fibers. The canonical bundle is the difference of the two half fibers; moreover, the universal cover $S \to T$ induces a *trivial* cover of every reduced fiber; indeed, if e is a primitive elliptic curve in T, then for any reduced curve $\Gamma \in |2e|$, $f^{-1}(e)$ is the disjoint union of two curve in the linear system $|f^*e|$. In fact, the covering $S \to T$ is induced by base change via a degree two morphism $\mathbb{P}^1 \to \mathbb{P}^1$ ramified at the two points corresponding to the non reduced fibers; as we have already mention, in the case of $\nu : N \to |C|$, the restriction of the universal cover to the fibers of ν is non trivial.

The rest of the section is devoted to prove that ω_N is trivial.

Consider the morphism $\nu : N \to \mathbb{P}^{g-1} = |C|$ of smooth projective varieties. We can write,

$$\omega_N = \nu^* \omega_{\mathbb{P}^{g-1}} \otimes \omega_\nu = \nu^* \mathcal{O}_{\mathbb{P}^{g-1}}(-g) \otimes \omega_\nu, \tag{4.29}$$

where ω_ν is the relative dualizing sheaf of ν. Clearly, for every $t \in U$, we have $\omega_{\nu|N_t} \cong \omega_{N_t} \cong \mathcal{O}_{N_t}$, which implies that

$$\omega_{\nu|N_t} \cong \mathcal{O}_{N_t}, \quad \text{for every } \ t \in W,$$

where W denotes the locus of reduced curves. Since $N_W \to W$ is flat, it follows that $\nu_* \omega_{\nu|W}$, is locally free, and the natural morphism,

$$\omega_{\nu_W} \to \nu^* \nu_* \omega_{\nu_W}, \tag{4.30}$$

is an isomorphism. Moreover, since $\operatorname{codim}(W, |C|) \geq 2$, we have

$$(\nu_* \omega_\nu)_{|W} \cong \mathcal{O}_W(k),$$

for some $k \in \mathbb{Z}$ to be determined. Assumption 4.1.5 now guarantees that

$$\operatorname{Pic}(N) \cong \operatorname{Pic}(N_W),$$

and thus that

$$\omega_\nu \cong \nu^* \mathcal{O}_{\mathbb{P}^{g-1}}(k). \tag{4.31}$$

The rest of this Section is devoted to show, under the assumption that the singular fibers of the

general pencil in $|C|$ are reduced and nodal, that

$$k = g, \tag{4.32}$$

which, by (4.29) proves the following Theorem

Theorem 4.4.4. *Under Assumption 4.1.5 and the assumption that the general pencil in $|C|$ has reduced and nodal members, the canonical bundle of N is trivial, i.e.,*

$$\omega_N \cong \mathcal{O}_N. \tag{4.33}$$

Corollary 4.4.5. *With the same assumptions as in the Theorem,*

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_N) = 0.$$

Proof. It is enough to prove that $\chi(\mathcal{O}_Y) = 0$. This follows from a calculation of Beauville [10] on the $\hat{A}$-genus of the fixed locus of an anti-symplectic involution on a Hyperkähler manifold which implies, in the case $\dim M_{v,H}(S) \equiv 2 \pmod 4$, that

$$\int_Y \mathrm{Todd}(Y) = 0.$$

□

The main ingredients in the proof of (4.32) are the following. First of all, the simple observation that to compute k we can restrict the family N to a general line in $\mathbb{P}^{g-1}$. Second, the theory of degeneration of Hodge bundles, as explained in [76] and Section 2 of [37], to understand the direct image of the relative canonical sheaf of $N \to \mathbb{P}^{g-1}$.

Let $\mathbb{P}^1 \subset \mathbb{P}^{g-1}$ be a general line, set

$$\nu' : N' = \nu^{-1}(\mathbb{P}^1) \to \mathbb{P}^1. \tag{4.34}$$

We can choose the $\mathbb{P}^1$ so that N' is smooth and so that the local monodromy around the singular fibers is unipotent. We let $t_1, \ldots, t_\delta$ be the points of the discriminent $\Delta' := \Delta \cap \mathbb{P}^1$ of ν. Notice that ν' is flat.

Lemma 4.4.6. *With the above notation, we have*

$$\omega_{\nu'} \cong \omega_{\nu|N'}, \quad \text{i.e.} \quad \omega_{\nu'} \cong \nu'^{*}\mathcal{O}_{\mathbb{P}^1}(k). \tag{4.35}$$

Proof. Notice that $\nu^*\mathcal{N}_{\mathbb{P}^1|\mathbb{P}^{g-1}} \cong \mathcal{N}_{N'|N}$[3]. Since $\det \mathcal{N}_{\mathbb{P}^1|\mathbb{P}^{g-1}} \cong \mathcal{O}_{\mathbb{P}^1}(g-2)$, it follows by adjunction that

$$\omega_{N'} \cong (\omega_\nu \otimes \nu^*\mathcal{O}_{\mathbb{P}^{g-1}}(-g))_{|N'} \otimes \nu'^{*}\mathcal{O}_{\mathbb{P}^1}(g-2), \tag{4.36}$$

and comparing this to $\omega_{N'} \cong \nu'^{*}\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \omega_{\nu'}$ we get,

$$\omega_{\nu|N'} \cong \omega_{\nu'}. \tag{4.37}$$

□

This allows us to restrict our attention to $N' \to \mathbb{P}^1$, where the situation is simpler. Indeed, by our choice of $\mathbb{P}^1 \subset |C|$, the restriction $p' : \mathcal{C}' \to \mathbb{P}^1$ of the universal family $\mathcal{C} \to \mathbb{P}^{g-1}$ to the line is just the blow up of the surface S at the $2g-2$ simple base points of the pencil.

We have now the following proposition whose proof will be given at the end of the Section, after a brief detour on degeneration of Hodge bundles.

Proposition 4.4.7. *With the above notation. Then*

$$\nu'_*\omega_{\nu'} \cong (\det R^1p_*\mathcal{O}_{\mathcal{C}'})^\vee. \tag{4.38}$$

The right hand side of (4.38) is readily calculated: by Theorem 2.11 of [68] the sheaf $R^1p_*\mathcal{O}_{\mathcal{C}'}$ is locally free of rank g. Hence, there exist g integers $a_1, \ldots, a_g$, such that

$$R^1p_*\mathcal{O}_{\mathcal{C}'} \cong \bigoplus_{i=1}^{g} \mathcal{O}_{\mathbb{P}^1}(a_i). \tag{4.39}$$

Since the base is one dimensional, the Leray spectral sequence degenerates and we can use the Hodge numbers of $\mathcal{C}'$ to calculate the a_i's. From

$$\begin{aligned} 0 = H^1(\mathcal{C}', \mathcal{O}) &\cong H^1(\mathbb{P}^1, \mathcal{O}) \oplus H^0(\mathbb{P}^1, R^1p_*\mathcal{O}_{\mathcal{C}'}), \quad \text{and} \\ 0 = H^2(\mathcal{C}', \mathcal{O}) &\cong H^2(\mathbb{P}^1, \mathcal{O}) \oplus H^1(\mathbb{P}^1, R^1p_*\mathcal{O}_{\mathcal{C}'}) \oplus H^0(\mathbb{P}^1, R^2p_*\mathcal{O}_{\mathcal{C}'}), \end{aligned} \tag{4.40}$$

[3]The natural surjective base change map between the conormal bundles $\nu'^*\mathcal{N}^\vee_{\mathbb{P}^1|\mathbb{P}^{g-1}} \to \mathcal{N}^\vee_{N'|N}$ is an isomorphism because the two bundles have the same rank.

we deduce that

$$a_i < 0, \quad \text{and} \quad a_i > -2.$$

We conclude that $a_i = -1$ for every i, so that

$$\nu'_* \omega_{\nu'} \cong \mathcal{O}_{\mathbb{P}^1}(g). \tag{4.41}$$

By relative Serre duality applied to $\nu' : N' \to \mathbb{P}^1$, we get an isomorphism

$$\nu'_* \omega_{\nu'} \cong (R^g \nu'_* \mathcal{O}_{N'})^\vee. \tag{4.42}$$

Moreover, if we let

$$V \subset \mathbb{P}^1,$$

be the locus where ν' is smooth, we have

$$R^1 \nu'_* \mathcal{O}_{N'|V} \cong R^1 p'_* \mathcal{O}_{\mathcal{C}'|V}, \quad \text{and} \quad R^g \nu'_* \mathcal{O}_{N'|V} \cong \wedge^g R^1 \nu'_* \mathcal{O}_{N'|V}. \tag{4.43}$$

To prove the Proposition we wish to extend these isomorphisms to the whole $\mathbb{P}^1$ in order to get an isomorphism

$$R^g \nu'_* \mathcal{O}_{N'} \cong \wedge^g R^1 p'_* \mathcal{O}_{\mathcal{C}'}.$$

Before talking about variation of Hodge structure and Hodge bundles, we wish to make the following remark.

Remark 4.4.8. From our assumption on the existence of Lefschetz pencils and classical results of Oda and Seshadri [53] the fiber of ν' and of p' over a point t_i in Δ' is a reduced divisor with normal crossings. Moreover, the local monodromy of each family around a point t_i is unipotent.

4.4.1 Degeneration of Hodge bundles

Following [76], [33], and [37] we start by briefly recalling some facts about variations of Hodge structures and Hodge bundles.

Let B be a smooth curve, and consider the degree i variation of Hodge structures (VHS for short) induced by a smooth projective morphism $f : Z \to B$. The local system underlying this VHS is

$R^i f_* \mathbb{C}$. By [19], Prop. I.2.28, the locally free sheaf

$$\mathcal{H}^i := R^i f_* \mathbb{C} \otimes \mathcal{O}_B \tag{4.44}$$

is isomorphic to the hypercohomology sheaf

$$R^i f_* \Omega^\bullet_{Z|B}, \tag{4.45}$$

where $\Omega^\bullet_{Z|B}$ is the complex of relative differentials of the family. Let

$$\nabla : \mathcal{H}^i \to \mathcal{H}^i \otimes \Omega^1_B \tag{4.46}$$

be the Gauss-Manin connection of the family $Z \to B$.

Consider the so called *filtration bête*

$$\mathcal{F}^p \Omega^\bullet_{Z|B} = \Omega^{\bullet \geq p}_{Z|B}, \tag{4.47}$$

of the complex of the relative differentials. The spectral sequence in hypercohomology associated to this filtration has $E_1^{p,q}$ term equal to

$$R^q f_* \Omega^p_{Z|B}, \tag{4.48}$$

and abuts to (the associated graded pieces of)

$$R^i f_* \Omega^\bullet_{Z|B} = \mathcal{H}^i. \tag{4.49}$$

Since f is smooth these sheaves are locally free, the spectral sequence degenerates at E_1 and the maps

$$R^i f_* \Omega^{\bullet \geq p}_{Z|B} \to R^i f_* \Omega^\bullet_{Z|B} \tag{4.50}$$

are injective. In particular, the filtration induced on $\mathcal{H}^i$ is

$$\mathcal{F}^p \mathcal{H}^i = R^i f_* \Omega^{\bullet \geq p}_{Z|B} \subset \mathcal{H}^i, \tag{4.51}$$

and the associated graded pieces of $(\mathcal{H}^i, \mathcal{F}^p(\mathcal{H}^i))$ are the sheaves

$$R^q f_* \Omega^p_{Z|B}, \quad \text{with} \quad p + q = i. \tag{4.52}$$

The filtration (4.51) is precisely the filtration of the variation of Hodge structures of the family $Z \to B$.

Now consider a smooth projective compactification

$$f\colon \overline{Z} \to \overline{B}$$

and suppose that $W := f^{-1}(\overline{B} \setminus B)$ is a reduced divisor with normal crossing. We let $\Omega^{\bullet}_{\overline{Z}|\overline{B}}(\log W)$ be the complex of relative logarithmic differentials.

By a classical theorem (Thm 7.9 of [19]), it is known that $(\mathcal{H}^i, \nabla)$ is an algebraic differential equation with regular singular points, in the sense that ∇ has logarithmic poles at every point b in $\overline{B} \setminus B$. That is to say, there exists a vector bundle extension $\overline{\mathcal{H}}^i$ of $\mathcal{H}^i$ to all of $\overline{B}$, such that the connection ∇ extends to a morphism

$$\overline{\nabla} : \overline{\mathcal{H}}^i \to \overline{\mathcal{H}}^i \otimes \Omega^1_{\overline{Z}|\overline{B}}(\log W). \tag{4.53}$$

By definition, the *residue* $\mathrm{Res}_b(\overline{\nabla})$ of $\overline{\nabla}$ at a point b in $\overline{B} \setminus B$ is the endomorphism of $\overline{\mathcal{H}}^i_b$ induced by taking the residue in b. In other words, if we fix a local trivialization of $\overline{\mathcal{H}}^i$ around b and if z is a local coordinate on $\overline{B}$, the residue is defined by $\overline{\nabla} = dz \otimes (\mathrm{Res}_b(\overline{\nabla})1/z + \dots)$.

By [68], Thm 2.11, (cf. also the exposition in [76]) the sheaves

$$R^i f_* \Omega^{\bullet}_{\overline{Z}|\overline{B}}(\log W)$$

and

$$R^q f_* \Omega^p_{\overline{Z}|\overline{B}}(\log W), \qquad p+q = i,$$

are *locally free* extensions of (4.49) and (4.48) respectively. Moreover, Katz proved in [33] that there is a morphism

$$\overline{\nabla} : R^i f_* \Omega^{\bullet}_{Z|B}(\log W) \to R^i f_* \Omega^{\bullet}_{Z|B}(\log W) \otimes \Omega^1_{\overline{X}|\overline{B}}(\log W), \tag{4.54}$$

which extends the Gauss-Manin connection. It follows that we can set

$$\overline{\mathcal{H}}^i \cong R^i f_* \Omega^{\bullet}_{Z|B}(\log W). \tag{4.55}$$

On the other hand, the filtration (4.47) extends to a filtration

$$\mathcal{F}^p\Omega^\bullet_{\overline{Z}|\overline{B}}(\log W) := \Omega^{\bullet\geq p}_{\overline{Z}|\overline{B}}(\log W), \tag{4.56}$$

which gives a spectral sequence whose $E_1^{p,q}$ terms are

$$R^q f_*\Omega^p_{\overline{Z}|\overline{B}}(\log W). \tag{4.57}$$

Since these sheaves are locally free and the differential is generically zero, the differential is identically zero. Hence the spectral sequence degenerates at E_1 and abuts to the associated graded pieces of

$$R^i f_*\Omega^\bullet_{Z|B}(\log W). \tag{4.58}$$

Since the graded pieces $R^q f_*\Omega^p_{\overline{Z}|\overline{B}}(\log W)$ are locally free, the extensions

$$\mathcal{F}^p(\overline{\mathcal{H}}^i) := R^i f_*\Omega^{\geq p}_{\overline{X}|\overline{B}}(\log W) \tag{4.59}$$

of the sheaves $\mathcal{F}^p(\mathcal{H}^i)$ are actually extensions as vector *sub-bundles* of $\overline{\mathcal{H}}^i$. This is a particular case of Schmid's Nilpotent Orbit Theorem.

The last ingredient is the following classical theorem

Theorem 4.4.9 (Manin, [19], Prop. 5.4)**.** *Let $(\mathcal{H},\nabla)$ be an algebraic differential equation with regular singular points on the pointed disc Δ^*. Then $(\mathcal{H},\nabla)$ admits a* unique *locally free extension $(\overline{\mathcal{H}},\overline{\nabla})$ to the disc Δ, which satisfies that following two properties*

1. *$\overline{\nabla} : \overline{\mathcal{H}} \to \overline{\mathcal{H}} \otimes \Omega^1_{\overline{B}}(\log 0)$ has logarithmic poles;*
2. *The eigenvalues λ of $\mathrm{Res}_0(\overline{\nabla})$ satisfy $0 \leq \mathrm{Re}(\lambda) < 1$.*

An extension of an algebraic differential equation $\mathcal{H}$ as in the Theorem is called the *canonical extension.*

Recall that we are assuming that $W \subset Z$ is a reduced divisor with normal crossing. By the regularity theorem of Katz [33] (see [67] for the case of normal crossing and not just simple normal crossing; (5.1) for the notation and (5.3) for the result), it follows that

$$R^i f_*\Omega^\bullet_{Z|B}(\log W),$$

satisfies the assumptions of Theorem 4.4.9 with the eigenvalues of the residue $\mathrm{Res}_b(\overline{\nabla})$ of the Gauss-Manin connection equal to zero for every $b \in \overline{B} \setminus B$.

Under this circumstance we say, by abuse of notation, that $R^i f_* \Omega^{\geq p}_{\overline{X}|\overline{B}}(\log W)$ and $R^q f_* \Omega^p_{\overline{Z}|\overline{B}}(\log W)$ are the *canonical extensions* of $R^i f_* \Omega^{\bullet \geq p}_{Z|B}$ and $R^q f_* \Omega^p_{Z|B}$ respectively.

Notice that we can say "canonical extension" also for $\mathcal{F}^p(\overline{\mathcal{H}}^i)$ and for $\mathrm{Gr}^p(\mathcal{H}^i)$ because they are unique. In fact, let $j : B \to \overline{B}$ be the open immersion, let $\mathcal{H}$ be a vector bundle on B and let $E \subset \mathcal{H}$ be a sub-bundle. Suppose we are given a vector bundle extension $\overline{\mathcal{H}}$ of $\mathcal{H}$ on the whole of $\overline{B}$. Any extension of E to $\overline{B}$ as a sub-bundle of $\overline{\mathcal{H}}$ is always contained in the saturation of $\overline{\mathcal{H}} \cap j_* E$ in $\overline{\mathcal{H}}$, and thus has to be isomorphic to the saturation itself. In particular, since the extension of E as a vector sub-bundle of $\overline{\mathcal{H}}$ is unique, so is the extension of the quotient $\mathcal{H}/E$.

We now go back to our situation applying these remarks to the families $\mathcal{C}' \to \mathbb{P}^1$ and $N' \to \mathbb{P}^1$.

Proof of Proposition 4.4.7. Recall that our aim is to prove that there is an isomorphism $R^g \nu_* \mathcal{O}_N \cong \wedge^g R^1 p'_* \mathcal{O}_{\mathcal{C}'}$ extending that of (4.43). As should be clear by now, we want to prove this isomorphism by showing that the sheaves in question are both isomorphic to canonical extensions.

Recall that by the choice of the pencil $\mathbb{P}^1 \subset |C|$, the families $\mathcal{C}' \to \mathbb{P}^1$ and $N' \to \mathbb{P}^1$ have singular fibers that are reduced and have normal crossing. The sheaves

$$\begin{aligned} \overline{\mathcal{H}}^1_{N'} &= R^1 \nu_* \Omega^{\bullet}_{N'|\mathbb{P}^1}(\log(N'_{\Delta'})) \\ \overline{\mathcal{H}}^1_{\mathcal{C}'} &= R^1 p_* \Omega^{\bullet}_{\mathcal{C}'|\mathbb{P}^1}(\log(\mathcal{C}'_{\Delta'})) \end{aligned} \tag{4.60}$$

both extend $R^1 \nu_* \mathbb{C} \otimes \mathcal{O}_{N'|V} \cong R^1 p_* \mathbb{C} \otimes \mathcal{O}_{\mathcal{C}'|V}$, and are isomorphic to the canonical extension. Hence, there is an isomorphism

$$\overline{\mathcal{H}}^1_{N'} \cong \overline{\mathcal{H}}^1_{\mathcal{C}'}, \tag{4.61}$$

which extends the isomorphism over V. As a consequence, we get an isomorphism of the canonical extension of $\mathcal{F}^1(\mathcal{H}^1_{N'}) \cong \mathcal{F}^1(\mathcal{H}^1_{\mathcal{C}})$, i.e.,

$$\mathcal{F}^1(\overline{\mathcal{H}}^1_{N'}) \cong \mathcal{F}^1(\overline{\mathcal{H}}^1_{\mathcal{C}'}),$$

which in turn implies that there is an isomorphism of the first graded pieces

$$\mathrm{Gr}^0(\overline{\mathcal{H}}^1_{N'}) \cong \mathrm{Gr}^0(\overline{\mathcal{H}}^1_{\mathcal{C}'}), \tag{4.62}$$

i.e., an isomorphism

$$R^1\nu_*\mathcal{O}_{N'} \cong R^1p_*\mathcal{O}_{\mathcal{C}'}. \tag{4.63}$$

We now want to prove that

$$R^g\nu_*\mathcal{O}_{N'} \cong \wedge^g R^1p_*\mathcal{O}_{\mathcal{C}'}.$$

From the general theory, the exterior powers $\wedge^j\mathcal{H}$ of an algebraic differential equation $(\mathcal{H},\nabla)$ with regular singular points is naturally an algebraic differential equation with regular singular points. Moreover, if the residues of $\overline{\nabla}_{\mathcal{H}}$ have real part equal to zero, then the same is true for the residues of the induced connection on $\wedge^j\overline{\mathcal{H}}$. In other words, the sheaf $\wedge^j\overline{\mathcal{H}}$ is the canonical extension of $\wedge^j\mathcal{H}$. We deduce that for any i,

$$\wedge^i\overline{\mathcal{H}}^1_{N'} \cong \wedge^i\overline{\mathcal{H}}^1_{\mathcal{C}}. \tag{4.64}$$

Moreover, since N'_U is a family of abelian varieties, we have an isomorphism of VHS $\wedge^j\mathcal{H}^1_{N'} \cong \mathcal{H}^g_{N'}$. Using again the result of Katz, we known that

$$\overline{\mathcal{H}}^g_{N'} := R^g\nu'_*\Omega^\bullet_{N'|\mathbb{P}^1}(\log {N'}_{\Delta'}) \tag{4.65}$$

is also the canonical extension. It follows that,

$$\wedge^g\overline{\mathcal{H}}^1_{\mathcal{C}} \cong \overline{\mathcal{H}}^g_{N'}. \tag{4.66}$$

This induces an isomorphism of the canonical extension of the respective filtrations, and thus of the respective graded pieces. Since $\mathrm{Gr}^0(\wedge^g\mathcal{H}^1_{\mathcal{C}'}) \cong \wedge^g\mathrm{Gr}^0(\mathcal{H}^1_{\mathcal{C}})$ we conclude

$$R^g\pi'_*\mathcal{O}_{N'} \cong \wedge^g R^1p'_*\mathcal{O}_{\mathcal{C}'}, \tag{4.67}$$

and the proposition is proved. □

Remark 4.4.10. We want to highlight a difference in behavior between other type of moduli spaces of sheaves on Enriques surfaces. In [58], Oguiso and Schröer prove that the Hilbert scheme of n points on a given Enriques surface T has the property that the canonical bundle is not trivial, but twice the canonical bundle is trivial. It would be interesting to know if this is a general phenomenon, that is to say if there is a different behavior of the canonical bundle of odd and even dimensional moduli spaces of sheaves on Enriques surfaces.

For example, given a genus g linear system $|C|$, it would be interesting to study the geometry of the rational Abel-Jacobi maps

$$T^{[g-1]} \dashrightarrow N^{g-1}, \qquad \text{and} \qquad N^g \dashrightarrow T^{[g]},$$

where we have denoted by N^d the degree d relative compactified Jacobian of $|C|$.

We end the section with the following theorem.

Theorem 4.4.11. *Under the same assumptions as Theorem 4.4.4, the relative compactified Jacobian N is an irreducible Calabi-Yau manifold, in the sense that*

$$H^p(N, \mathcal{O}_N) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, 2g-1, \\ 0 & \text{otherwise.} \end{cases}$$

The main step in proving the theorem is the following proposition which generalizes what was prove over a general Lefshetz pencil, and uses Matsushita's theorem [45] on the higher direct images of the morphism $M \to |D|$ (cf. 3.2.3).

Proposition 4.4.12.

$$R^i\nu_*\mathcal{O}_N \cong \wedge^i \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^{g-1}}(-1), \tag{4.68}$$

Using this proposition, the spectral sequence to calculate $H^p(N, \mathcal{O}_N)$ degenerates since

$$H^k(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(-p+k)) \cong \begin{cases} \mathbb{C} & (k,p) = (0,0) \quad \text{or } (k,p) = (g-1, 2g-1) \\ 0 & \text{otherwise.} \end{cases}$$

and the theorem easily follows.

Proof of Proposition 4.4.12. Since the canonical bundle of N and of Y is trivial, by Theorem 2.1 in [36] the sheaves $R^i\nu_*\mathcal{O}_N$ and $R^i\pi_*\mathcal{O}_Y$ are torsion free and by Corollary 3.9 in [37] they are reflexive. Since Φ is a finite morphism, the composition of functors spectral sequence yields an isomorphism (recall the definition (4.4) of L)

$$R^i\nu_*\mathcal{O}_N \cong R^i\pi_*\mathcal{O}_Y \oplus R^i\pi_*L.$$

However, $R^i\pi_*L$ is a supported on the discriminant locus of $|C|$ and since $R^i\nu_*\mathcal{O}_N$ is torsion free, it

is identically zero and hence $R^i\nu_*\mathcal{O}_N \cong R^i\pi_*\mathcal{O}_Y$.

We claim that there is open subset $W \subset \mathbb{P}^{g-1}$, whose complement has codimension greater or equal to two, that has the property that

$$(R^i\nu_*\mathcal{O}_N)_{|W} \cong (\wedge^i R^1\nu_*\mathcal{O}_N)_{|W}.$$

Indeed, the such an isomorphism exist on the smooth locus U is clear. However, by what we proved in the last subsection it extends to an isomorphism over a slightly bigger open subset containg the lines $\mathbb{P}^1 \subset |C|$ with properties as in the previous subsection, whose complement has codimension greater or equal to two. Since the sheaves in question are reflexive, these isomorphisms extends to global isomorphisms on the whole $\mathbb{P}^{g-1}$. We are thus reduced to prove that

$$R^1\pi_*\mathcal{O}_Y \cong \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^{g-1}}(-1).$$

To see this, we argue as follows. Let $\mathcal{I}$ denote the ideal sheaf of $|C|$ in $|D|$. The following exact sequence,

$$0 \to \mathcal{I}/\mathcal{I}^2 \to (\Omega^1_{|D|})\big|_{|C|} \to \Omega^1_{|C|} \to 0.$$

is split, and the involution ι^* acts on it. The sheaf $\mathcal{I}/\mathcal{I}^2 \cong \oplus_{i=1}^g \mathcal{O}_{\mathbb{P}^{g-1}}(-1)$ is the ι^*-anti invariant part of $(\Omega^1_{|D|})\big|_{|C|}$, whereas the sheaf $\Omega^1_{|C|}$ is the invariant part. By Theorem 1.3 of [45], there is an isomorphism

$$\Omega^1_{|D|} \cong R^1\pi_*\mathcal{O}_M, \tag{4.69}$$

which is induced by the symplectic form σ of M. On W, the restriction morphism

$$(R^1\pi_*\mathcal{O}_M)\big|_W \to (R^1\pi_*\mathcal{O}_Y)\big|_W,$$

is surjective, and $(R^1\pi_*\mathcal{O}_Y)\big|_W$ is exactly the ι^*-invariant part of $(R^1\pi_*\mathcal{O}_M)\big|_W$. Since $\iota^*\omega = -\omega$, isomorphism (4.69) interchanges the ι^*-invariant and the ι^*-anti invariant parts of $\Omega^1_{|D|}$ and of $R^1\pi_*\mathcal{O}_M$. We deduce that there is an isomorphism

$$(R^1\pi_*\mathcal{O}_Y)\big|_W \cong \mathcal{I}/\mathcal{I}^2\big|_W,$$

which, using the fact that these sheaves are reflexive, extends to an isomorphism

$$R^1\pi_*\mathcal{O}_Y \cong \mathcal{I}/\mathcal{I}^2 \cong \oplus_{i=1}^{g}\mathcal{O}_{\mathbb{P}^{g-1}}(-1).$$

□

Remark 4.4.13. *In the same way we can show that the higher direct images of $\mathcal{O}_N$ and of $\mathcal{O}_{\widetilde{N}}$ are isomorphic. Hence,*

$$H^p(\widetilde{N}, \mathcal{O}_{\widetilde{N}}) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, 2g-1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\widetilde{N}$ is an irreducible Calabi-Yau manifold.

4.5 An example: the three-fold case

In this section we describe in more detail the relative compactified Jacobian of a genus 2 linear system on a general Enriques. Where the arguments of Section 4.2 and 4.3 do not work, we give ad hoc arguments to compute the fundamental group and the second Betti number. Finally we show that the Hodge diamond is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 10 & & 0 & \\ 1 & & 10 & & 10 & & 1 \end{array}$$

We start with describing into detail the well known and beautiful construction ([16], [17]) of a non-special genus two pencils. We will use the construction to show that by putting ourselves in a general setting we can assume that $|C|$ is a Lefschetz pencil.

Let T be a general Enriques surface and let $e_1, e_2 \subset T$ be two primitive elliptic curves such that $e_1 \cdot e_2 = 1$. By [16] the general curve

$$C \in |e_1 + e_2|$$

is a smooth genus 2 curve. Recall from Section 3.4 that linear system $|e_1 + e_2|$ is a pencil with two simple base points, $p = e_1 \cap e_2'$ and $q = e_1' \cap e_2$, where the curves e_i' (notation as in (3.4)) are also smooth and elliptic. A genus two pencil is called non special, if it can be written as $|e_1 + e_2|$ for two

primitive elliptic curves meeting in one point. Following Cossec, we let

$$\varphi = (\varphi_{|C|}, \varphi_{|C'|}) \colon T \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

be the map associated to the linear systems $|C|$ and $|C'|$. It is a rational, generically 2: 1, map which is defined on the complement of the four base points. The four elliptic curves are contracted by φ.

Lemma 4.5.1 ([16]). *The ramification divisor of $\varphi : T \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ consists in four lines r_1, r_1', r_2, r_2' forming a square and a curve Γ of bidegree $(4,4)$ having simple nodes at the corners of the square. The branches of the nodes are not tangent to the sides of the square. If T is a general Enriques surface, then Γ is smooth outside the four nodes.*

Proof. Let $\phi : \widetilde{T} \to T$ be the blow up of T at the four base points, and let $\widetilde{\varphi} : \widetilde{T} \to \mathbb{P}^1 \times \mathbb{P}^1$ the induced morphism. Then $\widetilde{\varphi}$ contracts the proper transforms of the elliptic curves, and is ramified along the four exceptional divisors. The images of the exceptional divisors form a square in $\mathbb{P}^1 \times \mathbb{P}^1$. The curves in either of the linear systems double cover the lines of either of the two rulings. Since the generic curve is a smooth genus two curve, the intersection number of the lines in each pencil with the ramification curve has to be equal to six. We already know that the map $\widetilde{\varphi}$ ramifies along the sides of the square, so the remaining part Σ of the ramification divisor has to be of type $(4,4)$.

Now let P be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the corners of the square, let $l_1, \ldots, l_4$ be the four exceptional divisors, let R_1, R_1', R_2, R_2' be the proper transforms of the sides of the square, and finally let Γ be the proper transform of Δ. Since $\widetilde{\varphi}$ contracts the four elliptic curves to the four corners of the square, the induced rational map $\psi : \widetilde{T} \to B$ is a generically 2: 1 morphism, ramified over the disjoint union of Γ and R_1, R_1', R_2, R_2'. Hence ψ induces a double cover of the proper transform of the four elliptic curves onto the exceptional divisors. Notice also that such double covers are ramified at four distinct points, which are the intersection points of the exceptional divisor with the ramification curve. Since all of the exceptional divisor meets transversally two sides of the square, it follows that Γ has to meet each exceptional divisor in two distinct points, and thus Σ has a simple node at each corner of the square and that the branches of the nodes are not tangent to the sides of the square.
If we suppose that the only reducible members of the linear system $|C|$ are the curves $e_1 + e_2$ and $e_1' + e_2'$ (property that is achieved, for example, for a general T) then the morphism

$$\psi : \widetilde{T} \to P,$$

is finite, and since it is a degree two cover between two smooth surfaces, the ramification curve has to be a disconnected union of smooth curves. It follows that in the general case Γ is smooth, and the four nodes of Σ are its only singularities. $\square$

The following Lemma reverses the construction.

Lemma 4.5.2. *The general non special genus two pencil on an Enriques surface is obtained by reversing the construction of Lemma 4.5.1. Moreover, one can choose the ramification curve so that the singular curves in the linear system have exactly one node.*

Proof. Consider a square of lines r_1, r_1', r_2, r_2' in $\mathbb{P}^1 \times \mathbb{P}^1$, where for $i = 1, 2$ the lines r_i and r_i' belong to the same ruling. In the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)|$ consider the dimension 12 linear subspace W of curves having a node at the four corners of the square. The general curve in W is smooth outside the four nodes and, moreover, the branches of the nodes are not tangent to the sides of the square. Let

$$b : P \to \mathbb{P}^1 \times \mathbb{P}^1,$$

be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the four nodes of the square, let

$$l_1, \ldots, l_4,$$

be the four exceptional divisors, let

$$R_1, R_1', R_2, R_2',$$

be the proper transforms of the sides of the square, and finally let Γ be the proper transform of a curve $\Gamma' \in W$ that is smooth outside the four nodes. Now consider the double cover

$$z : Z \to B,$$

ramified along

$$\Gamma \cup R_1 \cup R_1' \cup R_2 \cup R_2'.$$

Then,

$$2K_Z = z^*(2K_B + \Gamma + R_1 + R_1' + R_2 + R_2'). \tag{4.70}$$

Since

$$K_B = b^*\mathcal{O}(-2,-2) + l_1 + l_1' + l_2 + l_2', \tag{4.71}$$

and

$$\Gamma = b^*\mathcal{O}(4,4) - 2(l_1 + l_1' + l_2 + l_2'), \tag{4.72}$$

we have

$$2K_Z = z^*(R_1 + R_1' + R_2 + R_2'). \tag{4.73}$$

For $i = 1, \ldots, 4$, there are smooth rational (-1)-curves S_i, S_i' , and have the property that $2S_i = z^*(R_i)$, $2S_i' = z^*(R_i')$. Let $\epsilon : Z \to T'$ be the contraction of these four rational curves. Then $K_Z = \epsilon^*\omega_T + S_1 + S_1' + S_2 + S_2'$, implies

$$2\epsilon^*\omega_{T'} = \mathcal{O}_Z. \tag{4.74}$$

Thus $2\omega_{T'} = \mathcal{O}_{T'}$ and then it is readily checked that T' is an Enriques surface.

Let $\phi : T' \to \mathbb{P}^1 \times \mathbb{P}^1$ be the induced rational morphism. Then $C = \phi^*\mathcal{O}(1,0)$ is a linear system of genus two, and the generic curve is a smooth hyperelliptic curve because it is a double cover of a line, ramified at 4 distinct points. Moreover, notice that we can choose the curve Γ so that the lines in $\mathcal{O}(1,0)$ are neither bitangent nor flex lines to Γ itself (this can be achieved because one can easily find degenerate curves with this property and, since it is an open condition, one can also find a curve satisfying this requirement and such that the only nodes are at the four corners of the square). If we choose such a curve Γ, then the curves in the linear system $|C|$ have at most one simple node. Since the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ fixing a square of lines is two dimensional and acts on the 12-dimensional sublinear system W, it follows that this construction depends on 10 parameters. □

Corollary 4.5.3. *Let $q : \mathcal{C} \to \mathbb{P}^1$ be the universal family of a general genus two pencil on a general Enriques surface T. Then q has exactly* 18 *singular fibers,* 16 *of which are irreducible with one node, and* 2 *of which are reducible, and consist of two elliptic curve meeting tranversely at one point.*

Proof. It is sufficient to apply a classical cut and paste Euler characteristic argument. □

For the rest of the section $|C|$ will be a general non special genus two pencil on a general Enriques surface T, and we will denote $\mathcal{C} \to \mathbb{P}^1$ be the universal curve. Let

$$t_0, t_1, \ldots, t_{17},$$

be the points in $\mathbb{P}^1$ parameterizing the irreducible fibers, with

$$t_0, t_1,$$

be the points corresponding to the curves $e_1 \cup e_2$ and $e_1' \cup e_2'$. Finally set

$$U = |C| \setminus \{t_0, t_1, \ldots, t_{17}\}, \quad \text{and } V = |C| \setminus \{t_2, \ldots, t_{17}\}.$$

Notice that the Lefschetz pencil $\mathcal{C} \to \mathbb{P}^1$ has 2 sections, twisting by one of them defines an isomorphism the relative compactified Jacobians of different degrees are all birational. For simplicity, we describe the degree zero case. Let us then set $w = (0, [C], -1)$, $v = (0, [D], -2)$, where $D = f^{-1}(C)$ and let N and Y be as in (4.1) and (4.2).

Proposition 4.5.4. *Let N and Y be as above. Then $\nu : N \to \mathbb{P}^1$ and $\pi : Y \to \mathbb{P}^1$ have exactly* 16 *singular fibers, each of which is a rank one degeneration of an abelian surface. Moreover, ν and π have a section and hence there are no multiple fibers.*

Proof. This follows from the fact that the Jacobian variety of a union of two smooth curves meeting transversally in one point is smooth and compact, and from Oda and Seshadri's description of the compactified Jacobian of an irreducible nodal curve. □

Corollary 4.5.5. *The Euler characteristics of N and of X are trivial.*

Proof. This follows from the above Lemma, since a rank one degeneration of an $n \geq 2$ abelian variety has trivial Euler number. By cutting and pasting, it follows that

$$\chi_{top}(X) = 0. \tag{4.75}$$

□

Theorem 4.5.6. *Let $|C|$ be a general non special genus* 2 *pencil on T. Let N and Y be as above. Then*

$$\pi_1(Y) = \mathbb{Z}/(2) \times \mathbb{Z}/(2), \quad \textit{and} \quad \pi_1(N) = \mathbb{Z}/(2).$$

Proof. Fix a base point $o \in U \subset |C|$ and let C_o be the curve that it parametrizes. For each of the points t_i, let λ_i be a generator of $\pi_1(|C| \setminus t_i, o)$.

The proof is the same as for Theorem 4.2.1. The only modification is due to the presence of the reducible curves above the points $t_0, t_1 \in |C|$. For $i = 0, 1$ let α_i be the simple closed loop in C_o whose vanishing generates the node of C_{t_i}. Since $\alpha_i = 0$ in $H^1(C_o, \mathbb{Z})$, $f^*(\alpha_i) = \beta_i + \iota\beta_i = 0$, where β_i is the class of a simple closed loop in D_o. Notice that D_o belongs to a component of the discriminant $\Sigma \subset |D|$ that parametrizes curves that are the union of two smooth elliptic curves meeting in two points. It follows that there is only one branch of Σ in s_i and that we can write

$$j_*\lambda_i = \eta_i,$$

where the local monodromy around η_i is $c \mapsto c + 2(c, \beta)\beta$. Hence, if c is invariant we have $c^{-1}PL_{\lambda_i \cdot \iota\lambda_i}(c) = 2(c, \beta)(\beta + \iota\beta) = 0$, which is in accordance with the fact that Y_{t_i} is smooth and so there is no monodromy around t_i. It follows, using the notation of Section 4.2, that

$$R'' = [\pi_1(Y_o), \pi_1(V)] \cong [\pi_1(Y_o), \pi_1(U)] \cong (1 + \iota)H^1(D_o, \mathbb{Z}),$$

and hence that

$$\pi_1(Y) = \pi_1(Y_o)/R'' \cong H^1(D_o, \mathbb{Z})^+/(1 + \iota)H^1(D_o, \mathbb{Z}) \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2).$$

□

Next, we have the computation of the second cohomology group.

Theorem 4.5.7. $H^2(Y, \mathbb{C}) \cong H^2(N, \mathbb{C}) \cong \mathbb{C}^{10}$.

Proof. First of all notice that since $\mathcal{C}$ is the blow up of T at the two simple base points, it follows that

$$H^2(\mathcal{C}) \cong \mathbb{C}^{12}.$$

The strategy is the same as in the general case that is to say that we compare the long exact sequence in cohomology associated to the pairs $(\mathcal{C}, \mathcal{C}_U)$ and (N, N_U). Using Lemma 4.3.2 we see that

$$H^1(\mathcal{C}_U) \cong H^1(N_U) \cong H^1(U) = \mathbb{C}^{17}.$$

As in Lemma 4.3.3 we compute that

$$h^2(\widetilde{N}, \widetilde{N}_U) = h^2(Y, Y_U) = h^2(N, N_U) = \#\text{ of irreducible components of } N_\Delta = 18,$$
$$h^2(\mathcal{C}, \mathcal{C}_U) = \#\text{ of irreducible components of } \mathcal{C}_\Delta = 20,$$

so that (4.14) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & H^2(N) & \xrightarrow{r} & H^2(N_U) \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & \mathbb{C}^3 & \longrightarrow & H^2(\mathcal{C}) & \xrightarrow{s} & H^2(\mathcal{C}_U). \end{array}$$

The second vertical arrow is induced by the Abel-Jacobi map $j : \mathcal{C}_U \subset N_U$ which in turn is induced by a section $\mathcal{C} \to \mathbb{P}^1$ of the family of curves. By Proposition 4.3.6 j^* is an isomorphism and the theorem is proved. □

Corollary 4.5.8. *The Hodge diamond of Y, N and $\widetilde{N}$ is*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 10 & & 0 & \\ 1 & & 10 & & 10 & & 1 \end{array}$$

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