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Stability results for nonlinear hyperbolic equations

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STABILITY RESULTS FOR NONLINEAR HYPERBOLIC EQUATIONS

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Abstract

This thesis is concerned with three stability results concerning nonlinear wave equations described over four Chapters, the first one being introductory. A common theme among all three results is that we must extend the vector field method to situations where we do not have direct access to the usual collection of operators for one reason or another. Of the three results, we provide summaries for the first two and complete proofs for the third.

The second Chapter in the thesis describes joint work with Federico Pasqualotto concerning global stability of the trivial solution to quasilinear wave equations satisfying the null condition under a novel class of perturbations. These perturbations are allowed to be localized around a finite collection of points, in contrast with the classical theory, which requires data to be localized around a single point. The proof uses an analysis of the interaction of waves originating from distant points and adapting the vector.

The third Chapter in this thesis concerns joint work with Samuel Zbarsky concerning the stability and instability of plane wave solutions to semilinear systems of wave equations satisfying the null condition. We identify sets of conditions leading to both stability and instability. The stability proof requires analyzing the geometry of the interaction between the background plane wave and the perturbation while the proof of the instability result is based on a geometric optics ansatz.

The fourth and most substantial Chapter of this thesis initiates the study of stability results for anisotropic systems of quasilinear wave equations. Equations of this kind arise naturally in the study of birefringent phenomena, and in particular in the study of crystal optics. The proof introduces a strategy for proving decay based on bilinear energy estimates that we interface with the vector field method and several geometric considerations. The study of this problem represents the first step in a research program envisioned by Klainerman involving the study of hyperbolic equations having multiple characteristics. In addition to providing proofs of the main results, we discuss the role of anisotropic systems of wave equations in this larger context.

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¹I must also thank Mihalis for his help with writing introductions, and I hope he will be amused by the presence of a footnote in the acknowledgements (the reader must note that the footnote is not in the abstract).

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To my family

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Chapter 1

Introduction

This thesis is concerned mainly with the *stability* properties of special solutions to hyperbolic partial differential equations (PDEs). For dynamical theories that predict the behavior of physical systems, this question is very natural for determining whether a particular solution ought to correspond to observation. A candidate solution which is supposed to be found in nature must be stable, for otherwise, the absence of perfect conditions in the world around us would push us away from the solution. The problems considered in this thesis involve applications and extensions of the *vector field method* in situations where, for a variety of reasons, we do not have complete access to all of the usual operators.

Nonlinear hyperbolic equations appear naturally in the study of various physical systems which exhibit properties commonly associated with waves. The wave like behavior of the Einstein equations was recently featured prominently with the celebrated observation of gravitational waves. Other physical examples include acoustic waves in the compressible Euler equations of gas dynamics and the behavior of light governed by the Maxwell equations. Situations involving electrically and magnetically charged fluids are often described using the equations of compressible magnetohydrodynamics (see [24]). The wave operator on Minkowski space $\mathbb{R}^{n+1}$ given by

$$\square = -\partial_t^2 + \sum_{i=1}^n \partial_i^2$$

is the standard example for understanding linear wave phenomena.

Much work has been done over the last roughly 40 years concerning stability. The use of energy estimates in the context of the vector field method (see Section 1.1.2) has been remarkably successful in a wide range of circumstances. This relies on the use of vector fields well adapted to the symmetries

of the problem at hand in order to control the solution. Typically, these vector fields have weights which capture the corresponding symmetries. The standard wave equation is singled out as the maximally symmetric case, that is, it is the one with access to all of the useful vector field corresponding to the continuous symmetries of Minkowski space.

Recently, much work has gone into understanding physically significant situations which are far from being maximally symmetric for one reason or another. This thesis concerns results which build in this direction. In each result, the setting of the problem breaks several of the symmetries, and this is the reason we lose access to weighted vector fields. Each problem breaks the symmetries in a different way. In all cases, the proofs require us to make effective use of the operators we do have access to, and the vector field method thus always places a decisive role.

One physically relevant way to remove symmetries is to remove the localization assumption on the initial data. This problem was studied together with Federico Pasqualotto in [11], and the result can roughly be stated by saying that the trivial solution to quasilinear wave equations satisfying the classical null condition in $3+1$ dimensions are globally nonlinearly stable with respect to perturbations which are localized around several points. In the case of only two points, the key novelty is that the size of the data is allowed to be uniform in the separation between these two points.

Another way to remove symmetries is to consider the stability of solutions which themselves break these symmetries. One particularly simple class of solutions which breaks symmetries (in particular, they are far from being spherically symmetric) are plane wave solutions, also called travelling waves. The study of the stability of plane wave solutions to nonlinear wave equation is a topic which has generated interest recently, and we in particular mention the works of Abberscia-Wong [1] and [3], and the work of Liu-Zhou [61]. These works were motivated by studying the stability of plane symmetric solutions to the Lorentzian minimal surface equation. Together with Samuel Zbarsky in [12], the stability and instability of plane wave solutions to semilinear systems wave equations satisfying the null condition were studied. We are able to find conditions guaranteeing both global nonlinear stability and linear instability. The stability results extend the works mentioned above by introducing new strategies to control the solutions in this setting.

Finally, symmetries can be removed by changing the way in which the waves themselves propagate. This is related to the physical phenomena of *birefringence*, and it can be stated as the property that “different waves move at different speeds in different directions.” The different waves then have different symmetries. If the equations for the various waves with different symmetries are then coupled, the interactions between the different waves makes it difficult to use the weighted vector

fields in the usual way. This kind of model was introduced to the author by Klainerman as a model for studying more complicated equations exhibiting birefringence, which require novel extensions of the vector field method. This setting is the main focus of this thesis, and is the topic of the main Theorem, Theorem 25. Theorem 25 gives conditions guaranteeing global nonlinear stability of the trivial solution for a class of coupled nonlinear systems of anisotropic wave equations. This problem fits in the larger context of a research program whose goal is to understand hyperbolic equations with multiply sheeted characteristics (see Section 4.1.6). A representative example is given by the Maxwell equations in a biaxial crystal. In terms of stability for biaxial crystals, the only work the author is aware of is due to Liess [57] and [56]. The results in this thesis are in some sense orthogonal to the work of Liess because we sidestep several of the issues present in biaxial crystals, such as conical refraction. Nonetheless, we do have to deal with nonlinear coupling between waves of different character, and moreover, anisotropic systems of wave equations arise when studying uniaxial crystals and also biaxial crystals under a symmetry reduction. We thus hope that the considerations here will be useful in future study (see Section 4.1.6 where this is discussed in more detail).

This thesis is divided into four Chapters. The first Chapter contains background information describing existing work on stability for nonlinear wave equations. This will allow us to better motivate the results described briefly above. The second Chapter consists of a discussion of results by the Pasqualotto and the author in [11]. After motivating the result, a precise version of the main Theorem in that work is given (see Theorem 1.2.1), and a description of the proof is given. This part is taken heavily from the work [11]. The third Chapter consists of a discussion of results by Zbarsky and the author in [12]. Once again, we motivate the results, provide a precise version of the main Theorems in that work (see Theorems 13 and 14), and then we describe the proof. This part draws heavily from [12].

The fourth Chapter of the thesis contains the precise statement of the main Theorem as well as its proof (see Theorem 25). Because this is by far the most involved part, there is a separate introduction describing the specific setting of anisotropic wave equations. In addition to providing a proof of Theorem 25, we include a discussion of how this work fits in a larger research program concerning the study of hyperbolic equations having multiple characteristics. This Chapter of the thesis is taken mostly from the forthcoming papers [9] and [10]. In [9], the author studies the stability of anisotropic wave equations, and the proof of the main Theorem (Theorem 25) in this thesis comes from this paper. In [10], Klainerman and the author describe a larger research program concerning the study of hyperbolic equations having multiple characteristics. A good model to keep in mind is the system of crystal optics, which we introduce in Section 1.1.1 (see (1.3)). The

problem of studying the stability of *anisotropic systems of wave equations* was originally posed to the author by Klainerman in the context of this larger research program. The paper [10] describes this research program and how the proof of the main Theorem in this thesis, Theorem 25, fits in with the bigger picture. We note in particular the presence of other operators discovered by Klainerman that the author hopes to exploit together with Klainerman in future work (see Section 4.1.6 for more discussion).

1.1 Background on stability for nonlinear hyperbolic equations

1.1.1 Local well posedness of hyperbolic equations

We are interested in understanding the *Cauchy problem*, also called the *initial value problem*, for nonlinear hyperbolic equations. This means we want to understand the evolution of some equation given initial data. More precisely, we want to understand properties of the solution map that takes initial data for a hyperbolic equation to its solution. We think of these initial conditions as being prescribed on $t = 0$. The applicability of this to physics is clear because physical theories are supposed to be predictive in some way, meaning that we ought to be able to predict what will happen to a given system in the future given the initial conditions of the system.

The first problem one is interested in when studying the Cauchy problem is proving *local well posedness*, which consists roughly of:

- Identifying function spaces X and Y such that, given initial data in X , there exists a solution to the equation in Y which evolves from the initial data.
- Ensuring that the solution is unique in Y , i.e., there is exactly one function in Y that arises as a solution for every choice of initial data in X .
- The “size” of the solution in Y should depend continuously on the “size” of the initial data in X . More precisely, this means that the norm of the solution in Y should depend continuously on the norm of the data in X . We in fact often demand that the solution mapping be even more regular.

These are informal descriptions of what it means to be *well posed* in the sense of Hadamard.

The function spaces X and Y are supposed to allow for functions which are not analytic. The celebrated *Cauchy-Kowalevski Theorem* (see also the *Holmgren Uniqueness Theorem*, and see [40]

for a description of both) proves an analogous result for analytic spaces. These results are extremely general and only use the fact that the coefficients and data are analytic (in particular, they do not use anything else about the structure of the equations). Physical motivations require us to consider spaces with less regularity, and examples of useful spaces for well posedness are the collection of Hölder spaces $C^{k,\alpha}$ and the collection of Sobolev spaces $W^{s,p}$. There are several other good spaces depending on the context. These other spaces require us to use the structure of the equations. For example, the presence of energy estimates (see below) is a characteristic feature of hyperbolic equations, and they are fundamental in proving local well posedness.

Now, in the context of local well posedness, we note that we only ask to prove that a solution exists for some finite interval of time. This interval of time is allowed to depend on the size of the initial data. Showing that some class of data admits a global solution (i.e., for all future times $t \geq 0$) is of course very interesting, but it is not the question that is directly studied by local well posedness.

There is a long history of studying local well posedness for hyperbolic equations. Several of these results are described in the work [33] by Friedrichs, where references can also be found. This work by Friedrichs studies a fairly general class of hyperbolic equations, called *symmetric hyperbolic systems*, which have a good local well posedness theory. This theory that Friedrichs developed (in addition to [33], see [66] for a description) shows that the H^s Sobolev spaces are good function spaces for data. Indeed, symmetric hyperbolic systems satisfy an energy estimate which tells us that the L^2 norm of the solution can be controlled for small time in terms of the L^2 norm of the data. Commuting the equation with translation vector fields can then upgrade this to a statement about the H^s norm. The nonlinearities can be appropriately controlled in terms of the H^s norm because H^s is closed under products for $s > \frac{n}{2}$.

For our purposes, symmetric hyperbolic systems are sufficiently general. However, we note that there are still several equations motivated by physics which cannot be reduced to a symmetric hyperbolic system. We note in particular that a very general framework was developed by Christodoulou in [20] allowing the study of local well posedness for the physically relevant case of equations arising from a certain Lagrangians. This includes a large class of hyperbolic equations from continuum mechanics.

We now mention two physically relevant examples which serve as the main motivation in this thesis. The first is the Einstein vacuum equations of general relativity, which can be succinctly written as

$$\text{Ric}(g) = 0. \tag{1.1}$$

Here, we should think of g as being a Lorentzian metric on a manifold M , and $Ric(g)$ denotes the Ricci curvature of g . Written in this form, the equations look rather simple. However, in the physical case of $3+1$ spacetime dimensions, this is a very complicated system of equations, and even writing them down in a way that makes their wave like character visible is nontrivial (see below).

The second is the equations governing electromagnetic fields in optical media. If we use E and B to denote the electric and magnetic fields, the usual Maxwell equations can be written in the form

$$\begin{aligned}\partial_t E - \text{curl}(B) &= 0 \\ -\partial_t B - \text{curl}(E) &= 0 \\ \text{div}(E) = \text{div}(B) &= 0.\end{aligned}\tag{1.2}$$

The divergence free conditions are easily seen to propagate from the flow of the evolutionary equations as long as they hold initially. Thus, these equations are thought of as constraint equations.

These equations govern the propagation of light in isotropic media. In order to include anisotropic media, we introduce the *dielectric tensor* and the *magnetic permeability*. We shall take these quantities to be constant, representing a homogeneous media. More precisely, we shall take the dielectric tensor ϵ to be the diagonal matrix with constant positive entries

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix},$$

and we shall take μ to be a constant multiple of the identity matrix. In this setting, the equations take the form

$$\begin{aligned}\partial_t(\epsilon E) - \text{curl}(B) &= 0 \\ \partial_t(\mu B) + \text{curl}(E) &= 0 \\ \text{div}(\epsilon E) = \text{div}(\mu B) &= 0.\end{aligned}\tag{1.3}$$

We can take $\mu = I$ without loss of generality. Then, these equations reduce to the usual Maxwell equations in isotropic media when $\epsilon_1 = \epsilon_2 = \epsilon_3$. These numbers, called the *dielectric constants*, and they describe how the media itself reacts to the electromagnetic field.

Let us briefly note that it is possible to consider, among other things, magnetically anisotropic materials (μ not a multiple of the identity), and it is more broadly possible to consider the (physically relevant) case where ϵ and μ depend on the electromagnetic field. This of course results in a quasilinear system of equations which takes into account how the electrical and magnetic properties

of the material react to the electromagnetic field. We refer the reader to [54] and [14] for more information about electromagnetic fields in different media.

Both of the Einstein vacuum equations and the equations governing crystal optics can be interpreted as symmetric hyperbolic systems, and the theory due to Friedrichs is applicable. For the equations of linear crystal optics as we have written them, this is rather straightforward (see [56] for more discussion). However, for the Einstein vacuum equations, this is highly nontrivial. In the 1950's, the seminal work by Choquet-Bruhat [32] showed that the Einstein vacuum equations are well posed as a suitably interpreted initial value problem. In this framework, we solve for a Lorentzian manifold (M, g) , and data are posed in the form of a complete Riemannian 3 manifold $(\Sigma, \bar{g})$ which is additionally equipped with a symmetric two tensor k . The metric $\bar{g}$ and symmetric two tensor k can be thought of as the initial values of g and $\partial_t g$, where $\partial_t g$ is purely schematic. The manifold (M, g) must be such that Σ embeds isometrically into M , and such that k is the second fundamental form of Σ as a submanifold of M . We note that because of the Gauss and Codazzi equations, we cannot freely specify both $\bar{g}$ and k , and they must in fact satisfy an underdetermined family of elliptic equations called the *constraint equations*.

Unfortunately, writing the Einstein vacuum equations in an arbitrary coordinate system results in a system of equations which is not hyperbolic (and, in fact, is not manifestly anything). This is related to the *gauge freedom* inherent in the Einstein vacuum equations (note that the pullback of a Ricci flat metric by any diffeomorphism gives another Ricci flat metric). Thus, it is fundamentally important to express the Einstein equations in an appropriate gauge in order to effectively study them. To a first approximation, we can think of picking a gauge as the same thing as picking coordinates.

In [32], Choquet-Bruhat wrote the Einstein vacuum equations in *harmonic coordinates*, resulting in a quasilinear system of wave equations she was then able to analyze. This coordinate condition is simply the condition that the coordinate functions themselves be solutions to the wave equation, meaning that

$$\square_g x^\mu = 0.$$

Coupling these equations with $Rig(g) = 0$ appropriately results in quasilinear wave equations for the components of g which we can write schematically as

$$\square_g g_{\mu\nu} = F_{\mu\nu}(g, \partial g),$$

where $\square_g$ corresponds to the scalar wave operator associated to the metric g , and where the

$F_{\mu\nu}(g, \partial g)$ are quadratic in the first derivatives of g (see [62]). Indeed, this can be seen to remove several terms in the principal part of the equations in an arbitrary coordinate system. A modern treatment of local well posedness for the Einstein equations then follows from casting these equations as a symmetric hyperbolic system and applying general local well posedness theory to construct a portion of a unique spacetime (see [66] where much of this is described).¹ In fact, Choquet-Bruhat later showed together with Geroch in [17] that there exists a unique *maximal globally hyperbolic development*. We refer the reader to [80] for the rigorous definition of *globally hyperbolic*, but for now, we merely remark that this is a natural condition to impose on a spacetime in order to talk about dynamics. There is also a notion of global hyperbolicity due to Leray (see [55]) which is applicable to a general class of hyperbolic equations. In the context of Lorentzian manifolds, the geometric definition in [80] (which is applicable to, say, the covariant wave operator and the Einstein equations) coincides with the notion due to Leray found in [55].

The other important part of maximal globally hyperbolic development is the word *maximal*. This refers to the fact that any other globally hyperbolic spacetime which is a solution to the Einstein vacuum equations arising from an initial data set must embed isometrically into the maximal globally hyperbolic development of that data set. It is in this sense that the maximal globally hyperbolic development is maximal. The original construction of the maximal globally hyperbolic development required the use of the axiom of choice, but we note that Sbierski has recently removed this requirement in [67].

1.1.2 Stability

After establishing this very general local theory for hyperbolic equations, it is natural to study more refined properties of solutions. There are several interesting problems to pursue, and several of them are concerned with global solutions (i.e., solutions which exist for all positive times $t \geq 0$) of some kind. One class of problems concerning global solutions are stability problems. As was mentioned at the beginning of Chapter 1, physically relevant solutions to hyperbolic equations ought to be stable. Two common forms of stability that are discussed are *orbital stability* and *asymptotic stability*. Orbital stability refers to the fact that a solution arising from a small perturbation of the initial conditions remains close for all time to the original solution. Asymptotic stability is the stronger statement that the solution arising from the perturbation in fact converges asymptotically as $t \rightarrow \infty$ to the original solution in some sense. For a large class of interesting quasilinear equations, we

¹It is interesting to note that the well posedness theory used by Choquet-Bruhat in [32] was based on the work of Sobolev (see [74]). This used the fundamental solution instead of energy methods.

note that it seems to be very difficult to prove an orbital stability result without in fact proving an asymptotic stability result. We shall focus on the problem of asymptotic stability (from now on, we shall simply refer to this as stability) of special solutions, with the guiding motivation being to understand the mechanisms that provide stability in the context of special solutions to physically relevant equations.

The problem of asymptotic stability depends heavily on the dimension of the space involved. The biggest change occurs between $1+1$ dimensions and $n+1$ dimensions for $n \geq 2$. It is well known that hyperbolic equations in $1+1$ dimensions can develop singularities (see [25]). In higher dimensions, the *dispersive properties* of the wave equation begin to have an effect. Because the waves spread out over time, there is decay, which is a stabilizing mechanism that can prevent singularity formation. This thesis focuses on these higher dimensional cases.

The first global result for nonlinear hyperbolic equations was the work [42] by Jörgens. This relied on very special algebraic structure of the semilinear equation. These equations were restricted to semilinear terms of a specific form involving only the solution itself (not its derivatives). Other authors then made important contributions in this direction. We mention in particular the work of Segal (see [68]), Strauss (see [77] and [78]), and von Wahl (see [79] and, jointly with Brenner, [15]). These works already made it clear that the dispersive properties of the wave equation are very important for understanding long time dynamics. All of these works dealt with semilinear equations of specific forms given by

$$\square u = f(u).$$

Moreover, the methods could not treat quadratic nonlinearities. Later, John realized that it is possible to use the dispersive properties of the wave equation to more robustly study long time dynamics (in particular, the results worked for quasilinear equations and allowed for more general nonlinearities). For example, he was able to establish some improved lifespan estimates in [38].

A major advance occurred when Klainerman was able to prove a true global stability result for certain quasilinear wave equations in [45]. This sparked the beginning of rapid development in the study of stability. In order to describe how these arguments work intuitively, we provide an overview of certain properties of the wave equation on Minkowski space. The two main properties that are used are energy conservation and dispersion.

We consider $3+1$ dimensional Minkowski space $\mathbb{R}^{3+1}$ with coordinates (t, x, y, z) , and we shall denote by Σ_t constant t hypersurfaces. We shall also denote the coordinates by x^μ sometimes, and we introduce polar coordinates (t, r, ω) .

The homogeneous wave equation on Minkowski space

$$\square\phi = 0$$

is well known to conserve energy. Mathematically, if we multiply the equation by $\partial_t\phi$ and integrate by parts, we get that

$$\int_{\Sigma_t} (\partial_t\phi)^2 + (\partial^i\phi)(\partial_i\phi) dx$$

is constant in t (note that we are using Einstein summation convention). Moreover, we roughly have that “everything travels at the speed of light.” This fact along with energy conservation leads to decay, a manifestation of the dispersive properties of the wave equation. Specifically, if we start out with smooth data supported in the unit ball, we expect that the solution will roughly be supported in the region where $t - 1 \leq r = \sqrt{x^2 + y^2 + z^2} \leq t + 1$. Moreover, because the equation is isotropic and the data are roughly isotropic as well, we expect the solution to be roughly evenly spread out in this annular region. In order to be consistent with conservation of energy, we see that the amplitude of first derivatives of ϕ must then decay at a rate of $\frac{1}{t}$. In $n + 1$ dimensions, similar considerations lead to a decay rate of $t^{-\frac{n-1}{2}}$. That solutions to the wave equation decay at this rate is the content of the *dispersive estimate* for the wave equation. This was what Klainerman took advantage of in [45], and this allowed him to control nonlinear interactions. Indeed, if we schematically have that $\square\phi = (\partial\phi)^p$, an energy estimate tells us that

$$\|\partial\phi\|_{L^2(\Sigma_s)} \leq \|\partial\phi\|_{L^2(\Sigma_0)} + \int_0^s \|\square\phi\|_{L^2(\Sigma_t)} dt,$$

and we can then write that

$$\int_0^s \|\square\phi\|_{L^2(\Sigma_t)} dt \leq \int_0^s \|(\partial\phi)^p\|_{L^2(\Sigma_t)} dt \leq \int_0^s \|(\partial\phi)^{p-1}\|_{L^\infty(\Sigma_t)} \|\partial\phi\|_{L^2(\Sigma_t)} dt,$$

and if the amplitude $\|\partial\phi\|_{L^\infty(\Sigma_t)}$ decays fast enough and p is sufficiently large, this integral converges. This is, very schematically, the strategy that global stability results follow, and it is the strategy that Klainerman was able to push to global stability in [45]. We also note that this work was later simplified independently by Shatah in [69] and by Klainerman–Ponce in [50].

In a breakthrough work [47], Klainerman introduced the *vector field method* in order to control solutions to wave equations. This method is based on using the family of weighted vector fields consisting of the scaling vector field, the rotation vector fields, and the Lorentz boost vector fields,

which can be succinctly written as

$$S = t\partial_t + r\partial_r \quad \text{and} \quad \Omega_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu. \quad (1.4)$$

Klainerman realized that these vector fields give a very efficient and robust way to show that solutions to the wave equation decay, introducing the *Klainerman-Sobolev inequality*

$$|f|(t, r, \omega) \leq \frac{C}{(1+t+r)(1+|t-r|^{\frac{1}{2}})} \sum_{|\alpha| \leq 2} \|\Gamma^\alpha f\|_{L^2(\Sigma_t)},$$

where we are using Γ^α to denote strings of the weighted vector fields above and the translation vector fields ∂_μ . These vector fields are useful because they satisfy good commutation properties with the wave equation, meaning that if ϕ solves a wave equation, so does $\Gamma\phi$ for any of the commutation fields Γ . We of course note that this is the version of the inequality in $3+1$ dimensions, and in $n+1$ dimensions, the factor of $1+t+r$ should have a weight of $\frac{n-1}{2}$ instead.

This inequality allowed Klainerman to show global stability for the trivial solution to a general class of quasilinear wave equations with quadratic nonlinearities in $n+1$ dimensions for $n \geq 4$, and it allowed him to give a much simpler proof of global stability in $3+1$ dimensions for cubic nonlinearities. As can be seen from the Klainerman-Sobolev inequalities, it is natural to use weighted energies instead of the regular H^s energies. These weighted energies are compatible with the local well posedness statements described in Section 1.1.1, and they also allow us to use decay statements that come from the weights.

Because several physically interesting equations have quadratic nonlinearities in $3+1$ dimensions (for example, the Einstein vacuum equations), it may seem a bit troublesome that Klainerman's method did not result in global stability for quadratic nonlinearities in $3+1$ dimensions. This restriction, however, is fundamental, as was shown by John in [39]. In this work, John showed that general quadratic nonlinearities in $3+1$ dimensions can blow up in finite time, even for arbitrarily small and smooth compactly supported data.² An example of such an equation is

$$\square\phi = -(\partial_t\phi)^2.$$

With this in mind, Klainerman introduced a condition on quadratic nonlinearities in $3+1$ dimensions, called the *null condition*, in [46]. Nonlinearities satisfying the null condition satisfy improved

²These results on singularities were later further understood by Sideris in [71] and. This is related to the phenomena of *shock formation*, which was studied by Alinhac in [6] and [5] and then understood essentially completely in the work [21] by Christodoulou (see also [76]).

estimates, and the trivial solution to nonlinear wave equations are in fact globally stable (see [48] and [18]), in contrast to the examples John studied in [39].

Let us briefly try to explain why the null condition is useful in the context of stability (see Section 1.3 for a precise definition). Using the vector fields from the vector field method, we can see that certain derivatives are better than others. If we use $\bar{\partial}$ to denote derivatives which are tangent to the level sets of $u = t - r$, we see that we have the following dichotomy for any smooth function f :

$$|\partial f| \leq \frac{C}{1 + |t - r|} \sum_{|\alpha|=1} |\Gamma^\alpha f| \quad \text{and} \quad |\bar{\partial} f| \leq \frac{C}{1 + t + r} \sum_{|\alpha|=1} |\Gamma^\alpha f|.$$

This shows us that the $\bar{\partial}$ derivatives are better than generic derivatives ∂ . In other words, derivatives tangent to the light cones $u = t - r$ are better behaved than derivatives which pierce the light cones. In the context of global stability results, the null condition guarantees that any “bad” derivative $\partial\phi$ is multiplied by a “good” derivative $\bar{\partial}\phi$ in the nonlinearity. This leads to improved estimates and allows one to show global stability of the trivial solution for such nonlinear equations. Both the vector field method and the identifying the presence of a null condition were key ingredients in the monumental work [22] by Christodoulou–Klainerman. In this work, Christodoulou–Klainerman were able to show that Minkowski space, which can be thought of as the trivial solution to the Einstein vacuum equations, is globally nonlinearly stable with respect to sufficiently small and localized perturbations.

These ideas have continued to be the subject of intense research in the time since the work of Christodoulou–Klainerman. One recent direction has involved studying situations in which we do not have direct access to all of the vector fields from the vector field method.³ One of the first instances of this was the work by Klainerman–Sideris in [52] in which they developed a vector field method avoiding the use of the Lorentz boosts. In recent years, there has also been a huge amount of exciting work regarding the stability of black holes. This has required analyzing the behavior of waves on nontrivial backgrounds. The family of vector fields one has access to is much smaller on these black hole spacetimes, and this requires the introduction of new vector fields to be used as both multipliers and commutators. This has required understanding several novel phenomena, and describing this in any detail would require its own book. We encourage the interested reader to look at the works of Dafermos–Rodnianski [27], [28] (see also their excellent notes [29]), Dafermos–Rodnianski–

³In addition to these purely physical space methods, we also mention that other strategies have successfully been used to study stability in hyperbolic and, more generally, dispersive equations. One such strategy is the method of spacetime resonances (see [34] for a general introduction and [65] and [31] for applications to wave equations). We mention also the wave packet approach to studying wave equations. Using phase space decompositions such as wave packets has proven to be very useful in the study of wave equations (we note in particular the celebrated result [73] of Smith–Tataru which provides a sharp local well posedness result in H^s spaces for general quasilinear wave equations).

Shlapentokh–Rothman [30], Dafermos–Holzegel–Rodnianski [26], Klainerman–Szeftel [53], and the references therein.

New manifestations of some kind of null condition have also been found in various equations. In the context of nonlinear wave equations, Lindblad–Rodnianski identified a weaker form of the null condition called the *weak null condition* which is present in the Einstein vacuum equations in harmonic coordinates. This allowed them to provide a different proof of the stability of Minkowski space based on harmonic coordinates in the works [58], [59], and [60] (see also recent generalizations by Keir in [44]).

Let us now return to the other example we introduced earlier, namely that of crystal optics. When the dielectric tensor ϵ is simply a multiple of the identity, the equations can be written as a system of wave equations with the usual $\square$ operator, and the study of stability to nonlinear perturbations reduces to the setting described above. In particular, we have access to the usual family of Lorentz vector fields and the scaling vector field used by Klainerman in the vector field method. More precisely, the equations become schematically of the form

$$\square E^i = \square B^i = 0$$

for every component of the electric and magnetic fields. Thus, the system of equations fully decouples.

We can, however, consider materials where the dielectric constants are not all equal to each other. When two of the dielectric constants are equal to each other but distinct from the third, the equations govern the propagation of light in a *uniaxial crystal*. Mathematically, the equations are governed by an anisotropic system of wave equations, which is the kind of equation whose stability properties are studied in the fourth Chapter of this thesis. The system of equations still fully decouple as in the isotropic case, but the wave operators associated to different components can change. For example, the equations can then look schematically of the form

$$\square E^1 = \square E^2 = \square' E^3 = 0,$$

where $\square'$ is now a different wave operator.

When all three of the dielectric constants are distinct, the system of equations no longer decouple into several scalar equations, and they must be thought of as truly being a system. These equations govern the propagation of light through *biaxial crystals*. Light propagation in biaxial crystals is

mathematically much more complicated than in uniaxial crystals. The author is aware of very limited work studying these equations. From the viewpoint of stability, we note the works [57] and [56] by Liess. In the work [57], Liess provides an analysis of the *dispersion relation* for the equations governing light propagation in a biaxial crystal. Then, in [56], he uses the resulting dispersive estimate to prove a global stability result for nonlinearities with sufficiently high power, similar to the work of Klainerman–Ponce in [50]. We refer the reader to Section 4.1.6 for a more in depth discussion of this circle of ideas related to hyperbolic equations with multiply sheeted characteristics, and we refer the reader to the forthcoming [10] in which this is discussed in much more detail. In particular, we note that a certain symmetry reduction for a biaxial crystal results in an anisotropic system of wave equations. This is described in Section 4.1.6.

The overall theme of this thesis is that it involves studying stability properties of solutions to wave equations in other regimes where we do not have access to all of the vector fields from the vector field method. In Chapter 2, we lose direct access to the vector fields because the data are no longer forced to be localized around a single point, and we instead allow for data to be localized around several points. In Chapter 3, we lose access to vector fields because we have to contend with plane symmetric solutions, which are, for example, far from being spherically symmetric. In Chapter 4, we lose access to vector fields because we are given a system of wave equations where different components in the system have different well adapted vector fields, and the coupling prevents us from taking advantage of them. In all three cases, we must figure out a way to take advantage of the vector fields we do have access to in an effective way. The examples discussed above provide physical motivation for much of the work that is described in this thesis.

1.2 Summary of the main results

We can now describe the remaining Chapters of this thesis and give rough versions of the main results.

1.2.1 Stability under multi-localized perturbations

We now state a rough version of the main Theorems in Chapter 2. Let us consider an initial configuration of points $\{p_i\}_{i \in \{1, \dots, N\}}$ along with the origin O in $\mathbb{R}^3$. For technical convenience, we can assume that the minimum pairwise distance between the points is 2 after possibly rescaling about the origin. We now scale the configuration by some $R \geq 1$ around the origin O , meaning that we consider the points Rp_i . The smallest and largest pairwise distances between the points are then

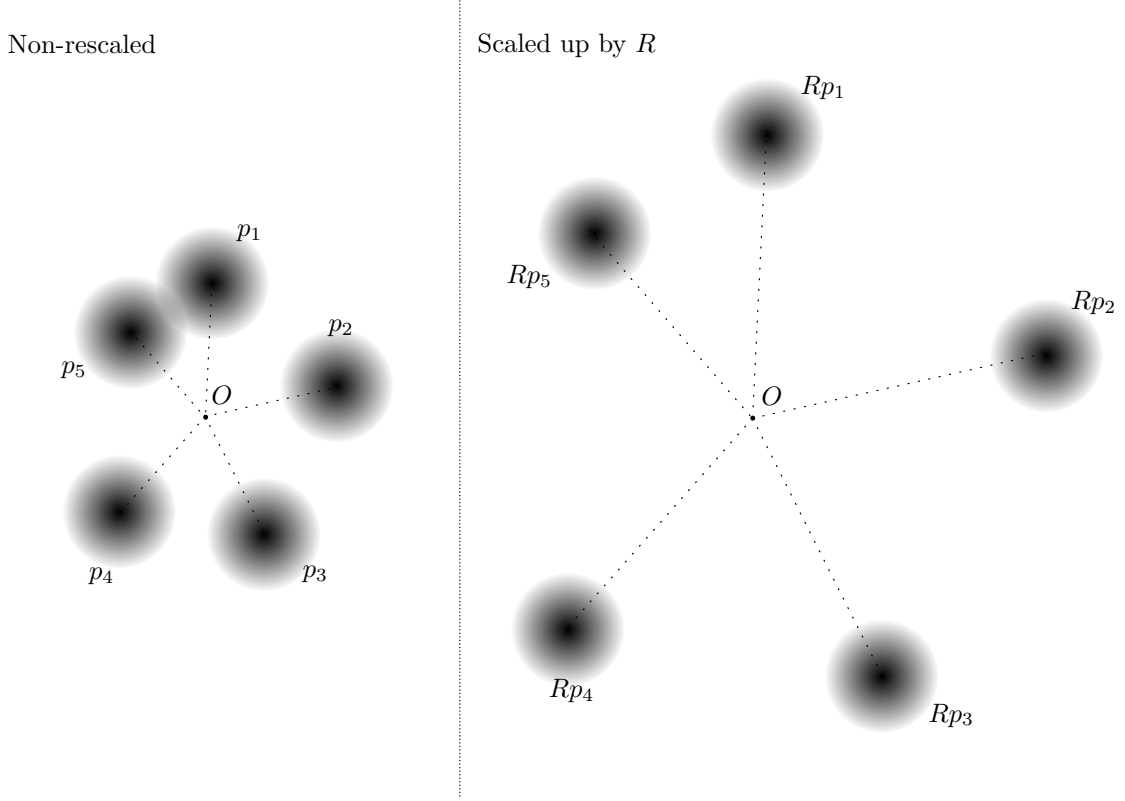


Figure 1.1: Initial data configuration and the parameter R .

proportional to R ; see Figure 1.1 to see how the configuration of our initial data looks.

The equations we are interested in studying are schematically of the form

$$\square\phi^I = Q_{JK}^I(d\phi^J, d\phi^K) + F_{JK}^I(d\phi^J, d^2\phi^K), \quad (1.5)$$

where the Q_{JK}^I and F_{JK}^I satisfy the classical null condition (see Section 1.3).

We shall prove global stability of the trivial solution for general quasilinear systems of wave equations satisfying the classical null condition independently of the parameter R . In other words, the size of the initial data will be allowed to depend only on N , where N is the number of points around which the data are localized, and d_Π , where d_Π is the ratio of the largest and smallest pairwise distances between the N points (see Section 3.2). It is in this sense that the smallness of the initial data does not depend on the scale of its configuration, which is measured by the parameter R . Summarizing, we have the following rough version of the global stability Theorem (Theorem 11).

Theorem 1 (Rough version of Theorem 11). *For all sufficiently small data which are localized around N points as in Figure 1.1, the system of nonlinear wave equations (1.5) admits global-in-*

time solutions. Moreover, the smallness of initial data is measured in a norm which does not depend on the scaling parameter R , and thus depends only on N and d_Π , which is the ratio of the largest and smallest pairwise distances between the N points.

The main obstacle in proving this result is showing decay. Indeed, we lose access to many of the vector fields from the vector field method because of the lack of localization. Nonetheless, we are able to establish in this setting because waves originating from distant localized sources disperse before interacting. The main tool used to prove this theorem is a trilinear estimate measuring the nonlinear interaction between two waves when the nonlinearity satisfies the null condition. This will allow us to show improved estimates on terms measuring the interaction between the distinct waves. The proof of these trilinear estimates involves an analysis of the geometry of the interaction of two waves which originate from distant sources. We give a more detailed description of this procedure in Section 2.3.3.

Another tool we shall need are modified Klainerman–Sobolev inequalities. These inequalities are designed to take advantage of the symmetry that exists when a pair of waves interact. This is described in more detail in Section 2.3.1.

We believe that the dependence of ϵ on the ratio d_Π of the largest and smallest pairwise distances between the points is purely technical. It arises from the fact that the pointwise estimates are far from sharp. We conjecture that the result is true with ϵ independent of the parameter d_Π , but the proof in [11] cannot directly be used to establish this result.⁴

As a corollary of the main Theorem described above, we are able to prove global existence for a class of initial data whose H^1 norm is arbitrarily large.⁵

The data we consider shall be localized around N points $\{w_i\}$ with $i \in \{1, \dots, N\}$. The energy can be made large because the pairwise distances between the w_i can be made even larger. Thus, the data localized around w_i and the data localized around w_j with $i \neq j$ will not interact for a long time, allowing the dispersive effects of the wave equation to still dominate. The rough version of this large data result is as follows. For the precise statement, see Theorem 12.

Theorem 2 (Rough version of Theorem 12). *Given any $L > 0$, there exist initial data with H^1 norm of size L supported in $N(L)$ distinct balls of unit size for which the system of nonlinear wave*

⁴We have not tried to optimize the proof in terms of the dependence on d_Π . However, we note that the proof can be used to remove the dependence of ϵ on d_Π when R is sufficiently large as a function of d_Π . This is similar to the proof of Theorem 12.

⁵The class of data considered here can be compared with the data considered in, say, [82] and [64]. In both cases, the large energy of the data is concentrated in outgoing waves. For our class of data, energy is not required to be outgoing in the same sense, and the largeness comes from the fact that the data are not well localized. However, the data are still mostly outgoing in the sense that there cannot be much focusing.

equations (1.5) admits a global-in-time solution.

We note, however, that the H^k norm of the data in any of the $N(L)$ balls is still small (here, we assume that $k \geq 19$).

1.2.2 Stability and instability of traveling waves

We now state rough versions of the main Theorems found in Chapter 3. There are two main results, one concerning stability and one concerning instability. Much of the following is adapted from [12].

We assume that we are given a semilinear system of wave equations satisfying the null condition. Letting capital Latin indices run from 1 to N , the N equations can be written as

$$\square \phi^I = Q_{JK}^I(d\phi^J, d\phi^K),$$

where each Q_{JK}^I is a null form (see Section 1.3). Each component of ϕ^I is a function from $\mathbb{R}^{3+1}$ to $\mathbb{R}$. We then assume that we are given a vector valued function f^I of a single variable which is smooth and compactly supported, and such that $f^I(t-x)$ is a solution to this system of equations. We then have the following schematic result for stability.

Theorem 3 (Rough version of stability theorem). *Let $f^I(t-x)$ be a smooth solution supported where $|t-x| \leq 1$ to the above semilinear system of wave equations. If the nonzero components of the vector valued function f only excite the standard null form $Q_0 = m$ where m is the Minkowski metric, then the solution f is globally nonlinearly stable under small perturbations supported in the unit ball. If f excites other null forms and if an additional genericity condition holds, then the solution f is linearly unstable.*

The instability result deals with the case when f does excite other null forms. We then have the following schematic result for instability.

Theorem 4 (Rough version of instability theorem). *Let $f^I(t-x)$ be a smooth solution supported where $|t-x| \leq 1$ to the above semilinear system of wave equations. If the nonzero components of the vector valued function f excite null forms other than $Q_0 = m$, then the solution f is linearly unstable.*

Both of these results are imprecise and slightly wrong as stated. The vector valued function f is actually allowed to excite a few other null forms as long as they look enough like the standard null forms when acting on f^I . Moreover, the instability result also requires a genericity condition.

However, we feel that these results give a good idea of the actual results to a first approximation. The precise versions of these statements can be found in Section 3.2.

The first main difficulty one encounters is that the linearization around the background plane wave f^I introduces several first order terms. It is well known that first order terms can be very problematic for the linear wave equation, potentially leading to growth. We are able to overcome this using a renormalization.

After this renormalization, the condition leading to stability guarantees that the linear part of the equation for the perturbation is the usual $\square$ operator. However, because the renormalization depends on f^I , the nonlinearity is no longer constant coefficient, but instead has coefficients that depend on f^I . We thus no longer have full access to the usual family of weighted vector fields because they do not interact well with f^I (for example, f^I is far from being spherically symmetric, meaning that the rotation vector fields will produce bad weights when applied to this function). We overcome this through a close examination of the geometry of the interaction between the background traveling wave f^I and the perturbation, and also by using L^p versions of the Klainerman-Sobolev inequality.

For the instability result, the renormalization does not remove all of the first order terms. We thus want to exploit these first order terms in order to show growth of the linearized equations. This is done using a geometric optics ansatz. In order to effectively use this ansatz, we must be able to do two things:

1. We must be able to effectively analyze the ordinary differential equations (ODEs) resulting from the ansatz.
2. We must be able to control the error from the ansatz.

Ensuring that the geometric optics expansion results in growth requires choosing a null geodesic carefully after considering the geometry of the support of the traveling wave f^I .

1.2.3 Stability for anisotropic systems of wave equations

Chapter 4 of this thesis concerns anisotropic systems of wave equations. This is by far the most substantial part of the thesis, and it contains the statement and proof of the main Theorem. Because of this, Chapter 4 has its own introduction (Section 4.1) which describes everything in much more detail. For now, we shall describe the setting in very broad terms.

We assume that we are given an anisotropic system of wave equations in $2 + 1$ dimensions. A

good example to keep in mind is

$$\begin{aligned}\square\psi &= -\partial_t^2\psi + \partial_x^2\psi + \partial_y^2\psi = (\partial_t\psi)^2(\partial_t\phi) \\ \square'\phi &= -\partial_t^2\phi + \lambda_1^{-2}\partial_x^2\phi + \lambda_2^{-2}\partial_y^2\phi = (\partial_t\phi)^2(\partial_t\psi)\end{aligned}\tag{1.6}$$

with fixed parameters λ_1 and λ_2 , although the class of equations we actually consider is more general (see Theorem 25 and the discussion thereafter).

We can already state a rough version of the main Theorem.

Theorem 5 (Rough version of main Theorem). *The trivial solution to the above system of (1.6) is globally nonlinearly asymptotically stable with respect to sufficiently small and smooth perturbations supported in the unit ball.*

Because the problem is in $2 + 1$ dimensions, cubic nonlinearities are critical in terms of decay, and we must use a kind of null condition. The null condition is that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$, and that the same wave not be cubed. Thus, cubic interactions between different waves are better than cubic interactions between a single wave and itself.

We also run into the difficulty that we cannot effectively use all of the vector fields in (1.4) because they do not commute with both equations. Thus, we cannot use the Klainerman-Sobolev inequality for decay. In order to overcome this, we shall prove decay using bilinear energy estimates. This says that

$$\int_{\Sigma_t} (\partial_t\psi)(\partial_t f) + (\partial^i\psi)(\partial_i f) dx$$

is independent in t up to nonlinear errors whenever $\square f = 0$. Thus, we may pick the data for f to give us an energy estimate for ψ that is localized to a scale 1 ball. This will allow us to show that the energy for ψ distributes in the way we expect, resulting in decay. Thus, we get decay by duality, testing against other solutions of the wave equation. This requires a decay statement for the homogeneous wave equation as a black box. In addition to studying anisotropic systems of wave equations, we shall further describe this method with some simple examples.

Finally, we note that the proof of global stability for the anisotropic system of wave equations will require us to use the scaling vector field $S = t\partial_t + r\partial_r$ in a fundamental way. Thus, we can still effectively use the operators we have in the context of these bilinear energy estimates. This follows the philosophy introduced by Klainerman in [47], which showed that it is very important to take advantage of all of the operators which commute with the equation. The way in which S is used involves careful geometric considerations. More generally, the geometry of the interaction of various

waves must be carefully understood in order to complete the proof. Everything described here is both further discussed and carried out in detail in Chapter 4.

Finally, as was mentioned in Section 1.1.2, these problems are motivated by studying hyperbolic equations with multiple characteristics. Although we focus on crystal optics, there are several physically relevant models involving equations with multiple characteristics such as compressible magnetohydrodynamics and the equations governing elasticity. Chapter 4 will also provide more discussion about the role that this problem plays in the larger context of a research program aiming to understand such equations. For a much more thorough discussion, we refer the reader to the forthcoming [10] joint with Klainerman.

1.3 Notation

We collect some notation that will be used throughout this thesis. We note that Chapter 4 has its own description of notation specific to that Chapter (see Section 4.4.2).

The main results in Chapters 2 and 3 involve results in $3 + 1$ dimensional Minkowski space, while the main result in Chapter 4 involves $2 + 1$ dimensional Minkowski space. We shall use coordinates (t, x, y, z) and (t, x, y) in $\mathbb{R}^{3+1}$ and $\mathbb{R}^{2+1}$ dimensions, respectively. We shall also sometimes use x^μ to denote the coordinates. In general, Greek indices in expressions range from 0 to n (i.e., both space and time), and Latin indices range from 1 to n (i.e., only space). We shall use Einstein summation convention, meaning that the same index appearing in a covariant and contravariant slot of a tensor implies contraction in those indices. We shall also raise and lower the indices with the Minkowski metric.

We shall often have to commute with collections of vector fields. Depending on the context, these admissible vector fields will often be denoted by Γ , and expressions of the form Γ^α will denote strings of these vector fields where α is a multiindex. This collection of vector fields will also be formed in some way from the translation vector fields ∂_μ , the Lorentz vector fields $\Omega_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, and the scaling vector field $S = t\partial_t + r\partial_r$.

We now discuss *null frames*, which are known to be an effective tool for studying wave equations. Given a point p , we can go into polar coordinates centered at p , denoted by (t, r, ω) . We can then form the vector fields $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, and e_A , where the e_A denote a frame of length comparable to 1 tangent to the spheres which are the level sets of t intersected with the level sets of r . We also introduce the null coordinates $v = t + r$ and $u = t - r$, giving us the coordinate system (u, v, ω) . We note that L and $\underline{L}$ are simply rescalings of the coordinate vector fields ∂_v and

∂_u , respectively. Following the discussion in Section 1.1.2, we shall use $\bar{\partial}$ to represent the good derivatives L and e_A , and we shall use ∂ to represent an arbitrary derivative.

Because several of our results will involve null forms, we now give the definitions and basic properties of null forms. The following is adapted from [11].

Definition 6. *Let k be a positive integer. We say that a k -linear form with constant coefficients $F : T\mathbb{R}^4 \times \dots \times T\mathbb{R}^4 \rightarrow \mathbb{R}$ is a classical null form if, for every null vector ξ with respect to the Minkowski metric, we have that*

$$F(\xi, \dots, \xi) = 0. \quad (1.7)$$

We recall the following lemma about null forms and commutation.

Lemma 7 ([35], Lemma 6.6.5). *Let Γ be some commutation field consisting of either a translation vector field, a Lorentz vector field, or the scaling vector field. Let also F be a classical trilinear null form with components $F_{\alpha\beta\gamma}$, and G be a classical bilinear null form with components $G_{\alpha\beta}$, both understood as per Definition 6. Then, we have, for functions $\eta, \zeta \in C^\infty(\mathbb{R}^4)$,*

$$\begin{aligned} \Gamma(F_{\alpha\beta\gamma} \partial^\alpha \eta \partial^\beta \partial^\gamma \zeta) &= F_{\alpha\beta\gamma} \partial^\alpha \Gamma \eta \partial^\beta \partial^\gamma \zeta + F_{\alpha\beta\gamma} \partial^\alpha \eta \partial^\beta \partial^\gamma \Gamma \zeta + F'_{\alpha\beta\gamma} \partial^\alpha \eta \partial^\beta \partial^\gamma \zeta, \\ \Gamma(G_{\alpha\beta} \partial^\alpha \eta \partial^\beta \zeta) &= G_{\alpha\beta} \partial^\alpha \Gamma \eta \partial^\beta \zeta + G_{\alpha\beta} \partial^\alpha \eta \partial^\beta \Gamma \zeta + G'_{\alpha\beta} \partial^\alpha \eta \partial^\beta \zeta. \end{aligned} \quad (1.8)$$

where F', G' are also classical null forms in the sense of Definition 6.

In what follows, to simplify notation, we will denote a classical trilinear null form F with components $F_{\alpha\beta\gamma}$ acting on two functions η, ζ by

$$F(d\eta, d^2\zeta) := F_{\alpha\beta\gamma} \partial^\alpha \eta \partial^\beta \partial^\gamma \zeta$$

(d^2 indicates that the null form is acting on the hessian of ζ). Similarly, we will denote the action of a bilinear null form G with components $G_{\alpha\beta}$ on two functions η, ζ by

$$G(d\eta, d\zeta) := G_{\alpha\beta} \partial^\alpha \eta \partial^\beta \zeta.$$

Iterating Lemma 7, we can prove the following

Lemma 8 (Null forms and commutation). *Let $K \in I_{K \cup K^{(i)} \cup K^{(j)}}^m$, with $m \in \mathbb{N}$, $m \geq 0$. Let also F be a trilinear null form, and G be a bilinear null form. In these conditions, for all smooth functions*

$\eta, \zeta \in \mathcal{C}^\infty(\mathbb{R}^4)$, we have

$$\Gamma^K F(d\eta, d^2\zeta) = \sum_{I+J \subset K} F_{IJ}(d\Gamma^I \eta, d^2 \Gamma^J \zeta), \quad (1.9)$$

$$\Gamma^K G(d\eta, d\zeta) = \sum_{I+J \subset K} G_{IJ}(d\Gamma^I \eta, d\Gamma^J \zeta). \quad (1.10)$$

Here, every F_{IJ} is a trilinear null form as in Definition 6, and every G_{IJ} is a bilinear null form as in Definition 6. The sum is taken over all I, J such that $I \in I_{\Gamma(i)}^{m_1}$, $J \in I_{\Gamma(i)}^{m_2}$, $m_1 + m_2 \leq m$, $I + J \subset K$.

We now recall a lemma on the structure of null forms.

Lemma 9 (Structure of null forms). *Let F, G be resp. a classical trilinear null form and a classical bilinear null form in the sense of Definition 6. Recall the definition of the good derivatives. Then, there exists a positive constant $C > 0$ such that the following holds. Let $\zeta, \eta \in \mathcal{C}^\infty(\mathbb{R}^4)$. We have the pointwise inequality*

$$|G(d\zeta, d\eta)| \leq C(|\bar{\partial}\zeta||\partial\eta| + |\partial\zeta||\bar{\partial}\eta|), \quad (1.11)$$

$$|F(d\zeta, d^2\eta)| \leq C(|\bar{\partial}\zeta||\partial^2\eta| + |\partial\zeta||\bar{\partial}\partial\eta|). \quad (1.12)$$

Recall the expressions $|\bar{\partial}\zeta|$ and $|\partial\zeta|$ from above, and recall that $h \in \{i, j\}$.

Combining Lemma 8 and Lemma 9, we obtain the following:

Lemma 10 (Fundamental null form inequality). *Let F, G be resp. a classical trilinear null form and a classical bilinear null form in the sense of Definition 6. Let also $R \geq 1$. We define the set of vector fields*

$$\mathbf{Z} := \mathbf{K} \cup \mathbf{K}_R \cup \bigcup_{h \in \{i, j\}} (\Gamma^{(h)} \cup \mathbf{K}^{(h)} \cup \mathbf{K}_R^{(h)}).$$

Let $m \in \mathbb{N}$, $m \geq 0$. Let furthermore I be a multi-index in $I_{\mathbf{Z}}^{\leq m}$. Then, there exists a positive constant $C > 0$ such that the following holds. Let $\zeta, \eta \in \mathcal{C}^\infty(\mathbb{R}^4)$. We have the pointwise inequalities

$$|\Gamma^I G(d\zeta, d\eta)| \leq C \sum_{H+K \subset I} (|\bar{\partial}\Gamma^H \zeta| |\partial\Gamma^K \eta| + |\partial\Gamma^H \zeta| |\bar{\partial}\Gamma^K \eta|), \quad (1.13)$$

$$|\Gamma^I F(d\zeta, d^2\eta)| \leq C \sum_{H+K \subset I} (|\bar{\partial}\Gamma^H \zeta| |\partial^2\Gamma^K \eta| + |\partial\Gamma^H \zeta| |\bar{\partial}\partial\Gamma^K \eta|). \quad (1.14)$$

Here, we used the expressions $|\bar{\partial}\zeta|$ and $|\partial\zeta|$ as in the above.

Chapter 2

Stability under multi-localized perturbations

2.1 Setting of the problem

We now turn to studying the problem of stability under multi-localized perturbations, which was joint work between Pasqualotto and the author (see [11]).¹ We start by motivating the problem in the present Section. In particular, we shall try to describe the two main mathematical obstructions that arise from trying to understand stability for nonlinear waves when the data are no longer localized. In Section 3.2, we shall then give a precise version of the main Theorem on stability of nonlinear wave equations under multi-localized perturbations, and in Section 3.3.3, we shall give an overview of the proof. We shall end with Section 2.4, which will describe some related directions. Much of this material in this Part is either taken directly from or adapted from [11], where a proof can be found in complete detail.

We recall the rough version of the main Theorem of this Chapter found in Section 1.2. It says that we can prove stability for nonlinear wave equations allowing for data localized near any finite collection of points, and we can take the smallness of the data to be uniform in the scale of the configuration (see Figure 1.1).

We recall from Section 1.1.2 that, in order to prove global stability for nonlinear wave equations, the two main ingredients are energy estimates and pointwise estimates. More precisely, weighted energies allow us to prove pointwise decay estimates, which in turn enable us to show that the accumulation

¹Pasqualotto and the author are both indebted to Yakov Shlapentokh-Rothman for recommending this problem.

of the nonlinear effects remains negligible. We recall the Klainerman–Sobolev inequality from [47] discussed in Section 1.1.2, which is given by

$$|f|(t, r, \omega) \leq \frac{C}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{|\alpha| \leq 2} \|\Gamma^\alpha f\|_{L^2(\Sigma_t)}. \quad (2.1)$$

Here, α is a multiindex with $|\alpha|$ its length, and Γ^α corresponds to a string of the weighted commutation fields given in (1.4) (see also Section 1.3). The function f is any smooth function decaying sufficiently rapidly at infinity in Σ_t . In practice, if we take $f = \partial\phi$ where ϕ is the solution to some wave equation, the RHS of (2.1) is precisely an energy term. The terms on the RHS can then be estimated by energies, exactly because these vector fields satisfy good commutation properties with $\square$. Schematically, the ∂_t energy estimate shows that

$$\|\partial\Gamma^\alpha\phi\|_{L^2(\Sigma_t)} \lesssim \|\partial\Gamma^\alpha\phi\|_{L^2(\Sigma_0)}, \quad (2.2)$$

where the RHS is determined by the initial data.

In our case, since we consider data localized around points $\{w_i\}_{i \in \{1, \dots, N\}}$ whose pairwise distances are comparable to R , we note that the initial norm $\|\partial\Gamma^\alpha\phi\|_{L^2(\Sigma_0)}$ will potentially be large as a function of R . Indeed, every Lorentz vector field having w_i as its origin will produce an R weight when applied to the piece of data localized at w_j , for $j \neq i$. This would result in an estimate of the form

$$|\partial\phi|(t, r, \omega) \leq \frac{CR^2}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \|\partial\phi\|_{H^2(\Sigma_0)}, \quad (2.3)$$

Even in the case of $N = 2$, and even if we force the data to be of size ϵ , the right hand side of (2.3) will be of size ϵR^2 . When R becomes of size $\frac{1}{\sqrt{\epsilon}}$, we are already of size 1, losing the necessary smallness for a bootstrap argument. The dependence on R in this estimate is thus a serious obstruction to proving global stability independently of R , and proving pointwise decay in a way that is compatible with this parameter R is the first serious obstruction to a proof.

The second serious obstruction is that we must find a way to use the null condition. We recall from Section 1.1.2 that general quadratic nonlinearities can experience blow up in $3+1$ dimensions, meaning that the nonlinear structure must be taken advantage of. From Section 1.1.2, we recall the following expressions

$$|\partial f|(t, r, \omega) \leq \frac{C}{1+|t-r|} \sum_{|\alpha|=1} |\Gamma^\alpha f|(t, r, \omega) \quad \text{and} \quad |\bar{\partial} f|(t, r, \omega) \leq \frac{C}{1+t+r} \sum_{|\alpha|=1} |\Gamma^\alpha f|(t, r, \omega), \quad (2.4)$$

where the “good” derivatives of f , denoted by $\bar{\partial}f$, are tangent to the outgoing light cones (i.e., the level sets of $u = t - r$). The null condition guarantees that bad derivatives (which are specifically ∂_u , the derivatives that pierce the light cone) always appeared multiplied by a good derivative $\bar{\partial}f$. Using (2.4), we see that this schematically means that the null condition guarantees that derivatives which decay at a rate of $\frac{1}{t}$ are always multiplied by derivatives which decay at a rate of $\frac{1}{t^2}$.

This property that a slow decaying derivative is always multiplied by a rapidly decaying derivative is false in our setting (see Section 2.3.4 where this is described in more detail). We indeed have nontrivial interaction between bad derivatives. We thus need to find out why this situation is different from the quadratic nonlinearities studied by John in [39]. A careful study of the geometry will allow us to exploit the null condition in an appropriate way.

Both of these obstructions require a careful analysis of the spacetime geometry of the configurations in question. We are then able to use this analysis in the context of a modified vector field method in order to prove the global stability statement.

After analyzing the above problem, we are in fact able to prove a large data global stability result as a corollary. We intuitively believe that waves emanating from points a distance R from each other will interact less and less as R increases. Our analysis results in a precise and quantitative version of this intuition. We are able to exploit this to prove that the trivial solution is stable with respect to perturbations which have arbitrarily large energy by placing the data in several mutually distant balls.

2.2 Precise version of the main Theorem on stability of wave equations under

Having motivated the problem, we now turn to the precise statement of the main Theorems from [11]. We begin with some notation that is necessary to state the Theorem precisely.

We consider $(\mathbb{R}^{3+1}, m)$ parametrized by the usual coordinates (t, x, y, z) , where m is the Minkowski metric. Let $\Pi := \{p_1, \dots, p_N\}$ be a collection of points in $\mathbb{R}^3$ satisfying the following properties:

$$\min_{\substack{i \neq j \\ i, j \in \{1, \dots, N\}}} |p_i - p_j| \geq 2.$$

We also define

$$d_\Pi := \left(\max_{i, j} |p_i - p_j| \right) / \left(\min_{\substack{i \neq j \\ i, j \in \{1, \dots, N\}}} |p_i - p_j| \right). \quad (2.5)$$

Let $R \geq 1$ be a parameter. We will consider initial data centered around the points $R \cdot p_i$, on the initial surface Σ_0 . Here, $|w|$ denotes the Euclidean length of $w \in \mathbb{R}^3$ as measured in $\mathbb{R}^3$.

We are now ready to state the main Theorem, which proves global existence for initial data whose size (measured in a suitable higher-order Sobolev norm) depends only on N and d_Π , and not on R .

Theorem 11 (Nonlinear wave equations with null condition and multi-localized initial data). *Let M, N, n_0 be non-negative integers, such that $n_0 \geq 19$. Let $\{F_A^{BC}\}_{A,B,C=1,\dots,M}$ and $\{G_A^{BC}\}_{A,B,C=1,\dots,M}$ be resp. a collection of trilinear and bilinear classical null forms, i. e. each of the F_A^{BC} and G_A^{BC} is as in Definition 6 and satisfies:*

$$F_A^{BC} : T\mathbb{R}^4 \times T\mathbb{R}^4 \times T\mathbb{R}^4 \rightarrow \mathbb{R}, \quad G_A^{BC} : T\mathbb{R}^4 \times T\mathbb{R}^4 \rightarrow \mathbb{R}.$$

Let furthermore $\Pi := \{p_1, \dots, p_N\}$ be a collection of N points in $\mathbb{R}^3$, with d_Π the ratio of the largest and smallest distances between them as defined in display (2.5).

Then, there exists $\varepsilon_0 = \varepsilon_0(d_\Pi, \{F_{BC}^A\}_{A,B,C=1,\dots,M}, \{G_{BC}^A\}_{A,B,C=1,\dots,M}, N) > 0$ such that the following holds true for all $R \geq 0$ and for all $0 < \varepsilon < \varepsilon_0$.

Consider a collection of functions

$$(\bar{\phi}_{A,i}^{(0)}, \bar{\phi}_{A,i}^{(1)})_{A \in \{1,\dots,M\}, i \in \{1,\dots,N\}} \quad (2.6)$$

which satisfy the following bounds, in the usual angular coordinates in $\mathbb{R}^3$:

$$|\widehat{\partial}^k \bar{\phi}_{A,i}^{(0)}| \leq \varepsilon(1+r)^{-k-2}, \quad |\widehat{\partial}^k \bar{\phi}_{A,i}^{(1)}| \leq \varepsilon(1+r)^{-k-3}, \quad (2.7)$$

for all $0 \leq k \leq n_0$, and $A \in \{1, \dots, M\}$. Let us then construct initial data as follows. Let w_i be the point $R \cdot p_i$. Let us furthermore define, for all $A \in \{1, \dots, M\}$,

$$\phi_A^{(0)} := \sum_{i=1}^N \bar{\phi}_{A,i}^{(0)}(x - w_i), \quad \phi_A^{(1)} := \sum_{i=1}^N \bar{\phi}_{A,i}^{(1)}(x - w_i). \quad (2.8)$$

Let us consider the initial value problem given by the following system of quasilinear wave equations

with the specified data:

$$\begin{aligned}
\Box \phi_A + \sum_{B,C=1}^M F_A^{BC}(d\phi_B, d^2\phi_C) &= \sum_{B,C=1}^M G_A^{BC}(d\phi_B, d\phi_C), \quad A = 1, \dots, M, \\
\phi_A|_{t=0} &= \phi_A^{(0)}, \quad A = 1, \dots, M, \\
\partial_t \phi_A|_{t=0} &= \phi_A^{(1)}, \quad A = 1, \dots, M.
\end{aligned} \tag{2.9}$$

Then, the initial value problem (2.9) admits a global-in-time solution ϕ_A , which furthermore decays with quantitative rates. Thus, the trivial solution to (2.9) is asymptotically stable under this class of non-localized perturbations uniformly in the scale R .

Using the analysis that goes into Theorem 11, we are able to prove the following large data global stability result.

Theorem 12. *Let $L \in \mathbb{R}$, $L \geq 0$, and $M \in \mathbb{N}$, $M \geq 0$ be given. There exist a real number ϵ_0 , an integer $N(L)$ and points $w_i \in \mathbb{R}^3$, $i \in \{1, \dots, N(L)\}$, such that the following holds. There exists a collection*

$$(\phi_{A,i}^{(0)}, \phi_{A,i}^{(1)})_{A \in \{1, \dots, M\}, i \in \{1, \dots, N(L)\}}$$

of pairs of smooth vector valued functions on $\mathbb{R}^3$ satisfying the following properties:

$$\text{supp}(\phi_{A,i}^{(0)}) \subset B(0, 1), \quad \text{supp}(\phi_{A,i}^{(1)}) \subset B(0, 1), \tag{2.10}$$

$$\|\phi_{A,i}^{(0)}\|_{H^{N_1}(B(0,1))} \leq \epsilon_0, \quad \|\phi_{A,i}^{(1)}\|_{H^{N_1-1}(B(0,1))} \leq \epsilon_0, \tag{2.11}$$

for all $i \in \{1, \dots, N(L)\}$ and all $A \in \{1, \dots, M\}$, where $N_1 \geq 19$. Moreover, under these conditions, the trivial solution to the initial value problem (2.9) is asymptotically stable under perturbations of the form

$$\phi_A^{(0)} = \sum_{i=1}^{N(L)} \phi_{A,i}^{(0)}(x - w_i), \quad \phi_A^{(1)} = \sum_{i=1}^{N(L)} \phi_{A,i}^{(1)}(x - w_i).$$

Moreover, we have that

$$\|\phi_A^{(0)}\|_{H^1(\mathbb{R}^3)} \geq L, \quad \|\phi_A^{(1)}\|_{L^2(\mathbb{R}^3)} \geq L,$$

meaning that the initial data are allowed to have arbitrarily large H^1 norm.

Of course, the data can also be made large in H^k for any k . The Theorem is stated in terms of H^1 just to emphasize that it is not a low regularity result.

2.3 Description of the proof

We shall now provide a sketch of the proof of Theorem 11. We use a careful analysis of the spacetime geometry of the above configuration of waves along with the vector field method. The main ingredients are modified Klainerman-Sobolev inequalities which are better adapted to the parameter R (see Section 2.3.1) and improved energy estimates coming from trilinear estimates which arise from geometric considerations. These improved energy estimates are discussed first in the cubic case in Section 2.3.2, and then in the more complicated quadratic case in Section 2.3.3 and Section 2.3.4. The geometric considerations described in Section 2.3.4 will also show us how we can take advantage of the null condition. We are then able to combine these ingredients and prove global stability using the vector field method, which we describe in Section 2.3.5.

Let us first give a broad overview of the ideas involved. In order to overcome the obstructions described in Section 2.1, we shall need to apply the vector field method more carefully. We first note that the main contribution to the solution should come from a superposition of the contributions arising from each of the compactly supported pieces of data considered individually. Subtracting off these N auxiliary solutions (which satisfy good asymptotics from existing theory, and which we can treat as known functions), we obtain a nonlinear equation with an inhomogeneity and vanishing initial data whose solutions have small energy in terms of the parameter R .

More precisely, the strategy can be divided into three steps.

1. First, we reduce to the case of N compactly supported pieces of initial data localized around N points, plus an additional error. This procedure is effective because the resulting error is small in terms of the parameter R .
2. As a second main step, we shall accomplish the task of showing that the main contribution to the solution comes from the superposition of the contributions arising from each individual compactly supported piece of data. This will be shown by proving that the resulting error is small in terms of R . This involves showing decay in R of the interactions between waves arising from points distance comparable to R from each other.
3. Finally, we shall improve (in terms of R) the pointwise estimates on the error arising from the first and second step. This involves using both the improved energy estimates from the second step and also modified Klainerman-Sobolev inequalities which are well adapted to the parameter R .

After finishing these three parts, we will be in a position to prove global stability.

In the case of cubic and higher nonlinearities (see Section 2.3.2), the heuristic reasoning presented here (in particular, the second step in the previous list) can be made quantitative in a straightforward way, and the resulting weights in R are particularly favorable. Such an improvement in energy creates good powers of R in the L^2 norms in the RHS of estimate (2.3). We now note the two main observations which go into showing this energy improvement in R for a cubic equation.

- The different solutions will not interact before time comparable to R .
- The individual solutions satisfy good quantitative decay estimates.

The second of these points follows from the existing theory discussed in Section 1.1.2. The first of these points can be seen by looking at Figure 2.1 below.

In order to close an argument for the cubic equation, we must then simply improve the L^∞ estimates, which amounts to lowering the power of R in the RHS of (2.3).

Both of these aspects (the improvement in energy and the improvement in the Klainerman–Sobolev estimates) are more complicated for the quadratic equation (see Sections 2.3.1, 2.3.3, and 2.3.4).

2.3.1 R -weighted Klainerman-Sobolev inequalities

We shall now describe the modified Klainerman-Sobolev inequalities that we need. The intuition can be summarized as follows. We will have a collection of vector fields having one of three kinds of weights. They will schematically either be t , 1, or $\frac{t}{R}$. Every direction with a weight of 1 provides no decay, every direction with a weight of t provides $t^{\frac{1}{2}}$ decay, and every direction with a weight of $\frac{t}{R}$ provides $\frac{t^{\frac{1}{2}}}{R^{\frac{1}{2}}}$ decay. For example, in the usual Klainerman-Sobolev inequality in $3+1$ dimensions, we see that along the light cone $t = r$, we only have two directions with a weight of t in the collection of vector fields. These are precisely the angular directions. This gives us two powers of $t^{\frac{1}{2}}$, leading to a decay rate of $\frac{1}{t}$ along the light cone (see (2.1)). Meanwhile, when $r \leq \frac{t}{2}$, the collection of vector fields actually span the tangent space of $\mathbb{R}^{3+1}$, meaning that they span the tangent space of Σ_t . They moreover all have a weight of t , meaning that we have 3 directions with a weight of t . This leads to three factors of $t^{\frac{1}{2}}$ decay, for a total decay rate of $t^{\frac{3}{2}}$ (see (2.1)). This intuition may be useful to keep in mind in the following discussion.

We first observe the following fact: the vector fields $\Gamma_R := \frac{1}{R}\Gamma$ do not introduce R weights on the initial data (recall that the distance between the supports of each individual piece of data is proportional to R). Repeating the proof of the Klainerman–Sobolev inequality using the vector

fields Γ_R instead of the vector fields Γ results in the following estimate:

$$|f|(t, r, \omega) \leq \frac{CR^{\frac{3}{2}}}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{|\alpha| \leq 3} \|\Gamma_R^\alpha f\|_{L^2(\Sigma_t)}. \quad (2.12)$$

Note that estimate (2.12) still depends on R . However, the constant on the RHS is already better in terms of R -weights than the corresponding one appearing in (2.3). Using a similar argument but not requiring decay in $u = t - r$ allows us to prove the following Klainerman–Sobolev inequality for $t \geq \frac{R}{10}$:

$$|f|(t, r, \omega) \leq \frac{CR}{(1+t+r)} \sum_{|\alpha| \leq 3} \|\Gamma_R^\alpha f\|_{L^2(\Sigma_t)}. \quad (2.13)$$

The estimate (2.13) along with the improved energy estimates is enough to prove global stability for nonlinear equations with cubic and higher nonlinearities.

Concerning the quadratic case, such improvement in the Sobolev estimates is not enough, and we will elaborate on this point in the present and following Section 2.3.2. The further complication is that we also need improved decay for good derivatives because we need to take advantage of the classical null condition (see Section 2.3.4 for a discussion on good derivatives).

To obtain better estimates, we realize that, restricting to $N = 2$ with both pieces of data supported in unit balls whose centers are on the x axis, then those rotations and Lorentz boosts which involve only y , z , and t do not introduce R -weights on the initial data. These are the following vector fields:

$$z\partial_y - y\partial_z, \quad t\partial_y + y\partial_t, \quad t\partial_z + z\partial_t. \quad (2.14)$$

Thus, we can use such vector fields without dividing by R in the RHS of the Klainerman–Sobolev estimates. This observation will allow us to get an estimate of the form

$$|f(t, r, \omega)| \leq \frac{CR^{\frac{1}{2}}}{(1+t+r)} \|\bar{\Gamma}^\alpha f\|, \quad (2.15)$$

where now the term $\|\bar{\Gamma}^\alpha f\|$ schematically refers to L^2 norms we expect to correspond to ∂_t energies after commuting by appropriately R weighted Lorentz fields. By appropriately R weighted, we mean the Γ_R vector fields used in (2.12) along with the vector fields that do not introduce bad weights given by (2.14).

This observation is useful even when $N > 2$. Indeed, for quadratic nonlinearities, all interactions occur between only two pieces of data up to an error that is better in R . This fact allows us to take advantage of (2.15) even when the data are localized around more than two different points.

2.3.2 The cubic case: subtracting off auxiliary solutions

In this section, we describe how we can improve the energy estimates (in terms of R -weights) for a cubic equation by subtracting off the main contribution to the solution arising from several localized pieces of initial data.

We take the cubic equation:

$$\square\phi = (\partial_t\phi)^3, \quad (2.16)$$

with initial data supported in two disjoint balls, one centered at $(-R, 0, 0)$, and the other centered at $(R, 0, 0)$. We should regard R as being a very large number, and the initial data for ϕ as being small in some $H^k(\mathbb{R}^3)$ norm.

As was discussed previously in Section 2.3, we want to take advantage of the following two facts: the individual pieces of data give rise to solutions that disperse, and the two solutions do not interact for time comparable to R . We do so by subtracting off the functions ϕ_1 and ϕ_2 , where ϕ_1 is the solution to the equation (2.16) with initial data equal to the portion of the initial data for ϕ supported in the ball centered at $(-R, 0, 0)$. Similarly, ϕ_2 is the solution to equation (2.16) with initial data equal to the portion of the initial data of ϕ that is supported in the ball centered at $(R, 0, 0)$.

By our assumptions on the data and by classical theory (see [47]), we know that ϕ_1 and ϕ_2 exist globally because their data are sufficiently small in $H^k(\mathbb{R}^3)$ (and we note that, here, the smallness does not depend on R , as ϕ_1 and ϕ_2 each only have localized data). Moreover, ϕ_1 and ϕ_2 decay at the same rates as solutions to the linear wave equation with compactly supported data. Therefore, quantitatively, we can assert that

$$\|\partial\phi_1\|_{L^\infty(\Sigma_t)} + \|\partial\phi_2\|_{L^\infty(\Sigma_t)} \leq \frac{C\epsilon}{1+t}. \quad (2.17)$$

Furthermore, we have the following control on the energy of each individual piece:

$$\|\partial\phi_1\|_{L^2(\Sigma_t)} + \|\partial\phi_2\|_{L^2(\Sigma_t)} \leq C\epsilon. \quad (2.18)$$

We now subtract off these functions from the solution ϕ , and we obtain an equation for $\psi := \phi - \phi_1 - \phi_2$. Schematically, the equation is as follows:

$$\square\psi \approx (\partial_t\psi)^3 + (\partial_t\psi)((\partial_t\phi_1)^2 + (\partial_t\phi_2)^2) + (\partial_t\phi_1)(\partial_t\phi_2)(\partial_t\phi_1 + \partial_t\phi_2). \quad (2.19)$$

This equation is inhomogeneous, and furthermore, the initial data for ψ vanishes. The crucial observation here is that $(\partial_t \phi_1)^3$ and $(\partial_t \phi_2)^3$ do not appear on the RHS, and this is consistent with having subtracted off the main contribution to the solution arising from both ϕ_1 and ϕ_2 . Indeed, because ψ has vanishing initial data, we expect that only the inhomogeneities are responsible for adding energy into ψ . Moreover, note that the remaining terms are all supported in $t \geq R - 2$ (see again Figure 2.1), which encodes the fact that ϕ_1 and ϕ_2 only interact after time comparable to R .

Now, multiplying equation (2.19) by $\partial_t \psi$ and integrating by parts, we get the following basic energy estimate:

$$\|\partial \psi\|_{L^2(\Sigma_t)} \leq 2 \int_{R-2}^t \|\square \psi\|_{L^2(\Sigma_s)} ds. \quad (2.20)$$

We now drop the terms containing ψ (which we expect to be better behaved in terms of R) in the expression for $\square \psi$ given by (2.19), we are left with the interaction terms involving ϕ_1 and ϕ_2 :

$$\int_{R-2}^t \|\partial_t \phi_1 \partial_t \phi_2 (\partial_t \phi_1 + \partial_t \phi_2)\|_{L^2(\Sigma_s)} ds \leq C \epsilon^3 \int_{R-2}^t \frac{1}{s^2} ds \leq C \frac{\epsilon^3}{R}. \quad (2.21)$$

Here, we have used the pointwise estimates (2.17) for two of the factors in the cubic expression, and we have used the energy estimate (2.18) for the remaining factor. Thus, we expect to be able to propagate an estimate of the form

$$\|\partial \psi\|_{L^2(\Sigma_t)} \leq \frac{C \epsilon^3}{R} \quad (2.22)$$

for the nonlinear problem. This is indeed possible, and taking (2.22) as a bootstrap assumption along with using (2.13) to control the nonlinearity allows us to close a bootstrap argument in the cubic case, yielding global stability of the trivial solution independently of the parameter R . We note that, in this procedure, we have estimated many derivatives of ϕ_1 and ϕ_2 in L^∞ . However, because we can consider ϕ_1 and ϕ_2 to be known functions with as much regularity as we want (as long as we only require boundedness in some H^k space), this loss of derivatives does not constitute an obstruction.

2.3.3 The quadratic case: trilinear estimates on null forms

Returning to the quadratic case, we still expect that the solutions arising from each individual piece of initial data will start interacting only after time comparable to R . Before this interaction, we expect the solution to exist and to decay at rates that are known from the classical theory (see [48]).

We will decompose the data into N compactly supported pieces which are localized near some $R p_i$, plus a remainder whose energy will be small in terms of R . The solution with data localized

near each Rp_i will be denoted by ϕ_i , and the solution with the remainder as initial data will be denoted by ϕ_0 .

For technical reasons, we cannot directly study the equation for the difference

$$\psi := \phi - \sum_{i=0}^N \phi_i,$$

as was done in the cubic case. This is because the weights in R prevent us from closing a bootstrap argument. The equation for ψ is a nonlinear wave equation with vanishing initial data and inhomogeneous terms containing ϕ_i . Because the initial data for ψ vanishes, we again expect the behavior of ψ to be dominated by the inhomogeneous terms, just as in the cubic case discussed in Section 2.3.2. Thus, it is natural for us to study linear inhomogeneous equations such as

$$\square\psi_{ij} = 2m(d\phi_i, d\phi_j). \quad (2.23)$$

The RHS of equation (2.23) is precisely one of the inhomogeneous terms that appears in the RHS of the equation for ψ . This term contains the information about the interaction between ϕ_i and ϕ_j . We shall first prove estimates for ψ_{ij} . This involves getting estimates for certain trilinear terms.

The trilinear estimates arise from using $\partial_t\psi_{ij}$ as a multiplier in (2.23). Thus, the estimates involve controlling spacetime integrals like

$$\int \int m(d\phi_i, d\phi_j) \partial_t\psi_{ij} dx dt, \quad (2.24)$$

and they show that the interaction between ϕ_i and ϕ_j is small. In general, m can be any null form.

As a consequence of these trilinear estimates, we shall get improved energy estimates for ψ_{ij} of the form

$$\|\partial\psi_{ij}\|_{L^2(\Sigma_t)} \leq C \frac{\epsilon^2}{R}, \quad (2.25)$$

which is analogous to the estimate (2.22) for the cubic equation. We can also get analogous improved fluxes of the ∂_t energy through outgoing cones.

After showing these improved estimates, we go one step further in decomposing ϕ by considering the equation for

$$\Psi := \phi - \sum_{i=0}^N \phi_i - \sum_{i \neq j} \psi_{ij}. \quad (2.26)$$

The resulting equation for Ψ is a nonlinear equation with vanishing initial data and inhomogeneous

terms on the RHS. Moreover, we expect Ψ to have very good estimates in terms of R weights. The improved estimates on ψ_{ij} will allow us to control Ψ in a sufficiently strong way.²

We shall now describe in more detail the trilinear estimates we need for ψ_{ij} and the strategy we follow to prove them.

2.3.4 Geometric tools for the trilinear estimates

We begin by recalling the dichotomy of good and bad derivatives. The derivatives $\bar{\partial}$ from (2.4) consist of ∂_v , e_A , and e_B while the bad derivative consists of ∂_u . We thus recall that $\bar{\partial}f$ decays like $\frac{1}{t^2}$ and $\partial_u f$ decays only like $\frac{1}{t}$, as we can see from (2.4).

Without loss of generality, we will take $i = 1$ and $j = 2$ in (2.23). Similar to the cubic case in Section 2.3.2, we want to prove decay in R of the energy of ψ_{12} . The ∂_t energy estimate on (2.23) is analogous to the schematic calculation in (2.21), as (2.23) is driven by only inhomogeneities. However, in the quadratic case, it is not enough to use directly the ∂_t energy estimate, Hölder's inequality, and the pointwise estimates for the known functions ϕ_1 and ϕ_2 as was done for the cubic case in estimate (2.21).

This is the main difference between the quadratic case and the cubic case where the estimates (2.17) sufficed to close a bootstrap argument. There, the inhomogeneity is cubic, and it is easy to gain $\frac{1}{R}$ (see equation (2.21)). For a quadratic nonlinearity, the null condition must be used (indeed, we recall that general quadratic nonlinearities lead to blowup).

However, directly using the null condition is not sufficient because there is a region with interaction between $\partial_{u_1}\phi_1$ and $\partial_{u_2}\phi_2$. Here, $\partial_{u_i} = \partial_t - \partial_{r_i}$, and r_i is the radial coordinate with Rp_i as its center. These are precisely the slowest decaying derivatives. Quadratic nonlinearities involving only terms that decay this slowly (specifically like $\frac{1}{t}$) are the obstruction to proving global existence when the null condition is not satisfied.

To see that this interaction must occur, we can consider the forward light cones centered at Rp_1 and Rp_2 (for simplicity, we can take $Rp_1 = (-R, 0, 0)$ and $Rp_2 = (R, 0, 0)$). Schematically, the interaction will then look as in Figure 2.1.

When the two light cones first intersect (which we can think of as being the first point of interaction of the distinct waves), a good derivative of ϕ_1 is exactly the bad derivative of ϕ_2 . More precisely, we have that $\partial_{v_1} = \partial_{u_2}$ at this point (see Figure 2.1). Here, we are taking $\partial_{v_i} = \partial_t + \partial_{r_i}$ with r_i as in the above. In terms of decay, this situation looks very similar to the nonlinearity

²We are not concerned about losing derivatives because we may take ψ_{ij} to be known functions with as much regularity as we want, similar to ϕ_i . Thus, we may estimate as many derivatives of ψ_{ij} in L^∞ as we want.

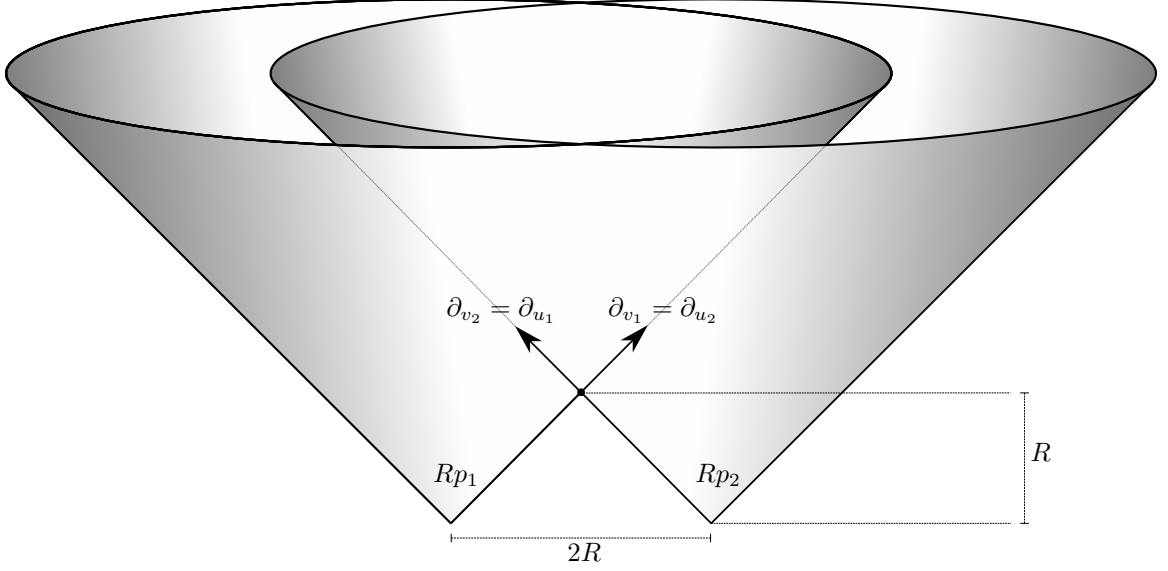


Figure 2.1: Interaction of two nonlinear waves.

$-(\partial_t \phi)^2$, because there is a region where the bad derivatives of ϕ_1 and ϕ_2 interact. Thus, in this region, the equation is similar to $\square \phi = -(\partial_t \phi)^2$, which blows up in finite time even for arbitrarily small and compactly supported data (see [39]).

In the cubic case, we only required the observations given in the bullet points in Section 2.3. We shall now list the properties we must take advantage of in the quadratic case, which include three additional observations:

- The distinct solutions ϕ_1 and ϕ_2 effectively will not interact for time comparable to R , just as in the cubic case.
- The distinct solutions ϕ_1 and ϕ_2 have sufficiently strong quantitative decay estimates, just as in the cubic case. However, we must now also use the fact that certain derivatives decay better after decomposing into a null frame.
- The measure of the set of interaction between $\partial_{u_1} \phi_1$ and $\partial_{u_2} \phi_2$ is small in terms of R . Thus, using Hölder's inequality to control interactions between ϕ_1 and ϕ_2 is wasteful, as the two functions are supported on sets with small intersection.
- The good derivatives of ϕ_1 and ϕ_2 become asymptotically aligned. This allows us to use the null condition for very large times, where very large means very large relative to R .
- The vector field ∂_t can be written in terms of the good derivatives of ϕ_1 and ϕ_2 in the region in which the interaction between ϕ_1 and ϕ_2 is largest. Even though the coefficients of this

decomposition degenerate as $t \rightarrow \infty$ as a consequence of the fact that the good derivatives of ϕ_1 and ϕ_2 become asymptotically aligned, it still gives an improvement.

Using these facts will allow us to get good energy estimates for ψ_{ij} in terms of R weights. We can then combine these good energy estimates with the modified Klainerman–Sobolev estimates described in Section 2.3.1 to get good pointwise estimates on ψ_{ij} .

2.3.5 Closing the argument

The linear estimates for ψ_{ij} described above are enough for us to prove global existence for the remainder Ψ (see equation (2.26)) using the vector field method as in, for example, [47], [48], [60], and [62].

Moreover, Ψ satisfies better energy bounds than ψ_{ij} in terms of R . Indeed, schematically, the ∂_t energy of Ψ will be of size $\epsilon^3 R^{-\frac{3}{2}}$. This implies that the ∂_t energy of $\Gamma_R^\alpha \Psi$ will be of size $\epsilon^3 R^{-\frac{3}{2}}$, where the vector fields Γ_R are as in Section 2.3.1. Applying (2.13) to $f = \partial \Gamma_R^\alpha \Psi$ and noting that the resulting norms on the right hand side of this estimate should be of size $\epsilon^3 R^{-\frac{3}{2}}$ tells us that we should expect decay estimates like

$$|\partial \Gamma_R^\alpha \Psi|(t, r, \omega) \leq C \epsilon^3 \frac{1}{R^{\frac{1}{2}} t + r}. \quad (2.27)$$

Moreover, using (2.1), we are able to obtain improved decay of the good derivatives like

$$|\bar{\partial} \Gamma_R^\alpha \Psi|(t, r, \omega) \leq C \epsilon^3 \frac{R^{\frac{3}{2}}}{t^{\frac{3}{2}}}. \quad (2.28)$$

These bounds are enough to prove global existence using a bootstrap argument for Ψ . We shall propagate the bootstrap assumption that the ∂_t energy of $\Gamma_R^\alpha \Psi$ for all $|\alpha|$ sufficiently large is schematically of size $\epsilon^3 R^{-\frac{3}{2}}$. We shall have to use the following observation. If we are interested in controlling nonlinear effects between time R and R^{20} , we note that better pointwise R decay is almost as good as better t decay. This is because, evaluating the integrals,

$$\int_R^{R^{20}} \frac{1}{R^\delta t} dt \leq C \frac{\log(R)}{R^\delta},$$

while

$$\int_R^{R^{20}} \frac{1}{t^{1+\delta}} dt \leq \frac{C_\delta}{R^\delta}.$$

These two expressions differ only by a factor of $\log R$, which we will be able to absorb using terms

with extra negative powers of R . Thus, for this time scale, we can use the estimates (2.27) which only have an improvement in R .

Then, from time R^{20} to ∞ , we use the estimates (2.28), which have worse R weights. This can be done because the fact that we are integrating from R^{20} will get rid of any bad R weights. More precisely, for $\alpha \in \mathbb{R}$ we have that

$$\int_{R^{20}}^{\infty} \frac{R^\alpha}{t^{\frac{3}{2}}} dt \leq CR^{\alpha-10}. \quad (2.29)$$

In practice, we will have α about 0, so this estimate is strong enough to propagate an energy estimate of size $\epsilon^3 R^{-\frac{3}{2}}$. This allows us to prove global existence for Ψ . After this, we have that $\Psi + \sum_{i \neq j} \psi_{ij} + \sum_i \phi_i$ is a solution to the original problem, as desired.

2.3.6 Large data stability

We now describe how we can use the analysis described above to prove Theorem 12. In fact, this follows almost immediately from the observation that the estimates we get end up having very good dependence on R . This allows us to use smallness coming from R in a bootstrap argument to compensate for largeness coming from the total energy.

More precisely, we pick some very small ϵ_0 such that the quasilinear wave equations in question are globally stable with respect to perturbations which are compactly supported in the unit ball and whose H^k norm is of size at most ϵ_0 . We then pick data of size $\frac{\epsilon_0}{2}$. Given some M representing what we want the energy of the initial data to be, we then pick N sufficiently large such that the total energy coming from N pieces of data of size $\frac{\epsilon_0}{2}$ is of size M . This is schematically picking $N = \frac{2M}{\epsilon_0}$.

Because the size of the data can no longer depend on N , we must use something else to get smallness. It turns out that the estimates described in Section 2.3.5 all have negative powers of R left over. Thus, we put data in N balls, and we pick R sufficiently large to absorb the bad dependence on N . This allows us to prove global stability for this class of large data.

2.4 Related directions

The results in [11] deal with quasilinear systems of wave equations satisfying the classical null condition, and we believe that there are several potentially interesting directions for further study. The most natural problem to pursue after [11] is to study the Einstein vacuum equations with the class of data considered in that paper, with a goal of proving stability of Minkowski space with this class of data. Together with Pasqualotto, the author hopes that this will be the topic of future work.

It is also natural to study the stability of other models in this context. One such model consists of nonlinear Klein-Gordon equations. Together with Iyer and Pasqualotto, the author hopes that this will also be the topic of future work. In this Section, we hope to describe a few aspects of these problems.

The monumental work [22] by Christodoulou–Klainerman showed that Minkowski space was stable in the framework for the initial value problem established by Choquet-Bruhat and Geroch (see Section 1.1.1 where we describe this). In the course of the proof, they had to deal with several issues related specifically to the Einstein vacuum equations in addition to the difficulties associated with wave equations. One of the main sources of difficulty came from the gauge freedom in the Einstein vacuum equations discussed in Section 1.1.1. They had to find a gauge in which they could effectively both use the vector field method and take advantage of some kind of null condition in order to avoid the behavior John studied in [39]. They developed a framework adapted to *optical functions*, which are solutions u to the Eikonal equation $g(du, du) = 0$. As a consequence of this equation, the level sets of u are guaranteed to be null hypersurfaces. Because waves intuitively propagate along null hypersurfaces, this allowed them to effectively understand the behavior of the waves. This framework allowed them to use the vector field method. Moreover, they were able to identify the presence of a null condition in the Einstein vacuum equations when they were expressed in this way. This circle of ideas was among the most important in completing the proof in [22] (we moreover note that these ideas related to optical functions have continued to be extremely valuable in the study of a wide variety of hyperbolic equations).

If we wish to study the stability of Minkowski space under multi-localized perturbations, the first obstacle we encounter are the constraint equations. This poses two difficulties. The first is that the constraint equations make it so that data cannot be too localized. Indeed, nontrivial data must have rather slower decay, slower than what Pasqualotto and the author considered in [11]. In addition, the constraint equations mean that even the existence of a data set which could be called multi-localized is nontrivial.

In ongoing work with Corvino and Pasqualotto, the author is in the process of proving that an appropriate class of data that can be called multi-localized exists. If we denote by r_i the Euclidean distance function on $\mathbb{R}^3$ away from some point p_i , this time symmetric class of data is schematically of the form

$$\left(1 + \sum_{i=1}^N \frac{m_i}{r_i}\right)^4 \delta_{ij}$$

in the region where $r_i \geq 1$, and it is a C^k perturbation of the flat metric within these balls. The

proof involves adapting a gluing construction due to Corvino (see [23]).

With the existence of this data in hand, the next problem involves picking a gauge. One possibility is to try to adapt the framework developed in Christodoulou–Klainerman. The equations are very well behaved in this framework, but the machinery is very involved. Working in harmonic coordinates (see Section 1.1.1) provides a relatively simple way of seeing that the Einstein vacuum equations are hyperbolic. However, the equations do not satisfy the classical null condition in this gauge, but they rather satisfy the *weak null condition*, introduced by Lindblad–Rodnianski in order to prove the stability of Minkowski space in harmonic coordinates in the works [58], [59], and [60]. The weak null condition will necessarily, however, lead to worse estimates in R than the ones we described in Section 2.3.4, meaning that the argument would have to be much more delicate. Another possibility would be to do a mixed approach, similar to what Keir does in [44]. In that work, Keir introduces harmonic coordinates but still also constructs optical functions on this background in order to improve the analysis.

For the problem involving Klein-Gordon, we believe that it is natural to study the problem for a quadratic nonlinearity in $2 + 1$ dimensions. This is because solutions to the Klein-Gordon equation decay faster than solutions to the wave equation (at a rate of $\frac{1}{t}$ in $2 + 1$ dimensions as opposed to $\frac{1}{\sqrt{t}}$), so this problem is critical in terms of decay. We would then have to figure out how to take advantage of a *normal form transformation* in the context of multi-localized data. These kinds of transformations (see [72] and [70]) have been seen to be fundamental when studying the stability of dispersive equations involving oscillation. In practice, it means that oscillations allow us to renormalize away certain nonlinear contributions because they are out of phase with the behavior that the linear equation wants to follow. Thus, for this problem, we no longer have to take advantage of the null condition, but must rather take advantage of this oscillation.

We end this Section by noting that, in both cases, we believe that our analysis will result in a large data global stability result analogous to Theorem 12. For Klein-Gordon, this would mean proving global stability for perturbations having arbitrarily large energy, while for the Einstein vacuum equations, this would mean proving global stability for perturbations having arbitrarily large mass.

Chapter 3

Stability and instability of traveling waves

3.1 Setting of the problem

We now turn to studying the stability and instability of traveling waves, which is based on work the author did together with Zbarsky (see [12]).¹ We shall first motivate the problem in the present Section and describe in more detail the mathematical obstructions present for both stability and instability. In Section 3.2, we shall then give a precise version of the main Theorems of this Chapter. Then, Section 3.3 will contain an overview of the proof of the stability Theorem, while Section 3.4 will contain an overview of the proof of the instability Theorem. We end with Section 2.4, which describes some potential directions for future study. Much of the material in this Chapter is either taken directly from or adapted from [12], where a proof can be found in full detail.

We recall the setting and rough version of the stability and instability Theorems of this Chapter from Section 1.2.2. The Theorems roughly say that a traveling wave solution to a semilinear system of wave equations satisfying the classical null condition is stable only when certain structural conditions are met regarding the traveling wave and the nonlinearity.

As was discussed in Section 1.2.2, linearizing the equations results in first order terms. First order terms for the wave equation are very worrisome from the point of view of stability, as can be

¹Zbarsky and the author are indebted to Igor Rodnianski for recommending this problem.

seen by considering the wave equation

$$\square\phi + \partial_t\phi = 0.$$

Taking an average in the spatial variables reveals that the 0 frequency of $\partial_t\phi$ experiences exponential growth. Thus, any stability result involving first order terms must avoid this phenomenon.

The renormalization described in Section 3.2.1 removes some of the first order terms, but it results in an equation where the nonlinearity depends on the background traveling wave. In order to see the difficulties this causes, we take a smooth function f of a single variable supported in the unit interval, and we examine $f(t - x)$ as a function from $\mathbb{R}^{3+1} \rightarrow \mathbb{R}$. The equation for the perturbation is then schematically

$$\square\phi = fQ(d\phi, d\phi),$$

see (3.10). Recalling the weighted vector fields (1.4), we see that they introduce weights when applied to f . Taking $x\partial_y - y\partial_x$ as a specific example, we apply this to $f(t - x)$ and we get that

$$x\partial_y - y\partial_x f(t - x) = yf'(t - x).$$

This has thus produced a weight in y , which can in general be large. Thus, having a nonlinearity whose coefficients are schematically of this form reduces our access to the usual collection of vector fields because commutation introduces these weights. This difficulty must be overcome.

For the linear instability, we are left with an equation for the perturbation whose linearization is schematically given by

$$\square\phi + f(t - x)\partial_y\phi = 0.$$

We shall use a geometric optics ansatz. The ansatz must be chosen such that growth occurs and the error can be controlled. We must ensure that both of these requirements hold and deal with the fact that the coefficients are not constant coefficient. The behavior of $f(t - x)$ is important in this procedure. Indeed, if the equation was instead

$$\square\phi + (\partial_t - \partial_x)\phi = 0,$$

we could rewrite this as

$$-(\partial_t + \partial_x)(\partial_t - \partial_x)\phi + \partial_y^2\phi + \partial_z^2\phi + (\partial_t - \partial_x)\phi = 0,$$

and a geometric optics ansatz would suppress the y and z derivatives, giving us an effective ODE that experiences exponential growth. We want to emulate this mechanism, but must deal with the different structure.

3.2 Precise version of the main Theorems

We now state precisely the main Theorems for this Chapter. In order to do so, we must first describe the equations for the perturbation of a given traveling wave more precisely. Then, in Section 3.2.1, we shall describe the renormalization procedure, which is the first step in the proof of both the stability and instability results.

We recall that we are considering a semilinear system of wave equations in $3 + 1$ dimensions satisfying the null condition. This is given by

$$\square \phi^I = Q_{JK}^I(d\phi^J, d\phi^K), \quad (3.1)$$

where the capital Latin indices run from 1 to N , denoting the different components of the equation. Let f^I be a smooth vector valued function supported in the unit interval. It is straightforward to see that $\square f^I(t - x) = 0$. Moreover, we can see that any null form Q has that $Q(d\phi^I, d\phi^I) = 0$. Thus, any function of this form is actually a solution to the equation (3.1). If we want to understand the stability of f^I , we take $f^I + \psi^I$ where ψ^I represents the perturbation. The data for ψ^I are restricted to being sufficiently small smooth functions which are supported in the unit ball in Σ_0 . The nonlinear stability result will give conditions for when the perturbation ψ decays to 0 and exists globally, whereas the linear instability result will give conditions for when the linearized equations for ψ result in growth.

We now give the stability condition.

Condition 1. *For every J and K such that $f^J \neq 0$ or $f^K \neq 0$, we have that*

$$Q_{JK}^I(dt - dx, dy) = Q_{JK}^I(dt - dx, dz) = 0.$$

When Condition 1 holds, there is still a dispersive mechanism for the linearized equations just like for the usual wave equation that we can take advantage of, and this dispersive mechanism gives us global stability. Condition 1 is satisfied for any f for certain null forms $m_{ij\ell}$, including the standard null form. However, the condition is phrased so as to allow other null forms, as long as

certain components of f are 0. Under this condition, we have the following stability result.

Theorem 13. *Suppose $f : \mathbb{R} \rightarrow \mathbb{R}^N$ is a smooth function supported on $[-1, 1]$. Moreover, suppose that the $m_{ij\ell}$ and f satisfy Condition 1. Then $f(t - x)$ is a globally asymptotically stable solution of (3.1). More precisely, there exists some $\epsilon > 0$ such that, whenever g, h are compactly supported in $B(0, 1)$ with $\|g\|_{H^3} + \|h\|_{H^2} < \epsilon$ and we are given the initial-value problem associated with the equations (3.1) and with initial data*

$$\begin{aligned}\phi|_{t=0} &= f(-x) + g \\ \partial_t \phi|_{t=0} &= f'(-x) + h\end{aligned}$$

then there is some constant C and a global in time solution $\phi = f(t - x) + \psi$ with

$$\begin{aligned}\|\partial\psi(t, \cdot)\|_{H^2} &< C(\|g\|_{H^3} + \|h\|_{H^2}) \\ |\partial\psi(t, r, \omega)| &\leq \frac{C\epsilon}{(1 + t + r)^{1-\delta}(1 + |t - r|)^{\frac{1}{2}}},\end{aligned}$$

where the constants are allowed to depend on f .

We also have a generic linear instability result when the equation does not satisfy Condition 1. We introduce the following condition for a real matrix M

Condition 2. *M has at least one eigenvalue with nonzero real part.*

$$\{A \in \mathbb{R}^{N \times N} \mid A \text{ does not satisfy Condition 2}\}$$

is lower-dimensional, so the condition is in a sense generic. At the beginning of Section 3.4, we will discuss in more detail the sense in which this condition is generic as applied to the following Theorem.

Theorem 14. *Suppose that we have an equation of the form*

$$\square \eta^I = B_y(t - x) \partial_y \eta^I + B_z(t - x) \partial_z \eta^I \tag{3.2}$$

and assume that there is some value u_0 so that some real linear combination of the matrices $B_y(u_0), B_z(u_0)$ satisfies Condition 2. Then there is some $K > 0$ so that for all $m > 0$, there is

some $c > 0$ so that for all $T > 0$, there is some solution η with

$$\|\eta(T, \cdot)\|_\infty \geq c \exp(K\sqrt{T}) \|\eta(0, \cdot)\|_{H^m}.$$

Furthermore, in the case, that $N = 1, B_z = 0, B_y = f'$ for some wave profile f , we have that we can take $K = \frac{1}{\sqrt{2}}|f|_{1/2} - \epsilon$ for any $\epsilon > 0$.

We note that a consequence of our analysis is that this $\exp(c\sqrt{t})$ growth is sharp (see Section 3.4.1).

Condition 2 may seem a bit artificial, but we shall give some intuition for it in Section 3.4.2.

3.2.1 Renormalization

We now describe the renormalization, which we shall do in a very simplified setting. We shall give two simple examples of such systems. The two examples will be different structurally, and they will show the structural difference which leads to either global nonlinear stability or linear instability. The first example is

$$\begin{aligned} \square\phi_1 &= m(d\phi_1, d\phi_2) \\ \square\phi_2 &= m(d\phi_1, d\phi_1), \end{aligned} \tag{3.3}$$

while the second example is

$$\begin{aligned} \square\phi_1 &= \partial_t\phi_1\partial_y\phi_2 - \partial_y\phi_1\partial_t\phi_2 \\ \square\phi_2 &= m(d\phi_1, d\phi_1) \end{aligned} \tag{3.4}$$

where m is the standard null form. In both cases, we note that $\phi_1 = 0$ and $\phi_2 = f(t - x)$ is a solution where $f(t - x)$ is smooth and supported when $|t - x| \leq 1$. The linearization of the first system around the background traveling wave is

$$\begin{aligned} \square\psi_1 &= 2(\partial_t + \partial_x)\psi_1 f'(t - x) \\ \square\psi_2 &= 0, \end{aligned} \tag{3.5}$$

while the nonlinear equation for the perturbation is

$$\begin{aligned} \square\psi_1 &= -2(\partial_t + \partial_x)\psi_1 f'(t - x) + m(d\psi_1, d\psi_2) \\ \square\psi_2 &= m(d\psi_1, d\psi_1). \end{aligned} \tag{3.6}$$

Meanwhile, the linearization of the second system around the background traveling wave is

$$\begin{aligned}\square\psi_1 &= -2\partial_y\psi_1 f'(t-x) \\ \square\psi_2 &= 0,\end{aligned}\tag{3.7}$$

and the nonlinear equation for the perturbation is

$$\begin{aligned}\square\psi_1 &= -2\partial_y\psi_1 f'(t-x) + \partial_t\psi_1\partial_y\psi_2 - \partial_y\psi_1\partial_t\psi_2 \\ \square\psi_2 &= m(d\psi_1, d\psi_1).\end{aligned}\tag{3.8}$$

The first example is representative of the structure present in systems where the plane wave solution is globally nonlinearly stable, while the second example is representative of the structure present where the traveling wave is linearly unstable.

The renormalization procedure takes these equations and returns a simplified equation. It turns out that the equations (3.8) are already in their renormalized forms (i.e., the renormalization is trivial). We note that we are indeed already in a setting that is covered by Theorem 14.

We thus examine the first equation (3.6). We simply take $\gamma_1 = e^{-f(t-x)}\psi_1$ and $\gamma_2 = \psi_2$. In terms of γ_1 and γ_2 , the linearized equations (3.5) take the form

$$\begin{aligned}\square\gamma_1 &= 0 \\ \square\gamma_2 &= 0.\end{aligned}\tag{3.9}$$

The nonlinear equations for the transformed quantities γ_1 and γ_2 become

$$\begin{aligned}\square\gamma_1 &= e^{-f(t-x)}m(d(e^{f(t-x)}\gamma_1), d\gamma_2) \\ \square\gamma_2 &= m(d(e^{f(t-x)}\gamma_1), d(e^{f(t-x)}\gamma_1)).\end{aligned}\tag{3.10}$$

We have thus renormalized away the first order terms, resulting in an equation whose linearization is the usual $\square$ operator. We can also easily see that the nonlinearity is no longer constant coefficient, but rather now depends on the background traveling wave. Condition 1 is exactly the condition that this renormalization procedure remove all first order terms and give $\square$ as the linearized operator.

The basic observation in this renormalization procedure is that an exponential involving f can remove the linear terms involving $(\partial_t + \partial_x)$. This is specifically because the Minkowski gradient of f is colinear with $\partial_t + \partial_x$, and because the $\square$ operator gives metric null form terms when applied to products of functions. The exponential factor for f can be thought of as solving an ODE in the $t-x$

direction whose coefficients are given in terms of f . The general procedure for a more complicated system is to solve a matrix ODE for a matrix A , and the coefficients of this matrix ODE will depend on the background traveling wave. The renormalization then comes from defining $\gamma^I = A_J^I \psi^J$. This renormalization is cooked up to remove the first order terms involving $\partial_t + \partial_x$ by cancelling them appropriately in an analogous way to the example considered above.

If there are no remaining first order terms after the renormalization, we are in a position to prove the global stability Theorem given by Theorem 13. If there are and if the coefficients satisfy the genericity condition from Theorem 14, we are in a position to prove the instability Theorem.

3.3 Description of the proof of the stability Theorem

We now describe the proof of the stability Theorem, which is Theorem 13. After the renormalization procedure, we are left with an equation that looks schematically like (3.10). For the linear part of the equation, we now have access to all of the usual commutation fields from (1.4) for $\square$. However, we must still be careful about them falling on f on the right hand side. The first two observations we use are geometric in nature. The first geometric observation is that the weights that appear from vector fields falling on f are not as bad as they first appear to be (this observation was already used in [1]). The second is that vector fields falling on f also gives an improvement. These observations are described in Section 3.3.1.

These observations will allow us to commute with weighted vector fields, but we will still have to be careful about the number of times we commute. Because of this, we cannot use the usual scheme to commute and use pointwise decay. Instead, we must also use L^p decay estimates which come in the form of L^p Klainerman-Sobolev inequalities. The proofs are relatively minor modifications of the proof of the usual Klainerman-Sobolev inequalities, and the inequalities are described in Section 3.3.2. The fact that requiring fewer derivatives could be useful in some way was described in the work [3].

Using these ingredients, we are then able to prove global stability, which we describe in Section 3.3.3.

3.3.1 Geometry

The geometric observations we shall originate with the following observation. In the equation for the perturbation η which is schematically as in (3.10), we are only afraid of weighted commutators hitting the coefficients depending on f . If they do not fall on the coefficients depending on f , they

fall on the renormalized perturbation, which is what we are controlling to begin with. Thus, we note that

$$(x\partial_y - y\partial_x)e^{-f(t-x)} = -yf'(t-x)e^{-f(t-x)}.$$

This term is of course quite worrisome because y can be of size t in the support of γ^I . However, we note that we also have the factor f' . Because this is supported where $|t-x| \leq 1$, we note that we have thus localized this error term to this region. It is thus natural to define the set S_t to be the set of points where $r \leq t+1$ and where $|t-x| \leq 1$. Because γ is supported where $r \leq t+1$ by finite speed of propagation, this represents the set of points where the background plane wave can interact with the perturbation in these terms. Because we are restricted to this set, the first observation is that the bad weights in this region are not allowed to be of size t , but are rather of size at most $\sqrt{t}$. Indeed, one can check that $y \leq C\sqrt{r}$ in the region in question, and because $r \leq t+1$, this implies that $y \leq C(1+\sqrt{t})$ in S_t . More generally, we can show that

$$|\Gamma f| \leq C(1+\sqrt{t})|f'|$$

in S_t , which is better than the a priori worst behavior of $C(1+t)$.

The second observation is that S_t leads to an improvement in the volume. In the usual situation, the region of worst interaction for a wave is in the set where roughly $t-1 \leq r \leq t+1$. We have made this set smaller by requiring in addition $|t-x| \leq 1$. If we compare the volume of the usual annular region with S_t in each Σ_t , we see that the first has volume of size t^2 while the second has volume of size t . Thus, although we have not gained any decay from the pointwise sense, we have in the integrated sense. Because the errors we encounter are integrated, we are able to use the fact that the region of integration is smaller. More generally, we have that

$$\|S_t\|_{L^p(\Sigma_t)} \leq (1+t)^{\frac{1}{p}}.$$

These observations are both very important in the proof of Theorem 13.

3.3.2 L^p Klainerman-Sobolev inequalities

Commuting additional times does not result in a further reduction in the volume. Because of this, we must still be careful about the number of times that we commute. Thus, we must make sure that are able to efficiently use the vector fields. In order to do this, we prove L^p versions of the

Klainerman-Sobolev inequalities. Specifically, we prove that

$$\|(1+t+r)^{k(n-1)/n}(1+|u|)^{k/n}h\|_{L^{1/((1/2)-k/n)}(\Sigma_t)} \leq C \sum_{|\alpha| \leq k} \|\Gamma^\alpha h\|_{L^2(\Sigma_t)}. \quad (3.11)$$

in $\mathbb{R}^{n+1}$ and for $k < \frac{n}{2}$ and h any smooth, compactly supported function.

In addition to this, we must prove L^p versions of an averaged characteristic energy estimate. This was used by Lindblad and Rodnianski in their proof of the stability of Minkowski space (see [58], [59], and [60]), and an estimate of this kind was first introduced by Alinhac in [7] where he called it a “ghost energy.”

These estimates can be shown by doing an energy estimate in the region bounded from below by Σ_0 and bounded from above by a level set of u (i.e., an outgoing cone). This gives control of the L^2 norm squared of good derivatives of the solution through the outgoing cone. Multiplying by $\frac{1}{1+|u|^{1+\delta}}$ and then integrating in u gives us that

$$\|(1+|u|)^{-\frac{1}{2}-\frac{\delta}{2}}\bar{\partial}\phi\|_{L_t^2 L_x^2}$$

is controlled by data. If we further allow ourselves to commute with weighted vector fields, this leads to the estimate

$$\|(1+|u|)^{-\frac{1}{6}-\frac{\delta}{2}}(1+t+r)^{\frac{2}{3}}\bar{\partial}\Gamma\gamma\|_{L_t^2 L_x^6}$$

in $3+1$ dimensions.

There are several other Sobolev inequalities similar to these that we need for technical reasons. These estimates give a flavor for how they look. The proofs are technical, and involve modifying the proof of the Klainerman-Sobolev inequality (see [75]). For example, (3.11) is proven using rescaled L^p Sobolev inequalities. The rescaling is different in the u and angular directions near the light cone because we control radial derivatives with a u weight and angular derivatives with an r weight in this region. In the regions far away from the light cone, we rescale with a factor of $t+r$ in every direction and use the fact that the commutation fields span the tangent space of Minkowski space in these regions.

3.3.3 Carrying out the proof of stability

We shall now briefly describe how these ingredients can be used to control error terms. In the proof, we only commute twice with weighted vector fields. If we commute the equation (3.10) twice with

weighted vector fields, and then do an energy estimate between Σ_0 and Σ_s , we see that we get an error terms of the form

$$\int_0^s \int_{\Sigma_t} (1+t) \chi_{S_t} |\partial\gamma| |\bar{\partial}\gamma| |\partial_t \Gamma^\alpha \gamma| dx dt,$$

where Γ^α contains two weighted vector fields. These error terms are actually of the worst kind (where both commutation fields fall on the coefficients). Note that we have used the fact that every weighted commutation field introduces a weight of at most $\sqrt{t}$ in the support of χ_{S_t} , which is the characteristic function of S_t . The general strategy is to put χ_{S_t} in as low an L^p space in order to take advantage of the measure and to put the other terms in as high an L^p space as possible. For the multiplier, we are essentially required to put this in $L_t^\infty L_x^2$. However, the other terms do not have any weighted commutators on them. This is because the commutators have been assumed to fall on the coefficients. Thus, we can put both of these terms in L^∞ .

Consider instead a term that is schematically given by

$$\int_0^s \int_{\Sigma_t} |\partial\Gamma\gamma| |\bar{\partial}\Gamma\gamma| |\partial_t \Gamma^\alpha \gamma| dx dt.$$

No weighted vector fields have fallen on the coefficients, meaning that this term is good in terms of t weights. However, we cannot use the usual Klainerman-Sobolev inequalities because they require losing two weighted vector fields (see (2.1)). Indeed, because Σ_t is 3 dimensional for us, we are schematically half a derivative away from being able to use a Sobolev embedding theorem. It is for terms of this kind that we use the L^p versions of the Klainerman-Sobolev inequalities. Instead of losing two weighted vector fields on a single factor in the nonlinearity, we lose one weighted vector field on each factor in the nonlinearity and use these L^p versions.

This is only a very schematic description of how to control two terms. There are several other terms, and handling them can be rather technical. However, we hope that this shows the general philosophy of how we use the geometry and L^p Klainerman-Sobolev inequalities.

3.4 Description of the proof of the instability Theorem

We now describe the proof of linear instability. Before doing so, we shall briefly discuss the sense in which Condition 2 is a genericity condition.

One may desire a statement of the form “whenever Condition 1 is not satisfied, we have that for a generic traveling wave profile, applying the transformation described in Section 3.2.1 and then linearizing gives an equation of the form (3.2) for which the conditions of Theorem 14 are

satisfied". However, that is false. In fact given any system of equations of the form (3.1) where Q_{JK}^I is antisymmetric in all three indices and has only $\partial_y \phi_j \partial_{u'} \phi_\ell - \partial_{u'} \phi_j \partial_y \phi_\ell$ terms, we have that B_y and B_z end up being antisymmetric, so any linear combination has purely imaginary eigenvalues, thus not satisfying Condition 2. More generally, we can take Q_{JK}^I to have terms antisymmetric in all three indices, plus standard null form terms. Then after applying the transformation described in Section 3.2.1, we get that B_y is similar to an antisymmetric matrix, so it still has imaginary eigenvalues. What is true is that the conditions of Theorem 14 are satisfied generically when we vary both the wave profile and the coefficients Q_{JK}^I on the right-hand side of (3.1). This can be seen from a small argument based on the fact that Condition 2 is a generic condition on matrices.

Finally, based on naive dimension-counting, we believe the following is true:

Conjecture 1. *For generic coefficients in the right-hand side of (3.1), all nonzero traveling wave profiles will satisfy the conditions of Theorem 14.*

We now describe the proof of the instability result. The argument is based on a geometric optics ansatz, which is motivated by the fact that for the equation

$$\square \eta + (\partial_t - \partial_x) \eta = 0 \tag{3.12}$$

experiences exponential growth. We want to emulate this mechanism, but things are made more difficult by the fact that the equation involves ∂_y and ∂_z derivatives instead of $\partial_t - \partial_x$, and by the fact that the coefficients depend on f . We thus take the geometric optics ansatz

$$\eta = \exp(i\mu \bar{L} \cdot \zeta) \sum_{j=0}^M \frac{\varphi_j}{(i\mu)^j} + \Pi. \tag{3.13}$$

We must now pick $\bar{L}$ appropriately, show growth for the amplitudes φ_j , and control the error Π .

3.4.1 Picking the null geodesic and controlling the ansatz error

We recall the equation

$$\square \eta + f(t-x) \partial_y \eta = 0. \tag{3.14}$$

For the equation (3.12), we can choose $\bar{L}$ to be the vector $\partial_t - \partial_x$ in (3.13). This corresponds to choosing to propagate in the $\partial_t + \partial_x$ direction. With this choice, it is easy to see that the amplitudes will grow exponentially for large times (where large is determined by the frequency μ). If we make the same choice for (3.14), the geometric optics ansatz will then suppress the ∂_y derivative. We

should thus turn the direction of propagation (and, therefore, $\underline{L}$). If we turn this too much, we run into the problem that we will exit the support of $f(t-x)$. Indeed, the only null geodesics that will remain in the support of f for all time are the integral curves of the null vector field $\partial_t + \partial_x$.

In order to balance these issues, we first fix a sufficiently large time scale T . Our goal is to show that, at time T , the solution has grown to size $\exp(K\sqrt{T})$ (see Theorem 14). We then pick L , the direction of propagation, to roughly satisfy

$$L \approx \partial_t + \left(1 - \frac{1}{T}\right) \partial_x + \frac{1}{\sqrt{T}} \partial_y.$$

This corresponds to schematically choosing

$$\bar{L} \approx \partial_t - \left(1 - \frac{1}{T}\right) \partial_x - \frac{1}{\sqrt{T}} \partial_y.$$

These vectors are not null as stated and we are ignoring constants, but the key point is the dependence on T . Because the ∂_y component is nonzero, this will excite the ∂_y derivative in (3.14). Moreover, it will only move through the support of $f(t-x)$ at a rate of $\frac{1}{T}$. Thus, we can expect the linear term $f(t-x)\partial_y\phi$ in (3.14) to contribute for time approximately T .

With this choice of $\bar{L}$ in hand, we must now control the ODE and control the error Π . Indeed, the ansatz gives several equations with various powers of the frequency μ . Matching the powers of μ gives the equations

$$\begin{aligned} 2\partial_L\varphi_0 + fL_y\varphi_0 &= 0 \\ 2\partial_L\varphi_j + fL_y\varphi_j &= -\square\varphi_{j-1} + f\partial_y\varphi_{j-1} \\ \square\Pi + f\partial_y\Pi &= \frac{\exp(i\mu\bar{L}\cdot\zeta)}{(i\mu)^M} (\square\varphi_M - f\partial_y\varphi_M) \end{aligned} \tag{3.15}$$

when applying the ansatz (3.13) to (3.14), and where L_y denotes the y component of the vector L described above.

We shall control the ODEs in Section 3.4.2, and shall describe how to control the error now.

In order to control the error Π in (3.15), we shall use energy estimates. Our goal is to show that the solution grows at the rate at most $\exp(C\sqrt{t})$. However, when doing an energy estimate with (3.14), we must deal with the bad first order terms. We thus pick a weighted multiplier to cancel this term. Let us first ignore the inhomogeneity on the right hand side of the equation for Π in (3.15). The claim is that we can take $\exp(h(t-x)\sqrt{1+t+x})(\partial_t + \partial_x)$ as a multiplier and cancel

these bad terms. This is motivated in part by the fact that we expect $\exp C\sqrt{t}$ growth. Doing so gives an effective energy estimate, but the energy estimate does not contain the $\partial_t - \partial_x$ term. However, because we control $\partial_t + \partial_x$, ∂_y , and ∂_z using this energy, we can treat $f(t-x)\partial_y\Pi$ as an inhomogeneity in the equation

$$\square\Pi = -f\partial_y\Pi.$$

We can then effectively use a usual ∂_t energy estimate on this equation.

In order to deal with the inhomogeneity that is present in (3.15), we simply use these bounds on solution operator along with the Duhamel principle. Now, in order for this term to be considered an error, it must be smaller than the amplitude φ_0 . We run into the problem that the constant C we have in the upper bound for the growth rate, $\exp(C\sqrt{t})$, is larger than the lower bound we shall get for the growth rate of φ_0 , which is $\exp(K\sqrt{t})$. In order to compensate for this, we must pick the frequency μ to be very large relative to the timescale we are trying to show growth for. Indeed, these far from optimal bounds on the error Π force us to take $\mu \approx \exp(\delta\sqrt{T})$ so that, taking M sufficiently large, the good powers of the frequency will allow us to control Π in (3.15).

3.4.2 Controlling the ODE

We finally briefly discuss how we control the ODEs for the amplitudes φ_j in (3.13). We must show upper bounds for the growth rates of all of the amplitudes because they feed into other amplitudes and into the error Π . Moreover, we must show a lower bound on the growth rate of φ_0 because this is the term which will result in growth. The bounds we get must be strong enough to show that this amplitude dominates over all of the other terms. This is true because of the μ weights present in the other terms (see 3.15).

Examining the ODEs in (3.15), we note that $L_y \approx \frac{1}{\sqrt{T}}$. Thus, we have ODEs which look like exponential growth at a rate of $\frac{1}{\sqrt{T}}$. However, the ODE's do not have constant coefficients because of the dependence on f . In order to control this, we note that integral curves of L move through the support of f at a rate of approximately $\frac{1}{T}$. Thus, we see that the coefficients should vary at a rate much slower than the solution itself because the solution should experience exponential growth at a rate of $\frac{1}{\sqrt{T}}$ while the coefficients should vary at scale $\frac{1}{T}$. Thus, the basic strategy is to compare with constant coefficient ODEs. The constant coefficient ODEs are easy to analyze, and we are able to control the error using Duhamel along with the fact that the coefficients vary so slowly. In practice, we discretize the interval $[0, T]$ on which we wish to control the ODE into intervals of length comparable to $\sqrt{T}$, and we compare with a constant coefficient ODE on each interval. We

then stitch all of the estimates together in order to get the desired results.

Condition 2 is used when proving a lower bound for the growth rate given by the ODE for φ_0 . In general, φ_0 satisfies a matrix ODE. The amplitude φ_0 can be seen to experience exponential growth at a rate that is approximately $\frac{1}{\sqrt{T}}\lambda$, where λ is the largest real part among all the eigenvalues of the matrix coefficients in the ODE. Condition 2 guarantees that λ is nonzero. Because φ_0 experiences exponential growth at a rate of roughly $\frac{1}{\sqrt{T}}$ for time T , we see that it will be of size roughly $\exp(\sqrt{T})$ at time T , as desired.

3.5 Related directions

In this Section, we shall describe problems related to the ones we have discussed in this Chapter. We shall discuss if and how the strategies followed above can be applied in these cases, and some of the additional difficulties that arise.

It is also natural to try to remove the assumptions of compact support for both the traveling wave and the perturbation. We believe that both of these assumptions in the paper are technical, and we conjecture that global nonlinear stability continues to be true as long as Condition 1 is satisfied and the traveling wave solution and perturbation each have data which decays sufficiently quickly (for the traveling wave, this means that the profile functions $f(t - x)$ have sufficiently fast decay rates away from 0, while for the perturbation, this means that the data have sufficiently fast decay rates away from the origin $r = 0$ in Σ_0). In fact, we note that in the work of Liu-Zhou in [61], they do not require compact support in the traveling wave f , and instead, they only require sufficiently fast decay.

Another natural problem is to try to extend these results to other dimensions. Because proving global stability for the trivial solution of nonlinear wave equations satisfying the null condition is much harder in $2 + 1$ dimensions, we shall first restrict ourselves to $n + 1$ dimensions with $n \geq 4$. We shall discuss a bit about $2 + 1$ dimensions after. We note that in $1 + 1$ dimensions, every solution is some kind of traveling wave, and there has been a lot of work on proving the stability of the trivial solution (see [81], [63], and [84]) and on proving general global existence results for certain equations (see [2]). We note that Shao and Zha prove global stability of suitable plane wave solutions in $1 + 1$ dimensions for any quasilinear wave equation satisfying the null condition in [16].

We believe that the analogous stability problem when Condition 1 is satisfied can be solved in $n + 1$ with $n \geq 4$ using the same strategy as in [12]. There would, however, be required modifications. One helpful fact in this case is that solutions to the wave equation decay faster in higher dimensions

(in fact, in dimensions $n + 1$ with $n \geq 4$, we recall from [47] that global stability holds for the trivial solution of general wave equations with quadratic nonlinearities). Another difference which works to your benefit is that the gain in the volume of the quantity analogous to S_t is larger. In $3 + 1$ dimensions, the volume goes from t^2 to t , which is a gain of t . In $n + 1$ dimensions, the volume goes from t^{n-1} to $t^{\frac{n-1}{2}}$, which is a gain of $t^{\frac{n-1}{2}}$. The difference which requires more care is that Sobolev embedding is worse in higher dimensions. This means that proving pointwise decay will require commuting with more vector fields (see Section 3.3.2), giving worse weights when the derivatives hit the traveling wave. From a rough calculation, we believe the gain in volume makes up for this. When Condition 1 is not satisfied but Condition 2 is satisfied, the proof we gave can be directly adapted to show linear instability.

In $2 + 1$ dimensions, the problem is harder due to the weaker decay of solutions to the linear wave equations. Indeed, in two dimensions, both quadratic and cubic terms must have special structure in order for existing proofs of global stability to work (see [43]) because general cubic nonlinearities just fail to result in sufficient decay. In $2 + 1$ dimensions, there is an instability statement analogous to Theorem 14 that can be proven in the same way as in [12]. For stability, we believe that the geometric observations described in Section 3.3.1 as well as certain aspects of the scheme used in the proof of Theorem 13 would be useful in studying stability of plane waves in $2 + 1$ dimensions. However, given the added difficulties in $2 + 1$ dimensions, there would have to be substantial modifications (see [43] and [83]).

Next, one may ask what happens when (3.1) and the traveling wave do not satisfy the conditions of either Theorem. The most interesting such case is when Q_{JK}^I in the right-hand side of (3.1) is antisymmetric, or the sum of antisymmetric terms and standard null forms (see the discussion at the beginning of Section 3.4). In that case, the geometric optics ansatz does not give growth, and we are not aware of a good construction to show instability. It is plausible that such traveling wave solutions are in fact stable, but we do not know how to prove this.

We finally mention the behavior of systems of semilinear wave equations admitting multiple traveling wave solutions which move in different directions. If we consider the system of equations

$$\begin{aligned}\square\phi_1 &= m(d\phi_2, d\phi_2) \\ \square\phi_2 &= m(d\phi_1, d\phi_2) + m(d\phi_3, d\phi_2) \\ \square\phi_3 &= m(d\phi_2, d\phi_2),\end{aligned}\tag{3.16}$$

we note that $\phi_2 = 0$, $\phi_1 = f(t - x)$, and $\phi_3 = f(t - (x + y)/\sqrt{2})$ is a solution to this system of

equations. Thus, this system admits vector valued solutions where different components of the vector are traveling waves moving in different directions. As a followup to the main results in Theorem 13 and Theorem 14, it is natural to study the stability and instability of these solutions.

The global stability of such solutions which satisfy a suitable version of Condition 1 (see below) is actually a corollary of Theorem 13. The smallness ϵ of the data will now have to depend on the background traveling wave along with the angles between the directions that each traveling wave is propagated along.

To make the setting more precise, we assume that we are given a semilinear system such as the one in (3.1), and we assume that we are given a collection of unit vectors $\omega_i^j \in S^2$ where $i \in \{1, \dots, k\}$. For each fixed i , j is allowed to range from 1 to some number N_i . The unit vectors ω_i^j represent the N_i different directions that the i th component of the background solution will propagate. If the i th component of the background solution is 0, we take $N_i = 0$ and the collection of numbers ω_i^j for that i is empty. Then, we assume that we are given a vector valued function f where the i th component f^i is a sum of functions $f^{i,j}$, each a function of $t - \omega_i^j \cdot x$ supported where $|t - \omega_i^j \cdot x| \leq 1$. We assume that f is a solution to the equation (3.1). We then have the following corollary.

Corollary 15. *As long as a suitable version of Condition 1 is satisfied in an appropriate sense, the solution f is globally nonlinearly stable under sufficiently small and smooth perturbations supported in the unit ball.*

We note that the required smallness will depend on the f^i , the ω^i , and the system of equations.

We shall first describe the appropriate version of Condition 1. Because we now have traveling waves moving in several different directions given by ω_i^j , we must assume that the null forms m_{ijl} vanish appropriately for every ω_a^b . More precisely, the appropriate version of Condition 1 is that $Q_{JK}^I(dt - d\bar{x}_a^b, d\bar{y}_a^b) = Q_{JK}^I(dt - d\bar{x}_a^b, d\bar{z}_a^b) = 0$ for every a and b where $(t, \bar{x}_a^b, \bar{y}_a^b, \bar{z}_a^b)$ is a flat coordinate system coming from a spatial rotation sending the x axis to ω_a^b .

To see that this is indeed follows from Theorem 13, we note that we can always pick $\epsilon(T)$ sufficiently small to give stability up until some fixed time T by Cauchy stability. Now, because the traveling waves are moving in different directions, their intersections will eventually exit the support of the perturbation. Thus, for sufficiently large T in terms of f and the equation, the perturbation will only ever interact with a single traveling wave at a time.

From here, the problem can be solved piece by piece, as we have reduced ourselves to a regime where the perturbation can only be influenced by traveling waves moving in a single direction at every point. Indeed, when $t \geq T$ and $|t - \omega_i^j \cdot x| \leq 1$, the equations for the perturbation look exactly

the same as the equations for the perturbation studied in Theorem 13 after a rotation sending ω_i^j to the unit vector pointing along the positive x axis. This is because the plane waves traveling in the $\omega_i^{j'}$ direction with $\omega_i^{j'} \neq \omega_i^j$ are 0 in this region. Thus, we can solve in the region between $t - \omega_i^j \cdot x = -1$ and $t - \omega_i^j \cdot x = 1$ taking the data inherited at $t = T$ in the same way as in Theorem 13. After this is done for each ω_i^j , we note that we now have inherited data on a Lipschitz surface $\mathcal{C}$ which is a union of null planes $t - \omega_i^j \cdot x = 1$ and the null cone $t = r - 1$. In this region, the equations for the perturbation become

$$\square \eta^I = Q_{JK}^I(\nabla \eta^J, \nabla \eta^K)$$

because $f = 0$ in this region. The problem can be solved in this remaining region using the bounds inherited from the previous step in the same way as Theorem 13 is established using the fact that the renormalized perturbations $\gamma^{i,j}$ coming from solving each problem in $|t - \omega_i^j \cdot x| \leq 1$ are equal to some constant times the original perturbation on $t - \omega_i^j \cdot x = 1$. This means that we have control over the appropriate quantities on $\mathcal{C}$ to solve the equation in the remaining region.

When Condition 1 is not satisfied but Condition 2 is satisfied, we are in a setting which is analogous to Theorem 14. For t sufficiently large, the traveling waves separate, and we can repeat the proof of Theorem 14 in order to prove linear instability with respect to perturbations arising from data at these later times. We conjecture that the solutions are still linearly unstable with respect to perturbations arising at $t = 0$.

Chapter 4

Stability results for anisotropic systems of wave equations

We now turn to the fourth and final Chapter of this thesis, which concerns anisotropic systems of wave equations. We recall that this Chapter contains the main Theorem of this thesis, Theorem 25. This Chapter is based on the forthcoming work [9] and [10].

4.1 Introduction

This work initiates the study of the global stability properties of anisotropic systems of wave equations. A good example to keep in mind is

$$\begin{aligned}\square\psi &= -\partial_t^2\psi + \partial_x^2\psi + \partial_y^2\psi = (\partial_t\psi)^2(\partial_t\phi) \\ \square'\phi &= -\partial_t^2\phi + \lambda_1^{-2}\partial_x^2\phi + \lambda_2^{-2}\partial_y^2\phi = (\partial_t\phi)^2(\partial_t\psi)\end{aligned}\tag{4.1}$$

in $2 + 1$ dimensions (see Theorem 25 and the discussion thereafter for a precise description of the kinds of admissible nonlinearities). We shall specifically be concerned with proving stability of the trivial solution. Because the problem is in $2 + 1$ dimensions and waves decay at a rate of $t^{-\frac{1}{2}}$, this system is critical in terms of decay. We show that, nonetheless, the trivial solution is globally stable assuming that a kind of *null condition* is present. This null condition will require that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ in the above, guaranteeing that the light cones for the equations intersect transversally. Interactions involving at least one factor of each wave will then satisfy a kind of null condition (see the discussion after Theorem 25).

This problem is motivated by trying to understand phenomena related to *birefringence*, which is associated with different waves moving at different speeds in different directions. The study of anisotropic systems of wave equations was introduced to the author by Sergiu Klainerman in the larger context of studying hyperbolic equations whose characteristics have multiple sheets (see Sections 4.1.4 and 4.1.6). The anisotropic system we study here is closely related to a system of equations that arises from applying a symmetry reduction to the equations governing the propagation of light in a *biaxial crystal* (see Section 4.1.6).

In order to prove global nonlinear stability, we shall introduce a strategy for proving estimates based on bilinear energy estimates and a duality argument. This paper also aims to describe this strategy. One of the main difficulties in this problem is that the equations do not share many symmetries, and even though both equations are in reality the $\square$ operator associated to the flat metric, the Lorentz operators associated to both equations are different. One can check that the only weighted commutator (see (4.2)) from the vector field method that can effectively be used is the scaling vector field $S = t\partial_t + r\partial_r$. Commuting with other weighted vector fields creates bad terms because, for example, the commutator adapted to ψ could hit ϕ , introducing growing weights in t .

We now briefly describe the role of bilinear energy estimates. If ψ and f are such that $\square\psi = \square f = 0$, then $\int_{\Sigma_t} \partial_t \psi \partial_t f + \partial^i \psi \partial_i f dx$ is independent of t . Thus, solutions of the homogeneous wave equation have infinitely many conservation laws because their inner products with other solutions are conserved in t . A more geometric way of saying this is that pairs of solutions to the wave equation obey bilinear spacetime integration by parts formulas, and these formulas are in fact available on general globally hyperbolic Lorentzian manifolds (see Section 4.2.1). Our strategy is, then, to prove estimates for ψ by using these bilinear estimates and making a good choice for the data for the other solution f (see Section 4.2 where this is described in more detail). We believe that this idea can be used in other settings, but we focus on using this to prove decay and stability statements in this work. The class of data chosen for f can be thought of as testing ψ against smoothed out fundamental solutions.

We require three ingredients in order to apply this strategy to global stability. The first tool consists of bilinear integration by parts formulas like the bilinear energy estimates (see Section 4.2.1). The second is decay rates for solutions to the homogeneous wave equation with a sufficiently large class of data. The third is a duality argument. We now elaborate.

1. The method is purely physical space, relying on bilinear energy estimates. Bilinear energy

estimates are well known, but we shall use them to track how much energy of the solution is present in various scale 1 balls (see Section 4.1.2). These estimates and their counterparts in general globally hyperbolic Lorentzian manifolds are discussed in Section 4.2.1.

2. The method uses decay for solutions to the homogeneous wave equation as a black box. Decay for the homogeneous wave equation may be established using whatever method one prefers. The rates must only be sufficiently strong for the application at hand.
3. Pointwise decay follows from the bilinear energy estimates and pointwise decay for solutions to the homogeneous wave equation arising from a sufficiently large class of test functions. Indeed, controlling the integral of ψ against a large class of test functions gives estimates on norms of ψ using a duality argument. We will also use the fact that we can freely commute with translation vector fields.

The proof of global stability for anisotropic systems of wave equations will require us to also use the scaling vector field $S = t\partial_t + r\partial_r$. This follows the philosophy introduced by Klainerman in [47] emphasizing the importance of taking advantage of operators that commute with the equation (see Sections 4.1.1, 4.1.4, and 4.1.6).

In practice, most of the work goes into controlling the nonlinear errors (see Section 4.2 to see how these terms appear). In order to better describe the method, we shall prove two easier stability results before turning to study anisotropic systems of wave equations. We believe that these examples will be instructive in how bilinear energy estimates can be effectively used.

The rest of the introduction consists of a discussion of existing work and how it relates to this this Chapter (Section 4.1.1), a more detailed description of the strategy used to prove decay (Section 4.1.2), a description of the first simple applications (Section 4.1.3), a description of anisotropic systems of wave equations (Section 4.1.4), and a discussion of some related directions (Section 4.1.5). We end this introductory part with a discussion in Section 4.1.6 specifically about how the study of anisotropic wave equations fits into a larger research program concerning hyperbolic equations with multiple characteristics. Then, Section 4.2 uses the bilinear estimates we have described and a duality argument to prove the decay estimates we shall use in the rest of the paper. Section 4.3 uses the decay estimates in two simple nonlinear settings. Finally, Section 4.4 proves global stability of the trivial solution for an anisotropic system of wave equations satisfying the null condition. This final section is the most substantial, and contains the main Theorem in the paper (Theorem 25).

4.1.1 Nonlinear wave equations and related results

At this point, the reader may wish to consult Section 1.1 (and, in particular, Section 1.1.2) in order to better understand the context of the problem we are considering. For ease of the reader, we recall the family of vector fields introduced by Klainerman in [47]. These are given by

$$S = t\partial_t + r\partial_r \quad \text{and} \quad \Omega_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad (4.2)$$

where x^μ represent coordinates on $n + 1$ dimensional Minkowski space. We recall that using these weighted vector fields leads to a weighted Sobolev inequality called the *Klainerman-Sobolev inequality*, which reads

$$|f|(t, r, \omega) \leq \frac{C}{(1+t+r)^{\frac{n-1}{2}}(1+|t-r|^{\frac{1}{2}})} \sum_{|\alpha| \leq \lceil n/2 \rceil} \|\Gamma^\alpha f\|_{L^2(\Sigma_t)}, \quad (4.3)$$

where Σ_t denotes the constant t hypersurface, and where Γ^α represents strings of the weighted vector fields introduced above along with the translation vector fields. This inequality allows one to effectively carry out the strategy for proving global stability that is described in Section 1.1.2. We will not have access to all of these vector fields, as was described in Section 1.2.3. Nonetheless, we do have access to the scaling vector field, and it plays a key role in studying anisotropic systems of wave equations (see the discussion in Section 1.1.2 on the vector field method). We also recall the role of the null condition for proving global stability for critical nonlinearities, including cubic nonlinearities in $2 + 1$ dimensions.

In order to overcome the fact that we do not have access to as many vector fields, we shall use a strategy based on bilinear energy estimates. This strategy has similarities with existing work, which we discuss to put it into context. One motivating idea comes from wave packets. Wave packets are often constructed by tilings in phase space which saturate the uncertainty principle, and the tilings are chosen such that the wave equation approximately becomes a transport equation on each piece for some time scale. Other constructions of wave packets are given in the work [51] of Klainerman–Rodnianski–Tao. This paper provides novel proofs of bilinear estimates which were motivated by applications to low regularity results, and it also contains two wave packet constructions different from the one described above. One of these wave packet constructions involves “two time cutoffs” in which a solution to the wave equation is truncated at both ends of a tube in order to construct a wave packet adapted to that tube. This construction provided inspiration for the strategy used in this work.

The work of Ifrim–Tataru in [36] describes how one can use wave packets to study global problems for nonlinear dispersive equations. They derive effective ODEs for inner products with wave packets, and this allows them to prove modified scattering for the solution. This philosophy of testing against other functions is also used in this work (see Section 4.2.3).

Another use of testing against wave packets is found in the paper [37] by Jeong–Oh. Unlike the other examples described above, this paper proves ill posedness for certain regimes of the Hall and electron magnetohydrodynamic equations, so this may seem out of place. However, the proof of ill posedness involves tracking inner products with wave packets solving the adjoint equation to a certain linearized equation, providing another example of the philosophy of studying an equation by testing its solutions against other functions (see Section 4.2.3).

As will be described in Section 4.4.1, a careful analysis of both the geometry of interaction (in particular the volume of interaction) and frame decompositions is important when studying anisotropic systems of wave equations (the geometry in general is important throughout the paper). We note that the author has previously worked on other problems where the same kinds of ideas have been useful. The work [11] joint with Pasqualotto studied the stability of the trivial solution to nonlinear wave equations with respect to perturbations localized around several points, while the work [12] joint with Zbarsky studied the stability and instability of plane wave solutions. Moreover, the nonlinear error integrals we encounter have similarities to those encountered by John–Klainerman in [41] where they were able to establish almost global existence results for nonlinear wave equations having quadratic nonlinearities in $3 + 1$ dimensions. This work involved using the fundamental solution for spherically symmetric solutions to the wave equation. The error from being spherically symmetric was controlled using the rotation vector fields, and this is one of the first papers where the usefulness of weighted vector fields for global problems was apparent. As was described above, the class of data we allow for the auxiliary multipliers makes them behave like smoothed out fundamental solutions, so the error integrals we must control have similarities with the integrals encountered by John–Klainerman in their work.

4.1.2 Proving decay using bilinear energy estimates

Let us now describe the strategy we will use to study the solutions to the equations in question. We shall use the notation Σ_t to denote the usual constant t spacelike hypersurfaces, and we shall denote by r the usual “spatial” radial coordinate. For this schematic discussion, we shall assume that we

want to study the solutions to the equation

$$\square\psi = F$$

in $n + 1$ dimensions. Because we are interested in proving decay for localized data, we shall assume that the data for ψ is smooth and compactly supported in the unit ball in Σ_0 . Moreover, because F is a nonlinearity involving ψ in practice, we shall take F to be an arbitrary smooth function supported where $t - r \geq -1$.

We shall use test functions and a duality argument (along with commuting the equation for ψ with translation vector fields) in order to prove statements about the solution to the equation. The goal is to show that ψ behaves like a solution to the homogeneous equation. Solutions to the homogeneous wave equation arising from sufficiently regular data supported in the unit ball (i.e., the case of $F = 0$ above) follow a specific profile. The solution is concentrated along the light cone $t = r$ in a spherically symmetric way that is consistent with conservation of ∂_t energy, where we recall that the ∂_t energy of f through Σ_t is given by

$$\frac{1}{2} \int_{\Sigma_t} (\partial_t \psi)^2 + \sum_{i=1}^n (\partial_i \psi)^2 dx.$$

An annular region $A_t \subset \Sigma_t$ of thickness 2 and inner radius $t - 1$ has volume comparable to t^{n-1} . This is precisely the region along the light cone $t = r$ in Σ_t where we expect all of the energy of the solution to be concentrated if we apply the heuristic principle that waves propagate at the speed of light. In order to be consistent with energy conservation and being almost spherically symmetric, the solution must be of size $r^{-\frac{n-1}{2}} \approx t^{-\frac{n-1}{2}}$ along the light cone. A spacelike ball $B \subset \Sigma_t$ of radius comparable to 1 whose center lies in A_t should, therefore, contain energy comparable to $t^{-(n-1)}$. This is because the ball has volume comparable to 1, and it would take roughly t^{n-1} such balls to cover the annular region. Thus, because the energy in each ball should be roughly the same, each ball should have energy roughly comparable to $t^{-(n-1)}$. See Figure 4.1 for a picture of this configuration. The fact that each such ball has energy comparable to $t^{-(n-1)}$ is usually a consequence of commuting the equation with rotation vector fields.

The motivation of the strategy used in this paper comes from trying to directly show that the energy will equidistribute among the balls in this way (every ball should roughly have energy t^{n-1}). Because this ball is of radius comparable to 1, a unit scale Sobolev embedding will then imply a pointwise decay rate of $t^{\frac{n-1}{2}}$, which is the expected rate. For applications to anisotropic systems of

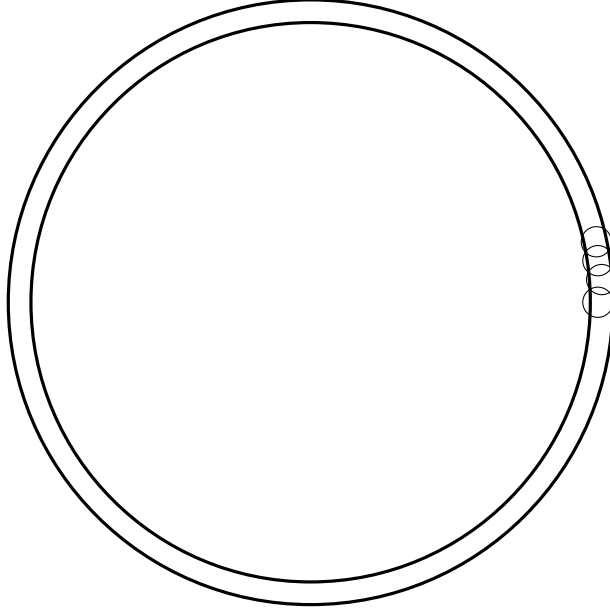


Figure 4.1: This annulus depicts the region containing most of the energy of a solution to the homogeneous wave equation in $2 + 1$ dimensions arising from smooth data supported in the unit disk in Σ_t . This annulus is a subset of Σ_t , has an outer radius of $t + 1$, and has an inner radius of $t - 1$. Thus, the annulus has area roughly t . We have covered a small portion of the annulus with disks of radius comparable to the thickness of the annulus, which is comparable to 1. It would take roughly t such disks to cover this annulus.

wave equations as in Section 4.4, this is a suitable way to show decay because we are still able to commute with translation vector fields. This procedure of showing that the energy equidistributes in this way among the balls can be thought of as replacing commuting with rotation vector fields.

In reality, the energy of the solution will not be located only along the light cone $t = r$, but it will also be present within the light cone. However, it is well known that the energy should decay away from the light cone. Thus, in addition to showing directly that the energy roughly equidistributes as if the solution was spherically symmetric, we shall also show that the energy in a unit sized ball decays as the center of the ball moves away from the light cone $t = r$. This replaces showing decay in u by commuting with weighted vector fields (although we shall have to figure out how to combine this method with commuting with the scaling vector field to study anisotropic wave equations in Section 4.4).

We now describe how bilinear estimates allow us to show that the energy of the solution behaves in this way. Suppose we are given a scale 1 ball $B \subset \Sigma_s$. We let τ denote the u coordinate of the center of this ball. Thus, the spacetime coordinates of the center of the ball B are $(s, x_0^1, \dots, x_0^n)$ where $s - \sqrt{(x_0^1)^2 + \dots + (x_0^n)^2} = \tau$. Now, suppose that we wish to show that the energy of the

solution present in this ball is of the expected size. This means that we wish to show that

$$\|\partial\psi\|_{L^2(B)}^2 \approx \frac{C}{(1+s)^{n-1}(1+|\tau|)^{2p}}, \quad (4.4)$$

where p measures the decay away from the light cone of the solution and depends on the application.

Now, let f be an arbitrary solution to the homogeneous wave equation $\square f = 0$. Because ψ solves the equation $\square\psi = F$, we note that $\square\partial^\alpha\psi = \partial^\alpha F$. The estimates we shall use are of the form

$$\begin{aligned} & \int_{\Sigma_s} (\partial_t \partial^\alpha \psi)(\partial_t f) + \sum_{i=1}^n (\partial_i \partial^\alpha \psi)(\partial_i f) dx \\ &= \int_{\Sigma_0} (\partial_t \partial^\alpha \psi)(\partial_t f) + \sum_{i=1}^n (\partial_i \partial^\alpha \psi)(\partial_i f) dx - \int_0^s \int_{\Sigma_t} (\partial^\alpha F)(\partial_t f) dx dt. \end{aligned} \quad (4.5)$$

These integration by parts identities are simply bilinear energy estimates, and we note that versions of this hold in general globally hyperbolic Lorentzian manifolds (see Section 4.2.3). When $F = 0$, we note that this identity reduces to the fact that the inner product in $\dot{H}^1$ is a conserved quantity for pairs of solutions to the homogeneous wave equation.

This identity contains a lot of freedom. Because we are interested in getting estimates for ψ , the data for ψ and F are both rigid. However, we have not specified the data for f , and this identity holds regardless of how we pick data for f . Because we are interested in proving estimates for ψ in B , it is natural to allow the data for f to be supported in, say, $2B$, which is the ball of twice the radius of B that is centered at the same point. Let us denote the trace of f in Σ_s by f_0 , and let us denote the trace of its time derivative by f_1 . The functions f_0 and f_1 can be freely specified to give a unique solution f to the homogeneous wave equation. We are thus restricting f_0 and f_1 to be supported in $2B$. In this case, the above identity becomes

$$\begin{aligned} & \int_{2B} (\partial_t \partial^\alpha \psi)(f_1) + \sum_{i=1}^n (\partial_i \partial^\alpha \psi)(\partial_i f_0) dx \\ &= \int_{\Sigma_0} (\partial_t \partial^\alpha \psi)(\partial_t f) + \sum_{i=1}^n (\partial_i \partial^\alpha \psi)(\partial_i f) dx - \int_0^s \int_{\Sigma_t} (\partial^\alpha F)(\partial_t f) dx dt. \end{aligned} \quad (4.6)$$

By this choice of data for the auxiliary multiplier f , we note that we can use a bilinear energy estimate to bound averages of derivatives of ψ against the functions f_0 and f_1 . If we allow f_0 and f_1 to vary among a suitable class of functions which are supported in $2B$, a duality argument will result in estimates for ψ and its derivatives localized to B (see Section 4.2.2). These localized estimates for ψ are what we set out to prove.

We now briefly discuss one way in which (4.6) can be used to show (4.4). If we allow for f_0 and f_1 to vary among all C^k and C^{k-1} functions, respectively, which are compactly supported in $2B$, then sufficient commutation with translation vector fields and a duality argument can lead to

$$\|\partial\psi\|_{L^2(B)} \leq C \sup_{f_0, f_1, |\alpha| \leq K(k)} \int_{2B} (\partial_t \partial^\alpha \psi) f_1 + \sum_{i=1}^n (\partial_i \partial^\alpha \psi) (\partial_i f_0) dx.$$

Thus, in order to show (4.4), it suffices to show that the right hand side of (4.6) is bounded by $\frac{C}{(1+s)^{\frac{n-1}{2}}(1+|\tau|)^p}$ for all admissible f_0, f_1 , and α .

Let us now assume that k is chosen sufficiently large so that we have pointwise decay rates for solutions to the homogeneous wave equation in that class. More precisely, we choose k so large such that we have that

$$|\partial f| \leq \frac{C}{(1+s-t)^{\frac{n-1}{2}}} \frac{1}{1+|s-t-\sqrt{(x^1-x_0^1)^2+\cdots+(x-x_0^n)^2}|^p}$$

for $t \leq s-t$. Such a rate can be proven in several different ways, and we recall that the sharp value of p depends on the dimension. For example, the Klainerman-Sobolev Inequality implies this with $p = \frac{1}{2}$ in arbitrary dimensions. As another example, in the case of $n = 2$, the sharp value of p is $\frac{3}{2}$, as can be seen using the fundamental solution.

After choosing k such that this is true, we can now examine the right hand side of (4.6). The integral over Σ_0 is controlled in terms of f , which we have information on, and the data for ψ . Because the data for ψ is compactly supported in the unit ball, an examination of the geometry involved tells us that

$$\frac{1}{(1+s-t)^{\frac{n-1}{2}}(1+|s-t-\sqrt{(x-x_0^1)^2+\cdots+(x-x_0^n)^2}|^p)}$$

is comparable to

$$\frac{1}{(1+s)^{\frac{n-1}{2}}(1+|\tau|)^p}$$

in the support of the data for ψ (see Figure 4.2 in Section 4.2.3). Thus, we see that

$$\int_{\Sigma_0} (\partial_t \partial^\alpha \psi) (\partial_t f) + \sum_{i=1}^n (\partial_i \partial^\alpha \psi) (\partial_i f) dx \leq \frac{C}{(1+s)^{\frac{n-1}{2}}(1+|\tau|)^p} \|\psi\|_{C^{|\alpha|+1}(\Sigma_0)}.$$

Thus, assuming that we can write

$$\int_0^s \int_{\Sigma_t} (\partial^\alpha F)(\partial_t f) dx dt \leq \frac{C}{(1+s)^{\frac{n-1}{2}}(1+|\tau|)^p} \|\psi\|_{C^{|\alpha|+1}(\Sigma_0)}, \quad (4.7)$$

we will have shown (4.4), which is the decay estimate we set out to prove.

These considerations are all carried out in detail in Section 4.2.3. In practice, the most difficult thing to do is of course to control the error integrals involving F . This is done in a few simple examples in Section 4.3, and it is done for a class of anisotropic systems of wave equations in Section 4.4. We note that several aspects of this strategy can be modified as needed (see Section 4.1.5 for a discussion of some of these potential modifications).

4.1.3 Simple applications

Before applying the ideas to anisotropic systems of wave equations, we shall discuss two simple applications which we believe to be instructive. The first consists of the following problem. Let χ be a smooth cutoff function equal to 1 for $|x| \leq 1$ and equal to 0 for $|x| \geq 2$. Then, working in $3+1$ dimensions, we take γ to be the null curve $(t, t, 0, 0)$ in the t, x plane, and we define $r_\gamma^2(t, x, y, z) = (x-t)^2 + y^2 + z^2$. We then take the semilinear wave equation

$$\square \phi = -\chi(r_\gamma)(\partial_t \phi)^2. \quad (4.8)$$

We have, thus, taken one of the examples studied by John in [39] (see Section 4.1.1 for more of a discussion on this) and have multiplied the nonlinearity by a function which localizes it to a tubular neighborhood of the null geodesic γ . The asymptotic system (see [35]) for this equation blows up in finite time, and yet we shall show in Section 4.3 that the trivial solution of this equation is globally nonlinearly stable. In addition to showing how the error integrals that arise when using this method can be controlled, this example shows the effect that angular localization can have when studying wave equations, a phenomenon which is present in anisotropic systems of wave equations (see Section 4.1.4). Indeed, we note that the cutoff function means that the nonlinearity only takes effect in a single angular direction from the origin. In fact, an analysis of the proof in Section 4.3 shows that we still have global stability for the equation

$$\square \phi = -\chi \left(r_\gamma \left(x, \frac{y}{(1+t)^\omega}, \frac{z}{(1+t)^\omega}, t \right) \right) (\partial_t \phi)^2$$

for any $\omega < \frac{1}{2}$.

The next simple application will be proving global stability for a cubic nonlinear wave equation in $3 + 1$ dimensions. The proof will actually give global stability and decay rates for an equation of the form

$$\square\phi = h(\partial_t\phi)^3, \quad (4.9)$$

where h is any smooth function with bounded C^k norm for k sufficiently large. This example will show how an analysis of null geometry is necessary for controlling the error integrals in a simpler setting.

In both cases, the proofs use a bootstrap argument. The bootstrap argument involves decay estimates and energy estimates. The decay estimates are shown using the bilinear energy estimates as described in Section 4.1.2. There is an energy level corresponding to N derivatives in L^2 , and we prove pointwise decay for only, say, the first $\frac{3N}{4}$ derivatives in order to close. The energy estimates are interpolated with the pointwise decay estimates in order to close the pointwise estimates. Closing the pointwise estimates requires an analysis of the geometry of the interaction between the nonlinearity and the auxiliary multipliers f described in Section 4.1.2. This is carried out in Section 4.3.

4.1.4 Applications to anisotropic systems of wave equations

Anisotropic systems of wave equations arise as the simplest model problem before studying more complicated equations having characteristics with multiple sheets. Notable physical examples of such hyperbolic equations include the equations of crystal optics (see [24]), compressible magnetohydrodynamics (see [24]), and crystalline materials (see [19]). Characteristics with multiple sheets are related to the physical phenomenon of birefringence. If we wish to understand the solutions of these equations, it is beneficial to try to study the effects of having multiple characteristics in a setting which does not require us to understand the other novel phenomena they exhibit (such as *conical refraction* in the case of a biaxial crystal in crystal optics). Thus, we study anisotropic wave equations in order to try to isolate the effects birefringence in these equations from the other phenomena they exhibit (see Section 4.1.6 for more discussion on the role of anisotropic wave equations in the larger context of studying hyperbolic equations with multiple characteristics). In fact, anisotropic systems of wave equations arise naturally in the study of *uniaxial crystals*, and also in the study of biaxial crystals under a certain symmetry reduction. We postpone a more thorough discussion of these issues and more generally of hyperbolic equations having multiply sheeted characteristics to Section 4.1.6.

For a nonlinear equation such as (4.1), one mathematical difficulty arises from the fact that the two wave equations only share the scaling symmetry. The proof of global stability will combine the ideas described in Section 4.1.2 with the vector field method. We shall only have access to the scaling vector field S . A rough version of the main Theorem is given below (see Theorem 25 for the precise statement):

Theorem 16 (rough version of the main Theorem). *The trivial solution to nonlinear wave equations of the form (4.1) is asymptotically stable as long as $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$.*

The main Theorem will actually cover equations more general than (4.1), and will allow for higher order nonlinearities, quasilinear terms, and other cubic nonlinearities (see Section 4.4). The null condition can be summarized as saying that cubic interactions do not have the same wave for all three factors, and that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$. This restriction on λ_1 and λ_2 guarantees that the light cones intersect transversally, and it will allow us to control the nonlinear interaction.

We shall now provide only a very brief description of the proof (see Section 4.4.1 for a more thorough description). The proof will once again follow from using a bootstrap argument. Pointwise estimates and L^2 based energy estimates will be propagated. There are two key ideas that must be used.

1. Controlling the error integrals will require a fine analysis of the geometry. This requires understanding the geometry of the intersections of various cones. The condition that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ will be required to prove estimates on the geometry which allow us to effectively control the nonlinear interactions.
2. The scaling vector field S must be used. It will help in getting good powers of τ when controlling the error integrals (see (4.7)). The use of this operator is fundamental to making the proof work, and this follows in the philosophy introduced by Klainerman in [47].

The details of all of these considerations are carried out in Section 4.4.

4.1.5 Related directions

We now briefly discuss some potential directions for further study related to this paper.

For the anisotropic system of wave equations studied in Section 4.4, there are several questions and improvements one may wish to pursue.

1. One natural improvement is to remove compact support and to instead have data decaying away from the origin. The estimates within the light cone from this paper should still be

applicable in this case, but they would have to be supplemented by an analysis of the geometry of the region outside of the light cones. Specifically, it seems as though the main new ingredient would be bounds on the behavior of the geometry of the scaling vector field and how it interacts with the geometry of the various cones in question in this region. We believe that the analysis in Section 4.4.5 could be useful in understanding the behavior of the scaling vector field in this region. We discuss this further in the remarks following the statement of Theorem 25.

2. Another natural direction is to investigate the problem in $3 + 1$ dimensions with a quadratic nonlinearity satisfying the analogous null condition. The analysis of the geometry becomes more complicated because of the increase in dimension. However, we believe that the techniques in this paper could be applicable, and we hope that this will be the topic of future work.
3. It is also natural to consider the problem consisting of a system of more than two wave equations with more than two different light cones. When the analogous null condition is satisfied, we believe that the techniques in this paper would be applicable. In fact, having a product of 3 waves with different light cones seems to be even better behaved, particularly when the three light cones do not intersect in the same four lines.
4. The fact that the null condition we consider suppresses most parallel interactions could lead to improved low regularity local well posedness results. This would be analogous to results concerning nonlinear wave equations satisfying the classical null condition (see, for example, [49] for the improved estimates that null forms satisfy in a low regularity setting, and for a description of how this can lead to improved local well posedness results).

The strategy of controlling the equation using bilinear estimates also has further flexibility that has not been used in this paper. An example of this flexibility is that we may control integrals over different regions using a similar strategy. Performing a bilinear energy estimate between data on Σ_0 and, say, a null hypersurface is one instance of this. The data for the auxiliary multiplier can be posed on this null hypersurface in an appropriate way. We also believe that this could be useful for studying solutions to wave equations in other contexts. Also, this kind of strategy could be used for other equations having some kind of bilinear energy structure.

Another possible use is that we can use the bilinear estimates in Section 4.2.1 in a more refined way in order to provide a finer analysis of the phase space behavior of solutions to wave equations (in order to show, for example, improved decay of good derivatives). We can also further interface

these ideas with existing techniques, such as more fully using weighted vector fields. We hope that exploiting this finer phase space analysis will be the topic of forthcoming work with Samuel Zbarsky.

4.1.6 Hyperbolic equations with characteristics having multiple sheets

The problem of studying the stability of systems of anisotropic wave equations was posed to the author by Sergiu Klainerman in the larger context of studying hyperbolic equations with multiple characteristics. In [10], Klainerman and the author discuss several interesting directions related to this program. In addition to the equations of crystal optics mentioned above, other physically relevant examples include compressible magnetohydrodynamics and the equations governing elasticity (see the work of Christodoulou [19] where he presents a general framework for these kinds of systems). In this introduction, we focus mostly on the role of anisotropic systems of wave equations in the context of studying crystal optics, but we encourage the reader to look at [10] for a much more thorough discussion.

In terms of stability, we run into obstacles when applying the vector field method in the usual way to hyperbolic systems of equations with multiple characteristics because different components in the system can have different natural operators associated to them. However, there are still some operators present for anisotropic systems of wave equations, namely the translation vector fields and the scaling vector field. The strategy of using bilinear energy estimates can interface with commuting operators from the vector field method, and we in fact have to use the scaling vector field in a fundamental way in the proof of the main Theorem in Section 4.4. This more broadly follows the philosophy introduced by Klainerman in [47] which emphasizes the importance of being able to take advantage of good commutators (and in particular weighted commutators for decay). Moreover, as is discussed in [10], physically significant hyperbolic systems with multiple characteristics (such as crystal optics) still often exhibit a compatible scaling symmetry. Thus, the fact that anisotropic systems of wave equations still have access to the scaling vector field is representative of interesting physical situations.

As is also discussed in [10], it is natural to look for other operators that satisfy good commutation properties with equations exhibiting this birefringent behavior. This is an important part of the larger goal of robustly extending the vector field method to these kinds of equations. Similar to vector fields, we believe that these operators could be very valuable in this more exotic context. These operators require much study themselves, and are a key part of providing a fine tuned analysis of the linear equations of, say, crystal optics. These operators may no longer be vector fields, but

may instead be, for example, second order operators. We note that second order operators were used by Andersson–Blue in [13] in the study of the wave equation on black hole spacetimes.

The system (4.1) admits such an operator. We now describe Klainerman’s procedure in this particular case. If we look at the symbols of $\square$ and $\square'$ by taking the spacetime Fourier transform, they are given by $\tau^2 - |\xi|^2$ and $\tau^2 - \lambda_1^{-2}\xi_1^2 - \lambda_2^{-2}\xi_2^2$. Level sets of these two functions are hyperboloids in frequency space. These hyperboloids are two dimensional in a three dimensional space, and they intersect transversally along families of curves. Because the symbol acts by multiplication, vector fields in frequency space which are tangent to level sets of the symbol annihilate the symbol, and thus commute with the equation because they annihilate the symbol. We can thus find an operator that commutes with both equations by looking at the vector field generated by the family of curves formed by intersecting these level sets. This vector field in frequency space can essentially be found by taking the cross product between the normal vector fields of the two families of hyperboloids. One can check that this results in the operator

$$-\Lambda_1\tau\xi_1\partial_{\xi_2} - \Lambda_2\tau\xi_2\partial_{\xi_1} - \Lambda_3\xi_1\xi_2\partial_\tau,$$

subject to the constraints that $\Lambda_1 + \Lambda_2 - \Lambda_3 = 0$ and $\lambda_2^{-2}\Lambda_1 + \lambda_1^{-2}\Lambda_2 - \Lambda_3 = 0$ (of course, multiplying this operator by any constant still commutes with both equations as well). This vector field in frequency space can be seen to correspond to the second order differential operator

$$\Lambda_1 y \partial_t \partial_x + \Lambda_2 x \partial_t \partial_y + \Lambda_3 t \partial_x \partial_y$$

in physical space. We can then hope to use operators of this kind to further study solutions to anisotropic systems of wave equations, and we can look for similar kinds of operators in other systems. For example, the weights in these operators lead to weighted Sobolev inequalities, similar to how the weights in the Lorentz and scaling vector fields lead to the weighted Klainerman-Sobolev inequality (see (4.2) and (4.3)). We refer the reader to [10] for a much more thorough discussion of this circle of ideas. For now, we just note that we hope to further explore uses of these operators together with Klainerman.

We end this Section with a more precise description of the relation between anisotropic systems of wave equations and crystal optics. These are a simpler counterpart to biaxial crystals. While they still exhibit birefringent behavior, they do not display other complicated phenomenon, such as conical refraction.

In the general case, it may be that the electrical and magnetic properties of the media themselves change as a function of the electric and magnetic fields. This corresponds to having the *dielectric tensor* ϵ and *magnetic permeability* μ can depend on the electric and magnetic fields. This leads to a nonlinear system of equations. For this discussion, we shall greatly simplify the situation by considering a constant, diagonal matrix ϵ and a constant μ . More precisely, we take the dielectric tensor ϵ to be given by

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}.$$

The three constants ϵ_i in this matrix are called the *dielectric constants*. When they are all equal, we are left with the usual Maxwell equations governing the propagation of light in an isotropic media. When two of the constants are equal to each other but distinct from the third, the resulting equations then govern the propagation of light in a uniaxial crystal. Finally, when all three of the constants are distinct from each other, the resulting equations govern the propagation of light in a biaxial crystal. The magnetic permeability μ is taken to be a multiple of the identity, corresponding to a magnetically isotropic material. This can be thought to be 1 for simplicity (see [24], [57], [56], and [10] for a more thorough discussion).

Let us now write down the equations in the most general form. We shall denote by E the electric field and B the magnetic field. Thus, in $3 + 1$ dimensions, both the electric and magnetic fields have 3 components. The Maxwell equations are then given by

$$\begin{aligned} -\partial_t(\epsilon E) + \text{curl}(B) &= 0 \\ -\partial_t(\mu B) - \text{curl}(E) &= 0 \\ \text{div}(\epsilon E) = \text{div}(\mu B) &= 0. \end{aligned} \tag{4.10}$$

It is easy to see that the last set of equation can be thought of as constraints because they propagate in the evolutionary equations as long as they hold initially.

Commuting the evolutionary equations in (4.10) with ∂_t , expanding them out in components,

and using the equations to get rid of the magnetic field B gives us the system of equations

$$\begin{aligned}
\epsilon_1 \partial_t^2 E^1 &= \partial_y^2 E^1 + \partial_z^2 E^2 - \partial_x \partial_y E^2 - \partial_x \partial_z E^3 \\
\epsilon_2 \partial_t^2 E^2 &= \partial_z^2 E^2 + \partial_x^2 E^2 - \partial_y \partial_z E^3 - \partial_y \partial_x E^1 \\
\epsilon_3 \partial_t^2 E^3 &= \partial_x^2 E^3 + \partial_y^2 E^3 - \partial_z \partial_x E^1 - \partial_z \partial_y E^2 \\
\epsilon_1 \partial_x E^1 + \epsilon_2 \partial_y E^2 + \epsilon_3 \partial_z E^3 &= 0.
\end{aligned} \tag{4.11}$$

Let us now make the symmetry assumption that the electric field does not depend on z , meaning that all ∂_z derivatives vanish. This results in the system of equations

$$\begin{aligned}
\epsilon_1 \partial_t^2 E^1 &= \partial_y^2 E^1 - \partial_x \partial_y E^2 \\
\epsilon_2 \partial_t^2 E^2 &= \partial_x^2 E^2 - \partial_y \partial_x E^1 \\
\epsilon_3 \partial_t^2 E^3 &= \partial_x^2 E^3 + \partial_y^2 E^3 \\
\epsilon_1 \partial_x E^1 + \epsilon_2 \partial_y E^2 &= 0.
\end{aligned} \tag{4.12}$$

We can now use the divergence free condition, which we know propagates in the evolution as long as it holds initially, to write the equations as

$$\begin{aligned}
-\partial_t^2 E^1 + \epsilon_2^{-1} \partial_x^2 E^1 + \epsilon_1^{-1} \partial_y^2 E^1 &= 0 \\
-\partial_t^2 E^2 + \epsilon_2^{-1} \partial_x^2 E^2 + \epsilon_1^{-1} \partial_y^2 E^2 &= 0 \\
-\partial_t^2 E^3 + \epsilon_3^{-1} \partial_x^2 E^3 + \epsilon_3^{-1} \partial_y^2 E^3 &= 0 \\
\epsilon_1 \partial_x E^1 + \epsilon_2 \partial_y E^2 &= 0.
\end{aligned} \tag{4.13}$$

The evolutionary equations listed here can be compared with (4.1), and we see that they are of a similar form. Indeed, we have a system of three equations. One equation (the one for E^3) has circular light cones while the other two equations (the ones for E^1 and E^2) have elliptical light cones. Thus, the equations we study in Section 4.4 are related to this symmetry reduction for biaxial crystals. Moreover, we can easily see that the null condition that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ is analogous to the condition that the ϵ_i all be distinct, meaning that the crystal is biaxial.¹ When the crystal is uniaxial, we see that there are two possibilities. If $\epsilon_1 = \epsilon_2$, the equations become an isotropic system of wave equations with multiple speeds. These kinds of equations can be treated using the techniques introduced by Klainerman–Sideris in [52] (see also [75]). If $\epsilon_1 = \epsilon_3$ or $\epsilon_2 = \epsilon_3$,

¹This is the first half of the null condition we need to prove global stability. It would be interesting to see if the second half, that the same wave not be cubed, corresponds to some physical property of the material.

the equations become anisotropic, but it is like the situation in which one of λ_1 or λ_2 is equal to 1.

Although the results in Section 4.4 thus seem to be relevant for the study of biaxial crystals under this symmetry reduction, we emphasize that the problem outside of symmetry is much more difficult and requires a much finer analysis. This symmetry reduction has made the equations for each component of the electric field decouple, which does not occur in the general case of a biaxial crystal. This decoupling removes many of the interesting difficulties present in biaxial crystals, such as conical refraction. The first step in analyzing these more complicated phenomena was undertaken by Liess in [57], resulting in a global stability statement in [56] arising from using machinery similar to that introduced by Klainerman–Ponce in [50]. The linear analysis provided by Liess in [57] shows that the situation is much more difficult for these kinds of systems than for the usual wave equation. The stability result in [56] does not cover the class of nonlinearities considered in this paper, and in particular, it does not exploit any kind of null condition (it treats fourth order and higher nonlinearities). To our knowledge, this is the only global stability result for a biaxial crystal outside of symmetry assumptions. Providing sharper global stability statements requires a finer analysis. We believe this to be an extremely interesting direction for future work, and we refer the reader once again to [10] for a more thorough discussion.

4.2 Decay estimates

In this section, we shall prove the estimates that will lead to decay. Decay comes from bilinear integration by parts formulas which show that certain quantities are conserved at the linear level. The energy estimate is a special case of these estimates. These bilinear estimates are simple, but they are quite general, with versions being true on general Lorentzian manifolds. We record these geometric estimates in Section 4.2.1. In this work, when we are interested in controlling ψ where $\square\psi = F$, the estimates are applied by taking an auxiliary solution of the homogeneous wave equation $\square f = 0$ and using this as a multiplier for ψ .

To go from these bilinear estimates to pointwise decay statements, we use a duality argument. Controlling integral averages of a function ψ against a sufficiently large class of test functions results in estimates on various norms for ψ . Thus, by allowing the data to vary appropriately for f , we prove that ψ decays pointwise through a duality argument. The duality argument is described in Section 4.2.2, and the duality argument and bilinear estimates are combined to prove pointwise estimates in Section 4.2.3. The decay rates this gives depend on decay rates for the homogeneous wave equation. Decay rates for solutions to the homogeneous wave equation can be taken as a black

box, and can be proven in whatever way one prefers.

4.2.1 Basic bilinear estimates

We begin with the estimates on Minkowski space $\mathbb{R}^{n+1}$. The first is a bilinear energy estimate and the second comes from integrating $\phi \square \psi$ by parts. In this work, we shall only use estimates between two time slices Σ_{t_1} and Σ_{t_2} , but we note that we are of course free to do estimates in regions with different geometry (such as, for example, null hypersurfaces). This is expressed in the more geometric versions of these integration by parts formulas below which simply show that we can integrate appropriate divergence and use Stokes' Theorem.

Lemma 17. *Let ψ and ϕ be smooth functions decaying sufficiently rapidly at infinity with $\square \psi = F$ and with $\square \phi = G$. Then, we have that*

$$\begin{aligned} & \int_{\Sigma_{t_2}} \partial_t \psi \partial_t \phi + \partial^i \psi \partial_i \phi dx + \int_{t_1}^{t_2} \int_{\Sigma_s} F \partial_t \phi dx ds \\ &= \int_{\Sigma_{t_1}} \partial_t \psi \partial_t \phi + \partial^i \psi \partial_i \phi dx - \int_{t_1}^{t_2} \int_{\Sigma_s} G \partial_t \psi dx ds. \end{aligned} \quad (4.14)$$

More generally, we have that

$$\begin{aligned} & \int_{\Sigma_{t_2}} \partial_t \partial^\alpha \psi \partial_t \phi + \partial^i \partial^\alpha \psi \partial_i \phi dx + \int_{t_1}^{t_2} \int_{\Sigma_s} \partial^\alpha F \partial_t \phi dx ds \\ &= \int_{\Sigma_{t_1}} \partial_t \partial^\alpha \psi \partial_t \phi + \partial^i \partial^\alpha \psi \partial_i \phi dx - \int_{t_1}^{t_2} \int_{\Sigma_s} G \partial^\alpha \partial_t \psi dx ds. \end{aligned} \quad (4.15)$$

Proof. We compute the following:

$$\begin{aligned} \square \psi \partial_t \phi &= -\partial_t^2 \psi \partial_t \phi + \partial_i \partial^i \psi \partial_t \phi = -\partial_t (\partial_t \psi \partial_t \phi) + \partial_t \psi \partial_t^2 \phi + \partial_i (\partial^i \psi \partial_t \phi) - \partial^i \psi \partial_i \partial_t \phi \\ &= \partial_t (-\partial_t \psi \partial_t \phi) + \partial_t \psi \partial_t^2 \phi + \partial_i (\partial^i \psi \partial_t \phi) - \partial_t (\partial^i \psi \partial_i \phi) + \partial_t \partial^i \psi \partial_i \phi \\ &= \partial_t (-\partial_t \psi \partial_t \phi) + \partial_t \psi \partial_t^2 \phi + \partial_i (\partial^i \psi \partial_t \phi) - \partial_t (\partial^i \psi \partial_i \phi) + \partial^i (\partial_t \psi \partial_i \phi) - \partial_t \psi \partial^i \partial_i \phi \\ &= \partial_t (-\partial_t \psi \partial_t \phi) + \partial_t (-\partial^i \psi \partial_i \phi) + \partial_i (\partial^i \psi \partial_t \phi) + \partial^i (\partial_t \psi \partial_i \phi) - \partial_t \psi \square \phi. \end{aligned} \quad (4.16)$$

Integrating between Σ_{t_1} and Σ_{t_2} gives us that

$$\begin{aligned} & \int_{\Sigma_{t_2}} \square \psi \partial_t \phi dx - \int_{\Sigma_{t_1}} \square \psi \partial_t \phi dx \\ &= \int_{\Sigma_{t_2}} -\partial_t \psi \partial_t \phi - \partial^i \psi \partial_i \psi dx - \int_{\Sigma_{t_1}} -\partial_t \psi \partial_t \phi - \partial^i \psi \partial_i \psi dx - \int_{t_1}^{t_2} \int_{\Sigma_s} \partial_t \psi \square \phi dx ds. \end{aligned} \quad (4.17)$$

Using the fact that $\square\psi = F$ and that $\square\phi = G$ gives us the first estimate. The second estimate comes from commuting the equation for ψ with ∂^α . $\square$

Lemma 18. *Let ψ and ϕ be smooth functions decaying sufficiently rapidly at infinity with $\square\psi = \square\phi = 0$. Then, we have that*

$$\begin{aligned} & \int_{\Sigma_{t_2}} \partial_t \psi \phi - \psi \partial_t \phi dx + \int_{t_1}^{t_2} F \phi dx ds \\ &= \int_{\Sigma_{t_1}} \partial_t \psi \phi - \psi \partial_t \phi dx + \int_{t_1}^{t_2} G \psi dx ds. \end{aligned} \quad (4.18)$$

More generally, we have that

$$\begin{aligned} & \int_{\Sigma_{t_2}} \partial_t \partial^\alpha \psi \phi - \partial^\alpha \psi \partial_t \phi dx + \int_{t_1}^{t_2} \partial^\alpha F \phi dx ds \\ &= \int_{\Sigma_{t_1}} \partial_t \partial^\alpha \psi \phi - \partial^\alpha \psi \partial_t \phi dx + \int_{t_1}^{t_2} G \partial^\alpha \psi dx ds. \end{aligned} \quad (4.19)$$

Proof. We compute the following:

$$\begin{aligned} \square\psi\phi &= -\partial_t^2 \psi \phi + \partial_i \partial^i \psi \phi = -\partial_t(\partial_t \psi \phi) + \partial_t \psi \partial_t \phi + \partial_i(\partial^i \psi \phi) - \partial^i \psi \partial_i \phi \\ &= -\partial_t(\partial_t \psi \phi) + \partial_t(\psi \partial_t \phi) - \psi \partial_t^2 \phi + \partial_i(\partial^i \psi \phi) - \partial^i(\psi \partial_i \phi) + \psi \partial^i \partial_i \phi \\ &= -\partial_t(\partial_t \psi \phi) + \partial_t(\psi \partial_t \phi) + \psi \square\phi + \partial_i(\partial^i \psi \phi) - \partial^i(\psi \partial_i \phi). \end{aligned} \quad (4.20)$$

Integrating between Σ_{t_1} and Σ_{t_2} gives us that

$$\int_{t_1}^{t_2} \int_{\Sigma_s} \square\psi\phi dx ds = \int_{\Sigma_{t_2}} -\partial_t \psi \phi + \psi \partial_t \phi dx - \int_{\Sigma_{t_1}} -\partial_t \psi \phi + \psi \partial_t \phi dx + \int_{t_1}^{t_2} \int_{\Sigma_s} \psi \square\phi dx ds. \quad (4.21)$$

Using the fact that $\square\psi = F$ and $\square\phi = G$ gives us the desired result. $\square$

In order to more accurately show the geometric character of these identities, we provide versions of these formulas which are available on arbitrary globally hyperbolic Lorentzian manifolds (M, g) .

Lemma 19. *Let D be a domain in M . Moreover, let $\square_g \psi = F$ and let $\square_g \phi = G$. We have that*

$$\int_D \operatorname{div}(\nabla \psi \phi) - \operatorname{div}(\nabla \phi \psi) d\operatorname{vol}(g) = \int_D \phi F - \psi G d\operatorname{vol}(g).$$

Proof. We can write the wave equation $\square_g \psi = F$ as $\operatorname{div}(\nabla \psi) = \operatorname{tr}(\nabla^2 \psi) = F$, where ∇ is the

covariant derivative operator associated with the Lorentzian metric g . Now, we have that

$$\operatorname{div}(\nabla\psi\phi) = \phi\operatorname{div}(\nabla\psi) + g(d\psi, d\phi) = \phi\Box_g\psi + g(d\psi, d\phi). \quad (4.22)$$

Similarly, we have that

$$\operatorname{div}(\nabla\phi\psi) = \psi\operatorname{div}(\nabla\phi) + g(d\psi, d\phi) = \psi\Box_g\phi + g(d\psi, d\phi) \quad (4.23)$$

Subtracting the second equation from the first gives us that

$$\phi\Box_g\psi - \psi\Box_g\phi = \operatorname{div}(\nabla\psi\phi) - \operatorname{div}(\nabla\phi\psi). \quad (4.24)$$

This gives us the desired result. $\square$

Integrating this identity and using the divergence theorem can give us fluxes analogous to Lemma 18 on a general Lorentzian manifold.

Lemma 20. *Let $T[\psi, \phi]$ denote the bilinear energy momentum tensor given by*

$$T[\psi, \phi] = d\psi \otimes d\phi + d\phi \otimes d\psi - g(d\psi, d\phi)g. \quad (4.25)$$

Then, we have that

$$\operatorname{div}(T[\psi, \phi]) = (\Box\psi)d\phi + (\Box\phi)d\psi.$$

Proof. We begin by noting that $T[\psi, \phi]$ is symmetric, so the divergence is well defined. This can be written as

$$T_{\mu\nu}[\psi, \phi] = \partial_\mu\psi\partial_\nu\phi + \partial_\nu\psi\partial_\mu\phi - g_{\alpha\beta}\partial^\alpha\psi\partial^\beta\phi g_{\mu\nu}. \quad (4.26)$$

Computing the divergence means contracting with ∇^μ , where ∇ is the connection associated with g . We have that

$$\begin{aligned} \nabla^\mu T_{\mu\nu}[\psi, \phi] &= (\nabla^\mu\partial_\mu\psi)\partial_\nu\phi + \partial_\mu\psi\nabla^\mu\partial_\nu\phi + \nabla^\mu\partial_\nu\psi\partial_\mu\phi + \partial_\nu\psi(\nabla^\mu\partial_\mu\phi) \\ &\quad - g_{\alpha\beta}\nabla^\mu\partial^\alpha\psi\partial^\beta\phi g_{\mu\nu} - g_{\alpha\beta}\partial^\alpha\psi\nabla^\mu\partial^\beta\phi g_{\mu\nu}. \end{aligned} \quad (4.27)$$

Now, we note that $\partial_\mu\psi\nabla^\mu\partial_\nu\phi = g_{\alpha\beta}\partial^\alpha\psi\nabla^\mu\partial^\beta\phi g_{\mu\nu} = \nabla_{\mu\nu}^2\phi\partial^\mu\psi = (\nabla^2\phi)(\nabla\psi, \cdot)$, where we have used the fact that the Hessian of a function is symmetric (we have also used the musical isomorphism induced by the metric between covariant and contravariant tensors). We have an analogous equality

for the other terms with ψ and ϕ interchanged. Putting everything together gives us that

$$\nabla^\mu T_{\mu\nu}[\psi, \phi] = (\nabla^\mu \partial_\mu \psi) \partial_\nu \phi + (\nabla^\mu \partial_\mu \phi) \partial_\nu \psi.$$

This is exactly the statement that

$$\text{div}(T[\psi, \phi]) = (\Box \psi) d\phi + (\Box \phi) d\psi,$$

as desired. $\square$

Because the divergence of $T[\psi, \phi]$ can be written in terms of $\Box_g \psi$ and $\Box_g \phi$, it can be contracted with appropriate vector fields to find useful identities after integrating over a spacetime domain and applying Stokes' Theorem, just as is the case with the usual energy momentum tensor $\frac{1}{2}T[\psi, \psi]$. The resulting identities will then also depend on the deformation tensors of the vector fields. For example, the identities in Lemma 17 which are specific to Minkowski space correspond to contracting this bilinear energy momentum tensor with the vector field ∂_t , which is a Killing field (i.e., it has a vanishing deformation tensor). We refer the reader to, for example, [8] for more thorough description of integrating the contraction of energy momentum tensors with vector fields in order to derive useful quantities.

4.2.2 Duality argument

The estimates from Section 4.2.1 lead us to think that we should be able to control the averages of ψ and its derivatives weighted by functions f as long as we can control the error integrals involving F in (4.5). Moreover, because we can commute with unit derivatives in practice, the same is true of $\partial^\alpha \psi$. If f is allowed to vary in a sufficiently large class and if $|\alpha|$ is allowed to be sufficiently large, it is well known that control over these averages will imply bounds for ψ and some of its derivatives. Because we require pointwise decay, we have chosen to immediately prove pointwise estimates using these averages. Of course, these averages can be used to prove estimates for other L^p spaces, including the energy space. Thus, we could modify the following argument to first prove an energy bound, and then prove pointwise decay using the usual Sobolev embedding.

We shall now provide a proof of this duality argument for completeness. Although this result is far from sharp, it suffices for our applications. We assume that we are on some Σ_t , and that we have that $\left| \int_{\Sigma_t} \partial^\alpha \psi f \right| dx \leq M$ where M is some explicit constant. Moreover, we assume that ψ is some smooth function, and that f is allowed to be any function with C^k norm bounded by another

constant D . Our goal is to show that we, in fact, have control over $\|\psi\|_{L^\infty}$ explicitly in terms of the constants M and D as long as we can take all ω with $|\omega|$ sufficiently large depending on k .

Proposition 21. *Let ψ be some smooth function on $\mathbb{R}^n$ such that, for any smooth function f supported in any ball of radius 1 having C^k norm at most D , we have that*

$$\left| \int \partial^\alpha \psi(x) f(x) dx \right| \leq M \quad (4.28)$$

for all $|\alpha| \leq k + n + 1$. Then, there exists some universal constant C depending only on k and n such that

$$\|\psi\|_{L^\infty} \leq \frac{CM}{D}. \quad (4.29)$$

Proof. It suffices to get control over ψ near the origin, as control over any other location follows in the same way. We fix a smooth, radially symmetric cutoff function χ equal to 1 in the ball of radius $\frac{1}{2}$, decaying monotonically to 0. The function is identically equal to 0 outside of the ball of radius 1. Let $\|\chi\|_{C^{2k+n+1}} = E$. Then, we have that $\frac{1}{E}\chi$ has C^{2k+n+1} norm controlled by 1.

Now, we take the cube centered at the origin with side lengths equal to 10. We note that this cube contains the support of χ . By identifying the faces of this cube in the usual way, we can think of the function $\chi\psi$ as living on the flat torus. We take the functions $e^{\frac{1}{5}\pi l \cdot x}$ for $l \in \mathbb{Z}^n$. Integrating against these functions give us the Fourier coefficients of a given function.

The bounds on integrals against test functions are not directly useful when integrating against the functions $e^{\frac{1}{5}\pi l \cdot x}$ because they have growing C^k norm as $\|l\|_{\ell^\infty}$ increases. Thus, we divide by $\|l\|_{\ell^\infty}^k$, giving us the functions $\frac{1}{\|l\|_{\ell^\infty}^k} e^{\frac{1}{5}\pi l \cdot x}$. These functions have C^k norm controlled by some constant A independent of l . Thus, multiplying by D and dividing by A , we get that the functions $f_l(x) := \frac{D}{A} \frac{1}{\|l\|_{\ell^\infty}^k} e^{\frac{1}{5}\pi l \cdot x}$ have C^k norm controlled by D . We now consider the Fourier coefficients of the function $\chi\psi$. We shall use the regularity of this function to show that its Fourier coefficients decay by integrating against the functions f_l .

By the assumptions given above, we have that

$$\frac{D}{EA\|l\|_{\ell^\infty}^k} |c_l| = \left| \int \frac{1}{E} \chi(x) \psi(x) f_l(x) dx \right| \leq MC_1, \quad (4.30)$$

where c_l is the Fourier coefficient of $\chi\psi$ associated to l , and where C_1 depends on k (this constant arises from controlling the C^k norm of $\frac{1}{E}\chi f_l$ in terms of the C^k norm of χ and the C^k norm of f_l).

Now, we have that

$$\left(\frac{\pi}{5}\right)^{|\alpha|} \frac{\|l\|_{\ell^\infty}^{|\alpha|} D}{EA \|l\|_{\ell^\infty}^k} |c_l| = \left| \int \frac{1}{E} \partial^\alpha (\chi(x) \psi(x)) f_l(x) dx \right| \leq MC_2, \quad (4.31)$$

where C_2 depends on $|\alpha|$ and k , and where the partial derivatives in ∂^α are all taken in the direction corresponding to the largest component of l . Taking $|\alpha| = k + n + 1$ gives us that

$$\left(\frac{\pi}{5}\right)^{|\alpha|} \|l\|_{\ell^\infty}^{n+1} \frac{D}{EA} |c_l| = \left| \int \frac{1}{E} \partial^\alpha (\chi(x) \psi(x)) f_l(x) dx \right| \leq MC''. \quad (4.32)$$

This gives us that

$$|c_l| \leq C \frac{MEA}{D \|l\|_{\ell^\infty}^{n+1}}, \quad (4.33)$$

where the constant C depends on the C^k norms of the f_l (we recall that this is A , which is independent of l), and the numbers k and $|\alpha| = k + n + 1$. Now, we have that

$$\chi(x) \psi(x) = \sum_{l \in \mathbb{Z}^n} c_l e^{\frac{1}{5} \pi l \cdot x}. \quad (4.34)$$

Plugging in the above estimates gives us the quantitative estimate

$$|\chi(x) \psi(x)| \leq \sum_{l \in \mathbb{Z}^n} |c_l| \leq \frac{CM}{D}, \quad (4.35)$$

where C is some absolute constant depending only on n , k , and the functions χ and f_l . This is the desired result. $\square$

4.2.3 Decay estimates

We now proceed to one specific application of the above considerations. We shall use the bilinear estimates in Minkowski space from Section 4.2.1 and the duality argument from Section 4.2.2 to prove pointwise decay for solutions to wave equations. We shall include inhomogeneities in the estimates. In our applications, these inhomogeneities are nonlinearities.

Let $\square \psi = F$, and suppose that ψ has compactly supported data in the unit ball. Suppose that we want to show pointwise decay for ψ . We then take auxiliary solutions of the homogeneous wave equation $\square f = 0$, and decay will be a result of applying the bilinear estimates from Section 4.2.1 to ψ and f , allowing the data for f to vary in an appropriate class, and then applying the duality argument from Section 4.2.2.

The data for f is chosen as follows. Suppose we want to show decay for ψ at some point (s, x_0^i) . We then consider the ball B of radius 1 centered at x_0^i in Σ_s , and we allow the data $f_0(x) = f(s, x)$ and $f_1(x) = -\partial_t f(s, x)$ for f to vary among all smooth functions whose C^k norm (where k will be specified shortly) is of size at most 10 and which are supported in $B \subset \Sigma_s$. More specifically, we have chosen the nonlinearities to only involve ∂_t derivatives for simplicity, so we shall always set $f_0 = 0$. If other derivatives are involved instead, this argument can be modified to treat those cases.

The estimate from Lemma 17 says that

$$\int_{\Sigma_s} \partial_t \psi f_1 dx = \int_{\Sigma_0} \partial_t \psi \partial_t f + \partial^i \psi \partial_i f dx - \int_0^s \int_{\Sigma_t} F \partial_t f dx dt.$$

Thus, we have that

$$\left| \int_{\Sigma_s} \partial_t \psi f_1 dx \right| \leq \left| \int_{\Sigma_0} \partial_t \psi \partial_t f + \partial^i \psi \partial_i f dx \right| + \left| \int_0^s \int_{\Sigma_t} F \partial_t f dx dt \right|$$

We denote by τ the u coordinate of (s, x_0^i) (i.e., $\tau = s - \sqrt{\sum_{i=1}^n (x_0^i)^2}$). In practice, we know that solutions to the homogeneous wave equation with compactly supported and sufficiently regular data (like f) decay in a certain way. Let us thus assume that k is chosen so large such that

$$|\partial f| \leq \frac{C}{(1+s-t)^{\frac{n-1}{2}}} \frac{1}{1 + |s-t - \sqrt{(x^1-x_0^1)^2 + \dots + (x-x_0^n)^2}|^p}$$

for $0 \leq t \leq s$. For example, when $n = 2$, we can take $k = 2$ and $p = \frac{3}{2}$ using the fundamental solution (this is of course not sharp).

As a consequence of this, we have that

$$\left| \int_{\Sigma_0} \partial_t \psi \partial_t f + \partial^i \psi \partial_i f dx \right| \leq \frac{C}{(1+s)^{\frac{n-1}{2}} (1+|\tau|^p)} \|\partial \psi\|_{L^1(\Sigma_0)}, \quad (4.36)$$

where $p > 0$ measures the decay away from the light cone. Here, we are using the fact that the data for ψ is compactly supported in the unit ball in Σ_0 . Indeed, the symmetry of the configuration guarantees that the distance from the center of the unit ball in Σ_0 to the light cone associated with f is comparable to τ , and clearly the height of this cone is given by s . This can be seen in Figure 4.2.

Thus, we have the estimate

$$\left| \int_{\Sigma_s} \partial_t \psi f_1 dx \right| \leq \frac{C}{(1+s)^{\frac{n-1}{2}} (1+|\tau|^p)} \|\partial \psi\|_{L^1(\Sigma_0)} + \left| \int_0^s \int_{\Sigma_t} F \partial_t f dx dt \right| \quad (4.37)$$

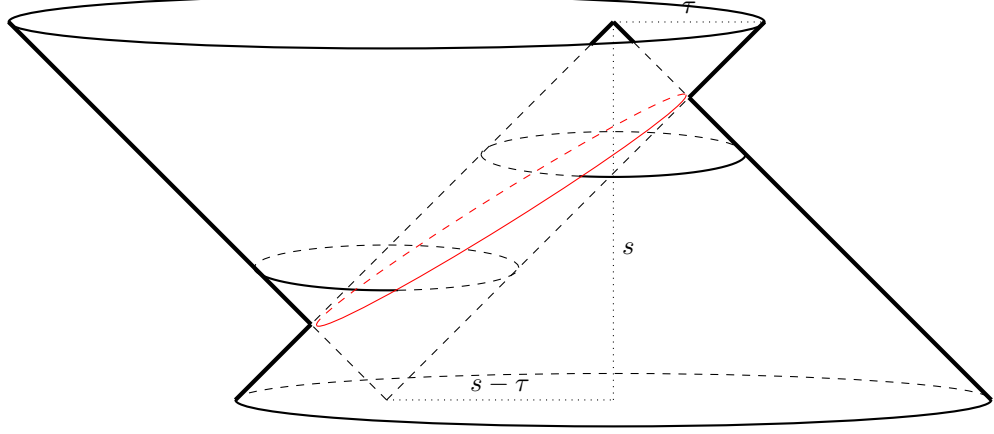


Figure 4.2: A configuration of light cones for ψ and f . The light cone for ψ is the upward opening cone, and the light cone for f is the downward opening cone. The vertical distance between the tips of the cones is s , and the horizontal distance from the tip of the downward opening cone and the upward opening cone is τ . The intersection of the two cones is a tilted spacelike ellipse whose major axis is comparable to s and whose minor axis is comparable to $\sqrt{\tau}\sqrt{s}$ (this follows from the results in Sections 4.4.4 and 4.4.5). This ellipse appears in red in the above figure. We also refer the reader to Figure 4.4 in Section 4.4.4 to see what this looks like in each Σ_t .

This is the form of the estimate we shall use. When combined with commutation with translation vector fields and the duality argument from Section 4.2.2, we get pointwise decay for $\partial_t \psi$ assuming that

$$\left| \int_0^s \int_{\Sigma_t} F \partial_t f dx dt \right| \leq \frac{C}{(1+s)^{\frac{n-1}{2}} (1+|\tau|^p)}. \quad (4.38)$$

In its uses in this work, we note that ψ is of size ϵ and F is schematically of size ϵ^q for some $q > 1$ in practice. We will be able to recover these estimates in a bootstrap argument.

The decay estimate resulting from (4.37) and the duality argument in Section 4.2.2 is recorded in the following proposition.

Proposition 22. *Let*

$$M[\psi](s, x_0^i) = \sup_{|\alpha| \leq k+n+1} \sup_{f_1} \left(\frac{C}{(1+s)^{\frac{n-1}{2}} (1+|\tau|^p)} \|\partial \partial^\alpha \psi\|_{L^1(\Sigma_0)} + \left| \int_0^s \int_{\Sigma_t} (\partial^\alpha F) \partial_t f dx dt \right| \right),$$

where (s, x_0^i) denotes the center of the unit ball in Σ_s where f_1 is supported, where $\tau = s - \sqrt{\sum_{i=1}^n (x_0^i)^2}$, and where the C^k norm of f_1 is bounded by 10. Then, we have that

$$|\partial_t \psi|(s, x_0^i) \leq CM[\psi](s, x_0^i).$$

The value of p will be specified in each application.

We end this Section by remarking that while Proposition 22 is based on using duality and the bilinear energy estimate from Lemma 17, there is a similar estimate coming from Lemma 19. Proposition 22 is used in Section 4.4 to study anisotropic wave equations. In Section 4.3, we shall actually use a version of Proposition 22 adapted to Lemma 19 instead. The statement and proof are analogous.

4.3 Simple applications

In this Section, we shall use the strategy described in Section 4.2 in order to prove global stability of the trivial solution $\psi = 0$ in a few specific cases. The problems are artificial, but they show how to use Section 4.2 in simplified settings, and they help us examine some of the behavior of solutions to nonlinear wave equations. Because the main purpose of these applications is to give an idea of how to handle anisotropic systems of wave equations, less detail will be given than is found in Section 4.4. The first result of this Section, Proposition 23, deals with a localized nonlinearity. The second result, Proposition 24, deals with a general cubic nonlinearity in $3 + 1$ dimensions. Both of these problems were described in Section 4.1.3.

Both of the following results are in $3 + 1$ dimensions. We shall use Proposition 22 for the application in Section 4.3.1, and we shall use the version of Proposition 4.2.3 adapted to Lemma 19 in Section 4.3.2 (this is described after Proposition 4.2.3). In order to do so, we need to assume pointwise decay of solutions to the homogeneous wave equation for a suitable class of initial data. We shall use the rates given by the fundamental solution, meaning that we shall assume that solutions arising from, say, C^2 initial data supported on the ball of radius R are supported where $|u| \leq R$ and decay like $\frac{1}{(1+t)}$. This is like taking $n = 3$ and, schematically, $p = \infty$ in (4.36). We do not require decay this strong in u for solutions to the homogeneous equation, but we shall assume it for simplicity.

Before turning to these applications, we note that these examples require similar analysis to that of anisotropic systems of wave equations, which are considered in Section 4.4. Both examples require an analysis of the geometry involved, which is the most technically involved part of Section 4.4. More precisely, the example in Section 4.3.1 exhibits how angular localization can help in control nonlinear effects, and the example in Section 4.3.2 exhibits how it can be important to understand the geometry of intersecting cones. There are analogous considerations that go into analyzing anisotropic systems of wave equations in Section 4.4.

4.3.1 Global stability with a localized nonlinearity

We recall the setting of (4.8). The function $r_\gamma^2(t, x, y, z) = (x-t)^2 + y^2 + z^2$ denotes a radial distance from the null geodesic in the t, x plane emanating from the spacetime origin. Then, for χ a smooth cutoff function equal to 1 for $|x| \leq 1$ and equal to 0 for $|x| \geq 2$, we are interested in solving (4.8), which we recall is given by

$$\square\psi = -\chi(r_\gamma)(\partial_t\psi)^2 \quad (4.39)$$

for small data compactly supported in the unit ball. We have the following Proposition.

Proposition 23. *There exists $\epsilon_0 > 0$ and $N \in \mathbb{N}$ such that, for all $\epsilon \leq \epsilon_0$, the trivial solution to (4.39) is globally nonlinearly stable with respect to perturbations $\psi(0, x) = \psi_0(x)$ and $\partial_t\psi(0, x) = \psi_1(x)$ which are compactly supported in the unit ball and have that $\|\partial\psi_0\|_{H^N} + \|\psi_1\|_{H^N} \leq \epsilon$. Moreover, the solutions to this equation satisfy the decay estimate*

$$|\partial_t\partial^i\psi| \leq \frac{C\epsilon}{1+t}$$

for all $|i| \leq \frac{3N}{4}$.

Proof. We shall take as bootstrap assumptions an energy bound on ψ as well as a pointwise bound. We let T be the maximal real number such that

1. We have that

$$\sup_{0 \leq t \leq T} \|\partial\psi\|_{H^N(\Sigma_t)} \leq C\epsilon^{\frac{3}{4}}(1+t)^{\sqrt{\epsilon}}. \quad (4.40)$$

2. We have that

$$|\partial_t\partial^i\psi|(t, x) \leq \frac{C\epsilon^{\frac{3}{4}}}{1+t} \quad (4.41)$$

for all $0 \leq t \leq T$ and $|i| \leq \frac{3N}{4}$.

We shall now recover both of these bootstrap assumptions. The energy will be recovered using an energy estimate, Gronwall's inequality, and the pointwise bootstrap assumptions, while the pointwise bootstrap assumptions will be recovered using Proposition 22.

We first recover the bootstrap assumptions for the energy. Let $0 \leq s \leq T$. We commute the equation (4.39) with ∂^i for $|i| \leq N$. Then, a ∂_t energy estimate gives us that

$$\|\partial\partial^i\psi\|_{L^2(\Sigma_s)}^2 = \|\partial\partial^i\psi\|_{L^2(\Sigma_0)}^2 + 2 \int_0^s \int_{\Sigma_t} -\partial^i(\chi(r_\gamma)(\partial_t\psi)^2)(\partial_t\partial^i\psi) dx dt. \quad (4.42)$$

We note that at least one factor of $\partial_t \psi$ will have fewer than $\frac{3N}{4}$ derivatives on it after applying the product rule to ∂^α . Thus, using the pointwise bootstrap assumptions in (4.42) gives us that

$$E(s) \leq E(0) + C\epsilon^{\frac{3}{4}} \int_0^s \frac{1}{1+t} E(t) dt,$$

where

$$E(t) = \sum_{j \leq i} \int_{\Sigma_t} (\partial_t \partial^j \psi)^2 + (\partial_x \partial^j \psi)^2 + (\partial_y \partial^j \psi)^2 + (\partial_z \partial^j \psi)^2 dx.$$

An application of Gronwall's inequality then gives us that $E(s) \leq 2\epsilon^2(1+t)^{\sqrt{\epsilon}}$ for ϵ sufficiently small, as desired.

We shall now recover the pointwise bootstrap assumptions. By Proposition 22, it suffices to show that

$$M[\partial^i \psi](s, x) \leq \frac{C\epsilon}{1+s}$$

for all $0 \leq s \leq T$ and $|i| \leq \frac{3N}{4}$, where M is as in Proposition 22. Now, because we are assuming (4.41) for the pointwise decay of the auxiliary functions f , we see that it suffices to control

$$\int_0^s \int_{\Sigma_t} -\partial^j \partial_i (\chi(r_\gamma) (\partial_t \psi)^2) (\partial_t f) dx dt$$

because the terms involving initial data in M in Proposition 22 are controlled by $\frac{C\epsilon}{1+t}$.

Now, after interpolating between the pointwise bootstrap assumptions and the energy bootstrap assumptions, we have that

$$|\partial^j \partial^i (\chi(r_\gamma) (\partial_t \psi)^2)|(t, x) \leq \frac{C\epsilon^{\frac{3}{2}}}{(1+t)^{2-\delta}}$$

for any $\delta > 0$ as long as N is taken sufficiently large as a function of δ . Thus, taking $\tilde{\chi}(x)$ to be a smooth, positive function equal to 1 for $|x| \leq 5$ and equal to 0 for $|x| \geq 10$, we have that

$$\begin{aligned} \left| \int_0^s \int_{\Sigma_t} -\partial^j \partial_i (\chi(r_\gamma) (\partial_t \psi)^2) (\partial_t f) dx dt \right| &\leq C\epsilon^{\frac{3}{2}} \int_0^s \int_{\Sigma_t} \tilde{\chi}(r_\gamma) \frac{1}{(1+t)^{2-\delta}} \frac{1}{1+s-t} dt \\ &\leq C\epsilon^{\frac{3}{2}} \|\tilde{\chi}\|_{L^1} \int_0^s \frac{1}{(1+t)^{2-\delta}} \frac{1}{1+s-t} dt \leq C\epsilon^{\frac{3}{2}} \frac{1}{1+s}, \end{aligned} \tag{4.43}$$

as desired. □

Examining this proof, we see that the localization given by χ is fundamental. It is interesting to

note that we can allow the support of χ to expand in t in the angular directions as long as the rate is smaller than $t^{\frac{1}{2}}$, which is the wave packet scaling. An examination of these terms in the absence of the cutoff shows that the wave packet scaling also appears naturally when analyzing the example

$$\square\phi = -(\partial_t\phi)^2$$

studied by John in [39] which leads to blow up. This gain in localization is similar to what will happen for anisotropic systems of wave equations in Section 4.4. We recall that this corresponds to an equation of the form

$$\square\phi = -\chi\left(r_\gamma\left(x, \frac{y}{(1+t)^\omega}, \frac{z}{(1+t)^\omega}, t\right)\right)(\partial_t\phi)^2,$$

where ω denotes the rate at which the support of χ expands.

If we allow the support of χ to expand at a rate of $t^{\frac{1}{2}}$ in the angular directions (i.e., if we take $\omega = \frac{1}{2}$), it still appears as though there is a substantial gain in the volume of interaction. Indeed, the total volume of worst interaction (i.e., the volume of intersection of the sphere with the cutoff) will be t instead of t^2 , which is what it usually is. However, wave packets that propagate along the null geodesic at the center of the cutoff will still experience amplification from interactions on a set of maximal measure. This can be seen by examining the integrals in (4.43) that come from picking auxiliary multipliers whose data are posed along this null geodesic. Thus, because these wave packets behave as if the nonlinearity was just $-(\partial_t\phi)^2$, the example that John studied, we cannot close the argument, and we believe that this may in fact still lead to blow up.

We note that we have allowed the support of χ to expand only in the angular directions in the sense that the y and z directions are roughly angular in the region of the support of χ . If we allow the support to expand in the radial direction as well (this corresponds roughly to x), the analysis must be slightly different, as the solution should decay faster away from the light cone.

4.3.2 Global stability with a general cubic nonlinearity

We now turn to the second of the simple applications. We have the following proposition.

Proposition 24. *Let ψ satisfy the nonlinear wave equation*

$$\square\psi = (\partial_t\psi)^3 \tag{4.44}$$

in $\mathbb{R}^{3+1}$ with initial data given by $\psi(0, x) = \psi_0(x)$ and $\partial_t \psi(0, x) = \psi_1(x)$. We assume that $\psi_0(x)$ and $\psi_1(x)$ are smooth functions supported in the unit ball in Σ_0 with H^{N+1} norm of size ϵ , where the minimum size of N is determined by the proof. As long as ϵ is sufficiently small, the above perturbations converge back to the trivial solution pointwise at a rate of $\frac{1}{1+t}$.

We note that the same proof works for a nonlinearity of $h(\partial_t \psi)^3$ where h is an arbitrary function whose C^N norm is bounded.

Proof. We shall take as bootstrap assumptions energy and pointwise bounds on ψ . Let $\delta > 0$ be some sufficiently small real number. We let T be the maximal real number such that, for all $0 \leq t \leq T$,

1. we have that

$$|\partial^\alpha \psi|(t, x) \leq \frac{C\epsilon^{\frac{3}{4}}}{(1+t)(1+|u|^{1-\delta})} \quad (4.45)$$

for $|\alpha| \leq \frac{N}{2} + 1$ and where ∂^α contains at most a single t derivative

2. we have that

$$\|\partial^\alpha \partial \psi\|_{L^2(\Sigma_t)} \leq C\epsilon^{\frac{3}{4}} \quad (4.46)$$

for all $|\alpha| \leq N$.

We shall recover that, for all t for which the above estimates hold true, we in fact have the stronger estimates coming from replacing $\epsilon^{\frac{3}{4}}$ in the bootstrap assumptions with $C\epsilon$. This will close the bootstrap argument, which will complete the proof of Proposition 24.

In order to effectively analyze this problem, it will be useful to introduce some notation. We shall denote by (t, r, θ) usual polar coordinates on $\mathbb{R}^{3+1}$ where t is the usual time coordinate, $r^2 = x^2 + y^2 + z^2$, and θ denotes coordinates on S^2 . There are of course null coordinates (v, u, θ) which are defined by $v = t + r$ and $u = t - r$.

Because we will use auxiliary multipliers as is described in Section 4.2.3, it will also be useful to introduce coordinates adapted to these auxiliary multipliers. We take some point (s, x_0, y_0, z_0) as the center of the ball in Σ_s where the data for the auxiliary multiplier will be supported. Following the notation in Section 4.2, we denote by τ the u coordinate of this point, meaning that $\tau = s - \sqrt{x_0^2 + y_0^2 + z_0^2}$. We shall denote by (t, r', θ') polar coordinates adapted to this point (meaning that $(r')^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$), and we shall denote by (v', u', θ') null coordinates adapted to this point. Because we are thinking of solving for f backwards in time, we are actually setting $v' = s - t + r'$ and $u' = s - t - r'$. Thus, for $0 \leq t \leq s$, we have that f is compactly supported in u' if we use the strong Huygens principle.

Because ψ decays in u and f decays in u' , it is additionally useful to introduce coordinates which are well adapted to how both of these functions decay. Thus, it makes sense to use coordinates given by (t, r, r', ϕ) and (t, u, u', ϕ) . Given level sets of r and r' , we have that the intersection of these two spheres is either empty, a circle, or a point. When it is a circle, ϕ denotes an angular coordinate on this circle. The coordinates degenerate when the intersection is a point, but this is a set of measure 0. The volume form of $\mathbb{R}^{3+1}$ is comparable to

$$\frac{rr'}{R} dt \wedge dr \wedge dr' \wedge d\phi, \quad (4.47)$$

and also to

$$\frac{rr'}{R} dt \wedge du \wedge du' \wedge d\phi, \quad (4.48)$$

where $R^2 = (x_0)^2 + (y_0)^2 + (z_0)^2$. We refer the reader to [11] where this fact is proved and where these coordinates are described in more detail.

With these considerations in hand, we can now turn to recovering the bootstrap assumptions.

It is clear that assumed bootstrap assumptions (4.45) and (4.46) directly recover the energy bootstrap assumptions. Indeed, if we commute the equation N times with unit partial derivatives in order to propagate $\|\partial^\alpha \partial \psi\|_{L^2(\Sigma_t)}$ for all $|\alpha| \leq N$, in the nonlinearity, there can be at most $\frac{N}{2} + 1$ derivatives falling on the term with the second most derivatives. We can, thus, put the term with the most derivatives in L^2 , and the pointwise decay for the two remaining terms is enough to give us a convergent integral. We must now recover the pointwise bootstrap assumptions.

We will now recover the pointwise bounds on some Σ_s with $0 \leq s \leq T$. We shall use a version of Proposition 22 in order to do so. In particular, we shall use one adapted to Lemma 18 instead of Lemma 17. This results in pointwise decay for ψ itself and not just $\partial_t \psi$. The necessary result analogous to Proposition 22 and the duality argument this requires follow in similar fashions to the ones proved in Section 4.2.3. We are, thus, using f as a multiplier for the wave equation as opposed to $\partial_t f$.

In order to implement this strategy, and following the notation above, we take some point (s, x_0, y_0, z_0) in Σ_s . We recall that τ is the u coordinate of this point. We consider initial data for an auxiliary solution to the wave equation f having initial data whose support is contained in a unit ball centered at this point in Σ_s . Now, we must recover pointwise bounds up to order $\frac{s}{2} + 1$.

More precisely, we must show that

$$|\partial^\alpha \psi| \leq \frac{C\epsilon}{(1+s)(1+|\tau|^{1-\delta})}, \quad (4.49)$$

where $|\alpha| \leq \frac{N}{2} + 1$. We must control error integrals of the kind found in (4.38) with f instead of $\partial_t f$. This provides pointwise bounds for $\partial^\alpha \psi$ after commuting with ∂^α , and then further commuting by ∂^β for all $|\beta| \leq k + n + 1 = 6$. Thus, in order to show decay for $|\partial^\alpha \psi|$, we must commute the equation by $\partial^\beta \partial^\alpha$ for all $|\beta| \leq 6$ where β consists of only spatial derivatives.

We consider the worst term in this commutation, which is when all of the derivatives fall on the same term in the nonlinearity. We have that

$$\square \partial^\beta \partial^\alpha \psi = N + (\partial_t \psi)^2 (\partial^\beta \partial^\alpha \partial_t \psi), \quad (4.50)$$

where the other terms in N can be treated in either the same way as we shall handle this remaining term, or in even easier ways.

Now, we know that we can write

$$\partial^\beta \partial^\alpha \partial_t \psi = \partial^\beta \partial^{\alpha'} \partial_i \partial_t \psi, \quad (4.51)$$

where ∂_i is some spatial derivative. If $\partial^{\alpha'}$ contains a time derivative (of which it can have at most one), we can rewrite this as

$$\partial^\beta \partial_i \partial^{\alpha''} \psi + N, \quad (4.52)$$

where N consists of higher order nonlinearities after using the equation. These terms are all better, so we disregard them. If it does not contain a time derivative, we can incorporate the ∂_t , giving us

$$\partial^\beta \partial_i \partial^{\alpha''} \psi. \quad (4.53)$$

In either case, we note that $\partial^{\alpha''} \psi$ is something we can assume pointwise bounds for using the bootstrap assumptions. This means, of course, that taking any ball B of radius 2 in some Σ_t where $t \leq s$ having center with some u coordinate u , we have that

$$\|\partial^{\alpha''} \psi\|_{L^2(B)} \leq \frac{C\epsilon^{\frac{3}{4}}}{(1+t)(1+|u|^{1-\delta})}. \quad (4.54)$$

Similarly, using the bootstrap assumptions on the energy, we have that

$$\|\partial^\gamma \partial^\beta \partial_i \partial^{\alpha''} \psi\|_{L^2(B)} \leq C\epsilon^{\frac{3}{4}}, \quad (4.55)$$

for all γ such that $|\gamma| + |\beta| + 1 + |\alpha''| \leq N$, where ∂^γ consists of only spatial derivatives. Now, using Sobolev embedding in $\mathbb{R}^3$, we know that

$$\|\partial^\beta \partial_i (\chi \partial^{\alpha''} \psi)\|_{L^\infty} \leq \|\partial^\beta \partial_i (\chi \partial^{\alpha''} \psi)\|_{H^2(\Sigma_t)}, \quad (4.56)$$

where χ is a smooth cutoff function equal to 1 in the ball of radius 1 concentric with B , decaying monotonically to 0, and equal to 0 outside of B . Moreover, we have that

$$\|\partial^\beta \partial_i (\chi \partial^{\alpha''} \psi)\|_{H^2(\Sigma_t)} \leq \|\chi \partial^{\alpha''} \psi\|_{H^9(\Sigma_t)}, \quad (4.57)$$

where we are using the fact that $|\beta| \leq 6$. Now, we have that $\|\chi \partial^{\alpha''} \psi\|_{L^2(\Sigma_t)} \leq \frac{C\epsilon^{\frac{3}{4}}}{(1+t)(1+|u|^{1-\delta})}$ by (4.54) above, and we additionally have that $\|\chi \partial^{\alpha''} \psi\|_{H^{\frac{N}{2}-1}} \leq C\epsilon^{\frac{3}{4}}$ by (4.55) above. Interpolating and with N sufficiently large in terms of δ gives us that

$$\|\chi \partial^{\alpha''} \psi\|_{H^9(\Sigma_t)} \leq \frac{C\epsilon}{(1+t)^{1-\delta} (1+|u|^{(1-\delta)^2})}, \quad (4.58)$$

meaning that we have that

$$|\partial^\beta \partial_i (\chi \partial^{\alpha''} \psi)| \leq \frac{C\epsilon}{(1+t)^{1-\delta} (1+|u|^{(1-\delta)^2})}. \quad (4.59)$$

Now, because $\chi = 1$ in a ball of radius 1 concentric with B , this implies that

$$|\partial^\beta \partial_i \partial^{\alpha''} \psi| \leq \frac{C\epsilon}{(1+t)^{1-\delta} (1+|u|^{(1-\delta)^2})} \quad (4.60)$$

on that same ball. Because this can be done for any such ball, we see that we can use the pointwise estimates for $\partial^\beta \partial^\alpha \psi$ up to a loss of t^δ and u^δ .

Thus, after commuting the equation by $\partial^\beta \partial^\alpha$ in order to apply Proposition 22, we see that the error integral we must control is of the form

$$\left| \int_0^s \int_{\Sigma_t} N' \phi dx dt \right| + \left| \int_0^s \int_{\Sigma_t} (\partial_t \psi)^2 (\partial^\beta \partial^\alpha \partial_t \psi) f dx dt \right|, \quad (4.61)$$

where, once again, the term N' can be controlled in the same way as the other. We can now plug in the pointwise bounds for $\partial_t \psi$ and $\partial^\beta \partial^\alpha \psi = \partial^\beta \partial_i \partial^{\alpha''} \psi$ from the above to get the desired result. Now, we note that $|f| \leq \frac{C}{1+s-t}$ by the asymptotics we have for the linear wave equation, and we note that it is additionally compactly supported in the u' coordinate. Thus, going into (t, r, r', ϕ) coordinates, we have that the error integral is controlled by

$$C\epsilon^{\frac{9}{4}} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_0^{s+2} \int_{s-t-5}^{s-t+5} \int_0^{2\pi} \frac{1}{(1+t)^{3-\delta}(1+|u|^{\frac{5}{2}})} \frac{1}{1+s-t} \frac{(1+s-t)(1+t)}{s+1} d\phi dr dr' dt \quad (4.62)$$

as long as $\tau \leq \frac{9s}{10}$ (we are using that, when this is satisfied, we have that R is comparable to s in 4.47). We are also assuming that δ is sufficiently small in order to bound the losses in decay in u , which depend on δ , by $\frac{1}{2}$. The integral bounds on r' come from the fact that f is compactly supported with respect to u' . The fact that the lower bound on the t integral is $\frac{\tau}{10}$ and the upper bound is $s - \frac{\tau}{10}$ deserves explanation. We are trying to show that $|\partial^\alpha \psi|$ is controlled on Σ_s at a point whose u coordinate is given by this τ . Thus, the initial data for f is prescribed in the ball of radius 1 centered at this point. Because of this, and using the strong Huygens principle for ϕ and a domain of dependence argument for ψ , the support of ϕ and the support of ψ do not intersect in Σ_t for $t \leq \frac{\tau}{10}$. These considerations hold only when $\tau \geq 20$. Otherwise, we simply take 0 as the lower bound for this integral. These considerations will be worked out more rigorously in an analogous setting in Lemma 28.

Using the fact that

$$\int_0^{s+2} \frac{1}{(1+|u|^{\frac{5}{2}})} dr \leq C, \quad (4.63)$$

we can control the above integral by

$$\frac{C\epsilon^{\frac{9}{4}}}{1+t} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \frac{1}{(1+t)^{2-\delta}} ds \leq \frac{C\epsilon^{\frac{9}{4}}}{(1+s)(1+|\tau|^{1-\delta})}. \quad (4.64)$$

We now consider the region $\tau \geq \frac{9t}{10}$. For this, we further decompose into two regions. Without loss of generality, we have that the data for f is supported in a ball whose center has y and z coordinate equal to 0, and we have that $(r')^2 = (x-a)^2 + y^2 + z^2$. We first consider the case where $a \geq 100$. Using the (t, u, u', ϕ) coordinate system, we have that the error integral is controlled by

$$C\epsilon^{\frac{9}{4}} \int \int \int \int \frac{1}{(1+t)^{3-\delta}(1+|u|^{\frac{5}{2}})} \frac{1}{1+s-t} \frac{(1+s-t)(1+t)}{a} d\phi du du' dt, \quad (4.65)$$

where we shall establish the bounds of integration shortly. We first note that the integral in u' is

can always be taken from -5 to 5 because f is compactly supported in u' . Then, we shall break up the integral in u into pieces of length 10. More specifically, we can control the above integral by

$$C\epsilon^{\frac{9}{4}} \sum_{i=0}^{\infty} \int_{5(i-1)}^{5(i+1)} \int \int \int_{-5}^5 \frac{1}{(s+1)^{3-\delta} (1+|5i|^{\frac{5}{2}})} \frac{1}{1+s-t} \frac{(1+s-t)(s+1)}{\bar{u}} du_2 ds d\theta du_1. \quad (4.66)$$

Now, we note that the bounds for ϕ are of course given by 0 and 2π . The final range we must establish is that of t . We take the line $t = 5(i+1) + x$ and $t = s + a - x + 5$ in the plane where $y = z = 0$, and we calculate the s coordinate of the intersection. This is given by $s = \frac{t+a+5(i+2)}{2}$. Similarly, we take the line $s = 5(i-1) - x$ and the line $s = t - a + x - 5$ and calculate the s coordinate of the intersection. This is given by $t = \frac{s-a+5(i-2)}{2}$. These give us the bounds of integration, as we know that the lowest and highest t coordinates must occur on the plane where $y = z = 0$ (this is considered more carefully in an analogous situation in Lemma 27). Thus, the integral is controlled by

$$C\epsilon^{\frac{9}{4}} \sum_{i=0}^{\infty} \int_{5(i-1)}^{5(i+1)} \int_{\frac{s-a+5(i-2)}{2}}^{\frac{s+a+5(i+1)}{2}} \int_0^{2\pi} \int_{-5}^5 \frac{1}{(1+t)^{3-\delta} (1+|5i|^{\frac{5}{2}})} \frac{1}{1+s-t} \frac{(1+s-t)(1+t)}{a} du' dt d\phi du. \quad (4.67)$$

Now, we note that the length of t integration is always comparable to a by the above bounds (note that $a \geq 100$). Moreover, in this region, both $1+t$ and $1+s-t$ are comparable to s . Thus, we have that we can control the above integral by

$$C\epsilon^{\frac{9}{4}} \sum_{i=0}^{\infty} \frac{1}{1+|5i|^{\frac{5}{2}}} \frac{1}{(1+s)^{2-\delta}}, \quad (4.68)$$

giving us the desired result. The final region where $\bar{u} \leq 100$ can be controlled by comparing with a downward pointing light cone of thickness 200 where the tip lies along the t axis. $\square$

4.4 Anisotropic systems of wave equations

We now precisely describe the setting of the main application for the techniques introduced in this paper. We recall that

$$\square = -\partial_t^2 + \partial_x^2 + \partial_y^2,$$

and we introduce the notation

$$\square' = -\partial_t^2 + \lambda_1^{-2} \partial_x^2 + \lambda_2^{-2} \partial_y^2.$$

We assume that $\frac{1}{4} \leq \lambda_1^2 \leq \frac{1}{2}$ and $2 \leq \lambda_2^2 \leq 4$.

We shall consider the following anisotropic system of wave equations in $2 + 1$ dimensions:

$$\begin{aligned}\square\psi + (\partial_t\phi)^2(\partial_t\psi)(\partial_t^2\psi) &= (\partial_t\phi)(\partial_t\psi)^2 + (\partial_t\psi)(\partial_t\phi)^2 + (\partial_t\psi)^4 \\ \square'\phi + (\partial_t\psi)^2(\partial_t\phi)(\partial_t^2\phi) &= (\partial_t\psi)(\partial_t\phi)^2 + (\partial_t\phi)(\partial_t\psi)^2 + (\partial_t\phi)^4.\end{aligned}\tag{4.69}$$

We take smooth data for these equations $\psi(0, x) = \psi_0(x)$, $\partial_t\psi(0, x) = \psi_1(x)$, $\phi(0, x) = \phi_0(x)$, and $\partial_t\phi(0, x) = \phi_1(x)$ supported in the ball of radius $\frac{1}{10}$. Moreover, we assume that the data are arbitrary functions with $\|\psi_0\|_{H^{N+1}} + \|\psi_1\|_{H^N} + \|\phi_0\|_{H^{N+1}} + \|\phi_1\|_{H^N} \leq \epsilon$ for some ϵ . With this, we are ready to state the main Theorem of this paper.

Theorem 25. *There exist $\epsilon_0 > 0$ sufficiently small and N_0 sufficiently large such that the trivial solution of (4.69) is globally nonlinearly asymptotically stable with respect to the perturbations described above whenever $\epsilon \leq \epsilon_0$ and $N \geq N_0$.*

Before proceeding to the proof of this result, we shall make some brief remarks.

1. **The fundamental restriction on λ_1 and λ_2 is that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$.** Indeed, the semilinear terms are cubic, which is critical in $2 + 1$ dimensions because solutions to the wave equation only decay like $t^{-\frac{1}{2}}$. The condition that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ produces a kind of null condition on interactions between these waves. When $\lambda_1 = 1$ or $\lambda_2 = 1$, the light cones are tangent instead of intersecting transversally, and parallel interactions in the tangential direction are not effectively suppressed. We are unsure what happens in this case, but would not be surprised if this situation would lead to blow up. In our setting, however, the null condition is thus that $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, and that the same wave not appear cubed.
2. The other restrictions on λ_1 and λ_2 are purely for convenience. We restricted to the case $\lambda_1 < 1 < \lambda_2$ to force the light cones to intersect. In case both λ_1 and λ_2 are either smaller than 1 or greater than 1, the argument seems to still work, with the integrals appearing to be easier to control. We further restricted to the case of $\frac{1}{4} \leq \lambda_1^2 \leq \frac{1}{2}$ and $2 \leq \lambda_2^2 \leq 4$ simply to make the equations symmetric under the change of coordinates described in Section 4.4.2. Indeed, an appropriate rescaling of the coordinates makes $\square'$ become $\square$ (see Section 4.4.2), and this restriction ensures that the form of the equation is preserved under this change of coordinates. We do, however, note that the estimates are not uniform in λ_1 and λ_2 , and a variety of constants may degenerate as λ_1 or λ_2 approach 0, 1, or ∞ . When λ_1 and λ_2 are uniformly away from these quantities, we believe the estimates can be made uniform.

3. The quasilinear terms in (4.69) are present to show that the method is applicable to quasilinear equations. They can be more general.
4. We have included the fourth order nonlinearity just to show that it can be handled. We can, of course, introduce higher order nonlinearities and higher order mixed nonlinearities and deal with them in the same way. We note, however, that the equation for ψ does not have a nonlinear term depending only on ϕ , and similarly for the equation for ϕ . While we believe the techniques used in this paper are applicable to study this problem, they would require a new analysis. In fact, because the light cones for ψ and ϕ have different causal structures, introducing these kinds of nonlinearities is related to removing the assumption of compact support, which we discuss below. Also, the derivatives in the nonlinearity are seen to all be ∂_t , and Proposition 22 is specifically adapted to showing decay for this derivative. Treating other derivatives in the nonlinearity requires relatively minor modifications to the considerations in Section 4.2.3.
5. We have not tried to optimize the results in this paper, and they are far from sharp in terms of, for example, regularity. Also, it seems as though the same strategy can work after weakening the assumptions. For example, it seems as though we can assume less decay for solutions to the homogeneous wave equation and a very similar argument will, at least until we assume less than integrable decay in u for first derivatives. With this assumption, the analysis may have to be changed. A very natural way to try to strengthen the result is to remove the assumption that the data are compactly supported (and to replace this with the assumption that the data are localized), and which we now discuss.
6. The proof uses that the data are compactly supported in the unit ball. However, we believe that the strategy followed in the proof of Theorem 25 can be used in the more general setting of data that decay sufficiently rapidly away from the origin. This would, however, require an analysis of the different geometry arising from this situation (see Section 4.4.7 to see where the geometry is necessary, and see Sections 4.4.4 and 4.4.5 to see where the geometry involved in the proof of Theorem 25 is studied). More precisely, we note that it seems as though all of the nonlinear terms except for $(\partial_t \psi)(\partial_t \phi)^2$ in the equation for ψ (and the analogous term in the equation for ϕ) can be treated using the considerations below along with the observation that, if $\tau < -100$, we in fact have that $|u|$ is comparable to τ in the entire support of the error integral (see Sections 4.2 and 4.4.2 where u and τ are described). For the remaining term, it seems as though an analysis of how the geometry of the scaling vector field interacts with the

light cone of f when $\tau < 0$ is necessary.

7. The estimates on the error integrals involving cubic interactions below are far from optimal.

In Section 4.4.1, some of the additional gains from studying the geometry more carefully will be described.

4.4.1 Description of the proof

The proof follows the same strategy as the proofs in Section 4.3. Indeed, we use the strategy outlined in Section 4.2 to prove decay. Thus, we have to control error integrals of the form

$$\int_0^s \int_{\Sigma_t} F(\psi, \phi) \partial_t f dx dt,$$

where f is the auxiliary solution to the wave equation (see Proposition 22). Once again, we shall interpolate between the energy and pointwise estimates in order to effectively take advantage of the transversal intersection of the cones. The main differences arise from the fact that we shall combine this method with commuting with weighted vector fields from the vector field method. We shall commute with the scaling vector field $S = t\partial_t + r\partial_r$. Because we need a norm that allows us to interpolate with scaling as well, we must use a Morawetz estimate. To avoid discussing this in detail, we shall actually use a simpler estimate that is very similar to a Morawetz estimate (see Lemma 26). The important fact is that this estimate, like the Morawetz estimate, is sharp in terms of decay (up to a loss of t^δ). Controlling the error integrals will require more geometric information than was needed for the simple applications in Section 4.3. However, as was noted in the remarks after Theorem 25, we are still not bounding at least some of the terms optimally. One example of this stems from the fact that tubes formed by considering u and $\bar{u}$ neighborhoods of thickness comparable to 1 intersect the downward pointing light cone of f in a set of small measure in τ (see Section 4.4.2 for a description of $\bar{u}$). Facts such as these, which can be seen by examining the intersections of the three cones more carefully, will not be needed in this argument. However, it may be useful to keep in mind for other applications that it seems to be possible to get stronger estimates.

We shall now describe the proof in a bit more detail. We shall first focus on how the scaling vector field is used in the context of the strategy described in Section 4.2.3. In order to apply Proposition 22 in a bootstrap argument, we must show that the nonlinear error integral has good powers of τ . The vector field S will help us in getting good powers of τ . The main observation is that the downward opening light cone associated to the auxiliary multiplier f is everywhere pierced

by S (see Figure 4.2). This means that good derivatives of f and S span the tangent space of $\mathbb{R}^{2+1}$. We may, thus, write any vector field in terms of the frame consisting of S and good derivatives of f . In particular, we may schematically write that $\partial_t f = \bar{\partial}_f f + \gamma S f$ where $\bar{\partial}_f$ denotes good derivatives of f . Thus, we may write that

$$\int_0^s \int_{\Sigma_t} F(\psi, \phi) \partial_t f dx dt = \int_0^s \int_{\Sigma_t} F(\psi, \phi) (\bar{\partial}_f f + \gamma S f) dx dt.$$

The term involving good derivatives of f is better than the one involving S , so to a first approximation, we can assume that this term satisfies improved estimates. For the term involving S , we can integrate by parts in S , giving us that, schematically,

$$\int_0^s \int_{\Sigma_t} F(\psi, \phi) \gamma S(f) dx dt = \int_0^s \int_{\Sigma_t} S(F(\psi, \phi)) \gamma f + S(\gamma) F(\psi, \phi) f + \gamma F(\psi, \phi) f dx dt.$$

Now, because we may commute the equations for ψ and ϕ with S , we note that $S(F(\psi, \phi)) \approx F(\psi, \phi)$ in the sense that any estimates that $F(\psi, \phi)$ satisfies are, to a first approximation, satisfied by $S(F(\psi, \phi))$. Thus, the gain comes from noting that γ and $S(\gamma)$ satisfy good estimates in certain regions. More specifically, they can be shown to be of size $\frac{1}{\tau}$. Thus, we can use S to gain a power of $\frac{1}{\tau}$, which helps in showing that the nonlinear error integrals are of the necessary size. Proving these estimates on γ and $S(\gamma)$ involves studying the geometry of S in relation to the various cones in question. This is carried out in Section 4.4.5.

We now describe how we estimate the geometry involved. To control quartic and higher order nonlinearities, we must estimate the geometry coming from the intersection of a forward opening and downward opening cone. This is because the error integrals are of the form

$$\int_0^s \int_{\Sigma_t} (\partial_t \psi)^4 (\partial_t f) dx dt,$$

and because ψ decays away from a forward opening cone while f decays away from a downward opening cone.

To a first approximation, ψ is supported in a forward pointing cone of thickness 1 and f is supported in a downward pointing cone of thickness 1. If we take the intersection of this configuration with Σ_t , we are left with two annuli of thickness 1, and it is natural to compute the volume of intersection order to control the interaction. Of course, these kinds of estimates depend on the parameters s and τ . These estimates are one example of the kind of analysis that must be done in order to control the nonlinearity. Such estimates are carried out in Section 4.4.4.

The interactions between ψ and ϕ can be controlled using the condition that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$. Geometrically, this means that either one cone is contained within the other, or that the two cones intersect transversally. This paper focuses on the case where the two light cones intersect transversally as this case is more interesting, although we note that a similar proof can be used in the other case. Thus, we restrict ourselves to this case.

There are now two forward opening cones to consider, the one associated to ψ and the one associated to ϕ . If we take two such cones of thickness 1, if we look in Σ_t , we are left with two annuli of thickness 1. One annulus is circular and the other is elliptical. When the cones intersect transversally, we note that the area of intersection of these annular regions is comparable to 1. Thus, decay away from the light cone gives localization to a small set, and the mechanism of improvement becomes similar to the case of the localized nonlinearity which was studied in Section 4.3.1 above. Indeed, the total area of the set of interaction in Σ_t becomes 1 instead of t , which is what it is between waves having the same light cone. Figure 4.3 shows the cross sections of the cones for different values of λ_1 and λ_2 . The left configuration is representative of the case considered in this paper (i.e., $\lambda_1 < 1$ and $\lambda_2 > 1$). The second corresponds to $\lambda_1 = 1$, which we have just discussed. The third is a configuration where $\lambda_1 > 1$ and $\lambda_2 > 1$, and although it is not treated in this paper, we believe that similar ideas can be used in this case.

These considerations also reveal why having $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ is important. If we consider the case where, say, $\lambda_1 = 1$, we can see that the area of intersection of the two annuli in Σ_t is comparable to $\sqrt{t}$. Given that cubic nonlinearities are critical in $2+1$ dimensions, one may ask why the gain over t is not enough. However, in this case, there exist wave packets which still experience amplification caused by interactions on sets with maximal measure. Indeed, when $\lambda_1 = 1$, we note that the two light cones intersect in two null lines. Wave packets propagating along those null lines will interact in essentially the same way as if the light cones were the same. This configuration looks very much like a $2+1$ dimensional version of the localized nonlinearity studied in Section 4.3.1 above, where we take $\omega = \frac{1}{2}$. Because of this behavior, we believe that this configuration is likely to lead to blow up in finite time.

We hope it is clear from the above considerations that the geometry of the cones must be effectively analyzed. This was already present in the example considered in Section 4.3.2, and we in fact use some of the same observations that are found therein. However, the geometry requires more substantial analysis in the case of anisotropic systems of wave equations which we are now considering.

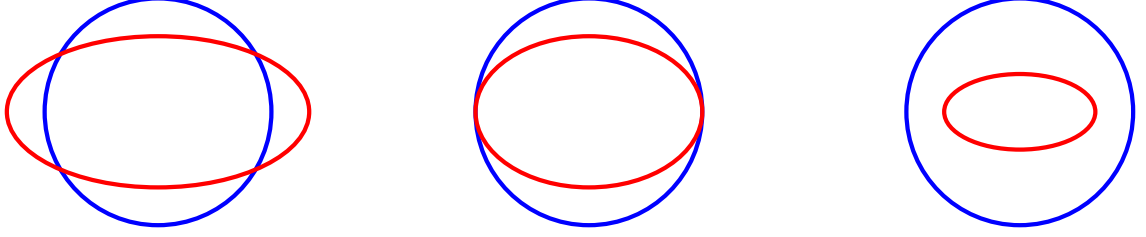


Figure 4.3: Cross sections of the light cones for ϕ and ψ for different λ_1 and λ_2 . More specifically, the blue circles represent intersections of the light cone of ψ with Σ_t , and the red ellipses represent intersections of the light cone of ϕ with Σ_t .

4.4.2 Notation, coordinates, and the parameters

We take $\mathbb{R}^{2+1}$ with the usual (t, x, y) coordinates. In these coordinates, we recall that the equations in question (see (4.69)) are of the form

$$\begin{aligned}\square\psi &= -\partial_t^2\psi + \partial_x^2\psi + \partial_y^2\psi = F(\partial_t\psi, \partial_t\phi, \partial_t^2\psi), \\ \square'\phi &= -\partial_t^2\psi + \lambda_1^{-2}\partial_x^2\psi + \lambda_2^{-2}\partial_y^2\psi = F(\partial_t\psi, \partial_t\phi, \partial_t^2\psi).\end{aligned}\tag{4.70}$$

We shall also use the coordinates (t, r, θ) and $(t, \bar{r}, \bar{\theta})$. These coordinates are defined as follows. The functions r and θ are the usual polar coordinates on $\mathbb{R}^2$, meaning that $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. We now consider the coordinates $(t, \bar{x}, \bar{y})$ given by $\bar{x} = \lambda_1 x$ and $\bar{y} = \lambda_2 y$. We then have that $\partial_{\bar{x}} = \lambda_1^{-1}\partial_x$ and that $\partial_{\bar{y}} = \lambda_2^{-1}\partial_y$. Thus, in the $(t, \bar{x}, \bar{y})$ coordinates, the equations can be written in the form

$$\begin{aligned}\square\psi &= -\partial_t^2\psi + \lambda_1^2\partial_{\bar{x}}^2\psi + \lambda_2^2\partial_{\bar{y}}^2\psi = F(\partial_t\psi, \partial_t\phi, \partial_t^2\psi), \\ \square'\phi &= -\partial_t^2\phi + \partial_{\bar{x}}^2\phi + \partial_{\bar{y}}^2\phi = G(\partial_t\psi, \partial_t\phi, \partial_t^2\phi).\end{aligned}\tag{4.71}$$

We define $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2} = \sqrt{\lambda_1^2 x^2 + \lambda_2^2 y^2}$, and we define $\bar{\theta} = \arctan\left(\frac{\bar{y}}{\bar{x}}\right) = \arctan\left(\frac{\lambda_2 y}{\lambda_1 x}\right)$. We also introduce the functions $u = t - r$ and $\bar{u} = t - \bar{r}$. We have that

$$S = t\partial_t + x\partial_x + y\partial_y = t\partial_t + r\partial_r = t\partial_t + \bar{r}\partial_{\bar{r}}.$$

This is a consequence of the fact that

$$r\partial_r = x\partial_x + y\partial_y = \lambda_1 x \lambda_1^{-1} \partial_x + \lambda_2 y \lambda_2^{-1} \partial_y = \bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}} = \bar{r}\partial_{\bar{r}}.$$

More geometrically, there is a metric g adapted to $\square$ and a metric $\bar{g}$ adapted to $\square'$. In (t, x, y)

coordinates, the metric g is given by the constant coefficient diagonal matrix whose entries are -1 , 1 , and 1 along the diagonal. The metric $\bar{g}$ is also constant coefficient and diagonal, but its entries are given by -1 , λ_1^2 , and λ_2^2 . In $(t, \bar{x}, \bar{y})$ coordinates, the metric $\bar{g}$ is constant coefficient and diagonal with entries -1 , 1 , and 1 , while the metric g is constant coefficient and diagonal with entries -1 , λ_1^{-2} , and λ_2^{-2} .

Because it will be necessary to consider the geometry involving the level sets of u and $\bar{u}$ (these are cones opening to the future with circular and elliptical cross sections, respectively), we note that the level set $u = 0$ and the level set $\bar{u} = 0$ intersect in four straight lines emanating from the spacetime origin. These four straight lines each have a constant θ coordinate depending only on λ_1 and λ_2 . These coordinates can be found by solving for $\tan(\theta) = \frac{y}{x}$ in the equations

$$t^2 = x^2 + y^2 = \lambda_1^2 x^2 + \lambda_2^2 y^2.$$

We shall denote the θ coordinates of these lines by Θ_1 , Θ_2 , Θ_3 , and Θ_4 , taking $0 < \Theta_1 < \frac{\pi}{2}$, $\frac{\pi}{2} < \Theta_2 < \pi$, $\pi < \Theta_3 < \frac{3\pi}{4}$, and $\frac{3\pi}{4} < \Theta_4 < 2\pi$. We note that all four of these values are bounded uniformly away from half integer multiples of π given the restrictions we have placed on λ_1 and λ_2 . We similarly use $\bar{\Theta}$ and $\bar{\Theta}_i$ for $1 \leq i \leq 4$ to denote the $\bar{\theta}$ coordinates of these quantities. These quantities are related to the Θ_i because

$$\tan(\bar{\theta}) = \frac{\bar{y}}{\bar{x}} = \frac{\lambda_2 y}{\lambda_1 x} = \frac{\lambda_2}{\lambda_1} \tan(\theta).$$

Once again, the function f will always denote an auxiliary solution to the homogeneous wave equation $\square f = 0$ or $\square' f = 0$. Which equation f solves will be clear from the context. Because f is the test function we use as a multiplier, it is natural to also introduce coordinates adapted to f . We introduce r' , θ' , and u' for f when f solves the equation $\square f = 0$. When it solves $\square' f = 0$, we introduce $\bar{r}'$, $\bar{\theta}'$, and $\bar{u}'$ instead, and these quantities are defined analogously (see below). Now, after rescaling x and y to $\bar{x}$ and $\bar{y}$, we note that we can always assume that f solves $\square f = 0$. Thus, in the following discussion, we shall make this assumption.

We denote by s the time slice in which data for f is posed. We then denote by τ the u coordinate of the center of the ball in Σ_s where the data for f is supported, and by Θ the θ coordinate of this point. If the center of this ball is given by (s, a, b) , we denote by r' the radial coordinate away from this point, that is, $(r')^2 = (x - a)^2 + (y - b)^2$, and we denote by θ' the angular coordinate for this point normalized such that $\theta' = 0$ along the line $\theta = \Theta$, meaning that $\theta' = \arctan\left(\frac{y-b}{x-a}\right) - \Theta$.

When proving geometric estimates that depend on τ and s , it will often be possible to assume that $b = 0$. When $b \neq 0$, we may of course redefine coordinates to make $b = 0$. We must be cautious, however, about circumstances where the geometry of the other light cone must be taken into account (rotating an ellipse does not produce the same ellipse because it changes the axes). Because we shall often not have to take the geometry of all three cones into account at the same time, this is still a useful observation. The only time we must consider the geometry of all three light cones for a geometric estimate (i.e., the geometry of the light cones for ψ , ϕ , and f) is in Lemma 33. Assuming that $b = 0$ is the same as assuming that $\Theta = 0$. The reader may also wish to consult Figure 4.4 where a diagram of some of these quantities is given.

We shall also introduce null coordinates adapted to ψ , ϕ , and f . For ψ , the null coordinates are given by (u, v, θ) where $u = t - r$, $v = t + r$, and $\theta = \theta$. The null coordinate systems $(\bar{u}, \bar{v}, \bar{\theta})$ and (u', v', θ') are defined analogously. More precisely, we note that $\bar{v} = t + \bar{r}$, $\bar{u} = t - \bar{r}$, $v' = s - t + r'$, and $u' = s - t - r'$. We note that

$$S = t\partial_t + x\partial_x + y\partial_y = v\partial_v + u\partial_u = \bar{v}\partial_{\bar{v}} + \bar{u}\partial_{\bar{u}}.$$

Because we often have to consider the functions ψ , ϕ , and f , we shall use $\bar{\partial}_\psi$, $\bar{\partial}_\phi$, and $\bar{\partial}_f$ to denote good derivatives for ψ , ϕ , and f , respectively. These are precisely the derivatives tangent to the light cones adapted to these functions. We shall also introduce the null frames $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, and $\partial_\theta = \partial_\theta$. The null frames $\bar{L}$, $\bar{\underline{L}}$, $\partial_{\bar{\theta}}$ and $\underline{L}'$, L' , $\partial_{\theta'}$ are defined analogously. We note that these are simply rescaled versions of the null coordinate vector fields (for example, L is a constant rescaling of ∂_v and $\underline{L}$ is a constant rescaling of ∂_u). The good derivatives $\bar{\partial}_\psi$ of ψ consist of L and ∂_θ , and similarly for the good derivatives of ϕ and f .

When doing estimates on the geometry in Sections 4.4.4 and 4.4.5, we can freely assume that s is larger than some fixed constant. In the proof of Theorem 25, we use ϵ to absorb all nonlinear contributions for $t \leq \frac{1}{\delta_0^{10}}$. Thus, when analyzing the geometry in those sections in order to recover the pointwise estimates, we can assume that $s \geq \frac{1}{\delta_0^{10}}$.

We shall sometimes use O notation, in which f is said to be $O(g)$ if $f \leq Cg$ for some C . There are some instances in which we will have expressions which are $O(\delta_0)$ (see below where the parameter δ_0 is described). In these cases, the constant C will be implied to not depend on δ_0 .

We shall have to commute with several vector fields. In the simple applications studied in Section 4.3, we only commuted with translation vector fields. However, we shall also have to use the scaling vector field. Thus, in this section, the admissible commutation fields will be strings of

translation vector fields and scaling vector fields. We shall denote by Γ^α such a string of scaling vector fields and translation vector fields. This collection of vector fields is closed under Lie brackets because the commutator of two commutation fields is another commutation field. Indeed, translation vector fields commute with each other and the scaling vector field commutes with itself. We also have that the commutator between S and a translation vector field is the negative of that translation vector field. Thus, for example, we have that

$$[S, \partial_t] = -\partial_t.$$

This shows that the commutator of any two of our commutation fields is another commutation field. Thus, by induction, we have that

$$\Gamma^\alpha(\partial_t h) = (\partial_t \Gamma^\alpha h) + \sum_{\gamma} c_{\gamma}(\partial_t \Gamma^{\gamma} h)$$

for an appropriate collection of γ and where $c_{\gamma} = \pm 1$, and we moreover have that $|\gamma| \leq |\alpha| - 1$. These facts will be used in the proof.

Now, given data supported in the ball of radius 1 whose C^2 norm is bounded by 10, we note that the solution to the linear wave equation $\square f = 0$ has that

$$|f| \leq \frac{C}{\sqrt{1+s-t}(1+|u'|)^{\frac{1}{2}}}. \quad (4.72)$$

This follows from the fundamental solution. Then, by commuting the homogeneous equation with Lorentz fields, we have that

$$|\partial f| \leq \frac{C}{\sqrt{1+s-t}(1+|u'|)^{\frac{3}{2}}}, \quad (4.73)$$

and we have that

$$|\bar{\partial}_f f| \leq \frac{C}{(1+s-t)^{\frac{3}{2}}(1+|u|^{\frac{1}{2}})}. \quad (4.74)$$

Thus, we shall take $n = 2$ and $p = \frac{3}{2}$ in Proposition 22. The improved decay of good derivatives is important because we shall decompose ∂_t in terms of S and $\bar{\partial}_f$, as was described in Section 4.4.1.

Similarly, we note that the solution to the linear wave equation $\square' f = 0$ with the same kind of data has that

$$|\partial_t f| \leq \frac{C}{\sqrt{1+s-t}(1+|\bar{u}'|)^{\frac{3}{2}}}, \quad (4.75)$$

and that

$$|\bar{\partial}_f f| \leq \frac{C}{(1+s-t)^{\frac{3}{2}}(1+|\bar{u}'|^{\frac{1}{2}})}. \quad (4.76)$$

We shall use these decay rates, but an examination of the proof below reveals that less decay would be enough. The main result that would require modification is Lemma 30, but even there, we believe the modification is not substantial. Moreover, we emphasize once again that it suffices to take decay rates as a black box. Indeed, all that is required is a proof of decay for the homogeneous equation with a sufficiently large class of test functions as data. The test functions we are taking are all smooth functions supported in a ball of radius 1 with C^2 norm bounded by 10.

We finally note that we make use of three smallness parameters in the following proof. They are ϵ , δ , and δ_0 . The parameter ϵ can be chosen to depend on δ , δ_0 , and the constants involved in the problem, and it controls the size of the initial data. It must be chosen sufficiently small for the bootstrap assumptions to be recovered. Because we are allowing ϵ to depend on δ and δ_0 , we also allow the expressions C and c to depend on δ and δ_0 as well. The expressions C and c represent positive numbers which may be very large or small depending on some universal constants and on δ and δ_0 . More specifically, C will represent something that may be very large (such as, for example, $\frac{1}{\delta_0}$), while c will represent something that may be very small (such as δ_0). There are a few places where it will be important to have constants that do not depend on δ_0 so that we can get smallness relative to them by making δ_0 smaller (see, for example, Lemma 33). Whenever this is necessary, we shall explicitly say so.

Now, the parameters $\delta \leq \frac{1}{2}$ and $\delta_0 \leq \frac{1}{2}$ must be chosen sufficiently small to make the following proof work. The parameter δ controls a loss in decay that comes from interpolation (see Lemma 41), and it is also the parameter we have used to denote the (necessary unless restricting to a dyadic region) loss in decay from a modified Morawetz estimate (see Lemma 26). Meanwhile, the parameter δ_0 is our universal smallness parameter for geometric estimates. For example, it will be beneficial to consider two regimes, one where τ is small compared to s and another where it is comparable to s . We shall thus consider the cases $\tau \leq \delta_0 s$ and $\tau \geq \delta_0 s$. There will be several other cases where we must compare quantities, and δ_0 is the parameter we shall use to make this comparison. There exist bounds for δ and δ_0 in the form of universal constants that make the following proof work, but we have not computed them. They can, however, be computed in principle.

We finally note the regularity parameter N . This denotes the L^2 Sobolev space for the initial data. This parameter is allowed to depend on δ , and we note that $N \rightarrow \infty$ as $\delta \rightarrow 0$. This is required in order to effectively interpolate (see Lemma 41). However, for every fixed $\delta > 0$, we note that N

is a finite positive integer. The dependence of N on δ can be explicitly computed (meaning that a lower bound for the regularity we require on the initial data can be computed), but we have not done so.

4.4.3 Additional linear estimate

We only require one additional linear estimate for this problem. In order to get a sharper result, it would be beneficial to use a Morawetz estimate, which schematically says that $\|r^{-\frac{1}{2}}\partial\psi\|_{L_t^2L_x^2}$ is controlled by the energy (the real estimate either requires a slight loss in the power of r or a restriction to a dyadic region). We shall not discuss this estimate here, but will rather use a very easy estimate which suffices for this problem.

Lemma 26. *We have that*

$$\begin{aligned} \int_{\Sigma_s} (1+t)^{-\frac{\delta}{2}} [(\partial_t\psi)^2 + (\partial_x\psi)^2 + (\partial_y\psi)^2] dx + \int_0^s \int_{\Sigma_t} \delta(1+t)^{-1-\delta} [(\partial_t\psi)^2 + (\partial_y\psi)^2 + (\partial_x\psi)^2] dx dt \\ = \int_{\Sigma_0} (1+t)^{-\frac{\delta}{2}} [(\partial_t\psi)^2 + (\partial_x\psi)^2 + (\partial_y\psi)^2] dx + \int_0^s \int_{\Sigma_t} (\Box\psi)(1+t)^{-\frac{\delta}{2}} (\partial_t\psi) dx dt. \end{aligned} \quad (4.77)$$

Proof. This follows from simply using $(1+t)^{-\frac{\delta}{2}}\partial_t\psi$ as a multiplier for the wave equation. $\square$

This estimate is useful because it is sharp in terms of decay up to a power of $t^{\frac{\delta}{2}}$ (the Morawetz estimate on dyadic regions is sharp). Moreover, it is a spacetime L^2 norm, meaning that it allows for interpolation with the scaling vector field S .

4.4.4 Estimates on geometry

We first record several observations which shall be used in Section 4.4.7. These geometric estimates shall be needed when using $\partial_t f$ as a multiplier for the equation for ψ . In order to apply Proposition 22, we must show that the error integrals are sufficiently small in terms of s and τ . These estimates will be used to control these quantities. Thus, several of the following estimates depend on the parameters s and τ . We recall that we take data for f supported in a unit disk whose center has coordinates (s, a, b) . The data consists of the trace of f and $\partial_t f$ in Σ_s . Moreover, we recall that $\tau = s - \sqrt{a^2 + b^2}$. Except in Lemma 33, we shall assume that $b = 0$, meaning that $a + \tau = s$. Lemma 33 is the only result which requires us to simultaneously compare the geometry of light cones adapted to ψ , ϕ , and f at the same time. In order to translate these results to the general case where b is potentially nonzero, we can simply replace instances of a with instances of $s - \tau$.

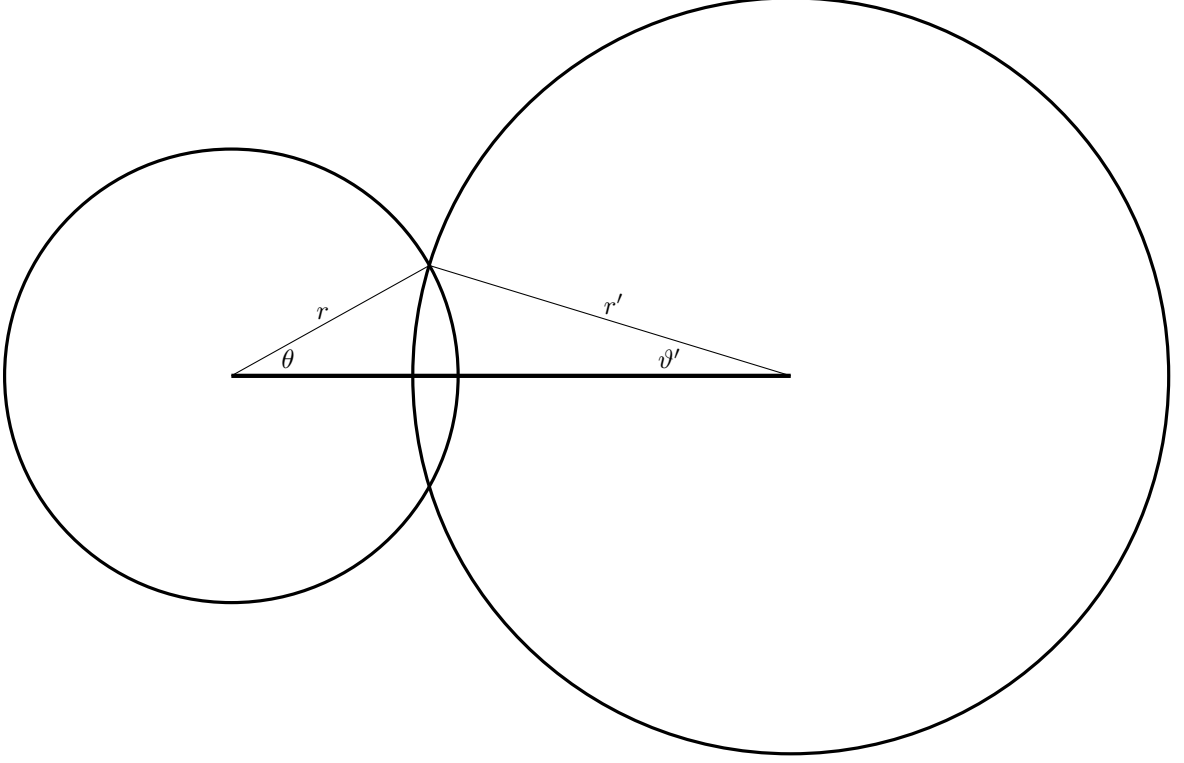


Figure 4.4: The circles denote the intersection of light cones adapted to ψ and f in some fixed Σ_t . The line segment connects the centers of the two resulting circles. The length of this line segment is $a = s - \tau$, where we note that we are assuming that $b = 0$. When $b \neq 0$, there is an appropriate modification for this formula. We note that $r = t - u$ and $r' = s - t - u'$. We also recall that $\vartheta = \pi - \theta$ and that $\vartheta' = \pi - \theta'$. For the spacetime picture, we refer the reader to Figure 4.2. The two points at which the circles intersect correspond to the two points of intersection of the red ellipse with some Σ_t . Of course, it is possible for this intersection to be a single point when the circles are tangent to each other, or to be empty when the light cones are no longer intersecting.

Many of the following lemmas will deal with a configuration arising from looking at the geometry of the light cone for ψ and the light cone for f in a single Σ_t slice. Figure 4.4 shows what this looks like. The reader may wish to keep this figure in mind throughout this Section, as it provides valuable intuition for why the results are true.

Throughout this Section, we recall that we can take $s \geq \frac{1}{\delta_0^{10}}$.

The first lemma says that r is comparable to t and r' is comparable to $s - t$ when we are close to the light cones for ψ and f .

Lemma 27. *Let $\tau \geq 100$.*

1. *We have that $r + r' - a \leq 2r$, and similarly that $r + r' - a \leq 2r'$.*
2. *We have that $r \geq \frac{\tau - u - u'}{2} \geq 0$, and similarly that $r' \geq \frac{\tau - u - u'}{2} \geq 0$.*
3. *In the region where $u \leq \delta_0 \tau$ and $u' \leq \delta_0 \tau$, we have that $r' \geq \frac{99}{100}(s - t) \geq \frac{\tau}{10}$ and that*

$$r \geq \frac{99}{100}t \geq \frac{\tau}{10}.$$

4. In the region where $u' \leq \delta_0\tau$ and $r \leq t+1$, we have that $t \geq \frac{\tau}{10}$. Similarly, in the region where $u \leq \delta_0\tau$ and $r' \leq s-t+1$, we have that $s-t \geq \frac{\tau}{10}$.

Moreover, in the region where $-1 \leq u \leq \delta_0\tau$ and $-1 \leq u' \leq \delta_0\tau$, we have that $r' \geq \frac{99}{100}(s-t) \geq \frac{\tau}{10}$ and that $r \geq \frac{99}{100}t \geq \frac{\tau}{10}$.

Proof. We begin by using the triangle inequality to note that

$$a - r' \leq r \leq a + r'.$$

Thus, we have that

$$0 \leq r + r' - a \leq 2r'.$$

Because the argument is symmetric in r and r' , this proves the first assertions.

Now, we note that

$$\tau - u - u' = \tau - (t - r) - (s - t - r') = \tau - s + r + r' = r + r' - a \leq 2r. \quad (4.78)$$

Similarly, we have that

$$\tau - u - u' = \tau - (t - r) - (s - t - r') = \tau - s + r + r' = r + r' - a \leq 2r'. \quad (4.79)$$

This implies the lower bounds on r and r' in terms of $\tau - u - u'$.

We now assume the further restrictions on u and u' , which imply that

$$\frac{\tau}{2} \leq \tau - u - u' = r + r' - a \leq 2r.$$

From this, it follows that

$$r \geq \frac{\tau}{4}.$$

Now, because $u \leq \delta_0\tau$, we have that

$$\frac{r}{t} = \frac{r}{r+u} \geq \frac{r}{r+\delta_0\tau} = \frac{1}{1+\frac{\delta_0\tau}{r}} \geq \frac{99}{100}$$

for δ_0 sufficiently small. This gives us the desired result for r and t , and by symmetry, the same

argument gives us the desired result for r' and $s - t$.

The fourth and final assertion follows from the second assertion, noting that $t = r + u$, and using that $u' \leq \delta_0 \tau$. This implies the lower bound on t , and the lower bound on $s - t$ follows in an analogous way.

□

The next lemma allows us to effectively use the fact that the light cones intersect transversally. It is here that the assumption that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ is fundamental. In the region along both light cones, we expect that the Jacobian of the $(r, \bar{r})$ coordinate system in Σ_t is comparable to 1 because the light cones intersect transversally (see Figure 4.3). This will allow us to effectively take advantage of decay in u and $\bar{u}$. The necessary properties of the $(r, \bar{r})$ coordinate system within each Σ_t along both light cones is established in the following lemma.

Lemma 28. *In the region where $t \geq 100$, $r \leq t + 1$, $\bar{r} \leq t + 1$, $|u| \leq \frac{1}{100}t$, and $|u'| \leq \frac{1}{100}t$, we have that $(r, \bar{r})$ is a well defined coordinate system on each Σ_t with uniformly bounded Jacobian. More precisely, we have that*

$$dx \wedge dy = \frac{r\bar{r}}{(\lambda_2^2 - \lambda_1^2)xy} dr \wedge d\bar{r}, \quad (4.80)$$

and we have that

$$\left| \frac{r\bar{r}}{(\lambda_2^2 - \lambda_1^2)xy} \right| \leq C,$$

where C depends only on λ_1 and λ_2 , and is uniform given our restrictions on λ_1 and λ_2 .

Proof. We begin by noting that r and $\bar{r}$ are comparable to t in this region. Thus, we have that r and $\bar{r}$ are comparable to each other. More precisely, we have that

$$\frac{9}{10} \leq \frac{r}{\bar{r}} \leq \frac{10}{9} \quad (4.81)$$

because the conditions on u and u' imply that

$$\frac{99}{100} \leq \frac{r}{t} \leq \frac{101}{100},$$

and similarly for $\frac{\bar{r}}{t}$.

We shall now establish the formula (4.80). We shall then show that x and y are also comparable to r and $\bar{r}$ in this region from which the desired result will follow.

We have that

$$rdr = xdx + ydy,$$

and we have that

$$\bar{r}d\bar{r} = \lambda_1^2 xdx + \lambda_2^2 ydy.$$

Thus, we have that

$$r\bar{r}dr \wedge d\bar{r} = (\lambda_2^2 - \lambda_1^2)xydx \wedge dy.$$

This implies (4.80), as desired.

Now, in order to show that x and y are comparable to r and $\bar{r}$ in this region, we go into polar coordinates (r, θ) . We shall show that θ is bounded uniformly away from 0 , $\frac{\pi}{2}$, π , and $\frac{3\pi}{2}$. This will imply the desired result.

Writing $\bar{r}$ in polar coordinates, we have that

$$\bar{r}^2 = \lambda_1^2 x^2 + \lambda_2^2 y^2 = \lambda_1^2 r^2 + (\lambda_2^2 - \lambda_1^2)y^2 = \lambda_1^2 r^2 + (\lambda_2^2 - \lambda_1^2)r^2 \sin^2(\theta).$$

Now, for θ sufficiently close to 0 or to π , we have that

$$\bar{r}^2 \leq \frac{11}{10}\lambda_1^2 r^2.$$

Thus, using the bounds on λ_1^2 , we have that

$$\frac{\bar{r}}{r} \leq \frac{3}{4}.$$

This contradicts the bounds (4.81). Similarly, for θ sufficiently close to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, we have that

$$\frac{\bar{r}}{r} \geq \frac{3}{2},$$

once again contradicting (4.81).

□

The next lemma writes x , $x - a$, and y in terms of r and r' when τ is small in the region along the light cones for ψ and f . The fact that τ is small and that we are along the light cones for ψ and f gives us smallness in $\frac{1}{a}$ and $\frac{\tau}{a}$, and it lets us treat u and u' as errors.

Lemma 29. *Let $\tau \leq \delta_0 s$.*

1. We have that

$$x = r \left(1 - \frac{\tau - u - u'}{r} + \frac{\tau - u - u'}{a} - \frac{(\tau - u - u')^2}{2ra} \right). \quad (4.82)$$

2. We have that

$$x - a = -r' \left(1 - \frac{\tau - u - u'}{r'} + \frac{\tau - u - u'}{a} - \frac{(\tau - u - u')^2}{2r'a} \right). \quad (4.83)$$

3. We have that

$$y^2 = r^2 - r^2 \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r} - \frac{(\tau - u - u')^2}{2ra} \right)^2 \leq Cr(\tau - u - u'). \quad (4.84)$$

4. We have that

$$y^2 = (r')^2 - (r')^2 \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r'} - \frac{(\tau - u - u')^2}{2r'a} \right)^2 \leq Cr'(\tau - u - u'). \quad (4.85)$$

Moreover, we have that $a \geq (1 - \delta_0)s$.

Proof. We have that $x^2 + y^2 = r^2$, and that $(x - a)^2 + y^2 = (r')^2$. This second expression expands out to $x^2 - 2xa + a^2 + y^2 = (r')^2$. Now, subtracting this from the first expression gives

$$2xa = r^2 - (r')^2 + a^2. \quad (4.86)$$

Now, we write $r = t - u$ and $r' = s - t - u'$. From this, it follows that

$$r + r' = s - u - u' = a + \tau - u - u'. \quad (4.87)$$

Solving for r' and plugging this in to (4.86) gives us

$$\begin{aligned} 2xa &= r^2 - (a + \tau - r - u - u')^2 + a^2 \\ &= r^2 - r^2 + 2ar + 2r(\tau - u - u') - a^2 - 2a(\tau - u - u') - (\tau - u - u')^2 + a^2 \\ &= 2ar + 2r(\tau - u - u') - 2a(\tau - u - u') - (\tau - u - u')^2. \end{aligned} \quad (4.88)$$

Thus, we have that

$$\begin{aligned} x &= r + \frac{r(\tau - u - u')}{a} - (\tau - u - u') - \frac{(\tau - u - u')^2}{2a} \\ &= r \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r} - \frac{(\tau - u - u')^2}{2ra} \right), \end{aligned} \tag{4.89}$$

giving us the first equality.

For $x - a$, we note that (4.86) implies that

$$x - a = \frac{r^2 - (r')^2 - a^2}{2a}.$$

Thus, using (4.87) to solve for r in terms of r' , a , τ , u , and u' gives us that

$$x - a = \frac{-2ar' + 2a(\tau - u - u') - 2r'(\tau - u - u') + (\tau - u - u')^2}{2a}.$$

Factoring out $-r'$ gives us that

$$x - a = -r' \left(1 - \frac{\tau - u - u'}{r'} + \frac{\tau - u - u'}{a} - \frac{(\tau - u - u')^2}{2ar'} \right),$$

as desired.

The expressions for y^2 now follow simply from using the equation relating r , x , and y , and the equation relating r' , $x - a$, and y . We have that

$$y^2 = r^2 - x^2 = r^2 - r^2 \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r} - \frac{(\tau - u - u')^2}{2ra} \right)^2,$$

and we have that

$$y^2 = (r')^2 - (x - a)^2 = (r')^2 - (r')^2 \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r'} - \frac{(\tau - u - u')^2}{2r'a} \right)^2.$$

Now, we note that

$$\tau - u - u' = r + r' - a \geq 0$$

by (4.78). Moreover, by Lemma 27, we have that

$$\frac{\tau - u - u'}{r} \leq C,$$

and we have that

$$\frac{\tau - u - u'}{r'} \leq C.$$

The desired bounds on the expression for y then follow after noting that $\frac{r}{a} \leq C$ and $\frac{r'}{a} \leq C$ because $\tau \leq \delta_0 s$.

The final point that $a \geq (1 - \delta_0)s$ follows immediately because $a = s - \tau$.

□

In order to control higher order nonlinearities such as $(\partial_t \psi)^4$, we must analyze the (r, r') coordinate system more carefully. We have the following lemma.

Lemma 30. *We have that*

$$rr' dr \wedge dr' = ay dx \wedge dy. \quad (4.90)$$

Moreover, let $\tau \geq 100$ and let $\tau \leq \delta_0 s$. We then fix any Σ_t with $0 \leq t \leq s$, and we take the annular region of thickness 1 within Σ_t given by $b \leq u \leq b + 1$ subject to the constraint that $-1 \leq b \leq \delta_0 \tau - 1$. Similarly, we take the annular region of thickness approximately $\delta_0 \tau$ within Σ_t given by $-1 \leq u' \leq \delta_0 \tau$. Then, we have that

$$\int_{\Sigma_t} \chi_{b \leq u \leq b+1} \chi_{-1 \leq u' \leq \delta_0 \tau} \frac{1}{1 + |u'|^{\frac{1}{2}}} dx \leq C \min(\sqrt{1+t} \log(1+t), \sqrt{1+s-t}). \quad (4.91)$$

Before turning to the proof, we mention that this integral estimate arises from trying to control interactions where $|u|$ and $|u'|$ are small. We are able to assume that we have compact support in $|u|$ because of the strong decay in $|u|$. However, an integration by parts in the scaling vector field will result in only $|u'|^{\frac{1}{2}}$ decay in $|u'|$ on the auxiliary multiplier f , so we must understand these integrals as opposed to simply understanding the intersections of two annular regions of thickness 1. Moreover, the conditions we have imposed on the regions may seem very arbitrary. The estimates are used to control quartic interactions in the nonlinearity. The motivation can be summarized by saying that we are most concerned with the region along the light cones for ψ and f , and this determines the restrictions above.

Proof. We have that

$$r dr = x dx + y dy,$$

and that

$$r' dr' = (x - a) dx + y dy.$$

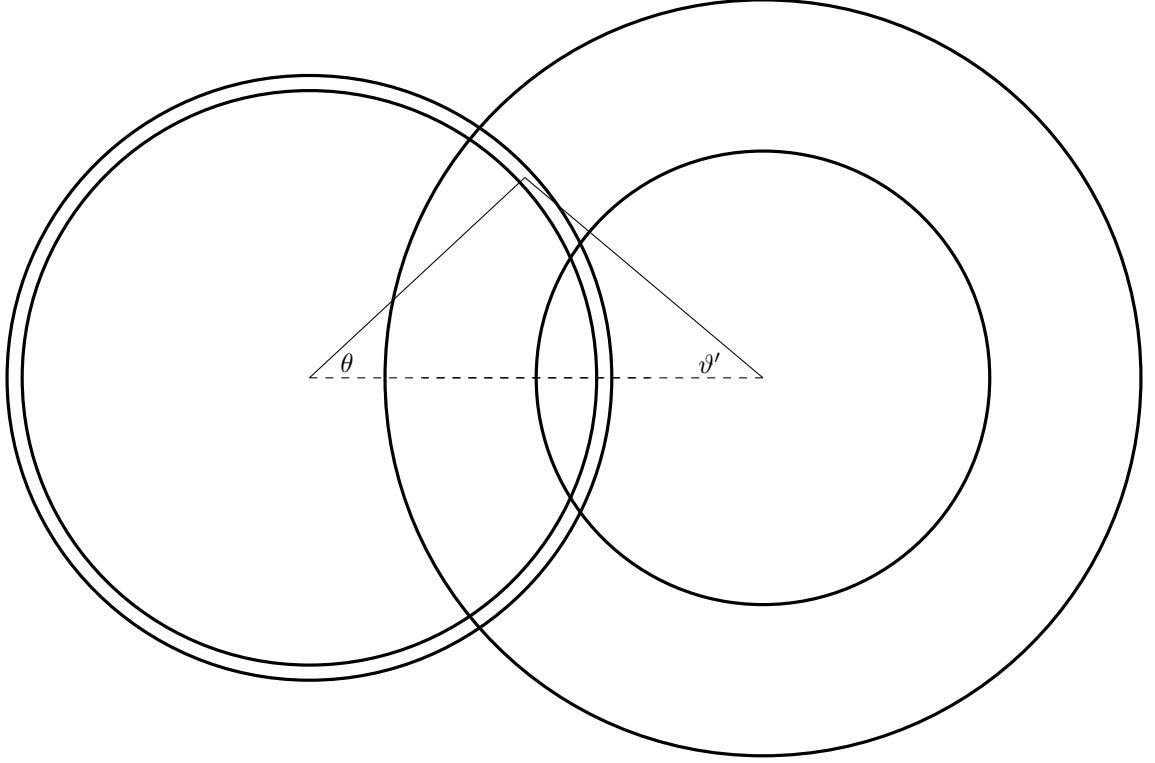


Figure 4.5: One possible configuration that schematically describes the regions where $t \leq \frac{3s}{4}$ and $|\vartheta| \geq \delta_0$, and also where $t \geq \frac{s}{4}$ and $|\theta'| \geq \delta_0$. The thin annulus should be thought of as being the annulus where $b \leq u \leq b+1$. The thick annulus should be thought of as being the annulus where $-1 \leq u' \leq \delta_0\tau$. The points in question are those which are in both annuli. The integral we are computing involves the function $\frac{1}{1+|u'|^2}$, which decays as we move away from the outer circle of the thick annulus towards the inner circle. In these regions, the important fact we are using is that $y\theta$ and $y\vartheta'$ satisfy good lower bounds (see Lemma 40), meaning that we have good bounds on the Jacobian (4.90). We are thus using that $u \leq \delta_0\tau$ and $u' \leq \delta_0\tau$.

From this, it immediately follows that

$$rr' dr \wedge dr' = ay dx \wedge dy,$$

as desired.

There are now 6 separate regions to consider. We first consider two cases depending on the size of t . When $t \leq \frac{3s}{4}$, we consider 3 regions adapted to r , and when $t \geq \frac{s}{4}$, we consider 3 regions adapted to r' . We first consider the regions adapted to r when $t \leq \frac{3s}{4}$. We also note that the integrand is 0 unless if $\frac{\tau}{10} \leq t \leq s - \frac{\tau}{10}$ by Lemma 27, and we note that $a \geq \frac{s}{2}$ because $\tau \leq \delta_0 s$.

The first region is where $|\theta| > \delta_0$ and $|\pi - \theta| > \delta_0$. Figure 4.5 shows schematically what this region looks like.

In this region, we have that $y = r \sin(\theta)$ is comparable to r , meaning that

$$\frac{rr'}{ay} \leq C.$$

Thus, the integral is controlled by

$$\int_{t-b-1}^{t-b} \int_{s-t-\delta_0\tau}^{s-t+1} \frac{1}{1 + |u'|^{\frac{1}{2}}} dr' dr \leq C(1 + \sqrt{\tau}) \leq C(1 + \sqrt{t}).$$

The second region is where $|\theta| \leq \delta_0$. This region also looks roughly like the one shown in Figure 4.5.

We begin by noting that we must have that $|\theta| \geq c\frac{\sqrt{\tau}}{\sqrt{s}}$ because $t \leq \frac{3}{4}s$ by Lemma 39. We thus also have that $y^2 \geq cr\tau$, meaning that $y \geq c\sqrt{r}\sqrt{\tau}$. Indeed, by Lemma 40, we have that $y\theta \geq \frac{\tau}{10}$ in this region, meaning that $y^2 \geq cry\theta \geq cr\tau$, where we are using that $r\theta$ is comparable to y because $|\theta| \leq \delta_0$.

Thus, we have that

$$\frac{rr'}{ay} \leq C\frac{\sqrt{r}}{\sqrt{\tau}}.$$

Thus, the integral is controlled by

$$\int_{t-b-1}^{t-b} \int_{s-t-\delta_0\tau}^{s-t+1} \frac{\sqrt{r}}{\sqrt{\tau} \left(1 + |u'|^{\frac{1}{2}}\right)} dr' dr \leq C\sqrt{1+t}.$$

We finally consider the region where $|\vartheta| = |\pi - \theta| \leq \delta_0$. Figure 4.6 shows schematically what this region looks like.

We now go into polar coordinates adapted to r . We now note the relation

$$r' \cos(\vartheta') = a - x = r \cos(\vartheta) + a.$$

We want to express r' and ϑ' in terms of r and ϑ . We take a derivative of this expression in ϑ , and we get

$$\frac{\partial r'}{\partial \vartheta} \cos(\vartheta') - r' \sin(\vartheta') \frac{\partial \vartheta'}{\partial \vartheta} = -r \sin(\vartheta). \quad (4.92)$$

Thus, we have that

$$\frac{\partial r'}{\partial \vartheta} \cos(\vartheta') = -y \left(1 - \frac{\partial \vartheta'}{\partial \vartheta}\right),$$

meaning that

$$\left| \frac{\partial r'}{\partial \vartheta} \right| \geq c|y| \left(1 - \frac{\partial \vartheta'}{\partial \vartheta}\right), \quad (4.93)$$

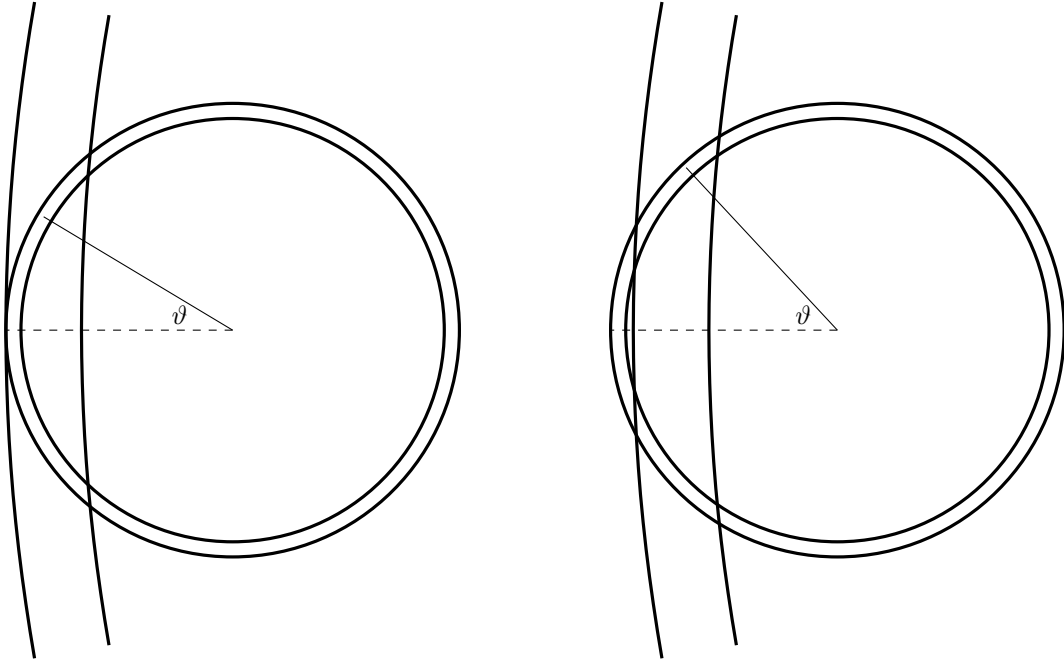


Figure 4.6: Two configurations giving a rough idea of the regime where $|\vartheta| = |\pi - \theta| \leq \delta_0$. The thin annulus should be thought of as being the annulus where $b \leq u \leq b + 1$. The long outer arc should be thought of as being the outer edge given by $r' = s - t + 1$. The distance between the long outer arc and the long inner arc is comparable to $\delta_0 \tau + 1$ and can be much larger than 1 in practice, which is why this thickness is much larger than the thickness of the annulus. The points in question are the points in the annulus, to the right of the outer long arc, and to the left of the inner long arc. Of course, the picture is a bit misleading because these points should also have very small $|\vartheta|$ value. The integral we are computing involves the function $\frac{1}{1+|u'|^{\frac{1}{2}}}$, which decays as we move away from the outer long arc towards the inner long arc.

where we have used the fact that $|\vartheta| \leq \delta_0$ implies that $|\vartheta'| \leq 10\delta_0$ by Lemma 40 (in fact, we have that $|\vartheta'| \leq |\vartheta|$ in this region) along with the fact that $y = r \sin(\vartheta) = r' \sin(\vartheta')$. We shall show that the derivative is approximately equal to $c - y$. We shall then be able to integrate $\frac{\partial u'}{\partial \vartheta}$ in order to control the size of $\frac{1}{1+|u'|^{\frac{1}{2}}}$ in the integrand.

We shall now bound $\frac{\partial \vartheta'}{\partial \vartheta}$ in this region. We recall that

$$\vartheta' = \arctan\left(\frac{y}{a-x}\right) = \arctan\left(\frac{r \sin(\vartheta)}{r \cos(\vartheta) + a}\right).$$

Thus, we have that

$$\frac{\partial \vartheta'}{\partial \vartheta} = \frac{1}{1 + \frac{r^2 \sin^2(\vartheta)}{(r \cos(\vartheta) + a)^2}} \left(\frac{r \cos(\vartheta)}{r \cos(\vartheta) + a} + \frac{r^2 \sin^2(\vartheta)}{(r \cos(\vartheta) + a)^2} \right)$$

Now, we have that

$$r \cos(\vartheta) + a = a \left(\frac{r}{a} \cos(\vartheta) + 1 \right) \geq \frac{a}{2},$$

where we have used the fact that

$$\frac{r}{a} \leq C \frac{\tau}{a} \tag{4.94}$$

for some C independent of δ_0 in the region in question, and also that $\tau \leq \delta_0 s \leq 2\delta_0 a$.

We shall now pause briefly to show that $\frac{r}{a} \leq C \frac{\tau}{a}$ for some C independent of δ_0 indeed holds. Because $|\vartheta| \leq \delta_0$, we have that $x = -r \cos(\vartheta) \leq -\frac{r}{2}$. Thus, using Lemma 29, we have that

$$r \left(1 - \frac{\tau - u - u'}{r} + \frac{\tau - u - u'}{a} - \frac{(\tau - u - u')^2}{2ra} \right) \leq -\frac{r}{2}.$$

Moreover, using that

$$\frac{\tau - u - u'}{a} \leq 100\delta_0,$$

we have that

$$-(\tau - u - u') \left(1 - \frac{r}{a} + O(\delta_0) \right) \leq -\frac{3}{2}r.$$

Thus, because $\frac{r}{a} \leq 10$, we have that

$$r \leq 100\tau,$$

for δ_0 sufficiently small. This allows us to say that, in fact, we have that

$$\frac{r}{a} \leq C \frac{\tau}{a}$$

for some C independent of δ_0 (for example, we can take $C = 100$), as desired.

Using this, we have that

$$\left| \frac{\partial \vartheta'}{\partial \vartheta} \right| \leq C \frac{r}{a} \leq C \frac{\tau}{a}. \quad (4.95)$$

Because the constant C in this expression does not depend on δ_0 and because $\frac{\tau}{a} \leq 2\delta_0$, this expression can be made arbitrarily small by picking δ_0 sufficiently small. From this and from (4.93), it follows that

$$\left| \frac{\partial r'}{\partial \vartheta} \right| \geq c|y| = cr|\sin(\vartheta)| \geq cr|\vartheta|. \quad (4.96)$$

This allows us to argue as follows. We have that the integral in (4.91) is controlled by

$$\int_{t-b-1}^{t-b} \int_{-\delta_0}^{\delta_0} \chi_{-1 \leq u' \leq \delta_0 \tau} \frac{1}{1 + |u'|^{\frac{1}{2}}} r d\vartheta dr.$$

Because the integrand is symmetric with respect to reflections over the x axis and because the region of integration is as well, it suffices to control the integral for ϑ from 0 to δ_0 instead. Now, for fixed r , we pick $\vartheta_0(r)$ such that u' is minimized under the constraint that $u' \geq -1$ (otherwise, the integrand is 0). We note then that the integrand is supported where $\vartheta \geq \vartheta_0(r)$ because we in fact have that $\frac{\partial r'}{\partial \vartheta} < 0$ for ϑ small and positive by (4.92). Thus, the integral we must control is given by

$$\int_{t-b-1}^{t-b} \int_{\vartheta_0(r)}^{\delta_0} \chi_{-1 \leq u' \leq \delta_0 \tau} \frac{1}{1 + |u'|^{\frac{1}{2}}} r d\vartheta dr.$$

Using (4.96), we then have that

$$u'(r, \vartheta) = u'(r, \vartheta_0) + \int_{\vartheta_0}^{\vartheta} \frac{\partial u'}{\partial \vartheta} d\vartheta_1 \geq -1 + c \int_{\vartheta_0}^{\vartheta} r \vartheta_1 d\vartheta_1 \geq -1 + cr(\vartheta_0 + \vartheta)(\vartheta - \vartheta_0),$$

where we have used that $u'(r, \vartheta_0) \geq -1$ by the presence of $\chi_{u' \geq -1}$. We now excise a neighborhood of thickness $\frac{10}{\sqrt{t}}$ around ϑ_0 . When $\vartheta - \vartheta_0 \geq \frac{10}{\sqrt{t}}$, we note that

$$1 + |u'(r, \vartheta)| \geq cr(\vartheta - \vartheta_0)^2 \geq c. \quad (4.97)$$

Now, we have that

$$\int_{t-b-1}^{t-b} \int_{\vartheta_0(r)}^{\delta_0} \frac{1}{1 + |u'|^{\frac{1}{2}}} r d\vartheta dr \leq \int_{t-b-1}^{t-b} \int_{\vartheta_0(r)}^{\vartheta_0(r) + \frac{10}{\sqrt{t}}} \frac{1}{1 + |u'|^{\frac{1}{2}}} r d\vartheta dr + \int_{t-b-1}^{t-b} \int_{\vartheta_0(r) + \frac{10}{\sqrt{t}}}^{\delta_0} \frac{1}{1 + |u'|^{\frac{1}{2}}} r d\vartheta dr,$$

where the second integral is taken to be 0 when $\vartheta_0(r) + \frac{10}{\sqrt{t}} \geq \delta_0$.

We now examine each of these integrals. For the first integral, we note that the inner integral is controlled by

$$C \int_{\vartheta_0(r)}^{\vartheta_0(r) + \frac{10}{\sqrt{t}}} d\vartheta \leq \frac{C}{\sqrt{t}},$$

meaning that the first integral is controlled by $C(1 + \sqrt{t})$, as desired. For the second integral, we use (4.97) to show that $|u'|$ is relatively large, giving us that

$$\int_{\vartheta_0(r) + \frac{10}{\sqrt{t}}}^{\delta_0} \frac{1}{1 + |u'|^{\frac{1}{2}}} d\vartheta \leq C \int_{\vartheta_0(r) + \frac{10}{\sqrt{t}}}^{\delta_0} \frac{1}{\sqrt{r}(\vartheta - \vartheta_0(r))} d\vartheta \leq \frac{C(1 + \log(1 + t))}{\sqrt{r}}.$$

From this, it follows that the second integral is controlled by $C(1 + \log(1 + t)\sqrt{t})$. This proves the correct upper bound in the region where $t \leq \frac{3}{4}s$.

We now turn to the regions where $t \geq \frac{s}{4}$, which are adapted to r' . The first region is where $|\theta'| > \delta_0$ and $|\pi - \theta'| > \delta_0$, the second region is where $|\pi - \theta'| \leq \delta_0$, and the third region is where $|\theta'| \leq \delta_0$. The bounds in the first two regions follow from the same argument to control the corresponding regions adapted to r . They both also roughly look like the configuration found in Figure 4.5.

We thus turn to control the final region where $|\theta'| \leq \delta_0$. Figure 4.7 gives an idea of what this region looks like.

We use the relation

$$r \cos(\theta) = r' \cos(\theta') + a.$$

We shall analyze this in the (r, θ) coordinate system. Taking a derivative with respect to θ gives us that

$$-r \sin(\theta) = \frac{\partial r'}{\partial \theta} \cos(\theta') - r' \sin(\theta') \frac{\partial \theta'}{\partial \theta}.$$

Thus, we have that

$$\frac{\partial r'}{\partial \theta} \cos(\theta) = y \left(\frac{\partial \theta'}{\partial \theta} - 1 \right). \quad (4.98)$$

Now, we have that

$$\theta' = \arctan \left(\frac{y}{x - a} \right) = \arctan \left(\frac{r \sin(\theta)}{r \cos(\theta) - a} \right)$$

From this, it follows that

$$\frac{\partial \theta'}{\partial \theta} = \frac{1}{1 + \frac{r^2 \sin^2(\theta)}{(r \cos(\theta) - a)^2}} \left(\frac{r \cos(\theta)}{r \cos(\theta) - a} + \frac{r^2 \sin^2(\theta)}{(r \cos(\theta) - a)^2} \right)$$

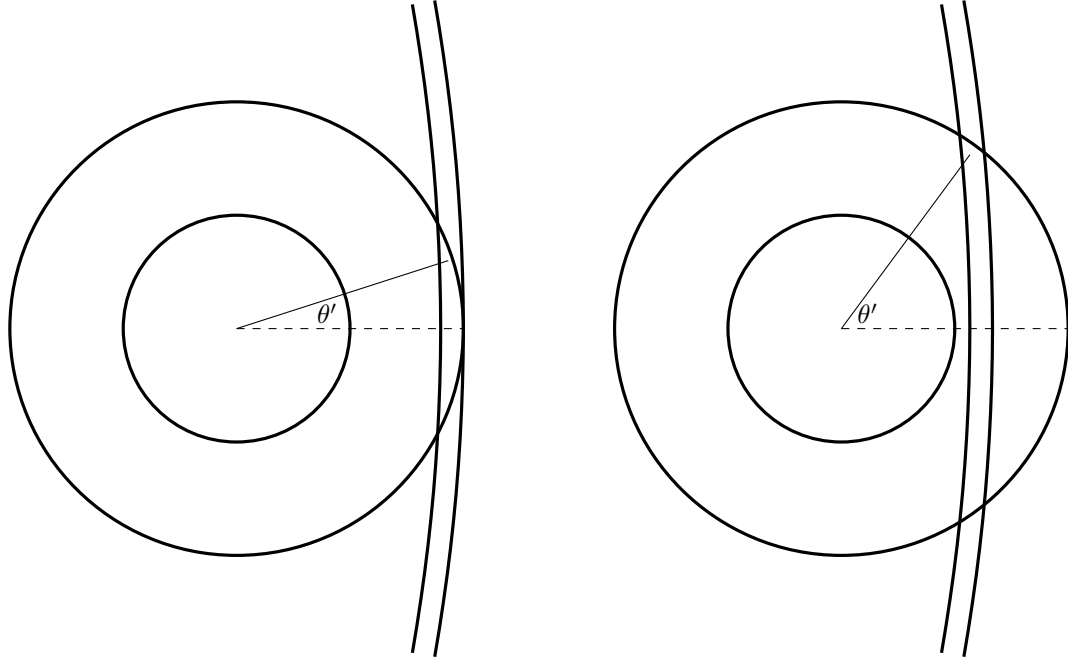


Figure 4.7: Two configurations giving a rough idea of the regime where $|\theta'| \leq \delta_0$. The two large arcs correspond to the region where $b \leq u \leq b + 1$. The thick annulus corresponds to $-1 \leq u' \leq \delta_0 \tau$, and it can in general be rather large. The points in question are the points in the annulus, to the left of the outer long arc, and to the right of the inner long arc. Just like in Figure 4.6, the picture is a bit misleading because these points should also have very small $|\theta'|$ value. The integral we are computing involves the function $\frac{1}{1+|u'|^2}$, which decays as we move away from the outer circle in the annulus towards the inner circle.

We now use the relations

$$r \cos(\theta) - a = r' \cos(\theta')$$

and

$$r \sin(\theta) = r' \sin(\theta')$$

in order to conclude that

$$\frac{\partial \theta'}{\partial \theta} = \frac{1}{1 + \frac{\sin^2(\theta')}{\cos^2(\theta')}} \left(\frac{r \cos(\theta)}{r' \cos(\theta')} + \frac{\sin^2(\theta')}{\cos^2(\theta')} \right).$$

Thus, we have that

$$\frac{1}{1 + \delta_0} \frac{r \cos(\theta)}{r' \cos(\theta')} \leq \frac{\partial \theta'}{\partial \theta} \leq (1 + \delta_0) \frac{r \cos(\theta)}{r' \cos(\theta')}. \quad (4.99)$$

Now, we have that the integral in (4.91) is controlled by

$$\int_{t-b-1}^{t-b} \int_{-\theta_1(r)}^{\theta_1(r)} \frac{1}{1 + |u'|^{\frac{1}{2}}} r d\theta dr \quad (4.100)$$

for some $\theta_1(r)$ (note that we have used that this region is symmetric with respect to reflections over the x axis). These points are determined by the restrictions that $r' \leq s - t + 1$ and $-\delta_0 \leq \theta' \leq \delta_0$.

We shall now bound the inner integral

$$\int_{-\theta_1(r)}^{\theta_1(r)} \frac{1}{1 + |u'|^{\frac{1}{2}}} d\theta$$

by $C \frac{\sqrt{s-t}}{t}$.

By symmetry, it suffices to control the integral from 0 to $\theta_1(r)$ instead. Now, we note that $\frac{\partial \theta'}{\partial \theta}$ is bounded from below by $\frac{1}{2} \frac{r}{r'}$ by (4.99) (note that $\frac{r}{r'} \geq \frac{1}{10\delta_0}$ in this region by an argument similar to the one used to show (4.94)). Thus, using this, (4.98), and Lemma 27, we have that

$$\frac{\partial r'}{\partial \theta} \geq c \frac{r^2}{r'} \theta \geq c \frac{t^2}{s-t} \theta$$

for some c in this region. Now, at $(r, \theta_1(r))$, we note that we have that $u' = -1$. Thus, for $0 \leq \theta \leq \theta_1(r)$, we have that

$$|u'(r, \theta)| = \left| u'(r, \theta_1(r)) - \int_{\theta}^{\theta_1(r)} \frac{\partial u'}{\partial \theta} d\theta_0 \right| \geq c \frac{t^2}{s-t} \theta_1(r) (\theta_1(r) - \theta) - 1.$$

Thus, in this region, we have that

$$\frac{1}{1 + |u'|^{\frac{1}{2}}} \leq \frac{C}{1 + \frac{t\sqrt{\theta_1(r)}}{\sqrt{s-t}} \sqrt{\theta_1(r) - \theta}}.$$

Now, we have that

$$\begin{aligned} \int_0^{\theta_1(r)} \frac{1}{1 + \frac{t\sqrt{\theta_1(r)}}{\sqrt{s-t}} \sqrt{\theta_1(r) - \theta}} d\theta &\leq \frac{\sqrt{s-t}}{t\sqrt{\theta_1(r)}} \int_0^{\theta_1(r)} \frac{1}{\sqrt{\theta_1(r) - \theta}} d\theta = \frac{\sqrt{s-t}}{t\sqrt{\theta_1(r)}} \int_0^{\theta_1(r)} \frac{1}{\sqrt{x}} dx \\ &\leq C \frac{\sqrt{s-t}}{t}. \end{aligned} \quad (4.101)$$

Plugging this into (4.100) and integrating in r then gives us the desired estimate in this region. $\square$

Before proceeding, we make a brief remark about the proof of Lemma 30. It is interesting to note the asymmetry in the logarithmic factor that occurs in t but not in $s - t$. This can be explained by comparing Figure 4.6 and Figure 4.7. As can be seen by examining (4.90), the worst contributions from the volume form come when the level sets of r and r' become tangent. Moreover, we note that we have a decaying factor in u' . In Figure 4.6, $|u'|$ is essentially minimized when these level sets are tangent, while in Figure 4.7, $|u'|$ is maximized when these level sets are tangent. Thus, the region of tangency and worst decay coincide in the region corresponding to Figure 4.6 (which occurs when t is relatively small), but not in the region corresponding to Figure 4.7.

We shall also need a simpler estimate involving the intersection of an annulus of thickness one and an annulus of thickness at most τ .

Lemma 31. *Let $100 \leq \tau \leq \delta_0 s$. Moreover, in Σ_t , let $-1 \leq b \leq 10\tau$ and let $-1 \leq b' \leq \tau$. Then, the area of intersection of an annulus of thickness 1 and radius $r = t - b \leq t + 1$ adapted to r and another annulus of thickness 10τ and radius $r' = s - t - b' \leq s - t + 1$ adapted to r' is at most $C(1 + \sqrt{t})\sqrt{\tau}$. In other words, we have that*

$$\int_{\Sigma_t} \chi_{b \leq u \leq b+1} \chi_{b' \leq u' \leq b'+100\tau} dx \leq C(1 + \sqrt{t})\sqrt{\tau} \quad (4.102)$$

Similarly, the intersection an annulus of thickness 1 and radius $r = t - b \leq t + 1$ and another annulus of thickness 1 and radius $r' = s - t - b' \leq s - t - 1$ has area at most $C \min(1 + \sqrt{t}, 1 + \sqrt{s-t})$. In

other words, we have that

$$\int_{\Sigma_t} \chi_{b \leq u \leq b+1} \chi_{b' \leq u' \leq b'+1} dx \leq C \min(1 + \sqrt{t}, 1 + \sqrt{s-t}).$$

Proof. The proof follows from similar considerations to those in the proof of Lemma 30. We shall now describe the geometric considerations.

We consider several different regimes depending on the relative size of t , $s-t$, and τ . We begin by noting that the result is clearly true when $t \leq \frac{1}{100\delta_0}\tau$ because $C(1 + \sqrt{t})\sqrt{\tau}$ controls the area of the annulus of radius t and thickness 1. We thus assume that $t \geq \frac{1}{100\delta_0}\tau$. This implies that $r = t - u \geq \frac{1}{100\delta_0}\tau - 10\tau - 1 \geq 1000\tau$, which implies that $|\vartheta| \geq \delta_0$ (see (4.94)). We then consider two additional cases, one where $|\theta| \geq \delta_0$ and one where $|\theta| \leq \delta_0$. When $|\theta| \geq \delta_0$, the Jacobian in the (r, r') coordinate system is well behaved (note that $\tau \leq \delta_0 s$), meaning that we may argue as in the proof of Lemma 30 (see Figure 4.5). We may thus restrict ourselves to the case where $|\theta| \leq \delta_0$. Now, we first assume that $s-t \leq \frac{1}{100\delta_0}\tau$. We then have that the diameter of the annulus adapted to r' is controlled by $C\tau$, and we have that $t \geq \frac{9}{10}s$ in this region. Thus, using Lemma 34, we get that the area of intersection is at most $C\tau \leq C(1 + \sqrt{t})\sqrt{\tau}$. Thus, we may assume that $s-t \geq \frac{1}{100\delta_0}\tau$. This forces us to have $|\theta'| \geq \delta_0$ (this is analogous to how having $t \geq \frac{1}{100\delta_0}\tau$ forces $|\vartheta| \geq \delta_0$). We now further consider two cases, one where $|\vartheta'| \geq \delta_0$ and another where $|\vartheta'| \leq \delta_0$. In the first case, the Jacobian in (r, r') coordinates is well behaved, and we can argue as in the proof of Lemma 30 (see Figure 4.5). We are thus left with the case where $|\theta| \leq \delta_0$ and $|\vartheta'| \leq \delta_0$. In this region, we consider the natural foliation of the r adapted annulus by circles. If we look at the intersection of any circle in this foliation with the r' annulus, we pick the point with largest x coordinate in this intersection, and we call this value x_0 . Because the region is symmetric in reflection across the x axis, we can assume that $y \geq 0$. We then note that the smallest possible x coordinate that is still in the intersection is greater than or equal to $x_0 - 100\tau$. We can simply compute that increasing θ by $1000\frac{\sqrt{\tau}}{\sqrt{r}}$ will always decrease the x coordinate by at least 100τ (note that $r \geq 1000\tau$ for δ_0 sufficiently small because $t \geq \delta_0^{-1}\tau$). This means that the arc length of the circle contained in this annulus is at most $1000r\frac{\sqrt{\tau}}{\sqrt{r}} \leq C(1 + \sqrt{t})\sqrt{\tau}$. This gives us the first desired result.

The second result follows from similar considerations. The most substantially different cases are when $|\vartheta| \leq \delta_0$ or when $|\theta'| \leq \delta_0$, so we shall now describe how to treat these regions. Because the integrals are now symmetric in r and r' after sending t to $s-t$ (this is because both annuli have thickness 1 now), we can assume that we are considering the case where $|\vartheta| \leq \delta_0$. Because the region is once again symmetric with respect to reflections over the x axis, we can restrict ourselves

to $\vartheta \geq 0$. We look at the natural foliation of the r annulus by circles. Taking any of these circles, we look at the minimum value of $\vartheta \geq 0$ for which the circle intersects the r' annulus. We can then simply compute that increasing the value of ϑ by $\frac{C}{\sqrt{r}}$ will case the value of r' to decrease by at least 1 (see (4.96)), meaning that after this change in ϑ , the circle must exit the support of the r' annulus, which has thickness 1. Thus, the arc length of the circle contained in the r' annulus is controlled by $C(1 + \sqrt{t})$, giving us the desired result. $\square$

We must now get estimates for the (θ, u, u') coordinate system when τ is large (meaning that $\tau \geq \delta_0 s$). This will allow us to effectively integrate in this region. The following result is effectively a consequence of the fact that an upward opening cone and a downward opening cone intersect uniformly transversally when τ is comparable to s (see Figure 4.2).

Lemma 32. *We have that*

$$-rd\theta \wedge du \wedge du' = (1 + \sin(\theta)\sin(\theta') + \cos(\theta)\cos(\theta'))dt \wedge dx \wedge dy.$$

Moreover, let $\tau \geq \delta_0 s$. Then, in the region where $-1 \leq u \leq \delta_0 \tau$ and $-1 \leq u' \leq \delta_0 \tau$, we have that

$$|1 + \sin(\theta)\sin(\theta') + \cos(\theta)\cos(\theta')| \geq c > 0,$$

meaning that

$$dt \wedge dx \wedge dy = J_{\theta, u, u'} rd\theta \wedge du \wedge du'$$

with $|J_{\theta, u, u'}| \leq C$.

Proof. We have that

$$-rd\theta = \frac{y}{r}dx - \frac{x}{r}dy.$$

Moreover, we have that $d\theta \wedge dr \wedge dr'$ vanishes because the one forms involved are all intrinsic to the Σ_t hypersurfaces, which are two dimensional. Thus, we have that

$$-rd\theta \wedge du \wedge du' = rd\theta \wedge dt \wedge (dr + dr') = dt \wedge (-rd\theta) \wedge (dr + dr').$$

We shall now estimate $-rd\theta \wedge (dr + dr')$.

We have that

$$dr + dr' = \left(\frac{x}{r}dx + \frac{x-a}{r'} \right) dx + \left(\frac{y}{r} + \frac{y}{r'} \right) dy.$$

Thus, we have that

$$\begin{aligned} -rd\theta \wedge (dr + dr') &= \left[\frac{y}{r} \left(\frac{y}{r} + \frac{y}{r'} \right) + \frac{x}{r} \left(\frac{x}{r} + \frac{x-a}{r'} \right) \right] dx \wedge dy \\ &= (1 + \sin(\theta) \sin(\theta') + \cos(\theta) \cos(\theta')) dx \wedge dy, \end{aligned} \tag{4.103}$$

as desired.

We shall now prove a lower bound on

$$|1 + \sin(\theta) \sin(\theta') + \cos(\theta) \cos(\theta')|,$$

from which the desired result will follow.

We begin by noting that $\sin(\theta)$ and $\sin(\theta')$ always have the same sign, meaning that $\sin(\theta) \sin(\theta')$ is always positive. Thus, when $x \leq 0$ or $x \geq a$, we have a lower bound of 1 because $\cos(\theta)$ and $\cos(\theta')$ also have the same sign, meaning that $\cos(\theta) \cos(\theta')$ is also positive. We may thus restrict ourselves to the region where $0 \leq x \leq a$.

Now, because $\tau \geq \delta_0 s$ and because of the bounds on $|u|$ and $|u'|$, we note that $|\theta| \geq c$ and $|\theta'| \geq c$ when $0 \leq x \leq x-a$ for some constant c depending only on δ_0 by Lemma 29. Indeed, when $0 \leq x \leq a$, y^2 is minimized when $x = 0$ and $x = a$. The only other critical point in this region corresponds to a maximum for y^2 (this behavior can be seen geometrically by examining the red ellipse in Figure 4.2, and it can be explicitly computed by calculating the intersection of the cones of constant u and constant u' subject to the constraint that $u \leq \delta_0 \tau$ and $u' \leq \delta_0 \tau$). Now, at $x = 0$, we still have that r is comparable to s by Lemma 27. This means that y^2 must still be comparable to s^2 at this point (we can in fact explicitly compute that it is equal to $(\tau - u - u')^2 \left(1 - \frac{\tau - u - u'}{s} + \frac{(\tau - u - u')^2}{s^2} \right) \geq cs^2$ for $\tau \geq \delta_0 s$ when $x = 0$ and $x = a$, see Lemma 29). By symmetry, y^2 must also be comparable to s^2 at $x = a$, meaning that y^2 is uniformly comparable to s^2 whenever $0 \leq x \leq a$, proving the lower bound on $|\theta|$ and $|\theta'|$ because $y^2 = r^2 \sin^2(\theta) = (r')^2 \sin^2(\theta')$. Thus, we have that $1 + \sin(\theta) \sin(\theta') \geq 1 + c^2$, and this implies the desired result. $\square$

Finally, we must get estimates for the $(\bar{r}, r')$ coordinate system. We shall only need these estimates when $\tau \leq \delta_0 s$, $r' \geq 10$, $t \geq (1 - 20\delta_0)s$, and $|\vartheta'| \leq \delta_0$. As a consequence of these assumptions, we note that the level sets of r' and r approximately coincide. Thus, the fact that this is true morally follows from Lemma 28.

Lemma 33. *Let $\tau \leq \delta_0 s$. Then, in the region where $t \geq (1 - 20\delta_0)s$, $|\vartheta'| \leq \delta_0$, $-1 \leq u$, $-1 \leq \bar{u} \leq$*

$\delta_0\tau$, and $-1 \leq u' \leq \delta_0\tau$, we have that

$$dx \wedge dy = J_{\bar{r},r'} d\bar{r} \wedge dr',$$

where

$$|J_{\bar{r},r'}| \leq C.$$

Proof. For this proof only, we shall denote by C a constant which is independent of δ_0 . Moreover, because we need to analyze the geometry of all three light cones in question at the same time, we shall not assume $b = 0$. Thus, the downward opening light cone has its tip at the point (s, a, b) (see Section 4.4.2).

We recall that Θ denotes the θ coordinate of the points where $r' = 0$. We also recall that Θ_i for $i = 1, 2, 3, 4$ denote the four lines of intersection of the cones $u = 0$ and $\bar{u} = 0$, and we recall the analogous five quantities in terms of $\bar{\theta}$. We shall first show that Θ must be very close to one of the Θ_i .

Without loss of generality, we can assume that $0 \leq \Theta \leq \frac{\pi}{2}$. We shall show that Θ must be very close to Θ_1 in this case, and this will give us the desired estimates. The arguments for the other cases follow in an analogous way by symmetry.

The first observation is that, in the region in question, we have that $\Theta_1 - C\delta_0 \leq \theta \leq \Theta_1 + C\delta_0$ where C depends only on λ_1 and λ_2 . Indeed, outside of this region, we have that $\bar{u}$ will no longer be between -1 and $\delta_0\tau$. This is because $\frac{\partial \bar{u}}{\partial \theta}(r, \theta) \geq cr$ for some c depending only on λ_1 and λ_2 . To see this, we recall that $\partial_\theta = x\partial_y - y\partial_x$, and we note that

$$2\bar{r} \frac{\partial \bar{r}}{\partial \theta}(r, \theta) = (x\partial_y - y\partial_x)(\lambda_1^2 x^2 + \lambda_2^2 y^2) = 2(\lambda_2^2 - \lambda_1^2)xy.$$

The desired result then follows because x and y are both comparable to r and $\bar{r}$ in this region. A similar argument works to show that $\bar{\Theta}_1 - C\delta_0 \leq \bar{\theta} \leq \bar{\Theta}_1 + C\delta_0$ because we have that $u \leq 100\delta_0 s$ in the region in question.

The second observation is that we must have that $|\Theta - \Theta_1| \leq C\delta_0$ where C depends only on λ_1 and λ_2 in order for the region to be nonempty. Indeed, given that we are restricting to the region where $t \geq (1 - 20\delta_0)s$, we note that the region in question requires that $r' \leq 20\delta_0 s + 1$. This restricts us to the region where $|\theta - \Theta| \leq 100\delta_0$ (see also Lemma 34). Thus, because θ is already restricted to being close to Θ_1 , we thus have that $|\Theta - \Theta_1| \leq C\delta_0$, as desired. See Figure 4.8 to see a geometric description of what is going on. These considerations will allow us to compare $\bar{\theta}$ and θ' with $\bar{\Theta}_1$ and

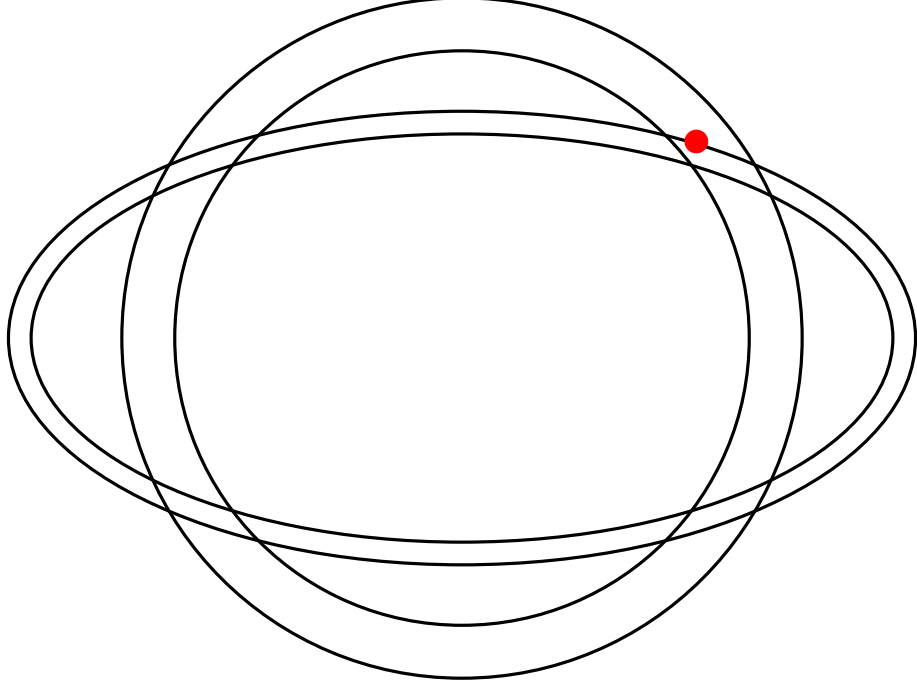


Figure 4.8: A figure showing why θ must be close to Θ and Θ_1 , and why $\bar{\theta}$ must be close to $\bar{\Theta}_1$, where close means within some constant multiplied by δ_0 . The red disk represents $r' \leq 20\delta_0 s + 1$, which is the largest r' can be when $t \geq (1 - 20\delta_0)s$. The line $\theta = \Theta$ is the line going from the center of the circular annulus to the center of the red disk. The circular annulus is $(1 - 50\delta_0)s \leq r \leq s + 1$ and the elliptical annulus is $(1 - 30\delta_0)s \leq \bar{r} \leq s + 1$. We are interested in points contained in the intersection of both annuli and the red disk. Because $\tau \leq \delta_0 s$, we can see that we are restricted to a small range of θ and $\bar{\theta}$ values. Of course, we additionally have the restriction that $|\vartheta'| \leq \delta_0$, but we have not drawn this restriction.

Θ_1 , respectively.

Because we have that $|\vartheta'| \leq \delta_0$ and that $|\bar{\theta} - \bar{\Theta}_1| \leq C\delta_0$, it is natural to get expressions for $d\bar{r}$ and dr' in terms of $\bar{\theta}$ and ϑ' . We have that

$$\bar{r}d\bar{r} = \lambda_1^2 x dx + \lambda_2^2 y dy.$$

Using that $x = \lambda_1^{-1}\bar{x} = \lambda_1^{-1}\bar{r}\cos(\bar{\theta})$ and that $y = \lambda_2^{-1}\bar{y} = \lambda_2^{-1}\bar{r}\sin(\bar{\theta})$, this gives us that

$$d\bar{r} = \lambda_1 \cos(\bar{\theta})dx + \lambda_2 \sin(\bar{\theta})dy.$$

Now, for dr' , we first note that

$$\theta' = \arctan\left(\frac{y-b}{x-a}\right) - \Theta.$$

Moreover, we have that

$$dr' = \frac{x-a}{r'}dx + \frac{y-b}{r'}dy.$$

Using that $\frac{x-a}{r'} = \cos\left(\arctan\left(\frac{y-b}{x-a}\right)\right)$ and that $\frac{y-b}{r'} = \sin\left(\arctan\left(\frac{y-b}{x-a}\right)\right)$ then gives us that

$$dr' = \cos(\theta' + \Theta)dx + \sin(\theta' + \Theta)dy.$$

Thus, we have that

$$d\bar{r} \wedge dr' = (\lambda_1 \cos(\bar{\theta}) \sin(\theta' + \Theta) - \lambda_2 \sin(\bar{\theta}) \cos(\theta' + \Theta))dx \wedge dy.$$

Now, we have that

$$\vartheta' = \pi - \theta' = \pi - \arctan\left(\frac{y-b}{x-a}\right) - \Theta.$$

From the bounds on ϑ' , it follows that

$$\left| \arctan\left(\frac{y-b}{x-a}\right) - \pi - \Theta \right| \leq \delta_0.$$

Thus, we have that $\theta' + \Theta = \pi + \Theta + O(\delta_0) = \pi + \Theta_1 + O(\delta_0)$.

From this, it follows that

$$d\bar{r} \wedge dr' = (\lambda_2 \sin(\bar{\Theta}_1) \cos(\Theta_1) - \lambda_1 \cos(\bar{\Theta}_1) \sin(\Theta_1) + O(\delta_0))dx \wedge dy.$$

This implies the desired result for δ_0 sufficiently small because the quantity is nonzero when $\delta_0 = 0$. Indeed, dividing by $\cos(\Theta_1)$ and $\cos(\bar{\Theta}_1)$ (which we can do because Θ_1 and $\bar{\Theta}_1$ are uniformly bounded away from half integer multiples of π given our restrictions on λ_1 and λ_2), we get that

$$\begin{aligned} \frac{1}{\cos(\Theta_1) \cos(\bar{\Theta}_1)} d\bar{r} \wedge dr' &= (\lambda_2 \tan(\bar{\Theta}_1) - \lambda_1 \tan(\Theta_1) + O(\delta_0))dx \wedge dy \\ &= \left(\frac{\lambda_2 \bar{y}}{\bar{x}} - \frac{\lambda_1 y}{x} + O(\delta_0) \right) dx \wedge dy = \left(\frac{(\lambda_2^2 - \lambda_1^2) \sqrt{1 - \lambda_1^2}}{\lambda_1 \sqrt{\lambda_2^2 - 1}} + O(\delta_0) \right) dx \wedge dy, \end{aligned} \tag{4.104}$$

which is bounded away from 0 uniformly by the restrictions we have placed on λ_1 and λ_2 for δ_0 sufficiently small. We note that we have simply solved for $\frac{y}{x}$ in the equations

$$t^2 = x^2 + y^2 = \lambda_1^2 x^2 + \lambda_2^2 y^2,$$

which gives the different values of $\tan(\theta)$ and $\tan(\bar{\theta})$ corresponding to the four lines of intersection of the two forward opening light cones. This gives us the desired result. $\square$

The final geometric result we shall need concerns the intersections of circles with sets having small diameter relative to the radius of the circle.

Lemma 34. *Let S be a circle of radius r , and let D be a region with diameter $d \leq \frac{1}{2}r$. Then, we have that the length of the portion of S contained in D is controlled by Cd .*

Proof. This follows immediately by going into polar coordinates adapted to S and noting that the set of all θ for which $S \cap D$ is nonempty must be contained in an interval of length at most $C\frac{d}{r}$. Indeed, if this were not true, it would contradict that D has diameter d because the arc length along S is comparable to the Euclidean distance at scales which are small compared to the radius of the circle. $\square$

We note that an analogous statement holds for ellipses.

4.4.5 Scaling vector field geometry

Throughout this Section, we recall that we can take $s \geq \frac{1}{\delta_{10}^0}$. Moreover, we shall once again be assuming without loss of generality that $b = 0$, meaning that $a = s - \tau$ (see Section 4.4.2).

We shall use the above ideas in addition with the scaling vector field $S = t\partial_t + r\partial_r = t\partial_t + x\partial_x + y\partial_y$. We first note that S satisfies good commutation properties with both $\square$ and $\square'$.

Lemma 35. *The vector field S has that $[\square, S] = 2\square$. Similarly, we have that $[\square', S] = 2\square'$.*

Proof. This follows easily from a computation. $\square$

Because of this, S can effectively be used as a commutator for this system of equations.

In order to use S in the way described in Section 4.4.1, we must find a way to write general vectors in the frame consisting of S and $\bar{\partial}_f$, the good derivatives of f . This is the content of the following lemma, which writes $\underline{L}'$ in terms of S and $\bar{\partial}_f$. We introduce the expression

$$\gamma = -\frac{1}{t + r' + \frac{a(x-a)}{r'}}. \quad (4.105)$$

This expression arises as a coefficient in frame decompositions, and it shall be very important in the following.

Lemma 36. *Let $\tau \geq 100$. In the region where $u' \leq \delta_0 \tau$, we have that*

$$\underline{L}'h = 2\gamma S(h) + \gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L'(h) + \gamma \frac{2ay}{(r')^2} \partial_{\theta'}(h) \quad (4.106)$$

where h is any smooth function.

Proof. We have that

$$S = t\partial_t + r\partial_r = -\frac{1}{2}t(L' + \underline{L}') + r'\partial_{r'} + a\partial_x = -\frac{1}{2}t(L' + \underline{L}') + \frac{1}{2}r'(L' - \underline{L}') + a\partial_x.$$

Moreover, we have that

$$\partial_x = \frac{\partial r'}{\partial x} \partial_{r'} + \frac{\partial \theta'}{\partial x} \partial_{\theta'} = \frac{x-a}{r'} \partial_{r'} - \frac{y}{(r')^2} \partial_{\theta'} = \frac{x-a}{2r'} (L' - \underline{L}') - \frac{y}{(r')^2} \partial_{\theta'}.$$

Thus, we have that

$$2S = -t(L' + \underline{L}') + r'(L' - \underline{L}') + \frac{a(x-a)}{r'} (L' - \underline{L}') - \frac{2ay}{(r')^2} \partial_{\theta'}.$$

Thus, we have that

$$-\left(t + r' + \frac{a(x-a)}{r'} \right) \underline{L}' = 2S + \left(t - r' - \frac{a(x-a)}{r'} \right) L' + \frac{2ay}{(r')^2} \partial_{\theta'}.$$

If we recall that $\gamma^{-1} = -\left(t + r' + \frac{a(x-a)}{r'} \right) = -\frac{tr' + (r')^2 + a(x-a)}{r'}$, we thus have that

$$\underline{L}' = 2\gamma S + \gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L' + \gamma \frac{2ay}{(r')^2} \partial_{\theta'},$$

as desired. $\square$

Terms not involving S and having the factor of γ will be improved because γ will be shown to be small. Moreover, they contain good derivatives of f , meaning that they gain a weight of $\frac{1}{r'}$, which is comparable to $\frac{1}{s-t}$ in the region in question. The term with S is dangerous because Sf can be large for the auxiliary multiplier. However, we can integrate it by parts and use the fact that we can commute the equation with S . The gain is then that the factor of γ remains. However, when integrating by parts, it is possible for S to hit γ , meaning that we must control $S(\gamma)$ as well.

Geometrically, the fact that the function γ is well behaved follows from the fact that S everywhere pierces the downward opening light cone of f , and from the fact that it has length comparable to

$t + r$. The angle between S and the normal to the tangent space of the downward pointing light cone for f may, however, be close to $\frac{\pi}{2}$, meaning that S may be almost tangent to the light cone for f . The reader may wish to keep Figure 4.2 in order to get an idea of the interplay of geometry between the scaling vector field and the light cones in question.

The smallness of γ and the control of $S(\gamma)$ are the content of the following lemma.

Lemma 37. *Let $\tau \geq 100$, let $-1 \leq u' \leq \delta_0 \tau$, and let $|u| \leq 100\tau$.*

1. *We have that*

$$|\gamma| \leq \frac{C}{\tau}. \quad (4.107)$$

2. *In the region where $|\vartheta'| = |\pi - \theta'| \geq \delta_0/2$ and $r' \geq 10$, we have that*

$$|S(\gamma)| \leq \frac{C}{r'}. \quad (4.108)$$

3. *In the region where in fact $-1 \leq u \leq \delta_0 \tau$, we have that*

$$|S(\gamma)| \leq \frac{C}{\tau}. \quad (4.109)$$

4. *In the region where $t \leq (1 - 10\delta_0)s$, we have that*

$$|S(\gamma)| \leq \frac{C}{\tau}. \quad (4.110)$$

5. *In the region where $\tau \geq \delta_0 s$ and $r' \geq 10$, we have that*

$$|S(\gamma)| \leq \frac{C}{r'}. \quad (4.111)$$

Before proceeding to the proof, let us try to briefly motivate these different regions. The choice of the regions is motivated by the fact that we have to take advantage of different things in different areas. We can then decompose the error integrals in Section 4.4.7 into sums of integrals over these localized regions, control each one individually, and then add together the (finitely many) resulting bounds.

When $\tau \geq \delta_0 s$, the scaling vector field uniformly pierces the light cone for f , so we expect to have good estimates in this region. When $s - t$ is comparable to s , we can use the fact that all of the expressions will end up having good powers of r' . Because r' is then comparable to $s - t$ along

the light cone for f , we can hope to take advantage of this. When $u \leq \delta_0 \tau$ and $u' \leq \delta_0 \tau$, we can control how tangent the two circles from Figure 4.4 are. This is important for controlling γ and $S(\gamma)$ because the coordinate y naturally appears, and preventing the circles from becoming too tangent is the same thing as getting a lower bound on the y coordinate, which is the vertical distance in Figure 4.4. When $|\vartheta'| \geq \delta_0$, we can get strong estimates for γ , allowing us to control everything.

Proof. We shall do several computations and specialize as necessary. The case where $\tau \leq \delta_0 s$ will require the most work.

We have that

$$\begin{aligned} \gamma &= -\frac{r'}{tr' + (r')^2 + a(x-a)} = -\frac{1}{t + r' + a \cos(\theta')} = -\frac{1}{t + s - t - u' + (s - \tau) \cos(\theta')} \\ &= -\frac{1}{s - u' + (s - \tau) \cos(\theta')}. \end{aligned} \quad (4.112)$$

Now, we note that

$$s - u' + (s - \tau) \cos(\theta') \geq \frac{\tau}{2}$$

because $u' \leq \delta_0 \tau$. This proves the first assertion (4.107) about γ .

Before proceeding, we shall make a few remarks. When $\tau \geq \delta_0 s$, the denominator in (4.112) is very large, and this makes the proof easier. Moreover, we note that it will be natural to consider $\vartheta' = \pi - \theta'$. This is because the most delicate region is where θ' is close to π . As we shall see below, we shall break up the expressions into regions where ϑ' is of different sizes depending on τ and s .

We note that

$$\begin{aligned} S(\gamma) &= -S\left(\frac{1}{t + r' + a \cos(\theta')}\right) = \frac{t}{(t + r' + a \cos(\theta'))^2} + \frac{r \partial_r r' + a r \partial_r \cos(\theta')}{(t + r' + a \cos(\theta'))^2} \\ &= (t + r \partial_r r' + a r \partial_r \cos(\theta')) \gamma^2. \end{aligned} \quad (4.113)$$

The way in which these terms can be controlled is geometrically motivated. We begin by noting that $\partial_r r' \approx -1$ on a large set when τ is small relative to s (when $\tau \geq \delta_0 s$, the proof is more immediate, as we shall see). Thus, we think that $\gamma^2(t + r \partial_r r') \approx \gamma^2 u$. Moreover y should be small relative to other quantities on a large set by Lemma 29. Thus, we expect to be able to control expressions with y in the numerator and other quantities in the denominator. More precisely, we note that $t + r \partial_r r' + a r \partial_r \cos(\theta')$ vanishes in the limit as $\tau \rightarrow 0$. Of course, when $\tau \rightarrow 0$, we have that γ blows up because the downward opening and upward opening light cones become tangent (see Figure 4.2). However, this gives us hope that this term can be controlled when τ is small relative

to s , and that (4.113) is small. We now make these ideas precise.

We shall first calculate the quantity $t + r\partial_r r' + ar\partial_r \cos(\theta')$. We have that

$$r\partial_r r' = x\partial_x(r') + y\partial_y(r') = \frac{x(x-a)}{r'} + \frac{y^2}{r'},$$

and that

$$r\partial_r \cos \theta' = x\partial_x \left(\frac{x-a}{r'} \right) + y\partial_y \left(\frac{x-a}{r'} \right) = \frac{x}{r'} - \frac{x(x-a)^2}{(r')^3} - \frac{y^2(x-a)}{(r')^3}.$$

Thus, we have that

$$S(\gamma) = \gamma^2 (t + r\partial_r r' + ar\partial_r \cos(\theta')) = \gamma^2 \left[t + \frac{x(x-a)}{r'} + \frac{y^2}{r'} + \frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3} - \frac{ay^2(x-a)}{(r')^3} \right]. \quad (4.114)$$

This already suffices to control $S(\gamma)$ when $\tau \geq \delta_0 s$ (we note that $\tau \leq s$ always holds). Indeed, in this region, we have that $\gamma^2 \leq \frac{C}{s^2}$. Thus, every term in (4.114) multiplied by γ^2 is bounded by $\frac{C}{r'}$. This implies (4.111), and it also implies (4.108), (4.109), and (4.110) in the region where $\tau \geq \delta_0 s$ (this requires comparing r' with τ using Lemma 27 for (4.109) and Lemma 39 for (4.110)). Thus, for the remainder of the proof, we shall assume that $\tau \leq \delta_0 s$.

Now, before proceeding, we shall explicitly group the terms in (4.114) into three different categories because they will be dealt with differently. Following the discussion at the beginning of the proof, we shall bound the terms in (4.114) with powers of y in the numerator directly. We shall treat $t + \frac{x(x-a)}{r'}$ as a single term and extract a power of u plus other terms which are well behaved. Then, we shall treat $\frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3}$ as a single term, noting that it is equal to $\frac{axy^2}{(r')^3}$. Thus, the first group consists of

$$\gamma^2 \left[t + \frac{x(x-a)}{r'} \right].$$

The second group consists of

$$\gamma^2 \left[\frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3} \right].$$

The third group consists of

$$\gamma^2 \left[\frac{y^2}{r'} - \frac{ay^2(x-a)}{(r')^3} \right].$$

The second group of terms is the hardest to control.

Following the discussion at the beginning of the proof, we shall bound the terms in (4.114) with powers of y in the numerator directly. We shall treat $t + \frac{x(x-a)}{r'}$ as a single term and extract a power of u plus other terms which are well behaved. Then, we shall treat $\frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3}$ as a single term,

noting that it is equal to $\frac{axy^2}{(r')^3}$.

Using Lemma 29, we have that

$$\begin{aligned} \frac{x(x-a)}{r'} &= -r \left(1 - \frac{\tau-u-u'}{r} + \frac{\tau-u-u'}{a} - \frac{(\tau-u-u')^2}{2ra} \right) \\ &\quad \times \left(1 - \frac{\tau-u-u'}{r'} + \frac{\tau-u-u'}{a} - \frac{(\tau-u-u')^2}{2r'a} \right). \end{aligned} \quad (4.115)$$

Expanding and grouping this with t in (4.114) as described above, we are able to extract a factor of $u = t - r$. Indeed, using (4.78), we have that

$$t + \frac{x(x-a)}{r'} = t - r + \frac{r}{r'}(\tau - u - u') + O(\tau - u - u'),$$

where we have used that $0 \leq \tau - u - u' \leq Cr'$ (see Lemma 27). We have that $\gamma^2(t-r) \leq C\gamma$ because $|u| \leq C\tau$. Moreover, we have that $\gamma^2 O(\tau - u - u') \leq C\gamma$ because $\gamma \leq \frac{C}{\tau}$. The only remaining term is $\frac{r}{r'}(\tau - u - u')$. Because $a \geq \frac{s}{2}$ when $\tau \leq \delta_0 s$, we have that

$$\frac{r}{r'}(\tau - u - u') \leq C \frac{ar(\tau - u - u')}{(r')^2},$$

and this term (see (4.117)) is controlled below. Altogether, we have so far shown that

$$\gamma^2 \left[t + \frac{x(x-a)}{r'} \right] \leq C\gamma + C\gamma^2 \frac{ar(\tau - u - u')}{(r')^2}. \quad (4.116)$$

This reduces controlling the first group of terms to controlling (4.117), which comes up in the second group of terms.

Now, by Lemma 29, we have that

$$y^2 = (r')^2 - (r')^2 \left(1 - \frac{\tau-u-u'}{r'} + \frac{\tau-u-u'}{a} - \frac{(\tau-u-u')^2}{2r'a} \right)^2 \leq Cr'(\tau - u - u')$$

because we are now assuming that $\tau \leq \delta_0 s$. Thus, we have that

$$\left| \frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3} \right| = \left| \frac{ax((r')^2 - (x-a)^2)}{(r')^3} \right| = \left| \frac{axy^2}{(r')^3} \right| \leq C \frac{ar(\tau - u - u')}{(r')^2}. \quad (4.117)$$

Multiplying by γ^2 , we have that

$$\left| \gamma^2 \left(\frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3} \right) \right| \leq C \frac{ar(\tau - u - u')}{(r')^2(t + r' + a \cos(\theta'))^2}.$$

We now have everything in place to establish (4.108). We first recall that

$$t + r' + a \cos(\theta') = t + s - t - u' + (s - \tau) \cos(\theta') = s - u' + (s - \tau) \cos(\theta').$$

Now, for $|\vartheta'| = |\theta' - \pi| \geq \delta_0/2$, we have that

$$|s - u' + (s - \tau) \cos(\theta')| \geq cs^2 \quad (4.118)$$

for some $c > 0$, resulting in the bound

$$|\gamma| \leq \frac{C}{s},$$

and also the bound

$$\left| \gamma^2 \left(\frac{ax}{r'} - \frac{ax(x-a)^2}{(r')^3} \right) \right| \leq C \frac{ar(\tau - u - u')}{s^2(r')^2} \leq \frac{C}{r'},$$

where we have used Lemma 27 to say that $\frac{|\tau - u - u'|}{r'} \leq C$. This controls the second group of terms in this region. We also note that

$$\gamma^2 \frac{y^2}{r'} \leq \frac{C}{r'},$$

and that

$$\gamma^2 \frac{ay^2(x-a)}{(r')^3} \leq \frac{C}{s},$$

which controls the third group of terms in this region. Along with (4.116), these considerations prove (4.108), and they also establish (4.109) in the region where $|\vartheta'| \geq \delta_0/2$ after using Lemma 27 to say that $r' \geq c\tau$ when $u \leq \delta_0\tau$, which is an assumption for the estimate (4.109).

All that remains is to establish (4.110) and (4.109). More precisely, we must only consider the case of $\tau \leq \delta_0 s$ in the region where $|\vartheta'| \leq \delta_0$ by what we have done above. We shall focus first on the most difficult term given by

$$\gamma^2 \frac{ar(\tau - u - u')}{(r')^2}.$$

In the grouping we defined earlier, this corresponds to controlling the second group of terms. The third group of terms will be controlled after.

Now, when $|\vartheta'| \leq \delta_0$, we compute the Taylor expansion around $\vartheta' = \pi - \theta' = 0$ for $t + r' + a \cos(\theta') = s - u' + (s - \tau) \cos(\theta')$, giving us that

$$\begin{aligned} s - u' + (s - \tau) \cos(\theta') &= s - u' - (s - \tau) + \frac{1}{2}(s - \tau)(\vartheta')^2 + (s - \tau)O((\vartheta')^4) \\ &= (\tau - u') + \frac{1}{2}(s - \tau)(\vartheta')^2 + (s - \tau)O((\vartheta')^4). \end{aligned} \quad (4.119)$$

Thus, it suffices to control

$$\frac{ar\tau}{(r')^2 \left(\tau - u' + \frac{1}{2}(s - \tau)(\vartheta')^2 + (s - \tau)O((\vartheta')^4) \right)^2}. \quad (4.120)$$

Let us first assume that $t \leq (1 - 10\delta_0 s)$. Then, the quantity (4.120) is controlled by

$$C \frac{ar\tau}{(r')^2 \tau^2}.$$

By Lemma 39, we have that $r' \geq cs$ in this region, meaning that this quantity is controlled by

$$\frac{C}{\tau}.$$

This controls this term in the region where $t \leq (1 - 10\delta_0 s)$.

More generally, there is a transition of dominant terms in the denominator between $\tau - u'$ and $\frac{1}{2}(s - \tau)(\vartheta')^2$ when they are equal. This occurs when $(\vartheta')^2 = 2\frac{\tau - u'}{s - \tau}$. Now, we have that $\tau - u' \geq (1 - \delta_0)\tau$ because $u' \leq \delta_0\tau$. Moreover, we are assuming that $\tau \leq \delta_0 s$. This means that $(\vartheta')^2$ must be comparable to $\frac{\tau}{s}$. Thus, it is natural to consider two regions, one where $(\vartheta')^2 \leq C\frac{\tau}{s}$ and the other where $(\vartheta')^2 > C\frac{\tau}{s}$ for some constant C . However, for technical reasons, it is easier to work instead with the region where $t \leq cs$ and $t \geq cs$ for some $0 < c < 1$. We prove in Lemma 39 that $t \geq (1 - 10\delta_0)s$, $u' \leq \delta_0\tau$, and $u \leq \delta_0\tau$ implies that $|\vartheta'| > c\frac{\sqrt{\tau}}{\sqrt{s}}$ when $\tau \leq \delta_0 s$. We note that we can now freely assume all of these conditions because all that remains to be shown for the second group of terms is that they are controlled by $\frac{C}{\tau}$ with these assumptions.

Now, in this region, we have that the quantity (4.120) is controlled by

$$C \frac{ar\tau}{(r')^2 s^2 (\vartheta')^4}. \quad (4.121)$$

Moreover, we note that $r'\vartheta'$ is comparable to y (note that we are assuming that $|\vartheta'| \leq \delta_0$). Then, because $y\vartheta'$ is comparable to τ by Lemma 40 below, we have that

$$\frac{ar\tau}{(r')^2 s^2 (\vartheta')^4} \leq C \frac{\tau}{(r')^2 (\vartheta')^4} \leq C \frac{\tau}{y^2 (\vartheta')^2} \leq \frac{C}{\tau}.$$

This completes the control of the second group of terms in every region.

We now proceed to the third and final group of terms. We have that

$$\gamma^2 \frac{ay^2(x-a)}{(r')^3} \leq C\gamma^2 \frac{a\tau}{r'},$$

where we have used Lemma 29 to control y^2 . Now, for $t \leq (1-10\delta_0)s$, we have that r' is comparable to s (see Lemma 39), meaning that this quantity is controlled by

$$C\gamma^2\tau \leq \frac{C}{\tau}.$$

When $t \geq (1-10\delta_0)s$, we have that $\frac{r}{r'} \geq 1$ because $\tau \leq \delta_0 s$, meaning that

$$C\gamma^2 \frac{a\tau}{r'} \leq C\gamma^2 \frac{ar\tau}{(r')^2},$$

and the bound follows in the same way as for the term (4.117) above.

Finally, for the last term in this third group of terms, we use Lemma 29 to control y^2 , giving us that

$$\gamma^2 \frac{y^2}{r'} \leq C\gamma^2\tau \leq \frac{C}{\tau},$$

as desired. □

We now control the other coefficients from the expansion in (4.106), which will allow us to finally control the error integrals. Now, because the L' derivative gains a weight of $\frac{1}{r'}$ when applied to f (see (4.74)), it is natural to consider instead the expression

$$2\gamma S(h) + \gamma \frac{1}{r'} \left(t - r' - \frac{a(x-a)}{r'} \right) r' L'(h) + \gamma \frac{2ay}{(r')^2} \partial_{\theta'}(h).$$

Note that $\partial_{\theta'}$ is already correctly normalized. We then have the following lemma which controls the coefficients of L' and $\partial_{\theta'}$.

Lemma 38. *Let $u \leq 100\tau$, and let $-1 \leq u' \leq \delta_0\tau$.*

1. *In the region where $u \leq \delta_0\tau$, we have that*

$$\left| \gamma \frac{1}{r'} \left(t - r' - \frac{a(x-a)}{r'} \right) \right| \leq \frac{C}{\tau},$$

and we have that

$$\left| \gamma \frac{2ay}{(r')^2} \right| \leq \frac{C}{\tau}.$$

2. In the region where $r' \geq 10$ and $|\vartheta'| = |\pi - \theta| \geq \delta_0/2$, we have that

$$\left| \gamma \frac{1}{r'} \left(t - r' - \frac{a(x-a)}{r'} \right) \right| \leq \frac{C}{r'},$$

and we have that

$$\left| \gamma \frac{2ay}{(r')^2} \right| \leq \frac{C}{r'}.$$

3. In the region where $t \leq (1 - 10\delta_0)s$ and $r' \geq 10$, we have that

$$\left| \gamma \frac{1}{r'} \left(t - r' - \frac{a(x-a)}{r'} \right) \right| \leq \frac{C}{\tau} + \frac{C}{r'},$$

and we have that

$$\left| \gamma \frac{2ay}{(r')^2} \right| \leq \frac{C}{\tau} + \frac{C}{r'}.$$

4. In the case where $\tau \geq \delta_0 s$, we have that

$$\left| \gamma \frac{1}{r'} \left(t - r' - \frac{a(x-a)}{r'} \right) \right| \leq \frac{C}{r'},$$

and we have that

$$\left| \gamma \frac{2ay}{(r')^2} \right| \leq \frac{C}{r'}.$$

Proof. We first consider the relative size of τ and s . When $\tau \geq \delta_0 s$, the result is immediate because γ is controlled by $\frac{C}{s}$ by Lemma 37. We may thus assume that $\tau \leq \delta_0 s$. Moreover, when $t \leq (1 - 10\delta_0)s$, we note that r' is comparable to s (see Lemma 39). Thus, in this region, we can bound the coefficients by $C\gamma$, as desired.

We now consider the region where $t \geq (1 - 20\delta_0)s$ and $|\vartheta'| \leq \delta_0$. With these conditions, we are only interested in the case where we additionally have that $u \leq \delta_0 \tau$. We note that the coefficient of $r'L'$ is bounded by

$$C\gamma + \frac{C\gamma a}{r'}.$$

The first of these terms is bounded by $\frac{C}{\tau}$ by Lemma 37. For the second term, we note that

$$\gamma \frac{a}{r'} \leq C \frac{a}{r' s (\vartheta')^2} \leq C \frac{1}{y \vartheta'},$$

where we have bounded γ in the same way as in (4.121). This quantity is then controlled by $\frac{C}{\tau}$ by

Lemma 40. Then, we have that

$$\left| \gamma \frac{2ay}{(r')^2} \right| \leq C \gamma \frac{a}{r'},$$

meaning that the same argument works here.

Finally, in the region where $|\vartheta'| \geq \delta_0/2$ and $r' \geq 10$, we note that γ is comparable to $\frac{1}{s}$. Thus, we have that

$$\gamma \frac{a}{r'} \leq \frac{C}{r'},$$

giving us the desired result. We note that we are using Lemma 27 to say that $r' \geq c\tau$ in the region where $u \leq \delta_0\tau$.

□

Lemma 39. *Let $\tau \leq \delta_0 s$.*

1. *In the region where $t \geq \frac{1}{4}s$, $u' \leq \delta_0\tau$, and $u \leq \delta_0\tau$, we have that $|\vartheta'| \geq c\frac{\sqrt{\tau}}{\sqrt{s}}$, and we have that $|\vartheta'| \geq \frac{1}{10}|\theta|$. An analogous statement holds when $t \leq \frac{3}{4}s$ with the roles of ϑ' and θ interchanged.*
2. *In the region where $t \leq (1 - 10\delta_0)s$ and $u' \leq \delta_0\tau$, we have that $r' \geq cs$. Similarly, in the region where $t \geq 10\delta_0s$ and $u \leq \delta_0\tau$, we have that $r \geq cs$.*

Proof. We first assume that $t \leq (1 - 10\delta_0)s$ and that $u' \leq \delta_0\tau$. We then have that $s - t \geq 10\delta_0s$. Moreover, because $u' \leq \delta_0\tau$ and $\tau \leq s$, we have that $u' \leq \delta_0s$. Thus, we have that

$$r' = s - t - u' \geq 9\delta_0s.$$

An analogous argument shows a similar lower bound for r instead of r' . This proves the second assertion.

We shall now prove the first assertion. We shall perform the argument to find a lower bound for $|\vartheta'|$. Finding a lower bound for $|\theta'|$ follows in an analogous way.

We assume that $t \geq \frac{1}{4}s$, $u' \leq \delta_0\tau$, and that $u \leq \delta_0\tau$. We then have that $r = t - u \geq \frac{1}{5}s$, and we have that

$$r' \leq s - t + 1 \leq \frac{3}{4}s + 1. \tag{4.122}$$

Then, because $y = r \sin(\theta) = r' \sin(\theta') = r' \sin(\vartheta')$, we must have that $|\vartheta'| \geq \frac{1}{10}|\theta|$. This can be seen by consulting Figure 4.4. More precisely, we have that

$$\frac{\sin(\vartheta')}{\sin(\theta)} = \frac{r}{r'} \geq \frac{1}{5}.$$

Moreover, without loss of generality, we have that $0 \leq \theta \leq \frac{\pi}{2}$ (we cannot have $\frac{\pi}{2} \leq \theta \leq \pi$ because we have that $|x - a| \leq r' \leq \frac{3}{4}s + 1$). We thus have that $\frac{2\theta}{\pi} \leq \sin(\theta) \leq \theta$. Without loss of generality, we may also assume that $0 \leq \vartheta' \leq \pi$. In the region where $\vartheta' \geq \frac{\pi}{2}$, the inequality is then obvious, so we may also assume that $0 \leq \vartheta' \leq \frac{\pi}{2}$. In this region, we once again have that $\frac{2\vartheta'}{\pi} \leq \sin(\vartheta') \leq \vartheta'$. Thus, we have that

$$\vartheta' \geq \sin(\vartheta') \geq \frac{1}{5} \sin(\theta) \geq \frac{1}{5} \frac{2}{\pi} \theta \geq \frac{1}{8} \theta.$$

This implies that $|\vartheta'| \geq \frac{1}{8}|\theta|$ in this region, giving us the desired result.

We finally turn to showing that $|\vartheta'| \geq c\frac{\sqrt{\tau}}{\sqrt{s}}$. By symmetry, we can take $\vartheta' > 0$, and without loss of generality, we may assume that $\vartheta' \leq \delta_0$ (otherwise, there is nothing left to show). By Lemma 29, we have that

$$\begin{aligned} a - x &= r' \left(1 - \frac{\tau - u - u'}{r'} + \frac{\tau - u - u'}{a} - \frac{(\tau - u - u')^2}{2r'a} \right) \\ &= r' - r' \frac{\tau - u - u'}{r'} \left(1 - \frac{r'}{a} + \frac{\tau - u - u'}{2a} \right). \end{aligned} \quad (4.123)$$

Because $r' \leq \frac{3}{4}s + 1$ and because $a \geq (1 - \delta_0)s$ (see Lemma 29), we have that

$$1 - \frac{r'}{a} + \frac{\tau - u - u'}{2a} \geq \frac{1}{10}. \quad (4.124)$$

Then, because $\vartheta' \leq \delta_0$, we have that $y \leq r'\delta_0$, meaning that the triangle inequality gives us that

$$r' - r'\delta_0 \leq a - x = r' - r' \frac{\tau - u - u'}{r'} \left(1 - \frac{r'}{a} + \frac{\tau - u - u'}{2a} \right) \leq r' - r' \frac{\tau - u - u'}{10r'}.$$

From this, it follows that

$$\frac{\tau - u - u'}{r'} \leq 10\delta_0.$$

Now, we have that

$$y^2 = (r')^2 - (r')^2 \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r'} - \frac{(\tau - u - u')^2}{2r'a} \right)^2.$$

Because $\vartheta' \leq \delta_0$ and $\sin(\vartheta')$ is comparable to ϑ' for ϑ' small, we can write that

$$(r')^2(\vartheta')^2 + (r')^2 O((\vartheta')^4) = (r')^2 - (r')^2 \left(1 + \frac{\tau - u - u'}{a} - \frac{\tau - u - u'}{r'} - \frac{(\tau - u - u')^2}{2r'a} \right)^2.$$

Thus, we have that

$$\begin{aligned}
(r')^2(\vartheta')^2 + (r')^2 O((\vartheta')^4) &= (r')^2 - (r')^2 \left[1 + \frac{\tau - u - u'}{r'} \left(\frac{r'}{a} - 1 - \frac{\tau - u - u'}{2a} \right) \right]^2 \\
&= (r')^2 - (r')^2 \\
&\times \left[1 + \frac{2(\tau - u - u')}{r'} \left(\frac{r'}{a} - 1 - \frac{\tau - u - u'}{2a} \right) + \frac{(\tau - u - u')^2}{(r')^2} \left(\frac{r'}{a} - 1 - \frac{(\tau - u - u')^2}{2r'a} \right)^2 \right] \quad (4.125) \\
&= 2r'(\tau - u - u') \left(1 - \frac{r'}{a} + \frac{\tau - u - u'}{2a} + O(\delta_0) \right).
\end{aligned}$$

Thus, using (4.124), we have that

$$(r')^2(\vartheta')^2 + (r')^2 O((\vartheta')^4) \geq \frac{1}{20} r'(\tau - u - u').$$

From this, it follows that

$$(\vartheta')^2 \geq c \frac{\tau - u - u'}{r'},$$

giving us the desired result. $\square$

Lemma 40. *Let $\tau \leq \delta_0 s$. In the region where $t \geq \frac{1}{4}s$, $|\vartheta'| \leq \delta_0$, $u \leq \delta_0 \tau$ and $u' \leq \delta_0 \tau$, we have that $y\vartheta' \geq \frac{1}{10}\tau$. Similarly, in the region where $t \leq \frac{3}{4}s$, $|\theta| \leq \delta_0$, $u \leq \delta_0 \tau$, and $u' \leq \delta_0 \tau$, we have that $y\theta \geq \frac{1}{10}\tau$.*

Proof. We shall prove the result for ϑ' . The result for θ follows in an analogous way.

We begin by noting that we must have that $0 \leq x \leq a = s - \tau$ in this region. Thus, we have that $r \cos(\theta) + r' \cos(\vartheta') = a = s - \tau$, where we recall that $\vartheta' = \pi - \theta'$ (see Figure 4.4 to see that this is true because $r \cos(\theta) + r' \cos(\theta')$ gives the horizontal distance between the center adapted to r and the center adapted to r' , and the line segment connecting these two points has length $s - \tau$). By symmetry, we can consider the region where $y \geq 0$. We have that $y = r \sin(\theta) = r' \sin(\theta') = r' \sin(\vartheta')$. Moreover, by Lemma 39, we have that $0 \leq \theta \leq 10\vartheta'$. Thus, we have that

$$r\sqrt{1 - \sin^2(\theta)} + r'\sqrt{1 - \sin^2(\vartheta')} = s - \tau.$$

Taylor expanding, this gives us that

$$r + r' - \frac{1}{2} \sin^2(\theta)r - \frac{1}{2} \sin^2(\vartheta')r' + rO(\sin^4(\theta)) + r'O(\sin^4(\vartheta')) = s - \tau.$$

Thus, we have that

$$-\frac{1}{2}y \sin(\theta) - \frac{1}{2}y \sin(\vartheta') + yO(\theta^3 + (\vartheta')^3) = s - \tau - r - r'.$$

We note that $r = t - u \geq t - \delta_0\tau$ and $r' = s - t - u' \geq s - t - \delta_0\tau$. Thus, we have that

$$s - \tau - r - r' \leq s - \tau - (t - u) - (s - t - u') = -\tau + u + u' \leq -\frac{9\tau}{10},$$

meaning that

$$-y \sin(\vartheta') + yO((\vartheta')^3) \leq -\frac{\tau}{7}.$$

From this, it follows that $y\vartheta' \geq \frac{\tau}{10}$ when δ_0 is sufficiently small, as desired.

□

With these estimates, we can now control all of the nonlinear error integrals effectively.

4.4.6 Bootstrap assumptions

We now list the bootstrap assumptions. For N a sufficiently large integer which will be chosen to make the proof work, we let T be the largest real number such that

$$\begin{aligned} \sum_{|\alpha| \leq N} \|(1+t)^{-\epsilon} \partial \Gamma^\alpha \psi\|_{L_t^\infty([0,T])L_x^2} &\leq \epsilon^{\frac{3}{4}}, \\ \sum_{|\alpha| \leq N} \|(1+t)^{-\frac{1}{2}-\frac{\delta}{4}} \partial \Gamma^\alpha \psi\|_{L_t^2([0,T])L_x^2} &\leq \epsilon^{\frac{3}{4}}, \\ \sum_{|\alpha| \leq \frac{3N}{4}} \|(1+t)^{\frac{1}{2}}(1+|u|)^{\frac{3}{2}-\delta} \partial \Gamma^\alpha \psi\|_{L_t^\infty([0,T])L_x^\infty} &\leq \epsilon^{\frac{3}{4}}. \end{aligned} \tag{4.126}$$

We shall show that assuming these estimates allows us to improve the constants on the right hand side of these norms to $C\epsilon$. This will prove global stability of the trivial solution. In the remainder of this section, we shall recover the energy bootstrap assumptions, which follow easily. We shall also interpolate between the pointwise estimates and the energy estimates. The next section will recover the pointwise bootstrap assumptions.

We shall now recover the bootstrap assumption for the energy. For every α with $|\alpha| \leq N$, it suffices to show that $\|\partial \Gamma^\alpha \psi\|_{L^2(\Sigma_t)} \leq C\epsilon$. Thus, we commute the equation with Γ^α . We have that

$$\square \Gamma^\alpha \psi + (\partial_t \phi)^2 (\partial_t \psi) (\partial_t^2 \Gamma^\alpha \psi) = F'$$

for an appropriate function F' . We shall now explicitly do the energy estimate at this level. If we use $\partial_t \Gamma^\alpha \psi$ as a multiplier, we see that

$$\begin{aligned} & \int_{\Sigma_s} (1 - (\partial_t \phi)^2 (\partial_t \psi)) (\partial_t \Gamma^\alpha \psi)^2 + (\partial_x \Gamma^\alpha \psi)^2 + (\partial_y \Gamma^\alpha \psi)^2 dx \\ & \leq \int_{\Sigma_s} (1 - (\partial_t \phi)^2 (\partial_t \psi)) (\partial_t \Gamma^\alpha \psi)^2 + (\partial_x \Gamma^\alpha \psi)^2 + (\partial_y \Gamma^\alpha \psi)^2 dx + \int_0^s \int_{\Sigma_t} |F| |\partial_t \Gamma^\alpha \psi| dx dt, \end{aligned} \quad (4.127)$$

where F is such that

$$|F| \leq C \sum_{\beta \leq \alpha} |\Gamma^\beta ((\partial_t \phi)(\partial_t \psi)^2)| + C \sum_{\beta \leq \alpha} |\Gamma^\beta ((\partial_t \phi)^2 (\partial_t \psi))| + C \sum_{\beta \leq \alpha} |\Gamma^\beta ((\partial_t \psi)^4)| = F_1 + F_2 + F_3. \quad (4.128)$$

Now, at least two of the factors in F_1 and F_2 must have fewer than $\frac{3N}{4}$ derivatives on them after applying the chain rule with the Γ^β . Similarly, at least three of the factors in F_3 must have fewer than $\frac{3N}{4}$ derivatives of them. Moreover, we note that

$$|\Gamma^\beta (\partial_t \psi)| \leq \sum_{\beta_1 \leq \beta} |\partial_t \Gamma^{\beta_1} \psi|$$

because $[\partial_t, \partial] = 0$ and $[\partial_t, S] = \partial_t$. Using the pointwise bootstrap assumptions on these factors with fewer derivatives thus tells us that

$$|F| \leq \frac{C\epsilon^{\frac{6}{4}}}{(1+t)} \sum_{\beta \leq \alpha} (|\partial \Gamma^\beta \psi| + |\partial \Gamma^\beta \phi|). \quad (4.129)$$

Repeating the same argument for ϕ gives us an expression analogous to (4.127) that satisfies the same estimate.

We now set

$$E(t) = \sum_{|\alpha| \leq N} \int_{\Sigma_t} (\partial_t \Gamma^\alpha \psi)^2 + (\partial_x \Gamma^\alpha \psi)^2 + (\partial_y \Gamma^\alpha \psi)^2 dx + \sum_{|\alpha| \leq N} \int_{\Sigma_t} (\partial_t \Gamma^\alpha \phi)^2 + (\partial_x \Gamma^\alpha \phi)^2 + (\partial_y \Gamma^\alpha \phi)^2 dx$$

By the bootstrap assumptions and (4.129), we have that (4.127) implies that

$$E(s) \leq 2E(0) + C\epsilon^{\frac{6}{4}} \int_0^s \frac{1}{(1+t)} \sum_{|\beta| \leq N} (|\partial \Gamma^\beta \psi| + |\partial \Gamma^\beta \phi|) |\partial_t \Gamma^\alpha \psi| dx dt \leq 2E(0) + C\epsilon^{\frac{6}{4}} \int_0^s \frac{1}{(1+t)} E(t) dt.$$

Thus, by Gronwall's inequality, we have that

$$E(t) \leq 2E(0)(1+t)^{2\epsilon} \leq 2\epsilon^2(1+t)^{2\epsilon}.$$

This recovers the bootstrap assumption for the energy.

Now, taking $(1+t)^{-\frac{\delta}{4}}\partial_t\psi$ as a multiplier and using the bootstrap assumptions, we see that the same argument shows that

$$\int_0^s \int_{\Sigma_t} (1+t)^{-1-\frac{\delta}{4}} [(\partial_t\Gamma^\gamma\psi)^2 + (\partial_x\Gamma^\gamma\psi)^2 + (\partial_y\Gamma^\gamma\psi)^2] dxdt \leq C\epsilon^2 + C\epsilon^3 \int_0^s \int_{\Sigma_t} (1+t)^{-1+2\epsilon-\frac{\delta}{4}}.$$

This last integral converges when $\delta > 8\epsilon$, giving us the desired result.

All that remains is to recover the pointwise bootstrap assumptions. In order to recover these, we shall first interpolate between the pointwise bootstrap assumptions and the energy bootstrap assumptions. The next section uses these interpolated estimates in order to recover the pointwise bootstrap assumptions, which will complete the proof of the main Theorem. The interpolated estimates are the content of the following proposition.

Lemma 41. *Let $|\gamma| \leq \frac{3N}{4}$. For N sufficiently large as a function of δ and for $|\beta| \leq 10$, we have that*

$$|\Gamma^\beta\partial\Gamma^\gamma\psi|(t, r, \theta) \leq \frac{C\epsilon^{\frac{3}{4}}}{(1+t)^{\frac{1}{2}-\frac{\delta}{2}}(1+|u|^{\frac{3}{2}-2\delta})}$$

Proof. We must only consider the region where $t \geq 10$ because the weights do not matter for small t . Let $h = \partial\Gamma^\gamma\psi$. We define a coordinate system on $t > 0$ in $\mathbb{R}^{2+1}$ as follows. We associate the coordinates $(\rho, \tilde{X}, \tilde{Y})$ to the point (t, x, y) in the usual flat coordinates by $\rho = t$, $\tilde{X} = \frac{x}{t}$, and $\tilde{Y} = \frac{y}{t}$. These coordinates are motivated simply by the fact that the scaling vector field points in the same direction as one of the coordinate vector fields (specifically, ∂_ρ , as is shown in (4.130)).

We shall use Sobolev embedding plus an interpolation result in these coordinates in order to prove the desired result. We note that these coordinates are motivated by the fact that we require a coordinate system which will allow us to effectively interpolate with $S = t\partial_t + r\partial_r$.

We must first examine the coordinate system a bit more carefully. We have that

$$d\rho = dt, \quad d\tilde{X} = \frac{1}{\rho}dx - \frac{x}{t^2}dt, \quad d\tilde{Y} = \frac{1}{\rho}dy - \frac{y}{t^2}dt.$$

Thus, we have that

$$dvol = dt \wedge dx \wedge dy = d\rho \wedge \left(\rho d\tilde{X} + \frac{\tilde{X}}{\rho}d\rho \right) \wedge \left(\rho d\tilde{Y} + \frac{\tilde{Y}}{\rho}d\rho \right) = \rho^2 d\rho \wedge d\tilde{X} \wedge d\tilde{Y}.$$

Similarly, we have that

$$\partial_\rho = \frac{\partial t}{\partial \rho} \partial_t + \frac{\partial x}{\partial \rho} \partial_x + \frac{\partial y}{\partial \rho} \partial_y = \partial_t + \tilde{X} \partial_x + \tilde{Y} \partial_y. \quad (4.130)$$

Thus, we have that

$$\rho \partial_\rho = \rho \partial_t + \rho \tilde{X} \partial_x + \rho \tilde{Y} \partial_y = t \partial_t + x \partial_x + y \partial_y = S.$$

We also have that

$$\partial_{\tilde{X}} = \frac{\partial t}{\partial \tilde{X}} \partial_t + \frac{\partial x}{\partial \tilde{X}} \partial_x + \frac{\partial y}{\partial \tilde{X}} \partial_y = \rho \partial_x,$$

and similarly that

$$\partial_{\tilde{Y}} = \rho \partial_y.$$

Let p be a point where we want to show the desired estimate. We shall let (t_0, x_0, y_0) denote its usual coordinates and $(\rho_0, \tilde{X}_0, \tilde{Y}_0)$ denote its coordinates in this modified coordinate system. We shall also denote by u_0 its u coordinate. Now, we take two smooth cutoff functions. The first $\chi_\rho(s)$ is defined to be 1 for $s \geq \frac{3}{4}$ and 0 for $s \leq \frac{1}{2}$. The second $\chi_{x,y}(s)$ is defined to be 1 for $|s| \leq \frac{1}{4}$ and 0 for $|s| \geq \frac{1}{2}$. We then consider the function $\tilde{h} = \chi_\rho\left(\frac{\rho}{\rho_0}\right) \chi_{x,y}(\rho_0 \sqrt{(\tilde{X} - \tilde{X}_0)^2 + (\tilde{Y} - \tilde{Y}_0)^2}) h$. This has localized h around the point $(\rho_0, \tilde{X}_0, \tilde{Y}_0)$ at scale 1 in the $\tilde{X}$ and $\tilde{Y}$ directions and at scale ρ_0 in the ρ direction (by scale, we mean relative to the (t, x, y) coordinate system). Setting $X = \rho_0 \tilde{X}$ and $Y = \rho_0 \tilde{Y}$ renormalizes the length of partial derivatives to have length comparable to 1. Indeed, in the support of $\tilde{h}$, we note that the volume form in (ρ, X, Y) coordinates is equal to

$$\frac{\rho^2}{\rho_0^2} d\rho \wedge dX \wedge dY.$$

This is comparable to $d\rho \wedge dX \wedge dY$. We also have that $\partial_X = \frac{\rho}{\rho_0} \partial_x$ is comparable to ∂_x are comparable, and similarly with ∂_Y and ∂_y .

Now, the function $\tilde{h}$ can be thought of as being defined in (ρ, X, Y) in the region where $\rho \leq \rho_0$. Moreover, the pointwise bootstrap assumptions tell us that

$$\int_{\rho \leq \rho_0} \rho_0^{-1-\frac{\delta}{2}} \tilde{h}^2 \frac{\rho^2}{\rho_0^2} d\rho dX dY \leq C \epsilon^{\frac{3}{2}} \frac{1}{(1+t_0)^{1+\frac{\delta}{2}} (1+|u_0|^{3-2\delta})},$$

where we have used the fact that t and $1+|u|$ are comparable to t_0 and $1+|u_0|$, respectively, in the support of $\tilde{h}$, and where we have used the fact that the volume of the support of h is comparable to

t_0 . Moreover, the energy bootstrap assumptions tell us that

$$\sum_{|\beta| \leq N-|\alpha|} \int_{\rho \leq \rho_0} \rho_0^{-1-\frac{\delta}{2}} (\Gamma^\beta \tilde{h})^2 \frac{\rho^2}{\rho_0^2} d\rho dx'' dy'' \leq C \sum_{|\beta| \leq N-|\alpha|} \int_0^{t_0} \int_{\Sigma_t} (1+t)^{-1-\frac{\delta}{2}} (\Gamma^\beta \partial \psi)^2 dx dt \leq C \epsilon^{\frac{3}{2}}.$$

Let us consider the function

$$H = \tilde{h}^2 \frac{\rho^2}{\rho_0^2}.$$

We shall show that

$$\int_{\rho \leq \rho_0} (\Gamma_{\rho_0}^\alpha H)^2 d\rho dX dY \leq C \sum_{|\beta| \leq |\alpha|} \int_{\rho \leq \rho_0} (\Gamma^\beta \tilde{h})^2 \frac{\rho^2}{\rho_0^2} d\rho dX dY,$$

where $\Gamma_{\rho_0}^\alpha$ corresponds to strings of the operators $\rho_0 \partial_\rho$, ∂_X , and ∂_Y . Indeed, we recall that Γ consists of translation vector fields along with $S = t\partial_t + r\partial_r$. This means that $\partial_X = \frac{t}{\rho_0} \partial_x$, $\partial_Y = \frac{t}{\rho_0} \partial_y$, and $\rho_0 \partial_\rho = \frac{\rho_0}{t} S = \rho_0 \partial_t + \frac{r\rho_0}{t} \partial_r$. This means that $\Gamma_{\rho_0} = Q\Gamma$ for an appropriate coefficient Q . Because ρ_0 is comparable to t in the support of H , we see that we have that

$$|\Gamma_{\rho_0} H| \leq C |\Gamma H|,$$

i.e., these coefficients are bounded. Moreover, we see that we have that

$$\Gamma^\beta \left(\frac{t}{\rho_0} \right) \leq C,$$

and that

$$\Gamma^\beta \left(\frac{\rho_0}{t} \right) \leq C.$$

These facts imply the desired result by expanding the Γ_{ρ_0} operators in terms of the Γ operators. Indeed, in the case of two operators, we schematically have that

$$|\Gamma_{\rho_0}(\Gamma_{\rho_0} H)| = |Q\Gamma(Q\Gamma H)| \leq |Q||\Gamma(Q)||\Gamma(H)| + |Q||Q||\Gamma(\Gamma(H))|,$$

and the bounds on the coefficients above implies the desired result. Thus, we have in fact shown that

$$|\Gamma_{\rho_0}^\alpha H| \leq C \sum_{|\beta| \leq |\alpha|} |\Gamma^\beta H|.$$

A similar argument shows us that

$$|\Gamma^\alpha H| \leq C \sum_{|\beta| \leq |\alpha|} |\Gamma^\beta H|. \quad (4.131)$$

Thus, interpolating between these L^2 based bounds (we can use an interpolation result in the half spaces $\rho \leq \rho_0$ in (ρ, X, Y) coordinate space because they have a nice boundary, see [4]) tells us that, for N sufficiently large in terms of δ , we have that

$$\sum_{|\beta| \leq 10} \int_{\rho \leq \rho_0} \rho_0^{-1-\frac{\delta}{2}} (\Gamma_{\rho_0}^\beta H)^2 d\rho dX dY \leq C \epsilon^{\frac{3}{2}} \frac{1}{(1+t_0)(1+|u_0|^{3-3\delta})}.$$

Then, applying a Sobolev inequality that is weighted in the ρ direction by ρ_0 in the (ρ, X, Y) coordinate system tells us that

$$\sum_{|\beta| \leq 8} |\rho_0^{-\frac{1}{2}-\frac{\delta}{4}} \Gamma_{\rho_0}^\beta \tilde{h}| \leq C \epsilon^{\frac{3}{4}} \frac{1}{(1+t_0)(1+|u_0|^{\frac{3}{2}-\frac{3}{2}\delta})}.$$

Thus, we altogether have that

$$\sum_{|\beta| \leq 8} |\Gamma_{\rho_0}^\beta \tilde{h}| \leq C \epsilon^{\frac{3}{4}} \frac{1}{(1+t_0)^{\frac{1}{2}-\frac{\delta}{2}} (1+|u_0|^{\frac{3}{2}-2\delta})}.$$

Using (4.131) above then gives us the desired result. $\square$

We shall also use the fact that ψ is supported where $u \geq -1$ and ϕ is supported where $\bar{u} \geq -1$. As discussed in Section 4.1.5, we do not believe that having compact support is fundamental to the argument working. We also note that this is trivial when the equation is semilinear instead of quasilinear.

Proposition 42. *Let $0 \leq t \leq T$. The bootstrap assumptions imply that ψ is supported where $u \geq -1$ and that ϕ is supported where $\bar{u} \geq -1$.*

Proof. Let $0 \leq s \leq T$. We take a point (s, x_0, y_0) in Σ_s with $s - \sqrt{x_0^2 - y_0^2} \leq -1$. We then take the ball of radius δ around this point in Σ_s . We look at the domain of influence of this ball intersected with Σ_t for $0 \leq t \leq s$. In every Σ_t , this is a ball whose radius is equal to $s - t + \delta$ and whose center is $(s - t, x_0, y_0)$. By construction, at $t = 0$, this does not intersect the support of the data for ψ for δ sufficiently small.

We set

$$E(t) = \int_{B_{s-t+\delta}(x_0, y_0)} (\partial_t \psi)^2 + (\partial_x \psi)^2 + (\partial_y \psi)^2 dx.$$

This is the energy in the intersection of the domain of influence of the small ball chosen above and the time slice Σ_t . We now think of the equation for ψ as a wave equation whose principal part is $\square$, putting both the quasilinear and semilinear terms on the right hand side. We do a ∂_t energy estimate on the truncated cone determined by this construction and ignore the positive terms on the side of the cone, giving us that

$$E(s) \leq E(0) + \int_0^s \frac{C\epsilon^{\frac{3}{2}}}{1+t} E(t) dt,$$

where we have used the pointwise bootstrap assumptions. Gronwall's inequality then gives us that $E(s) = 0$ because $E(0) = 0$. An analogous argument works for ϕ . $\square$

4.4.7 Closing the pointwise estimates

In order to recover the bootstrap assumptions for the pointwise estimates, we must show that

$$|\partial_t \Gamma^{\sigma_1} \psi|(t, x) \leq \frac{C\epsilon}{(1+t)^{\frac{1}{2}}(1+|u|)^{\frac{3}{2}-\delta}}$$

for all $|\sigma_1| \leq \frac{3N}{4}$. Commuting the equation with Γ^α , we get the equation

$$\square \Gamma^\alpha \psi + (\partial_t \psi)(\partial_t^2 \phi)(\partial_t \Gamma^\alpha \psi) = F^\alpha$$

for an appropriate F^α which contains only semilinear terms.

Now, if we want to apply Proposition 22 in order to show pointwise decay for this quantity, we must additionally commute with the operators ∂^{σ_2} tangent to Σ_t for all $|\sigma_2| \leq 6$. Thus, it suffices to commute with Γ^{σ_2} for all $|\sigma_2| \leq 6$. We thus commute the equation with $\Gamma^{\sigma_2} \Gamma^{\sigma_1}$. Thus, in order to recover the pointwise bootstrap assumptions, it suffices to show that

$$M[\Gamma^\sigma \psi](s, x_0^i) \leq \frac{C\epsilon}{(1+s)^{\frac{1}{2}}(1+|\tau|^{\frac{3}{2}-\delta})}$$

for all $|\sigma| \leq \frac{3N}{4} + 6$ and for all $s \leq T$ (see Proposition 22 for a description of M).

Now, when controlling M , we note that the estimate on the term from data follows immediately from assuming linear decay for the auxiliary multipliers f . Thus, for any admissible auxiliary

multiplier f , we must simply control the error integrals

$$\int_0^s \int_{\Sigma_t} F^\sigma(\partial_t f) dx dt - \int_0^s \int_{\Sigma_t} (\partial_t \phi)^2 (\partial_t \psi) (\partial_t^2 \Gamma^\sigma \psi) (\partial_t f) dx dt. \quad (4.132)$$

Now, by examining the equations (4.69), we note that F^σ arises from applying vector fields to one of three different semilinear terms. The first is

$$(\partial_t \psi)^2 (\partial_t \phi),$$

the second is

$$(\partial_t \psi) (\partial_t \phi)^2,$$

and the third is

$$(\partial_t \psi)^4.$$

Thus, in addition to the quasilinear term in (4.132), we must control the error integrals arising from these semilinear terms. We discussed in the remarks following the statement of Theorem 25 that the same proof works for other nonlinearities. The other nonlinear terms can all be controlled in essentially the same way as one of these terms.

The error integral (4.132) can thus be controlled by

$$\begin{aligned} & \sum_{\sigma' \leq \sigma} \left| \int_0^s \int_{\Sigma_t} \Gamma^{\sigma'} ((\partial_t \psi) (\partial_t \phi)^2) (\partial_t f) dx dt \right| + \sum_{\sigma' \leq \sigma} \left| \int_0^s \int_{\Sigma_t} \Gamma^{\sigma'} ((\partial_t \psi)^4) (\partial_t f) dx dt \right| \\ & + \sum_{\sigma' \leq \sigma} \left| \int_0^s \int_{\Sigma_t} \Gamma^{\sigma'} ((\partial_t \psi)^2 (\partial_t \phi)) (\partial_t f) dx dt \right| + \left| \int_0^s \int_{\Sigma_t} (\partial_t \phi)^2 (\partial_t \psi) (\partial_t^2 \Gamma^\sigma \psi) (\partial_t f) dx dt \right| \end{aligned} \quad (4.133)$$

We recall that the data for f is contained in the ball of radius 1 in Σ_s whose center is (s, x_0, y_0) . Moreover, we recall that the u coordinate of the center of this ball is given by τ , meaning that $\tau = s - \sqrt{x_0^2 + y_0^2}$.

In the following, we shall always use the pointwise estimates for every term up to a $\frac{\delta}{2}$ loss. These interpolation arguments allowing the use of pointwise estimates (with small losses) on all factors in the nonlinearity when recovering estimates that do not involve the most number of derivatives are standard, but we describe it here for completeness. We note that this argument is far from optimal in terms of the number of derivatives required.

We recall that using Proposition 22 to recover the pointwise bootstrap assumptions requires commuting with a total of $\frac{3N}{4} + 6$ derivatives. This means that cubic nonlinearities will have at

least two factors having no more than $\frac{3N}{8} + 5$ derivatives on them while quartic nonlinearities will have at least three factors with this property. Thus, for $\frac{3N}{8} + 5 \leq \frac{3N}{4}$, we can directly apply the bootstrap assumptions to use pointwise estimates on these factors. Moreover, we note that there can be at most one factor in every nonlinearity having at most $\frac{3N}{4} + 8$ derivatives on it. For this factor, we can use the interpolation result in Lemma 41 to give us pointwise decay up to $\frac{\delta}{2}$ losses for this factor. We see that the resulting pointwise bounds are worst for one of the three semilinear terms described above. Thus, it suffices to control the error integrals involving these terms in (4.133). We shall now consider each kind of term separately. Moreover, we shall assume throughout that $\tau \geq 100$ and that $s \geq \frac{1}{\delta_0^{10}}$. When $s \leq \frac{1}{\delta_0^{10}}$, we may control terms simply by choosing ϵ sufficiently small (the decay does not really matter at this point for controlling the nonlinearity because we can use the higher power of ϵ and the fact that s is uniformly bounded to absorb everything). Similarly, when $\tau \leq 100$, the dependence on τ no longer matters, as it can be absorbed by choosing ϵ smaller. The estimates when $\tau \leq 100$ can be recovered from similar considerations as in the following (and by taking ϵ sufficiently small).

We begin with terms of the form

$$\left| \int_0^s \int_{\Sigma_t} \Gamma^\sigma ((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right|. \quad (4.134)$$

The error integrals must be broken up into several regions. Broadly speaking, there are three main regions. The first and most delicate region is the region close to the light cones for both ψ and f , where “close” is measured with respect to τ . In this region, we integrate by parts using the scaling vector field S and use the results of Section 4.4.5. The second region is the region away from the light cone for ψ , and the third region is away from the light cone for f . These regions are better because of decay in u and u' . All three of these main regions must be further decomposed mainly for the technical fact that Lemma 28 means that the Jacobian in $(t, r, \bar{r})$ coordinates on $\mathbb{R}^{2+1}$ is bounded only when we are comparable to t close to the light cones for both ψ and ϕ . Thus, we must for example decompose depending on how far away we are from the light cone associated to ϕ .

We must introduce several cutoff functions in the proof, so let us now describe the notation a bit. These cutoff functions will allow us to effectively use the geometric estimates from Sections 4.4.4 and 4.4.5. There will be times when these cutoff functions will be hit by derivatives, and it will be appropriate to bound them in terms of cutoff functions adapted to slightly larger regions. In these cases, we shall abuse notation and simply use the same cutoff functions. These cutoff functions will be pullbacks of cutoff functions on $\mathbb{R}$. Thus, for example, a cutoff function may be of the form $\chi\left(\frac{u}{t}\right)$

where χ is a smooth function from $\mathbb{R} \rightarrow \mathbb{R}$ and u and t are the usual coordinates. Thus, given such a cutoff function χ , a vector field V hitting χ will always result in something of the form $h\chi'$ for some smooth function h . In the example above, if $V = S$, we have that

$$S\chi\left(\frac{u}{t}\right) = -\frac{u}{t}\chi'\left(\frac{u}{t}\right) - \frac{r}{t}\chi'\left(\frac{u}{t}\right) = -\frac{1}{t}\chi'\left(\frac{u}{t}\right).$$

Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cutoff function equal to 1 for $x \leq \frac{\delta_0}{2}$ and equal to 0 for $x \geq \delta_0$. Then, we take the functions $\chi_{\phi,C} = \chi\left(\frac{\bar{u}}{t}\right)$, $\chi_\psi = \chi\left(\frac{u}{\tau}\right)$, $\chi_{\psi,C} = \chi\left(\frac{u}{t}\right)$, and $\chi_f = \chi\left(\frac{u'}{\tau}\right)$. We also define the cutoff functions $\chi_\psi^c = 1 - \chi_\psi$, and similarly for the others. We shall sometimes need other analogous cutoffs, such as $\chi_\phi = \chi\left(\frac{\bar{u}}{\tau}\right)$, and we shall introduce them following the notational conventions set above. These cutoff functions will be used to localize the error integral to the regions described above. The most delicate regions are those along the light cones for both ψ and f because they require decomposition $\partial_{u'}$ in terms of S and $\bar{\partial}_f$. The other regions can be bounded directly using the bootstrap assumptions. We note that the region along all three light cones will be the only one that returns $\frac{1}{(1+s)^{\frac{1}{2}}\tau^{\frac{3}{2}-\delta}}$ up to a power of δ . The other regions are all better.

We first describe the region away from the light cone for f . This is given by the integral

$$\left| \int_0^s \int_{\Sigma_t} \chi_f^c \Gamma^\sigma ((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right|. \quad (4.135)$$

When also along the light cones for ψ and ϕ , we use the bootstrap assumptions, the interpolation result Lemma 41, and the assumed decay rates for f (see (4.72) and (4.73)) to give us that this integral is controlled by

$$\begin{aligned} & \left| \int_0^s \int_{\Sigma_t} \chi_{\psi,C} \chi_{\phi,C} \chi_f^c \Gamma^\sigma ((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right| \\ & \leq \frac{C\epsilon^{\frac{9}{4}}}{\tau^{\frac{3}{2}}} \int_0^s \int_{\Sigma_t} \chi_{r \leq t+1} \chi_{\psi,C} \chi_{\phi,C} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt. \end{aligned} \quad (4.136)$$

We ignore the power of ϵ because it is already large enough to recover the bootstrap assumption. Going into $r\bar{r}$ coordinates as in Lemma 28 and integrating in each Σ_t , the remaining quantity is controlled by

$$\frac{C}{\tau^{\frac{3}{2}}} \int_0^s \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C}{(1+s)^{\frac{1}{2}}\tau^{\frac{3}{2}}}.$$

Now, in the region which is instead away from the light cone for ϕ , we have that

$$\begin{aligned}
& \left| \int_0^s \int_{\Sigma_t} \chi_{\psi,C} \chi_{\phi,C}^c \chi_f^c \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right| \\
& \leq \frac{C}{\tau^{\frac{3}{2}}} \int_0^s \int_{\Sigma_t} \chi_{r \leq t+1} \chi_{\psi,C} \chi_{\phi,C}^c \frac{1}{(1+t)^{3-3\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt \\
& \leq \frac{C}{\tau^{\frac{3}{2}}} \int_0^s \frac{1}{(1+t)^{2-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C}{(1+s)^{\frac{1}{2}} \tau^{\frac{3}{2}}}.
\end{aligned} \tag{4.137}$$

Then, in the region which is instead away from the light cone for ψ , we have that

$$\begin{aligned}
& \left| \int_0^s \int_{\Sigma_t} \chi_{\psi,C}^c \chi_f^c \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right| \\
& \leq \frac{C\epsilon^{\frac{9}{4}}}{\tau^{\frac{3}{2}}} \int_0^s \int_{\Sigma_t} \chi_{\psi,C}^c \frac{1}{(1+t)^{\frac{9}{2}-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt \\
& \leq \frac{C\epsilon^{\frac{9}{4}}}{\tau^{\frac{3}{2}}} \int_0^s \frac{1}{(1+t)^{\frac{7}{2}-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C\epsilon^{\frac{9}{4}}}{(1+s)^{\frac{1}{2}} \tau^{\frac{3}{2}}}.
\end{aligned} \tag{4.138}$$

This has completely treated the integrals in the region away from the light cone for f . We now consider the region away from the light cone for ψ . We must in fact decompose this into two regions depending on how far away we are. We first consider the regions that are far from the light cone for ψ , but not too far. This corresponds to using the cutoffs χ_ψ^c and $\chi_{\psi,C}$. When also close to the light cone for ϕ , we can write

$$\begin{aligned}
& \left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{\psi,C} \chi_{\phi,C} \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right| \\
& \leq C\epsilon^{\frac{9}{4}} \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{\psi,C} \chi_{\phi,C} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt.
\end{aligned} \tag{4.139}$$

Going into $r\bar{r}$ coordinates once again and using the fact that $u \geq \frac{\delta_0}{10}\tau$ in this region gives us that the integral is in fact controlled by

$$\frac{C}{\tau^{2-3\delta}} \int_0^s \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C}{(1+s)^{\frac{1}{2}} \tau^{2-3\delta}}.$$

Now, when far from the light cone for ϕ , we have that

$$\begin{aligned}
& \left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{\psi,C} \chi_{\phi,C}^c \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right| \\
& \leq C\epsilon^{\frac{9}{4}} \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{\psi,C} \chi_{\phi,C}^c \frac{1}{(1+t)^{3-3\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt \\
& \leq \frac{C\epsilon^{\frac{9}{4}}}{\tau^{3-3\delta}} \int_0^s \frac{1}{(1+t)^{2-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C\epsilon^{\frac{9}{4}}}{(1+s)^{\frac{1}{2}} \tau^{3-3\delta}}.
\end{aligned} \tag{4.140}$$

We now consider the same region except with $\chi_{\psi,C}^c$ instead. Using that $\frac{1}{1+|u|^{3-3\delta}} \leq C \frac{1}{(1+t)} \frac{1}{\tau^{2-3\delta}}$ in this region, we then have that

$$\begin{aligned} & \left| \int_0^s \int_{\Sigma_t} \chi_{\psi}^c \chi_{\psi,C}^c \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt \right| \\ & \leq \frac{C\epsilon^{\frac{9}{4}}}{\tau^{2-3\delta}} \int_0^s \int_{\Sigma_t} \chi_{\bar{r} \leq t+1} \frac{1}{(1+t)^{\frac{5}{2}-\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt. \end{aligned} \quad (4.141)$$

Going into $(t, \bar{r}, \bar{\theta})$ coordinates and using the decay in $\bar{u}$ to control the integrals on Σ_t by $C(1+t)$, the integral is then bounded by

$$\frac{C\epsilon^{\frac{9}{4}}}{\tau^{2-3\delta}} \int_0^s \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C\epsilon^{\frac{9}{4}}}{(1+s)^{\frac{1}{2}} \tau^{2-3\delta}}.$$

The only region that remains is the region along the light cones for both ψ and f . This integral is given by

$$\int_0^s \int_{\Sigma_t} \chi_{\psi} \chi_f \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\partial_t f) dx dt. \quad (4.142)$$

We first pick coordinates where $b = 0$, meaning that the center adapted to r' lies on the new x axis as is described in Section 4.4.2 (we shall do this freely below without mentioning it again). Now, we write

$$\partial_t = -\frac{1}{2}(L' + \underline{L}') = -\frac{1}{2}L' - \gamma S - \frac{1}{2}\gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L' - \gamma \frac{ay}{(r')^2} \partial_{\theta'}, \quad (4.143)$$

where we have used Lemma 36. This results in the integral

$$\int_0^s \int_{\Sigma_t} \chi_{\psi} \chi_f \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2) \left[-\frac{1}{2}L' - \gamma S - \frac{1}{2}\gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L' - \gamma \frac{ay}{(r')^2} \partial_{\theta'} \right] f dx dt. \quad (4.144)$$

We focus first on the term with S . The integral we must control is

$$\int_0^s \int_{\Sigma_t} \chi_{\psi} \chi_f \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(\gamma S f) dx dt.$$

We integrate this term by parts to put the scaling vector field on the other terms. Doing so shows that it suffices to control

$$\begin{aligned} & \left| \int_0^s \int_{\Sigma_t} \chi_{\psi} \chi_f \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2)(S\gamma) f dx dt \right| + \left| \int_0^s \int_{\Sigma_t} \gamma \chi_{\psi} \chi_f S \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2) f dx dt \right| \\ & + \left| \int_0^s \int_{\Sigma_t} \gamma S(\chi_{\psi} \chi_f) \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2) f dx dt \right| + \left| \int_0^s \int_{\Sigma_t} \gamma \chi_{\psi} \chi_f \Gamma^\sigma((\partial_t \phi)(\partial_t \psi)^2) f dx dt \right|, \end{aligned} \quad (4.145)$$

where we have used the fact that the integrand is compactly supported where $\frac{\tau}{10} \leq t \leq s - \frac{\tau}{10}$, meaning that there are no boundary terms from integrating by parts. We begin by estimating $\gamma S(\chi_\psi)$ and $\gamma S(\chi_f)$. We note that $\partial_v \chi_\psi = 0$ and $|\partial_u \chi_\psi| \leq \frac{C}{\tau}$. Thus, we have that

$$|\gamma S(\chi_\psi)| = |\gamma| |v \partial_v \chi_\psi + u \partial_u \chi_\psi| \leq C |\gamma|.$$

An analogous argument works to control γS applied to cutoffs adapted to ϕ (see also Section 4.4.2), and a similar argument works for the other kinds of cutoffs adapted to ψ and ϕ . Now, for χ_f , we shall use the background Euclidean structure on $\mathbb{R}^3$. We note that $\nabla_e \chi_f = h \underline{L}'$ for some smooth function h with $|h| \leq \frac{C}{\tau}$ where $\nabla_e \chi_f$ is the Euclidean gradient of χ_f . We have that

$$|S(\chi_f)| = |\langle S, \nabla_e \chi_f \rangle_e| = |h| |\langle S, \underline{L}' \rangle_e|,$$

where $\langle \cdot, \cdot \rangle_e$ denotes the Euclidean inner product on $\mathbb{R}^3$. Now, using Lemma 36 to write S in terms of the frame $L', \underline{L}'$, and $\frac{1}{r} \partial_{\theta'}$, and noting that these vectors are mutually orthogonal with respect to the Euclidean inner product, we have that

$$|\langle S, \underline{L}' \rangle_e| \leq \frac{C}{\gamma}.$$

Thus, we altogether have that

$$|S(\chi_f)| \leq \frac{C}{\tau \gamma},$$

meaning that

$$|\gamma S(\chi_f)| \leq \frac{C}{\tau}.$$

Using these estimates along with Lemma 37 gives us that these integrals are all controlled by

$$\frac{C \epsilon^{\frac{9}{4}}}{\tau} \int_{\frac{\tau}{10}}^{s - \frac{\tau}{10}} \int_{\Sigma_t} \chi_{r \leq t+1} \chi_f \chi_\psi \frac{1}{(1+t)^{\frac{3}{2}-2\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt. \quad (4.146)$$

We note that we have used the fact that the integrand is only supported for $\frac{\tau}{10} \leq t \leq s - \frac{\tau}{10}$ (see Lemma 27).

Now, we further decompose this integral depending on the distance to the light cone for ϕ . When close to the light cone for ϕ , we can go into $r\bar{r}$ coordinates and the Jacobian will be bounded by

Lemma 28. We thus have that

$$\begin{aligned}
& \frac{1}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_{r \leq t+1} \chi_f \chi_\psi \chi_{\phi, C} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt \\
& \leq \frac{C}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_{r \leq t+1} \chi_f \chi_\psi \chi_{\phi, C} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dr d\bar{r} dt \quad (4.147) \\
& \leq \frac{C}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C}{(1+s)^{\frac{1}{2}} \tau^{\frac{3}{2}-\delta}}.
\end{aligned}$$

We now consider the region far from the light cone of ϕ . We have that

$$\begin{aligned}
& \frac{C}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_{r \leq t+1} \chi_f \chi_\psi \chi_{\phi, C}^c \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt \\
& \leq \frac{C}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_{r \leq t+1} \chi_f \chi_\psi \frac{1}{(1+t)^{3-3\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt \quad (4.148) \\
& \leq \frac{C}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \frac{1}{(1+t)^{2-3\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq \frac{C}{(1+s)^{\frac{1}{2}} \tau^{2-3\delta}}
\end{aligned}$$

We now consider the terms without S in (4.144). Using Lemma 38 and improved decay of good derivatives of f (see (4.74)), we see that the integrals are once again controlled by

$$\frac{C\epsilon^{\frac{9}{4}}}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_{r \leq t+1} \chi_\psi \chi_f \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|u|^{3-3\delta}} \frac{1}{1+|\bar{u}|^{\frac{3}{2}-2\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt.$$

Thus, the bounds for these terms follow in the same way as for (4.146).

We now turn to control the terms of the form

$$\left| \int_0^s \int_{\Sigma_t} \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right|.$$

We once again introduce cutoff functions adapted to the light cones for ψ , ϕ , and f . We first consider the region along all three light cones. This is given by

$$\left| \int_0^s \int_{\Sigma_t} \chi_\psi \chi_\phi \chi_f \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right|$$

This integral can be handled in exactly the same way as (4.142). Moreover, the integral over the region away from the light cone for f can be controlled in a similar way to the term (4.135), and the integral away from the light cone for ϕ can be controlled in a similar way to the terms (4.139), (4.140), and (4.141).

We now turn to the final region (which is the most difficult new region for these terms compared

to the terms of the form (4.134)). This is the region away from the light cone of ψ but still along the light cones for ϕ and f . This is given by

$$\left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_\phi \chi_f \Gamma^\sigma ((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right|$$

Let us briefly describe why this term is problematic for this cubic nonlinearity but not the other one. The worst region is when we are along the light cone for f and the linear factor in the nonlinearity (which is composed of one linear and one quadratic factor in both cases). In this region, we want to decompose ∂_t in terms of S and $\bar{\partial}_f$, as was done for the term (4.144). For that term, because we are restricted to u being small relative to τ , we have good estimates for S by Lemma 37. However, in this term, u is comparable to τ , so we do not have good estimates for S everywhere. We do, however, have a good power of τ (this is in fact almost enough already to close the argument). Thus, we must decompose into regions where S is still useful and other regions where the geometry helps us.

For technical reasons (to avoid $r' = 0$), we must further decompose this into a region where r' is comparable to $s - t$ and a region where r' is much smaller than $s - t$. We thus define the function

$$\chi_{f,I} = \chi \left(\frac{r' - 100}{1 + s - t} \right).$$

Then, in the support of $\chi_{f,I}$ and where $s - t \geq 0$, we note that $1 + |u'| \geq c(1 + s - t)$. We now consider first the region which is within the light cone for f in this sense. This is given by

$$\left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I} \chi_\phi \chi_f \Gamma^\sigma ((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right|. \quad (4.149)$$

Now, using that $1 + |u'| \geq c(1 + s - t)$ in this region, we have that this integral is controlled by

$$C \epsilon^{\frac{9}{4}} \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I} \chi_\phi \chi_f \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \frac{1}{1+|\bar{u}|^{3-2\delta}} \frac{1}{(1+s-t)^{\frac{3}{2}}} dx dt.$$

We in fact could improve the power of $1 + s - t$ from $\frac{3}{2}$ to 2, but we use $\frac{3}{2}$ in order to treat this term the same way as terms that follow. We ignore the power of ϵ which is larger than $\epsilon^{\frac{3}{4}}$, and we focus on the powers of s and τ . We break this integral up into two regions, one where $t \geq (1 - 10\delta_0)s$ and another where $t \leq (1 - 10\delta_0)s$. In the first region, we note that the support of $\chi_f \chi_{f,I}$ is a set of very small diameter compared to the scale of the ellipses which are the level sets of r' in the region where $u' \leq \delta_0 \tau$. Thus, we use Lemma 34 (or more specifically, the version for an ellipse described

after Lemma 34) to note that the integral is controlled by

$$C \frac{1}{(1+s)^{\frac{3}{2}-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \int_{(1-10\delta_0)s}^s \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq C \frac{1}{(1+s)^{1-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}}$$

Then, in the region where $t \leq (1-10\delta_0)s$, we note that $1+s-t \geq c(1+s)$. Thus, the integral is controlled by

$$C \frac{1}{(1+s)^{\frac{3}{2}}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \int_0^{(1-10\delta_0)s} \int_{\Sigma_t} \chi_{\bar{r} \leq t+1} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{1+|\bar{u}|^{3-2\delta}} dx dt.$$

Going into $(t, \bar{r}, \bar{\theta})$ coordinates then gives us that this is controlled by

$$C \frac{1}{(1+s)^{\frac{3}{2}}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \int_0^{(1-10\delta_0)s} \frac{1}{(1+t)^{\frac{1}{2}-\delta}} dt \leq C \frac{1}{(1+s)^{1-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}}.$$

We now consider the region associated with $\chi_{f,I}^c$ instead, and we note that $r' \geq 10$ and $s-t \geq 10$ in the support of $\chi_{f,I}^c$. The integral we must control is then

$$\left| \int_0^s \int_{\Sigma_t} \chi_{\psi}^c \chi_{f,I}^c \chi_{\phi} \chi_f \Gamma^{\sigma}((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right|$$

Now, using (4.143) to decompose $\partial_t f$, we are left with an integral of the form

$$\int_0^s \int_{\Sigma_t} \chi_{\psi}^c \chi_{f,I}^c \chi_{\phi} \chi_f \Gamma^{\sigma}((\partial_t \phi)^2(\partial_t \psi)) \left[-\frac{1}{2} L' - \gamma S - \frac{1}{2} \gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L' - \gamma \frac{ay}{(r')^2} \partial_{\theta'} \right] f dx dt. \quad (4.150)$$

Most of the regions that follow will use this decomposition of ∂_t in terms of S and $\bar{\partial}_f$. There is only one region that will not.

We must now consider two cases, one in which $\tau \leq \delta_0 s$ and one in which $\tau \geq \delta_0 s$. We begin with the easier case of $\tau \geq \delta_0 s$. We first consider the terms not involving S . By Lemma 38, we know that the integral of these terms is controlled by

$$C \int_0^s \int_{\Sigma_t} \chi_{\psi}^c \chi_{f,I}^c \chi_{\phi} \chi_f |\Gamma^{\sigma}((\partial_t \phi)^2(\partial_t \psi))| \frac{1}{r'} \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{1}{2}}} dx dt.$$

These terms can thus be controlled in the same way as (4.149) above because r' is comparable to $1+s-t$ in the support of the integrand.

For the term involving S , we integrate by parts in S as in (4.145) and note that, by Lemma 37,

we have that $|S(\gamma)| \leq \frac{C}{r'}$. Thus, the integral is controlled by

$$\begin{aligned}
& C \int_0^s \int_{\Sigma_t} |\gamma| |S(\chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f)| |\Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| |f| dx dt \\
& + C \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f |\Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| \frac{1}{r'} |f| dx dt \\
& + C \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f |S \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| \frac{1}{r'} |f| dx dt.
\end{aligned} \tag{4.151}$$

We note that S is well behaved when applied to any of the cutoff functions involved. Indeed, the discussion after (4.145) shows that S is well behaved when applied to some of the cutoff functions involved, and controlling the others follows similarly after writing S in terms of L , $\underline{L}'$, and $\partial_{\theta'}$ and then using Lemma 38. More precisely, we have that

$$|\gamma| |S(\chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f)| \leq \left(\frac{C}{\tau} + \frac{C}{r'} \right) \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f, \tag{4.152}$$

where we recall that we are abusing notation and using the cutoffs on the right hand side to represent appropriate cutoffs over slightly larger regions. Thus, these terms can be controlled in the same way as (4.149) above. We also note that there are no boundary terms from the integration by parts because the integrand is compactly supported where $s - t \geq 10$ (because of the cutoff $\chi_{f,I}^c$) and $t \geq \frac{\tau}{10}$ (by Lemma 27).

When $\tau \leq \delta_0 s$, we must further decompose. We first consider the case of small t (where $s - t$ is comparable to s) and the case of large t . In the case of large t , we will have to further decompose in terms of the behavior of f . This motivates us to introduce the cutoff function $\chi_t = \chi\left(\frac{s-t}{20s}\right)$. This function is supported where $t \geq (1 - 20\delta_0)s$, and it is equal to 1 when $t \geq (1 - 10\delta_0)s$. Thus, the function $\chi_t^c = 1 - \chi_t$ is supported where $t \leq (1 - 10\delta_0)s$.

When $t \leq (1 - 10\delta_0)s$, we note that $|S(\gamma)| \leq \frac{C}{\tau}$ by Lemma 37. We then decompose ∂_t in terms of S as we have done several times before, giving us the integral

$$\begin{aligned}
& \int_0^s \int_{\Sigma_t} \chi_t^c \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \Gamma^\sigma((\partial_t \phi)^2 (\partial_t \psi)) \\
& \times \left[-\frac{1}{2} L' - \gamma S - \frac{1}{2} \gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L' - \gamma \frac{ay}{(r')^2} \partial_{\theta'} \right] f dx dt.
\end{aligned} \tag{4.153}$$

For the term involving S , we integrate by parts as we have done before, and the bound on $|\gamma|$ and $|S(\gamma)|$ given by Lemma 37 in the support of the integrand gives us that the resulting integral is

controlled by

$$\begin{aligned}
& \int_0^s \int_{\Sigma_t} |\gamma| |S(\chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_t^c)| |\Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| |f| dx dt \\
& + \frac{C}{\tau} \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_t^c |S \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| |f| dx dt \\
& + \frac{C}{\tau} \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_t^c |\Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| |f| dx dt.
\end{aligned} \tag{4.154}$$

We recall that we have good control over $|\gamma| |S(\chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_t^c)|$ (see (4.152) and note that $S\chi_t \leq C$).

Now, by Lemma 27, we note that these integrals are supported in the region where $t \geq \frac{\tau}{10}$. We may thus control these terms in a similar way as either (4.146) or (4.149) above, giving us that the integrals are controlled by

$$C\epsilon^{\frac{9}{4}} \frac{1}{(1+s)^{\frac{1}{2}}(1+\tau^{\frac{3}{2}-\delta})},$$

as desired (we note that we are in fact throwing away good powers of τ that we do not need). We can then use Lemma 38 and improved decay of good derivatives of f to control the other terms in the error integral which do not involve S in a similar way. Indeed, this gives us that those integrals are controlled by

$$C\epsilon^{\frac{9}{4}} \int_0^s \int_{\Sigma_t} \left(\frac{1}{\tau} + \frac{1}{r'} \right) \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_t^c |\Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{1}{2}}} dx dt,$$

which can be dealt with in the same way as the terms above.

In the remaining region where $t \geq (1-20\delta_0)s$, we introduce more cutoff functions adapted to f . We take $\chi_{f,\theta'}(\theta') = \chi(|\vartheta'|) = \chi(|\pi - \theta'|)$. This localizes us to the set of points where $|\vartheta'| \leq \delta_0$.

We note that, in the support of $\chi'_{f,I}$ and $(\chi_{f,I}^c)'$, we have that there exists some constant c such that $1+|u'| \geq c(1+s-t)$.

Now, decomposing ∂_t in terms of S as we have done before, we have that

$$\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c \partial_t f = -\frac{1}{2} \chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c (L' + \underline{L}') f.$$

The term with the L' derivative on f is better, so we focus on the other term. We have that

$$\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c \underline{L}' f = \underline{L}'(\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c f) - \underline{L}'(\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c) f.$$

Now, we have that

$$|\underline{L}'(\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c)| \leq \frac{C}{\tau} + \frac{C}{s} + \frac{C}{1+s-t}.$$

Let $h = \chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c f$. We now use Lemma 36 to write that

$$\underline{L}'(h) = 2\gamma S(h) + \frac{\gamma}{r'} \left(t - r' - \frac{a(x-a)}{r'} \right) r' L'(h) + \gamma \frac{2ay}{(r')^2} \partial_{\theta'}(h)$$

Now, we note that

$$|r' L'(\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c)| \leq C,$$

and similarly, we note that

$$|\partial_{\theta'}(\chi_f \chi_t \chi_{f,\theta'}^c \chi_{f,I}^c)| \leq C.$$

Thus, by Lemma 38 (note that $\vartheta' \geq \delta_0/2$ in the support of the integrand), we have that

$$\begin{aligned} & \left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right| \\ & \leq C \left| \int_0^s \int_{\Sigma_t} \chi_\phi \chi_\psi^c \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2) \gamma S(\chi_f \chi_{f,I}^c \chi_t \chi_{f,\theta'}^c f) dx dt \right| \\ & + \int_0^s \int_{\Sigma_t} \left(\frac{C}{\tau} + \frac{C}{1+s-t} \right) \chi_\psi^c \chi_\phi \chi_f \chi_{f,I}^c \chi_t \chi_{f,\theta'}^c |\Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)| \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{1}{2}}} dx dt. \end{aligned} \quad (4.155)$$

The second of these terms can be controlled in the same way as the terms above (see (4.146) and (4.149)). For the term involving S , we integrate by parts in S just as before. Because we have appropriate control over $S(\chi_\phi)$, $S(\chi_\psi^c)$, and $S(\gamma)$ in the region in question by Lemma 37, the desired result follows in an analogous way as the terms above (more precisely, the integral can once again be controlled in the same way as either (4.146) or (4.149)).

We now consider the same region where $|\vartheta'| \leq \delta_0$. For this term, we shall not decompose ∂_t in terms of S and $\bar{\partial}_f$, and the integral is instead given by

$$\int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_{f,\theta'} \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt.$$

In this region, we note that the Jacobian in $(\bar{r}, r')$ coordinates is well behaved by Lemma 33. Thus,

we have that

$$\begin{aligned}
& \left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_t \chi_f \chi_{f,\theta'} \Gamma^\sigma((\partial_t \psi)(\partial_t \phi)^2)(\partial_t f) dx dt \right| \\
& \leq C \epsilon^{\frac{9}{4}} \frac{1}{(1+s)^{\frac{3}{2}-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \int_{\frac{s}{2}}^s \int_{\Sigma_t} \chi_\psi^c \chi_{f,I}^c \chi_\phi \chi_f \chi_{f,\theta'} \frac{1}{1+|\bar{u}|^{3-4\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{3}{2}}} dx dt \\
& \leq C \epsilon^{\frac{9}{4}} \frac{1}{(1+s)^{\frac{3}{2}-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \int_{\frac{s}{2}}^s \int_{\Sigma_t} \frac{1}{1+|\bar{u}|^{3-4\delta}} \frac{1}{1+|u'|^{\frac{3}{2}}} \frac{1}{(1+s-t)^{\frac{1}{2}}} d\bar{r} dr' dt \\
& \leq C \epsilon^{\frac{9}{4}} \frac{1}{(1+s)^{\frac{3}{2}-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}} \int_{\frac{s}{2}}^s \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq C \epsilon^{\frac{9}{4}} \frac{1}{(1+s)^{1-\delta}} \frac{1}{1+\tau^{\frac{3}{2}-2\delta}},
\end{aligned} \tag{4.156}$$

as desired.

We finally consider quartic terms. The integrals we must control are of the form

$$\int_0^s \int_{\Sigma_t} \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt.$$

We begin with the region away from the light cone for ψ , which is the easiest to control.

We first assume that $\tau \leq \delta_0 s$. Let j_τ denote the greatest integer which is less than or equal to $\frac{\delta_0 \tau}{10}$. After discretizing the integral within each Σ_t in terms of u and u' , we have that

$$\begin{aligned}
& \left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt \right| \\
& \leq C \epsilon^3 \int_0^s \sum_{j=j_\tau}^\infty \sum_{k=1}^\infty \int_{\Sigma_t} \chi_{j-1 \leq u \leq j+1} \chi_{k-1 \leq u' \leq k+1} \frac{1}{(1+j)^{6-5\delta}} \frac{1}{(1+k)^{\frac{3}{2}}} \frac{1}{(1+t)^{2-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt.
\end{aligned} \tag{4.157}$$

We note that the lower bound $j = j_\tau$ comes from the fact that $\chi_\psi^c = 0$ outside of this region.

Now, by Lemma 31, we have that

$$\int_{\Sigma_t} \chi_{j-1 \leq u \leq j+1} \chi_{k-1 \leq u' \leq k+1} dx \leq C(1+t)^{\frac{1}{2}}.$$

Thus, we have that

$$\begin{aligned}
\left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt \right| & \leq C \epsilon^3 \frac{1}{(1+\tau)^{5-5\delta}} \int_0^s \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \\
& \leq C \epsilon^3 \frac{1}{(1+\tau)^{5-5\delta}} \frac{1}{(1+s)^{\frac{1}{2}}}.
\end{aligned} \tag{4.158}$$

We now assume that $\tau \geq \delta_0 s$. We then have that $1 + |u| \geq c(1 + s)$, meaning that we have that

$$\begin{aligned} \left| \int_0^s \int_{\Sigma_t} \chi_\psi^c \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt \right| &\leq C\epsilon^3 \frac{1}{(1+s)^{5-5\delta}} \int_0^s \frac{1}{(1+t)^{1-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \\ &\leq C\epsilon^3 \frac{1}{(1+s)^{\frac{11}{2}-6\delta}}, \end{aligned} \quad (4.159)$$

as desired.

We now consider the integral in the region determined by χ_ψ . We must once again consider two cases depending on the size of τ relative to s . We begin with the easier case given by $\tau \geq \delta_0 s$ when τ is comparable to s .

When $\tau \geq \delta_0 s$, we further consider two cases depending on how large u' is compared to τ in the usual way. When $u' \leq \delta_0 \tau$, we must control the integral

$$\left| \int_0^s \int_{\Sigma_t} \chi_\psi \chi_f \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt \right|.$$

Decomposing ∂_t in terms of the frame consisting of S and good derivatives for f as we have done several times, we are left with

$$\left| \int_0^s \int_{\Sigma_t} \chi_\psi \chi_f \Gamma^\sigma((\partial_t \psi)^4) \left[-\frac{1}{2} L' - \gamma S - \frac{1}{2} \gamma \left(t - r' - \frac{a(x-a)}{r'} \right) L' - \gamma \frac{ay}{(r')^2} \partial_{\theta'} \right] dx dt \right|.$$

We integrate by parts in S as we have done several times before and use Lemma 38 to bound the other terms directly. Using in addition Lemma 37, we get that the resulting integrals are controlled by

$$C\epsilon^3 \frac{1}{s} \int_0^s \int_{\Sigma_t} \chi_\psi \chi_f \frac{1}{(1+t)^{2-\delta}} \frac{1}{1+|u|^{6-8\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{1}{2}}} dx dt,$$

where we have used that r and r' are comparable to s where the integrand is supported by Lemma 27 (note that this also implies that t and $s-t$ are comparable to s in this region). We then go into (θ, u, u') coordinates as in Lemma 32, giving us that the integral is controlled by

$$C\epsilon^3 \frac{1}{(1+s)^{\frac{5}{2}-\delta}} \int_0^{2\pi} \int_{-1}^{\delta_0 \tau} \int_{-1}^{\delta_0 \tau} \frac{1}{1+|u|^{6-8\delta}} \frac{1}{1+|u'|^{\frac{1}{2}}} du' du d\theta \leq C\epsilon^3 \frac{1}{(1+s)^{2-\delta}},$$

as desired.

Now, when $u' \geq \delta_0 \tau$, we note that the integral is controlled by

$$C\epsilon^3 \frac{1}{(1+s)^{\frac{3}{2}}} \int_0^s \int_{\Sigma_t} \chi_{r \leq t+1} \frac{1}{(1+t)^{2-\delta}} \frac{1}{1+|u|^{6-8\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt,$$

where we are using that $\tau \geq \delta_0 s$. Thus, using the decay in u to bound the integral on each Σ_t by $C(1+t)$, we have that this is controlled by

$$C\epsilon^3 \frac{1}{(1+s)^{\frac{3}{2}}} \int_0^s \frac{1}{(1+t)^{1-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq C\epsilon^3 \frac{1}{(1+s)^{2-\delta}},$$

as desired.

We finally consider the case where $\tau \leq \delta_0 s$. We once again decompose relative to the distance to the light cone adapted to f . We first consider the region where $u' \geq \delta_0 \tau$. Now, we note that when $u' \geq 10\tau$, we have that ψ is identically 0 by domain of dependence. Thus, we have that the integrand in question is supported where $\delta_0 \tau \leq u' \leq 10\tau$. We shall use the fact that this interval in u' is comparable to τ in length.

The integral we must control is given by

$$\int_0^s \int_{\Sigma_t} \chi_\psi \chi_{\delta_0 \tau \leq u' \leq 10\tau} \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt.$$

This integral is controlled by

$$C\epsilon^3 \frac{1}{\tau^{\frac{3}{2}}} \int_0^s \int_{\Sigma_t} \chi_\psi \chi_{\delta_0 \tau \leq u' \leq 10\tau} \frac{1}{(1+t)^{2-\delta}} \frac{1}{1+|u|^{6-8\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt.$$

We break up the integral into two regions in t , one from 0 to τ and the other from τ to s . In the first region, we use the decay in u to bound the integral on each Σ_t by $C(1+t)$, giving us that the integral is controlled by

$$C\epsilon^3 \frac{1}{(1+s)^{\frac{1}{2}}} \frac{1}{\tau^{\frac{3}{2}}} \int_0^\tau \frac{1}{(1+t)^{1-\delta}} dt \leq C\epsilon^3 \frac{1}{(1+s)^{\frac{1}{2}}} \frac{1}{\tau^{\frac{3}{2}-\delta}}.$$

The remaining integral is from τ to s . Discretizing this integral in u within each Σ_t then gives us that this is controlled by

$$C\epsilon^3 \sum_{j=-1}^{\infty} \frac{1}{\tau^{\frac{3}{2}}} \int_\tau^s \int_{\Sigma_t} \chi_\psi \chi_{\delta_0 \tau \leq u' \leq 10\tau} \chi_{j \leq u \leq j+1} \frac{1}{(1+t)^{2-\delta}} \frac{1}{(2+j)^{6-8\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt.$$

Now, for j fixed, we note that the inner integral in x in the expression

$$\int_\tau^s \int_{\Sigma_t} \chi_\psi \chi_{u'} \chi_{j \leq u \leq j+1} \frac{1}{(1+t)^{2-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dx dt$$

is an integral over the intersection of two annular regions, one adapted to ψ having thickness 1 and the other adapted to f having thickness comparable to τ . Thus, by Lemma 31, we have that

$$\int_{\Sigma_t} \chi_\psi \chi_{u'} \chi_{j \leq u \leq j+1} dx \leq C(1 + \sqrt{t})\sqrt{\tau}$$

Integrating the remaining integral in t and summing in j then gives us that the integral we desired to bound is controlled by

$$C\epsilon^3 \frac{1}{(1+s)^{\frac{1}{2}}} \frac{1}{\tau^{\frac{3}{2}-\delta}},$$

as desired.

The only remaining region is where $u' \leq \delta_0 \tau$. The integral we must control is given by

$$\int_0^s \int_{\Sigma_t} \chi_\psi \chi_f \Gamma^\sigma((\partial_t \psi)^4)(\partial_t f) dx dt.$$

Decomposing ∂_t in terms of S and good derivatives for f as we have done before results in the integral

$$\int_0^s \int_{\Sigma_t} \chi_\psi \chi_f \Gamma^\sigma((\partial_t \psi)^4) \left[-\frac{1}{2}L' - \gamma S - \frac{1}{2}\gamma \left(t - r' - \frac{a(x-a)}{r'} \right) \partial_{v'} - \gamma \frac{ay}{(r')^2} \partial_{\theta'} \right] f dx dt.$$

We integrate by parts the term having S and bound the others directly. We first focus on the term with S . Integrating by parts as we have several times before, we have that the integral is controlled by

$$C\epsilon^3 \frac{1}{\tau} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_\psi \chi_f \frac{1}{(1+t)^{2-\frac{\delta}{2}}} \frac{1}{1+|u|^{6-8\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{1}{2}}} dx dt.$$

We note that we are now using the slightly improved interpolation for the power of t (i.e., a $\frac{\delta}{2}$ loss instead of a δ loss). This is because this integral will have a further logarithmic loss coming from Lemma 30. Indeed, discretizing the integral in u (and taking j_τ to be the smallest integer such that $j_\tau \geq \delta_0 \tau$) gives us that this is controlled by

$$C\epsilon^3 \frac{1}{\tau} \sum_{j=-1}^{j_\tau} \frac{1}{(2+j)^{6-8\delta}} \int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \int_{\Sigma_t} \chi_f \chi_{j \leq u \leq j+1} \frac{1}{(1+t)^{2-\frac{\delta}{2}}} \frac{1}{(1+s-t)^{\frac{1}{2}}} \frac{1}{1+|u'|^{\frac{1}{2}}} dx dt.$$

Now, by Lemma 30, we note that

$$\int_{\Sigma_t} \chi_f \chi_{j \leq u \leq j+1} \frac{1}{1+|u'|^{\frac{1}{2}}} dx \leq C(1 + \sqrt{t}) \log(1+t) \leq C(1+t)^{\frac{1}{2}+\frac{\delta}{2}}.$$

Thus, the remaining integral is controlled by

$$\int_{\frac{\tau}{10}}^{s-\frac{\tau}{10}} \frac{1}{(1+t)^{\frac{3}{2}-\delta}} \frac{1}{(1+s-t)^{\frac{1}{2}}} dt \leq C(1+s)^{-\frac{1}{2}} \tau^{-\frac{1}{2}+\delta}.$$

Summing in j then gives us that the whole integral is controlled by

$$C\epsilon^3 \frac{1}{(1+s)^{\frac{1}{2}} \tau^{\frac{3}{2}-\delta}},$$

as desired. The terms not involving S can be controlled in a similar way after using Lemma 38.

Because we have now controlled all of the error integrals required to use Proposition 22, we have recovered the pointwise bootstrap assumptions, and we have thus completed the proof of Theorem 25.

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