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BOUNDARY REGULARITY FOR AREA-MINIMIZING CURRENTS

IAN FLECHLER

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

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Abstract

In this thesis, we settle an old question of William Allard's thesis from 1969, on boundary regularity of area-minimizing m -currents in arbitrary codimension, with higher boundary multiplicity and a convexity assumption. This setting generalizes Allard's celebrated theorem from 1975 on boundary regularity of area-minimizing currents to a higher boundary multiplicity minimizing setting. We develop a regularity theory that answers Allard's 1969 question by proving the $(m - 3)$ -rectifiability of the singular set. We show this regularity theory to be dimensionally sharp by constructing an example of an area-minimizing current with a boundary singularity for $m = 3$. The starting point of the regularity theory is the uniqueness of the tangent cone at minimum density points, which we also establish in this thesis. The question of regularity for higher multiplicity boundaries, which we address in Allard's original form, was raised in a broader framework by Brian White in the Proceedings of the 1984 AMS Summer Institute, a famous collection of open problems in Geometric Measure Theory. Part of the regularity theory is in collaboration with Reinaldo Resende.

Additionally, in collaboration with Camillo de Lellis, we generalize a theorem of Besicovitch from 1956 on classical rectifiability for curves in the two-dimensional plane to arbitrary dimension of the set and of the ambient space. As an application, we simplify the setup of the Naber-Valtorta technique, an extremely powerful and flexible tool for proving the rectifiability of singular sets in geometric analysis, which has been used for a variety of different problems. Together with Reinaldo Resende, we use this theorem to simplify part of the boundary regularity theory we develop in this thesis.

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Chapter 1

Introduction

An area minimizing integral current is an oriented generalized surface with boundary which minimizes area amongst those with the same fixed smooth boundary. Integral currents, rigorously introduced by Federer and Fleming, generalize classical smooth oriented surfaces, enabling the study of possibly singular geometric objects while maintaining core geometric concepts such as area, tangent planes, and orientation in a measure theoretic sense. The orientation framework ensures the validity of Stokes' theorem, aligning integral currents closely with classical differential geometry. Specifically, for a current T with boundary Γ , the relation $T(d\omega) = Q \int_{\Gamma} \omega$ holds for a positive integer Q and any smooth differential form Ω . In this case, we say that the boundary of T is Γ with multiplicity Q .

Another important concept is the definition of a rectifiable set: a set $E \subset \mathbb{R}^n$ is k rectifiable if there exist countably many lipschitz maps $f_i : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that $E \setminus \bigcup f_i(\mathbb{R}^k)$ has k -dimensional measure zero.

The appropriate class of surfaces T comes from rectifiable sets taken with integer multiplicity and assigned an orientation.

We say that T is locally area minimizing at its boundary Γ , taken with multiplicity Q , if there is no competitor T' , with the same boundary, which agrees with T outside of the ball $\mathbf{B}_R$ for some R and $\text{Area}(T', \mathbf{B}_R) < \text{Area}(T, \mathbf{B}_R)$.

Density and Tangent Cones A key tool in the study of singularities is the *density* of a current. Given an m -dimensional area-minimizing current T , the density at a point $p \in \text{spt}(T)$ and scale $r > 0$ is defined by

$$\Theta(T, p, r) := \frac{\|T\|(B_r(p))}{\omega_m r^m},$$

where ω_m denotes the volume of the unit ball in $\mathbb{R}^m$, where $\|T\|$ denotes the measure associated to the current. This quantity measures how the mass of the current accumulates near the point p , normalized by the Euclidean volume growth.

The *monotonicity formula* ensures that for area minimizing currents $\Theta(T, p, r)$ is a non-decreasing function of r . More precisely, for $r > s > 0$, one has

$$\Theta(T, p, r) - \Theta(T, p, s) = \int_{B_r(p) \setminus B_s(p)} \frac{|p_{\bar{T}(q)}(q - p)|^2}{|q - p|^{m+2}} d|T|(q) > 0,$$

where $p_{\bar{T}(q)}(q - p)$ denotes the orthogonal projection of $q - p$ onto the normal complement of the approximate tangent plane to T at q . This identity implies that the limit $\Theta(T, p, 0) := \lim_{r \rightarrow 0} \Theta(T, p, r)$ exists and is finite.

A central consequence is that *blow-up limits are cones*: if one rescales T around a point p by setting $T_r := (\eta_{p,r})_{\#} T$ where $\eta_{p,r}(x) = \frac{x-p}{r}$, then any sequential limit of T_r as $r \rightarrow 0$ is an area-minimizing cone. Since, if $\Theta(T, p, r)$ is constant in r , then the current itself is conical near p .

1.1 Historical overview

For a more detailed historical overview until 2021 see [18].

Interior regularity

An interior point is said to be regular if the current agrees with a smooth minimal surface with multiplicity in a neighborhood of it.

In codimension 1, the sharp dimensional bound on the singular set was established in the 1970 through combined contributions of De Giorgi, Federer, Fleming, Almgren, Simons (for a detailed survey see [18]). The rectifiability of the singular set was proven by Simon in 1993 [73] and later reproved by Naber-Valtorta [67] in 2015. The theorem is as follows:

Theorem 1.1.1 (Codimension 1). *If T is an area minimizing current in $\mathbf{B}_1 \subset \mathbb{R}^{m+1}$ without boundary then:*

- *The Hausdorff dimension of the singular set is at most $m-7$. In particular if $1 \leq m \leq 6$, the singular set is empty and if $m = 7$, it is discrete.*
- *The singular set is $m-7$ rectifiable.*

In arbitrary codimension, there is a theorem of Allard that gives under a density condition regularity in any dimension and codimension. The theorem only needs a stationarity condition, but we state it in the same minimizing framework of the rest of the discussion.

Theorem 1.1.2 (Allard's Interior Regularity [2]). *If T is an area minimizing current in $\mathbf{B}_1 \subset \mathbb{R}^{m+n}$ without boundary then points of density 1 are regular (i.e. at a point of the surface where the associated measure has density 1, the surface agrees in a neighborhood with a smooth classical minimal surface).*

Allard's Interior Regularity theorem is a consequence of an excess-decay type lemma which shows that if a current is sufficiently flat (close to a plane of multiplicity one) at unit scale, then at smaller scales it becomes progressively flatter, relative to a best-approximating plane that may vary with scale. This decay is obtained by linearizing the area functional and comparing the current to the graph of a harmonic function over that plane, allowing the regularity theory of harmonic functions to be transferred back to the current.

Sketch of the Codimension One Interior Regularity Results

A central feature of codimension one area-minimizing currents is that every point with a flat tangent cone (some blow-up limit is a plane with multiplicity), must be regular.

This is a consequence of Allard's interior regularity combined by a decomposition theorem into boundaries of sets of finite perimeter. The proof of the interior regularity theorem in codimension 1 can be understood through a two-step strategy: first, using density estimates and compactness to show that under some quantitative control on the density gaps the current is regular at a ball, and second, applying a measure-theoretic approximation lemma to bound the dimension of the singular set.

The mass-ratio monotonicity formula ensures that blow-up limits at any point are area-minimizing cones. In particular, if the density $\Theta(T, p, r)$ varies very little between two scales at enough well-separated points, one can conclude that the current is quantitatively close to an m -dimensional plane.

The key quantitative step is the following lemma:

Lemma 1.1.3. *For every $\eta, \varepsilon > 0$, there exists $\delta = \delta(m, \eta, \varepsilon) > 0$ such that the following holds. Suppose that $p_0, \dots, p_{m-6} \in \mathbf{B}_1 \cap \text{spt}(T)$ are η -separated points (i.e. $\min_{0 \leq i \neq j \leq m-6} |p_i - p_j| \geq \eta$) satisfying*

$$\Theta(T, p_i, 2) - \Theta(T, p_i, 1) < \delta \quad \text{for all } 0 \leq i \leq m-6.$$

Then there exists an m -dimensional plane P such that

$$\text{dist}(T|_{\mathbf{B}_2}, P) < \varepsilon,$$

and thus T is regular in $\mathbf{B}_1$ by Allard's interior regularity theorem.

Sketch of the proof of Lemma 1.1.3. The proof proceeds by contradiction: if no such approximating plane exists, blow-up limits at the points p_i would yield an area-minimizing cone T_∞ invariant under translations along an $(m-6)$ -dimensional plane. This would force T_∞ to split off a factor $\mathbb{R}^{m-6}$ and reduce to a 6-dimensional area-minimizing cone. By the classification of such cones, T_∞ must be flat, contradicting the assumed non-flatness. $\square$

To pass from quantitative flatness to a dimension estimate, we use the following measure-theoretic lemma:

Lemma 1.1.4 (Weak η -Approximation). *There exists a continuous increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(0) = 0$ such that the following holds. Suppose $E \subset \mathbf{B}_1$ satisfies: for every ball $\mathbf{B}_r(p) \subset \mathbf{B}_1$, there exists a k -dimensional plane L such that*

$$E \cap \mathbf{B}_r(p) \subset \text{an } (\eta r)\text{-neighborhood of } L.$$

Then

$$\dim_{\mathcal{H}}(E) \leq k + f(\eta).$$

In our setting, E is the singular set $\text{Sing}(T)$ and $k = m - 7$. If the weak approximation condition fails at some scale, then there must exist a ball where the density drops by at least δ between scales r and $2r$. However, the mass bound controls how often such failures can occur: roughly, this can happen at most $C(\Theta(T, 0, 2)/\delta)$ times. An iterative covering argument then yields that the singular set has Hausdorff dimension at most $m - 7$.

Quantitative Rectifiability and Rectifiability of the Singular Set in Codimension one

A central development in the regularity theory for area-minimizing currents has been the introduction of quantitative flatness measurements.

Definition 1.1.5 (Jones' β -numbers). Given a measure μ , the Jones' β -numbers are defined by

$$\beta_{2,\mu}(x, r)^2 := \inf_{L \text{ } m\text{-plane}} \frac{1}{r^m} \int_{B(x,r)} \left(\frac{\text{dist}(y, L)}{r} \right)^2 d\mu(y),$$

capturing how far the measure is, on average, from being concentrated near an m -dimensional plane at the given scale.

An essential theorem in rectifiability, due to Azzam–Tolsa and independently to Naber–Valtorta and Edelen–Naber–Valtorta, gives a general powerful quantitative rectifiability criterion:

Theorem 1.1.6 (Azzam - Tolsa [9, 76]/ Edelen-Naber-Valtorta [40]). *A measure is rectifiable (i.e., it is obtained by integrating a non-negative function over a rectifiable set) if and only if the β -numbers are square-integrable across scales:*

$$\int_0^1 \beta_{2,\mu}(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-almost every } x.$$

This provides a flexible and powerful criterion for rectifiability based purely on multi-scale geometric information.

In the context of area-minimizing currents, Naber and Valtorta established a key estimate linking the density of the density function to the behavior of the β -numbers. Precisely, for measures μ supported on the singular set $\text{Sing}(T)$, they proved

$$\beta_{2,\mu}(p, r)^2 \leq Cr^{-m} \int_{B_r(p)} (\Theta(T, x, 8r) - \Theta(T, x, r)) d\mu(x), \quad (1.1.1)$$

where $\Theta(T, x, r)$ is the mass ratio at scale r . Since the monotonicity formula controls the variation of Θ , this estimate implies that

$$\int_0^1 \beta_{2,\mu}(p, r)^2 \frac{dr}{r} < \infty$$

for μ -almost every p in $\text{Sing}(T)$, thus establishing the rectifiability of the singular set.

This connection between mass ratio decay, quantitative flatness, and rectifiability represents a conceptual turning point in the understanding of singularities for area-minimizing currents and other geometric variational problems.

Higher codimension

In higher codimension, Almgren [4] proved the sharp dimensional bound in 1983, with his detailed proof reprinted in 2000, occupying over 1000 pages. De Lellis and Spadaro in 2015 [31, 33, 34] majorly simplified the proof of the dimensional bound. Additionally, in 1988 [14], Chang proved that 2-dimensional area minimizing integral currents have a discrete interior singular set. Later, De Lellis, Spadaro, and Spolaor [35, 36, 38, 37] revisited Chang's work and completed the missing part of the proof regarding the existence of a branched center manifold, which Chang had only briefly sketched. Recently, using the Naber-Valtorta technique and Simon's method for proving the uniqueness of the tangent cone, De Lellis, Minter, and Skorobogatova [28, 29, 24] have proven the rectifiability of the singular set in a series of works. In parallel, Krummel and Wickramasekera independently established the rectifiability of the singular set, relying on Simon's method.

Theorem 1.1.7 (Higher codimension). *If T is an area minimizing current in $\mathbf{B}_1 \subset \mathbb{R}^{m+n}$ without boundary then:*

- *The Hausdorff dimension of the singular set is at most $m - 2$. In particular if $m = 2$, it is discrete.*
- *The singular set is $m - 2$ rectifiable.*

A brief discussion on interior regularity in higher codimension

Almgren's multivalued functions In higher codimension, the regularity theory is built on Almgren's framework of Q -valued functions. For a positive integer Q , the space of unordered Q -tuples of points in $\mathbb{R}^n$ is defined as

$$\mathcal{A}_Q(\mathbb{R}^n) := (\mathbb{R}^n)^Q / S_Q,$$

equipped with the metric

$$d\left(\sum_{i=1}^Q \llbracket p_i \rrbracket, \sum_{i=1}^Q \llbracket q_i \rrbracket\right)^2 := \inf_{\sigma \in S_Q} \sum_{i=1}^Q |p_i - q_{\sigma(i)}|^2.$$

Within this setting, one considers Dirichlet minimizers $u : \Omega \subset \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ of the energy

$$\int_{\Omega} |Du|^2,$$

subject to prescribed boundary conditions. Dirichlet-minimizing Q -valued functions serve as the appropriate linear model for analyzing singularities of area-minimizing currents in higher codimension. They play a role analogous to that of harmonic functions in the codimension-one setting.

Almgren's frequency function A key monotone quantity in this theory is Almgren's frequency function, defined by

$$I(r) := \frac{r \int_{B_r} |Du|^2}{\int_{\partial B_r} |u|^2},$$

which satisfies the identity

$$I'(r) = \frac{I(r)}{r} \int_{\partial B_r} \left| \partial_n u - \frac{I(r)}{r} u \right|^2.$$

In particular, I is monotone increasing in r , and constant if and only if u is homogeneous.

The full interior regularity theory requires the following ingredients:

- **Multivalued Approximations:** Lipschitz Q -valued approximations exist when the current is close to a flat plane with multiplicity Q .
- **Center Manifold:** Singularities arise as higher-order perturbations of a smooth average, known as the center manifold.
- **Blowups:** Limits of the multivalued approximations are energy minimizing.

- **Frequency function Control:** Ensures blowups are nontrivial.
- **Singular Set Bound:** If $\dim$ of the singular set were $> m-2$, a minimizing multivalued function would also satisfy this bound.

The rectifiability of the linear problem was established by De Lellis, Marchese, Spadaro, and Valtorta [23] (and independently Krummel and Wickramasekera [56]) using a similar estimate to (1.1.1)

This was further refined by De Lellis and Skorobogatova in [28, 29], who defined a universal frequency $I(T, p)$ at flat multiplicity- Q points p of a current T and showed that the set $\{p : I(T, p) > 1\}$ is rectifiable. Compared with [23], De Lellis and Skorobogatova have to address significant error terms arising from errors in the multivalued approximations and changes in center manifolds.

In later work with Minter [24], they established uniqueness of the tangent cone at $\mathcal{H}^{m-2}$ -almost every such point and the set where $I = 1$ has zero $\mathcal{H}^{m-2}$ measure, completing the interior rectifiability theory in higher codimension. They use Simon's techniques on the uniqueness of cylindrical tangent cones [72].

The uniqueness of tangent cones and rectifiability was also obtained by Krummel and Wickramasekera, by an application of Simon's techniques on the uniqueness of cylindrical tangent cones in [57, 57] combined with a third upcoming paper.

Boundary regularity

The starting point for the boundary regularity theory for area minimizing currents was Allard's boundary regularity theorem, which gives under a density condition regularity at the boundary in any dimension and codimension. The theorem only needs a stationarity condition, but we state it in the same minimizing framework of the rest of the discussion.

Theorem 1.1.8 (Allard's Boundary Regularity [3]). *If T is an area minimizing current with a smooth boundary Γ then points of density $1/2$ are regular (i.e. at a point of the current*

where the associated measure has density $1/2$, the surface agrees in a neighborhood with a smooth classical minimal surface that attaches smoothly to the boundary). Moreover, this holds at every point when $\Gamma \subset \partial\Omega$ lies at the boundary of a uniformly convex set.

In his PhD thesis in 1969, Allard [1] proved the above theorem in the convexity setting, where he posed the same question in a higher multiplicity setting, where he highlights the difficulty of the problem.

In codimension 1, the problem of boundary regularity was settled by Hardt and Simon in 1979.

Theorem 1.1.9 (Hardt-Simon [51]). *If T is an m dimensional area minimizing current in $\mathbb{R}^{m+1}$ with a smooth boundary Γ , then every boundary point is a regular point. In this setting, regular means one of two valid options:*

- *T is a smooth minimal surface which attaches smoothly at the boundary.*
- *T is a smooth minimal surface which is split in two by Γ , where it switches orientation. In order to get the correct boundary multiplicity one side has multiplicity $Q + 1$ the other Q for some positive integer Q .*

In arbitrary codimension, the closest analogue to Hardt-Simon, requiring Almgren's regularity techniques in a boundary setting, is the following theorem:

Theorem 1.1.10 (De Lellis- De Philipis - Hirsch - Massaccesi [21]). *If T is an area minimizing current with a smooth boundary Γ , then the set of regular points is open and dense. Here the definition of regular is the same one of Hardt-Simon. This result is sharp, meaning that the singular set can have a Hausdorff dimension equal to that of the full boundary.*

The remaining question, which was asked by Brian White in 1984 in the famous list of open questions [13], is what happens when there is an arbitrary boundary multiplicity.

In dimension 2, when the multiplicity is arbitrary there are two results:

Theorem 1.1.11 (De Lellis- Steinbruchel - Nardulli [26, 27]). *If T is a 2 dimensional area minimizing current with a smooth boundary Γ taken with multiplicity Q , then every density $Q/2$ point is a regular point: the surface consists of a union of smooth minimal surfaces which multiplicity attaching smoothly to the boundary.*

Theorem 1.1.12 (Nardulli - Resende [68]). *If T is a 2 dimensional area minimizing current with a smooth boundary Γ taken with multiplicity Q , then the regular set is open and dense. A regular point is again, defined to be one as those considered by Hardt and Simon.*

1.2 Contributions of this thesis

In this thesis we extend Allard's boundary regularity to a higher multiplicity setting.

1.2.1 Uniqueness of the tangent cone

As a result of the monotonicity formula for area, we know that blowup limits, which are suitable rescalings of the surface along a sequence, converge to an area minimizing cone. These sequential blowup limits are called tangent cones. However, the tangent cone may depend on the specific sequence taken.

In 1993, Leon Simon [72] developed a method using the error term in the area monotonicity formula to study the uniqueness of the tangent cone. His approach reduces this problem to analyzing the decay rate of energy for the associated linear problem. This method has been used by Wickramasekera [82], De Lellis-Minter-Skorobogatova [24] and Krummel-Wickramasekera [57, 58] for the relevant settings at interior points.

The situation and techniques are rather different in the context of 2d area minimizing currents, since in that setting there is an available isoperimetric inequality which is exclusive of dimension 2. For relevant literature in 2 dimensions from White, Hirsch-Marini, De Lellis-Nardulli-Steinbrüchel see [79, 54, 26], where the latter is the uniqueness of the tangent cone in the setting we are considering, but for 2-dimensional area minimizing currents.

The theorem we obtain is the following:

Theorem A (Fleschler - Chapter 3). Let T be an area minimizing current with boundary Γ taken with multiplicity Q . At points of density $Q/2$, the tangent cone is unique. Moreover, there exists a well-defined normal map, which is multivalued (taking Q normal vectors that define the tangent cone). This normal map is Hölder continuous as a multivalued function. Additionally, if Γ lies on the boundary of a uniformly convex set, then T has a unique tangent cone at every point on Γ , and the multivalued normal derivative is well-defined and Hölder continuous along the entire boundary.

1.2.2 Regularity theory

Building on the uniqueness of the tangent cone we prove in Chapter 3, we develop a regularity theory for the boundary singular set of area minimizing currents with higher boundary multiplicity. Building on the uniqueness of the tangent cone at density $Q/2$ points, we adapt the techniques of De Lellis and Skorobogatova [28, 29] to the boundary setting. Our approach relies on quantitative rectifiability tools introduced by Naber and Valtorta, which we discussed in the interior setting of this introduction, and necessitates a careful adaptation of the multivalued approximation and error control framework to the boundary context.

Theorem B (Fleschler - Resende - Chapter 4). *Let T be an m -dimensional area minimizing current with boundary Γ taken with multiplicity Q . The set of singular density $Q/2$ points is $m - 3$ rectifiable (countable when $m = 3$). Moreover, if Γ lies on the boundary of a uniformly convex set, then the whole boundary singular set of T is $m - 3$ -rectifiable (countable when $m = 3$).*

We also obtain the analogue of [21] in our setting. This provides the best possible extension of the Hardt–Simon regularity theorem to the setting of higher boundary multiplicity. This theorem, as [21], is sharp with respect to the Hausdorff dimension of the singular set.

Theorem C (Fleschler - Resende - Chapter 4). *Let T be an m -dimensional area minimizing current with a smooth boundary Γ taken with multiplicity Q , then the regular set is open and dense. A regular point is again, defined to be one as those considered by Hardt and Simon.*

Before our work, not a single boundary point in a higher boundary multiplicity setting was known to be regular for $m \geq 3$ dimensional integral currents (with codimension at least 2), even under convexity assumptions.

Technical considerations In order to run the regularity theory, we define the universal frequency $I(T, p)$ of De Lellis and Skorobogatova, at flat boundary singular points p . We obtain $I(T, p) \geq 1 + \alpha(m, n)$ and prove rectifiability via the Naber–Valtorta technique getting an estimate like (1.1.1).

We need to address the following:

- **Multivalued Approximations:** The approximations and center manifold that Almgren constructs must be adapted to our boundary setting.
- **Error Terms:** We need to control new boundary error terms, not present in [28, 29].
- **Dimensional Bound:** Requires adaptations of the [28, 29] estimates on the frequency function, highlighting a significant contrast with the interior case.

1.2.3 Singular example

The construction of singular examples of area minimizing currents presents substantial challenges, typically requiring the construction of a calibration, which is a specific type of differential form adapted to the surface whose existence implies the area minimizing property.

In codimension 1, the proof of the minimality of the Simons cone in 1969 [11] was a turning point in the history of regularity of area minimizing currents since it provided the first example of a singular area minimizing integral current in codimension 1. The proof was later simplified in [39]. In higher codimension, a large family of examples is given by

complex varieties due to Federer [41], since they are automatically calibrated and thus area minimizing.

In the context of boundary regularity for area minimizing currents, White in 1997 showed that in a parametric context there are no true boundary branch points for a 2-dimensional area minimizing current with a real analytic boundary curve [80]. A true branch point is one where the current cannot be separated into smooth minimal surfaces. In that work, he posed the question of whether an area minimizing current with a real analytic boundary has finite topology, which is equivalent to finiteness of the interior and boundary singular sets and to the area minimizing current being a Douglas - Radò solution. In contrast, for a smooth boundary, Gulliver in 1991 [50] provided an example of a true boundary branch point for a minimal surface (not minimizing) in $\mathbb{R}^3$.

In [53], Hirsch shows an example of a holomorphic function vanishing to infinite order at the boundary in the half-space. This example is used in [21] to show that if the boundary curve is merely allowed to be smooth then the boundary singular set can have Hausdorff dimension 1. Furthermore, in [19], De Lellis, De Philipis and Hirsch provide an example of an area minimizing current in a Riemannian manifold whose boundary is a smooth curve and the associated surface has infinite topology. These examples show that the behaviour when the boundary is smooth but not real analytic can be quite singular - unlike the regularity predicted by White's conjecture in the real analytic case.

Theorem D (Fleschler - Chapter 5). *There exists a 3-dimensional area minimizing current T in $\mathbf{B}_1$ with a real analytic boundary taken with multiplicity 2, where every boundary point is taken with density 1, and which has an essential singularity: a point where the surface cannot be decomposed locally as two smooth minimal surfaces attaching to the boundary.*

This theorem demonstrates the dimensional sharpness of the regularity theory we developed 4, in collaboration with Reinaldo Resende. The example we developed in Chapter 5 shows that, even if the boundary is required to be real analytic, there are counterexamples of essential singularities, analogous to the true branch points considered by White and Gulliver,

but in the higher multiplicity setting corresponding to Allard's PhD thesis [1] (multiplicity 2, density 1).

The example comes from a more general theorem, which roughly speaking says that if for T a 3d area minimizing current, ∂T looks like the picture on the right then it has an essential boundary singularity. The defining features of the picture are that ∂T consists of two pieces, one which is topologically a disk Γ^0 (taken with multiplicity 2) and the other which is a Möbius strip Γ^1 which passes through $\partial\Gamma^0$.

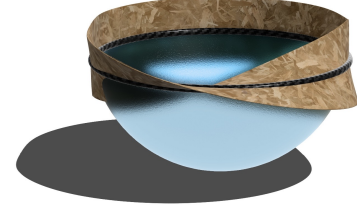


Figure 1.1: Boundary of example

The theorem is a consequence of the Hölder continuity of the multivalued normal map, which we show to hold on the spherical cap in the picture up to its boundary given by the black wire. This relies in the multivalued Hölder estimate at smooth boundary points established in Chapter 3 and employs at the corner points (the black wire), Simon's method for the uniqueness of tangent cones [72].

1.2.4 A theorem in classical rectifiability

The field of rectifiability started in the 1920's by Besicovitch with his study of sets of finite 1 dimensional measure. He proved a decomposition of any 1 dimensional set of finite measure into two pieces, a rectifiable part and a purely unrectifiable part. The rectifiable part, up to a set of 1 dimensional zero, can be covered by a union of countably many dimensional curves of finite length, the unrectifiable part intersects every curve of finite length in a set of 1-dimensional measure zero. Rectifiability was later extended into higher dimensions with significant contributions spanning from the 1920s to the 1990s by Besicovitch, Federer, Marstrand, Mattila and Preiss. In higher dimensions, a set $E \subset \mathbb{R}^n$ is k rectifiable if there exist countably many Lipschitz maps $f_i : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that $E \setminus \bigcup f_i(\mathbb{R}^k)$ has k -dimensional

measure zero. A set is said to be k -purely unrectifiable, if its intersection with any rectifiable set has k -dimensional measure zero.

In 1956, Besicovitch [10] showed, in more modern terms, that a set contained in the two-dimensional plane with one-dimensional non σ -finite measure, meaning it cannot be decomposed into countable sets of finite one-dimensional measure, has a purely unrectifiable subset. This is equivalent to stating that a set with every subset of one-dimensional finite measure being rectifiable must have σ -finite measure and be rectifiable in the classical sense. In collaboration with Camillo de Lellis, we generalize Besicovitch's theorem to a higher dimension and codimension setting.

Theorem E (De Lellis - Fleschler - Chapter 2). *A borel set with non σ -finite k -dimensional measure contains a purely unrectifiable subset with positive and finite k -dimensional measure. Equivalently, a set whose finite measure subsets are rectifiable, has σ -finite measure and is rectifiable in the classical sense.*

This generalization not only extends Besicovitch's classical theorem, but also serves to simplify the use of the Naber-Valtorta technique, a powerful tool widely used in geometric analysis for proving the rectifiability of singular sets in various settings, as already discussed in the context of regularity of area minimizing currents. We also utilize this result in the thesis to develop the boundary regularity theory with Reinaldo Resende in Chapter 4.

Organization of the thesis

This thesis is structured as follows. Chapter 2, written in collaboration with Camillo de Lellis, presents our rectifiability result and discusses its broader implications within geometric measure theory. Chapter 3 establishes the uniqueness of boundary tangent cones in the higher multiplicity Allard's setting. Chapter 4, jointly authored with Reinaldo Resende, develops the boundary regularity theory established in this thesis. Finally, Chapter 5 constructs examples of boundary singularities, demonstrating the optimality and sharpness of our theory.

Chapter 2

An elementary rectifiability lemma

This chapter is in collaboration with Camillo de Lellis and has been published in [22].

2.1 Introduction

The purpose of this chapter is to show two facts in geometric measure theory which have a rather elementary flavor and can have several interesting applications, but nonetheless seem to have gone unnoticed in the literature. As usual $\mathcal{H}^k$ denotes the Hausdorff k -dimensional measure and when k is an integer, we will call a set $R \subset \mathbb{R}^n$ k -rectifiable if it is $\mathcal{H}^k$ measurable and it can be covered by countably many Lipschitz k -dimensional graphs up to an $\mathcal{H}^k$ -null set. Moreover, we will call a Radon measure μ k -rectifiable if there is a k -rectifiable set R and a Borel function f such that $\mu = f\mathcal{H}^k \llcorner R$. We are then ready to state the first lemma.

Theorem 2.1.1. *Let $1 \leq k < n$ be two given integers. Assume $E \subset \mathbb{R}^n$ is a Borel subset and consider the family $\mathcal{F}(E)$ of all Radon measures μ supported in E such that:*

(a) *μ is a Frostman measure, namely there is a constant C such that*

$$\mu(B_r(x)) \leq Cr^k \quad \text{for every } x \text{ and every } r. \quad (2.1.1)$$

(b) the upper k -dimensional density of μ is positive μ -almost everywhere, namely

$$\limsup_{r \downarrow 0} r^{-k} \mu(B_r(x)) > 0 \quad \text{for } \mu\text{-a.e. } x.$$

Then E is k -rectifiable if and only if every measure $\mu \in \mathcal{F}(E)$ is k -rectifiable.

In the applications we have in mind E will typically be a closed set, in particular the assumption that E is Borel is more than enough. However understanding what is the minimal measurability assumption on E seems an interesting question. Given the relation with Radon measures, we conjecture that the most natural assumption on E is “universal measurability”, namely that for every Radon measure μ on $\mathbb{R}^n$ E is Borel up to some μ -null set. While we do not wish to dwell on this question, the theory available in the literature combined with our arguments give immediately the statements of Theorem 2.1.1 and Theorem 2.1.2 for an intermediate class of sets, called Souslin or analytic, which are universally measurable and contain the Borel sets. From now on we will therefore state our results for this more general class of sets. Since however this is not the main point of our note and rather a remark of technical nature, we just refer to [42, Section 2.2.10] for the definition of Souslin sets. Moreover, while we will point to the references in which one finds the necessary statements about them needed in our proofs, we will not make any further comments: the reader who is not familiar with the theory of Souslin sets and is not interested in fine measurability questions can simply substitute the word Borel to Souslin, and then all our statements would be covered by “classical” measure theory.

We are now ready to state the second lemma.

Theorem 2.1.2. *Let $1 \leq k < n$ be two given integers. Assume $E \subset \mathbb{R}^n$ is a Souslin set which is not $\mathcal{H}^k$ σ -finite. Then E contains a closed subset F such that $0 < \mathcal{H}^k(F) < \infty$ and which is purely k -unrectifiable, namely $\mathcal{H}^k(F \cap \Gamma) = 0$ for every Lipschitz k -dimensional graph Γ .*

Theorem 2.1.1 follows then from Theorem 2.1.2 rather quickly. First of all, if E is k -rectifiable, then it is obvious that any measure $\mu \in \mathcal{F}(E)$ is k -rectifiable, since condition (a)

implies that $\mu \ll \mathcal{H}^k \llcorner E$. The converse is particularly simple to prove when E is $\mathcal{H}^k$ - σ -finite. In this case if E were not k -rectifiable, then it is a classical fact that it contains a purely k -unrectifiable closed set F with $0 < \mathcal{H}^k(F) < \infty$. We can then appeal to the classical Frostman lemma (cf. [62]) to find a nontrivial Frostman measure μ which is supported in F . The Frostman condition also implies that $\mu \ll \mathcal{H}^k \llcorner F$ leading to the existence of a Borel function f such that $\mu = f\mathcal{H}^k \llcorner F$. In particular $\mu \in \mathcal{F}(E)$ and it is not k -rectifiable. The latter makes also clear why Theorem 2.1.2 provides the general converse in Theorem 2.1.1: it just guarantees the existence of a set F as above even when E is not $\mathcal{H}^k$ - σ -finite.

Remark 2.1.3. The theorem also holds when $E \subset (X, d)$ is a Souslin subset of a σ -compact doubling metric space, i.e. a metric space for which there exists M with the property that every ball of radius r can be covered by at most M balls of radius $r/2$. Since our main interest is in the Euclidean space we will only briefly sketch how to handle this more general case.

Before coming to a discussion of the interesting applications that the theorems above have, we point out that the case $n = 2$ and $k = 1$ of Theorem 2.1.2 is in fact contained in a classical work of Besicovitch, cf. [10, Theorem 6]. However, in spite of its rather elementary nature, we have not been able to find Theorem 2.1.2 in the literature, nor we have found any work asking the fairly obvious question of whether Besicovitch's theorem can be generalized to arbitrary dimension and codimension.

2.1.1 Rectifiability questions in geometric analysis

Our interest has been partially spurred by the groundbreaking work of Naber and Valtorta [67]. In several problems of geometric nature, for instance in the theory of minimal surfaces, that of harmonic maps, or that of Ricci limits, one is naturally lead to study solutions of PDEs, or minimal/critical points of variational problems, or generalized spaces, which are not everywhere smooth, but for which a suitable regularity theory gives upper bounds on the

dimension of the singular set (defined as the complement of all points at which the object of interest is smooth). When the upper bound is optimal, namely it matches the dimension of the singular set for some known examples, the next natural question is whether the singular set has some more structure. In some of the examples mentioned above, i.e. in the case of singularities of harmonic maps and of minimal surfaces, showing even just the rectifiability of the singular set has been surprisingly hard. In some notable cases (minimizing harmonic maps, area-minimizing hypersurfaces, and mod-2 minimizing surfaces in any dimension and codimension) this was achieved in pioneering works of Leon Simon in the nineties. We refer to chapter 15 of [63] for a more detailed overview of questions on the rectifiability of singularities of solutions to geometric variational problems.

A few years ago Naber and Valtorta introduced in [67] a rather powerful and flexible technique to recover and improve Simon's rectifiability results. The approach of [67] has in fact been extended to several other situations, especially due to its flexibility (cf. for instance [5, 6, 15, 23, 29, 40, 47, 48, 49, 66, 65]): while Simon's proof uses quite hard PDE techniques, the approach of Naber and Valtorta uses very little of the problem at hand, the real key point being the availability of a monotonicity formula with some suitable structure, a property which is common to a variety of situations in geometric analysis.

Roughly speaking the approach of [67] can be subdivided into three steps. To fix ideas we assume that the singular set we are interested in has Hausdorff dimension k .

- (i) A first general theorem, which has been independently discovered by Azzam and Tolsa in [8], ensures the k -rectifiability of a k -dimensional measure μ under a suitable sharp control of what in the literature is called L^2 β -number. For a particular case of the statement see Theorem 2.1.4 below (the version of [40] is slightly stronger, the version we refer to was already proved combining [8] and [77]).
- (ii) The sharp control needed in (i) is then achieved for every Frostman k -dimensional measure μ which is supported in the singular set using in a careful and clever way the monotonicity formula available for the problem.

(iii) If one knew the σ -finiteness of the singular set with respect to the $\mathcal{H}^k$ measure, (i) and (ii) would then immediately imply its rectifiability. However we only know a priori that its Hausdorff dimension is k . To overcome this difficulty, Naber and Valtorta use quite subtle covering arguments and an approximate version of (i)-(ii) for suitably discretized measures. An alternative approach is given in [15] through what the authors call “neck regions”.

The reason of our interest in Theorem 2.1.2 should now be obvious: with the latter at hand, we can just completely bypass point (iii) in the Naber-Valtorta strategy, and conclude immediately the σ -finiteness and rectifiability from (i) and (ii) at an “abstract level”. It must be however noted that, while Theorem 2.1.2 is thus a useful general tool, which allows to bypass a quite sizable and difficult argument in (iii), it does not have its exact same power. In fact, under certain additional assumptions, Naber and Valtorta are able to use their arguments in step (iii) to conclude also the local $\mathcal{H}^k$ -finiteness of the singular set. We do not exclude the possibility that, once we know the rectifiability through Theorem 2.1.2, the argument of Naber-Valtorta for proving $\mathcal{H}^k$ local finiteness could be simplified and streamlined: we have simply not tried to do it.

For completeness we end this paragraph by stating the Azzam-Naber-Tolsa-Valtorta Theorem referenced in point (i) above.

Theorem 2.1.4. *Let $S \subset \mathbb{R}^n$ be an $\mathcal{H}^k$ -measurable set with $0 < \mathcal{H}^k(S) < \infty$ and consider $\mu := \mathcal{H}^k \llcorner S$. Then S is k -rectifiable if and only if*

$$\int_0^1 \beta_{2,\mu}^k(x, s)^2 \frac{ds}{s} < \infty \quad \text{for } \mu\text{-a.e. } x,$$

where

$$\beta_{2,\mu}^k(x, s) := \inf \left(\left\{ r^{-k-2} \int_{B_r(x)} \text{dist}^2(y, L) d\mu(y) \right\}^{1/2} : L \text{ is a } k\text{-dim. affine space} \right).$$

2.1.2 Quantitative rectifiability

We are also interested in Theorem 2.1.2 in the broader context of quantitative rectifiability. In quantitative rectifiability many questions (which a posteriori can have many fruitful applications) can be phrased in the following form:

(Q) Consider a quantity $\alpha_E(x, r)$ which measures “how close to k -rectifiable a set E is” (such as some form of flatness or symmetry) at the scale r around the point x . Having fixed E consider

$$A_{\alpha_E} := \left\{ x \in E : \int_0^1 \alpha_E(x, r)^2 \frac{dr}{r} < \infty \right\}.$$

Is A rectifiable? More specifically, if we knew that $A_{\alpha_E} = E$, is E rectifiable?

The idea is then to use Theorem 2.1.2, in appropriate situations, to reduce the question above to the much more favorable case where E has finite $\mathcal{H}^k$ measure and $A_{\alpha_E} = E$.

For concreteness we will give two examples. We define the β coefficient for the Hausdorff content as:

$$\tilde{\beta}_{2,E}^k(x, r) := \inf \left\{ \left(r^{-k-2} \int_{B_r(x) \cap E} \text{dist}^2(y, L) d\mathcal{H}_\infty^k(y) \right)^{1/2} : L \text{ is a } k\text{-dim. affine space} \right\}.$$

The integral above is with respect to the Hausdorff content $\mathcal{H}_\infty^k$, which is *not a measure*. In particular we define such integral as

$$\int_A f(x)^p d\mathcal{H}_\infty^k(x) := \int_0^\infty \mathcal{H}_\infty^k(\{x \in A | f(x) > t\}) t^{p-1} dt.$$

Let E be a compact set. We first remark that $A_{\tilde{\beta}_{2,E}^k}$ is Borel. The coefficient $\tilde{\beta}_{2,E}^k$ is upper semicontinuous on (x, r) and the integral can be equivalently taken as a discrete sum over dyadic scales. This information is enough to verify that the set is Borel since the coefficient is a Borel measurable function of x .

Let μ be any Frostman (or k upper regular) measure supported on E . It is rather easy to see then that:

$$\beta_{2,\mu}^k(x, r) \lesssim \tilde{\beta}_{2,E}^k(x, r).$$

This means that, if $F \subseteq A_{\tilde{\beta}_{2,E}^k}$ is a Borel set such that $\mu = c\mathcal{H}^k \llcorner F$ (for some constant $c > 0$) is Frostman, then μ has finite $\beta_{2,\mu}^k$ square function. In particular we can use Theorem 2.1.4 to say that every such subset is rectifiable. Fix now any $F \subset A_{\tilde{\beta}_{2,E}^k}$ with finite Hausdorff measure. A simple exercise shows that F can be written as a countable union $Z \cup \bigcup_i F_i$, where the Borel sets F_i enjoy the above property and Z is $\mathcal{H}^k$ -null. We can apply Theorem 2.1.2 and conclude that the answer to (Q) is positive. The coefficients $\tilde{\beta}_{2,E}^k$ were introduced by Azzam and Schul in their work [7] on the analyst's traveling salesman theorem in the context of lower content regular sets. The study of the set $A_{\tilde{\beta}_{2,E}^k}$ has been done in the context of tangent points to lower content regular sets by Villa in [78]. The argument above gives that the latter set is rectifiable even when the lower content regularity is dropped.

The second author was also interested in Theorem 2.1.2 due to his work on the higher dimensional Carleson ε^2 conjecture, joint with Tolsa and Villa. Let $\Omega^+, \Omega^- \subset \mathbb{R}^{n+1}$ be open and disjoint and let $B(x, r) \subset \mathbb{R}^{n+1}$ be a ball. Let $H \subset \mathbb{R}^{n+1}$ be a halfspace such that $x \in \partial H$. Denote $S_H^+ = \partial B(x, r) \cap H$ and $S_H^- = \partial B(x, r) \setminus \overline{H}$. Define

$$\varepsilon_n(x, r) = \frac{1}{r^n} \inf_H \sum_{i=+,-} \mathcal{H}^n(S_H^i \setminus \Omega^i).$$

Theorem 2.1.5. *Let $\Omega^+, \Omega^- \subset \mathbb{R}^{n+1}$ be open and disjoint and let*

$$E := \left\{ x \in \mathbb{R}^{n+1} : \int_0^1 \varepsilon_n(x, r)^2 \frac{dr}{r} < \infty \right\}.$$

Then E is n -rectifiable.

The strategy used in [46] is to prove that every subset of this set with finite measure is rectifiable. In order to conclude that the whole set has σ -finite measure (and hence is

n -rectifiable) [46] uses a specific argument which still relies on Lemma 2.3.4. Alternatively we can now use Theorem 2.1.2 and Corollary 2.1.1. We only need to check that A_{ε_n} is Souslin, but indeed it is Borel. This is already remarked in [46], but we include a brief discussion here. Indeed the coefficients $\varepsilon_n(x, r)$ are upper semicontinuous on (x, r) and thus, for every positive s , the map $x \mapsto \int_s^1 \varepsilon_n(x, r)^2 \frac{dr}{r}$ is upper semicontinuous. This is enough to check that the A_{ε_n} is Borel.

2.2 Geometric preliminaries

We denote by $\mathcal{Q}_j$ the family of closed dyadic cubes of $\mathbb{R}^n$ with sidelength 2^{-j} .

Definition 2.2.1. We say that a set E is sparse if there is a sequence of integers $l_j \uparrow \infty$ with the following property. For every $Q \in \mathcal{Q}_{l_j}$ there is a subcube $Q' \subset Q$ with $Q' \in \mathcal{Q}_{l_j+j}$ such that

$$E \subset \bigcup_{Q \in \mathcal{Q}_{l_j}} Q'. \quad (2.2.1)$$

For the purpose of Theorem 2.1.2 the latter definition would be enough and we would need the rather standard fact that any subset of a sparse set cannot be k -rectifiable for any $k \geq 1$. In order to handle the more general case of doubling metric spaces it is however useful to consider a version of the sparsity defined above which uses balls in place of cubes.

Definition 2.2.2. We say that a set E is weakly sparse if there is a sequence of real numbers $r_j \downarrow 0$ and a universal constant N with the following property. For every ball B of radius r_j there exist at most N balls $\{B_l\}_{1 \leq l \leq N}$ of radius $2^{-j}r_j$ such that

$$E \cap B \subset \bigcup_{1 \leq l \leq N} B_l. \quad (2.2.2)$$

Remark 2.2.3. Notice that if E is sparse then it is weakly sparse with $N = 3^n$. This follows from the fact that a ball of radius 2^{-j+1} can intersect 3^n cubes of sidelength 2^{-j} .

The main tool we need from this section towards Theorem 2.1.2 is the following proposition.

Proposition 2.2.4. *Let k be a positive integer and (X, d) a metric space. If $F \subset X$ is an $\mathcal{H}^k$ -measurable set which is weakly sparse, then F is purely k -unrectifiable.*

Proof of Proposition 2.2.4. We can reduce to the case where X is Banach space by embedding the metric space X isometrically into some Banach space.

Suppose that F is not purely k -unrectifiable. This means there exists a Borel set $\mathbb{B} \subset \mathbb{R}^k$ and a Lipschitz function $f : \mathbb{B} \rightarrow X$ such that $\mathcal{H}^k(f(\mathbb{B}) \cap F) > 0$. We will show that this cannot happen. It will be enough to show this to be the case when f is a bi-Lipschitz map on $\mathbb{B}$. Indeed, a rectifiable subset of a Banach space can be covered, up to a set of $\mathcal{H}^k$ measure zero, by a countable union of bi-Lipschitz images of Borel sets. We appeal to chapter 7 of [63], page 56, for this particular characterization of rectifiable subsets of Banach spaces.

We will denote the bi-Lipschitz constant by L . We can further restrict $\mathbb{B}$ to an $\mathcal{H}^k$ -measurable subset, such that its image via f is now contained in F . We claim that $\mathbb{B}$ must be a weakly sparse subset of $\mathbb{R}^k$. Let $\hat{B}$ be any ball of radius $(2L)^{-1}r_j$.

Then $f(\hat{B}) \subseteq B$ where B is a ball of radius r_j . This means that there exist $\{B_l\}_{1 \leq l \leq N}$ balls of radius $2^{-j}r_j$ such that

$$\mathbb{B} \cap \hat{B} \subseteq \bigcup_{1 \leq l \leq N} f^{-1}(B_l).$$

There exists balls $\hat{B}_l$ of radius $L2^{-j}r_j$ that contain the ball $f^{-1}(B_l) \subset \hat{B}_l$. This is indeed the weakly sparse condition.

Now the weakly sparse condition will give us that for every $x \in \mathbb{B}$, the lower Lebesgue density is zero (i.e. $\Theta_*^k(x, \mathbb{B}) = 0$). This implies that $\mathbb{B}$ has zero Lebesgue measure, and thus $f(\mathbb{B})$ has zero $\mathcal{H}^k$ measure. This completes the proof of the proposition. $\square$

2.3 Generalized Hausdorff measures

In this section we collect the other tool which is needed to prove Theorem 2.1.2. We follow [71] and introduce the following generalization of the Hausdorff measure $\mathcal{H}^k$.

Definition 2.3.1. Given a continuous increasing function $r \mapsto h(r)$ with $h(0) = 0$ and a set E we define

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam}(C_i)) \mid \text{diam}(C_i) \leq \delta, \bigcup_{i=1}^{\infty} C_i \supseteq E \right\}.$$

We also define $\mathcal{H}^h(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(E)$.

In particular the usual Hausdorff measure $\mathcal{H}^k$ amounts to the choice of $h(r) = \omega_k \left(\frac{r}{2}\right)^k$. All the classical facts about “standard Hausdorff measures”, e.g. that they are an outer measure, that they are Borel regular, and many of the tools available for them generalize to the case above. Since it is not in our interest to review the general theory, which is indeed extensively treated in [71], we will just limit ourselves to provide precise references to the facts which are needed in our proof.

We will need three key lemmas. The following is a generalization of Frostman’s lemma.

Lemma 2.3.2. *Let $r \mapsto h(r)$ be a continuous increasing function with $h(0) = 0$. Given an analytic set E such that $\mathcal{H}^h(E) > 0$ there is a Radon measure μ such that $\mu(E) > 0$ and $\mu(Q) \leq h(\text{diam}(Q))$ for every dyadic cube Q .*

Proof. We can use [71, Corollary 2, Section 6.2] to find an F_σ set $E' \subset E$ with $\mathcal{H}^h(E') = \mathcal{H}^h(E)$, which in turn reduces the lemma to the case of a compact E . We can then follow the proof of [62, Theorem 8.8]. We report it here for the reader’s convenience to highlight that, even though the proof in [62, Theorem 8.8] is written for the case $h(r) = \omega_s \left(\frac{r}{2}\right)^s$, namely that of classical Hausdorff measures, it works for general h .

Without loss of generality we assume that $E \subset [0, 1]^n$. We fix a positive integer m and define the measure $\mu_{m,m}$ as

- $\mu_{m,m} \llcorner Q = h(\text{diam}(Q))2^{-nm} \mathcal{L}^n \llcorner Q$ for every $Q \in \mathcal{Q}_m$ such that $Q \cap E \neq \emptyset$;

- $\mu_{m,m} \llcorner Q = 0$ otherwise.

We then define $\mu_{m,l}$ inductively with l decreasing till $\mu_{m,0}$. Assuming that $\mu_{m,l}$ has been defined and $l > 0$, we define $\mu_{m,l-1}$ on each cube $Q \in \mathcal{Q}_{l-1}$ as follows:

- $\mu_{m,l-1} \llcorner Q = \mu_{m,l} \llcorner Q$ if $\mu_{m,l}(Q) \leq h(\text{diam}(Q))$;
- $\mu_{m,l-1} \llcorner Q = h(\text{diam}(Q))(\mu_{m,l}(Q))^{-1} \mu_{m,l} \llcorner Q$ otherwise.

We then set $\mu_m := \mu_{m,0}$ and observe that $\mu_{m,0}(\mathbb{R}^n) \leq h(\text{diam}([0,1]^n)) = h(\sqrt{n})$. In particular we can assume the existence of a subsequence, not relabeled, converging weakly* to some Radon measure μ . Observe that $\mu_m(Q) \leq h(\text{diam}(Q))$ for every dyadic cube Q with $\text{diam}(Q) \geq \sqrt{n}2^{-m}$. However, since Q is closed, we have the inequality $\mu(Q) \geq \limsup_{m \uparrow \infty} \mu_m(Q)$, which of course cannot be used to get an upper bound on $\mu(Q)$. Rather, for every fixed Q , consider the 3^n dyadic cubes $Q_1, \dots, Q_{3^n}$ with same diameter which have nonempty intersection with it. If we let U be the interior of $\bigcup_i Q_i$, then obviously $\mu_m(U) \leq 3^n h(\text{diam}(Q))$ whenever $\sqrt{n}2^{-m} \leq \text{diam}(Q)$. On the other hand U is open and contains Q , so we can conclude

$$\mu(Q) \leq \mu(U) \leq \liminf_{m \rightarrow \infty} \mu_m(U) \leq 3^n h(\text{diam}(Q)).$$

In particular, up to dividing by 3^n , we achieve the inequality $\mu(Q) \leq h(\text{diam}(Q))$ for every dyadic Q .

Next consider that μ_m is, by construction, supported in the union of cubes of diameter $\sqrt{n}2^{-m}$ which intersect E , and thus in the $\sqrt{n}2^{-m}$ -neighborhood of E . Since E is closed, μ is therefore supported in E . Hence, as soon as we show that $\mu(\mathbb{R}^n) > 0$, we are done. On the other hand μ is supported in $[0,1]^n$ and therefore $\mu(\mathbb{R}^n) = \lim_m \mu_m(\mathbb{R}^n)$. It thus suffices to find a lower bound for $\mu_m(\mathbb{R}^n)$. Fix m and observe that each $x \in E$ is contained in a cube Q of diameter at least $\sqrt{n}2^{-m}$ such that $\mu_m(Q) = h(\text{diam}(Q))$ and $Q \subset [0,1]^n$. Denote it by Q_x . $\{Q_x\}_{x \in E}$ is then a finite collection of dyadic cubes. So, given two different x and y in E , either $Q_x \subset Q_y$, or the intersection of Q_x and Q_y is contained in ∂Q_x . In particular

if we let $\mathcal{C}$ denote the family of those Q_x which are maximal under inclusion, we find that the elements of $\mathcal{C}$ have disjoint interiors, while they form a covering of E . Note that μ_m is absolutely continuous with respect to the Lebesgue measure: though two distinct cubes $Q, Q' \in \mathcal{C}$ might have nonempty intersection, the latter has zero Lebesgue measure, and hence zero μ_m measure. We therefore have

$$\sum_{Q \in \mathcal{C}} h(\text{diam}(Q)) = \sum_{Q \in \mathcal{C}} \mu_m(Q) \leq \mu_m(\mathbb{R}^n).$$

However, because $\mathcal{C}$ is a cover of E , we find

$$\mathcal{H}_\infty^h(E) \leq \sum_{Q \in \mathcal{C}} h(\text{diam}(Q)) \leq \mu_m(\mathbb{R}^n).$$

Now, since $\mathcal{H}^h(E) > 0$, we must have $\mathcal{H}_\infty^h(E) > 0$ as well, in particular we have found the desired positive lower bound for $\mu_m(\mathbb{R}^n)$. $\square$

The second is a classical result for “usual” Hausdorff measures, which follows from combining [62, Theorem 8.8] with [62, Theorem 8.13].

Lemma 2.3.3. *Assume ν is a (nontrivial) Radon measure on $\mathbb{R}^n$ and $s > 0$ a real number such that $\nu(Q) \leq (\text{diam}(Q))^s$ for every dyadic cube Q . Then there is a closed subset $F \subset \text{spt}(\nu)$ such that $0 < \mathcal{H}^s(F) < \infty$.*

The third is perhaps the most critical ingredient. In the case of a closed set E it is a theorem of Rogers in [70], generalizing a classical one by Besicovitch in the plane, cf. [10, Theorem 3]. The case of an analytic set is achieved combining Roger’s Theorem with [75, Theorem 6.6], which implies that any analytic set which is not $\mathcal{H}^h$ σ -finite contains a compact set which is not $\mathcal{H}^h$ - σ -finite (the case $h(r) = r^s$ of the latter theorem is due to Davies in [16]).

Lemma 2.3.4. *Let $r \mapsto h(r)$ be a continuous increasing function with $h(0) = 0$ and let $E \subset \mathbb{R}^n$ be an analytic set which is not $\mathcal{H}^h$ - σ -finite. Then there is a continuous function*

$r \mapsto g(r)$ with $g(0) = 0$ such that E is still not $\mathcal{H}^g$ - σ -finite and

$$\lim_{r \downarrow 0} \frac{g(r)}{h(r)} = 0. \quad (2.3.1)$$

In the converse direction one has that if $\mathcal{H}^g(E) > 0$ and g, h satisfy 2.3.1 then $\mathcal{H}^h(E)$ is non σ -finite. This is easily seen to be true since if $\mathcal{H}^h(E)$ was σ finite we can cut it into countable pieces with finite measure, each of those having zero $\mathcal{H}^g$ measure.

2.4 Proof of Theorem 2.1.2

We start with a set E as in Theorem 2.1.2 and we apply Lemma 2.3.4 to find a continuous function h such that $h(0) = 0$, $r \mapsto h(r)$ is increasing,

$$\lim_{r \downarrow 0} \frac{h(r)}{r^k} = 0, \quad (2.4.1)$$

and E is not $\mathcal{H}^h$ σ -finite. We next apply Lemma 2.3.2 to find a corresponding measure μ . We can, without loss of generality, assume $\mu(\mathbb{R}^n \setminus E) = 0$, since we can restrict μ to E (note that, since E is analytic, E is also universally measurable, which in particular implies that it is μ -measurable). Since μ is a Radon measure, we can find a compact subset E' such that $\mu(E') > 0$. Without loss of generality we assume that μ is in fact supported in E' (again, it suffices to substitute μ with $\mu \llcorner E'$). We next prove the following lemma.

Lemma 2.4.1. *Let h be a continuous increasing function such that (2.4.1) holds (and thus $k < n$) and μ a nontrivial Radon measure such that $\mu(Q) \leq h(\text{diam}(Q))$ for every dyadic cube Q . There is a nontrivial Radon measure ν with the following properties:*

- (a) $\nu(Q) \leq (\text{diam}(Q))^k$ for every dyadic cube Q ;
- (b) the support of ν is contained in the support of μ ;
- (c) the support of ν is sparse.

Having Lemma 2.4.1 at our disposal, we find a corresponding measure ν . We can then apply Lemma 2.3.3 to find a set F in the support of ν such that $0 < \mathcal{H}^k(F) < \infty$. It follows from our discussion so far that F is contained in the set E given at the very beginning. On the other hand such set F must be sparse and hence, by Proposition 2.2.4, it is purely k -unrectifiable, thus completing the proof of Theorem 2.1.2. We are thus left with proving Lemma 2.4.1.

Proof. We fix μ as in the statement and, without loss of generality, we assume that μ is supported in $[0, 1]^n$ and that μ is a probability measure. We also assume, without loss of generality, that

$$\mu(\partial Q) = 0 \quad \text{for every dyadic cube } Q. \quad (2.4.2)$$

In fact, if the condition (2.4.2) does not hold, it means that $\mu(F) > 0$ for some face F of ∂Q . We can then restrict μ to the latter face, and reduce the statement to the same one in $\mathbb{R}^{n-1}$. The measure μ is still a Frostman measure over F with the dimension function $\tilde{h}(r) := h(\sqrt{\frac{n}{n-1}}r)$. We then get that $\text{spt}(\mu) \cap F$ has positive $\mathcal{H}^{\tilde{h}}$ measure. Of course, $\tilde{h}$ still satisfies 2.4.1, and thus $\text{spt}(\mu) \cap F$ still has non σ -finite $\mathcal{H}^k$ measure. We iterate until (2.4.2) holds and we observe that this must happen because, if we were to reach $k = n - 1$, given that the dimension function h satisfies (2.4.1), any $\mathcal{H}^{n-1}$ -finite Borel set would be μ -null.

The measure ν will be constructed as weak* limit of an appropriate sequence ν_j of measures. The latter, which will be constructed inductively from $\mu = \nu_0$, will all be probability measures satisfying the following properties.

- (i) The support of ν_j is contained in the support of ν_{j-1} and (2.4.2) holds with ν_j in place of μ .
- (ii) $\nu_j(Q) \leq \min\{2^{j^2 n} h(\text{diam}(Q)), C_0(\text{diam}(Q))^k\}$ for all dyadic cubes Q , where the constant $C_0 \geq 1$ is chosen so that $h(r) \leq C_0 r^k$ for every $r \leq \sqrt{n}$.

Moreover, there will be a sequence of integers l_j satisfying $l_{j+1} \geq l_j + j$ and $l_0 = 0$, such that ν_j satisfies the following additional properties:

(iii) $\nu_j(Q) = \nu_{j-1}(Q)$ for all dyadic cubes $Q \in \mathcal{Q}_{l_j}$.

(iv) For $j \geq 1$ there is a family $\mathcal{F}_j \subset \mathcal{Q}_{l_j+j}$ such that

(iv') ν_j is supported in $\bigcup_{Q' \in \mathcal{F}_j} Q'$;

(iv'') For each cube $Q \in \mathcal{Q}_{l_j}$ there is at most one cube $Q' \in \mathcal{F}_j$ such that $Q' \subset Q$.

Observe that, combining (iii) and (i) we actually know that

(iii') $\nu_j(Q) = \nu_{j-1}(Q)$ for all dyadic cubes $Q \in \mathcal{Q}_l$ with $l \leq l_j$.

Moreover $\nu_0 = \mu$ satisfies all the requirements.

Before coming to the inductive construction of ν_j , let us observe that ν_j converges weakly* to some measure ν (we could in fact appeal to the weak* compactness of probability measures to conclude the existence of such a limit for an appropriate subsequence, but actually by (iii') there is no need for such an extraction) and that such ν satisfies the requirements of the lemma. In fact $\text{spt}(\nu) \subset \text{spt}(\nu_j)$ for every j by (i), which in particular implies that $\text{spt}(\nu) \subset \text{spt}(\nu_0) = \text{spt}(\mu)$. Moreover, because of (iv), we conclude that $\text{spt}(\nu)$ is sparse. Observe that, because of (ii), $\nu_j([0, 1]^n) = \mu([0, 1]^n) = 1$ for every j , and in particular ν must be a probability measure. Finally, from (ii) and (iii) we also conclude (a) after rescaling.

We now come to the construction of ν_j from ν_{j-1} . First we pick $l_j \geq l_{j-1} + j$ so that

$$\frac{h(\text{diam}(Q))}{(\text{diam}(Q))^k} \leq 2^{-nj^2} \quad \forall Q \in \mathcal{Q}_l \text{ with } l \geq l_j, \quad (2.4.3)$$

which obviously is possible because of (2.4.1). Next, select all the cubes $Q \in \mathcal{Q}_{l_j}$ for which $\nu_{j-1}(Q) > 0$ and call this collection $\mathcal{G}$. Secondly, subdivide each cube Q into 2^{nj} cubes belonging to $\mathcal{Q}_{l_j+j}$ and select one Q' among them so that

$$\nu_{j-1}(Q') \geq 2^{-nj} \nu_{j-1}(Q).$$

We next define $\mathcal{F}_j$ to be the collection of such Q' and we define ν_j on each cube $Q \in \mathcal{G}$ as

$$\nu_j \llcorner Q = \frac{\nu_{j-1}(Q)}{\nu_{j-1}(Q')} \nu_{j-1} \llcorner Q'. \quad (2.4.4)$$

Note that the latter gives a well defined measure because we know $\nu_{j-1}(\partial Q) = 0$ for every dyadic cube Q .

Observe that (i), (iii), and (iv) all hold by construction. As for (ii), because of (iii'), the inequality holds for any cube $Q \in \mathcal{Q}_l$ with $l \leq l_j$. We thus need to show it for any cube $Q \in \mathcal{Q}_l$ with $l > l_j$. But by construction such a cube must be contained in an element $\bar{Q} \in \mathcal{G}$. If we denote by $\bar{Q}'$ the element of $\mathcal{F}_j$ contained in $\bar{Q}$, (2.4.4) yields the inequality

$$\nu_j(Q) \leq \frac{\nu_{j-1}(\bar{Q})}{\nu_{j-1}(\bar{Q}')} \nu_{j-1}(Q) \leq 2^{nj} \nu_{j-1}(Q).$$

Next, observe that, since (iii) holds for ν_{j-1} , we have

$$\begin{aligned} \nu_j(Q) &\leq 2^{nj} \min\{2^{(j-1)2^n} h(\text{diam}(Q)), C_0(\text{diam}(Q))^k\} \\ &\leq 2^{j2^n} h(\text{diam}(Q)) = \min\{2^{j2^n} h(\text{diam}(Q)), C_0(\text{diam}(Q))^k\}, \end{aligned}$$

where we have used (2.4.3), which we can apply because $Q \in \mathcal{Q}_l$ and $l > l_j$. In particular this shows the estimate (ii) for any dyadic cube, and hence completes the proof. $\square$

Sketch of the proof of Remark 2.1.3. We provide a brief sketch of the modifications to be made. The interested reader can fill the details.

First of all, any σ -compact metric space X can be written as the countable union of compact subsets X_i of X , while if $E \subset X$ is not σ -finite for the Hausdorff k -dimensional measure, then for some i , $E \cap X_i$ must be non σ -finite for the Hausdorff k -dimensional measure on X_i . This shows that our statement reduces to compact metric spaces.

Lemma 2.3.4 can be used in the same way as before, in particular in the paper [75] the lemma is proved in compact metric spaces.

The proofs of Lemma [2.4.1](#) and Lemma [2.3.2](#) rely on the dyadic cube structure of the Euclidean space, which still exists on doubling metric spaces. Note that the constant N will depend precisely on the doubling constant of the metric space.

The dyadic cube structure of doubling metric spaces has been frequently exploited in harmonic analysis and there is much literature on the topic. The reader can consult for instance [\[55\]](#).

We can then complete the proof using Proposition [2.2.4](#). □

Chapter 3

Uniqueness of tangent cones at minimum density boundary points

This chapter has been published as a preprint in [\[43\]](#).

3.1 Introduction

In this chapter, we prove the uniqueness of the tangent cone to an area minimizing current with smooth multiplicity Q boundary, at boundary points where the density is the lowest possible, that is, $Q/2$. This is the first paper in a series of three where we extend Allard's Boundary Regularity theorem to a higher multiplicity boundary setting. In collaboration with Reinaldo Resende, in Chapter [4](#), we show that for an m -dimensional area minimizing current, the set of boundary singular points with density $Q/2$ is $\mathcal{H}^{m-3}$ rectifiable. Moreover, we show that the set of boundary regular points, without any density restriction, is both open and dense. We show the regularity theory for density $Q/2$ points to be dimensionally sharp in Chapter [5](#).

As a consequence of the uniqueness of the tangent cone, we show a decomposition theorem that serves as the foundation for the regularity theory in Chapter [4](#).

Additionally, under a convex barrier hypothesis - a setting previously considered by Allard in the multiplicity 1 case - we show that all boundary points will have density $Q/2$. Thus, our theorems also apply in this context.

In Chapter 5 we use the results and methods of this Chapter to prove an excess decay type theorem at points of "corner type". The boundaries we consider in that setting consist of smooth surfaces meeting at a smooth $m - 2$ dimensional common interface and we obtain decay to the tangent cone at the common interface. This allows us to construct a very general family of examples of essential boundary singularities of density 1 for a 3 dimensional area minimizing current in $\mathbb{R}^5$ with a multiplicity 2 boundary that can be taken real analytic. This family of examples shows dimensional sharpness of the regularity theory developed in [45].

3.1.1 History

In his 1993 paper [72], Simon investigated the uniqueness of tangent cones for a class of stationary varifolds with multiplicity 1. In this work, he developed what are now referred to as Simon's estimates, using the remainder of the monotonicity formula under the presence of points of suitable density, often called the "no holes" condition. These estimates show that the L^2 height of the varifold does not concentrate at the spine of the cone, thus reducing the task of proving uniqueness of the tangent cone to studying the energy decay for the associated linear problem.

Also in an interior regularity setting, Wickramasekera in [82] extended Simon's work to a higher multiplicity setting in codimension 1. He studied stationary varifolds that are stable in the regular part and have no classical singularities. More recently, De Lellis, Minter, and Skorobogatova [24], along with Krummel and Wickramasekera [57, 58], independently used Simon's techniques at the interior for area minimizing currents with higher multiplicity and higher codimension.

The recent work [25] studies area minimizing currents mod p , where the tangent cones under consideration include open books (composed of planes meeting at a common spine)

and classical planes with multiplicity. In the present paper, we study, for the first time, the case of area minimizing currents with boundary taken with integer multiplicity $Q > 1$ for an integral current of arbitrary dimension and codimension. We mention that in this context, the "no holes" condition is naturally satisfied at every point, allowing us to establish the uniqueness of tangent cones at **every** boundary point of minimum density.

An important difference from [25] is that, while their work also considers tangent cones that includes open books in the $m - 1$ strata, the currents do not have boundary mod p . Their results instead address, in a sense, an interior regularity problem for a class of stationary varifolds. Furthermore, their cones satisfy a balancing condition: $\sum Q_i \nu_i = 0$, where ν_i are the normals to the spine, and the multiplicities Q_i fulfill $Q_i \leq p/2$ and $\sum Q_i = p$. For a related uniqueness of tangent cones result concerning a subclass of open books in codimension 1, see [64].

The approaches to studying the uniqueness of the tangent cone differ substantially in the context of 2d area minimizing currents since they rely on the isoperimetric inequality which is only available in 2d. Indeed in [79], White showed the uniqueness of tangent cones at interior points in this setting using an isoperimetric inequality. Hirsch and Marini later adapted White's result to the case of *boundary* uniqueness for 2d area minimizing currents with boundary multiplicity 1 [54]. More recently, De Lellis, Nardulli, and Steinbrüchel [26] extended these results to arbitrary boundary multiplicity.

3.1.2 Results

This chapter studies regularity properties at the boundary of an area-minimizing current at points of minimum density. We assume the current has a C^2 boundary and is area minimizing in a C^2 manifold.

Assumption 3.1.1. Let $m, n \geq 2$, $\bar{n} \geq 1$, $Q \geq 1$ be integers. Consider Σ an $(m + \bar{n})$ -manifold of class C^2 in $\mathbb{R}^{m+n}$, Γ a C^2 oriented $(m - 1)$ -submanifold of Σ without boundary, and assume

$0 \in \Gamma$. Let T be an integral m -dimensional area minimizing current in $\Sigma \cap \mathbf{B}_1$ with boundary $\partial T \llcorner \mathbf{B}_1 = Q \llbracket \Gamma \cap \mathbf{B}_1 \rrbracket$.

The convex barrier condition in Euclidean space is the following:

Assumption 3.1.2 (Convex Barrier). Let $\Omega \subset \mathbb{R}^{m+n}$ be a domain such that $\partial\Omega$ is a C^2 uniformly convex submanifold of $\mathbb{R}^{m+n}$. We say that $\sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$ has a convex barrier if Q_i are positive integers and $\Gamma_i \subset \partial\Omega$ are disjoint C^2 closed oriented submanifolds of $\partial\Omega$. In this setting we consider T an area minimizing current with $\partial T = \sum_i Q_i \llbracket \Gamma_i \rrbracket$.

We will also consider Assumption 3.4.7, a more general version of the convex barrier condition, later in the paper. This condition is local in nature and includes the possibility of an ambient Riemannian manifold.

We start the program by obtaining a classification of the area minimizing cones. We define open books as follows:

Definition 3.1.3. A cone C is an *open book* if it can be written as $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ where H_i are distinct half-planes with $\partial \llbracket H_i \rrbracket = \llbracket V \rrbracket$ where V is an $m-1$ dimensional plane, referred to as the spine of the cone, and Q_i are positive integers.

Theorem F (Classification of the tangent cones). Let C be an m -dimensional oriented area minimizing cone in $\mathbb{R}^{m+n}$ with $\partial \llbracket C \rrbracket = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$. Then $\Theta(C, 0) \geq Q/2$. The equality case holds if and only if C is an open book. Moreover, there is a density jump: if C is not an open book then $\Theta(C, 0) \geq Q/2 + \varepsilon(m, n)$ where $\varepsilon(m, n) > 0$.

Under Assumption 3.1.1, a monotonicity formula for the density holds at boundary points (Theorem 3.2.4). As a consequence of the classification of Theorem F, the boundary density must be at least $Q/2$ and equality only holds when every tangent cone is an open book. The density jump mentioned is only obtained at the end of the program as a consequence of the techniques for the main theorem rather than at the start.

Moreover, if T is an area minimizing current with $\partial T = \sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$ which satisfies the convex barrier Assumption 3.1.2, then for any $p \in \Gamma_i$, $\Theta(T, p) = Q_i/2$, and thus it belongs to the class of minimum density points, which we are studying.

The main theorem of the work is the following:

Theorem G (Uniqueness of the tangent cone). Let T, Γ, Σ be as in Assumption 3.1.1. Further assume that $\Theta(T, 0) = Q/2$. Then, the tangent cone at 0 is an open book and it is unique, meaning it does not depend on the blow up sequence. Moreover, the current T converges to the unique tangent cone with a power rate.

The previous statement in dimension 2 was proved in [26]. We consider the convex barrier assumption considered in [1, 3] and obtain the following structural consequence of the uniqueness of the tangent cone:

Theorem H. Let T be as in either Assumption 3.1.2 (convex barrier) or Assumption 3.4.7 (Local convex barrier), then every tangent cone to T at a boundary point q is an open book. Moreover, we have:

- (a) if $q \in \Gamma_i$ in the first case, then $\Theta(T, q) = Q_i/2$,
- (b) if $q \in \Gamma$ in the second case, then $\Theta(T, q) = Q/2$.

Thus there is a unique tangent cone for T in both settings, convex barrier 3.1.2 and local convex barrier 3.4.1, at every boundary point. Moreover, the current T has a power rate of convergence to the unique tangent cone.

In order to reduce the above theorem to Theorem G, we need to classify tangent cones with Allard's convex barrier assumption. In 3.4.8, we classify those tangent cones relying on Lemma 5.1 [3].

An essential consequence of the uniqueness of tangent cones, which will enable us to run the regularity theory in [45], is a decomposition theorem. This decomposition theorem allows us to reduce the study of boundary singularities to the study of **flat** boundary singularities. In dimension 2, the decomposition was proved by [26, 27].

Theorem I (Decomposition theorem). Let T, Γ, Σ be as in Assumption 3.1.1. Further assume that $\Theta(T, 0) = Q/2$. Let $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ be the unique tangent cone to T at 0, where the half-planes H_i in the representation are assumed to be distinct. Then there exists $\rho > 0$ and area minimizing currents $T_1, T_2, \dots, T_N$ in $\mathbf{B}_\rho$ such that

$$T \llcorner \mathbf{B}_\rho = \sum_{i=1}^N T_i$$

where the supports of T_i only intersect at Γ , $\partial T_i \llcorner \mathbf{B}_\rho = Q_i \llbracket \Gamma \rrbracket$, and the (unique) tangent cone at 0 of T_i is $Q_i \llbracket H_i \rrbracket$.

We will work with multivalued functions, understood as in [4] and [30]. Another consequence of the uniqueness of tangent cones that plays a key role in constructing the example is the Hölder continuity of the multivalued normal map.

Theorem J. Let T, Γ, Σ be as in Assumption 3.1.1. There is a well-defined multivalued normal map induced by the tangent cone, which is defined in 3.5.11. The normal map is a locally Hölder map in $U := \{x \in \Gamma : \Theta(T, x) = Q/2\}$, which forms a relative open set of Γ . The set U is relatively since there exists $\theta(Q, m, n) > 0$ since there exists $\theta(Q, m, n) > 0$ such that $\Theta(T, x) < Q/2 + \theta$ implies $\Theta(T, x) = Q/2$.

We also comment that there are analogous theorems for the linear problem. The general linear problem is defined as:

Assumption 3.1.4. Given a Dir-minimizing function $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. The boundary linear problem consists of understanding functions u such that $u \llcorner \partial\Omega \cap \mathbf{B}_1 = Q \llbracket 0 \rrbracket$.

We denote the frequency function by I in the theorem below using a suitably smoothed distance function. For the precise definition of the frequency function we refer to Section 10.

Theorem K. Assume that Ω is a C^2 domain and u as in Assumption 3.1.4.

- For all $q \in \partial\Omega$ the frequency can be defined in a neighbourhood of q and is almost monotone. This means,

$$e^{C\mathbf{A}_{\partial\Omega}r}I(r)$$

is monotone in r for appropriate constant C , where $\mathbf{A}_{\partial\Omega}$ denotes the C^0 norm of the second fundamental form of Ω .

- At the limit, the frequency is independent of the choice of a distance function and $\forall q \in \partial\Omega, I(q) \geq 1$.
- Either $I(q) = 1$ or $I(q) \geq 1 + \alpha(Q, m, n)$ for a positive constant $\alpha(Q, m, n)$. If $I(q) = 1$ then the blowups at q are unique and linear, in the sense they are of the form

$$f(x) = \sum_{i=1}^N Q_i \llbracket v_i(x \cdot \boldsymbol{\eta}_{\partial\Omega, q}) \rrbracket$$

with $\boldsymbol{\eta}_{\partial\Omega, q}$ the normal to $\partial\Omega$ at q , and unique. The uniqueness is quantitative and thus there is a well-defined multivalued normal derivative at the boundary which is Hölder continuous.

3.1.3 Basic definitions

Throughout the paper, we use the notation ' $\lesssim$ ' to denote inequality up to a multiplicative constant, which may depend only on $m, n, \bar{n}$, and Q . We also use $C(\alpha_1, \dots, \alpha_k)$ to denote a constant which depends on $\alpha_1, \dots, \alpha_k$.

We define $\mathbf{B}_r(p) := \{x \in \mathbb{R}^{m+n} : |x-p| < r\}$ and typically use $V := \mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^{m+n}$ as the spine of our cones. For a plane π , let $\mathbf{p}_\pi$ denote the projection onto π . For any $\varepsilon > 0$, we use $B_\varepsilon(V)$ to represent the ε -neighborhood of V , that is, $B_\varepsilon(V) = \{x \in \mathbb{R}^{m+n} : \text{dist}(x, V) < \varepsilon\}$.

We assume each k -dimensional linear subspace π of $\mathbb{R}^{m+n}$ is oriented by a k -vector $\vec{\pi} = v_1 \wedge \dots \wedge v_k$, where $(v_i)_{i=1}^k$ forms an orthonormal basis for π . Additionally, we adopt the notation $|\pi_2 - \pi_1|$ for $|\vec{\pi}_2 - \vec{\pi}_1|$, where $|\cdot|$ represents the norm tied to the standard inner

product of k -vectors. The k -dimensional Hausdorff measure on $\mathbb{R}^{m+n}$ is written as $\mathcal{H}^k$. We denote $\mathcal{A}_Q(\mathbb{R}^{m+n})$ the Q -tuples in $\mathbb{R}^{m+n}$ for some integer $Q \geq 1$. We denote by $\mathcal{G}(p_1, p_2)$, the Wasserstein distance between a pair of points p_1, p_2 on the space $\mathcal{A}_Q(\mathbb{R}^{m+n})$. For more on multi-valued functions, see [30]. We denote by $\mathbf{A}_M$ the C^0 -norm of the second fundamental form of a smooth submanifold M of $\mathbb{R}^{m+n}$.

For further basic definitions and standard notations, we refer to [42] and the survey [17].

We will use $\mathbf{E}$ to denote the L^2 height excess, and local versions which we introduce later. The L^2 excess relative to an open book C is defined as

$$\mathbf{E}(T, C, \mathbf{B}_r(p)) := \frac{1}{r^{m+2}} \int_{\mathbf{B}_r(p)} \text{dist}(x, C)^2 d|T|(x).$$

We will use $\mathbb{E}$ for a suitable measure-theoretic excess, which will be defined later in equation (3.2.3). Finally, we introduce the circular projection $\pi_\circ : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$, defined as $\pi_\circ(x, y) = (x, |y|)$ for $(x, y) \in \mathbb{R}^{m-1} \times \mathbb{R}^{n+1}$.

Definition 3.1.5. An m -dimensional integral current T in U with $\text{spt}(T) \subset \Sigma$ is *area minimizing* in $\Sigma \cap U$, if it holds that $\mathbf{M}(T) \leq \mathbf{M}(T + \partial \tilde{T})$, for all $\tilde{T}$ $m+1$ -dimensional integral current in U with $\text{spt}(\tilde{T}) \subset \Sigma \cap U$.

Cones

We remind the definition of an open book and introduce their angle:

Definition 3.1.6. A cone C is an *open book* if it can be written as $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ where H_i are distinct half-planes with $\partial \llbracket H_i \rrbracket = \llbracket V \rrbracket$ where V is an $m-1$ dimensional plane, referred to as the spine of the cone. We define the minimum angle of an open book as

$$\alpha(C) := \min_{i \neq j} \langle H_i, H_j \rangle$$

where $\alpha(C) := 1$ if $C = Q \llbracket H \rrbracket$ for H a half-plane.

Definition 3.1.7. A cone C is a *two-sided flat cone* if

$$C = (Q/2 + Q^*) \llbracket H \rrbracket - (Q^*) \llbracket -H \rrbracket$$

for Q and Q^* integers and H a half plane.

We introduce a distance between open books with the same spine.

Definition 3.1.8 (Distance Open Books). Given a pair of open books C^j for $j = 0, 1$ which are of the form

$$C^j := \sum_i Q_i^j \llbracket H_i^j \rrbracket$$

where $\partial \llbracket H_i^j \rrbracket = \llbracket V \rrbracket$. We take η_i^j to be the unit normal to V that determines H_i . We define $\eta^j \in \mathcal{A}_Q(V^\perp)$ as $\eta^j = \sum Q_i \llbracket \eta_i^j \rrbracket$. We define

$$\mathcal{G}(C^1, C^2) := \mathcal{G}(\eta^1, \eta^2). \quad (3.1.1)$$

Tangent cones.

For $x \in \mathbb{R}^{m+n}$ and $r > 0$, define $\iota_{x,r}(y) := \frac{y-x}{r}$. For any integral current T , we define the rescaled current T at x at scale r as $\iota_{x,r} T =: T_{x,r}$ and $T_r =: \iota_{0,r} T$. The current T_x is called a blow-up of T at x , if there exists a sequence of radii $r_j \rightarrow 0$ such that $T_{x,r_j} \rightarrow T_x$ in the weak topology. By [12], it follows from the monotonicity formula that if T is area minimizing in Σ and $\partial T = Q \llbracket \Gamma \rrbracket$ for $\Gamma \in C^{1,\kappa}$, every convergent subsequence of T_{p,r_j} converges to an oriented cone about p for any $p \in \Gamma$ and any $r_j \downarrow 0$. An oriented cone about p is understood as a current C such that $C_{p,r} = C$ for any $r > 0$. See Theorem 3.2.4 for the precise statements of the monotonicity formula and first variation formula for an area minimizing current T in Σ with $\partial T = Q \llbracket \Gamma \rrbracket$ where Σ and Γ are C^2 submanifolds.

3.1.4 Structure of the chapter

- In Section 2, we define a measure-theoretic/stronger excess, which allows us to track not only the closeness to an open book but also the multiplicity to which it is close. We also prove the monotonicity formula with the corresponding error terms suitable for our use in the later estimates.
- In Section 3, we show that cones of density $Q/2$ with a flat boundary with multiplicity Q are open books. We provide two different arguments.
- In Section 4, we examine the convex barrier property and the local convex barrier property. We show that they are boundary points of the type we consider, i.e., of density $Q/2$, whose tangent cones are open books.
- In Section 5, we perform the main technical reductions of the theorems in this chapter, reducing them to an excess decay lemma, which the remainder of the chapter focuses on proving.
- In Section 6, we construct a Lipschitz approximation and show a reverse excess estimate, which allows us to control the measure-theoretic excess introduced in Section 2 by the L^2 excess.
- In Section 7, we prove Simon's non-concentration estimates and their corollaries in our setting.
- In Section 8, we study the linear problem. We show that homogeneous frequency 1 Dir-minimizers are linear, and that local Dir-minimizers away from the boundary are global Dir-minimizers. We provide a decay lemma to a linear average that is suitable to run the excess decay lemma.
- In Section 9, we prove the excess decay lemma stated in Section 5, thus concluding the uniqueness of the tangent cone.

- In Section 10, we discuss important consequences of the theory developed in this chapter for the linear problem, which mirror the properties we have discussed throughout the chapter for the nonlinear problem.

3.2 Preliminaries

3.2.1 A measure theoretic excess

We introduce a stronger version of the L^2 -excess using a variant of the Wasserstein L^2 distance, which we recall below.

Given two measures μ^1 and μ^2 , an admissible transport plan σ is a measure on $\mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$ satisfying $(\pi_1)_\# \sigma = \mu^1$ and $(\pi_2)_\# \sigma = \mu^2$ where π_1 and π_2 are the projections in the first and second component, respectively.

The Wasserstein L^2 distance between two measures μ^1 and μ^2 is defined as

$$W_2(\mu^1, \mu^2)^2 := \inf \left\{ \int d(x, y)^2 d\sigma(x, y) : (\pi_1)_\# \sigma = \mu^1 \text{ and } (\pi_2)_\# \sigma = \mu^2 \right\} \quad (3.2.1)$$

where we set $W_2(\mu^1, \mu^2) = \infty$ if $\mu^1(\mathbb{R}^{m+n}) \neq \mu^2(\mathbb{R}^{m+n})$.

Given a fixed $m-1$ dimensional plane V , we define the distance d_V between two measures μ^1 and μ^2 as follows:

$$d_V(\mu^1, \mu^2) := \inf \left\{ W_2(\mu_1^1, \mu_1^2)^2 + \sum_{i=1}^2 \int \text{dist}(x, V)^2 d|\mu_2^i| \mid \mu_1^i + \mu_2^i = \mu^i, i = 1, 2. \right\}. \quad (3.2.2)$$

In this definition, μ^1 , μ^2 , μ_1^1 , and μ_1^2 are Radon measures, while μ_2^1 and μ_2^2 are signed measures. This allows us to add or remove a part of the measure μ^1 to transport it to μ^2 , with a cost proportional to the weighted total variation of the leftover measures.

In the rest of the chapter we will drop V from the notation and just use d , since it is more convenient and the plane V will be clear from the context (and fixed in most arguments).

If $d(\mu^1, \mu^2) = 0$ then $\text{spt}(|\mu^1 - \mu^2|) \subset V$. We remark that in the cases of interest the measures π_1 and π_2 considered are always absolutely continuous with respect to $\mathcal{H}^m$ and thus $d(\mu^1, \mu^2) = 0$ implies $\mu^1 = \mu^2$ in that case.

Let ϕ be a smooth, canonical bump function supported in $\mathbf{B}_1(0)$, constantly equal to 1 in $\mathbf{B}_{1/2}(0)$, and satisfying $\phi(x) \lesssim \text{dist}(x, \partial\mathbf{B}_1)^2$. We define $\phi_r := \phi(\cdot/r)$.

The stronger excess relative to an open book $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ with $\partial H_i = V$ as

$$\mathbb{E}(T, C, \mathbf{B}_r) := r^{-(m+2)} d(\phi_r d|T|, \phi_r d|C|). \quad (3.2.3)$$

The bump function is introduced for technical reasons: even if the current were a classical smooth graph over the cone, the map given by the graphical structure might extend beyond the ball $\mathbf{B}_r$ near the boundary.

Remark 3.2.1 (Strong excess versus classical L^2 and reverse L^2). The classical L^2 excess and the reverse L^2 excess are both controlled by this stronger excess. This follows immediately from the admissibility of the transport plan and Fubini's theorem. Precisely we have:

$$\frac{1}{r^{m+2}} \int \phi_r(x) \text{dist}(x, C)^2 d\|T\|(x) + \frac{1}{r^{m+2}} \int \phi_r(x) \text{dist}(x, \text{spt}(T))^2 d\|C\|(x) \leq \mathbb{E}(T, C, \mathbf{B}_r). \quad (3.2.4)$$

To motivate the definition of the excess, we show that it is bounded in terms of standard quantities that we can control in practice. More precisely we have the following remark which will give the desired motivation.

Remark 3.2.2 (Lipschitz Approximation and Strong Excess). Let $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ and let $u_i \in \mathcal{A}_{Q_i}(H_i \cap \mathbf{B}_2)$ be multi-valued Lipschitz graphs over C of small enough Lipschitz constant. Let K_i be a subset of H_i and $T_i = \mathbf{Gr}(u_i \llcorner K_i)$. Assume that $\text{spt}(T_i) \cap \mathbf{B}_1 \subseteq \text{spt}(T) \cap \mathbf{B}_2$ and that $\forall x \in \text{spt}(T_i) \cap \mathbf{B}_1$ $\Theta(T_i, x) \leq \Theta(T, x)$.

Then

$$\mathbb{E}(T, C, \mathbf{B}_1) \lesssim \max_i |u_i|_{L^\infty(\mathbf{B}_1)}^3 + \sum_{i=1}^N \int_{\mathbf{B}_1 \cap (H_i \setminus K_i)} \text{dist}(x, V)^2 d\|H_i\|(x) \quad (3.2.5)$$

$$+ \int_{\mathbf{B}_1} \text{dist}(x, V)^2 d\|T - \sum_{i=1}^N T_i\|(x) + \sum_{i=1}^N \int_{\mathbf{B}_1} \text{dist}(x, C)^2 d\|T_i\|(x) \quad (3.2.6)$$

$$+ \sum_{i=1}^N \int_{K_i} \text{dist}(x, V)^2 |Du_i|^2. \quad (3.2.7)$$

Proof. Let A_i be the subset of K_i such that $|x + f_i^j(x)| > 1$ for some $1 \leq j \leq Q_i$ (i.e. the set where the Lipschitz graph escapes the ball).

We must have that

$$\mathbb{E}(T, C, \mathbf{B}_1) \lesssim \sum_{i=1}^N \int_{\mathbf{B}_1 \cap ((H_i \setminus K_i) \cup A_i)} \text{dist}(x, V)^2 d\|H_i\|(x) + \int_{\mathbf{B}_1} \text{dist}(x, V)^2 d\|T - \sum_{i=1}^k T_i\|(x) \quad (3.2.8)$$

$$+ W_2(\|T_i\|, (\pi_{H_i})_\# \|T_i\|)^2 + \int_{K_i} \text{dist}(x, V)^2 (d(\pi_{H_i})_\#(|T_i|) - d\|H_i\|(x)) \quad (3.2.9)$$

This inequality comes from the pairing of $\|T_i\|$ with $(\pi_{H_i})_\# \|T_i\|$ and getting as error terms the measures $\|T - \sum T_i\|$, $\sum_{i=1}^N d\|\mathbf{B}_1 \setminus (A_i \cup (H_i \setminus K_i))\|$ and $(\pi_{H_i})_\# \|T_i\| - d\|K_i \setminus A_i\|$.

We must have that A_i is contained in an annulus whose outer boundary is $\partial \mathbf{B}_1$ and its width is bounded by $\|u_i\|_{L^\infty}$, which gives us the bound $\|u_i\|_{L^\infty}^3$. The linear contribution comes from the area of the region and the quadratic contribution comes from the bump function.

We observe that

$$W_2(|T_i|, (\pi_{H_i})_\# |T_i|)^2 \leq \int_{K_i} \text{dist}(x, C)^2 d|T_i|. \quad (3.2.10)$$

Finally we notice that

$$\int_{K_i} \text{dist}(x, V)^2 d(\pi_{H_i})_\# (T_i) - d|H_i| \lesssim \int_{K_i} \text{dist}(x, V)^2 |Du_i|^2. \quad (3.2.11)$$

□

Remark 3.2.3. The question whether we can control excess from one scale to a smaller one is technically non trivial and requires the construction of the Lipschitz approximation and the smallness of the excess. We will also need to be able to switch center.

These issues are subtle because the transport plan for a given scale might not be suitable for a smaller scale. See Lemma 3.6.14.

3.2.2 Monotonicity formula

Given T, Γ, Σ as in Assumption 3.1.1. We define, for some $C_0(m, n, \bar{n})$ a suitable large constant

$$\Theta_i(T, p, r) := \exp(C_0 \mathbf{A}_\Sigma^2 r^2) \frac{\|T\|(\mathbf{B}_r(p))}{\omega_m r^m} \quad (3.2.12)$$

and

$$\Theta_b(T, p, r) := \exp(C_0 (\mathbf{A}_\Sigma + \mathbf{A}_\Gamma) r) \frac{\|T\|(\mathbf{B}_r(p))}{\omega_m r^m}. \quad (3.2.13)$$

Theorem 3.2.4 (Monotonicity formula). *Let T, Γ, Σ be as in Assumption 3.1.1.*

- *If $p \in \mathbf{B}_1 \setminus \Gamma$ then $r \rightarrow \Theta_i(T, p, r)$ is monotone on the interval $(0, \min \{ \text{dist}(p, \Gamma), 1 - |p| \})$.*
- *If $p \in \mathbf{B}_1 \cap \Gamma$, then $r \rightarrow \Theta_b(T, p, r)$ is monotone on $(0, 1 - |p|)$.*

Thus the density exists at every point. Moreover, the restrictions of the map $p \rightarrow \Theta(T, p)$ to $\mathbf{B}_1 \cap \Gamma$ and $\mathbf{B}_1 \setminus \Gamma$ are upper semicontinuous.

Additionally, this implies that blowup limits are cones, both at the interior and at the boundary.

If $X \in C_c^1(\mathbf{B}_1, \mathbb{R}^n)$, then

$$\delta T(X) = - \int_{\mathbf{B}_1} X \cdot \vec{H}_T(x) d\|T\|(x) + Q \int_{\Gamma \cap \mathbf{B}_1} X \cdot \vec{\eta}(x) d\mathcal{H}^{m-1}(x) \quad (3.2.14)$$

where $\vec{H}_T = \sum_{i=1}^m A_\Sigma(e_i, e_i)$ where $\vec{T} = \wedge_{i=1}^m e_i$ is an orthonormal basis of $\vec{T}$. The vector $\vec{\eta}$ is defined $\mathcal{H}^{m-1}$ almost everywhere as:

- For every tangent cone C at $p \in \Gamma$ which is an open book $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$,

$$\vec{\eta} = \sum_{i=1}^N \frac{Q_i \vec{\eta}_i}{Q} \quad (3.2.15)$$

where $\vec{\eta}_i$ is the normal to $T_p(\Gamma)$ which determines the half plane H_i . The vector $\vec{\eta}$ is independent of the open book C at p , but the open book might depend on the blowup sequence.

- For every tangent cone $C = (Q/2 + Q^*) \llbracket H \rrbracket - (Q^*) \llbracket -H \rrbracket$ at $p \in \Gamma$ which is a two-sided flat cone

$$\vec{\eta} = \eta_H$$

where η_H is the normal to $T_p(\Gamma)$ which determines the half plane H .

Corollary 3.2.5 (Monotonicity formula with error terms). *Under the hypothesis of Theorem 3.2.4 for every $p \in \mathbf{B}_1$ and $0 < s < r < 1 - |p|$,*

$$\begin{aligned} r^{-m} \|T\|(\mathbf{B}_r(p)) - s^{-m} \|T\|(\mathbf{B}_s(p)) &= \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_s(p)} \frac{|(x-p)^\perp|^2}{|x-p|^{m+2}} d\|T\|(x) \\ &+ \int_s^r \rho^{-m-1} \left[\int_{\mathbf{B}_\rho(p)} (x-p)^\perp \cdot \vec{H}_T(x) d\|T\|(x) + Q \int_{\Gamma \cap \mathbf{B}_\rho(p)} (x-p) \cdot \vec{\eta} d\mathcal{H}^{m-1} \right] d\rho \end{aligned} \quad (3.2.16)$$

Proof of Theorem 3.2.4 and Corollary 3.2.5. The proof follows the same steps as Theorem 3.2 in [21]. However, the key difference is that we allow for arbitrary boundary multiplicity, whereas [21] considers only boundaries with multiplicity 1. The first two statements are a consequence of Allard interior and boundary monotonicity formulas for stationary varifolds.

The first variation formula is established in the same way as in [21], which we briefly sketch. We first obtain the expression $\delta T = \vec{H}_T \|T\| + \delta T_s$ where δT_s is a singular vector

valued measure supported in Γ . This leads to the identity

$$\delta T(x) = - \int_{\mathbf{B}_1} X \cdot \vec{H}_T(x) d\|T\|(x) + Q \int_{\Gamma \cap \mathbf{B}_1} X \cdot \vec{\eta}(x) d\|\delta T_s\|. \quad (3.2.17)$$

As a consequence of the monotonicity formula we get that $\|\delta T_s\|$ is an absolutely continuous measure with respect to $\mathcal{H}^{m-1}$. The density of the singular measure and vector $\vec{\eta}$ are determined by the $\mathcal{H}^{m-1}$ normal and density at the blowups.

In this setting, there exists a stratification of the tangent cones at the boundary analogous to all other stratification theorems from [80]. See [21, Section 3.2] for more details. In this setting, we obtain $\mathcal{H}^{m-1}$ -almost everywhere there exists a tangent cone with an $m - 1$ dimensional spine. By the classification of 1d area minimizing cones, we conclude that $\mathcal{H}^{m-1}$ almost everywhere, there exists a tangent cone which is either an open book or a flat two sided cone.

We consider the first variation formula in the case that C is an open book or C is two-sided flat.

If $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ where V is the common spine and $\vec{\eta}_i$ is the normal vector to the spine that determines H_i , then

$$\delta C(X) = \sum_{i=1}^N Q_i \int_V (X \cdot \vec{\eta}_i) d\mathcal{H}^{m-1}. \quad (3.2.18)$$

If $C = (Q + Q^*) \llbracket H \rrbracket - Q^* \llbracket H^- \rrbracket$ then

$$\delta C(X) = Q \int_V (X \cdot \eta_H) d\mathcal{H}^{m-1}. \quad (3.2.19)$$

Where η_H is the normal to V which determines H .

In the first case, we define $\vec{\eta} = \sum_{i=1}^N \frac{Q_i \eta}{Q}$. In the second case, $\vec{\eta} = \eta_H$.

By combining (3.2.17) with the absolute continuity of $\|\delta T_s\|$ with respect to $\mathcal{H}^{m-1}$ we obtain $\mathcal{H}^{m-1}$ almost everywhere by Lebesgue differentiation theorem that $\vec{\eta}(p)$ agrees with the vector $\vec{\eta}$ for the tangent cones C in the $m-1$ - strata.

From the above discussion, we conclude Theorem 3.2.4. The proof of Corollary 3.2.5 as a consequence of Theorem 3.2.4 is analogous to [21]. $\square$

3.3 Classification of area minimizing cones of density $Q/2$ with boundary

In this section, we prove the following classification of area minimizing cones of density $Q/2$ with boundary:

Theorem 3.3.1 (Classification of area minimizing cones with density $Q/2$). *Let C be an m -dimensional area minimizing cone in $\mathbb{R}^{m+n}$. Assume that $\partial C = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$ for some positive integer Q . Then $\Theta(C, 0) \geq Q/2$. Moreover, equality holds if and only if C is an open book.*

The first proof follows an argument inspired by Federer's dimension reduction, while the second proof uses structural information gained from the projection $\pi_o : \mathbb{R}^{m+n-1} \rightarrow \mathbb{R}^{m-1} \times [0, \infty)$ defined as

$$\pi_o(x, y) = (x, |y|) \text{ where } (x, y) \in (\mathbb{R}^{m-1} \times \mathbb{R}^{n+1}).$$

This projection is well adapted to the this boundary setting, $\pi_{o\#}C = Q \llbracket \mathbb{R}^{m-1} \times \mathbb{R}^+ \rrbracket$ for any area minimizing cone C with $\partial C = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$.

Proof of Theorem 3.3.1. Let C be an area minimizing cone with $\partial C = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$. We denote $V = \mathbb{R}^{m-1} \times \{0\}$.

By Theorem 3.2.4, the classical monotonicity formula holds without error terms for any area minimizing current T on $\mathbb{R}^{m+n}$ with $\partial T = Q \llbracket V \rrbracket$. Indeed, in the formulas of Theorem

3.2.4 we have $\mathbf{A}_V = \mathbf{A}_\Sigma = 0$. Alternatively, if we consider T as a varifold and reflect through V we obtain a stationary varifold, which has twice the mass at every ball centered at V , for which the classical monotonicity formula for stationary varifolds holds. Moreover we also have, as usual, that if $p \in V$ and

$$\|T\|(\mathbf{B}_r(p)) = \Theta_b(T, p) \quad (3.3.1)$$

for all $r \leq r_0$, then T is a cone at p .

To establish $\Theta(C, 0) \geq Q/2$, assume by contradiction that $\Theta(C, 0) = \Theta < Q/2$. Since C is a cone then $\Theta(C, p) = \Theta(C, rp)$ for every $p \in V$ such that $p \neq 0$ and $r > 0$. By upper semi-continuity of the density for every $p \in V$ we must have that $\Theta(C, p) \leq \Theta(C, 0)$. Consequently, any blowup at p results in another area minimizing cone C_1 with density strictly lower than Θ . However C_1 must be a product of a cone $C'_1 \times \llbracket L \rrbracket$ where up to rotation $\partial C'_1 = \llbracket \mathbb{R}^{m-2} \times \{0\} \rrbracket$.

By repeating the argument, we are lowering the density but increasing the number of symmetries. We obtain an area minimizing cone C_{m-1} with

$$C_{m-1} = \llbracket \mathbb{R}^{m-1} \rrbracket \times C'_{m-1} \quad (3.3.2)$$

where C'_{m-1} is a 1d area minimizing cone with boundary $Q \llbracket 0 \rrbracket$.

The only options for such cones are a union of half-lines whose multiplicities sum up to Q . The cone C_{m-1} must have density equal to $Q/2$, which is a contradiction since the density is at most Θ which is strictly lower than $Q/2$.

We note that if C is an open book then trivially $\Theta(C, 0) = Q/2$. Thus it remains to show that if $\Theta(C, 0) = Q/2$ then C is an open book.

We know that for every $p \in V$

$$\lim_{r \rightarrow \infty} \Theta(C, p, r) = \lim_{r \rightarrow \infty} \frac{|C|(\mathbf{B}_r(p))}{\omega_m r^m} \leq \lim_{r \rightarrow \infty} \frac{(r + |p|)^m}{r^m} \frac{|C|(\mathbf{B}_{r+|p|})}{\omega_m (r + |p|)^m} = Q/2. \quad (3.3.3)$$

This implies that we must have the equality in the monotonicity formula centered at every $p \in V$, which implies that C is independent of the directions of V and thus C comes from a 1d area minimizing cone. As we have seen, this implies C is an open book. $\square$

Alternative Proof of Theorem 3.3.1. Define $\pi_{\circ} : \mathbb{R}^{m+n-1} \rightarrow \mathbb{R}^{m-1} \times [0, \infty)$ as

$$\pi_{\circ}(x, y) = (x, |y|) \text{ where } (x, y) \in (\mathbb{R}^{m-1} \times \mathbb{R}^{n+1}).$$

We observe that $\pi_{\circ}$ is 1-Lipschitz:

$$\begin{aligned} |\pi_{\circ}(x_1, y_1) - \pi_{\circ}(x_2, y_2)|^2 &= |(x_1 - x_2, |y_1| - |y_2|)|^2 = |x_1 - x_2|^2 + (|y_1| - |y_2|)^2 \\ &\leq |x_1 - x_2|^2 + (|y_1 - y_2|)^2 = |(x_1 - x_2, y_1 - y_2)|^2. \end{aligned}$$

By the constancy theorem, $\pi_{\circ\#}C = Q_0 \llbracket \mathbb{R}^{m-1} \times \mathbb{R}^+ \rrbracket$ where Q_0 is a positive integer. Since $\partial\pi_{\circ\#}T = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$, it follows that $Q = Q_0$.

Since $\pi_{\circ}$ is 1-Lipschitz

$$\Theta(C, 0) = \frac{|C|(\mathbf{B}_1)}{\omega_m} \geq \frac{|\pi_{\circ\#}C|(\pi(\mathbf{B}_1))}{\omega_m} = \frac{Q}{2}. \quad (3.3.4)$$

This proves the desired inequality. The rest of this proof will be spent studying the equality case. We note that if C is an open book then trivially $\Theta(C, 0) = Q/2$. Thus it remains to show that if $\Theta(C, 0) = Q/2$ then C is an open book.

We consider the differential form

$$\omega := dr \wedge de_1 \wedge \dots \wedge de_{m-1} \quad (3.3.5)$$

where given $(x, y) \in \mathbb{R}^{m-1} \times \mathbb{R}^{n+1}$ we take $r = |y|$. By definition $d\omega = 0$ outside of V and $\|\omega\| \leq 1$. Let Ω be the volume form in $\mathbb{R}^{m-1} \times [0, \infty)$. We notice that $\omega = \pi_{\circ}^{\#}\Omega$ and $\|\omega\| = 1$ outside of V .

We have that

$$|C|(\mathbf{B}_1) = \frac{Q\omega_m}{2} = \int_{\pi_o(\mathbf{B}_1)} \langle \vec{\Omega}, \overrightarrow{\pi_{o\#}(C)} \rangle d|\pi_{o\#}(C)| = \int_{\mathbf{B}_1} \langle \omega, \vec{C} \rangle d|C|. \quad (3.3.6)$$

Given that $\|\omega\| = 1$ and $\|\vec{C}\| = 1$ we must have that $\langle \omega, \vec{C} \rangle \leq 1$. Since the equality holds this implies that $\langle \omega(p), \vec{C}_p \rangle = 1$ for $|C|$ almost every p and thus ω is a calibration for $\vec{C}$.

We see that this implies that

$$\vec{C}_p = \left(\frac{\sum_{k=0}^n x_{m+k} e_{m+k}}{r} \right) \wedge e_1 \wedge \dots \wedge e_{m-1}$$

for almost every p . This is because both of them have norm 1 and their inner product is equal to 1, by Cauchy Schwartz they must be a multiple of each other (which implies they are equal up to a sign, but once we know this we conclude that they must have the same orientation).

In order to conclude we consider a new map $\sigma : \mathbb{R}^{m-1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \rightarrow \mathbb{R}^{n+1}$ defined as

$$\sigma(x, y) = \frac{y}{|y|} \text{ where } (x, y) \in \left(\mathbb{R}^{m-1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \right).$$

We note that since $T_p(C) = V \oplus \mathbb{R}^+ p$ for almost every $p \in C$ we must have that the tangential gradient of σ along C vanishes (i.e., $D_C \sigma \equiv 0$). This implies that $\sigma \lrcorner C$ is locally constant and thus every connected component of C outside of V agrees with a half-plane passing through V . We easily conclude from this that C is an open book. See the proof of [3.4.9](#). □

3.4 Convex barrier assumption

In this section, we study the convex barrier assumption and we classify the tangent cones we obtain in that setting.

We recall the definition of the convex barrier in Euclidean space.

Assumption 3.4.1 (Convex Barrier). Let $\Omega \subset \mathbb{R}^{m+n}$ be a domain such that $\partial\Omega$ is a C^2 uniformly convex submanifold of $\mathbb{R}^{m+n}$. We say that $\sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$ has a convex barrier if Q_i are positive integers and $\Gamma_i \subset \partial\Omega$ are disjoint C^2 closed oriented submanifolds of $\partial\Omega$. In this setting we consider T an area minimizing current with $\partial T = \sum_i Q_i \llbracket \Gamma_i \rrbracket$.

The main theorem of this section will be

Theorem 3.4.2. *Let T be as in either Assumption 3.1.2 (convex barrier) or Assumption 3.4.7 (local convex barrier), then every tangent cone to T at a boundary point q is an open book. Moreover, we have:*

- (a) *if $q \in \Gamma_i$ in the first case, then $\Theta(T, q) = Q_i/2$,*
- (b) *if $q \in \Gamma$ in the second case, then $\Theta(T, q) = Q/2$.*

3.4.1 The convex hull property

In Euclidean ambient space under 3.4.1 we must have

$$\text{spt}(T) \subset \text{ConvexHull}(\cup_{i=1}^N \Gamma_i).$$

However, this property does not necessarily hold in arbitrary Riemannian manifolds. Indeed some global conditions of the geometry are required to reach a conclusion of that nature.

The following lemma 3.4.3 illustrates a wider context, that of manifolds non-positive sectional curvature, where the mentioned property holds. The argument does not rely on the differentiable structure and thus it is also translatable to the context of metric spaces with non-positive sectional curvature. For another relevant reference on the convex barrier property see Chapter 4 [74] Lemma 6.1 and Theorem 6.2.

Lemma 3.4.3 (Convex barrier with curvature conditions). *Let Σ be a manifold with non-positive sectional curvature. Let Γ be an integral current in Σ with $\partial\Gamma = 0$. Let T an area minimizing current in Σ with $\partial T = \Gamma$. Then*

$$\text{spt}(T) \subset \text{ConvexHull}(\text{spt}(\Gamma)).$$

Proof. Given a geodesic ball $\mathbf{B}_r^\Sigma = \{x : d(x, p) < r\}$ we take the projection f_r from Σ to $\mathbf{B}_r^\Sigma$. This is well-defined since Σ is uniquely geodesic. Since the convex hull of a set is the intersection of balls which contain it, it is enough to show for every geodesic ball $\mathbf{B}_r^\Sigma$, if $\text{spt}(\Gamma) \subset \mathbf{B}_r^\Sigma$, then $\text{spt}(T) \subset \mathbf{B}_r^\Sigma$.

We wish to show that for every ball B_r we have

$$\mathbf{M}((f_r)_\#(T)) \leq \mathbf{M}(T)$$

with equality iff $\text{spt}(T) \subset \mathbf{B}_r^\Sigma$.

We will show that f_r is 1-Lipschitz and that if $A \subset \partial\mathbf{B}_s^\Sigma$ then

$$\mathcal{H}^m(f_r(A)) \leq \left(\frac{r}{s}\right)^m \mathcal{H}^m(A).$$

Assume $d(p, x) \geq d(p, y)$. Let x' be such that $d(p, x') = \max\{d(p, y), r\}$. We take a comparison triangle p_0, x_0, y_0 in Euclidean space and x'_0 is such that $|x'_0 - p_0| = d(x', p_0)$. By basic Euclidean geometry, $|x'_0 - y_0| \leq |x_0 - y_0|$. By comparison $d(x', y) \leq |x'_0 - y_0|$ and thus $d(x', y) \leq d(x, y)$.

We reduce then to show the 1-Lipschitz property of f_r to the case where $d(p, x) = d(p, y)$.

Given two points x, y , we take the two geodesics parameterized by arclength α and β connecting p to x and p to y . We must have $f_r(x) = \alpha(r)$ and $f_r(y) = \beta(r)$.

The non-positive sectional curvature condition implies that $d(\alpha(t), \beta(t))$ is a convex function and thus using $d(\alpha(0), \beta(0)) = 0$ we get

$$\begin{aligned} d(f_r(x), f_r(y)) &= d(\alpha(r), \beta(r)) \leq \left(\frac{s-r}{s}\right) d(\alpha(0), \beta(0)) + \left(\frac{r}{s}\right) d(\alpha(s), \beta(s)) \\ &= \left(\frac{r}{s}\right) d(x, y). \end{aligned}$$

This implies the desired mass bound.

Assume that

$$\mathbf{M}((f_r)_\#(T)) = \mathbf{M}(T)$$

then we must have that for every s with $r \leq s$

$$\mathbf{M}((f_r)_\#(T)) = \mathbf{M}((f_s)_\#(T))$$

this can only happen if $\text{spt}(T) \subset \mathbf{B}_r^\Sigma$. □

3.4.2 A local version of the convex barrier

The condition in Assumption 3.4.1 is global and we will instead consider local versions of this condition. We introduce some notation on wedges.

Definition 3.4.4. Given an $(m-1)$ -dimensional plane $V \subset \mathbb{R}^{m+n}$ we denote by $\mathbf{p}_V$ the orthogonal projection onto V . Given additionally a unit vector ν normal to V and an angle $\vartheta \in (0, \frac{\pi}{2})$ we then define the wedge with spine V , axis ν and opening angle ϑ as the set

$$W(V, \nu, \vartheta) := \{y : |y - \mathbf{p}_V(y) - (y \cdot \nu)\nu| \leq (\tan \vartheta)y \cdot \nu\}.$$

Lemma 2.3 of [27], which we cite below, gives us the desired local geometrical information.

Lemma 3.4.5. *Let $T, \sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket, \Omega$ be as in 3.4.1. Then there is a $0 < \vartheta < \frac{\pi}{2}$ (which is independent of the point in $\cup_{i=1}^N \Gamma_i$) such that the convex hull of $\cup_{i=1}^N \Gamma$ satisfies*

$$\text{ConvexHull}(\cup_{i=1}^N \Gamma_i) \subset \bigcap_{q \in \Gamma} (q + W(T_q \Gamma, \nu(q), \vartheta))$$

The former condition forces an additional property for area minimizing cones.

Definition 3.4.6 (Convex barrier cone). We say that an m -dimensional area minimizing cone C in $\mathbb{R}^{m+n}$ satisfies the convex barrier condition if, up to rotation, $\partial C = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$ and $\exists \theta > 0$:

$$\text{spt}(C) \subseteq \{p = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |y| \leq (\tan \theta) x_m\}.$$

Clearly if T satisfies assumption 3.4.1 then, by the conclusions of 3.4.5 we must have that for every i , a tangent cone to T at Γ_i must satisfy 3.4.6 for $Q = Q_i$.

We introduce an analogue of Assumption 2.4 in [27] for this setting:

Assumption 3.4.7. Let T, Γ, Σ be as in 3.1.1. Assume that there exists $\nu : \Gamma \rightarrow \mathbb{S}^{n+1} \cap T\Sigma$ a map such that $\nu(q) \perp T_q \Gamma$ and $\nu(q) \in T_q \Sigma$. We say T satisfies the local convex barrier assumption if

$$\text{spt}(T) \subset \bigcap_{q \in \Gamma} (q + W(T_q \Gamma, \nu(q), \vartheta)) \cap \Sigma.$$

As seen by Lemma 3.4.5, convex barrier implies local convex barrier. Moreover, local convex barrier still implies that every tangent cone is a convex barrier cone.

3.4.3 Convex barrier cones

We wish to classify the convex barrier cones defined in 3.4.6. We show this in the following theorem:

Theorem 3.4.8 (Classification of area minimizing cones with convex barrier). *Assume that C is an area minimizing cone in $\mathbb{R}^{m+n}$ with*

$$\partial C = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$$

and for some $\theta > 0$

$$\text{spt}(C) \subseteq \{p = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |y| \leq (\tan \theta) x_m\} =: W.$$

Then C is an open book with each half-plane H_i satisfying $H_i \subset W$.

Notice that the set W plays the role of the convex barrier hypothesis at the level of the tangent cone.

We notice that $\text{spt}(C) \cap \{x_m = 0\} \cap W \subset \mathbb{R}^{m-1}$ and thus $|C|(\{x_m = 0\}) = 0$. Thus the above theorem will be an immediate consequence of the following slightly more general theorem:

Theorem 3.4.9. *Assume that C is an area minimizing cone in $\mathbb{R}^{m+n}$ with*

$$\partial C = Q \llbracket \mathbb{R}^{m-1} \times \{0\} \rrbracket$$

and

$$\text{spt}(C) \subseteq \{x_m \geq 0\} \tag{3.4.1}$$

Then $C \llcorner \{x_m > 0\}$ is an open book. This means, there exist half-planes $H_1, \dots, H_N$ and positive integer multiplicities Q_i such that $\partial \llbracket H_i \rrbracket = \llbracket V \rrbracket$ and $H_i \subset \{x_m > 0\}$, such that:

$$C \llcorner \{x_m > 0\} = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket. \tag{3.4.2}$$

In particular, if $|C|(\{x_m = 0\}) = 0$, then C is an open book and $\sum_{i=1}^N Q_i = Q$.

Allard shows a version of the classification of tangent cones with a convex barrier for the multiplicity $Q = 1$ case in his boundary regularity paper [3]. We will show that his ideas extend to higher multiplicities $Q > 1$.

We remind the notation used in the alternative proof of the classification of tangent cones $\sigma : \mathbb{R}^{m-1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \rightarrow \mathbb{R}^{n+1}$ defined as

$$\sigma(x, y) = \frac{y}{|y|} \text{ where } (x, y) \in \left(\mathbb{R}^{m-1} \times \left(\mathbb{R}^{n+1} \setminus \{0\} \right) \right). \quad (3.4.3)$$

The following is a modification of a general result about varifolds from Allard in his boundary regularity paper Section 5 [3].

Theorem 3.4.10. *Fix a 2-dimensional plane $\pi_0 \subset \mathbb{R}^{m+n}$ and assume that:*

- (i) $\mathcal{V}$ is an m -dimensional varifold which is stationary on $\mathbb{R}^{m+n} \setminus \pi_0^\perp$.
- (ii) $\Theta(\mathcal{V}, x) \geq c_0 > 0$ for some positive number c_0 and for $\|\mathcal{V}\|$ -almost every $x \in \text{spt}(\mathcal{V}) \setminus \pi_0^\perp$.
- (iii) $\mathcal{V}$ is a cone (i.e. $(\lambda_{0,r})_\# \mathcal{V} = \mathcal{V}$).
- (iv) The orthogonal projection of $\text{spt}(\mathcal{V})$ onto π_0 is contained on a half-space. This means there is a vector $e \in \pi_0 \cap \mathbb{S}^{m+n-1}$ such that

$$\mathbf{p}_{\pi_0}(\text{spt}(\mathcal{V})) \subset \{x : 0 \leq \langle x, e \rangle\}. \quad (3.4.4)$$

The following conclusions holds:

- (a) There exist finitely many $v_1, \dots, v_{k(\rho)} \in \pi_0 \cap \mathbb{S}^{m+n-1}$ such that

$$(\mathbf{B}_1 \cap \text{spt}(\mathcal{V})) \setminus B_\rho(\pi_0^\perp) \subseteq \bigcup_{i=1}^{k(\rho)} \pi_0^\perp + \mathbb{R}^+ v_i \quad (3.4.5)$$

for every $\rho > 0$.

- (b) We denote $(x, y) = z$, $x^\perp = (-x_2, x_1)$ if $x = \mathbf{p}_{\pi_0}(z)$. For $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$ we have $\mathbf{p}_\pi(x^\perp) = 0$.

We remind the notation used in the alternative proof of the classification of tangent cones $\sigma : \mathbb{R}^{m-1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \rightarrow \mathbb{R}^{n+1}$ is defined as

$$\sigma(x, y) = \frac{y}{|y|} \text{ where } (x, y) \in \left(\mathbb{R}^{m-1} \times \left(\mathbb{R}^{n+1} \setminus \{0\} \right) \right).$$

Corollary 3.4.11. *Assume that*

- (i) $\mathcal{V}$ is an m -dimensional varifold which is stationary on $\mathbb{R}^{m+n} \setminus (\mathbb{R}^{m-1} \times \{0\})$.
- (ii) $\Theta(\mathcal{V}, x) \geq c_0 > 0$ for some positive number c_0 and for $\|\mathcal{V}\|$ -almost every $x \in \text{spt}(\mathcal{V}) \setminus (\mathbb{R}^{m-1} \times \{0\})$.
- (iii) $\mathcal{V}$ is a cone (i.e. $(\lambda_{0,r})_\# \mathcal{V} = \mathcal{V}$).
- (iv) The varifold $\mathcal{V}$ is contained in a hemisphere

$$\text{spt}(\mathcal{V}) \subset \{x : x_m \geq 0\}.$$

Then for $\mathcal{V}$ a.e. (x, π) where $x \in \mathbb{R}^{m+n} \setminus \{x_m = 0\}$

$$D_\pi \sigma(x) = 0.$$

Proof of Corollary 3.4.11. The proof of this Corollary immediate by applying the Theorem 3.4.10 for the planes $\pi_{0,k} = \mathbb{R}e_m \oplus \mathbb{R}e_{m+k}$ where $1 \leq k \leq n$. Since $\pi_{0,k}^\perp \subset \{x_m = 0\}$ we have that for $\mathcal{V}$ a.e. (x, π) with $x_m > 0$, $\mathbf{p}_\pi(-x_{m+k}e_m + x_me_{m+k}) = 0$ which implies $D_\pi \sigma(x) = 0$.

□

Proof of Theorem 3.4.8. We prove the decomposition of $C \llcorner \{x_m > 0\}$ first with, not necessarily positive, integer multiplicities:

We apply Corollary 3.4.11. This implies that $|C|$ almost everywhere in $C \setminus \{x_m > 0\}$ we have

$$D_C \sigma \equiv 0.$$

By Almgren [4], De Lellis - Spadaro [31, 33, 34], the area minimizing cone C is regular up to a set of codimension 2. For every regular point $p \in \text{spt}(C) \cap \{x_m > 0\}$ there is a unique half-plane H_p which passes through p and $\mathbb{R}^{m-1} \times \{0\}$. Since the tangential derivative of σ along C is zero, by unique continuation $C \setminus H_p = Q_p \llbracket H_p \rrbracket$ where $Q_p \neq 0$. We choose the orientations of H_p so that $\partial \llbracket H_p \rrbracket = \llbracket V \rrbracket$.

We choose distinct half-planes H_i which satisfy that property. Suppose there are at least N of them. Then

$$|C|(\mathbf{B}_1) \geq |C|(\{x_m = 0\} \cap \mathbf{B}_1) + \sum_{i=1}^N |C|(H_i \cap \mathbf{B}_1) \geq \frac{N\omega_m}{2}.$$

Since N needs to be bounded, there are only finitely many possible such planes and thus we get

$$C \setminus \{x_m > 0\} = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket.$$

We show that the multiplicities must be all positive and conclude

It remains to show all Q_i have the same sign. If this was not the case assume without loss of generality that $Q_1 > 0$ and $Q_2 < 0$.

We define $S := \llbracket H_1 \rrbracket - \llbracket H_2 \rrbracket$ and $T := (Q_1 - 1) \llbracket H_1 \rrbracket + (Q_2 + 1) \llbracket H_2 \rrbracket + \sum_{i=3}^k Q_i \llbracket H_i \rrbracket + C \setminus \{x_m = 0\}$. Since $S + T = C$ and $\|S\| + \|T\| = \|C\|$ it must be that both S and T must be area minimizing.

By definition $\partial S = 0$, since we oriented H_1 and H_2 so that $\partial \llbracket H_1 \rrbracket = \partial \llbracket H_2 \rrbracket = \llbracket V \rrbracket$. We know that $S = \llbracket \mathbb{R}^{m-1} \rrbracket \times (\llbracket L_1 \rrbracket - \llbracket L_2 \rrbracket)$ where L_1, L_2 are half-lines in $\mathbb{R}^{n+1}$. The only way that S is an area minimizing cone is that $\llbracket L_1 \rrbracket - \llbracket L_2 \rrbracket$ is an area minimizing cone with no boundary $\mathbb{R}^{n+1}$. This can only happen when $L_1 = -L_2$ (i.e. they form a full line), which is

impossible in the setting we are considering because both lines have positive x_m coordinate with opposite orientations.

We can also see that Q_i must be positive. If Q_i are all negative, we study the tangent cones at points in V . By the Almgren stratification, we can take a point in V such that the tangent cone splits V . The tangent cone C' at a point $p \in V$ must be

$$C' = \mathbb{R}^{m-1} \times \llbracket S \rrbracket$$

where S is a 1d area minimizing cone, which either has flat support or is a sum of half-lines all with the same orientation. By assumption, since Q_i are negative the orientation of the half-lines is opposite that of V and thus the support would need to be flat. The flat case is impossible since the support of C' must be on $x_m \geq 0$ and include a half-plane with positive x_m coordinate.

Thus, we obtain the desired decomposition. When $|C|(\{x_m = 0\}) = 0$, we must have $\sum_{i=1}^N Q_i = Q$ so that it takes the right boundary data. $\square$

We will use the theory of general varifolds and the monotonicity formula for stationary (not necessarily integral) varifolds. See Simon's lecture notes [74] for a reference on the theory of general varifolds.

Assume $\pi_0 = \{(x_1, x_2, 0, \dots, 0) \in \mathbb{R}^{m+n}\}$ and we denote $z = (x, y) \in \pi_0 \times \pi_0^\perp$. The following lemma implies 3.4.10:

Lemma 3.4.12. *Assume $\mathcal{V}$ is a varifold which is stationary on $\mathbb{R}^{m+n} \setminus \pi_0^\perp$ and is a cone (conditions (1) and (3) of Theorem 3.4.10). Consider the following Radon measure μ on $\mathbb{S}^1 := \{x \in \pi_0 : |x| = 1\}$*

$$\int \varphi d\mu = \int_{\mathbf{B}_1 \setminus \pi_0^\perp} \varphi \left(\frac{x}{|x|} \right) |x|^{-2} |\mathbf{p}_\pi(x^\perp)|^2 d\mathcal{V}(z, \pi) \quad (3.4.6)$$

where $(x, y) = z$, $x^\perp = (-x_2, x_1)$. Then $\mu = c\mathcal{H}^1$ for some $c \in \mathbb{R}$.

In addition, if $\mu = 0$ and we have conditions (2) and (4) in Theorem 3.4.10 then:

1. $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$.
2. The set $\mathbf{p}_{\pi_0}(\text{spt}(\mathcal{V}) \setminus B_\rho(\pi_0^\perp)) \cap S^1$ is finite and it has at most $\frac{2^m \Theta(\mathcal{V}, 0)}{\rho^m c_0}$ points.

Proof of Theorem 3.4.10. The condition (3.4.4) implies that the support of μ must be contained in the arc of $\mathbb{S}^1$ with positive e component. Thus $\mu \equiv 0$ and we conclude Theorem 3.4.10. $\square$

In order to conclude we prove the remaining lemma for varifolds

Proof of Lemma 3.4.12. To prove that $\mu = c\mathcal{H}^1$ we must show that if $\varphi \in C(S^1)$ and has zero average, i.e. $\int_{\mathbb{S}^1} \varphi d\mathcal{H}^1 = 0$, then

$$\int \varphi d\mu = 0.$$

We consider φ as a 2π periodic function on $\mathbb{R}$ and we take $\psi(t) := \int_0^t \varphi(\tau) d\tau$ to be a primitive of φ on $\mathbb{R}$. Since φ has zero average, ψ is also 2π periodic and thus it induces a map $\psi : \mathbb{S}^1 \rightarrow \mathbb{R}$. We extend ψ to a 0-homogeneous function on $\mathbb{R}^2$. We get then that

$$\nabla \psi(x) = \varphi\left(\frac{x}{|x|}\right) \frac{x^\perp}{|x|^2}. \quad (3.4.7)$$

We need to show that

$$\int_{\mathbf{B}_1 \setminus \pi_0^\perp} \varphi\left(\frac{x}{|x|}\right) \left| \mathbf{p}_\pi\left(\frac{x^\perp}{|x|}\right) \right|^2 d\mathcal{V}(z, \pi) = 0. \quad (3.4.8)$$

We will deduce this from the fact that $\mathcal{V}$ is stationary (i.e. $\int \text{div}_\pi X(z) d\mathcal{V}(z, \pi) = 0$ for every vector field $X \in C^1(\mathbb{R}^n)$). We will want (3.4.8) to get the identity by using the first variation formula over a specific vector field. The fact that the integral is computed on $\mathbf{B}_1(0) \setminus \pi_0^\perp$ is a technical issue that we will resolve by introducing a cutoff. In order to get the gradient of ψ , it is reasonable to consider the vector field $\psi(x)x^\perp$ (in fact this is what Allard does).

We consider a vector field of the form

$$X(z) = \alpha(|z|)\beta(|x|)\psi(x)x^\perp.$$

Where $\alpha, \beta \in C^\infty(\mathbb{R})$ are defined with the following properties: $\alpha \equiv 1$ for $t \leq 1 - \delta$, $\alpha \equiv 0$ for $t \geq 1 - \delta/2$, and it is decreasing. $\beta \equiv 1$ for $t \geq \delta$, $\beta \equiv 0$ for $t < \delta/2$, and it is increasing with $\|\beta'\|_{C^0} \lesssim \delta^{-1}$. Of course, for each δ we will have a different vector field, but we will abuse notation and not write the δ dependence.

We will compute $\text{div}_\pi(X)$. We fix an orthonormal basis $v_1, \dots, v_m$ for π and compute in this coordinates the following:

$$\text{div}_\pi(X) = (\text{I}) + (\text{II}) + (\text{III}) + (\text{IV}),$$

where

$$(\text{I}) := \sum_{i=1}^m \alpha'(|z|) \left(\frac{z}{|z|} \cdot v_i \right) \beta(|x|) \psi(x) (x^\perp \cdot v_i),$$

$$(\text{II}) := \sum_{i=1}^m \alpha(|z|) \beta'(|x|) \left(\frac{x}{|x|} \cdot v_i \right) \psi(x) (x^\perp \cdot v_i),$$

$$(\text{III}) := \sum_{i=1}^m \alpha(|z|) \beta(|x|) (\nabla \psi(x) \cdot v_i) (x^\perp \cdot v_i),$$

$$(\text{IV}) := \alpha(|z|) \beta(|x|) \psi(x) \sum_{i=1}^m (D_{v_i} x^\perp \cdot v_i).$$

We deal with every term separately.

As for the first term: Since $\mathcal{V}$ is a cone, $z \in \pi$ for $\mathcal{V}$ a.e. (z, π) . This means $\sum_{i=1}^m (z \cdot v_i) v_i = z$ and we can rewrite

$$(\text{I}) = \frac{\alpha'(|z|)}{|z|} \beta(|x|) \psi(x) (z \cdot x^\perp).$$

But $z \cdot x^\perp = 0$, so (I) = 0.

As for the second term: Here we just use $|\beta'| \lesssim \delta^{-1}$, $\text{spt}(\beta') \subseteq \{|x| < \delta\}$ and $|x^\perp \cdot v_i| \leq \delta$ to conclude:

$$\int |(\text{II})| d\mathcal{V} \lesssim \|\mathcal{V}\| \left(B_1 \cap \left(B_\delta(\pi_0^\perp) \setminus \pi_0^\perp \right) \right).$$

In particular, since

$$\bigcap_{\delta > 0} B_\delta(\pi_0^\perp) = \emptyset.$$

We get $\lim_{\delta \rightarrow 0} \int |(\text{II})| d\mathcal{V} = 0$.

As for the third term: We recall

$$\nabla \psi(x) = \varphi\left(\frac{x}{|x|}\right) \frac{x^\perp}{|x|^2}.$$

Thus

$$\begin{aligned} (\text{III}) &= \frac{1}{|x|^2} \alpha(|z|) \beta(|x|) \varphi\left(\frac{x}{|x|}\right) \sum_{i=1}^m (x^\perp \cdot v_i)^2 = \frac{1}{|x|^2} \alpha(|z|) \beta(|x|) \varphi\left(\frac{x}{|x|}\right) |\mathbf{p}_\pi(x^\perp)|^2 \\ &= \alpha(|z|) \beta(|x|) \varphi\left(\frac{x}{|x|}\right) \left| \mathbf{p}_\pi\left(\frac{x^\perp}{|x|}\right) \right|^2. \end{aligned}$$

This implies that

$$\lim_{\delta \rightarrow 0} \int (\text{III}) d\mathcal{V} = \int_{\mathbf{B}_1(0) \setminus \pi_0^\perp} \varphi\left(\frac{x}{|x|}\right) \left| \mathbf{p}_\pi\left(\frac{x^\perp}{|x|}\right) \right|^2 d\mathcal{V}(z, \pi).$$

As for the last term: We have

$$Dx^\perp = -e_1 \otimes e_2 + e_2 \otimes e_1 = \begin{pmatrix} 0 & -1 & 0_{(m+n-2) \times 1} \\ 1 & 0 & 0_{(m+n-2) \times 1} \\ 0_{1 \times (m+n-2)} & 0_{1 \times (m+n-2)} & 0_{(m+n-2) \times (m+n-2)} \end{pmatrix} \quad (3.4.9)$$

The former matrix is antisymmetric matrix. Notice

$$\sum_{i=1}^m D_{v_i} X^\perp \cdot v_i = \left\langle Dx^\perp : \left(\sum_{i=1}^m v_i \otimes v_i \right) \right\rangle.$$

Given that the matrix $\sum_{i=1}^m v_i \otimes v_i$ is symmetric and the fact that the Hilbert-Schmidt product of a symmetric matrix and an antisymmetric matrix vanishes, $(IV) = 0$.

Since $\mathcal{V}$ is a stationary varifold,

$$\int \operatorname{div}_\pi X(z) d\mathcal{V}(z, \pi) = 0.$$

We conclude taking $\delta \rightarrow 0$ that (3.4.8) holds.

We conclude the theorem studying the case $\mu \equiv 0$. Notice that if $\mu = 0$ then $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. (z, π) where $z \in \mathbf{B}_1 \setminus \pi_0^\perp$ and since $\mathcal{V}$ is a cone, $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$.

We need to bound the cardinality of the set

$$\mathbf{p}_{\pi_0}(\operatorname{spt}(\mathcal{V}) \setminus B_\rho(\pi_0^\perp)) \cap \mathbb{S}^1 := F.$$

Pick k distinct points $\xi_1, \dots, \xi_k \in F \subseteq \mathbb{S}^1$. We consider k bump functions $\chi_1, \dots, \chi_k \in C^\infty(\mathbb{S}^1)$, identically 1 in a neighborhood of ξ_i and such that $\sum_i \chi_i = 1$.

We define the truncated varifolds $\mathcal{V}_i$ by

$$\int \varphi(z, \pi) d\mathcal{V}_i(z, \pi) = \int \varphi(z, \pi) \chi_i \left(\frac{x}{|x|} \right) d\mathcal{V}(z, \pi).$$

We claim that each $\mathcal{V}_i$ is stationary on $\mathbb{R}^{m+n} \setminus \pi_0^\perp$:

Given a vector field $X \in C_c^\infty(\mathbb{R}^{m+n} \setminus \pi_0^\perp)$ then

$$\begin{aligned} \int \operatorname{div}_\pi X(z) d\mathcal{V}_i(z, \pi) &= \int \operatorname{div}_\pi \left(X(z) \chi_i \left(\frac{x}{|x|} \right) \right) d\mathcal{V}(z, \pi) \\ &\quad - \int \sum_j (X(z) \cdot v_j) D_{v_j} \left(\chi_i \left(\frac{x}{|x|} \right) \right) d\mathcal{V}(z, \pi). \end{aligned}$$

The first integral on the right hand side vanishes because $\mathcal{V}$ is stationary. For the second integral on the right hand side we observe that

$$D_{v_j} \left(\chi_i \left(\frac{x}{|x|} \right) \right) = \varphi_i \left(\frac{x}{|x|} \right) \frac{x^\perp}{|x|^2}$$

for an appropriate function $\varphi_i \in C_c^\infty(\mathbb{S}^1)$. In particular, we have

$$\begin{aligned} \sum_j (X(z) \cdot v_j) D_{v_j} \left(\chi_i \left(\frac{x}{|x|} \right) \right) &= \varphi_i \left(\frac{x}{|x|} \right) \frac{1}{|x|^2} \sum_j (X(z) \cdot v_j) (x^\perp \cdot v_j) \\ &= \varphi_i \left(\frac{x}{|x|} \right) \frac{1}{|x|^2} (\mathbf{p}_\pi(x^\perp) \cdot X(z)). \end{aligned}$$

This term must be zero since we already establish that $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$. This shows that $\mathcal{V}_i$ is stationary.

Now we consider $\zeta_i \in \operatorname{spt}(\mathcal{V}) \cap \mathbf{B}_1 \setminus B_\rho(\pi_0^\perp)$ such that

$$\frac{\mathbf{p}_\pi(\zeta_i)}{|\mathbf{p}_\pi(\zeta_i)|} = \xi_i$$

Then $\Theta(\mathcal{V}_i, \zeta_i) = \Theta(\mathcal{V}, \zeta_i) \geq c_0$ and since $B_\rho(\zeta_i) \subseteq \mathbb{R}^{m+n} \setminus \pi_0^\perp$, by the monotonicity formula

$$\|\mathcal{V}_i\|(B_\rho(\zeta_i)) \geq c_0 \omega_m \rho^m.$$

Now

$$\|\mathcal{V}\|(B_2(0)) \geq \sum_i \|\mathcal{V}_i\|(B_2(0)) \geq \sum_i \|\mathcal{V}_i\|(B_\rho(\zeta_i)) \geq k \omega_m c_0 \rho^m.$$

Since $\|\mathcal{V}\|(B_2(0)) = \omega_m 2^m \Theta(\mathcal{V}, 0)$, the last inequality gives us the desired bound. $\square$

3.5 Main reductions

In this section we will reduce the proofs of the main theorems in this chapter to a proof of an excess decay type lemma. We show the uniqueness of the tangent cone (Theorem 3.5.10), the Hölder continuity of the multivalued normal (Corollary 3.5.12), and the decomposition Theorem 3.5.13.

3.5.1 Proof of small excess uniqueness of the tangent cone

Definition 3.5.1. We say an open book C is admissible for $Q \llbracket \Gamma \rrbracket$ at a point $p \in \Gamma$ if $C = \sum Q_i \llbracket H_i \rrbracket$ where $\sum Q_i = Q$ and $\partial H_i = T_p(\Gamma)$. Additionally, we require that for every i $H_i \subset T_p(\Sigma)$.

Assumption 3.5.2 (Main Reductions Assumption). Let T and Γ be as in 3.1.1 with $\|T\|(\mathbf{B}_2) \leq (2^{m-1}Q + 1)\omega_m$, $\mathbf{A}_\Gamma + \mathbf{A}_\Sigma \leq 1$. We assume furthermore that C is an admissible open book.

Remark 3.5.3 (Choices of parameters). There will be Q versions of excess decay depending the number of sheets of the cone. The four relevant parameters will be $\theta_N \in \mathbb{R}^+$, $\eta_N \in (0, 1/2)$, $\varepsilon_N \in \mathbb{R}^+$, $\gamma(Q, m, n, \bar{n})$. The parameter $\gamma(Q, m, n, \bar{n})$ is purely dimensional and depends on the comparison between excess and Dirichlet energy and not on N . The constant θ_N is chosen so $\theta_N \ll \min_{N+1 \leq i \leq Q} \varepsilon_i$ and that $\theta_N \ll \gamma(Q, m, n, \bar{n})$. The parameters η_N and ε_N are chosen to ensure excess decay on a cone with N sheets with parameter θ_N . Additionally, we require that $\varepsilon_N \ll \gamma(Q, m, n, \bar{n})$.

Lemma 3.5.4 (N th excess decay with small angle and small second fundamental forms). *Let T and C be chosen as in Assumption 3.5.2. For every $\theta_N \in \mathbb{R}^+$ there exist $\varepsilon_N \in \mathbb{R}^+$,*

$\eta_N \in (0, 1/2)$ such that if

$$\mathbb{E}(T, C, \mathbf{B}_1) \leq \varepsilon_N \alpha(C)^2 \text{ and } \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \leq \varepsilon_N \mathbb{E}(T, C, \mathbf{B}_1)$$

(when $N = 1$ the first inequality becomes $\mathbb{E}(T, C, \mathbf{B}_1) \leq \varepsilon_1$) there exist a radius $r_0 \in [\eta_N, 1/2]$ and an admissible open book C' such that

$$\mathcal{G}(C', C)^2 \leq \gamma(Q, m, n, \bar{n}) \mathbb{E}(T, C, \mathbf{B}_1) \quad (3.5.1)$$

and

$$\mathbb{E}(T, C', \mathbf{B}_{r_0}) \leq \theta_N \min \left\{ \alpha(C')^2, \mathbb{E}(T, C, \mathbf{B}_1) \right\}.$$

Remark 3.5.5. Under the hypothesis above, (3.5.1) remains valid up to modifying the constant $\gamma(Q, m, n, \bar{n})$, with the following

$$C' = \sum_{i,j} Q_{i,j} \llbracket H_{i,j} \rrbracket$$

has $\sum_j Q_{i,j} = Q_i$ and the angle condition

$$\angle(H_i, H_{i,j}) \leq \gamma(Q, m, n, \bar{n}) \mathbb{E}(T, C, \mathbf{B}_1)^{1/2} \quad (3.5.2)$$

In particular, we know that one of the following two statements holds:

1. C' has the same number of sheets than C and

$$\alpha(C') \geq \alpha(C) - \gamma(Q, m, n, \bar{n})^{-1/2} \mathbb{E}(T, C, \mathbf{B}_1)^{1/2}. \quad (3.5.3)$$

2. C' has more sheets than C (this case will never happen when C consists of Q multiplicity 1 sheets).

Our excess decay needs a control on the angle of the cone, which we require for our graphical approximations to be valid later. We require a combinatorial argument that controls the angle. If excess is large relative to the angle we will prune the cone by collapsing the two closest sheets into one and adjusting the multiplicities appropriately. This is collected in the following proposition:

Proposition 3.5.6. *Let T, Γ, C be as in 3.5.2. Given positive parameters $\{\varepsilon_i\}_{1 \leq i \leq Q}$, there exists $\varepsilon > 0$ such that if $\mathbb{E}(T, C, \mathbf{B}_1) < \varepsilon$ then we have the following:*

- *An integer $1 \leq N' \leq N$ and an admissible cone C' with N' sheets.*
- *A partition function $h : \{1, \dots, N\} \rightarrow \{A \subseteq \{1, \dots, N\}\}$ (i.e. $h(i)$ are disjoint, possibly empty sets and their union is $\{1, \dots, N\}$).*
- *$C' = \sum \sum_{j \in h(i)} Q_j \llbracket H_i \rrbracket$. In particular, if $N = N'$ then $C' = C$.*

•

$$\mathbb{E}(T, C', \mathbf{B}_1) \leq C(m) \left(\min_{1 \leq k \leq Q} \varepsilon_k \right)^{-Q} \mathbb{E}(T, C, \mathbf{B}_1).$$

- *We have*

$$\mathbb{E}(T, C', \mathbf{B}_1) \leq \varepsilon_{N'} \alpha(C')^2$$

where we remind that if C' is a single half-plane then $\alpha(C') = 1$.

•

$$\mathcal{G}(C', C)^2 \leq C(m) \left(\min_{1 \leq k \leq Q} \varepsilon_k \right)^{-Q} \mathbb{E}(T, C, \mathbf{B}_1).$$

Proof. We outline an iterative procedure. Let $C_0 = C$. In step k if we have

$$\mathbb{E}(T, C_k, \mathbf{B}_1) \leq \varepsilon_{N-k} \alpha(C_k)^2$$

we take $C' = C_k$, $N' = N - k$ and we stop.

Otherwise assume that

$$\mathbb{E}(T, C_k, \mathbf{B}_1) > \varepsilon_{N-k} \alpha(C_k)^2.$$

Then we select two half-planes of C_k , H_i and H_j , that form an angle of $\alpha(C_k)$. We remove H_j and change the multiplicity of H_i to $Q_i + Q_j$. This cone is defined to be C_{k+1} . We must have

$$\mathbb{E}(T, C_{k+1}, \mathbf{B}_1) \leq c(m) \frac{1}{\delta_{N-k}} \mathbb{E}(T, C_k, \mathbf{B}_1).$$

If the process never stops, we define $C' = C_N$ and $N' = 1$. We must have that

$$\mathbb{E}(T, C', \mathbf{B}_1) \leq c(m) \prod_{N'+1 \leq k \leq N} \frac{1}{\varepsilon_k} \mathbb{E}(T, C, \mathbf{B}_1).$$

This implies that if ε is chosen small enough then in the case we didn't stop we also have

$$\mathbb{E}(T, C', \mathbf{B}_1) \leq \varepsilon_1 \alpha(C')^2.$$

The last inequality of the proposition is a straightforward consequence of the procedure. $\square$

We are now ready to prove the uniqueness of the tangent cone.

Definition 3.5.7. We define the modified excess as

$$\hat{\mathbb{E}}(T, C, \mathbf{B}_r(p)) := \max \left\{ \mathbb{E}(T, C, \mathbf{B}_r(p)), \kappa^{-1} \mathbf{A}_\Gamma r, \kappa^{-1} \mathbf{A}_\Sigma^2 r^2 \right\}$$

The dependence on κ is implicit in the notation; in Theorem ?? we will choose κ as κ_{decay} .

Theorem 3.5.8 (Small Excess Uniqueness of the Tangent Cone). *Let T and C be as in 3.5.2.*

There exist $\varepsilon_{\text{decay}} > 0$, $\kappa_{\text{decay}} > 0$ such that if

$$\hat{\mathbb{E}}(T, C, \mathbf{B}_1(p)) < \varepsilon_{\text{decay}},$$

then the tangent cone, C_p , at p is unique and is an admissible open book. Here, $\hat{\mathbb{E}}$ is defined using $\kappa = \kappa_{\text{decay}}$. Moreover, there exists $\alpha(Q, m, n, \bar{n}) > 0$ such that for every $\rho < 1/2$,

$$\hat{\mathbb{E}}(T, C_p, \mathbf{B}_\rho(p)) \leq \rho^\alpha \hat{\mathbb{E}}(T, C, \mathbf{B}_1(p)).$$

We also have

$$\mathcal{G}(C_p, C)^2 + \text{dist}_H(C_p, C)^2 \leq c(Q, m, n, \bar{n}) \hat{\mathbb{E}}(T, C, \mathbf{B}_1(p)).$$

Proof. Before proceeding we apply, we apply the pruning proposition 3.5.6 to the cone C , obtaining a cone $\hat{C}$ which satisfies

$$\mathbb{E}(T, \hat{C}, \mathbf{B}_1) \leq C(m) \left(\min_{1 \leq k \leq Q} \varepsilon_k \right)^{-Q} \mathbb{E}(T, C, \mathbf{B}_1).$$

Since $\mathcal{G}(\hat{C}, C) \lesssim \mathbb{E}(T, C, \mathbf{B}_1)^{1/2}$, if we prove the theorem to hold for $\hat{C}$ then it will hold for C . By choosing ε small enough, we may assume that the starting cone C is pruned (without relabeling) and thus it satisfies the conclusions of Proposition 3.5.6.

We will start by proving the following iterative excess decay starting with a pruned cone C . There exists a sequence of admissible open books $\mathbf{C}^l$ and radii $r_{l+1}r_l^{-1} \in [\eta_{N_l}, 1/2]$, where $\mathbf{C}_0 := C$ and $r_0 := 1$, such that

- The number of distinct sheets N_l of cone $\mathbf{C}^l$ is non-decreasing.
- For every l we have $\hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}) \leq \varepsilon_{N_l} \alpha(\mathbf{C}_l)^2$.
- If $\hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}) > \mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l})$ then $\mathbf{C}^{l+1} := \mathbf{C}^l$ and $r_{l+1} = \frac{1}{2}r_l$.
- If $\hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}) = \mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l})$ we apply excess decay 3.5.4. Here $r_{l+1}r_l^{-1} \in [\eta_{N_l}, 1/2]$ and

$$\mathcal{G}(\mathbf{C}^{l+1}, \mathbf{C}^l) \leq \gamma(Q, m, n, \bar{n})^{1/2} \mathbb{E}(T, C, \mathbf{B}_{r_l})^{1/2}.$$

- For every non-negative integer l

$$\hat{\mathbb{E}}(T, \mathbf{C}_{l+1}, \mathbf{B}_{r_{l+1}}) \leq \frac{1}{2} \hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}).$$

We prove we can do the above. We chose $\kappa \ll \min_{1 \leq k \leq Q} \varepsilon_k$.

The case of large second fundamental forms: If

$$\hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}) > C(Q, m, n, \bar{n}) \mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l})$$

we chose $r_{l+1} := \frac{1}{2}r_l$, $\mathbf{C}_{l+1} := \mathbf{C}_l$, then

$$\hat{\mathbb{E}}(T, \mathbf{C}_{l+1}, \mathbf{B}_{r_{l+1}}) \leq \frac{1}{2} \hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}).$$

In the last inequality we use Lemma 3.6.14.

The case of small second fundamental forms: We must have

$$\mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l}) \leq \varepsilon_N \alpha(\mathbf{C}_l)^2 \text{ and } \mathbf{A}_\Gamma r_l + \mathbf{A}_\Sigma^2 r_l^2 \leq \varepsilon_N \mathbb{E}(T, C, \mathbf{B}_{r_l}).$$

We are in the hypothesis to apply Lemma 3.5.4. This gives us the cone $\mathbf{C}_{l+1}$, which satisfies

$$\mathbb{E}(T, \mathbf{C}_{l+1}, \mathbf{B}_{r_{l+1}}) \leq \theta_N \mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l}).$$

If $\mathbf{C}_{l+1}$ has more sheets than $\mathbf{C}_l$ then

$$\mathbb{E}(T, \mathbf{C}_{l+1}, \mathbf{B}_{r_{l+1}}) \leq \theta_N \alpha(\mathbf{C}_{l+1})^2 \leq \varepsilon_{N_{l+1}} \alpha(\mathbf{C}_{l+1})^2$$

the last step by our choice of parameters 3.5.3.

In the case that $\mathbf{C}_{l+1}$ and $\mathbf{C}_l$ have the same number of sheets we have that

$$\begin{aligned}
\alpha(\mathbf{C}_{l+1}) &\geq \alpha(\mathbf{C}_l) - C(Q, m) \mathcal{G}(\mathbf{C}_{l+1}, \mathbf{C}_l) \\
&\geq \alpha(\mathbf{C}_l) - C(Q, m) \gamma(Q, m, n, \bar{n}) \mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l})^{1/2} \\
&\geq (1 - \varepsilon_N \gamma(Q, m, n, \bar{n})) \alpha(\mathbf{C}_l) \geq \frac{1}{\sqrt{2}} \alpha(\mathbf{C}_l).
\end{aligned}$$

This implies that

$$\mathbb{E}(T, \mathbf{C}_{l+1}, \mathbf{B}_{r_{l+1}}) \leq \frac{1}{2} \mathbb{E}(T, \mathbf{C}_l, \mathbf{B}_{r_l}) \leq \frac{1}{2} \varepsilon_{N_l} \alpha(\mathbf{C}_l)^2 \leq \varepsilon_{N_{l+1}} \alpha(\mathbf{C}_{l+1})^2$$

since $N_l = N_{l+1}$. Additionally since $r_{l+1} r_l^{-1} < \frac{1}{2}$ the second fundamental form terms must also shrink by a factor of 1/2 and thus we must have that

$$\hat{\mathbb{E}}(T, \mathbf{C}_{l+1}, \mathbf{B}_{r_{l+1}}) \leq \frac{1}{2} \hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l}).$$

Convergence of the cones $\mathbf{C}_l$ and uniqueness of the tangent cone:

We must have

$$\mathcal{G}(\mathbf{C}_{l+1}, \mathbf{C}_l) \leq A \hat{\mathbb{E}}(T, \mathbf{C}_l, \mathbf{B}_{r_l})^{1/2} \leq A \frac{1}{2^l} \mathbb{E}(T, C, \mathbf{B}_r)^{1/2}$$

for a universal parameter A depending on all the other parameters. This implies that there exists a limit C_p in $\mathcal{G}$ of the sequence $\mathbf{C}_l$ and

$$\mathcal{G}(C_p, \mathbf{C}_l) \leq 2A \frac{1}{2^l} \mathbb{E}(T, C, \mathbf{B}_r)^{1/2}.$$

This implies

$$\mathbb{E}(T, C_p, \mathbf{B}_{r_l}) \leq C(Q, m, n, \bar{n}, \min_{1 \leq k \leq Q} \varepsilon_k) \frac{1}{2^l} \mathbb{E}(T, C, \mathbf{B}_r)^{1/2}.$$

Given ρ we can choose l such that $\rho \in (r_{l+1}, r_l]$. Let $\eta := \min_{1 \leq i \leq Q} \eta_i$. Then

$$\hat{\mathbb{E}}(T, C_p, \mathbf{B}_\rho) \leq C(m) \frac{1}{\eta^{m+2}} \hat{\mathbb{E}}(T, C_p, \mathbf{B}_{r_l}) \leq C(\eta, \varepsilon_1) \frac{1}{2^l} \hat{\mathbb{E}}(T, C, \mathbf{B}_r)$$

We notice that if $\rho < 1/2r$ then using that $\eta^{l+1}r \leq r_{l+1} < \rho$ and defining $\alpha := \log(2)/\log(\eta)$ we get

$$\frac{1}{2^{l+1}} \leq \left(\frac{\rho}{r}\right)^\alpha.$$

This implies as desired

$$\hat{\mathbb{E}}(T, C_p, \mathbf{B}_\rho) \leq C(Q, m, n, \bar{n}, \min_{1 \leq k \leq Q} \varepsilon_k, \eta) \left(\frac{\rho}{r}\right)^\alpha \hat{\mathbb{E}}(T, C, \mathbf{B}_r(p)).$$

In order to get the estimate on the intermediate scales see Proposition 3.6.14.

The last inequality implies the uniqueness of the tangent cone. Indeed, let $r_j \rightarrow 0$ a sequence of scales such that $(\lambda_{r_j})_\# T \rightarrow C'$. Then

$$\mathbb{E}(C', C_p, \mathbf{B}_1) = \lim_{j \rightarrow \infty} \mathbb{E}((\lambda_{r_j})_\# T, C_p, \mathbf{B}_1) = 0.$$

This implies $C' = C_p$. □

3.5.2 Consequences of the excess decay lemma

As an immediate consequence we get an upgrade on the classification of the tangent cones and thus on the minimum density.

Theorem 3.5.9 (Density jump). *Let C be an area minimizing cone with $\partial C = Q[\{0\} \times \mathbb{R}^{n+1}]$. There exists $\theta(Q, m, n) > 0$ such that if $\Theta(C, 0) < Q/2 + \theta$ then $\Theta(C, 0) = Q/2$ (and thus C is an open book).*

Proof. Suppose by contradiction there exists a sequence of area minimizing cones C_j with $\Theta(C_j, 0) \rightarrow Q/2$. Up to a subsequence we can assume $C_j \rightarrow C$ for C an area minimizing cone. We must have then that $\Theta(C, 0) = Q/2$ and thus C is an open book, by Theorem 3.3.1.

We must have that for j large enough $\mathbb{E}(C_j, C, \mathbf{B}_1) < \varepsilon$ for ε in 3.5.8. This implies that the tangent cone to C_j at 0 for some j large enough must be unique and an open book. Thus C_j must be an open book for j large enough. $\square$

Theorem 3.5.10 (Uniqueness of the tangent cone). *Let T, Γ, Σ be as in Assumption 3.1.1 or the local convex barrier 3.4.7. Assume in the first case that $\Theta(T, 0) \leq Q/2 + \theta$. Then $\Theta(T, 0) = Q/2$ and the tangent cone at is a unique open book.*

Proof. In the first case, since $\Theta(T, 0) < Q/2 + \theta$ then $\Theta(T, 0) = Q/2$. Thus in this case, every tangent cone must be an open book. Analogously, in the second case, by Theorem 3.4.2 every tangent cone at 0 must be an open book. We take a tangent cone C . We must then have that

$$\hat{\mathbb{E}}(T, C, \mathbf{B}_{r_j}) \rightarrow 0$$

which means that there exists j for which $\hat{\mathbb{E}}(T, C, \mathbf{B}_{r_j}) < \varepsilon$ with ε as in 3.5.8. This implies that T has a unique tangent cone at 0 by 3.5.8. $\square$

Let $N\Gamma$ be the normal bundle of Γ .

Definition 3.5.11. Given $p \in \Gamma$ let $C_p = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ be the tangent cone at p . We define the normal map $\eta : \Gamma \rightarrow \mathcal{A}_Q(N\Gamma)$ given by

$$\eta(p) = \sum_{i=1}^N Q_i \llbracket \mathbf{p}_{T_p(\Gamma)^\perp}(H_i) \rrbracket.$$

Corollary 3.5.12 (Multivalued Normal Map Hölder Estimate). *There exists ε such that if*

$$\hat{\mathbb{E}}(T, C, \mathbf{B}_1(p)) < \varepsilon$$

then the following quantitative estimate holds $\forall q_1, q_2 \in B(p, 1/16)$

$$\mathcal{G}(\eta(q_1), \eta(q_2))^2 \leq c(Q, m, n, \bar{n}) \hat{\mathbb{E}}(T, C, \mathbf{B}_1) |q_1 - q_2|^\alpha.$$

Here, $\hat{\mathbb{E}}$ is defined using $\kappa = \kappa_{\text{decay}}$. Moreover, let $U := \{x \in \Gamma : \Theta(x) < Q/2 + \theta\}$, which forms a relatively open set of Γ , and let $K \subset U$ a compact subset of U . Then $\eta \in C^\alpha(K)$.

Proof. By 3.6.14, in combination with 3.5.6 if ε and κ are small enough then for every $q_1 \in \mathbf{B}_{1/4}(p)$

$$\hat{\mathbb{E}}(T, C_{q_1}, \mathbf{B}_{1/2}(q_1)) \leq C(Q, m, n, \bar{n}) \hat{\mathbb{E}}(T, C, \mathbf{B}_1).$$

The open book C_{q_1} is defined by translating C so that it passes through p and tilting it so its spine is $T_{q_1}(\Gamma)$ and it is still contained on $T_{q_1}(\Sigma)$.

Up to taking ε smaller we must have that for $\mathbf{C}_{q_1}$ the tangent cone at q_1 , for every $0 < r_1 < 1/4$

$$\hat{\mathbb{E}}(T, \mathbf{C}_{q_1}, \mathbf{B}_{r_1}) \leq C(Q, m, n, \bar{n}) r_1^\alpha \hat{\mathbb{E}}(T, C, \mathbf{B}_1).$$

We define $C_{q_1}^{q_2}$ as the cone obtained by translating and tilting $\mathbf{C}_{q_1}$ so that it is centered at q_2 and has a spine tangent to Γ and it is still contained on $T_{q_2}(\Sigma)$. If q_1 and q_2 are in $\mathbf{B}_{1/16}(p)$ and $r_1 = 2|q_1 - q_2|$ then

$$\hat{\mathbb{E}}(T, C_{q_1}^{q_2}, \mathbf{B}_{r_1}(q_1)) \leq C(Q, m, n, \bar{n}) r_1^\alpha \hat{\mathbb{E}}(T, C, \mathbf{B}_1).$$

Thus if ε is small enough then $\hat{\mathbb{E}}(T, C_{q_1}^{q_2}, \mathbf{B}_{r_1}(q_1)) < \varepsilon_{\text{decay}}$ where $\varepsilon_{\text{decay}}$ is the one needed to run theorem 3.5.8. This allows us to apply the theorem and obtain the desired estimate

$$\mathcal{G}(\eta(q_1), \eta(q_2))^2 \lesssim \mathcal{G}(C_{q_1}^{q_2}, C_{q_1})^2 \lesssim |q_1 - q_2|^\alpha \hat{\mathbb{E}}(T, C, \mathbf{B}_1)$$

The qualitative conclusion follows by compactness and continuity. By the quantitative statement we must have that the normal map is a continuous map of U . We must show that

$$\frac{\mathcal{G}(\eta(p), \eta(q))}{|p - q|^\alpha}$$

is bounded in $K \times K \subset \Gamma \times \Gamma$. It is clearly bounded away from the diagonal by continuity. We can cover the diagonal by balls on which the quantitative result applies and conclude by compactness. $\square$

Theorem 3.5.13 (Decomposition lemma). *Let T and Γ be under Assumption 3.1.1 and $q \in \Gamma$. Assume that C is a non-flat open book. There exists $\varepsilon_{decomp} > 0$ such that the following holds. If $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$*

$$\hat{\mathbb{E}}(T, C, \mathbf{B}_1(p)) < \varepsilon_{decomp} \alpha(C)^2. \quad (3.5.4)$$

There exist $T_1, T_2, \dots, T_N$ area minimizing currents on $\mathbf{B}_{1/32} \cap \Sigma$ with

- $\partial T_i = Q_i \llbracket \Gamma \rrbracket$

-

$$T \llcorner \mathbf{B}_{1/32}(p) = \sum_{i=1}^N T_i \quad (3.5.5)$$

- The supports of T_i only intersect at Γ .

Theorem 3.5.14 (Qualitative decomposition theorem). *Let T be an area minimizing current as in Assumption 3.1.1. Further assume that $\Theta(T, 0) = Q/2$. Let $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ be the unique tangent cone to T at 0 (where the representation of C is such that the half-planes are distinct). Then there exists $\rho > 0$ and area minimizing currents $T_1, T_2, \dots, T_N$ in $\mathbf{B}_\rho$ such that*

$$T \llcorner \mathbf{B}_\rho = \sum_{i=1}^N T_i$$

where the supports of T_i only intersect at Γ , $\partial T_i \perp \mathbf{B}_\rho = Q_i \llbracket \Gamma \rrbracket$, and the (unique) tangent cone at 0 of T_i is $Q_i \llbracket H_i \rrbracket$.

The proof of Theorem 3.5.14 follows immediately from Theorem 3.5.13 at some scale small enough from Theorem 3.5.8.

Proof of Theorem 3.5.13. Let C_q be the tangent at q . If $\mathbf{A}_\Gamma$ is small enough then the projection $\mathbf{p}_\Gamma$ is well-defined.

By assuming that $\hat{\mathbb{E}}(T, C, \mathbf{B}_1) < \varepsilon_{Holder}$. We start by coupling the estimates from the proof of 3.5.12 with the L^2 - L^∞ height bound of De Lellis-Minter-Skorobogatova (Theorem 3.2 [24]).

If $q \in \text{spt}(T) \cap \mathbf{B}_{1/16}$ we have

$$\frac{\text{dist}(q, C_{\mathbf{p}_\Gamma(q)})}{|q - \pi(q)|} \leq C(Q, m, n, \bar{n}) |q - \pi(q)|^{\alpha/2} \hat{\mathbb{E}}(T, C, \mathbf{B}_r(p))^{1/2}. \quad (3.5.6)$$

We will also have that $\forall q_1, q_2 \in B(p, 1/16)$

$$\mathcal{G}(\eta(q_2), \eta(q_1)) \leq C(Q, m, n, \bar{n}) \varepsilon^{1/2} \alpha(C).$$

We also have

$$\mathcal{G}(C_p, C) \leq C(Q, m, n, \bar{n}) \hat{\mathbb{E}}(T, C, \mathbf{B}_1)^{1/2} \leq \varepsilon_{decomp} \alpha(C).$$

This implies that the tangent cones C_q have at least as many sheets as C .

We parameterize $\mathbf{B}_1$ as a subset of the normal bundle of Γ . We consider the function $d : \mathbf{B}_1 \setminus \Gamma \rightarrow \mathbb{R}$

$$d(q) := \frac{\text{dist}(q, C_{\pi(q)})}{|q - \pi(q)|}.$$

As a consequence of (3.5.6) we get $d(q) \ll \alpha(C_p)$ on $\text{spt}(T) \cap \mathbf{B}_{1/16}$.

This means that $\text{spt}(T) \setminus \Gamma$ is split into N disjoint open sets, each of which is non-empty. If one of the sets was empty, one of the half-planes in the support of C could not be in the support of the tangent cone to T at 0.

This allows us to decompose T into N currents as desired. $\square$

Theorem 3.5.15. *Let T , Γ and Σ be as in 3.5.2. There exists ε such that if either*

1.

$$\max \left\{ \mathbf{E}(T, C, \mathbf{B}_1), \kappa^{-1} \mathbf{A}_\Gamma, \kappa^{-1} \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

2.

$$\max \left\{ \frac{\|T\|(\mathbf{B}_1)}{\omega_m} - Q/2, \kappa^{-1} \mathbf{A}_\Gamma, \kappa^{-1} \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

then there exists C' such that

$$\hat{\mathbb{E}}(T, C', \mathbf{B}_{1/2}) < \varepsilon_{\text{decomp}}.$$

In the first case, we have $\text{spt}(C') \subset C$ and if

$$\int_{\mathbf{B}_1 \setminus \mathbf{B}_{1/4}} \text{dist}(x, \text{spt}(T))^2 d|C| < \varepsilon$$

then we have $\text{spt}(C') = C$.

Proof. We argue in both cases by compactness, by taking a sequence of currents T_j which satisfies the assumptions of the theorem with a vanishing sequence of parameters $\varepsilon = \varepsilon_j$. We assume after rescaling the radius is 1 and up to rotation $T_0(\Gamma) = \mathbb{R}^{m-1} \times \{0\}$. Up to subsequence, $T_j \rightarrow T$, $\Gamma_j \rightarrow \mathbb{R}^{m-1} \times \{0\}$, $\Sigma_j \rightarrow \mathbb{R}^{m+\bar{n}}$.

In the first case up to subsequence also $C_j \rightarrow C$. We must have that $\mathbf{E}(T, C, \mathbf{B}_1) = 0$. This implies $\text{spt}(T) \subset \text{spt}(C)$ and thus there exists C' such that $\mathbb{E}(T, C', \mathbf{B}_1) = 0$ which is a contradiction since $\hat{\mathbb{E}}(T_j, C', \mathbf{B}_1) < \varepsilon_{\text{decay}}$ for j large enough.

In the second case T must have density constantly equal to $Q/2$ which means that it must be a cone by the monotonicity formula and it must be an open book by 3.3.1. We must then have $\hat{\mathbb{E}}(T_j, T, \mathbf{B}_1) < \varepsilon_{\text{decay}}$ for j large enough which is a contradiction. $\square$

We consider for a plane π the tilt excess

$$\overline{\mathbf{E}}(T, \mathbf{B}_r(p), \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{B}_r(p)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2} d\|T\|(x),$$

Theorem 3.5.16. *Let T, Γ, Σ be as in Assumption 3.1.1 and assume $\Theta(T, p) = Q/2$. Assume that T has a flat one-sided tangent cone $Q \llbracket \pi \rrbracket$ at p . There exists $\varepsilon(Q, m, n, \bar{n}) > 0$ such that if*

$$\max \left\{ \overline{\mathbf{E}}(T, \mathbf{B}_1(p), \pi), \mathbf{A}_\Gamma, \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

then there exists $C(Q, m, n, \bar{n})$ and $\alpha(Q, m, n, \bar{n}) > 0$ such that

$$\overline{\mathbf{E}}(T, \mathbf{B}_r(p), \pi) \leq C(Q, m, n, \bar{n}) r^\alpha, \text{ for every } r \in (0, 1).$$

Proof. It is enough to show that there exists ε such that if

$$\max \left\{ \overline{\mathbf{E}}(T, \mathbf{B}_1(p), \pi), \mathbf{A}_\Gamma, \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

then

$$\hat{\mathbb{E}}(T, \mathbf{B}_{1/2}(p), \pi) < \varepsilon_{\text{decay}}.$$

This follows by compactness. Since $Q \llbracket \pi \rrbracket$ is a tangent cone, we obtain the decay towards $Q \llbracket \pi \rrbracket$.

We thus establish for $r < 1/8$ that

$$\overline{\mathbf{E}}(T, \mathbf{B}_r(p), \pi) \lesssim \mathbf{E}(T, \mathbf{B}_{2r}(p), \pi) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \lesssim \hat{\mathbb{E}}(T, \mathbf{B}_{4r}(p), Q \llbracket \pi \rrbracket) \lesssim r^\alpha.$$

The first inequality is the classical L^2 - tilt excess bound. The second inequality is the strong excess to L^2 bound. The last inequality is the decay to the tangent cone from Theorem 3.5.10. $\square$

3.6 Lipschitz approximation

In this section, we construct the Lipschitz approximation and conclude with some technical estimates regarding measure theoretic excess and its relationship to L^2 excess.

We introduce an assumption that enables the construction of a suitable Lipschitz approximation defined on the ball $\mathbf{B}_1$.

Assumption 3.6.1. Let T , Γ and Σ be as in Assumption 3.1.1, but instead considered within the ball $\mathbf{B}_8$. Suppose C is an admissible open book. We assume for a small constant $\varepsilon(Q, m, n, \bar{n})$ that

$$\mathbb{E}(T, C, \mathbf{B}_8) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \leq \varepsilon \alpha(C)^2.$$

Moreover, we assume that for every $p \in \Gamma \cap \mathbf{B}_2$ and all $0 < r < 1/8$

$$\|T\|(\mathbf{B}_r(p)) \leq (Q/2 + 1/8) \omega_m r^m.$$

The Lipschitz approximation will be defined on the ball $\mathbf{B}_1$.

3.6.1 Lipschitz approximation away from the spine

We cite the relevant results on the Lipschitz approximation from [24] and outline the necessary modifications. The following result follows from the $L^2 - L^\infty$ height bound with respect to parallel planes (Theorem 3.2 of [24]).

Lemma 3.6.2 (Crude splitting). *Let T and Σ be as in Assumption 3.1.1. Let C be an open book with $\partial C = Q \llbracket V \rrbracket$ with $0 \in V$. Assume that $\partial T \llcorner (\mathbf{B}_4 \setminus B_{1/32}(V)) = 0$. There are*

constants $\delta(Q, m, n, \bar{n}) > 0$ and $\rho(Q, m, n, \bar{n}) > 0$ with the following property. Assume that

$$\int_{\mathbf{B}_4 \setminus B_{1/32}(V)} \text{dist}(p, C)^2 + \mathbf{A}_\Sigma^2 < \delta^2 \alpha(C)^2$$

Then

1. The sets $W_i := (\mathbf{B}_4 \setminus B_{1/32}(V)) \cap \{\text{dist}(\cdot, H_i) < \rho\sigma\}$ are pairwise disjoint.

2.

$$\text{spt}(T) \cap \mathbf{B}_3 \setminus B_{1/16}(V) \subseteq \cup_i W_i.$$

The proof of this lemma is the same as in Lemma 8.5 in [24] with the additional information that the boundary does not appear at the desired scales.

We will also denote $W'_i := \mathbf{B}_{5/2} \setminus B_{1/20}(V) \cap \{\text{dist}(\cdot, H_i) < 2\rho\sigma\}$, $\Omega'_i = \mathbf{B}_{5/2} \setminus B_{1/20}(V)$ and we suppose that ρ is such that W'_i are also disjoint (by possibly taking δ smaller).

Proposition 3.6.3. *Let T , C , W_i as in Lemma 3.6.2. Consider $\Omega_i := \mathbf{B}_2 \cap H_i \setminus B_{1/16}(V)$ and $\Omega_i = \mathbf{B}_3 \cap \mathbf{p}_{H_i}^{-1}(\Omega_i)$. Set $T_i := T \llcorner (W_i \cap \Omega_i)$ and*

$$E_i := \int_{\mathbf{B}_3 \setminus B_{1/32}(V)} \text{dist}(p, H_i)^2 d\|T_i\|(p).$$

Assume that $(\pi_o)_\#(T \llcorner \mathbf{B}_4 \setminus B_{1/32}(V)) = Q \llbracket \mathbf{B}_4 \cap \{x_m > 1/32\} \rrbracket$ and $\|T\|(\mathbf{B}_4) \leq (Q/2 + 1/4) 4^m \omega_m$.

Then, there exist non-negative integers $Q'_1, \dots, Q'_k$ such that $\sum Q'_i = Q$ and the following properties hold:

1. $\partial T_i \llcorner \Omega_i = 0$;

2. $(\mathbf{p}_{H_i})_\#(T_i) = Q'_i \llbracket \Omega_i \rrbracket$;

3. The following estimate holds

$$\text{dist}^2(q, H_i) = |\mathbf{p}_{H_i}^\perp(q)|^2 \lesssim E_i + \mathbf{A}_\Sigma^2 \quad \forall q \in \text{spt}(T_i) \cap \Omega_i :$$

4. For all i with $Q'_i \geq 1$, there are Lipschitz multi-valued maps $u_i : \Omega_i \rightarrow A_{Q'_i}(H_i^\perp)$ and closed sets $K_i \subseteq \Omega_i$ such that $\text{gr}(u_i) \subseteq \Sigma$, $T_i \llcorner \mathbf{p}_{H_i}^{-1}(K_i) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{H_i}^{-1}(K_i)$ and the following estimates hold

$$\|u_i\|_\infty^2 + \|Du_i\|_{L^2}^2 \leq CE_i + C\mathbf{A}_\Sigma^2$$

$$\text{Lip}(u_i) \leq C \left(E_i + \mathbf{A}_\Sigma^2 \right)^\gamma$$

$$|\Omega_i \setminus K_i| + \|T\|(\Omega_i \setminus \mathbf{p}_{H_i}^{-1}(K_i)) \leq C \left(E_i + \mathbf{A}_\Sigma^2 \right)^{1+\gamma}$$

5. $Q'_i = 0$ if and only if $T_i = 0$;

6. Finally, if in addition

$$\mathbb{E}(T, C, \mathbf{B}_8) < \delta^2 \alpha(C)^2$$

then $Q'_i = Q_i$ for every i .

Proof. The proof of the common statements follows [24]. The additional properties we establish are the bound $\sum Q'_i \geq Q$ and (6).

The bound $\sum Q'_i \geq Q$ is an immediate consequence (coupled with the information coming from the same proposition in [24]) of the assumption on the circular projection

$$(\pi_\circ)_\#(T \llcorner \mathbf{B}_4 \setminus B_{1/32}(V)) = Q \llbracket \mathbf{B}_4 \cap \{x_m > 1/32\} \rrbracket.$$

It remains to show (6).

We know that

$$|T_i|(\Omega_i) - Q_i|\Omega_i| \lesssim \left(E_i + \mathbf{A}_\Sigma^2 \right)^{1+\gamma}.$$

In particular

$$|T_i|(\Omega_i) - Q_i|\Omega_i| \ll 1/2|\Omega_i|.$$

Assume that $Q'_i > Q_i$ for some i , then $|T|(\Omega_i) - |C|(\Omega'_i) \geq c(m)$ if excess is small enough.

Assume that $|T| = |T|^1 + |T|^2$ and $|C| = |C|^1 + |C|^2$ with a transport plan σ between $|T|^1$ and $|T|^2$. We recall that W'_i is a fattening of W_i and Ω'_i is a fattening of Ω_i and thus exiting Ω'_i from Ω_i means traveling a constant distance.

Since

$$\int \text{dist}(x, V)^2 d|T|^2 + \int \text{dist}(x, V)^2 d|C|^2 \ll 1$$

then we must have $||T|^1 - |T||(\Omega'_i) \ll 1$ and $||C|^1 - |C||(\Omega'_i) \ll 1$. Thus

$$c(m) \leq |T|(\Omega_i) - |C|(\Omega'_i) \lesssim |T|^1(\Omega_i) - |C|^1(\Omega'_i) \leq \sigma(\Omega_i \cap W_i, 4B \setminus \Omega'_i \cap W'_i).$$

Since $|x - y| \geq c(m)\alpha(C)$ for $x \in \Omega_i \cap W_i$ and $y \in 4B \setminus \Omega'_i \cap W'_i$ we have

$$c(m)\alpha(C) \leq \int |x - y|^2 d\sigma(x, y) \leq \mathbb{E}(T, C, \mathbf{B}_4).$$

This is a contradiction to the assumption that $Q'_i > Q_i$ for some i which implies $Q'_i = Q_i$ and thus we conclude (6). $\square$

Notice that we don't separate between the case of a single and multiple planes (as in [24]) since we define $\alpha(C) := 1$ when C is a single half-plane.

3.6.2 Whitney regions and dyadic cubes

To construct the Lipschitz Approximation, we define a suitable Whitney region. We assume $V = \mathbb{R}^{m-1} \times \{0\}$. We let L_0 be a cube with side length $\frac{2}{\sqrt{m-1}}$ centered at 0.

We define the regions as in [24]. Let $R := \{p : \mathbf{p}_V(p) \in L_0 \text{ and } 0 < |\mathbf{p}_{V^\perp}(p)| \leq 1\}$. The cubes $\mathcal{G}_l$ are cubes of sidelength $\frac{2^{1-l}}{\sqrt{m-1}}$ obtained by subdividing L_0 into $2^{l(m-1)}$ cubes. The integer l , the generation of the cube, is denoted by $l(L)$. If $L \subset L'$ and $\ell(L') = \ell(L) + 1$, we then call L' the parent of L , and L a child of L' . When $\ell(L') > \ell(L)$, we say that L' is an ancestor of L and L is a descendant of L' .

For every cube L we denote by y_L its center and $\mathbf{B}(L)$ the ball $\mathbf{B}_{2^{-l(L)}}(y_L)$ (in $\mathbb{R}^{m+n}$) and by $\mathbf{B}^h(L)$ the set $\mathbf{B}(L) \setminus B_{2^{-5-l(L)}}(V)$.

We define for $L \in \mathcal{G}_l$

$$R(L) := \left\{ p : \mathbf{p}_V(p) \in L \text{ and } 2^{-l-1} \leq |p_{V^\perp}(p)| \leq 2^{-l} \right\}.$$

We also define λL as the cube concentric with L with side length $\lambda \frac{2^{1-l}}{\sqrt{m-1}}$ and

$$\lambda R(L) := \left\{ p : \mathbf{p}_V(p) \in \lambda L \text{ and } \lambda^{-1} 2^{-l-1} \leq |p_{V^\perp}(p)| \leq \lambda 2^{-l} \right\}.$$

We also define $\lambda L_i := \lambda R(L) \cap H_i$. We define the circular projection $\pi_\circ : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m-1} \times \mathbb{R}^+$ as $\pi_\circ(q) := (\mathbf{p}_V(q), |\mathbf{p}_{V^\perp}(q)|)$. We remind the basic properties of the regions.

Lemma 3.6.4. *Consider the collection of cubes $\mathcal{G}$ introduced above and its elements L . Then the following properties hold:*

1. *Given any pair of distinct $L, L' \in \mathcal{G}$ the interiors of $R(L)$ and $R(L')$ are pairwise disjoint and $R(L) \cap R(L') \neq \emptyset$ if and only if $L \cap L' \neq \emptyset$ and $|\ell(L) - \ell(L')| \leq 1$, while the interiors of L and L' are disjoint if $\ell(L) \leq \ell(L')$ and L' is not an ancestor of L .*
2. *The union of $R(L)$ ranging over all $L \in \mathcal{G}$ is the whole set R .*
3. *The diameters of the sets $L, R(L), \lambda L, \lambda R(L), L_i, \lambda L_i$, and $\mathbf{B}^h(L)$ are all comparable to $2^{-\ell(L)}$ and, with the exception of $L, \lambda L$, all comparable to the distance between an arbitrarily element within them and V ; more precisely, any such diameter and distance is bounded above by $C 2^{-\ell(L)}$ and bounded below by $C^{-1} 2^{-\ell(L)}$ for some constant C which depends only on m and n .*
4. *There is a constant $C = C(m, n)$ such that, if $\mathbf{B}^h(L) \cap \mathbf{B}^h(L') \neq \emptyset$, then $|\ell(L) - \ell(L')| \leq C$ and $\text{dist}(L, L') \leq C 2^{-\ell(L)}$. In particular, for every $L \in \mathcal{G}$, the subset of $L' \in \mathcal{G}$ for which $\mathbf{B}^h(L)$ and $\mathbf{B}^h(L')$ have nonempty intersection is bounded by a constant.*

5. $\sum_{L \in \mathcal{G}_\ell} \mathcal{H}^{m-1}(L) = C(m)$ for any ℓ and therefore, for any $\kappa > 0$,

$$\sum_{L \in \mathcal{G}} 2^{-(m-1+\kappa)\ell(L)} \leq C(\kappa, m).$$

We cite the Layer subdivision of [24], which will determine a sequence of subcones for a given open book C .

Lemma 3.6.5 (Layer subdivision). *For every integer $N \geq 2$ and every $0 < \delta \leq 1$, there is $\eta = \eta(\delta, N) > 0$ with the following properties. Let C be an open book. Then, there is $\kappa \in \mathbb{N}$ and subcollections $I(0) \supsetneq I(1) \supsetneq \cdots \supsetneq I(\kappa)$ of the integers $\{1, \dots, N\}$, each of cardinality at least 2 and with $I(0) = \{1, \dots, N\}$, so that the numbers*

$$m(s) := \min_{i < j \in I(s)} \angle(H_i, H_j)$$

$$d(s) := \max_{i \in I(0)} \min_{j \in I(s)} \angle(H_i, H_j)$$

$$M(s) := \max_{i < j \in I(s)} \angle(H_i, H_j)$$

satisfy the following requirements

1. $M(\kappa) = M(0)$
2. $\eta M(\kappa) \leq m(\kappa)$
3. $d(s) \leq \delta m(s)$ and $\eta d(s) \leq m(s-1)$ for every $1 \leq s \leq \kappa$;
4. $m(s-1) \leq \delta m(s)$ for every $1 \leq s \leq \kappa$.

If $M(\kappa) < \delta$, we extend the above by $\bar{\kappa} = \kappa + 1$ and $I(\bar{\kappa}) := \min \{I(\kappa)\}$. Otherwise $\bar{\kappa} := \kappa$.

Remark 3.6.6. We can assign multiplicities to the sequence of cones $C_0 \supsetneq C_1 \supsetneq \dots \supsetneq C_{\bar{k}}$, given by the layering procedure Lemma 3.6.5. We write

$$C_k = \sum_{i \in I(k)} Q_i^k \llbracket H_i \rrbracket.$$

We define inductively starting from assigning the multiplicities for $I(0)$. We define for $i \in I(k+1)$, the multiplicity Q_i^{k+1} as the sum of Q_j^k over $j \in I(k)$ such that

$$\angle(H_j, H_i) = \min_{l \in I(k+1)} \angle(H_l, H_i).$$

By construction, this minimum will only be achieved by a single sheet of the cone which means that the multiplicity arrangement is well-defined.

The multiplicities in remark 3.6.6 are defined to be used for Lemma 3.6.13 but in the other statements the cones are considered without having an assigned multiplicity.

Assumption 3.6.7 (Selection of parameters). We fix $\tau(m, n, \bar{n}, Q)$ small enough. The parameter $\bar{\delta} = \bar{\delta}(m, n, \bar{n}, Q, \tau)$ must be chosen much smaller than τ . Finally $\varepsilon = \varepsilon(Q, m, n, \bar{\delta}, \tau)$ in Assumption 3.6.1 will be smaller $\bar{\delta}$.

The parameters are chosen to satisfy lemma 8.3 of [24]. We define for the open books C_k given by

$$\mathbf{E}(L, k) := 2^{(m+2)l(L)} \int_{\mathbf{B}^h(L)} \text{dist}^2(q, C_k) d\|T\|(q).$$

Given an open book C_k we define

$$\mathbf{s}(k) := \min_{i < j \in I(k)} \text{dist}(H_i \cap B_1, H_j \cap B_1)$$

and $\mathbf{s}(k) = \bar{\delta}$ if $I(k)$ is a singleton. We will have a division into outer, central and inner cube as in [24] with the addition the respective balls don't intersect the boundary.

Definition 3.6.8. Let $L \in \mathcal{G}$. We say that L

1. L is an *interior cube* if $\mathbf{B}^h(L') \cap \Gamma = \emptyset$ for every ancestor L' of L (including L) and

$$|T|(\mathbf{B}(L)) \leq (Q/2 + 1/2)2^{m(2-l(L))}. \quad (3.6.1)$$

2. L is a *boundary stopping cube* if $\mathbf{B}^h(L) \cap \Gamma \neq \emptyset$ or $|T|(\mathbf{B}(L)) > (Q/2 + 1/2)2^{m(2-l(L))}$ and its parent is an interior cube.
3. A cube L is a k -type cube if it is an interior cube and $\mathbf{E}(L', 0)/\mathbf{s}(k)^2 \leq \tau^2$ for every ancestor L' of L (including L), but it is not a $(k-1)$ -type cube (if $k=0$ this condition is not apply).
4. A cube L is an *outer cube* if it is a 0-type cube. We denote by $\mathcal{G}^o$ the outer cubes. A cube L is a *central cube* if it is k -type for some $1 \leq k \leq \bar{\kappa}$. We denote by $\mathcal{G}^c$ the central cubes.
5. A cube L is a *inner cube* if it is an interior cube, it is neither an outer nor a central cube, but its parent is an outer or a central cube.

We note that for every interior cube $\pi_{\circ\#}(T \upharpoonright_{\mathbf{B}^h(L')}) = Q \llbracket \pi_{\circ}(\mathbf{B}^h(L')) \rrbracket$. If the density condition (3.6.1) fails then we must have $\text{dist}(\Gamma, y_L) \leq C(m)\ell(L)$ since otherwise (3.6.1) would be obtained as a consequence of the monotonicity formula at a boundary point.

We define the following regions that will be useful for the non-concentration estimates.

- The *boundary region* denoted R^b which is the union of $R(L)$ over all cubes which are not interior cubes.
- The *outer region* denoted R^o , is the union of $R(L)$ over all outer cubes.
- The *central region* denoted R^c , is the union of $R(L)$ over all central cubes.
- The *inner region* denoted R^i , is the union of $R(L)$ over all interior cubes which are not outer or central cubes.

3.6.3 Local lipschitz approximation and coherent outer approximation

We conclude this section with local Lipschitz approximations and a coherent outer approximation defined on the outer region. The proofs are the same as in [24] since the definition of these regions is such that they don't encounter the boundary and satisfy the appropriate upper density estimate.

We obtain the local approximations by 3.6.3 for the current $T_{y_L, 2^{-\ell(L)}}$ and the cone $C_{k(L)}$. The domains of definition will be $\Omega(L) := \left(\mathbf{B}_{2^{1-\ell(L)}}(y_L) \setminus \overline{B}_{2^{-4\ell(L)}}(V) \right) \cap C_{k(L)}$. We also denote $\Omega_i(L) := \Omega(L) \cap H_i$, $u_{L,i}$ the corresponding multi-valued approximation (obtaining by translating and rescaling to use the appropriate lemma) $\Omega_i(L)$ the sets $2^{\ell(L)}\Omega_i + y_L$ and by $K_i(L)$ the sets $2^{-\ell(L)}K_i + y_L$, where Ω_i and K_i are given by proposition 3.6.3. When $Q_{L,i} = 0$, the map $u_{L,i}$ does not exist and we will not change the notation to take that into consideration.

Proposition 3.6.9. *Let T, Σ, Γ, C as in Assumption 3.6.1. There exists $1 < \lambda = \lambda(m) \leq \frac{3}{2}$ and $\bar{C} = \bar{C}(Q, m, n, \bar{n})$ such that the local approximations satisfy the following properties.*

1. $Q_{L,i} = Q_i$ for every $L \in \mathcal{G}^o$.
2. $\sum_{i=1}^N Q_{L,i} = Q$ for every $L \in \mathcal{G}^o \cup \mathcal{G}^c$.
3. For every $L \in \mathcal{G}^o \cup \mathcal{G}^c$ we have $\text{spt}(T) \cap \lambda R(L) \subset \bigcup_i \Omega_i(L)$ and

$$2^{2\ell(L)} |q - \mathbf{p}_{H_i}(q)|^2 \leq \bar{C} \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_{\Sigma}^2 \right) \quad \forall q \in \text{spt}(T) \cap \Omega_i(L);$$

4. For $L \in \mathcal{G}^o \cup \mathcal{G}^c$, if we set

$$T_{L,i} := T \llcorner \Omega_i(L) \cap \left\{ \text{dist}(\cdot, H_i) < \bar{C} 2^{-\ell(L)} \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_{\Sigma}^2 \right)^{1/2} \right\}$$

(with $\bar{C}$ larger than the constant in the previous estimate, then, for $K_i(L) = 2^{-\ell(L)}K_i + y_L \subset \Omega_i(L)$ as above,

$$T_{L,i} \lrcorner \mathbf{p}_{H_i}^{-1}(K_i(L)) = \mathbf{G}_{u_{L,i}} \lrcorner \mathbf{p}_{H_i}^{-1}(K_i(L)),$$

$gr(u_{L,i}) \subset \Sigma$, and the following estimates hold:

$$\begin{aligned} 2^{2\ell(L)} \|u_{L,i}\|_\infty^2 + 2^{m\ell(L)} \|Du_{L,i}\|_{L^2}^2 &\leq \bar{C} \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2 \right) \\ \text{Lip}(u_{L,i}) &\leq C \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2 \right)^\gamma \\ |\Omega_i(L) \setminus K_i(L)| + \|T_{L,i}\| \left(\Omega_i(L) \setminus \mathbf{p}_{H_i}^{-1}(K_i(L)) \right) &\leq \bar{C} 2^{-m\ell(L)} \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2 \right)^{1+\gamma}. \end{aligned}$$

Definition 3.6.10. Let T, Σ, Γ, C be as in Proposition 3.6.9. Fix $L \in \mathcal{G}^o$. We define

$$\bar{\mathbf{E}}(L) := \max \{ \mathbf{E}(L') : L' \in \mathcal{G}^o \text{ with } R(L) \cap R(L') \neq \emptyset \}$$

Proposition 3.6.11 (Coherent outer approximation). *Let T, Γ, Σ, C be as in Assumption 3.6.1. We define*

$$R_i^o := \cup_{L \in \mathcal{G}^o} L_i \equiv H_i \cap \cup_{L \in \mathcal{G}^o} R(L).$$

Then, there exist Lipschitz multivalued maps $u_i : R_i^o \rightarrow A_{Q_i}(H_i^\perp)$ and closed subsets $\bar{K}_i(L) \subset L_i$ satisfying the following properties:

1. $gr(u_i) \subset \Sigma$ and $T_{L,i} \lrcorner \mathbf{p}_{H_i}^{-1}(\bar{K}_i(L)) = \mathbf{G}_{u_i} \lrcorner \mathbf{p}_{H_i}^{-1}(\bar{K}_i(L))$ for every $L \in \mathcal{G}^o$, where $T_{L,i}$ is defined as in the proposition above.

2.

$$2^{2\ell(L)} \|u_i\|_{L^\infty(L_i)}^2 + 2^{m\ell(L)} \|Du_i\|_{L^2(L_i)}^2 \lesssim \bar{\mathbf{E}}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2$$

$$\|Du_i\|_{L^\infty(L_i)} \lesssim \left(\bar{\mathbf{E}}(L) + \mathbf{A}_\Sigma^2 \right)^\gamma$$

$$|L_i \setminus K_i(L)| + \|T_{L,i}\|(\mathbf{p}_{H_i}^{-1}(L_i \setminus K_i(L))) \lesssim 2^{-m\ell(L)} \left(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2 \right)^{1+\gamma}$$

Proposition 3.6.12 (Blowup). *Let T, Σ, Γ, C satisfy Assumption 3.6.1. Then, for every $\sigma, \varsigma > 0$ there are constants $C = C(m, n, Q, \bar{\delta}, \tau) > 0$ and $\varepsilon = \varepsilon(m, n, Q, \bar{\delta}, \tau, \sigma, \varsigma) > 0$ such that the following holds:*

1. $R \setminus B_\sigma(V) \subset R^\circ$
2. $u_i, R_i := (R \setminus B_\sigma(V)) \cap H_i$ then

$$\int_{R_i} |Du_i|^2 \leq C\sigma^{-2} \mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Sigma^2$$

3. If additionally $\mathbf{A}_\Sigma^2 \leq \varepsilon^2 \mathbf{E}(T, C, \mathbf{B}_4)$, then there is a map $w_i : R_i \rightarrow A_{Q_i}(H_i^\perp)$ which is Dir-minimizing and such that

$$d_{W^{1,2}}(v_i, w_i) \leq \varsigma$$

where $d_{W^{1,2}}$ is the $W^{1,2}$ distance defined in [30].

3.6.4 Technical estimates on measure theoretic excess

In the following lemma, we use the definitions of the different type of cubes as 3.6.8, but do not require the global Assumption 3.6.1. We still consider the definition of types of cubes as above with respect to the cone C and their associated Lipschitz approximation.

Lemma 3.6.13. *Let T, Γ, Σ be as in Assumption 3.1.1, but instead within $\mathbf{B}_4$. Assume that every cube that intersects $\mathbf{B}_{2^{-10}}$ and their ancestors are outer cubes whose multiplicities of their local Lipschitz approximations agree with those of C . Then*

$$\mathbb{E}(T, C, \mathbf{B}_1) \leq C(Q, m, n, \bar{n}, \tau, \bar{\delta}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma^2 \right) \quad (3.6.2)$$

Proof. Let φ_L be a partition of unity such that φ_L is supported in $\mathbf{B}^h(L)$. We can split into cubes L

$$\mathbb{E}(T, C, \mathbf{B}_1) \leq \sum_{L \text{ interior}} d(\varphi_L |T|, \varphi_L C) + C \mathbf{A}_\Gamma^2.$$

In the boundary term we are also including the cubes such that the density

Now we further split by estimating

$$\sum_{L \text{ interior}} d(\varphi_L|T|, \varphi_L C) \leq (\text{I}) + (\text{II}) + (\text{III}) \quad (3.6.3)$$

with

$$\begin{aligned} (\text{I}) &= \sum_{L \text{ inner cube}} C \ell(L)^{m+2}, \quad (\text{II}) = \sum_k \sum_{L \text{ } k\text{-type}} d(\varphi_L|T|, \varphi_L C_L) \\ (\text{III}) &= \sum_k \sum_{L \text{ } k\text{-type}} d(\varphi_L C_L, \varphi_L C). \end{aligned}$$

In the above equations we define the cone C_L as the multiplicity arrangement of C given by constructing the local Lipschitz approximations over C .

We consider the term (I):

The bound for (I) is an immediate consequence of the definition. Indeed for any interior cube L we have

$$l(L)^{m+2} \leq C 2^{-(m+2)\ell(L)} \mathbf{E}(L),$$

which implies

$$(\text{I}) \leq C \mathbf{E}(T, C, \mathbf{B}_4).$$

We consider the term (II) : We obtain, in a way similar to Remark 3.2.2, for L of k -type the following inequality

$$d(\varphi_L|T|, \varphi_L|C_L|) \leq C 2^{-(m+2)\ell(L)} \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2 \right). \quad (3.6.4)$$

We can conclude

$$(\text{II}) \leq C \mathbf{E}(T, C, \mathbf{B}_4) + C \mathbf{A}_\Sigma^2.$$

We consider the term (III) : By definition of k -type, every ancestor to L must have a bound on the excess $\mathbf{E}(L', 0)/s(k)^2 \leq c(k)\tau^2$ and thus the $L^2 - L^\infty$ height bound is valid

with respect to C_k for L and every ancestor of L . This coupled with the fact that the regions over which the Lipschitz approximations for L and it's father overlap, implies that they can all be modelled over the cone C_k with the same multiplicities given by

$$Q_i^k = \sum_{j \in J(i)} Q_{L,j}$$

where

$$J(i) := \left\{ j \in I(0) : \langle (H_j, H_i) = \min_{l \in I(k)} \langle (H_j, H_l) \right\}.$$

Thus, we can bound the distance of the cones by the angle between them

$$d(\varphi_L C_L, \varphi_L C) \leq C \mathbf{s}(k)^2 \ell(L)^{m+2}$$

for any cube of k -type. We wish to bound then the contribution of all the cubes of k -type.

We call first k -type a central cube which doesn't have an ancestor of k -type.

It is easy to see that

$$\sum_{L \text{ } k\text{-type}} \mathbf{s}(k)^2 \ell(L)^{m+2} \leq \sum_{L \text{ first } k\text{-type}} \mathbf{s}(k)^2 \ell(L)^{m+2}.$$

This is because the sum of $\ell(L)^{m+2}$ of the descendants of a fixed cube R is comparable to $\ell(R)^{m+2}$.

Moreover for a first k -type cube L , by definition of the layer subdivision and being the first of k -type (and thus having a father which has large excess relative to C_{k-1}) implies

$$\mathbf{s}(k)^2 \ell(L)^{m+2} \leq C \mathbf{s}(k-1)^2 \ell(L)^{m+2} \leq C 2^{-(m+2)\ell(L)} \mathbf{E}(L).$$

Now by packing the last estimate over all cubes, we get that

$$\sum_{L \text{ first } k\text{-type}} \mathbf{s}(k)^2 \ell(L)^{m+2} \leq C \mathbf{E}(T, C, \mathbf{B}_4)$$

and thus $(II) \leq C\mathbf{E}(T, C, \mathbf{B}_4)$. The proof of the remaining estimate of (III) follows analogously to the one of (II) by reducing the bound to summing over first k -type cubes and using the stopping condition over those. $\square$

Lemma 3.6.14. *Let T, Γ, Σ, C be as in Assumption 3.6.1 for the ball $\mathbf{B}_1$. Let $\eta \in (0, 1/2)$ and assume that ε is sufficiently small with respect to η . For every $\rho \in [\eta, 1/2]$:*

$$\mathbb{E}(T, C, \mathbf{B}_\rho) \leq C(Q, m, n, \bar{n}, \eta) \left(\mathbb{E}(T, C, \mathbf{B}_1) + \mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma^2 \right).$$

Moreover for every $p \in \mathbf{B}_{1/4}(0)$ and $\rho \in [\eta, 1/2]$:

$$\mathbb{E}(T, C_p, \mathbf{B}_\rho(p)) \leq C(Q, m, n, \bar{n}, \eta) \left(\mathbb{E}(T, C, \mathbf{B}_1) + \mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma^2 \right).$$

The open book C_p is obtained by a suitable translation and tilting so that the spine of C_p is tangent to Γ at p and $C_p \subset T_p(\Sigma)$.

Proof. The first part of the statement is a consequence of the previous lemma when $\rho \in [\eta, 1/4]$. In order to get it to go to $[\eta, 1/2]$ we do need to take the constant of Assumption 3.6.1 smaller (and to construct the Lipschitz approximations in regions that get closer to the boundary of the ball).

The second part follows analogously as in Lemma 3.6.13 with some minor technical modifications. When bounding excess relative to a ball centered at $p \neq 0$, the bump function used in the definition of the excess is taken to be centered at p so this requires us to take this into consideration when estimating the transport plan. $\square$

3.7 Simon's non-concentration estimates

In this section we prove Simon's non-concentration estimates - the key technical machinery we require in this chapter.

Theorem 11.2 of [24] goes back to Leon Simon's work on the uniqueness of cylindrical tangent cones. The bound on the error term of the monotonicity formula implies, with delicate technical work, the Simon non-concentration inequality. In our setting, we will also have an error term coming from the fact that we have a boundary. This technique has also been used in [57] within the same setting. We will follow [24] taking care of the additional error terms. Throughout this section let $r = \frac{1}{3\sqrt{m-1}}$.

Theorem 3.7.1 (Simon's Estimates). *Let T, Γ, Σ and C as in assumption 3.6.1 for ε small enough. For any $r \leq \frac{1}{3\sqrt{m-1}}$ we have*

$$\int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\| \leq C(Q, m, n, \bar{n}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right) \quad (3.7.1)$$

here $q^\perp := q - \mathbf{p}_{\vec{T}}(q)$ at $\mathcal{H}^m$ a.e. $q \in \text{spt}(T)$ ($\mathbf{p}_{\vec{T}}$ is the orthogonal projection to the span of $\vec{T}(q)$). We also have

$$\int_{\mathbf{B}_r} |\mathbf{p}_V \circ \mathbf{p}_T^\perp|^2 d\|T\| \leq C(Q, m, n, \bar{n}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right). \quad (3.7.2)$$

Corollary 3.7.2. *Let T, Γ, Σ and C as in Assumption 3.6.1 for ε small enough. Then for all $\kappa \in (0, m+2)$,*

$$\int_{\mathbf{B}_r} \frac{\text{dist}(q, C)^2}{|q|^{m+2-\kappa}} \leq C_\kappa \left(\mathbf{E}(T, \mathbf{B}_4, C) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right) \quad (3.7.3)$$

The proof of Corollary 3.7.2 is an immediate consequence of Theorem 3.7.1 and the following Lemma 3.7.3.

Lemma 3.7.3. *Let T, Γ, Σ and C as in assumption 3.6.1 for ε small enough. Then for every $\kappa > 0$ we have*

$$\int_{\mathbf{B}_1} \frac{\text{dist}(q, C)^2}{|q|^{m+2-\kappa}} d\|T\|(q) \leq C_\kappa \int_{\mathbf{B}_1} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q) + C_\kappa \mathbf{E}(T, C, \mathbf{B}_1) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2. \quad (3.7.4)$$

Proof of Theorem 3.7.1. By the monotonicity formula from Corollary 3.2.5

$$\begin{aligned} \int_{\mathbf{B}_\rho} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q) &= \frac{\|T\|(B_\rho(0))}{\omega_m \rho^m} - Q/2 - \frac{1}{m} \int_{B_\rho} (q^\perp \cdot \vec{H}_T(q)) (|q|^{-m} - \rho^{-m}) \\ &\quad - Q \int_0^\rho t^{-m-1} \int_{\Gamma \cap B_t} (q \cdot \vec{\eta}) d\mathcal{H}^{m-1}(q) dt \end{aligned} \quad (3.7.5)$$

We let H_i be the half planes C and Q_i their respective multiplicities. We have then that

$$\frac{Q\omega_m r^m}{2} = \sum Q_i |H_i \cap B_r|.$$

We later use the following formula, which holds for any Radon measure μ and any non-negative function f :

$$\int f(|q|) d\mu = \int_0^\infty f(r) \frac{d}{dr} (\mu(B_r)) dr.$$

We start fixing a smooth χ , monotone, non-increasing, such that $\mathbb{1}_{[0,r]} \leq \chi \leq \mathbb{1}_{[0,2r]}$.

Define $\Gamma(t) := - \int_s^\infty \frac{d}{ds} (\chi(s)^2) s^m ds$. We take a weighted version of the identity (3.7.5) by multiply both sides by ρ^m take the derivative on ρ and then integrate over $\chi(\rho)^2$. We can then get estimate (11.10) in [24] with a boundary term. We explain carefully how to deal with the boundary terms.

We bound

$$\int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q) \leq C \left[\int \chi^2(|q|) d\|T\|(q) - \sum_{i=1}^N Q_i \int_{H_i} \chi^2(|q|) d|H_i| \right] \quad (3.7.6)$$

$$+ C \int \frac{|\Gamma(|q|) q^\perp \cdot \vec{H}_T|}{|q|^m} d\|T\|(q) \quad (3.7.7)$$

$$+ CQ \int_0^4 \chi^2(|\rho|) \rho^{-1} \int_{\Gamma \cap \mathbf{B}_\rho} (q \cdot \vec{\eta}) d\mathcal{H}^{m-1} d\rho \quad (3.7.8)$$

$$+ CQ \int_0^4 \chi^2(|\rho|) \rho^{m-1} \int_0^\rho t^{-m-1} \int_{\Gamma \cap \mathbf{B}_t} (q \cdot \vec{\eta}) d\mathcal{H}^{m-1} dt d\rho \quad (3.7.9)$$

The term (3.7.7) is dealt analogously. The terms (3.7.8), (3.7.9) are dealt by observing that

$$\int_{\Gamma \cap \mathbf{B}_\rho} |q \cdot \vec{\eta}| d\mathcal{H}^{m-1} = \int_{\Gamma \cap \mathbf{B}_\rho} |\mathbf{p}_{T_q(\Gamma)^\perp}(q) \cdot \vec{\eta}| d\mathcal{H}^{m-1} \lesssim \rho^{m-1} \sup_{q \in \Gamma \cap \mathbf{B}_\rho} |\mathbf{p}_{T_q(\Gamma)^\perp}(q)| \lesssim \rho^{m+1} \mathbf{A}_\Gamma.$$

Thus the terms (3.7.8), (3.7.9) are bounded by $\mathbf{A}_\Gamma$. As for the term (3.7.6), we take the vector field

$$X(q) = \chi(|q|) p_V^\perp(q).$$

The first variation formula gives us

$$\int \operatorname{div}_{\vec{T}}(X) = \mathbf{O}(\mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma).$$

This gives us

$$\begin{aligned} \int \chi^2(|q|) d\|T\|(q) - \int \chi^2(|q|) d\|C\|(q) &\leq C \mathbf{A}_\Sigma^2 + C \mathbf{A}_\Gamma + C \int_{B_{2r}} |x^\perp|^2 d\|T\|(q) \\ &\quad + \int \chi(|q|) x \cdot \nabla_{V^\perp} \chi(|q|) d\|C\|(q). \end{aligned}$$

The proof then proceeds almost identically as in [24] since we have the same estimates for inner, central and outer regions. The only term that remains are the boundary regions which can be bounded by $\mathbf{A}_\Gamma$. $\square$

Proof of Lemma 3.7.3. See the proof of lemma 11.6 in [24]. It follows from a first variation argument on the vector field

$$X(q) := \operatorname{dist}^2(q, C) \left(\max(r, |q|)^{-m-2-\kappa} - 1 \right)_+ q$$

where $g_+(q) := \max(g(q), 0)$.

The only required modification is controlling the boundary term

$$\int \operatorname{div}_{\vec{T}}(X) d\|T\| = Q \int_{\Gamma} (X \cdot \vec{\eta}) d\mathcal{H}^{m-1} - (X \cdot H_T) d\|T\|.$$

We estimate as in the boundary error term in the proof of Theorem 3.7.1

$$\int_{\Gamma} (X \cdot \vec{\eta}) d\mathcal{H}^{m-1} \leq \int_{\Gamma \cap \mathbf{B}_1} |q|^{-m-\kappa} |\mathbf{p}_{T_q(\Gamma)^\perp}(q) \cdot \vec{\eta}| d\mathcal{H}^{m-1} \leq C_\kappa \mathbf{A}_\Gamma.$$

□

Theorem 3.7.4. *Let T, Γ, Σ, C be as in Assumption 3.6.1. For every $\alpha > 0$*

1. For every cube L in the cubical decomposition from the previous section

$$\mathbf{E}(L) \lesssim_\alpha 2^{-\alpha \ell(L)} \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right)$$

2. For every σ and $r \leq \frac{1}{6\sqrt{m-1}}$:

$$\frac{1}{r^{m+2}} \int_{\mathbf{B}_r \cap B_{\sigma r}(V)} \operatorname{dist}(q, C)^2 d\|T\| \leq C_\alpha \sigma^{3-\alpha} \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right).$$

3. Moreover, if $\mathbf{B}_r \setminus B_{\sigma r}(V) \subset R^{\text{out}}$ then

$$\begin{aligned} \frac{1}{r^{m+2}} \int_{\mathbf{B}_r} \operatorname{dist}(q, C)^2 d\|T\| &\leq C(\alpha, Q, m, n, \bar{n}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right) \sigma^{3-\alpha} \\ &\quad + \frac{1}{r^{m+2}} \sum_i \int_{\mathbf{B}_r \cap H_i \setminus B_{\sigma r}(V)} |u_i|^2 \\ &\quad + C(Q, m, n, \bar{n}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + r^2 \mathbf{A}_\Sigma^2 \right) \left(\mathbf{E}(T, C, \mathbf{B}_4) \sigma^{-m-2} + r^2 \mathbf{A}_\Sigma^2 \right)^\gamma \end{aligned}$$

Proof. The first inequality is a consequence of Theorem 3.7.1 Simon's non-concentration estimate at suitable boundary points.

We prove the second inequality. We start covering $\mathbf{B}_r \cap B_{\sigma r}(V)$ with balls of size comparable to σr . For every such ball we have

$$\int_{B_i} \text{dist}(q, C)^2 \leq (\sigma r)^{m+2-\alpha} \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right).$$

Part of the error, which is already accounted for corresponds to the fact that the earlier non-concentration estimates require the cone to be admissible. We thus have to tilt and translate the cone which introduces an error of $\mathbf{A}_\Gamma^2 + \mathbf{A}_\Sigma^2$.

By taking a Besicovitch cover the balls will have bounded overlap. The number of balls will be thus bounded by $C(m)\sigma^{-m+1}$. This implies the second inequality in the lemma.

We estimate the remaining part of the integral in order to conclude the third inequality:

$$\begin{aligned} r^{m+2} \mathbf{E}(T, C, \mathbf{B}_r \setminus B_{\sigma r}(V)) &\leq \sum_{L: \sigma/2 \leq 2^{-\ell(L)} \leq 4r} \sum_i \int_{\Omega(L)} \text{dist}(p, H_i)^2 d|T_{i,L}| \\ &\leq \sum_{L: \sigma/2 \leq 2^{-\ell(L)} \leq 4r} \sum_i \int_{\Omega_i(L)^2} |u_i|^2 + C 2^{-(m+2)\ell(L)} \left(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}_\Sigma^2 \right)^{1+\gamma} \\ &\leq \sum_i \int_{\mathbf{B}_{4r} \cap H_i \setminus B_{\sigma r}(V)} |u_i|^2 + C \left(\mathbf{E}(T, C, \mathbf{B}_4) + r^2 \mathbf{A}_\Sigma^2 \right) \left(\mathbf{E}(T, C, \mathbf{B}_4) \sigma^{-m-2} + r^2 \mathbf{A}_\Sigma^2 \right)^\gamma. \end{aligned}$$

This type of estimate comes from Proposition 3.6.9. There is an additional issue we deal with here which is that there is a slight mismatch between the current on the cylindrical region $\mathbf{R}(L)$ and where the graphical approximation is defined. This issue does appear in the above estimate and is addressed in Lemma 8.16 and Remark 8.17 of [24]. $\square$

3.8 The boundary linear problem

In this section, we study the boundary linear problem of multi-valued Dirichlet minimizing functions on the half-space. We show this problem to have similar properties to the interior problem of Dirichlet minimizing functions treated in [30]. We show the 1-homogeneous

Dir-minimizers to be linear and a suitable decay lemma. We conclude the section with a technical result which will later allow us to conclude that blowups are Dir-minimizing up to the boundary.

3.8.1 Preliminaries

We refer to [30] for the detailed set up for the linear problem.

Let $\mathbb{R}_+^m = \{(y, t) \in \mathbb{R}^{m-1} \times \mathbb{R}^+\}$. Let $H = \{x : x_m = 0\}$ and if $x \in H$ then $\mathbf{B}_r^+(x)$ denote the half ball $\mathbf{B}_r(x) \cap \mathbb{R}_+^m$. We denote $\partial\mathbf{B}_r^+(x) = \partial\mathbf{B}_r(x) \cap \{x_m > 0\}$. Given $\Omega \subset \mathbb{R}_+^m$ be a Lipschitz domain there is a well-defined continuous trace operator

$$\circ|_{\partial\Omega} : W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \rightarrow L^2(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n)).$$

Definition 3.8.1. Let Ω be a Lipschitz domain. We say u is Dir-minimizing in Ω if

$$u \in W^{1,2}(\overline{\Omega}, \mathcal{A}_Q(\mathbb{R}^n))$$

and for every $v \in W^{1,2}(\overline{\Omega}, \mathcal{A}_Q(\mathbb{R}^n))$, we have $u \llcorner \partial\Omega = v \llcorner \partial\Omega$

$$\int_{\Omega} |Du|^2 \leq \int_{\Omega} |Dv|^2.$$

The boundary linear problem is the following:

Assumption 3.8.2 (Linear problem). Given a Dir-minimizing function $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$.

The boundary linear problem concerns functions u satisfying $u \llcorner \partial\Omega \cap \mathbf{B}_1 = Q \llbracket 0 \rrbracket$.

In this chapter it will be enough for us to consider the linear problem on the half space (i.e., the following):

Assumption 3.8.3 (Linear problem on the half space). Given a Dir-minimizing function $u \in W^{1,2}(\mathbf{B}_1^+, \mathcal{A}_Q(\mathbb{R}^n))$. The boundary linear problem on the half space consists of functions u such that $u|_{H \cap \mathbf{B}_1} = Q \llbracket 0 \rrbracket$.

In [52] Hirsch proved Hölder regularity for Dir minimizers with Hölder regular boundary data on C^1 domains. A relevant consequence of his work for us is:

Theorem 3.8.4. *Let u be as in Assumption 3.8.2 for a C^1 domain Ω . There exists $\alpha(Q, m, n, \bar{n}) > 0$ such that every $r \in (0, 1)$, $u \in C^{0,\alpha}(\overline{\mathbf{B}_r} \cap \Omega)$.*

His work can handle any Hölder continuous boundary data over the boundary of a domain, and even though the main statement of [52] is written in a global way, the proof of it is purely local and implies the above theorem.

We follow the main properties of the frequency function as in [30]

Definition 3.8.5. For $x \in H \cap \mathbf{B}_1$, $0 < r < 1 - |x|$, we define the Dirichlet energy, L^2 spherical height and the boundary frequency function:

$$D(x, r) = \int_{\mathbf{B}_r^+(x)} |Du|^2, \quad H(x, r) = \int_{\partial \mathbf{B}_r^+(x)} |u|^2 \quad \text{and} \quad I(x, r) = \frac{rD(x, r)}{H(x, r)}$$

Proposition 3.8.6. *Let u be as in Assumption 3.8.3. For every $x \in H \cap \mathbf{B}_1$ and a.e. $r \in (0, 1 - |x|)$*

$$(m-2) \int_{\mathbf{B}_r^+(x)} |Du|^2 = r \int_{\partial \mathbf{B}_r^+(x)} |Du|^2 - 2r \int_{\partial \mathbf{B}_r^+(x)} \sum_i |\partial_\nu u_i|^2$$

$$\int_{\mathbf{B}_r^+(x)} |Du|^2 = \int_{\partial \mathbf{B}_r^+(x)} \sum \langle \partial_\nu u_i, u_i \rangle$$

Proof. The proof is analogous to the corresponding version for classical Dir-minimizing functions (see Proposition 3.2. [30]).

For the first formula we use the same specific families of inner variations, which are diffeomorphisms that move H tangentially (and thus preserve the zero boundary data condition). These families are of the form $\varphi(x) = \phi(|x|)x$.

For the second formula we use the same specific families of outer variations. The competitors are of the form

$$\psi_\varepsilon(x) = \sum \llbracket u_i(x) + \varepsilon \phi(|x|)u_i(x) \rrbracket,$$

which preserve the zero boundary data condition along H and thus they are a valid 1-parameter family of competitors. $\square$

We can prove with the above the monotonicity of the frequency.

Theorem 3.8.7. *The frequency function for a function u satisfying Assumption 3.8.3 is monotone. For any $x \in H \cap \mathbf{B}_1$ either there exists ρ such that $u|_{B_\rho(x)} \equiv 0$ or $I(x, r)$ is an absolutely continuous non decreasing positive function of r on $(0, 1 - |x|)$.*

Corollary 3.8.8. *We have $I(0, r) \equiv r$ for $r \in (0, 1)$ if and only if u is α -homogeneous i.e. there is f such that*

$$u(y) = |y|^\alpha f\left(\frac{y}{|y|}\right)$$

Similar statements at different centers and scales can be obtained by suitable rescaling.

Corollary 3.8.9. *We set $x = 0$ and consider radii up to 1. The following estimates hold:*

1. *For almost every $r \leq s \leq t$,*

$$\frac{d}{d\tau}\Big|_{\tau=s} \left[\ln \left(\frac{H(\tau)}{\tau^{m-1}} \right) \right] = \frac{2I(s)}{s}$$

and

$$\left(\frac{r}{t}\right)^{2I(t)} \frac{H(t)}{t^{m-1}} \leq \frac{H(\tau)}{r^{m-1}} \leq \left(\frac{r}{t}\right)^{2I(r)} \frac{H(t)}{t^{m-1}};$$

2. *if $I(t) > 0$, then*

$$\frac{I(r)}{I(t)} \left(\frac{r}{t}\right)^{2I(t)} \frac{D(t)}{t^{m-2}} \leq \frac{D(r)}{r^{m-2}} \leq \left(\frac{r}{t}\right)^{2I(r)} \frac{D(t)}{t^{m-2}}. \quad (3.8.1)$$

The proof of Theorem 3.8.7, Corollary 3.8.8 and Corollary 3.8.9 follow analogously doing the same computations as in the interior setting in the Memoir [30]. For example for the proof of Theorem 3.8.7 we analogously get

$$I'(x, r) = \frac{(2 - m)D(x, r) + rD'(x, r)}{H(x, r)} - 2r \frac{D(x, r)^2}{H(x, r)^2}$$

for a.e. $0 < r < 1 - |x|$ by using identities 3.8.6, which allows us to conclude $I'(x, r)$ is non-negative by Cauchy Schwartz.

Proposition 3.8.10. *The frequency function*

$$I : H \cap \mathbf{B}_1 \rightarrow \mathbb{R}^+ \text{ defined by } I(x) := \lim_{r \rightarrow 0} I(x, r)$$

is an upper semicontinuous function on $H \cap \mathbf{B}_1$.

Proof. We take a sequence $x_k \rightarrow x$ and $r_k \rightarrow r$ we further assume $r_k < \frac{1}{2}(1 - |x|)$. For k large enough $r_k < 1 - |x_k|$, which makes the frequency well-defined.

We know that

$$I(x_k, r_k) \rightarrow I(x, r).$$

By the monotonicity formula

$$I(x, r) \geq \limsup_{k \rightarrow \infty} I(x_k)$$

for every $r \in (0, 1 - |x|)$. By taking $r \rightarrow 0$ we get the desired upper semicontinuity. $\square$

Proposition 3.8.11. *If $x \in H$ and $I(x) = \alpha$ then for every radius $r < \frac{1 - |x|}{2}$ we have that*

$$\int_{\mathbf{B}_r^+(x)} |Du|^2 \leq C_x r^{m-2+2\alpha}.$$

Proof. We plug $t = r_x := \frac{1 - |x|}{2}$ in (3.8.1) which gives us for $r \leq r_x$

$$\int_{\mathbf{B}_r^+(x)} |Du|^2 \leq r^{m-2} \times \left(\frac{r}{r_x}\right)^{2I_x(r)} \int_{\mathbf{B}_{r_x}^+(x)} |Du|^2 \leq C_x r^{m-2+2\alpha}.$$

In the last step we used the monotonicity formula. □

3.8.2 Blowup of Dir-minimizing Q valued functions

Let $y \in H \cap \mathbf{B}_1$ assume that $\text{Dir}(f, B_\rho(y)) > 0$ for every $\rho \leq 1 - |y|$. The blowups of f at y are defined as:

$$f_{y,\rho}(x) = \frac{\rho^{\frac{m-2}{2}} f(\rho x + y)}{\sqrt{\text{Dir}(f, B_\rho(y))}}$$

Theorem 3.8.12. *Let $f \in W^{1,2}(\mathbf{B}_1^+, \mathcal{A}_Q)$ be Dir-minimizing with zero boundary data and $\text{Dir}(f, B_\rho) > 0$ for every $\rho \leq 1$. Then for any sequence $\rho_k \rightarrow 0$ there exists a subsequence, not relabeled, such that $f_{0,\rho}$ covers locally uniformly and in weak $W^{1,2}(\mathbb{R}_+^m)$ to a Dir-minimizing function $g : \mathbb{R}_+^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ with the following properties:*

1. $\text{Dir}(g, \mathbf{B}_1^+) = 1$ and $g|_\Omega$ is Dir-Minimizing for any bounded Ω .
2. $g(x) = |x|^\alpha g\left(\frac{x}{|x|}\right)$, where $\alpha = I_{0,f}(0) > 0$ is the frequency of f at 0.

The proof is analogous to the one from [30].

Lemma 3.8.13. *(Cylindrical Blowups)*

Let $g \in W^{1,2}(\mathbb{R}_+^{m+1}, \mathcal{A}_Q(\mathbb{R}^n))$ be an α homogeneous and Dir-Minimizing function with zero boundary value, $\text{Dir}(g, \mathbf{B}_1^+) > 0$ and $\beta = I_{z,g}(0)$. Suppose $z = \frac{e_1}{2}$. Then, the tangent functions h to g at z satisfy

$$h(x_1, x_2, \dots, x_m) = \hat{h}(x_2, \dots, x_m)$$

where $\hat{h} \in W^{1,2}(\mathbb{R}_+^{m-1}, \mathcal{A}_Q(\mathbb{R}^n))$ is a Dir-Minimizing Q valued function with zero boundary values on $\mathbb{R}_+^{m-1}$

The proof is analogous to the interior version of this statement. We do require a small modification for the construction of the competitor, which we will do as part of the next lemma. We recall here the interpolation result that we will use to construct the competitor in next section.

Proposition 3.8.14. *There is a constant $C = C(m, n, Q)$ with the following property. For every $g \in W^{1,2}(\partial\mathbf{B}_1, \mathcal{A}_Q)$, there is $h \in W^{1,2}(\mathbf{B}_1, \mathcal{A}_Q)$ with $h|_{\partial\mathbf{B}_1} = g$ and*

$$\text{Dir}(h, \mathbf{B}_1) \leq C \text{Dir}(g, \partial\mathbf{B}_1) + C \int_{\partial\mathbf{B}_1} \mathfrak{g}(g, Q \llbracket 0 \rrbracket)^2.$$

This is Corollary 2.16 of [30].

3.8.3 Classification of 1 homogeneous Dir-minimizers

Lemma 3.8.15. *Let h be a α homogeneous Dir-minimizing function on $\mathbb{R}_+^m$ with zero boundary value. Suppose that $I(z) = \alpha$ for $z = e_1/2$. Then $h(s, x) = \hat{h}(x)$ for a Dir-minimizing function $\hat{h}$ on $\mathbb{R}_+^{m-1}$.*

Proof. **We show that $y \rightarrow h(y - z)$ must be α homogeneous.**

We know that for $r \leq t$

$$\frac{I(z, r)}{I(z, t)} \left(\frac{r}{t}\right)^{2I(z, t)} \frac{D(z, t)}{t^{m-2}} \leq \frac{D(z, r)}{r^{m-2}}.$$

Moreover by the α homogeneity

$$\int_{\mathbf{B}_r(0)} |Dh|^2 = r^{m-2+2\alpha} \int_{\mathbf{B}_1(0)} |Dh|^2.$$

This implies that $D(z, t) \gtrsim t^{m-2+2\alpha}$ for $t \geq 2$. Moreover we also know that for $r \leq 1$

$$D(z, r) \lesssim r^{m-2+2\alpha}.$$

This implies

$$\left(\frac{t}{r}\right)^{2I(z, t)-2\alpha} \lesssim \frac{I(z, t)}{I(z, r)} \lesssim \frac{I(z, t)}{\alpha}.$$

For this inequality to hold we must have $I(z, t) \equiv \alpha$ since when $r \rightarrow 0$ the left hand side blowsup otherwise.

As for the interior we can get invariance of the function on the z direction. If $x \in \mathbb{R}_+^{m-1}$ then for $s > 0$

$$h(se_1 + x') = (2s)^\alpha h\left(\frac{e_1}{2} + \frac{x'}{2s}\right) = h\left(\frac{e_1}{2} + x'\right) = \hat{h}(x').$$

For $s < 0$ the equality is analogous. This gets us invariance in the direction of z .

We show that $\hat{h}$ is Dir-minimizing. If it is not, there exists a competitor h' on a $m - 1$ dimensional half ball $\mathbf{B}^+$:

$$\int_{\mathbf{B}^+} |D\tilde{h}|^2 < \int_{\mathbf{B}^+} |D\hat{h}|^2 \text{ and } \tilde{h}|_{\partial\mathbf{B}^+} = \hat{h}|_{\partial\mathbf{B}^+}.$$

where h' also has zero boundary value along H .

We can as in the proof of lemma 3.24 in [30] use $\tilde{h}$ to construct a competitor for h in a sufficiently large half cylinder. The proof is analogous but we still give a brief sketch on the construction of the competitor.

We assume $\mathbf{B}^+$ is centered at 0 and is of radius 1 (up to rescaling and translating). Given $L \in \mathbb{R}$ we take a competitor $\{\tilde{h} \neq h'\} \subset [-L, L] \times \mathbf{B}_1^+$. The competitor h' is chosen so that it agrees with $\tilde{h}$ in $[-L + 1, L - 1] \times \mathbf{B}_1^+$.

The transition regions are two half-cylinders of the form $[0, 1] \times \mathbf{B}_1^+$. In order to get the interpolating functions over the cylinders, which will be independent of L , we can use the interpolation result 3.8.14 to get the desired interpolating function in the transition regions. The fact that the unit ball and the half-cylinder are bi-lipschitz equivalent allows to use the lemma despite it is stated originally for the unit ball.

When L is large enough the gain obtained by using a less energy competitor for the slices dominates the loss of the interpolating function. The precise computation is the same as in lemma 3.24 in [30]. $\square$

Lemma 3.8.16. *Let $u \in W^{1,2}(\mathbb{R}_+^m, \mathcal{A}_Q(\mathbb{R}^n))$ be Dir-minimizing with zero boundary value along H . Assume u is α -homogeneous. Then $\alpha \geq 1$. If $\alpha = 1$, then u is linear: $\exists v_1, \dots, v_Q \in \mathbb{R}^n$*

such that

$$u = \sum_{i=1}^Q \llbracket v_i x_m \rrbracket.$$

Proof. In order to prove $\alpha \geq 1$, we argue by contradiction. Suppose $\alpha < 1$. Given $z \neq 0$ by homogeneity $I(rz) = I(z)$ for all $r > 0$. By upper semicontinuity of the frequency $I(z) < I(0) < 1$. We take a blowup at z and get an $I(z)$ homogeneous Dir minimizing function on $\mathbb{R}_+^{m-1}$ with zero boundary data. We can repeat this procedure to get a Dir-minimizing β homogeneous function with $\beta < 1$ on $\mathbb{R}^+$ with zero boundary data. The only Dir minimizers with zero boundary data in 1d are linear functions so $\beta = 1$ which reaches a contradiction.

We now show that if $\alpha = 1$, then u must be linear. By upper semicontinuity of the frequency $I(z) \leq I(0) = 1$ for every $z \in H$. By the last proposition, $I(z) \geq 1$ for every $z \in H$. Thus $I(z) = 1$ for every $z \in H$.

By Lemma 3.8.15, the function u is invariant in all directions parallel to the boundary by Lemma 3.8.15. Thus there exists u' a 1 dimensional Dir-minimizing function on $\mathbb{R}^+$ $u(y, t) = u'(t)$. Since all 1 dimensional Dir-minimizing functions are linear, u' must be linear:

$$u'(t) = \sum_{i=1}^Q \llbracket v_i t \rrbracket$$

with $v_i \in \mathbb{R}^n$. Thus we get the desired expression for u . □

3.8.4 A decay lemma

Proposition 3.8.17. *If $I(0) = \alpha$ then*

$$\int_{\mathbf{B}_r^+} |u|^2 \leq \frac{1}{m+1} r^{m+2\alpha} \int_{\partial \mathbf{B}_1^+} |u|^2.$$

In particular, if $I(0) > 1$ then

$$\lim_{r \rightarrow 0} \frac{1}{r^{m+2}} \int_{\mathbf{B}_r^+} |u|^2 = 0.$$

Proof. We have that

$$\int_{\partial \mathbf{B}_r^+} |u|^2 \leq r^{m-1+2\alpha} \int_{\partial \mathbf{B}_1^+} |u|^2$$

from Corollary 3.8.9. We integrate over the radius and get since $\alpha \geq 1$

$$\int_{\mathbf{B}_r^+} |u|^2 \leq \left(\int_0^r s^{m+1} \int_{\partial \mathbf{B}_1^+} |u|^2 \right) r^{2(\alpha-1)} = \frac{1}{m+1} r^{m+2\alpha} \int_{\partial \mathbf{B}_1^+} |u|^2.$$

This gives us the desired estimate. The conclusion at the end of the proposition is immediate from the fact that $\alpha > 1$ implies the exponent on r is greater than $m+2$. $\square$

The following lemma will allow us to get decay to either a sequence of linear functions or to zero.

Lemma 3.8.18. *Let u be as in Assumption 3.8.2. Then for every sequence of radii $r_j \rightarrow 0$, there exists a subsequence, not relabeled, and linear maps L_j , each with zero boundary value, such that the following properties hold. We require for every j*

$$\int_{\mathbf{B}_1} |DL_j|^2 \leq \int_{\mathbf{B}_1} |Du|^2.$$

If $L_j = \sum_{i=1}^N \llbracket v_{i,j} x_n \rrbracket$, we define the angle

$$\alpha(L_j) := \min_{\substack{i_1, i_2 \\ v_{i_1,j} \neq v_{i_2,j}}} |v_{i_1,j} - v_{i_2,j}|$$

If $L_j = Q \llbracket L'_j \rrbracket$, then $\alpha(L_j) := 1$. One of the following cases holds:

1. $\liminf_{j \rightarrow \infty} \alpha(L_j) > 0$ and

$$\lim_{j \rightarrow \infty} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} \mathcal{G}(u, L_j)^2 = 0,$$

2. $L_j = Q \llbracket L' \rrbracket$ and

$$\lim_{j \rightarrow 0} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} |u \ominus Q \llbracket L' \rrbracket|^2 = 0.$$

Proof. We start by proving the lemma first for the case that u has zero average. If $I(u) > 1$ or $u \equiv 0$ we have by Proposition 3.8.17 that

$$\lim_{j \rightarrow 0} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} |u|^2 = 0$$

so we conclude.

If $I(u) = 1$, given the sequence r_j , we can choose a subsequence, not relabeled, so that

$$u_{0,r_j} \rightarrow L$$

is linear average free and with zero boundary value. This follows from Lemma 3.8.16, since L must be Dir minimizing 1-homogeneous with zero boundary value function and thus it is linear. We have

$$\begin{aligned} \int_{\mathbf{B}_1} \mathcal{G}(u_{0,r_j}, L)^2 &= \int_{\mathbf{B}_1} \mathcal{G} \left(\frac{r_j^{m-2/2}}{\sqrt{\text{Dir}(u, \mathbf{B}_{r_j})}}, u(r_j x), L(x) \right)^2 dx \\ &= \frac{1}{r_j^{m+2}} \frac{r_j^m}{\text{Dir}(u, \mathbf{B}_{r_j})} \int_{\mathbf{B}_{r_j}} \mathcal{G} \left(u, \frac{\sqrt{\text{Dir}(u, \mathbf{B}_{r_j})}}{r_j^{m/2}} L \right)^2. \end{aligned}$$

We denote

$$c_j := \frac{\sqrt{\text{Dir}(u, \mathbf{B}_{r_j})}}{r_j^{m/2}}.$$

Thus

$$\lim_{j \rightarrow 0} \frac{1}{r_j^{m+2}} \frac{1}{c_j^2} \int_{\mathbf{B}_{r_j}} \mathcal{G}(u, c_j L)^2 = 0.$$

We notice that

$$\frac{\sqrt{\text{Dir}(u, \mathbf{B}_r)}}{r^{m/2}} \leq \sqrt{\text{Dir}(u, \mathbf{B}_1)}.$$

since the left hand side is monotone for a Dir-minimizing function with frequency at least 1 (See Lemma 3.8.11). This implies $c_j \leq \sqrt{\text{Dir}(u, \mathbf{B}_1)}$ and thus if $L_j = c_j L$, then $\int_{\mathbf{B}_1} |DL_j|^2 \leq \int_{\mathbf{B}_1} |Du|^2$. We must have by definition that $\alpha(L_j) = c_j \alpha(L)$.

We must show that if

$$\liminf_{j \rightarrow \infty} \alpha(L_j) = 0$$

then up to a subsequence

$$\lim_{j \rightarrow 0} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} |u|^2 = 0.$$

This comes from

$$\lim_{j \rightarrow 0} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} |u|^2 \leq C \lim_{j \rightarrow 0} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} \mathcal{G}(u, c_j L)^2 + C \lim_{j \rightarrow 0} \alpha(L_j)^2 = 0.$$

This comes from the triangle inequality coupled with the fact that

$$\frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} |c_j L|^2 \leq C c_j^2 \alpha(L)^2 = \alpha(L_j)^2.$$

Thus we conclude with the proof of the lemma for the case of zero average.

We extend it for the case when u is not average free. We denote $\eta \circ u$ the average of u . In the first case, we define

$$L_j = c_j L \oplus Q \llbracket D(\eta \circ u)(0) \rrbracket.$$

In the second case, we define

$$L_j = Q \llbracket D(\eta \circ u)(0) \rrbracket.$$

We obtain the same properties for the average

$$\int_{\mathbf{B}_1} |D(\eta \circ u)(0)|^2 \leq \int_{\mathbf{B}_1} |D(\eta \circ u)|^2$$

and

$$\lim_{j \rightarrow \infty} \frac{1}{r_j^{m+2}} \int_{\mathbf{B}_{r_j}} |\eta \circ u(x) - D(\eta \circ u)(0)x|^2 dx = 0.$$

We obtain the desired conclusion by summing the average free estimate with the estimate with the average. $\square$

3.8.5 Local Dir-minimizers away from the halfplane

This subsection proves a useful result for later. Namely a function with zero boundary data on $\mathbf{B}_1^+$ which is Dir-minimizing away from the spine is Dir-minimizing on $\mathbf{B}_1^+$ with zero boundary value along H .

Theorem 3.8.19. *Let $u \in W^{1,2}(\mathbf{B}_1^+, \mathcal{A}_Q(\mathbb{R}^n))$ have zero boundary value along H . Assume that u is Dir-minimizing on every subdomain of $\mathbf{B}_1^+$ with smooth boundary. Then u is Dir-minimizing in $\mathbf{B}_1^+$ with zero boundary value along H .*

Proof. Assume by contradiction u is not Dir-minimizing in $\mathbf{B}_1^+$. Then there exists v with $u|_{\partial \mathbf{B}_1^+} = v|_{\partial \mathbf{B}_1^+}$ and

$$\int_{\mathbf{B}_1^+} |Dv|^2 < \int_{\mathbf{B}_1^+} |Du|^2.$$

We note that v must also have zero boundary data along H .

If ξ is the bi-Lipschitz embedding from [30] then the above inequality is equivalent to

$$\int_{\mathbf{B}_1^+} |D\xi(v)|^2 < \int_{\mathbf{B}_1^+} |D\xi(u)|^2.$$

We interpolate between u and v so that they agree on a fattened neighborhood of the boundary.

We consider α be a bump function increasing and smooth with $\alpha(t) = 0, \forall t \in [0, 1/2]$, $\alpha(t) = 1$ for every $t \in [1, \infty)$. The bump function we care about will be $\phi_\epsilon(x) = \alpha(\text{dist}(x, \partial \mathbf{B}_1^+)/\epsilon)$. It is supported in

$$U_\epsilon := \{x : \text{dist}(x, \partial \mathbf{B}_1^+) > \epsilon\}.$$

We define the embeddings $\bar{v} = \xi(v)$ and $\bar{u} = \xi(u)$. Then consider thus

$$\bar{v}_\epsilon = (\Phi_\epsilon \xi(v) + (1 - \Phi_\epsilon) \xi(u)).$$

We define the interpolations v_ϵ as $v_\epsilon = \rho(\bar{v}_\epsilon)$.

For every ϵ , v_ϵ agrees with u at $\mathbf{B}_1^+ \setminus U_\epsilon$. We consider V_ϵ a subdomain of $\mathbf{B}_1^+$ with smooth boundary such that $U_\epsilon \subset V_\epsilon$.

By the Dir-minimizing property on V_ϵ

$$\int_{\mathbf{B}_1^+} |Du|^2 \leq \int_{\mathbf{B}_1^+} |D\xi(v_\epsilon)|^2.$$

We will show that

$$\int |D\bar{v}|^2 = \lim_{\epsilon \rightarrow 0} \int |D\bar{v}_\epsilon|^2.$$

This will allow us to conclude that u is Dir-minimizing in $\mathbf{B}_1^+$.

We compute

$$D\xi(v_\epsilon) = D(\xi \circ \rho)(v_\epsilon) \cdot (\Phi_\epsilon D\xi(v) + (1 - \Phi_\epsilon) D\xi(u) + D\Phi_\epsilon(\xi(v) - \xi(u))).$$

We will show strong convergence in L^2 of the above $\epsilon \rightarrow 0$ to

$$D(\xi \circ \rho)(\xi(v)) \cdot D\xi(v).$$

It is enough to show that $D(\xi \circ \rho)(v_\epsilon)$ converges strongly in L^2 to $D(\xi \circ \rho)(\xi(v))$ and $\Phi_\epsilon D\xi(v) + (1 - \Phi_\epsilon) D\xi(u) + D\Phi_\epsilon(\xi(v) - \xi(u))$ converges weakly in L^2 to $D\xi(v)$.

For the first term we have

$$\int_{\mathbf{B}_1^+} |D(\xi \circ \rho)(\bar{v}_\epsilon) - D(\xi \circ \rho)(\bar{v})|^2 \lesssim |\{\bar{v}_\epsilon \neq \bar{v}\}|$$

In the above, we use that $\xi \circ \rho$ is bi-lipschitz. The L^2 convergence follows from construction since the right hand side of the inequality are neighborhoods of the boundary that converge to zero.

As for the weak L^2 convergence, it is an immediate consequence of the weak L^2 convergence of Φ_ϵ to 1.

We conclude from the strong and weak convergence the Dir-minimizing property of u . $\square$

3.9 Proof of the excess decay lemma

In this section we prove the excess decay lemma, introduced in Section 5, from which we deduced the main theorems in this chapter.

Lemma 3.9.1 (*Nth excess decay with small angle and small second fundamental forms*).

Let T and C be chosen as in Assumption 3.5.2. For every $\theta_N \in \mathbb{R}^+$ there exists $\varepsilon_N \in \mathbb{R}^+$, $\eta_N \in (0, 1/2)$ such that if

$$\mathbb{E}(T, C, \mathbf{B}_1) \leq \varepsilon_N \alpha(C)^2 \text{ and } \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \leq \varepsilon_N \mathbb{E}(T, C, \mathbf{B}_1)$$

(when $N = 1$ the first inequality becomes $\mathbb{E}(T, C, \mathbf{B}_1) \leq \varepsilon_1$) there exists a radius $r_0 \in [\eta_N, 1/2]$ and an admissible open book C' such that

$$\mathcal{G}(C', C)^2 \leq \gamma(Q, m, n, \bar{n}) \mathbb{E}(T, C, \mathbf{B}_1)$$

and

$$\mathbb{E}(T, C', \mathbf{B}_{r_0}) \leq \theta_N \min \left\{ \alpha(C')^2, \mathbb{E}(T, C, \mathbf{B}_1) \right\}.$$

We will prove the above lemma by contradiction. If the above lemma does not hold we will have to study the properties of certain blowup sequences. We rescale the ball $\mathbf{B}_1$ to $\mathbf{B}_8$ and we consider the following definition of a blowup sequence:

Assumption 3.9.2 (Blowup Sequence). A sequence of currents T_k is said to be a blowup sequence if $T_k, \Gamma_k, \Sigma_k, C_k$ satisfy Assumption 3.1.1 with $T_0(\Gamma_k) = \mathbb{R}^{m-1} \times \{0\}$, and $C_k = \sum_{i=1}^N Q_i \llbracket H_{i,k} \rrbracket$ being open books with the same number of sheets and labeling. Moreover, the following limits hold:

$$\frac{\mathbb{E}(T_k, C_k, \mathbf{B}_8)}{\alpha(C_k)^2} \rightarrow 0 \text{ and } \frac{\mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} \rightarrow 0.$$

We wish to take a Dir-minimizing blowup out of a blowup sequence of currents. Up to a subsequence $H_{i,k} \rightarrow H_i$. Define $C := \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ then $C_k \rightarrow C$. We also have $T_k \rightarrow C$, $\Sigma_k \rightarrow \mathbb{R}^{m+\bar{n}}$, $\Gamma_k \rightarrow \mathbb{R}^{m-1} \times \{0\}$.

If k is large enough, T_k satisfy the hypothesis 3.6.1. The excess bound is obviously satisfied so it only remains to check the mass bound. This is a consequence of the monotonicity formula and the fact that $T_k \rightarrow C$ with C an open book of multiplicity Q .

We take the Lipschitz approximations u_i^k defined on the outer regions, which are subdomains of $(\mathbf{B}_1 \setminus B_{1/k}(V)) \cap H_{i,k}$. We do need them to be defined on $\mathbf{B}_1 \setminus B_{1/k}(V) \cap H_i$. This is easy to achieve by applying a rotation we define for i, k a rotation $\sigma_{i,k}$ on $\mathbb{R}^{m+n}$ that takes $H_{i,k}$ to H_i , while fixing $\mathbb{R}^{m-1} \times \{0\}$. We choose $\sigma_{i,k}$ to be the admissible rotation closest to the identity. Clearly $\sigma_{i,k} \rightarrow_k \text{Id}$. We define

$$\bar{u}_i^k := \frac{\mathbf{p}_{H_i^\perp} \circ \sigma_{i,k} \circ u_i^k \circ \sigma_{i,k}^{-1}}{\sqrt{\mathbb{E}(T_k, C_k, \mathbf{B}_8)}}.$$

Up to a subsequence $\bar{u}_i^k$ converges weakly to a function $\bar{u}_i$. We will be able to estimate the functions $\bar{u}_i$ at the scale $r = \frac{1}{12\sqrt{m-1}}$ since we have the non-concentration estimate at scale $\frac{1}{6\sqrt{m-1}}$. The functions $\bar{u}_i$ will be the Dir-minimizing blowup of the blowup sequence.

Lemma 3.9.3 (Dir-minimizing blowup). *Let $r = \frac{1}{12\sqrt{m-1}}$. The following properties hold for the sequence $\bar{u}_i^k$ and their weak limit $\bar{u}_i$:*

- *The functions $\bar{u}_i^k$ have equibounded Dirichlet energy.*
- *$\bar{u}_i^k$ converge strongly in $W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_{Q_i}(\mathbb{R}^n))$ (up to a subsequence) to a function $\bar{u}_i$ in $W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_{Q_i}(\mathbb{R}^n))$.*
- *The functions $\bar{u}_i$ are Dir-minimizing on $\mathbf{B}_r^+$ with zero boundary value along V .*

Proof. We will show in order the following facts about the functions $\bar{u}_i^k$:

1. $\bar{u}_i^k$ converge locally strongly in $W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_{Q_i}(\mathbb{R}^n))$ (up to a subsequence) to a function $\bar{u}_i \in W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_Q(\mathbb{R}^n))$.
2. The function $\bar{u}_i$ is Dir-minimizer on subdomains of $\mathbf{B}_r^+$ with smooth boundary.
3. $|\bar{u}_i| \in W^{1,2}(\mathbf{B}_r^+, \mathbb{R})$ are sub-harmonic.
4. The functions $\bar{u}_i^k$ have equi-bounded Dirichlet Energy, i.e.,

$$\int |\bar{u}_i^k|^2 \lesssim 1.$$

5. The functions $\bar{u}_i^k$ have zero boundary value at V in the classical sense.
6. $\bar{u}_i^k$ converge strongly in $W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_{Q_i}(\mathbb{R}^n))$ (up to a subsequence) to $\bar{u}_i$.
7. $\bar{u}_i$ are Dir-minimizing in $\mathbf{B}_r^+$.

(1) By Lemma 3.6.12, the functions $\bar{u}_i^k$ converge locally strongly in $W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_Q(\mathbb{R}^n))$ to a function which is locally Dir-minimizing in certain regions.

(2) Given a subdomain of $\Omega \subset \mathbf{B}_r^+$ with smooth boundary, the functions $\bar{u}_i^k$ converge strongly in $W^{1,2}(\mathbf{B}_r^+, \mathcal{A}_{Q_i}(\mathbb{R}^n))$ to $\bar{u}_i$ and $\bar{u}_i$ is Dir-minimizer in Ω . This is a consequence of the proof of Theorem 5.2 on the first harmonic approximation [31]. The theorem uses balls as

domains but it can be easily modified to be any domain with smooth boundary. We remark that at this stage we have not yet claimed the functions $\bar{u}_i$ to be non trivial.

We also know by [31] that

$$\int_{L_i \setminus \overline{K_i}(L)} |D\bar{u}_i^k|^2 \rightarrow_{k \rightarrow \infty} 0. \quad (3.9.1)$$

(3) Since Dir-Minimizers have sub-harmonic norm, the local Dir-Minimizer property implies that $|\bar{u}_i|$ are sub-harmonic functions.

(4) We have for every $L \subseteq R^{out}$ that

$$2^{2L(L)} \|\bar{u}_i^k\|_{L^\infty(L)}^2 + 2^{ml(L)} \int_L |D\bar{u}_i^k|^2 \lesssim \frac{\mathbf{E}(T_k, C_k, L)}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} + 2^{-2l(L)} \frac{\mathbf{A}_{\Sigma_k}^2}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)}.$$

By taking the limit as $k \rightarrow \infty$, we get that for any $\alpha > 0$ there exists C_α such that $|\bar{u}_i| \leq C_\alpha 2^{-(1-\alpha/2)\ell(L)}$. This implies that $\bar{u}_i$ have zero boundary value along V (and in the classical sense).

Given $0 < \beta < 1$ we have the weighted estimate

$$\begin{aligned} \int_{\mathbf{B}_r^+} \text{dist}(x, V)^{-\beta} |D\bar{u}_i|^2 &\leq C \sum_L 2^{\ell(L)\beta} \limsup_{k \rightarrow \infty} \int_L |D\bar{u}_i^k|^2 \\ &\leq C(\alpha, Q, m, n, \bar{n}) \sum_{l=0}^{\infty} \frac{1}{2^{-l(m-1)}} 2^{-l(m-\alpha+\beta)} = \sum_{l=0}^{\infty} 2^{-l(1-\alpha+\beta)} \lesssim_{\alpha, \beta} 1. \end{aligned}$$

The second inequality follows from 3.7.4 coupled with properties of the cubical decomposition.

By taking $1 - \beta < \alpha < 1$ we get the desired convergence.

(5) The strong convergence is a classical consequence of equiboundedness in energy.

(6) We proved in Theorem 3.8.19 that, in this context, we are able to obtain the Dir-minimizing property of the function if the function is locally Dir-minimizing in smooth subdomains. $\square$

Proof of Excess Decay 3.5.4. We rescale the ball $\mathbf{B}_1$ to $\mathbf{B}_8$. If the excess decay lemma does not hold there is sequence of currents as in the Assumption 3.5.2 that fails the conclusions of the lemma. We can assume up to rotation that they satisfy Assumption 3.9.2. The sequence of currents can be chosen such that

$$\mathbb{E}(T_k, C'_k, \mathbf{B}_r) \geq \theta_N \left\{ \alpha(C')^2, \mathbb{E}(T, C, \mathbf{B}_1) \right\}.$$

for every $r \in [1/k, 4]$ and every cone C'_k such that

$$\mathcal{G}(C'_k, C_k)^2 \leq \gamma(Q, m, n, \bar{n}) \mathbb{E}(T_k, C'_k, \mathbf{B}_8).$$

We will show that either there exists a sequence of radii $r_j \rightarrow 0$ and a sequence of cones $C'_{k,j}$, which are obtained by adjusting the open books C_k satisfying the constraint

$$\mathcal{G}(C'_{k,j}, C_k)^2 \leq \gamma(Q, m, n, \bar{n}) \mathbb{E}(T_k, C'_k, \mathbf{B}_8)$$

such that

$$\lim_{j \rightarrow 0} \limsup_{k \rightarrow 0} \frac{\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{r_j})}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} = 0$$

and

$$\lim_{j \rightarrow 0} \limsup_{k \rightarrow 0} \frac{\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{r_j})}{\alpha(C'_{k,j})^2} = 0.$$

which will be a contradiction.

The construction of the cones $C'_{k,j}$ We take any sequence $r_j \rightarrow 0$ and choose $L_{i,j}$ as in Lemma 3.8.18 so that

$$\lim_{j \rightarrow 0} \frac{1}{\min_{1 \leq i \leq N} \alpha(L_{i,j})^2} \sum_{i=1}^N \frac{1}{(4r_j)^{m+2}} \int_{\mathbf{B}_{4r_j}} \mathcal{G}(\bar{u}_i, L_{i,j})^2 = 0 \quad (3.9.2)$$

where if the second case of Lemma 3.8.18 holds we define $L_{i,j} := 0$ in order to unify the notation. We can apply Lemma 3.8.18 so that the same sequence subsequence of r_j works simultaneously for every i .

We define then the adjusted cone as

$$C'_{k,j} := \sum_i \llbracket \sigma_{i,k}^{-1}(H_k \oplus \mathbb{E}(T_k, C_k, \mathbf{B}_8)^{1/2} L_{i,j}) \rrbracket.$$

We have that

$$\mathcal{G}(C'_{k,j}, C_k)^2 \leq \gamma(Q, m, n, \bar{n}) \mathbb{E}(T_k, C_k, \mathbf{B}_8). \quad (3.9.3)$$

The above equation follows from

$$\int_{\mathbf{B}_1} |DL_{i,j}|^2 \leq \int_{\mathbf{B}_1} |D\bar{u}_i|^2 \leq C(Q, m, n, \bar{n}).$$

There are two possible cases, the case where the cones $C'_{k,j}$ have all N collapsed sheets and the case where $C'_{k,j}$ has more than N sheets (which happens when for some i the first case of the conclusions of Lemma 3.8.18 hold).

In the second case we have that for k large enough

$$\alpha(C'_{k,j}) \simeq \min_{1 \leq i \leq N} \alpha(L_{i,j}) \mathbb{E}(T_k, C_k, \mathbf{B}_8)^{1/2}. \quad (3.9.4)$$

Bounding $\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_r)$ and $\mathbb{E}(T_k, C_k, \mathbf{B}_r)$ with respect to $\mathbb{E}(T_k, C_k, \mathbf{B}_8)$:

We remind that $\mathbb{E}(T_k, C_k, \mathbf{B}_4) \lesssim \mathbb{E}(T_k, C_k, \mathbf{B}_8)$. This allows us to estimate everything in terms of $\mathbb{E}(T_k, C_k, \mathbf{B}_8)$.

We estimate for $r \leq \frac{1}{6\sqrt{m-1}}$:

$$\begin{aligned} \mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_r) &\leq C\mathbf{E}(T_k, C_k, \mathbf{B}_r \cap B_\sigma(V)) + C\mathcal{G}(C'_{k,j}, C_k)^2\sigma^2 + \mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_r \setminus B_\sigma(V)) \\ &\leq C\sigma^{3-\alpha} \left(\mathbf{E}(T_k, C_k, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right) + C\sigma^2 \mathbf{E}(T_k, C_k, \mathbf{B}_8) \\ &\quad + \mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_r \setminus B_\sigma(V)). \end{aligned}$$

The first inequality comes from using the triangle inequality on the L^2 excess in the region $\mathbf{B}_r \cap B_\sigma(V)$. In the second inequality the first term we used the non concentration estimate Theorem 3.7.4, the second term we used equation (3.9.3).

We wish to bound the third term. It is possible to do so by a modification of the argument on Theorem 3.7.4. In the same fashion we use that the current in each Whitney region is agrees up to a superlinear error the graphical approximation. We get the following bound for k large enough

$$\begin{aligned} \mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_{4r} \setminus B_\sigma(V)) &\leq \frac{1}{r^{m+2}} \sum_i \int_{\mathbf{B}_{4r} \setminus B_\sigma(V)} \mathcal{G}(u_i^k, \mathbb{E}(T_k, C_k, \mathbf{B}_8)^{1/2} L_{i,j})^2 \\ &\quad + C \left(\mathbf{E}(T_k, C_k, \mathbf{B}_4) + r^2 \mathbf{A}_\Sigma^2 \right) \left(\mathbf{E}(T_k, C_k, \mathbf{B}_4) \sigma^{-m-2} + r^2 \mathbf{A}_\Sigma^2 \right)^\gamma \end{aligned}$$

We take first $\limsup$ over k and then $\sigma \rightarrow 0$ to get

$$\limsup_{k \rightarrow \infty} \frac{\mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_r)}{\mathbf{E}(T_k, C_k, \mathbf{B}_8)} \leq C \sum_{i=1}^N \frac{1}{r^{m+2}} \int_{\mathbf{B}_r} \mathcal{G}(\bar{u}_i, L_{i,j})^2.$$

The last term will disappear when taking $k \rightarrow \infty$ since it is superlinear with respect to the excess.

Moreover, by our choice of $L_{i,j}$ we have that

$$\lim_{j \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{\mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_{4r_j})}{\alpha(C'_{k,j})^2} = 0 \tag{3.9.5}$$

and

$$\lim_{j \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{\mathbf{E}(T_k, C'_{k,j}, \mathbf{B}_{4r_j})}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} = 0.$$

In the case that $C'_{k,j}$ has more sheets than C_k , (3.9.5) follows from (3.9.4) and (3.9.2).

In the case that $C'_{k,j}$ has the same number of sheets as C_k : we obtain

$$\limsup_{k \rightarrow \infty} \frac{\mathbb{E}(T_k, C_k, \mathbf{B}_8)}{\alpha(C_k)^2} \rightarrow 0,$$

$$|\alpha(C_k) - \alpha(C'_{k,j})| \leq C(Q, m) \mathcal{G}(C'_{k,j}, C_k).$$

We can conclude from the above and (3.9.3) that for every j

$$\lim_{k \rightarrow 0} \frac{\alpha(C_k)}{\alpha(C'_{k,j})} = 1.$$

This immediately implies (3.9.5).

Separation estimates in the first case at scales comparable to r_j :

As a consequence of (3.9.5), for j large enough and k large enough depending on j we must have that the $L^2 - L^\infty$ height bound applies at scale $4r_j$ and thus for some δ which can be taken arbitrarily small

$$\text{spt}(T_k) \cap (\mathbf{B}_{3r_j} \setminus \mathbf{B}_{10^{-10}r_j}) \subset \left\{ x \in \mathbf{B}_{3r_j} \setminus \mathbf{B}_{10^{-10}r_j} : \text{dist}(x, C'_{k,j}) \leq \delta \alpha(C'_{k,j}) \right\}$$

This must also hold at the level of the Dir-minimizers. Given $\delta > 0$ there if j is large enough

$$\frac{1}{r_j^{m+2}} \sum_{i=1}^N \int_{\mathbf{B}_{4r_j}} \mathcal{G}(\bar{u}_i, L_{i,j})^2 \leq \delta^2 \min_i \alpha(L_{i,j})^2.$$

This implies that

$$\sup_{x \in \mathbf{B}_{3r_j} \setminus \mathbf{B}_{10^{-10}r_j}} \mathcal{G}(\bar{u}_i, L_{i,j}) \leq C(Q, m, n, \bar{n}) \delta \min_i \alpha(L_{i,j})^2.$$

The graph $\bar{u}_i$ must separate at scales comparable to r_j if j is taken to be small enough. This implies that for j large enough the graph of u_i^k must separate into disjoint neighborhoods of $C'_{k,j}$ at scales comparable to r_j .

Bounding $\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{r_j})$ with respect to $\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{4r_j})$

We wish to use Lemma 3.6.13 to prove the reverse excess bound. In order to do that we must show that all large enough Whitney cubes with respect to $C'_{k,j}$ inside $\mathbf{B}_{4r_j}$ are outer cubes whose multiplicities are correct. We give a brief sketch of what is left to conclude that from the separation we already obtained the outer cube property.

Let L be a cube of size comparable to r_j inside $\mathbf{B}_{4r_j}$. For a bump function φ_L we must have that

$$d(\varphi_L |T_k|, \varphi_L \sum_{i,l} \mathbf{G}_{u_i^k}) \leq C \left(\mathbb{E}(T_k, C_k, \mathbf{B}_4) + r_j^2 \mathbf{A}_\Sigma^2 \right) \left(\mathbb{E}(T_k, C_k, \mathbf{B}_4) r_j^{-m-2} + r_j^2 \mathbf{A}_\Sigma^2 \right)^\gamma.$$

Thus for k large enough, the separation estimates allow us to write u_i^k as a graph over the cone $C'_{k,j}$. We must have that, up to reparameterization u_i^k is a strong Lipschitz approximation of T_k on top of the cone $C'_{k,j}$ with the correct multiplicity arrangement. This implies the outer cube property and allows us to conclude.

This implies that

$$\lim_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \frac{\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{r_j})}{\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{4r_j})} \leq C(Q, m, n, \bar{n})$$

which concludes with the desired estimate on $C'_{k,j}$ and thus gives us the desired estimates:

$$\lim_{j \rightarrow 0} \limsup_{k \rightarrow 0} \frac{\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{r_j})}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} = 0$$

and

$$\lim_{j \rightarrow 0} \limsup_{k \rightarrow 0} \frac{\mathbb{E}(T_k, C'_{k,j}, \mathbf{B}_{r_j})}{\alpha(C'_{k,j})^2} = 0.$$

□

3.10 A further study of the linear problem

In this section, we provide refined statements for theorems applicable to the linear problem. We will omit the proof, as they follow the same strategy as the proofs already performed on the nonlinear problem. In order to reduce the nonlinear problem to the decay of the linear problem we only needed to study the Dir-minimizing functions $u \in W^{1,2}(\mathbf{B}_1^+, \mathcal{A}_Q(\mathbb{R}^n))$ with zero boundary value along $\mathbf{B}_1 \cap (\mathbb{R}^{m-1} \times \{0\})$.

The methods used before allow us to treat a more general version of the linear problem which we remind.

Assumption 3.10.1. Given a Dir-minimizing function $u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$. The boundary linear problem consists of functions u such that $u \llcorner \partial\Omega \cap \mathbf{B}_1 = Q \llbracket 0 \rrbracket$.

This is equivalent to understanding the case in which a $u \llcorner \partial\Omega \cap \mathbf{B}_1 = Q \llbracket \varphi \rrbracket$ with suitable smoothness assumptions on φ since removing the average for u makes it have zero boundary value.

The frequency function must be defined in a different way since we have a curved boundary. If $d(x, q)$ is a good distance function like in [21] and

$$\varphi(t) := \begin{cases} 1, & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 2(1-t), & \text{for } \frac{1}{2} \leq t \leq 1, \\ 0, & \text{for } t \geq 1. \end{cases}$$

We define

$$D(q, r) := \int \varphi\left(\frac{d(q, x)}{r}\right) |Du(x)|^2 dx$$

and

$$H(q, r) := - \int \varphi\left(\frac{d(q, x)}{r}\right) |\nabla d(x, q)|^2 \frac{|u(x)|^2}{d(x, q)} dx.$$

We define

$$I(q, r) := \frac{rD(q, r)}{H(q, r)}.$$

Theorem 3.10.2 (Almost monotonicity of the frequency function). *Assume that $\mathbf{A}_{\partial\Omega}$ is small enough. Then, for every $q \in \partial\Omega \cap \mathbf{B}_{1/2}$ $0 < s < r < 1/2$*

$$I(q, r) \geq e^{-C\mathbf{A}_{\partial\Omega}(r/s)} I(q, s). \quad (3.10.1)$$

In particular

$$I(q) := \lim_{r \rightarrow 0} I(q, r)$$

exists.

In loose terms we get the following

Theorem 3.10.3. *Assume that Ω is a C^2 domain and u as in Assumption 3.8.2.*

- *For all $q \in \partial\Omega$ the frequency can be defined in a neighbourhood of q and is almost monotone. This means,*

$$e^{C\mathbf{A}_{\partial\Omega}r} I(r)$$

is monotone in r for an appropriate constant C .

- *The frequency at the limit is independent of the choice of a distance function and $\forall q \in \partial\Omega$, $I(q) \geq 1$.*
- *Either $I(q) = 1$ or $I(q) \geq 1 + \alpha(Q, m, n)$ for a positive constant $\alpha(Q, m, n)$. If $I(q) = 1$ then the blowups at q are linear, of the form*

$$f(x) = \sum_{i=1}^N Q_i \llbracket v_i(x \cdot \boldsymbol{\eta}_{\partial\Omega, q}) \rrbracket \quad (3.10.2)$$

with $\boldsymbol{\eta}_{\partial\Omega, q}$ the normal to $\partial\Omega$ at q , and unique. The uniqueness is quantitative and guarantees the existence of a well-defined multivalued normal derivative at the boundary, which is Hölder continuous.

The theorem above is a consequence of an excess decay type theorem for the linear problem. More precisely the excess decay theorem is the following

Theorem 3.10.4. *Given a constant M , assume that $\int_{\mathbf{B}_1} |Du|^2 \leq M$. Further assume that $0 \in \partial\Omega$. There exists $\varepsilon > 0$ and $\kappa > 0$ such that if $L : \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ is linear with $L \perp T_0(\partial\Omega) = Q \llbracket 0 \rrbracket$ and*

$$\max \left\{ \int_{\mathbf{B}_1} \mathcal{G}(u(p), L(p))^2 dp, \kappa^{-1} \mathbf{A}_{\partial\Omega} \right\} < \varepsilon$$

then there exists a linear map $\mathbf{L}$ with $L \perp T_0(\partial\Omega) = Q \llbracket 0 \rrbracket$ such that

$$\max \left\{ \frac{1}{r^{m+2}} \int_{\mathbf{B}_r} \mathcal{G}(u(p), \mathbf{L}(p))^2 dp, \kappa^{-1} \mathbf{A}_{\partial\Omega} r \right\} \leq r^\alpha \max \left\{ \int_{\mathbf{B}_1} \mathcal{G}(u(p), L(p))^2 dp, \kappa^{-1} \mathbf{A}_{\partial\Omega} \right\}.$$

We are not going to give a detailed proof of these facts, since they are the direct analogue of the existing results and the proofs are in fact even simpler.

We do a brief discussion of the structure of the proof in the particular case of the linear problem on the half-space and comment on some of the simplifications that occur in this setting.

1. As we observed already, the lower bound of the frequency by 1 gives monotonicity of the normalized Dirichlet energy

$$\frac{1}{r^m} \int_{\mathbf{B}_r(x)^+} |Du|^2.$$

This plays the role of the monotonicity formula for the area. If we considered a more general domain instead, the same monotonicity formula holds up to an error.

2. In this setting we do also need to have a family of excess decay theorems depending on the number of "sheets" of L , where with the number of sheets of L we mean N such that

$$L = \sum_{i=1}^N Q_i \llbracket x_n v_i \rrbracket$$

where $v_i \in \mathbb{R}^n$ are distinct and Q_i are positive integers with $\sum Q_i = Q$.

3. There is of course an $L^2 - L^\infty$ height bound in this setting as in [24] (in fact the one we used in the nonlinear setting is a consequence of the one for the linear problem). Thus if $L = \sum_{i=1}^N Q_i \llbracket x_n v_i \rrbracket$ there exists ε_k such that if

$$\int_{\mathbf{B}_4} \mathcal{G}(u(p), L(p))^2 dp + \mathbf{A}_{\partial\Omega} < \varepsilon_k \alpha(L)^2$$

then

$$\max_{p \in \mathbf{B}_2^+ \setminus B_{1/32}(V)} \mathcal{G}(u(p), L(p)) \ll \alpha(L).$$

This implies that the function u decomposes in $\mathbf{B}_2^+ \setminus B_{1/32}(V)$ as a sum $u = \sum_{i=1}^k (u_i + x_n v_i)$ with $u_i \in W^{1,2}(\mathbf{B}_2 \setminus B_{1/32}(V), \mathcal{A}_{Q_i}(\mathbb{R}^n))$.

The u_i are Dir-minimizing and "parameterized" on top of $\llbracket x_n v_i \rrbracket$. We perform the same Whitney decomposition into regions satisfying the same estimates established in Section 6.

4. The Simon's non-concentration estimates still hold in this setting and allows us to get energy convergence in a contradiction limit to the excess decay lemma. We consider a contradiction sequence to the excess decay theorem and obtain, Dir-minimizing limits $\bar{u}_i \in W^{1,2}(\mathbf{B}_1^+, \mathcal{A}_{Q_i})$ with zero boundary value.
5. We can conclude by, if necessary, adjusting L at suitable scales (as in Lemma 3.8.18) so $Q_i \llbracket x_n v_i \rrbracket$ is replaced by the linear function $Q_i \llbracket x_n v_i \rrbracket + \mathbf{E}(u_k, L)^{1/2} \bar{L}_{i,j}$ where we denote $\mathbf{E}(u, L) := \int_{\mathbf{B}_1} \mathcal{G}(u(p), L(p))^2 dp$.

The strategy described above was carried out for the interior in [56] to prove $\mathcal{H}^{m-2}$ almost everywhere uniqueness of the blowup. The advantage we have in this setting is that in the same way that for the nonlinear problem we obtain a no holes condition since $\Theta(T, q) \geq Q/2$ for every $q \in \Gamma$, we have that $I(q) \geq 1$ for every $q \in \partial\Omega$.

Chapter 4

On the regularity of area minimizing currents at boundaries with arbitrary multiplicity

This chapter is in collaboration with Reinaldo Resende and has been published as a preprint in [\[45\]](#).

4.1 Introduction

This work concerns the boundary regularity of area minimizing integral m -currents T within a $(m + \bar{n})$ -submanifold $\Sigma \in C^{3,\kappa}$ of $\mathbb{R}^{m+n}$, $n \geq \bar{n} \geq 0$, and $m \geq 2$. The boundary $\Gamma \subset \Sigma$ is also of class $C^{3,\kappa}$ and T takes the boundary Γ with multiplicity $Q \in \mathbb{N} \setminus \{0\}$. We refer the reader to Section [4.2.1](#) for precise definitions.

Before providing a historical overview of the problem, it is important to note that the regularity properties of higher multiplicity boundaries in arbitrary codimensions have been a long-standing challenge, originating after Allard's works [\[3, 1\]](#) for the case of $Q = 1$ and density close to $1/2$. In these works, the author emphasized the complexity of generalizing

his results to the setting of higher multiplicity, where the above density assumption must be replaced by closeness to $Q/2$ when $Q \in \mathbb{N} \setminus \{0\}$.

These questions have since been revisited in various forms, for instance, in [13, Problem 4.19]. White posed a general question about regularity properties of area minimizing currents at boundaries with arbitrary multiplicity. Subsequently, partial answers have been provided in specific cases—see De Lellis-De Philippis-Hirsch-Massaccesi [21] for $Q = 1$, as well as De Lellis-Nardulli-Steinbruechel [27] and Nardulli and Resende [68] for the case of $m = 2$. In De Lellis’ ICM survey ([18, Section 16.6]), the author reviews these questions and developments, highlighting how the particular cases addressed in these works still fall short of resolving the problem in full generality.

This thesis provides a complete solution to the questions raised in [18, Section 16.6]. To our knowledge, these results are the first to deal with arbitrary integer values of Q, m , and n , thus addressing the previously mentioned issues for area minimizing currents highlighted by Allard in [1], White in [13, Problem 4.19], and De Lellis in [18].

Historical overview. In the masterpiece [51], Hardt and Simon prove that there are *no* boundary singular points in the context of codimension and multiplicity 1, i.e., $n = Q = 1$. By a decomposition argument (which only works for $n = 1$), in [81], White shows that all boundary points are regular in the setting of codimension 1 and arbitrary multiplicity, i.e., $n = 1$ and $Q \geq 1$.

It is well known that the theory becomes significantly more involved when one takes a step forward and considers the higher codimension setting, i.e., $n > 1$. In fact, in [21, Theorem 1.8 (b)], De Lellis, De Philippis, Hirsch, and Massaccesi, provide an example of a 2 dimensional area-minimizing integral current T in $\mathbb{R}^4$ with a boundary $\Gamma \in C^\infty$ (consisting of two disjoint closed curves) of multiplicity $Q = 1$, such that the Hausdorff dimension of the boundary singular set $\text{Sing}_b(T)$ is 1. However, in this example the singular set has $\mathcal{H}^1$ -measure zero, leaving open the question of whether or not it is possible to prove that $\mathcal{H}^{m-1}(\text{Sing}_b(T)) = 0$ in general.

Assuming $\partial T = Q \llbracket \Gamma \rrbracket$, we define a point $p \in \Gamma$ as *one-sided* if the density of T at p is $Q/2$, and a point $q \in \Gamma$ as *two-sided* if the density of T restricted to an open neighborhood of q in Γ and intersected with Γ is constant equal to $Q/2 + Q^*$ for some $Q^* \in \mathbb{N} \setminus \{0\}$. We refer the reader to Definition 4.1.2 and the accompanying pictures.

In the celebrated paper [3], Allard proves that if only points of density $1/2$ are allowed and $Q = 1$, i.e., one-sided points in the multiplicity 1 case, then all boundary points of T are regular for any codimension $n \geq 1$. In particular, when the boundary lies on a convex barrier, defined as the boundary of a strictly convex open set, all boundary points are regular. However, the situation becomes more intricate if other types of boundary points are also allowed—for instance, *two-sided* points. Indeed, the example in [21] mentioned above highlights how much less is expected (compared to [3] and the case $n = 1$) in terms of describing the boundary regular set of T . Nonetheless, in the groundbreaking work [21], the authors adapt all the machinery introduced in [4, 30, 32, 31, 33, 34] (see also [69] for $\Sigma \in C^{2,\kappa}$) to prove that the boundary regular set of T is open and dense in Γ , provided that the boundary multiplicity is $Q = 1$ (cf. Theorem N). In light of the example in [21], this result is close to the best one can hope for.

When studying the case of arbitrary boundary multiplicity $Q \geq 1$ and arbitrary codimension $n \geq 1$, recent progress has been made in the context of 2-dimensional currents. Specifically, in [27], for $m = 2$ and considering *only one-sided* points, De Lellis, Nardulli, and Steinbrüchel, showed that all boundary points of T are regular (cf. Theorem M). Their work relies fundamentally on two key results proven by them: (1) the uniqueness of tangent cones in this dimension $m = 2$ setting, and (2) the regularity theory for Q -valued functions minimizing the Dirichlet energy with a prescribed C^3 boundary. Furthermore, in [68], Nardulli and Resende allow *two-sided* points to exist in Γ and rely on a regularity theory for $(Q - Q^*/2)$ -valued Dirichlet minimizers, along with the uniqueness of tangent cones in [26], to prove that the boundary regular set of T is open and dense in the setting $Q \geq 1$, $n \geq 1$, and $m = 2$.

4.1.1 Description of the main results

This work aims at better understanding the general case, that is, Q , m , and n , all arbitrary. One of the novelties here is that there is no boundary regularity theory developed for Q -valued Dirichlet minimizers for $m > 2$ and also *one-sided singular* points of T may exist (see Theorem 4.1.9). This differs dramatically from every other previous investigation due to the fact that *one-sided* points are always regular in the settings described above (i.e., $m = 2$ or $Q = 1$). Meanwhile, the uniqueness of tangent cones for $m = 2$ obtained in [27] is now known for general dimensions m , as well as an excess decay-type estimate with exponential decay rate, due to work in Chapter 3 (see Theorems 4.2.4 and 4.2.5). Such facts are crucial for the present article and will be exhaustively used in this work.

Summarizing, for m, n, Q arbitrary, the following results are proved in this article:

- (1) the set of regular boundary points is open and dense in $\Gamma \in C^{3,\kappa}$, see Theorem N,
- (2) the set of singular boundary points with density $Q/2$ of T is $\mathcal{H}^{m-3}$ -rectifiable, see Theorem M,
- (3) the set of boundary singular points of Q -valued Dirichlet minimizers is $\mathcal{H}^{m-3}$ -rectifiable, see Theorem 4.3.6,
- (4) if p is a density $Q/2$ flat point which has an open neighborhood in the boundary of density $Q/2$ flat points, then p is regular, see Theorem L.

Item (1) is *optimal* in the sense that there are examples of boundary singular sets with the same Hausdorff dimension as the boundary (see Theorem 4.1.12). The results (2) and (3) are *optimal* with respect to the Hausdorff dimensional bound (see Theorem 4.1.9) and, in addition to proving that the regular set has Hausdorff dimension bounded by $m - 3$, we also establish a structural property (rectifiability) of the singular set. Item (4) is likewise *optimal* in the sense that the assumption of an open neighborhood of *flat* one-sided points is essential and cannot be removed (cf. Remark 4.6.2).

The investigation of rectifiability of singular sets has been recently developed in the context of higher codimensions $n > 1$ to tackle *interior* regularity in [28, 29, 24], see also [57, 58]. The approach used in this work resembles that of [28, 29], which, while inspired by the seminal work [67] by Naber and Valtorta, differs *significantly* from it.

For the first time, this machinery is applied to study boundary regularity and it introduces significant conceptual differences compared to the interior setting.

The estimates derived through the Naber-Valtorta framework are identical for both linear (Dirichlet minimizing) and non-linear (area minimizing) problems. In fact, the $\mathcal{H}^{m-3}$ -rectifiability for the linear problem is also established in this article. However, we will not provide an extensive treatment of the linear problem before presenting the proofs for the non-linear case.

It must be remarked that the novelty of this work, which the authors find surprising, is that in this framework proving the $(m - 3)$ -Hausdorff dimensional bound for the one-sided singular set does not seem significantly easier to prove than proving the $\mathcal{H}^{m-3}$ -rectifiability of the boundary singular set.

The results, loosely stated in this Introduction, are described in a precise manner in what follows.

4.1.2 Definition of area minimality and one- and two-sided regular points

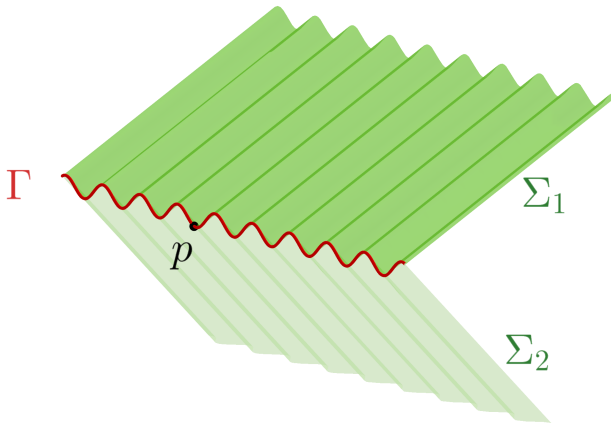
The following assumptions will always be in place: Σ and Γ are a $(m + \bar{n})$ -submanifold and a $(m - 1)$ -submanifold of $\mathbb{R}^{m+n}$, respectively. They are both of class $C^{3,\kappa}$, for some $\kappa > 0$, and satisfy $\Gamma \subset \Sigma$ and $\bar{n} \geq 1$. The space of integral k -dimensional currents in $\mathbb{R}^{m+n}$ will be denoted by $\mathbf{I}_k(U)$ for $k \in \{0, \dots, m + n\}$, the mass of a current T is denoted by $\mathbf{M}(T)$, and $\text{spt}(T)$ denote its support.

Definition 4.1.1. Let A current $T \in \mathbf{I}_m(U)$ with $\text{spt}(T) \subset \Sigma$ is *area minimizing* in $\Sigma \cap U$, if it holds that $\mathbf{M}(T) \leq \mathbf{M}(T + \partial \tilde{T})$, for all $\tilde{T} \in \mathbf{I}_{m+1}(U)$ with $\text{spt}(\tilde{T}) \subset \Sigma \cap U$.

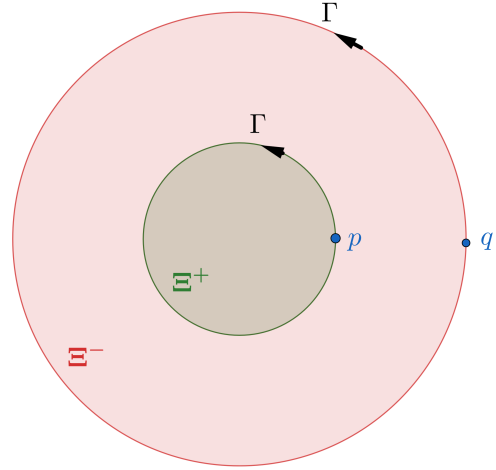
The definitions of one-sided and two-sided points are provided below. Here, for a smooth orientable k -manifold, we denote the naturally induced k -current by $\llbracket M \rrbracket$, and $\Theta^m(T, p)$ denotes the density of $\|T\|$ (the total variation of T) at p with respect to the m -dimensional Hausdorff measure. For precise basic definitions, we refer the reader to Section 4.2.1.

Definition 4.1.2. Let $T \in \mathbf{I}_m(U)$ with $\text{spt}(T) \subset \Sigma$, $\partial T = Q \llbracket \Gamma \rrbracket$, $Q \in \mathbb{N} \setminus \{0\}$, then $p \in \Gamma$ is called

- (1) an *one-sided point* for T if $\Theta^m(T, p) = Q/2$,
- (2) a *two-sided point* for T if there exist $Q^* \in \mathbb{N} \setminus \{0\}$ and an open neighborhood Z of p such that $\Theta^m(T, q) = Q/2 + Q^*$ for every $q \in \Gamma \cap Z$.



(a) If T is given by $\llbracket \Sigma_1 \rrbracket + \llbracket \Sigma_2 \rrbracket$, then $p \in \Gamma$ is a *one-sided*.



(b) If T is given by $2 \llbracket \Xi^+ \rrbracket + \llbracket \Xi^- \rrbracket$, then $p \in \Gamma$ is *two-sided* and q is *one-sided*.

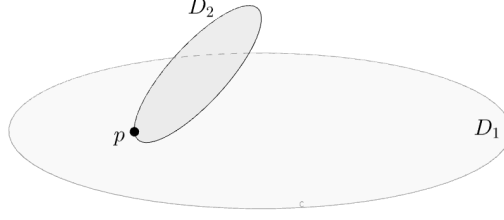


Figure 4.2: If T is given by $\llbracket D_1 \rrbracket + \llbracket D_2 \rrbracket$, then p is neither a one-sided nor a two-sided point.

Regular one-sided points have been studied in [3] for $Q = 1$ and in [27] for $m = 2$ and regular two-sided points have been explored in [21] for $Q = 1$ and in [68]. The two notions of regular boundary points in an arbitrary boundary multiplicity setting are defined below.

Definition 4.1.3 (Regular one-sided boundary points). Let $T \in \mathbf{I}_m(U)$ with $\text{spt}(T) \subset \Sigma$, $\partial T = Q \llbracket \Gamma \rrbracket$, $Q \in \mathbb{N} \setminus \{0\}$. Then $p \in \Gamma$ is called an *one-sided regular boundary point* if there are:

- (i) a finite number $\Sigma_1, \dots, \Sigma_J$ of oriented embedded m -submanifolds of $U \cap \Sigma$,
- (ii) and a finite number of positive integers $k_1, \dots, k_J$ such that:
 - (a) $\partial \Sigma_j \cap U = \Gamma \cap U = \Gamma_i \cap U$ (in the sense of differential topology) for every j ,
 - (b) $\Sigma_j \cap \Sigma_l = \Gamma \cap U$ for every $j \neq l$,
 - (c) for all $j \neq l$ and at each $q \in \Gamma$ the tangent cones to Λ_j and Λ_l are distinct,
 - (d) $T \llcorner U = \sum_j k_j \llbracket \Sigma_j \rrbracket$ and $\sum_j k_j = Q$.

The set $\text{Reg}_b^1(T)$ of *one-sided regular boundary points* is a relatively open subset of Γ and the set of *one-sided singular boundary points* is defined as

$$\text{Sing}_b^1(T) := \text{Sing}_b(T) \cap \{q \in \Gamma : \Theta^m(T, q) = Q/2\}.$$

Definition 4.1.4 (Regular two-sided boundary points). Let $T \in \mathbf{I}_m(U)$ with $\text{spt}(T) \subset \Sigma$, $\partial T = Q \llbracket \Gamma \rrbracket$, $Q \in \mathbb{N} \setminus \{0\}$. Then we say that $p \in \Gamma$ is a *two-sided regular boundary point* for T if there exist a neighborhood $U \ni p$ and a smooth manifold $\Xi \subset U \cap \Sigma$ such that $\text{spt}(T) \cap U = \Xi$. The set of such points will be denoted by $\text{Reg}_b^2(T)$.

Remark 4.1.5. We record the following consequence of the Constancy Lemma ([74, Theorem 2.34]) and the definition above (see also Picture (B) in Definition 4.1.2). Let $p \in \text{Reg}_b^2(T) \subset \Gamma$, and let U and Ξ be as in the definition above. Then, there exists $Q^* \in \mathbb{N} \setminus \{0\}$ such that $\Gamma \cap U$ divides Ξ into two disjoint regular submanifolds, Ξ^+ and Ξ^- , of Ξ with boundaries $(Q + Q^*) \llbracket \Gamma \rrbracket$ and $-Q^* \llbracket \Gamma \rrbracket$, respectively. Moreover, we have

$$T \llcorner U = (Q + Q^*) \llbracket \Xi^+ \rrbracket + Q^* \llbracket \Xi^- \rrbracket.$$

We now collect all singular and regular points in the definition below.

Definition 4.1.6 (Singular points). Let $T \in \mathbf{I}_m(U)$ with $\text{spt}(T) \subset \Sigma$, $\partial T = Q \llbracket \Gamma \rrbracket$, $Q \in \mathbb{N} \setminus \{0\}$. We denote the set of regular boundary points by $\text{Reg}_b(T) := \text{Reg}_b^1(T) \dot{\cup} \text{Reg}_b^2(T)$ and the set of singular boundary points by $\text{Sing}_b(T) := \Gamma \setminus (\text{Reg}_b^1(T) \dot{\cup} \text{Reg}_b^2(T))$.

Remark 4.1.7. Notice that we only have $\text{Sing}_b^1(T) \subset \text{Sing}_b(T)$.

4.1.3 Main results of this work

Next, in this subsection, the central assumptions and main results of this work will be stated.

Assumption 4.1.8. Let $\kappa \in (0, 1]$ and integers $m, n \geq 2$, $\bar{n} \geq 1$, $Q \geq 1$. Consider Σ an embedded $(m + \bar{n})$ -manifold of class $C^{3,\kappa}$ in $\mathbb{R}^{m+n}$, Γ a $C^{3,\kappa}$ embedded oriented $(m - 1)$ -submanifold of Σ without boundary, and assume $p \in \Gamma$. Let T be an integral m -dimensional area minimizing current in $\Sigma \cap \mathbf{B}_R(p)$ with boundary $\partial T \llcorner \mathbf{B}_R(p) = Q \llbracket \Gamma \cap \mathbf{B}_R(p) \rrbracket$.

We will prove a version of the boundary regularity theorem obtained in by Allard in [3] (for $Q = 1$ and only allowing for *one-sided* points) on a higher multiplicity boundary, that is, $Q > 1$. We call $q \in \Gamma$ a *flat* boundary point if the tangent cone to T at q is supported in a half m -plane (cf. Definition 4.2.3).

Theorem L. Under Assumption 4.1.8, assume that $q \in \Gamma$ is a one-sided *flat* point of T and there is a neighborhood of q in $\Gamma \cap \mathbf{B}_r(p)$ containing *only* one-sided *flat* points, then q is a regular point.

Theorem L is close to the best one can hope for, i.e., it is not possible to get an entire counterpart of [3] for $Q > 1$, since one-sided singular points may exist. In fact, in ?? we provide an example of this behavior which is stated below.

In order to state it, define a point $p \in \Gamma$ to be an *essential one-sided singularity* for T if $\Theta(T, 0) = Q/2$ and T cannot be written as $T \llcorner \mathbf{B}_\rho = \sum Q_i \llbracket \Sigma_i \rrbracket$, where Σ_i are smooth minimal surfaces and $\partial \Sigma_i = \Gamma \cap \mathbf{B}_\rho$, for any radius $\rho > 0$.

Theorem 4.1.9. *There exists a 3d area minimizing current T in $\mathbf{B}_1 \subset \mathbb{R}^5$ (i.e., $n = 2$) with T and Γ satisfying Assumption 4.1.8 with Γ analytic (i.e. Γ is a real-analytic submanifold of $\mathbb{R}^5$), and $Q = 2$, such that $\Theta^3(T, p) = Q/2 = 1$ for every $p \in \Gamma \cap \mathbf{B}_1$ and 0 is an essential one-sided boundary singularity. Moreover, the tangent cone to T at 0 is flat.*

Remark 4.1.10. As observed in Remark 4.4.5, the set of one-sided points is open. This example then demonstrates that, in Theorem L, it is not possible to remove the assumption that *all* points in the open neighborhood of q in Γ are *flat*.

In light of this example, the natural follow-up question to be asked is: What can we say about the boundary singular set? In fact, this question is posed in [18] and is fully answered in the subsequent theorems.

Theorem M. Under Assumption 4.1.8. If $m \geq 3$, the following holds:

$$\text{Sing}_b^1(T) \text{ is } \mathcal{H}^{m-3}\text{-rectifiable.}$$

In particular, if $m = 3$, we have that $\text{Sing}_b^1(T)$ is countable. Moreover, if $m = 2$, we have $\text{Sing}_b^1(T) = \emptyset$.

Remark 4.1.11. The fact that there are *no* one-sided singular points in the two dimensional case is proved in [27].

Recalling that in the interior setting, the Hausdorff dimension of the singular set is bounded by $m - 2$, we observe that in the first dimension where singularities appear, i.e., $m = 2$, the singular set is discrete. As proved in the series of papers [35, 36, 38, 37], addressing this problem is *considerably* more intricate, requiring the introduction of several new techniques.

In our boundary setting, singularities first appear for $m = 3$, as shown in Theorem 4.1.9, and we establish the countability of the one-sided singular set in Theorem M. The problem that Theorem M leaves open is the following.

Open question 1. Under Assumption 4.1.8. If $m = 3$, is $\text{Sing}_b^1(T)$ a discrete set?

We do not pursue further details on this question here, but it is worth noting that even in the linearized setting (i.e., for Q -valued Dirichlet minimizers with collapsed and regular boundary data), the problem remains entirely open.

A consequence of Theorem M, along with streamlined versions of the theorems Nardulli and Resende in [68], is that the whole regular boundary set is open and dense.

Theorem N. Under Assumption 4.1.8, it follows that

$$\text{Reg}_b(T) \text{ is an open and dense subset of } \Gamma \cap \mathbf{B}_R(p) .$$

As mentioned in the introduction, the theorem above is optimal in the sense that there cannot be a Hausdorff dimensional estimate for the singular set in this general context. Indeed, as shown in [21, Theorem 1.8 (b)], we have the following example.

Theorem 4.1.12. *There exists a 2d area minimizing current T in $\mathbf{B}_1 \subset \mathbb{R}^4$ with a boundary Γ (consisting of two disjoint closed C^∞ curves) and $\partial T = \llbracket \Gamma \rrbracket$, i.e., $Q = 1$, such that the Hausdorff dimension of the boundary singular set equals 1 and $\mathcal{H}^1(\text{Sing}_b(T)) = 0$.*

Although this example has a “large” singular set, it remains an open problem to understand, even when $Q = 1$, the following question.

Open question 2. Under Assumption 4.1.8, is it true that $\text{Sing}_b(T)$ is $\mathcal{H}^{m-1}$ -null?

This was originally conjectured in [21, Conjecture 2.8]; however, it remains a widely open question, as Theorem 4.1.12 is known not to generalize in a way that produces a singular set of positive measure. Even the counterpart of this question in the linearized problem (i.e., for $(Q - Q^*/2)$ -valued Dirichlet minimizers with regular interface) remains completely open.

4.2 Preliminaries

4.2.1 Basic definitions and notation

Denote $\mathbf{B}_r(p) := \{x \in \mathbb{R}^{m+n} : |x - p| < r\}$, fix $\pi_0 := \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$ and $\mathbf{p} := \mathbf{p}_{\pi_0}$. For any linear subspace $\pi \subset \mathbb{R}^{m+n}$, $\pi^\perp$ is its orthogonal complement in $\mathbb{R}^{m+n}$, $\mathbf{p}_\pi$ is the orthogonal projection on π . The tilted disk is defined as $\mathbf{B}_r(p, \pi) := \mathbf{B}_r(p) \cap (p + \pi)$ and the tilted cylinder as the set

$$\mathbf{C}_r(p, \pi) := \{(x + y) : x \in \mathbf{B}_r(p, \pi), y \in \pi^\perp\}.$$

Set $\mathbf{C}_r(p) := \mathbf{C}_r(p, \pi_0)$ and $\mathbf{B}_r(p) := \mathbf{B}_r(p, \pi_0)$. The half-ball will be defined as

$$\mathbf{B}_r^+(p) := \mathbf{B}_r(p) \cap \{x \in \mathbb{R}^m : x_m \geq 0\}.$$

Moreover, $B_r := B_r(0)$, $B_r^+ := B_r^+(0)$, and $\mathbf{C}_r := \mathbf{C}_r(0)$. For any vector subspace of $\mathbb{R}^{m+n}$, define its ε -neighborhood by

$$\mathbf{B}_\varepsilon(V) := \{x \in \mathbb{R}^{m+n} : \text{dist}(x, V) < \varepsilon\}.$$

It will always be assumed that each linear k -dimensional subspace π of $\mathbb{R}^{m+n}$ is oriented by a k -vector $\vec{\pi} := v_1 \wedge \cdots \wedge v_k$, where $(v_i)_{i \in \{1, \dots, k\}}$ is an orthonormal base of π and, with abuse of notation, we write $|\pi_2 - \pi_1|$ standing for $|\vec{\pi}_2 - \vec{\pi}_1|$, where $|\cdot|$ is the norm associated with the canonical inner product of k -vectors.

Denote the k -dimensional Hausdorff measure in $\mathbb{R}^{m+n}$ by $\mathcal{H}^k$. For notation and basic results on multivalued functions, the reader is referred to [30], here $\mathcal{A}_Q(\mathbb{R}^{m+n})$ denotes the set of Q -points (i.e., the sum of Q Dirac masses) in $\mathbb{R}^{m+n}$ for $Q \geq 1$.

Let us fix some notation that will be used throughout this Chapter regarding currents and refer the reader to the treatise [42]. The space of k -dimensional integral currents in $U \subset \mathbb{R}^{m+n}$ is denoted by $\mathbf{I}_k(U)$ for $k \in \{1, \dots, m+n\}$. Define the upper and lower d -density of a nonnegative Borel measure at $p \in \mathbb{R}^{m+n}$ as follows:

$$\Theta^{d,*}(\mu, p) := \limsup_{r \rightarrow 0} \frac{\mu(\mathbf{B}_r(p))}{\omega_d r^d}, \text{ and } \Theta_*^d(\mu, p) := \liminf_{r \rightarrow 0} \frac{\mu(\mathbf{B}_r(p))}{\omega_d r^d}.$$

Whenever the limit exists, the d -density of μ at p is $\Theta^d(\mu, p) = \Theta^{d,*}(\mu, p) = \Theta_*^d(\mu, p)$, where $\omega_m := \mathcal{H}^m(B_1)$. If $T \in \mathbf{I}_m(U)$, $p \in U$, and $\|\cdot\|$ denotes the mass of the current T , we denote $\Theta^m(T, p) := \Theta^m(\|T\|, p)$. A measure μ is said to be d -upper regular if for some universal constant C , $\mu(\mathbf{B}_r(p)) \leq Cr^d$ for every ball $\mathbf{B}_r(p)$.

Given a k -rectifiable set $M \subset \mathbb{R}^{m+n}$, it naturally induces a k -current which is denoted by $\llbracket M \rrbracket$. The boundary of a m -current T is taken with multiplicity Q^* if $T = Q^* \llbracket \Gamma \rrbracket$ for some $(m-1)$ -rectifiable set Γ .

We denote by $\mathbf{A}_M$ the C^0 -norm of the second fundamental form of a smooth submanifold M of $\mathbb{R}^{m+n}$.

The classical definitions for the excess and height that will be throughly used in this work are given below. Let $T \in \mathbf{I}_m(U)$ with $\text{spt}(T) \subset \Sigma$, $\Gamma \subset \Sigma$, $\partial T = Q \llbracket \Gamma \rrbracket$, π a m -dimensional plane, $p \in \mathbb{R}^{m+n}$, and $r > 0$. The *cylindrical excess relative to the plane* π is the quantity

$$\mathbf{E}(T, \mathbf{C}_r(p), \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{C}_r(p)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2} d\|T\|(x),$$

and, for $p \in \Gamma$, the *cylindrical excess* the quantity

$$\mathbf{E}^b(T, \mathbf{C}_r(p)) := \min\{\mathbf{E}(T, \mathbf{C}_r(p), \pi) : T_p \Gamma \subset \pi \subset T_p \Sigma\}.$$

Define *spherical excess relative to the plane* π to be the quantity

$$\mathbf{E}(T, \mathbf{B}_r(p), \pi) := \frac{1}{\omega_m r^m} \int_{\mathbf{B}_r(p)} \frac{|\vec{T}(x) - \vec{\pi}|^2}{2} d\|T\|(x),$$

and, for $p \in \Gamma$, the *spherical excess* to be the number

$$\mathbf{E}^b(T, \mathbf{B}_r(p)) := \min\{\mathbf{E}(T, \mathbf{B}_r(p), \pi) : T_p \Gamma \subset \pi \subset \mathbb{R}^{m+n}\}.$$

The *height of T in a set $G \subset \mathbb{R}^{m+n}$ with respect to a plane π* is defined as

$$\mathbf{h}(T, G, \pi) := \text{diam}(\mathbf{p}_\pi^\perp(\text{spt}(T) \cap G)) = \sup \left\{ \left| \mathbf{p}_\pi^\perp(q - p) \right| : q, p \in \text{spt}(T) \cap G \right\},$$

where $\mathbf{p}_\pi^\perp$ denotes the orthogonal projection onto $\pi^\perp$.

Tangent cones.

For $x \in \mathbb{R}^{m+n}$ and $r > 0$, define $\iota_{x,r}(y) := \frac{y-x}{r}$. For any current $T \in \mathbf{I}_m(U)$, let us define the rescaled current T at x at scale r as $\iota_{x,r} T =: T_{x,r}$ and $T_r =: \iota_{0,r} T$. The current T_x is called a blow-up of T at x , if there exists a sequence of radii $r_j \rightarrow 0$ such that $T_{x,r_j} \rightarrow T_x$ in the weak topology. A well-known consequence of the monotonicity formula tells us that if

T is area minimizing in Σ and $\partial T = Q \llbracket \Gamma \rrbracket$ for $\Gamma \in C^{1,\kappa}$, then T_{p,r_j} converges to an oriented cone about p for any $r_j \downarrow 0$. An oriented cone about p is understood as a current C such that $C_{p,r} = C$ for any $r > 0$.

Some classes of cones obtained by limits of area-minimizing currents will be of special interest in what follows; essentially, they are: one-sided flat cones (half m -planes), two-sided flat cones (m -planes), and open books (a union of half m -planes). Their precise definitions are given below.

Definition 4.2.1 (Open books). We say that C is an *open book with boundary $\llbracket V \rrbracket$ and multiplicity Q* , if $\partial C = Q \llbracket V \rrbracket$ and there exist $N \in \mathbb{N} \setminus \{0\}$, $Q_1, \dots, Q_N \in \mathbb{N} \setminus \{0\}$ and $\pi_1, \dots, \pi_N$ distinct m -dimensional half-planes such that

- (a) $\partial \llbracket \pi_i \rrbracket = \llbracket V \rrbracket, \forall i \in \{1, \dots, N\}$,
- (b) $T_0 = \sum_{i=1}^N Q_i \llbracket \pi_i \rrbracket$ with $Q = \sum_{i=1}^N Q_i$.

If there exist $i \neq j$ such that $\pi_i \neq \pi_j$, we say that C is a *genuine open book with boundary $\llbracket V \rrbracket$ and multiplicity Q* .

One-sided boundary flat cones will also be called *non-genuine open books*.

Definition 4.2.2 (Flat cones). We say that C is a *flat cone with boundary $\llbracket V \rrbracket$ and multiplicity Q* , if $\partial C = Q \llbracket V \rrbracket$ and there exist a m -dimensional closed plane π , $Q^{\text{int}} \in \mathbb{N}$ and $Q, Q^* \in \mathbb{N} \setminus \{0\}$, $Q \leq Q^*$, such that

- (a) $\text{spt}(C) = \pi$ is an m -dimensional subspace,
- (b) $\partial \llbracket \pi^+ \rrbracket = -\partial \llbracket \pi^- \rrbracket = \llbracket V \rrbracket$,
- (c) $C = Q^{\text{int}} \llbracket \pi \rrbracket + Q^* \llbracket \pi^+ \rrbracket + (-1)^k (Q^* - Q) \llbracket \pi^- \rrbracket$, $k \in \{0, 1\}$.

If $Q^{\text{int}} = 0$ and either $Q = Q^*$ or $k = 1$, we call C an *one-sided boundary flat cone with multiplicity Q* . If $k = 0$ and C is not a one-sided boundary flat cone, we say that C is a *two-sided boundary flat cone with multiplicity Q* .

For $x \in \mathbb{R}^{m+n}$ and $r > 0$, define $\iota_{x,r}(y) := \frac{y-x}{r}$ and, for any $T \in \mathbf{I}_m(U)$, the rescaling of T at x at scale r as $\iota_{x,r\#}T =: T_{x,r}$ and $T_r =: \iota_{0,r\#}T$.

Definition 4.2.3. Let $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ with $\text{spt}(T) \subset \Sigma$, $\partial T = Q \llbracket \Gamma \rrbracket$, and $p \in \Gamma$. Assume that $r_j \downarrow 0$ and T_{p,r_j} weakly converge to an oriented cone C with $\partial C = Q \llbracket T_p \Gamma \rrbracket$. We say that

- (a) p is a *boundary flat point* of T provided C is a flat cone,
- (b) p is a *one-sided boundary flat point* of T provided C is an open book which is non genuine,
- (c) p is a *two-sided boundary flat point* of T provided C is a two-sided boundary flat cone.

Jones' β_2 coefficients.

We define the Jones' β_2 coefficient that measures how close the support of μ is to a plane in $L^2(\mu)$ as follows.

Given a finite non-negative Radon measure $\mu \in \mathbb{R}^{m+n}$ and $k \in \{1, \dots, m+n-1\}$, we define the k -dimensional Jones' β_2 coefficient of μ as

$$\beta_{2,\mu}^k(x, r) := \inf_{\text{affine } k\text{-planes } L} \left[r^{-k-2} \int_{\mathbf{B}_r(x)} \text{dist}(y, L)^2 d\mu(y) \right]^{1/2}.$$

We will often omit μ from the notation.

4.2.2 Outline of the proofs

The results established in Chapter 3 are pivotal for the development of this Chapter, and they are presented here for completeness. Although these results were originally proven under more general assumptions, we have opted to state them in a specific form that aligns with our current application.

Theorem 4.2.4 (Uniqueness of the tangent cone). *Assume Assumption 4.1.8 and, in addition, $\Theta(T, p) = Q/2$. The tangent cone at p is an open book and is unique, i.e. it does not depend*

on the blow-up sequence. Moreover, $T_{p,r}$ converges (in flat distance, cf. [74, §7, Chapter 6]) as $r \rightarrow 0$ to the unique open book faster than r^α for some $\alpha \in (0, 1)$.

The following excess decay, whose proof is essentially contained in Chapter 3, will be repeatedly used.

Theorem 4.2.5. *Assume Assumption 4.1.8 and, in addition, $\Theta(T, p) = Q/2$. Assume that T has a flat one-sided tangent cone $Q \llbracket \pi \rrbracket$ at p . There exists $\varepsilon(Q, m, n) > 0$ such that if*

$$\max \left\{ \mathbf{E}(T, \mathbf{B}_R(p), \pi), \mathbf{A}_\Gamma, \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

then there exists $C(Q, m, n)$ and $\alpha(Q, m, n) > 0$ such that

$$\mathbf{E}(T, \mathbf{B}_r(p), \pi) \leq C(Q, m, n)r^\alpha, \text{ for every } r \in (0, R).$$

To investigate the $\mathcal{H}^{m-3}$ -rectifiability of the singular set and prove Theorem M, the first step is to employ a suitable decomposition of T (see Theorem 4.4.6), which allows us to split the current into different sheets. Each of these sheets has a boundary multiplicity smaller than that of T and does not share the same tangent cone.

Using this decomposition, the problem of proving the $\mathcal{H}^{m-3}$ -rectifiability of the set of one-sided singular points (those with density $Q/2$) is reduced, in Section 4.4.4, to establishing the $\mathcal{H}^{m-3}$ -rectifiability of the subset of one-sided *flat* singular points (which possess at least one tangent cone supported on a plane, cf. Definition 4.2.3). Relying on the result of Chapter 2, one can reduce the problem of proving rectifiability of a set to showing the rectifiability of Frostman (upper-regular) measures supported on it. Then, an application of Azzam and Tolsa's main theorem of [8] (see Theorem 4.3.1) reduces the proof of $\mathcal{H}^{m-3}$ -rectifiability to establishing a Dini-type condition for the Jones' β_2^{m-3} coefficient for Frostman measures supported on pieces of the set of one-sided flat singular points.

To analyze one-sided flat singular boundary points, we utilize the standard machinery to address regularity in this setting—namely, the study of Lipschitz approximations, center manifolds, and normal approximation (cf. Proposition 4.5.1 and Theorems 4.5.6 and 4.5.10)—adapting the approach used for $m = 2$ in [27]. As observed in [28, 29, 24] (for flat interior singular points), the “classical” estimates in [33, 34] are insufficient. The same holds in the boundary case: the estimates in [27] are not sharp enough to yield the necessary bounds. To address this, we provide finer estimates in Section 4.5.2 for quantities related to the universal frequency function (Section 4.5.2), improving on those in [27, 21, 68].

When working with the universal frequency function, we must consider the intervals of flattening, following the approach in [34, 27]. In contrast, in boundary settings considered in [21] and in the work of Nardulli and Resende [68]—where they study collapsed two-sided points—there is no need to introduce intervals of flattening, as prior results ensure an almost quadratic excess decay rate. However, for one-sided points, the excess decay rate established in Chapter 3 is certainly not almost quadratic, and there is no reason to expect such a rate in general.

The analysis of the fine properties of the universal frequency function begins by establishing the almost monotonicity of the frequency (see Theorem 4.5.19), with errors quantitatively controlled by the excess and the norms of both the second fundamental form of the boundary and the second fundamental form of the ambient manifold. In contrast to [28], by the work of Chapter 3, we already know that every boundary one-sided point has a unique tangent cone, which serves as the starting point for the regularity theory. The frequency values $\mathbf{I}(T, p)$ are then established in Theorem 4.6.5 to be always bounded below by a universal geometric constant $1 + \alpha/2 > 1$, where α is the exponent obtained from the excess decay result in 3.

Although the presence of the boundary introduces both technical and conceptual challenges, we manage to follow the approach in [29] in an analogous way. Taking into account the lower bound for the frequency values of $1 + \alpha/2 > 1$, the procedure from [29] can be implemented to

bound Jones' coefficient β_2^{m-3} (for certain Frostman measures) in terms of frequency pinching, up to an error (see Theorems 4.8.8 and 4.8.9).

The bounds for β_2^{m-3} mentioned above and established Theorems 4.8.8 and 4.8.9 are a consequence of the classification of homogeneous 2-dimensional solutions for the linear problem, proved in [27]. This classification asserts that every 2-dimensional homogeneous multivalued Dirichlet minimizer with zero boundary data on a flat boundary must be linear (see Theorem 4.3.3).

In fact, if the set of flat one-sided singular points is not sufficiently close to an $(m-3)$ -dimensional plane (i.e., β_2^{m-3} is "large"), we can construct a solution to the linear problem that depends only on *two* variables and has a frequency function at least $1 + \alpha/2 > 1$ (see Theorem 4.8.8). This contradicts the aforementioned classification, which implies that the frequency must be exactly 1 (Theorem 4.3.3). Consequently, we obtain the desired control on the Jones' β_2^{m-3} coefficient (up to an error) via the frequency pinching (Theorems 4.8.8 and 4.8.9).

Another key ingredient in obtaining the bound mentioned above in β_2^{m-3} is achieving a precise understanding of the frequency values along different center manifolds—i.e., the fine monotonicity formula for the universal frequency—and, moreover, being able to compare the frequency values at different points at the same scale. These properties have been absent in previous works that addressed boundary regularity, namely [21, 27, 68].

4.2.3 Convex barrier

The convex barrier condition in Euclidean spaces, defined in the following, ensures the applicability of Theorem M (see Theorem 4.2.9) for the whole boundary singular set.

Assumption 4.2.6 (Convex barrier condition). Let $\Omega \subset \mathbb{R}^{m+n}$ be a domain such that $\partial\Omega$ is a C^2 uniformly convex submanifold of $\mathbb{R}^{m+n}$. The set $\partial\Omega$ is a *convex barrier for T* if $\partial T = \sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$, Q_i are positive integers, and $\Gamma_i \subset \partial\Omega$ are disjoint C^2 closed oriented submanifolds of $\partial\Omega$.

The following notation for wedges will be used from now on.

Definition 4.2.7. Given an $(m - 1)$ -dimensional plane $V \subset \mathbb{R}^{m+n}$, we denote by $\mathbf{p}_V$ the orthogonal projection onto V . Given a unit vector ν normal to V and an angle $\vartheta \in (0, \pi/2)$, the *wedge with spine V , axis ν , and opening angle ϑ* , is defined as the set

$$W(V, \nu, \vartheta) := \left\{ y \in \mathbb{R}^{m+n} : |y - \mathbf{p}_V(y) - \langle y, \nu \rangle \nu| \leq (\tan \vartheta) \langle y, \nu \rangle \right\}.$$

A local version of the convex barrier condition is defined below and is actually implied by the global version defined above in Assumption 4.2.6.

Assumption 4.2.8 (Local convex barrier condition). Let T, Σ and Γ be as in Assumption 4.1.8. Assume that there exists $\nu : \Gamma \rightarrow \mathbb{S}^{n+1} \cap T\Sigma$ a map such that $\nu(q) \perp T_q\Gamma$ and $\nu(q) \in T_q\Sigma$. Then T satisfies the local convex barrier assumption if

$$\text{spt}(T) \subset \bigcap_{q \in \Gamma} (q + W(T_q\Gamma, \nu(q), \vartheta)) \cap \Sigma.$$

A much more detailed discussion of both convex barrier conditions has been explored in Section 3.4 where we prove Theorem 4.2.9, which shows that the theory in the present work can be applied to the convex barrier setting.

Theorem 4.2.9. *Let T be as in Assumption 4.2.6 (convex barrier) or Assumption 4.2.8 (local convex barrier), then every tangent cone to T at a boundary point q is an open book. Moreover, it holds:*

- if $q \in \Gamma_i$ in the first case, then $\Theta(T, q) = Q_i/2$,
- if $q \in \Gamma$ in the second case, then $\Theta(T, q) = Q/2$.

4.2.4 Structural properties

Although, in general, the regular boundary set is only dense in Γ (Theorem N), certain assumptions may allow us to derive additional information about boundary points. A question

originally posed by Almgren and answered, for $Q = 1$ in [21, Theorem 1.9 and Corollary 1.10], is: is it possible for the set of density $1/2$ points to be empty when Γ is connected?

Here, we present the counterpart of these results in the arbitrary boundary multiplicity setting $Q > 1$, with proofs provided at the end of this subsection.

Proposition 4.2.10. *Let T , Q , Σ , and Γ , be as in Assumption 4.1.8 and $\Gamma' \subset \Gamma$ be a connected component of Γ . In addition, suppose that Γ and $\text{spt}(T)$ are compactly supported in $\mathbf{B}_R(p)$. If*

$$\text{there exists } p \in \Gamma' \cap \text{Reg}_b(T) \text{ with } \Theta(T, p) > \frac{Q}{2},$$

then there is no one-sided point in Γ' and $\dim_H(\text{Sing}_b(T) \cap \Gamma') \leq m - 2$. Moreover, if $m = 2$, then $\text{Sing}_b(T) \cap \Gamma'$ consists of finitely many points.

Remark 4.2.11. In the case $\Sigma = \mathbb{R}^{m+n}$, a portion of Γ is contained within the real analytic minimal surface $\text{Reg}_i(T)$. This condition is quite rigid, and indeed, under reasonable definitions for generic C^∞ boundaries, a generic boundary data will not have any “two-sided” regular point.

Next, we establish that the connectedness of the boundary Γ guarantees only the existence of one-sided points.

Proposition 4.2.12. *Let T , Q , Σ , and Γ , be as in Assumption 4.1.8. In addition, suppose that*

$$\Gamma \text{ is connected and both } \Gamma \text{ and } \text{spt}(T) \text{ are compactly supported in } \mathbf{B}_R(p).$$

Then Γ only contains one-sided points and, in particular, $\text{Reg}_b(T)$ is contained in the set of density $\frac{Q}{2}$ points.

Both of the above theorems will follow as consequences of the following structural decomposition argument. This decomposition follows from the density of the boundary regular set (cf. Theorem N) and provides a decomposition of the area-minimizing current T into its *irreducible* components.

Theorem 4.2.13 (Irreducible components of T). *Let T , Q , Σ , and Γ , be as in Assumption 4.1.8. In addition, suppose that*

$$\Gamma \text{ and } \text{spt}(T) \text{ are compactly supported in } \mathbf{B}_R(p).$$

Let $\Gamma_1, \dots, \Gamma_{L_0}$ be the connected components of Γ . Then there exist $L \in \mathbb{N}$, $Q_j \in \mathbb{N} \setminus \{0\}$, and area minimizing integral m -currents T_j satisfying $T = \sum_{j=1}^L Q_j T_j$ where, for each $j \in \{1, \dots, L\}$, we have

$$(i) \quad \partial T_j = \sum_{i=1}^{L_0} \sigma_{ij} [\Gamma_i] \text{ with } \sigma_{ij} \in \{-1, 0, 1, \dots, Q\},$$

$$(ii) \quad T_j = [\Lambda_j] \text{ with } \Lambda_1, \dots, \Lambda_L \text{ being the connected components (accordingly oriented) of } \text{Reg}_i(T).$$

Moreover, for each $i \in \{1, \dots, L_0\}$, only one of the two following alternatives holds:

$$(iii) \quad \Gamma_i \text{ is one-sided, i.e. there is } J_i \subset \{1, \dots, L\} \text{ such that } \sigma_{ij} \geq 1 \text{ for all } j \in J_i, \sigma_{ij} = 0 \text{ for all } j \notin J_i, \text{ and } \sum_{j \in J_i} \sigma_{ij} Q_j = Q. \text{ Moreover, it holds}$$

$$\Gamma_i \cap \text{Reg}_b(T) \subset \text{Reg}_b^1(T).$$

$$(iv) \quad \Gamma_i \text{ is two-sided, i.e. there exist } p(i), n(i) \in \{1, \dots, L\} \text{ such that } \sigma_{ip(i)} = -\sigma_{in(i)} = 1 \text{ and } \sigma_{ij} = 0, \text{ for all } j \notin \{p(i), n(i)\}.$$

In this case, $Q_{p(i)} - Q_{n(i)} = Q$ and $T_{p(i)} + T_{n(i)}$ is area minimizing. Moreover, it holds

$$\Gamma_i \cap \text{Reg}_b(T) \subset \text{Reg}_b^2(T).$$

Proof of Theorem 4.2.13. We omit the proof of this theorem, as it follows [21, Theorem 2.1] mutatis mutandis, but now relying on Theorem N. □

Proof of Proposition 4.2.12. The proof follows [21, Corollary 1.10] with a minor modification that we describe now. Since Γ is connected, we have $L_0 = 1$ in Theorem 4.2.13 and, thus, the

decomposition therein consists of *exactly* two currents $T_{p(i)}$ and $T_{n(i)}$ in case Γ is two-sided (as in Theorem 4.2.13(iv)). However, in this case, $S := (Q_{p(i)} - Q_{n(i)})T_{p(i)}$ would satisfy $\partial S = Q \llbracket \Gamma \rrbracket$ and, by Theorem 4.2.13(ii), the mass of S would be strictly smaller than the mass of T which contradicts the minimality of T . Hence, the only possibility is that Γ is one-sided (as in Theorem 4.2.13(iii)). $\square$

Proof of Proposition 4.2.10. The proof is the same as in [21, Theorem 1.9]. By Theorem 4.2.13(iv), Γ' is necessarily two-sided, therefore $S := T_{p(1)} + T_{n(1)}$ is area-minimizing. Since all points of Γ' are interior points of S , by the interior regularity theory, the interior singular set of S has Hausdorff dimension at most $m - 2$ (and is discrete if $m = 2$, since Γ is compactly supported, it has to be finite). The boundary regularity is then a straightforward consequence of Definition 4.1.3 and Remark 4.1.5. $\square$

4.3 Rectifiability for the linear problem

The goal of this section is to present the rectifiability criteria based on the Jones' β_2 coefficients (Theorem 4.3.1), establish key results on the linear problem (i.e., Dirichlet minimizers) that are fundamental for the development of the rest of the paper (Section 4.3.1), and state the rectifiability of the boundary singular set of Dirichlet minimizers (Theorem 4.3.6).

The novelty in the Jones' β_2 coefficient (see Section 4.2.1 for the definition) lies in the fact that its finiteness is equivalent to the rectifiability of the measure μ , this is a celebrated result proved in [8].

Theorem 4.3.1 (Theorem 1.1 of [8]). *Let μ be a finite Borel measure in $\mathbb{R}^{m+n}$ such that $0 < \Theta^{k,*}(\mu, x) < \infty$ for μ -a.e. $x \in \mathbb{R}^{m+n}$. If*

$$\int_0^1 \beta_{2,\mu}^k(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^{m+n},$$

then μ is d -rectifiable.

Our main interest is to apply Theorem 4.3.1 for Frostman measures μ , as in [22, Theorem 1.1], supported on pieces of the boundary singular set of an area minimizing current T or a Q -valued function.

Next, we define the frequency function as introduced in [21, Chapter 4] for boundary points, see also [27] and [68]. Let $u : \Omega \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ and $\Gamma := \partial\Omega$, then define $I_{u,p}(r)$ is the value of the smoothed frequency function of u at $p \in \Gamma$ at scale $r > 0$, i.e.,

$$I_{u,p}(r) := \frac{rD_{u,p}(r)}{H_{u,p}(r)}, \quad D_{u,p}(r) := \int_{\Omega} |Du(y)|^2 \phi\left(\frac{d(y,p)}{r}\right),$$

$$\text{and } H_{u,p}(r) := - \int_{\Omega} \frac{|u(y)|^2}{d(y,p)} \phi'\left(\frac{d(y,p)}{r}\right) |\nabla d(y,p)|^2,$$

where the cut-off ϕ is chosen as before, i.e.,

$$\phi(t) := \begin{cases} 1, & t \in [0, 1/2], \\ 2 - 2t, & t \in [1/2, 1], \\ 0, & \text{otherwise,} \end{cases}$$

and the function d is a good distorted distance function with good asymptotics and such that $\nabla_y d$ is tangent to Γ , as given in [21, Lemma 4.25]. Set $I_{u,p}(0) := \lim_{r \downarrow 0} I_{u,p}(r)$ whenever the limit exists.

4.3.1 Q -valued Dir-minimizers with zero boundary data on a flat boundary

We develop a set of tools for the case where the Dirichlet minimizer has zero boundary data on a flat boundary. These results will be utilized throughout the paper. In particular, we establish the following counterpart (to the boundary setting) of [23, Lemma 5.4].

Lemma 4.3.2. *Let u be a Dir minimizing function in B_s^+ with zero boundary data on $B_s \cap \{x \in \mathbb{R}^m : x_m = 0\}$ and $B_r^+(p) \subset B_s^+$, $p \in B_s^+$, $s > r$. If $u|_{B_r^+(p)}$ is a function of only the*

two variables in $\text{span}(v, x_m)$ with $v \in \mathbb{R}^{m-1}$, then u is a function of only $\text{span}(v, x_m)$ in the whole ball B_s^+ .

Proof. We denote by $\text{Sing}_i(u)$ the interior singular set of u . By interior regularity theory, see [30], we know that $\dim_H(\text{Sing}_i(u)) \leq m - 2$. Thus, $B_s^+ \setminus \text{Sing}_i(u)$ is a path-wise connected set. Let $V = \text{span}(v, x_m)$. By assumption, we deduce that $\nabla_{V^\perp} u(x) = 0$ for any $x \in B_r^+(p)$. Since u is given by Q harmonic sheets on $B_s^+ \setminus \text{Sing}_i(u)$, the gradient $\nabla_{V^\perp} u(x)$ vanishes for any $x \in B_s^+ \setminus \text{Sing}_i(u)$. Since $u \in W^{1,2}(B_s^+)$, we easily obtain that $\nabla_{V^\perp} u \equiv 0$ in B_s^+ which ensures that u is a Q -valued function which only depends on the variables in V . $\square$

We will need to use that, in the 2-dimensional case, the only possible frequency value for the linear problem is 1. We state this result in a slightly different form than [27, Theorem 5.9], assuming homogeneity in our formulation.

Theorem 4.3.3 (Theorem 5.9 of [27]). *Let u be a nontrivial homogeneous Dirichlet minimizing function on $B_s^+ \subset \mathbb{R}^2$ with zero boundary value on $[-1, 1] \times \{0\}$ and assume that $\boldsymbol{\eta} \circ u \equiv 0$. Then $I_u(r) \equiv 1$ and u is linear, i.e., $u = \sum_i Q_i \llbracket L_i \rrbracket$ where each $L_i : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ is linear, $L_i|_{\{x_2=0\}} \equiv 0$, $\sum_i Q_i = Q$, and $L_i \neq L_j$ on $\mathbb{R}^2 \setminus \{x_2 = 0\}$.*

We use Theorem 4.3.3 to show a crucial lower bound for the derivatives of Dirichlet minimizing functions with zero boundary data on a flat boundary that will play a crucial role within our program to prove rectifiability.

Lemma 4.3.4. *Let u be a nontrivial Dir minimizing function in B_s^+ . Let $\alpha > 0$ and $\Lambda > 0$. Assume that*

$$I_u(s) < \Lambda, \quad \int_{B_s^+} |u|^2 = 1, \quad \text{and} \quad I_u(0) \geq 1 + \alpha/2.$$

If $\{v_1, \dots, v_{m-1}\}$ is an orthonormal basis of $\mathbb{R}^{m-1}$ and $s > r > 0$, we obtain that

$$\sum_{i=1}^{m-2} \int_{B_s^+ \setminus \overline{B_r^+}} |\partial_{v_i} u|^2 \geq c(\Lambda) > 0.$$

Proof. We argue by contradiction, take a contradiction sequence u_k as in the statement and assume that

$$\sum_{i=1}^{m-2} \int_{B_s^+ \setminus \overline{B_r^+}} |\partial_{v_i} u_k|^2 \leq 1/k. \quad (4.3.1)$$

Up to subsequences, $\{u_k\}$ converge in $W^{1,2}$ to u_0 such that

$$\int_{B_s^+} |u_0|^2 = 1, \int_{B_s^+} |Du_0|^2 \leq C\Lambda, \text{ and } I_{u_0}(0) \geq 1 + \alpha/2.$$

Additionally, from (4.3.1), we also have

$$\sum_{i=1}^{m-2} \int_{B_s^+ \setminus \overline{B_r^+}} |\partial_{v_i} u_0|^2 = 0,$$

which, in turn, implies that u_0 is a function of only $\text{span}(v_{m-1}, x_m)$ in $B_s^+ \setminus \overline{B_r^+}$. Noting that the proof of Lemma 4.3.2 goes through with $B_s^+ \setminus \overline{B_r^+}$ in place of $B_r^+(p)$, we obtain that u_0 must be a function of only two variables, namely $\text{span}(v_{m-1}, x_m)$, on B_s^+ .

To conclude, we take h as a blow-up of u_0 at 0. The function h is a homogeneous Dirichlet minimizing function in B_s^+ that is still a function of only the two variables in $\text{span}(v_{m-1}, x_m)$. Thus, h is a homogeneous Dirichlet minimizer on $\text{span}(v_{m-1}, x_m)$ with zero boundary value on $\{x_m = 0\}$. By Theorem 4.3.3, $I_h(0) = 1$. However, since $I_{u_0}(0) \geq 1 + \alpha/2$, we obtain $I_h(0) \geq 1 + \alpha/2$, which gives a contradiction. $\square$

We state a lemma that will be useful in some later arguments.

Lemma 4.3.5. *Let u be a nontrivial Dirichlet minimizer in B_1^+ with $u \equiv Q \llbracket 0 \rrbracket$ on $B_1 \cap \{x_m = 0\}$. Suppose further that $I_u(s) = I_u(t)$ for $0 < s < t < 1$. Then, the function u is $I_u(s)$ -homogeneous in B_1^+ .*

Proof. Since I_u is monotone, it must be constant in $[s, t]$. As in [30, Corollary 3.16], we then have, for almost every $r \in [s, t]$ and almost every y , that

$$|y| = u_i(y) = \lambda_r \partial_\nu u_i(y),$$

with λ_r being a constant. We recall that the classical frequency I (that is, given the choice $\phi \equiv 1$) is monotone with a flat boundary as in the situation in Chapter 3. The above equality implies $I(s) = I(t)$, but as in [30, Corollary 3.16], we obtain that

$$I(r) = \frac{r \int_{\partial B_r} \sum_i \langle \partial_\nu u_i, u_i \rangle}{\int_{\partial B_r} \sum_i |u_i|^2} = r \lambda_r^{-1}.$$

Thus, λ_r must be the same constant for almost every r in $[s, t]$. Set $I_0 := I(s)$. This in turn leads to $I_0 u_i(y) = r \partial_\nu u_i(y)$. Defining $\sigma_y := \{ry : s \leq y \leq t\}$, we get

$$(u_i|_{\sigma_y})'(r) = \frac{I_0}{r} u_i|_{\sigma_y}(r).$$

This implies that $u_i|_{\sigma_y}$ is I_0 -homogeneous in $[s, t]$. By unique continuation in the interior regular set of u (where it agrees with a classical harmonic function), the function u must be I_0 -homogeneous in B_1^+ . $\square$

4.3.2 Rectifiability for the linear problem

We now state the boundary rectifiability result for Dirichlet minimizers. The counterpart of this result for the *interior* singular set is stated in [23] and in [56].

Theorem 4.3.6. *Let $u : \Omega \subset \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ be a Dirichlet minimizer, $u|_{\partial\Omega} = Q \llbracket g \rrbracket$, $\Omega \in C^2$, and $g \in C^2$. Then, the boundary singular set of u is $\mathcal{H}^{m-3}$ -rectifiable.*

The proof of this result follows by repeating the arguments used to establish the rectifiability of the nonlinear problem (Theorem M), but with many of the error terms absent. Since this proof represents a simpler case of the analysis we carry out later and proceeds in an entirely analogous manner, we omit its details.

4.4 Reduction of main theorems to rectifiability of one-sided flat singular points

4.4.1 Excess decay and decomposition for two-sided points with trapped density

For two-sided points, we establish an excess decay at an almost-quadratic rate and, as a consequence, the uniqueness of tangent cones. This result holds under a density condition (see (4.4.1)) that is satisfied in a relatively *open* neighborhood in Γ , which is, a priori, more general than the condition assumed in [21, 68]. This refinement strengthens previously known results in [68, Theorems 4.3 and 4.5] and, for $Q = 1$, in [21, Theorem 6.3].

Building on this, we derive a decomposition of the current at two-sided points under the same density assumption; see Proposition 4.4.2.

Proposition 4.4.1 (Excess decay and uniqueness of tangent cones at two-sided points). *Let T, Σ , and Γ be as in Assumption 4.1.8, $Q^* \in \mathbb{N} \setminus \{0\}$, and $p_0 \in \Gamma$. For any $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon, Q, Q^*, m, n, \bar{n}) > 0$ with the following property. If, for some $r_0 > 0$, the following holds*

$$\left| \frac{\|T\|(\mathbf{B}_\rho(q))}{\omega_m \rho^m} - \left(\frac{Q}{2} + Q^* \right) \right| < \delta, \quad \forall q \in \Gamma \cap \mathbf{B}_{r_0}(p_0), \forall \rho \in (0, r_0), \quad (4.4.1)$$

then, for any $q \in \Gamma \cap \mathbf{B}_{r_0}(p_0)$ and any $\rho \in (0, r_0/2)$, the following is true:

(i) *there exists a m -dimensional plane $\pi(q)$ with $T_q \Gamma \subset \pi(q) \subset T_q \Sigma$ and*

$$\mathbf{E}^b(T, \mathbf{B}_\rho(q), \pi(q)) = \mathbf{E}^b(T, \mathbf{B}_\rho(q)) \leq C \left(\frac{\rho}{r_0} \right)^{2-2\varepsilon} \mathbf{E}^b(T, \mathbf{B}_{r_0}(p_0)) + C \rho^{2-2\varepsilon} r^{2\varepsilon} \mathbf{A}^2,$$

- (ii) $T_q := (Q + Q^*) \llbracket \pi(q)^+ \rrbracket + Q^* \llbracket \pi(q)^- \rrbracket$ is the unique tangent cone to T at q where $\pi(q)^+$ and $\pi(q)^-$ are the two connected components of $\pi(q) \setminus T_q \Gamma$ oriented such that $\partial \llbracket \pi(q)^+ \rrbracket = -\partial \llbracket \pi(q)^- \rrbracket = \llbracket T_q \Gamma \rrbracket$.

Proof. To prove [68, Lemma 4.2] the only property that is used is (4.4.1), the full strength of the definition of collapsed points (i.e., lower bound for the density and existence of a flat tangent cone) is not used. We state here the consequence of [68, Lemma 4.2]: for every $\varepsilon > 0$, there exist $\varepsilon_0 = \varepsilon_0(\varepsilon, Q, Q^*, m, n) > 0$ and $M_0 = M_0(\varepsilon, Q, Q^*, m, n) > 0$ with the following property. If we define the function θ as $\theta(\sigma) := \max\{\mathbf{E}^b(T, \mathbf{B}_\sigma(q)), M_0 \mathbf{A}^2 \sigma^2\}$ for $\sigma \in (0, r_0)$, we obtain

$$\mathbf{A}^2 \sigma^2 + \mathbf{E}^b(T, \mathbf{B}_{4\sigma}(q)) < \varepsilon_0 \quad \Rightarrow \quad \theta(\sigma) \leq \max\{2^{-4+4\varepsilon} \theta(4\sigma), 2^{-2+2\varepsilon} \theta(2\sigma)\}, \quad (4.4.2)$$

where $\mathbf{A} := \max\{\mathbf{A}_\Gamma^2, \mathbf{A}_\Sigma^2\}$.

We will show that (4.4.1) implies the smallness condition on the left-hand side of (4.4.2). Assume by contradiction that there is a sequence of currents T'_k such that (4.4.1) holds for T'_k with $\delta = 1/k$, and for some $\sigma_k \in (0, r_0)$, and

$$\mathbf{A}_k^2 \sigma_k^2 + \mathbf{E}^b(T'_k, \mathbf{B}_{4\sigma_k}(q_k)) \geq \varepsilon_0 \text{ for some } \varepsilon_0 > 0 \text{ and } q_k \in \Gamma_k \cap \mathbf{B}_{r_0}(p_0).$$

Define $T_k = (T'_k)_{0,4\sigma_k}$ and recall that $\mathbf{E}^b(T_k, \mathbf{B}_1(q_k)) = \mathbf{E}^b(T'_k, \mathbf{B}_{4\sigma_k}(q_k)) \geq \varepsilon_0$. Moreover, by (4.4.1) with $\rho = s\sigma_k$ and rescaling, we obtain

$$\left| \frac{\|T_k\|(\mathbf{B}_s(q))}{\omega_m s^m} - \left(\frac{Q}{2} + Q^*\right) \right| < \frac{1}{k}, \quad \forall q \in \Gamma \cap \mathbf{B}_{r_0}(p_0), \forall s \in \left(0, \frac{1}{4}\right), \quad (4.4.3)$$

We now take a limit of the sequence T_k , the boundaries Γ_k , and the ambient manifolds Σ_k . The currents will converge to a current T_∞ and the boundaries Γ_k converge to a flat boundary L_∞ . Since we have (4.4.1), the mass ratio of T_∞ will be constantly equal to $Q/2 + Q^*$ at every point of L_∞ and in every radius $r < \rho$. Thus, it must be a two-sided flat cone. Up to a

subsequence, $q_k \rightarrow q_\infty$ and then $\mathbf{E}^b(T_\infty, \mathbf{B}_{4r_0}(q_\infty)) \geq \varepsilon_0$, which is a contradiction since T_∞ is a flat two-sided cone, i.e., supported in a m -plane.

Hence, for any $\varepsilon > 0$, there is a choice of $\delta = \delta(\varepsilon)$ such that the left-hand side of (4.4.2) holds. Thus, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, Q, Q^*, m, n, \bar{n}) > 0$ such that (4.4.1) imply

$$\theta(\sigma) \leq \max\{2^{-4+4\varepsilon}\theta(4\sigma), 2^{-2+2\varepsilon}\theta(2\sigma)\}.$$

From this, the very same proof of [68, Theorem 4.3] provides (i) and (ii). $\square$

We establish a decomposition for T at two-sided points as a direct consequence of the uniqueness of tangent cones at such points (Proposition 4.4.1). Although this result *is not* used elsewhere in this article, it is of independent interest. Notably, the significance of this decomposition lies in its elementary nature. Indeed, it does not rely on the full regularity theory at two-sided points with density close to $Q/2 + Q^*$ (i.e., constructing center manifolds, defining frequency functions, etc., see Proposition 4.4.3); rather, it follows directly from an application of Proposition 4.4.1.

Proposition 4.4.2 (Decomposition at two-sided points). *Let T, Σ , and Γ be as in Assumption 4.1.8, $Q^* \in \mathbb{N} \setminus \{0\}$, and $p_0 \in \Gamma$. There exists $\delta > 0$ with the following property. If, for some $r_0 \in (0, r/4)$, the following holds*

$$\left| \frac{\|T\|(\mathbf{B}_\rho(q))}{\omega_m \rho^m} - \left(\frac{Q}{2} + Q^* \right) \right| < \delta, \quad \forall q \in \Gamma \cap \mathbf{B}_{r_0}(p_0), \forall \rho \in (0, r_0), \quad (4.4.4)$$

then $T \llcorner \mathbf{B}_{r_0}(p_0) = T_1 + T_2$ where $\partial T_1 = (Q + Q^) \llbracket \Gamma \rrbracket$, $\partial T_2 = -Q^* \llbracket \Gamma \rrbracket$ with $\text{spt}(T_1) \cap \text{spt}(T_2) = \Gamma$, and every $q \in \Gamma \cap \mathbf{B}_{r_0}(p_0)$ is a one-sided flat boundary point for T_1 and T_2 .*

Proof. For $\theta_1, \theta_2 \in \mathbb{R}, \theta_2 > \theta_1$, we define the wedge region:

$$W_{\theta_1}^{\theta_2} := \left\{ x \in \mathbf{B}_{r_0}(p_0) : \left\langle \frac{x - \mathbf{p}_\Gamma(x)}{|x - \mathbf{p}_\Gamma(x)|}, N_{\mathbf{p}_\Gamma(x)} \Gamma \right\rangle \in [\theta_1, \theta_2] \right\},$$

where, for any $q \in \Gamma$, $N_q \Gamma := (T_p \Gamma)^\perp$. Since Γ is $C^{1,1}$, we find that $\mathbf{p}_\Gamma$ is well-defined and $\text{dist}(\cdot, \Gamma)$ is differentiable provided that r_0 is small enough. We will show that

$$\left(W_{-\pi/4}^{\pi/4} \cup W_{5\pi/4}^{3\pi/4}\right) \cap \text{spt}(T) = \emptyset, \quad (4.4.5)$$

which then allow us to conclude the proof since this means that

$$\text{spt}(T) \cap \mathbf{B}_{r_0}(p_0) \subset W_{\pi/4}^{3\pi/4} \cup W_{5\pi/4}^{7\pi/4}$$

and, hence, we can write $T = T_1 + T_2$ with

$$\text{spt}(T_1) \setminus \Gamma \subset W_{\pi/4}^{3\pi/4} \text{ and } \text{spt}(T_2) \setminus \Gamma \subset W_{5\pi/4}^{7\pi/4}.$$

Additionally, we obtain that $\text{spt}(\partial T_1) \subset \Gamma$ and $\text{spt}(\partial T_2) \subset \Gamma$ and thus $\partial T_1 = (Q + Q^*) \llbracket \Gamma \rrbracket$ and $\partial T_2 = -Q^* \llbracket \Gamma \rrbracket$.

To prove (4.4.5), we proceed by contradiction as follows. Assume the existence of

$$x \in \left(W_{-\pi/4}^{\pi/4} \cap W_{5\pi/4}^{3\pi/4}\right) \cap \text{spt}(T).$$

We set $p := \mathbf{p}_\Gamma(x)$, let $\rho := |x - p|$ and denote $\pi(p)$ the unique tangent cone to T at p (given by Proposition 4.4.1). By the height bound in [68, Corollary 4.7], denoting the projection onto $\pi^\perp$ by $\mathbf{p}_\pi^\perp$ for any m -plane π , we have that

$$\left|\mathbf{p}_{\pi(q)}^\perp(x) - p\right| \leq C(r_0^{-1} \mathbf{E}^b(T, \mathbf{B}_{4r_0}(p)))^{\frac{1}{2}} + \mathbf{A})^{\frac{1}{2}} \rho^{3/2}.$$

Since $x \in W_{-\pi/4}^{\pi/4} \cap W_{5\pi/4}^{3\pi/4}$, we know that $|x - \mathbf{p}_{\pi(p)}(x)| \leq C|x - \mathbf{p}_{\pi(p)}^\perp(x)|$. This together with the last displayed inequality imply that

$$\rho = |x - p| \leq C|x - \mathbf{p}_{\pi(p)}(x)| \leq C(r_0^{-1} \mathbf{E}^b(T, \mathbf{B}_{4r_0}(p)))^{\frac{1}{2}} + \mathbf{A})^{\frac{1}{2}} \rho^{3/2},$$

which gives us a contradiction. $\square$

4.4.2 Regularity of two-sided points with trapped density

The following theorem builds on the theory developed in [68, 21], where the authors study the notion of collapsed points and establish that such points are always regular for T . However, we provide a refined version of this result. Specifically, the definition of collapsed points in [68] requires two conditions: (1) the existence of a flat tangent cone at the given point, say p_0 , and (2) p_0 to attain the minimum density in some relatively *open* neighborhood of Γ , with this density being $\frac{Q}{2} + Q^*$ for integers $Q, Q^* \in \mathbb{N} \setminus \{0\}$. Instead of assuming both (1) and (2), we *only* require that the density is δ -close to $\frac{Q}{2} + Q^*$ to establish regularity.

Proposition 4.4.3 (Regularity of two-sided points with density close to $(\frac{Q}{2} + Q^*)$). *Let T, Σ , and Γ be as in Assumption 4.1.8, $Q^* \in \mathbb{N} \setminus \{0\}$, and $p_0 \in \Gamma$. For any $\varepsilon > 0$, there exists a constant $\delta = \delta(\varepsilon) > 0$ with the following property. If, for some $r_0 > 0$, the following holds*

$$\left| \frac{\|T\|(\mathbf{B}_\rho(q))}{\omega_m \rho^m} - \left(\frac{Q}{2} + Q^* \right) \right| < \delta, \quad \forall q \in \Gamma \cap \mathbf{B}_{r_0}(p_0), \forall \rho \in (0, r_0),$$

then p_0 must be a two-sided boundary regular point for T .

Proof. Appealing to Proposition 4.4.1, we can readily obtain that

$$\Theta^m(T, p_0) = \min \left\{ \Theta^m(T, q) : q \in \mathbf{B}_{r_p}(p) \cap \Gamma \cap \mathcal{P}_{m-1}^* \right\} > \frac{Q}{2},$$

where we define $\mathcal{P}_{m-1}^*$ to be only the points in the top $(m-1)$ -stratum of Γ with respect to T , i.e.,

$$\mathcal{P}_{m-1}^* = \mathcal{P}_{m-1}(T, \Gamma) \setminus \bigcup_{k=1}^{m-2} \mathcal{P}_k(T, \Gamma), \quad (4.4.6)$$

we refer the reader to [68, Section 3]. Since p_0 is two-sided and belongs to $\mathcal{P}_{m-1}^*$, by [68, Lemma 2.12], we see that $T_{p_0,r}$ blows up to a two-sided boundary flat cone. Hence,

$$\Theta^m(T, p_0) = \frac{Q}{2} + Q^*, \text{ for some } Q^* \in \mathbb{N} \setminus \{0\}.$$

We now prove that p_0 is a collapsed point as described above, see also [68, Definition 2.17]. Given any $x \in \Gamma \cap \mathbf{B}_{r_p}(p)$, we have $\Theta^m(T, x) \geq \Theta^m(T, p_0)$ if $x \in \mathcal{P}_{m-1}^*$, since p_0 is a minimum density point in $\mathcal{P}_{m-1}^*$. If $x \notin \mathcal{P}_{m-1}^*$, we use [80, Theorem 3.2] (see also [68, Section 3]) to obtain that $\Gamma \cap \mathbf{B}_{r_{p_0}}(p_0)$ and $\mathcal{P}_{m-1}^* \cap \mathbf{B}_{r_{p_0}}(p_0)$ are $\mathcal{H}^{m-1}$ -equivalent, then we can take $x_k \in \mathcal{P}_{m-1}^* \cap \mathbf{B}_{r_{p_0}}(p_0)$ to converge to x . Next, using the upper semicontinuity of the density restricted to the boundary ([68, Proposition 1.8]), it holds that

$$\Theta^m(T, x) \geq \limsup_{k \rightarrow +\infty} \Theta^m(T, x_k) \geq \Theta^m(T, p).$$

Therefore, p_0 is a collapsed boundary point of T . We can now apply [68, Theorem 2.21] to see that p_0 is a regular boundary point for T and conclude the proof. Note that [68, Theorem 2.21] is stated for $\Sigma = \mathbb{R}^{m+n}$, one can see (cf. [21]) that the proof works for any T minimizing area in an arbitrary $C^{3,\kappa}$ -submanifold Σ of $\mathbb{R}^{m+n}$. $\square$

4.4.3 Decomposition at one-sided points

In this section, we first develop the essential machinery for one-sided points, which will enable us to carry out our main reduction argument. In Section 4.4.4, we then apply these tools to reduce the proof of the main theorems to the study of one-sided flat points.

Decomposition theorem

We present the following result, which ensures that the density is forced to be equal to $Q/2$, provided that it is sufficiently close to this value (with respect to a uniform constant). This result is established in Chapter 3.

Theorem 4.4.4 (Density jump). *Under Assumption 4.1.8, we have $\Theta^m(T, p) \geq Q/2$. The equality case holds if and only if the tangent cone to T at p is an open book and unique. Moreover, there exists a constant $\varepsilon = \varepsilon(m, n, Q) > 0$ such that if $\Theta^m(T, p) > Q/2$ then $\Theta^m(T, p) > Q/2 + \varepsilon$.*

Remark 4.4.5. Notice that this result, together with the upper semicontinuity of the density restricted to Γ ([68, Proposition 1.8]) implies that the set of one-sided boundary points is a relatively open set in Γ .

The following decomposition result is stated in [27] for 2-dimensional area minimizing integral currents and generalized in Chapter 3 to arbitrary dimensions.

Theorem 4.4.6 (Decomposition at one-sided points theorem). *Let T be an area minimizing current as in Assumption 4.1.8, assume that $\Theta(T, 0) = Q/2$, and let $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ be the unique tangent cone to T at 0 (where the representation of C is such that the half-planes are distinct). Then there exists $\rho > 0$ and area minimizing currents $T_1, T_2, \dots, T_N$ in $\mathbf{B}_\rho$ such that*

$$T \llcorner \mathbf{B}_\rho = \sum_{i=1}^N T_i, \text{ with } \partial T_i \llcorner \mathbf{B}_\rho = Q_i \llbracket \Gamma \rrbracket,$$

and where the supports of T_i only intersect at Γ . Moreover, the (unique) tangent cone at 0 of T_i is $Q_i \llbracket H_i \rrbracket$.

4.4.4 Main reduction argument and proof of the main results

We are now prepared to reduce the main results to establish the rectifiability of the set of singular *flat* one-sided points. For the readers' convenience, we restate the main results here.

Theorem (Theorem M). *Under Assumption 4.1.8, we have that $\text{Sing}_b^1(T)$ is $\mathcal{H}^{m-3}$ -rectifiable.*

This theorem will be derived as a corollary of the following.

Proposition 4.4.7 (Rectifiability of the set of one-sided flat singular points). *Under Assumption 4.1.8, we have*

$$\text{Sing}_b^1(T) \cap \{q \in \Gamma : q \text{ is flat}\} \text{ is } \mathcal{H}^{m-3}\text{-rectifiable.}$$

The proof of Proposition 4.4.7 will be the focus of the remaining sections of this article. We now demonstrate how Theorem M follows from Proposition 4.4.7.

Proof of Theorem M. We argue by induction. The case $Q = 1$ is given by Allard's boundary regularity result, see [3]. Indeed, in this case, we do not have any singular boundary point. Assume now that the result holds for $Q' \geq 1$ and take $Q > Q'$. We define

$$E_i := \text{Sing}_b(T) \cap \{q \in \Gamma : \Theta^m(T, q) = Q/2 \text{ and } T_q \text{ has } i \text{ sheets}\},$$

where T_q is the tangent cone to T at q . We can cover each $E_i, i > 2$, with a countable family of open balls whose centers belong to E_i , namely $\{B_j^i\}_{j \in \mathbb{N}}$, each of which is given by Theorem 4.4.6. Indeed, since $i > 2$, any $q \in E_i$ is a one-sided singular point which is not flat and, thus, we can apply Theorem 4.4.6. Notice that for every $q \in \Gamma \cap B_j^i$, we have $\Theta^m(T, q) = Q/2$. From this covering, we can write T locally in each of these balls as

$$T \llcorner B_j^i = T_1^{j,i} + T_2^{j,i}, \quad j \in \mathbb{N},$$

where $T_1^{j,i}$ and $T_2^{j,i}$ are area minimizing in B_j^i , $Q_1^{j,i}, Q_2^{j,i} > 0$ are two integers, $\partial T_1^{j,i} = Q_1^{j,i} \llbracket \Gamma \cap B_j^i \rrbracket$, $\partial T_2^{j,i} = Q_2^{j,i} \llbracket \Gamma \cap B_j^i \rrbracket$, and $Q_1^{j,i} + Q_2^{j,i} = Q$. Now, we can apply our inductive hypothesis to each $T_1^{j,i}$ and $T_2^{j,i}$ with $i > 2$ to get the $\mathcal{H}^{m-3}$ -rectifiability of the corresponding set of singular one-sided points. Given that

$$E_i = \bigcup_{j \geq 1} \left(\text{Sing}_b(T_1^{j,i}) \cup \text{Sing}_b(T_2^{j,i}) \right) \cap B_j^i, \text{ and } E = \bigcup_{i=1}^Q E_i,$$

to obtain the rectifiability of E , we are only left to prove rectifiability of E_1 which is exactly what we claim in Proposition 4.4.7. Thus, we conclude the proof. $\square$

As a consequence of Theorem M, we can establish an Allard-type regularity result, provided there exists a relatively *open* neighborhood of Γ of *flat* one-sided points. Furthermore, given the counterexample in Theorem 4.1.9, we know that this assumption cannot be eliminated.

Additionally, we recall that for $m = 2$, Allard's theorem holds for any Q , meaning that one-sided points are automatically regular, as shown in [27].

For the readers' convenience, we restate Theorem L here.

Theorem (Theorem L). *Under Assumption 4.1.8, assume that $q \in \Gamma$ is a one-sided flat point of T and there is a neighborhood of q in Γ containing only one-sided flat points, then q is a regular point.*

Proof. Let q be a one-sided flat point and $r > 0$ such that $\mathbf{B}_r(q) \cap \Gamma$ consists of one-sided flat points. By Theorem M, there is a one-sided regular point $x \in \mathbf{B}_r(q) \cap \Gamma$. Using the fact that x is a flat and regular point, there exist $\rho > 0$ and a *single* smooth manifold M such that $T \llcorner \mathbf{B}_\rho(x) = Q \llbracket M \rrbracket$ (cf. Definition 4.1.3).

Next, we extend M to $\tilde{M}$, a connected component of $\text{spt}(T) \setminus (\Gamma \cup \text{Sing}_i(T))$ outside $\mathbf{B}_\rho(x)$ where $\text{Sing}_i(T)$ denotes the interior singular set of T . Since $\dim_H(\text{Sing}_i(T)) \leq m - 2$, we have that $\text{spt}(\partial \llbracket \tilde{M} \rrbracket) \subset \Gamma$. By the Constancy Lemma, $\partial \llbracket \tilde{M} \rrbracket = \llbracket \Gamma \cap \mathbf{B}_r(q) \rrbracket$.

We will show that T agrees with $Q \llbracket \tilde{M} \rrbracket$ on a neighborhood of q . It suffices to show that

$$\mathbf{B}_r(q) \cap \text{spt}(T - Q \llbracket \tilde{M} \rrbracket) = \emptyset.$$

Assume by contradiction that y belongs to such a set. Since $T - Q \llbracket \tilde{M} \rrbracket$ is area minimizing, by the monotonicity formula, we have $\Theta^m(T - Q \llbracket \tilde{M} \rrbracket, y) \geq 1$. Additionally, $\Theta^m(T, y) = \Theta^m(Q \llbracket M \rrbracket, y) = Q$, which follows by construction $y \in \text{spt}(T) \cap \tilde{M}$ and the Hausdorff dimensional bound for the interior singular set. These two facts are in contradiction, and hence we conclude the proof. $\square$

We employ a reduction argument to demonstrate that it suffices to prove Theorem N by assuming that every boundary point is a one-sided flat point for T within a small ball, and then apply Proposition 4.4.7. For the readers' convenience, we restate it here.

Theorem (Theorem N). *Under Assumption 4.1.8, we have that $\text{Reg}_b(T)$ is an open and dense subset of $\Gamma \cap \mathbf{B}_R(p)$.*

Proof of Theorem N. Assume that there is a singular boundary point x for T and $r > 0$ such that $\mathbf{B}_r(x) \cap \Gamma \subset \text{Sing}_b(T)$. We choose $p_0 \in \mathbf{B}_r(x) \cap \Gamma$ to have minimum density in the $(m-1)$ -stratum (this is possible due to [68, Lemma 2.12]), i.e.,

$$\Theta^m(T, p_0) = \min \left\{ \Theta^m(T, q) : q \in \mathbf{B}_r(x) \cap \Gamma \cap \mathcal{P}_{m-1}^* \right\}, \quad (4.4.7)$$

where $\mathcal{P}_{m-1}^*$ is defined as above in (4.4.6).

We divide into two cases. If $\Theta^m(T, p_0) = Q/2 + Q^*$, for some $Q^* \in \mathbb{N} \setminus \{0\}$, then there exists $r_1 > 0$ such that

$$\Theta^m(T, q) \geq Q/2 + Q^* \text{ for any } q \in \mathbf{B}_{r_1}(p_0) \cap \Gamma \subset \mathbf{B}_r(x). \quad (4.4.8)$$

In fact, if not, there is $\{x_k\}_k \subset \mathbf{B}_{1/k}(p_0) \cap \Gamma \subset \mathbf{B}_r(x)$ such that $\Theta^m(T, x_k) < Q/2 + Q^*$ and $\Theta^m(T, x_k) \rightarrow Q/2 + Q^*$. Recall that $\mathcal{P}_{m-1}^*$ is $\mathcal{H}^{m-1}$ -equivalent to Γ , and take a sequence $\{y_j^k\}_j \subset \mathcal{P}_{m-1}^* \cap \mathbf{B}_r(x)$ such that $y_j^k \rightarrow x_k$ as j grows. Therefore, we have

$$Q/2 + Q^* > \Theta^m(T, x_k) \geq \limsup_j \Theta^m(T, y_j^k) \geq Q/2 + Q^*,$$

where the last inequality follows from $y_j^k \in \mathcal{P}_{m-1}^* \cap \mathbf{B}_r(x)$ and that p_0 is a minimum density point within this set, cf. (4.4.7). The last displayed inequality provides the contradiction and concludes the proof of (4.4.8).

By the upper semicontinuity of the density restricted to Γ and (4.4.8), given $\delta > 0$, there is an open neighborhood U of p_0 such that $\Theta^m(T, q)$ is δ -close to $\Theta^m(T, p_0) = Q/2 + Q^*$ for

any $q \in U \cap \Gamma$ and thus Proposition 4.4.3 implies that p_0 is regular, which contradicts the fact that $\mathbf{B}_r(x) \cap \Gamma$ only has singular points.

Next, assume $\Theta^m(T, p_0) = Q/2$. Recall that, by upper semicontinuity of the density ([68, Proposition 1.8]) and the density jump (Theorem 4.4.4), we obtain that the set of one-sided boundary points is relatively open in Γ and, therefore, there exists $r_0 > 0$ such that $\mathbf{B}_{r_0}(p_0) \subset \mathbf{B}_r(x)$ and

$$\Theta^m(T, q) = \frac{Q}{2}, \text{ for every } q \in \mathbf{B}_{r_0}(p_0) \cap \Gamma \subset \text{Sing}_b(T).$$

Now, by Theorem M, the Hausdorff dimension of $\text{Sing}_b(T) \cap \{q \in \Gamma : \Theta^m(T, q) = Q/2\}$ is capped at $m - 3$ which then gives us a contradiction since $\mathbf{B}_{r_0}(p_0) \cap \Gamma$ is supposed to only have flat singular points of T . Hence, we conclude the proof of Theorem N. $\square$

4.5 Universal frequency function I at one-sided flat points

4.5.1 Center manifold and normal approximation at one-sided flat points

In this subsection, we construct the center manifold (Theorem 4.5.6) and the normal approximation (Theorem 4.5.10) around a flat one-sided singular point of an area-minimizing current T . These constructions are well established and have previously been formulated in different contexts.

Next, we introduce the intervals of flattening (Section 4.5.2), which identify the scales at which T decays almost quadratically toward its tangent cone. Using these intervals, we define the universal frequency function (Section 4.5.2), a fundamental tool for analyzing regularity in this setting. Although most of the frequency function estimates in this section are well

known in the two-dimensional case—see [27]—certain refinements are necessary to integrate them into the framework used to address the rectifiability problem.

We begin by establishing the existence of a Lipschitz approximation for an area-minimizing current in a neighborhood of a one-sided flat singular point.

We first recall and introduce the following notation. The function ϕ is defined on $\mathbf{p}_{\pi_0}(\Gamma)$ and it satisfies $\text{graph}(\phi) \cap \mathbf{B}_2 = \Gamma \cap \mathbf{B}_2$. We define $\mathbf{A} := \max\{\mathbf{A}_\Sigma, \mathbf{A}_\Gamma\}$ where $\mathbf{A}_\Gamma$ and $\mathbf{A}_\Sigma$ are the C^0 -norms of the second fundamental forms of Γ and Σ .

Proposition 4.5.1. *Let T, Σ , and Γ be as in Assumption 4.1.8 with $p = 0$ and $R = 2$, assume in addition that*

- $T_0\Gamma = \mathbb{R}^{m-1} \times \{0\} \subset \pi_0 \subset T_0\Sigma$,
- $\mathbf{p}_{\pi_0}(\Gamma)$ splits $\mathbf{B}_2$ into two disjoint connected open sets, which we denote by $\mathbf{B}_2^+$ and $\mathbf{B}_2^-$,
- $(\mathbf{p}_{\pi_0})_\# T \llcorner \mathbf{C}_2 = Q \llbracket \mathbf{B}_2^+ \rrbracket$.

Then there are geometric constants $\gamma_{la} > 0$, $\varepsilon_{la} > 0$, and $C > 0$ with the following properties. If $E := \mathbf{E}(T, \mathbf{C}_2, \pi_0) < \varepsilon_{la}$, there exist a set $K \subset \mathbf{B}_{3/2}^+$ and a function f defined on $\mathbf{B}_{3/2}^+$ taking values in $\mathcal{A}_Q(\mathbb{R}^n)$ such that

$$f|_{\partial \mathbf{B}_2^+} = Q \llbracket \phi \rrbracket \text{ and } \text{spt}(\mathbf{G}_f) \subset \Sigma, \quad (4.5.1)$$

$$\text{Lip}(f) \leq C(E + r^2 \mathbf{A}^2)^{\gamma_{la}}, \quad (4.5.2)$$

$$\text{osc}(f) \leq C\mathbf{h}(T, \mathbf{C}_2) + Cr(E + r^2 \mathbf{A}^2)^{\frac{1}{2}}, \quad (4.5.3)$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n), \quad (4.5.4)$$

$$\mathcal{H}^m(\mathbf{B}_1 \setminus K) + \mathbf{e}_T(\mathbf{B}_1 \setminus K) \leq Cr^m(E + r^2 \mathbf{A}^2)^{1+\gamma_{la}}, \quad (4.5.5)$$

$$\left| \|T\|(A \times \mathbb{R}^n) - Q\mathcal{H}^m(A \setminus \mathbf{B}_2) - \frac{1}{2} \int_A |Df|^2 \right| \leq Cr^m(E + r^2 \mathbf{A}^2)^{1+\gamma_{la}}. \quad (4.5.6)$$

Proof. The theory in [27, Sections 6, 7, 8, and 9] is already developed in an arbitrary dimension. Note also that [27, Lemma 7.6] (stated in 2d) can be replaced by its counterpart in any dimension [26, Lemma 7.1]. □

Before starting the construction of the center manifold and normal approximation at flat one-sided points, we note that [27] will be frequently referenced throughout. Although the authors consider $\Sigma = \mathbb{R}^{m+n}$, it is straightforward to verify that all their results remain valid for an arbitrary $\Sigma \subset \mathbb{R}^{m+n}$ of class $C^{3,\kappa}$, introducing certain errors that we will explicitly state below.

Assumption 4.5.2. Under Assumption 4.1.8 with $p = 0$ and $R = 2$, we assume that $\Sigma \cap \mathbf{B}_2 = \text{graph}(\Psi)$ and $\Gamma \cap \mathbf{B}_2(0) = \text{graph}(\phi)$ where $\Psi : \mathbf{B}_2 \cap \mathbb{R}^{m+\bar{n}} \rightarrow \mathbb{R}^{n-\bar{n}}$ and $\phi : \mathbf{B}_2 \cap \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{n+1}$. We assume that

- (i) $\max \left\{ \mathbf{E}^b(T, \mathbf{B}_2), \|\Psi\|_{C^{3,\kappa}}^2, \|\phi\|_{C^{3,\kappa}}^2 \right\} =: \mathbf{m}_0 < \varepsilon_{cm}$, where $\varepsilon_{cm} = \varepsilon_{cm}(m, n, \bar{n}, Q) > 0$ is a small constant,
- (ii) Assume 0 is a one-sided flat boundary point for T and the tangent cone to T at 0 is $T_0 = Q \llbracket \pi_0^+ \rrbracket$.

We begin by describing a Whitney-type decomposition of $\mathbf{B}_{3/2}^+$, which denotes the connected component of $\mathbf{B}_{3/2} \setminus \mathbf{p}_{\pi_0}(\Gamma)$ with nonempty intersection with $\mathbf{p}_{\pi_0\#}(T \llcorner \mathbf{B}_{3/2})$, with cubes whose sides are parallel to the coordinate axes and have side length $2\ell(L)$. The center of any such cube L will be denoted by c_L and its sidelength will be denoted by $2\ell(L)$. We introduce a family of dyadic cubes $L \subset \pi_0$ in the following way: for $j \geq N_0$, where N_0 is an integer whose choice will be specified below, we introduce the families

$$\mathcal{C}_j := \{L : L \text{ is a dyadic cube of side } \ell(L) = 2^{-j} \text{ and } \mathbf{B}_{3/2}^+ \cap L \neq \emptyset\},$$

For each L define a radius $r_L := M_0 \sqrt{m} \ell(L)$, with $M_0 \geq 1$ to be chosen later. We then subdivide $\mathcal{C} := \cup_j \mathcal{C}_j$ into, respectively, *boundary cubes* and *non-boundary cubes*

$$\begin{aligned} \mathcal{C}^b &:= \{L \in \mathcal{C} : \text{dist}(c_L, \mathbf{p}_{\pi_0}(\Gamma)) < 64r_L\}, & \mathcal{C}_j^b &= \mathcal{C}^b \cap \mathcal{C}_j, \\ \mathcal{C}^\natural &:= \{L \in \mathcal{C} : \text{dist}(c_L, \mathbf{p}_{\pi_0}(\Gamma)) \geq 64r_L\}, & \mathcal{C}_j^\natural &= \mathcal{C}^\natural \cap \mathcal{C}_j. \end{aligned}$$

If $H, L \in \mathcal{C}$ we say that:

- H is a *descendant* of L and L is an *ancestor* of H , if $H \subset L$;
- H is a *child* of L and L is the *parent* of H , if $H \subset L$ and $\ell(H) = \frac{1}{2}\ell(L)$;
- H and L are *neighbors* if $\frac{1}{2}\ell(L) \leq \ell(H) \leq \ell(L)$ and $H \cap L \neq \emptyset$.

Lemma 4.5.3. *Let H be a boundary cube. Then any ancestor L and any neighbor L with $\ell(L) = 2\ell(H)$ are necessarily a boundary cube. In particular: the descendant of a non-boundary cube is a non-boundary cube.*

- If $L \in \mathcal{C}^{\natural}$, then we define the *non-boundary satellite ball* $\mathbf{B}_L = \mathbf{B}_{64r_L}(p_L)$ where $p_L \in \text{spt}(T)$ such that $\mathbf{p}_{\pi_0}(p_L) = c_L$, such p_L is a priori not unique, and π_L is a plane which minimizes the excess in $\mathbf{B}_L$, namely $\mathbf{E}(T, \mathbf{B}_L) = \mathbf{E}(T, \mathbf{B}_L, \pi_L)$ and $\pi_L \subset T_{p_L}\Sigma$,
- If $L \in \mathcal{C}^{\flat}$, then we define the *boundary satellite ball* $\mathbf{B}_L^{\flat} = \mathbf{B}_{2^7 64r_L}(p_L^{\flat})$ where $p_L^{\flat} \in \Gamma$ is such that $|\mathbf{p}_{\pi_0}(p_L^{\flat}) - c_L| = \text{dist}(c_L, \mathbf{p}_{\pi_0}(\Gamma))$. Note that in this case, the point $p_L^{\flat}$ is uniquely determined because Γ is $C^{3,\kappa}$ and $\mathbf{m}_0$ is assumed to be sufficiently small. Likewise π_L is a plane that minimizes the excess $\mathbf{E}^{\flat}$, that is, such that $\mathbf{E}^{\flat}(T, \mathbf{B}_L^{\flat}) = \mathbf{E}(T, \mathbf{B}_L^{\flat}, \pi_L)$ and $T_{p_L^{\flat}}\Gamma \subset \pi_L \subset T_{p_L^{\flat}}\Sigma$.

Stopping conditions

We introduce the set up to decide whether or not a cube is such that T has good enough properties around it. These properties are mainly based upon how small the excess or height of T are with respect to the size of a given cube.

Let $C_{\mathbf{e}}, C_{\mathbf{h}}$ be two large positive constants that will be specified later. We define

- (i) $\mathcal{W}_{N_0}^{\mathbf{e}} := \{L \in \mathcal{C}_{N_0}^{\natural} \cup \mathcal{C}_{N_0}^{\flat} : \mathbf{E}(T, \mathbf{B}_L) > C_{\mathbf{e}}\mathbf{m}_0\ell(L)^{2-\alpha_{\mathbf{e}}}\},$
- (ii) $\mathcal{W}_{N_0}^{\mathbf{h}} := \{L \in \mathcal{C}_{N_0}^{\natural} \cup \mathcal{C}_{N_0}^{\flat} : \mathbf{h}(T, \mathbf{B}_L, \pi_L) > C_{\mathbf{h}}\mathbf{m}_0^{1/2m}\ell(L)^{1+\alpha_{\mathbf{h}}}\},$
- (iii) $\mathcal{S}_{N_0} := \mathcal{C}_{N_0} \setminus (\mathcal{W}_{N_0}^{\mathbf{e}} \cup \mathcal{W}_{N_0}^{\mathbf{h}}).$

Assume inductively that for a certain step $j \geq N_0 + 1$, we have defined the families $\mathcal{W}_{j-1} := \mathcal{W}_{j-1}^{\mathbf{e}} \cup \mathcal{W}_{j-1}^{\mathbf{h}}$ and $\mathcal{S}_{j-1}$. We set the following stopping criteria:

- (i) $\mathcal{W}_j^{\mathbf{e}} := \{L \text{ child of } K \in \mathcal{S}_{j-1} : \mathbf{E}(T, \mathbf{B}_L) > C_{\mathbf{e}} \mathbf{m}_0 \ell(L)^{2-2\alpha_{\mathbf{e}}}\},$
- (ii) $\mathcal{W}_j^{\mathbf{h}} := \{L \text{ child of } K \in \mathcal{S}_{j-1} : L \notin \mathcal{W}_j^{\mathbf{e}} \text{ and } \mathbf{h}(T, \mathbf{B}_L, \pi_L) > C_{\mathbf{h}} \mathbf{m}_0^{1/2m} \ell(L)^{1+\alpha_{\mathbf{h}}}\},$
- (iii) $\mathcal{W}_j^{\mathbf{n}} := \{L \text{ child of } K \in \mathcal{S}_{j-1} : L \notin \mathcal{W}_j^{\mathbf{e}} \cup \mathcal{W}_j^{\mathbf{h}} \text{ but } \exists L' \in \mathcal{W}_{j-1} \text{ with } L \cap L' \neq \emptyset\}.$

Thus, we define $\mathcal{S}_j := \{L \in \mathcal{C}_j \text{ child of } K \in \mathcal{S}_{j-1}\} \setminus \mathcal{W}_j$. Furthermore, we set

$$\mathcal{W} := \bigcup_{j \geq N_0} \mathcal{W}_j, \quad \mathcal{S} := \bigcup_{j \geq N_0} \mathcal{S}_j, \quad \mathbf{S} := \bigcap_{j \geq N_0} \left(\bigcup_{L \in \mathcal{S}_j} L \right) = \mathbf{B}_{3/2}^+ \setminus \bigcup_{H \in \mathcal{W}} H.$$

Remark 4.5.4. Unlike the two-sided case, as in [21, 68], we cannot state that every boundary cube is *nonstopping* since we do not have an almost-quadratic excess decay (which is strongly used in [21, 68] to prove this affirmation). Hence, we have to include boundary cubes in the stopping conditions.

Since $(\mathcal{W}, \mathbf{S})$ is a Whitney decomposition of $[-3/2, 3/2]^m$, it is not hard to see that

$$\text{sep}(\mathbf{S}, L) := \inf \{|x - y| : x \in L, y \in \mathbf{S}\} \geq 2\ell(L), \text{ for any } L \in \mathcal{W}. \quad (4.5.7)$$

We now assume an order of hierarchy for all the parameters involved in the construction of the center manifold that will follow.

Assumption 4.5.5. T and Γ are as in Assumptions 4.5.2 and we also assume that

- (i) $0 < \alpha_{\mathbf{h}} \leq 1/2m$ and $\alpha_{\mathbf{e}} > 0$ is positive but small, depending only on $\alpha_{\mathbf{h}}$,
- (ii) M_0 is larger than a suitable constant, depending only upon $\alpha_{\mathbf{e}}$,
- (iii) $2^{N_0} \geq C(m, n, M_0)$,
- (iv) $C_{\mathbf{e}}$ is sufficiently large depending upon $\alpha_{\mathbf{e}}$, $\alpha_{\mathbf{h}}$, M_0 and N_0 ,

- (v) $C_{\mathbf{h}}$ is sufficiently large depending upon $\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0$ and $C_{\mathbf{e}}$,
- (vi) Item (i) of Assumption 4.5.2 holds with ε_{par} sufficiently small depending upon all the other parameters above.

Existence of the center manifold

In this subsection, we state the existence of the center manifold which acts as an “average” of the sheets of the current T . Furthermore, the center manifold $\mathcal{M}$ is a classical submanifold of Σ of regularity $C^{3,\kappa}$ which allows us to study further properties of the current T .

Theorem 4.5.6 (Theorem 10.16, [27]). *Under Assumptions 4.5.2 and 4.5.5, there is a $\kappa := \kappa(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, Q, Q^*, m, \bar{n}, n) > 0$, such that*

- (i) $\varphi_j \in C^{3,\kappa}$ with $\|\varphi_j\|_{C^{3,\kappa}(B_{3/2}^+)} \leq C \mathbf{m}_0^{1/2}$, for some $C := C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$,
- (ii) If $i \leq j$, $L \in \mathcal{W}_{i-1}$ and H is a cube concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$, then $\varphi_j = \varphi_i$ on H ,
- (iii) φ_j converges in the C^3 -topology to a map $\varphi : B_{3/2}^+ \rightarrow \mathbb{R}^n$, whose graph is a $C^{3,\kappa}$ -submanifold $\mathcal{M}$ of Σ , which will be called the center manifold,
- (iv) $\partial(\text{graph}(\varphi)) = \Gamma$, i.e., $\partial\mathcal{M} \cap \mathbf{C}_{3/2} = \Gamma \cap \mathbf{C}_{3/2}$,
- (v) For any $q \in \partial\mathcal{M} \cap \mathbf{C}_{3/2}$, we have $T_q\mathcal{M} = \pi^+(q)$ where $\pi(q)$ is the supporting m -plane for the unique tangent cone to T at q .

Proof. We will omit the proof because it follows the same lines as in [27, Theorem 10.16]. In fact, the authors, in [68], do not use the fact that $m = 2$ to prove the existence of $\mathcal{M}$ as above. □

Normal approximations

With the center manifold $\mathcal{M}$ in hand, our objective is to construct approximations of T —akin to those in Proposition 4.5.1—by functions defined on $\mathcal{M}$ and taking values in its normal bundle. These approximations, known as $\mathcal{M}$ -normal approximations, are formally defined in Definition 4.5.9 and their existence is established in Theorem 4.5.10.

Let us define the graph parametrization map of $\mathcal{M}$ as $\Phi(x) := (x, \varphi(x))$. We will call *contact set* the subset $\Phi(\mathbf{S})$. For every cube $L \in \mathscr{W}$ we associate a *Whitney region* $\mathcal{L}$ on $\mathcal{M}$ as follows:

- $\mathcal{L} := \Phi(H \cap B_1)$ where H is the cube concentric to L such that $\ell(H) = \frac{17}{16}\ell(L)$.

Assumption 4.5.7. Under Assumptions 4.5.2 and 4.5.5. We possibly further reduce ε_{cm} , call it ε_{na} , and assume $\mathbf{m}_0 < \varepsilon_{na}$, to be such that, if

$$\mathbf{U} := \{q \in \mathbb{R}^{m+n} : \exists! q' = \mathbf{p}(q) \in \mathcal{M} \text{ s.t. } |q - q'| < 1 \text{ and } q - q' \in T_{q'}^\perp \mathcal{M}\}$$

where $T_{q'}^\perp \mathcal{M} := (T_{q'} \mathcal{M})^\perp$, then the map $\mathbf{p}$ extends to a Lipschitz map to the closure $\overline{\mathbf{U}}$ which is $C^{2,\kappa}$ on $\mathbf{U} \setminus \mathbf{p}^{-1}(\Gamma)$ and $\mathbf{p}^{-1}(q') = q' + \overline{B_1(0, (T_{q'} \mathcal{M})^\perp)}$ for all $q' \in \overline{\mathcal{M}}$.

The following corollary states strong properties on the behavior of the current T whenever we look at points on the contact set.

Corollary 4.5.8. *Under Assumptions 4.5.2 and 4.5.5,, we have*

- (i) $\text{spt}(\partial(T \llcorner \mathbf{U})) \subset \partial_t \mathbf{U}$, $\text{spt}_\#(T \llcorner [-\frac{7}{2}, \frac{7}{2}] \times \mathbb{R}^n) \subset \mathbf{U}$, and $\mathbf{p}(T \llcorner \mathbf{U}) = Q \llbracket \mathcal{M} \rrbracket$,
- (ii) $\text{spt}(\langle T, \mathbf{p}, \Phi(q) \rangle) \subset \left\{ y : |\Phi(q) - y| \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\alpha_h} \right\}$ for every $q \in L \in \mathscr{W}$,
- (iii) $\langle T, \mathbf{p}, p \rangle = Q \llbracket p \rrbracket$ for every $p \in \Phi(\mathbf{S}) \cup (\Gamma \cap \partial \mathcal{M})$.

Next, we define the $\mathcal{M}$ -normal approximations of T defined on $\mathcal{M}$ and taking values in its normal bundle.

Definition 4.5.9. Let $\mathcal{M}$ be the center manifold under Assumption 4.5.7. We say that $(\mathcal{K}, F)$ is an $\mathcal{M}$ -normal approximation of T , if

(i) there exists a Lipschitz function

$$\begin{aligned} \mathcal{N} : \mathcal{M} \cap \mathbf{C}_1 &\rightarrow \mathcal{A}_Q(T^\perp \mathcal{M}) \\ x &\mapsto \mathcal{N}(p) := \sum_{i=1}^Q \llbracket \mathcal{N}_i(p) \rrbracket, \end{aligned}$$

where $(\mathcal{N}_i : \mathcal{M} \cap \mathbf{C}_1 \rightarrow T^\perp \mathcal{M})_{i \in \{1, \dots, Q\}}$ are measurable sections of the normal bundle satisfying $p + \mathcal{N}_i(p) \in \Sigma$. We then define the Lipschitz function given by

$$\begin{aligned} F : \mathcal{M} \cap \mathbf{C}_1 &\rightarrow \mathcal{A}_Q(T^\perp \mathcal{M}_k) \\ x &\mapsto (\mathcal{N} \oplus \text{id})(x). \end{aligned}$$

(ii) $\mathcal{K} \subset \mathcal{M}$ is closed and $\mathbf{T}_F \mathbb{L} \mathbf{p}^{-1}(\mathcal{K} \cap \mathcal{M}_k) = T \mathbb{L} \mathbf{p}^{-1}(\mathcal{K} \cap \mathcal{M}_k)$, where $\mathbf{T}_F := (F)_\# \llbracket \mathcal{M} \rrbracket$, according to [32, Definition 1.3],

(iii) $\Phi(\mathbf{S}) \cap \Gamma \subset \mathcal{K}$, $\mathcal{N}|_{\mathcal{K}} \equiv Q \llbracket 0 \rrbracket$, and then $F(p) = Q \llbracket p \rrbracket$ on $\mathcal{K}$.

We finally state the existence and local properties of a normal Lipschitz approximation of T .

Theorem 4.5.10 (Local behavior of $\mathcal{M}$ -normal approximations. Theorem 10.21 of [27]). *Under Assumption 4.5.7, there is a constant $\alpha_{\mathbf{L}} = \alpha_{\mathbf{L}}(m, n, \bar{n}, Q) > 0$ such that there exists an $\mathcal{M}$ -normal approximation $(\mathcal{K}, F)$ satisfying the following estimates on any Whitney region $\mathcal{L} \subset \mathcal{M}$ associated to a cube $L \in \mathcal{W}$:*

$$\begin{aligned} \text{Lip}(\mathcal{N}|_{\mathcal{L}}) &\leq C \mathbf{m}_0^{\alpha_{\mathbf{L}}} \ell(L)^{\alpha_{\mathbf{L}}} \text{ and } \|\mathcal{N}|_{\mathcal{L}}\|_0 \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\alpha_{\mathbf{h}}}, \\ \mathcal{H}^m(\mathcal{L} \setminus \mathcal{K}) + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) &\leq C \mathbf{m}_0^{1+\alpha_{\mathbf{L}}} \ell(L)^{m+2+\alpha_{\mathbf{L}}}, \\ \int_{\mathcal{L}} |D\mathcal{N}|^2 &\leq C \mathbf{m}_0 \ell(L)^{m+2-2\alpha_{\mathbf{e}}}, \end{aligned} \tag{4.5.8}$$

for a constant $C = C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$. Moreover, for any $a > 0$ and any Borel $\mathcal{V} \subset \mathcal{L}$,

$$\int_{\mathcal{V}} |\eta \circ \mathcal{N}| \leq C \mathbf{m}_0 \left(\ell(L)^{m+3+\frac{\alpha_{\mathbf{h}}}{3}} + a \ell(L)^{2+\frac{\alpha_{\mathbf{L}}}{2}} \mathcal{H}^m(\mathcal{V}) \right) + \frac{C}{a} \int_{\mathcal{V}} \mathcal{G}(\mathcal{N}, Q[\eta \circ \mathcal{N}])^{2+\alpha_{\mathbf{L}}}.$$

Next, we state global estimates that are derived from summing up all the local estimates in the previous theorem over all cubes of the Whitney decomposition.

Corollary 4.5.11 (Global estimates for $\mathcal{M}$ -normal approximations on Whitney regions. Corollary 10.22 of [27]). *Under the same assumptions and notation of Theorem 4.5.10, and setting $\mathcal{M}' = \Phi(B_1^+ \cap [-7/2, 7/2]^m)$, we have:*

$$\begin{aligned} \text{Lip}(\mathcal{N}|_{\mathcal{M}'}) &\leq C \mathbf{m}_0^{\alpha_{\mathbf{L}}} \text{ and } \|\mathcal{N}|_{\mathcal{M}'}\|_0 \leq C \mathbf{m}_0^{1/2m}, \\ \mathcal{H}^m(\mathcal{M}' \setminus \mathcal{K}) + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{M}')) &\leq C \mathbf{m}_0^{1+\alpha_{\mathbf{L}}} \text{ and } \int_{\mathcal{M}'} |D\mathcal{N}|^2 \leq C \mathbf{m}_0. \end{aligned}$$

for a constant $C = C(\alpha_{\mathbf{e}}, \alpha_{\mathbf{h}}, M_0, N_0, C_{\mathbf{e}}, C_{\mathbf{h}}) > 0$. Moreover, $\mathcal{N} \equiv Q[0]$ on $\Phi(\mathbf{S}) \cap \mathcal{M}'$ and $\int_{\mathcal{M}'} |\mathcal{N}|^2 \leq C \mathbf{m}_0$.

We now state some properties on the normal Lipschitz approximations concerning their “splitting” behavior on stopping cubes.

Proposition 4.5.12 (Splitting, Proposition 14.4 of [27]). *Under the same assumptions and notation as Theorem 4.5.10, there is a constant $C = C(m, n, \bar{n}, M_0, \delta_1) > 0$ with the following property. If $L \in \mathcal{W}^e$, $q \in \pi_0$ with $\text{dist}(q, L) \leq 4\sqrt{m}\ell(L)$, $B_{\ell(L)/4}(q) \subset B_2$, and $\Omega := \Phi(B_{\ell(L)/4}(q))$, then we have that*

$$C \mathbf{m}_0 \ell(L)^{m+2-2\delta_1} \leq \ell(L)^m \mathbf{E}(T, \mathbf{B}_L^b) \leq C \int_{\Omega} |D\mathcal{N}|^2, \quad (4.5.9)$$

$$\int_{\mathcal{L}} |D\mathcal{N}|^2 \leq C \ell(L)^m \mathbf{E}(T, \mathbf{B}_L^b) \leq C \ell(L)^{-m} \int_{\Omega} |\mathcal{N}|^2. \quad (4.5.10)$$

4.5.2 Intervals of flattening and definition of I

We begin by defining the intervals of flattening, which allow us to transition between center manifolds whenever they cease to be good approximations for the “average” of the sheets of T . Before that, we make a few remarks that are important for comparison with the interior setting and that also highlight the significance of the excess decay available for one-sided points.

Remark 4.5.13. Our approach closely follows the strategy in [27], as they also do not have an *a priori* almost-quadratic decay of the excess. In contrast, when dealing with two-sided boundary points, the authors of [21] and [68] establish almost quadratic excess decay through a blow-up argument combined with strong properties of the limiting two-sided Dir-minimizing functions. The availability of such a decay allows them to bypass the need for intervals of flattening.

Remark 4.5.14. In the interior case ([34]), analogous to the boundary setting, the center manifold can only be constructed when the excess is below a certain threshold, namely ε_{cm} given in Assumption 4.5.2; for the interior, see [34, Eq. 2.4]. This requirement explains the gaps that may exist between the intervals of flattening in [28]: At scales within these gaps, the excess may be too large to construct the center manifold. However, in our setting, this issue does not arise, as we rely on an excess decay theorem (with exponent $\alpha \in (0, 1)$) established in Chapter 3).

Remark 4.5.15. We note that [27] will be frequently referenced, where the setting assumes $\Sigma = \mathbb{R}^{m+n}$. Nevertheless, it is straightforward to verify that all the results remain valid for an arbitrary $\Sigma \subset \mathbb{R}^{m+n}$ of class $C^{3,\kappa}$, taking into account the errors we explicitly state below.

Interval of flattening

Assume Assumption 4.5.2, Assumption 4.5.5, and Assumption 4.5.7, with $p \in \Gamma$ in place of 0, and fix

$$\max \left\{ \mathbf{E}^b(T, \mathbf{B}_2(p)), \mathbf{A}_{p,\Sigma}^2, \mathbf{A}_{p,\Gamma}^2 \right\} =: \mathbf{m}_p < \varepsilon_{cm}.$$

We set $t_1 = 2$, $\overline{\mathcal{M}}_{p,1} = \mathcal{M}_{p,1}$ and $\overline{\mathcal{N}}_{p,1} = \mathcal{N}_{p,1}$ to be the center manifold and the $\mathcal{M}_{p,1}$ -normal approximation given by Theorem 4.5.6 and Theorem 4.5.10 when applied the translation of T by $p \in \Gamma$, i.e., $(\iota_{p,1})_{\#}T$, respectively.

Denote by $\mathscr{W}^{p,1}$ the Whitney decomposition as described in Section 4.5.1. Notice that the construction of the intervals of flattening below depend on p , we will however omit this dependence for the sake of notation.

We now assume by induction that $t_{j-1} > 0$ is defined with $t_{i-1} < t_i$, for any $i \leq j-1$ and consider:

- $T_{p,j} := \left((\iota_{p,t_j})_{\#} T \right) \lrcorner \mathbf{B}_2$, $\overline{\Sigma}_{p,j} := \iota_{p,t_j}(\Sigma) \cap \mathbf{B}_2$, $\overline{\Gamma}_{p,j} := \iota_{p,t_j}(\Gamma) \cap \mathbf{B}_2$, $\overline{\mathcal{M}}_{p,j}$ and $\overline{\mathcal{N}}_{p,j}$ to be the center manifold and the normal approximation given by Theorem 4.5.6 and Theorem 4.5.10 now applied to the current $T_{p,j}$, the boundary $\overline{\Gamma}_{p,j}$, and the ambient manifold $\overline{\Sigma}_{p,j}$. We also denote by $\mathscr{W}^{p,j}$ the Whitney decomposition coupled with $\overline{\mathcal{M}}_{p,j}$ as described above.

Next, we inductively define:

$$t_j = t_{j-1} \max \left(\left\{ 64\sqrt{m}\ell(L) : L \in \mathscr{W}^{p,j} \text{ and } 64\sqrt{m}\ell(L) \geq \text{dist}(0, L) \right\} \cup \{0\} \right).$$

Observe that $t_j/t_{j-1} \leq 2^{-5}$ which ensures that $[t_{j-1}, t_j]$ is a nontrivial interval. If $t_j = 0$, we stop the induction; otherwise we keep going and end up with infinitely many intervals. We also fix the following notations:

$$\mathcal{M}_{p,j} := t_j \overline{\mathcal{M}}_{p,j}, \quad \mathcal{N}_{p,j}(q) := t_j \overline{\mathcal{N}}_{p,j} \left(\frac{q}{t_j} \right), \quad \Gamma_{p,j} := t_j \overline{\Gamma}_{p,j}, \text{ and } \Sigma_{p,j} := t_j \overline{\Sigma}_{p,j}.$$

Notice that $\Gamma_{p,j} = (\Gamma \cap \mathbf{B}_{t_j}(p)) - p$ and $\Sigma_{p,j} = (\Sigma \cap \mathbf{B}_{t_j}(p)) - p$.

Lemma 4.5.16. *Let L be a cube such that $L \in \mathcal{W}^{p,j}$ and $64\sqrt{m}\ell(L) \geq \text{dist}(0, L)$ and $64\sqrt{m}\ell(L) = \max(\{64\sqrt{m}\ell(L) : L \in \mathcal{W}^{p,j} \text{ and } 64\sqrt{m}\ell(L) \geq \text{dist}(0, L)\} \cup \{0\})$. Then $L \in (\mathcal{W}^{p,j})^e$, i.e., it has to stop for the excess.*

Proof. Observe that if L is in $(\mathcal{W}^{p,j})^n$, then there is a neighboring cube H of double side length that also belongs to $\mathcal{W}^{p,j}$ and it is straightforward to check that it satisfies $H \in \mathcal{W}^{p,j}$ and $64\sqrt{m}\ell(H) \geq \text{dist}(0, H)$ too, thus violating the maximality of L . Since L satisfies $64\sqrt{m}\ell(L) \geq \text{dist}(0, L)$, L is a boundary cube and thus cannot belong to $(\mathcal{W}^{p,j})^h$, see [27, Lemma 10.4]. $\square$

Universal frequency function at one-sided flat points

Before defining the frequency function, we first introduce a well-suited distance function d that accounts for the curvature of Γ in our calculations. Specifically, d serves as the direction of the vector field used in the inner and outer variations, requiring ∇d to be tangent to Γ . The function d is constructed in the following lemma.

Lemma 4.5.17 (Good distorted distance function). *There exists a continuous function $d : \mathbf{B}_1 \times \mathbf{B}_1 \rightarrow \mathbb{R}^+$ which we will call good distorted distance function satisfying, for any $q \in \Gamma_{p,1}$, the following:*

- (i) $x \mapsto d(x, q)$ is C^2 on $\mathbf{B}_1 \setminus \{q\}$,
- (ii) $d(x, q) = |x - q| + O(\mathbf{m}_0|x - q|^2)$,
- (iii) $\nabla_x d(x, q) = \frac{x-q}{|x-q|} + O(\mathbf{m}_0|x - q|)$,
- (iv) $D_x^2 d(x, q) = |x - q|^{-1} (\text{Id} - |x - q|^{-2}(x - q) \otimes (x - q)) + O(\mathbf{m}_0)$,
- (v) $\nabla_x d(x, q)$ is tangent to $\Gamma_{p,1}$ at $x \in \Gamma_{p,1}$.

Proof. The proof is given in [21, Lemma 4.25]. Even though slightly worse estimates are stated, i.e., there is no $\mathbf{m}_0$ in [21, Lemma 4.25], their proof is robust enough and immediately gives the improvements that we claim in this lemma. $\square$

We first fix a cut-off function as follows:

$$\phi(t) := \begin{cases} 1, & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 2(1-t), & \text{for } \frac{1}{2} \leq t \leq 1, \\ 0, & \text{for } t \geq 1. \end{cases}$$

Having fixed a good distorted distance function d , a cut-off ϕ , and the intervals of flattening, for any $q \in \Gamma_{p,j}$ and $r \in (t_{j+1}, t_j]$, we define the following three quantities:

$$\begin{aligned} \mathbf{D}_p(q, r) &= \mathbf{D}_p(\phi, d, q, r) := \int_{\mathcal{M}_{p,j}} \phi\left(\frac{d(x, q)}{r}\right) |D\mathcal{N}_{p,j}|^2(x) \, d\mathcal{H}^m(x), \\ \mathbf{H}_p(q, r) &= \mathbf{H}_p(\phi, d, q, r) := - \int_{\mathcal{M}_{p,j}} \phi'\left(\frac{d(x, q)}{r}\right) |\nabla_{\mathcal{M}_{p,j}} d(x, q)|^2 \frac{|\mathcal{N}_{p,j}(x)|^2}{d(x, q)} \, d\mathcal{H}^m(x), \\ \mathbf{S}_p(q, r) &= \mathbf{S}_p(\phi, d, q, r) := \int_{\mathcal{M}_{p,j}} \phi\left(\frac{d(x, q)}{r}\right) |\mathcal{N}_{p,j}(x)|^2 \, d\mathcal{H}^m(x). \end{aligned}$$

Next, we finally define the *universal frequency function* as follows:

$$\mathbf{I}_p(q, r) = \mathbf{I}_p(\phi, d, q, r) := \frac{r\mathbf{D}_p(q, r)}{\mathbf{H}_p(q, r)}, \quad \text{for any } r \in (0, 2], q \in \mathbf{B}_2.$$

We will also fix $d_j(x, q) := t_j^{-1}d(t_j x, t_j q)$ and $q \in \mathbf{B}_2$ to define the rescaled functions:

$$\begin{aligned} \overline{\mathbf{D}}_{p,j}(q, r) &= \overline{\mathbf{D}}_{p,j}(\phi, d, q, r) := \int_{\overline{\mathcal{M}}_{p,j}} \phi\left(\frac{d_j(x, q)}{r}\right) |D\overline{\mathcal{N}}_{p,j}|^2(x) \, d\mathcal{H}^m(x), \\ \overline{\mathbf{H}}_{p,j}(q, r) &= \overline{\mathbf{H}}_{p,j}(\phi, d, q, r) := - \int_{\overline{\mathcal{M}}_{p,j}} \phi'\left(\frac{d_j(x, q)}{r}\right) |\nabla_{\overline{\mathcal{M}}_{p,j}} d_j(x, q)|^2 \frac{|\overline{\mathcal{N}}_{p,j}(x)|^2}{d_j(x, q)} \, d\mathcal{H}^m(x), \\ \overline{\mathbf{S}}_{p,j}(q, r) &= \overline{\mathbf{S}}_{p,j}(\phi, d, q, r) := \int_{\overline{\mathcal{M}}_{p,j}} \phi\left(\frac{d_j(x, q)}{r}\right) |\overline{\mathcal{N}}_{p,j}(x)|^2 \, d\mathcal{H}^m(x). \end{aligned}$$

We can then straightforwardly check that:

$$\overline{\mathbf{D}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m}\mathbf{D}_p(q, r), \quad \partial_r \overline{\mathbf{D}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m+1}\partial_r \mathbf{D}_p(q, r), \quad (4.5.11)$$

$$\overline{\mathbf{H}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m-1}\mathbf{H}_p(q, r), \quad \partial_r \overline{\mathbf{H}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m}\partial_r \mathbf{H}_p(q, r), \quad (4.5.12)$$

$$\overline{\mathbf{S}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m-2}\mathbf{S}_p(q, r), \quad \partial_r \overline{\mathbf{S}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m-1}\partial_r \mathbf{S}_p(q, r). \quad (4.5.13)$$

We also define the following quantities:

$$\begin{aligned} \mathbf{E}_p(q, r) &:= -\frac{1}{r} \int_{\mathcal{M}_j} \phi' \left(\frac{d(x, q)}{r} \right) \sum_i \langle (\mathcal{N}_{p,j})_i(x), D(\mathcal{N}_{p,j})_i(x) \nabla_{\mathcal{M}_{p,j}} d(x, q) \rangle \, d\mathcal{H}^m(x), \\ \mathbf{G}_{p,j}(q, r) &:= -\frac{1}{r^2} \int_{\mathcal{M}_{p,j}} \phi \left(\frac{d(x, q)}{r} \right) \frac{d(x, q)}{|\nabla_{\mathcal{M}_{p,j}} d(x, q)|^2} |D\mathcal{N}_{p,j} \cdot \nabla_{\mathcal{M}_{p,j}} d(x, q)|^2 \, d\mathcal{H}^m(x), \\ \overline{\mathbf{E}}_{p,j}(q, r) &:= -\frac{1}{r} \int_{\overline{\mathcal{M}}_{p,j}} \phi' \left(\frac{d_j(x, q)}{r} \right) \sum_i \langle (\overline{\mathcal{N}}_{p,j})_i(x), D(\overline{\mathcal{N}}_{p,j})_i(x) \cdot \nabla_{\overline{\mathcal{M}}_{p,j}} d_j(x, q) \rangle \, d\mathcal{H}^m(x), \\ \overline{\mathbf{G}}_{p,j}(q, r) &:= -\frac{1}{r^2} \int_{\overline{\mathcal{M}}_{p,j}} \phi \left(\frac{d_j(x, q)}{r} \right) \frac{d_j(x, q)}{|\nabla_{\overline{\mathcal{M}}_{p,j}} d_j(x, q)|^2} |D\overline{\mathcal{N}}_{p,j} \cdot \nabla_{\overline{\mathcal{M}}_{p,j}} d_j(x, q)|^2 \, d\mathcal{H}^m(x), \end{aligned}$$

then, we also have the following rescalings:

$$\overline{\mathbf{E}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m}\mathbf{E}_p(q, r) \text{ and } \overline{\mathbf{G}}_{p,j}(t_j^{-1}q, t_j^{-1}r) = t_j^{-m+1}\mathbf{G}_p(q, r). \quad (4.5.14)$$

4.5.3 Radial derivatives in the realm of the frequency function

The main goal of this subsection is to derive a series of inequalities involving the radial derivatives of the various quantities defined above. These inequalities will be central in computing the radial derivative of $\mathbf{I}$ and in establishing further properties of $\mathbf{I}$ (such as its almost monotonicity), which in turn encapsulates significant information about the current T .

Lemma 4.5.18 (Radial variations and list of bounds). *Let T, Γ , and Σ , satisfy Assumptions 4.5.2, 4.5.5, and 4.5.7, with p in place of 0. Then either $T = Q \llbracket p + \mathcal{M}_{p,j} \rrbracket$ in a neighborhood of p , for some $j \in \mathbb{N}$, or $\mathbf{H}_p(p, s) > 0$ and $\mathbf{D}_p(p, s) > 0$, for every $s \in (0, 2]$. In the latter case, there exist $C > 0$ and $\tau > 0$ (which are not dependant on p, j , T , nor ε_{na}) such that,*

for all $r > 0$, we have

$$\begin{aligned}\mathbf{H}_p(p, r) &\leq Cr\mathbf{D}_p(p, r), \quad \mathbf{I}_p(p, r) \geq C^{-1}, \\ \partial_r \mathbf{S}_p(p, r) &\leq Cr\mathbf{D}_p(p, r), \quad \text{and } \mathbf{S}_p(p, r) \leq Cr^2\mathbf{D}_p(p, r),\end{aligned}\tag{4.5.15}$$

Additionally, we obtain that

$$\mathbf{D}_p(p, r) \leq Cr^{m+2\kappa}.\tag{4.5.16}$$

We also have the following formulas for any $r \in (t_{j+1}, t_j)$:

$$\partial_r \mathbf{D}_p(p, r) = - \int \phi' \left(\frac{d(x, p)}{r} \right) \frac{d(x, p)}{r^2} |D\mathcal{N}_{p,j}(x)|^2 d\mathcal{H}^m(x),\tag{4.5.17}$$

$$\left| \partial_r \mathbf{H}_p(p, r) - \frac{m-1}{r} \mathbf{H}_p(p, r) + 2\mathbf{E}_p(p, r) \right| \leq C\mathbf{m}_{p,j} \mathbf{H}_p(p, r).\tag{4.5.18}$$

Moreover, for any $r \in (t_{j+1}, t_j)$, we have:

$$|\partial_r \mathbf{D}_p(p, r)| \leq Cr^{-1} \mathbf{D}_p(p, r),\tag{4.5.19}$$

$$|\mathbf{D}_p(p, r) - \mathbf{E}_p(p, r)| \leq C\mathbf{m}_p^\tau \mathbf{D}_p(p, r)^{1+\tau} + C\mathbf{m}_p t_j^{-2+2\kappa} \mathbf{S}_p(p, r),\tag{4.5.20}$$

$$\begin{aligned}|\partial_r \mathbf{D}_p(p, r) - \frac{m-2}{r} \mathbf{D}_p(p, r) - 2\mathbf{G}_p(p, r)| &\leq C\mathbf{m}_p^\tau \mathbf{D}_p^\tau(p, r) \left(\frac{\mathbf{D}_p(p, r)}{r} + \partial_r \mathbf{D}_p(p, r) \right) \\ &\quad + C\mathbf{m}_p t_j^{2\kappa-1} \mathbf{D}_p(p, r),\end{aligned}\tag{4.5.21}$$

$$\left| \frac{1}{\mathbf{D}_p(p, r)} - \frac{1}{\mathbf{E}_p(p, r)} \right| \leq C\mathbf{m}_p^\tau \mathbf{D}_p(p, r)^{\tau-1} + C\mathbf{m}_p t_j^{-2+2\kappa} \frac{\mathbf{S}_p(p, r)}{\mathbf{D}_p(p, r)^2}.\tag{4.5.22}$$

Proof. These estimates depend on the outer variation and an inner variation on specific vector fields as is well-known. We denote by $\mathbf{p}_j$ the projection onto the center manifold $\mathcal{M}_{p,j}$.

The vector fields are the following:

$$X_o^j(q) := \phi \left(\frac{d_j(\mathbf{p}_j(q))}{r} \right) (q - \mathbf{p}_j(q)), \quad X_i^j(q) := -Y_j(\mathbf{p}_j(q)) \quad \text{with } Y_j := \frac{1}{2} \phi \left(\frac{d_j}{r} \right) \frac{\nabla d_j}{|\nabla d_j|^2},$$

where in the formula above, ∇ denotes the tangent gradient to $\mathcal{M}_{p,j}$. The vector field X_o^j vanishes at Γ and the vector field X_i^j is tangent to $\mathcal{M}_{p,j}$ and to $\Gamma_{p,j}$ by properties of d . Let us drop the dependence on p in the notation and assume without loss of generality that $p = 0$. The proof of (4.5.15) for the rescaled quantities is given in [21, Proposition 9.4] from which, together with the rescalings above, i.e., (4.5.11), (4.5.13), and (4.5.12), we get (4.5.15). In order to prove (4.5.16), we note that [27, Eq. (16.29)] gives $\overline{\mathbf{D}}_j(r) \leq C_0 t_j^{2\kappa} r^{m+2-2\alpha_e}$, thus

$$\mathbf{D}(r) \stackrel{(4.5.11)}{=} t_j^m \overline{\mathbf{D}}_j(t_j^{-1}r) \leq C_0 t_j^{m+2\kappa} (t_j^{-1}r)^{m+2-2\alpha_e} \stackrel{r < t_j}{<} C_0 r^{m+2\kappa}.$$

The proof of (4.5.17) is a direct computation. For (4.5.18), we can mimic the proof of [27, Proposition 16.2] using the slightly improved estimates for d given by Lemma 4.5.17. Inequality (4.5.19) is a consequence of the definition of $\mathbf{D}$ and (4.5.17). To prove (4.5.20) we will do as in [21, Section 9]. Indeed,

$$|\overline{\mathbf{D}}_j(r) - \overline{\mathbf{E}}_j(r)| \leq \sum_{j=1}^5 |\text{Err}_j^o|,$$

where Err_k^o are as in [21, Section 9] and

$$|\overline{\mathbf{D}}'_j(r) - O(t_j^{\tau_1})\overline{\mathbf{D}}_j(r) - 2\overline{\mathbf{G}}_j(r)| \leq \frac{2}{r} \sum_{k=1}^5 |\text{Err}_k^i|,$$

where Err_k^i are as in [21, Section 9]. Now, revisiting the argument of [21, Proposition 9.14], one can easily keep track that $\mathbf{m}_0$ can be carried throughout the proof to ensure

$$|\overline{\mathbf{D}}_j(r) - \overline{\mathbf{E}}_j(r)| \leq C \mathbf{m}_0^{\tau_1} \overline{\mathbf{D}}_j(r)^{1+\tau_1} + C \mathbf{m}_0 t_j^{2\kappa} \overline{\mathbf{S}}_j(r), \text{ and} \quad (4.5.23)$$

$$|\overline{\mathbf{D}}'_j(r) - \frac{m-2}{r} \overline{\mathbf{D}}_j(r) - 2\overline{\mathbf{G}}_j(r)| \leq C \mathbf{m}_0^{\tau_1} \overline{\mathbf{D}}_j(r)^{\tau_1} \left(\frac{\overline{\mathbf{D}}_j(r)}{r} + \overline{\mathbf{D}}'_j(r) \right) + C \mathbf{m}_0 t_j^{2\kappa} \overline{\mathbf{D}}_j(r). \quad (4.5.24)$$

We use now the rescalings (4.5.11), (4.5.13), and (4.5.14), in (4.5.23) to get

$$\begin{aligned} t_j^{-m} |\mathbf{D}(r) - \mathbf{E}(r)| &= |\overline{\mathbf{D}}_j(t_j^{-1}r) - \overline{\mathbf{E}}_j(t_j^{-1}r)| \leq C\mathbf{m}_0^{\tau_1} \overline{\mathbf{D}}_j(t_j^{-1}r)^{1+\tau_1} + C\mathbf{m}_0 t_j^{2\kappa} \overline{\mathbf{S}}_j(t_j^{-1}r) \\ &= C\mathbf{m}_0^{\tau_1} t_j^{-m(1+\tau_1)} \mathbf{D}(r)^{1+\tau_1} + C\mathbf{m}_0 t_j^{-m-2+2\kappa} \mathbf{S}(r). \end{aligned}$$

Using (4.5.16) and $r \leq t_j$, we get $t_j^{-m} \mathbf{D}(r) \leq C t_j^{-m} r^m \mathbf{D}(r)^{1-\frac{m}{m+2\kappa}} \leq C \mathbf{D}(r)^{1-\frac{m}{m+2\kappa}}$ which together with the last displayed inequality provide

$$\begin{aligned} |\mathbf{D}(r) - \mathbf{E}(r)| &\leq C\mathbf{m}_0^{\tau_1} t_j^{-m\tau_1} \mathbf{D}(r)^{1+\tau_1} + C\mathbf{m}_0 t_j^{-2+2\kappa} \mathbf{S}(r) \\ &= C\mathbf{m}_0^{\tau_1} \left(t_j^{-m} \mathbf{D}(r) \right)^{\tau_1} \mathbf{D}(r) + C\mathbf{m}_0 t_j^{-2+2\kappa} \mathbf{S}(r) \\ &\leq C\mathbf{m}_0^{\tau_1} \mathbf{D}(r)^{1+\tau_1 \left(1-\frac{m}{m+2\kappa}\right)} + C\mathbf{m}_0 t_j^{-2+2\kappa} \mathbf{S}(r). \end{aligned} \tag{4.5.25}$$

We can choose τ_1 slightly smaller accordingly with the exponent $\tau_1 \left(1 - \frac{m}{m+2\kappa}\right)$ to verify (4.5.20). We argue analogously to get (4.5.21) from (4.5.24), we use (4.5.24) and the rescalings (4.5.11), (4.5.14) to obtain that

$$\begin{aligned} t_j^{-m+1} |\mathbf{D}'(r) - \frac{m-2}{r} \mathbf{D}(r) - 2\mathbf{G}(r)| &= |\overline{\mathbf{D}}'_j(t_j^{-1}r) - \frac{(m-2)t_j^{2\kappa}}{r} \overline{\mathbf{D}}_j(t_j^{-1}r) - 2\overline{\mathbf{G}}_j(t_j^{-1}r)| \\ &\leq C\mathbf{m}_0^{\tau_1} \overline{\mathbf{D}}(t_j^{-1}r)^{\tau_1} \left(\frac{\overline{\mathbf{D}}(t_j^{-1}r)}{t_j^{-1}r} + \overline{\mathbf{D}}'(t_j^{-1}r) \right) + C\mathbf{m}_0 t_j^{2\kappa} \overline{\mathbf{D}}(t_j^{-1}r) \\ &= t_j^{-m+1} \left(C\mathbf{m}_0^{\tau_1} t_j^{-m\tau} \mathbf{D}(r)^{\tau_1} \left(\frac{\mathbf{D}(r)}{r} + \mathbf{D}'(r) \right) + C\mathbf{m}_0 t_j^{2\kappa-1} \mathbf{D}(r) \right). \end{aligned}$$

We argue as in (4.5.25) to obtain (4.5.21), for a possibly smaller $\tau_1 > 0$. To prove (4.5.22), we apply (4.5.15) and the fact that $r \leq t_j$ to achieve, for any $\sigma \in (0, 1)$, that

$$t_j^{2\sigma-2} \mathbf{S}(r) \leq t_j^{2\sigma-2} C r^2 \mathbf{D}(r) \leq C r^{2\sigma} \mathbf{D}(r).$$

The last displayed inequality guarantees

$$\begin{aligned}
|\mathbf{D}(r) - \mathbf{E}(r)| &\stackrel{(4.5.25)}{\leq} C\mathbf{m}_0^\tau \mathbf{D}(r)^{1+\tau} + C\mathbf{m}_0 t_j^{-2+2\kappa} \mathbf{S}(r) \\
&\leq C\mathbf{m}_0^\tau \mathbf{D}(r)^{1+\tau} + C\mathbf{m}_0 r^\tau \mathbf{D}(r) \\
&\stackrel{(4.5.16)}{\leq} C\mathbf{m}_0^\tau r^{\tau(m+2\kappa)} \mathbf{D}(r) + C\mathbf{m}_0 r^\tau \mathbf{D}(r) \\
&\leq C\mathbf{m}_0^\tau r^\tau \mathbf{D}(r),
\end{aligned}$$

which in turn ensure that $\mathbf{D}(r)/2 \leq \mathbf{E}(r) \leq 2\mathbf{D}(r)$ provided r is small enough. Hence, we get that (4.5.25) can be turned into

$$\left| \frac{1}{\mathbf{D}(r)} - \frac{1}{\mathbf{E}(r)} \right| \leq C\mathbf{m}_0^\tau \mathbf{D}(r)^{\tau-1} + C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r)}{\mathbf{D}(r)^2}.$$

□

4.5.4 Almost monotonicity of $\mathbf{I}$ at one-sided flat singular points

We are now in a position to prove the almost monotonicity of the universal frequency function $\mathbf{I}$, with improved bounds compared to [27] and in arbitrary dimensions.

Theorem 4.5.19 (Almost monotonicity and jumps of $\mathbf{I}$). *Let T, Γ , and Σ , satisfy Assumptions 4.5.2, 4.5.5, and 4.5.7, with p in place of 0. Then either $T = Q \llbracket p + \mathcal{M}_{p,j} \rrbracket$ in a neighborhood of q , for some $j \in \mathbb{N}$, or $\mathbf{H}_p(p, r) > 0$ and $\mathbf{D}_p(p, r) > 0$, for every $r > 0$. In the latter case, $\mathbf{I}$ is absolutely continuous on each (t_{j+1}, t_j) and there exist $C, \tau > 0$ (not depending on p, j, T , nor ε_{na}) such that for a.e. $r \in (t_{j+1}, t_j)$ it holds*

$$\frac{d}{dr} \left(\log \mathbf{I}_p(p, r) + C\mathbf{m}_p^\tau \mathbf{D}_p(p, r)^\tau - C\mathbf{m}_p t_j^{-2+2\kappa} \frac{\mathbf{S}_p(p, r)}{\mathbf{D}_p(p, r)} \right) \geq -C\mathbf{m}_p^\tau r^{-1+\tau}. \quad (4.5.26)$$

Moreover, we have a bound on the jumps of $\mathbf{I}$ given by

$$\left| \lim_{s \downarrow t_j} \mathbf{I}_p(p, s) - \lim_{s \uparrow t_j} \mathbf{I}_p(p, s) \right| \leq C \mathbf{m}_p^\tau t_j^\tau, \quad \sum_j \left| \lim_{s \downarrow t_j} \mathbf{I}_p(p, s) - \lim_{s \uparrow t_j} \mathbf{I}_p(p, s) \right| \leq C \mathbf{m}_p^\tau, \quad (4.5.27)$$

$$\text{and } \mathbf{I}_p(p, r) \geq e^{-C \mathbf{m}_p^\tau t_j^\tau} \mathbf{I}_p(p, a), \quad (4.5.28)$$

where the last inequality holds for $r \in (t_{j+1}, t_j]$ and any $a \in (0, r)$.

Remark 4.5.20. Due to the excess decay in Chapter 3 at one-sided points, we obtain excellent estimates for the jumps of $\mathbf{I}$, as stated in (4.5.27), which mitigate several difficulties compared to the interior case. In the interior setting, where the uniqueness of tangent cones and excess decay are not known *a priori*, the authors must establish a significantly more delicate and labor-intensive BV-estimate for $\log(\mathbf{I} + 1)$, ensuring similar bounds on the jumps of $\mathbf{I}$ (cf. [28, Proposition 6.2]).

Proof. We drop the dependences on p and reduce to $p = 0$ as in Lemma 4.5.18. Applying now (4.5.18), we get

$$\frac{d}{dr} \log \mathbf{I}(r) = \frac{1}{r} + \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} = \frac{m-2}{r} \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{2\mathbf{E}(r)}{\mathbf{H}(r)} - C \mathbf{m}_0.$$

Plugging the displayed equality above to (4.5.22) and (4.5.21), we get

$$\begin{aligned}
\frac{d}{dr} \log \mathbf{I}(r) &\stackrel{(4.5.22)}{\geq} \frac{\mathbf{D}'(r)}{\mathbf{E}(r)} - C\mathbf{m}_0^\tau \mathbf{D}(r)^{\tau-1} \mathbf{D}'(r) - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r) \mathbf{D}'(r)}{\mathbf{D}(r)^2} \\
&\quad - \frac{2\mathbf{E}(r)}{\mathbf{H}(r)} - C\mathbf{m}_0 \\
&\stackrel{(4.5.21)}{\geq} \frac{2\mathbf{G}(r) - C\mathbf{m}_0^\tau \mathbf{D}(r)^\tau \left(\frac{\mathbf{D}(r)}{r} + \mathbf{D}'(r) \right) - C\mathbf{m}_0 t_j^{2\kappa-1} \mathbf{D}(r)}{\mathbf{E}(r)} \\
&\quad - C\mathbf{m}_0^\tau \mathbf{D}(r)^{\tau-1} \mathbf{D}'(r) - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r) \mathbf{D}'(r)}{\mathbf{D}(r)^2} \\
&\quad - \frac{2\mathbf{E}(r)}{\mathbf{H}(r)} - O(\mathbf{m}_0) \\
&\geq \frac{-C\mathbf{m}_0^\tau \mathbf{D}(r)^\tau \left(\frac{\mathbf{D}(r)}{r} + \mathbf{D}'(r) \right) - C\mathbf{m}_0 t_j^{2\kappa-1} \mathbf{D}(r)}{\mathbf{E}(r)} \\
&\quad - C\mathbf{m}_0^\tau \mathbf{D}(r)^{\tau-1} \mathbf{D}'(r) - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r) \mathbf{D}'(r)}{\mathbf{D}(r)^2} - C\mathbf{m}_0,
\end{aligned}$$

where in the last inequality we applied Cauchy-Schwarz to have $\mathbf{G}(r)\mathbf{H}(r) \geq \mathbf{E}(r)^2$, and hence $2\mathbf{G}(r)/\mathbf{E}(r) - 2\mathbf{E}(r)/\mathbf{H}(r) \geq 0$. By (4.5.20), we have $\mathbf{D}(r)/\mathbf{E}(r) \leq C_0$ which ensures, by the last displayed inequality, that

$$\begin{aligned}
\frac{d}{dr} \log \mathbf{I}(r) &\geq -C\mathbf{m}_0^\tau \left(\frac{\mathbf{D}(r)^\tau}{r} + \frac{\mathbf{D}'(r)}{\mathbf{D}(r)^{1-\tau}} \right) - C\mathbf{m}_0 t_j^{2\kappa-1} - C\mathbf{m}_0^\tau \frac{\mathbf{D}'(r)}{\mathbf{D}(r)^{1-\tau}} \\
&\quad - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r) \mathbf{D}'(r)}{\mathbf{D}(r)^2} - C\mathbf{m}_0.
\end{aligned}$$

We absorb the $C\mathbf{m}_0$ term in the $\mathbf{m}_0 t_j^{2\kappa-1}$ term, $-\frac{\mathbf{S}(r) \mathbf{D}'(r)}{\mathbf{D}(r)^2} = \frac{d}{dr} \frac{\mathbf{S}(r)}{\mathbf{D}(r)} - \frac{\mathbf{S}'(r)}{\mathbf{D}(r)}$ and $\frac{d}{dr} \mathbf{D}(r)^\tau = \tau \frac{\mathbf{D}'(r)}{\mathbf{D}(r)^{1-\tau}}$, we derive

$$\begin{aligned}
\frac{d}{dr} \left(\log \mathbf{I}(r) + C\mathbf{m}_0^\tau \mathbf{D}(r)^\tau - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r)}{\mathbf{D}(r)} \right) &\geq -C\mathbf{m}_0^\tau \frac{\mathbf{D}(r)^\tau}{r} - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}'(r)}{\mathbf{D}(r)} \\
&\quad - C\mathbf{m}_0 t_j^{2\kappa-1}.
\end{aligned}$$

Using (4.5.16) and (4.5.15), possibly taking τ smaller, we get

$$\begin{aligned} \frac{d}{dr} \left(\log \mathbf{I}(r) + C\mathbf{m}_0^\tau \mathbf{D}(r)^\tau - C\mathbf{m}_0 t_j^{-2+2\kappa} \frac{\mathbf{S}(r)}{\mathbf{D}(r)} \right) &\geq -C\mathbf{m}_0^\tau r^{-1+\tau} - C\mathbf{m}_0 t_j^{2\kappa-1} \\ &\quad - C\mathbf{m}_0 t_j^{-2+2\kappa} r \\ &\stackrel{t_j > r}{\geq} -C\mathbf{m}_0^\tau r^{-1+\tau}. \end{aligned}$$

This finishes the proof of (4.5.26). The proof of the first estimate in (4.5.27) is given in [27, Theorem 15.5, Part II], where the authors do not use $m = 2$ in their proof. More precisely the desired estimate is [27, Equation 17.7], which is a consequence of [27, Proposition 17.1] and [27, Proposition 17.2]. The only difference in our case is that we have the additional term coming from Σ , precisely $|\mathbf{I}(t_j^+) - \mathbf{I}(t_j^-)|$ is controlled by $\mathbf{E}(T, 6t_j)^\tau + \mathbf{A}_\Sigma^{2\tau}$ which is bounded by $\mathbf{m}_0^\tau t_j^{\tau\alpha}$ due to the excess decay in Chapter 3. Here, the superscripts $+$ and $-$ denote the right- and left-hand side limits, respectively. The second estimate of the jumps in (4.5.27) follows immediately from $t_{j+1}/t_j \leq \frac{1}{2}$.

We now prove (4.5.28) using (4.5.27). Fix $r \in (t_{j_r+1}, t_{j_r}]$, $a \in (t_{j_a+1}, t_{j_a}]$, $r > a$, and integrate (4.5.27) as follows

$$\int_a^r \frac{d}{ds} \left(\log \mathbf{I}(s) + C\mathbf{m}_0^\tau \mathbf{D}(s)^\tau - C\mathbf{m}_0 t_{i(s)}^{-2+2\kappa} \frac{\mathbf{S}(s)}{\mathbf{D}(s)} \right) ds \geq -C\mathbf{m}_0^\tau (r^\tau - a^\tau), \quad (4.5.29)$$

where $i(s)$ denote the index for which $s \in (t_{j(s)+1}, t_{j(s)})$. We now notice that

$$\begin{aligned} \int_a^r \frac{d}{ds} \left(-t_{i(s)}^{-2+2\kappa} \frac{\mathbf{S}(s)}{\mathbf{D}(s)} \right) ds &= - \sum_{i=j_r}^{j_a-1} \left(-t_i^{-2+2\kappa} \frac{\mathbf{S}(t_i^-)}{\mathbf{D}(t_i^-)} + t_{i+1}^{-2+2\kappa} \frac{\mathbf{S}(t_{i+1}^+)}{\mathbf{D}(t_{i+1}^+)} \right) \\ &\quad - t_{j_r}^{-2+2\kappa} \frac{\mathbf{S}(r)}{\mathbf{D}(r)} + t_{j_r}^{-2+2\kappa} \frac{\mathbf{S}(t_{j_r}^+)}{\mathbf{D}(t_{j_r}^+)} \\ &\quad - t_{j_a}^{-2+2\kappa} \frac{\mathbf{S}(t_{j_a}^-)}{\mathbf{D}(t_{j_a}^-)} + t_{j_a}^{-2+2\kappa} \frac{\mathbf{S}(a)}{\mathbf{D}(a)}. \end{aligned}$$

We discard the negative terms and we use (4.5.15) to turn the last displayed equation into

$$\int_a^r \frac{d}{ds} \left(-C\mathbf{m}_0 t_{i(s)}^{-2+2\kappa} \frac{\mathbf{S}(s)}{\mathbf{D}(s)} \right) ds \leq C\mathbf{m}_0 \left(\sum_{i=j_r}^{j_a-1} t_i^{2\kappa} + t_{j_r}^{2\kappa} + t_{j_a}^{2\kappa} \right) \leq C\mathbf{m}_0 t_{j_r}^{2\kappa}, \quad (4.5.30)$$

where we used that $t_{j_a} \leq t_i < r \leq t_{j_r}$, $i \in \{j_r + 1, \dots, j_a\}$ and $t_{i+1}/t_i < 2^{-5}$ (by construction of the intervals of flattening). We argue analogously, this time using (4.5.16), to obtain

$$\begin{aligned} \int_a^r \frac{d}{ds} (C\mathbf{m}_0^\tau \mathbf{D}(s)^\tau) ds &= C\mathbf{m}_0^\tau \left(\mathbf{D}(r)^\tau - (\mathbf{D}(t_{j_r}^+))^\tau + \sum_{i=j_r}^{j_a-1} (\mathbf{D}(t_i^-))^\tau - (\mathbf{D}(t_{i+1}^+))^\tau \right. \\ &\quad \left. (\mathbf{D}(t_{j_a}^-))^\tau - \mathbf{D}(a)^\tau \right) \\ &\leq C\mathbf{m}_0^\tau \left(r^{m+2\kappa} + \sum_{i=j_r}^{j_a} t_i^{m+2\kappa} \right) \leq C\mathbf{m}_0^\tau t_{j_r}^{m+2\kappa}. \end{aligned} \quad (4.5.31)$$

We plug (4.5.30) and (4.5.31) into (4.5.29) to obtain (possibly reducing $\tau > 0$) that

$$\int_a^r \frac{d}{ds} (\log \mathbf{I}(s)) ds \geq -C\mathbf{m}_0^\tau (r^\tau - a^\tau) - C\mathbf{m}_0^\tau t_{j_r}^{2\kappa} \geq -C\mathbf{m}_0^\tau t_{j_r}^\tau.$$

Finally, we have that

$$\log \mathbf{I}(r) - \log \mathbf{I}(a) = \int_a^r \frac{d}{ds} (\log \mathbf{I}(s)) ds + \sum_{i=j_r}^{j_a} (\log \mathbf{I}(t_i^+) - \log \mathbf{I}(t_i^-)).$$

Since $|\log(x) - \log(y)| \leq C \frac{|x-y|}{\min\{|x|, |y|\}}$ and, by (4.5.15), $\mathbf{I}(s) \geq C^{-1}$, we derive that

$$\begin{aligned} \sum_{i=j_r}^{j_a} \log \mathbf{I}(t_i^+) - \log \mathbf{I}(t_i^-) &\geq -C \sum_{i=j_r}^{j_a} |\mathbf{I}(t_i^+) - \mathbf{I}(t_i^-)| \\ &\stackrel{(4.5.27)}{\geq} -C\mathbf{m}_0^\tau \sum_{i=j_r}^{j_a} t_i^\tau \geq -C\mathbf{m}_0^\tau t_{j_r}^\tau. \end{aligned}$$

Putting the last three displayed inequalities together, we guarantee that $\log \mathbf{I}(r) - \log \mathbf{I}(a) \geq -C\mathbf{m}_0^\tau t_j^\tau$ which in turn ensures (4.5.28). $\square$

4.6 Uniform lower bound and the frequency regimes

4.6.1 Uniform lower bound for I at one-sided flat singular points

In this section, the main result is Theorem 4.6.5, which, among other implications, establishes a *uniform* lower bound of $1 + \alpha/2$ for the frequency values, where $\alpha = \alpha(m, n, \bar{n}, Q) > 0$ is the constant from the excess decay result in Chapter 3. This lower bound marks a significant distinction between the interior setting studied in [28, 29, 24] and the boundary setting considered here.

We now lay the groundwork for the proof of this crucial lower bound. For a small constant $\bar{\varepsilon} > 0$ (to be specified later) satisfying $\bar{\varepsilon} > \max\{\mathbf{A}_\Sigma, \mathbf{A}_\Gamma\}$, and adopting the notation from Section 4.5.2, we set

$$\mathbf{m}_{p,j} := \max\left\{\mathbf{E}^b(T_{p,j}, \mathbf{B}_2), \bar{\varepsilon} t_j^{2-2\alpha_e}\right\}.$$

Assumption 4.6.1. Let T, Σ and Γ be as in Assumption 4.1.8 and p be a flat one-sided singular point for T .

Remark 4.6.2. Due to Chapter 3, we can guarantee for a sufficiently small scale $\bar{r}$ all Assumptions 4.5.2, 4.5.5, and 4.5.7 are satisfied for $r \leq \bar{r}$. We can also ensure that $\bar{\varepsilon} \bar{r}^{2-2\alpha_e} < \varepsilon_{na}$.

Definition of fine blowups

Let T be as in Assumption 4.6.1 and let $\{t_j\}_{j \in \mathbb{N}}$ be the endpoints of the intervals of flattening as constructed in Section 4.5.2. Now, assume that $r_k \leq \bar{r}$, $r_k \in [t_{j(k)+1}, t_{j(k)}]$. We will also use all notation for the rescaled objects of Section 4.5.2.

Next, we define the following sequence of Q -valued functions:

$$u_k := \frac{\bar{N}_k \circ \exp_{0, \bar{M}_k}}{\mathbf{h}_k}, \text{ where } \mathbf{h}_k := \left\| \bar{N}_k \right\|_{L^2\left(\exp_{0, \bar{M}_k}(\mathbf{B}_{3/2} \cap \pi_k^+)\right)},$$

where $\exp_{0, \bar{M}_k}$ is the exponential map and π_k is the optimal m -plane for $\mathbf{E}^b(T_{p,j(k)}, \mathbf{B}_2)$.

Under these conditions, there exists a subsequence of $\{r_k\}_k$ and a Dirichlet minimizing map $u \in W^{1,2}(B_{3/2}^+)$ with $\boldsymbol{\eta} \circ u = 0$ and $\|u\|_{L^2(B_{3/2}^+)} = 1$ satisfying, up to rotation, that

$$u_k \rightarrow u \text{ strongly on } W_{loc}^{1,2}(B_{3/2}^+, \mathcal{A}_Q(\mathbb{R}^n)) \cap L^2(B_{3/2}^+, \mathcal{A}_Q(\mathbb{R}^n)).$$

Here, convergence is understood after applying a suitable rotation that maps the domains of u_k into π_0 .

We also have that Γ_k converges to a flat boundary and the function u vanishes on that flat boundary (indeed, this is a direct consequence of the convergence of Γ_k and Definition 4.5.9(iii)).

Definition 4.6.3 (Fine blowups). Any function u generated by the procedure above applied to T_{p,r_k} will be called a *fine blowup with respect to $r_k \downarrow 0$ at p* . Moreover, we define the *singularity degree of T at the flat singular point p* as

$$\mathbf{I}(T, p) := \inf \{I_u(0) : u \text{ is a fine blowup at } p \text{ w.r.t. to some } r_k \downarrow 0\}.$$

In the definition above, $I_u(0) := \lim_{r \downarrow 0} I_u(r)$ and $I_u(r)$ is the value of the smoothed frequency function of u at 0, i.e.,

$$I_u(r) := \frac{r D_u(r)}{H_u(r)}, \quad D_u(r) := \int |Du(y)|^2 \phi\left(\frac{d(y)}{r}\right),$$

$$\text{and } H_u(r) := - \int \frac{|u(y)|^2}{d(y)} \phi'\left(\frac{d(y)}{r}\right) |\nabla d(y)|^2,$$

where the cut-off ϕ is chosen as before, i.e.,

$$\phi(t) := \begin{cases} 1, & t \in [0, 1/2], \\ 2 - 2t, & t \in [1/2, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.6.4. We also consider a sequence of currents T_k that does not consist solely of rescalings of the same current, as required in some compactness arguments in [29]. For our purposes (and also for [29]), it suffices that the limit u is a solution to the corresponding linear problem and satisfies the following properties: it is Dirichlet minimizer, average-free (i.e., $\boldsymbol{\eta} \circ u \equiv 0$), nontrivial, and has a flat boundary with zero boundary data.

Main lower bound for the universal frequency function

We now provide a necessary and sufficient condition for having finitely many intervals of flattening, based on the frequency value. We also prove the uniform lower bound for $\mathbf{I}(T, p)$ and demonstrate how to derive the decay rate of the excess, depending on $\mathbf{I}(T, p)$.

Theorem 4.6.5 (Universal lower bound for the frequency function). *Let T, Γ, Σ , and p , satisfy Assumption 4.6.1. Then, the following holds:*

- (i) *for any fine blowup u at p , u is $\mathbf{I}(T, p)$ -homogeneous and $\lim_{r \downarrow 0} \mathbf{I}_p(p, r) = \mathbf{I}(T, p) = I_u(0)$,*
- (ii) *if $\mathbf{I}(T, p) > 2 - \alpha_{\mathbf{e}}$, then $t_{J+1} = 0$ for some $J \in \mathbb{N}$. Conversely, if $t_{J+1} = 0$ for some $J \in \mathbb{N}$, then $\mathbf{I}(T, p) \geq 2 - \alpha_{\mathbf{e}}$.*
- (iii) *$\mathbf{I}(T, p) \geq 1 + \alpha/2$, where $\alpha = \alpha(m, n, \bar{n}, Q) > 0$ is the excess decay exponent in Chapter 3, and $\inf_j \frac{t_{j+1}}{t_j} > 0$ unless $\{t_j\}_j$ is finite.*

In order to prove the above theorem, we first establish an improved excess decay estimate and an intermediate lower bound (by 1) for the frequency values.

Lemma 4.6.6. *Let T, Γ, Σ , and p , satisfy Assumption 4.6.1. For any fine blow-up u at p , we have $\lim_{r \rightarrow 0} \mathbf{I}_p(p, r) = I_u(0) = \mathbf{I}(T, p) \geq 1$ and u being $\mathbf{I}(T, p)$ -homogeneous. Moreover, if $\mathbf{I}(T, p) > 1$, then*

$$\mathbf{E}(T, \mathbf{B}_r) \leq C t_j^\beta, \text{ for any } \beta < \min\{2(\mathbf{I}(T, p) - 1), 2 - 2\alpha_{\mathbf{e}}\} \text{ and } r \in (t_{j+1}, t_j).$$

Remark 4.6.7. The moreover part of Lemma 4.6.6 is not used to prove ((iii)) in Theorem 4.6.5, we only use the work of Chapter 3 instead. Hence, *a fortiori*, using ((iii)) in Theorem 4.6.5, we are always under the assumption $\mathbf{I}(T, p) > 1$.

Proof. Assume $p = 0$. Due to Lemma 3.8.16, for any fine blow-up u , we have $I_u(0) \geq 1$, thus $\mathbf{I}(T, 0) \geq 1$. Notice that $I_0 := \lim_{r \rightarrow 0} \mathbf{I}_p(p, r)$ exists due to (4.5.28) in Theorem 4.5.19. We take any sequence $s_k \in (\frac{t_{j(k)+1}}{r_k}, \frac{t_{j(k)}}{r_k}]$, if $s_k \rightarrow s$, we must have $\mathbf{I}_p(p, s_k r_k) \rightarrow I_u(s)$ and thus, since $s_k r_k \rightarrow 0$, we have $I_u(s) = I_0$. This implies that if

$$s^+ := \limsup_{k \rightarrow \infty} \frac{t_{j(k)}}{r_k}, \quad s^- := \liminf_{k \rightarrow \infty} \frac{t_{j(k)+1}}{r_k}.$$

then $I_u(s) = I_0$ for any $s \in [s^-, s^+]$. By construction of the intervals of flattening, we have that $\frac{s^-}{s^+} \leq \frac{1}{2}$. By Lemma 4.3.5, we must have that u is I_0 -homogeneous, since its frequency is constant on the radii in $[s^-, s^+]$. For the moreover part, we fix $0 < \beta < \min\{2(I_0 - 1), 2 - 2\alpha_e\}$.

We show that it is enough to prove

$$\mathbf{E}(T, \mathbf{B}_{2t_{j+\kappa}}) \leq \left(\frac{t_{j+\kappa}}{t_j} \right)^\beta \mathbf{m}_{0,j}. \quad (4.6.1)$$

Indeed, if $\mathbf{m}_{0,j} = \bar{\varepsilon}^2 t_j^{2-2\alpha_e}$, since $2 - 2\alpha_e > \beta$, we obtain

$$\mathbf{m}_{0,j+\kappa} = \max\{\mathbf{E}(T, \mathbf{B}_{2t_{j+\kappa}}), \bar{\varepsilon}^2 t_{j+\kappa}^{2-2\alpha_e}\} \stackrel{(4.6.1)}{\leq} \left(\frac{t_{j+\kappa}}{t_j} \right)^\beta \bar{\varepsilon}^2 t_j^{2-2\alpha_e} = \left(\frac{t_{j+\kappa}}{t_j} \right)^\beta \mathbf{m}_{0,j}.$$

If $\mathbf{m}_{0,j} = \mathbf{E}(T, \mathbf{B}_{2t_j})$, then $\mathbf{E}(T, \mathbf{B}_{2t_j}) \geq \bar{\varepsilon}^2 t_j^{2-2\alpha_e}$ and, since $2 - 2\alpha_e > \beta$, thus

$$\bar{\varepsilon}^2 t_{j+\kappa}^{2-2\alpha_e} = \bar{\varepsilon}^2 \left(\frac{t_{j+\kappa}}{t_j} \right)^\beta \left(\frac{t_{j+\kappa}}{t_j} \right)^{2-2\alpha_e-\beta} t_j^{2-2\alpha_e} \leq \left(\frac{t_{j+\kappa}}{t_j} \right)^\beta \mathbf{E}(T, \mathbf{B}_{2t_j}) = \left(\frac{t_{j+\kappa}}{t_j} \right)^\beta \mathbf{m}_{0,j}.$$

The proof of (4.6.1) goes along the same lines as the proof of [28, Equation (88)]. $\square$

We now state the following inequality that will be crucial to prove Theorem 4.6.5 and is a consequence of a compactness argument demonstrated below.

Lemma 4.6.8. *Let T, Γ, Σ , and p , satisfy Assumption 4.6.1. For any $\beta > \mathbf{I}(T, p)$, we have, for j sufficiently large and any $\rho, r \in (t_{j+1}, t_j]$ with $\rho < r$, that*

$$\left(\frac{\rho}{r}\right)^{m+2(\beta-1)} \int_{\mathcal{M}_{p,j} \cap \mathbf{B}_r} |D\mathcal{N}_{p,j}|^2 \leq \int_{\mathcal{M}_{p,j} \cap \mathbf{B}_\rho} |D\mathcal{N}_{p,j}|^2. \quad (4.6.2)$$

Proof. Assume $p = 0$. It is enough to show that this inequality holds for r small enough with $t_{j+1} < \rho = r/2 < r \leq t_j$. Assume by contradiction that there exists $\beta_0 > \mathbf{I}(T, 0)$ such that for every $k \in \mathbb{N}$, there exists $j(k) \geq k$ and $r_{j(k)} \in (t_{j(k)+1}, t_{j(k)}]$ such that

$$\left(\frac{1}{2}\right)^{m+2(\beta_0-1)} \int_{\mathcal{M}_{j(k)} \cap \mathbf{B}_{r_{j(k)}}} |D\mathcal{N}_{j(k)}|^2 > \int_{\mathcal{M}_{j(k)} \cap \mathbf{B}_{r_{j(k)}/2}} |D\mathcal{N}_{j(k)}|^2.$$

Take the fine blowup u_0 with respect to $r_{j(k)}$, by Lemma 4.6.6, we obtain that u_0 is $I_{u_0}(0)$ -homogeneous and $I_{u_0}(0) = \mathbf{I}(T, 0)$. By the natural rescaling of the Dirichlet energy, see [30] and Section 3.8 of this thesis, and taking the limit in the last displayed inequality, we would force that $m + 2(\beta_0 - 1) \leq m + 2(I_{u_0}(0) - 1)$ which gives $\beta_0 \leq I_{u_0}(0) = \mathbf{I}(T, 0) < \beta_0$ which gives the contradiction. $\square$

We are now ready to apply the estimates from the construction of the center manifolds and normal approximations in Section 4.5.1, and combine them with the estimates from the previous lemma to prove the main theorem of this subsection (Theorem 4.6.5).

Proof of Theorem 4.6.5. Assume $p = 0$. The item (i) is settled in Lemma 4.6.6. We first prove (ii) assuming that the intervals of flattening are finitely many. So, take $J \in \mathbb{N}$ such that $t_{J+1} = 0$. We argue as in [34, Remark 3.4] (using Theorem 4.5.10) to obtain that

$$\int_{\mathbf{B}_\rho \cap \mathcal{M}_J} |D\mathcal{N}_J|^2 \leq C \rho^{m+2-2\alpha_e} \mathbf{m}_{0,J}, \text{ for every } \rho \in (0, t_J].$$

By Lemma 4.6.8, we get $2 - 2\alpha_e < 2(\beta - 1)$, for any $\beta > \mathbf{I}(T, 0)$. This then gives $\mathbf{I}(T, 0) \geq 2 - \alpha_e$. We prove the opposite implication to conclude the proof of (ii). We

assume $\mathbf{I}(T, 0) > 2 - \alpha_e$ and that the intervals of flattening are infinitely many. By (4.5.9), we obtain

$$\int_{\mathbf{B}_{t_{j+1}} \cap \mathcal{M}_j} |D\mathcal{N}_j|^2 \geq C \mathbf{m}_{0,j} \left(\frac{t_{j+1}}{t_j} \right)^{m+2-2\alpha_e}. \quad (4.6.3)$$

By Lemma 4.6.6, we have $\mathbf{E}(T, \mathbf{B}_r) \leq Cr^\beta$ and, since $\mathbf{I}(T, 0) > 2 - \alpha_e$, we can choose $\beta > 2 - 2\alpha_e$. This in turn guarantees that, for a sufficiently large j , $\mathbf{m}_{0,j} = \bar{\varepsilon}^2 t_j^{2-2\alpha_e}$. Then, we derive

$$\mathbf{E}(T, \mathbf{B}_r) \leq Cr^{\beta-(2-2\alpha_e)} \frac{r^{2-\alpha_e}}{(\bar{\varepsilon}^2 t_j)^{2-2\alpha_e}} \mathbf{m}_{0,j} \leq Cr^{\beta-(2-2\alpha_e)} \left(\frac{t_{j+1}}{t_j} \right)^{2-2\alpha_e} \mathbf{m}_{0,j}, \forall r \in (t_{j+1}, t_j].$$

Additionally, the last displayed inequality implies

$$\int_{\mathcal{M}_j \cap \mathbf{B}_{t_{j+1}}} |D\mathcal{N}_j|^2 \stackrel{(4.5.10)}{\leq} \tilde{C} \left(\frac{t_{j+1}}{t_j} \right)^m \mathbf{E}(T_j, \mathbf{B}_L) \leq Ct_{j+1}^{\beta-(2-2\alpha_e)} \left(\frac{t_{j+1}}{t_j} \right)^{m+2-2\alpha_e} \mathbf{m}_{0,j}.$$

By the last displayed inequality and (4.6.3), we get that

$$C \mathbf{m}_{0,j} \left(\frac{t_{j+1}}{t_j} \right)^{m+2-2\alpha_e} \leq \int_{\mathbf{B}_{t_{j+1}} \cap \mathcal{M}_j} |D\mathcal{N}_j|^2 \leq Cr^{\beta-(2-2\alpha_e)} \left(\frac{t_{j+1}}{t_j} \right)^{m+2-2\alpha_e} \mathbf{m}_{0,j},$$

for any $r \in (t_{j+1}, t_j]$. Due to the choice of β , which ensures $\beta - (2 - 2\alpha_e) > 0$, we obtain a contradiction from the last displayed inequality for j sufficiently large. Thus, $\{t_j\}_j$ has to be finite, and this concludes the proof of ((ii)).

Let us prove ((iii)), note that if we have finitely many intervals of flattening, ((ii)) implies ((iii)). Assume then that $t_j \neq 0$ for all j . We apply (4.5.9) and the results of Chapter 3 to get

$$\int_{\mathcal{M}_j \cap \mathbf{B}_{t_{j+1}}} |D\mathcal{N}_j|^2 \leq \int_{\mathcal{M}_j \cap \mathbf{B}_{\frac{t_{j+1}}{t_j}}} |D\mathcal{N}|^2 \leq C \left(\frac{t_{j+1}}{t_j} \right)^m \mathbf{E}(T, \mathbf{B}_L^b) \leq \tilde{C} \mathbf{m}_{0,j} \left(\frac{t_{j+1}}{t_j} \right)^{m+\alpha}.$$

We apply (4.5.10) and Lemma 4.6.8 to obtain, for any $\rho \in (t_{j+1}, t_j]$ and any $\beta > \mathbf{I}(T, 0)$, that

$$C\mathbf{m}_{0,j} \leq \int_{\mathcal{M}_j \cap \mathbf{B}_{t_j}} |D\mathcal{N}|^2 \leq \left(\frac{\rho}{t_j}\right)^{-m-2(\beta-1)} \int_{\mathcal{M}_j \cap \mathbf{B}_\rho} |D\mathcal{N}|^2. \quad (4.6.4)$$

Combining the last two displayed inequalities, we arrive at

$$C\mathbf{m}_{0,j} \left(\frac{\rho}{t_j}\right)^{m+2(\beta-1)} \leq \tilde{C}\mathbf{m}_{0,j} \left(\frac{t_{j+1}}{t_j}\right)^{m+\alpha}, \text{ for any } \rho \in (t_{j+1}, t_j] \text{ and any } \beta > \mathbf{I}(T, 0),$$

which, for j large enough, guarantees that $\beta \geq 1 + \alpha/2$ for any $\beta > \mathbf{I}(T, 0)$. Thus, $\mathbf{I}(T, 0) \geq 1 + \alpha/2$. We now need to show $\inf_j \frac{t_{j+1}}{t_j} > 0$ assuming that $\{t_j\}_j$ is infinite. In light of ((ii)) and the first part of ((iii)), we have $1 + \alpha/2 \leq \mathbf{I}(T, 0) < 2 - \alpha_e$. Let us assume by contradiction that $\inf_j \frac{t_{j+1}}{t_j} = 0$. Arguing as in (4.6.4) and [34, Remark 3.4] (same argument performed above), we obtain that

$$C\mathbf{m}_{0,j} \left(\frac{t_{j+1}}{t_j}\right)^{m+2(\beta-1)} \leq \int_{\mathbf{B}_{t_{j+1}} \cap \mathcal{M}_j} |D\mathcal{N}_j|^2 \leq \tilde{C}\mathbf{m}_{0,j} \left(\frac{t_{j+1}}{t_j}\right)^{m+2-2\alpha_e},$$

for any $\beta > \mathbf{I}(T, 0)$, hence $\mathbf{I}(T, 0) \geq 2 - \alpha_e$. This is a contradiction for j sufficiently large, since $\inf_j \frac{t_{j+1}}{t_j} = 0$ and $\mathbf{I}(T, 0) < 2 - \alpha_e$. Thus, we conclude the proof of ((iii)). $\square$

4.6.2 High-frequency regime

This subsection addresses the high-frequency regime, where the frequency is at least almost quadratic and, by Lemma 4.6.6, an almost-quadratic excess decay holds. In this regime, we define the frequency pinching and prove the main lemma of this subsection (Lemma 4.6.14).

To this end, we introduce a new exponent, denoted by δ_e (which corresponds to δ_3 in the notation of [29]), and require that $2 - 2\delta_e < 2 - 2\alpha_e$. This exponent δ_e will replace α_e in the “stopping by the excess condition” (see Section 4.5.1), while α_h in the “stopping by the height condition” remains unchanged.

The exponent δ_e will then be used to construct the center manifold and the normal approximation of Section 4.5.1, namely $\mathcal{M}_{p,J}^{\delta_e}$ and $\mathcal{N}_{p,J}^{\delta_e}$. For convenience, we will drop the dependence on δ_e in the notation.

Assumption 4.6.9. Let T, Γ, Σ , and p , satisfy Assumption 4.6.1, $\mathbf{I}(T, p) > 2 - \delta_e$, and let $J + 1$ be the index of Theorem 4.6.5 ((ii)) for which $t_{J+1} = 0$. Assume also that $q \in \Gamma_{p,J}$.

In the high-frequency regime, given the quasi-quadratic excess decay in Lemma 4.6.6, there are no boundary-stopping cubes. In other words, the contact set contains the boundary, as stated below.

Lemma 4.6.10 (Boundary cubes never stop in the high-frequency regime). $\mathcal{C}_j^b \cap \mathcal{W}_i = \emptyset$ for every $i, j \geq N_0$ and, in particular, $\Gamma_{p,J} \subset \Phi_{p,J}(\mathbf{S}_{p,J})$.

Proof. This proof is entirely analogous to [68, Lemma 6.5] using $\mathbf{I}(T, p) > 2 - \delta_e$ and the almost quadratic excess decay provided by Lemma 4.6.6. $\square$

Lemma 4.6.11 (Radial variations). *Under Assumption 4.6.9, all the consequences stated in Lemma 4.5.18 are valid for p, q , and every radius $r \in (0, t_J)$.*

Proof. The proof will be omitted here, since it consists of a mere repetition of the arguments in Lemma 4.5.18, Theorem 4.5.19, and [27], taking into consideration that they only depend on the following property: every cube $L \in \mathcal{W}$ such that $L \cap B_r(\mathbf{p}(q)) \neq \emptyset$ cannot have side length larger than $c_0 r$ for a dimensional constant $c_0 > 0$. This property in turn follows directly from Lemma 4.6.10 and (4.5.7). $\square$

We define frequency pinching, which estimates the error in the frequency when measured at two different scales at the same boundary point q .

Definition 4.6.12 (Frequency pinching). Under Assumption 4.6.9. Let ρ, r be two radii which satisfy the inequalities $0 < \rho \leq r \leq t_J$. We then define the *frequency pinching* $W_\rho^r(p, q)$ around q between the scales ρ and r to be the following quantity

$$W_\rho^r(p, q) := |\mathbf{I}_p(q, r) - \mathbf{I}_p(q, \rho)|, \text{ for any } q \in \Gamma_{p,J}.$$

With the very same argument of [29, Lemma 4.1], we can ensure the existence of Λ depending on T such that

$$C^{-1} \leq \mathbf{I}_p(q, r) \leq \Lambda, \quad \forall q \in \Gamma_{p,J}, \quad \forall r \in (0, t_J]. \quad (4.6.5)$$

We will now prove a lemma that will be useful in establishing the dimensional bound for the set of one-sided flat singular boundary points in Section 4.7 and also for proving the rectifiability results in Section 4.8. To do this, we define the following.

Definition 4.6.13. We say that a set $S = \{x_0, x_1, \dots, x_k\} \subset \mathbf{B}_r(x)$ is *ρr -linearly independent* if

$$d(x_i, \text{span}(\{x_0, \dots, x_{i-1}\})) \geq \rho r, \quad \text{for all } i = 1, \dots, k.$$

We say that a set $F \subset \mathbf{B}_r(x)$ *ρr -spans an k -dimensional affine subspace V* if there is a ρr -linearly independent set of points $S = \{x_i\}_{i=0}^k \subset F$ such that $V = \text{span}(S)$.

We now demonstrate that it is not possible to have $m-1$ quantitatively linearly independent points that exhibit small pinching simultaneously.

Lemma 4.6.14. *Assume Assumption 4.6.9 and $\rho \in (0, 1]$. There exists a suitable threshold $\delta_1 = \delta_1(m, n, Q, \rho) > 0$, with the following property. Take $r \in (0, \bar{r})$ (where $\bar{r}$ is from Remark 4.6.2) and an ρr -linearly independent set*

$$S = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(p) \cap \text{Sing}_b(T) \cap \{q : \Theta^m(T, q) = Q/2 \text{ and } q \text{ flat}\}.$$

Then, there exist $i \in \{0, \dots, m-2\}$ and $r_i \in (0, r)$ such that, for any $y \in \mathbf{B}_{r_i}(x_i) \cap \text{Sing}_b(T) \cap \{q : \Theta^m(T, q) = Q/2 \text{ and } q \text{ flat}\}$, we have

$$W_{r_i}^{2r_i}(p, y) \geq \delta_1.$$

Proof. The proof of the lemma follows the same lines as the contradiction argument in [29, Lemma 6.2]. The main difference being that the authors contradict the *interior* regularity for the linear problem (in [30, 23]) with their argument, we instead run the very same contradiction argument in order to get a fine blow-up u that is I_0 -homogeneous with respect to every point in S and $I_0 \geq 1 + \alpha/2$. Then Lemma 4.3.2 gives us a function f which is I_0 -homogeneous 2d Dirichlet minimizer with zero boundary data on a flat boundary and $I_0 \geq 1 + \alpha/2$. This contradicts Theorem 4.3.3, which says that the frequency of f should be exactly 1. $\square$

4.6.3 Low-frequency regime

We now address the case of low-frequency, where we do not have an almost-quadratic decay. The proofs in the low-frequency regime follow a similar approach to those in the high-frequency regime from the previous section. However, one of the main challenges in this regime is that the intervals of flattening are infinitely many (Theorem 4.6.5((ii))). Although this introduces several technical difficulties, which will be explained and addressed in the following, it does not fundamentally alter the conceptual approach for proving rectifiability.

Setting up uniform bounds in a suitable partition

We define the following sets for $K \in \mathbb{N} \setminus \{0\}$:

$$\mathfrak{G}_K := \text{Sing}_b(T) \cap \left\{ q : \Theta^m(T, q) = Q/2, q \text{ is flat, and } \mathbf{I}_q(T, q) \leq 2 - \alpha_e - \frac{1}{2K} \right\} \cap \mathbf{B}_2(p).$$

Since the main goal of this work is to prove the rectifiability of the set of one-sided flat singular boundary points (Proposition 4.8.6), which is equivalent to proving the rectifiability of $\cup_{1 \leq K \in \mathbb{N}} \mathfrak{G}_K$, we will begin by studying each $\mathfrak{G}_K$ individually.

Assumption 4.6.15. Let T, Γ , and Σ satisfy Assumption 4.1.8, $p \in \mathfrak{G}_K$, and η be a given small number to be specified later.

We take $\gamma \in (0, 1/2]$ to be a geometric constant, depending on m, n, Q , as in [29, Section 10.1], and denote by $\mathcal{M}_{p, \gamma^j}$ and $\mathcal{N}_{p, \gamma^j}$ the center manifold and normal approximation, respectively, with respect to $T_{p, \gamma^j}, \Gamma_{p, \gamma^j}$, and the intervals $(\gamma^{j+1}, \gamma^j]$ (which replace the intervals of flattening; see also [29, Section 10.1]).

This constant γ ensures the uniformity of certain estimates used below. For example, an important feature now is that $\inf_j \frac{\gamma^{j+1}}{\gamma^j} = \gamma > 0$, as opposed to Theorem 4.6.5((iii)), which only guarantees that $\inf_j \frac{s_{j+1}}{s_j} > 0$ with no uniform lower bound.

Lemma 4.6.16 (Radial variations). *Under Assumption 4.6.15. All assertions stated in Lemma 4.5.18 are valid for p, q , and*

$$4\eta\gamma^{j+1} < \rho \leq r \leq 2\gamma^j \text{ and } q \in \Gamma_{p, \gamma^j}.$$

Proof. The proof will be omitted here, since it consists of a mere repetition of the arguments in Lemma 4.5.18, Theorem 4.5.19, and [27], taking into consideration that they only depend on the following property: every cube $L \in \mathcal{W}$ such that $L \cap B_r(\mathbf{p}(q)) \neq \emptyset$ cannot have side length larger than $c_0 r$ for a dimensional constant $c_0 > 0$. This property in turn follows directly from the assumption that $r > 4\eta\gamma^{j+1}$ and (4.5.7). $\square$

We state here the counterpart, for the low-frequency regime, of Lemma 4.6.14. In order to do this, we need to define the frequency pinching in this regime.

Definition 4.6.17 (Frequency pinching). Under Assumption 4.6.15. Let ρ, r be two radii which satisfy the inequalities $4\eta\gamma^{j+1} < \rho \leq r \leq \gamma^j$. We then define the *frequency pinching* $W_\rho^r(p, j, q)$ around q between the scales ρ and r to be the following quantity

$$W_\rho^r(p, j, q) := |\mathbf{I}_p(q, r) - \mathbf{I}_p(q, \rho)|, \text{ for any } q \in \Gamma_{p, \gamma^j}.$$

The proof of the following lemma is entirely analogous to Lemma 4.6.14, this time following the contradiction argument in [29, Lemma 12.2].

Lemma 4.6.18. *Assume Assumption 4.6.9 and $\rho \in (0, 1]$. There exists a suitable threshold $\delta_2 = \delta_2(m, n, Q, \gamma, K, \rho) > 0$, with the following property. Take $r \in (\gamma^{j+1}, \gamma^j)$ and an ρ -linearly independent set $S = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(p) \cap \mathfrak{G}_K$. Then, there exist $i \in \{0, \dots, m-2\}$ and $r_i \in (0, r)$ such that, for any $y \in \mathbf{B}_{r_i}(x_i) \cap \mathfrak{G}_K$, we have*

$$W_{r_i}^{2r_i}(p, y) \geq \delta_1.$$

4.7 Dimensional bound and countability of the singular set in 3d

In this section, we show how to obtain a Hausdorff dimensional bound for the set of one-sided singular boundary points, i.e., with density $Q/2$. We will split the proof into two cases: $m > 3$ (Theorem 4.7.2) and $m = 3$ (Theorem 4.7.3). Additionally, we note that when proving the rectifiability of the set of one-sided singular points for $m > 3$ in Section 4.8, the Hausdorff dimensional bound follows as a by-product of the finer analysis conducted therein, which is entirely independent of this section.

We first recall a standard lemma in geometric measure theory regarding the weak ϵ -approximation property, for which we provide a proof for the sake of completeness.

Lemma 4.7.1 (Weak ϵ -approximation property). *Let $0 \in S \subset \mathbb{R}^{m+n}$, $k \in \{1, \dots, m+n-1\}$, and $\epsilon \in (0, 1/4)$. Assume that S satisfies the weak ϵ -approximation property in $\mathbf{B}_1$, meaning that for each $r \in (0, 1]$ and each $x \in S \cap \mathbf{B}_1$, there exists a k -plane $L = L(r, x)$ containing x and satisfying*

$$S \cap \mathbf{B}_r(x) \subset (\epsilon r)\text{-neighborhood of } L \cap \mathbf{B}_r(x). \quad (4.7.1)$$

Then, there exists a function $g : (0, 1) \rightarrow (0, +\infty)$ with $\lim_{s \rightarrow 0} g(s) = 0$ such that $\mathcal{H}^{k+g(\epsilon)}(S \cap \mathbf{B}_1) = 0$.

Proof. Given a ball $\mathbf{B}_r(x)$, by (4.7.1), we can cover the set $S \cap \mathbf{B}_r(x)$ with at most $C(k)\epsilon^{-k}$ balls of radius ϵr . We fix the cover $\mathfrak{C}_0 = \mathbf{B}_1$. The cover $\mathfrak{C}_{j+1}$ is then obtained by replacing each ball B in $\mathfrak{C}_j$ with $C(k)\epsilon^{-k}$ balls of radius $\epsilon r(B)$.

For any $\alpha > 0$, we observe that

$$\mathcal{H}_\infty^\alpha(S \cap \mathbf{B}_1) \leq \sum_{B \in \mathfrak{C}_{j+1}} \omega_\alpha r(B)^\alpha \leq \sum_{B \in \mathfrak{C}_j} \omega_\alpha C(k)\epsilon^{-k}(\epsilon r(B))^\alpha = C(k)\epsilon^{\alpha-k} \sum_{B \in \mathfrak{C}_j} \omega_\alpha r(B)^\alpha.$$

By iterating on j , we obtain $\mathcal{H}^\alpha(S \cap \mathbf{B}_1) \leq (C(k)\epsilon^{\alpha-k})^j \omega_\alpha$. If $C(k)\epsilon^{\alpha-k} < 1$, it follows that $\mathcal{H}^\alpha(S \cap \mathbf{B}_1) = 0$.

We define g for any $s > 0$ by setting $C(k)s^{g(s)} = 2^{-1}$. It is straightforward to verify that g is continuous, satisfies $\lim_{s \rightarrow 0} g(s) = 0$, and ensures $\mathcal{H}^{k+g(\epsilon)}(S \cap \mathbf{B}_1) = 0$. $\square$

We now assume that $m > 3$ to prove the following dimensional bound.

Theorem 4.7.2. *Let T, Σ , and Γ , be as in Assumption 4.1.8. Suppose $m > 3$, then*

$$\dim_{\mathcal{H}}(\text{Sing}_b^1(T)) \leq m - 3.$$

Proof. In the proof of Theorem M (see Section 4.4.4), we constructed a decomposition of the singular set into countable pieces, each consisting of flat boundary singularities for T . By applying the same induction argument as in that proof, we readily reduce the problem to proving Theorem 4.7.2 for flat boundary singularities.

For the remainder of the proof, we consider the set S (which will be a Borel set) to consist of either the one-sided flat singularities of high frequency or the one-sided flat singularities of low frequency, i.e.,

$$S := \mathfrak{G}_K \text{ or } S := (\text{Sing}_b^1(T) \cap \{q \in \Gamma : q \text{ is flat}\}) \setminus \mathfrak{G}_K.$$

Assume, for the sake of contradiction, that $\dim_{\mathcal{H}}(S) =: M > m - 3$. Let $\epsilon_0 > 0$ be sufficiently small so that if a set satisfies the weak ϵ_0 -approximation property, then

$\mathcal{H}^M(S) = 0$. Removing a set of $\mathcal{H}^M$ -measure zero if necessary, we may assume that, for all $q \in S$, $\Theta^M(S, q) > 0$. Consequently, for every $q \in S$ and $r > 0$, the set S fails to satisfy the weak ϵ_0 -approximation property in $\mathbf{B}_r(q)$. This, in turn, implies the existence of a point $x_0 \in S \cap \mathbf{B}_r(q)$ and a radius $r_0 > 0$ such that *no* $(m-3)$ -plane L satisfies (4.7.1), i.e.,

$$S \cap \mathbf{B}_{r_0}(x_0) \not\subset (\epsilon_0 r_0)\text{-neighborhood of } L, \text{ for any } (m-3)\text{-plane } L.$$

We now aim to construct a set of $m-1$ points in S that are $(r_0 \epsilon_0)$ -linearly independent. Beginning with x_0 , we proceed inductively: suppose we have already found an $(r_0 \epsilon_0)$ -linearly independent set $\{x_0, \dots, x_k\} \subset S$. Then, we define $x_{k+1} \in S$ as a point satisfying

$$\text{dist}(x_{k+1}, \text{span}(x_0, x_1, \dots, x_k)) > \epsilon_0 r_0.$$

If this process terminates before reaching $k+1 = m-2$, i.e., we fail to find such an $x_{j+1} \in S \cap \mathbf{B}_{r_0}(x_0)$ for some $j < m-2$, then every $x \in S \cap \mathbf{B}_{r_0}(x_0)$ must satisfy

$$\text{dist}(x, \text{span}(x_0, x_1, \dots, x_j)) \leq \epsilon_0 r_0.$$

Since $L_j := \text{span}(x_0, x_1, \dots, x_j)$ has dimension $j < m-2$, this contradicts our earlier assumption about x_0 and r_0 , as $S \cap \mathbf{B}_{r_0}(x_0)$ would then satisfy (4.7.1).

Thus, we obtain an $(r_0 \epsilon_0)$ -linearly independent set $\{x_0, \dots, x_{m-2}\} \subset S \cap \mathbf{B}_{r_0}(x_0)$. To derive a contradiction, we proceed as follows. Since $\Theta^M(S, x_i) > 0$ for each index i given by Lemma 4.6.14 (for $S = \mathfrak{G}_K$) or Lemma 4.6.18 (otherwise), we can repeat the above process in $\mathbf{B}_{r_i}(x_i)$ to find another set $\{x_0^{(1)}, \dots, x_{m-2}^{(1)}\} \subset S \cap \mathbf{B}_{r_i}(x_i)$ that is $(r_i \epsilon_0)$ -linearly independent. Applying Lemma 4.6.14 (if $S = \mathfrak{G}_K$) or Lemma 4.6.18 (otherwise) with $\rho = \epsilon_0$, we obtain an index $i_1 \in 0, \dots, m-2$ and a radius $r_{i_1} > 0$ such that

$$W_{r_{i_1}/2}^{2r_{i_1}}(p, y) \geq \delta(\epsilon_0) > 0, \quad \forall y \in S \cap \mathbf{B}_{r_{i_1}}(x_{i_1}^{(1)}). \quad (4.7.2)$$

Since the frequency function is uniformly bounded above (see (4.6.5)) and non-negative, the iterative process must terminate after finitely many steps, contradicting the assumption that $\Theta^M(S, x) > 0$ for all $x \in S$ up to a $\mathcal{H}^M$ -null set. This contradiction implies that $M \leq m - 3$, completing the proof. $\square$

In the case $m = 3$, we establish the countability of the one-sided singular set. The argument used in Theorem 4.7.2 must be replaced with a different approach, as there exist uncountable sets with Hausdorff dimension equal to zero.

Theorem 4.7.3. *Let T, Σ , and Γ , be as in Assumption 4.1.8. Suppose $m = 3$, then*

$$\text{Sing}_b^1(T) \text{ is a countable set.}$$

Proof. In the proof of Theorem M (see Section 4.4.4), we provide a decomposition of the boundary singular set into countable pieces, each consisting of flat boundary singularities for T . By applying the same induction argument used therein, we can reduce the proof of Theorem 4.7.3 to establishing countability for flat boundary singularities. Specifically, by demonstrating that each piece in the decomposition is countable, we conclude Theorem 4.7.3.

For the remainder of the proof, we consider the set S to be either the one-sided flat singularities of high frequency or those of low frequency, i.e.,

$$S := \mathfrak{G}_K \text{ or } S := \left(\text{Sing}_b^1(T) \cap \{q \in \Gamma : q \text{ is flat}\} \right) \setminus \mathfrak{G}_K.$$

If the set of one-sided flat singularities were uncountable, then either the high-frequency or the low-frequency points would form an uncountable set. Without loss of generality, we may assume that every point in S is an accumulation point of S . Indeed, the set of isolated points in S is at most countable, and removing them does not affect our argument.

Next, take two points $x_0, x_1 \in S$ and set $|x_0 - x_1| =: \epsilon_0 > 0$. We can now apply either Lemma 4.6.14 ($S = \mathfrak{G}_K$) or Lemma 4.6.18 (otherwise) with $\rho = \epsilon_0$, to obtain that, without

loss of generality,

$$W_{r_0}^{2r_0}(p, y) \geq \delta_1(\epsilon_0) > 0, \quad \forall y \in S \cap \mathbf{B}_{r_0}(x_0), \quad \text{for some } r_0 \in (0, \epsilon_0/4).$$

Since x_0 is an accumulation point of S , we can now reapply this procedure to obtain two points $x_0^{(1)}, x_1^{(1)} \in S \cap \mathbf{B}_{r_0}(x_0)$ with $|x_0^{(1)} - x_1^{(1)}| =: \epsilon_0 \rho_1 > 0$. Again, without loss of generality, we have

$$W_{r_1}^{2r_1}(p, y) \geq \delta(\epsilon_0) > 0, \quad \forall y \in S \cap \mathbf{B}_{r_1}(x_0), \quad \text{for some } r_1 \in (0, \rho_1/4). \quad (4.7.3)$$

Since the frequency function is uniformly bounded from above (see (4.6.5)) and is non-negative, this process can only be iterated finitely many times. However, this contradicts (4.7.3), as the assumption that all points in S are accumulation points would allow us to repeat the procedure indefinitely. Hence, S must be countable. $\square$

4.8 Rectifiability for the nonlinear problem

We are now aiming at proving the main rectifiability results of this paper. We mention that the results in this section are independent of the dimensional bound in Section 4.7 when $m > 3$. In the case that $m = 3$, the problem is already settled in Theorem 4.7.3.

4.8.1 Spatial variations

In previous sections, we computed several different quantities related to $\mathbf{I}$ in which estimates for their radial derivatives were achieved, see Lemma 4.5.18. These estimates give us the almost monotonicity of the frequency. However, in order to compare the frequency amongst different points, we will need a better understanding of spatial derivatives.

Spatial variations in the high-frequency regime

The relevant quantities and their properties are stated in the following lemma.

Lemma 4.8.1 (Spatial variations). *Under Assumption 4.6.9. Then either $T_{p,J} = Q \llbracket \mathcal{M}_{p,J} \rrbracket$ in a neighborhood of q , or*

$$\mathbf{H}_p(q, r) > 0 \text{ and } \mathbf{D}_p(q, r) > 0, \text{ for every } r \in (0, t_J].$$

In the latter case, there exists a constant $\gamma_1 = \gamma_1(m, \delta_{\mathbf{e}}, \alpha_{\mathbf{h}}, \kappa, \tau) > 0$ with the following property. Let v be a vector field on $\mathcal{M}_{p,J}$ that is tangent to $\Gamma_{p,J}$. We obtain

$$\begin{aligned} \partial_v \mathbf{D}_p(q, r) &= \frac{2t_J}{r} \int_{\mathcal{M}_{p,J}} \phi' \left(\frac{d(q, z)}{r} \right) \sum_i \partial_{\nabla_z d(q, z)} (\mathcal{N}_{q,J})_i(z) \cdot \partial_v (\mathcal{N}_{q,J})_i(z) dz \\ &\quad + O \left(\mathbf{m}_{p,J}^{\gamma_1} \right) r^{\gamma_1} t_J^{\gamma_1-1} \mathbf{D}_p(q, r), \\ \partial_v \mathbf{H}_p(q, r) &= -2 \sum_i \int_{\mathcal{M}_{p,J}} \frac{|\nabla d(q, z)|^2}{d(q, z)} \phi' \left(\frac{d(q, z)}{r} \right) \langle \partial_v (\mathcal{N}_{p,J})_i(z), (\mathcal{N}_{p,J})_i(z) \rangle dz, \end{aligned}$$

for any $r \in (0, t_J]$.

Proof. The proof is analogous to [29, Lemma 5.5] using that v is tangent to the boundary and the machinery from [27] which is in turn based upon [21]. We briefly sketch it here to show the main differences. To prove the first claimed equality, we take $r \in (0, 2)$ and define $\bar{v}(z) := t_J^{-1} v(t_J z)$ on $\overline{\mathcal{M}}_{p,J}$ which is tangent to $\Gamma_{p,J}$ and

$$Y(z) := \phi \left(\frac{d_J(q, z)}{r} \right) \bar{v}(z), \text{ for } z \in \overline{\mathcal{M}}_{p,J}, \text{ where } v \text{ is tangent to } \Gamma_{p,J}.$$

We then have that

$$\begin{aligned} \operatorname{div}_{\mathcal{M}_{p,J}} Y(z) &= \frac{1}{r} \phi' \left(\frac{d_J(q, z)}{r} \right) \langle \nabla d_J(q, z), \bar{v}(z) \rangle - \langle Y(z), H_{\mathcal{M}_{p,J}} \rangle, \\ D_{\mathcal{M}_{p,J}} Y(z) &= \frac{1}{r} \phi' \left(\frac{d_J(q, z)}{r} \right) \bar{v}(z) \otimes \nabla d_J(q, z) - \phi \left(\frac{d_J(q, z)}{r} \right) \sum_k A_{\mathcal{M}_{p,J}}(e_k, \bar{v}(z)) \otimes e_k, \end{aligned}$$

where $\{e_k\}_k$ is an orthonormal frame of $\overline{\mathcal{M}}_{p,J}$ with $e_1 = \bar{v}$. By a direct computation, we also obtain that

$$\partial_{\bar{v}} \overline{\mathbf{D}}_p(q, r) = \frac{1}{r} \int_{\overline{\mathcal{M}}_{p,J}} \phi' \left(\frac{d_J(q, z)}{r} \right) \langle \nabla d_J(q, z), \bar{v}(z) \rangle |D\overline{\mathcal{N}}_{p,J}(z)|^2 dz.$$

Arguing as in [29, Lemma 5.5], we plug the vector field Y into the inner variation formulas in [27, Proposition 16.4] and using their notation for $\text{Err}_k^i, k \in \{1, \dots, 4\}$, we obtain that

$$\begin{aligned} \partial_{\bar{v}} \overline{\mathbf{D}}_p(q, r) &= \frac{2}{r} \int_{\overline{\mathcal{M}}_{p,J}} \phi' \left(\frac{d_J(q, z)}{r} \right) \sum_i \left\langle \partial_{\nabla d_J(q, z)} (\overline{\mathcal{N}}_{p,J})_i(z), \partial_{\bar{v}(z)} (\overline{\mathcal{N}}_{p,J})_i(z) \right\rangle dz \\ &= O(\mathbf{m}_{p,J}^{1/2}) r^{\frac{1-\delta_{\mathbf{e}}}{2}} \overline{\mathbf{D}}_p(q, r) + \sum_{k=1}^4 \overline{\text{Err}_k^i}. \end{aligned}$$

Now, as in [27, Proposition 16.7], we control the error terms as follows:

$$\sum_{k=1}^4 |\overline{\text{Err}_k^i}| \leq C \mathbf{m}_{p,J}^\tau \overline{\mathbf{D}}_{p,J}(q, r)^\tau \left(\overline{\mathbf{D}}_{p,J}(q, r) + r \partial_r \overline{\mathbf{D}}_{p,J}(q, r) \right) + C \mathbf{m}_{p,J} t_J^{2\kappa} r \overline{\mathbf{D}}_{p,J}(q, r).$$

We now use the rescaling (4.5.11) to obtain

$$\begin{aligned} \sum_{k=1}^4 |\overline{\text{Err}_k^i}| &\leq C \mathbf{m}_{p,J}^\tau t_J^{-m(1+\tau)} \mathbf{D}_p(t_J q, t_J r)^\tau (\mathbf{D}_p(t_J q, t_J r) + r t_J \partial_r \mathbf{D}_p(t_J q, t_J r)) \\ &\quad + C \mathbf{m}_{p,J} t_J^{-m+2\kappa} r \mathbf{D}_p(t_J q, t_J r). \end{aligned}$$

The last displayed inequalities together with $\partial_{\bar{v}(z)} \overline{\mathbf{D}}_p(q, r) = t_J^{-m} \partial_{v(t_J z)} \mathbf{D}_p(t_J q, t_J r)$ gives our claimed inequality. Indeed:

$$\begin{aligned} \partial_{v(z')} \mathbf{D}_p(q', s) &= \frac{2t_J^{m+1}}{s} \int_{\overline{\mathcal{M}}_{p,J}} \phi' \left(\frac{d_J(q, z)}{s} \right) \sum_i \left\langle \partial_{\nabla d_J(q, z)} (\overline{\mathcal{N}}_{p,J})_i(z), \partial_{\bar{v}(z)} (\overline{\mathcal{N}}_{p,J})_i(z) \right\rangle dz \\ &\leq O(\mathbf{m}_{p,J}^{1/2}) \left(\frac{s}{t_J} \right)^{\frac{1-\delta_{\mathbf{e}}}{2}} \mathbf{D}_p(q', s) + C \mathbf{m}_{p,J} t_J^{2\kappa-1} s \mathbf{D}_p(q', s) \\ &\quad + C \mathbf{m}_{p,J}^\tau t_J^{-(1+\tau)} \mathbf{D}_p(q', s)^\tau (\mathbf{D}_p(q', s) + \partial_r \mathbf{D}_p(q', s)) \\ &\stackrel{(4.5.16), (4.5.19), s < t_J}{\leq} O \left(\mathbf{m}_{p,J}^{\min\{1/2, \tau\}} \right) s^{\min\{\frac{1-\delta_{\mathbf{e}}}{2}, \tau(m-2+2\kappa)\}} t_J^{\min\{-1+\tau, 2\kappa-1, -\frac{1-\delta_{\mathbf{e}}}{2}\}} \mathbf{D}_p(q', s), \end{aligned}$$

for any $q' \in \Gamma_{p,J}$, $z' \in \mathcal{M}_{p,J}$, and $s \in (0, t_J)$. Rescaling the integrand above and choosing γ_1 appropriately, we conclude the proof. The spatial derivative of $\mathbf{H}$ is a direct computation, as performed in [23, Proposition 3.1]. $\square$

It is possible to measure how much the normal approximation $\mathcal{N}_{p,J}$ deviates from being homogeneous in terms of the frequency pinching. This is stated in the following lemma. We omit the proof of the lemma since it follows exactly the same argument as in [29, Proposition 5.2].

Lemma 4.8.2 (Deviation from homogeneity of $\mathcal{N}_{p,J}$). *Under Assumption 4.6.9. There exist $C = C(m, n, Q, \Lambda) > 0$ and $\gamma_2 = \gamma_2(m, n, Q, \Lambda) > 0$ such that the following estimate holds for every pair ρ, r with $0 < \rho \leq r \leq t_J$. We define the annuli*

$$\mathcal{A}_{\rho/4}^{2r}(q) := \left(\mathbf{B}_{2r}(q) \setminus \overline{\mathbf{B}_{\rho/4}(q)} \right) \cap \mathcal{M}_{p,J},$$

then

$$\begin{aligned} \int_{\mathcal{A}_{\rho/4}^{2r}(q)} \sum_i \left| D(\mathcal{N}_{p,J})_i(z) \frac{d(q, z) \nabla d(q, z)}{|\nabla d(q, z)|} - \mathbf{I}_p(q, d(q, z)) (\mathcal{N}_{p,J})_i(z) |\nabla d(q, z)| \right|^2 \frac{dz}{d(q, z)} \\ \leq C \mathbf{H}_p(q, 2r) \left(W_{\rho/4}^{2r}(p, q) + \mathbf{m}_{p,J}^{\gamma_2} r^{\gamma_2} \log \left(\frac{4r}{\rho} \right) \right). \end{aligned}$$

We now show how to estimate the frequency function $\mathbf{I}$ evaluated at two possibly different boundary points, p and q , in terms of the frequency pinching. We omit the proof of the following lemma since it follows entirely analogously to [29, Lemma 5.4] using Lemma 4.8.1.

Lemma 4.8.3. *Under Assumption 4.6.9. Let $q_1, q_2 \in \Gamma_{p,J}$ with $d(q_1, q_2) \leq r/8$, where $r \in (0, t_J]$. There exist $C = C(m, n, Q, \Lambda) > 0$ and $\gamma_3 = \gamma_3(m, n, Q, \Lambda) > 0$ such that for any $z, w \in [q_1, q_2]$ we have*

$$|\mathbf{I}_p(w, r) - \mathbf{I}_p(z, r)| \leq C \left[\left(W_{r/8}^{4r}(p, q_1) \right)^{1/2} + \left(W_{r/8}^{4r}(p, q_2) \right)^{1/2} + \mathbf{m}_{p,J}^{\gamma_3} r^{\gamma_3} \right] \frac{d(z, w)}{r}.$$

Spatial variations in the low-frequency regime

We record here the counterpart for the low-frequency regime of the results in the high-frequency regime.

Lemma 4.8.4. *Under Assumption 4.6.15, all the conclusions of Lemma 4.8.1, Lemma 4.8.2 and Lemma 4.8.3 are valid for*

$$4\eta\gamma^{j+1} \leq \rho \leq r < 2\gamma^j \text{ and } q, q_1, q_2 \in \Gamma_{p, \gamma^j}.$$

We omit the proof of this lemma here, as it is a repetition of the arguments for the high-frequency regime, along with the observations made in [29, Section 10.1].

4.8.2 Proof of the rectifiability result

In this section, we prove Proposition 4.4.7, which states the $\mathcal{H}^{m-3}$ -rectifiability of the set of one-sided flat singular points. We divide the proof into the high and low-frequency regimes.

Proposition 4.8.5 (Rectifiability of high-frequency points). *Let T, Σ , and Γ , be as in Assumption 4.1.8. Then*

$$\text{Sing}_b(T) \cap \{q \in \Gamma : \Theta^m(T, q) = Q/2, q \text{ is flat, and } \mathbf{I}(T, q) \geq 2 - \alpha_e\}$$

is $\mathcal{H}^{m-3}$ -rectifiable.

Proposition 4.8.6 (Rectifiability of low-frequency points). *Let T, Σ , and Γ , be as in Assumption 4.1.8. Then*

$$\text{Sing}_b(T) \cap \{q \in \Gamma : \Theta^m(T, q) = Q/2, q \text{ is flat, and } \mathbf{I}(T, q) < 2 - \alpha_e\}$$

is $\mathcal{H}^{m-3}$ -rectifiable.

Remark 4.8.7. We note that the Hausdorff dimensional bound obtained in Section 4.7 is not needed to prove either of the two theorems above. In fact, it can be derived as a consequence of the proof.

Paralleling the setting of interior points from [29], we encounter a crucial difference. Singular points for the boundary linear problem are not necessarily points where the multi-valued function vanishes. In contrast, in the interior setting, after applying an argument involving the average of the multi-valued function, we reduce the problem to studying points where the function vanishes.

Instead, we focus on the set of points with frequency greater than $1 + \alpha/2$. To prove the bound on β_2 , we must be able to preserve a frequency bound of $1 + \alpha/2$ in the limit. While this is straightforward in the high-frequency regime, as the almost monotonicity formula for the frequency holds without changing the center manifold, it is more challenging in the low-frequency regime. In contrast, in the low-frequency regime in [29], the authors only need to preserve the existence of Q points without controlling their frequency.

Bound on the Jones' β_2 coefficient in the high-frequency regime

We now prove Theorem 4.8.8 which gives an upper bound for the $(m - 3)$ -dimensional Jones' β_2 coefficient in terms of the frequency pinching.

Theorem 4.8.8 (Frequency pinching controls the Jones' β_2 coefficient). *Assume $m > 3$. There exist constants $\eta_0 = \eta_0(m) > 0$, $\epsilon_0 = \epsilon_0(m, n, Q, \eta_0, \Lambda) > 0$, $\gamma_4 = \gamma_4(m, n, Q, \Lambda) > 0$, and $C = C(m, n, Q, \Lambda) > 0$, such that the following holds. Under Assumption 4.6.9, and assume further that $\eta < \eta_0$ and $\mathbf{m}_{p,J} < \epsilon_0^2$. If μ is a finite non-negative Radon measure with $\text{spt}(\mu) \subset \Gamma_{p,J}$. Then for all $r \in (0, t_J]$ and every $q \in \mathbf{B}_{r/8} \cap \Gamma_{p,J}$ we have*

$$\left[\beta_{2,\mu}^{m-3}(q, r/8) \right]^2 \leq \frac{C}{r^{m-3}} \int_{\mathbf{B}_{r/8}(q)} W_{r/8}^{4r}(p, z) d\mu(z) + C \mathbf{m}_{p,J}^{\gamma_4} \frac{\mu(\mathbf{B}_{r/8}(q))}{r^{m-3-\gamma_4}}.$$

With this control, we are in a position to apply Theorem 4.3.1 for Frostman measures, as done in [29], to conclude the proof of the $\mathcal{H}^{m-3}$ -rectifiability in the high-frequency regime, i.e., Proposition 4.8.5. We are now left with proving Theorem 4.8.8.

Proof of Theorem 4.8.8. We assume without loss of generality that $\mu(\mathbf{B}_{r/8}(q)) > 0$, otherwise the inequality is trivial. As in [29, Proposition 7.2], denote the barycenter of μ in $\mathbf{B}_{r/8}(q) \cap T_q\mathcal{M}_J$ by

$$\bar{q}_r := \frac{1}{\mu(\mathbf{B}_{r/8}(q))} \int_{\mathbf{B}_{r/8}(q)} z d\mu(z),$$

where $\mathbf{p}_q$ is the orthogonal projection of $\mathbb{R}^{m+n}$ onto $T_q\mathcal{M}_J$. Next, define the symmetric bilinear form $b_{q,r} : T_q\mathcal{M}_J \times T_q\mathcal{M}_J \rightarrow \mathbb{R}$ as

$$b_{q,r}(v, w) := \int_{\mathbf{B}_{r/8}(q)} ((z - \bar{q}_r) \cdot v) ((z - \bar{q}_r) \cdot w) d\mu(z).$$

Diagonalizing $b_{q,r}$ yields an orthonormal basis $\{v_i\}_{i=1}^m$ of eigenvectors and a corresponding family of eigenvalues $0 \leq \lambda_m \leq \dots \leq \lambda_1$ for the linear map

$$L(v) := \int_{\mathbf{B}_{r/8}(q)} ((z - \bar{q}_r) \cdot v) \mathbf{p}_q(z) d\mu(z),$$

which diagonalize $b_{q,r}$, i.e., $L(v_i) = \lambda_i v_i$ and $b_{q,r}(v_i, v_i) = \lambda_i$. This gives the following characterization:

$$\inf_{\substack{\text{affine (m-3)-planes } \pi \\ \pi \subset T_q\mathcal{M}_J}} \left[\left(\frac{r}{8} \right)^{-m+1} \int_{\mathbf{B}_{r/8}(q)} \text{dist}(y, L)^2 d\mu(y) \right]^{1/2} = \left(\frac{r}{8} \right)^{-m+1} (\lambda_{m-2} + \lambda_{m-1} + \lambda_m).$$

We now have the following simplification of our claimed inequality:

$$\left[\beta_{2,\mu}^{m-3}(q, r/8) \right]^2 \leq 3 \left(\frac{8}{r} \right)^{m-1} \lambda_{m-2}.$$

The argument differs from [29, Proposition 4.2] for two reasons. Firstly, ∇d needs to be tangent to $\Gamma_{p,J}$, which we ensure through Lemma 4.5.17. Once this is established, we can apply Lemma 4.8.1 with any compactly supported vector field pointing in the same direction as ∇d , following the approach in [29, Proposition 4.2]. Secondly, we need to prove that:

$$\int_{\mathcal{A}_{r/4}^{2r}(q)} \sum_{h=1}^{m-2} |D\mathcal{N}(z) \cdot \ell_z(v_h)|^2 dz \geq c(\Lambda) \frac{\mathbf{H}_p(q, 2r)}{r} \text{ for some } c(\Lambda) > 0. \quad (4.8.1)$$

Note that we are using the $(m-3)$ -dimensional Jones' β_2 coefficient. The sum in the expression above is taken over the $m-2$ vectors derived from the bilinear form $b_{q,r}$. We proceed by contradiction to prove (4.8.1). We rescale and translate the current such that we have $q = 0$ and $r = 1$. If (4.8.1) fails, we construct a contradiction sequence of currents T_k with $\mathbf{m}_{p,k} \leq \varepsilon_k \rightarrow 0$. From (4.5.15) and (4.5.16), the right-hand side of (4.8.1) is bounded above by a constant C . Thus, we obtain the inequality:

$$\int_{\mathcal{A}_{r/4}^{2r}(0)} \sum_{h=1}^{m-2} |D\mathcal{N}_k(z) \cdot \ell_z^k(v_h^k)|^2 dz < \frac{1}{k}. \quad (4.8.2)$$

For the contradiction sequence, we have the corresponding center manifolds, boundaries, and normalized normal approximations, denoted by $\mathcal{M}_k$, Γ_k , and $\mathcal{N}_k$, respectively. Additionally, we have sequences ℓ_z^k and $\{v_h^k\}_k$ that converge to Id and, for each $h \in \{1, \dots, m-2\}$, to v_h , respectively. Moreover, $\mathcal{M}_k$ converges to some m -dimensional half-plane π_∞^+ , Γ_k converges to $\partial_{\pi_\infty} \pi_\infty^+$, and $\boldsymbol{\eta} \circ \mathcal{N}_k \rightarrow 0$.

From this, we can extract a weak limit of $\mathcal{N}_k$ in $W^{1,2}$, denoted by $u_0 : B_2 \cap x_m \geq 0 \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$, which is a Dirichlet minimizer with Dirichlet energy equal to 1, $\boldsymbol{\eta} \circ u_0 = 0$, and u_0 vanishes identically at its boundary (note that u_0 is not necessarily a fine blow-up). The same arguments as in [21, Section 10.3] (see also [27, Section 18.1]) allow us to upgrade, up to a subsequence, the weak convergence of $\mathcal{N}_k$ to u_0 to *strong* convergence in $W^{1,2}(B_2)$.

Such strong convergence ensures that $\mathbf{I}_p^{T_k}(0, s)$ converges to $I_{u_0}(s)$ for any $s \in (0, t_J]$, which implies:

$$I_{u_0}(0) \geq 2 - \delta_e \geq 1 + \alpha/2.$$

We also obtain using (4.8.2) that

$$\int_{(\mathbf{B}_2 \setminus \overline{\mathbf{B}_{1/4}}) \cap \{x_m \geq 0\}} \sum_{h=1}^{m-2} |\nabla_{v_h} u_0|^2 = 0.$$

We reach a contradiction in the same manner as in the argument in Lemma 4.3.4. Specifically, we can apply Lemma 4.3.2 to obtain a nontrivial $I_{u_0}(0)$ -homogeneous Dirichlet minimizer in 2 dimensions, with zero boundary value along a straight line. This, however, contradicts Theorem 4.3.3, since we have $I_{u_0}(0) \geq 1 + \alpha/2$. $\square$

Bound on the Jones' β_2 coefficient in the low-frequency regime

We now establish the same control in the low-frequency regime as given in Theorem 4.8.8. With this control (Theorem 4.8.9), we are able to apply Theorem 4.3.1 for Frostman measures, as done in [29], to conclude the proof of the $\mathcal{H}^{m-3}$ -rectifiability in the low-frequency regime, i.e., Proposition 4.8.6. This, in turn, completes the proof of Proposition 4.4.7, as explained above.

Theorem 4.8.9 (Frequency pinching controls the Jones' β_2 coefficient). *Assume $m > 3$. There exist $\eta_1 = \eta_1(m) > 0$, $\epsilon_1 = \epsilon_1(m, n, Q, \eta_1, K) > 0$, $\gamma_5 = \gamma_5(m, n, Q, \Lambda) > 0$, and $C = C(m, n, Q, \Lambda, \gamma) > 0$, such that the following holds.*

Under Assumption 4.6.15, and assume further that $\eta < \eta_0$ and $\mathbf{m}_{p,j} < \epsilon_1^2$. If μ is a finite non-negative upper $(m-3)$ -regular Radon measure with $\text{spt}(\mu) \subset \mathfrak{G}_K$. Then for all $r \in (8\eta\gamma^{j+1}, \gamma^j]$ and every $q \in \mathfrak{G}_K$ we have

$$\left[\beta_{2,\mu}^{m-3}(q, r/8) \right]^2 \leq \frac{C}{r^{m-3}} \int_{\mathbf{B}_{r/8}(q)} W_{r/8}^{4r}(p, j, z) d\mu(z) + C \mathbf{m}_{p,j}^{\gamma_5} \frac{\mu(\mathbf{B}_{r/8}(q))}{r^{m-3-\gamma_5}}.$$

Proof. Since the measure μ is upper $(m-3)$ -regular, the coefficient $\beta_{2,\mu}^{m-3}(q, r/8)$ is always bounded. Therefore, it is sufficient to prove the theorem under the assumption that

$$\frac{1}{r^{m-3}} \int_{\mathbf{B}_{r/8}(q)} W_{r/8}^{4r}(p, j, z) d\mu(z) \leq C\delta_3.$$

This proof follows the same structure as [29, Proposition 13.3], with the main difference being the proof of the following inequality:

$$\int_{\mathcal{A}_{r/4}^{2r}(q)} \sum_{h=1}^{m-2} |D\mathcal{N}_j(z) \cdot \ell_z(v_h)|^2 dz \geq c(\Lambda) \frac{\mathbf{H}_p(q, 2r)}{r} \text{ for some } c(\Lambda) > 0. \quad (4.8.3)$$

We prove this estimate by contradiction, following the same approach as in Proposition 4.8.5, but in this case, we deal with infinitely many center manifolds and intervals of flattening. By rescaling and translating, we assume $q = 0$ and $r = 1$. If the inequality fails, it must fail for every $\epsilon_1 > 0$ and $\delta > 0$. We can thus take a contradiction sequence of currents T_k such that $\mathbf{m}_p^{(k)} \rightarrow 0$, where $\mathbf{m}_p^{(k)}$ is defined with respect to T_k .

Up to rotation, we can assume that $T_0 \Gamma_k = \mathbb{R}^{m-1} \times \{0\}$ and $\pi_k = \mathbb{R}^m \times \{0\}$, and that

$$\int_{\mathbf{B}_{1/8}} W_{1/4}^2(p, j(k), z) d\mu_k(z) \leq \frac{1}{k}.$$

This allows us to select points $x_k \in \Gamma_{p,1} \cap \mathbf{B}_{1/8}$ such that $W_{1/4}^2(p, j(k), x_k) \leq k^{-1}$. Up to subsequences, $\mu_k \rightharpoonup \mu$ with $\text{spt}(\mu)$ being a finite non-negative Radon measure supported in $\mathbb{R}^{m-1} \times \{0\}$, and $x_k \rightarrow x \in \text{spt}(\mu)$.

We let $\theta_k = \gamma^{j(k)} \in [1, \frac{1}{8\eta\gamma})$ be such that the modified interval of flattening given by $(\gamma^{j(k)+1}, \gamma^{j(k)})$ becomes $(8\eta\gamma\theta_k, \theta_k]$. Up to a subsequence, we assume $\theta_k \rightarrow \theta$.

For each T_k , we construct the corresponding center manifold $\mathcal{M}_k$ and normal approximations $\mathcal{N}_k$. Arguing as in Theorem 4.8.8, we obtain a limiting function $u_0 : (\mathbf{B}_\theta \setminus \overline{\mathbf{B}_{8\eta\gamma\theta}}) \cap \{x_m \geq 0\} \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ that is a Dirichlet minimizer with Dirichlet energy equal to 1, $\boldsymbol{\eta} \circ u_0 = 0$, and u_0 is identically zero at its boundary (note that u_0 is not necessarily a fine blow-up).

Moreover, as in Theorem 4.8.8, the convergence of $\mathcal{N}_k$ to u_0 is *strong* in $W^{1,2}(\mathbf{B}_\theta \setminus \overline{\mathbf{B}_{8\eta\gamma\theta}})$ up to subsequences. This strong convergence furnishes

$$\mathbf{I}_p^{T_k}(x_k, \theta_k) \rightarrow I_{u_0}(x, \theta), \quad \mathbf{I}_p^{T_k}(x_k, 8\eta\gamma\theta_k) \rightarrow I_{u_0}(x, 8\eta\gamma\theta), \quad \text{and} \quad I_{u_0}(x, \theta) \geq 1 + \alpha/2,$$

where $\mathbf{I}^{T_k}$ denotes the frequency defined with respect to T_k .

Since $W_{1/4}^2(p, j(k), x_k) \rightarrow 0$, we have that $I_{u_0}(x, 8\eta\gamma\theta) = I_{u_0}(x, \theta)$. Hence, Lemma 4.3.5 implies that u_0 is I_0 -homogeneous on $\mathbf{B}_\theta \cap \{x_m \geq 0\}$ with $I_0 \geq 1 + \alpha/2$. Since the contradiction sequence is violating (4.8.3), we arrive again at the identity

$$\int_{(\mathbf{B}_2 \setminus \mathbf{B}_{1/4}) \cap \{x_m \geq 0\}} \sum_{h=1}^{m-2} |\nabla_{v_h} u_0|^2 = 0.$$

Again, by the same argument given in Lemma 4.3.4, we conclude, using Lemma 4.3.2, that we obtain an I_0 -homogeneous Dirichlet minimizer u in 2d with zero boundary value along a straight line and $I_0 \geq 1 + \alpha/2$, which violates Theorem 4.3.3. This concludes the proof of the inequality (4.8.3), and, arguing as in [29, Proposition 13.3], we conclude the proof of the proposition. $\square$

Chapter 5

An essential one sided boundary singularity for a 3-dimensional area minimizing current in $\mathbb{R}^5$

This chapter has been published as a preprint in [\[44\]](#).

5.1 Introduction

In this thesis we address the question: "To what extent does Allard's boundary regularity theory hold in a higher multiplicity setting?". In this chapter, we give an example that shows the dimensional sharpness of the theory we developed in [Chapter 4](#).

We prescribe a boundary surface that has non-trivial topology and will induce an essential singularity at the boundary of an area minimizing current for topological reasons. These methods are extremely flexible to construct essential boundary singularities and do not rely on constructing a calibration.

In order to show the construction of the family of examples we prove an excess decay type theorem at points of "corner type". The boundaries we consider in that setting consist of smooth surfaces with multiplicity meeting at a smooth $m - 2$ dimensional common interface

and we get decay to the tangent cone at the points in the common interface. Under a suitable "multiplicity 1" assumption, every corner point is regular.

We will give a statement of the example provided in this chapter. In order to do this, we must first define the notion of essential singularity.

Definition 5.1.1. Let T, Γ in $\mathbf{B}_1 \subset \mathbb{R}^{m+n}$ such that T is an m -dimensional area minimizing integral current, Γ an $m - 1$ -dimensional submanifold of $\mathbb{R}^{m+n}$ with $\partial T \llcorner \mathbf{B}_1 = Q \llbracket \Gamma \rrbracket$. We say that 0 is an *essential one-sided singularity* if $\Theta(T, 0) = Q/2$ and there is no radius $\rho > 0$ such that

$$T \llcorner \mathbf{B}_\rho = \sum Q_i \llbracket \Sigma_i \rrbracket \quad (5.1.1)$$

where Σ_i are smooth minimal surfaces in $\mathbf{B}_\rho$ and $\partial \Sigma_i = \Gamma \cap \mathbf{B}_\rho$ (but they are not required to be disjoint outside of the boundary or meet transversal at the boundary).

Theorem O. There exist a 3-dimensional area minimizing current T in $\mathbf{B}_1 \subset \mathbb{R}^5$ and a real analytic surface Γ such that $\partial T = 2 \llbracket \Gamma \rrbracket$, $\Theta(T, p) = 1$ for every $p \in \Gamma \cap \mathbf{B}_1$, and 0 is an essential one-sided boundary singularity.

5.1.1 Historical overview

The construction of singular examples of area minimizing currents presents substantial challenges, typically requiring the construction of a calibration.

In codimension 1, the proof of the minimality of the Simons cone in 1969 [11] was a turning point in the history of regularity of area minimizing currents since it provided the first example of a singular area minimizing integral current in codimension 1. The proof was later simplified in [39]. In higher codimension, a large family of examples is given by complex varieties due to Federer [41], since they are automatically calibrated and thus area minimizing.

It is worth mentioning that recently Liu, in a series of papers, constructed examples of interior singularities of area minimizing currents, of fractal type, of non-smoothable type, and examples of non-calibrated area minimizing currents, in [59, 60, 61].

In the context of boundary regularity of area minimizing currents, White in 1997 showed that in a parametric context there are no true boundary branch points for a 2-dimensional area minimizing current with a real analytic boundary curve [80]. A true branch point is one where the current cannot be separated into smooth minimal surfaces. In that work, he posed the question of whether an area minimizing current with a real analytic boundary has finite topology, which is equivalent to finiteness of the interior and boundary singular sets and to the area minimizing current being a Douglas - Radò solution. In contrast, for a smooth boundary, Gulliver in 1991 [50] provided an example of a true boundary branch point for a minimal surface (not minimizing) in $\mathbb{R}^3$.

In [53], Hirsch shows an example of a holomorphic function vanishing to infinite order at the boundary in the half-space. This example is used in [21] to show that if the boundary curve is merely allowed to be smooth then the boundary singular set can have Hausdorff dimension 1. Furthermore, in [19], De Lellis, De Phillipis and Hirsch provide an example of an area minimizing current in a Riemannian manifold whose boundary is a smooth curve and the associated surface has infinite topology. These examples show that the behaviour when the boundary is smooth but not real analytic can be quite singular - unlike the regularity predicted by White's conjecture in the real analytic case.

5.1.2 Brief description of the example

Theorem O demonstrates the dimensional sharpness of the regularity theory that we developed in Chapter 4. This example shows that, even if the boundary is required to be real analytic, there are counterexamples of essential singularities, analogous to the true branch points considered by White and Gulliver, but in the higher multiplicity setting corresponding to density 1, multiplicity 2 points.

The example comes from a more general theorem, which roughly speaking says that if for T an area minimizing current and ∂T looks like Figure 5.1, then it has an essential boundary singularity. The defining features of the picture are that ∂T consists of two pieces, one which is topologically a disk Γ^0 (taken with multiplicity 2) and the other which is a Möbius strip Γ^1 which contains $\partial\Gamma^0$ and twists around it. There are additional, technical requirements which can be seen in Theorem R.

The theorem is a consequence of the Hölder continuity of the multivalued normal map, which we show to hold on the spherical cap in the picture up to its boundary given by the black wire. This relies on the multivalued Hölder estimate at smooth boundary points established in Chapter 3 and employs, at the corner points (the black wire), Simon's method for the uniqueness of tangent cones. The Hölder continuity of the multivalued normal map in the spherical cap up to the boundary forces the map to agree at the black wire to the normal map induced by the Möbius strip. The non-trivial topology induced by the boundary data of this multivalued map generates an essential boundary singularity for the spherical cap away from the black wire.

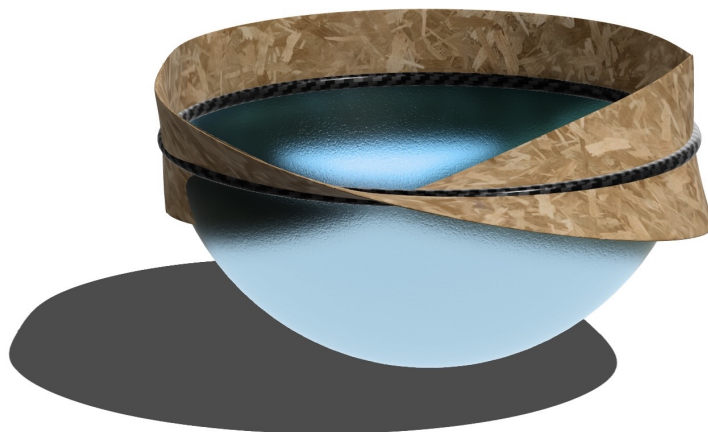


Figure 5.1: Boundary of example

See the following sketch of the proof for a more detailed discussion of the argument in the simplified setting of the linear problem.

5.1.3 Sketch of the proof

We explain the procedure we use to construct the example in the particular case of the linear problem. The linear problem consists of multi-valued functions u which vanish at the boundary of a certain domain. We will work with multivalued functions, understood as in [4] and [30].

Assumption 5.1.2 (Linear Problem). Given a Dir-minimizing function $u \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))$ for an open set $\Omega \subset \mathbb{R}^m$. The boundary linear problem consists of functions u such that $u \perp \partial\Omega \cap \mathbf{B}_1 = Q \llbracket 0 \rrbracket$.

In the interior setting, a simple example of an essential singularity is the two-valued square root $u \in W^{1,2}(\mathbb{R}^2, \mathcal{A}_2(\mathbb{R}^2))$

$$u(z) = \sum_{w^2=z} \llbracket w \rrbracket$$

where we identify $\mathbb{C}$ with $\mathbb{R}^2$.

A natural singular candidate for the boundary linear problem that we would be interested whether it is energy minimizing is $f \in W^{1,2}(\mathbb{R}^2 \times \mathbb{R}^+, \mathcal{A}_2(\mathbb{R}^2))$ defined by

$$f(z, t) = \sum_{w^2=z} \llbracket wt \rrbracket$$

where we identify $\mathbb{R}^2$ with $\mathbb{C}$ both in the domain and in the image.

This function can be shown to be stationary and it is unclear whether it is energy minimizing. Moreover, even if one were able to demonstrate, in a complicated way, that f is energy minimizing, there seems to be no reason to expect that such a strategy would translate

directly into the construction of boundary singularities for an area minimizing current of the desired type.

Construction of the singular Dir-minimizer

We consider the cylinder $\mathbf{C} := \{z \in \mathbb{R}^2 : |z| \leq 1\} \times [0, 1]$. We can just take u a solution for the linear problem with boundary data f as above. This is

$$u \in W^{1,2}(\mathbf{C}, \mathcal{A}_2(\mathbb{R}^2)) \text{ Dir-minimizing with } u \llcorner \partial \mathbf{C} = f \llcorner \partial \mathbf{C}.$$

We might have $u = f$ if the function f were energy minimizing. We can show that regardless of what u is, it will have an essential boundary singularity at some non-corner boundary point (i.e., in $\{|z| < 1\} \times \{0\}$).

As we mentioned earlier, showing that $u = f$ or that f is energy minimizing is a technically challenging question which would likely (if true) have a very rigid proof. On the other hand, the proof we provide of the singular behaviour is very flexible.

The normal derivative as a multivalued map

By Theorem 3.10.3, we know that there is a unique blowup for the linear problem. This implies that there exists a 2-valued "normal derivative" map that is well defined at the boundary $\{|z| < 1, t = 0\}$. The map $\eta : \{|z| < 1, t = 0\} \rightarrow \mathcal{A}_2(\mathbb{R}^2)$ turns out to locally be a Hölder continuous map in $\{|z| < 1, t = 0\}$, by the theory developed in Chapter 3.

The theory that we develop in this chapter allows us to extend the Hölder continuity to the boundary of the domain where the multivalued normal derivative is defined. We show an excess decay type theorem at points in corners, which for the linear setting corresponds to understanding the rate of decay of u at $\mathbb{S}^1 \times \{0\}$. Moreover, those points can be shown to be regular points. This is, the function u decouples in a neighborhood of $\mathbb{S}^1 \times \{0\}$ as two classical harmonic functions whose graphs do not intersect in the interior.

This allows us to extend the definition of η to $\{|z| \leq 1, t = 0\}$ in a Hölder continuous way.

By definition, u agrees with f in ∂C . Furthermore, since u is regular at $\mathbb{S}^1 \times \{0\}$

$$\eta(z) = \sum_{w^2=z} \llbracket w \rrbracket.$$

Topological existence of an essential singularity We concluded that η is a continuous 2-valued function on the unit disk $\{|z| \leq 1, t = 0\}$, which at the boundary $\{|z| = 1, t = 0\}$ must take the complex square root. This implies that η must have at least one essential singularity $p \in \{|z| < 1, t = 0\}$. The argument goes as follows: if all the points where either regular or non-essentially singular, then in a neighborhood of every point η splits as two smooth functions. Since $\{|z| \leq 1, t = 0\}$ is simply connected, we must thus have that η splits globally on $\{|z| \leq 1, t = 0\}$ as two smooth graphs. This gives a contradiction to the fact that $\eta(\{|z| = 1, t = 0\})$, as a subset of $\mathbb{R}^5$, is a smooth connected curve with no-self intersections which is taken with multiplicity 1. See Proposition [5.11.12](#).

A point p at which η has an essential singularity must be an essential boundary singularity for u . Indeed, if p was not an essential singularity for u then there exists ρ such that

$$u \llcorner \mathbf{B}_\rho(p) = \llbracket u_1 \rrbracket + \llbracket u_2 \rrbracket$$

where u_1 and u_2 are two classical harmonic functions. This would imply that the normal derivative at the flat piece of the boundary of $\mathbf{C}$ (i.e., $\{(z, t) \in \mathbb{C} \times \mathbb{R} : |z| \leq 1, t = 0\}$) can be written as

$$\eta \llcorner \mathbf{B}_\rho(p) = \left[\left[\frac{\partial}{\partial t} \right]_{t=0} u_1 \right] + \left[\left[\frac{\partial}{\partial t} \right]_{t=0} u_2 \right].$$

This contradicts the fact that p is an essential singularity for η .

As we have seen, we only require topological information and this construction does not rely on a calibration. Indeed, these methods work on any sufficiently smooth boundary data which induces non-trivial topology.

5.1.4 Higher multiplicity Allard boundary regularity

We remind the general type of assumption we considered in Chapter 3 and Chapter 4.

Assumption 5.1.3 (Assumption $C^{k,\alpha}$). Let integers $m, n \geq 2$, $Q \geq 1$, $k \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and $\alpha \in [0, 1]$. Consider Γ a $C^{k,\alpha}$ oriented $(m-1)$ -submanifold of $\mathbb{R}^{m+n}$ without boundary, and assume $0 \in \Gamma$ (when $k = \omega$, Γ is a real-analytic submanifold of $\mathbb{R}^{m+n}$). Let T be an integral m -dimensional area minimizing current in $\mathbf{B}_1$ with boundary $\partial T \llcorner \mathbf{B}_1 = Q \llbracket \Gamma \cap \mathbf{B}_1 \rrbracket$.

The convex barrier assumption is the following:

Assumption 5.1.4 (Convex Barrier). Let $\Omega \subset \mathbb{R}^{m+n}$ be a domain such that $\partial\Omega$ is a C^2 uniformly convex submanifold of $\mathbb{R}^{m+n}$. We say that $\sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$ has a convex barrier if Q_i are positive integers and $\Gamma_i \subset \partial\Omega$ are disjoint C^2 closed oriented submanifolds of $\partial\Omega$. In this setting we consider T an area minimizing current with $\partial T = \sum_i Q_i \llbracket \Gamma_i \rrbracket$.

Theorem O shows that our regularity theory is dimensionally sharp and moreover that in this setting (i.e., density $Q/2$) the situation is not dimensionally better if we wish to study only essential singularities. Moreover, the situation does not get better with the assumption of a real analytic boundary. The theorem we obtained in Chapter 4 is the following:

Theorem 5.1.5 (Rectifiability of the one-sided singular set - Chapter 4). *Let T be an area minimizing current which satisfies Assumption 5.1.3 with $k = 3$, $\alpha > 0$ and $m \geq 3$. Further assume that $\Theta(T, p) = Q/2$ for every $p \in \mathbf{B}_1 \cap \Gamma$. The boundary singular set (the set of points on $\mathbf{B}_1 \cap \Gamma$ which are not one-sided regular points) is $\mathcal{H}^{m-3}$ rectifiable (countable if $m = 3$). Furthermore, if $\partial T = \sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$ has a convex barrier (i.e. it satisfies Assumption 5.1.4), the boundary singular set is $\mathcal{H}^{m-3}$ rectifiable (countable if $m = 3$).*

In the context of $Q = 1$ and dimension $m = 2$, based on known examples of [21], the situation with essential boundary singularities seems to be different with that of general boundary singularities. The general boundary singular set can be of Hausdorff dimension 1,

but those singularities are of removable type, in their context it means there exists $\rho > 0$ such that

$$T \llcorner \mathbf{B}_\rho = \llbracket \Sigma_1 \rrbracket + \llbracket \Sigma_2 \rrbracket$$

with Σ_1 and Σ_2 two smooth minimal surfaces with $\partial \llbracket \Sigma_1 \rrbracket = \llbracket \Gamma \rrbracket$ and $\partial \llbracket \Sigma_2 \rrbracket = 0$. There is no known example in the literature of a smooth embedded loop $\Gamma \subset \mathbb{R}^{2+n}$ and a 2-dimensional area-minimizing current T with $\partial T = \llbracket \Gamma \rrbracket$ such that T has infinitely many essential boundary singularities.

5.1.5 Main theorems of this chapter

In order to construct the example we develop a regularity theory for currents whose boundaries have corners.

Assumption 5.1.6. Consider the following:

- (i) Let $j \in \{0, 1\}$, N^j be two positive integers. Let Σ be a C^2 ambient manifold in $\mathbb{R}^{m+n}$.
- (ii) Let Γ_i^j , $1 \leq i \leq N^j$ be $m - 1$ dimensional C^2 submanifolds of Σ such that $\partial \llbracket \Gamma_i^j \rrbracket = (-1)^j \llbracket M \rrbracket$ for M a $m - 2$ dimensional C^2 submanifold of Σ with $0 \in M$. We further assume that Γ_i^j are transversal to each other at M . We also denote $\mathbf{\Gamma} := \cup_{i,j} \Gamma_i^j$.
- (iii) Let Q_i^j be positive integer multiplicities with $\sum (-1)^j Q_i^j = 0$. We also denote $Q = \sum_i Q_i^0 = \sum_i Q_i^1$.
- (iv) Every point in M is a good corner point, as defined in later in Definition 5.2.4. This is a technical condition that ensures a bound on the mass.

The current T is area minimizing in $\mathbf{B}_1 \cap \Sigma$ with cornered boundaries Γ_i^j if

$$\partial T = \sum_{0 \leq j \leq 1} \sum_{1 \leq i \leq N^j} Q_i^j \llbracket \Gamma_i^j \rrbracket.$$

The boundary pieces Γ_i^0 are those with positive orientation with respect to the orientation of M . The boundary pieces Γ_i^1 are those with negative orientation with respect to M .

We provide an example of a current taking the boundary data.

We consider $T = H_1 + H_2$,
where H_1 and H_2 are quarter
pieces of a plane which are
oriented such that:

$$\partial[H_1] = [\Gamma_1^0] - [\Gamma_1^1]$$

$$\partial[H_2] = [\Gamma_1^0] - [\Gamma_2^1]$$

The boundary pieces Γ_0^1, Γ_1^1 ,
and Γ_2^1 are half-planes.

$$N^0 = 1, N^1 = 2, Q_1^0 = 2,$$

$$Q_1^1 = Q_2^1 = 1.$$

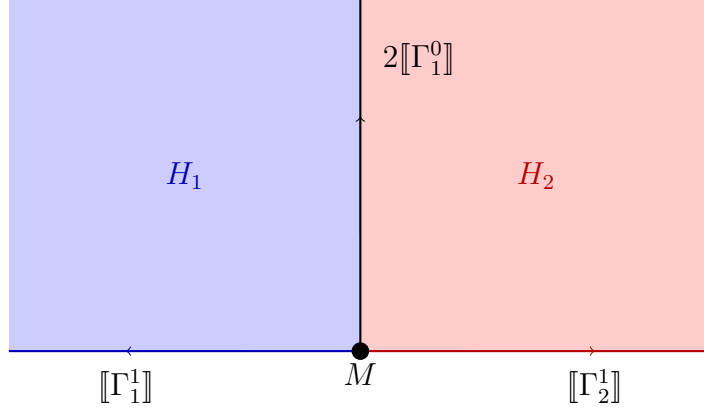


Figure 5.2: Example of Corner

We simplify the notation of the main theorem for cornered boundaries. For the precise statements see Assumption 5.4.1 and Theorem 5.7.2

Theorem P (Excess decay). We consider $T, \Gamma_{i,j}, M, \Sigma$ as in Assumption 5.1.6.

- Assume for every $1 \leq i_0 \leq N^0, 1 \leq i_1 \leq N^1$ we have $\Gamma_{i_0}^0$ and $\Gamma_{i_1}^1$ meet perpendicularly at M .
- Assume, in addition, that for any pair of surfaces with the same orientation $j = 0, 1$, $\Gamma_{i_0}^j$ and $\Gamma_{i_1}^j$, meet at M with an angle which is uniformly bounded.

Then there is a unique tangent cone at every point of M at which T decays with a power rate.

Theorem Q (Regularity in the multiplicity 1 case). Assume that we are in the hypothesis of Theorem P. Additionally assume that there is only one classical boundary piece with positive orientation and all the sheets with negative orientation have multiplicity 1 (i.e., $N^0 = 1$, $N^1 = N$, $Q_1^0 = N$, $Q_1^1 = Q_2^1 = \dots = 1$).

Then T is regular along M : for every $p \in M$ there exists ρ such that

$$T \llcorner \mathbf{B}_\rho(p) = \sum_{i=1}^N \llbracket \Sigma_i \rrbracket$$

where Σ_i are minimal surfaces, which intersect only at M and $\partial \llbracket \Sigma_i \rrbracket = \llbracket \Gamma_0^0 \rrbracket - \llbracket \Gamma_i^1 \rrbracket$. The minimal surface Σ_i is smooth away from M and attaches smoothly to Γ_0^0 and Γ_i^1 .

These assumptions are not restrictive, as the theorems above suffice for our applications. Given that we have a decomposition theorem, the regularity theory with higher multiplicity is equivalent to the case when $N^0 = 1$, $N^1 = 1$, $Q^0 = Q^1 = Q$. It would be reasonable to expect the same type of regularity theory we developed in Chapter 4 to apply for this setting, using the same methods. Assuming the same boundary and ambient manifold smoothness which allows us to run the regularity theory in Chapter 4, one would expect the singular set of T along M to be $\mathcal{H}^{m-3}$ rectifiable.

As a corollary, we get the following theorem which holds for a flexible family of boundaries

Theorem R. Let Ω be a C^2 uniformly convex open set.

- Let $M \subset \partial\Omega$ be a C^2 simple closed curve.
- Let Γ^0 be a C^2 surface with $\partial \llbracket \Gamma^0 \rrbracket = M$ diffeomorphic to a 2 dimensional disk.
- Let $\Gamma^1 \subset \partial\Omega$ be an integral 2 dimensional current. We assume that $M \subset \text{spt}(\Gamma^1)$ and in a neighborhood U of M , $U \cap \text{spt}(\Gamma^1)$ is a C^2 surface diffeomorphic to a Möbius strip which twists once around M . Indeed, $U \cap \text{spt}(\Gamma^1) \setminus M$ will be connected. We assign multiplicity 1 to $\text{spt}(\Gamma^1)$ in U . We suppose that Γ^1 is oriented such that $\partial \llbracket \Gamma^1 \rrbracket = -2 \llbracket M \rrbracket$. The surface $U \cap \Gamma^1$ is a smooth surface whose orientation switches across M .

Let T be an area minimizing current with $\partial T = 2 \llbracket \Gamma^0 \rrbracket + \llbracket \Gamma^1 \rrbracket$. Then T is regular in a neighborhood of M and has an essential boundary singularity at some point of $\Gamma^0 \setminus M$.

We show that, at the disk Γ^0 , we can define a 2-valued normal map, induced by the unique tangent cone to T . The normal map, defined at Γ^0 , is Hölder all the way to the boundary M . At the boundary M , the normal map is given by the tangent to Γ^1 at M , which is topologically non-trivial since $\text{spt}(\Gamma^1)$ is a Möbius strip in a neighborhood of M . The topological non-triviality implies, as we saw in the linear problem, that there is an essential singularity at some interior point of Γ^0 .

We note that Γ^0 and Γ^1 are required to lie on a convex barrier in order to be able to apply the theorems of Chapter 3 as well as the corresponding analogues at points of M established in this chapter.

5.1.6 Basic definitions

Throughout the chapter, we use the notation ' $\lesssim$ ' to denote inequality up to a multiplicative constant, which may depend only on m , n , $\bar{n}$, and Q . We also use $C(\alpha_1, \dots, \alpha_k)$ to denote a constant which depends on $\alpha_1, \dots, \alpha_k$.

We define $\mathbf{B}_r(p) := \{x \in \mathbb{R}^{m+n} : |x - p| < r\}$. We typically use V for $m - 1$ dimensional planes. We use $L := \mathbb{R}^{m-2} \times \{0\} \subset \mathbb{R}^{m+n}$ as the spine of our cones. For a plane π , let $\mathbf{p}_\pi$ denote the projection onto π . For any $\varepsilon > 0$, we use $B_\varepsilon(L)$ to represent the ε -neighborhood of L , that is, $B_\varepsilon(l) = \{x \in \mathbb{R}^{m+n} : \text{dist}(x, L) < \varepsilon\}$.

We assume each k -dimensional linear subspace π of $\mathbb{R}^{m+n}$ is oriented by a k -vector $\vec{\pi} = v_1 \wedge \dots \wedge v_k$, where $(v_i)_{i=1}^k$ forms an orthonormal basis for π . Additionally, we adopt the notation $|\pi_2 - \pi_1|$ for $|\vec{\pi}_2 - \vec{\pi}_1|$, where $|\cdot|$ represents the norm tied to the standard inner product of k -vectors. The k -dimensional Hausdorff measure on $\mathbb{R}^{m+n}$ is written as $\mathcal{H}^k$. We denote $\mathcal{A}_Q(\mathbb{R}^{m+n})$ the Q -tuples in $\mathbb{R}^{m+n}$ for some integer $Q \geq 1$. We denote by $\mathcal{G}(p_1, p_2)$, the Wasserstein distance between a pair of points p_1, p_2 on the space $\mathcal{A}_Q(\mathbb{R}^{m+n})$. For more on multi-valued functions, see [30]. We denote by $\mathbf{A}_M$ the C^0 -norm of the second

fundamental form of a smooth submanifold M of $\mathbb{R}^{m+n}$. We also denote under Assumption 5.1.6, $\mathbf{A}_\Gamma := \max_{i,j} \mathbf{A}_{\Gamma_i^j}$.

For further basic definitions and standard notations, we refer to [42] and the survey [17].

We will use $\mathbf{E}$ to denote the L^2 excess, and local versions which we introduce later. The L^2 excess relative to an open book C or cornered open book is defined as

$$\mathbf{E}(T, C, \mathbf{B}_r(p)) := \frac{1}{r^{m+2}} \int_{\mathbf{B}_r(p)} \text{dist}(x, C)^2 d\|T\|(x). \quad (5.1.2)$$

We will use $\mathbb{E}$ for a suitable measure-theoretic excess, which will be defined later in equation (5.4). Finally, we introduce the circular projection $\pi_\circ : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$, defined as $\pi_\circ(x, y) = (x, |y|)$ for $(x, y) \in \mathbb{R}^{m-1} \times \mathbb{R}^{n+1}$.

Definition 5.1.7. An m -dimensional integral current T in U with $\text{spt}(T) \subset \Sigma$ is *area minimizing in $\Sigma \cap U$* , if it holds that $\mathbf{M}(T) \leq \mathbf{M}(T + \partial \tilde{T})$, for all $\tilde{T}$ $(m+1)$ -dimensional integral current in U with $\text{spt}(\tilde{T}) \subset \Sigma \cap U$.

Cones

We recall the definition of an open book and introduce their angle:

Definition 5.1.8. A cone C is an *open book* if it can be written as $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ where H_i are distinct half-planes with $\partial \llbracket H_i \rrbracket = \llbracket V \rrbracket$ where V is an $m-1$ dimensional plane, referred to as the spine of the cone. We define the minimum angle of an open book as

$$\alpha(C) := \min_{i \neq j} \langle H_i, H_j \rangle$$

where $\alpha(C) := 1$ if $C = Q \llbracket H \rrbracket$ for H a half-plane.

Definition 5.1.9. A cone C is a *two-sided flat cone* if

$$C = (Q/2 + Q^*) \llbracket H \rrbracket - (Q^*) \llbracket -H \rrbracket$$

for Q and Q^* integers and H a half plane with $\partial H = V$ where $-H$ denotes the reflection of H along V .

We introduce a distance between open books with the same spine.

Definition 5.1.10 (Distance Open Books). Given a pair of open books C^j for $j = 0, 1$ which are of the form

$$C^j := \sum_i Q_i^j \llbracket H_i^j \rrbracket$$

where $\partial \llbracket H_i^j \rrbracket = \llbracket V \rrbracket$. We take η_i^j to be the unit normal to V that determines H_i . We define $\eta^j \in \mathcal{A}_Q(V^\perp)$ as $\eta_i^j = \sum_i Q_i \llbracket \eta_i^j \rrbracket$. We define

$$\mathcal{G}(C^1, C^2) := \mathcal{G}(\eta^1, \eta^2). \quad (5.1.3)$$

Definition 5.1.11. A cornered area minimizing cone C with spine L (of dimension $m - 2$) is one that satisfies

$$\partial C = \sum_{j=0}^1 \sum_{1 \leq i \leq N^j} Q_i^j \llbracket V_i^j \rrbracket \quad (5.1.4)$$

where V_i^j are $m - 1$ dimensional half-planes with $\partial \llbracket V_i^j \rrbracket = (-1)^j \llbracket L \rrbracket$ and Q_i^j are positive integers with $\sum_i Q_i^0 = \sum_i Q_i^1$.

Definition 5.1.12 (Quadrant). We define H to be an m -quadrant (or just quadrant if the dimension is easily understood) if $H \subset \mathbb{R}^{m+n}$ is an orthonormal transformation of the set $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \times \mathbb{R}^{m-2}$. We define $\hat{H}$ to be the full m -dimensional plane containing H .

Definition 5.1.13. A cornered open book with spine L (of dimension $m - 2$) is

$$C = \sum_{1 \leq i \leq N} Q_i \llbracket H_i \rrbracket$$

where H_i is a quadrant with $\partial \llbracket H_i \rrbracket = \llbracket V_i^0 \rrbracket - \llbracket V_i^1 \rrbracket$ where V_i^0 and V_i^1 are orthogonal half-planes of dimension $m - 1$ with $\partial \llbracket V_i^j \rrbracket = (-1)^j \llbracket L \rrbracket$. We denote $Q = \sum_{1 \leq i \leq N} Q_i$.

Remark 5.1.14. Cornered open books as we consider will automatically have, if C is centered at 0, $\Theta(C, 0) = Q/4$. This setting is appropriate for the desired applications. More general settings could be considered, such as not requiring H_i to be quadrants and just be pieces of planes.

The simplest cornered open book is a single quadrant with multiplicity. The main example for the construction of the example later is the one given by Figure 5.2.

We define $\mathcal{C}$ as the set of cornered open books which have the right boundary data.

Definition 5.1.15. Let Γ_i^j , M as in 5.1.6. We define $\mathcal{C} : M \rightarrow \{\text{Cornered open books}\}$. For each $p \in M$ we will denote $\mathcal{C}_p$ the set of generalized cornered open books with the following conditions.

Given the cornered open book

$$C = \sum_{1 \leq i \leq N} Q'_i \llbracket H_i \rrbracket$$

it belongs to $\mathcal{C}_p$ if

- $L = T_p(M)$.
- There exists an index selection $k(\cdot, 0) : \{1, \dots, N\} \rightarrow \{1, \dots, N^0\}$, $k(\cdot, 1) : \{1, \dots, N\} \rightarrow \{1, \dots, N^1\}$.

•

$$V_i^j = T_p \left(\Gamma_{k(i,j)}^j \right).$$

- For every $j \in \{0, 1\}$, $1 \leq k \leq N^j$

$$Q_k^j = \sum_{k(i,j)=k} Q'_i.$$

Tangent cones

For $x \in \mathbb{R}^{m+n}$ and $r > 0$, define $\iota_{x,r}(y) := \frac{y-x}{r}$. For any integral current T , we define the rescaled current T at x at scale r as $\iota_{x,r} T =: T_{x,r}$ and $T_r =: \iota_{0,r} T$. The current T_x is called a blow-up of T at x , if there exists a sequence of radii $r_j \rightarrow 0$ such that $T_{x,r_j} \rightarrow T_x$ in the weak topology. By [12], it follows from the monotonicity formula that if T is area minimizing in Σ and $\partial T = Q \llbracket \Gamma \rrbracket$ for $\Gamma \in C^{1,\kappa}$, every convergent subsequence of T_{p,r_j} converges to an oriented cone about p for any $p \in \Gamma$ and any $r_j \downarrow 0$. An oriented cone about p is understood as a current C such that $C_{p,r} = C$ for any $r > 0$. See Theorem 5.2.2 for the precise statements of the monotonicity formula and first variation formula for an area minimizing current T in Σ with $\partial T = Q \llbracket \Gamma \rrbracket$ where Σ and Γ are C^2 submanifolds. In this chapter, we will also need to consider a type boundaries which consist of smooth $(m-1)$ -dimensional submanifolds meeting at a smooth $(m-2)$ -dimensional smooth submanifold. In Theorem 5.2.7 we show the first variation formula to hold at the $(m-2)$ -dimensional smooth submanifold, and a monotonicity formula with error terms Corollary 5.2.8. This implies existence of tangent cones at corner points.

Part 1: The study of corner points

5.2 Monotonicity formula at corner points

In this section, we show the validity of the first variation formula in this context and the monotonicity formula with error terms under appropriate assumptions.

5.2.1 Smooth boundary setup

We remind the setup for the monotonicity formula used in Chapter 3.

Assumption 5.2.1. Let $m, n \geq 2$, $\bar{n} \geq 1$, $Q \geq 1$ be integers. Consider Σ an $(m+\bar{n})$ -manifold of class C^2 in $\mathbb{R}^{m+n}$, Γ a C^2 oriented $(m-1)$ -submanifold of Σ without boundary, and assume

$0 \in \Gamma$. Let T be an integral m -dimensional area minimizing current in $\Sigma \cap \mathbf{B}_1$ with boundary $\partial T \llcorner \mathbf{B}_1 = Q \llbracket \Gamma \cap \mathbf{B}_1 \rrbracket$.

Given T, Γ, Σ as in 5.2.1. We define

$$\Theta_i(T, p, r) := \exp(C_0 \mathbf{A}_\Sigma^2 r^2) \frac{\|T\|(\mathbf{B}_r(p))}{\omega_m r^m}$$

and

$$\Theta_b(T, p, r) := \exp(C_0 (\mathbf{A}_\Sigma + \mathbf{A}_\Gamma) r) \frac{\|T\|(\mathbf{B}_r(p))}{\omega_m r^m}$$

where $\mathbf{A}_\Sigma$ and $\mathbf{A}_\Gamma$ denote the C^0 norm of the second fundamental form.

Theorem 5.2.2 (Monotonicity formula - Theorem 3.2.4). *Let T, Γ, Σ be as in Assumption 5.1.3.*

- *If $p \in \mathbf{B}_1 \setminus \Gamma$ then $r \rightarrow \Theta_i(T, p, r)$ is monotone on the interval $(0, \min \{ \text{dist}(p, \Gamma), 1 - |p| \})$.*
- *If $p \in \mathbf{B}_1 \cap \Gamma$, then $r \rightarrow \Theta_b(T, p, r)$ is monotone on $(0, 1 - |p|)$.*

Thus the density exists at every point. Moreover, the restrictions of the map $p \rightarrow \Theta(T, p)$ to $\mathbf{B}_1 \cap \Gamma$ and $\mathbf{B}_1 \setminus \Gamma$ are upper semi-continuous.

Additionally, this implies that blowup limits are cones, both at the interior and at the boundary.

If $X \in C_c^1(\mathbf{B}_1, \mathbb{R}^n)$, then

$$\delta T(X) = - \int_{\mathbf{B}_1} X \cdot \vec{H}_T(x) d\|T\|(x) + Q \int_{\Gamma \cap \mathbf{B}_1} X \cdot \vec{\eta}(x) d\mathcal{H}^{m-1}(x) \quad (5.2.1)$$

where $\vec{H}_T = \sum_{i=1}^m A_\Sigma(e_i, e_i)$ where $\vec{T} = \wedge_{i=1}^m e_i$ is an orthonormal basis of $\vec{T}$. The vector $\vec{\eta}$ is defined $\mathcal{H}^{m-1}$ almost everywhere as:

- For every tangent cone C at $p \in \Gamma$ which is an open book $C = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$,

$$\vec{\eta} = \sum_{i=1}^N \frac{Q_i \vec{\eta}_i}{Q} \quad (5.2.2)$$

where $\vec{\eta}_i$ is the normal to $T_p(\Gamma)$ which determines the half plane H_i . The vector $\vec{\eta}$ is independent of the open book C at p , but the open book might depend on the blowup sequence.

- For every tangent cone $C = (Q/2 + Q^*) \llbracket H \rrbracket - (Q^*) \llbracket -H \rrbracket$ at $p \in \Gamma$ which is a two-sided flat cone

$$\vec{\eta} = \eta_H \quad (5.2.3)$$

where η_H is the normal to $T_p(\Gamma)$ which determines the half plane H .

Corollary 5.2.3 (Monotonicity formula with error terms). *Under the hypothesis of the above theorem*

$$\begin{aligned} r^{-m} \|T\|(\mathbf{B}_r(p)) - s^{-m} \|T\|(\mathbf{B}_s(p)) &= \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_s(p)} \frac{|(x-p)^\perp|^2}{|x-p|^{m+2}} d\|T\|(x) \\ &+ \int_s^r \rho^{-m-1} \left[\int_{\mathbf{B}_\rho(p)} (x-p)^\perp \cdot \vec{H}_T(x) d\|T\|(x) + Q \int_{\Gamma \cap \mathbf{B}_\rho(p)} (x-p) \cdot \vec{\eta} d\mathcal{H}^{m-1} \right] d\rho \end{aligned}$$

5.2.2 Cornered boundary

We introduce the notion of a good corner point, which is responsible for a mass bound at corner points, and allows us to obtain the first variation formula and monotonicity formula.

Definition 5.2.4 (Good corner point). Under Assumption 5.1.6[(i-ii-iii)], a point $q \in M$ is a good corner point if there exists $r_q > 0$ such that for every $p \in \mathbf{B}_{r_q/2}(q) \cap M$, there exists a diffeomorphism $\Phi_p \in C^2(\mathbf{B}_{r_q}(p), \mathbb{R}^{m+n})$ such that

- $\Phi_p(p) = p$ and $D\Phi_p(p) = I$

- $\Phi_p(\Gamma_i^j) \subset T_p(\Gamma_i^j)$ (abusing notation and considering the full $m - 1$ dimensional plane) and $\Phi_p(M) = T_p(M)$.
- $D\Phi_p$ maps normal vectors to M into normal vectors to $T_p(M)$. $D\Phi_p$ maps normal vectors to Γ_i^j into normal vectors to $T_p(\Gamma_i^j)$.
- We have the following uniform C^2 bound

$$\sup_{p \in \mathbf{B}_{r_q/2}(q) \cap M} \|\Phi_p\|_{C^2(\mathbf{B}_{r_q}(p))} < \infty.$$

The following lemma is an appropriate modification of [21, Lemma 4.25] and thus we omit its proof.

Lemma 5.2.5. *Under Assumption 5.1.6, we define $d_p(x) := |\Phi_p(x)|$. There exists $r_0 > 0$ such that the following holds:*

- For every $p \in \mathbf{B}_{1/2} \cap M$, d_p is in $C^2(\mathbf{B}_{r_0} \setminus \{0\})$ and for $x \in \mathbf{B}_{r_0/2}(p) \setminus \{0\}$ $D^j d(x) = D^j |x - p| + O(|x - p|^{2-j})$ for $j = 0, 1, 2$ with uniform constants on p in $\mathbf{B}_{1/2}(0)$.
- ∇d_p is tangent to M at points of $M \cap \mathbf{B}_{r_0}(p)$ and tangent to $\Gamma_i^j \cap \mathbf{B}_{r_0}(p)$ at points of $\Gamma_i^j \cap \mathbf{B}_{r_0}(p)$.

We will also need sufficient conditions for M to form a good corner.

Lemma 5.2.6. *Let N_i be C^2 surfaces in $\mathbb{R}^{m+n}$ with a common boundary $0 \in M$. Assume that any pair of surfaces N_i and N_j their normal vectors to M are orthogonal. Then there exists a neighborhood of M such that the analogous properties of Definition 5.2.4 hold.*

Proof. We will construct a diffeomorphism Φ_p with the desired properties by using the Whitney extension theorem.

Since the construction is similar for p in a neighborhood of M with a uniform C^2 norm, we assume without loss of generality that $p = 0$.

We start defining Φ on M . We consider a $f_M : M \cap \mathbf{B}_r \rightarrow T_0(M)$ which is a C^2 diffeomorphism with $f_M(0) = 0$ and $Df_M = Id$.

We wish to extend the diffeomorphism f_M to diffeomorphisms f_{N_i} of N_i . We consider extensions of the diffeomorphisms $f_{N_i} : N_i \cap \mathbf{B}_r \rightarrow T_0(N_i)$ which extend f_M (possibly reducing r). The extensions will be C^2 and satisfy $Df_{N_i} = Id$. The extensions can be constructed for instance by using the Whitney extension theorem.

We wish to extend the maps f_{N_i} to a map Φ with the desired properties which is defined on a ball. In order to construct the desired extension we use the Whitney extension theorem. The condition of the orthogonal intersection between N_i and N_j ensures that the first derivatives and second derivatives are compatible with being derivatives of a C^2 function in the ball.

In order to make Φ map normal vectors to N_i , to normal vectors to $T_0(N_i)$ it is enough to prescribe, as part of the data to use the Whitney extension theorem, the derivatives of Φ along normal vectors to N_i appropriately. Since N_i are C^2 manifolds, the appropriate normal derivatives can be taken C^1 and chosen to satisfy the compatibility conditions.

Since the C^2 norm is bounded in a neighborhood then up to shrinking the radius we obtain that Φ is a diffeomorphism. $\square$

Theorem 5.2.7. *Let $T, \Gamma_{i,j}, M, \Sigma$ be as in Assumption 5.1.6. If $X \in C_c^1(\mathbf{B}_1, \mathbb{R}^{m+n})$, then*

$$\delta T(X) = - \int_{\mathbf{B}_1} X \cdot \vec{H}_T(x) d\|T\|(x) + \sum_{i,j} Q_i^j \int_{\Gamma_i^j \cap \mathbf{B}_1} X \cdot \vec{\eta}_{i,j}(x) d\mathcal{H}^{m-1}(x) \quad (5.2.4)$$

where $\vec{H}_T = \sum_{i=1}^m A_\Sigma(e_i, e_i)$ where $\vec{T} = \wedge_{i=1}^m e_i$ is an orthonormal basis of $\vec{T}$. The vectors $\vec{\eta}_{i,j}$ satisfy $|\vec{\eta}_{i,j}| \leq 1$ and are chosen according to equations (5.2.2) and (5.2.3).

Corollary 5.2.8 (Monotonicity formula with error terms). *Under the hypothesis of the above theorem for every $p \in M \cap \mathbf{B}_1$ and $0 < s < r < 1 - |p|$,*

$$r^{-m} \|T\|(\mathbf{B}_r(p)) - s^{-m} \|T\|(\mathbf{B}_s(p)) = \int_{\mathbf{B}_r(p) \setminus \mathbf{B}_s(p)} \frac{|(x-p)^\perp|^2}{|x-p|^{m+2}} d\|T\|(x) \quad (5.2.5)$$

$$+ \int_s^r \rho^{-m-1} \left[\int_{\mathbf{B}_\rho(p)} (x-p)^\perp \cdot \vec{H}_T(x) d\|T\|(x) + \sum_{i,j} Q_i^j \int_{\Gamma_i^j \cap \mathbf{B}_\rho(p)} (x-p) \cdot \vec{\eta}_{i,j} d\mathcal{H}^{m-1} \right] d\rho \quad (5.2.6)$$

As a consequence, for every $p \in M \cap \mathbf{B}_1$ there exist tangent cones. This means every sequence of radii $r_i \rightarrow 0$ such that $T_{p,r_i} \rightarrow T_p$ in the weak topology, must have T_p is an area minimizing cone whose boundary is an union of half-planes with multiplicities.

Proof of Theorem 5.2.7 and Corollary 5.2.8. We start by proving Theorem 5.2.7. We wish to show that for a vector field $X \in C_c^1(\mathbf{B}_1, \mathbb{R}^{m+n})$ the identity (5.2.4) holds.

Since we have the first variation formula at smooth boundaries with multiplicities (Theorem 5.2.2, equation (5.2.1)), equation (5.2.4) holds for any $X \in C_c^1(\mathbf{B}_1, \mathbb{R}^{m+n})$ such that $\text{spt}(X) \cap M = \emptyset$. We can reach that conclusion by using local versions of the first variation formula at interior points and smooth boundary points and patching them with a partition of unity.

Given $h > 0$, we consider a bump function φ_h supported in an h -neighborhood of M . We get the following

$$\begin{aligned} \delta T(X\varphi_h) + \delta T(X(1-\varphi_h)) &= \int_{\mathbf{B}_1} X \cdot \nabla_{\vec{T}} \varphi_h d\|T\|(x) + \int_{\mathbf{B}_1} \text{div}_{\vec{T}}(X) \varphi_h d\|T\|(x) \\ &- \int_{\mathbf{B}_1} X(1-\varphi_h) \cdot \vec{H}_T(x) d\|T\|(x) + \sum_{i,j} Q_i^j \int_{\Gamma_i^j \cap \mathbf{B}_1} X \cdot \vec{\eta}_{i,j}(x) (1-\varphi_h) d\mathcal{H}^{m-1}(x). \end{aligned}$$

In the former equation, we expanded the divergence of the vector field $X\varphi_h$ and used the first variation identity for the vector field $X(1-\varphi_h)$, for which equation (5.2.4) holds as $\text{spt}(X(1-\varphi_h)) \cap M = \emptyset$.

In order to conclude equation (5.2.4) for the vector field X , we take a limit of h to zero. The second term is easily seen to converge to zero. The third and fourth converge to the desired quantities of the right hand side of equation (5.2.4).

To conclude, it remains to control the first term involving $\nabla_{\vec{T}}\varphi_h$. It is enough to show that for every $\varepsilon > 0$

$$\lim_{h \rightarrow 0} \frac{1}{h} \|T\|(\{x \in \mathbf{B}_{1-\varepsilon} : \text{dist}(x, M) < h\}) = 0. \quad (5.2.7)$$

This requires a control of the mass along the points on M .

We can use similar arguments to [20, Theorem 4.4] to show that there is a constant C such that for every $p \in M \cap \mathbf{B}_{1/2}$ the functions

$$s \rightarrow e^{Cs} \frac{\|T\|(\{d_p < s\})}{s^m} \quad (5.2.8)$$

are monotone in some interval $[0, r_0]$. This follows from using the first variation formula for the vector field ∇d_p , which has the right regularity to use the first variation formula and maps the corner M onto itself and Γ_i^j onto themselves.

The monotonicity formula (5.2.8) gives us a uniform mass bound at balls centered at M . Thus we can split the sets

$$\{x \in \mathbf{B}_{1-\varepsilon} : \text{dist}(x, M) < h\}$$

into $C \frac{1}{h^{m-2}}$ balls of size comparable with h , each of them, for small enough h , with mass bounded by Ch^m by the monotonicity formula (5.2.8). This implies that for small enough h

$$\frac{1}{h} \|T\|(\{x \in \mathbf{B}_{1-\varepsilon} : \text{dist}(x, M) < h\}) \leq Ch$$

which allows us to conclude (5.2.7).

The monotonicity formula with error term follows from the first variation formula as in [21, Theorem 3.2].

The existence of tangent cones follows from the usual argument, which we sketch below. If $p \in M$, $r_i \rightarrow 0$ and $T_{p,r_i} \rightarrow T_p$ in the weak topology. We must have $\partial T_p = \sum_{i,j} Q_i^j \llbracket V_i^j \rrbracket$ where $V_i^j = T_p(\Gamma_i^j)$. For the current T_p , we obtain as a consequence of the monotonicity of (5.2.8) that $\Theta(T_p, r) = \lim_{r \rightarrow 0} \Theta(T, p, r)$. In the limit, for the current T_p , the contribution from the second fundamental form vanishes, and the boundary term disappears since each $\vec{\eta}_{i,j}$ becomes a constant vector orthogonal to V_i^j .

This implies that

$$\int_{\mathbf{B}_1(p)} \frac{|x^\perp|^2}{|x|^{m+2}} d\|T_p\|(x) = 0$$

which shows that T_p is a cone. □

5.3 Classification of certain area minimizing cornered cones with density $Q/4$

In this section, we classify area minimizing cones with density $Q/4$ and cornered boundary. In general, we also obtain a lower on the density of $Q/4$ which ensures a no-holes condition when applying Leon Simon's uniqueness of cylindrical tangent cones strategy [72].

We refer to Definitions 5.1.11 and 5.1.13 for cornered cones and cornered open books, respectively.

Lemma 5.3.1. *Let C be a cornered area minimizing cone as in Definition 5.1.11. Assume that for every i_0, i_1 ,*

$$V_{i_0}^0 \perp V_{i_1}^1. \tag{5.3.1}$$

Then $\Theta(C, 0) \geq Q/4$ with equality if and only if C is a cornered open book.

We will need the following technical proposition.

Proposition 5.3.2. *Let C be a 2 dimensional area minimizing cornered cone as in Lemma 5.3.1. Then $\Theta(C, 0) \geq Q/4$ with equality if and only if C is a cornered open book.*

Proof of Proposition 5.3.2. Assume that $\Theta(C, 0)$ is the minimum possible among all cones with the same boundary data. We will show that this is $Q/4$ and the equality case only holds when C is a cornered open book.

By interior regularity for area minimizing currents due to Almgren [4] and De Lellis-Spadaro [31, 33, 34], C being a 2d area minimizing cone, must be fully regular in the interior.

The link of C (i.e., its restriction to the sphere $\mathbb{S}^{n+1}$) must be stationary outside of $\mathbb{S}^{n+1} \cap \left(\bigcup_{i,j} V_i^j\right)$. This implies that C in $\mathbb{S}^{n+1}$ outside of the boundary must consist of geodesic segments. The geodesic segments can either be a full geodesic or a geodesic joining two boundary points of C .

We must then have that

$$\text{spt}(C) \subset \bigcup_k P_k$$

where P_k are planes passing through $\{0\}$.

By the constancy theorem applied to the planes P_k , C can be decomposed as a sum of finitely many $Q_k \llbracket H_k \rrbracket$ where H_k are parts of planes and $Q_k > 0$ are positive integers. If the orientations and multiplicities are the ones assigned by the constancy theorem we obtain

$$C = \sum_k Q_k \llbracket H_k \rrbracket, \quad \|C\|(\mathbf{B}_1) = \sum_k Q_k |H_k|(\mathbf{B}_1).$$

We classify the pieces of planes we obtain in types - interior and 3 different types of planar pieces:

1. Interior - H_k is a full plane: $\partial \llbracket H_k \rrbracket = 0$.
2. Piece of Type 1 - H_k is a piece of a plane (an open subset of an m -dimensional plane) which respects the orientation: There exist i_k^0 and i_k^1 such that $\partial \llbracket H_k \rrbracket = \llbracket V_{i_k^0}^0 \rrbracket - \llbracket V_{i_k^1}^1 \rrbracket$.
3. Piece of Type 2 - H_k is a piece of a plane (an open subset of an m -dimensional plane) which reverses the orientation: There exist i_k^0 and i_k^1 such that $\partial \llbracket H_k \rrbracket = -\llbracket V_{i_k^0}^0 \rrbracket + \llbracket V_{i_k^1}^1 \rrbracket$.

4. Piece of Type 3 - H_k is a piece of a plane (an open subset of an m -dimensional plane) which connects two pieces of the same orientation: There exist i_k^0, i_k^1 and j_k such that
- $$\partial \llbracket H_k \rrbracket = \llbracket V_{i_k^0}^{j_k} \rrbracket - \llbracket V_{i_k^1}^{j_k} \rrbracket.$$

There can't be interior pieces with non-zero multiplicity. Removing such pieces keeps the boundary of C but reduces $\Theta(C, 0)$ and this would contradict the minimality of $\Theta(C, 0)$.

We argue in what remains by induction. Let K be the number of planar pieces H_k in the decomposition above and Q as usual is the total multiplicity of the boundary. Assume we have shown that for every cone with $K - 1$ sheets in the decomposition and any multiplicity Q we must have $\Theta(C, 0) \geq Q/4$ and equality on the cornered open book case. Then the same holds for a cone with K sheets. The base case is $K = 1$, which is trivial since it is a single quarter plane.

If C has a piece of type 1 H_K , then removing the piece changes Q to $Q - Q_k$ and K to $K - 1$. We can conclude this case by the inductive step, since $\Theta(H_K, 0) \geq 1/4$ and the equality holds when H_K is a quadrant.

If C has a piece of type 3, then we can remove it obtaining a cone C' with total multiplicity Q and $K - 1$ sheets. This implies that $\Theta(C', 0) \geq Q/4$ and thus $\Theta(C, 0) > Q/4$.

If C has a piece of type 2 and no piece of type 3, then there must be two pieces H_1 and H_2 with $\partial \llbracket H_1 \rrbracket + \partial \llbracket H_2 \rrbracket = 0$, but this implies that $\llbracket H_1 \rrbracket + \llbracket H_2 \rrbracket$ is area minimizing without boundary which gives us the same contradiction as in the interior case. $\square$

Proof of Lemma 5.3.1. In the context of higher multiplicity boundaries, the analogous to this lemma in Chapter 3 is Theorem 3.3.1.

We wish to use monotonicity formula without error terms at points of L . If T is any area minimizing current with boundary $\partial T = \sum_{i,j} (-1)^j Q_i^j \llbracket V_i^j \rrbracket$ then the monotonicity formula will have no error terms: indeed, if we take in the monotonicity formula with the error term (5.2.8) for the case $V_i^j = \Gamma_i^j$ then $\overline{n_{i,j}} \perp V_{i,j}$. Thus the boundary term vanishes. We denote

for $p \in L$

$$\Theta(T, p) = \lim_{r \rightarrow 0} \frac{\|T\|(\mathbf{B}_r)}{\omega_m r^m}.$$

Since the monotonicity formula holds without error terms at such points, this limit exists and defines the density.

As usual, if $\frac{\|T\|(\mathbf{B}_r)}{\omega_m r^m} = \Theta(T, p)$ for all $r_0 > r > 0$ (for some r_0) and $p \in L$ then T is a cone at p . We obtain

$$\int_{\mathbf{B}_1(p)} \frac{|x^\perp|^2}{|x|^{m+2}} d\|T\| = 0,$$

which implies that T is a cone. This argument was carried out already as part of the proof of Corollary 5.2.8 to show existence of tangent cones, but we include it here for completeness.

Suppose that $\Theta(C, 0) < Q/4$. Let $\Theta(C, 0) = \Theta < Q/4$. Since C is a cone then $\Theta(C, p) = \Theta(C, rp)$ for every $p \in L$ such that $p \neq 0$ and $r > 0$. Since the density is upper semi-continuous, then $\Theta(C, p) \leq \Theta$ for every $p \in L$. By taking iterative blowups at points of L we obtain an area minimizing cone $\hat{C}$ which is of the form

$$\hat{C} = \mathbb{R}^{m-2} \times C'$$

where C' is a $2d$ area-minimizing cone with

$$\partial C' = \sum_{j=0}^1 \sum_{1 \leq i \leq N^j} Q_i^j \llbracket \mathbf{p}_{L^\perp}(V_i^j) \rrbracket$$

We conclude by the inequality of Proposition 5.3.2.

Now we wish to study the equality case. As a consequence of the following inequality for every $p \in L$

$$\lim_{r \rightarrow \infty} \Theta(C, p, r) = \lim_{r \rightarrow \infty} \frac{\|C\|(\mathbf{B}_r(p))}{\omega_m r^m} \leq \lim_{r \rightarrow \infty} \frac{(r + |p|)^m}{r^m} \frac{\|C\|(\mathbf{B}_{r+|p|})}{\omega_m (r + |p|)^m} = Q/4.$$

We must have that the equality in the monotonicity formula holds for every point. Then it must be a cone with respect to every point in L .

Thus

$$C = \llbracket \mathbb{R}^{m-2} \rrbracket \times C'$$

where C' is a 2d area minimizing cone in $\mathbb{R}^{n+2}$ with cornered boundary

$$\partial C' = \sum_{i,j} (-1)^j Q_i^j \llbracket \mathbf{p}_{L^\perp}(V_i^j) \rrbracket.$$

We conclude from the equality case of Proposition 5.3.2. □

5.4 Soft uniqueness of tangent cones at corner points

In this section, we show uniqueness of tangent cones to hold at corner points with soft arguments. This will not be enough for the desired applications. The uniqueness of tangent cones will be made quantitative in the rest of the chapter.

We will define a measure theoretic version of excess as in Subsection 3.2.1, with respect to cones $C = \sum_{1 \leq i \leq N} \llbracket H_i \rrbracket$ with $\partial \llbracket H_i \rrbracket = \llbracket V_i^0 \rrbracket - \llbracket V_i^1 \rrbracket$ where V_i^0 and V_i^1 are halfplanes with $\partial V_i^j = (-1)^j \llbracket L \rrbracket$ (and L is a common $m - 2$ dimensional plane).

The Wasserstein-2 distance between two measures μ^1 and μ^2 is defined as

$$W_2(\mu^1, \mu^2)^2 := \inf \left\{ \int d(x, y)^2 d\sigma(x, y) : (\pi_1)_\# \sigma = \mu^1 \text{ and } (\pi_2)_\# \sigma = \mu^2 \right\},$$

where σ is a transport plan between μ^1 and μ^2 , meaning that σ is a measure on $\mathbb{R}^{m+n} \times \mathbb{R}^{m+n}$ such that $(\pi_1)_\# \sigma = \mu^1$ and $(\pi_2)_\# \sigma = \mu^2$.

Given two measures μ^1 and μ^2 , we define the following distance between them:

$$d(\mu^1, \mu^2) := \inf \left\{ W_2(\mu_1^1, \mu_1^2)^2 + \sum_{l=1}^2 \int \text{dist} \left(x, \bigcup_{i,j} V_{i,j} \right)^2 d|\mu_2^l| \mid \mu_1^l + \mu_2^l = \mu^l, l = 1, 2. \right\}.$$

Let ϕ be a smooth, canonical bump function supported in $B_1(0)$, constantly equal to 1 in $B_{1/2}(0)$, and satisfying $\phi(x) \lesssim \text{dist}(x, \partial B_1)^2$. We define $\phi_r := \phi(\cdot/r)$.

We define the stronger excess as

$$\mathbb{E}(T, C, \mathbf{B}_r) := r^{-(m+2)} d(\phi_r d\|T\|, \phi_r d\|C\|).$$

We are going to restrict ourselves to a narrower setup which will be more than enough to produce an example of an essential boundary singularity.

Assumption 5.4.1. Suppose we are under the Assumptions of 5.1.6. Let $N_{\Gamma_i^j}(p)$ denote the vector in $T_p(\Gamma_i^j)$ which is normal M .

- We assume that for every $p \in M \cap \mathbf{B}_1$, $\forall 1 \leq i_0 \leq N^0$ and $\forall 1 \leq i_1 \leq N^1$

$$N_{\Gamma_{i_0}^0}(p) \perp N_{\Gamma_{i_1}^1}(p).$$

- We also assume that there is a universal constant α_0 such that for every $(i_0, j_0) \neq (i_1, j_1)$ and $1 \leq i_0 \leq N^{j_0}$, $1 \leq i_1 \leq N^{j_1}$, $0 \leq j_0 \leq j_1$

$$2^{1000} \alpha_0 \leq \left\langle N_{\Gamma_{i_0}^{j_0}}(p), N_{\Gamma_{i_1}^{j_1}}(p) \right\rangle.$$

The angle condition is a reasonable assumption since Γ_i^j are assumed to be smooth and transversal. In practice this only will mean that we will not track the fine dependence of the angle in the Lipschitz approximations and decay, which is irrelevant for the applications we prove in this chapter. This setting is different than the one we consider in Chapter 3, since the tangent cone is automatically unique by soft arguments since the boundary data does not give any degree of freedom. The interesting question in this setting instead is the rate of decay which we do need for the application in mind.

Assumption 5.4.1 provides a good setup for this problem because of the following lemma.

Lemma 5.4.2. *Let T , Γ_i^j , M , Σ , be as in Assumption 5.4.1. Then $\Theta(T, p) \geq Q/4$, in the equality case there is a unique tangent cone at T which is a cornered open book.*

The importance of obtaining the lower density bound is that we will use Leon Simon's uniqueness of cylindrical tangent cones strategy [72] and this provides us a no good density gaps condition.

Remark 5.4.3. A slightly more general version of Lemma 5.4.2 holds with no modification. Under Assumption 5.1.6, if every tangent cone at p is of the form

$$C = \sum_{i=1}^Q \llbracket H_i \rrbracket$$

where each H_i is an open subset of a plane, then the tangent cone is unique.

Proof of Lemma 5.4.2. We refer to Definition 5.1.15 of cornered open books with the right boundary data.

Given $C \in \mathcal{C}_p$, it is easy to see that $\mathbb{E}(T, C, \mathbf{B}_r)$ is continuous on r . We know that $\mathcal{C}_p$ is a finite set, since there are finitely many possibilities for choosing Q quadrants with the right boundary data. Each of them corresponds to a function $l : \{1, \dots, Q\} \rightarrow \{1, \dots, N^0\} \times \{1, \dots, N^1\}$ which associates a quadrant H_i with its boundary $T_p(\Gamma_{l(i)_0}^0)$ with positive orientation and $T_p(\Gamma_{l(i)_1}^1)$ with negative orientation (where $l(i)_0, l(i)_1$ are the coordinates of $l(i)$). The function uniquely determines the cornered open book. This implies that $\mathcal{C}_p$ is a finite set and thus it must be disconnected in the Wasserstein distance.

We consider the function $f : (0, 1) \rightarrow \mathbb{R}^{\#\mathcal{C}_p}$ given by

$$f(r) := (\mathbb{E}(T, C, \mathbf{B}_r))_{C \in \mathcal{C}_p}.$$

We consider all the possible limits of the function $f(r)$ along a sequence of radius going to zero. This must be a connected subset of $\mathbb{R}^{\#\mathcal{C}_p}$ because f is a continuous function of r and the limit set of a continuous function is connected.

We know that there are $\#\mathcal{C}_p$ possible values the limit of f can take which is

$$(\mathbb{E}(C, C', \mathbf{B}_1))_{C \in \mathcal{C}_p}.$$

This is a disconnected set of $\mathbb{R}^{\#\mathcal{C}_p}$ and thus the only option is for it to have a single element and thus a single tangent cone. $\square$

5.5 Linear problem

In this section, we introduce an appropriate linear model for the nonlinear problem with singular boundaries of the type considered in this chapter. We refer to [30] for the detailed setup for the linear problem in the interior and 3.8 of this thesis for the boundary setting.

Given $\Omega \subset \mathbb{R}_+^m$ a Lipschitz domain there is a well defined continuous trace operator

$$\circ|_{\partial\Omega} : W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)) \rightarrow L^2(\partial\Omega, \mathcal{A}_Q(\mathbb{R}^n)).$$

Definition 5.5.1. Let Ω be a Lipschitz domain. We say u is Dir-minimizing in Ω if

$$u \in W^{1,2}(\overline{\Omega}, \mathcal{A}_Q(\mathbb{R}^n))$$

and for every $v \in W^{1,2}(\overline{\Omega}, \mathcal{A}_Q(\mathbb{R}^n)) : u \llcorner \partial\Omega = v \llcorner \partial\Omega$

$$\int_{\Omega} |Du|^2 \leq \int_{\Omega} |Dv|^2.$$

We define the quarter balls

$$\mathbf{B}_r^{1/4} := \mathbf{B}_r \cap (\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{m-2})$$

In order to keep the notation consistent with the rest of this work, we define the quarter plane $H := \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{m-2}$. We denote the boundary as $V^0 = \{0\} \times \mathbb{R}^+ \times \mathbb{R}^{m-2}$ and $V^1 := \mathbb{R}^+ \times \{0\} \times \mathbb{R}^{m-2}$. We denote the corner as $L := \{0\} \times \{0\} \times \mathbb{R}^{m-2}$.

In this chapter it will be enough for us to consider the linear problem on the quarter space. More precisely we consider the following setup.

Assumption 5.5.2 (Linear problem on the quarter space). Given a Dir-minimizing function $u \in W^{1,2}(\mathbf{B}_1^{1/4}, \mathcal{A}_Q(\mathbb{R}^n))$. The boundary linear problem on the quarter space consists of functions u such that $u \llcorner (V^0 \cup V^1) \cap \mathbf{B}_1 = Q \llbracket 0 \rrbracket$.

We adapt the classical Almgren frequency function to this quarter-domain setting at points in the corner L .

Definition 5.5.3. For $x \in L \cap \mathbf{B}_1$, $0 < r < 1 - |x|$, we define the Dirichlet energy, L^2 spherical height and the frequency function:

$$D(x, r) = \int_{\mathbf{B}_r^{1/4}(x)} |Du|^2, \quad H(x, r) = \int_{\partial \mathbf{B}_r^{1/4}(x)} |u|^2 \quad \text{and} \quad I(x, r) = \frac{rD(x, r)}{H(x, r)}$$

The arguments here follow those of Section 3.8, which studies the higher multiplicity boundary linear problem.

Theorem 5.5.4. *The frequency function for a function u satisfying Assumption 5.5.2 is monotone. For any $x \in L \cap \mathbf{B}_1$ either there exists ρ such that $u|_{B_\rho(x)} \equiv 0$ or $I(x, r)$ is an absolutely continuous non decreasing positive function of r on $(0, 1 - |x|)$.*

Observe a minor change here with Chapter 3 and [30]. In this setting, we don't claim uniform convergence, since we have not proved Hölder continuity of u before this. We are not going to address the Hölder continuity for functions u satisfying Assumption 5.5.2 since we won't need it.

Theorem 5.5.5. *Let $f \in W^{1,2}(\mathbf{B}_1^{1/4}, \mathcal{A}_Q)$ be Dir-minimizing with zero boundary data and $\text{Dir}(f, B_\rho) > 0$ for every $\rho \leq 1$. Then for any sequence $\rho_k \rightarrow 0$ there exists a subsequence, not*

reabeled, such that $f_{0,\rho}$ converges locally strongly in $W^{1,2}(H)$ to a Dir-minimizing function $g : H \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ with the following properties:

1. $\text{Dir}(g, \mathbf{B}_1^{1/4}) = 1$ and $g|_\Omega$ is Dir-Minimizing for any bounded Ω .
2. $g(x) = |x|^\alpha g\left(\frac{x}{|x|}\right)$, where $\alpha = I_{0,f}(0)$ is the frequency of f at 0.

The following two lemmas have analogous proofs to those in Section 3.8, which considers the boundary linear problem in the half-space. We will use them to lower bound the frequency in Lemma 5.5.8.

Lemma 5.5.6. (Cylindrical Blowups) Assume that $m \geq 3$. Let $g \in W^{1,2}(H, \mathcal{A}_Q(\mathbb{R}^n))$ be an α homogeneous and Dir-Minimizing function with zero boundary value at $V^0 \cup V^1$, $\text{Dir}(g, \mathbf{B}_1^{1/4}) > 0$ and $\beta = I_{z,g}(0)$. Suppose $z = \frac{e_m}{2}$ ($z \in L$). Then, the tangent functions h to g at z satisfy

$$h(x_1, \dots, x_{m-1}, x_m) = \hat{h}(x_1, \dots, x_{m-1})$$

where $\hat{h} \in W^{1,2}(H \cap \mathbb{R}^{m-1} \times \{0\}, \mathcal{A}_Q(\mathbb{R}^n))$ is a Dir-Minimizing Q valued function with zero boundary values on $\mathbb{R}_+^{m-1}$

Lemma 5.5.7. Let h be a α homogeneous Dir-minimizing function as in Assumption 5.5.2. Suppose that $I(z) = \alpha$ for $z = e_m/2$. Then $h(x, s) = \hat{h}(x)$ for a Dir-minimizing function $\hat{h}$ on $H \cap \mathbb{R}^{m-1} \times \{0\}$ with zero boundary data at $(V^0 \cup V^1) \cap (\mathbb{R}^{m-1} \times \{0\})$.

Lemma 5.5.8. Let u be as in Assumptions of the linear problem on the quarter space 5.5.2. Assume u is α -homogeneous. Then $\alpha \geq 2$. Furthermore if u is 2-homogeneous then if θ denotes the angle on the quarter-plane (in the first two components)

$$u(x) = \sum_{i=1}^k \llbracket v_i \cos(2\theta) |x|^2 \rrbracket.$$

Proof. Assume by contradiction there exists u an α homogeneous Dir-minimizing function with $\alpha < 2$.

Let $z \in L$. By homogeneity, if $z \neq 0$ then for all $r > 0$, $I(rz) = I(z)$. Thus by upper-semi-continuity $\alpha \geq I(z)$. We take a blowup at z which gives us an $I(z) \leq \alpha < 2$ homogeneous Dir-minimizer on $H \cap \mathbb{R}^{m-1}$. By iterating and applying Lemma 5.5.6, we obtain a β homogeneous Dir-minimizing function u on $\mathbb{R}^+ \times \mathbb{R}^+$ with zero boundary data on $\{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}$.

By the interior regularity for Dir-minimizing functions u [30] in 2d the function u needs to be smooth outside of its boundary. Thus it must decouple as classical harmonic function that has zero boundary value along $\{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}$. A classical harmonic function which vanishes on the boundary of the quarter 2d plane must have frequency at least 2. This is because it's restriction to S^1 must solve

$$f''(\theta) + \lambda^2 f(\theta) = 0$$

with $f(0) = f(\pi/2) = 0$. We can extend periodically, which makes it a classical homogeneous harmonic function in the plane. Since f constant or f linear are not options for f to be non-trivial, the homogeneity for f must be at least 2. Moreover we must have that $\lambda = k$ is an even integer and

$$f(\theta) = \sum_{i=1}^Q \llbracket v_i \cos(k\theta) \rrbracket.$$

If $k \neq 2$ then f has interior singularities where $\cos$ vanishes and thus f must be 2-homogeneous and in this case for $v_i \in \mathbb{R}^n$

$$f(\theta) = \sum_{i=1}^Q \llbracket v_i \cos(2\theta) \rrbracket.$$

When u is 2-homogeneous, an argument analogous to the one at the end of the proof of Lemma 5.3.1 shows that u arises from a two-dimensional solution in a quarter space. $\square$

We have an analogue of 3.8.19 in this setting. We omit the proof since it is analogous.

Theorem 5.5.9 (Local Dir-minimizers). *Let $u \in W^{1,2}(\mathbf{B}_1^{1/4}, \mathcal{A}_Q(\mathbb{R}^n))$ have zero boundary value along $V_0 \cup V_1$. Assume that u is Dir-minimizing on every subdomain of $\mathbf{B}_1^{1/4}$ with smooth boundary. Then u is Dir-minimizing in $\mathbf{B}_1^{1/4}$ with zero boundary value along $V_0 \cup V_1$.*

Proposition 5.5.10. *If $I(0) = \alpha$ then*

$$\int_{\mathbf{B}_r^{1/4}} |u|^2 \leq \frac{1}{m+1} r^{m-1+2\alpha} \int_{\partial \mathbf{B}_1^{1/4}} |u|^2.$$

In particular since $I(0) \geq 2$ then

$$\limsup_{r \rightarrow 0} \frac{1}{r^{m+4}} \int_{\mathbf{B}_r^{1/4}} |u|^2 < \infty.$$

Proof. The proof is analogous to Proposition 3.8.11. It follows from the estimates on the spherical height which are a consequence of the monotonicity of the frequency. See Chapter 3.4 of [30]. □

5.6 Lipschitz approximation

In this section, we construct the Lipschitz approximation and conclude with some technical estimates regarding measure theoretic excess and its relationship to L^2 excess.

The general assumption for the Lipschitz approximation is the following:

Assumption 5.6.1. Suppose that T , Γ_i^j , Σ , M are as in Assumption 5.4.1, but now defined in the ball $\mathbf{B}_8$. Let C be the unique tangent cone at 0. For a small constant $\varepsilon(Q, m, n, \bar{n})$ we assume

$$\mathbb{E}(T, C, \mathbf{B}_8) + \mathbf{A}_{\mathbf{r}} + \mathbf{A}_{\Sigma}^2 \leq \varepsilon \alpha(C)^2.$$

Moreover we assume that for every $p \in M \cap \mathbf{B}_2$ and all $0 < r < 1/8$

$$\|T\|(\mathbf{B}_r(p)) \leq (Q/4 + 1/16) \omega_m r^m.$$

We consider a parameter $\delta > 0$ which will be taken small enough depending on Q_i^j , m , n , α_0 . In order to construct the Lipschitz approximation we will need to define a suitable Whitney region. Let $L = \mathbb{R}^{m-2} \times \{0\}$. Let P_0 be a cube with side length $\frac{2}{\sqrt{m-2}}$ centered at 0.

We define the regions as in [24]. The set $R := \{p : \mathbf{p}_L(p) \in P_0 \text{ and } 0 < |\mathbf{p}_{V^\perp}(p)| \leq 1\}$. The cubes $\mathcal{G}_l$ are cubes of sidelength $\frac{2^{1-l}}{\sqrt{m-2}}$ obtained by subdividing P_0 into $2^{l(m-2)}$ cubes. The integer l , the generation of the cube, is denoted by $l(P)$. If $P \subset P'$ and $\ell(P') = \ell(P) + 1$, we then call P' the parent of P , and P a child of P' . When $\ell(P') > \ell(P)$, we say that P' is an ancestor of P and P is a descendant of P' .

For every cube P we denote by y_P its center and $\mathbf{B}(P)$ the ball $\mathbf{B}_{2^{-l(P)}}(y_P)$ (in $\mathbb{R}^{m+n}$) and by $\mathbf{B}^h(P)$ the set $\mathbf{B}(P) \setminus B_{2^{-5-l(P)}}(L)$.

We define for $P \in \mathcal{G}_l$

$$R(P) := \left\{ p : \mathbf{p}_L(p) \in P \text{ and } 2^{-l-1} \leq |p_{L^\perp}(p)| \leq 2^{-l} \right\}.$$

We also define λP as the cube concentric with P with side length $\lambda \frac{2^{1-l}}{\sqrt{m-2}}$ and

$$\lambda R(P) := \left\{ p : \mathbf{p}_L(p) \in \lambda P \text{ and } \lambda^{-1} 2^{-l-1} \leq |p_{L^\perp}(p)| \leq \lambda 2^{-l} \right\}.$$

We also define $\lambda P_i := \lambda R(P) \cap H_i$. We notice that $\mathbf{p}_{\hat{H}_i}(\Gamma_{l(i)}^0)$ and $\mathbf{p}_{\hat{H}_i}(\Gamma_{l(i)}^1)$ delimit a domain in $\hat{H}_i$. When $\mathbf{A}_\Gamma$ is small, we must have that this domain must be close to H_i , a quarter of the plane $\hat{H}_i$. We denote D_i the described domain.

Lemma 5.6.2 (Splitting). *There exists a constant $\delta(Q, m, n, \alpha_0)$ with the following property.*

Assume that

$$\int_{\mathbf{B}_4 \setminus B_{1/32}(L)} \text{dist}(x, C)^2 d\|T\| + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \leq \delta^2.$$

There exist $T_1, T_2, \dots, T_N$ such that

$$T \llcorner (\mathbf{B}_{3.5} \setminus B_{3/64}(L)) = \sum_{i=1}^N T_i \text{ and } \forall i : \partial T_i = Q'_i \llbracket \Gamma_{l(i)}^0 \rrbracket - Q'_i \llbracket \Gamma_{l(i)}^1 \rrbracket.$$

The multiplicity of Q'_i is the one obtained by constancy over the projection on $(\mathbf{p}_{H_i})_{\#} T_i$. Furthermore, if instead we assume

$$\mathbb{E}(T, C, \mathbf{B}_8) + \mathbf{A}_{\Gamma} + \mathbf{A}_{\Sigma}^2 < \delta^2$$

then $Q'_i = Q_i$.

Proof. We will show that if δ is small enough there exist disjoint open sets $U_i \subset \mathbf{B}_{3.5} \setminus B_{3/64}(L)$ that satisfy

$$\partial U_i \cap \partial U_j \subset \Gamma \cap (\mathbf{B}_3 \setminus B_{1/32}(L))$$

and

$$\text{spt}(T) \cap \mathbf{B}_{3.5} \setminus B_{3/64}(L) \subset \cup_{1 \leq i \leq N} U_i.$$

We start by taking a cover of $\Gamma \cap \mathbf{B}_{3.5} \setminus B_{3/64}(L)$ by balls $\mathbf{B}_{2^{-12}\alpha_0}(p_k)$, which we assume to be a Besicovitch covering. By the choice of the radii, if $\mathbf{A}_{\Gamma}$ is small enough no ball will intersect two different boundary pieces.

We consider

$$A := \left(\text{spt}(T) \cap (\mathbf{B}_{3.5} \setminus B_{3/64}(V)) \right) \setminus B_{2^{-20}\alpha_0}(\Gamma).$$

We must have assuming that $\mathbf{A}_{\Gamma}$ is small enough, by the height bound that

$$\sup_{x \in A} \text{dist}(x, C) \leq C\delta^2.$$

If δ is small enough with respect to α_0 and m, n then A is split into disjoint neighborhoods $U_{i,0}$ of H_i . The neighborhoods $U_{i,0}$ serve to decompose $\text{spt}(T)$ away from Γ .

By the bounded overlap property, we know that

$$\sum_l \int_{\mathbf{B}_{2^{-12}\alpha_0}(p_k)} \text{dist}(x, C)^2 d\|T\| \leq C(\alpha_0) \int_{\mathbf{B}_4 \setminus B_{1/32}(L)} \text{dist}(x, C)^2 d\|T\| \leq \delta^2. \quad (5.6.1)$$

At the boundary Γ in $\mathbf{B}_3 \setminus B_{3/64}(V)$, we use the following Theorem which we state combining two results:

Theorem 5.6.3 (Decomposition: Theorem 3.5.13 + Theorem 3.5.15). *Let T and Γ be under 5.2.1. Assume that C is a non-flat open book whose spine is $T_0(\Gamma)$ There exists $\varepsilon > 0$ such that if*

$$\max \left\{ \mathbf{E}(T, C, \mathbf{B}_1(0)), \varepsilon^{-1} \mathbf{A}_\Gamma, \varepsilon^{-1} \mathbf{A}_\Sigma^2 \right\} \leq \varepsilon \alpha(C)^2.$$

There there exist $T_1, T_2, \dots, T_N$ area minimizing currents and Q_i integers on $\mathbf{B}_{1/64} \cap \Sigma$ with

- $\partial T_i = Q_i \llbracket \Gamma \rrbracket$

-

$$T \llcorner \mathbf{B}_{1/64}(0) = \sum_{i=1}^N T_i \tag{5.6.2}$$

- The supports of T_i only intersect at Γ .

In our setting, we will apply it to a modification of C the starting cornered open book. All cornered open books with the right boundary data C' (recall Definition 5.1.15) under Assumption 5.4.1 satisfy $\alpha(C') \gtrsim \alpha_0$. We need to translate and adjust the cone C so that it passes through the center where we apply this result, but that only introduces errors that are controlled by $\mathbf{A}_\Gamma$ and keeps $\alpha(C) \gtrsim \alpha_0$.

If δ is small enough then we will be able to apply the theorem above and get such decomposition at scale $\mathbf{B}_{2^{-18}}(p_k)$. This means that $T \llcorner \mathbf{B}_{2^{-18}}(p_k) = \sum_{1 \leq i \leq N} T_i$ and we can choose $U_{i,l}$ disjoint neighborhoods of $\mathbf{B}_{2^{-18}}(p_k)$ each of which containing $(H_i \cup \text{spt}(T_i)) \cap \mathbf{B}_{2^{-18}}(p_k)$.

We define

$$U_i = U_{i,0} \bigcup_l U_{i,l}$$

which satisfy the desired properties.

The last two remarks follow are a consequence of the construction of the strong Lipschitz approximation in the interior and of the definition of the excess. See Proposition 3.6.3 and Lemma 8.5 [24]. $\square$

Lemma 5.6.4 (Lipschitz Approximation). *Let T and C as in the above Lemma 5.6.2. Let*

$$E_i := \int_{\mathbf{B}_3 \setminus B_{1/32}(L)} \text{dist}(p, H_i)^2 d\|T_i\|(p).$$

Let $\Omega_i := D_i \cap \mathbf{B}_2 \setminus B_{1/16}(L)$ and $\mathbf{\Omega}_i := \mathbf{B}_3 \cap \mathbf{p}_{\hat{H}_i}^{-1}(\Omega_i)$.

1. We have $(p_{\hat{H}_i})_{\#}(T_i) = Q'_i \llbracket D_i \rrbracket$.
2. Then there exist Lipschitz multivalued functions $u_i : \Omega_i \rightarrow A_{Q'_i}(\hat{H}_i^\perp)$ and a closed set $K_i \subset \Omega_i$ such that $\text{gr}(u_i) \subseteq \Sigma$, $T_i \llcorner \mathbf{p}_{H_i}^{-1}(K_i) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{H_i}^{-1}(K_i)$ and the following estimates hold

$$u_i \llcorner (\mathbf{p}_{\hat{H}_i}(\Gamma_{l(i)}^j) \cap \mathbf{B}_2 \setminus B_{1/16}(L)) = Q'_i \llbracket \mathbf{p}_{\hat{H}_i^\perp}(\Gamma_{l(i)}^j) \rrbracket \quad j = 0, 1.$$

$$\|u_i\|_\infty^2 + \|Du_i\|_{L^2}^2 \leq CE_i + C\mathbf{A}_\Gamma + C\mathbf{A}_\Sigma^2$$

$$\text{Lip}(u_i) \leq C(E_i + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2)^\gamma$$

$$|\Omega_i \setminus K_i| + \|T\|(\mathbf{\Omega}_i \setminus \mathbf{p}_{H_i}^{-1}(K_i)) \leq C(E_i + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2)^{1+\gamma}.$$

3. If additionally $\mathbb{E}(T, C, \mathbf{B}_8) < \delta^2$ then $Q_i = Q'_i$ for all i .

Proof. The proof is standard so we will just give some basic considerations. The new element in this context is that we do take the boundary data. The proof follows as in Proposition 8.6 [24], but we need to incorporate the boundary and for that we use Proposition 4.1 4.5.1. The Lipschitz approximation gets defined in balls which overlap and then patched together. Given a point $p \in \Gamma \cap \mathbf{B}_{3.5} \setminus B_{3/64}(L)$ in order to use Proposition 4.5.1 we need to slightly tilt the plane to a new plane $\hat{H}_{i,p}$ so that for the appropriate l, j , $T_p(\Gamma_l^j) \subset \hat{H}_{i,p} \subset T_p(\Sigma)$. $\square$

The analogous statement holds locally over Whitney regions associated to a cube P . We define

$$\mathbf{E}(P) := 2^{(m+2)\ell(P)} \int_{\mathbf{B}^h(P)} \text{dist}(x, C)^2 d\|T\|.$$

Lemma 5.6.5 (Local Splitting). *There exists a constant δ such that if*

$$\mathbf{E}(P) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \leq \delta^2.$$

Then there exist $T_{1,P}, \dots, T_{N,P}$ such that

$$T \llcorner (\mathbf{B}_{3 \cdot 2^{-\ell(P)}}(y_p) \setminus B_{2^{-4-\ell(P)}}(L)) = \sum_{i=1}^N T_i \text{ and } \forall i : \partial T_i = Q'_i \llbracket \Gamma_1^0 \rrbracket - Q'_i \llbracket \Gamma_i^1 \rrbracket.$$

The multiplicity of Q'_i is the one obtained by constancy over the projection on $(\mathbf{p}_{H_i})_\# T_i$.

Furthermore, if we assume

$$\mathbf{E}(T, C, \mathbf{B}_8) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 < \delta^2$$

then $Q'_i = Q_i$ for every P with $\ell(P) \sim 1$.

Definition 5.6.6. Let $\delta_0 = \delta(Q, m, n, \bar{n})$ be so that Lemma 5.6.4 and 5.6.5 hold.

Definition 5.6.7. Let $P \in \mathcal{G}$. We say that

1. A cube P is a *classical boundary cube* if $\mathbf{B}^h(P') \cap M = \emptyset$ for every ancestor L' of L (including L) and

$$\|T\|(\mathbf{B}(L)) \leq (Q/4 + 1/4)2^{m(2-l(P))}. \quad (5.6.3)$$

2. A cube P is a *corner stopping cube* if $\mathbf{B}^h(P) \cap M \neq \emptyset$ or $\|T\|(\mathbf{B}(P)) > (Q/4 + 1/4)2^{m(2-l(P))}$ and it's parent is a classical boundary cube.
3. A cube P is an *outer cube* if it is of classical boundary type and $\mathbf{E}(P') \leq \delta_0^2$ for every ancestor P' of P (including P). We denote $\mathcal{G}^\circ$ the outer cubes.
4. A cube P is a *inner cube* if it is a classical boundary cube, but it an outer cube.

It is convenient to define the following regions

- The *corner region* denoted R^{corner} which is the union of $R(P)$ over all cubes which are not classical boundary cubes.
- The *outer region* denoted R^o , is the union of $R(P)$ over all outer cubes.
- The *inner region* denoted R^i , is the union of $R(P)$ over all classical boundary cubes which are not outer cubes.

Definition 5.6.8. Let $T, \Gamma_i^j, \Sigma, M, C$ be as in Assumption 5.6.1. Fix $P \in \mathcal{G}^o$. We denote

$$\bar{\mathbf{E}}(P) := \max \{ \mathbf{E}(P') : P' \in \mathcal{G}^o \text{ with } R(P) \cap R(P') \neq \emptyset \}$$

We use Lipschitz approximation locally and then patch the pieces as in [24] to get a coherent outer approximation.

Proposition 5.6.9 (Coherent outer approximation). *Let T, Γ, Σ, C as in Assumption 5.6.1. We define*

$$R_i^o := \cup_{L \in \mathcal{G}^o} L_i \equiv H_i \cap \cup_{L \in \mathcal{G}^o} R(L).$$

Then, there are Lipschitz multivalued maps $u_i : R_i^o \rightarrow A_{Q_i}(\widehat{H}_i^\perp)$ and closed subsets $\bar{K}_i(P) \subset P_i$ satisfying the following properties:

1.

$$u_i \llcorner (\mathbf{p}_{\hat{H}_i}(\Gamma_{l(i)}^j) \cap R_i^o) = Q_i \llbracket \mathbf{p}_{\widehat{H}_i^\perp}(\Gamma_{l(i)}^j) \rrbracket \quad j = 0, 1.$$

2. *$gr(u_i) \subset \Sigma$ and $T_{P,i} \llcorner \mathbf{p}_{H_i}^{-1}(\bar{K}_i(P)) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{H_i}^{-1}(\bar{K}_i(P))$ for every $L \in \mathcal{G}^o$, where $T_{P,i}$ is defined as in the proposition above.*

3.

$$2^{2l(P)} \|u_i\|_{L^\infty(L_i)}^2 + 2^{ml(P)} \|Du_i\|_{L^2(L_i)}^2 \lesssim \bar{\mathbf{E}}(L) + 2^{l(P)} \mathbf{A}_\Gamma + 2^{-2l(P)} \mathbf{A}_\Sigma^2$$

$$\|Du_i\|_{L^\infty(L_i)} \lesssim (\bar{\mathbf{E}}(L) + \mathbf{A}_\Sigma^2)^\gamma$$

$$|L_i \setminus K_i(L)| + \|T_{L,i}\|(\mathbf{p}_{H_i}^{-1}(L_i \setminus K_i(L))) \lesssim 2^{-ml(P)} \left(\overline{\mathbf{E}}(L) + 2^{-2l(P)} \mathbf{A}_\Sigma^2 \right)^{1+\gamma}$$

Proof. The proof is the same as Proposition 8.19 in [24]. The coincidence set is taken to be $\overline{K}_i(P) := \cap_{P': R(P') \cap R(P) \neq \emptyset} K_i(P')$. The Lipschitz function u_i is obtained by extending the function from $\cup_{P \in R^\circ} \overline{K}_i$. $\square$

In the following lemma we consider the definitions of the different type of cubes as above, but we do not require the global Assumption 5.6.1. We still consider the definition of types of cubes as above with respect to the cone C and their associated Lipschitz approximation.

Lemma 5.6.10. *Let T, Γ, Σ be as in Assumption 5.4.1 in $\mathbf{B}_4$. Assume that every cube that intersects $\mathbf{B}_{2^{-10}}$ and their ancestors are outer cubes whose multiplicities of their local Lipschitz approximations agree with those of C .*

$$\mathbb{E}(T, C, \mathbf{B}_1) \leq C(Q, m, n, \bar{n}, \delta_0) \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma^2 \right)$$

5.7 Proof of main theorems at corner points

In this section, we will show a decomposition theorem, excess decay and Hölder Continuity of the normal map. We will define what we mean by a regular point at M .

Definition 5.7.1. Let $T, \Gamma_i^j, \Sigma, Q_i^j$ be as in Assumption 5.4.1. A point $p \in M$ is *regular* if

$$T \llcorner \mathbf{B}_\rho(p) = \sum_{i=1}^N Q_i \llbracket \Xi_i \rrbracket$$

where $\sum_{i=1}^N Q_i = Q$ and

- Ξ_i is an area minimizing surface in $\mathbf{B}_\rho(p)$ with

$$\partial \Xi_i = \llbracket \Gamma_{l(i)}^0 \rrbracket - \llbracket \Gamma_{l(i)}^1 \rrbracket.$$

Moreover, Ξ_i in $\mathbf{B}_\rho(p)$ is smooth outside of M and attaches smoothly to both $\Gamma_{l(i)}^0$ and $\Gamma_{l(i)}^1$.

- For $i_1 \neq i_2$ we have that

$$\text{spt}(\Xi_{i_1}) \cap \text{spt}(\Xi_{i_2}) \subset \left(\bigcup_{j \in \{0,1\}} \Gamma_{l(i_1)}^j \cap \Gamma_{l(i_2)}^j \cap \mathbf{B}_\rho(p) \right)$$

where if $\Gamma_{l(i_1)}^j = \Gamma_{l(i_2)}^j$ then $\Gamma_{l(i_1)}^j \cap \Gamma_{l(i_2)}^j = \Gamma_{l(i_1)}^j = \Gamma_{l(i_2)}^j$, otherwise $\Gamma_{l(i_1)}^j \cap \Gamma_{l(i_2)}^j = M$.

Theorem 5.7.2. *Let $T, \Gamma_i^j, \Sigma, Q_i^j$ be as in Assumption 5.4.1. Assume that $\Theta(T, 0) = Q/4$. By Lemma 5.4.2 there is a unique tangent cone, which is a cornered open book $C \in \mathcal{C}_0$. Then there is a neighborhood $\mathbf{B}_\rho$ of 0 such that $T \lfloor \mathbf{B}_\rho = \sum_{1 \leq i \leq N} T_i$, each T_i with corresponding flat tangent cone $Q_i \llbracket H_i \rrbracket$ (each H_i a single quadrant from the cornered open book), with*

$$\partial T_i = Q'_i \llbracket \Gamma_{k(i,0)}^0 \rrbracket - Q'_i \llbracket \Gamma_{k(i,1)}^1 \rrbracket.$$

In the case $N^0 = 1$, $N^1 = N$, $Q_1^0 = N$, $Q_1^1 = Q_2^1 = \dots = 1$, 0 must be a regular point for T .

Lemma 5.7.3 (Excess decay with small second fundamental forms). *Let $T, \Gamma_i^j, \Sigma, Q_i^j$ be as in Assumption 5.4.1. There ε and $\rho < 1/2$ such that if $\mathbb{E}(T, C, \mathbf{B}_1) < \varepsilon$,*

$$\mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma < \varepsilon \mathbb{E}(T, C, \mathbf{B}_1)$$

for C a cornered open book then

$$\mathbb{E}(T, C, \mathbf{B}_\rho) \leq \frac{1}{2} \mathbb{E}(T, C, \mathbf{B}_1).$$

Lemma 5.7.4 (Rate of convergence to the the tangent cone). *Let $T, \Gamma_i^j, \Sigma, Q_i^j$ be as in Assumption 5.4.1. There is $\varepsilon > 0$, $\alpha > 0$ and $\kappa > 0$ such that if*

$$\max \left\{ \mathbb{E}(T, C_p, \mathbf{B}_4(p)), \kappa^{-1} \mathbf{A}_\Gamma, \kappa^{-1} \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

then, for any $q \in \mathbf{B}_1(p) \cap M$ and $r < 1/4$

$$\max \left\{ \mathbb{E}(T, C_q, \mathbf{B}_r), \kappa^{-1} \mathbf{A}_\Gamma r, \kappa^{-1} \mathbf{A}_\Sigma^2 r^2 \right\} \leq Cr^\alpha \max \left\{ \mathbb{E}(T, C_p, \mathbf{B}_4), \kappa^{-1} \mathbf{A}_\Gamma, \kappa^{-1} \mathbf{A}_\Sigma^2 \right\},$$

where C_q denotes the unique tangent cone to T at q .

Corollary 5.7.5 (Hölder estimate at the corner). *Let $T, \Gamma_i^j, \Sigma, Q_i^j$ be as in Assumption 5.4.1. Then the function $p \rightarrow C_p$ which assigns at points in $M \cap \mathbf{B}_1$ the tangent cone of T at p is locally Hölder continuous in the Hausdorff distance. This is, for every $\varepsilon > 0$ and every $p, q \in M \cap \mathbf{B}_{1-\varepsilon}$*

$$\text{dist}_H(C_p, C_q) \leq C|p - q|^\alpha.$$

Remark 5.7.6. The Hölder continuity of C_p as a function of p in $\mathbf{B}_1 \cap M$ serves to locally determine the cornered open book when there are multiple admissible cornered open books. This is interesting outside of the multiplicity 1 case in which there is a unique cornered open book which takes the right boundary data.

Proof of Lemma 5.7.4 and Corollary 5.7.5. If q is required to be p then we need to extend the small second fundamental forms case to the general case as in Theorem 3.5.8.

The error of replacing p to q is the error of changing the cornered open book, which we did in detail in Corollary 3.5.12.

The proof of Corollary 5.7.5 holds analogously to Corollary 5.12 3.5.12. □

Now we can prove Theorem 5.7.2.

Proof of Theorem 5.7.2. We assume $p = 0$. Since C_p is the tangent cone, there is a radius $r > 0$ such that $\mathbf{B}_r$ is in the hypothesis of the above Lemma 5.7.4. This is

$$\max \left\{ \mathbb{E}(T, C_0, \mathbf{B}_4), \kappa^{-1} \mathbf{A}_\Gamma, \kappa^{-1} \mathbf{A}_\Sigma^2 \right\} < \varepsilon$$

Moreover, possibly shrinking the radius, we assume that M is parameterized by bi-lipschitz map whose bi-lipschitz constant is close to 1. That is, since $L = T_p(M)$, there exists $\psi : L \cap \mathbf{B}_r \rightarrow \mathbb{R}^{m+n}$ such that

$$M \cap \mathbf{B}_{2r} = \psi(L \cap \mathbf{B}_{2r}) \cap \mathbf{B}_{2r}.$$

We need to construct a cubical decomposition. We use a variation of the notation introduced for the Lipschitz approximation at the beginning of Section 6.

We define $\mathcal{G}_l^M$ to consist of "cubes" $P = \varphi(\hat{P})$, where $\hat{P}$ is a dyadic cube of size $\frac{1}{\sqrt{m-2}}2^{-l}$ in L . We define λP as $\psi(\lambda\hat{P})$.

We recall the notation

$$2R(P) := \left\{ p : \mathbf{p}_M(p) \in P \text{ and } 2^{-l-2} \leq \text{dist}(x, M) \leq 2^{-l+1} \right\}.$$

Let y_P be the center of the cube (i.e. $\psi(y_{\hat{P}})$ where $y_{\hat{P}}$ is the center of the original cube).

We notice that there exists a universal constant λ such that given $P \in \mathcal{G}_l$ then region is contained in some ball

$$2R(P) \subset \mathbf{B}_{\lambda 2^{-l}}(y_P).$$

Thus if $\lambda 2^{-l(P)} \leq r/4$ then $\mathbf{B}_{\lambda 2^{-l}}(y_P) \subset \mathbf{B}_{r/4}(y_P) \subset \mathbf{B}_r$ which implies

$$\max \left\{ \frac{1}{2^{-l(m+2)}} \int_{\mathbf{B}_{\lambda 2^{-l}}(y_P)} \text{dist}(x, C_{p_l})^2 d|T|, \kappa^{-1} 2^{-l+1} \lambda \mathbf{A}_{\mathbf{F}}, \kappa^{-1} \lambda^2 2^{-2l+2} \mathbf{A}_{\Sigma} \right\} \leq C(\lambda, m, n) 2^{-l\alpha} \varepsilon. \quad (5.7.1)$$

By the separation Lemma 5.6.2, we obtain that in the region $2R(P)$ the current T decomposes as $\sum_{i=1}^N T_i$, whose supports only intersect at the boundary Γ_i^j . The regions $2R(P)$ for varying cubes P are chosen to have overlap and additionally they cover $\mathbf{B}_{r_0}$ for some $r_0 < r$. This implies that in $\mathbf{B}_{r_0}$ the current T can be decomposed as $T = \sum_{i=1}^N T_i$ whose supports only intersect at $\cup_{i,j} \Gamma_i^j \cup M$.

To conclude, we must show the regularity of T_i when the multiplicities are 1. We can cover $2R(P)$ with balls of size comparable to 2^{-l} some centered at the boundary, some at the interior, having some overlap. If ε is small enough, we can apply Allard's interior regularity [2] for the interior balls and Allard's boundary regularity [3] for the balls centered at the boundary (here we are using that the number of balls is a dimensional constant). $\square$

Assume that $N^0 = 1$, $N^1 = N$, $Q_1^0 = N$, $Q_1^1 = Q_2^1 = \dots = 1$.

Definition 5.7.7. Given $p \in \Gamma_1^0$ let $C_p = \sum_{i=1}^N Q_i \llbracket H_i \rrbracket$ be the tangent cone at p . We denote $N\Gamma_1^0$ the normal bundle of Γ_1^0 . We define the normal map $\eta : \Gamma_1^0 \rightarrow \mathcal{A}_{Q_1^0}(N\Gamma_1^0)$ as

$$\eta(p) = \sum_{i=1}^N Q_i \llbracket \mathbf{p}_{T_p(\Gamma)^\perp}(H_i) \rrbracket. \quad (5.7.2)$$

We define it analogously at points of M (where it is required to take the value of the unique cornered open book).

Theorem 5.7.8. *Let T, Γ_i^j, Σ, M be as Assumption 5.4.1. Assume that $N^0 = 1$, $N^1 = N$, $Q_1^0 = N$, $Q_1^1 = Q_2^1 = \dots = 1$. Let $p \in \mathbf{B}_r$. Then there exists a radius $r > 0$ such that η is Hölder in $\mathbf{B}_r(p) \cap \Gamma_1^0$ up to the boundary M .*

Proof. We give a sketch of the argument, which is a modification of the proof of Corollary 3.5.12. By the Theorem 5.7.2, p is a regular point and thus it is equivalent to show it for the case that $N^0 = 1$, $N^1 = 1$, $Q_1^0 = Q_1^1 = 1$ (i.e., two boundary pieces of multiplicity 1).

We assume that we are on a neighborhood such that we have the setup of Theorem 5.7.2.

Let H_q be the tangent cone at q which is a single quarter plane of multiplicity 1. By Lemma 5.7.4 by restricting to a suitable neighborhood of p we have

$$\hat{\mathbb{E}}(T, H_q, \mathbf{B}_r(q)) \leq Cr^\alpha.$$

We use the same cubical decomposition as before. For $x \in 2R(P) \cap \Gamma_0^0$ we have that if we translate and tilt H_q to a new quadrant H_q^x such that $\partial \llbracket H_q^x \rrbracket = \llbracket T_x(\Gamma_0^0) \rrbracket$ and $H_q^x \subset T_x(\Sigma)$

then

$$\mathbb{E}(T, H_q^x, \mathbf{B}_{2^{-l-4}}) + \mathbf{A}_\Gamma 2^{-l} + \mathbf{A}_\Sigma^2 2^{-2l} \leq C \hat{\mathbb{E}}(T, H_q, \mathbf{B}_{\lambda 2^{-l}}(y_p)) \leq C \lambda^\alpha 2^{-l\alpha} \varepsilon.$$

As a consequence of Allard Boundary regularity [3] we have excess decay and thus if H_x is the tangent cone at x

$$\angle(H_x, H_q^x) \leq C \lambda^\alpha 2^{-l\alpha} \varepsilon. \quad (5.7.3)$$

Moreover, our control of the excess we have also implies, up to changing the exponent α , the Hölder estimate

$$\|\eta\|_{C^\alpha(2R(P))} \leq C\varepsilon. \quad (5.7.4)$$

By patching together (5.7.3) and (5.7.4) we obtain the desired conclusion. $\square$

Corollary 5.7.9. *Let T, Γ_i^j, Σ, M be as in Theorem 5.7.8. Then η is Hölder continuous in $\Gamma_0^0 \cup M$.*

Proof. By the Theorem 5.7.8, we have the Hölder continuity locally at M . This fact, coupled with the Hölder continuity at a single boundary piece Corollary 3.5.12 implies the Hölder continuity estimate. $\square$

5.8 Simon's non concentration estimates

In this section, we prove Simon's non-concentration estimates - the key technical machinery we require to prove the excess decay theorem.

We can then show

Lemma 5.8.1. *Let T, Γ_i^j, Σ, M be as Assumption 5.6.1 for ε small enough. Let $r = \frac{1}{3\sqrt{m-2}}$*

$$\int_{B_r} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q) \leq C\mathbf{E}(T, C, \mathbf{B}_4) + C\mathbf{A}_\Sigma^2 + C\mathbf{A}_\Gamma \quad (5.8.1)$$

where $q^\perp$ is the projection of q into the tangent plane determined by $\vec{T}_q$.

Proof. The proof is done in the same way as in Theorem 3.7.1, in that proof we explain how to take care of the boundary terms which are the same that we have here. The error terms appear when using the monotonicity formula with errors Corollary 5.2.8 and the more general first variation formula from Theorem 5.2.7. In order to take care of the interior terms we use Theorem 11.2 [24].

This involves a careful application of the first variation formula of 5.2.2 and the monotonicity formula (5.2.8). For any χ , monotone, non-increasing, such that $\mathbb{1}_{[0,r]} \leq \chi \leq \mathbb{1}_{[0,2r]}$. We also define $\Gamma(t) := - \int_t^\infty \frac{d}{ds}(\chi(s)^2) s^m ds$.

We bound

$$\int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q) \leq C \left[\int \chi^2(|q|) d\|T\|(q) - \sum_i Q_i \int_{H_i} \chi^2(|q|) d|H_i| \right] \quad (5.8.2)$$

$$+ C \int \frac{|\Gamma(|q|) q^\perp \cdot \vec{H}_T|}{|q|^m} d\|T\|(q) \quad (5.8.3)$$

$$+ C \sum_{i,j} Q_i^j \left[\int_0^4 \chi(|\rho|)^2 \rho^{-1} \int_{\Gamma_i^j \cap B_\rho} (q \cdot \vec{\eta}_{i,j}) d\mathcal{H}^{m-1} d\rho \right] \quad (5.8.4)$$

$$+ C \sum_{i,j} Q_i^j \left[\int_0^4 \chi(|\rho|)^2 \rho^{m-1} \int_0^\rho t^{-m-1} \int_{\Gamma_{i,j} \cap B_t} (q \cdot \vec{\eta}_{i,j}) d\mathcal{H}^{m-1} dt d\rho \right] \quad (5.8.5)$$

The term (5.8.3) is dealt analogously to [24]. The terms (5.8.4), (5.8.5) are dealt by observing that

$$\int_{\Gamma_i^j \cap B_\rho} |q \cdot \vec{\eta}_{i,j}| d\mathcal{H}^{m-1} = \int_{\Gamma_i^j \cap B_\rho} |\mathbf{p}_{T_q(\Gamma_i^j)}^\perp(q) \cdot \vec{\eta}_{i,j}| d\mathcal{H}^{m-1} \lesssim \rho^{m-1} \sup_{q \in \Gamma \cap B_\rho} |\mathbf{p}_{T_q(\Gamma_i^j)}^\perp(q)| \lesssim \rho^{m+1} \mathbf{A}_\Gamma.$$

In order to bound the first term we can proceed again as in [24], but when we use the first variation formula a boundary term gets introduced.

The vector field that we have to test with is the vector field

$$X(q) = \chi(|q|) p_L^\perp(q).$$

The first variation formula gives us

$$\int \operatorname{div}_{\vec{T}}(X) = \mathbf{O}(\mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma).$$

The remaining estimates follow as in [24], using quantities under control relating the Lipschitz approximation. $\square$

Lemma 5.8.2. *Let T, Γ_i^j, Σ, M be as Assumption 5.6.1 for ε small enough. For every $\kappa > 0$*

$$\int_{\mathbf{B}_1} \frac{\operatorname{dist}(q, C)^2}{|q|^{m+2-\kappa}} d\|T\| \leq C_\kappa \int_{\mathbf{B}_1} \frac{|q^\perp|^2}{|q|^{m+2}} d\|T\| + C_\kappa (E(T, C, \mathbf{B}_4) + \mathbf{A}_\Sigma^2 + \mathbf{A}_\Gamma). \quad (5.8.6)$$

Proof. This follows again exactly as in Lemma 3.7.3. For completeness, we again include the boundary term estimates, reducing the rest of the proof to an adaptation of the proof Lemma 11.6 in [24].

This lemma follows from a first variation argument on the vector field

$$X(q) := \operatorname{dist}^2(q, C) \left(\max(r, |q|)^{-m-2-\kappa} - 1 \right)_+ q$$

where $g_+(q) := \max(g(q), 0)$.

The only required modification is the control of the boundary term.

$$\int \operatorname{div}_{\vec{T}}(X) d\|T\| = \sum_{i,j} Q_i^j \int_{\Gamma_i^j} (X \cdot \vec{\eta}_{i,j}) d\mathcal{H}^{m-1} - \int (X \cdot H_T) d\|T\|.$$

We estimate as in the boundary error term in the proof of Lemma 5.8.1

$$\int_{\Gamma_i^j} (X \cdot \eta) d\mathcal{H}^{m-1} \leq \int_{\Gamma_i^j \cap \mathbf{B}_1} |q|^{-m-\kappa} |\mathbf{p}_{T_q(\Gamma_i^j)^\perp}(q) \cdot \eta_{i,j}| d\mathcal{H}^{m-1} \leq C_\kappa \mathbf{A}_\Gamma.$$

$\square$

Theorem 5.8.3. *Let T, Γ, Σ, C be as in Assumption 5.6.1. For every $\bar{\alpha} > 0$*

1. For every cube L in the cubical decomposition from the previous chapter

$$\mathbf{E}(P) \lesssim_{\bar{\alpha}} \ell(P)^{-\bar{\alpha}} \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_{\mathbf{r}} + \mathbf{A}_{\Sigma}^2 \right)$$

2. For every σ and $r \leq \frac{1}{6\sqrt{m-2}}$:

$$\frac{1}{r^{m+2}} \int_{\mathbf{B}_r \cap B_{\sigma r}(L)} \text{dist}(q, C)^2 d\|T\| \leq C_{\delta} \sigma^{4-\bar{\alpha}} \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_{\mathbf{r}} + \mathbf{A}_{\Sigma}^2 \right).$$

3. Moreover, if $\mathbf{B}_r \setminus B_{\sigma r}(L) \subset R^{out}$ then

$$\begin{aligned} \frac{1}{r^{m+2}} \int_{\mathbf{B}_r} \text{dist}(q, C)^2 d\|T\| &\leq C(\delta, Q, m, n, \bar{n}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + \mathbf{A}_{\mathbf{r}} + \mathbf{A}_{\Sigma}^2 \right) \sigma^{4-\bar{\alpha}} \\ &\quad + \frac{1}{r^{m+2}} \sum_i \int_{\mathbf{B}_r \cap H_i \setminus B_{\sigma r}(L)} |u_i|^2 \\ &\quad + C(Q, m, n, \bar{n}) \left(\mathbf{E}(T, C, \mathbf{B}_4) + r \mathbf{A}_{\mathbf{r}} + r^2 \mathbf{A}_{\Sigma}^2 \right) \left(\mathbf{E}(T, C, \mathbf{B}_4) \sigma^{-m-2} + r \mathbf{A}_{\mathbf{r}} + r^2 \mathbf{A}_{\Sigma}^2 \right)^{\gamma} \end{aligned}$$

Proof. The proof is the same as Theorem 3.7.4. □

5.9 Proof of excess decay

In this section we prove the excess decay lemma, introduced in Section 7, from which we deduce the Hölder continuity of the multivalued normal and the regularity at multiplicity 1 corner points.

Assumption 5.9.1 (Blowup Sequence). A sequence of currents T_k is said to be a blowup sequence if they satisfy Assumption 5.1.3, $T_0(L) = \mathbb{R}^{m-2} \times \{0\}$, $C_k = \sum_{i=1}^N Q_i \llbracket H_{i,k} \rrbracket$ cornered open books (with the same number of sheets and labeling) and:

$$\mathbb{E}(T_k, C_k, \mathbf{B}_8) \leq \frac{1}{k} \text{ and } \mathbf{A}_{\mathbf{r}}^2 + \mathbf{A}_{\Sigma}^2 \leq \frac{1}{k} \mathbb{E}(T_k, C_k, \mathbf{B}_8).$$

We wish to take a Dir-minimizing blowup out of a blowup sequence of currents. Up to a subsequence, $H_{i,k} \rightarrow H_i$. If $C := \sum Q_i \llbracket H_i \rrbracket$ with $\partial \llbracket H_i \rrbracket = \llbracket V_{l(i)}^0 \rrbracket - \llbracket V_{l(i)}^1 \rrbracket$ then $C_k \rightarrow C$. We also have $T_k \rightarrow C$, $\Sigma_k \rightarrow \mathbb{R}^{m+n}$, $\Gamma_{i,k}^j \rightarrow V_i^j$. The cornered boundaries satisfy $M_k \rightarrow L$.

If k is large enough, T_k satisfy the Assumption 5.6.1. The excess bound is obviously satisfied so it only remains to check the mass bound. This is a consequence of the monotonicity formula and the fact that $T_k \rightarrow C$ with C an open book of multiplicity Q .

We take the Lipschitz approximations u_i^k whose domains of definition are the outer region which are subdomains of $(\mathbf{B}_1 \setminus B_{1/k}(L)) \cap D_{i,k}$. We need them to be defined on $\mathbf{B}_1 \setminus B_{1/k}(L) \cap D_{i,k}$. This is easy to achieve by applying a rotation we define for i, k a rotation $\sigma_{i,k}$ on $\mathbb{R}^{m+n}$ that takes $H_{i,k}$ to H_i , while fixing $\mathbb{R}^{m-1} \times \{0\}$. Clearly, $\sigma_{i,k} \rightarrow_k \text{Id}$. We define

$$\bar{u}_i^k := \frac{\mathbf{p}_{H_i^\perp} \circ \sigma_{i,k} \circ u_i^k \circ \sigma_{i,k}^{-1}}{\sqrt{\mathbb{E}(T_k, C_k, \mathbf{B}_8)}}.$$

Up to a subsequence $\bar{u}_i^k$ converges weakly to a function $\bar{u}_i$. We will be able to estimate the functions $\bar{u}_i$ at the scale $r = \frac{1}{12\sqrt{m-1}}$ since we have the non-concentration estimate at scale $\frac{1}{6\sqrt{m-1}}$. The functions $\bar{u}_i$ will be the Dir-minimizing blowup of the blowup sequence.

Lemma 5.9.2 (Dir-minimizing blowup). *Let $r = \frac{1}{12\sqrt{m-1}}$. We list properties of the sequence $\bar{u}_i^k$ and their weak limit $\bar{u}_i$:*

- *The functions $\bar{u}_i^k$ have equibounded Dirichlet energy.*
- *$\bar{u}_i^k$ converge strongly in $W^{1,2}(\mathbf{B}_r^+, A_{Q_i}(\mathbb{R}^n))$ (up to a subsequence) to a function $\bar{u}_i$ in $W^{1,2}(\mathbf{B}_r^+, A_{Q_i}(\mathbb{R}^n))$.*
- *The functions $\bar{u}_i$ are Dir-minimizing on $\mathbf{B}_r^+$ with zero boundary value along $V_{l(i)}^0 \cup V_{l(i)}^1$.*

Proof. The proof is the same as Lemma 3.9.3. The only new aspect is that the zero boundary data holds through the classical boundary parts, which follows from the fact that

$$\frac{\mathbf{A}_r}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} \rightarrow 0. \quad \square$$

Proof of Lemma 5.7.3. We rescale the ball $\mathbf{B}_1$ to $\mathbf{B}_8$ and get a sequence of currents as in Assumption 5.9.1.

We prove the lemma by contradiction. We claim it suffices to show that

$$\limsup_{r \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{\mathbb{E}(T_k, C_k, \mathbf{B}_r)}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} = 0. \quad (5.9.1)$$

If the lemma was false there exists θ and a sequence of radii $r_{k_l} \rightarrow 0$ such that

$$\frac{\mathbb{E}(T_{k_l}, C_{k_l}, \mathbf{B}_{r_{k_l}})}{\mathbb{E}(T_{k_l}, C_{k_l}, \mathbf{B}_8)} > \theta.$$

This would contradict equation (5.9.1) and conclude the proof of the lemma.

It remains to prove (5.9.1).

We have that

$$\mathbf{E}(T_k, C_k, \mathbf{B}_4) \leq C \mathbb{E}(T_k, C_k, \mathbf{B}_8).$$

By the non-concentration estimate for every σ if $r \leq \frac{1}{6\sqrt{m-2}}$ then

$$\begin{aligned} \frac{1}{r^{m+2}} \int_{\mathbf{B}_r} \text{dist}(q, C_k)^2 d|T_k| &\leq C(\delta, Q, m, n, \bar{n}) \left(\mathbf{E}(T_k, C_k, \mathbf{B}_4) + \mathbf{A}_\Gamma + \mathbf{A}_\Sigma^2 \right) \sigma^{4-\delta} \\ &\quad + \frac{1}{r^{m+2}} \sum_i \int_{\mathbf{B}_r \cap H_i \setminus B_{\sigma r}(L)} |u_{i,k}|^2 \\ &+ C(Q, m, n, \bar{n}) \left(\mathbf{E}(T_k, C_k, \mathbf{B}_4) + r \mathbf{A}_\Gamma + r^2 \mathbf{A}_\Sigma^2 \right) \left(\mathbf{E}(T_k, C_k, \mathbf{B}_4) \sigma^{-m-2} + r \mathbf{A}_\Gamma + r^2 \mathbf{A}_\Sigma^2 \right)^\gamma \end{aligned}$$

We divide by $\mathbb{E}(T_k, C_k, \mathbf{B}_8)$. We take $\limsup_{k \rightarrow \infty}$. Then, as $\sigma \rightarrow 0$, the last term vanishes since it is superlinear with respect to the excess and the first term will disappear because we took σ to zero. Therefore, for $r \leq \frac{1}{12\sqrt{m-2}}$

$$\limsup_{k \rightarrow \infty} \frac{\mathbf{E}(T_k, C_k, \mathbf{B}_r)}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} \leq \frac{1}{r^{m+2}} \sum_i \int_{\mathbf{B}_r} |\bar{u}_i|^2.$$

We conclude that

$$\lim_{r \rightarrow 0} \limsup_{k \rightarrow 0} \frac{\mathbf{E}(T_k, C_k, \mathbf{B}_r)}{\mathbf{E}(T_k, C_k, \mathbf{B}_8)} = 0.$$

Thus, by the reverse excess Lemma 5.6.10, we conclude

$$\lim_{r \rightarrow 0} \limsup_{k \rightarrow 0} \frac{\mathbb{E}(T_k, C_k, \mathbf{B}_r)}{\mathbb{E}(T_k, C_k, \mathbf{B}_8)} = 0$$

which gives a contradiction and proves the excess decay lemma. $\square$

Part 2: Construction of the example

5.10 Classification of certain cornered tangent cones under a convexity assumption

In this section, we will classify some cornered tangent cones which are necessary to provide our example. The tangent cones we classify are those we will obtain in corners under a convex barrier hypothesis similar to Assumption 5.1.4.

Theorem 5.10.1. *Let C satisfy the following hypothesis in $\mathbb{R}^{3+n}$:*

1. *We consider the half-planes $V^0 = \mathbb{R}e_3 \oplus \mathbb{R}^+e_2$, $V_i^1 = \mathbb{R}e_3 \oplus \mathbb{R}^+(-1)^i e_1$ which we orient so that $\partial \llbracket V^0 \rrbracket = -\partial \llbracket V_i^1 \rrbracket$.*
2. *Assume that C is an area minimizing cone such that $\partial C = 2 \llbracket V^0 \rrbracket + \sum_{i=1}^2 \llbracket V_i^1 \rrbracket$.*
3. *Assume that the support of the cone satisfies*

$$\text{spt}(C) \subset W := \left\{ p = (x_1, x_2, x_3, x_4, y) \in (\mathbb{R})^4 \times \mathbb{R}^{n-1} : |x_2^-| + |y| \leq C_0 x_4 \right\} \quad (5.10.1)$$

where $x_2^- = -\max\{0, x_2\}$ and C_0 is a positive constant.

Then the cone C is a cornered open book of the form shown Figure 5.2:

$$C = [\mathbb{R}^+ e_1 \oplus \mathbb{R}^+ e_2 \oplus \mathbb{R} e_3] + [\mathbb{R}^+ (-e_1) \oplus \mathbb{R}^+ e_2 \oplus \mathbb{R} e_3]$$

when the orientations of the quarter spaces match appropriately the orientation of the boundary data.

We depict the boundary of the cone we consider in the following picture:

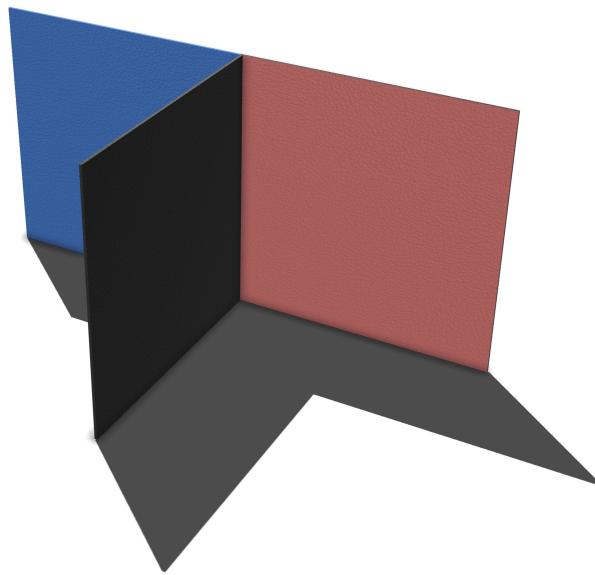


Figure 5.3: Boundary of the Cone

The half-plane V^0 is depicted in black, the half-planes V_i^1 are depicted in red and blue respectively. The theorem says that a cone C with the boundary as in Figure 5.3 is a cornered open book of the type depicted in Figure 5.2.

We remind of the following theorem originally due to Allard in [3] which we expanded on in Subsection 3.4.3 and applied it to a higher multiplicity setting.

Theorem 5.10.2. *Fix a 2-dimensional plane $\pi_0 \subset \mathbb{R}^{m+n}$ and assume that:*

1. $\mathcal{V}$ is an m -dimensional varifold which is stationary on $\mathbb{R}^{m+n} \setminus \pi_0^\perp$.
2. $\Theta(\mathcal{V}, x) \geq c_0 > 0$ for some positive numbers c_0 and for $\|\mathcal{V}\|$ -almost every $x \in \text{spt}(\mathcal{V}) \setminus \pi_0^\perp$.

3. $\mathcal{V}$ is a cone (i.e. $(\lambda_{0,r})_{\#} \mathcal{V} = \mathcal{V}$).

4. The orthogonal projection of $\text{spt}(\mathcal{V})$ onto π_0 is contained on a half-space). This means there is a vector $e \in \pi_0 \cap \mathbb{S}^{m+n-1}$ such that

$$\mathbf{p}_{\pi_0}(\text{spt}(\mathcal{V})) \subset \{x : 0 \leq \langle x, e \rangle\}.$$

• Then $\exists$ finitely many $v_1, \dots, v_k \in \pi_0 \cap \mathbb{S}^{m+n-1}$ such that

$$(\mathbf{B}_1 \cap \text{spt}(\mathcal{V})) \setminus B_\rho(\pi_0^\perp) \subseteq \bigcup_{i=1}^{k(\rho)} \pi_0^\perp + \mathbb{R}^+ v_i$$

for every $\rho > 0$.

• We denote $(x, y) = z$, $x^\perp = (-x_2, x_1)$ if $x = \mathbf{p}_{\pi_0}(z)$. For $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$ we have $\mathbf{p}_\pi(x^\perp) = 0$.

Assume $\pi_0 = \{(x_1, x_2, 0, \dots, 0) \in \mathbb{R}^{m+n}\}$ and we denote $z = (x, y) \in \pi_0 \times \pi_0^\perp$. The following lemma implies Theorem 5.10.2:

Lemma 5.10.3. Assume $\mathcal{V}$ is a varifold which is stationary on $\mathbb{R}^{m+n} \setminus \pi_0^\perp$ and is a cone (conditions (1) and (3) of Theorem 5.10.2). Consider the following Radon measure μ on $\mathbb{S}^1 := \{x \in \pi_0 : |x| = 1\}$

$$\int \varphi d\mu = \int_{\mathbf{B}_1 \setminus \pi_0^\perp} \varphi \left(\frac{x}{|x|} \right) |x|^{-2} |\mathbf{p}_\pi(x^\perp)|^2 d\mathcal{V}(z, \pi) \quad (5.10.2)$$

where $(x, y) = z$, $x^\perp = (-x_2, x_1)$. Then $\mu = c\mathcal{H}^1$ for some $c \in \mathbb{R}$.

In addition, if $\mu = 0$ and we have conditions (2) and (4) in Theorem 5.10.2 then:

1. $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$.

2. The set $\mathbf{p}_{\pi_0}(\text{spt}(\mathcal{V}) \setminus B_\rho(\pi_0^\perp)) \cap \mathbb{S}^1$ is finite and it has at most $\frac{2^m \Theta(\mathcal{V}, 0)}{\rho^m c_0}$ points.

The theorem above is a direct consequence of this lemma and we reproved it in Chapter 3.

Proof of Theorem 5.10.1. We denote $V^1 := V_0^1 \cup V_1^1 = \mathbb{R}e_1 \oplus \mathbb{R}e_3$. We can apply $n - 1$ times the above Lemma 5.10.3, each time taking π_0 as $\mathbb{R}e_4 \oplus \mathbb{R}e_{4+k}$, for every $1 \leq k \leq n - 1$.

We define the vector fields $v_k := -x_4 e_{4+k} + x_{4+k} e_4$. Thus we get that for $\|C\|$ - almost every $p \in \text{spt}(C) \cap \{x_4 > 0\}$ we have

$$\mathbf{p}_{\vec{C}}(v_k) = 0 \quad \forall 1 \leq k \leq n - 1.$$

We will show in Lemma 5.10.4 that a variant of Lemma 5.10.3 holds for $\pi_0 = \mathbb{R}e_2 \oplus \mathbb{R}e_4$. This implies $v_0 := x_2 e_4 - x_4 e_2$ then

$$\mathbf{p}_{\vec{C}}(v_0) = 0 \text{ for } \|C\| - a.e. \quad (5.10.3)$$

This allows us to conclude that if we define $\sigma : \mathbb{R}^{3+n} \setminus V^1 \rightarrow V^{1\perp}$ as

$$\sigma(x) := \frac{\mathbf{p}_{V^{1\perp}}(x)}{|\mathbf{p}_{V^{1\perp}}(x)|} \quad (5.10.4)$$

then $\|C\|$ almost everywhere $D_C(\sigma) \equiv 0$.

We conclude assuming Lemma 5.10.4 and thus its consequence, that $\|C\|$ almost everywhere $D_C(\sigma) \equiv 0$. By interior regularity theory due to Almgren [4] and De Lellis - Spadaro [31, 33, 34], C is regular in the interior up to a set of codimension 2. If q is an interior regular point for C and $P_q := \sigma^{-1}(\sigma(q))$ is independent of the spine L , and moreover by constancy

$$|C|(P_q \cap \mathbf{B}_1) \geq \frac{\omega_m}{4}.$$

Given such a plane P_q , it must part of it must lie in $\{x_4 \geq 0\}$ since C assigns mass to it. There are two options: either $P_q \subset \{x_4 = 0\}$ holds or it doesn't.

We define $H_0 = \mathbb{R}e_1 \oplus \mathbb{R}^+e_2 \oplus \mathbb{R}e_3$. We split H_0 into $H_0^+ = H_0 \cap \{x_1 \geq 0\}$ and $H_0^- = H_0 \cap \{x_1 \leq 0\}$.

- If $|P_q|(\{x_4 = 0\}) = 0$ then we consider $H_p := P_q \cap \{x_4 \geq 0\}$ must be a half-plane with boundary $[\mathbb{R}e_1 \oplus \mathbb{R}e_3]$. We have that $|C|(P_q \setminus H_p) = 0$ and $|C|(H_p \cap \mathbf{B}_1) \geq \frac{\omega_m}{2}$.
- If $P_q \subset \{x_4 = 0\}$ (which is equivalent to $|P_q|(\{x_4 > 0\}) = 0$ since $V^1 \subset P_q$), we must have that $q \in \mathbb{R}e_1 \oplus \mathbb{R}^+e_2 \oplus \mathbb{R}e_3$. In this case we must have by constancy that the multiplicity is constantly nonzero either in H_0^+ or in H_0^- and thus $|C|(P_q \cap \mathbf{B}_1) \geq \frac{\omega_m}{4}$. We orient $[[H_0^\pm]]$ such that $\partial [[H_0^+]] = [[V^0]] + [[V_2^1]]$ and $\partial [[H_0^-]] = [[V^-]] + [[V_1^1]]$.

Assume that there are at least N distinct planes P_i containing a regular point. Then

$$|C|(\mathbf{B}_1) \geq \sum_{i=1}^N |C|(P_i \cap \mathbf{B}_1) \geq \frac{N\omega_m}{4}$$

this implies there are only finitely many such planes.

We must have that for Q_i integers

$$C \llcorner \{x_4 > 0\} = \sum_{i=1}^N Q_i [[H_k]].$$

Since $\text{spt}(C) \cap \{x_4 = 0\} \subset H_0$, by constancy we obtain for $Q^\pm$ integers

$$C = \sum_{\pm} Q^\pm [[H_0^\pm]] + \sum_{i=1}^N Q_i [[H_k]].$$

Up to reversing the orientations of $[[H_i]]$ we can assume that $Q_i > 0$ for every $1 \leq i \leq N$. Given that $\text{spt}(\partial [[H_i]]) \subset V^1$, we will show that $\partial [[H_i]] = \partial [[H_j]]$ for all $1 \leq i < j \leq N$. Assume that instead for some $1 \leq i < j \leq N$, we have $\partial [[H_i]] + \partial [[H_j]] = 0$.

By taking $S = [[H_i]] + [[H_j]]$, we can write C as $C = S + T$ where $\|C\| = \|S\| + \|T\|$. This implies that S is area minimizing without boundary. The only option for this to hold in this setting given that the planes have a codimension 1 spine is if $H_i \cup H_j$ is a full plane. This is impossible since both of them are contained in $\{x_4 \geq 0\}$.

In order to take the boundary data the following three equations must be satisfied

1. $Q^+ + Q^- = 2$
2. $Q^+ + \sum_{i=1}^N Q_i = 1$
3. $Q^- + \sum_{i=1}^N Q_i = 1$.

It follows that $Q^+ = Q^- = 1$ and $\sum Q_i = 0$. Since all Q_i have the same sign, they must be zero.

Allard's lemma for the plane $\mathbb{R}e_2 \oplus \mathbb{R}e_4$

In order to conclude we must show that we can apply a variation of Lemma 5.10.3 to the present case with the plane $\pi_0 = \langle e_2, e_4 \rangle$. We will show that an appropriate version of the lemma holds in the hypothesis that we have. Inside the proofs of these lemmas we will adopt the notation $z = (x, y) \in \pi_0$ and $x^\perp = (-x_4, x_2)$.

Lemma 5.10.4. *We consider $\pi_0 = \langle e_2, e_4 \rangle$ and $\mathcal{V}$ the varifold induced by C . Then the measure μ defined by (5.10.2) satisfies $\mu = c\mathcal{H}^1$, which implies in the present context $\mu \equiv 0$. Thus, $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. (z, π) where $z \in \mathbb{R}^{m+n} \setminus (\pi_0^\perp \cup \text{spt}(\partial C))$.*

Proof. To prove that $\mu = c\mathcal{H}^1$ we must show that if $\varphi \in C(S^1)$ and has zero average, i.e. $\int_{S^1} \varphi d\mathcal{H}^1 = 0$, then

$$\int \varphi d\mu = 0.$$

We consider φ as a 2π periodic function on $\mathbb{R}$ and we take $\psi(t) := \int_0^t \varphi(\tau) d\tau$ a primitive of φ on $\mathbb{R}$.

We consider a vector field of the form

$$X(z) = \alpha(|z|)\beta(|x|)\psi(x)x^\perp.$$

Where $\alpha, \beta \in C^\infty(\mathbb{R})$ are defined with the following properties: $\alpha \equiv 1$ for $t \leq 1 - \delta$, $\alpha \equiv 0$ for $t \geq 1 - \delta/2$, and it is decreasing. $\beta \equiv 1$ for $t \geq \delta$, $\beta \equiv 0$ for $t < \delta/2$, and it is increasing with $\|\beta'\|_{C^0} \lesssim \delta^{-1}$.

We will have that

$$\int \operatorname{div}_\pi X(z) d\mathcal{V}(z, \pi) = 2 \int_{V^0} \int X \cdot \eta^0 d\mathcal{H}^2.$$

We take $\delta \rightarrow 0$ which implies by the arguments in Lemma 3.4.12 that

$$\int \varphi d\mu = \lim_{\delta \rightarrow 0} \int \operatorname{div}_\pi X d\mathcal{V}(z, \pi) = \psi(0, 1) 2 \int_{V^0 \cap B_1(0)} (x^\perp \cdot \eta^0) d\mathcal{H}^2.$$

Since the left hand side of the equality is independent of the choice of ψ , the primitive of φ , then the right hand side must be independent of the choice ψ . This implies that the right hand side equals zero.

Alternatively, the vanishing of the right-hand side can be seen directly by testing with the vector field

$$Y(z) = \alpha(|z|)\beta(|x|)x^\perp$$

with α and β as above and taking the limit $\delta \rightarrow 0$.

Since

$$\operatorname{spt}(\mu) \subset \{|x_2^-| \leq Cx_4\}$$

then $\mu \equiv 0$ and thus $\mathbf{p}_\pi(x^\perp) = 0$ for $\mathcal{V}$ a.e. $(z, \pi) \in \mathbb{R}^{m+n} \setminus \pi_0^\perp$. □

Lemma 5.10.4 implies 5.10.3, which in turn implies $D_C(\sigma) \equiv 0$ $\|C\|$ -almost everywhere, allowing us to conclude. □

5.11 Construction

In this section, we construct a general family of examples of area minimizing currents with essential one-sided boundary singularities. We start the section by providing an example of a specific boundary which is real analytic to which the theorems will apply, which allows us to conclude Theorem O from Theorem 5.11.5.

5.11.1 Concrete example in $\mathbb{R}^5$

The lower part of the boundary Γ^0 is defined to be the 2-dimensional lower hemisphere

$$\Gamma^0 := \{(x_1, x_2, x_3, 0, 0) \in \mathbb{S}^4 : x_3 < 0\}.$$

The upper part of the boundary Γ^1 , is described by a two-valued map $\Phi^1 : \mathbb{S}^1 \times [0, 1] \rightarrow \mathcal{A}_2(\mathbb{S}^4) \subset \mathcal{A}_2(\mathbb{R}^5)$.

We consider $\mathbb{S}^1 \subset \mathbb{C}$ so that the complex multiplication and thus the 2-valued square root makes sense.

Let

$$\Phi^1(z, t) := \sum_{w^2=z} \left[\left(\sqrt{1-t^2-t^4}z, t^2, tw \right) \right].$$

We define

$$\Gamma^1 := (\Phi^1)_{\#} (\mathbb{S}^1 \times [0, 1]).$$

Let $M := \{(z, 0, 0) \in \mathbb{C} \times \mathbb{R} \times \mathbb{C} : |z| = 1\}$. We orient Γ^0 , Γ^1 and M so that $2\partial \llbracket \Gamma^0 \rrbracket = 2 \llbracket M \rrbracket = -\partial \llbracket \Gamma^1 \rrbracket$.

Theorem 5.11.1. *Let T be a 3-dimensional area minimizing current such that $\partial T = 2 \llbracket \Gamma^0 \rrbracket + \llbracket \Gamma^1 \rrbracket$ with Γ^0 and Γ^1 , being the 2d integral currents defined as above. Then T has an essential boundary one-sided singularity at some interior point of the boundary surface Γ^0 .*

By localizing to a small ball centered at an essential singular point, we obtain the following Theorem:

Theorem 5.11.2. *There exist a 3-dimensional area minimizing current T in $\mathbf{B}_1 \subset \mathbb{R}^5$ and a real analytic surface Γ such that $\partial T = 2 \llbracket \Gamma \rrbracket$, $\Theta(T, p) = 1$ for every $p \in \Gamma \cap \mathbf{B}_1$, and 0 is an essential one-sided boundary singularity.*

5.11.2 A class of general examples

Assumption 5.11.3 (Density hypothesis). We say that two $m \geq 3$ dimensional integral currents satisfy Γ^0 and Γ^1 with $\partial \llbracket \Gamma^0 \rrbracket = \llbracket M \rrbracket$, $\partial \llbracket \Gamma^1 \rrbracket = -2 \llbracket M \rrbracket$ and $\text{spt}(\Gamma^0) \cap \text{spt}(\Gamma^1) = M$ satisfy the density hypothesis if:

- There is a C^2 open set Ω which has uniformly convex boundary and $M \subset \partial\Omega$ is a smooth closed $m - 2$ dimensional submanifold.
- $\text{spt}(\Gamma^0) \subset \partial\Omega$ is a C^2 surface with boundary M .
- $\text{spt}(\Gamma^1)$ is a smooth surface in a tubular neighborhood U of M .
- $\text{spt}(\Gamma^0)$ and $\text{spt}(\Gamma^1)$ only intersect at M and they have orthogonal normals along M .

An area minimizing current T with $\partial T = 2 \llbracket \Gamma^0 \rrbracket + \llbracket \Gamma^1 \rrbracket$ is said to satisfy the density hypothesis if Γ^0, Γ^1 satisfy the density hypothesis.

The convexity implies that $\text{spt}(T) \subset \overline{\Omega}$. The convexity allows us to say that if $p \in \Gamma^0 \setminus M$ then $\Theta(T, p) = 1$, if $p \in U \cap \text{spt}(\Gamma^1) \setminus M$ then $\Theta(T, p) = 1/2$ and if $p \in M$ then $\Theta(T, p) = 1/2$. This allows us to use the developed regularity theory.

Assumption 5.11.4 (Additional Topological hypothesis). We say that Γ^0 and Γ^1 , two 2 dimensional integral currents, satisfy the additional topological hypothesis if it satisfies the density hypothesis 5.11.3 for $m = 3$ and

- Γ^0 is diffeomorphic to a disk with boundary M (which is a closed simple curve).
- In a tubular neighborhood U of M , the surface $\text{spt}(\Gamma^1)$ is non-orientable: $U \cap \text{spt}(\Gamma^1)$ is a C^2 surface diffeomorphic to a Möbius strip twisting once around M , and its orientation reverses across M .

A 3 dimensional area minimizing current T with $\partial T = 2 \llbracket \Gamma^0 \rrbracket + \llbracket \Gamma^1 \rrbracket$ is said to satisfy the additional topological hypothesis if Γ^0, Γ^1 satisfy the additional topological hypothesis.

We show the following theorem, which provides a large family of examples of essential one-sided boundary singularities, by splitting it into two parts.

Theorem 5.11.5. *Let Ω be a C^2 uniformly convex open set.*

- *Let $M \subset \partial\Omega$ be a C^2 simple closed curve.*
- *Let Γ^0 be a C^2 surface with $\partial \llbracket \Gamma^0 \rrbracket = M$ diffeomorphic to a 2 dimensional disk.*
- *Let $\Gamma^1 \subset \partial\Omega$ be an integral 2 dimensional current. We assume that $M \subset \text{spt}(\Gamma^1)$ and in a neighborhood U of M , $U \cap \text{spt}(\Gamma^1)$ is a C^2 surface diffeomorphic to a Möbius strip which twists once around M . Indeed, $U \cap \text{spt}(\Gamma^1) \setminus M$ will be connected. We assign multiplicity 1 to $\text{spt}(\Gamma^1)$ in U . We suppose that Γ^1 is oriented such that $\partial \llbracket \Gamma^1 \rrbracket = -2 \llbracket M \rrbracket$. The surface $U \cap \Gamma^1$ is a smooth surface whose orientation switches across M .*
- *Further assume Γ^0 and Γ^1 meet perpendicularly at M and $\Gamma^0 \cap \Gamma^1 = M$.*

Let T be an area minimizing current with $\partial T = 2 \llbracket \Gamma^0 \rrbracket + \llbracket \Gamma^1 \rrbracket$. Then T is regular in a neighborhood of M and has an essential boundary singularity at some point of $\Gamma^0 \setminus M$.

The first part is the control on the density and the regularity theory:

Theorem 5.11.6. *Let $T, \Omega, \Gamma^0, \Gamma^1$ be as in the density hypothesis Assumption 5.11.3. Then*

1. $\forall p \in \Gamma^0 \setminus M, \Theta(T, p) = 1$.
2. $\forall p \in \Gamma^1 \cap U \setminus M, \Theta(T, p) = 1/2$.
3. $\forall p \in M, \Theta(T, p) = 1/2$ and thus T is regular at M .

The tangent cone is unique at all the above points, and the multivalued normal map at Γ^0 is Hölder continuous up to the boundary M (where it agrees with the cornered open book). Moreover, T is regular in a neighborhood of M .

Remark 5.11.7. The density $\Theta(T, p)$ is defined differently at interior points p , at classical boundary points $p \in \Gamma^0 \setminus M$ or $p \in \Gamma^1 \cap U \setminus M$ and at the corner points M . It is upper semi-continuous in each of these pieces.

We start with a quick discussion of why we consider the convex barrier property. The convex barrier property that we considered in Chapter 3 and was considered in [27] is the following:

Assumption 5.11.8 (Convex Barrier). Let $\Omega \subset \mathbb{R}^{m+n}$ be a domain such that $\partial\Omega$ is a C^2 uniformly convex submanifold of $\mathbb{R}^{m+n}$. We say that $\sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$ has a convex barrier if Q_i are positive integers and $\Gamma_i \subset \partial\Omega$ are disjoint C^2 closed oriented submanifolds of $\partial\Omega$. In this setting we consider T an area minimizing current with $\partial T = \sum_i Q_i \llbracket \Gamma_i \rrbracket$.

In order to get to the local information we define the wedge regions

Definition 5.11.9. Given an $(m-1)$ -dimensional plane $V \subset \mathbb{R}^{m+n}$ we denote by $\mathbf{p}_V$ the orthogonal projection onto V . Given additionally a unit vector ν normal to V and an angle $\vartheta \in (0, \frac{\pi}{2})$ we then define the wedge with spine V , axis ν and opening angle ϑ as the set

$$W(V, \nu, \vartheta) := \{y : |y - \mathbf{p}_V(y) - (y \cdot \nu)\nu| \leq (\tan \vartheta)y \cdot \nu\}.$$

Lemma 2.3 of [27], which we cite below, gives us the desired local geometrical information.

Lemma 5.11.10. *Let T , $\sum_{i=1}^N Q_i \llbracket \Gamma_i \rrbracket$, Ω be as in 5.1.4. Then there is a $0 < \vartheta < \frac{\pi}{2}$ (which is independent of the point in $\cup_{i=1}^N \Gamma_i$) such that the convex hull of $\cup_{i=1}^N \Gamma_i$ satisfies*

$$\text{ConvexHull}(\cup_{i=1}^N \Gamma_i) \subset \bigcap_{q \in \Gamma} (q + W(T_q \Gamma, \nu(q), \vartheta))$$

Proof of Theorem 5.11.6. At points of $\Gamma^0 \setminus M$ or $\Gamma^1 \cap U \setminus M$, we apply a version of Lemma 2.3 of [27], which we just cited above Lemma 5.11.10 obtaining: At a point p of $\Gamma^0 \setminus M$ the tangent cone C_p must satisfy

$$\partial C_p = 2 \llbracket T_p(\Gamma^0) \rrbracket$$

and

$$C_p \subset \left\{ W(T_p \Gamma^0, \nu(p), \vartheta) \right\}.$$

At a point p of $\Gamma^1 \cap U \setminus M$ the tangent cone must satisfy

$$\partial C_p = \llbracket T_p(\Gamma^1) \rrbracket$$

and

$$C_p \subset \left\{ W(T_p \Gamma^0, \nu(p), \vartheta) \right\}.$$

Thus for every point $p \in \Gamma^0 \setminus M$ or $p \in \Gamma^1 \cap U \setminus M$ we must have that the tangent cone must be of the convex barrier type. These are the convex barrier cones that we classified in Theorem 3.4.8 for arbitrary multiplicity, and Allard classified them for multiplicity 1. In the first cases it is an open book and thus $\Theta(T, p) = 1$, in the second case it is a single half-plane and thus $\Theta(T, p) = 1/2$.

Let $p \in M$, v_0, v_1 be the normal vectors to Γ^0 and Γ^1 respectively. At M the arguments from Lemma 2.3 [27], imply that

$$C_p \subset \left\{ y : |y - \mathbf{p}_{T_p(M)}(y) - (y \cdot v_0)^+ v_0 - (y \cdot \nu(p))| \leq \tan(\theta)(y \cdot \nu(p)) \right\}.$$

This implies that C_p is of the type of cones we classified in Theorem 5.10.1 and thus C_p is a cornered open book. Thus C_p is as in Figure 5.2 and $\Theta(T, p) = 1$. The regularity at M follows from 5.7.2 and the Hölder continuity of the normal follows from Corollary 5.7.9. $\square$

We now turn to the second part: the conclusion.

Theorem 5.11.11. *Let T, Γ^0, Γ^1 satisfy the topological Assumption 5.11.4. Then T has an essential boundary singularity at some point of Γ^0 .*

Proof of 5.11.1. Since Γ^0 is diffeomorphic to a disk with boundary M then there exist $\varphi : \{z \in \mathbb{R}^2 : |z| \leq 1\} \rightarrow \mathbb{R}^{m+n}$ which is a smooth parameterization of Γ^0 and $\varphi(\mathbb{S}^1) = M$.

Then we can define the normal as a function of the disk as

$$\eta \circ \varphi : \{z \in \mathbb{R}^2 : |z| \leq 1\} \rightarrow \mathcal{A}_2(N\Gamma^0).$$

We must have that at M , $\eta(p) = \llbracket v_1^p \rrbracket + \llbracket -v_1^p \rrbracket$ where v_1^p is one of the normals of Γ^1 at M . Since Γ^1 is a Möbius strip then the graph of $\eta \circ \varphi \llcorner \mathbb{S}^1$ induces a single continuous connected curve.

Since the normal bundle of Γ^0 is trivial, up to a diffeomorphism it agrees with the normal bundle of

$$\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^n : |x| \leq 1, y = 0\} \subset \mathbb{R}^{2+n}.$$

After changing the image space by a diffeomorphism we conclude the proof with a classical elementary observation, which we collect in the following proposition:

Proposition 5.11.12. *Let $f : \{z \in \mathbb{C} : |z| \leq 1\} \rightarrow A_2(\mathbb{R}^n)$ be a continuous multivalued function such that $f \llcorner \mathbb{S}^1$ is a continuous connected curve of multiplicity 1. Then f has an essential singularity.*

Proof. This is a fairly classical consequence of the theory of covers in algebraic topology and that the boundary loop is non trivial.

For the reader's convenience, we sketch the argument. Suppose, for contradiction, that f has no interior essential singularities. Then, at every point p , f admits a continuous local selection into two single-valued branches f_1 and f_2 . This remains true at boundary points by continuity, since there are no double points at the boundary. By compactness, we can cover the closed disk with finitely many open sets over which such selections exist. If a continuous selection exists for two overlapping open sets U and V then they must agree on $U \cap V$, and thus we can extend the selection to $U \cup V$. Iterating this, we obtain a global continuous selection on the whole disk. But this leads to a contradiction: the trace of f on the boundary

is a connected loop of multiplicity 1, which cannot be written as the disjoint union of two continuous branches. Therefore, f must have at least one interior essential singularity. $\square$

The above proposition implies that f must have an essential singularity, which in turn forces T to have an essential boundary singularity at some interior point of Γ^0 . Otherwise, f would have no interior essential singularities, which contradicts the proposition. $\square$

5.12 Essential singularities for the linear problem

In the introduction, we sketched the validity of a general result for Dir-minimizing functions. We point their validity in this section without giving a proof, since the main interest of this chapter is the area minimizing setting and the proof of the result for the linear problem is analogous to those for the nonlinear case.

Theorem 5.12.1. *Let Ω be a compact domain in $\mathbb{R}^3$ which is C^2 -diffeomorphic to a cylinder. Thus $\partial\Omega$ consists of three two dimensional surfaces, D_0, D_1, C , where D_0 and D_1 are disks and C is a cylinder which connects ∂D_0 with ∂D_1 . Let $u \in W^{1,2}(\overline{\Omega}, A_2(\mathbb{R}^n))$. Assume that:*

- *C meets D_0 orthogonally.*
- *The function $u|_C$ is C^2 in a neighborhood of ∂D_0 , and it is constantly equal to $2 \llbracket 0 \rrbracket$ on D_0 .*
- *Assume that the normal derivative of $u|_C$ to ∂D_0 is a continuous connected curve of multiplicity 1.*

Then u has an essential boundary singularity at an interior point of the 2-disk D_0 .

Corollary 5.12.2. *There exists $u : W^{1,2}(\mathbb{R}^2 \times \mathbb{R}^+) \rightarrow \mathcal{A}_2(\mathbb{R}^2)$ which is $I > 1$ homogeneous Dir-minimizer.*

Proof of Corollary 5.12.2. We consider u as in the introduction or u as above with Ω the unit cylinder. Then, there exists a point p with an essential boundary singularity of u . If

the frequency at p is 1, then, by Theorem 3.10.3 there is a unique linear blowup and thus a neighborhood of p , u can be separated as two classical harmonic functions. This is a contradiction since at p , u has an essential singularity.

Thus, the tangent map at p is an $I > 1$ homogeneous Dir-minimizer in the half-space which allows us to conclude. $\square$

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