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Long time behavior of some nonlinear dispersive equations

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LONG TIME BEHAVIOR OF SOME NONLINEAR DISPERSIVE
EQUATIONS

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Abstract

This thesis is divided into two parts. The first part consists of Chapters 2 and 3, in which we study the random data theory for the Benjamin-Ono equation on the periodic domain. In Chapter 2 we shall prove the invariance of the Gibbs measure associated to the Hamiltonian E_1 of the equation, which was constructed in [49]. Despite the fact that the support of the Gibbs measure contains very rough functions that are not even in L^2 , we have successfully established the global dynamics by combining probabilistic arguments, $X^{s,b}$ type estimates and the hidden structure of the equation. In Chapter 3, which is joint work with N. Tzvetkov and N. Visciglia, we extend this invariance result to the weighted Gaussian measures associated with the higher order conservation laws E^2 and E^3 , thus completing the collection of invariant measures (except for the white noise), given the result of [51].

The second part concerns the global behavior of solutions to quasilinear dispersive systems in $\mathbb{R}^d$ with suitably small data. In Chapter 4 we shall prove global existence and scattering for small data solutions to systems of quasilinear Klein-Gordon equations with arbitrary speed and mass in $3D$, which extends the results in [20] and [32]. Moreover, the methods introduced here are quite general, and can be applied in a number of different situations. In Chapter 5, we briefly discuss how these methods, together with other techniques, are used in recent joint work with A. Ionescu and B. Pausader to study the $2D$ Euler-Maxwell system.

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I am also grateful to my collaborators, Nikolay Tzvetkov, Nicola Visciglia, Benoit Pausader and Fabio Pusateri; they are kind people and have provided so much help at the beginning of my career. Collaboration with them have turned into Chapters 3 and 5 of this thesis, and I am sure that there will be more excellent works done in the future.

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To my parents.

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Chapter 1

Overview

One of the most challenging problems in the theory of partial differential equations is the study of long time behavior of solutions. Depending on the structures of equations, the properties of underlying manifolds and the sizes of solutions, one can have very different asymptotic behavior in different situations. In the current thesis we shall explore two typical cases in the context of nonlinear dispersive equations; the first one is the periodic Benjamin-Ono equation, which is a semilinear PDE posed on a compact domain, and we will show a long time recurrent behavior for generic, arbitrarily large data. The other is a quasilinear multi-speed Klein-Gordon system on the Euclidean domain, for which we have asymptotically linear scattering for small data. The ideas and techniques involved in the two problems are very different, but they both feature, to some extent, the notion of “resonance”.

1.1 The Benjamin-Ono equation

We first discuss the periodic Benjamin-Ono equation. It is a semilinear equation posed on the one-dimensional torus $\mathbb{T}$; this in particular means that the corresponding linear flow is periodic, and thus recurrent. One may ask whether the nonlinear flow behaves similarly; this will depend on the exact structure of the equation and is not true in general.

In the case of Benjamin-Ono, one has a favorable completely integrable Hamiltonian structure that leads to an infinite number of conservation laws, thus heuristically one may expect to have recurrent behavior by constructing formally invariant measures. This construction was done in [50] and [51], and the goal of Chapters 2 and 3 is to prove that they are rigorously invariant. This, via Poincaré’s theorem, implies long time recurrence for the equation.

The main idea of the proof is a combination of robust local analysis and probabilistic arguments. In Chapter 2 where the invariance of Gibbs measure is proved, the major part of the analysis concerns local theory, since the typical regularity is below L^2 . Here we shall use $X^{s,b}$ type estimates, thus the so-called “division problem” will play an important role. In Chapter 3 the local analysis is much easier, and the main difficulty comes from the absence of exact invariance for finite dimensional approximations. To overcome this we need to construct suitable approximate energy functionals, by using a variant of the famous I -method.

1.2 The Multi-speed Klein-Gordon system

The other topic discussed in this thesis is the nonlinear Klein-Gordon system. Here the goal is to show asymptotic linear behavior and decay in L^∞ , at least for small data, which is expected due to the well-known dispersion effect for free Klein-Gordon equations. In three dimensions, such results were proved in Klainerman [36], Shatah [46] for the case of a single Klein-Gordon equation, and Hayashi-Naumkin-Wibowo [29] for systems with the same speed. In the case of multiple speeds, Germain [20] and Ionescu-Pausader [32] solved the problem under additional restrictions on the parameters.

In Chapter 4 we shall prove global existence and scattering for small data in dimension three, without making any restriction on the parameters. Here the division problem will again come in, under the name of “time resonance”; we also need to study the related “spacetime resonance”, in the terminology of Germain-Masmoudi-Shatah [22], which accounts for the main contribution of the nonlinearity. Another important feature is the use of rotation vector fields, which improves both linear and bilinear estimates.

In Chapter 5 we will extend the analysis to dimension two, and discuss one particular instance of multi-speed Klein-Gordon systems, namely the irrotational Euler-Maxwell system. Here new ingredients have to be introduced, namely a quasilinear I -method and a crucial L^2 lemma that allow us to close the energy estimate even with inadequate decay. Unfortunately the proof in [18] is rather long and technical, so in this chapter we will only sketch the proof; for details see [18].

Chapter 2

Invariance of the Gibbs measure for the Benjamin-Ono equation

2.1 Introduction

In this and the next chapter we consider the periodic Benjamin-Ono equation

$$u_t + Hu_{xx} = uu_x, \quad (t, x) \in I \times \mathbb{T}, \quad (2.1)$$

which was introduced in [1] and [45] to model one dimensional internal waves. Here I is a time interval, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and the Hilbert transform H is defined by $\widehat{Hu}(n) = -i \cdot \text{sgn}(n)\widehat{u}(n)$, where we understand that $\text{sgn}(0) = 0$. We will assume throughout that u is real valued and has zero average, or $\int_{\mathbb{T}} u = 0$; this will be preserved by the flow.

In these two chapters, we will focus on the so-called “random-data theory” for (2.1). In the general case of semilinear dispersive equations, this line of study was initiated in Lebowitz-Rose-Speer [40] and Bourgain [3], and followed in many subsequent works, such as Bourgain [4], Burq-Tzvetkov [9], [10], [11], Tzvetkov [49], Oh [44], Colliander-Oh [13], Nahmod-Oh-Rey-Bellet-Staffilani [43], Bourgain-Bulut [6]. The main idea is to use probabilistic methods to analyze long time behavior of generic solutions, which may have regularity lower than what is needed for deterministic well-posedness. A central concept here is that of formally *invariant measures*; this is related to the Hamiltonian structure and conserved quantities of the equation, which we discuss below.

The equation (2.1) has the standard Hamiltonian structure with the symplectic pairing

$$\omega(u, v) := 2 \int_{\mathbb{T}} u \cdot \partial_x^{-1} v$$

and the Hamiltonian

$$E^1[u] = \|u\|_{\dot{H}^{1/2}}^2 - \frac{1}{3} \int_{\mathbb{T}} u^3. \quad (2.2)$$

It is also notable for being completely integrable, thus having an infinite collection of conserved quantities of form $E^\sigma[u] = \|u\|_{\dot{H}^{\sigma/2}}^2 + R^\sigma[u]$, for $0 \leq \sigma \in \mathbb{Z}$. In the current work we will only need the first four of them; except for the Hamiltonian E^1 which is written in (2.2) above, the other three are respectively $E^0[u] = \|u\|_{L^2}^2$ and

$$\begin{aligned} E^2[u] &= \|u\|_{\dot{H}^1}^2 - \frac{3}{4} \int_{\mathbb{T}} u^2 H u_x + \frac{1}{8} \int_{\mathbb{T}} u^4; \\ E^3[u] &= \|u\|_{\dot{H}^{3/2}}^2 + \int_{\mathbb{T}} \left[\left(\frac{3}{2} u u_x^2 + \frac{1}{2} u (H u_x)^2 \right) - \left(\frac{1}{3} u^3 H u_x + \frac{1}{4} u^2 H (u u_x) \right) + \frac{1}{20} u^5 \right]. \end{aligned} \quad (2.3)$$

For each σ , there exists a measure ν^σ associated with the quantity E^σ , formally written as

$$d\nu^\sigma(f) := e^{-E^\sigma[f]} \prod_{x \in \mathbb{T}} df(x). \quad (2.4)$$

To make this formal definition rigorous, one has to recast it as a weighted Gaussian measure and insert suitable cutoff functions, as was done in [50] for $\sigma = 1$ (ν^1 is also called the Gibbs measure, as it is connected with the Hamiltonian E^1), and [51] for $\sigma \geq 2$. These constructions are also reviewed in Sections 2.2.1 and 3.2 below. The measures ν^σ so defined are formally invariant under (2.1); they are also expected to be genuinely invariant when $\sigma \geq 1$, see for example [50], where the author made several observations supporting a possible probabilistic well-posedness result on the support of ν^1 .

In the current chapter we will present, as in [16], the proof of invariance of the Gibbs measure ν^1 , which we will henceforth denote by ν . The main task is to establish well-behaved global dynamics on the support of ν , since the measure ν^σ is known to be supported on $H^{(\sigma-1)/2-\epsilon}$ for any $\epsilon > 0$ but not $H^{(\sigma-1)/2}$, and thus the support of ν barely misses L^2 . This prevents us from using the L^2 well-posedness result of Molinet [41], and requires us to perform an extremely delicate analysis in some Besov-type space Z_1 below L^2 . Due to the length of the proof, here we will only present the most important parts; for full details see [16].

2.1.1 Preliminaries

Notations

The notations introduced here will be used throughout this and the next chapter.

For a function u on $\mathbb{R} \times \mathbb{T}$, we define its spacetime Fourier transform $\widehat{u}_{n,\widehat{\xi}}$ by

$$u(t, x) = \sum_n \int_{\mathbb{R}} \widehat{u}_{n,\widehat{\xi}} \times e^{i(nx + \widehat{\xi}t)} d\widehat{\xi},$$

and denote $\widetilde{u}_{n,\xi} = \widehat{u}_{n,\widehat{\xi}} := \widehat{u}_{n,\xi - |n|n}$. Thus we have three ways to represent u : $u(t, x)$ as a function of t and x , $\widehat{u}_{n,\widehat{\xi}}$ as a function of n and $\widehat{\xi}$, and $\widetilde{u}_{n,\xi}$ as a function of n and ξ , where the ξ and $\widehat{\xi}$ are always related by $\widehat{\xi} = \xi - |n|n$. Since we will be dealing with more than one function, n and ξ may be replaced with other letters possibly with subscripts, say m_1 or β_2 . To simplify the notation, when there is no confusion, we will omit the “hat” and “tilde” symbols; for example, if we talk about an expression involving $u_{m,\widehat{\alpha}}$, it will actually mean $\widehat{u}_{m,\widehat{\alpha}}$. The appearance of functions f defined on $\mathbb{T}$ will not be too frequent, but when they do appear, we will adopt the same convention and write for example f_n instead of $\widehat{f}(n)$. We also use the notations

$$u^+ = u, u^- = \overline{u}; \quad (\Re u)_{n,\xi} = |u_{n,\xi}|, \quad (u^{(\lambda,\mu)})_{n,\xi} = \langle n \rangle^\lambda \langle \xi \rangle^\mu u_{n,\xi}. \quad (2.5)$$

Let $\mathbb{P}$ denote (spatial) frequency projections; for example $\mathbb{P}_+$ (or $\mathbb{P}_{\leq 0}$) will be the projection onto strictly positive (or non-positive) frequencies and $\mathbb{P}_{\gtrsim \lambda}$ will be the projection onto frequencies with absolute value $\gtrsim \lambda$. We also fix a smooth, even cutoff function ψ on $\mathbb{R}$ which equals 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes outside $[-\frac{3}{4}, \frac{3}{4}]$; let $\psi_0 = 1 - \psi$.

Define $\mathcal{V}$ to be the space of distributions on $\mathbb{T}$ that are real-valued and have zero average; in other words $f \in \mathcal{V}$ if and only if $f_{-n} = \overline{f_n}$ and $f_0 = 0$. Let $\mathcal{V}_N$ be the subspace of $\mathcal{V}$ containing functions of frequency not exceeding N (so that $\mathcal{V}_N$ is identified with $\mathbb{R}^{2N}$), and $\mathcal{V}_N^\perp$ be its orthogonal complement. Let $\Pi_N = \mathbb{P}_{\leq N}$ and $\Pi_N^\perp = \mathbb{P}_{> N}$ be projections to the corresponding spaces.

Parameters and the space Z_1

We fix a number $s > 0$ that is sufficiently small. The large constants C and small constants c may depend on s ; any situation in which they are independent of s will be easily recognized. We choose

a few other parameters, namely $(p, r, b, \tau, q, \kappa, \gamma, \epsilon)$, as follows:

$$\begin{aligned} p &= \frac{2}{1-2s} + s^2, & r &= \frac{1}{2} - \frac{1}{p}, & b &= \frac{1}{2} - s^{15/8}, & \tau &= 8 - s^{13/8}, \\ q &= 1 + s^{3/2}, & \kappa &= 1 - s^{5/4}, & \gamma &= 2 - s^{5/2}, & \epsilon &= s^{7/4}. \end{aligned} \quad (2.6)$$

When s is small enough, we have the following hierarchy of smallness factors:

$$s^3 \ll 2 - \gamma \ll r - s = \frac{1}{2} - \frac{1}{p} - s \ll \frac{1}{2} - b \ll \epsilon \ll 8 - \tau \ll q - 1 \ll 1 - \kappa \ll s \ll s^{1/2}. \quad (2.7)$$

In (2.7) each $\ll$ symbol connects two numbers that actually differ in scale by a power of s . We will also use $0+$ to denote some small positive number (whether it depends on s will be clear from the context); the meanings of $0-$, and $a+$, $a-$ are then obvious. Finally, using these parameters, we can define the space Z_1 by

$$\|f\|_{Z_1} = \sup_{d \geq 0} \left(\sum_{n \sim 2^d} 2^{rpd} |f_n|^p \right)^{\frac{1}{p}}. \quad (2.8)$$

Note that we are including $n = 0$ when $d = 0$.

Next we will introduce finite dimensional truncations of (2.1). For a positive integer N , we define the multiplier S_N by

$$(S_N f)_n = \psi\left(\frac{n}{N}\right) f_n. \quad (2.9)$$

We also allow $N = \infty$, in which case $S_\infty = 1$. The truncated equations are then

$$u_t + H u_{xx} = S_N (S_N u \cdot S_N u_x). \quad (2.10)$$

Notice that (2.10) conserves the L^2 mass of u ; also, if u is a solution of (2.10) with u_n supported in $|n| \leq N$ for one time t , then this automatically holds for all time.

2.1.2 The main results, and major difficulties

With these preparations, we can now state the main results in this chapter. The most precise and detailed versions are somewhat technical, and will be postponed to Section 2.6.

Theorem 2.1.1 (Local well-posedness). *For any $A > 0$, let $T = T(A) = C^{-1}e^{-CA}$; for the metric space $\mathcal{BO}^T$ (see Definition 2.5.17) containing $B^0([-T, T] \rightarrow Z_1)$, which denotes the space of bounded functions on $[-T, T]$ valued in Z_1 , we have the following. For any f with $\|f\|_{Z_1} \leq A$, there*

exists a unique function $u \in \mathcal{BO}^T$, such that $u(0) = f$, and u verifies (2.1) on $[-T, T]$ in the sense of distributions (we may define uu_x as a distribution for all $u \in \mathcal{BO}^T$; for details see Remark 2.5.18). Moreover, if we write $u = \Phi f$, then the map Φ , from the ball $\{f : \|f\|_{Z_1} \leq A\}$ to the metric space $\mathcal{BO}^T$, will be a Lipschitz extension of the classical solution map for regular data, and its image is bounded away from the zero element in $\mathcal{BO}^T$ by Ce^{CA} .

Theorem 2.1.2 (Measure invariance). *Recall the Gibbs measure ν on $\mathcal{V}$ defined in [50], which is absolutely continuous with respect to a Wiener measure ρ (see Section 2.2.1 for details). There exists a subset Σ of $\mathcal{V}$ with full ρ measure such that for each $f \in \Sigma$, the equation (2.1) has a unique solution $u \in \cap_{T>0} \mathcal{BO}^T$ (in the sense described in Remark 2.5.18) with initial data f . If we denote $u = \Phi f = (\Phi_t f)_t$, then for each $t \in \mathbb{R}$ we get a map $f \mapsto \Phi_t f$ from Σ to itself. These maps form a one parameter group, and each of them keeps invariant the Gibbs measure ν .*

Since we are solving (2.1) in Z_1 , we would like to have that the solution u is continuous in t with value in Z_1 ; this is not true. The discontinuity, which already exhibits the subtlety of (2.1) below L^2 , is due to a modulation factor needed to eliminate one logarithmically growing term (see Section 2.4.3), and can be characterized explicitly.

Theorem 2.1.3 (Modified continuity). *(1) Let $u \in \mathcal{BO}^T$ be the local solution described in Theorem 2.1.1. Let $u_k(t)$ denote the k -th Fourier coefficient of u at time t and define*

$$\Delta_n(t) = \int_0^t \frac{1}{2} \sum_{k=0}^n |u_k(t')|^2 dt' \quad (2.11)$$

for $n > 0$ and extended to be odd for $n \leq 0$, then Δ_n will grow at most logarithmically in n , and the function u^* , defined by

$$(u^*)_n(t) = e^{-i\Delta_n(t)} u_n(t)$$

for each time, will be continuous in t with value in Z_1 .

(2) Let $f \in \Sigma$ and u be the global solution described in Theorem 2.1.2. Let the function $u^\#$, real-valued and having zero average, be defined by

$$(u^\#)_n(t) = e^{-i(t \log n)/8\pi} u_n(t)$$

for all $n > 0$ and t , then $u^\#$ is continuous in t with value in Z_1 .

Next we will briefly discuss the proof of the above theorems. The first step of solving (2.1), see [41], is to use the gauge transform $w = \mathbb{P}_+(ue^{-i\partial_x^{-1}u/2})$ introduced in [48] to obtain a more favorable

nonlinearity; this already becomes problematic with infinite L^2 mass. In fact, with this transform, the evolution equation satisfied by w would be

$$(\partial_t - i\partial_{xx})w = \frac{i}{2}\partial_x\mathbb{P}_+(\partial_x^{-1}w \cdot \partial_x\mathbb{P}_-(\overline{w}\partial_x^{-1}w)) + \frac{i}{4}\mathbb{P}_0(u^2)w + GT, \quad (2.12)$$

where GT represents good terms. Here one can recognize the term $\mathbb{P}_0(u^2)w$ that can be infinite for $u \in Z_1$. However, when we further analyze the cubic term above, we find another contribution, namely the “resonant” one, which is basically some constant multiple of $\mathbb{P}_0(|w|^2)w$. It then turns out that the coefficients match exactly to give a multiple of $\|w\|_{L^2}^2 - \|\mathbb{P}_+u\|_{L^2}^2$. Since (at least heuristically)

$$w = \mathbb{P}_+(ue^{-i\partial_x^{-1}u/2}) = \mathbb{P}_+u \cdot e^{-i\partial_x^{-1}u/2} + GT, \quad (2.13)$$

this expression will be finite even if u is only in Z_1 .

The next obstacle to local theory is the failure of standard multilinear $X^{s,b}$ estimates, which play a crucial role in [41]. Recall from (2.12) that a typical nonlinearity of the transformed equation looks like

$$\partial_x\mathbb{P}_+(\partial_x^{-1}w \cdot \partial_x\mathbb{P}_-(\overline{w}\partial_x^{-1}w)). \quad (2.14)$$

If the frequency of $\partial_x^{-1}w$ appearing in $\overline{w}\partial_x^{-1}w$ is low, we may pretend that this frequency is zero, obtaining a quadratic nonlinearity which is similar to the KdV equation. In fact, there is a similar failure of bilinear estimates for the KdV equation below $H^{-\frac{1}{2}}$, which is necessary in proving the invariance of white noise. This problem was solved in [44] by considering the second iteration, a strategy already used in [5]. We will use the same method, though the fact that our nonlinearity is only quadratic “to the first order” makes the argument a little more involved.

Passing from local theory to global well-posedness and measure invariance is another challenge. The only known method is to produce finite dimensional truncations like (2.9), exploit the invariance of the (finite dimensional) truncated Gibbs measures, and use a limiting procedure to pass to the original equation. This requires, among other things, uniform estimates for solutions to (2.10). The obstacle here is that the gauge transform in [41] is now inadequate for eliminating all bad interactions. To see this, recall that when $w = \mathbb{P}_+(Mu)$ with some function M , then

$$\begin{aligned} (\partial_t - i\partial_{xx})w &= \mathbb{P}_+[-2iM \cdot \partial_{xx}\mathbb{P}_-u + u \cdot (\partial_t - i\partial_{xx})M] \\ &+ \mathbb{P}_+[M \cdot S(Su \cdot Su_x) - 2iM_x u_x], \end{aligned}$$

where we assume that u verifies (2.10) with $S = S_N$. The terms in the first bracket enjoy a smoothing effect and are (more or less) easier to bound, and those in the second bracket will be most troublesome. If $S = 1$, this second bracket can be made zero by choosing $M = e^{-i\partial_x^{-1}u/2}$, but it is impossible when $S = S_N$ with N finite but large. However, note that we only need to eliminate the “high-low” interactions where the factor Su contributes very low frequency and Su_x contributes high frequency, and this is indeed possible if we replace M with some carefully chosen *operator* defined from a combination of S (which is a Fourier multiplier) and suitable multiplication operators. See Section 2.4.1.

Finally, in order for the limiting procedure to work out, we must compare a solution to (2.10) with a solution to (2.1). Since $\psi(n/N)$ equals 1 only for $|n| \leq N/2$, the difference will contain some term involving factors like $\mathbb{P}_{\gtrsim N}u$, which does not decay for large N due to the l^∞ nature of our norm Z_1 . Nevertheless, these bad terms eventually add up to zero, at least to first order, which is enough for our analysis. Note that the bad terms involve ψ factors which are unique to (2.10) and are not found in (2.1), this cancellation is really something of a miracle. See Section 2.4.2.

The rest of this chapter is organized as follows. In Section 2.2 we provide the basic probabilistic arguments, and in Section 2.3 we will introduce the major spacetime norms and corresponding linear estimates. We next define the gauge transform for (2.10) in Section 2.4 and use it to derive the new equations; then in Section 2.5 we will prove the main *a priori* estimates. Finally, in Section 2.6 we will prove our main results, which are (local and almost sure global) well-posedness for (2.1), invariance of Gibbs measure, and modified continuity.

2.2 Relevant probabilistic results

2.2.1 Review of the construction of ν

In this section we briefly review the construction of the Gibbs measure ν as done in [50]. As stated in Section 2.1, ν will be defined by attaching a weight to some Gaussian measure ρ , so we first discuss this Gaussian measure.

Consider a sequence of independent complex Gaussian random variables $\{g_n\}_{n>0}$ living on some ambient probability space $(\Omega, \mathcal{B}, \mathbb{P})$ which are normalized so that $\mathbb{E}(|g_n|^2) = 1$. We may also assume that $|g_n| = O(\langle n \rangle^{10})$ holds everywhere on Ω ; this assumption is just in order to define the map $\mathbf{f}$

and is irrelevant otherwise. Letting $g_{-n} = \overline{g_n}$, we define the random series

$$\mathbf{f} : \Omega \ni \omega \mapsto \sum_{n \neq 0} \frac{g_n(\omega)}{2\sqrt{\pi|n|}} e^{inx} \in \mathcal{V} \quad (2.15)$$

as a map from Ω to $\mathcal{V}$ (recall that $\mathcal{V}$ is the subset of $\mathcal{D}'(\mathbb{T})$ containing real-valued distributions with zero average). This then defines the Wiener measure ρ^1 on $\mathcal{V}$ by $\rho(E) = \mathbb{P}(\mathbf{f}^{-1}(E))$. For each positive integer N , if we identify $\mathcal{V}$ with $\mathcal{V}_N \times \mathcal{V}_N^\perp$, then the measure ρ is identified with $\rho_N \times \rho_N^\perp$, where the latter two measures are defined by

$$\rho_N(E) = \mathbb{P}((\Pi_N \mathbf{f})^{-1}(E)), \quad \rho_N^\perp(E) = \mathbb{P}((\Pi_N^\perp \mathbf{f})^{-1}(E)).$$

Fix a compactly supported smooth cutoff ζ , $0 \leq \zeta \leq 1$, which equals 1 on some neighborhood of 0. Consider for each N the functions

$$\theta_N(f) = \zeta(\|\Pi_N f\|_{L^2}^2 - \alpha_N) e^{\frac{1}{3} \int_{\mathbb{T}} (S_N f)^3}; \quad (2.16)$$

$$\theta_N^\sharp(f) = \zeta(\|\Pi_N f\|_{L^2}^2 - \alpha_N) e^{\frac{1}{3} \int_{\mathbb{T}} (\Pi_N f)^3}, \quad (2.17)$$

where we recall $\Pi_N = \mathbb{P}_{\leq N}$ as in Section 2.1.1, S_N as in (2.9), and

$$\alpha_N = \sum_{n=1}^N \frac{1}{n} = \mathbb{E}(\|\Pi_N \mathbf{f}_1\|_{L^2}^2).$$

Clearly θ_N and $\theta_N^\sharp$ only depend on $\Pi_N f$, thus they can also be understood as functions on $\mathcal{V}_N$. Define the measures

$$d\nu_N = \theta_N d\rho, \quad d\nu_N^\circ = \theta_N d\rho_N; \quad d\nu_N^\sharp = \theta_N^\sharp d\rho, \quad d\nu_N^\circ = \theta_N^\sharp d\rho_N.$$

Then we could identify ν_N and $\nu_N^\sharp$ with $\nu_N^\circ \times \rho_N^\perp$ and $\nu_N^\circ \times \rho_N^\perp$, respectively. Moreover, if we identify $\mathcal{V}_N$ with $\mathbb{R}^{2N}$ and thus denote the measure on $\mathcal{V}_N$ corresponding to the Lebesgue measure on $\mathbb{R}^{2N}$ by $\mathcal{L}_N$, then with some constant C_N

$$d\nu_N^\circ = C_N \zeta(\|f\|_{L^2}^2 - \alpha_N) e^{-E[f]} d\mathcal{L}_N; \quad (2.18)$$

$$d\nu_N^\circ = C_N \zeta(\|f\|_{L^2}^2 - \alpha_N) e^{-E[f]} d\mathcal{L}_N, \quad (2.19)$$

with the f here denoting some element of $\mathcal{V}_N$, the Hamiltonian $E = E_1$ as in (2.2), and the truncated version E_N defined by

$$E_N[f] = \|f\|_{\dot{H}^{1/2}}^2 - \frac{1}{3} \int_{\mathbb{T}} (S_N f)^3. \quad (2.20)$$

It is important to notice, as compared to the situation to be discussed in Chapter 3 below, that ν_N° is invariant under the truncated equation (2.10). This is because (2.10) is also a Hamiltonian system with Hamiltonian E_N , a property which is specific to the Gibbs measure.

The main result of [50] now reads as follows.

Proposition 2.2.1 ([50], Theorem 1). *The sequence $\theta_N^\sharp$ converges in $L^r(d\rho)$ to some function θ for all $1 \leq r < \infty$, and if we define ν by $d\nu = \theta d\rho$, then $\nu_N^\sharp$ converges strongly to ν in the sense that the total variation of their difference tends to zero. This ν is defined to be the Gibbs measure for (2.1).*

Remark 2.2.2. *Only weak convergence was claimed in [50], but an easy elaboration of the arguments there actually gives a much stronger convergence as stated in Proposition 2.2.1 above.*

Remark 2.2.3. *We note that the measure ν depends on the choice of ζ . In this regard we have the following easy observation: there exists a countable collection $\{\zeta^R\}_{R \in \mathbb{N}}$ with corresponding θ^R such that the union of $\mathcal{A}^R = \{f : \theta^R(f) > 0\}$ has full ρ measure. Note that $\mathcal{A}^R$ is the largest set on which ρ and ν^R are mutually absolutely continuous.*

The finite dimensional approximations we will actually use are ν_N instead of $\nu_N^\sharp$, thus we still need to prove the convergence of ν_N . However, the proof is essentially the same as the proof of Proposition 2.2.1, so we shall omit it here and only state the result.

Proposition 2.2.4. *The sequence θ_N converges in $L^r(d\rho)$ to the θ defined in Proposition 2.2.1 for all $1 \leq r < \infty$, and ν_N converges strongly to the ν defined in Proposition 2.2.1 in the sense that the total variation of their difference tends to zero.*

2.2.2 Compatibility with the Besov space

By elementary probabilistic arguments we can see that

$$\rho(f \in \mathcal{V} : \|f\|_{L^2} < \infty) = 0; \quad (2.21)$$

$$\rho(f \in \mathcal{V} : \|f\|_{H^{-\delta}} < \infty) = 1, \quad (2.22)$$

for all $\delta > 0$. Namely, the Wiener measure $d\rho$ (and hence the Gibbs measure $d\nu$) is compatible with $H^{-\delta}$ but not L^2 , which is the essential difficulty in establishing the invariance result. In this section

we show that this difficulty may be resolved by using the Besov space Z_1 defined in Section 2.1.1. First we prove a lemma.

Lemma 2.2.5. *Suppose that $g_j (1 \leq j \leq N)$ are independent normalized complex Gaussian random variables. Then we have*

$$\mathbb{P}\left(\sum_{j=1}^N |g_j|^4 \geq \alpha N\right) \leq 4e^{\sqrt{\alpha N}/120}, \quad (2.23)$$

for all $\alpha > 1600$ and positive integer N .

Proof. Let $X = \sum_{j=1}^N |g_j|^4$. Since $\mathbb{E}(|g_j|^{4m}) = (2m)!$, we can estimate, for each integer $k \geq 1$, the k -th moment of X by

$$\begin{aligned} \mathbb{E}(X^k) &= \sum_{m_1 + \dots + m_N = k} \frac{k!}{m_1! \dots m_N!} \mathbb{E}(|g_1|^{4m_1} \dots |g_N|^{4m_N}) = k! \sum_{m_1 + \dots + m_N = k} \prod_{j=1}^N \frac{(2m_j)!}{m_j!} \\ &\leq k! 4^k \sum_{m_1 + \dots + m_N = k} \prod_{j=1}^N m_j!, \end{aligned}$$

since $\binom{2m}{m} \leq 4^m$. From this, we have that (for $\epsilon > 0$)

$$\mathbb{E}(e^{\sqrt{\epsilon}X}) \leq 2\mathbb{E}(\cosh \sqrt{\epsilon}X) \leq 2 + 2 \sum_{k \geq 1} \frac{\epsilon^k}{(2k)!} \mathbb{E}(X^k) \leq 2 + \sum_{k \geq 1} \frac{(8\epsilon)^k}{k!} S_{N,k},$$

where

$$S_{N,k} = \sum_{m_1 + \dots + m_N = k} \prod_{j=1}^N m_j!, \quad (2.24)$$

which we shall now estimate. By identifying the nonzero terms in $(m_1, \dots, m_N)$, we can rewrite $S_{N,k}$ as

$$S_{N,k} = \sum_{1 \leq r \leq \min\{N, k\}} \binom{N}{r} S'_{k,r}, \quad (2.25)$$

where

$$S'_{k,r} = \sum_{m_1 + \dots + m_r = k, m_j \geq 1} \prod_{j=1}^r m_j!.$$

Clearly the number of choices of $(m_1, \dots, m_r)$ is at most $\binom{k-1}{r-1} \leq 2^k$, and for each choice of $(m_1, \dots, m_r)$, we have

$$\begin{aligned} \prod_{j=1}^r m_j! &\leq m_1 \dots m_r \times \prod_{j=1}^r (m_j - 1)! \leq (k/r)^r \left(\sum_{j=1}^r (m_j - 1) \right)! \\ &\leq e^{r \frac{k}{r}} (k - r)! \leq 3^k (k - r)!. \end{aligned}$$

Therefore we know that $S'_{k,r} \leq 6^k(k-r)!$. Next, notice that there are at most $k \leq 2^k$ choices of r , and that $\binom{N}{r} \leq N^r/r!$, we have

$$S_{N,k} \leq 12^k \max_{1 \leq r \leq k} \frac{N^r(k-r)!}{r!}. \quad (2.26)$$

If the maximum in (2.26) is attained at $r = k$, it will be bounded by $\frac{N^k}{k!}$; otherwise it is attained at some $r < k$, from which we know $N \leq (r+1)(k-r) \leq 2r(k-r)$. Therefore the maximum in this case is bounded by

$$\frac{N^r(k-r)!}{r!} \leq \frac{2^k r^r (k-r)^r (k-r)^{k-r}}{r!} \leq (6k)^k \leq 18^k k!.$$

Altogether we have

$$S_{N,k} \leq 216^k k! + \frac{(12N)^k}{k!},$$

and hence

$$\mathbb{E}(e^{\sqrt{\epsilon}X}) \leq 2 + \sum_{k \geq 1} (1728\epsilon)^k + \sum_{k \geq 1} \frac{(384\epsilon N)^k}{(2k)!}, \quad (2.27)$$

which is clearly bounded by $4e^{20\sqrt{\epsilon}N}$ if we choose $\epsilon = 1/3456$. Now if $\alpha > 1600$, we will have

$$\mathbb{P}(X \geq \alpha N) \leq e^{-\sqrt{\epsilon\alpha}N} \mathbb{E}(e^{\sqrt{\epsilon}X}) \leq 4e^{-\sqrt{\alpha}N/120},$$

as desired. $\square$

Now we can prove that the Wiener measure $d\rho$ is compatible with our Besov space Z_1 . Namely, we have

Proposition 2.2.6. *With the measure ρ defined in Section 2.2.1, we have $\rho(Z_1) = 1$; more precisely we have*

$$\rho(\{f \in \mathcal{V} : \|f\|_{Z_1} \leq K\}) \geq 1 - Ce^{-C^{-1}K^2} \quad (2.28)$$

for all $K > 0$.

Proof. We only need to prove (2.28). Setting C large, this inequality will be trivial when $K \leq 100$. When $K > 100$, we get from the definition that

$$\rho(\{f \in \mathcal{V} : \|f\|_{Z_1} > 100K\}) \leq \sum_{j \geq 0} \mathbb{P}\left(\sum_{0 < n \sim 2^j} |g_n|^p \geq K^p 2^j\right). \quad (2.29)$$

By Hölder,

$$\sum_{0 < n \sim 2^j} |g_n|^p \geq K^p 2^j$$

implies

$$\sum_{0 < n \sim 2^j} |g_n|^4 \geq K^4 2^j.$$

By lemma 2.2.5, this has probability not exceeding $Ce^{-C^{-1}K^2 2^{j/2}}$ provided $K > 100$. Summing up over j , we see that

$$\rho(\{f \in \mathcal{V} : \|f\|_{Z_1} > K\}) \leq \sum_{j \geq 0} Ce^{-C^{-1}K^2 2^{j/2}} \leq Ce^{-C^{-1}K^2}.$$

This completes the proof. □

2.3 Linear theory

2.3.1 Spacetime norms

Throughout the proof, we will use a number of spacetime norms based on the notations introduced in Section 2.1.1. In general, when we write a norm as $l^2 L^1$, this will mean the $l_n^2 L_\xi^1$ norm for some $\tilde{u}$ (which equals the $l_n^2 L_{\hat{\xi}}^1$ norm for $\hat{u}$); the meaning of $L^1 l^2$ will thus be clear. The space-time Lebesgue norms will be denoted by $L^6 L^6$ etc. For example, under this convention the expression $\|u\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^p L^1}$ actually means

$$\sup_{d \geq 0} \left(\sum_{n \sim 2^d} \|\tilde{u}_{n,\xi}\|_{L_\xi^1}^p \right)^{1/p}.$$

Next, observe that up to a constant,

$$\|u\|_{L^6 L^6}^6 = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \left| \sum_{n_1 + n_2 + n_3 = n} \int_{\hat{\xi}_1 + \hat{\xi}_2 + \hat{\xi}_3 = \hat{\xi}} \prod_{i=1}^3 u_{n_i, \hat{\xi}_i} \right|^2 d\hat{\xi}. \quad (2.30)$$

It follows that if $|u_{n,\xi}| \leq v_{n,\xi}$, then $\|u\|_{L^6 L^6} \leq \|v\|_{L^6 L^6}$; thus $\|\mathfrak{N}u\|_{L^6 L^6}$ is a norm of u .

The basic norms

Now we list the easier norms we use:

$$\|u\|_{X_1} = \|\langle n \rangle^s \langle \xi \rangle^b u\|_{l^p L^2} = \|u^{(s,b)}\|_{l^p L^2}; \quad (2.31)$$

$$\|u\|_{X_2} = \|\langle n \rangle^r u\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^p L^1} = \|u^{(r,0)}\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^p L^1}; \quad (2.32)$$

$$\|u\|_{X_3} = \|\langle n \rangle^{-\epsilon} \mathfrak{N}u\|_{L^6 L^6} = \|\mathfrak{N}u^{(-\epsilon,0)}\|_{L^6 L^6}; \quad (2.33)$$

$$\|u\|_{X_4} = \|\langle n \rangle^{-1} \langle \xi \rangle^\kappa u\|_{l^\gamma L^2} = \|u^{(-1,\kappa)}\|_{l^\gamma L^2}; \quad (2.34)$$

$$\|u\|_{X_5} = \|u\|_{l_{d \geq 0}^\infty L^q l_{\sim 2^d}^2}; \quad (2.35)$$

$$\|u\|_{X_6} = \|\langle n \rangle^r \langle \xi \rangle^{1/2+s^2} u\|_{l^2 L^2} = \|u^{(r,1/2+s^2)}\|_{l^2 L^2}; \quad (2.36)$$

$$\|u\|_{X_7} = \|\langle n \rangle^r \langle \xi \rangle^{1/8} u\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^p L^2} = \|u^{(r,1/8)}\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^p L^2}. \quad (2.37)$$

We also recall the norm Z_1 defined in Section 2.1.1, and rewrite it as

$$\|f\|_{Z_1} = \|\langle n \rangle^r f\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^p}. \quad (2.38)$$

We make a few comments about the norms we use. X_1 is a substitution for the standard $X^{s,b}$ norm which is familiar in this kind of problems; however it is not sharp in terms of spatial regularity, and this is compensated by X_2 . X_3 is the spacetime Lebesgue norm that is to be estimated via Strichartz estimates, and X_4 is a special norm designed to control u since it does not belong to the standard $X^{s,b}$ spaces; X_5 is another “sharp” norm which, together with X_7 , is of auxiliary use. Finally X_6 can be seen as an “envelope” norm that controls four of the other norms (see Proposition 2.3.2 below), and will simplify the proof in relatively easy cases.

The space in which we work

Define

$$\|u\|_{Y_1} = \|u\|_{X_1} + \|u\|_{X_2} + \|u\|_{X_4} + \|u\|_{X_5} + \|u\|_{X_7}; \quad (2.39)$$

$$\|u\|_{Y_2} = \|u\|_{X_2} + \|u\|_{X_3} + \|u\|_{X_4}. \quad (2.40)$$

Moreover, for each space $\mathcal{Z}$ (which can be Y_1 , Y_2 or any other space) we define

$$\|u\|_{\mathcal{Z}^T} = \inf \{ \|v\|_{\mathcal{Z}} : v|_{[-T,T]} = u|_{[-T,T]} \}. \quad (2.41)$$

This $[-T, T]$ may also be replaced by any interval I . In the main proof, the gauged variable w will be estimated in the space Y_1^T , and the original unknown u in Y_2^T , while other norms will be introduced whenever necessary.

2.3.2 Linear estimates, and more

Relations between norms

Lemma 2.3.1 (Strichartz estimates). *For any function u , we have*

$$\|u\|_{L^k L^k} \lesssim \|u^{(\sigma, \beta)}\|_{l^2 L^2}, \quad (2.42)$$

provided that the parameters are set as

$$(k, \sigma, \beta) \in \left\{ (2, 0, 0), (4, 0, \frac{3}{8}), (6, s^5, \frac{1}{2} + s^5), (\infty, \frac{1}{2} + s^5, \frac{1}{2} + s^5) \right\}. \quad (2.43)$$

Proof. When $(k, \sigma, \beta) = (2, 0, 0)$, the inequality (2.42) is simply Plancherel; when instead $(k, \sigma, \beta) = (\infty, 1/2 + s^5, 1/2 + s^5)$, this can also be easily proved by combining Hausdorff-Young and Hölder. When $(k, \sigma, \beta) = (4, 0, 3/8)$, the inequality reduces, after separating positive and negative frequencies and using time inversion, to the L^4 Strichartz estimate for the linear Schrödinger equation on $\mathbb{T}$ which is well-known; see for example [2], Proposition 2.6. Finally, when $(k, \sigma, \beta) = (6, s^5, 1/2 + s^5)$, (2.42) basically reduces to the L^6 Strichartz estimate that is proved in [2], Proposition 2.36; for a complete proof, see [16], Proposition 3.1. $\square$

By Lemma 2.3.1 and interpolation, we get a series of $L^k L^k$ Strichartz estimates for $2 \leq k \leq \infty$. It is these that we will actually use in the proof; we will not care too much about the exact numerology because there will be enough room whenever we use them.

Proposition 2.3.2 (Relations between norms). *We have the following inequalities:*

$$\|u\|_{X_3} \lesssim \|u\|_{X_1} + \|u\|_{X_4}, \quad \|u\|_{X_1} + \|u\|_{X_2} + \|u\|_{X_5} + \|u\|_{X_7} \lesssim \|u\|_{X_6}. \quad (2.44)$$

Note that this in particular implies $\|u\|_{X_j} \lesssim \|u\|_{Y_1}$ if $1 \leq j \leq 7$ and $j \neq 6$.

Proof. By Lemma 2.3.1 and hierarchy (2.7) we know that

$$\|u\|_{X_3} \lesssim \|u^{(-\epsilon/2, 1/2 + s^5)}\|_{l^2 L^2}. \quad (2.45)$$

Comparing this with the definition of X_1 and X_4 , and noticing that $\gamma < 2$ and by (2.7) and Hölder,

$$\|u\|_{X_1} \gtrsim \|u^{(-\epsilon/4, b)}\|_{l^2 L^2},$$

we will be able to prove the first inequality in (2.44) provided we can show

$$\langle n \rangle^{-\epsilon/2} \langle \xi \rangle^{1/2+s^5} \lesssim \langle n \rangle^{-1} \langle \xi \rangle^\kappa + \langle n \rangle^{-e\epsilon/4} \langle \xi \rangle^b. \quad (2.46)$$

But this is clear since by (2.7), the left hand side is controlled by the first term on the right hand side if $\langle \xi \rangle \geq \langle n \rangle^{100}$, and by the second term if $\langle \xi \rangle < \langle n \rangle^{100}$.

Now we prove the second inequality in (2.44). The X_1 norm is controlled by X_6 norm because $s < r$, $b < \frac{1}{2} + s^2$, and $2 < p$. For basically the same reason we can use Hölder to show $\|u\|_{X_2} + \|u\|_{X_7} \lesssim \|u\|_{X_6}$. Finally, to prove $\|u\|_{X_5} \lesssim \|u\|_{X_6}$, we only need to show that $\|g_\xi\|_{L^q} \lesssim \|\langle \xi \rangle^{1/2+s^2} g_\xi\|_{L^2}$, but this again follows from Hölder since $q > 1$. $\square$

Bounds for linear evolution

Next, we introduce the (cut-off) Duhamel operator $\mathcal{E}$ defined by

$$\mathcal{E}u(t, x) = \chi(t) \int_0^t \chi(t') (e^{-(t-t')H\partial_{xx}} u(t'))(x) dt', \quad (2.47)$$

where $\chi(t)$ is a cutoff function (compactly supported and equals 1 in a neighborhood of 0) in t . Here and below we shall use many such functions, but unless really necessary, we will not distinguish them and will denote them all by χ (for example, we write $\chi^2 = \chi$). We shall summarize the linear estimates for $\mathcal{E}$ in Proposition 2.3.4 below, but before doing so, we need to introduce two more norms, namely:

$$\|u\|_{X_8} = \|u^{(r,0)}\|_{l_{d \geq 0}^\infty L^{q'} l_{\sim 2^d}^p}, \quad (2.48)$$

$$\|u\|_{X_9} = \|u^{(r,-1/8)}\|_{l_{d \geq 0}^\infty L^\tau l_{\sim 2^d}^p}. \quad (2.49)$$

Lemma 2.3.3. *With constants c_j , we have*

$$(\mathcal{E}u)_{n,\xi} = c_1 (\widehat{\chi} * (\eta^{-1} (\widehat{\chi} * u_{n,*})_\eta))_\xi + c_2 \left(\int_{\mathbb{R}} \frac{(\widehat{\chi} * u_{n,*})_\eta}{\eta} d\eta \right) \cdot \widehat{\chi}_\xi. \quad (2.50)$$

Here the $\frac{1}{\eta}$ is to be understood as the principal value distribution. This operator obeys the following

basic estimates, valid for all $\sigma, \beta \in \mathbb{R}$ and $1 \leq h, k \leq \infty$:

$$\|(\mathcal{E}u)^{(\sigma, \beta)}\|_{L^h l^k} \lesssim \|u^{(\sigma, \beta-1)}\|_{L^h l^k} + \|u^{(\sigma, -1)}\|_{l^k L^1}; \quad (2.51)$$

$$\|(\mathcal{E}u)^{(\sigma, \beta)}\|_{l^k L^h} \lesssim \|u^{(\sigma, \beta-1)}\|_{l^k L^h} + \|u^{(\sigma, -1)}\|_{l^k L^1}. \quad (2.52)$$

Note the reversed order of norms in the second term on the right hand side of (2.51). If moreover $\beta > 1 - \frac{1}{h}$, we can remove the $l^k L^1$ norms. Finally, by commuting with $\mathbb{P}$ projections, we get similar estimates for norms like X_2 and X_5 .

Proof. The computation (2.50) is basically done in [5]. In our case, noticing that multiplication by $\chi(t)$ corresponds to convolution by $\widehat{\chi}$ on the “tilde” side, we only need to express the Fourier transform of $\int_0^t u(t') dt'$ (which is exactly the Duhamel operator on the “tilde” side) in terms of $u(t)$. We compute

$$\int_0^t u(t') dt' = \frac{1}{2} u * \text{sgn}(t) + \frac{1}{2} \int_{\mathbb{R}} u(t') \text{sgn}(t') dt'. \quad (2.53)$$

On the Fourier side, these two terms gives exactly the two terms in (2.50) after another convolution with $\widehat{\chi}$.

We will only prove (2.51), since the proof of (2.52) will be basically the same; also notice that if $\beta > 1 - \frac{1}{h}$, then

$$\|w\|_{l^k L^1} \lesssim \|w\|_{L^1 l^k} \lesssim \min \{ \|\langle \xi \rangle^\beta w\|_{l^k L^h}, \|\langle \xi \rangle^\beta w\|_{L^h l^k} \}$$

for $w = u^{(\sigma, -1)}$, by Hölder. Now to prove (2.51), we first consider the second term of (2.50). Due to its structure, we only need to prove for any function $z = z_\xi$ that

$$\left| \int_{\mathbb{R}} \frac{(z * \widehat{\chi})_\eta}{\eta} d\eta \right| \lesssim \|\langle \xi \rangle^{-1} z\|_{L^1}. \quad (2.54)$$

By considering $|\eta| \gtrsim 1$ and $|\eta| \lesssim 1$ separately and using the cancelation coming from the $\frac{1}{\eta}$ factor, we can control the left hand side by $\|\langle \eta \rangle^{-1} (z * \widehat{\chi})_\eta\|_{L^1}$ (which is easily bounded by the right hand side of (2.51)), plus another term bounded by $\|\langle \eta \rangle^{-1} \partial_\eta (z * \widehat{\chi})\|_{L^\infty}$. If we shift the derivative to $\widehat{\chi}$ to get rid of it, we can again bound this expression by the right hand side of (2.51).

Next, we consider the first term of (2.50). Again we consider the terms with $|\eta| \gtrsim 1$ and $|\eta| \lesssim 1$ separately (by introducing a smooth, even cutoff ϕ_η , say). The part where $|\eta| \gtrsim 1$ is easy, since convolution by $\widehat{\chi}_\xi$ is bounded on any weighted mixed norm Lebesgue space that appears here, and $\frac{1}{\eta}$ is comparable to $\langle \eta \rangle^{-1}$ when restricted to the region $|\eta| \gtrsim 1$. Now for the region $|\eta| \lesssim 1$, we can

actually prove for $y = y_\xi$ and arbitrary $K > 0$ that

$$\left| \left(\widehat{\chi} * \left(\frac{\phi_\eta}{\eta} (\widehat{\chi} * y)_\eta \right) \right)_\tau \right| \lesssim \langle \tau \rangle^{-K} \|\langle \xi \rangle^{-K} y\|_{L^1}, \quad (2.55)$$

which easily implies our inequality. To prove this, let $\widehat{\chi} * y = z$, and compute

$$\left(\widehat{\chi} * \left(\frac{\phi(\eta)}{\eta} z_\eta \right) \right)_\tau = \int_{|\eta| \lesssim 1} \widehat{\chi}_\tau \frac{\phi_\eta z_\eta - z_0}{\eta} d\eta + \int_{|\eta| \lesssim 1} \frac{\widehat{\chi}_{\tau-\eta} - \widehat{\chi}_\tau}{\eta} \phi_\eta z_\eta d\eta.$$

From this we already have $\langle \tau \rangle^{-K}$ decay, and it will suffice to prove that $\sup_{|\eta| \lesssim 1} |z_\eta| \lesssim \|\langle \xi \rangle^{-K} y\|_{L^1}$, but this will be clear from the definition of z . $\square$

Proposition 2.3.4. *We have the following estimates:*

$$\|\mathcal{E}u\|_{X_6} \lesssim \|u^{(0,-1)}\|_{X_6}, \|\mathcal{E}u\|_{X_4} \lesssim \|u^{(0,-1)}\|_{X_4}; \quad (2.56)$$

$$\|\mathcal{E}u\|_{X_1} + \|\mathcal{E}u\|_{X_2} \lesssim \|u^{(0,-1)}\|_{X_1} + \|u^{(0,-1)}\|_{X_2} \lesssim \|u\|_{X_9}; \quad (2.57)$$

$$\|\mathcal{E}u\|_{X_7} \lesssim \|u\|_{X_9} \lesssim \|u\|_{X_8}, \|\mathcal{E}u\|_{X_5} \lesssim \|u\|_{X_9}. \quad (2.58)$$

Moreover, suppose u is such that $u_{n,\xi}$ is supported in $\{(n, \xi) : n \sim 2^d, \xi \gtrsim 2^d\}$ for some d , then

$$\|\mathcal{E}u\|_{X_5} + \|\mathcal{E}u\|_{X_7} \lesssim \|u^{(0,-1)}\|_{X_1} + \|u^{(0,-1)}\|_{X_2}. \quad (2.59)$$

Notice that these estimates automatically imply the dual versions about the boundedness of $\mathcal{E}'$.

Proof. By checking the numerology, we see that (2.56) is a direct consequence of Lemma 2.3.3. To prove the first inequality in (2.57), we use Lemma 2.3.3 to conclude

$$\|\mathcal{E}u\|_{X_1} + \|\mathcal{E}u\|_{X_2} \lesssim \|u^{(0,-1)}\|_{X_1} + \|u^{(0,-1)}\|_{X_2} + \|u^{(s,-1)}\|_{l^p L^1}, \quad (2.60)$$

and note that the last term can be controlled by $\|u^{(0,-1)}\|_{X_2}$ also. To prove that $\|u^{(0,-1)}\|_{X_2} \lesssim \|u\|_{X_9}$, one first commute with $\mathbb{P}_{\sim 2^d}$, then control the $l^p L^1$ norm by the $L^1 l^p$ norm, then use Hölder (note the hierarchy (2.7)). To prove that $\|u^{(0,-1)}\|_{X_1} \lesssim \|u\|_{X_9}$, one first replace the $\|\langle n \rangle^s * \|_{l^p}$ norm by the larger $\|\langle n \rangle^r * \|_{l_{d \geq 0} l_{n \sim 2^d}^p}$ norm, then commute with $\mathbb{P}_{\sim 2^d}$, and control the $l^p L^2$ norm by the $L^2 l^p$ norm and use Hölder again. Along the same lines, we have

$$\|\mathcal{E}u\|_{X_7} \lesssim \|u^{(0,-1)}\|_{X_2} + \|u^{(0,-1)}\|_{X_7} \quad (2.61)$$

as well as

$$\|\mathcal{E}u\|_{X_5} \lesssim \|u^{(0,-1)}\|_{l_{d \geq 0}^\infty l_{\sim 2^d}^2 L^1} + \|u^{(0,-1)}\|_{X_5}, \quad (2.62)$$

where the first term on the right hand side of (2.62) is bounded by $\|u^{(0,-1)}\|_{X_2}$, and the second terms on the right hand side of both (2.61) and (2.62) are bounded by the X_{10} norm, by controlling the $l^p L^2$ norm by the $L^2 l^p$ norm and using Hölder. Also we have $\|u\|_{X_{10}} \lesssim \|u\|_{X_9}$ by Hölder. This proves (2.58).

Let us now prove (2.59). For the X_7 norm we use (2.61), and the support condition will easily allow us to control the second term on the right hand side of (2.61) by $\|u^{(0,-1)}\|_{X_1}$. For the X_5 norm, we only need to bound the second term on the right hand side of (2.62) by $\|u^{(0,-1)}\|_{X_1}$. Since we can restrict to $|n| \sim 2^d$ and $|\xi| \gtrsim 2^d$, we can bound this term by

$$\begin{aligned} \|u^{(0,-1)}\|_{L^q l^2} &\lesssim \|u^{(0,\sigma)}\|_{L^2 l^2} = \|u^{(0,\sigma)}\|_{l^2 L^2} \\ &\lesssim 2^{(\sigma-b+1)d} \|u^{(0,b-1)}\|_{l^2 L^2} \lesssim 2^{(\sigma+\sigma'-b+1)d} \|u^{(0,-1)}\|_{X_1}, \end{aligned}$$

where $\sigma' = \frac{1}{2} - \frac{1}{p} - s > 0$, $\sigma = -\frac{1}{2} - \frac{1}{2q'}$ so that $\sigma + \sigma' - b + 1 < 0$ by (2.7). $\square$

Quasi-continuity of norms

Next we will prove two auxiliary results about our norms Y_j and Y_j^T , as defined in Section 2.3.1. They are only used in the qualitative part of the bootstrap argument, so we will only state the results here. For the (not so trivial) proof, The reader may refer to [16], Propositions 3.5 and 3.6.

Proposition 2.3.5. *Suppose $j \in \{1, 2\}$, and $u = u(t, x) \in Y_j$ is a function that vanishes at $t = 0$, then with a time cutoff χ (recall our convention about such functions) we have, uniformly in $T \lesssim 1$, that*

$$\|\chi(T^{-1}t)u\|_{Y_j} \lesssim \|u\|_{Y_j}. \quad (2.63)$$

If u is smooth, then we also have

$$\lim_{T \rightarrow 0} \|\chi(T^{-1}t)u\|_{Y_j} = 0. \quad (2.64)$$

Proposition 2.3.6. *Suppose $u = u(t, x)$ is a smooth function on $\mathbb{R} \times \mathbb{T}$, then for $j \in \{1, 2\}$, the function $T > 0 \mapsto \mathcal{M}(T) = \|u\|_{Y_j^T}$ satisfies $\mathcal{M}(T+0) \leq C\mathcal{M}(T-0)$ for all $0 < T \lesssim 1$, and $\mathcal{M}(0+) \leq C\|u(0)\|_{Z_1}$.*

2.4 The gauge transform

We now introduce the gauge transform adapted to (2.10), which will play a crucial role in the proof. We will fix a large positive integer N throughout, and drop the subscript N in S_N (we allow $N = \infty$, in which case the arguments should be modified slightly but no essential difference occurs). We also fix a smooth solution u to (2.10); note that smooth solutions are automatically global. When N is finite, we will assume that $\hat{u}$ is supported in $|n| \leq N$ for all time.

Below, we will use Λ to denote various functions of the n_i and m_j variables, that are (linear combinations of) products of ψ factors. They may be different when appearing in different places, and usually we do not write out the variables these Λ depend on; when we do so, however, it will be understood that the involved Λ functions are the same, and that the variables not appearing are regarded as parameters.

It is also important that we actually have explicit formulas for all these Λ factors (though for simplicity we do not present them here; see [16], Sections 5 to 7 for the rather long and tedious computations), which are of no use in other places, but will be vital in exploiting the miraculous cancellation below (see Section 2.4.2).

2.4.1 Definition of w and the first reduction

Let F be the unique mean-zero antiderivative of u , namely $F_n = \frac{1}{in} u_n$ for $n \neq 0$ and $F_0 = 0$. Define the operators $Q_0 : \phi \mapsto (Su) \cdot \phi$ and $P_0 : \phi \mapsto (SF) \cdot \phi$, as well as $Q = SQ_0S$ and $P = SP_0S$. Further, define the operator

$$M = e^{-\frac{iP}{2}} = \sum_{\mu \geq 0} \frac{1}{\mu!} \left(-\frac{i}{2} \right)^\mu P^\mu. \quad (2.65)$$

The function w will be defined by

$$w = \mathbb{P}_+(Mu). \quad (2.66)$$

We also define $v = Mu$, so $w_n = v_n$ when $n > 0$, and $w_n = 0$ otherwise. The evolution equation satisfied by w can be computed as follows:

$$\begin{aligned}
(\partial_t - i\partial_{xx})w &= \mathbb{P}_+ M(\partial_t - i\partial_{xx})u + \mathbb{P}_+ [\partial_t, M]u - i\mathbb{P}_+ [\partial_{xx}, M]u \\
&= -2i\mathbb{P}_+ (M\mathbb{P}_- u_{xx}) + \mathbb{P}_+ ([\partial_t, M]u - i[\partial_x, [\partial_x, M]]u) \\
&+ \mathbb{P}_+ (MS(Su \cdot Su_x) - 2i[\partial_x, M]u_x) \\
&= \mathbb{P}_+ \partial_x (-2iM\mathbb{P}_- u_x) \tag{2.67}
\end{aligned}$$

$$+ \mathbb{P}_+ (-2i[\partial_x, M] + MQ)u_x \tag{2.68}$$

$$+ 2i\mathbb{P}_+ [\partial_x, M]\mathbb{P}_- u_x + \mathbb{P}_+ ([\partial_t, M] - i[\partial_x, [\partial_x, M]])u. \tag{2.69}$$

In the first reduction, we will compute (2.67), (2.68) and (2.69), and reduce them to two typical terms, a “quadratic” one and a “cubic” one (see below for the terminologies).

The term in (2.67)

This term is where we have the “smoothing estimate” which is the reason why the gauge transform is introduced. This is exhibited in the $\mathbb{P}_+$ and $\mathbb{P}_-$ projections, which are so arranged that the top frequency must be among the functions involved in M , and thus appear in the denominator. By expanding M using (2.65), we can write the term in (2.67) as

$$(2.67)_{n_0} = 2i \sum_{\mu} \frac{(-1)^{\mu}}{2^{\mu} \mu!} \sum_{\substack{n_0 > 0, n_1 < 0 \\ n_1 + m_1 + \dots + m_{\mu} = n_0}} n_0 n_1 \Lambda \cdot u_{n_1} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}. \tag{2.70}$$

Notice the input appearing as u_{n_1} (which we may call the major input) and those appearing as u_{m_i}/m_i (which we call the minor inputs). Since minor inputs satisfy very good estimates, our analysis will mainly focus on major inputs; thus (2.70) can be seen as “linear” in this regard. However, notice that $\sum_{i=1}^{\mu} m_i = |n_0| + |n_1|$, we can always “upgrade” one of the minor inputs u_{m_i}/m_i . Actually, using

$$\frac{1}{m_1 \dots m_{\mu}} = \frac{1}{|n_0| + |n_1|} \sum_{i=1}^{\mu} \frac{m_i}{m_1 \dots m_{\mu}}, \tag{2.71}$$

we can rewrite

$$(2.70) = i \sum_{\mu} \frac{(-1)^{\mu+1}}{2^{\mu} (\mu+1)!} \sum_{\substack{n_0 > 0, n_1 < 0; \\ m_1 + \dots + m_{\mu} + n_1 + n_2 = n_0}} \frac{n_0 n_1}{|n_0| + |n_1|} \Lambda \cdot u_{n_1} u_{n_2} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}, \tag{2.72}$$

where the upgraded variable is now called n_2 . Now, if $|m_i| \gtrsim \min(|n_0|, |n_1|)$ for some i , we may upgrade this u_{m_i}/m_i input and obtain a “cubic” term of form

$$(\mathcal{N}_3)_{n_0} = \sum_{n_1+n_2+n_3+m_1+\dots+m_\mu=n_0} O(1) u_{n_1} u_{n_2} u_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}, \quad (2.73)$$

where the μ summation is temporarily omitted. If instead $|m_i| \ll \min(|n_0|, |n_1|)$ for each i , then we will have a “quadratic” term of form

$$(\mathcal{N}_2)_{n_0} = \sum_{n_1+n_2+m_1+\dots+m_\mu=n_0} O(1) \min(|n_0|, |n_1|, |n_2|) u_{n_1} u_{n_2} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}. \quad (2.74)$$

The term in (2.68)

This term will not be present if $N = \infty$; it is precisely due to this term that we have to define M in the current way. Since $[\partial_x, P] = Q$, we may compute

$$\begin{aligned} [\partial_x, M] &= \sum_{\mu} \frac{1}{\mu!} \left(-\frac{i}{2} \right)^{\mu} [\partial_x, P^{\mu}] \\ &= \sum_{\mu, \nu} \frac{1}{(\mu + \nu + 1)!} \left(-\frac{i}{2} \right)^{\mu + \nu + 1} P^{\mu} Q P^{\nu} \\ &= -\frac{i}{2} M Q - \frac{i}{2} \sum_{\mu, \nu} \frac{1}{(\mu + \nu + 1)!} \left(-\frac{i}{2} \right)^{\mu + \nu} P^{\mu} [Q, P^{\nu}]. \end{aligned} \quad (2.75)$$

By expanding the commutator in (2.75), we can write the term in (2.68) as

$$(2.68) = - \sum_{\mu, \nu} \frac{\mu + 1}{(\mu + \nu + 2)!} \left(-\frac{i}{2} \right)^{\mu + \nu + 1} \mathbb{P}_+ P^{\mu} [Q, P] P^{\nu} u_x. \quad (2.76)$$

Notice that

$$[Q, P] = S(Q_0 S^2 P_0 - P_0 S^2 Q_0) S, \quad (2.77)$$

we can thus write

$$(2.68)_{n_0} = \sum_{\mu, \nu} \frac{(-1)^{\mu + \nu} (\mu + 1)}{2^{\mu + \nu + 1} (\mu + \nu + 2)!} \sum_{\substack{n_0 > 0; n_1 + n_2 + n_3 \\ + m_1 + \dots + m_{\mu + \nu} = n_0}} \frac{n_3}{n_2} (\Lambda(n_2, n_3) - \Lambda(n_1, n_3)) u_{n_1} u_{n_2} u_{n_3} \prod_{i=1}^{\mu + \nu} \frac{u_{m_i}}{m_i}. \quad (2.78)$$

After exploiting the symmetry in n_1 and n_3 , we can replace the weight in (2.78) by

$$\frac{1}{2n_2} (n_3 \Lambda(n_2, n_3) - n_3 \Lambda(n_1, n_3) + n_1 \Lambda(n_2, n_1) - n_1 \Lambda(n_3, n_1)). \quad (2.79)$$

Now we analyze the weight (2.79); by symmetry we may assume $|n_1| \geq |n_3|$. If $|n_3| \lesssim |n_0|$ also, then for the first two terms in (2.79), we can extract the n_3 factor, downgrade n_2 to a minor input, and notice that $|n_3| \lesssim \min(|n_0|, |n_1|, |n_3|)$ to obtain $\mathcal{N}_2$ as in (2.74). For the last two terms, by mean value theorem

$$|n_1\Lambda(n_2, n_1) - n_1\Lambda(n_3, n_1)| \lesssim \frac{|n_1|(|n_2| + |n_3|)}{N}.$$

Since also $|n_1| \lesssim N$, we obtain a combination of $\mathcal{N}_2$ and $\mathcal{N}_3$. Now if $|n_3| \gg |n_0|$, then either $\max(|n_2|, |m_i|) \gtrsim |n_1|$ so that we have $\mathcal{N}_3$, or $\max(|n_2|, |m_i|) \ll |n_1|$ so that (2.79) is bounded by

$$\frac{1}{2n_2} (|n_3\Lambda(n_2, n_3) + n_1\Lambda(n_2, n_1)| + |n_3\Lambda(n_1, n_3) + n_1\Lambda(n_3, n_3)|) \lesssim \frac{|n_1|}{N} \frac{\max(|n_0|, |n_2|, |m_i|)}{n_2}$$

again by mean value theorem, using the fact that Λ is a product of even functions. Thus we again get either $\mathcal{N}_2$ or $\mathcal{N}_3$.

The term in (2.69)

Clearly we have

$$[\partial_t, M] = \sum_{\mu, \nu} \frac{1}{(\mu + \nu + 1)!} \left(-\frac{i}{2}\right)^{\mu + \nu + 1} P^\mu [\partial_t, P] P^\nu, \quad (2.80)$$

where

$$[\partial_t, P] : \psi \mapsto S(SF_t \cdot S\psi); \quad (2.81)$$

also we may compute

$$\begin{aligned} [\partial_x, [\partial_x, M]] &= \sum_{\mu, \nu} \frac{1}{(\mu + \nu + 1)!} \left(-\frac{i}{2}\right)^{\mu + \nu + 1} [\partial_x, P^\mu Q P^{\mu_2}] \\ &= \sum_{\mu, \nu} \frac{1}{(\mu + \nu + 1)!} \left(-\frac{i}{2}\right)^{\mu + \nu + 1} P^\mu [\partial_x, Q] P^\nu \\ &+ 2 \sum_{\mu, \nu, \sigma} \frac{1}{(\mu + \nu + \sigma + 2)!} \left(-\frac{i}{2}\right)^{\mu + \nu + \sigma + 2} P^\mu Q P^\nu Q P^\sigma. \end{aligned}$$

Using the fact that

$$[\partial_t, P] - i[\partial_x, Q] : \psi \mapsto S(SG \cdot S\psi) \quad (2.82)$$

where

$$G = F_t - iF_{xx} = -2i\mathbb{P}_- u_x + \frac{1}{2} \left(S((Su)^2) - \mathbb{P}_0((Su)^2) \right), \quad (2.83)$$

we may write

$$\begin{aligned}
(2.69)_{n_0} = & \sum_{\mu, \nu, \sigma} \frac{(-1)^{\mu+\nu+\sigma+1}}{2^{\mu+\nu+\sigma+2}(\mu+\nu+\sigma+2)!} \sum_{\substack{n_0 > 0; n_1+n_2+n_3 \\ +m_1+\dots+m_{\mu+\nu+\sigma}=n_0}} \Lambda u_{n_1} u_{n_2} u_{n_3} \prod_{i=1}^{\mu+\nu+\sigma} \frac{u_{m_i}}{m_i} \\
& + \sum_{\mu, \nu} \frac{(-1)^{\mu+\nu}}{2^{\mu+\nu}(\mu+\nu+1)!} \left[\sum_{\substack{n_0 > 0, n_2 < 0; n_1+n_2 \\ +m_1+\dots+m_{\mu+\nu}=n_0}} n_2(\Lambda(n_2) - \Lambda(n_1)) u_{n_1} u_{n_2} \prod_{i=1}^{\mu+\nu} \frac{u_{m_i}}{m_i} \right. \\
& \left. + \frac{i}{4} \mathbb{P}_0((Su)^2) \sum_{\substack{n_0 > 0; n_1+m_1 \\ +\dots+m_{\mu+\nu}=n_0}} \Lambda \cdot u_{n_1} \prod_{i=1}^{\mu+\nu} \frac{u_{m_i}}{m_i} \right]. \quad (2.84)
\end{aligned}$$

Now the first and third terms on the right hand side of (2.84) are clearly $\mathcal{N}_3$; as for the second term, we may assume $|m_i| \ll |n_2|$ or we again obtain $\mathcal{N}_3$. Since also $n_0 > 0$ and $n_2 < 0$, we must also have $|n_2| \lesssim |n_1|$. Since by mean value theorem

$$|n_2(\Lambda(n_2) - \Lambda(n_1))| \lesssim \frac{|n_0 n_2|}{N} + \frac{|n_2|}{N} \max |m_i|$$

because Λ is even, we also have $\mathcal{N}_2$ or $\mathcal{N}_3$ here.

Summary

Now we have obtained a first (crude) version of the equation satisfied by w , namely that

$$\begin{aligned}
(\partial_t - i\partial_{xx})w_{n_0} = & \sum_{\mu} C_{\mu} \left[\sum_{n_1+n_2+m_1+\dots+m_{\mu}=n_0} O(1) \min(|n_0|, |n_1|, |n_2|) u_{n_1} u_{n_2} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \right. \\
& \left. + \sum_{n_1+n_2+n_3+m_1+\dots+m_{\mu}=n_0} O(1) u_{n_1} u_{n_2} u_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \right], \quad (2.85)
\end{aligned}$$

where $|C_{\mu}| \lesssim 1/\mu!$. Again it is important to notice that we actually have explicit formulas for these $O(1)$ here, and that they will be important in the discussion below.

2.4.2 The miraculous cancellation and the second reduction

The miraculous cancellation

The $\mathcal{N}_2$ and $\mathcal{N}_3$ nonlinearities in (2.85) are good enough if we were solving the problem in L^2 . In the current situation, the “resonant” part in $\mathcal{N}_3$ where (say) $n_2 + n_3 = 0$ will be dangerous, since

they contain a factor

$$\sum_{|n_2| \lesssim N} |u_{n_2}|^2 \approx \log N$$

which cannot be controlled uniformly in N . However, this growth is rather weak, and will be trumped by any nontrivial gain in the weight. For example, if the weight $O(1)$ is bounded by $\min(|n_0|, |n_2|)/|n_2|$ when $n_2 + n_3 = 0$, we will not have this growth anymore. Therefore, the strategy in this section is to identify all the resonant terms, then throw away any acceptable parts and make use of the explicit formulas for the $O(1)$ weights to get an exact expression for this bad contribution. It turns out that this final expression is exactly zero, so we are left with only acceptable contributions. This was done in [16], and we will make a brief overview below.

First, note that we may assume $\max |m_i| \ll \min(|n_0|, |n_1|, |n_2|)$ in any $\mathcal{N}_2$ term in (2.85), since otherwise we can upgrade some m_i and obtain an $\mathcal{N}_3$ term. Now, in $\mathcal{N}_3$ we will focus on the contribution when $n_2 + n_3 = 0$ (and others by symmetry), and $|n_2| \gg \max(|n_0|, |m_i|)$, since other parts do not lead to $\log N$ divergence. Denoting the weight $O(1)$ by Γ now, this term would be

$$\sum_{\mu} C_{\mu} \sum_k |u_k|^2 \sum_{n_1 + m_1 + \dots + m_{\mu} = n_0} \Gamma(k; n_0, m_1, \dots, m_{\mu}) u_{n_1} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i},$$

where C_{μ} equals $(-i/2)^{\mu}/\mu!$ times some rational function of μ . Note that Γ involves some simple fractions and factors of ψ or $k\psi'/N$, with the last one appearing when we use mean value theorem in obtaining an $\mathcal{N}_3$ term.

Now, in each of these factors, we will replace any variable other than k by 0; due to the structure of Γ , the error terms so produced will be bounded by $\max(|n_0|, |m_i|)/|n_2|$ which is acceptable. In this way, we can reduce Γ to a function $\Gamma_{\mu}(k)$ of k alone; it will in fact be a polynomial of $\psi(k/N)$ (depending on μ), possibly multiplied by factors like $k\psi'(k/N)/N$. Therefore we can reduce this bad term to

$$\sum_k \sum_{\mu} C_{\mu} \Gamma_{\mu}(k) |u_k|^2 \sum_{n_1 + m_1 + \dots + m_{\mu} = n_0} u_{n_1} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} = \sum_k \sum_{\mu} \frac{Q(\mu) |u_k|^2}{\mu!} (u(-iF/2)^{\mu})_{n_0} \Gamma_{\mu}(k) \quad (2.86)$$

with some function $\Gamma_{\mu}(k)$ and some rational function $Q(\mu)$.

We may absorb the $Q(\mu)$ factor into $\Gamma_{\mu}(k)$; we then do this for each part of $\mathcal{N}_3$, and explicitly verify that these $\Gamma_{\mu}(k)$ terms add up to zero for each μ (this is just a polynomial identity), and conclude that the resonant terms do not contribute. For details see [16], Section 6.

We have thus obtained the following result (here the “quartic” term appears when one $|m_i|$ in

$\mathcal{N}_3$ is not negligible, and will generally be easy to deal with).

Proposition 2.4.1. *We have, with $|C_\mu| \lesssim 1/\mu!$, that*

$$\begin{aligned}
(\partial_t - i\partial_{xx})w_{n_0} = & \sum_{\mu} C_{\mu} \sum_{\substack{n_1+n_2+n_3+m_1+\dots+m_{\mu}=n_0; \\ (n_1+n_2)(n_2+n_3)(n_3+n_1) \neq 0}} O(1)u_{n_1}u_{n_2}u_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
& + \sum_{\mu} C_{\mu} \sum_{\substack{n_2; n_1+m_1 \\ +\dots+m_{\mu}=n_0}} O(1) \frac{\min(|n_2|, |n_0| + |n_1|)}{|n_2|} u_{n_1}|u_{n_2}|^2 \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
& + \sum_{\mu} C_{\mu} \sum_{\substack{n_1+n_2+n_3+n_4 \\ +m_1+\dots+m_{\mu}=n_0}} O(1) \left(\max_{0 \leq i \leq 4} |n_i| \right)^{-1} u_{n_1}u_{n_2}u_{n_3}u_{n_4} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
& + \sum_{\mu} C_{\mu} \sum_{\substack{n_1+n_2+m_1+\dots+m_{\mu}=n_0; \\ \max |m_i| \ll \min(|n_0|, |n_1|, |n_2|)}} O(1) \min(|n_0|, |n_1|, |n_2|) u_{n_1}u_{n_2} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}.
\end{aligned} \tag{2.87}$$

Remark 2.4.2. *An easier situation is when $N = \infty$; in this case, the cancellation can already be seen from the computations in [41]. Here however, all terms involve the arbitrary cutoff function ψ that is in no way linked to the original equation (2.1), thus it is not to be expected that the miraculous cancellation actually happens (of course if it were to be expected it would not be called miraculous).*

It is also worth mentioning that, the recent work [6] provides a (relatively soft) way to handle the logarithmic growth issue without this cancellation, in the context of the nonlinear Schrödinger equation.

The second reduction

The nonlinearities in (2.87) are enough for the purpose of solving the equation in Z_1 . However, these terms still involve factors of u , which are not bounded in usual $X^{s,b}$ type spaces. Thus we have to perform a second reduction by replacing u with w whenever possible.

Notice that $v = Mu$ and $w = \mathbb{P}_+v$, so that for $n > 0$ we have

$$u_n = \sum_{\mu} \frac{i^{\mu}}{2^{\mu}\mu!} (P^{\mu}v)_n = \sum_{\mu} \frac{1}{2^{\mu}\mu!} \sum_{n_1+m_1+\dots+m_{\mu}=n} \Psi_{\mu} \cdot v_{n_1} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}, \tag{2.88}$$

where $\Psi_{\mu} = \Psi_{\mu}(n, n_1, m_1, \dots, m_{\mu})$ is a product of ψ factors. When $n < 0$, since $u_n = \overline{u_{-n}}$, we have instead

$$u_n = \sum_{\mu} \frac{(-1)^{\mu}}{2^{\mu}\mu!} \sum_{n_1+m_1+\dots+m_{\mu}=n} \Psi_{\mu} \cdot (\bar{v})_{n_1} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}, \tag{2.89}$$

where we note $(\bar{v})_n = \overline{v_{-n}}$. We now make the substitutions (2.88) or (2.89) for each premier input

u_{n_i} in the “quadratic” and “cubic” terms in (2.87), and leave the “quartic” terms as they are.

We first study the cubic terms; after substituting we get

$$\sum_{\mu} \prod_{l=1}^3 \sum_{\mu_l} C_{\mu} \prod_{l=1}^3 \frac{(\pm 1)^{\mu_l}}{2^{\mu_l} \mu_l!} \sum_{n'_1 + n'_2 + n'_3 + \sum m_i^l = n_0} \Gamma \cdot \prod_{l=1}^3 \Psi_{\mu_l} \cdot (y^1)_{n'_1} (y^2)_{n'_2} (y^3)_{n'_3} \prod_{l=1}^3 \prod_{i=1}^{\mu_l} \frac{u_{m_i^l}}{m_i^l},$$

where Γ represents the coefficient in (2.87) and y^l equals either v or $\bar{v}$. If for some l and i we have $|m_i^l| \gtrsim |n'_l|$, then we can upgrade this $u_{m_i^l}/m_i^l$ to major input and obtain a well-behaved “quartic” term; see the statement of Proposition 2.4.3 below (there is actually still some subtlety here, but is easily dealt with using the same arguments as below; see [16], Section 7.1). Now suppose $|m_i^l| \ll |n'_l|$ for each i and l , then we may always replace y^l by w or $\bar{w}$, so we have already obtained a “cubic” term with major inputs being w or $\bar{w}$; next we investigate the occasion when (say) $n'_1 + n'_2 = 0$. Let $n'_1 = k$ and $n'_2 = -k$ with $k > 0$, we must have $y^1 = w$ and $y^2 = \bar{w}$. Here we shall, in all the factors involved in Γ , Ψ_{μ_1} and Ψ_{μ_2} , replace any m_i^1 and m_i^2 by zero, noting that any error term so produced will be a good “quartic” term. Now fix $\mu_1 + \mu_2 := \mu$ and any other variable other than μ_1 and μ_2 , so that the factor $\prod_{i=1}^{\mu_1} (u_{m_i^1}/m_i^1) \prod_{i=1}^{\mu_2} (u_{m_i^2}/m_i^2)$ can be extracted after rearrangement of variables. We then have a sum

$$\sum_{\mu_1 + \mu_2 = \mu} \frac{1}{2^{\mu_1} \mu_1!} \frac{(-1)^{\mu_1}}{2^{\mu_2} \mu_2!},$$

which is zero unless $\mu = 0$. Finally if $\mu_1 = \mu_2 = 0$, then as in Proposition 2.4.1, we must have the improved bound $|\Gamma| \lesssim \min(|n_1|, |n_0| + |n_3|)/|n_1|$.

Next, we will study the “quadratic” terms after the substitution in the same way. Basically, if $|m_i^l| \ll |n'_l|$ for each i and l , then we get a term of the same form as the “quadratic” term in (2.85), but with the major inputs u replaced by w or $\bar{w}$. When $|m_i^l| \gtrsim |n'_l|$ for some i and l we will upgrade this $u_{m_i^l}/m_i^l$, and it turns out that we get a term of form

$$\sum_{\mu} C_{\mu} \sum_{n_1 + n_2 + n_3 + m_1 + \dots + m_{\mu} = n_0} O(1) \frac{\min(|n_0|, |n_1|, |n_2 + n_3|)}{\max(|n_2|, |n_3|)} (y^1)_{n_1} (y^2)_{n_2} (y^3)_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i},$$

where each y^l maybe u , v or $\bar{v}$. Then we substitute v back by u using the relation $v = Mu$ and use (2.88) and (2.89) again, so that we get either a good quartic term, or the same term as above.

All these lead to the following

Proposition 2.4.3. *We have, with $|C_\mu| \lesssim 1/\mu!$, that*

$$\begin{aligned}
(\partial_t - i\partial_{xx})w_{n_0} &= \sum_{\mu} C_{\mu} \sum_{\substack{n_2; n_1+m_1 \\ +\dots+m_{\mu}=n_0}} O(1) \frac{\min(|n_2|, |n_0| + |n_1|)}{|n_2|} w_{n_1} |w_{n_2}|^2 \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
&+ \sum_{\mu} C_{\mu} \sum_{\substack{n_2; n_3 \\ +n_4+m_1+\dots+m_{\mu}=n_0}} O(1) \left(\max_{2 \leq l \leq 4} |n_l| \right)^{-1} |u_{n_2}|^2 u_{n_3} u_{n_4} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
&+ \sum_{\mu} C_{\mu} \sum_{\substack{n_1+n_2+n_3+m_1+\dots+m_{\mu}=n_0; \\ (n_1+n_2)(n_2+n_3)(n_3+n_1) \neq 0}} O(1) (w^{\pm})_{n_1} (w^{\pm})_{n_2} (w^{\pm})_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
&+ \sum_{\mu} C_{\mu} \sum_{\substack{n_1+n_2 \neq 0; n_1+n_2+n_3 \\ +n_4+m_1+\dots+m_{\mu}=n_0}} O(1) (|n_3| + |n_4|)^{-1} u_{n_1} u_{n_2} u_{n_3} u_{n_4} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \\
&+ \sum_{\mu} C_{\mu} \sum_{\substack{n_1+n_2+m_1+\dots+m_{\mu}=n_0; \\ \max |m_i| \ll \min(|n_0|, |n_1|, |n_2|)}} O(1) \min(|n_0|, |n_1|, |n_2|) (w^{\pm})_{n_1} (w^{\pm})_{n_2} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i}.
\end{aligned} \tag{2.90}$$

Remark 2.4.4. *Here we have simplified the discussion in [16] by removing what is called the $\mathcal{J}^{3.5}$ contribution in that paper. In fact, we observed that by one more substitution, this term can be reduced to a part of $\mathcal{J}^3$ in the terminology there, which is just the “cubic” nonlinearity in the current section. Therefore, we will not need the X_8 norm (which is specifically designed to attack the $\mathcal{J}^{3.5}$ term) in that paper, and we can get rid of Section 10 of that paper completely.*

2.4.3 The modulation factor and the third reduction

The nonlinearities in (2.90) are almost enough for prove the estimates; however, there is still the “resonant” contribution, which will now not grow like $\log N$, but still needs to be handled. In fact, consider the “cubic” term in (2.90) where $n_0 = n_1$ and $n_2 + n_3 = 0$. Again letting the weight be Γ , we get a term

$$\left(\sum_{\mu} C_{\mu} \sum_{n_2} \sum_{m_1+\dots+m_{\mu}=0} \Gamma(n_0, n_2; m_1, \dots, m_{\mu}) |w_{n_2}|^2 \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \right) w_{n_0}, \tag{2.91}$$

where $|\Gamma| \lesssim \min(1, |n_0|/|n_2|)$. The sum here grows like $\log(n_0)$ instead of $\log N$, which still prevents us from estimating it directly, but due to its structure, we can eliminate it by introducing an additional modulation factor. Note that this factor also explains the modified continuity result in Theorem 2.1.3.

To be precise, in (2.91), we further replace each m_i by 0, and also replace n_2 by 0 (since here we

are mainly interested in the contribution when $|n_2| \ll |n_0|$ to obtain a term of form

$$w_{n_0} \sum_{\mu} C_{\mu} \Gamma_{\mu}(n_0) \sum_{|n_2| \ll |n_0|} |w_{n_2}|^2 \sum_{m_1 + \dots + m_{\mu} = 0} \prod_{i=1}^{\mu} \frac{w_{m_i}}{m_i} = w_{n_0} \sum_{\mu} C_{\mu} \Gamma_{\mu}(n_0) \sum_{|n_2| \ll |n_0|} |w_{n_2}|^2 (-iF/2)^{\mu} \Big|_0.$$

With fixed n_0 , this is then simply

$$(\mathcal{L}_{n_0}(F))_0 \sum_{|n_2| \ll |n_0|} |w_{n_2}|^2 w_{n_0},$$

where $\mathcal{L}_{n_0}(F)$ is some function of F depending on n_0 , which is obtained from analyzing the explicit expressions of all the “cubic” terms in (2.90). Through a similar computation as in Section 2.4.2, we find out that that $\mathcal{L}_{n_0}$ is actually a (purely imaginary) constant

$$\mathcal{L}_{n_0}(F) = i \left[\frac{1}{2} \psi^4 \left(\frac{n_0}{N} \right) + \frac{2n_0}{N} \psi^3 \left(\frac{n_0}{N} \right) \psi' \left(\frac{n_0}{N} \right) \right].$$

Thus we finally arrived at the following

Proposition 2.4.5. *We can define (u^*, w^*) , for each fixed time, by*

$$(u^*, w^*)_n = e^{-i\Delta_n} (u, w)_n; \quad \Delta_n(t) = \int_0^t \delta_n(t') dt', \quad (2.92)$$

where the δ factors are

$$\delta_n = \left[\frac{1}{2} \psi^4 \left(\frac{n}{N} \right) + \frac{2n}{N} \psi^3 \left(\frac{n}{N} \right) \psi' \left(\frac{n}{N} \right) \right] \sum_{k=0}^n |w_k|^2 \quad (2.93)$$

for $n > 0$ (notice we may replace the w_k by $(w^*)_k$ in this expression), and extended to $n \leq 0$ so that they are odd in n . With these definitions, we have

$$(\partial_t - i\partial_{xx})w^* = \mathcal{N}_2^*((w^*)^{\pm}, (w^*)^{\pm}) + \mathcal{N}_3^* + \mathcal{N}_4^*, \quad (2.94)$$

where with $|C_{\mu}| \leq \mu!$, we have

$$(\mathcal{N}_2^*(f, g))_{n_0} = \sum_{\mu} C_{\mu} \sum_{\substack{n_1 + n_2 + m_1 + \dots + m_{\mu} = n_0; \\ \max |m_i| \ll \min(|n_0|, |n_1|, |n_2|)}} e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})} \Phi^2 f_{n_1} g_{n_2} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i};$$

$$(\mathcal{N}_3^*)_{n_0} = \sum_{n_1 + n_2 + n_3 + m_1 + \dots + m_{\mu} = n_0} e^{-i\Delta_{n_0}} \Phi^3 \prod_{l=1}^3 (w^{\pm})_{n_l} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i};$$

$$(\mathcal{N}_4^*)_{n_0} = \sum_{n_1 + \dots + n_4 + m_1 + \dots + m_\mu = n_0} e^{-i\Delta_{n_0}} \Phi^4 \prod_{l=1}^4 u_{n_l} \prod_{i=1}^\mu \frac{u_{m_i}}{m_i}.$$

The weights will satisfy

$$|\Phi^2| \lesssim \min(|n_0|, |n_1|, |n_2|), \quad |\Phi^3| \lesssim 1, \quad |\Phi^4| \lesssim (|n_3| + |n_4|)^{-1}.$$

Moreover, for Φ^3 we have the stronger bound

$$|\Phi^3| \lesssim \frac{\min(|n_2|, |n_0| + |n_1|)}{|n_2|} \text{ if } n_2 + n_3 = 0; \quad |\Phi^3| \lesssim \frac{\min(|n_0|, |n_2|)}{\max(|n_0|, |n_2|)} \text{ if } n_0 = n_1, n_2 + n_3 = 0. \quad (2.95)$$

Similarly, for Φ^4 we have the bound

$$|\Phi^4| \lesssim \left(\max_{2 \leq l \leq 4} |n_l| \right)^{-1} \text{ if } n_1 + n_2 = 0. \quad (2.96)$$

All these hold also for other permutations.

It is important to notice the bound $|\Phi^3| \lesssim \min(|n_0|, |n_2|)/\max(|n_0|, |n_2|)$ in (2.95). This implies that, with n_0 fixed, the sum in n_2 will thus not have *any* growth, so that it can be bounded directly in the bootstrap arguments.

Remark 2.4.6. *In fact, all we need for the estimates below is that δ_n grows at most logarithmically in n , and that it is real valued. The first property does not require any computation and follows immediately from (2.90), while the second property is to be expected due to conservation of mass (and Gibbs measure).*

2.5 The a priori estimates

Based on Proposition 2.4.5, in this section we state and prove our main *a priori* estimates. Basically we shall bound w^* in the space Y_1^T , u^* in Y_2^T , and u in some space slightly weaker than Y_2^T ; there will thus be many extension arguments, which we summarize in Section 2.5.1. Then we prove the bounds for w^* in Section 2.5.2 and the bounds for u^* and u in Section 2.5.3. Finally, in Section 2.5.4, we state (without proof) a version of these main estimates that works for the difference of two solutions.

2.5.1 The bootstrap

Let us fix a smooth solution u , defined on $\mathbb{R} \times \mathbb{T}$, to the equation (2.10), with the parameter $1 \ll N \leq \infty$. In what follows we will assume $N < \infty$, since the case $N = \infty$ will follow from similar (and simpler) arguments. The main estimate can then be stated as follows.

Proposition 2.5.1. *There exists an absolute constant C such that the following holds. Suppose $\|u(0)\|_{Z_1} \leq A$ for some large A , then within a short time $T = C^{-1}e^{-CA}$, for the functions v and w defined in Section 2.4, and the functions u^* and w^* defined in Section 2.4.3, we have*

$$\|w^*\|_{Y_1^T} + \|u^*\|_{Y_2^T} \leq Ce^{CA}; \quad \|u^{(-s^3, 0)}\|_{Y_2^T} \leq CA. \quad (2.97)$$

Remark 2.5.2. *The constant C will depend on the constants in the inequalities in earlier sections, such as Proposition 2.3.6 and Proposition 2.4.5. To make this clear, we will now use C_0 to denote any (large) constant that can be bounded by the constants appearing in those inequalities.*

In the proof of Proposition 2.5.1 we will use a bootstrap argument. The starting point is

Proposition 2.5.3. *The estimate (2.97) is true, with C replaced by C_0 , when $T > 0$ is sufficiently small.*

Proof. Note that $u^*(0) = u(0)$ and the same holds for w^* , and that $w(0) = \mathbb{P}_+v(0)$; by invoking Proposition 2.3.6, we only need to prove that $\|u(0)\|_{Z_1} \leq C_0A$ and $\|v(0)\|_{Z_1} \leq C_0e^{C_0A}$. The first inequality follows from our assumption, so we only need to prove that $\|Mu\|_{Z_1} \lesssim C_0e^{C_0\|u\|_{Z_1}}$. By the definition of M , we only need to prove that $\|P^\mu u\|_{Z_1} \leq C_0^\mu \|u\|_{Z_1}^{\mu+1}$ for all μ . Now we clearly have

$$|(P^\mu u)_{n_0}| \lesssim \sum_{n_1} |u_{n_1}| \cdot |z_{n_0-n_1}|, \quad \text{where } z_m = \sum_{m_1+\dots+m_\mu=m} \prod_{i=1}^\mu \frac{|u_{m_i}|}{\langle m_i \rangle}. \quad (2.98)$$

Since when $m = m_1 + \dots + m_\mu$ we have $\langle m \rangle \leq C_0^\mu \langle m_1 \rangle \dots \langle m_\mu \rangle$, we conclude that

$$\sum_m \langle m \rangle^{1/4} |z_m| \lesssim C_0^\mu \prod_{i=1}^\mu \sum_{m_i} \frac{|u_{m_i}|}{\langle m_i \rangle^{3/4}} \lesssim (C_0 \|u\|_{Z_1})^\mu, \quad (2.99)$$

where the last inequality is because

$$\begin{aligned} \sum_m \frac{|u_m|}{\langle m \rangle^{3/4}} &\lesssim \sum_d 2^{-3d/4} \sum_{m \sim 2^d} |u_m| \lesssim \sum_d 2^{(-3/4+1-1/p-r)d} \|\langle m \rangle^r u_m\|_{l^p_{m \sim 2^d}} \\ &\lesssim \sum_d 2^{-d/4} \|u\|_{Z_1} \lesssim \|u\|_{Z_1}. \end{aligned}$$

Now using (2.99), we will be able to complete the proof once we have

$$\|(u_{n+m})_{n \in \mathbb{Z}}\|_{Z_1} \lesssim \langle m \rangle^{1/4} \|u\|_{Z_1};$$

but this is easily proved by using the definition of Z_1 and comparing coefficients for each u_n . $\square$

Starting from Proposition 2.5.3 and with the help of Proposition 2.3.6, it is easily seen that we only need to prove the following

Proposition 2.5.4. *Suppose C_j is large enough depending on C_{j-1} for $1 \leq j \leq 2$, and $0 < T \leq C_2^{-1}e^{-C_2A}$. Then if the inequalities*

$$\|w^*\|_{Y_1^T} + \|u^*\|_{Y_2^T} \leq C_1 e^{C_1 A}; \quad \|u^{(-s^3, 0)}\|_{Y_2^T} \leq C_1 \quad (2.100)$$

hold, these inequalities must be true with C_1 replaced by C_0 .

The extensions

By the definition of Y_j^T norms, we have globally defined functions u'' , w'' and u''' which agree with u^* , w^* and u on $[-T, T]$ respectively, and verify the inequalities in (2.100) with the superscript T in the norms removed. By inserting a time cutoff $\chi(t)$, we may assume that they are all supported in $|t| \leq 1$. We then define the factors δ_n and Δ_n for all time as in (2.92) and (2.93), with w^* and u replaced by w'' and u''' respectively. We may also define functions u' and w' by $(u')_n = e^{i\Delta_n}(u'')_n$ and $(w')_n = e^{i\Delta_n}(w'')_n$.

Now we could interpret the bilinear form $\mathcal{N}_2^*$ and terms $\mathcal{N}_j^*$ on the right hand side of (2.94), by replacing each minor input u with u''' , each w with w' , each major input u with u' , each δ_n and Δ_n with what we defined above. If we then choose some $0 < \mathcal{T} \leq T$ and define the function z by $z(t) = w''(t)$ for $t \in [-\mathcal{T}, \mathcal{T}]$ and $(\partial_t - i\partial_{xx})z(t) = 0$ on both $(-\infty, -\mathcal{T}]$ and $[\mathcal{T}, +\infty)$, then we can check that this function z verifies the equation

$$(\partial_t - i\partial_{xx})z = \mathbf{1}_{[-\mathcal{T}, \mathcal{T}]}(t) \mathcal{N}_2^*(z^\pm, z^\pm) + \mathbf{1}_{[-\mathcal{T}, \mathcal{T}]}(t) \sum_{j \in \{3, 4\}} \mathcal{N}_j^*, \quad (2.101)$$

with initial data $z(0) = w(0)$. Using the time cutoff $\chi(t)$, we can define $y(t) = \chi(t)z(t)$. From

(2.101) we conclude that

$$y = \chi(t)e^{it\partial_{xx}}w(0) + \mathcal{E}(\mathbf{1}_{[-\mathcal{T}, \mathcal{T}]} \cdot \mathcal{N}_2^*(y^\pm, y^\pm)) + \sum_{j \in \{3, 4\}} \mathcal{E}(\mathbf{1}_{[-\mathcal{T}, \mathcal{T}]} \cdot \mathcal{N}_j^*). \quad (2.102)$$

Since w'' is smooth on the interval $[-T, T]$, we may conclude that $\mathcal{T} \mapsto y$ is a continuous map from $(0, T]$ to Y_1 ; also it is clear that when $\mathcal{T}$ is sufficiently small we have $\|y\|_{Y_1} \leq C_0 e^{C_0 A}$. Thus in order to prove the estimate for w^* , we only need to prove the following

Proposition 2.5.5. *Suppose $y \in Y_1$ is a function verifying (2.102) with $0 < \mathcal{T} \leq C_2^{-1} e^{-C_2 A}$, and $\|y\|_{Y_1} \leq C_1 e^{C_1 A}$, then we must have $\|y\|_{Y_1} \leq C_0 e^{C_0 A}$.*

In what follows, we will use T instead of $\mathcal{T}$ for simplicity; note that $T \leq C_2^{-1} e^{-C_2 A}$.

Information about the $e^{i\Delta_n}$ factor

Before proceeding, we need an additional result, Proposition 2.5.8, concerning the exponential factors $e^{\pm i\Delta_n(t)}$. It basically states that this factor loses only an arbitrarily small power, which follows from Proposition 2.5.6 and Lemma 2.5.7 below. Here we only prove Proposition 2.5.6; the reader may refer to [16], Section 8.2 for the full proof.

Proposition 2.5.6. *We have*

$$\|\widehat{\delta}_n\|_{L^1} \leq C_0 C_1 e^{C_0 C_1 A} \log(2 + |n|); \quad \|\mathcal{F}(\delta_{n+1} - \delta_n)\|_{L^1} \leq C_0 C_1 e^{C_0 C_1 A} \langle n \rangle^{-1} \log(2 + |n|). \quad (2.103)$$

Proof. Recall from Proposition 2.4.5 that

$$\delta_n = C(n) \cdot \sum_{k=0}^n |(w'')_k|^2, \quad (2.104)$$

where clearly we have $|C(n)| \lesssim 1$ and $|C(n+1) - C(n)| \lesssim \langle n \rangle^{-1}$. Now, using the fact that $\|\widehat{fg}\|_{L^1} \leq \|\widehat{f}\|_{L^1} \|\widehat{g}\|_{L^1}$, we obtain

$$\begin{aligned} \|\widehat{\delta}_n\|_{L^1} &\lesssim \sum_{k \lesssim \langle n \rangle} \|\widehat{(w'')_k}\|_{L^1} \|\widehat{(w'')_{-k}}\|_{L^1} \lesssim \|w''\|_{l_{k \lesssim \langle n \rangle}^2 L^1} \| \overline{w''} \|_{l_{k \lesssim \langle n \rangle}^2 L^1} \\ &\lesssim C_0 C_1 e^{C_0 C_1 A} \log(2 + |n|). \end{aligned} \quad (2.105)$$

Here we have used the fact that

$$\|w''\|_{l_{k \lesssim \langle n \rangle}^2 L^1} \lesssim \log(2 + |n|) \cdot \|w''\|_{l_{d \geq 0}^\infty l_{k \sim 2^d}^2 L^1} \lesssim \log(2 + |n|) \|w''\|_{X_2}, \quad (2.106)$$

and the same estimate for $\overline{w''}$. The estimate for the difference is proved in the same way. $\square$

Lemma 2.5.7. *Suppose $h_j = h_j(t)$, $j \in \{0, 1\}$ are two functions of t , and define $J_j(t) = \chi(t)e^{iH_j(t)}$, where $H_j(t) = \int_0^t h_j(t')dt'$, then we have the estimate*

$$\|\langle \xi \rangle \mathcal{F}(J_1 - J_0)(\xi)\|_{L^k} \lesssim \|\mathcal{F}(h_1 - h_0)\|_{L^1} (1 + \|\widehat{h_1}\|_{L^1} + \|\widehat{h_0}\|_{L^1})^2 \quad (2.107)$$

for all $1 \leq k \leq \infty$.

Proposition 2.5.8. *For any function h , let h' be defined by $(h')_n = \chi(t)e^{\pm i\Delta_n} h_n$ for each fixed time. We then have*

$$\|(h')^{(-s^3, 0)}\|_{X_j} \leq O_{C_1}(1)e^{C_0 C_1 A} \|h\|_{X_j} \quad (2.108)$$

for $1 \leq j \leq 7$.

2.5.2 The estimates for w^*

In this section we prove Proposition 2.5.5, starting from (2.102). The linear term is clearly bounded in Y_1 by $C_0 e^{C_0 A}$, so we only need to bound the $\mathcal{N}_j^*$ terms. It turns out that the $\mathcal{N}_4^*$ term is always easier to handle, in the sense that either it can be bounded trivially, or the same obstacle encountered in estimating $\mathcal{N}_4^*$ also appears in the estimate for $\mathcal{N}_3^*$. Thus, below we will focus on $\mathcal{N}_2^*$ and $\mathcal{N}_3^*$ only, and divide the proof into three main propositions.

Proposition 2.5.9. *For each $j \in \{2, 3, 4\}$, define*

$$\mathcal{M}_j = \mathcal{E}(\mathbf{1}_{[-T, T]} \mathcal{N}_j^*), \quad (2.109)$$

where we may write $\mathcal{N}_2^*$ or $\mathcal{N}_2^*(y^\pm, y^\pm)$ depending on the context. We then have

$$\|\mathcal{M}_2\|_{X_4} \leq O_{C_1}(1)e^{C_0 C_1 A} T^{0+}, \quad (2.110)$$

as well as

$$\sum_{j \in \{3, 4\}} \|(\mathcal{M}_j)^{(-1/20, \kappa)}\|_{l^2 L^2} \leq O_{C_1}(1)e^{C_0 C_1 A} T^{0+}. \quad (2.111)$$

Remark 2.5.10. *Since we have $\|u^{(-1, \kappa)}\|_{l^\gamma L^2} \leq C_0 \|u^{(-1/20, \kappa)}\|_{l^2 L^2}$ by Hölder, the inequalities (2.110) and (2.111) will imply $\|y\|_{X_4} \leq C_0 e^{C_0 A}$, due to the restriction $T \leq C_2^{-1} e^{-C_2 A}$.*

Proof. Below we will omit the implicit constants, understanding that they are always under control. We need to be careful with the *sharp* cutoff $\mathbf{1}_{[-T, T]}$; denote by $\phi_\xi = \frac{e^{iT\xi} - e^{-iT\xi}}{i\xi}$ the Fourier transform of $\mathbf{1}_{[-T, T]}$, then we have $|\phi_\xi| \lesssim \min(T, \frac{1}{\langle \xi \rangle})$, and that $\|\phi\|_{L^1(\{|\xi| \geq K\})} \lesssim T^{0+} \langle K \rangle^{0-}$.

First let us prove $\|\mathcal{M}^2\|_{X_4} \lesssim T^{0+}$. This will be an easy part, so we will use this as an example to demonstrate our general scheme. Choose a function g such that $\|g\|_{X'_4} \leq 1$ and define $f = \mathcal{E}'g$. Also define f' by $(f')_n = e^{i\Delta_n} f_n$ and y' similarly; these notations will be used throughout the proof below. Note that since f has compact time support, we may insert $\chi(t)$ in the definition of f' , so that we can use Proposition 2.5.8. The same comment applies also for later discussions.

From the bound $\|g\|_{X'_4} \leq 1$ we obtain by Proposition 2.3.4 that $\|f^{(1, 1-\kappa)}\|_{l^2 L^2} \lesssim 1$, which then implies, thanks to (Hölder and) Proposition 2.5.8, that

$$\|(f')^{(1-O(2-\gamma), 1-\kappa)}\|_{l^2 L^2} \lesssim 1. \quad (2.112)$$

Using Plancherel, we now only need to bound the expression

$$\begin{aligned} \mathcal{S} &= \sum_{n_0} \int_{\mathbb{R}} \overline{f_{n_0, \alpha_0}} \cdot (\mathbf{1}_{[-T, T]} \mathcal{N}_2^*)_{n_0, \alpha_0} d\alpha_0 \\ &= \sum_{n_0=n_1+n_2+m_1+\dots+m_\mu} \int_{\mathbb{R}} \Phi^2 \overline{f_{n_0, \alpha_0}} \mathcal{F} \left(\mathbf{1}_{[-T, T]} e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})} \prod_{l=1}^2 (y^\pm)_{n_l} \prod_{i=1}^\mu \frac{(u''')_{m_i}}{m_i} \right) (\alpha_0 - |n_0|n_0) d\alpha_0 \\ &= \sum_{n_0=n_1+n_2+m_1+\dots+m_\mu} \int_{\mathbb{R}} \Phi^2 \overline{(f')_{n_0, \alpha_0}} \mathcal{F} \left(\mathbf{1}_{[-T, T]} \prod_{l=1}^2 (y')_{n_l}^\pm \prod_{i=1}^\mu \frac{(u''')_{m_i}}{m_i} \right) (\alpha_0 - |n_0|n_0) d\alpha_0 \\ &= \sum_{n_0=n_1+n_2+\dots+m_\mu} \int_{(T)} \Phi^2 \cdot \overline{(f')_{n_0, \alpha_0}} \prod_{l=1}^2 (y')_{n_l, \alpha_l}^\pm \cdot \phi_{\alpha_3} \prod_{i=1}^\mu \frac{(u''')_{m_i, \beta_i}}{m_i}. \end{aligned}$$

Here the integration (T) is interpreted as the integration over the set

$$\left\{ (\alpha_0, \dots, \alpha_3, \beta_1, \dots, \beta_\mu) : \alpha_0 = \sum_{l=1}^3 \alpha_l + \sum_{i=1}^\mu \beta_i + \Xi \right\}, \quad \Xi := |n_0|n_0 - \sum_{l=1}^2 |n_l|n_l - \sum_{i=1}^\mu |m_i|m_i,$$

which is a hyperplane in $\mathbb{R}^{\mu+4}$ (recall the notation that $\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3$), with respect to the standard measure $\prod_{l=1}^3 d\alpha_l \cdot \prod_{i=1}^\mu d\beta_i$.

Let $\sum_{l=0}^2 \langle n_l \rangle \sim 2^d$ and fix d ; we may replace f' by $(f')^{1-O(2-\gamma)}$, so that (2.112) reduces to $\|(f')^{(0, 1-\kappa)}\|_{l^2 L^2} \lesssim 1$. Since $|\Phi^2| \lesssim |n_0|$, the gain from this reduction can cancel the Φ^2 weight, up to a loss of $2^{O(2-\gamma)d}$. Below we will use the reduced bound for f' and the new weight $|\Phi^2| \lesssim 1$.

Notice that $|m_i| \ll \min_{0 \leq l \leq 2} \langle n_l \rangle$, we can check algebraically that

$$|\Xi| \sim \min_{0 \leq l \leq 2} \langle n_l \rangle \cdot \max_{0 \leq l \leq 2} \langle n_l \rangle, \quad (2.113)$$

thus at least one of the α and β variables must be $\gtrsim 2^d$. First assume $\langle \alpha_0 \rangle \gtrsim 2^d$, then by considering $(f')^{(0,1-\kappa)}$ we can gain a power $2^{(1-\kappa)d}$, and reduce the bound to $\|f'\|_{l^2 L^2} \lesssim 1$. In the same way, we can use the X_1 and X_4 bounds for y to deduce some bound for y' (see Proposition 2.5.8), and strengthen the bound to $\|(y')^{(s^2, 1/2+s^2)}\|_{l^2 L^2} \lesssim 1$ at a price of at most $2^{O(1/2-b)d}$.

We then fix all the m and β variables to get a partial expression that is bounded by (with C being irrelevant constants)

$$\begin{aligned} \mathcal{S}_{sub} &\lesssim \sum_{n_0=n_1+n_2+C} \int_{\widehat{\alpha}_0=\widehat{\alpha}_1+\widehat{\alpha}_2+\widehat{\alpha}_3+C} |(f')_{n_0, \widehat{\alpha}_0}| |\phi_{0, \widehat{\alpha}_3}| \cdot \prod_{l=1}^2 |(y')_{n_l, \widehat{\alpha}_l}^\pm| \\ &\lesssim \left\| |\widehat{f'}| * |\widehat{(y')^\pm}| * |\widehat{(y')^\pm}| * |\widehat{\phi}| \right\|_{l^\infty L^\infty} \\ &\lesssim \|f'\|_{l^2 L^2} \cdot \|\mathfrak{N}y'\|_{L^{6+} L^{6+}} \cdot \|\mathfrak{N}y'\|_{L^3 L^3} \cdot \|\widehat{\phi}\|_{l^1 L^1}. \end{aligned} \quad (2.114)$$

Here $\widehat{\alpha}_l = \alpha_l - |n_l|n_l$, and ϕ is viewed as a function of (t, x) that is supported at $n = 0$ (so that $\widehat{\alpha}_3 = \alpha_3$). The right hand side will be bounded by T^{0+} by our (reduced) assumptions and Strichartz estimates, provided we choose the $6+$ to be $6 + cs^2$ with some small c , and choose $1+$ accordingly.

Now we sum over m_i and integrate β_i , exploiting the bound $\|(u''')^{(-1,0)}\|_{l^1 L^1} \leq C_1 A$, to bound the whole expression for a single d ; taking into account the gains and losses from the reductions made before and exploiting (2.7), we conclude that the part of $\mathcal{S}$ considered above is bounded by $T^{0+} 2^{(0-)^d}$, which allows us to sum over d . This concludes the case when $\langle \alpha_0 \rangle \gtrsim 2^d$. The cases when $\langle \alpha_1 \rangle \gtrsim 2^d$ or $\langle \alpha_2 \rangle \gtrsim 2^d$ are similar, just that we use the bound $\|(f')^{(0,1-\kappa)}\|_{l^2 L^2} \lesssim 1$ to control $\|\mathfrak{N}f'\|_{L^{2+} L^{2+}}$ (with $2+$ being some $2 + c(1-\kappa)$), and then use the bound $\|(y')^{(s^2, \frac{1}{2}+s^2)}\|_{l^2 L^2} \lesssim 1$ to control $\|\mathfrak{N}y'\|_{L^{6-} L^{6-}}$ (with $6-$ being some $6 - c(1-\kappa)$). Choosing the constants c appropriately, we can close these case in the same way as above.

Next, assume that $\langle \beta_i \rangle \gtrsim 2^d$ for some i . If for this i we also have $\langle m_i \rangle \gtrsim 2^{d/30}$, then we would bound $|m_i|^{-1} \lesssim 2^{-\frac{d}{90}} \langle m_i \rangle^{-2/3}$ to gain a power of 2^{cd} and proceed as above, since we still have

$$\|(u''')^{(-2/3,0)}\|_{l^1 L^1} \lesssim \|(u''')^{(-s^3,0)}\|_{X_2} \leq C_0 C_1 A \quad (2.115)$$

which allows us to sum over m_i and integrate over β_i . If $\langle m_i \rangle \lesssim 2^{d/30}$, we could use the X_4 bound of $(u''')^{(-s^3,0)}$ and Proposition 2.5.8 to bound $\|(y')^{(-3/2, 9/10)}\|_{l^2 L^2} \lesssim 1$, and exploit the largeness of

β_i to gain a power $2^{d/20}$ and reduce the above bound to $\|(y')^{(2,3/5)}\|_{l^2 L^2} \lesssim 1$, which would imply $\|y'\|_{l^1 L^1} \lesssim 1$ so that we can still apply the argument above, sum over m_i and integrate over β_i . This concludes the proof of (2.110).

Now let us prove (2.111). As said before, we will only do this for $\mathcal{N}_3^*$. Using exactly the same scheme, we only need to bound

$$\mathcal{S} = \sum_{n_0=n_1+n_2+n_3+\dots+m_\mu} \int_{(T)} \Phi^3 \overline{(f')_{n_0, \alpha_0}} \prod_{l=1}^3 (w')_{n_l, \alpha_l}^\pm \phi_{\alpha_4} \prod_{i=1}^\mu \frac{(u''')_{m_i, \beta_i}}{m_i}, \quad (2.116)$$

with the integration (T) interpreted as the integral over the set

$$\left\{ (\alpha_0, \dots, \alpha_4, \beta_1, \dots, \beta_\mu) : \alpha_0 = \sum_{l=1}^4 \alpha_l + \sum_{i=1}^\mu \beta_i + \Xi \right\}, \quad \Xi = |n_0|n_0 - \sum_{l=1}^3 |n_l|n_l - \sum_{i=1}^\mu |m_i|m_i \quad (2.117)$$

with respect to the standard measure, where we now have $\|(f')^{(1/30, 1-\kappa)}\|_{l^2 L^2} \lesssim 1$.

The estimate of this term goes in basically the same way as with the cubic Schrödinger equation in $1D$; however, here we have the power $|n_0|^{1/30}$ at our disposal. Now if $|n_0| \gtrsim 2^{cd}$, then we gain a small power 2^{cd} which is enough to cover any possible $2^{O(s)d}$ loss, thus we can proceed in the standard way by bounding every term in appropriate spacetime Lebesgue spaces.

If instead $|n_0| \lesssim 2^{cd}$, then either $n_1 + n_2 = 0$ (in which case we again gain a power 2^{cd} from the stronger bound for Φ^3 in (2.95)) or we again have $|\Xi| \gtrsim 2^d$, so at least one α_i or β_i must be at least 2^d , which allows us to gain a suitable power of 2^d from that factor and again close the estimate. Note that if $\langle \beta_i \rangle \gtrsim 2^d$ we have to use the X_4 bound of $(u''')^{(-s^3, 0)}$ and argue as we do with $\mathcal{M}_2$ above. $\square$

Proposition 2.5.11. *We have*

$$\sum_{j \in \{3, 4\}} \sum_{k \in \{1, 2, 5, 7\}} \|\mathcal{M}_j\|_{X_k} \lesssim T^{0+}. \quad (2.118)$$

Proof. Again we only consider $\mathcal{M}_3$. Note that it involves a sum over the n_l and m_i variables; we shall first prove the bound for the terms where $n_0 \notin \{n_1, n_2, n_3\}$, since they will be bounded in the “envelope norm” X_6 .

Let the functions g and f, f' be as before, with $\|g\|_{X'_6} \leq 1$; this would imply

$$\|(f')^{(-s-O(s^3), 1/2-O(s^2))}\|_{l^2 L^2} \lesssim 1.$$

What we need to control is the same quantity $\mathcal{S}$ as in (2.116), and we assume the maximal n_l variable is $\sim 2^d$ as usual. Moreover, with a loss of $2^{O(\epsilon)d}$, we may assume that $\|w'^{(s^2, 1/2+s^2)}\|_{l^2 L^2} \lesssim 1$.

Now to bound $\mathcal{S}$ we take absolute values and get

$$|\mathcal{S}| \lesssim \sum_{n_0=n_1+n_2+n_3+m_1+\dots+m_\mu} \int_{(T)} |(f')_{n_0, \alpha_0}| \prod_{l=1}^3 |(w')_{n_l, \alpha_l}^\pm| \cdot |\phi_{\alpha_4}| \prod_{i=1}^\mu \left| \frac{(u''')_{m_i, \beta_i}}{m_i} \right|. \quad (2.119)$$

If some of the α_i or β_i is at least $2^{d/90}$, then we can gain 2^{cd} from this factor, and proceed in the same way as the proof of Proposition 2.5.9 before; thus we now assume that $|\alpha_i| \ll 2^{d/90}$ and $|\beta_i| \ll 2^{d/90}$, and of course $|m_i| \ll 2^{d/90}$ also.

In this situation we do not have any 2^{cd} gain to cover the various loss which is due to the fact that we are below L^2 , so we have to use the power $\langle n \rangle^s$ in the X_1 norm for w' to gain a power 2^{csd} . This forces us to estimate only in the weaker l^p with $p > 2$; in this regard we have the following lemma.

Lemma 2.5.12. *Suppose that n_0, n_1, n_2, n_3 satisfy*

$$n_0 + n_1 + n_2 + n_3 = K_1; \quad |n_0|n_0 + |n_1|n_1 + |n_2|n_2 + |n_3|n_3 = K_2,$$

where K_j are constants such that

$$|K_1| + |K_2| \ll 2^{d/40}, \quad \max_{0 \leq l \leq 3} \langle n_l \rangle \sim 2^d,$$

then one of the followings must hold.

(i) Up to some permutation, we have $n_0 + n_1 = n_2 + n_3 = 0$. In particular, this can happen only if $K_1 = K_2 = 0$.

(ii) Up to some permutation, we have $n_0 + n_1 = 0$, $\langle n_0 \rangle \sim 2^d$, and that $\langle n_2 \rangle + \langle n_3 \rangle \ll 2^{d/40}$. Note that it is possible that (say) $n_1 + n_2 = 0$ and n_0, n_3 are small.

(iii) No two of n_l add to zero. Under this restriction we must have $\langle n_l \rangle \gtrsim 2^{0.9d}$ for each l ; moreover, if we fix K_1, K_2 and any n_l , there will be at most $\lesssim 2^{s^3d}$ choices for the quadruple (n_0, n_1, n_2, n_3) .

Proof of Lemma 2.5.12. Suppose some $\langle n_l \rangle \ll 2^{0.9d}$ (say $l = 0$), then one of $\langle n_l \rangle$ for $1 \leq l \leq 3$ must also be $\ll 2^{0.9d}$, since otherwise we would have

$$|n_1|n_1 + |n_2|n_2 + |n_3|n_3| \gtrsim \max_{1 \leq l \leq 3} \langle n_l \rangle \cdot \min_{1 \leq l \leq 3} \langle n_l \rangle \gtrsim 2^{1.9d}$$

while $|n_0|^2 \lesssim 2^{1.8d}$, which is impossible. Now assume that $\langle n_1 \rangle \ll 2^{0.9d}$, then in particular $\langle n_2 + n_3 \rangle \ll 2^d$, thus $n_2 n_3 < 0$ as well as $|n_2 - n_3| \sim 2^d$. Suppose $n_0 + n_1 = k$ and $n_2 + n_3 = l$, we will have $|k + l| \leq c2^{d/40}$ and

$$2^d |l| \leq 2^{0.9d} \langle k \rangle + 2^{d/40}.$$

Now if $l \neq 0$, this inequality cannot hold, since it would require $|k| \gg |l|$, which implies $\langle k \rangle \lesssim 2^{d/40}$, so that the right hand side will be at most $2^{(0.9+1/40)d}$ and the left hand side is at least 2^d . Therefore we must have $n_2 + n_3 = 0$. If also $n_0 + n_1 = 0$, we will be in case (i); otherwise $k \neq 0$, so that we always have $||n_0|n_0 + |n_1|n_1| \gtrsim |n_0| + |n_1|$, which then implies that $\langle n_0 \rangle + \langle n_1 \rangle \ll 2^{d/40}$ and we will be in case (ii).

Now assume that $\langle n_l \rangle \gtrsim 2^{0.9d}$ for each l . By the discussion above, we cannot have any $n_h + n_l = 0$ (unless we are in case (i)), so we will be in case (iii). Finally, suppose we fix K_1, K_2 and n_0 . The requirement $n_h + n_l \neq 0$ implies that each $\langle n_l \rangle \gtrsim 2^{0.9d}$, so without loss of generality we may assume $n_0 > 0 > n_1$. Now n_2 and n_3 cannot have the same sign since $|K_1| \lesssim 2^{d/40}$, thus we may assume $n_2 > 0$ and $n_3 < 0$. Therefore we will have

$$n_0 + n_1 + n_2 + n_3 = K_1, \quad n_0^2 - n_1^2 + n_2^2 - n_3^2 = K_2,$$

which implies

$$(n_2 + n_1)(n_2 + n_3) = \frac{1}{2}(K_1^2 - 2K_1 n_0 + K_2).$$

By our assumptions, the right hand side is a nonzero constant whose absolute value does not exceed 2^{2d} . The result now follows from the standard divisor estimate, since knowing $n_2 + n_1$ and $n_2 + n_3$ will allow us to recover the whole quadruple. $\square$

Proceeding to the estimate of $\mathcal{M}_3$, we can see that the only possibility is case (iii) in Lemma 2.5.12 (since we have required $n_0 \notin \{n_1, n_2, n_3\}$; also if $n_1 + n_2 = 0$ and n_0, n_3 are small, we will gain a power 2^{cd} from the weight Φ so we can argue as above to close the estimate).

Recall that up to a loss of $2^{O(s^3)d}$, with small c , we have $\|(f')^{(-s, 1/2-c)}\|_{l^2 L^2} \lesssim 1$; also by invoking the X_1 norm of w we obtain the estimate $\|(w')^{(s, 1/2-c)}\|_{l^p L^2} \lesssim 1$ with a loss of at most $2^{O(s^3)d}$. Since we now have $2^{0.9d} \lesssim n_l \lesssim 2^d$, we may replace the $\pm s$ parameters in the above two bounds by 0, and through this process we gain at least $2^{1.7sd}$. Therefore, by fixing m_i and β_i first, we will be

able to get the desired result if we can prove the following inequality:

$$\mathcal{S}_{sub} = \sum_{n_0=n_1+n_2+n_3+K_1} \int_{(T)} \prod_{l=0}^3 |(A^l)_{n_l, \alpha_l}| \cdot \min \left\{ T, \frac{1}{\langle \alpha_4 \rangle} \right\} \lesssim 2^{O(s^3)d} T^{0+} \prod_{l=0}^3 \|(A^l)^{(0, 1/2-c)}\|_{l^{2+c}L^2}, \quad (2.120)$$

provided c is a small absolute constant, where the integral (T) is taken over the set

$$\left\{ (\alpha_0, \dots, \alpha_4) : \alpha_0 = \alpha_{14} + |n_0|n_0 - \sum_{l=1}^3 |n_l|n_l + K_2 \right\}, \quad (2.121)$$

and we restrict to the region where no two of $(-n_0, n_1, n_2, n_3)$ add to zero, $\sum_l \langle n_l \rangle \sim 2^d$, the NR factor $|n_0|n_0 - \sum_{l=1}^3 |n_l|n_l| \ll 2^{\frac{d}{40}}$ and $|K_1| + |K_2| \ll 2^{\frac{d}{40}}$.

We will use an interpolation argument to prove (2.120); in fact, it will suffice to prove the estimate when we replace the parameter set $(1/2, 2+c)$ with $(2/5, 2)$ or $(3, 4)$. When we have $(\frac{2}{5}, 2)$ we will be able to control $\Re A^l$ in $L^{4+}L^{4+}$ for each l , so that we can control the α_4 factor in $l^{1+}L^{1+}$ to conclude. When we have $(3, 4)$, assuming the norm of each A^l is one, we will get that

$$\|\langle \hat{\alpha}_l + |n_l|n_l \rangle \mathcal{F}(A^l)_{n_l}(\hat{\alpha}_l)\|_{L^1} \lesssim \|\langle \hat{\alpha}_l + |n_l|n_l \rangle^3 \mathcal{F}(A^l)_{n_l}(\hat{\alpha}_l)\|_{L^2} := A_{n_l}^l,$$

with

$$\|A_{n_l}^l\|_{l^4} \lesssim \|(A^l)^{(0,3)}\|_{l^4L^2} \lesssim 1. \quad (2.122)$$

Therefore when we fix $(n_0, \dots, n_3)$ and integrate over $(\alpha_0, \dots, \alpha_4)$, we get

$$\begin{aligned} \mathcal{S}'_{sub} &\lesssim T^{0+} \int_{\mathbb{R}^4} \left\langle \hat{\alpha}_0 - \sum_{l=1}^3 \hat{\alpha}_l - K_2 \right\rangle^{-1+s^3} \prod_{l=0}^3 \langle \hat{\alpha}_l + |n_l|n_l \rangle^{-1} \prod_{l=0}^3 \langle \hat{\alpha}_l + |n_l|n_l \rangle |\mathcal{F}(A^l)_{n_l}(\hat{\alpha}_l)| \prod_{l=1}^4 d\hat{\alpha}_l \\ &\lesssim T^{0+} \left\langle |n_0|n_0 - \sum_{l=1}^3 |n_l|n_l + K_2 \right\rangle^{-1+s^3} \prod_{l=0}^3 A_{n_l}^l. \end{aligned}$$

We then sum this over (n_l) ; by Hölder, we only need to bound the sum

$$\sum_{(n_0, \dots, n_3)} \left\langle |n_0|n_0 - \sum_{l=1}^3 |n_l|n_l + K_2 \right\rangle^{-1+s^4} (A_{n_0}^0)^4.$$

If we fix $|n_0|n_0 - \sum_{l=1}^3 |n_l|n_l = K_3$ with $|K_3| \ll 2^{\frac{d}{40}}$, the corresponding sum will be $\lesssim 2^{O(s^3)d}$, since each n_0 appears at most this many times due to Lemma 2.5.12; also the sum over K_3 will contribute at most $\sum_{|K_3| \lesssim 2^{\frac{d}{40}}} \langle K_3 - K_2 \rangle^{-1+s^3} = 2^{O(s^3)d}$. This completes the interpolation, and thus the proof for this part of $\mathcal{M}_3$.

What remains is when (say) $n_0 = n_1$. Here we will use the expression (2.116), but with f' replaced by f , and $(w')^\pm$ replaced by $(w'')^\pm$. This is easily justified by definition and the fact that $n_0 = n_1$. We will assume $\langle n_2 \rangle + \langle n_3 \rangle \sim 2^{d'}$, then fix d and d' . We will also use a new bound for f ; recall from Proposition 2.3.4 that $\|g\|_{X'_j} \lesssim 1$ for some $j \in \{1, 2, 5, 7\}$ implies $\|f\|_{X'_8} \lesssim 1$, or equivalently

$$\|f\|_{L^q_{l^{p'}_{\sim 2^d}}} \lesssim 2^{rd} T_d, \quad (2.123)$$

where the T_d is such that $\sum_{d \geq 0} T_d \lesssim 1$.

The bound (2.123), together with the bound $\|w''\|_{X^2} \lesssim 1$, basically implies that in the expression $\mathcal{S}$, the sum over n_1 will be exactly bounded, *without any loss in terms of 2^d* . We may only lose some power in terms of $2^{d'}$, thus we can assume $|m_i| \ll 2^{d'/90}$ and $|\beta_i| \ll 2^{d'/90}$, or we gain $2^{cd'}$ and can close the proof with ease. In the same way, we may assume $\langle \alpha_4 \rangle \ll 2^{d'/90}$ in (2.116). Now if $n_2 + n_3 = 0$, we must have $|\Phi| \lesssim 2^{-|d-d'|}$ due to (2.95). Also we may replace the other two w' factors in (2.116) by w'' (in the same way we replace f' and the first w' by f and w''). Then we may fix m_i and β_i and bound $\mathcal{S}_{sub}$ by

$$\begin{aligned} \mathcal{S}_{sub} &\lesssim 2^{-|d-d'|} \sum_{n_0, n_2} \int_{\alpha_0 = \alpha_1 + \dots + \alpha_4 + c_2} |f_{n_0, \alpha_0}| |(w'')^\pm_{n_0, \alpha_1}| \cdot |(w'')^\pm_{n_2, \alpha_2}| \cdot |(w'')^\pm_{-n_2, \alpha_3}| \cdot |\phi_{\alpha_4}| \\ &\lesssim T^{0+} 2^{-|d-d'|} \sum_{n_0, n_2} \|\widehat{f_{n_0}}\|_{L^q} \|\widehat{(w'')^\pm_{n_0}}\|_{L^1} \|\widehat{(w'')^\pm_{n_2}}\|_{L^1} \|\widehat{(w'')^\pm_{-n_2}}\|_{L^1} \\ &\lesssim T^{0+} 2^{-|d-d'|} \|f\|_{l^{p'}_{\sim 2^d} L^q} \|w''\|_{l^p_{\sim 2^d} L^1} \|w''\|_{l^2_{\sim 2^{d'}} L^1} \|w''\|_{l^2_{\sim 2^{d'}} L^1} \\ &\lesssim T^{0+} 2^{-|d-d'|} T_d, \end{aligned}$$

where c_j are constants, and we are restricting to $n_0 \sim 2^d$, $n_2 \sim 2^{d'}$. Then we may sum and integrate over (m_i, β_i) , and sum over d, d' to bound this part by T^{0+} .

Finally, when $n_2 + n_3 \neq 0$, we must have $n_2 \sim n_3 \sim 2^{d'}$ and $|\Xi'| \gtrsim 2^{d'}$ where $\Xi' = |n_2|n_2 + |n_3|n_3$, so at least one $\langle \alpha_l \rangle$ must be at least $2^{d'}$. To exploit this, we then simply use the bound

$$\|f\|_{X'_9} = \|f^{(-r, 1/8)}\|_{l^1_{d \geq 0} L^{r'}_{l^{p'}_{n_0 \sim 2^d}}} \lesssim 1 \quad (2.124)$$

for f , which is deduced from Proposition 2.3.4 and the fact that $\|g\|_{X'_j} \leq 1$ for some $j \in \{1, 2, 5, 7\}$, and X_7 bound for the w'' corresponding to n_1 . In this way we gain $2^{cd'}$, which (since we do not lose any power in terms of 2^d) easily allows us to close the estimate. This completes the proof for the $n_0 = n_1$ case. $\square$

Proposition 2.5.13. *We have*

$$\sum_{j \in \{1, 2, 5, 7\}} \|\mathcal{M}_2\|_{X^j} \lesssim T^{0+}. \quad (2.125)$$

Proof. Fixing the functions g, f, f' and the scale d as before, we need to bound the expression

$$\mathcal{S} = \sum_{n_0=n_1+n_2+m_1+\dots+m_\mu} \int_{(T)} \Phi^2 \cdot \overline{f_{n_0, \alpha_0}} \prod_{l=1}^2 (y^\pm)_{n_l, \alpha_l} \cdot \phi_{\alpha_3} \mathcal{F}(\chi e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})})(\alpha_4) \prod_{i=1}^\mu \frac{(u''')_{m_i, \beta_i}}{m_i}. \quad (2.126)$$

Here the integration (T) is over the set

$$\left\{ (\alpha_0, \dots, \alpha_4, \beta_1, \dots, \beta_\mu) : \alpha_0 = \sum_{l=1}^4 \alpha_l + \sum_{i=1}^\mu \beta_i + \Xi \right\}, \quad \Xi = |n_0|n_0 - \sum_{l=1}^2 |n_l|n_l - \sum_{i=1}^\mu |m_i|m_i.$$

Note that we may insert χ since f has compact time support.

The main difficulty here comes from the weight Φ^2 , which we plan to cancel by some α_l or β_i which is comparable to Ξ . Suppose that $\min_{0 \leq l \leq 2} \langle n_l \rangle \sim 2^h$ and also fix h , then we have $\langle m_i \rangle \ll 2^h$, so that $|\Xi| \gtrsim 2^{d+h}$; also we have $h \leq d + O(1)$ and $|\Phi^2| \lesssim 2^h$. Note that one of α or β variables must be $\gtrsim 2^{d+h}$; we first treat the easy cases, which we collect in the following lemma.

Lemma 2.5.14. *Let $\mathcal{S}$ be defined in (2.126), where all the restrictions made above are assumed.*

Then if we have $h < 0.9d$, or

$$\langle \alpha_0 \rangle + \langle \alpha_3 \rangle + \langle \alpha_4 \rangle + \sum_{i=1}^\mu (\langle m_i \rangle + \langle \beta_i \rangle) \gtrsim 2^{d/90}, \quad (2.127)$$

the corresponding contribution will be bounded by $T^{0+}2^{(0-)d}$.

Proof. First assume $h \geq 0.9d$. If $\langle \beta_i \rangle \gtrsim 2^{d+h}$ for some i , we may use the X_4 bound for $(u''')^{(-s^3, 0)}$ to gain a power $2^{0.99(d+h)}$ and then estimate this $(u''')_{m_i, \beta_i}$ factor in $L^2 L^2$. Next we may bound

$$|\mathcal{F}(\chi e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})})(\alpha_4)| \lesssim 2^{s^3 d} \langle \alpha_4 \rangle^{-1} \quad (2.128)$$

by Proposition 2.5.6 and Lemma 2.5.7, estimate the right hand side (again, viewed as a function of space-time supported at $n = 0$) and the ϕ_{α_3} factor in $l^{1+} L^{1+}$. We then fix (m_j, β_j) for $j \neq i$ to produce an $\mathcal{S}_{sub}$ involving $(u''')_{m_i, \beta_i}$, which we estimate by controlling $\mathfrak{N}f$ in $L^{6-} L^{6-}$, $\mathfrak{N}y$ in $L^{6+} L^{6+}$ (using the X_1 bound for y and the norm for f deduced from the X'_6 bound for g ; here the $6-$ and $6+$ are $6 + O(s)$). In this process we lose at most $2^{O(s)d}$, but the gain $2^{0.99(d+h)}$ (after canceling the 2^h loss from the Φ^2 weight) will allow us to cancel the Φ^2 factor and still gain 2^{cd} .

Next, suppose $\langle \alpha_3 \rangle \gtrsim 2^{d+h}$. By using (2.128) and losing a harmless $2^{O(s)d}$ factor, the argument for α_4 can be done in the same way. Let c_j be constants (or functions of n_l), and recall we are restricting to $\sum_l \langle n_l \rangle \sim 2^d$, we may fix m_i and β_i , and bound the $\mathcal{S}_{sub}$ term by

$$\begin{aligned}
\mathcal{S}_{sub} &\lesssim T^{0+} \sum_{n_0=n_1+n_2+c_1} \int_{\alpha_0=\alpha_1+\dots+\alpha_4+c_2(n_0,n_1,n_2)} 2^h |f_{n_0,\alpha_0}| \cdot \prod_{l=1}^2 |y_{n_l,\alpha_l}^\pm| \cdot \frac{\mathbf{1}_{\{|\alpha_3| \gtrsim 2^{d+h}\}}}{\langle \alpha_3 \rangle^{0.9} \langle \alpha_4 \rangle} \\
&\lesssim T^{0+} \sum_{n_0=n_1+n_2+c_1} 2^{-0.62d} \|\widehat{f_{n_0}}\|_{L^q} \|\widehat{y_{n_1}^\pm}\|_{L^1} \|\widehat{y_{n_2}^\pm}\|_{L^1} \\
&\lesssim T^{0+} 2^{-cd} \|f^{(-0.2,0)}\|_{l^{3/2}L^q} \|y^{(-0.2,0)}\|_{l^{3/2}L^1} \|y^{(-0.2,0)}\|_{l^{3/2}L^1} \\
&\lesssim T^{0+} 2^{-cd},
\end{aligned}$$

thus this term is also acceptable.

Next, assume that $\langle \alpha_1 \rangle \gtrsim 2^{d+h}$ (the α_2 case is proved in the same way), and that one of $\alpha_0, \alpha_3, \alpha_4, m_i$ or β_i is $\gtrsim 2^{\frac{d}{90}}$. We then cancel the Φ^2 weight by using the X_1 norm for the $y^\pm$ factor associated with n_1 and using the first assumption (so that we lose $2^{O(s)d}$), and gain 2^{cd} from the second assumption. We then bound the exponential factor by (2.128) and each function in appropriate spacetime Lebesgue spaces, and exploit this 2^{cd} gain to conclude.

The only remaining case is when $\langle \alpha_0 \rangle \gtrsim 2^{d+h}$. By basically the same argument as above, we may assume that the other α_l and (m_i, β_i) are all $\ll 2^{d/90}$. Also recall that two of n_l ($0 \leq l \leq 2$) are $\sim 2^d$ and the third is $\sim 2^h$. Now we may use the bound (2.128), then fix (α_3, α_4) and all (m_i, β_i) to produce

$$|\mathcal{S}_{sub}| \lesssim 2^{(b-s)h-(1-b)d} \sum_{n_0=n_1+n_2+c_1} \int_{\alpha_0=\alpha_1+\alpha_2+\Xi'+c_2} A_{n_0,\alpha_0} B_{n_1,\alpha_1} C_{n_2,\alpha_2}, \quad (2.129)$$

where $c_j \ll 2^{d/10}$ are constants, $\Xi' = |n_0|n_0 - |n_1|n_1 - |n_2|n_2$, and the relevant functions are defined by

$$A = \mathfrak{N}f^{(-s,1-b)}, \quad B = \mathfrak{N}(y^{\omega_1})^{(s,0)}, \quad C = \mathfrak{N}(y^{\omega_2})^{(s,0)}. \quad (2.130)$$

Also note that when we sum over m_i , integrate over β_i and (α_3, α_4) , we will gain T^{0+} and lose at most $2^{O(s^3)d}$.

Now we estimate $\mathcal{S}_{sub}$. If $\|g\|_{X'_m} \leq 1$ for some $m \in \{1, 2\}$, by using Proposition 2.3.4, we may assume $\|f^{(0,1)}\|_{X'_j} \lesssim 1$ for some $j \in \{1, 2\}$ (this relies on the fact that E can be written as the sum of two linear operators that are bounded from each W_j to X_1 separately, where $\|u\|_{W_j} = \|u^{(0,-1)}\|_{X_j}$). If $\|g\|_{X'_m} \lesssim 1$ for some $m \in \{5, 7\}$, since we may insert a $\mathbf{1}_E$ factor to $\overline{f_{n_0,\alpha_0}}$ with

$E = \{|n_0| \sim 2^{d'}, |\alpha_0| \gtrsim 2^{d'}\}$ with $d' \in \{d, h\}$, we can use (2.59) and again assume $\|f^{(0,1)}\|_{X'_j} \lesssim 1$ for some $j \in \{1, 2\}$. Next, notice that $|\alpha_0 - \Xi'| \lesssim 2^{d/10}$, so α_0 is also restricted to some set of measure $O(2^{1.1d})$ for each fixed n_0 . Since α_0 is restricted to be $\gtrsim 2^{1.9d}$ and $|n_0| \lesssim 2^d$, we will have

$$\begin{aligned} \|f^{(0,1)}\|_{X'_1} &\lesssim 2^{O(s)d} \|f^{(0,0.6)}\|_{l^2 L^2} \lesssim 2^{(O(s)+0.55)d} \|f^{(0,0.6)}\|_{l^2 L^\infty} \\ &\lesssim 2^{(0.6-0.4 \times 1.9)d} \|f^{(0,1)}\|_{l^2 L^\infty} \lesssim 2^{-cd} \|f^{(0,1)}\|_{X'_2}, \end{aligned}$$

thus we may furthermore assume $j = 1$.

Now, using this bound for f and the X_1 bound for y , we deduce that

$$\|A\|_{l^{p'} L^2} + \|B^{(0,b)}\|_{l^p L^2} + \|C^{(0,b)}\|_{l^p L^2} \lesssim 2^{O(s^2)d}.$$

Let us define

$$\mathbf{B}_{n_1} = \|\langle \widehat{\alpha_1} + |n_1|n_1 \rangle^b \widehat{B_{n_1}}(\widehat{\alpha_1})\|_{L^2}$$

and $\mathbf{C}_{n_3}$ similarly, so that $\|\mathbf{B}\|_{l^p} + \|\mathbf{C}\|_{l^p} \lesssim 2^{O(s^2)d}$, then we will have the estimate

$$\|\langle \alpha_0 - \Xi' - c_2 \rangle^{2b-\frac{1}{2}} (\widehat{B_{n_1}} * \widehat{C_{n_2}})(\alpha_0 - |n_0|n_0 - c_2)\|_{L^2_{\alpha_0}} \lesssim \mathbf{B}_{n_1} \mathbf{C}_{n_2},$$

which, after taking Fourier transform, follows from the standard one dimensional inequality $\|fg\|_{H^{2b-\frac{1}{2}}} \lesssim \|f\|_{H^b} \|g\|_{H^b}$. Now we will be able to control $\mathcal{S}_{sub}$ by

$$\mathcal{S}_{sub} \lesssim 2^\lambda \left(\sum_{n_0} \left\| \sum_{n_1+n_2=n_0-c_1} (\widehat{B_{n_1}} * \widehat{C_{n_2}})(\alpha_0 - |n_0|n_0 - c_2) \right\|_{L^2_{\alpha_0}}^p \right)^{\frac{1}{p}},$$

where $\lambda = (b-s)h + (b-1 + O(s^2))d$, and the square of the inner L^2 norm is bounded by

$$\begin{aligned}
\mathcal{J}^2 &= \int_{\mathbb{R}} \left| \sum_{n_1+n_2=n_0-c_1} (\widehat{B_{n_1}} * \widehat{C_{n_2}})(\alpha_0 - |n_0|n_0 - c_2) \right|^2 d\alpha_0 \\
&\lesssim \int_{\mathbb{R}} \left(\sum_{n_1+n_2=n_0-c_1} \langle \alpha_0 - \Xi' - c_2 \rangle^{1-4b} \right) \\
&\quad \times \left(\sum_{n_1+n_2=n_0-c_1} \langle \alpha_0 - \Xi' - c_2 \rangle^{4b-1} |(\widehat{B_{n_1}} * \widehat{C_{n_2}})(\alpha_0 - |n_0|n_0 - c_2)|^2 \right) \\
&\lesssim \sup_{\alpha_0} \left(\sum_{n_1+n_2=n_0-c_1} \langle \alpha_0 - \Xi' - c_2 \rangle^{1-4b} \right) \\
&\quad \times \sum_{n_1+n_2=n_0-c_1} \int_{\mathbb{R}} \langle \alpha_0 - \Xi' - c_2 \rangle^{4b-1} |(\widehat{B_{n_1}} * \widehat{C_{n_2}})(\alpha_0 - |n_0|n_0 - c_2)|^2 d\alpha_0 \\
&\lesssim \sup_{\alpha_0} \left(\sum_{n_1+n_2=n_0-c_1} \langle \alpha_0 - \Xi' - c_2 \rangle^{1-4b} \right) \cdot \sum_{n_1+n_2=n_0-c_1} \mathbf{B}_{n_1}^2 \mathbf{C}_{n_2}^2.
\end{aligned}$$

Next we claim that for fixed n_0 and α_0 we have

$$\sum_{n_1+n_2=n_0-c_1} \langle \alpha_0 - \Xi' - c_2 \rangle^{-\frac{3}{4}} \lesssim 1. \quad (2.131)$$

In fact, if $n_1 n_2 < 0$, then $\alpha_0 - \Xi' - c_2$ is a linear expression in n_1 with leading coefficient $k = \pm(n_0 - c_1)/2 \gtrsim 2^{0.9d}$ (in absolute value), so any two summands in (2.131) differ by at least k , while there are $\lesssim 2^d$ summands. The sum is thus bounded by

$$1 + \sum_{h=1}^{2^d} (kh)^{-\frac{3}{4}} \lesssim 1 + k^{-\frac{3}{4}} 2^{\frac{d}{4}} \lesssim 1. \quad (2.132)$$

If $n_1 n_2 > 0$, then $\alpha_0 - \Xi' - c_2$ equals $\pm \frac{1}{2}(n_1 - n_2)^2$ plus a constant, so similarly we only need to prove

$$\sum_{k \in \mathbb{Z}} \langle \alpha - k^2 \rangle^{-\frac{3}{4}} \lesssim 1$$

for each α , but this is again easily proved by separating the cases $\langle k \rangle^2 \lesssim \langle \alpha \rangle$ and otherwise, and applying elementary inequalities.

Now we are able to bound

$$\begin{aligned}
\mathcal{S}_{sub} &\lesssim 2^\lambda \left(\sum_{n_0} \left(\sum_{n_1+n_2=n_0-c_1} \mathbf{B}_{n_1}^2 \mathbf{C}_{n_2}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \lesssim 2^{\lambda + (\frac{1}{2} - \frac{1}{p})d} \left(\sum_{n_0} \sum_{n_1+n_2=n_0-c_1} \mathbf{B}_{n_1}^p \mathbf{C}_{n_2}^p \right)^{\frac{1}{p}} \\
&\lesssim 2^{\lambda + (\frac{1}{2} - \frac{1}{p})d} \|\mathbf{B}\|_{l^p} \|\mathbf{C}\|_{l^p},
\end{aligned}$$

where we notice that

$$\lambda + \left(\frac{1}{2} - \frac{1}{p}\right)d = (b-s)h + \left(b - \frac{1}{2} - \frac{1}{p} + O(s^2)\right)d \leq (2b-1)d + \left(\frac{1}{2} - s - \frac{1}{p} + O(s^2)\right)d,$$

and this is $\leq -c(1/2 - b)d$ by (2.7). We may then sum and integrate over the previously fixed variables to get a desirable estimate for $\mathcal{S}$.

Finally, suppose $h < 0.9d$. Since at least one α_l or β_i will be $\gtrsim 2^{d+h}$, we may repeat the arguments above; using the inequality $2^{b(d+h)} \gtrsim 2^{cd+h}$ that holds for $h < 0.9d$, we will be able to gain an additional power of 2^{cd} after canceling the Φ^2 weight, which will allow us to close the estimate as above. This completes the proof. $\square$

With Lemma 2.5.14, what remains to be bounded, denoted by $\mathcal{S}^E$, is actually the same expression as $\mathcal{S}$, but restricted to the region $h \geq 0.9d$ and with the additional factor $\mathbf{1}_E$, where

$$E = \{\langle \alpha_0 \rangle \vee \langle \alpha_3 \rangle \vee \langle \alpha_4 \rangle \vee \langle m_i \rangle \vee \langle \beta_i \rangle \ll 2^{\frac{d}{90}}, \forall i\},$$

with $a \vee b := \max\{a, b\}$. Now let $E_l = \{\langle \alpha_l \rangle \ll 2^{\frac{d}{90}}\}$ for $l \in \{1, 2\}$, we have

$$\mathbf{1}_E = \mathbf{1}_{E \cap E_1} + \mathbf{1}_{E \cap E_2} + \mathbf{1}_{E - (E_1 \cup E_2)}. \quad (2.133)$$

By symmetry, we need to bound $\mathcal{S}^{E \cap E_1}$ and $\mathcal{S}^{E - (E_1 \cup E_2)}$ (whose meaning is obvious). In the latter case, we may assume that $|\alpha_1| \gtrsim 2^{d+h}$, and also $|\alpha_2| \gtrsim 2^{\frac{d}{90}}$, so we can estimate this part in the same was as in the proof of Proposition 2.5.14.

It remains to bound $\mathcal{S}^{E \cap E_1}$. Let $E \cap E_1 = F$, using (2.50) and (2.102) we may compute

$$\begin{aligned} (y^\pm)_{n_2, \alpha_2} &= (\chi(t)e^{-H\partial_{xx}}w^\pm(0))_{n_2, \alpha_2} + (\mathcal{E}(\mathbf{1}_{[-T, T]} \cdot \mathcal{N}_2^*(y^\pm, y^\pm))^\pm)_{n_2, \alpha_2} + \sum_{j \in \{3, 4\}} (\mathcal{E}(\mathbf{1}_{[-T, T]} \mathcal{N}_j^*)^\pm)_{n_2, \alpha_2} \\ &= \sum_{j \in \{0, 3, 4\}} ((\mathcal{M}^j)^\pm)_{n_2, \alpha_2} + (\mathcal{L}^1)_{n_2, \alpha_2} + (\mathcal{L}^2)_{n_2, \alpha_2}. \end{aligned} \quad (2.134)$$

Here we denote $\mathcal{M}^0 = \chi(t)e^{i\partial_{xx}}w(0)$, and

$$\begin{aligned} (\mathcal{L}^1)_{n_2, \alpha_2} &= c_1 \int_{\mathbb{R}^2} \frac{\widehat{\chi}(\alpha_2 - \gamma_2) \widehat{\chi}(\gamma_2 - \gamma_1)}{\gamma_2} \mathcal{I}_{n_2, \gamma_1} d\gamma_1 d\gamma_2; \\ (\mathcal{L}^2)_{n_2, \alpha_2} &= c_2 \widehat{\chi}(\alpha_2) \cdot \int_{\mathbb{R}^2} \frac{\widehat{\chi}(\gamma_2 - \gamma_1)}{\gamma_2} \mathcal{I}_{n_2, \gamma_1} d\gamma_1 d\gamma_2, \end{aligned}$$

where $\mathcal{I} = (\mathbf{1}_{[-T, T]} \mathcal{N}_2^*(y^\pm, y^\pm))^\pm$. Interpreting the singular integral as a principal value, we may

compute that

$$\begin{aligned} \left| \int_{\mathbb{R}} \frac{\widehat{\chi}(\gamma_2 - \gamma_1)}{\gamma_2} d\gamma_2 \right| &\lesssim \frac{1}{\langle \gamma_1 \rangle}; \\ \left| \int_{\mathbb{R}} \frac{\widehat{\chi}(\alpha_2 - \gamma_2) \widehat{\chi}(\gamma_2 - \gamma_1)}{\gamma_2} d\gamma_2 \right| &\lesssim \frac{1}{(\langle \alpha_2 \rangle + \langle \gamma_1 \rangle) \langle \alpha_2 - \gamma_1 \rangle^{1/s}}; \\ \left| \nabla_{\alpha_2, \gamma_1} \int_{\mathbb{R}} \frac{\widehat{\chi}(\alpha_2 - \gamma_2) \widehat{\chi}(\gamma_2 - \gamma_1)}{\gamma_2} d\gamma_2 \right| &\lesssim \frac{1}{(\langle \alpha_2 \rangle + \langle \gamma_1 \rangle)^2 \langle \alpha_2 - \gamma_1 \rangle^{1/s}}, \end{aligned}$$

where the third inequality can be proved by integrating by parts in γ_2 . Now, to treat the first three terms in (2.134), we may use Proposition 2.5.9, the easy observation that

$$\|(\mathcal{M}^0)^{(-1/20, \kappa)}\|_{l^2 L^2} \lesssim \|\langle n \rangle^{-1/20} (w(0))_n\|_{l^2} \lesssim 1,$$

together with the following

Lemma 2.5.15. *If we consider the sum (2.126) with the factor $\mathbf{1}_F$, and one of the $y^\pm$ replaced by some function ζ verifying $\|\zeta^{(-1/20, \kappa)}\|_{l^2 L^2} \lesssim 1$, then this contribution can be bounded by T^{0+} .*

Proof. Since in F we will have $\langle \alpha_2 \rangle \gtrsim 2^{d+h}$, we can gain a power $2^{0.999(d+h)}$ from the $\langle \alpha_2 \rangle^\kappa$ factor in the bound for ζ . After exploiting this, we may then estimate ζ in $L^2 L^2$ with a loss $2^{(\frac{1}{20} + O(s))d}$. Then we fix (m_i, β_i) as usual, and use the inequality (2.128) to bound the factor involving α_4 . To bound the resulting $\mathcal{S}_{sub}$ term, we estimate ζ in $L^2 L^2$, $\mathfrak{N}f$ and $\mathfrak{N}y$ in $L^{4+} L^{4+}$ (where $4+$ equals $4 + c$) with a loss of $2^{O(s)d}$, the α_3 and α_4 factors in $l^{1+} L^{1+}$. Note that here we will gain a power T^{0+} , and the total power of 2^d we may lose is at most $2^{(1.1 + O(s))d}$, which is smaller than the gain $2^{0.999(d+h)}$. Then we sum over m_i and integrate over β_i to conclude. $\square$

Next consider the contribution of $\mathcal{L}^2$. Since we are in F (thus $|\alpha_2| \gtrsim 2^d$), the gain from $\widehat{\chi}(\alpha_2)$ will overwhelm any possible loss in terms of 2^d . Therefore we may even fix all the n, m and β variables and estimate the integral in α variables and γ_1 only; but we can easily estimate this integral by controlling all the factors except $\langle \gamma_1 \rangle^{-1} |\mathcal{I}_{n_2, \gamma_1}|$ in L^{1+} (since the expression now has a convolution structure in the α variables), and estimate the $\langle \gamma_1 \rangle^{-1} |\mathcal{I}_{n_2, \gamma_1}|$ factor in L^1 . This last estimate is due to (the proof of) Proposition 2.5.9, which implies

$$\|\langle \gamma_1 \rangle^{-1} \mathcal{I}_{n_2, \gamma_1}\|_{L^1} \lesssim \|\langle \gamma_1 \rangle^{\kappa-1} \mathcal{I}_{n_2, \gamma_1}\|_{L^2} \lesssim 2^{O(1)d}.$$

It then remains to bound the $\mathcal{L}^1$ contribution. After integrating over γ_2 , we may rename the

variable $\alpha_2 - \gamma_1$ as γ_2 , and reduce to estimating (up to a constant)

$$\begin{aligned} \mathcal{S}^F = & \sum_{n_0=n_1+n_2+m_1+\dots+m_\mu} \int_{(T)} \mathbf{1}_F \cdot \Phi^2 \cdot \overline{f_{n_0, \alpha_0}} \cdot (y^\pm)_{n_1, \alpha_1} \cdot \phi_{\alpha_3} \\ & \times \mathcal{F}(\chi e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})})(\alpha_4) \cdot \prod_{i=1}^{\mu} \frac{(u''')_{m_i, \beta_i}}{m_i} \cdot \eta(\gamma_1, \gamma_2) \cdot \mathcal{I}_{n_2, \gamma_1}, \end{aligned}$$

where η is some function bounded by

$$|\eta(\gamma_1, \gamma_2)| \lesssim \frac{1}{\langle \gamma_1 \rangle \langle \gamma_2 \rangle^{1/s}}; \quad |\partial_{\gamma_1} \eta(\gamma_1, \gamma_2)| \lesssim \frac{1}{\langle \gamma_1 \rangle^2 \langle \gamma_2 \rangle^{1/s}}.$$

and the integral (T) is taken over the set

$$\left\{ (\alpha_0, \alpha_1, \alpha_3, \alpha_4, \beta_1, \dots, \beta_\mu, \gamma_1, \gamma_2) : \alpha_0 = \alpha_1 + \alpha_3 + \alpha_4 + \sum_{i=1}^{\mu} \beta_i + \gamma_1 + \gamma_2 + \Xi \right\},$$

with the NR factor

$$\Xi = |n_0|n_0 - \sum_{l=1}^2 |n_l|n_l - \sum_{i=1}^{\mu} |m_i|m_i. \quad (2.135)$$

Clearly we may also assume $\langle \gamma_2 \rangle \ll 2^{d/90}$ and add this restriction into F , so we now have $F \subset \{\langle \gamma_1 \rangle \gtrsim 2^{d+h}\}$.

Next, note that $\overline{\mathcal{N}_2^*(y, y)} = \overline{\mathcal{N}_2^*(\bar{y}, \bar{y})}$, where $\overline{\mathcal{N}_2^*}$ is another bilinear form with the same properties as $\mathcal{N}_2^*$; thus we only need to bound the above expression with $\mathcal{I}_{n_2, \gamma_1}$ replaced by $(\mathbf{1}_{[-T, T]} \mathcal{N}_2^*(y^\pm, y^\pm))_{n_2, \gamma_1}$.

Thus we reduce to estimating

$$\begin{aligned} \mathcal{S}' = & \sum_{n_0=n_1+n_5+n_6+m_1+\dots+m_\nu} \int_{(T)} \Phi^2 (\Phi^2)' \cdot \overline{f_{n_0, \alpha_0}} \prod_{l \in \{1, 5, 6\}} (y^\pm)_{n_l, \alpha_l} \cdot \phi_{\alpha_3} \phi_{\alpha_7} \\ & \times \prod_{i=1}^{\nu} \frac{(u''')_{m_i, \beta_i}}{m_i} \int_{\alpha_4 + \alpha_8 = \alpha_9} \mathbf{1}_F \eta(\gamma_1, \gamma_2) \cdot \mathcal{F}(\chi e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})})(\alpha_4) \mathcal{F}(\chi e^{i(\Delta_{n_5} + \Delta_{n_6} - \Delta_{n_2})})(\alpha_8), \end{aligned}$$

where $\nu = \mu + \mu'$, the integral (T) is taken over the set

$$\left\{ (\alpha_0, \alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \beta_1, \dots, \beta_\nu, \gamma_2) : \alpha_0 = \alpha_1 + \alpha_3 + \sum_{l=5}^7 \alpha_l + \alpha_9 + \sum_{i=1}^{\mu} \beta_i + \gamma_2 + \Xi' \right\},$$

with the new NR factor

$$\Xi' = |n_0|n_0 - |n_1|n_1 - |n_5|n_5 - |n_6|n_6 - \sum_{i=1}^{\nu} |m_i|m_i.$$

The Φ^2 and $(\Phi^2)'$ are functions of the n and m variables that are bounded by $\min_{l \in \{0,1,2\}} \langle n_l \rangle$ and $\min_{l \in \{2,5,6\}} \langle n_l \rangle$ respectively. The other implicit variables are $n_2 = n_5 + n_6 + m_{\mu+1,\nu}$ and

$$\gamma_1 = \alpha_0 - \alpha_1 - \alpha_3 - \alpha_4 - \sum_{i=1}^{\mu} \beta_i - \gamma_2 - \Xi = \sum_{l=5}^8 \alpha_l + \sum_{i=\mu+1}^{\nu} \beta_i + (\Xi' - \Xi),$$

where Ξ is the same as in (2.135). Also recall from the definition of $\mathcal{N}_2^*$ that $n_0 \neq n_1$ and $n_5 + n_6 \neq 0$.

The point with the expression $\mathcal{S}'$ is that the $\Phi^2(\Phi^2)'$ weight is now cancelled by the η factor (which is at most $2^{-(d+h)}$) with *no* loss. Note that this cannot be achieved without going to this second iteration. Now the remaining part of the expression has basically the same form as that appeared in the proof of Proposition 2.5.11 when we estimate $\mathcal{M}_3$, namely (2.119). Therefore, unless $n_0 = n_5$ and $n_1 + n_6 = 0$ (or with n_5 and n_6 switched), we can bound $\mathcal{S}'$ by repeating the same argument in that part of the proof of Proposition 2.5.11. In particular, we may also assume that

$$\langle m_i \rangle + \langle \beta_i \rangle + \langle \alpha_j \rangle \ll 2^{d/70} \quad (2.136)$$

for all i and each $j \in \{5, 6, 7, 9\}$.

Now suppose $n_0 = n_5$ and $n_1 + n_6 = 0$. We will first replace the $\eta(\gamma_1, \gamma_2)$ factor appearing in the expression of $\mathcal{S}'$ by $\eta(\gamma'_1, \gamma_2)$, where $\gamma'_1 = \gamma_1 - \alpha_8$. Note that γ'_1 depends on α_4 only through $\alpha_9 = \alpha_4 + \alpha_8$. When we estimate the difference caused by this substitution, since we still have the restriction $\mathbf{1}_F$, we will have $|\gamma_1| \sim 2^{d+h}$, so we will gain a power $2^{2(d+h)-d/70}$, which is much more than enough to cancel $\Phi^2(\Phi^2)'$, thus this part will be acceptable. We also note that the assumption (2.136) allows us to insert another characteristic function which depends on α_4 only through α_9 ; the presence of this function (as well as the part of $\mathbf{1}_F$ independent of α_4) will allow us to conclude $\langle \gamma'_1 \rangle \sim 2^{d+h}$. Therefore, if we *remove* the part in $\mathbf{1}_F$ depending on α_4 , the error we create will be an expression of the type $\mathcal{S}'$, but restricted to some set on which we have $|\eta| \lesssim 2^{-(d+h)} \langle \gamma_2 \rangle^{-10}$ (note that here we already have $\eta(\gamma'_1, \gamma_2)$ instead of $\eta(\gamma_1, \gamma_2)$), *as well as* $\langle \alpha_4 \rangle \gtrsim 2^{\frac{d'}{90}}$. Then we will be able to take absolute values, cancel $\Phi^2(\Phi^2)'$ by the η factor, and gain a power $2^{cd'}$ from the assumption about α_4 , and proceed exactly as above.

After we have made the above substitutions, the integral with respect to α_4 (or α_8) will be *exactly*

$$\int_{\mathbb{R}} \mathcal{F}(\chi e^{i(\Delta_{n_1} + \Delta_{n_2} - \Delta_{n_0})})(\alpha_4) \mathcal{F}(\chi e^{i(\Delta_{n_0} - \Delta_{n_1} - \Delta_{n_2})})(\alpha_9 - \alpha_4) d\alpha_4 = \widehat{\chi^2}(\alpha_9).$$

Note that there is now *no* log loss which are supposed to be associated with $e^{i\Delta_n}$ factors; we then

get rid of this integration, take absolute values, and fix (m_i, β_i) to obtain an expression

$$\begin{aligned} \mathcal{S}'_{sub} &\lesssim \sum_{n_0, n_1} \int_{(T)} 2^{2h} |f_{n_0, \alpha_0}| \cdot \prod_{l \in \{1, 5, 6\}} |(y^\pm)_{n_l, \alpha_l}| \prod_{l \in \{3, 7\}} \min \left\{ T, \frac{1}{\langle \alpha_l \rangle} \right\} \cdot 2^{-d-h} \langle \alpha_9 \rangle^{-10} \langle \gamma_2 \rangle^{-10} \\ &\lesssim 2^{-|d-h|} T^{0+} \sum_{n_0, n_1} \|\widehat{f_{n_0}}\|_{L^q} \prod_{l \in \{1, 5, 6\}} \|\widehat{(y^\pm)_{n_l}}\|_{L^1}. \end{aligned}$$

Here c_j are constants, $n_5 = n_0$, $n_6 = -n_1$, and that the summation and integration are restricted to suitable sets. Since we now have the $2^{-|d-h|}$ factor which plays the same role as the improved weight in (2.95) does in the proof of Proposition 2.5.11, the rest of the proof will go exactly as the proof of that proposition. $\square$

2.5.3 The estimates for u^* and u

In this section we will construct appropriate extensions of u^* and u so that the improved version of (2.100) holds. Note that we have already constructed a function, denoted by $w^{(4)}$, that coincides with w^* on $[-T, T]$, and verifies $\|w^{(4)}\|_{Y_1} \leq C_0 e^{C_0 A}$. We will fix this function in later discussions. In particular, we may (starting from this point) redefine the δ_n and Δ_n factors as in (2.92) and (2.93) by replacing w^* with $w^{(4)}$ (instead of w'') and u with u''' .

We will also need an extension v'' of v^* , where $(v^*)_n = e^{-i\Delta_n} v_n$. This is not included in the bootstrap assumptions, but can easily be constructed using a similar argument as in the proof of Proposition 2.5.16 below (which itself involves only bounds for u). Note that v'' is bounded in Y_2 as u'' does, and with the (trivial) bound $O_{C_1}(1) e^{C_0 C_1 A}$.

The extension of u

Fix a scale K so that $K = C_{1.5} e^{C_{1.5} A}$ where $C_{1.5}$ is large enough depending on C_1 , and the C_2 defined before is large enough depending on $C_{1.5}$. In order to construct a function $u^{(5)}$ that coincides with u on $[-T, T]$ and satisfies $\|(u^{(5)})^{(-s^3, 0)}\|_{Y_2} \leq C_0 A$, we need to construct $\mathbb{P}_{>K} u^{(5)}$ and $\mathbb{P}_{\leq K} u^{(5)}$ separately.

To construct $\mathbb{P}_{>K} u^{(5)}$, simply note that u'' coincides with u^* on $[-T, T]$, and we have $\|u''\|_{Y_2} \leq C_1 e^{C_1 A}$; thus if we define $(u^{(5)})_n = e^{i\Delta_n} (u'')_n$, where Δ_n is redefined as above, then $\mathbb{P}_{>K} u^{(5)}$ will equal $\mathbb{P}_{>K} u$ on $[-T, T]$, and we have

$$\|(u^{(5)})^{(-s^4, 0)}\|_{X_j} \lesssim O_{C_1}(1) e^{C_0 C_1 A}, \quad (2.137)$$

for $j \in \{2, 3, 4\}$, thanks to Proposition 2.5.8. Here note that the s^3 exponent in that proposition can actually be replaced by s^4 (which is clear from the proof), and the current (δ_n, Δ_n) also verifies Proposition 2.5.6 (in the same way as the (δ_n, Δ_n) defined in Section 2.5.1 does). Since we are restricting to high frequencies, the inequality (2.137) will easily imply

$$\|\mathbb{P}_{>K}(u^{(5)})^{(-s^3, 0)}\|_{X_j} \leq A$$

for $j \in \{2, 3, 4\}$, which is what we need for $\mathbb{P}_{>K}u^{(5)}$.

Now let us construct $\mathbb{P}_{\leq K}u^{(5)}$. Recall that the function u verifies the equation (2.10), and the Y_2 norm of $\chi(t)e^{-tH\partial_{xx}}u(0)$ is clearly bounded by C_0A , we only need to prove

$$\left\| \int_0^t e^{-(t-t')H\partial_{xx}} \mathbb{P}_{\neq 0}((S_N u(t'))^2) dt' \right\|_{(X^{-1/s, \kappa})^T} \lesssim T^{0+}, \quad (2.138)$$

with the implicit constants bounded by $O_{C_{1.5}}(1)e^{C_0 C_{1.5} A}$, where $X^{\sigma, \beta}$ is the standard space normed by $\|u\|_{X^{\sigma, \beta}} = \|u^{(\sigma, \beta)}\|_{l^2 L^2}$. Define the function $u^{(7)}$ by equations (2.88) and (2.89), with the u appearing on the right hand side replaced by u''' , and v replaced by $\mathbb{P}_{\leq 0}v''' + w'''$ with v''' defined by $(v''')_n = e^{i\Delta_n}(v'')_n$ and w''' similarly, so that $u^{(7)}$ coincides with u on $[-T, T]$ (note that the Δ_n here is different from the Δ_n defined in Section 2.5.1; later we will further modify the definition of Δ_n , and this will be clearly stated at that time), then (2.138) follows from the bound

$$\|\mathcal{E}(\mathbf{1}_{[-T, T]} \mathbb{P}_{\neq 0}((S_N u^{(7)})^2))\|_{X^{-1/s, \kappa}} \lesssim T^{0+}, \quad (2.139)$$

since the two functions on the left hand side of (2.138) and (2.139) coincide on $[-T, T]$.

The proof of (2.139) is easy and will be omitted here; basically we shall write out the substitutions (2.88) and (2.89) as in the definition of $u^{(7)}$ and then estimate the resulting expression in $X^{-1/s, \kappa}$ in almost the same way as the part of estimating $\mathcal{M}_3$ in the proof of Proposition 2.5.9; in this process we shall need a strong lower bound for the non-resonance factor Ξ , which is guaranteed by the $\mathbb{P}_{\neq 0}$ projector here, together with the same cancellation as in Section 2.4.2.

In addition, note that $(u^*)_n = e^{-i\Delta_n}u_n$ on the interval $[-T, T]$, we have

$$(\partial_t + H\partial_{xx})(u^*)_n = e^{-i\Delta_n}(\partial_t + H\partial_{xx})u_n - ie^{-i\Delta_n}(\delta_n u_n).$$

The first term on the right hand side can be bounded in $X^{-2/s, \kappa-1}$ using Proposition 2.5.8 and what we proved above, while the second term is easily bounded in the stronger space $X^{-10, 0}$, by

$O_{C_{1.5}}(1)e^{O_{C_{1.5}}(1)A}$. Therefore by the same argument, we can construct an extension of $\mathbb{P}_{\leq K}u^*$ that verifies (2.100).

The extensions of u^*

Now, in order to construct an appropriate extension of $\mathbb{P}_{>K}u^*$, we need the following

Proposition 2.5.16. *Let δ_n and Δ_n be redefined using (2.92) and (2.93), this time with w^* replaced by $w^{(4)}$ and u replaced by $u^{(5)}$, then the new factors will verify Proposition 2.5.6 with the constants being $C_0e^{C_0A}$ instead of $O_{C_1}(1)e^{C_0C_1A}$.*

Now suppose h, k and h', k' are four functions, supported in $|t| \lesssim 1$, that are related by $(h')_n = e^{i\Delta_n}h_n$ and $(k')_n = e^{i\Delta_n}k_n$. Assume that

$$(h')_{n_0} = \sum_{\mu} C_{\mu} \sum_{n_0=n_1+m_1+\dots+m_{\mu}} \Psi \cdot (k')_{n_1} \prod_{i=1}^{\mu} \frac{(u^{(5)})_{m_i}}{m_i}, \quad (2.140)$$

with Ψ bounded, we will have $\|h\|_{Y_2} \lesssim C_0e^{C_0A}\|k\|_{Y_2}$. Moreover, if Ψ is nonzero only when $\langle m_i \rangle \gtrsim K$ for some i (again, the constant here may involve polynomial factors of μ), then we have $\|h\|_{Y_2} \lesssim K^{0-}\|k\|_{Y_2}$.

Proof. The estimates about δ_n and Δ_n are proved in the same way as in Proposition 2.5.6; notice that all the relevant norms bounded by $O_{C_1}(1)e^{C_0C_1A}$ there are now bounded by $C_0e^{C_0A}$ in this updated version, thanks to the construction of $w^{(4)}$ in previous sections and the construction of $u^{(5)}$ above.

Now we need to bound $\|h\|_{X_j}$ for $j \in \{2, 3, 4\}$. However, the bounds for $j \in \{2, 3\}$ are quite easy, since translation in Fourier space acts well with these norms. Thus we now assume $j = 4$. By fixing and then summing over μ , we may assume that

$$|h_{n_0, \alpha_0}| \leq C_0 \sum_{n_0=n_1+m_1+\dots+m_{\mu}} \int_{(T)} |k_{n_1, \alpha_1}| \cdot \mathcal{F}(\chi e^{i(\Delta_{n_1}-\Delta_{n_0})})(\alpha_2) \prod_{i=1}^{\mu} \left| \frac{(u^{(5)})_{m_i, \beta_i}}{m_i} \right|,$$

where the integration is taken over the set

$$\left\{ (\alpha_1, \alpha_2, \beta_1, \dots, \beta_{\mu}) : \alpha_0 = \alpha_1 + \alpha_2 + \sum_{i=1}^{\mu} \beta_i + \Xi \right\}, \quad \Xi = |n_0|n_0 - |n_1|n_1 - \sum_{i=1}^{\mu} |m_i|m_i.$$

Throughout the proof we will only use the Y_2 norm for $(u^{(5)})^{(-s^3, 0)}$, and notice that these norms are bounded by C_0A instead of C_1A .

Let g be such that $\|g\|_{X'_4} \lesssim 1$, then we only need to estimate $\mathcal{S} := (g, h)$. This is an expression we have seen many times before; to analyze it, we notice that either $\langle \Xi \rangle$, or one of $\langle \alpha_l \rangle$ (where $l \in \{1, 2\}$) or $\langle \beta_i \rangle$, must be $\gtrsim \langle \alpha_0 \rangle$.

Suppose $\langle \alpha_0 \rangle \lesssim \langle \Xi \rangle$. Let the maximum of $\langle n_0 \rangle$, $\langle n_1 \rangle$ and all $\langle m_i \rangle$ be $\sim 2^d$ (and we fix d), then $\langle \alpha_0 \rangle \lesssim 2^{2d}$. If among the variables n_0 and m_i , at least two are $\gtrsim 2^{(1-s^2)d}$, then we will gain a net power $2^{c(1-\kappa)d}$ from the weights in the X'_4 bound for g , or from the $|m_i|^{-1}$ weights appearing in $\mathcal{S}$. Then we will be able to bound the $\mathcal{F}(\chi e^{i(\Delta_{n_1} - \Delta_{n_0})})(\alpha_2)$ factor using some inequality similar to (2.128), fix the irrelevant (m_j, β_j) variables to produce $\mathcal{S}_{sub}$, then estimate it by bounding $\mathfrak{N}g$ in $L^{2+}L^{2+}$, $\mathfrak{N}k$ and the two $\mathfrak{N}u^{(5)}$ factors in L^6L^6 and the $\mathcal{F}(\chi e^{i(\Delta_{n_1} - \Delta_{n_0})})(\alpha_2)$ factor in $l^{1+}L^{1+}$, where $2+$ is some $2 + cs^2$, with a further loss of at most $2^{O(\epsilon)d}$. We then sum over the (m_j, β_j) variables and sum over d to conclude the estimate for $\mathcal{S}$. If instead only one of them can be $\gtrsim 2^{(1-s^2)d}$ (again, assume d is large enough), then this variable and n_1 must both be $\sim 2^d$. Let the maximum of all the remaining variables be $\sim 2^{d'}$ where $d' \leq (1-s^2)d$ is also fixed, then we will have $|\alpha_0| \lesssim 2^{d+d'}$. Since we will be able to gain a power $2^{c(1-\kappa)(d+d')}$ from the weights, we can proceed in the same way as above.

Next, suppose $\langle \alpha_0 \rangle \lesssim \langle \alpha_2 \rangle$. By invoking (2.103) we may get an estimate better than (2.128) for the α_2 factor, namely

$$\|\langle \alpha_2 \rangle \mathcal{F}(\chi e^{i(\Delta_{n_1} - \Delta_{n_0})})(\alpha_2)\|_{L^\sigma} \lesssim C_0 e^{C_0 A} \sum_{i=1}^{\mu} \langle m_i \rangle^{s^5}, \quad (2.141)$$

for all $1 \leq \sigma \leq \infty$; the $\lesssim$ here allows for a polynomial factor in μ . Therefore, by losing a tiny power of some m_i , we may cancel the α_0 weight in the X'_4 bound for g and still bound the α_2 factor in L^2 , then fix (m_i, β_i) and produce $\mathcal{S}_{sub}$, and estimate it by

$$\mathcal{S}_{sub} \lesssim \sum_{n_0} \langle n_0 \rangle^{-1} \|(g^{(1, -\kappa)})_{n_0, \alpha_0}\|_{L_{\alpha_0}^2} \|k_{n_0 + c_1, \alpha_1}\|_{L_{\alpha_1}^1} \|g^{(0, -\kappa)}\|_{l^{\frac{3}{2}} L^2} \cdot \|k_{n_1, \alpha_1}\|_{l^3 L^1} \lesssim 1,$$

where c_j are constants. If instead $\langle \alpha_0 \rangle \lesssim \langle \alpha_1 \rangle$, we can invoke the α_1 weight in X_4 norm for k to cancel the α_0 weight, then notice that $\langle n_1 \rangle \lesssim \langle n_0 \rangle + \langle m_i \rangle$ for some i , then bound the α_2 factor in L^1 and fix all the other (m_j, β_j) to produce $\mathcal{S}_{sub}$. If $\langle n_1 \rangle \lesssim \langle m_i \rangle$ we will estimate $\mathcal{S}_{sub}$ by

$$\begin{aligned} \mathcal{S}_{sub} &\lesssim \sum_{n_0 = n_1 + m_i + c_1} \frac{\langle n_1 \rangle}{\langle n_0 \rangle \langle m_i \rangle} \|(g^{(1, -\kappa)})_{n_0, \alpha_0}\|_{L_{\alpha_0}^2} \|(k^{(-1, \kappa)})_{n_1, \alpha_1}\|_{L_{\alpha_1}^2} \|(u^{(5)})_{m_i, \beta_i}\|_{L_{\beta_i}^1} \\ &\lesssim \|g^{(0, -\kappa)}\|_{l^1 L^2} \cdot \|k^{(-1, \kappa)}\|_{l^\gamma L^2} \cdot \|u^{(5)}\|_{l^{\gamma'} L^1} \lesssim 1, \end{aligned}$$

where c_j are constants; note that $\|u^{(5)}\|_{l^{\gamma'}L^1}$ can be controlled by the X_2 norm of $(u^{(5)})^{(-4s^3,0)}$ due to (2.7). If $\langle n_1 \rangle \lesssim \langle n_0 \rangle$ we will instead estimate the g factor above in $l^{\gamma'}L^2$, the k factor in $l^{\gamma}L^2$, and the $u^{(5)}$ factor with weight $\langle m_i \rangle^{-1}$ in l^1L^1 . Finally, if $\langle \alpha_0 \rangle \lesssim \langle \beta_i \rangle$ for some i , we will cancel the $\langle \alpha_0 \rangle$ weight by the $\langle \beta_i \rangle$ weight, then fix (m_j, β_j) and again get $\mathcal{S}_{sub}$, which we estimate by

$$\begin{aligned} \mathcal{S}_{sub} &\lesssim \sum_{n_0=n_1+m_i+c_1} \langle c_1 \rangle^{-s} \langle n_0 \rangle^{-s} \langle n_1 \rangle^{-c(2-\gamma)} \| (g^{(s,-\kappa)})_{n_0, \alpha_0} \|_{L_{\alpha_0}^2} \| (k^{(c(2-\gamma),0)})_{n_1, \alpha_1} \|_{L_{\alpha_1}^1} \| ((u^{(5)})^{(-1,\kappa)})_{m_i, \beta_i} \|_{L_{\beta_i}^2} \\ &\lesssim \| g^{(s,-\kappa)} \|_{l^1L^2} \| k^{(c(2-\gamma),0)} \|_{l^{\gamma'}L^1} \cdot \| (u^{(5)})^{(-4s^3,0)} \|_{X_4} \lesssim 1, \end{aligned}$$

where c_j are constants, and again note that we can gain any small power of c_1 , since $\pm c_1$ is the sum of all m_j where $j \neq i$.

Finally, we may check that throughout the above proof, we only need to use the X'_j norms of $\langle \partial_x \rangle^{-2s^3} u^{(5)}$ instead of $\langle \partial_x \rangle^{-s^3} u^{(5)}$; thus we will gain a power K^{0+} if we make the restriction $m_i \gtrsim K$ for some i . $\square$

To see how Proposition 2.5.16 allows us to construct the extension of $\mathbb{P}_{>K} u^*$, we first note that u^* is real valued, so we only need to construct an extension of $\mathbb{P}_{>+K} u^*$ (which is an abbreviation of $\mathbb{P}_+ \mathbb{P}_{>K} u^*$). Now, in Proposition 2.5.16 we may choose k to be an arbitrary extension of v^* and h to be some extension of u^* (and choose h' and k' accordingly) so that (2.140) holds with appropriate coefficients (cf. (2.88) and (2.89)).

Exploiting the freedom in the choice of k , we will set $\mathbb{P}_+ k = w^{(4)}$ and $\mathbb{P}_{\leq 0} k = \mathbb{P}_{\leq 0} v''$. The part coming from $\mathbb{P}_+ k$ is bounded in Y_2 (before or after the $\mathbb{P}_{>+K}$ projection) by $C_0 e^{C_0 A}$ due to Proposition 2.5.16, since we already have $\|w^{(4)}\|_{Y_2} \lesssim \|w^{(4)}\|_{Y_1} \leq C_0 e^{C_0 A}$. As for the part coming from $\mathbb{P}_{\leq 0} k$, we must have $n_0 > K$ and $n_1 \leq 0$ in (2.140), so the Ψ factor will be nonzero only when $\langle m_i \rangle \gtrsim K$ for some i , thus we may again use Proposition 2.5.16 to bound this part in Y_2 by $O_{C_1}(1) e^{C_0 C_1 A} K^{0-} \leq 1$, since we have $\|v''\|_{Y_2} \leq O_{C_1}(1) e^{C_0 C_1 A}$. This completes the construction for the extension of u^* and finally finishes the proof of Proposition 2.5.1.

2.5.4 Subtracting two solutions

The main purpose of this section is to provide necessary estimates for differences of two solutions to (2.10). The statement of these estimates are complicated, and we need to introduce some new spaces; on the other hand, the proof of these results is mostly technical and differs from the estimates in Sections 2.5.2 and 2.5.3 only at minor places. Therefore, we choose to simply state the results without proof. The full proof can be found in [16], Section 12.

Definition 2.5.17. Suppose $\mathcal{Q} = (u'', w'', u''')$ and $\mathcal{Q}' = (u^{\dagger\dagger}, w^{\dagger\dagger}, u^{\dagger\dagger\dagger})$ are two triples of functions defined on $\mathbb{R} \times \mathbb{T}$, we define their distance by

$$\mathfrak{D}_\sigma(\mathcal{Q}, \mathcal{Q}') = \|(w'' - w^{\dagger\dagger})^{(-\sigma, 0)}\|_{Y_1} + \|(u'' - u^{\dagger\dagger})^{(-\sigma, 0)}\|_{Y_2} + \|(u''' - u^{\dagger\dagger\dagger})^{(-s^3 - \sigma, 0)}\|_{Y_2},$$

for $\sigma \in \{0, s^5\}$, where recall by our notation that $\langle \partial_x \rangle^{-\sigma} u = u^{(-\sigma, 0)}$. In particular, if $\sigma = \mathcal{Q}' = 0$, we define the triple norm

$$\|\mathcal{Q}\| := \mathfrak{D}_0(\mathcal{Q}, 0) = \|w''\|_{Y_1} + \|u''\|_{Y_2} + \|(u''')^{(-s^3, 0)}\|_{Y_2}.$$

Next, suppose u and $u^\times$ are functions defined on $I \times \mathbb{T}$ for some interval I , we will define the functions (u^*, w^*) corresponding to u and some M , and $(u^\diamond, w^\diamond)$ corresponding to $u^\times$ and some N (note the definition depends on the choice of the origin in $\Delta_n(t) = \int^t \delta_n(t') dt'$, but this will not affect the triple norm $\|\cdot\|$; this does affect estimates for differences, but we need them only when $I = [-T, T]$ or its translation, in which case the choice of origin is canonical), as in Sections 2.4 and 2.4.3, and then define

$$\mathfrak{D}_\sigma^{I, MN}(u, u^\times) = \inf_{\mathcal{Q}, \mathcal{Q}'} \mathfrak{D}_\sigma(\mathcal{Q}, \mathcal{Q}'), \quad (2.142)$$

where the infimum is taken over all triples $\mathcal{Q}$ and $\mathcal{Q}'$ that extends (u^*, w^*, u) and $(u^\diamond, w^\diamond, u^\times)$ from $I \times \mathbb{T}$ to $\mathbb{R} \times \mathbb{T}$, respectively. We will also define $\|u\|_I^M = \mathfrak{D}_0^{I, MM}(u, 0) = \inf_{\mathcal{Q}} \|\mathcal{Q}\|$; these notations can be written in a more familiar way as

$$\mathfrak{D}_\sigma^{I, MN}(u, u^\times) = \|(w^* - w^\diamond)^{(-\sigma, 0)}\|_{Y_1^I} + \|(u^* - u^\diamond)^{(-\sigma, 0)}\|_{Y_2^I} + \|(u - u^\times)^{(-s^3 - \sigma, 0)}\|_{Y_2^I};$$

$$\|u\|_I^M = \|w^*\|_{Y_1^I} + \|u^*\|_{Y_2^I} + \|u^{(-s^3, 0)}\|_{Y_2^I}.$$

Also, if $M = N = \infty$ we will omit them. Now we can define the metric space

$$\mathcal{BO}^I = \{u : \|u\|_I = \|u\|_I^\infty < \infty\}, \quad (2.143)$$

with the distance function given by $\mathfrak{D}_0^I$ (we will also use $\mathfrak{D}_{s^5}^I$, which is also well-defined on $\mathcal{BO}^I$). Finally, when $I = [-T, T]$, we may use T in place of I in sub- or superscripts, so this contains the definition of $\mathcal{BO}^T$.

Remark 2.5.18. If $u \in \mathcal{BO}^T$, we may define $uu_x = \frac{1}{2}\partial_x(\mathbb{P}_{\neq 0}u^2)$ as a distribution on $[-T, T]$ through an argument similar to the one in Section 2.5.3. More precisely, we may uniquely define the function

$$h(t) = \int_0^t e^{-(t-t')H\partial_{xx}}(u(t')\partial_x u(t')) dt' \quad (2.144)$$

as an element of $(X^{-1/s, \kappa})^T$.

In particular, we may define $u \in \mathcal{BO}^T$ to be a solution to (2.1) on $[-T, T]$, if u verifies the integral version of (2.1) with the evolution term defined as in (2.144). Clearly this definition is independent of the choice of origin, and $[-T, T]$ may be replaced by any interval I .

Moreover, since the arguments in Section 2.5.3 allow for some room, the map sending u to h in (2.144) is continuous with respect to the weak distance function $\mathfrak{D}_{s^5}^T$ (or $\mathfrak{D}_{s^5}$ if we consider the map sending the triple $\mathcal{Q}$ to h). This fact will be important in the proof of Theorem 2.6.1.

Proposition 2.5.19 (Embedding into B_t^0). *Let B_t^0 be the space of bounded functions of t into some Banach space. Suppose u and $u^\times$ are two functions defined on $I \times \mathbb{T}$, and choose corresponding extensions $\mathcal{Q} = (u'', w'', u''')$ and $\mathcal{Q}' = (u^{\dagger\dagger}, w^{\dagger\dagger}, u^{\dagger\dagger\dagger})$ corresponding to M and N , where $M \geq N$. We then have*

$$\|u\|_{B_t^0(I \rightarrow Z_1)} \lesssim \|\mathcal{Q}\|; \quad \|u\|_{B_t^0(I \rightarrow Z_1)} \lesssim \|u\|_I^M. \quad (2.145)$$

Concerning differences, we only have the weaker estimate

$$\|\langle \partial_x \rangle^{-s^5}(u - u^\times)\|_{C_t^0(I \rightarrow Z_1)} \lesssim O_{\|\mathcal{Q}\|, \|\mathcal{Q}'\|}(1) \cdot (\mathfrak{D}_{s^5}(\mathcal{Q}, \mathcal{Q}') + N^{0-}); \quad (2.146)$$

$$\|\langle \partial_x \rangle^{-\theta}(u - u^\times)\|_{C_t^0(I \rightarrow Z_1)} \lesssim O_{\theta, \|\mathcal{Q}\|, \|\mathcal{Q}'\|}(1) \cdot (\mathfrak{D}_0(\mathcal{Q}, \mathcal{Q}') + N^{0-}), \quad (2.147)$$

for all $\theta > 0$, where the constant may also depend on the upper bound of the length of I .

Now suppose u is a smooth function solving (2.10) on $[-T, T]$. The arguments in the above sections actually give us a way to update a given triple $\mathcal{Q} = (u'', w'', u''')$ extending (u^*, w^*, u) to a new triple $\mathcal{Q}' = (u^{(4)}, w^{(4)}, u^{(5)})$, which remains to be an extension, and verifies better bounds. We will define $\mathfrak{J}$ to be the map from the set of extensions to itself, that sends $\mathcal{Q}$ to $\mathcal{Q}'$. We then have the following two properties for $\mathfrak{J}$, which are the versions of Proposition 2.5.1 that apply to differences, and are the main results of this section.

Proposition 2.5.20 (Closedness under $\mathfrak{J}$). *Let C_1 be large enough, C_2 large enough depending on C_1 , and $0 < T \leq C_2^{-1}e^{C_2A}$. Suppose u is a smooth function solving (2.10) on $[-T, T]$, and $\mathcal{Q}$ is an*

extension satisfying

$$\|\mathcal{Q}\| \leq C_1 e^{C_1 A}, \quad \|(u''')^{(-s^3, 0)}\|_{Y_2} \leq C_1 A, \quad (2.148)$$

then the same estimate will hold if we replace $\mathcal{Q}$ by $\mathfrak{I}\mathcal{Q}$.

Proposition 2.5.21 (Lipschitz bounds for $\mathfrak{I}$). *Let C_1 , C_2 and T be as in Proposition 2.5.20. Suppose u and $u^\times$ are two smooth functions solving (2.10) with truncation S_N and S_M respectively, where $1 \ll N \leq M \leq \infty$, $\mathcal{Q}$ and $\mathcal{Q}'$ are two triples corresponding to u and $u^\times$ respectively, such that (2.148) holds, and that*

$$\mathfrak{D}_{s^5}(\mathcal{Q}, \mathcal{Q}') \leq B \quad (2.149)$$

for some $B > 0$, then we have

$$\mathfrak{D}_{s^5}(\mathfrak{I}\mathcal{Q}, \mathfrak{I}\mathcal{Q}') \leq \frac{B}{2} + O_{C_1}(1) e^{C_0 C_1 A} (\|\langle \partial_x \rangle^{-s^5} (u(0) - u^\times(0))\|_{Z_1} + N^{0-}), \quad (2.150)$$

where C_0 is any constant appearing in previous sections. In particular, we have

$$\mathfrak{D}_{s^5}^{T, NM}(u, u^\times) \leq O_{C_2, A}(1) (\|\langle \partial_x \rangle^{-s^5} (u(0) - u^\times(0))\|_{Z_1} + N^{0-}), \quad (2.151)$$

provided $\|u(0)\|_{Z_1} + \|u^\times(0)\|_{Z_1} \leq A$ for some large A . Moreover, if $M = N$, we may replace the $\mathfrak{D}_{s^5}$ distance by the $\mathfrak{D}_0$ distance and remove the N^{0-} term on the right hand side of (2.151).

2.6 Proof of the main results

With Propositions 2.5.1 and 2.5.21, it is now easy to prove our main results. Since the argument in this section will be more or less standard, we may present only the most important steps.

2.6.1 Local well-posedness and approximation

Theorem 2.6.1 (Precise version of Theorem 2.1.1). *There exists a constant C such that, when we choose any $A > 0$ and $0 < T \leq C^{-1} e^{-CA}$, the followings will hold.*

(1) *Existence: for any $f \in \mathcal{V}$ with $\|f\|_{Z_1} \leq A$, there exists some $u \in \mathcal{BO}^T$ such that $\|u\|_T \leq C e^{CA}$, and it verifies the equation (2.1), in the sense described in Remark 2.5.18, with initial data $u(0) = f$.*

(2) *Continuity: let the solution described in part (1) be $u = \Phi f = (\Phi_t f)_t$. Suppose $\|f\|_{Z_1} \leq A$*

and $\|g\|_{Z_1} \leq A$, then each $\varepsilon > 0$, we have

$$\sup_{|t| \leq T} \|\langle \partial_x \rangle^{-s^5} (\Phi_t f - \Phi_t g)\|_{Z_1} + \mathfrak{D}_{s^5}^T(\Phi f, \Phi g) \leq O_{C,A}(1) \|\langle \partial_x \rangle^{-s^5} (f - g)\|_{Z_1};$$

$$\sup_{|t| \leq T} \|\langle \partial_x \rangle^{-\varepsilon} (\Phi_t f - \Phi_t g)\|_{Z_1} + \mathfrak{D}_0^T(\Phi f, \Phi g) \leq O_{\varepsilon,C,A}(1) \|f - g\|_{Z_1}.$$

(3) *Approximation for short time:* let $u = \Phi f$ as in part (2), and let Φ^N be the solution flow of (2.10) and $u^N = \Phi^N \Pi_N f$. Then we have

$$\lim_{N \rightarrow \infty} \left(\mathfrak{D}_{s^5}^{T,N\infty}(u^N, u) + \sup_{|t| \leq T} \|\langle \partial_x \rangle^{-s^5} (u^N(t) - u(t))\|_{Z_1} \right) = 0.$$

(4) *Uniqueness:* for any other time T' , suppose u and $u^\times$ are two elements of $\mathcal{BO}^{T'}$ with the same initial data, and they both solve (2.1), then we must have $u = u^\times$ (on $[-T', T']$).

(5) *Long-time existence:* consider any $f \in Z_1$, and define the functions u^N as in part (3). Suppose for some other time T' and some subsequence $\{N_k\}$ that

$$\sup_k \|u^{N_k}\|_{T'}^{N_k} < \infty, \quad (2.152)$$

then there exists a solution $u \in \mathcal{BO}^{T'}$ to (2.1) with initial data f .

Proof. Suppose $f \in Z_1$ and $\|f\|_{Z_1} \leq A$, and let $0 < T \leq C_2^{-1} e^{-C_2 A}$ with constants as in Propositions 2.5.20 and 2.5.21. Consider u^N as defined in (3); using Proposition 2.5.1, we may choose for each N some triple $\mathcal{Q}_N$ corresponding to u^N that verifies (2.148). We then define

$$\mathcal{Q}^N = \mathfrak{J}^N \mathcal{Q}_N, \quad (2.153)$$

then it will be clear from Propositions 2.5.20 and 2.5.21 that

$$\|\mathcal{Q}^N\| \leq C_1 e^{C_1 A}, \quad (2.154)$$

$$\lim_{N, M \rightarrow \infty} \mathfrak{D}_{s^5}(\mathcal{Q}^M, \mathcal{Q}^N) = 0. \quad (2.155)$$

By a simple completeness argument we can then find some $\mathcal{Q}$ so that $\mathfrak{D}_{s^5}(\mathcal{Q}^N, \mathcal{Q}) \rightarrow 0$ (in particular $\mathcal{Q}$ will have initial data f), and by an argument similar to the proof of Proposition 2.3.6 we deduce that $\|\mathcal{Q}\| \leq C_1 e^{C_1 A}$. By using Remark 2.5.18, we can now pass to the limit and show that the triple $\mathcal{Q}$ gives a solution $u \in \mathcal{BO}^T$ of (2.1) on the interval $[-T, T]$. This proves existence.

Parts (2) and (3) will follow from basically the same argument. In fact, for each (f, g) , we may construct $\mathcal{Q}^N$ and $\mathcal{Q}^{N^\times}$ corresponding to $\Phi^N \Pi_N f$ and $\Phi^N \Pi_N g$ as above, so that they have uniformly bounded triple norm, and moreover

$$\mathfrak{D}_{s^5}(\mathcal{Q}^N, \mathcal{Q}^{N^\times}) \lesssim \|\langle \partial_x \rangle^{-s^5} (f - g)\|_{Z_1} + N^{0-}.$$

Using Proposition 2.5.19 and passing to the limit, we obtain the result in (2). The result in (3) follows from comparing $\mathcal{Q}^N$ with $\mathcal{Q}$ and using Proposition 2.5.19 also.

As for part (5), we will deduce it merely from the condition that $\|u^{N_k}\|_{T'}^{N_k} \leq A$ and

$$\|\langle \partial_x \rangle^{-s^5} (u^{N_k} - u)(0)\|_{Z_1} \rightarrow 0, \quad (2.156)$$

which is clearly satisfied in our setting. Choose some τ small enough depending on A , then $\|u(0)\|_{Z_1} \leq C_0 A$ implies we can solve (2.1) on $[-\tau, \tau]$, and from (Proposition 2.5.21 and) what we just proved, we also have

$$\|\langle \partial_x \rangle^{-s^5} (u^{N_k} - u)(\pm\tau)\|_{Z_1} \rightarrow 0, \quad (2.157)$$

and therefore

$$\|u(\pm\tau)\|_{Z_1} \leq \limsup_{N \rightarrow \infty} \|u^{N_k}(\pm\tau)\|_{Z_1} \leq C_0 A. \quad (2.158)$$

This information will allow us to restart from time $\pm\tau$, and thus obtain a solution to (2.1) on $[-2\tau, 2\tau]$. Repeating this, we will finally get a solution on $[-T', T']$, which we can prove to be in $\mathcal{BO}^{T'}$ using partitions of unity. This proves (conditional) global existence.

Finally, to prove uniqueness, let u and $u^\times$ be to solutions to (2.1) that both belong to $\mathcal{BO}^{T'}$ and have the same initial data. Let their strong norms be bounded by A , and choose τ small enough depending on A . To prove that $u = u^\times$ on $[-\tau, \tau]$, we need to prove the following claim: if for triples $\mathcal{Q}$ and $\mathcal{Q}'$ corresponding to u and $u^\times$ respectively, we have

$$\|\mathcal{Q}\| + \|\mathcal{Q}'\| \leq A, \quad \mathfrak{D}_{s^5}(\mathcal{Q}, \mathcal{Q}') \leq K, \quad (2.159)$$

then with $\mathcal{Q}$ replaced by $\mathfrak{I}\mathcal{Q}$ and $\mathcal{Q}'$ by $\mathfrak{I}\mathcal{Q}'$, the inequalities will hold with A unchanged and K replaced by $K/2$. This can be done by repeating the whole argument from Section 2.5.1 to Section 2.5.4. There is some subtlety since we do not have a priori smoothness assumption here, but this can be resolved by slight change of one part in the proof. See [15], Section 13.1 for details. $\square$

2.6.2 Global well-posedness and measure invariance

Using Theorem 2.6.1 and the exact invariance of the approximation measure ν_N° under the approximation equation (2.10) (see Section 2.2.1), we can extend the local solutions constructed in Theorem 2.6.1 almost surely to global solutions, and prove invariance of the Gibbs measure ν . The arguments in this section are completely standard in the literature (except that the time of local existence is exponentially decaying in the size of the initial data; but this will not change the proof as long as this decay speed is slower than that involved in the large deviation estimate in (2.28), which is Gaussian), so we will only state the results here.

Theorem 2.6.2 (Restatement of Theorem 2.1.2). *Let the Wiener measure ρ be defined as in Section 2.2.1. There exists a subset $\Sigma \subset \mathcal{V}$ such that $\rho(\mathcal{V} - \Sigma) = 0$, and the following holds: for any $f \in \Sigma$ there exists a unique global solution u to (2.1) with initial data f such that $u \in \mathcal{BO}^T$ for all $T > 0$. Moreover, let $u = \Phi f = (\Phi_t f)_t$, then these Φ_t form a measurable transformation group from Σ to itself. Finally, suppose the Gibbs measure ν is defined as in Section 2.2.1 (using some cutoff function ζ), then each Φ_t keeps ν invariant.*

2.6.3 Modified continuity

In this section we prove Theorem 2.1.3; this modified continuity result is essentially due to the introduction of the $e^{i\Delta_n}$ factor in Section 2.4.3.

To prove part (1), note that $u \in \mathcal{BO}^T$, we know

$$u^* \in Y_2^T \subset C_t^0([-T, T] \rightarrow Z_1)$$

using the notation in Proposition 2.4.5. Recall from (2.93) that $\Delta_n(t) = \int_0^t \delta_n(t') dt'$ and

$$\delta_n(t) = \frac{1}{2} \sum_{k=1}^n |w_k|^2 = \frac{1}{2} \sum_{k=1}^n |u_k|^2 + R$$

for $n > 0$, where R does not grow with n (this is easy using $w = \mathbb{P}_+(Mu)$ and the assumptions about u). Now, if it were not for the logarithmic factor on the right hand side of (2.103) which these factors verify, $\Delta_n(t)$ would be continuous in t *uniformly* in n , and u would be in $C_t^0([-T, T] \rightarrow Z_1)$; this shows that we may disregard R and pretend that Δ_n is defined as in (2.11), and this proves part (1).

For part (2) we need another probabilistic argument. Recall that $u = \Phi f = (\Phi_t f)_t$ is defined for

$f \in \Sigma$ which is equipped with the Gaussian measure ρ . In order to use part (1) we just proved, we will define

$$\tilde{\delta}_n(t) = \sum_{k=1}^n (|u_k(t)|^2 - \frac{1}{4\pi k}) = \sum_{d \leq \lfloor \log_2 n \rfloor} \widetilde{\delta_{(d)}}(t) + R,$$

where

$$\widetilde{\delta_{(d)}}(t) = \sum_{0 < k \sim 2^d} (|u_k(t)|^2 - \frac{1}{4\pi k}) \quad (2.160)$$

and $\widetilde{\Delta}_n$ similarly, where R is already bounded in n and can be neglected. We only need to prove continuity in any interval $[-T, T]$; for simplicity assume $T = 1$. If we define

$$Y_{(d)} = \int_{-1}^1 |\widetilde{\delta_{(d)}}(t)|^2 dt$$

as a random variable on Σ for each d , then part (2) will follow if we can show that

$$\limsup_{d \rightarrow \infty} 2^{d/2} Y_{(d)} \leq 1 \quad (2.161)$$

for ρ -almost all $f \in \Sigma$. Fix one Gibbs measure ν , we have

$$\mathbb{E}_\nu(2^d Y_{(d)}) \lesssim \int_{-1}^1 \left[\mathbb{E}_\nu \left(\exp \left(2^{d/2} \widetilde{\delta_{(d)}}(t) \right) \right) + \mathbb{E}_\nu \left(\exp \left(-2^{d/2} \widetilde{\delta_{(d)}}(t) \right) \right) \right] dt. \quad (2.162)$$

Now *using the invariance of ν* , we only need to consider $t = 0$; also we will study only the first term. Since $(\mathbb{E}_\nu H)^2 \lesssim \mathbb{E}_\nu H^2$ by Cauchy-Schwartz, this is bounded by

$$\mathbb{E}_\omega \left(\exp \left(2^{d/2} \sum_{0 < k \sim 2^d} \frac{|g_k(\omega)|^2 - 1}{2\pi k} \right) \right),$$

which can be easily computed and is $O(1)$ due to our choice of parameters. Then (2.161) follows by standard measure theoretic arguments (for any ν , and thus for ρ).

Chapter 3

Invariance of higher order weighted Gaussian measures for the Benjamin-Ono equation

3.1 Introduction

In this chapter we continue the study of the Benjamin-Ono equation in Chapter 2, where the invariance of the Gibbs measure ν^1 is proved, and prove the invariance of ν^2 and ν^3 . The content of this chapter is joint work with N. Tzvetkov and N. Visciglia.

3.1.1 Description of the result

As explained in Section 2.1, the Gaussian measure ν^σ associated with each conserved quantity E^σ of the Benjamin-Ono equation (2.1) are formally invariant under (2.1), and this invariance is expected to be rigorous for¹ $\sigma \geq 1$; in fact, the whole Chapter 2 is devoted to the proof of the fact that the Gibbs measure ν^1 is invariant. Recently, in [52] and [53], Tzvetkov-Visciglia developed a new method that allowed them to prove the invariance of ν^σ for $\sigma \geq 4$.

In the current chapter we will fill in the gap and establish the invariance of ν^2 and ν^3 . Namely we will prove the following

¹Notice that, when $\sigma = 0$ we have the white noise, which is supported in $H^{-1/2-\epsilon}$. Proving even a local well-posedness result at this regularity seems to be out of reach with current techniques.

Theorem 3.1.1. *Let $\sigma \in \{2, 3\}$, the weighted gaussian measures ν^σ be defined as in [51]. Then ν^σ is absolutely continuous with respect to the Gaussian measure ρ^σ ; moreover, almost surely with respect to ρ^σ , the equation (2.1) has global solution, and this global flow keeps ν^σ invariant.*

The difficulty with truncated equations

Since ν^2 and ν^3 is supported in $H^{1/2-\epsilon}$, we automatically have almost sure global well-posedness for (2.1) on this support; in the proof we also need an approximation result concerning (2.10), which can be proved by using (a small subset of) the arguments in Chapter 2.

However, unlike the Gibbs measure case, there is one essential difficulty, namely that there is no approximation measure ν_N^σ that is invariant under (2.10). To overcome this, we shall use the technique introduced in [52], which roughly works like below.

Recall the notations introduced in Section 2.1.1. Let Φ_t^N and Φ_t be the solution flow to (2.10) and (2.1) respectively, and ν_N^σ be some truncated version of ν^σ , it is easily seen that the invariance result would be a consequence of the following approximation result, namely

$$\lim_{N \rightarrow \infty} \sup_{t, A} \left| \frac{d}{dt} \int \mathbf{1}_{\Phi_t^N \Pi_N(A) \times \mathcal{V}_N^\perp} d\nu_N^\sigma \right| = 0; \quad (3.1)$$

note that (3.1) is trivial in the case of the Gibbs measure. Recall (ignoring for now the cutoff factors) that

$$d\nu_N^\sigma = (e^{-E_N^\sigma[f]} d\mathcal{L}_N) \times (\text{standard Gaussian})$$

with some truncated version E_N^σ of E^σ and $\mathcal{L}_N$ the Lebesgue measure on $\mathcal{V}_N$. We now notice that the *Lebesgue measure* is invariant under Φ_t^N , so we can make a change of variables using $(\Phi_t^N)^{-1}$, so that the derivative falls on $e^{-E_N^\sigma[f]}$. If we use the invariance of the Lebesgue measure once again, and combine this with Hölder's inequality, we can reduce (3.1) to the following limit result,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\rho^\sigma} \left(\left(\frac{d}{dt} E_N^\sigma[\Phi_t^N \Pi_N f] \Big|_{t=0} \right)^2 \right) = 0, \quad (3.2)$$

where ρ^σ is the “background” Gaussian measure in Theorem 3.1.1. Note that (3.2) is a fixed-time estimate and does not rely on the evolution of (2.10).

An I -method in energy truncations

We then need to prove (3.2). This turns out, at least when $\sigma \in \{2, 3\}$, to require a careful construction of the approximate energy E_N^σ . Basically,

$$E_N^\sigma[u] = \|u\|_{H^{\sigma/2}}^2 + \mathcal{E}_3[u] + \mathcal{E}_4[u] + \cdots$$

will be composed of $\mathcal{E}_j$ which is of degree j and contains totally $(\sigma - j + 2)$ derivative in general. Using (2.10), we therefore have

$$\partial_t E_N^\sigma[u] = \mathcal{R}_3[u] + \mathcal{R}_4[u] + \cdots,$$

where $\mathcal{R}_j$ is of degree j and contains totally $(\sigma - j + 3)$ derivatives. Since the unknown function u itself is fairly smooth (belonging to $H^{(\sigma-1)/2-\epsilon}$), the optimal strategy would be to design those $\mathcal{E}_j$ so that the $\mathcal{R}_j$ with lower degree vanish. There is a well-known algorithm that exactly solves this problem, namely the (corrected) I -method; see for example [12].

In our case, we can actually make $\mathcal{R}_3$ and $\mathcal{R}_4$ vanish, before encountering singularities. These will in fact be adequate, since $\sigma \in \{2, 3\}$ gives us enough room, such that all quintic terms in $\partial_t E_N^\sigma$ will satisfy (3.2) provided that their coefficients satisfy some simple cancellation condition, which is always easy to verify.

The rest of this chapter is organized as follows. In Section 3.2, we will review the definition of ν^2 and ν^3 as in [51]; in Section 3.3 we will construct the truncated energy E_N^σ using the I -method, and compute its time derivative. This expression is then analyzed, which allows us to prove (3.2). In Section 3.4 we briefly sketch the deterministic theory; much of the estimate is already contained in Chapter 2, so we shall not go over all details. Finally, in Section 3.5 we will prove Theorem 3.1.1.

3.2 The measures ν^2 and ν^3

This section is arranged in basically the same way as Section 2.2.1. In particular, we will use all the notations defined in that section (and also Section 2.1.1). For $\sigma \in \{2, 3\}$, let ρ^σ be the distribution of the random Fourier series

$$\sum_{n \neq 0} \frac{g_n(\omega)}{2\sqrt{\pi}|n|^{\sigma/2}} e^{inx},$$

which is identified with $\rho_N^\sigma \times (\rho_N^\sigma)^\perp$, with ρ_N^σ and $(\rho_N^\sigma)^\perp$ being corresponding Gaussian measures on $\mathcal{V}_N$ and $\mathcal{V}_N^\perp$ respectively. Note that $\rho^\sigma(H^{(\sigma-1)/2}) = 0$ and $\rho^\sigma(H^{(\sigma-1)/2-\theta}) = 1$ for all $\theta > 0$. Moreover, define the weights

$$(\theta_N^2)^\sharp(f) := \zeta(\|\Pi_N f\|_{L^2}^2) \zeta(E^1[\Pi_N f] - \alpha_N) e^{-R^2[\Pi_N f]}, \quad (3.3)$$

$$(\theta_N^3)^\sharp(f) := \zeta(\|\Pi_N f\|_{L^2}^2) \zeta(E^1[\Pi_N f]) \zeta(E^2[\Pi_N f] - \alpha_N) e^{-R^3[\Pi_N f]}, \quad (3.4)$$

where ζ is a smooth cutoff, E^σ and R^σ are defined as in Section 2.1, and

$$\alpha_N = \sum_{n=1}^N \frac{1}{n}; \quad \mathbb{E}_{\rho^\sigma}(E^{\sigma-1}[\Pi_N f]) = \alpha_N + O(1).$$

We then define the measures $(\nu_N^\sigma)^\sharp$, $(\nu_N^\sigma)^\phi$ exactly as in Section 2.2.1, and restate the result of [51] as follows.

Proposition 3.2.1. *For $\sigma \in \{2, 3\}$, the sequence $(\theta_N^\sigma)^\sharp$ converges in $L^r(d\rho^\sigma)$ to some function θ^σ for all $1 \leq r < \infty$, and if we define ν^σ by $d\nu^\sigma = \theta^\sigma d\rho^\sigma$, then $(\nu_N^\sigma)^\sharp$ converges strongly to ν^σ in the sense that the total variation of their difference tends to zero.*

As with the $\nu = 1$ case, here we cannot use the $(\theta_N^\sigma)^\sharp$ truncation since they are not compatible with (2.10). Instead we will define the weights

$$\theta_N^2(f) := \zeta(\|\Pi_N f\|_{L^2}^2) \zeta(E_N^1[\Pi_N f] - \alpha_N) e^{-R_N^2[f]}, \quad (3.5)$$

$$\theta_N^3(f) := \zeta(\|\Pi_N f\|_{L^2}^2) \zeta(E_N^1[\Pi_N f]) \zeta(E_N^2[\Pi_N f] - \alpha_N) e^{-R_N^3[f]}, \quad (3.6)$$

where $R_N^\sigma[f]$ are suitably chosen truncated versions of $\mathcal{R}^\sigma$, and $E_N^\sigma[f] = \|f\|_{\dot{H}^{\sigma/2}}^2 + R_N^\sigma[f]$ as in Section 2.1; in particular E_N^1 will be the same as the E_N in (2.20).

The precise definitions of R_N^2 and R_N^3 are somewhat involved and will be postponed to the next section. However, the arguments in [51] has enough flexibility so that, as long as R_N^σ is “reasonably” defined (see the next section for the precise conditions), we always have the following

Proposition 3.2.2. *The sequence θ_N^σ converges in $L^r(d\rho^\sigma)$ to the θ^σ defined in Proposition 3.2.1 for all $1 \leq r < \infty$, and ν_N^σ converges strongly to ν^σ defined in Proposition 3.2.1 in the sense that the total variation of their difference tends to zero.*

3.3 The truncated energy E_N^2 and E_N^3

3.3.1 The I -method

We seek for quantities of the form

$$E_N^\sigma[u] = \|u\|_{\dot{H}^{\sigma/2}}^2 + \sum_{j=3}^{\sigma+2} \sum_{a_1+\dots+a_j=0} m_N^{\sigma j}(a_1, \dots, a_j) \prod_{i=1}^j (Su)_{a_i}, \quad (3.7)$$

where each $m_N^{\sigma j}$ is *symmetric*. By (2.10), we can compute that

$$\partial_t E_N^\sigma[u] = \sum_{j=3}^{\sigma+3} \sum_{a_1+\dots+a_j=0} n_N^{\sigma j}(a_1, \dots, a_j) \prod_{i=1}^j (Su)_{a_i}, \quad (3.8)$$

where each $n_N^{\sigma j}$ is also assumed to be symmetric, and is given by

$$n_N^{\sigma 3}(a_1, a_2, a_3) = -i \left(\sum_{i=1}^3 |a_i| a_i \right) m_N^{\sigma 3}(a_1, a_2, a_3) - i \sum_{i=1}^3 \frac{|a_i|^\sigma a_i}{3} \quad (3.9)$$

for $j = 3$, and

$$\begin{aligned} n_N^{\sigma j}(a_1, \dots, a_j) = & -i \left(\sum_{i=1}^j |a_i| a_i \right) m_N^{\sigma j}(a_1, \dots, a_j) + i \sum_{1 \leq k < l \leq j} \frac{a_k + a_l}{j} \psi^2 \left(\frac{a_k + a_l}{N} \right) \\ & \times m_N^{\sigma, j-1}(a_1, \dots, \widehat{a_k}, \dots, \widehat{a_l}, \dots, a_j, a_k + a_l), \end{aligned} \quad (3.10)$$

for $j \geq 4$, with the understanding that $m_N^{\sigma, \sigma+3} \equiv 0$.

By solving the equation $n_N^{\sigma j} \equiv 0$ for $j \in \{3, 4\}$, we obtain that

$$m_N^{\sigma j}(a_1, \dots, a_j) = (-1)^{\sigma-j+2} m_N^{\sigma j}(-a_1, \dots, -a_j); \quad (3.11)$$

$$m_N^{23}(a, b, c) = \frac{c}{2}, \quad m_N^{33}(a, b, c) = ab - 2c^2 \quad \text{if } a > 0, b > 0, c < 0; \quad (3.12)$$

$$m_N^{24}(a, b, c, d) = \frac{1}{16} \left(\psi^2 \left(\frac{a+c}{N} \right) + \psi^2 \left(\frac{a+d}{N} \right) \right) \quad \text{if } a > 0, b > 0, c < 0, d < 0; \quad (3.13)$$

$$m_N^{24}(a, b, c, d) = \frac{1}{16(ab + bc + ca)} \sum_{\text{cyc}} \psi^2 \left(\frac{b+c}{N} \right) a(b+c) \quad \text{if } a > 0, b > 0, c > 0, d < 0; \quad (3.14)$$

$$m_N^{34}(a, b, c, d) = -\frac{1}{24} \left\{ \psi^2 \left(\frac{a+b}{N} \right) (a+b) + \psi^2 \left(\frac{a+c}{N} \right) (2a+2b+|a+c|) + \psi^2 \left(\frac{a+d}{N} \right) (2a+2b+|a+d|) \right\}$$

if $a > 0, b > 0, c < 0, d < 0$; (3.15)

$$m_N^{34}(a, b, c, d) = -\frac{1}{24(ab+bc+ca)} \sum_{\text{cyc}} \psi^2 \left(\frac{b+c}{N} \right) (b+c)(2a^2+3ab+3ac+bc)$$

if $a > 0, b > 0, c > 0, d < 0$, (3.16)

where the sums in (3.14) and (3.16) are cyclic in a, b and c . In addition, we will choose $m_N^{35} \equiv 1/20$ in order to match R^3 for large N . This completes the construction of E_N^σ ; notice that they satisfy the following three conditions, namely

- (1) $R_N^\sigma[f]$ depends only on $\Pi_N f$, that is, $R_N^\sigma[f] = R_N^\sigma[\Pi_N f]$;
- (2) $E_N^\sigma[f] = E^\sigma[f]$ if f is supported in frequency not exceeding $N/4$, that is when $f = \Pi_{N/4} f$;
- (3) $|m_N^{\sigma j}(a_1, \dots, a_j)| \lesssim (\max |a_i|)^{\sigma-j+2}$, as with the terms in E^σ .

By similar arguments as in [51], one can prove that these three conditions will guarantee that Proposition 3.2.2 holds; we omit the proof here.

3.3.2 Quintic terms and the probabilistic bound

Now we know that $\partial_t E_N^\sigma$ contains only quintic and sextic terms. Namely, we have

$$\partial_t E_N^2 = \sum_{a_1+\dots+a_5=0} n_N^{25}(a_1, \dots, a_5) (Su)_{a_1} \cdots (Su)_{a_5}; \quad (3.17)$$

$$\partial_t E_N^3 = \sum_{a_1+\dots+a_5=0} n_N^{35}(a_1, \dots, a_5) (Su)_{a_1} \cdots (Su)_{a_5} + \sum_{a_1+\dots+a_6=0} n_N^{36}(a_1, \dots, a_6) (Su)_{a_1} \cdots (Su)_{a_6}, \quad (3.18)$$

where $n_N^{\sigma 5}$ and $n_N^{\sigma 6}$ are defined by (3.10).

By conditions (2) and (3) in Section 3.3.1 above, we know that $n_N^{\sigma j} = 0$ if $|a_i| \leq N/4$ for all i , and that

$$|n_N^{25}| \lesssim \max_{1 \leq i \leq 5} |a_i|; \quad |n_N^{35}| \lesssim \max_{1 \leq i \leq 5} |a_i|^2, \quad |n_N^{36}| \lesssim \max_{1 \leq i \leq 6} |a_i|.$$

We next claim that actually a stronger bound holds due to cancellation, namely

$$|n_N^{25}| \lesssim |a'_3|, \quad |n_N^{35}| \lesssim |a'_1 a'_3|, \quad \text{if } \{a'_1, \dots, a'_5\} = \{a_1, \dots, a_5\}, |a'_1| \geq \dots \geq |a'_5|. \quad (3.19)$$

In fact, consider say n_N^{35} . Since m_N^{35} is constant, and $|\sum |a_i|a_i| \lesssim |a'_1 a'_3|$, the first term in (3.10) will satisfy (3.19). As for the second term, we only need to study the case where $1 \leq k \leq 2$ and $3 \leq l \leq 5$ (the other terms already satisfy (3.19)). For these we simply pair the two terms with the same l , and use the bound for derivatives of m_N^{34} , which can be checked explicitly, to achieve the cancellation. The same argument works also for n_N^{25} .

We can now prove the crucial probabilistic bound (3.2), which is restated in the following

Proposition 3.3.1. *We have, for $\sigma \in \{2, 3\}$, that*

$$\mathbb{E}_{\rho^\sigma} \left(\left. \frac{d}{dt} E_N^\sigma [\Phi_t^N \Pi_N f] \right|_{t=0} \right)^2 \lesssim N^{-2} \log^2 N. \quad (3.20)$$

Proof. The time derivative in (3.20) equals the right hand side of (3.17) or (3.18) with u replaced by f . Since the expectation is taken with respect to the measure ρ^σ , we only need to bound

$$E^{\sigma j} := \mathbb{E} \left| \sum_{a_1 + \dots + a_j = 0} n_N^{\sigma j}(a_1, \dots, a_j) \prod_{i=1}^j \psi\left(\frac{a_i}{N}\right) \frac{g_{a_i}(\omega)}{|a_i|^{\sigma/2}} \right|^2,$$

with $n_N^{\sigma j}$ satisfying the requirements above. Below we will only prove the bound for E^{25} , since the other two cases are similar (and E^{36} is much easier).

To bound E^{25} , first consider the sum where no two a_i add up to zero. By the property of Gaussians, we know that the summands here are actually pairwise orthogonal, thus this contribution is bounded by

$$E_1^{25} \lesssim \sum_{\substack{a_1 + \dots + a_5 = 0, \\ N/4 \leq \max |a_i| \leq N}} |a'_3|^2 \prod_{i=1}^5 \frac{1}{|a_i|^2} \lesssim \prod_{i=1}^2 \sum_{|a'_i| \gtrsim N} \frac{1}{|a'_i|^2} \prod_{i=4}^5 \sum_{a'_i} \frac{1}{|a'_i|^2} \lesssim N^{-2}.$$

Now suppose (say) $a_1 + a_2 = 0$, then we reduce the the sum

$$\sum_{a_1} \frac{|g_{a_1}(\omega)|^2}{|a_1|^2} \sum_{a_3 + a_4 + a_5 = 0} \widetilde{n_N^{25}}(a_1, a_3, a_4, a_5) \prod_{i=3}^5 \frac{g_{a_i}(\omega)}{|a_i|}.$$

To bound this sum in L_ω^2 , we notice that the summands with a fixed a_1 are now pairwise orthogonal, thus this contribution is bounded by

$$E_2^{25} \lesssim \left[\sum_{a_1} \frac{1}{|a_1|^2} \left(\sum_{a_3 + a_4 + a_5 = 0} \frac{|a'_3|^2}{|a_3 a_4 a_5|^2} \right)^{1/2} \right]^2.$$

Suppose $|a_3| \geq |a_4| \geq |a_5|$, we know that either $|a_1| \gtrsim N/4$ and $|a'_3| \lesssim |a_3|$, or $\min(|a_3|, |a_4|) \gtrsim N$

and $|a'_3| \lesssim \max(|a_1|, |a_5|)$. Therefore

$$E_2^{25} \lesssim \left(\sum_{|a_1| \gtrsim N} \frac{1}{|a_1|^2} \right)^2 + \sum_{|a_3|, |a_4| \gtrsim N} \frac{1}{|a_3 a_4|^2} + \sum_{|a_3|, |a_4| \gtrsim N} \frac{1}{|a_3 a_4|^2} \sum_{a_1} \frac{1}{|a_1|} \lesssim N^{-2} \log^2 N.$$

□

3.4 Deterministic Theory

In this section we review the deterministic theory for (2.1) and (2.10) on the support of ν^2 and ν^3 .

We shall prove the following

Proposition 3.4.1. *Let $0 < \theta < \theta'$ small enough and $A > 0$ be fixed. We have, for some $T = T(\theta, \theta', A) > 0$, $C = C(\theta, \theta', A) > 0$ that:*

$$\begin{aligned} \|\Phi_t^N \Pi_N f\|_{H^{1/2-\theta}} + \|\Phi_t f\|_{H^{1/2-\theta}} &\leq C, \quad \|\Phi_t^N \Pi_N f - \Phi_t f\|_{H^{1/2-\theta'}} \leq CN^{-c}, \\ &\text{provided that } \|f\|_{H^{1/2-\theta}} \leq A \text{ and } |t| \leq T. \end{aligned} \quad (3.21)$$

Note that, though much easier than Theorem 2.6.1 in Chapter 2, the result here is not its logical consequence, since preservation of regularity is not discussed there.

Proof. First, we shall prove the bound for $\Phi_t^N f$. The bound for $\Phi_t f$ is proved similarly, and the bound for the difference follows from a standard procedure of taking differences, which will be briefly described at the end of the proof. Therefore from now on we will fix one N . Let $u(t) = \Phi_t^N f$.

Recall the notations in Section 2.1.1. Let the standard spaces be

$$\|u\|_{X^{s,b}} = \|u^{(s,b)}\|_{l^2 L^2}; \quad \|u\|_{Y^s} = \|u^{(s,0)}\|_{l^2 L^1},$$

and define

$$\|u\|_U = \|u\|_{X^{-1/2-\theta, 12/25}} + \|u\|_{Y^{1/2-\theta}}; \quad \|w\|_W = \|w\|_{X^{1/2-\theta, 1/2+\theta}}, \quad (3.22)$$

and the localized versions U^T and W^T in the same way as Section 2.3.1. Let the Duhamel evolution operator (with time truncation) be $\mathcal{E}$, we shall use Strichartz and evolution bounds for these norms, whose proofs can be found in Lemmas 2.3.1 and 2.3.3.

In the proof below we will denote

$$s = 1/2 - \theta, \quad s' = 1/2 - \theta', \quad r = 1/2 + \theta = 1 - s,$$

and we shall control the gauged variable w in W^T , and the original function u in U^T .

Our starting point is the first reduction of (2.1) using the gauge transform, namely (2.85). Using the substitutions (2.88) and (2.89) in the same way as in Section 2.4.2, we can replace the major inputs u_{n_l} in the “quadratic” part of (2.85) by $w_{n_l}^\pm$; for convenience we rewrite the equation below, which is

$$\begin{aligned} (\partial_t - i\partial_{xx})w_{n_0} = & \sum_{\mu} C_{\mu} \left[\sum_{\substack{n_1+n_2+m_1+\dots+m_{\mu}=n_0, \\ \max |m_i| \ll \min(|n_0|, |n_1|, |n_2|)}} O(1) \min(|n_0|, |n_1|, |n_2|) w_{n_1}^\pm w_{n_2}^\pm \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \right. \\ & \left. + \sum_{n_1+n_2+n_3+m_1+\dots+m_{\mu}=n_0} O(1) u_{n_1} u_{n_2} u_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i} \right]. \end{aligned} \quad (3.23)$$

We make the bootstrap assumptions

$$\|u\|_{U^T} + \|w\|_{W^T} \leq A', \quad (3.24)$$

where A' is large enough depending on A . This holds for short time T , due to the fact that $\|w(0)\|_{H^s} \leq O_A(1)$, which can be proved easily using the same arguments as in Proposition 2.5.3.

Now we only need to improve the right hand side of (3.24) to $O_A(1)$. Let u' and w' be extensions of w and u such that $\|u'\|_U \leq 2A'$ and $\|w'\|_W \leq 2A'$, we shall first construct another extension w^* of w by

$$w^* = \chi(t) e^{it\partial_{xx}} w(0) + \mathcal{E}(\chi(T^{-1}t) \cdot (\mathcal{N}_2 + \mathcal{N}_3)),$$

where χ is a cutoff, $\mathcal{N}_2$ and $\mathcal{N}_3$ are respectively the “quadratic” and “cubic” terms on the right hand side of (3.23) with w replaced by w' and u replaced by u' . Now to bound w^* , we only need to bound $\|\chi(T^{-1}t)\mathcal{N}_j\|_{X^{s,r-1}}$ for $j \in \{2, 3\}$; by duality, it suffices to bound

$$\mathcal{S}_2 := \sum_{(n,m)} \int_{(\xi,\eta)} O(1) \min_{0 \leq l \leq 2} |n_l| \cdot (h')_{n_0, \xi_0} \prod_{l=1}^2 (w')_{n_l, \xi_l}^\pm \prod_{i=1}^{\mu} \frac{(u')_{m_i, \eta_i}}{m_i},$$

and

$$\mathcal{S}_3 := \sum_{(n,m)} \int_{(\xi,\eta)} O(1) \cdot (h')_{n_0, \xi_0} \prod_{l=1}^3 (u')_{n_l, \xi_l} \prod_{i=1}^{\mu} \frac{(u')_{m_i, \eta_i}}{m_i},$$

where the summation and integration are taken over the set

$$\left\{ n_0 = \sum_{l=1}^2 n_l + \sum_{i=1}^{\mu} m_i, \quad \xi_0 = \sum_{l=1}^2 \xi_l + \sum_{i=1}^{\mu} \eta_i + \Xi \right\}, \quad \Xi = |n_0|n_0 - \sum_{l=1}^2 |n_l|n_l - \sum_{i=1}^{\mu} |m_i|m_i$$

for $\mathcal{S}_2$, over

$$\left\{ n_0 = \sum_{l=1}^3 n_l + \sum_{i=1}^{\mu} m_i, \quad \xi_0 = \sum_{l=1}^3 \xi_l + \sum_{i=1}^{\mu} \eta_i + \Xi \right\}, \quad \Xi = |n_0|n_0 - \sum_{l=1}^3 |n_l|n_l - \sum_{i=1}^{\mu} |m_i|m_i$$

for $\mathcal{S}_3$, and h' is such that $\|h'\|_{X^{-s,1/2-2\theta}} \lesssim T^c$.

Consider now $\mathcal{S}_3$; by symmetry we may assume $|n_0| \lesssim |n_1|$ or $|n_0| \lesssim |m_1|$. In the former case

$$|\mathcal{S}_3| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |n_0|^s |n_1|^{-s} (\mathfrak{N}(h')^{(-s,0)})_{n_0,\xi_0} (\mathfrak{N}(u')^{(s,0)})_{n_1,\xi_1} \prod_{l=2}^3 (\mathfrak{N}u')_{n_l,\xi_l} \prod_{i=1}^{\mu} (\mathfrak{N}(u')^{(-1,0)})_{m_i,\eta_i},$$

which is then bounded (using Hölder) by

$$\|\mathfrak{N}(h')^{(-s,0)}\|_{L_{t,x}^4} \|\mathfrak{N}(u')^{(s,0)}\|_{L_{t,x}^2} \|\mathfrak{N}(u')\|_{L_{t,x}^8}^2 \|\mathfrak{N}(u')^{(-1,0)}\|_{L_{t,x}^\infty}^\mu \lesssim T^\theta (O_{A'}(1))^\mu,$$

which is acceptable. In the latter case

$$|\mathcal{S}_3| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |n_0|^s |m_1|^{-s} (\mathfrak{N}(h')^{(-s,0)})_{n_0,\xi_0} (\mathfrak{N}(u')^{(-1+s,0)})_{m_1,\eta_1} \prod_{l=1}^3 (\mathfrak{N}u')_{n_l,\xi_l} \prod_{i=2}^{\mu} (\mathfrak{N}(u')^{(-1,0)})_{m_i,\eta_i},$$

which is then bounded (using Hölder) by

$$\|\mathfrak{N}(h')^{(-s,0)}\|_{L_{t,x}^4} \|\mathfrak{N}(u')^{(-1+s,0)}\|_{L_{t,x}^\infty} \|\mathfrak{N}(u')\|_{L_{t,x}^4}^3 \|\mathfrak{N}(u')^{(-1,0)}\|_{L_{t,x}^\infty}^\mu \lesssim T^\theta (O_{A'}(1))^\mu.$$

Next consider $\mathcal{S}_2$. We may assume that $\min |n_l| \gg \max |m_i|$; otherwise we will reduce it to $\mathcal{S}_3$.

Now by elementary algebra we have

$$\sum_{l=0}^2 |\xi_l| + \sum_{i=1}^{\mu} |\eta_i| \gtrsim PQ$$

where

$$P = \max_{0 \leq l \leq 2} |n_l|, Q = \min_{0 \leq l \leq 2} |n_l|; \quad |n_0|^s |n_1|^{-s+\theta} |n_2|^{-s+\theta} \min_{0 \leq l \leq 2} |n_l| \lesssim P^\theta Q^{1/2+2\theta} \lesssim (PQ)^{13/50}.$$

Now if $|\beta_1| \gtrsim PQ$, we have

$$|\mathcal{S}_2| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |n_0|^s |n_1 n_2|^{-s+\theta} \min_{0 \leq l \leq 2} |n_l| \cdot |\eta_1|^{-13/50} (\mathfrak{N}(h')^{(-s,0)})_{n_0, \xi_0} (\mathfrak{N}(w')^{(s-\theta,0)})_{n_1, \xi_1}^{\pm} \\ \times (\mathfrak{N}(w')^{(s-\theta,0)})_{n_2, \xi_2}^{\pm} (\mathfrak{N}(u')^{(-1,13/50)})_{m_1, \eta_1} \prod_{j=2}^{\mu} (\mathfrak{N}(u')^{(-1,0)})_{m_j, \eta_j}. \quad (3.25)$$

Notice that $\mathfrak{N}(u')^{(-1,0)} \in L_{t,x}^{\infty}$, and that

$$\mathfrak{N}(h')^{(-s,0)} \in X^{0,1/2-2\theta} \subset L_{t,x}^4, \quad \mathfrak{N}(w')^{(s-\theta,0)} \in X^{\theta,r} \subset L_{t,x}^6, \quad \mathfrak{N}(u')^{(-1,13/50)} \in X^{1/2-\theta,11/50} \subset L_{t,x}^{12/5}$$

by interpolation, we can use Hölder to bound $|\mathcal{S}_2| \lesssim T^c (O_{A'}(1))^{\mu+2}$ with some positive c .

If (say) $|\xi_1| \gtrsim PQ$, we have

$$|\mathcal{S}_1| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |n_0|^s |n_1 n_2|^{-s+\theta} \min_{0 \leq l \leq 2} |n_l| \cdot |\xi_1|^{-13/50} (\mathfrak{N}(h')^{(-s,0)})_{n_0, \xi_0} \\ \times (\mathfrak{N}(w')^{(s-\theta,13/50)})_{n_1, \xi_1}^{\pm} (\mathfrak{N}(w')^{(s-\theta,0)})_{n_2, \xi_2}^{\pm} \prod_{j=1}^{\mu} (\mathfrak{N}(u')^{(-1,0)})_{m_j, \eta_j}, \quad (3.26)$$

which is bounded by $|\mathcal{S}_2| \lesssim T^c (O_{A'}(1))^{\mu+2}$ using that

$$\mathfrak{N}(h')^{(-s,0)} \in X^{0,1/2-2\theta} \subset L_{t,x}^4, \quad \mathfrak{N}(w')^{(s-\theta,0)} \in X^{\theta,r} \subset L_{t,x}^4, \quad \mathfrak{N}(w')^{(s-\theta,13/50)} \in X^{\theta,11/50} \subset L_{t,x}^2.$$

If $|\xi_0| \gtrsim PQ$, then

$$|\mathcal{S}_2| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |n_0|^s |n_1 n_2|^{-s+\theta} \min_{0 \leq l \leq 2} |n_l| \cdot |\xi_0|^{-13/50} (\mathfrak{N}(h')^{(-s,13/50)})_{n_0, \xi_0} \\ \times (\mathfrak{N}(w')^{(s-\theta,0)})_{n_1, \xi_1}^{\pm} (\mathfrak{N}(w')^{(s-\theta,0)})_{n_2, \xi_2}^{\pm} \prod_{j=1}^{\mu} (\mathfrak{N}(u')^{(-1,0)})_{m_j, \eta_j}, \quad (3.27)$$

and use that

$$\mathfrak{N}(h')^{(-s,13/50)} \in X^{0,1/5} \subset L_{t,x}^2, \quad \mathfrak{N}(w')^{(s-\theta,0)} \in X^{\theta,r} \subset L_{t,x}^4$$

to bound $|J| \lesssim T^{\theta} (O_{A'}(1))^{\mu+2}$. This completes the estimate for w ; in fact, we have constructed some w^* such that $w^* = w$ on $[-T, T]$, and that

$$\|w^*\|_{X^{s',r}} \leq O_A(1) + T^{\theta} O_{A'}(1) = O_A(1).$$

Now we turn to the bound for u ; clearly we only need to consider $\pi_{>0}u$. First, let u^* be defined by $\pi_{>K}u^* = \pi_{>K}u'$, and

$$\pi_{[0,K]}u^* = \chi(t)e^{-\mathcal{H}\partial_{xx}}\pi_{[0,K]}u(0) + \mathcal{E}(\chi(T^{-1}t) \cdot \pi_{[0,K]}S(Su' \cdot S(u')_x)),$$

where K is a quantity which is large compared to A' , but small compared to T^{-1} . We claim that $\|\langle \partial_x \rangle^{-\sigma} u^*\|_V = O_A(1)$; in fact, first we have

$$\|\langle \partial_x \rangle^{-\sigma} \pi_{>K}u^*\|_V \lesssim K^{-\sigma} \|u'\|_V \lesssim K^{-\sigma} O_{A'}(1) = O_A(1).$$

For $\pi_{[0,K]}u^*$, the initial data term is also bounded trivially, and the Duhamel term is also bounded since

$$\begin{aligned} \|\chi(T^{-1}t)\pi_{[0,K]}S(Su' \cdot S(u')_x)\|_{X^{2,r-1}} &\lesssim T^\theta K^3 \|\pi_{[0,K]}S(Su')^2\|_{X^{0,2\theta-1/2}} \\ &\lesssim T^\theta K^3 \|(Su')^2\|_{L_{t,x}^{4/3}} \lesssim T^\theta K^3 O_{A'}(1) = O_A(1). \end{aligned}$$

In the same way, we also have that $\|u^*\|_V = O_{A'}(1)$. Next, define u^{**} by $\pi_{[0,K]}u^{**} = \pi_{[0,K]}u^*$, and

$$\pi_{>K}u^{**} = \pi_{>K}(M^*)^{-1}(w^* + \pi_{\leq 0}M^*u^*).$$

where M^* is an operator defined in the same way as M , but with u replaced by u^* . Clearly $u = u^{**}$ on $[-T, T]$, so we only need to show that $\|\pi_{>K}u^{**}\|_V \lesssim O_A(1)$.

The Y^s norm is easily bounded using the structure of the operator M^* and the simple fact that translation in Fourier space acts well in Y^s . If we are considering the term $\pi_{>K}(M^*)^{-1}w^*$, then we get a bound of $O_A(1)$ in Y^s norm due to the control for w^* we just obtained; as for $\pi_{>K}(M^*)^{-1}\pi_{\leq 0}M^*u^*$, we have a bound of $O_{A'}(1)$, but due to the two projectors $\pi_{>K}$ and $\pi_{\leq 0}$, we must have at least one $|m_i| \gtrsim K$ in the sum; this gains us a small power K^{-c} , which improves our bound to $K^{-c}O_{A'}(1) = O_A(1)$.

Next we prove that

$$\|\pi_{>K}(M^*)^{-1}w^*\|_{X^{-1/2-\theta, 12/25}} \leq O_A(1).$$

Again by duality, we only need to bound

$$\mathcal{S}_1 = \sum_{(n,m)} \int_{(\xi,\eta)} O(1) \cdot h_{n_0,\xi_0}(w^*)_{n_1,\xi_1} \prod_{i=1}^{\mu} \frac{(u^*)_{m_i,\eta_i}}{m_i}, \quad (3.28)$$

where $\|h\|_{X^{1/2+\theta,-12/25}} \leq 1$, and the summation and integration are taken over the set

$$\left\{ n_0 = n_1 + \sum_{i=1}^{\mu} m_i, \quad \xi_0 = \xi_1 + \sum_{i=1}^{\mu} \eta_i + \Xi \right\}, \quad \Xi = |n_0|n_0 - |n_1|n_1 - \sum_{i=1}^{\mu} |m_i|m_i.$$

In (3.28), assume first that $|m_i| \ll \max(|n_0|, |n_1|)$, then

$$|\xi_0| \lesssim |\Xi| + |\xi_1| + \sum_{i=1}^{\mu} |\eta_i|; \quad |\Xi| \lesssim |n_0| \cdot \max_{1 \leq i \leq \mu} |m_i|.$$

If $|\xi_0| \lesssim |n_0 m_1|$, then we can bound

$$\begin{aligned} |\mathcal{S}_1| &\lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |n_0 m_1|^{-1/2} |\xi_0|^{12/25} (\mathfrak{N} h^{(1/2,-12/25)})_{n_0, \xi_0} \\ &\quad \times (\mathfrak{N} w^*)_{n_1, \xi_1} (\mathfrak{N} (u^*)^{(-1/2,0)})_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathfrak{N} (u^*)^{(-1,0)})_{m_i, \eta_i} \end{aligned} \quad (3.29)$$

which is bounded by $O_A(1)$ by estimating h and w^* factors in $L_{t,x}^2$ and other factors in $L_{t,x}^\infty$, using the bound $\|\langle \partial_x \rangle^{-\theta} u^*\|_{Y^{s'}} \leq O_A(1)$. If $|\xi_0| \lesssim |\xi_1|$, then we will bound

$$|\mathcal{S}_1| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |\xi_0|^{12/25} |\xi_1|^{-r} (\mathfrak{N} h^{(0,-12/25)})_{n_0, \xi_0} (\mathfrak{N} (w^*)^{(0,r)})_{n_1, \xi_1} \prod_{i=1}^{\mu} (\mathfrak{N} (u^*)^{(-1,0)})_{m_i, \eta_i}, \quad (3.30)$$

and estimate the h and w^* factors in $L_{t,x}^2$ and other factors in $L_{t,x}^\infty$. If $|\xi_0| \lesssim |\eta_1|$, then we will bound

$$\begin{aligned} |\mathcal{S}_1| &\lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |\xi_0|^{12/25} |\eta_1|^{-12/25} (\mathfrak{N} h^{(0,-12/25)})_{n_0, \xi_0} \\ &\quad \times (\mathfrak{N} w^*)_{n_1, \xi_1} (\mathfrak{N} (u^*)^{(-1,12/25)})_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathfrak{N} (u^*)^{(-1,0)})_{m_i, \eta_i}. \end{aligned} \quad (3.31)$$

We then estimate in Fourier space and bound the h factor in $l_n^{4/3} L_\xi^2$, the w^* factor in $l_n^2 L_\xi^1$, the separate u^* factor in $l_m^{4/3} L_\eta^2$, and all the other u^* factors in $l_m^1 L_\xi^1$.

Now assume in (3.28) that $R := |m_1| \gtrsim \max(|n_0|, |n_1|, |m_i|)$ for $i > 1$, then we have $|\Xi| \lesssim R^2$. If $|\xi_0| \lesssim \max(|\xi_1|, |\eta_1|)$, then we can use exactly the same arguments as above (they do not require any information on the relative sizes of n_j and m_i). Now let us assume that $|\xi_0| \lesssim R^2$, then we have

$$|\mathcal{S}_1| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |\xi_0|^{12/25} |m_1|^{-1} (\mathfrak{N} h^{(0,-12/25)})_{n_0, \xi_0} (\mathfrak{N} w^*)_{n_1, \xi_1} (\mathfrak{N} u^*)_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathfrak{N} (u^*)^{(-1,0)})_{m_i, \eta_i}, \quad (3.32)$$

which is bounded by $O_A(1)$, by estimating the h factor in $l_n^{4/3} L_\xi^2$, the w^* factor in $l_n^2 L_\xi^2$, the separate

u^* factor in $l_n^{4/3}L_\xi^1$, and all the other u^* factors in $l_m^1L_\xi^1$.

The estimate for

$$\|\pi_{>K}(M^*)^{-1}\pi_{\leq 0}M^*u^*\|_{X_{-1/2-\sigma',12/25}}$$

can be done basically in the same way as above. Notice that in the above argument, except in the case when $|\xi_0| \lesssim |\xi_1|$, we can close the estimate using only the V norm for w^* ; thus this proof will also work for u^* . The bound we get for now is only $O_{A'}(1)$, but due to the projections $\pi_{>K}$ and $\pi_{\leq 0}$, there must be one m_i (say m_1) such that $|m_i| \gtrsim \max(|n_1|, K)$, thus we gain a power of K (from the fact that the bounds used involving m_1 are not sharp) and thus recover the $O_A(1)$ bound.

If $|\xi_0| \lesssim |\xi_1|$, then again there must be one m_i (say m_1) such that $|m_1| \gtrsim \max(|n_1|, K)$. Now instead of (3.30) we have

$$\begin{aligned} |\mathcal{S}_1| &\lesssim \sum_{(n,m)} \int_{(\xi,\eta)} |\xi_0|^{12/25} |\xi_1|^{-12/25} |n_1|^{1/2+\sigma'} |m_1|^{-1/2-\theta} (\mathfrak{N}h^{(0,-12/25)})_{n_0,\xi_0} \\ &\quad \times (\mathfrak{N}(u^*)^{(-1/2-\theta,12/25)})_{n_1,\xi_1} (\mathfrak{N}(u^*)^{(-1/2+\theta,0)})_{m_1,\eta_1} \prod_{i=2}^{\mu} (\mathfrak{N}(u^*)^{(-1,0)})_{m_i,\eta_i} \end{aligned} \quad (3.33)$$

so we can bound the h factor and the first u^* factor in $L_{t,x}^2$, and other factors in $L_{t,x}^\infty$. Again, here we gain a power of K and thus can improve the $O_{A'}(1)$ bound to an $O_A(1)$ bound.

Finally, we show how to estimate the differences. Introduce the spaces

$$\|u\|_{U'} = \|u\|_{X_{-1/2-\theta',12/25}} + \|u\|_{Y^{1/2-\theta'}}; \quad \|w\|_{W'} = \|w\|_{X^{1/2-\theta',1/2+\theta}}, \quad (3.34)$$

and denote now $u^N = \Phi_t^N f$ and $u = \Phi_t f$. Let $((u^N)', u'')$ and $((w^N)', w')$ be constructed like above, so that they are bounded in U and W respectively. Note that all the above arguments remain valid if U is replaced by U' and W is replaced by W' ; we will use this to analyze the difference

$$\gamma := \|(u^N)' - u'\|_{U'} + \|(w^N)' - w'\|_{W'}.$$

In fact, when doing this, the nonlinearity will involve two kinds of terms: one involves a factor of γ , and the other involves terms where at least one of n_j or m_i is $\gtrsim N$. When bounding the second kind of terms, we gain a small power of N since the corresponding factor is bounded in a stronger space, and we only estimate it in a weaker space. In this way we get that

$$\gamma = T^\theta O_{A'}(1)\gamma + O(N^{-c})$$

for some positive c , thus $\gamma = O(N^{-c})$ as $N \rightarrow \infty$. $\square$

3.5 Proof of the Theorem 3.1.1

In this section we prove Theorem 3.1.1. First we need a lemma, which is basically (3.1) and essentially follows from Proposition 3.3.1.

Lemma 3.5.1. *Recall that each Φ_t^N is a map from $\mathcal{V}_N$ to itself; let $(\nu_N^\sigma)^\circ$ be such that $\nu_N^\sigma = (\nu_N^\sigma)^\circ \times (\rho_N^\sigma)^\perp$ as in Section 2.2.1. For each Borel set A of $H^{1/2-\theta}$ and each time t , we have*

$$\left| \frac{d}{dt} \int \mathbf{1}_{\Phi_t^N \Pi_N(A)} d(\nu_N^\sigma)^\circ \right| \lesssim N^{-1} \log N. \quad (3.35)$$

Proof. Let $B = \Pi_N A$; first assume here $\sigma = 2$. Recall that

$$\int \mathbf{1}_{\Phi_t^N(B)} d(\nu_N^\sigma)^\circ = \int_{\Phi_t^N(B)} \zeta(\|f\|_{L^2}^2) \zeta(E_N^1[f] - \alpha_N) e^{-E_N^2[f]} d\mathcal{L}_N,$$

where $\mathcal{L}_N$ is the Lebesgue measure; note that it is invariant under $\Psi = \Phi_t^N$. By changing variables and noticing that $\|f\|_{L^2}^2$ and $E_N^1[f]$ are conserved under Ψ , we have

$$\begin{aligned} \frac{d}{dt} \int \mathbf{1}_{\Phi_t^N \Pi_N(A)} d(\nu_N^\sigma)^\circ &= \frac{d}{dt} \int_B \zeta(\|\Psi f\|_{L^2}^2) \zeta(E_N^1[\Psi f] - \alpha_N) e^{-E_N^2[\Psi f]} d\mathcal{L}_N \\ &= \int_B (-\partial_t E_N^2[\Psi f]) \zeta(\|\Psi f\|_{L^2}^2) \zeta(E_N^1[\Psi f] - \alpha_N) e^{-E_N^2[\Psi f]} d\mathcal{L}_N \\ &= \int_{\Psi B} (-\partial_t E_N^2[\Psi f]|_{t=0}) \theta_N^2(f) d\rho_N, \end{aligned} \quad (3.36)$$

where in the last step we have used again the invariance of $\mathcal{L}_N$. Using Cauchy-Schwartz, Proposition 3.3.1 and the uniform L^2 bound for θ_N^2 , we can bound this expression by $O(N^{-1} \log N)$.

The proof for $\sigma = 3$ goes in the same way, except that there will now be a term involving $\partial_t E_N^2$, but this is again easily estimated using the arguments in the proof of Proposition 3.3.1. $\square$

Now we complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Given a time T and a set $A \subset H^{1/2-\theta}$, we need to prove that $\nu^\sigma(A) \leq \nu^\sigma(\Phi_T A)$. By approximation we may assume A is compact in $H^{1/2-\theta}$. Fix $L > 0$, guaranteed by [41] to exist, such that $\Phi_t A \subset B_L(H^{1/2-\theta})$ for every $t \in [-T, T]$.

By Lemma 3.5.1 we have that

$$\nu_N^\sigma(A) \leq \nu_N^\sigma(\Pi_N A \times \mathcal{V}_N^\perp) \leq \nu_N^\sigma(\Phi_t^N \Pi_N A \times \mathcal{V}_N^\perp) + CN^{-1} \log N \cdot |t|; \quad (3.37)$$

on the other hand we have by Proposition 3.4.1 that there exist $t_1 = t_1(L) > 0$ and $C = C(L) > 0$ such that

$$\|\Phi_t^N \Pi_N f - \Pi_N \Phi_t f\|_{H^{1/2-\theta'}} \leq CN^{-c}$$

for some $c > 0$ provided that $\|u\|_{H^{1/2-\theta}} \leq L$ and $|t| \leq t_1$, which implies for $|t| \leq t_1$ that

$$\Phi_t^N \Pi_N A \times \mathcal{V}_N^\perp \subset Y_N := \Pi_N \Phi_t A \times \mathcal{V}_N^\perp + B_{CN^{-c}}(H^{-1/2-\theta'})$$

and thus $\nu_N^\sigma(A) \leq \nu_N^\sigma(Y_N) + o_{N \rightarrow \infty}(1)$. By compactness of A (and also $\Phi_t A$) it is easy to see that $\{Y_N\}$ is decreasing and converges to $\Phi_t A$, thus by taking the $N \rightarrow \infty$ limit and using Proposition 3.2.2 we have that $\nu^\sigma(A) \leq \nu^\sigma(\Phi_t A)$ for $|t| \leq t_1$. Choosing $t = T/K$ for some large enough positive integer K , we may iterate this inequality to get $\nu^\sigma(A) \leq \nu^\sigma(\Phi_T A)$. This completes the proof. $\square$

Chapter 4

Multi-speed Klein-Gordon systems in dimension three

4.1 Introduction

We now turn to the study of dispersive equations on Euclidean spaces. The goal of this and the next chapter is to analyze global behavior of small solutions to quasilinear dispersive systems in $\mathbb{R}^d$; in particular, we will develop a useful method of proving small data scattering, based on the general scheme of *spacetime resonance* introduced by Germain-Masmoudi-Shatah and the *Z-norm method* introduced by Ionescu-Pausader.

In this chapter we shall consider a system of quasilinear Klein-Gordon equations in space dimension *three*, namely

$$(\partial_t^2 - c_\alpha^2 \Delta + b_\alpha^2)u_\alpha = \mathcal{Q}_\alpha(u, \partial u, \partial^2 u), \quad 1 \leq \alpha \leq d, \quad (4.1)$$

where the speeds c_α and masses b_α are arbitrary, and $\mathcal{Q}$ is a quadratic, quasilinear nonlinearity satisfying a suitable symmetry condition.

The system (4.1) models many important equations from physics; one particular example is the irrotational *Euler-Maxwell* system in plasma physics, which reduces to (4.1) with $d = 2$ and $b_1 = b_2 = 1$; with this specific choice of masses, we can actually solve the (much harder) $2D$ problem, and this will be briefly discussed in the next chapter.

The problem of small data global existence for (4.1) has been studied in many previous works; the difficulty of this problem decreases with higher space dimension, and also depends on the exact

choices of speeds c_α and masses b_α . The easiest case is a *single* Klein-Gordon equation in $3D$, and was first solved independently by Klainerman [36] and Shatah [46]; in the case of (4.1) with the *same* speed ($c_\alpha = 1$) and multiple masses, Hayashi-Naumkin-Wibowo [29] showed global existence in $3D$, and Delort-Fang-Xue [14] proved the same result in $2D$ under a non-resonance condition. Note that the above works are more or less based on physical space analysis.

The case of (4.1) with multiple speeds is significantly harder, even in $3D$. In recent years there have been works developing new, Fourier-based techniques to solve these problems; these include for example Germain [20] where a semilinear version of (4.1) with the same mass was considered in $3D$, and Ionescu-Pausader [32], where the authors treated the general system (4.1) in $3D$, with the speeds and masses (b_α, c_α) satisfying two nondegeneracy conditions. Moreover, the techniques involved in all these works have also been used in analyzing many other dispersive equations or systems which are not necessarily Klein-Gordon, see for example [21], [23], [24], [28], [26], [33], [34].

In this chapter we will completely solve the small data problem of (4.1) in $3D$, by removing the assumptions made in [32]. More precisely, we will prove the following

Theorem 4.1.1. *We make the following assumptions on (4.1).*

1. *The nonlinearity $\mathcal{Q}_\alpha$ in (4.1) has the form*

$$\mathcal{Q}_\alpha(u, \partial u, \partial^2 u) = \sum_{\beta, \gamma=1}^d \sum_{j,k,l=1}^3 (A_{\alpha\beta\gamma}^{jk} u_\gamma + B_{\alpha\beta\gamma}^{jkl} \partial_l u_\gamma) \partial_j \partial_k u_\beta + \mathcal{Q}'_\alpha(u, \partial u), \quad (4.2)$$

where A and B are tensors symmetric in α and β with norm not exceeding one, and $\mathcal{Q}'_\alpha$ is an arbitrary quadratic form of u and ∂u ;

2. *The initial data $u(0) = g, \partial_t u(0) = h$ satisfy the bound*

$$\|(g, \partial_x g, h)\|_{\mathcal{H}^N} + \|(g, \partial_t g, h)\|_Z \leq \varepsilon \leq \varepsilon_0, \quad (4.3)$$

where $N = 1000$, the norm

$$\|u\|_{\mathcal{H}^k} = \sup_{|\beta| \leq k} \|\Gamma^\beta u\|_{L^2}, \quad \Gamma = (\partial_i, x_i \partial_j - x_j \partial_i)_{1 \leq i < j \leq 3}, \quad (4.4)$$

the Z norm is defined in Definition 4.2.4 below, and ε is small enough depending on b_α and c_α .

Then we have the followings:

1. The system (4.1) has a unique solution u , with prescribed initial data, such that

$$u \in C_t^1 \mathcal{H}_x^N(\mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^d), \quad \partial_x u \in C_t^0 \mathcal{H}_x^N(\mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^d); \quad (4.5)$$

2. We have the estimates

$$\|u(t)\|_{\mathcal{H}^N} + \|(\partial_x, \partial_t)u(t)\|_{\mathcal{H}^N} \lesssim \varepsilon, \quad (4.6)$$

$$\sum_{|\mu| \leq N/2} (\|\Gamma^\mu u(t)\|_{L^\infty} + \|\Gamma^\mu (\partial_x, \partial_t)u(t)\|_{L^\infty}) \lesssim \frac{\varepsilon}{(1+|t|) \log^{20}(2+|t|)}; \quad (4.7)$$

3. There exist $\mathbb{R}^d$ valued functions $w^\pm$ verifying the linear equation

$$(\partial_t^2 - c_\alpha^2 \Delta + b_\alpha^2)w_\alpha^\pm = 0, \quad (4.8)$$

such that we have scattering in slightly weaker spaces.

$$\lim_{t \rightarrow \pm\infty} (\|u(t) - w^\pm(t)\|_{\mathcal{H}^{N-1}} + \|(\partial_x, \partial_t)(u(t) - w^\pm(t))\|_{\mathcal{H}^{N-1}}) = 0. \quad (4.9)$$

Remark 4.1.2. Note that ϵ_0 depends on (b_α, c_α) in a way that is not necessarily continuous. The reason is that, in the proof, we will use the fact that (say)

$$\text{either } b_\alpha + b_\beta - b_\gamma = 0, \quad \text{or } |b_\alpha + b_\beta - b_\gamma| = \theta > 0,$$

and that the ϵ_0 in the proof depends on θ . We believe that this annoyance is of technical nature, and can be avoided by more careful analysis. However, since this will make the proof too complicated and is not very relevant to the main idea and techniques of this chapter, we have chosen to state Theorem 4.1.1 as it is.

4.1.1 Notations and choice of parameters

Notations

The notations defined in this section will be used throughout this and the next chapter.

We fix an even smooth function $\varphi : \mathbb{R} \rightarrow [0, 1]$ that is supported in $[-8/5, 8/5]$ and equals 1 in $[-5/4, 5/4]$. For simplicity of notation, we also let φ denote the corresponding radial function on

$\mathbb{R}^3$. Let

$$\varphi_k(x) = \varphi_k(x) := \varphi(|x|/2^k) - \varphi(|x|/2^{k-1}) \quad \text{for any } k \in \mathbb{Z}, \quad \varphi_I := \sum_{m \in I \cap \mathbb{Z}} \varphi_m \quad \text{for any } I \subseteq \mathbb{R},$$

and define functions like $\varphi_{\leq B}$ in the standard way. For any $x \in \mathbb{Z}$ let $x_+ = x^+ := \max(x, 0)$ and $x_- = x^- := -\min(x, 0)$. Let

$$\mathcal{J} := \{(k, j) \in \mathbb{Z} \times \mathbb{Z}_+ : k + j \geq 0\},$$

and for any $(k, j) \in \mathcal{J}$ let

$$\varphi_j^{(k)}(x) = \varphi_j(x), \quad \text{if } j > \max(0, -k); \quad \varphi_j^{(k)}(x) = \varphi_{\leq j}(x), \quad \text{if } j = \max(0, -k).$$

Let P_k denote the operator on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \rightarrow \varphi_k(\xi)$; let $P_{\leq B}$ (respectively $P_{> B}$) denote the operators on $\mathbb{R}^3$ defined by the Fourier multipliers $\xi \rightarrow \varphi_{\leq B}(\xi)$ (respectively $\xi \rightarrow \varphi_{> B}(\xi)$). For $(k, j) \in \mathcal{J}$ let

$$(Q_{jk}f)(x) := \varphi_j^{(k)}(x) \cdot P_k f(x); \quad f_{jk} := Q_{jk}f, \quad f_{jk}^* := P_{[k-2, k+2]} Q_{jk}f. \quad (4.10)$$

Let $\Lambda_\alpha = \sqrt{b_\alpha^2 - c_\alpha^2} \Delta$ be the linear phase, and define

$$\Lambda_\alpha(\xi) := \sqrt{c_\alpha^2 |\xi|^2 + b_\alpha^2}, \quad 1 \leq \alpha \leq d; \quad b_{-\alpha} = -b_\alpha, \quad c_{-\alpha} = c_\alpha, \quad \Lambda_{-\alpha} = -\Lambda_\alpha.$$

Let $\mathcal{P} = \mathbb{Z} \cap ([1, d] \cup [-d, -1])$; for $\sigma, \mu, \nu \in \mathcal{P}$, we define the associated nonlinear phase

$$\Phi_{\sigma\mu\nu}(\xi, \eta) := \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta), \quad (4.11)$$

and the corresponding function

$$\Phi_{\sigma\mu\nu}^+(\alpha, \beta) := \Phi_{\sigma\mu\nu}(\alpha e, \beta e) = \Lambda_\sigma(\alpha) - \Lambda_\mu(\alpha - \beta) - \Lambda_\nu(\beta),$$

where $e \in \mathbb{S}^2$ and $\alpha, \beta \in \mathbb{R}$.

For later purposes, we also need the spherical harmonics decomposition, which is described below. Let (r, θ, φ) are polar coordinates for ξ and Y_q^m be spherical harmonics, and note $\Delta_{\mathbb{S}^2} Y_q^m =$

$-q(q+1)Y_q^m$. For any function f , we have the expansion

$$f(\xi) = \sum_{q=0}^{\infty} \sum_{m=-q}^q f_q^m(r) Y_q^m(\theta, \varphi), \quad (4.12)$$

and define

$$S_l f := \sum_{q=0}^{\infty} \sum_{m=-q}^q \varphi_l(q) f_q^m(r) Y_q^m(\theta, \varphi) \quad (4.13)$$

for $l \geq 1$, and with φ_0 replaced by $\varphi_{\leq 0}$ for $l = 0$. Note that S_l commutes with each P_k and Q_{jk} .

Choice of parameters

Throughout the proof, we will fix some parameters as follows:

$$\begin{aligned} N &= 1000, & \delta &= 1/1000, & \delta' &= 1/980, & N_0 &\gg 1; \\ D_0 &\gg_{\mathcal{B}} 1, & K_0 &\gg_{D_0} 1, & \varepsilon_0 &\ll_{K_0} 1, \end{aligned} \quad (4.14)$$

where $\mathcal{B} := \{(b_\alpha, c_\alpha)\}$. Note that $\delta = 1/N$ and $\delta' = 1/(N-20)$. In addition, let A denote any large absolute constant such that $A \ll N_0$, whose exact value may vary at different places; in the same way, let o denote any small constant, so for example $o \ll \delta^2$.

4.1.2 Description of the methods

General strategy, and the Z norm

Letting $v_\sigma = (\partial_t - i\Lambda_\sigma)u_\sigma$, we can reduce (4.1) to

$$(\partial_t + i\Lambda_\sigma)v_\sigma = \mathcal{N}_\sigma(v, v), \quad \nu \in \mathcal{P}, \quad (4.15)$$

where $\mathcal{N}_\sigma$ is some quasilinear quadratic term.

Following the now standard strategy, the proof of small data global existence of (4.15) can be schematically described as follows.

Step 1. Produce an energy estimate of form

$$\partial_t \mathcal{E} \approx \int_{\mathbb{R}^3} \partial^N v \cdot \partial^N v \cdot \partial^2 v, \quad \mathcal{E} \approx \|v\|_{H_N}^2. \quad (4.16)$$

Since there is enough decay in $3D$, (4.16) will be enough for our needs; however, we shall see later in Chapter 5 that (4.16) is inadequate in $2D$ and that becomes a major obstacle.

Step 2. Prove a linear decay estimate of form

$$\|v(t)\|_{L^\infty} \lesssim t^{-1} \|f(t)\|_Z, \quad f_\sigma(t) = e^{it\Lambda_\sigma} v_\sigma(t) \quad (4.17)$$

with some localization norm Z . This combined with (4.16) guarantees that the top order energy $\mathcal{E}$ remains bounded (or grows slowly) provided that one can bound $\|f(t)\|_Z$. The notion of Z norm was first introduced in [32], though similar localization estimates had been used before in [20] and [21].

Step 3. Prove the $\|f(t)\|_Z$ bound by using the Duhamel formula

$$\widehat{f}_\sigma(t, \xi) = \widehat{f}_\sigma(0, \xi) + \int_0^t \int_{\mathbb{R}^3} e^{is\Phi_{\sigma\mu\nu}(\xi, \eta)} m(\xi, \eta) \widehat{f}_\mu(s, \xi - \eta) \widehat{f}_\nu(s, \eta) d\eta ds. \quad (4.18)$$

Here one can notice that, at least in the first approximation when the inputs f_μ and f_ν are Schwartz, the main contribution of the integral (4.18) comes from the vicinity of the set

$$\mathcal{R} := \{(\xi, \eta) : \Phi_{\sigma\mu\nu}(\xi, \eta) = \nabla_\eta \Phi_{\sigma\mu\nu}(\xi, \eta) = 0\}.$$

This set $\mathcal{R}$, which is called the *spacetime resonance* set, was first introduced in [22] and plays a crucial role in the analysis.

Using this strategy, in [32], the authors were able to perform a robust stationery phase analysis near $\mathcal{R}$, and obtained a scattering result under the additional assumptions that

$$(c_\alpha - c_\beta)(c_\alpha^2 b_\alpha - c_\beta^2 b_\beta) \geq 0, \quad 1 \leq \alpha, \beta \leq d; \quad (4.19)$$

$$b_\alpha + b_\beta - b_\gamma \neq 0, \quad 1 \leq \alpha, \beta, \gamma \leq d. \quad (4.20)$$

In the current chapter we shall discuss how to remove these two assumptions. This requires improvements to both Step 2 and Step 3 in the above scheme, which we detail below.

Spherical symmetry and rotation vector fields

It is clear that the choice of the Z norm is important in both Step 2 and Step 3 above; that (4.17) holds determines the weakest Z norm one could choose, and that (4.18) can be bounded determines the strongest Z norm one could have.

Let

$$\|f\|_Z = \sup_{j,k} \|f_{jk}\|_{Z_{jk}};$$

if we set Z_{jk} to be a simple weighted L^2 norm as in [20] or [32], then it is easy to check that it has to be (almost) as strong as $\|f\|_{Z_{jk}} = 2^j \|f\|_{L^2}$. Thus one also need to recover this norm, which, by a simple volume counting argument, is possible only when $\det(\nabla_\eta^2 \Phi_{\sigma\mu\nu}) \neq 0$ on the spacetime resonance set $\mathcal{R}$. This is guaranteed by the first assumption (4.19) made in [32], and explains why that condition has to be there.

In the absence of (4.19), the integral (4.18) can be degenerate near $\mathcal{R}$, and in this case, the strongest possible Z norm one could control is only $\|f\|_{Z_{jk}} = 2^{5j/6} \|f\|_{L^2}$, which is not strong enough to imply (4.17). In this chapter we shall overcome this difficulty by introducing the rotation vector fields, and use the vector field set Γ as in (4.4). The idea of using such vector fields goes back to Klainerman [36], [37], [38] (see also [39]); in the current situation, we have

$$\|e^{-it\Lambda} f_{jk}\|_{L^\infty} \lesssim (1+t)^{-1-\epsilon} 2^{(1/2+\epsilon)j} (\|f_{jk}\|_{L^2} + \|\Omega f_{jk}\|_{L^2}), \quad (4.21)$$

at least when $j \leq (1-\epsilon)m$ and $|k| \lesssim 1$, where Ω is the rotational part of Γ . Note that we gain a power $2^{(1/2-\epsilon)j}$ at the price of using one Ω , which in particular allows us to close with the weak $5j/6$ power in the Z norm. For the proof of the improved linear dispersion inequality (4.21), see Proposition 4.2.7 below.

A bilinear lemma

In analyzing (4.18), with a weaker Z norm, the information we have on the input functions f_μ and f_ν is also weaker. Thus, it is still challenging to bound (4.18), in particular when one of f_μ or f_ν has a large physical support (i.e. far from being Schwartz). In particular, the orthogonality trick use in [32] will not be sufficient here. This issue is resolved again by using rotation vector fields in Ω ; this time note that at least for medium frequencies (i.e. frequencies ~ 1 , which is where the spacetime resonance takes place), we have

$$\sup_{\theta \in \mathbb{S}^2} \|f(\rho\theta)\|_{L_\rho^2} \lesssim \sup_{|\alpha| \leq 4} \|\Omega_0^\alpha f\|_{L^2},$$

thus we can gain a factor of δ^2 , provided that we can somehow restrict the direction vector of η in (4.18) to a region of size δ in $\mathbb{S}^2$. This motivates the second main technical tool in this chapter, namely Lemma 4.6.1 below, which holds under very mild conditions, and can be used to restrict

$|\sin \angle(\xi, \eta)| \lesssim 2^{-(1/2-\epsilon)m}$ for medium frequencies. With this additional gain, we can then close the Z norm estimate using Schur's lemma; see Section 4.6.

Low frequencies, and sharp integration by parts

The other assumption (4.20) in [32] was used to ensure that $(0, 0) \notin \mathcal{R}$. This also greatly simplifies the analysis, since one can then integrate by parts in s whenever one is close to $(0, 0)$.

Without (4.20), the point $(0, 0)$ can be spacetime resonant; moreover it can be very degenerate, meaning that it is possible to have $(\nabla_\eta^\alpha \Phi)(0, 0) = 0$ for $|\alpha| \leq 3$; for example when

$$\Phi(\xi, \eta) = \sqrt{|\xi|^2 + 1} - \sqrt{2|\xi - \eta|^2 + 4} + \sqrt{|\eta|^2 + 1} = \xi \cdot \eta / 2 + O(|\xi|^4 + |\eta|^4).$$

To see the effect of this degeneracy, we consider the “rescaled Schwartz” component $f_{j, -j}$ with $k = -j$. Suppose that $|s| \sim 2^m$ in (4.18), and that the inputs are rescaled Schwartz functions

$$\widehat{f}_\mu(s, \xi - \eta) = 2^{cj} \chi(2^j(\xi - \eta)), \quad \widehat{f}_\nu(s, \eta) = 2^{cj} \chi(2^j(\eta))$$

with $j = m/4$ and some constant c , then in the region where $|\eta| \sim |\xi - \eta| \sim 2^{-j}$ and $|\xi| \sim 2^{-3j}$, we have $|\Phi| \lesssim 2^{-m}$, so the oscillation factor $e^{is\Phi}$ is irrelevant, thus the output will be a rescaled Schwartz function supported at the scale $|\xi| \sim 2^{-3j}$ with L_ξ^∞ norm bounded by

$$2^m 2^{-3j} 2^{2cj} = 2^{(3j)(2c+1)/3}.$$

If we start with $c = 0$ (imagine cutting off a Schwartz function at scale $|\xi| \sim 2^{-j}$), then in finitely many iterations we obtain a profile with form $2^{cj} \chi(2^j \xi)$, where c can be arbitrarily close to 1 (which is the unique fixed point of the map $c \mapsto (2c + 1)/3$). Therefore, if we were to choose $\|f\|_{Z_{jk}} = 2^j 2^{\lambda k} \|f\|_{L^2}$ as in [32], then we *must* have $\lambda \geq 1/2$. On the other hand, it can be proved that when $\lambda \geq 1/2$, the best decay rate for $\|e^{-it\Lambda_\nu} f\|_{L^\infty}$ is precisely t^{-1} , which is *not integrable*.

In this chapter, we circumvent this difficulty by adding a log factor and choosing

$$\|f\|_{Z_{jk}} = 2^j 2^{k/2} \langle j \rangle^{N_0} \|f\|_{L^2}.$$

Note that this makes the decay rate integrable, and does *not* violate the above heuristics. The drawback is that, when integrating by parts with low frequencies, we have to be precise “up to log factors”. In order to achieve this, we use a sharp integration by parts lemma, Proposition 4.2.5,

which requires that we integrate by parts A times with A depending on the functions themselves. This lemma is proved in Section 4.2, and the analysis of low frequencies is carried out in Section 4.5.

Plan of this chapter

In Section 4.2 we define the relevant notations and in particular the Z norm, and prove the crucial linear dispersion bound. In Section 4.3 we prove the energy bound, and reduce Theorem 4.1.1 to the main Z norm estimate, namely Proposition 4.3.4. In Section 4.4 we make several reductions and get rid of some easy instances of Proposition 4.3.4; we then discuss the low frequency case in Section 4.5, and the medium frequency case, which requires more care, in Section 4.6. The high frequency case is much easier and is dealt with in Section 4.7. Finally, in Section 4.8 we collect some auxiliary facts about the phase function and the spherical harmonics decomposition that will be used throughout this chapter.

4.2 Norms and basic estimates

4.2.1 Suitable cutoff functions

Definition 4.2.1. For $C \geq 1$, define $\mathcal{Z}_C$ to be the space of smooth functions χ on some $\mathbb{R}^d$ (with value in some other $\mathbb{R}^{d'}$), such that

$$|\partial_x^\mu \chi(x)| \lesssim (CN)!, \quad (4.22)$$

for each $N \geq 0$ and each $|\mu| \leq N$, uniformly on each compact set. Let $\mathcal{Z}$ be the union of all $\mathcal{Z}_C$.

An immediate consequence of the above definition is that, when $\chi \in \mathcal{Z}_C$ has compact support, we will have $\|\partial_x^\mu \chi\|_{L^\infty} \lesssim (C|\mu|)!$. In practice χ will be used as a cutoff function, and the existence of such functions is guaranteed by the following two lemmas.

Lemma 4.2.2. Suppose $\chi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ and $\chi_2 : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d''}$ are two smooth functions that are in $\mathcal{Z}$. Then the composition $\chi_2 \circ \chi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^{d''}$ also belongs to $\mathcal{Z}$.

Proof. By working componentwise, we may assume $d'' = 1$. Suppose $\chi_1 \in \mathcal{Z}_{C_1}$ and $\chi_2 \in \mathcal{Z}_{C_2}$. Fix any compact set $K \subset \mathbb{R}^d$, and let $\chi_1(K) = K' \subset \mathbb{R}^{d'}$. For each $x \in K$ and multi-index μ with $|\mu| = N$, we have the general formula such that

$$\partial_x^\mu (\chi_2 \circ \chi_1)(x) = \sum_{r=0}^N \sum_{\mathcal{T}} A(\mathcal{T}) \cdot (\partial_{j'_1} \cdots \partial_{j'_r} \chi_2)(\chi_1(x)) \cdot \prod_{q=1}^r \partial_x^{\mu_q} \partial_{j_q} (\chi_1)_{j'_q}(x). \quad (4.23)$$

Here the second sum is taken over all tuples

$$\mathcal{T} := (\mu_1, \dots, \mu_r, j_1, \dots, j_r, j'_1, \dots, j'_r)$$

where each μ_i is a d -tuple, each j_q ranges from 1 to d , each j'_q ranges from 1 to d' , and

$$\sum_{i=1}^r |\mu_i| = N - r; \quad |A(\mathcal{T})| \lesssim (N+1)!.$$

To prove (4.23) we will induct in N . When $N = 0$ we can only have $r = 0$ and $A = 1$, so (4.23) holds; suppose it holds for $N < k$, then for $N = k$ we may choose μ' so that $\partial_x^\mu = \partial_j \partial_x^{\mu'}$. Replace μ by μ' in (4.23) and then take ∂_j , we obtain a product of similar form, but with either one $\partial_x^{\mu_q}$ replaced by $\partial_j \partial_x^{\mu_q}$, or r replaced by $r+1$, a new parameter j'_{r+1} appears and a new factor occurs with $j_{r+1} = j$ and $\mu_{r+1} = 0$. In all cases (4.23) still holds the same form; moreover each new term in (4.23) comes from at most $r+1$ old terms, so $|A(\mathcal{T})| \lesssim (N+1)!$ by induction hypothesis.

Now, using (4.23), the fact that $\chi_1 \in \mathcal{Z}_{C_1}$ and $\chi_2 \in \mathcal{Z}_{C_2}$, and the simple inequality $M!N! \leq (M+N)!$, we can bound (very roughly)

$$|\partial_x^\mu(\chi_2 \circ \chi_1)(x)| \lesssim (N+1)(2dd')^N \cdot (N+1)!A_1^r A_2(C_1 N)!(C_2 r)!,$$

where A_1 depends only on K and A_2 depends only on K' . This shows that $\chi_2 \circ \chi_1 \in \mathcal{Z}_C$ for any $C > C_1 + C_2 + 1$. $\square$

Lemma 4.2.3. *For a fixed compact set K and a fixed open set U such that $K \subset U \subset \mathbb{R}^d$, there exists a real valued function $\chi \in \mathcal{Z}_6$ such that $\chi = 0$ outside U , and $\chi = 1$ on K . Moreover we can assume χ is radial if both K and U have spherical symmetry.*

Proof. Fix a small ϵ , there exists a (compact) set E so that

$$\text{dist}(E, U^c) \geq 2\epsilon, \quad \text{dist}(E^c, K) \geq 2\epsilon.$$

Define

$$\psi(\eta) = c \cdot \exp\left(\frac{-1}{\epsilon^2 - |\eta|^2}\right)$$

for $|\eta| < \epsilon$ and $\psi_1(\eta) = 0$ otherwise, and choose the appropriate constant c so that $\|\psi\|_{L^1(\mathbb{R}^3)} = 1$. Let $\chi = (\mathbf{1}_E) * \psi$ where $\mathbf{1}$ denotes characteristic function. Since $\psi(\eta) = 0$ for $|\eta| \geq \epsilon$ and $\int_{\mathbb{R}^d} \psi = 1$, we know that $\chi = 0$ outside U , and $\chi = 1$ on K . If both K and U has spherical symmetry, we may

assume the same thing for E , thus this χ will also be radial.

Finally, notice that $\partial_x^\mu \chi = (\mathbf{1}_E) * (\partial_x^\mu \psi)$, we only need to prove that $\psi \in \mathcal{Z}$. Since $\eta \mapsto \epsilon^2 - |\eta|^2$ is clearly in $\mathcal{Z}_0$, using Lemma 4.2.2, we only need to prove that

$$\left| \frac{d^N}{d\rho^N} e^{-\frac{1}{\rho}} \right| \lesssim (4N)!$$

for all $\rho > 0$. Now it is easily seen by induction that

$$\frac{d^N}{d\rho^N} e^{-\frac{1}{\rho}} = P_N(\rho) \rho^{-2N} e^{-\frac{1}{\rho}},$$

where P_N is a polynomial such that

$$P_{N+1}(\rho) = \rho^2 P'_N(\rho) - (1 + 2N\rho) P_N(\rho).$$

By induction we can prove that $\deg(P_N) \leq N$ and all the coefficients of P_N are bounded by $3^N(N+1)!$. When $\rho \geq 1$ the estimate is thus obvious; when $0 < \rho \leq 1$ we can use the inequality $\lambda^{2N} e^{-\lambda} \leq (2N)^{2N}$ to conclude. $\square$

4.2.2 The definition of Z norm

Recall the notations defined in Section 4.1.1, and also the vector field set Γ as in (4.4).

Definition 4.2.4. Fix $(j, k) \in \mathcal{J}$, define the norm

$$\|f\|_{Z_{jk}} = \begin{cases} \sup_{|\mu| \leq N/2+3} \langle j \rangle^{N_0} 2^{k/2+j} \|\Gamma^\mu f\|_{L^2}, & |k| \geq K_0; \\ \sup_{|\mu| \leq N/2+3} (2^{5j/6} \langle j \rangle^{-N_0} \|\Gamma^\mu f\|_{L^2} + \langle j \rangle^{N_0} 2^j \|\widehat{\Gamma^\mu f}\|_{L^1}), & |k| < K_0. \end{cases} \quad (4.24)$$

Also define the full Z norm by

$$\|f\|_Z = \sup_{(j,k) \in \mathcal{J}} \|f_{jk}\|_{Z_{jk}}, \quad (4.25)$$

and the X norm by

$$\|f\|_X = \sup_{|\mu| \leq N} \|\Omega^\mu f\|_{L^2} + \|f\|_Z.$$

Notice that

$$\|f\|_{Z_{jk}} \lesssim \|f\|_{Z_{jk'}}, \quad |k| < K_0 \leq |k'| \leq O_{K_0}(1), \quad (4.26)$$

if $\widehat{f}$ is supported in $|\xi| \lesssim 1$; this simple observation will later be convenient.

4.2.3 Linear and bilinear estimates

Proposition 4.2.5 (Sharp integration by parts). *Suppose $K, \lambda \geq 1$ and ϵ_j are positive parameters, and $h(\eta)$ is some compact supported function on $\mathbb{R}^3$, verifying*

$$\|\partial_\eta^\mu h(\eta)\|_{L^1} \lesssim (CN)! \lambda^N \quad (4.27)$$

for some $C \geq 1$ and all $|\mu| \leq N$. Moreover, suppose $\Phi = \Phi(\eta) \in \mathcal{Z}_C$ is some function such that

$$|\partial_\eta \Phi(\eta)| \geq \epsilon_1; \quad |\partial_\eta^\mu \Phi(\eta)| \leq \epsilon_{|\mu|} \quad (4.28)$$

holds for each $2 \leq |\mu| \leq n$ and at each point where h or one of its derivatives is nonzero, where $k \in \mathbb{N}^+$, then we have the estimate

$$\left| \int_{\mathbb{R}^3} e^{iK\Phi(\eta)} h(\eta) d\eta \right| \lesssim e^{-\gamma M^\gamma}, \quad (4.29)$$

where

$$M = \min(K\epsilon_1^2/\epsilon_2, K\epsilon_1\epsilon_2/\epsilon_3, \dots, K\epsilon_1\epsilon_{n-1}/\epsilon_n, K\epsilon_1\epsilon_n, K\epsilon_1/\lambda), \quad (4.30)$$

and γ is small enough depending on C . In particular, if we can take $\epsilon_j = \epsilon^{\frac{n-j+1}{l}}$, then we have

$$M = \min(K\epsilon^{\frac{n+1}{n}}, K\epsilon/\lambda).$$

If we fix a direction $\theta \in \mathbb{S}^2$ and replace the ∂_η in (4.27) and (4.28) by the directional derivative $\theta \cdot \partial_\eta$ (in which case μ becomes an integer instead of a multi-index), then the same result will remain true (uniformly in θ).

Proof. First consider the case when we have a directional derivative, and assume that it is ∂_1 . We will integrate by parts in η_1 a total of N times, where N is a large integer to be determined. Let the differential operator D be defined by

$$Du = \frac{\partial_1 u}{\partial_1 \Phi}; \quad D(e^{iK\Phi}) = iK e^{iK\Phi},$$

then its dual D' would be

$$D'u = -\partial_1 \left(\frac{u}{\partial_1 \Phi} \right).$$

Therefore, integrating by parts, we obtain

$$\left| \int_{\mathbb{R}^3} e^{iK\Phi(\eta)} h(\eta) d\eta \right| \leq K^{-N} \|(D')^N h\|_{L^1}. \quad (4.31)$$

In order to bound $(D')^N h$, we will use the following explicit formula, namely

$$(D')^N h = \sum_{r=0}^N \sum_{\alpha_0 + \dots + \alpha_r = N-r} \Omega(n, r; \alpha_0, \dots, \alpha_r) \cdot \partial_1^{\alpha_0} h \cdot \frac{\partial_1^{\alpha_1+2} \Phi \dots \partial_1^{\alpha_r+2} \Phi}{(\partial_1 \Phi)^{r+N}}, \quad (4.32)$$

where the coefficient

$$|\Omega| \leq 3^N (N+1)!.$$

The formula (4.32) is proved by induction, in the same way as (4.23).

Now, using (4.28), (4.27) and (4.32), we have

$$K^{-N} \|(D')^N h\|_{L^1} \lesssim ((3C+3)N)! \cdot K^{-N} \lambda^{\alpha_0} \cdot \epsilon_1^{-N-r} \prod_{j=0}^{n-2} \epsilon_{j+2}^{r_j}. \quad (4.33)$$

Here we denote by r_j the number of $q \geq 1$ such that $\alpha_q = j$. Let

$$\rho := r - \sum_{j=0}^{n-2} r_j \geq 0; \quad \alpha_0 + \sum_{j=0}^{n-2} j r_j \leq N - r - (n-1)\rho. \quad (4.34)$$

We now denote $K\epsilon_1/\lambda = M_0$ and $K\epsilon_1\epsilon_j/\epsilon_{j+1} = M_j$, where $\epsilon_{n+1} = 1$. We could then solve that

$$\epsilon_j = (M_1 \dots M_{j-1} K)^{\frac{j-n-1}{n+1}} (M_j \dots M_n)^{\frac{j}{n+1}};$$

$$\lambda = (M_1 \dots M_n K)^{\frac{1}{n+1}} M_0^{-1}.$$

Plugging into (4.33) we obtain

$$\begin{aligned} K^{-N} \|(D')^N h\|_{L^1} &\lesssim ((3C+3)N)! K^{-N} (M_1 \dots M_n K)^{\frac{\alpha_0}{n+1}} M_0^{-\alpha_0} \times (M_1 \dots M_n)^{\frac{-N-r}{n+1}} \times \\ &\times K^{\frac{n(N+r)}{n+1}} \prod_{j=0}^{n-2} (M_1 \dots M_{j+1} K)^{\frac{r_j(j-n+1)}{n+1}} (M_{j+2} \dots M_n)^{\frac{r_j(j+2)}{n+1}} \\ &= ((3C+3)N)! K^\sigma \prod_{j=0}^n M_j^{T_j}, \end{aligned} \quad (4.35)$$

where

$$\sigma = -N + \frac{\alpha_0 + n(N+r)}{n+1} + \sum_{j=0}^{n-2} \frac{r_j(j-n+1)}{n+1} \leq 0,$$

and $\tau_0 = -\alpha_0 \leq 0$, and

$$\tau_j = \frac{\alpha_0 - N - r}{n+1} - \sum_{i=j-1}^{n-2} r_i + \frac{1}{n+1} \sum_{i=0}^{n-2} r_i(i+2) \leq -\rho \leq 0$$

for $1 \leq j \leq n$, and

$$\begin{aligned} \sum_{j=0}^n \tau_j &= \frac{-\alpha_0 - n(N+r)}{n+1} + \frac{n}{n+1} \sum_{i=0}^{n-2} r_i(i+2) - \sum_{i=0}^{n-2} r_i(i+1) \\ &= -\frac{nN}{n+1} - \frac{1}{n+1} \left(\alpha_0 + \sum_{i=0}^{n-2} r_i(i+1) + n\rho \right) \leq -\frac{n}{n+1} N, \end{aligned}$$

all the inequalities being consequences of (4.34). This then implies that

$$K^{-N} \|(D')^N h\|_{L^1} \lesssim ((3C+3)N)! \min(M_0, \dots, M_l)^{-\frac{nN}{n+1}}.$$

If we now choose N appropriately, an easy computation will show that

$$\left| \int_{\mathbb{R}^3} e^{iK\Phi(\eta)} h(\eta) d\eta \right| \lesssim e^{-\gamma M^\gamma} \quad (4.36)$$

for γ small enough.

Now suppose the directional derivative is replaced by the full gradient. By (4.28), we have that

$$h(\eta) = h(\eta) \cdot \left(1 - \prod_{j=1}^3 \varphi_0(2\epsilon_1^{-1} \partial_j \Phi(\eta)) \right),$$

so we only need to bound the integral with h replaced by $h_1 = h \cdot \varphi_0(2\epsilon_1^{-1} \partial_1 \Phi) \varphi_0(2\epsilon_1^{-1} \partial_2 \Phi)$ (the other similar terms are bounded in the same way).

Since (4.28) now holds with ∂_η replaced by ∂_1 at each point where h_1 or one of its derivative is nonzero (possibly with different constants), we only need to show that h_1 also verifies the bound (4.27), but with λ replaced by the maximum of λ and each $\epsilon_{j+1}/\epsilon_j$ (again, let $\epsilon_{n+1} = 1$). Using Leibniz rule, we can further reduce to proving the same result for $\varphi_0(2\epsilon_1^{-1} \partial_2 \Phi)$ but with λ replaced by the maximum of $\epsilon_{j+1}/\epsilon_j$, the L^1 norm replaced by the L^∞ norm, and restrict to the subset where h or one of its derivatives is nonzero (note that $\varphi_0(2\epsilon_1^{-1} \partial_1 \Phi)$ is estimated in the same way).

Now, using (4.23) we have

$$\partial_\eta^\mu(\varphi_0(2\epsilon_1^{-1}\partial_2\Phi)) = \sum_{r=0}^N (2\epsilon_1^{-1})^r \sum_{\mathcal{T}} \Omega(\mathcal{T}) \cdot (\partial_{j_1} \cdots \partial_{j_r} \varphi_0)(2\epsilon_1^{-1}\partial_2\Phi) \cdot \prod_{q=1}^r \partial_x^{\mu_q} \partial_{j_q} \partial_2\Phi, \quad (4.37)$$

here the summation is taken over all tuples

$$\mathcal{T} = (\mu_1, \dots, \mu_d, j_1, \dots, j_d),$$

and we have

$$\sum_{i=1}^r |\mu_i| = N - r, \quad |\Omega(\mathcal{T})| \leq (N+1)!.$$

Let, for each $0 \leq j \leq n-2$, the number of q 's such that $|\mu_q| = j$ be r_j , then we will have

$$\rho = r - \sum_{j=0}^{n-2} r_j \geq 0, \quad \sum_{j=0}^{n-2} j r_j \leq N - r - (n-1)\rho.$$

Therefore, since we are in the set where the second part of (4.28) holds, we can bound

$$\begin{aligned} \|\partial_\eta^\mu(\varphi_0(2\epsilon_1^{-1}\partial_2\Phi))\|_{L^\infty} &\lesssim ((3C+8)N)! \cdot \epsilon_1^{-r} \prod_{j=0}^{n-2} \epsilon_{j+2}^{r_j} \lesssim ((3C+8)N)! \epsilon_1^{-r} \prod_{j=0}^{n-2} (\epsilon_1(\lambda')^{j+1})^{r_j} \\ &\lesssim ((3C+8)N)! \epsilon_1^{-\rho} (\lambda')^{N-n\rho} \\ &\lesssim ((3C+8)N)! (\lambda')^N, \end{aligned}$$

where λ' is the maximum of $\epsilon_{j+1}/\epsilon_j$, since we have $\epsilon_1^{-1} \lesssim (\lambda')^n$. This completes the proof. $\square$

Remark 4.2.6. When $n = 1$, we will also use a slightly more general version that allow

$$|\partial_\eta^\mu \Phi(\eta)| \lesssim (CN)! (\lambda')^N$$

for some $\lambda' \geq 1$. In this case we simply rescale to reduce to the case proved above, and obtain that (4.29) holds with $M = \min(K(\lambda')^{-2}\epsilon^2, K\epsilon/\lambda)$.

Proposition 4.2.7 (Improved dispersion decay). *Suppose $\|f\|_X \lesssim 1$, and that $\langle t \rangle \sim 2^m$. Also let $\Lambda = \Lambda_\nu$ with $\nu \in \mathcal{P}$ and fix $(j, k) \in \mathcal{J}$.*

1. If $k \leq -K_0$, we have

$$\sup_{|\mu| \leq N/2} \|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim \langle j \rangle^{-N_0} \min(2^{-j+k}, 2^{-(3m-j+k)/2}). \quad (4.38)$$

2. If $|k| < K_0$, and $|j - m| \geq A \log m$, we have

$$\sup_{|\mu| \leq N/2} \|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim 2^{-3m/2-j/3} \langle m \rangle^{N_0+A}. \quad (4.39)$$

3. If $|k| < K_0$ and $|j - m| \leq A \log m$, we have

$$\sup_{|\mu| \leq N/2} \|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim 2^{-j} \langle j \rangle^{-N_0}. \quad (4.40)$$

4. If $k \geq K_0$, we have

$$\sup_{|\mu| \leq N/2} \|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim 2^{-3k} \langle m \rangle^{-N_0} \min(2^{-j+k}, 2^{-(3m-j-5k)/2}). \quad (4.41)$$

5. If $k \geq K_0$ and $|j - m| \geq A \log m$, we have

$$\sup_{|\mu| \leq N/2} \|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim 2^{3k/2} \langle j \rangle^{-N_0+A} 2^{-(3m+j)/2}. \quad (4.42)$$

Proof. First, (4.40) is a direct consequence of Hausdorff-Young and the definition of the Z norm; the bounds (4.38) and (4.41) are also easily deduced from definition. In fact, when $(j, k) \in \mathcal{J}$ and $k \leq -K_0$, from Hausdorff-Young we have

$$\|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim \|\widehat{\Gamma^\mu f_{jk}^*}\|_{L^1} \lesssim \|\varphi_{[k-2, k+2]}(\xi)\|_{L^2} \|\widehat{\Gamma^\mu f_{jk}}\|_{L^2} \lesssim \langle j \rangle^{-N_0} 2^{-j+k}, \quad (4.43)$$

while from the standard dispersion estimate (see for example [32], Lemma 5.2) we have

$$\|e^{it\Lambda} \Gamma^\mu f_{jk}^*\|_{L^\infty} \lesssim 2^{-3m/2} \|\Gamma^\mu f_{jk}\|_{L^1} \lesssim 2^{-3(m-j)/2} \|\Gamma^\mu f_{jk}\|_{L^2} \lesssim \langle j \rangle^{-N_0} 2^{-(3m-j+k)/2}. \quad (4.44)$$

The inequality (4.41) is proved using basically the same argument, but with the Z_{jk} norm for $k \geq K_0$ (the 2^{-3k} factor comes from the three extra vector fields in Definition 4.2.4), and the proof is omitted here.

Now we will prove (4.39) in the case $j \leq m - A \log m$. The case $j \geq m + A \log m$ only needs minor changes. Let $\Gamma^\mu f_{jk} = F$ and $\Gamma^\mu f_{jk}^* = F^*$; if $|x| \leq 2^m \langle m \rangle^{-A}$, recall that

$$(e^{it\Lambda} F^*)(x) = \int_{\mathbb{R}^3} e^{it\Lambda(\xi) + ix \cdot \xi} \psi(\xi) \widehat{F}(\xi) d\xi, \quad (4.45)$$

where $\psi(\xi) = \varphi_{[k-2, k+2]}(\xi)$ is a cutoff because $|k| \leq K_0$. Since $F(z)$ is supported in $|z| \sim 2^j$ (unless $j = \max(-k, 0)$, in which case $j = O(1)$ and the estimate will be trivial), we may rewrite the above expression as

$$(e^{it\Lambda}F^*)(x) = \int_{\mathbb{R}^3} \psi_0(2^{-j}z)F(z) dz \int_{\mathbb{R}^3} e^{i(t\Lambda(\xi) + (x-z)\cdot\xi)} \psi(\xi) d\xi, \quad (4.46)$$

with another cutoff $\psi_0(z)$ supported in the region $|z| \sim 1$. Now fix any z such that $|z| \sim 2^j$, we can use Proposition 4.2.5 with

$$K = 2^{\max(m, j)}, \quad n = 1, \quad \epsilon, \lambda \sim 1$$

to bound the ξ -integral by 2^{-10m} , which is clearly enough for (4.39). The same argument also applies for $|x| \geq 2^m \langle m \rangle^A$.

Therefore, to prove (4.39), we may assume $|x| \sim 2^r$ where $|r - m| \leq A \log m$; without loss of generality we may also assume $x = (\alpha, 0, 0)$, where $\alpha = |x|$. Decomposing F into $S_l F$ as in (4.13), we may also assume $l \leq 7\delta m$, since otherwise we have

$$\|S_l F\|_{L^2} \lesssim 2^{-Nl/2} \sup_{|\nu| \leq N/2} \|\Omega^\nu S_l F\|_{L^2} \lesssim 2^{-7m/2},$$

from which (4.41) follows easily. Since the ψ in (4.45) is radial, we may write $\psi(|\xi|)\widehat{S_l F}(\xi) = g(|\xi|, \xi/|\xi|)$, then we have

$$(e^{it\Lambda}S_l F^*)(x) = \int_{\mathbb{R}} \int_{\mathbb{S}^2} e^{i(t\Lambda(\rho) + \alpha\rho\theta^1)} \rho^2 \psi_1(\rho) g(\rho, \theta) d\rho d\omega(\theta), \quad (4.47)$$

where $d\omega$ is the surface measure on $\mathbb{S}^2$ and ψ_1 is another cutoff.

Since g is (qualitatively) a Schwartz function of ρ and θ , we may decompose $g = g_1 + g_2$, where for each θ , $\mathcal{F}_\rho g_1(\rho, \theta)(\tau)$ is supported in the region $|\tau| \lesssim M_0$, and $\mathcal{F}_\rho g_2(\rho, \theta)(\tau)$ is supported in the region $|\tau| \gg M_0$, where $M_0 = 2^j \langle m \rangle^{A/8}$. To estimate g_2 , we only need to estimate

$$\int_{|\tau| \gg M_0} |(\mathcal{F}_\rho g(\rho, \theta))(\tau)| d\tau$$

uniformly in θ . But for each fixed τ with $|\tau| \gg M_0$ we have

$$(\mathcal{F}_\rho g(\rho, \theta))(\tau) = \int_{\mathbb{R}} e^{-i\rho\tau} g(\rho, \theta) d\rho = \int_{\mathbb{R}} e^{-i\rho\tau} \psi(\rho) d\rho \int_{\mathbb{R}^3} \psi_0(2^{-j}z) S_l F(z) e^{-i\rho(\theta \cdot z)} dz.$$

Now if we fix z , then the integral in ρ can be bounded by $\min(|\tau|^{-10}, 2^{-10m})$, which will be acceptable. Therefore in (4.47) we may replace the function g by g_1 . Using (4.131) and also noticing the bounded

Fourier support of $g_1(\cdot, \theta)$ for each θ , we have

$$\|g_1\|_{L_\theta^\infty L_\rho^\infty} \lesssim M_0^{1/2} \|g_1\|_{L_\theta^\infty L_\rho^2} \lesssim M_0^{1/2} \|g_1\|_{L_\rho^2 L_\theta^\infty} \lesssim 2^l M_0^{1/2} \|g_1\|_{L_\rho^2 L_\theta^2}. \quad (4.48)$$

If, in the integral (4.47), we restrict to the region $|(\theta^1)^2 - 1| \geq 1/100$, since the function $\theta \mapsto \theta^1$ has no critical point in this region, we can integrate by parts in θ many times to bound the left hand side of (4.47) by 2^{-10m} , which implies (4.41). Therefore, in (4.47), after replacing g by g_1 , we may also cutoff in the region $|\theta^1 - 1| \leq 1/50$ (the case $|\theta^1 + 1| \leq 1/50$ is treated in the same way), on which we can use (θ^2, θ^3) as local coordinates, so that the integral (4.47) reduces to

$$I = \int_{\mathbb{R} \times \mathbb{R}^2} e^{i(t\Lambda(\rho) + \alpha\rho\sqrt{1-(\theta^2)^2 - (\theta^3)^2})} \psi_1(\rho) \psi_2(\theta^2, \theta^3) h(\rho, \theta^2, \theta^3) d\rho d\theta_2 d\theta_3, \quad (4.49)$$

where ψ_1 and ψ_2 are cutoff functions, and h is obtained from g_1 after change of variables.

Now, recall $\alpha \sim 2^r$, let

$$\epsilon' = 2^{-r/2} \langle m \rangle^{2A}; \quad \epsilon'' = 2^{\max(j-m, -m/2)} \langle m \rangle^{2A}, \quad (4.50)$$

we proceed to estimate (4.49) in the region where $|\theta^2| + |\theta^3| \gtrsim \epsilon'$. After inserting a cutoff $(1 - \varphi_0)((\epsilon')^{-1}\theta^2, (\epsilon')^{-1}\theta^3)$, we will use Proposition 4.2.5 to integrate by parts in (θ^2, θ^3) , for any fixed ρ , choosing

$$K = \alpha, \quad n = 1, \quad \epsilon \sim \epsilon', \quad \lambda \sim (\epsilon')^{-1},$$

so that I is bounded by $\exp(-\gamma \langle m \rangle^{A\gamma/2})$, which is $\lesssim 2^{-10m}$ if A is large enough. Here we have used that $2^l \leq 2^{7\delta' m/2} \leq (\epsilon')^{-1}$ and $\|\partial_\theta^\mu h(\rho, \theta)\|_{L_\theta^2} \lesssim 2^{|\mu|l}$, which is because $g_1(\rho, \cdot)$ is still a linear combination of spherical harmonics of degree $\lesssim 2^l$.

Now, in (4.49), we will restrict to the region where $|\theta^2| + |\theta^3| \ll \epsilon'$, and then fix θ^2 and θ^3 . Let

$$\Xi(\rho) := \partial_\rho(\Lambda(\rho) + t^{-1}\alpha\rho\theta^1) = \Lambda'(\rho) + t^{-1}\alpha\theta^1,$$

we will then consider the part where $|\Xi(\rho)| \gtrsim \epsilon''$. In this case, we will use Proposition 4.2.5, and set

$$K = t, \quad n = 1, \quad \epsilon \sim \epsilon'', \quad \lambda \sim \max(M_0, (\epsilon'')^{-1})$$

to bound $I \lesssim 2^{-10m}$. Here we have used $\|\partial_\rho^\mu h(\rho, \theta)\|_{L^2} \lesssim M_0^{|\mu|}$, because $\mathcal{F}_\rho g_1(\rho, \theta)(\tau)$ is supported in $|\tau| \lesssim M_0$.

Therefore, we can restrict to the region $|\Xi(\rho)| \lesssim \epsilon''$. Using Hölder and (4.48), we have

$$|I| \lesssim (\epsilon')^2 \min(\epsilon'' \cdot \|g_1\|_{L_\theta^\infty L_\rho^\infty}, (\epsilon'')^{1/2} \|g_1\|_{L_\theta^\infty L_\rho^2}) \lesssim (\epsilon')^2 2^l \min(\epsilon'' M_0^{1/2}, (\epsilon'')^{1/2}) \|g_1\|_{L_\rho^2 L_\theta^2},$$

which implies

$$|I| \lesssim \langle m \rangle^A 2^{(j-3m)/2} (\|F\|_{L^2} + \|\Omega F\|_{L^2}) \lesssim 2^{-3m/2-j/3} \langle m \rangle^{N_0+A} \quad (4.51)$$

by (4.24). This proves (4.39).

Finally, (4.42) is proved in the same way as (4.39). Since $k \geq K_0$, and $\rho = |\xi| \sim 2^k$, in (4.47) the factor ρ^2 will count as 2^{2k} , and also that the measure of the set where $|\Xi(\rho)| \lesssim \epsilon''$ is now $2^{3k}\epsilon''$ since $|\Lambda''(\rho)| \sim 2^{-3k}$. Moreover in (4.50) we can set $\epsilon' = 2^{-(r+k)/2} \langle m \rangle^{2A}$, thus in total we have a loss of 2^{4k} compared with the proof of (4.39), but we can recover 2^{2k} using the three extra vector fields from Definition 4.2.4 (where one of them has to be Ω ; see (4.51)), and recover $2^{k/2}$ from Definition 4.2.4, thus we lose $2^{3k/2}$ in the end and hence (4.42). $\square$

Corollary 4.2.8. *Suppose $f = f(x)$ is a function with $\|f\|_X \lesssim \epsilon$, and let $v = e^{it\Lambda} f$ with Λ equaling one of the Λ_α , then we have*

$$\sum_{|\mu| \leq N/2} \|\Omega^\mu v\|_{L^\infty} \lesssim \frac{\epsilon}{(1+|t|) \log^{(N_0-2)}(2+|t|)}. \quad (4.52)$$

Proof. Let $1+|t| \sim 2^m$, and we decompose

$$v = \sum_{(j,k) \in \mathcal{J}} e^{it\Lambda} f_{jk}^*. \quad (4.53)$$

Now (4.38)-(4.41) in particular implies

$$\|e^{it\Lambda} \Omega^\mu f_{jk}^*\|_{L^\infty} \lesssim \epsilon (1+2^{k/2})^{-1} (\max\langle m \rangle, \langle j \rangle)^{-N_0} 2^{-\max(m,j)}$$

for each $|\mu| \leq N/2$. Therefore

$$\begin{aligned}
\|\Omega^\mu v\|_{L^\infty} &\lesssim \sum_{(j,k) \in \mathcal{J}} \epsilon (1 + 2^{k/2})^{-1} \max(\langle m \rangle, \langle j \rangle)^{-N_0} \cdot 2^{-\max(m,j)} \\
&\lesssim \epsilon \sum_{j \geq 0} 2^{-\max(m,j)} \max(\langle m \rangle, \langle j \rangle)^{-N_0} \cdot \sum_{k=-j}^{\infty} (1 + 2^{k/2})^{-1} \\
&\lesssim \epsilon \sum_{j \geq 0} 2^{-\max(m,j)} (\max\langle m \rangle, \langle j \rangle)^{-(N_0-1)} \\
&\lesssim \epsilon 2^{-m} \langle m \rangle^{-(N_0-2)},
\end{aligned} \tag{4.54}$$

which is what we need. $\square$

Proposition 4.2.9 (Basic bilinear estimates). *For the bilinear operator $\mathcal{T}$ defined by*

$$\mathcal{F}T(f, g)(\xi) = \int_{\mathbb{R}^3} K(\xi, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta, \tag{4.55}$$

we have the followings:

1. *If $\|\mathcal{F}_{\xi, \eta}^{-1} K\|_{L^1} \leq 1$, then*

$$\|T(f, g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}, \tag{4.56}$$

when $p, q, r \in [1, \infty]$ and $1/r = 1/p + 1/q$.

2. *If*

$$\sup_{\xi} \int_{\mathbb{R}^3} |K(\xi, \eta)|^2 d\eta + \sup_{\eta} \int_{\mathbb{R}^3} |K(\xi, \eta)|^2 d\xi \lesssim 1,$$

then we have

$$\|T(f, g)\|_{L^2} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{4.57}$$

3. *Suppose f and g are radial functions, and that K is bounded by 1 and supported in the region*

$$|\xi| \sim 2^k, \quad |\xi - \eta| \sim 2^{k_1}, \quad |\eta| \sim 2^{k_2}; \quad |\Phi(\xi, \eta)| \leq \epsilon,$$

where $\Phi = \Phi_{\sigma\mu\nu}$ is defined in (4.11). Then we have

$$\|T(f, g)\|_{L^2} \lesssim \min(2^{k/2}, 2^{k^-/2} \epsilon^{1/2}) 2^{-(k_1+k_2)} \|\widehat{f}\|_{L^1} \|\widehat{g}\|_{L^1}. \tag{4.58}$$

Proof. (1) This is standard; see [32].

(2) We may assume that $F = \widehat{f}$ and $G = \widehat{g}$ are nonnegative. Let $\mathcal{FT}(f, g) = H$, by Cauchy-Schwartz we have

$$|H(\xi)|^2 \lesssim \int_{\mathbb{R}^3} K(\xi, \eta) F(\xi - \eta) G^2(\eta) d\eta \cdot \int_{\mathbb{R}^3} K(\xi, \eta) F(\xi - \eta) d\eta.$$

The second factor is bounded by $\|F\|_{L^2}$ by Cahchy-Schwartz again, thus we have

$$\|H\|_{L^2}^2 \lesssim \|f\|_{L^2} \cdot \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(\xi, \eta) F(\xi - \eta) G^2(\eta) d\eta d\xi,$$

where the last integral equals

$$\int_{\mathbb{R}^3} G^2(\eta) d\eta \int_{\mathbb{R}^3} K(\xi, \eta) F(\xi - \eta) d\xi \lesssim \int_{\mathbb{R}^3} G^2(\eta) \cdot \|f\|_{L^2} d\eta = \|f\|_{L^2} \cdot \|g\|_{L^2}^2,$$

so this proves (2).

(3) Let $|\widehat{f}(\xi - \eta)| = F(|\xi - \eta|)$ and $|\widehat{g}(\eta)| = G(|\eta|)$, we may assume $\xi = (\lambda, 0, 0)$ and $\xi - \eta = (x, y, z)$, so that

$$|\mathcal{FT}(f, g)(\xi)| \leq \int_{\mathbb{R}^3} F(\sqrt{x^2 + y^2 + z^2}) G(\sqrt{(\lambda - x)^2 + y^2 + z^2}) dx dy dz.$$

Make the change of variables

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tau = \sqrt{(\lambda - x)^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \frac{z}{y},$$

the (inverse) Jacobian being

$$J^{-1} := \begin{vmatrix} \rho_x & \rho_y & \rho_z \\ \tau_x & \tau_y & \tau_z \\ \theta_x & \theta_y & \theta_z \end{vmatrix} = \frac{1}{\rho\tau(y^2 + z^2)} \begin{vmatrix} x & y & z \\ x - \lambda & y & z \\ 0 & -z & y \end{vmatrix} = \frac{\lambda}{\rho\tau},$$

so we have

$$|\mathcal{FT}(f, g)(\xi)| \leq \frac{2\pi}{\lambda} \int_{|\Phi| \leq \epsilon} \rho\tau F(\rho) G(\tau) d\rho d\tau. \quad (4.59)$$

Note that

$$\|F(\rho)\|_{L_\rho^1} \sim 2^{-2k_1} \|\widehat{f}\|_{L^1}; \quad \|G(\tau)\|_{L_\tau^1} \sim 2^{-2k_2} \|\widehat{g}\|_{L^1},$$

this implies that

$$|\mathcal{FT}(f, g)(\xi)| \lesssim 2^{-k-k_1-k_2} \|\widehat{f}\|_{L^1} \|\widehat{g}\|_{L^1},$$

which implies the first part of (4.58) by Hölder; as for the second part, choose a suitable function $J(\lambda)$ with $\|J\|_{L^2} = 1$, we have

$$\|T(f, g)\|_{L^2} \lesssim 2^{k_1+k_2} \int_{|\Phi| \leq \epsilon} F(\rho) G(\tau) J(\lambda) d\rho d\tau d\lambda,$$

then we fix ρ and τ and notice $|\partial_\lambda \Phi| \sim 2^{-k^-}$, so the measure of $\{\lambda : |\Phi| \leq \epsilon\}$ is bounded by $2^{k^-} \epsilon$, then use Hölder to conclude. $\square$

4.3 Proof of Theorem 4.1.1: Reduction to Z -norm estimate

Let u be a solution to (4.1) on $[0, T]$. We define the function v with value in $\mathbb{C}^{2d}$ by

$$v_\nu = (\partial_t - i\Lambda_\nu)u_\nu, \quad \nu \in \mathcal{P}, \quad (4.60)$$

where $u_{-\alpha} = u_\alpha$; so we have $v_{-\alpha} = \overline{v_\alpha}$. Moreover, let the corresponding profile f be

$$f_\nu(t) = e^{it\Lambda_\nu} v_\nu(t), \quad 0 \leq t \leq T. \quad (4.61)$$

Proposition 4.3.1. *Suppose $g, h : \mathbb{R}^3 \rightarrow \mathbb{R}^d$ are such that $\|(g, \partial_x g, h)\|_{\mathcal{H}^N} \leq \varepsilon_0$, then there exists a unique solution u to (4.1) such that*

$$u \in C_t^1 \mathcal{H}_x^N([0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}^d), \quad \partial_x u \in C_t^0 \mathcal{H}_x^N([0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}^d); \quad u(0) = g, \partial_t u(0) = h. \quad (4.62)$$

Moreover, if $\|(g, \partial_x g, h)\|_Z \leq \varepsilon_0$, then we have $f(t) \in C([0, 1] \rightarrow Z)$, where $f(t) = (f_\nu(t))$ is defined as in (4.61) above.

Proof. This is proved, with slightly different parameters, in [32], Proposition 2.1 and 2.4; the proof in our case is basically the same. $\square$

With Proposition 4.3.1, we can reduce the proof of Theorem 4.1.1 to the following a priori estimate.

Proposition 4.3.2. *Suppose u is a solution to (4.1) on a time interval $[0, T]$ with initial data $u(0) = g$ and $u_t(0) = h$ such that*

$$u \in C_t^1 \mathcal{H}_x^N([0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^d), \quad \partial_x u \in C_t^0 \mathcal{H}_x^N([0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^d), \quad (4.63)$$

and let $f(t)$ be defined accordingly. Assume

$$\|(g, \partial_x g, h)\|_X \leq \varepsilon \leq \varepsilon_0, \quad \sup_{0 \leq t \leq T} \|f(t)\|_X \leq \varepsilon_1 \ll 1,$$

then we have

$$\sup_{0 \leq t \leq T} \|f(t)\|_X \lesssim \varepsilon_1^{3/2} + \varepsilon.$$

4.3.1 Control of energy

From now on we will fix a solution u as described in Proposition 4.3.2, and the corresponding f ; in this section we will recover the energy bounds.

Proposition 4.3.3. *We have*

$$\sup_{0 \leq t \leq T} \sup_{|\mu| \leq N} \|\Gamma^\mu f(t)\|_{L^2} \lesssim \varepsilon_1^{3/2} + \varepsilon. \quad (4.64)$$

Proof. Recall that

$$\|\Gamma^\mu f(t)\|_{L^2} \sim \|\Gamma^\mu u(t)\|_{L^2} + \|\Gamma^\mu(\partial_x, \partial_t)u(t)\|_{L^2},$$

the proof is basically the same as in [32]; we will present it here since the norms involved are different.

Define the energy

$$\begin{aligned} \mathcal{E}(t) &= \sum_{|\mu| \leq N} \int_{\mathbb{R}^3} \left(\sum_{\alpha=1}^d (|\partial_t \Gamma^\mu u_\alpha|^2 + b_\alpha^2 |\Gamma^\mu u_\alpha|^2 + c_\alpha^2 |\nabla \Gamma^\mu u_\alpha|^2) + \right. \\ &\quad \left. + \sum_{\alpha, \beta=1}^d \sum_{j, k=1}^3 S_{\alpha\beta}^{jk}(u, \partial u) \partial_j \Gamma^\mu u_\alpha \cdot \partial_k \Gamma^\mu u_\beta \right) dx, \end{aligned} \quad (4.65)$$

where

$$S_{\alpha\beta}^{jk}(u, \partial u) = \sum_{\gamma=1}^d \sum_{l=1}^3 (A_{\alpha\beta\gamma}^{jk} u_\gamma + B_{\alpha\beta\gamma}^{jkl} \partial_l u_\gamma).$$

Note that in the whole time interval $[0, T]$ the H^N based norms are small, we thus have

$$\mathcal{E}(t) \sim \|\Gamma^\mu u(t)\|_{L^2}^2 + \|\Gamma^\mu(\partial_x, \partial_t)u(t)\|_{L^2}^2.$$

Now using (4.1) and the symmetry assumption of S , and integrating by parts, we may compute that

$$\begin{aligned}
\frac{1}{2}\partial_t \mathcal{E}(t) &= \frac{1}{2} \sum_{|\mu| \leq N} \sum_{\alpha, \beta=1}^d \sum_{j,k=1}^3 \int_{\mathbb{R}^3} \partial_t S_{\alpha\beta}^{jk}(u, \partial u) \cdot \partial_j \Gamma^\mu u_\alpha \cdot \partial_k \Gamma^\mu u_\beta \\
&- \sum_{|\mu| \leq N} \sum_{\alpha, \beta=1}^d \sum_{j,k=1}^3 \int_{\mathbb{R}^3} \partial_j S_{\alpha\beta}^{jk}(u, \partial u) \cdot \partial_t \Gamma^\mu u_\alpha \cdot \partial_k \Gamma^\mu u_\beta \\
&+ \sum_{|\mu| \leq N} \sum_{\alpha, \beta=1}^d \sum_{j,k=1}^3 \partial_t \Gamma^\mu u_\alpha \cdot [\Gamma^\mu (S_{\alpha\beta}^{jk}(u, \partial u) \partial_j \partial_k u_\beta) - S_{\alpha\beta}^{jk}(u, \partial u) \Gamma^\mu \partial_j \partial_k u_\beta] \\
&+ \sum_{|\mu| \leq N} \sum_{\alpha=1}^d \partial_t \Omega^\mu u_\alpha \cdot \Gamma^\mu \mathcal{Q}'_\alpha(u, \partial u).
\end{aligned}$$

Now, using the equation (4.1) again to eliminate ∂_t^2 terms and using Leibniz rule, we can bound the time derivative by

$$|\partial_t \mathcal{E}(t)| \lesssim \mathcal{E}(t) \cdot \left(\sup_{|\mu| \leq N/2} \|\Gamma^\mu u(t)\|_{L^\infty} + \sup_{|\mu| \leq N/2} \|\Gamma^\mu (\partial_t, \partial_x) u(t)\|_{L^\infty} \right).$$

Next, since we have

$$\partial_t u_\alpha = \frac{v_\alpha + v_{-\alpha}}{2}; \quad u_\alpha = \frac{i}{2} \Lambda_\alpha^{-1} (v_\alpha - v_{-\alpha}), \quad (4.66)$$

and also $v_\sigma(t) = e^{-it\Lambda_\sigma} f_\sigma(t)$ for $\nu \in \mathcal{P}$ with the bound $\|f_\sigma(t)\|_X \lesssim \varepsilon_1$, we can use Corollary 4.2.8 to deduce that

$$\sup_{|\mu| \leq N/2} \|\Gamma^\mu u(t)\|_{L^\infty} + \sup_{|\mu| \leq N/2} \|\Gamma^\mu (\partial_t, \partial_x) u(t)\|_{L^\infty} \lesssim \frac{\varepsilon_1}{(1+|t|) \log^{20}(2+|t|)}, \quad (4.67)$$

and hence (note that $\mathcal{E}(t) \lesssim \varepsilon_1^2$)

$$|\partial_t \mathcal{E}(t)| \lesssim \frac{\varepsilon_1^3}{(1+|t|) \log^{20}(2+|t|)}.$$

Since $\mathcal{E}(0) \lesssim \varepsilon^2$ and the weight is integrable in t , this proves (4.64). $\square$

4.3.2 Duhamel formula, and control of Z norm

By definition of v and (4.1), we know that $(\partial_t - i\Lambda_\sigma)v_\sigma$ equals a (constant coefficient) quadratic form of u , ∂u or $\partial \partial_x u$. Using also (4.66) and Duhamel formula, we will obtain

$$\widehat{f}_\sigma(t, \xi) - \widehat{f}_\sigma(0, \xi) = \sum_{\mu, \nu \in \mathcal{P}} \int_0^t \int_{\mathbb{R}^3} e^{is\Phi_{\sigma\mu\nu}(\xi, \eta)} m_{\sigma\mu\nu}(\xi, \eta) \widehat{f}_\mu(s, \xi - \eta) \widehat{f}_\nu(s, \eta) d\eta ds \quad (4.68)$$

for each $\sigma \in \mathcal{P}$. The weight

$$m_{\sigma\mu\nu} = \sum_{i=1}^{20} \sum_{k, k_1, k_2} (1 + 2^{\max(k, k_1, k_2)}) \psi_{kk_1k_2}^{\sigma\mu\nu, i0} \left(\frac{\xi}{2^k} \right) \psi_{kk_1k_2}^{\sigma\mu\nu, i1} \left(\frac{\xi - \eta}{2^{k_1}} \right) \psi_{kk_1k_2}^{\sigma\mu\nu, i2} \left(\frac{\eta}{2^{k_2}} \right), \quad (4.69)$$

where the ψ 's have uniform compact support and belong to $\mathcal{Z}_6$ with uniform bounds.

The proof of Proposition 4.3.2 is now reduced to the following Z norm estimate.

Proposition 4.3.4. *Fix a choice of (σ, μ, ν) . Suppose*

$$m \geq 0, (j, k), (j_1, k_1), (j_2, k_2) \in \mathcal{J}; \quad \max(m, j, |k|, j_1, j_2, |k_1|, |k_2|) := M.$$

Let $2^m \leq a \leq b \leq 2^{m+1}$, and define the quantity J by

$$\widehat{J}(\xi) = \int_a^b \int_{\mathbb{R}^3} e^{is\Phi_{\sigma\mu\nu}(\xi, \eta)} m_{\sigma\mu\nu}(\xi, \eta) \mathcal{F}_x(f_\mu)_{j_1k_1}^*(s, \xi - \eta) \mathcal{F}_x(f_\nu)_{j_2k_2}^*(s, \eta) d\eta ds, \quad (4.70)$$

then we have

$$\|J_{jk}\|_{Z_{jk}} \lesssim \langle M \rangle^{-20} \varepsilon_1^2. \quad (4.71)$$

4.4 Proof of Proposition 4.3.4: The setup

First, for each $|\beta| \leq N/2 + 3$ we have¹

$$\widehat{\Gamma^\beta J}(\xi) = \sum_{|\beta_1| + |\beta_2| \leq |\beta|} \int_a^b \int_{\mathbb{R}^3} e^{is\Phi_{\sigma\mu\nu}(\xi, \eta)} m_{\sigma\mu\nu}^{\beta_1\beta_2}(\xi, \eta) \mathcal{F}_x(\Gamma^{\beta_1} f_\mu)_{j_1k_1}^*(s, \xi - \eta) \mathcal{F}_x(\Gamma^{\beta_2} f_\nu)_{j_2k_2}^*(s, \eta) d\eta ds, \quad (4.72)$$

where $m_{\sigma\mu\nu}^{\beta_1\beta_2}$ is obtained from $m_{\sigma\mu\nu}$ by applying Ω , and has the same form as (4.69) with uniform bounds for each $|\mu| \leq N/2 + 3$. In (4.72), we further decompose f_μ into $S_{l_1} f_\mu$ and f_ν into $S_{l_2} f_\nu$, and reduce to estimating I , where

$$\widehat{I}(\xi) = \int_a^b \int_{\mathbb{R}^3} e^{is\Phi(\xi, \eta)} m(\xi, \eta) \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta ds, \quad (4.73)$$

where the (σ, μ, ν) subindices are omitted, and

$$F = (\Gamma^{\beta_1} S_{l_1} f_\mu)_{j_1k_1}^*, \quad G = (\Gamma^{\beta_2} S_{l_2} f_\nu)_{j_2k_2}^*.$$

¹Strictly speaking we should commute Γ^σ with Q_{jk} here, but the commutators are estimated easily so we omit them.

4.4.1 Bounds for F , G and their time derivatives

Proposition 4.4.1. *We have the following bounds for F and $\partial_t F$; similar bounds will hold for G and $\partial_t G$.*

1. For F we have

$$\|F\|_{L^2} \lesssim 2^{-(N-6)\max(k_1, l_1)/2} \varepsilon_1; \quad (4.74)$$

$$\|\widehat{F}\|_{L^\infty} \lesssim 2^{-j_1/4} \varepsilon_1, \quad \text{if } k_1 \geq -K_0^2; \quad (4.75)$$

$$\|\widehat{F}\|_{L^\infty} \lesssim 2^{2l_1} 2^{(-3k_1-j_1)/2} \langle j_1 \rangle^{-N_0} \varepsilon_1, \quad \text{if } k_1 \leq -K_0^2. \quad (4.76)$$

2. If $k_1 \leq -K_0$ we have

$$\|F\|_{L^2} \lesssim \langle j_1 \rangle^{-N_0} 2^{-j_1-k_1/2} \varepsilon_1 \lesssim \langle j_1 \rangle^{-N_0} 2^{-j_1/2} \varepsilon_1; \quad (4.77)$$

$$\|e^{-i\Lambda_\mu} F\|_{L^\infty} \lesssim \langle j_1 \rangle^{-N_0} \min(2^{-j_1+k_1}, 2^{-(3m-j_1+k_1)/2}) \varepsilon_1; \quad (4.78)$$

$$\|e^{-i\Lambda_\mu} F\|_{L^\infty} \lesssim \max(\langle m \rangle, \langle j_1 \rangle)^{-N_0} 2^{-\max(m, j_1)} \varepsilon_1. \quad (4.79)$$

3. If $|k_1| < K_0$ we have

$$\|F\|_{L^2} \lesssim 2^{-5j_1/6} \langle j_1 \rangle^{N_0} \varepsilon_1, \quad \|\widehat{F}\|_{L^1} \lesssim 2^{-j_1} \langle j_1 \rangle^{-N_0} \varepsilon_1; \quad (4.80)$$

$$\|e^{-i\Lambda_\mu} F\|_{L^\infty} \lesssim 2^{l_1} 2^{-3m/2-j_1/3} \langle m \rangle^{N_0+A} \varepsilon_1, \quad \text{if } |j_1 - m| \geq A \log m; \quad (4.81)$$

$$\|e^{-i\Lambda_\mu} F\|_{L^\infty} \lesssim \langle m \rangle^{-N_0} 2^{-j_1} \varepsilon_1, \quad \text{if } |j_1 - m| \leq A \log m. \quad (4.82)$$

4. If $k_1 \geq K_0$ we have

$$\|F\|_{L^2} \lesssim \langle j_1 \rangle^{-N_0} 2^{-j_1-k_1/2} \varepsilon_1 \lesssim \langle j_1 \rangle^{-N_0} 2^{-j_1} \varepsilon_1; \quad (4.83)$$

$$\|e^{-i\Lambda_\mu} F\|_{L^\infty} \lesssim \langle j_1 \rangle^{-N_0+A} 2^{7k_1/2+l_1} 2^{-(3m+j_1)/2} \varepsilon_1, \quad \text{if } |j_1 - m| \geq A \log m; \quad (4.84)$$

$$\|e^{-i\Lambda_\mu} F\|_{L^\infty} \lesssim \langle m \rangle^{-N_0} 2^{-j_1} 2^{5k_1/2} \varepsilon_1, \quad \text{if } |j_1 - m| \leq A \log m. \quad (4.85)$$

5. For $\partial_t F$ we have

$$\|\partial_t F\|_{L^2} \lesssim \langle m \rangle^A 2^{\min(-(3m+k_1)/2, 3k_1/2)} \varepsilon_1^2 \lesssim \langle m \rangle^A 2^{-9m/8} \varepsilon_1^2. \quad (4.86)$$

Proof. The inequality (4.74) follows from the energy bound; for (4.75), we may assume $|k_1| \leq 4\delta'j_1$ (otherwise we use (4.74)), and decompose $\widehat{F}$ into spherical harmonics as in (4.12), assuming that $l_1 \leq 4\delta'j_1$ due to (4.74); since for each radial function $g(|\xi|)$ appearing in this expansion we have $\|g\|_{L^\infty} \lesssim 2^{j_1/2}\|g\|_{L^2}$ by one-dimensional Sobolev, we obtain that

$$\|\widehat{F}\|_{L^\infty} \lesssim 2^{j_1/2+2l_1} 2^{-(5/6-o)j_1} \varepsilon_1 \lesssim 2^{-j_1/4} \varepsilon_1.$$

(4.76) is proved in the same way, the only difference being that now $\|G(|\xi|)\|_{L_\xi^2} \sim 2^{-k_1}\|G\|_{L_{\mathbb{R}}^2}$ for radial G supported in $|\xi| \sim 2^{k_1}$, which introduces an additional factor of 2^{-k_1} .

Next, everything from (4.77) to (4.85) follows from either Definition 4.2.4 and Hölder, or Proposition 4.2.7; note that now we do not have the three extra derivatives, so the corresponding estimates have to be changed; for example we have a factor 2^{l_1} in (4.81) due to the presence of Ω in (4.51).

Next we prove (4.86); we only need to prove the first inequality. Recall from (4.68) we have that

$$\begin{aligned} \widehat{\partial_t F}(t, \xi) &= S_{l_1} P_{[k_1-2, k_1+2]} Q_{j_1 k_1} \sum_{\gamma, \tau \in \mathcal{P}} \sum_{j_2, k_2, j_3, k_3} \sum_{|\beta_2|+|\beta_3| \leq |\beta_1|} \int_{\mathbb{R}^3} e^{is\Phi_{\mu\gamma\tau}(\xi, \eta)} \\ &\quad \times m_{\mu\gamma\tau}^{\beta_2\beta_3}(\xi, \eta) \mathcal{F}_x(\Gamma^{\beta_2} f_\gamma)_{j_2 k_2}^*(s, \xi - \eta) \mathcal{F}_x(\Gamma^{\beta_3} f_\tau)_{j_3 k_3}^*(s, \eta) d\eta, \end{aligned}$$

similar to (4.72). Again we make the further S_{l_2} and S_{l_3} decompositions and reduce to estimating

$$I' = \varphi_{[k_1-2, k_1+2]}(\xi) \int_{\mathbb{R}^3} e^{is\Phi(\xi, \eta)} m(\xi, \eta) \widehat{K}(s, \xi - \eta) \widehat{L}(s, \eta) d\eta,$$

where we assume $\max(k_2, k_3, l_2, l_3) \leq 2\delta'm$, and

$$K = (\Gamma^{\beta_2} S_{l_2} f_\gamma)_{j_2 k_2}^*, \quad L = (\Gamma^{\beta_3} S_{l_3} f_\tau)_{j_3 k_3}^*,$$

and they satisfy the bounds in (4.74)-(4.85) above. First note that

$$\|I'\|_{L^2} \lesssim 2^{3k_1/2} \|I'\|_{L^\infty} \lesssim 2^{3k_1/2} \|K\|_{L^2} \|L\|_{L^2} \lesssim 2^{3k_1/2} \varepsilon_1^2.$$

Next, using Proposition 4.2.9, we have

$$\|I'\|_{L^2} \lesssim 2^{\max(k_2^+, k_3^+)} \min(\|K\|_{L^2} \|e^{-is\Lambda_\tau} L\|_{L^\infty}, \|L\|_{L^2} \|e^{-is\Lambda_\gamma} K\|_{L^\infty}). \quad (4.87)$$

Now suppose $\max(k_1, k_2) \leq -K_0^2$. We have $\|I'\|_{L^2} \lesssim 2^{\min(\kappa_1, \kappa_2)} \varepsilon_1^2$, where

$$\kappa_1 = -3m/2 + (j_2 - k_2)/2 - j_3 - k_3/2, \quad \kappa_2 = -3m/2 + (j_3 - k_3)/2 - j_2 - k_2/2.$$

Assume $j_2 \geq j_3$, then

$$\kappa_2 \leq -3m/2 - j_2/2 - (k_2 + k_3)/2 \leq -3m/2 - \max(k_2, k_3) \leq -3m/2 - k_1/2,$$

which proves (4.86).

If $\min(k_2, k_3) \geq -K_0^2$, then if $j_2 \geq (1-o)m$, we can bound K in L^2 and $e^{-is\Lambda_\tau} L$ in L^∞ to get $\|I'\|_{L^2} \lesssim 2^{-1.9m} \varepsilon_1^2$. If $\max(j_2, j_3) \leq (1-o)m$, we then choose to bound one factor in L^∞ and the other factor in L^2 to get

$$\|I'\|_{L^2} \lesssim \min \{ 2^{4k_2+l_2} 2^{-3m/2} 2^{-10 \max(k_3, l_3)}, 2^{4k_3+l_3} 2^{-3m/2} 2^{-10 \max(k_2, l_2)} \} \varepsilon_1^2,$$

which implies (4.86). If $k_2 \geq -K_0^2 \geq k_3$, we may also assume $k_2 > -K_0$. If $j_2 \geq (1-o)m$ we can argue as before, and if $j_2 \leq (1-o)m$ we can again use (4.78) and (4.81) to bound

$$\|I'\|_{L^2} \lesssim \min \{ 2^{4k_2+l_2} 2^{-3m/2} 2^{-j_3/2}, 2^{-3m/2} 2^{j_3} 2^{-10 \max(k_2, l_2)} \} \varepsilon_1^2,$$

which also implies (4.86). □

Remark 4.4.2. Note that, in proving (4.75) we have actually used the following bound

$$\|\mathcal{F} S_l f_{jk}^*\|_{L^\infty} \lesssim 2^{2l+j/2-k} \|f_{jk}\|_{L^2};$$

combining this with (4.86) allows us to bound $\|\widehat{\partial_t F}\|_{L^\infty}$, which will be used later in the estimates below.

4.4.2 Easy cases

We now begin to prove the estimate of I . By (4.87), we have the following basic estimate

$$\|I\|_{L^2} \lesssim 2^m \sup_{a \leq s \leq b} 2^{\max(k_1^+, k_2^+)} \min(\|F\|_{L^2} \|e^{-s\Lambda_\tau} G\|_{L^\infty}, \|G\|_{L^2} \|e^{-s\Lambda_\sigma} F\|_{L^\infty}), \quad (4.88)$$

which will be used repeatedly below.

First, we shall get rid of the case when

$$j \geq \max(-k, \min(j_1, j_2), m - \min(k_1^-, k_2^-)) + A \log M,$$

where recall that $A \ll N_0$ is some absolute constant. In fact, assume $j_1 \leq j_2$, we have

$$I_{jk}(x) = \int_{\mathbb{R}^3} \varphi_k(\xi) e^{ix \cdot \xi} d\xi \int_a^b \int_{\mathbb{R}^3} e^{is\Phi(\xi, \eta)} m(\xi, \eta) \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta ds \quad (4.89)$$

for $|x| \sim 2^j$. Fixing s and η , we analyze the integral in ξ , which is

$$\int_{\mathbb{R}^3} \varphi_k(\xi) e^{i(x \cdot \xi + s\Phi(\xi, \eta))} m(\xi, \eta) \widehat{F}(s, \xi - \eta) d\xi,$$

by using Proposition 4.2.5, choosing

$$K = |x|, \quad n = 1, \quad \epsilon \sim 1, \quad \lambda \sim 2^{\max(0, -k, j_1)},$$

and noticing $|\nabla_\xi \Phi| \lesssim 2^{-\min(k_1^-, k_2^-)}$, so that we can bound the output $\|I_{jk}\|_{L^2} \lesssim 2^{-10D} \epsilon_1^2$.

Next, suppose $j \geq 3m$. From above, we may also assume that either $M \geq m^2$ (in which case $\max(j_1, k_2, k_1, k_2) \geq M$ and the estimates are trivial), or

$$j \leq |k| + A \log m, \quad \text{or} \quad \min(j_1, j_2) \geq j - A \log m.$$

In the former case, we only need to prove $\|\widehat{I}\|_{L^\infty} \lesssim 2^j \epsilon_1^2$, which is trivial by estimating both F and G factors in L^2 in (4.73), using (4.74). In the latter case, we simply exploit the fact $j \geq 3m$, then use (4.88) and Proposition 4.4.1 to conclude.

Now, assuming $j \leq 3m$, we can easily treat the case when² $\max(k_1, l_2, k_2, l_2) \geq 5\delta'm$ or when $\max(j_1, j_2) \geq 9m$, using (4.74) and (4.79) respectively. Thus from now on we will assume the inequalities

$$j \leq 3m, \quad \max(k_1, k_2, l_1, l_2) \leq 5\delta'm, \quad M \leq 9m, \quad (4.90)$$

and

$$j \leq \max(-k, \min(j_1, j_2), m - \min(k_1^-, k_2^-)) + A \log m. \quad (4.91)$$

Moreover, if we have $j \leq (1 + o)m$, we can also assume that $\max(k_1, k_2, l_1, l_2) \leq (2 + o)\delta'm$. For

²In this case we will actually use a bound stronger than (4.74), using the simple fact that $\min(|\beta_1|, |\beta_2|) \leq (N+6)/4$.

simplicity, from now on we will use the symbol $X \preceq Y$ to denote $X \leq Y + A \log m$ (and similarly for $X \succ Y$), so for example we have $j \preceq m$ from (4.90).

4.5 Low frequencies

In this section we consider the case $\max(k_1, k_2) \leq -K_0$. By (4.26), we may assume $k_0 \leq -K_0$ also. *In this section only*, we will replace F and G by $\langle j_1 \rangle^{N_0} F$ and $\langle j_2 \rangle^{N_0} G$ respectively, so that our goal is to prove

$$L = 2^\kappa := 2^{j+k/2} \varepsilon_1^{-2} \|I_{jk}\|_{L^2} \lesssim \langle M \rangle^{-30} \left(\frac{\langle j_1 \rangle \cdot \langle j_2 \rangle}{\langle j_0 \rangle} \right)^{N_0}. \quad (4.92)$$

In each case below, we will prove that either $L \lesssim 2^{-o \cdot m}$, or $L \lesssim \langle M \rangle^A$ and $\min(j_1, j_2) \geq o \cdot m$; either will imply (4.92). Below we will always denote $\rho := b_\nu/b_\sigma$, while ρ_i will be various constants.

4.5.1 First reduction

Suppose $j \succ m + \max(k_1, k_2)$, then we must have either $j \preceq \min(j_1, j_2)$, or $j + k \preceq 0$. In the first situation we may, without loss of generality, assume $k \leq k_1 + 6$, and bound using (4.88) that

$$\kappa \preceq j + \frac{k}{2} + m - j_1 - \frac{k_1}{2} - m \preceq 0.$$

Also, using Proposition 4.4.1, we will have $\kappa \leq -m/10$, unless $|j_2 - k_2 - m| \leq m/5$ and $|j_0 - j_1| \leq m/10$, in which case we must have $\min(j_1, j_2) \geq o \cdot D$, so (4.92) is proved anyway.

In the second situation we have $j + k \preceq 0$, thus we only need to bound $\|\widehat{P_{k_0} I}\|_{L^\infty} \leq 2^{\kappa'} \varepsilon_1^2$ with $\kappa' \preceq j$. Since we have assumed that $j \succeq m + \frac{k_1 + k_2}{2}$, we can directly use Young's inequality to bound

$$\kappa' \preceq m - j_1 - \frac{k_1}{2} - j_2 - \frac{k_2}{2} \preceq m + \frac{k_1 + k_2}{2} \preceq j.$$

Again, we would actually have $\kappa' \leq -o \cdot m$ from above, unless $|j_1 + k_1| + |j_2 + k_2| + |k_1 - k_2| \leq o \cdot m$, which means we only need to consider the case

$$j_1, k_1, j_2, k_2 = o \cdot m, \quad j_0, -k_0 = (1 + o)m. \quad (4.93)$$

This would imply that that F and G are in Schwartz space with suitable some Schwartz norm bounded by $2^{o \cdot m} \varepsilon_1$, so we will treat the integral in η in (4.73) as a standard oscillatory integral with phase $s\Phi(\xi, \eta)$. Using Van der Corput lemma (see Proposition 4.8.1), we can bound this integral for

each ξ by $2^{-m/10}\epsilon_1^2$ (which would imply $\kappa' \preceq j - m/10$ since $j \geq (1 - o)m$), unless $\Lambda_\mu + \Lambda_\nu = 0$, in which case we have $|\Phi(\xi, \eta)| \gtrsim 1$ and thus will integrate by parts in s once to conclude; see below.

Now we can assume $j \preceq m + \max(k_1, k_2)$. Note that from the above proof, we can also assume $j \succ \min(j_1, j_2)$. If $j + k \preceq 0$, we may assume $j_1 \geq j_2$ and compute, by (4.88), that

$$\kappa \preceq \frac{j}{2} + m - j_1 - \frac{k_1}{2} - \frac{3m}{2} + \frac{j_2 - k_2}{2} \preceq 0,$$

which is because

$$-j_1 + \frac{j_2}{2} - \frac{k_1 + k_2}{2} \leq -\frac{\max(j_1, j_2)}{2} - \frac{k_1 + k_2}{2} \leq -\frac{\max(k_1, k_2)}{2}.$$

Moreover, if $j_2 \leq o \cdot m$ we will either have the stronger bound $\kappa \leq -o \cdot m$, or we will be in the case (4.93); if instead $j_1 \sim j_2 \sim m$, then we can recover the logarithmic loss from the Z norm bound.

Now we already have

$$\max(-k, \min(j_1, j_2)) \prec j \preceq m + \max(k_1, k_2).$$

Next, suppose $b_\nu - b_\sigma - b_\tau \neq 0$, so $|\Phi| \gtrsim 1$. Integrating by parts in s , we obtain

$$\begin{aligned} \widehat{I}(\xi) &= \int_{\mathbb{R}^3} e^{i\Phi(\xi, \eta)} \frac{m(\xi, \eta)}{i\Phi(\xi, \eta)} \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta \Big|_{s=a}^{s=b} \\ &\quad - \int_a^b \int_{\mathbb{R}^3} e^{is\Phi(\xi, \eta)} \frac{m(\xi, \eta)}{i\Phi(\xi, \eta)} \widehat{\partial_s F}(s, \xi - \eta) \widehat{\partial_s G}(s, \eta) ds d\eta \\ &\quad - \int_a^b \int_{\mathbb{R}^3} e^{is\Phi(\xi, \eta)} \frac{m(\xi, \eta)}{i\Phi(\xi, \eta)} \widehat{F}(s, \xi - \eta) \widehat{\partial_s G}(s, \eta) ds d\eta. \end{aligned} \quad (4.94)$$

Denote the three terms by I_0 , I_1 and I_2 respectively, by symmetry, we only need to consider I_0 and I_1 . Since $|\Phi| \gtrsim 1$, we can use a version of (4.88) together with Proposition 4.4.1 to conclude $\kappa \preceq \max(\kappa_1, \kappa_2)$ where

$$\begin{aligned} \kappa_1 &\preceq j + \frac{k}{2} - \max\left(m + \frac{j_1}{2}, \frac{3m - j_1 + k_1}{2}\right) \preceq -\frac{m}{6}, \\ \kappa_2 &\preceq j + \frac{k}{2} + m + \min\left(\frac{-3m - k_1}{2}, \frac{3k_1}{2}\right) - m \preceq -\frac{m}{8}. \end{aligned}$$

Thus this contribution will also be acceptable.

Now we will assume $b_\sigma - b_\mu - b_\nu = 0$. The easier case would be when $k_2 - k \leq -D_0$, so we will

have $|k_1 - k| \leq 6$ and hence $|\nabla_\eta \Phi(\xi, \eta)| \sim 2^k$. We may then assume that $\max(j_1, j_2) \succeq m + k$ or we could integrate by parts in η and choose

$$K = 2^m, \quad n = 1, \quad \epsilon \sim 2^k, \quad \lambda \sim 2^{\max(j_1, j_2)}$$

in Proposition 4.2.5 to bound $\|I\|_{L^2} \lesssim 2^{-10m} \varepsilon_1^2$.

Next, suppose $j_1 \succeq \max(j_2, m + k)$. Since we also have $j \preceq m + k$, we will use (4.88) and estimate the F factor in L^2 and the G factor in L^∞ , so we get

$$\kappa \preceq j + \frac{k}{2} + m - j_1 - \frac{k}{2} - m \preceq 0,$$

and again we will have $\kappa \leq -D/10$ unless $j_2 \geq m/10$, thus this contribution will also be acceptable.

If instead $j_2 \succeq \max(j_1, m + k)$, we will have $\kappa \preceq \min(\kappa_1, \kappa_2)$, where

$$\kappa_1 \preceq j + \frac{k}{2} + m - j_1 - \frac{k}{2} - j_2 + k_2 \preceq m - j_1 + k_2$$

is obtained from estimating the F factor in L^2 and the G factor in L^∞ , and

$$\kappa_2 \preceq j + \frac{k}{2} + m - \frac{3m}{2} + \frac{j_1 - k}{2} - j_2 - \frac{k_2}{2} \preceq \frac{-m + j_1 - k_2}{2}$$

is obtained the other way. Now at least one of κ_1 and κ_2 must be $\preceq 0$, therefore we will get the desired bound up to a logarithmic loss, which we can always recover unless

$$j_1, k = o \cdot m, \quad j_0, j_2, -k_2 = (1 + o)m.$$

In this final scenario, we will integrate by parts in s , to produce I_0 , I_1 and I_2 terms. Notice that $|\zeta| := |\xi - \eta| \sim 2^k$, we have $\Phi(\xi, \eta) = \Phi(\xi, 0) + \eta \cdot \Psi(\xi, \eta)$, where $\Phi(\xi, 0)$ is a fixed nonzero analytic function of $|\xi|^2$. Using Taylor expansion and a scaling argument, we can see that

$$\tilde{m}(\xi, \eta) = \frac{m(\xi, \eta)}{\Phi(\xi, \eta)} \psi_1(2^{-k}(\xi - \eta)) \psi_2(2^{-k_2} \eta),$$

where ψ_1 is supported in $|\xi| \sim 1$ and ψ_2 in $|\xi| \lesssim 1$, verifies the bound $\|\mathcal{F}\tilde{m}\|_{L^1} \lesssim 2^{o(m)}$, so we can close the estimate by repeating the estimate of I_j above.

From now on, under the assumption $\max(k_0, k_1, k_2) \leq -K_0$, we can assume

$$\max(\min(j_1, j_2), -k) \prec j \preceq m + k_1; \quad |k_1 - k_2| \leq D_0; \quad k \leq k_1 + D_0.$$

Recall that at point $(0, 0)$, we may expand the phase function as

$$\Phi(\xi, \eta) = \frac{c_\sigma^2}{2b_\sigma} |\xi|^2 - \frac{c_\mu^2}{2b_\mu} |\xi - \eta|^2 - \frac{c_\nu^2}{2b_\nu} |\eta|^2 + O(|\xi|^4 + |\eta|^4). \quad (4.95)$$

4.5.2 Second reduction

Here we assume $c_\mu^2/b_\mu + c_\nu^2/b_\nu = 0$. Since $b_\mu + b_\nu \neq 0$, we must have $\rho_2 := -c_\mu^4/(8b_\mu^3) - c_\nu^4/(8b_\nu^3) \neq 0$.

The phase function is now expanded as

$$\Phi(\xi, \eta) = \rho_0 |\xi|^2 + \rho_1 (\xi \cdot \eta) + \rho_2 |\eta|^4 + O(|\eta|^6 + |\xi||\eta|^3), \quad (4.96)$$

where $\rho_1 \rho_2 \neq 0$. Note that $|\eta| \sim |\xi - \eta| \sim 2^{k_1}$, we will first exclude the case $k_1 \leq k + D_0$. In fact, in this case we would have $|\nabla_\eta \Phi| \sim 2^k$ in the region of interest, under which assumption every estimate can be done in exactly the same way as the case when $k_2 - k \leq -D_0$, which was treated at the end of Section 4.5.1 above. Therefore, we may assume further $k_1 \geq k + D_0$.

(1) First, assume $3k_1 \leq k - D_0^2$, then we will have $|\nabla_\eta \Phi(\xi, \eta)| \sim 2^k$ in the region of interest. Moreover, We are having $|\nabla_\eta^\nu \Phi(\xi, \eta)| \lesssim 2^{(4-|\nu|)k_1}$ in the region of interest for $2 \leq |\nu| \leq 4$. Now if $\max(j_1, j_2) \prec m + k$, we will use Proposition 4.2.5 and take

$$K = 2^m, \quad n = 3, \quad \epsilon \sim 2^k, \quad \lambda \sim 2^{\max(j_1, j_2)}$$

and deduce that

$$\|I_{jk}\|_{L^2} \lesssim \exp(-c(\min(2^{m+4k/3}, 2^{m+k-j_1}, 2^{m+k-j_2}))^c) \epsilon_1^2$$

which will be sufficient, since

$$m + 4k_0/3 \succeq m + k + k_1 \succeq m + k - j_1 \succ 0.$$

Therefore by symmetry, we may assume $j_1 \succeq \max(j_2, m + k)$. We then use (4.88), and bound the F

factor in L^2 norm, the F factor in L^∞ norm, to obtain an estimate (note also that $j \preceq m + k_1$)

$$\kappa \preceq (m + k_1) + \frac{k}{2} + m - j_1 - \frac{k_1}{2} - \frac{3m - j_2 + k_1}{2} \preceq 0. \quad (4.97)$$

Moreover, we have $\kappa \leq -m/10$ unless $j_2 \geq m/10$, this proves (4.92).

(2) Next, assume $3k_1 \geq k + D_0^2$. then we will have $|\nabla_\eta \Phi(\xi, \eta)| \sim 2^{3k_1}$ in the region of interest. Moreover, We are having $|\nabla_\eta^\nu \Phi(\xi, \eta)| \lesssim 2^{(4-|\nu|)k_1}$ in the region of interest for $2 \leq |\nu| \leq 4$. Now if $\max(j_1, j_2) \prec m + 3k_1$, we will use Proposition 4.2.5 and take

$$K = 2^m, \quad n = 3, \quad \epsilon \sim 2^{3k_1}, \quad \lambda \sim 2^{\max(j_1, j_2)}$$

and deduce that

$$\|I_{jk}\|_{L^2} \lesssim \epsilon_1^2 \exp(-\gamma(\min(2^{m+3k_1-j_1}, 2^{m+3k_1-j_2}))^\gamma) \epsilon_1^2$$

which will be sufficient. Therefore by symmetry, we may assume $j_1 \succeq \max(j_2, m + 3k_1)$. We then use (4.88), bound the F factor in L^2 and G factor in L^∞ to obtain the estimate (note also that $j \preceq m + k_1$)

$$\kappa \preceq (m + k_1) + \frac{3k_1}{2} + m - j_1 - \frac{k_1}{2} - \frac{3m - j_2 + k_1}{2} \preceq 0, \quad (4.98)$$

and also either $j_2 \geq o \cdot m$ or $\kappa \leq -o \cdot m$, so (4.92) is proved in this case.

(3) Now assume $|3k_1 - k| \leq D_0^2$. We may also assume that $k_1 \succ -m/4$, since otherwise we may assume $j_1 \geq j_2 \succeq -k_1$, and directly use (4.88) to bound

$$\kappa \preceq m + k_1 + \frac{3k_1}{2} + m - j_1 - \frac{k_1}{2} - \frac{3m - j_2 + k_1}{2} \preceq \frac{m}{2} + 2k_1 \preceq 0,$$

which implies (4.92) since $\min(j_1, j_2) \geq |k_1| \geq m/5$. Now we will have

$$\nabla_\eta \Phi(\xi, \eta) = \rho_1 \xi + 4\rho_2 |\eta|^2 \eta + O(2^{5k_1}), \quad (4.99)$$

as well as

$$\Phi(\xi, \eta) = \rho_1 (\eta \cdot \xi) + \rho_2 |\eta|^4 + O(2^{6k_1}). \quad (4.100)$$

Choose a cutoff $\psi_1(\tau)$ supported in $|\tau| \ll 1$ such that $1 - \psi_1 = \psi_2$ is supported in $|\tau| \gtrsim 1$ (this may depend on $\mathcal{B}$).

We first consider the contribution where the factor $\psi_2(2^{-3k_1}(\rho_1 \xi + 4\rho_2 |\eta|^2 \eta))$ is attached to

the weight $m(\xi, \eta)$. In this situation we will have $|\nabla_\eta \Phi| \sim 2^{3k_1}$ and $|\nabla_\eta^\nu \Phi(\xi, \eta)| \lesssim 2^{(4-|\nu|)k_1}$ for $2 \leq |\nu| \leq 4$, thus when $\max(j_1, j_2) \prec m + 3k_1$, we will be able to use Proposition 4.2.5 with

$$K = 2^m, \quad n = 3, \quad \epsilon \sim 2^{3k_1}, \quad \lambda \sim 2^{\max(j_1, j_2)}$$

to obtain sufficient decay. Note that a new difficulty arises with the introduction of the ψ_2 factor, but one have

$$|\partial_\eta^\nu \psi_2(2^{-3k_1}(\rho_1 \xi + 4\rho_2 |\eta|^2 \eta))| \lesssim (C|\nu|)! 2^{-k_1 |\nu|}$$

which can be proved by rescaling. Therefore, we may assume that $\max(j_1, j_2) \succeq m + 3k_1$, which allows us to repeat the proof in part (2) above. Here one should note that (4.88) still holds, because the function

$$\chi(2^{-3k_1} \xi) \chi(2^{-k_1} \eta) \psi_2(2^{-3k_1}(\rho_1 \xi + 4\rho_2 |\eta|^2 \eta))$$

has its inverse Fourier transform bounded in L^1 , which is again easily seen by rescaling, so that we can still use Proposition 4.2.9.

Now suppose that $\psi_1(2^{-3k_1}(\rho_1 \xi + 4\rho_2 |\eta|^2 \eta))$ is attached to $m(\xi, \eta)$. In this contribution we always have $|\Phi| \sim 2^{4k_1}$. Moreover, if we consider the function

$$\tilde{m}(\xi, \eta) = \frac{2^{4k_1} m(\xi, \eta)}{\Phi(\xi, \eta)} \psi_1(2^{-3k_1}(\rho_1 \xi + 4\rho_2 |\eta|^2 \eta)) \chi(2^{-3k_1} \xi) \chi(2^{-k_1} \eta), \quad (4.101)$$

then it will have inverse Fourier transform bounded in L^1 (simply by rescaling and using the expansion (4.100)), thus we will be able to integrate by parts in s to produce the I_0 , I_1 and I_2 terms but with additional cutoff factors, then use a version of (4.88) and Proposition 4.4.1 to obtain (by symmetry) $\kappa \preceq \max(\kappa_1, \kappa_2)$, where

$$\kappa_1 \preceq m + k_1 + \frac{3k_1}{2} - 4k_1 - \max(j_1, j_2) - \frac{k_1}{2} - \frac{3m - \min(j_1, j_2) + k_1}{2} \preceq 0$$

corresponds to I_0 and

$$\kappa_2 \preceq m + k_1 + \frac{3k_1}{2} + m - 4k_1 - \frac{3m}{2} - \frac{k_1}{2} - m = -\frac{m}{2} - 2k_1 \preceq 0$$

corresponds to I_1 . Moreover, we have either $\mu \leq -m/10$ or $\min(j_1, j_2) \geq |k_1| \geq m/10$. Therefore we have finished the proof in the case when $c_\mu^2/b_\mu + c_\nu^2/b_\nu = 0$.

4.5.3 The final case

Assume $c_\sigma^2/b_\sigma + c_\tau^2/b_\tau \neq 0$. In this case we have

$$\Phi(\xi, \eta) = \alpha|\xi|^2 + \beta\xi \cdot \eta + \gamma|\eta|^2 + O(|\xi|^4 + |\eta|^4) \quad (4.102)$$

with $\beta\gamma \neq 0$. If $k_1 \geq k + D_0^2$, then we will have $|\nabla_\eta \Phi| \sim 2^{k_1}$, so we can argue exactly as in Section 4.5.2 above; the situation will be the same if $k_1 \leq k + D_0^2$, and we insert certain cutoff to restrict to the region $|\beta\xi + 2\gamma\eta| \gtrsim 2^k$. Note in the latter case, the introduction of this cutoff will not affect the use of Proposition 4.2.9 as in the derivation of (4.88), as will be shown below.

Next, suppose $k_1 - k \leq D_0^2$, and we attach some cutoff supported in the region where $|\beta\xi + 2\gamma\eta|$ is small, and $\beta^2 - 4\alpha\gamma \neq 0$. Then we will have $|\Phi| \sim 2^{2k}$, so we can integrate by parts in s and argue as in Section 4.5.2 above. Here one should note that (4.88) holds, because the function

$$\tilde{m}(\xi, \eta) = \frac{2^{2k}m(\xi, \eta)}{\Phi(\xi, \eta)}\psi_1(2^{-k}(\beta\xi + 2\gamma\eta))\psi_2(2^{-k}\xi)\psi_3(2^{-k}\eta)$$

will have its inverse Fourier transform bounded in L^1 due to scaling (with or without the $2^{2k}/\Phi$ factor).

In the only remaining case we have $\beta^2 - 4\alpha\gamma = 0$, so after re-parametrization we will have $\Phi(\xi, \eta) = \rho_3|\eta - \rho\xi|^2 + O(2^{4k})$. We have also assumed that $k_1 - k \leq D_0^2$ and $|\eta - \rho\xi| \ll 2^k$. (again these may depend on $\mathcal{B}$).

(1) Suppose $|k| \geq m/6$. If $|k| \geq m/2$, then we can directly use (4.88) to bound

$$\kappa \leq (m + k) + \frac{k}{2} + m - j_1 - \frac{k}{2} - j_2 + k \leq 2m + 4k \leq 0,$$

which implies (4.92) because $\min(j_1, j_2) \geq |k| \geq m/3$. Thus we may assume $-m/2 \prec k \leq -m/6$. By inserting suitable cutoff functions to the weight $m(\xi, \eta)$, we may consider the integral in the regions where $|\eta - \rho\xi| \sim 2^{-r}$ for $|k| \leq l \prec m/2$, and where $|\eta - \rho\xi| \lesssim 2^{-m/2}\langle m \rangle^A$.

In the first situation we have $|\nabla_\eta \Phi| \sim |\nabla_\xi \Phi| \sim 2^{-r}$. Suppose $\max(j_1, j_2) \prec m - r$, we will use Proposition 4.2.5, setting

$$K = 2^m, \quad n = 1, \quad \epsilon \sim 2^{-r}, \quad \lambda \sim 2^{\max(j_1, j_2, r)},$$

to bound (say) $\kappa \leq -10m$ (here the restriction $\lambda \geq 2^r$ is due to the newly introduced cutoff factor $\psi(2^r(\eta - \rho\xi))$). Moreover, if $j \succ m - r$, we may also assume $j \succ j_1$, so we can integrate by parts in

ξ with fixed η exactly as in the estimate of (4.89), where we choose, in Proposition 4.2.5,

$$K = 2^m, \quad n = 1, \quad \epsilon \sim 2^{j-m}, \quad \lambda \sim 2^{\max(r, j_1)},$$

to obtain sufficient decay (note that $j_0 \succ m - l \succ \max(r, m/2)$).

Now we have $j \preceq m - r \preceq \max(j_1, j_2)$. Suppose $j_1 \geq j_2$, we then use (4.88) to bound

$$\kappa \preceq (m - r) + \frac{k}{2} + m - (m - r) - \frac{k}{2} - m = 0,$$

and in this case we can recover the logarithmic loss because $\min(j_1, j_2) \geq |k| \geq m/7$. Here we still need to verify the validity of (4.88) with weight

$$\tilde{m}(\xi, \eta) = m(\xi, \eta) \psi_1(2^{-k_0} \xi) \psi_2(2^{-k_0} \eta) \psi_3(2^l(\eta - \rho \xi)).$$

Now by the algebra property of the norm $\|\mathcal{F}^{-1}M\|_{L^1}$, we only need to bound this norm for the function

$$\mathbf{n}(\xi, \eta) = \psi_1(2^{-k_0} \xi) \psi_3(2^l(\eta - \rho \xi)),$$

which is easily estimated by a linear transformation and rescaling.

In the second situation above, we must have $j \preceq m/2$, otherwise we could bound the integral in the same way as in the estimate of (4.89). Therefore we may assume $j_1 \geq j_2$, and use (4.88) to bound

$$\kappa \preceq \frac{m}{2} + \frac{k}{2} + m - j_1 - \frac{k}{2} - \frac{3m}{2} + \frac{j_2 - k}{2} \preceq 0,$$

also with $\min(j_1, j_2) \succeq m/6$.

(2) Suppose $|k| \prec m/6$. By inserting suitable cutoff functions, we will consider the integral in regions where $|\eta - \rho \xi| \sim 2^{-r}$, for $|k| \leq r \prec 3|k|$, and where $|\eta - \rho \xi| \lesssim 2^{3k} \langle m \rangle^A$. First suppose $r \prec 3|k|$, then we will have $|\nabla_\eta \Phi| \sim 2^{-r}$ as well as $|\nabla_\xi \Phi| \sim 2^{-l}$, and note that $r \prec m/2$. This will allow us to assume that $\max(j_1, j_2) \succeq m - r \succeq j_0$, since if $\max(j_1, j_2) \prec m - r$, we will be able to use Proposition 4.2.5 to integrate by parts in η , setting

$$K = 2^m, \quad k = 1, \quad \epsilon \sim 2^{-l}, \quad \lambda \sim 2^{\max(j_1, j_2, r)}$$

to obtain sufficient decay; also since we have assumed that $j_0 \succ \min(j_1, j_2)$, if $j_0 \succ m - r$ we will be able to integrate by parts in ξ just as above. Now that we have $\max(j_1, j_2) \succeq m - r \succeq j_0$, we may

assume $j_1 \geq j_2$ (otherwise just switch F and G) and use (4.88) to bound

$$\kappa \preceq j + \frac{k}{2} + m - j_1 - \frac{k}{2} - m \preceq 0,$$

and also either $\mu \leq -m/10$, or $j_1 \geq j_2 \geq m/20$, which proves (4.92).

Next, let us assume $|k| \prec m/6$ and we are in the region $|\eta - \rho\xi| \lesssim 2^{3k}\langle m \rangle^A$. Since now $|\nabla_\eta \Phi| \lesssim 2^{3k}\langle m \rangle^A$, we must have $j \preceq m + 3k$ (otherwise we integrate by parts in ξ as before; note also that $3k \prec m/2$). In this situation we need much more careful analysis of the phase function, which we will carry out now.

recall that

$$\Phi(\xi, \eta) = \sum_{\alpha=1}^2 \sum_{\beta=0}^2 \rho_{\alpha\beta} |\zeta_\beta|^{2\alpha} + O(2^{6k_0}), \quad (4.103)$$

where $\zeta_0 = \xi$, $\zeta_1 = \xi - \eta$ and $\zeta_2 = \eta$, and the coefficients are

$$\rho_{10} = \frac{c_\sigma^2}{2b_\sigma}, \quad \rho_{11} = -\frac{c_\mu^2}{2b_\mu}, \quad \rho_{12} = -\frac{c_\nu^2}{2b_\nu}; \quad (4.104)$$

$$\rho_{20} = \frac{c_\sigma^4}{8b_\sigma^3}, \quad \rho_{21} = -\frac{c_\mu^4}{8b_\mu^3}, \quad \rho_{22} = -\frac{c_\nu^4}{8b_\nu^3}. \quad (4.105)$$

Now, in order for the quadratic term to be a perfect square, we must have $\rho_{10}\rho_{11} + \rho_{11}\rho_{12} + \rho_{12}\rho_{10} = 0$, or equivalently

$$\frac{b_\sigma}{c_\sigma^2} - \frac{b_\mu}{c_\mu^2} - \frac{b_\nu}{c_\nu^2} = 0. \quad (4.106)$$

Recall also that

$$b_\sigma - b_\mu - b_\nu = 0. \quad (4.107)$$

Now we compute the fourth-order term; note that apart from an error of at most $O(2^{6k})$, we may assume $\eta = \rho_4 \xi$, where $\rho_4 = -\rho_{10}/\rho_{12}$. We again have two situations.

(A) Suppose the c 's are not all equal, then in particular no two of the c 's can be equal, and we can compute $\sigma_2 = (c_\sigma^2 - c_\mu^2)/(c_\nu^2 - c_\mu^2)$. Therefore we may assume

$$\xi = (c_\nu^2 - c_\mu^2)v, \quad \eta = (c_\sigma^2 - c_\mu^2)v; \quad \xi - \eta = (c_\nu^2 - c_\sigma^2)v,$$

where $|v| \sim 2^k$. Up to a constant we may also assume

$$b_\sigma = \frac{1}{c_\nu^2} - \frac{1}{c_\mu^2}, \quad b_\mu = \frac{1}{c_\nu^2} - \frac{1}{c_\sigma^2}, \quad b_\nu = \frac{1}{c_\sigma^2} - \frac{1}{c_\mu^2}.$$

Therefore we may compute

$$\sum_{\beta=0}^2 \rho_{2\beta} |\zeta_\beta|^4 = \lambda |v|^4,$$

where up to a constant

$$\lambda = (c_\sigma c_\mu c_\nu)^6 \sum_{\text{cyclic}} \frac{c_\mu^2 - c_\nu^3}{c_\sigma^2} = -(c_\sigma c_\mu c_\nu)^4 (c_\sigma^2 - c_\mu^2)(c_\mu^2 - c_\nu^2)(c_\nu^2 - c_\sigma^2) \neq 0. \quad (4.108)$$

This means that $\Phi(\xi, \eta) = \sigma_3 |\xi|^4 + O(2^{6k})$, and also the function

$$\tilde{m}(\xi, \eta) = \frac{2^{4k} m(\xi, \eta)}{\Phi(\xi, \eta)} \psi(2^{-k} \xi) \psi_1(2^{-3k}(\eta - \sigma_2 \xi)),$$

where ψ is supported in $|\xi| \sim 1$ and ψ_1 supported in $|\xi| \lesssim 1$, has inverse Fourier transform bounded in L^1 (with bounds depending on $\mathcal{B}$), by a linear transformation and rescaling argument. Therefore, we can integrate by parts in s , producing I_0 , I_1 and I_2 terms, and use a version of (4.88) to bound $\kappa \preceq \max(\kappa_1, \kappa_2)$, where $(\kappa_1$ corresponds to I_0 , in which we may assume $j_1 \geq j_2$)

$$\kappa_1 \preceq (m + 3k) + \frac{k}{2} - 4k - j_1 - \frac{k}{2} - 3m/2 + \frac{j_2 - k}{2} \preceq -\frac{m}{2} - k \leq -\frac{m}{4};$$

$$\kappa_2 \preceq (m + 3k) + \frac{k}{2} + m - 4k - 3m/2 - \frac{k}{2} - m = -\frac{m}{2} - k \leq -\frac{m}{4}.$$

(B) Suppose that $c_\sigma = c_\mu = c_\nu$, and $b_\sigma - b_\mu - b_\nu = 0$. By extracting a constant we may assume $c_\sigma = 1$; by symmetry we may assume $b_\mu, b_\nu > 0$, so

$$\Phi(\xi, \eta) = \sqrt{|\xi|^2 + b_\sigma^2} - \sqrt{|\xi - \eta|^2 + b_\mu^2} - \sqrt{|\eta|^2 + b_\nu^2},$$

which implies that $\sigma_2 = b_\nu/b_\sigma$ and that $\Phi(\xi, \eta) = \partial_\eta \Phi(\xi, \eta) = 0$ at the point $\eta = \rho\xi$ for all ξ . Now, if we expand Φ as a power series of ξ and $\eta - \rho\xi$, the constant term will be 0 and the second order term will be $\rho_5 |\eta - \rho\xi|^2$; moreover, each term of degree four or higher must contain at least *two* factors of $\eta - \rho\xi$. Therefore, if we restrict to regions where $|\eta - \rho\xi| \sim 2^{-r}$ for $r \prec m/2$, we will have $|\nabla_\eta \Phi| \sim |\nabla_\xi \Phi| \sim 2^{-r}$, regardless of the relationship between r and k . In this situation we thus can repeat the argument before and assume $j \preceq m - r \preceq \max(j_1, j_2)$ and close the estimate using (4.88) as before. Now we may restrict to the region where $|\eta - \rho\xi| \lesssim 2^{-m/2} \langle m \rangle^A$, so we have $j \preceq m/2$.

Under this assumption we may then assume $j_1 \geq j_2$ and deduce

$$\kappa \preceq \frac{m}{2} + \frac{k}{2} + m - j_1 - \frac{k}{2} - \frac{3m}{2} + \frac{j_2}{2} - \frac{k}{2} \preceq 0.$$

In the final case above, we will have either $\kappa \leq -o \cdot m$ or $\min(j_1, j_2) \geq o \cdot m$, unless $j_1, j_2, k = o \cdot m$ and $j = (1/2 + o)m$. If so, we will *not* insert any cutoff to the integral as above; instead we will use a special argument as follows.

Note that

$$I_{jk}(x) = \int_a^b \int_{(\mathbb{R}^3)^2} e^{i(s\Phi(\xi, \eta) + x \cdot \xi)} \varphi_{j_0}(x) \varphi_{k_0}(\xi) m(\xi, \eta) \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\xi d\eta ds.$$

Since j_1, j_2 and k are all $o(m)$ in absolute value, the function $\chi(2^{-k}\xi)m(\xi, \eta)\widehat{F}(\xi - \eta)\widehat{G}(\eta)$ will be bounded in a suitable Schwartz norm by $2^{o \cdot m}$. Now we make a change of variables and let $\eta - \rho\xi = \zeta$, then we will be estimating, for fixed s , an integral of form

$$\int_{(\mathbb{R}^3)^2} e^{i(s\Psi(\xi, \zeta) + x \cdot \xi)} \phi(\xi, \zeta) d\xi d\zeta, \quad (4.109)$$

where ϕ is a function with Schwartz norm bounded by $2^{o \cdot m}$ and

$$\Psi(\xi, \zeta) = \rho_6 |\zeta|^2 + \sum_{q,r=1}^3 \zeta^q \zeta^r h_{qr}(\xi, \zeta),$$

with $h_{q,r}(0, 0) = 0$, so that we have $|\nabla_\zeta \Psi| \sim |\zeta|$ where $|\xi| + |\zeta|$ is small. Now, if we cutoff in the region $|\zeta| \gtrsim 2^{-2m/5}$, we will be able to integrate by parts in ζ to obtain sufficient decay; if we cutoff in the region $|\zeta| \ll 2^{-2m/5}$, we will then fix ζ and integrate by parts in ξ , noting that $|s\zeta^q \zeta^r| \lesssim 2^{m/5}$ and $|x| \sim 2^j \gtrsim 2^{2m/5}$, to obtain sufficient decay for this integral. This completes the proof in the final scenario and thus finishes the proof of Proposition 4.3.4 in the case $\max(k_0, k_1, k_2) \leq -K_0$.

4.6 Medium frequencies

In this section we consider the case when $\max(k, k_1, k_2) < K_0^2$, and $\max(k_1, k_2) > -K_0$. By (4.26), we may also assume $\max(k_1, k_2) < K_0$.

First we shall prove the crucial bilinear lemma introduced in Section 4.1.2. This allows us to restrict the angular variable under very mild conditions, and will be used frequently in the proof below.

Lemma 4.6.1. *Restrict the integral (4.73) by adding two cutoff functions and forming*

$$I' := \int_{\mathbb{R}^3} e^{it\Phi(\xi, \eta)} \psi_1(2^\kappa \Phi(\xi, \eta)) \varphi_k(\xi) m(\xi, \eta) \psi_2(2^v \sin \angle(\xi, \eta)) \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta, \quad (4.110)$$

where $|t| \sim 2^m$, ψ_1 is Schwartz, $\psi_2(z)$ is supported in $|z| \sim 1$. Assume that

$$\max(k_1, k_2) \geq -2K_0^2, \quad \kappa \prec m, \quad \max(0, k_1, k_2) := \bar{k}, \quad \max(l_1, l_2) := \bar{l} < m/10, \quad j_1 - k_1 \prec m,$$

then we have $|I'| \lesssim 2^{-20m} \varepsilon_1^2$, in each of the following three cases:

1. When $|k_1 - k_2| \leq 6$, $v \prec (m + k)/2 - \bar{k}$, and $m + k \succ 2\bar{l}$.
2. When $|k - k_1| \leq 6$, $v \prec (m + k_2)/2 - \bar{k}$, and $m + k_2 \succ 2\bar{l}$.
3. When $|k - k_2| \leq 6$, $|k_1| \prec (m - \bar{l})/2$ and $v \prec (m - \bar{l})/2$.

Proof. Let $\max(k_1, k_2) := k_3$. By symmetry, we may assume $\xi = |\xi|e_1$ where $|\xi| \sim 2^k$, and e_i are coordinate vectors; we then make several reductions. Fix a time s , let

$$\rho = |\eta|, \quad \theta = \angle(e_1, \eta), \quad \phi = \angle(\eta - (\eta \cdot e_1)e_1, e_2);$$

$$\rho' = |\xi - \eta|, \quad \theta' = \angle(e_1, \xi - \eta), \quad \phi' = \angle(\xi - \eta - ((\xi - \eta) \cdot e_1)e_1, e_2)$$

be the spherical coordinates, we expand as in (4.12)

$$F(s, \xi - \eta) = \sum_{q \lesssim 2^{l_1}} \sum_{m=-q}^1 f_q^m(\rho') Y_q^m(\theta', \phi'); \quad G(s, \eta) = \sum_{q' \lesssim 2^{l_2}} \sum_{m'=-q'}^{q'} g_{q'}^{m'}(\rho) Y_{q'}^{m'}(\theta, \phi),$$

and fix a choice (ρ, q, q', m, m') . Using the Fourier transform of ψ_1 we can write

$$e^{it\Phi(\xi, \eta)} \psi_1(2^\kappa \Phi(\xi, \eta)) = \int_{\mathbb{R}} 2^{-\kappa} \widehat{\psi_1}(2^{-\kappa} r) e^{i(t+r)\Phi(\xi, \eta)} dr \quad (4.111)$$

and restrict $|r| \lesssim 2^{(m+\kappa)/2}$, then fix one r ; similarly decomposing

$$f_q^m(\rho') = \int_{\mathbb{R}} e^{i\sigma\rho'} \widehat{f_q^m}(\sigma) d\sigma,$$

we may assume $|\sigma| \lesssim 2^{(m+j_1)/2}$ (otherwise $\widehat{f_q^m}(\sigma)$ decays rapidly), and then fix σ . After absorbing

$m(\xi, \eta)$ into cutoff functions, we reduce to a θ integral

$$I'' = \int_{\mathbb{S}^2} e^{i(t+r)\Lambda(\rho') + \sigma\rho'} \psi_2(2^v \sin \theta) Y_q^m(\theta', \phi') Y_{q'}^{m'}(\theta, \phi) d\theta d\phi, \quad (4.112)$$

where $\Lambda = \Lambda_\nu$ for some $\nu \in \mathcal{P}$, ρ' , θ' and ϕ' are functions of θ and ϕ with fixed ρ . Let $H = (t+r)\Lambda(\rho') + \sigma\rho'$, note that (ρ', θ', ϕ') is smooth in θ , and $\partial_\theta((\rho')^2) = -2|\xi|\rho \sin \theta$ by the law of cosines. Using the formula

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{p=0}^n \sum_{\alpha_1 + \dots + \alpha_p = n-p} A(n, p; \alpha_1, \dots, \alpha_p) \cdot f^{(p)}(g(x)) \prod_{i=1}^p g^{(\alpha_i+1)}(x), \quad (4.113)$$

which is a special case of (4.23), we deduce that

$$\begin{aligned} |\partial_\theta \rho'| &\sim 2^{k-v}, & |\partial_\theta H| &\sim 2^{m+k-v}, & |\partial_\theta^\alpha(\theta', \phi')| &\lesssim (C\alpha)! 2^{\alpha k_3}; \\ |\partial_\theta^\alpha H| &\lesssim (C\alpha)! 2^{\alpha k_3} 2^{m+k}, & |\partial_\theta^\alpha \rho'| &\lesssim (C\alpha)! 2^{\alpha k_3} 2^k \end{aligned} \quad (4.114)$$

in case (1), and that

$$\begin{aligned} |\partial_\theta \rho'| &\sim 2^{k_2-v}, & |\partial_\theta H| &\sim 2^{m+k_2-v}, & |\partial_\theta^\alpha(\theta', \phi')| &\lesssim (C\alpha)! 2^{\alpha k_3}; \\ |\partial_\theta^\alpha H| &\lesssim (C\alpha)! 2^{\alpha k_3} 2^{m+k_2}, & |\partial_\theta^\alpha \rho'| &\lesssim (C\alpha)! 2^{\alpha k_3} 2^{k_2} \end{aligned} \quad (4.115)$$

in case (2), and that

$$\begin{aligned} |\partial_\theta \rho'| &\gtrsim 2^{2k_3-k_1-v}, & |\partial_\theta H| &\gtrsim 2^{m+2k_3-v}, & |\partial_\theta^\alpha(\theta', \phi')| &\lesssim (C\alpha)! 2^{\alpha(2k_3+n)}; \\ |\partial_\theta^\alpha \rho'| &\lesssim (C\alpha)! 2^{k_1} 2^{\alpha(2k_3+n)}, & |\partial_\theta^\alpha H| &\lesssim (C\alpha)! 2^{m+2k_1} 2^{\alpha(2k_3+n)}, & n &= \max(-k_1, -2k_1-v) \end{aligned} \quad (4.116)$$

in case (3). We then integrate by parts in θ , using Proposition 4.2.5 and Remark 4.2.6, setting

$$K = 2^{m+k}, \quad n = 1, \quad \epsilon \sim 2^{-v}, \quad \lambda \sim 2^{\max(l_1+k_3, l_2, v)}, \quad \lambda' = 2^{k_3}$$

in case (1),

$$K = 2^{m+k_2}, \quad n = 1, \quad \epsilon \sim 2^{-v}, \quad \lambda \sim 2^{\max(l_1+k_3, l_2, v)}, \quad \lambda' = 2^{k_3}$$

in case (2), and

$$K = 2^{m+2k_1}, \quad n = 1, \quad \epsilon \sim 2^{2k_3-2k_1-v}, \quad \lambda \sim 2^{\max(v, l_2, l_1+2k_3+n)}, \quad \lambda' \sim 2^{2k_3+n}$$

in case (3), so that we can bound $|J'| \lesssim \exp(-\gamma \langle m \rangle^{A\gamma})$ with some fixed constant γ and conclude the proof, provided A is chosen large enough. $\square$

4.6.1 Mixed inputs

Suppose $\min(k_1, k_2) \leq -K_0^2$. By symmetry, we may assume $-K_0 < k_1 < K_0^2$ and $k_2 \leq -K_0^2$. If $k \leq -K_0$, using Proposition 4.8.1 we have $|\Phi(\xi, \eta)| \gtrsim 1$, so we can integrate by parts in s , then use Proposition 4.4.1 to conclude, in the same way as estimating (4.94); thus we will now assume $k \geq -K_0$, so we need to prove

$$\|I_{jk}\|_{L^2} \lesssim \langle m \rangle^{-100} \langle j \rangle^{N_0} 2^{-5j/6} \varepsilon_1^2, \quad \|\widehat{I}_{jk}\|_{L^1} \lesssim 2^{-j} \langle j \rangle^{-N_0} \langle m \rangle^{-100} \varepsilon_1^2. \quad (4.117)$$

First note that, if $j \leq (1+o)m$ and we use a cutoff to restrict $|\Phi(\xi, \eta)| \gtrsim 2^{-m/9}$, we can then integrate by parts in s and form I_0 , I_1 and I_2 terms as in (4.94). Consider for example

$$\widehat{I}_1(\xi) = \int_a^b \int_{\mathbb{R}^3} \frac{e^{is\Phi(\xi, \eta)}}{\Phi(\xi, \eta)} \psi(2^{m/9}\Phi(\xi, \eta)) m(\xi, \eta) \widehat{\partial_s F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta ds,$$

where ψ is some cutoff; recall as in the proof of Lemma 4.6.1 that

$$\Phi^{-1}\psi(2^{m/9}\Phi) = \int_{\mathbb{R}} e^{ir\Phi} \chi(2^{-m/9}r) dr \quad (4.118)$$

for some $\chi \in \mathcal{S}$, and that we can restrict $|r| \lesssim \langle m \rangle^A 2^{m/9}$, so we can include the cutoff $\psi(2^{m/9}\Phi)$ into the phase, then use (4.88) to bound

$$\begin{aligned} \|I_1\|_{L^2} &\lesssim 2^{m/9} \sup_{|r| \lesssim 2^{m/8}} \left\| \int_a^b \int_{\mathbb{R}^3} e^{i(s+r)\Phi} m(\xi, \eta) \widehat{\partial_s F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta ds \right\|_{L^2} \\ &\lesssim 2^{(10/9+o)m} 2^{-9m/8} 2^{-m} \varepsilon_1^2 \lesssim 2^{-(1+o)m} \varepsilon_1^2. \end{aligned}$$

In the same way, we can bound I_0 and I_2 using (4.77) and (4.86) respectively, so this will be acceptable.

Next suppose $j_2 \prec j$; by (4.91) we may also assume $j \preceq m$ and $j_2 \prec m$, so we have $\max(k_1, k_2, l_1, l_2) \leq (2+o)\delta'm$. since $|\nabla_\eta \Phi(\xi, \eta)| \sim 1$, using Proposition 4.2.5, we must have $j_1 \succeq m$. Using (4.77), (4.81) and (4.88), we obtain

$$\|I\|_{L^2} \lesssim 2^m \langle m \rangle^A \cdot 2^{-m-k_1/2} \cdot 2^{2\delta'm} 2^{-3m/2} \varepsilon_1^2 \lesssim 2^{-8j/9} \varepsilon_1^2,$$

which proves the first half of (4.117); as for the second half, note the above estimate would actually give $\|I\|_{L^2} \lesssim 2^{-(1+o)m} \varepsilon_1^2$ if $|k_1| \leq (1 - 1/20)m$, and when $|k_1| \geq (1 - 1/20)m$, we would have

$$\|\widehat{I}\|_{L^\infty} \lesssim \varepsilon_1 2^m 2^{3k_1/2} \|F\|_{L^2} \lesssim 2^{k_1} \langle m \rangle^A \varepsilon_1^2$$

using (4.75) and (4.77). Note that from above we can also assume $|\Phi(\xi, \eta)| \lesssim 2^{-m/9}$ and hence $|\Phi(\xi, \xi)| \lesssim 2^{-m/9}$ (which restricts ξ to some set of volume at most $2^{-m/18}$), so we could close by Hölder, using also Proposition 4.8.1.

Now suppose $j \preceq j_2$ (note that this includes the case when $j \succ m$, in which case we have $\min(j_1, j_2) \succeq j$, and we can use similar arguments as below), we simply use (4.79) and (4.88) to bound

$$\|I\|_{L^2} \lesssim 2^m \cdot \langle m \rangle^{-N_0} 2^{-m} \cdot \langle j_2 \rangle^{N_0} 2^{-5j_2/6} \varepsilon_1^2 \lesssim \langle j \rangle^{N_0/2} 2^{-5j/6} \varepsilon_1^2$$

to prove the first part of (4.117); now we work on the Fourier L^1 bound. If $j_1 - k_1 \succeq m$, this would follow from estimating both F and G factors in Fourier L^1 norm, using (4.77), (4.80) and Hölder; so we will assume $j_1 - k_1 \prec m$ (so in particular $j_1 \preceq m$ and hence $j \preceq m$).

Note that we may assume $|\Phi| \leq 2^{-m/9}$ as above; actually since $|k_1| \leq m/2$, using (4.86), the bound can be improved to $|\Phi| \leq 2^{-m/4}$ by the same argument. Next we treat the case when $|\Phi| \lesssim 2^{k_1 - (8\delta' + o)m}$. If $k_1 + m/2 \preceq \bar{k}$, this means $|k_1| \geq (1/2 - 2\delta')m$, which restricts $|\Phi(\xi, \xi)| \lesssim 2^{-2m/5}$. If we moreover assume $|\nabla_\xi \Phi| \sim 2^{-r}$, then this restricts ξ , by Proposition 4.8.1, to a set of volume $\lesssim 2^{\min(-r, r-2m/5)}$. Moreover we may assume $j \preceq m - r$ if $r \prec m/2$ (or otherwise we integrate by parts in ξ as before), thus we have

$$\|\widehat{I}_{jk}\|_{L^1} \lesssim 2^{-(5/6-o)m} 2^{\min(-r, r-2m/5)/2} \varepsilon_1^2 \lesssim 2^{-(1+o)(m-r)} \varepsilon_1^2 \lesssim 2^{-(1+o)j} \varepsilon_1^2.$$

If $k_1 + m/2 \succ \bar{k}$, using Proposition 4.6.1, we can restrict that $|\sin \angle(\xi, \eta)| \lesssim 2^{-\rho}$, where $\rho = (1/2 - \delta' - o)m$. Now we write $\alpha = \pm|\xi|$, $\beta = \pm|\eta|$, and fix s and the direction vectors $\widehat{\xi} = \theta$ and $\widehat{\eta} = \phi$, so that we reduce to an integral

$$2^m 2^{-2\rho} \int_{\mathbb{R}} \widehat{F}(\alpha\theta - \beta\phi) \widehat{G}(\beta\phi) m(\alpha, \beta) d\beta,$$

where $m(\alpha, \beta)$ is bounded and supported in the region where $|\Phi^+(\alpha, \beta)| \lesssim 2^{\rho_0}$ where $\rho_0 = k_1 - (8\delta' + o)m$, and also $|\alpha - \beta| \lesssim 2^{k_1}$; note that we have omitted the exponential factor. By Schur's

test, we only need to bound the integrals

$$\int_{\mathbb{R}} m(\alpha, \beta) \widehat{F}(\alpha\theta - \beta\phi) d\alpha, \quad \int_{\mathbb{R}} m(\alpha, \beta) \widehat{F}(\alpha\theta - \beta\phi) d\beta.$$

By (4.76) and the fact that $|\partial_\beta \Phi_+| \sim 1$, we know that the first integral is trivially bounded by ε_1 , and the second integral is bounded by $2^{-k_1+\rho_0}\varepsilon_1$. This gives the bound

$$\|\widehat{I}_{jk}\|_{L^2} \lesssim 2^m 2^{2\delta' m} 2^{-2\rho} (2^{-k_1+\rho_0})^{1/2} 2^{-j_2} \varepsilon_1^2 \lesssim 2^{-(1+o)j} \varepsilon_1^2.$$

Now we are left with the case when $|\Phi| \gtrsim 2^{\rho_0}$ with ρ_0 as above. Since also $|\Phi| \lesssim 2^{-m/4}$, we must have $|k_1| \geq 2m/9$, so ξ is already restricted to a region of volume $\lesssim 2^{-m/9}$; by Hölder, we only need to bound $\|I\|_{L^2} \lesssim 2^{-(17/18+o)m} \varepsilon_1^2$. Notice that $|k_1| \leq m/2$ (since $j_1 - k_1 \leq m$), we will integrate by parts in s . The boundary term I_0 is clearly acceptable by (4.88); the term I_1 where the derivative falls on F is estimated in the same way as I_0 , where one uses (4.38) to deduce that

$$\|e^{is\Lambda} \partial_t F\|_{L^\infty} \lesssim 2^{-3m/2} 2^{3j_1/2} 2^{-3m/2} 2^{-k_1/2} \lesssim 2^{-5m/2} 2^{j_1} \lesssim 2^{-3m/2} 2^{k_1} \varepsilon_1^2$$

and get

$$\|I_1\|_{L^2} \lesssim 2^{m-\rho_0} 2^{-5m/6} 2^{-3m/2} 2^{k_1} \varepsilon_1^2 \lesssim 2^{-m} \varepsilon_1^2.$$

Finally, using (4.86) and (4.88) as well as the trick used in (4.118), we can bound $\|I_2\|_{L^2} \lesssim 2^\kappa \varepsilon_1^2$ with

$$\kappa \leq -\rho_0 + m - m - 3m/2 \leq -(17/18 + o)m.$$

4.6.2 Medium frequency output

Here we assume $k > -K_0$, so we need to prove the (slightly stronger) bound

$$\|I_{jk}\|_{L^2} \lesssim 2^{-5j/6} \langle j \rangle^A \varepsilon_1^2, \quad \|\widehat{I}_{jk}\|_{L^1} \lesssim 2^{-(1+o)j} \varepsilon_1^2. \quad (4.119)$$

The case when $j \succ m$ or $\min(j_1, j_2) \succeq m$ can be treated in the same way as in Section 4.6.1 above; if $|\Phi| \gtrsim 1$, we can integrate by parts in s . Both cases are easy, so we will now assume $j \preceq m$, $\min(j_1, j_2) \prec m$ and $|\Phi| \ll 1$. We next separate two different situations.

The case of large j 's

Here we assume (say) $j_2 = \max(j_1, j_2) \geq m/3$. Note that $j \leq m$; we will insert a cutoff to restrict $|\nabla_\xi \Phi(\xi, \eta)| \sim 2^{-r}$. If $r \succ \min(m - j, m/2)$, then either $j \leq \min(j_1, j_2)$ (in which case we then use Proposition 4.6.1 to restrict $\angle(\xi, \eta)$, and the remaining estimates will be easy), or we can fix η (or $\xi - \eta$) and integrate by parts in ξ as we have done many times before. Below we will assume $r \leq \min(m - j, m/2)$.

Using Lemma 4.6.1, we can assume $|\sin \angle(\xi, \eta)| \lesssim 2^{-(1/2-o)m}$ (note that when $|k| \lesssim 1$, the insertion of a cutoff of form $\psi(2^r \nabla_\xi \Phi)$ will not affect this, as seen in the proof of Lemma 4.6.1). Fix a time s , the direction vectors ξ_e and η_e of ξ and η , let $\alpha = \pm|\xi|$ and $\beta = \pm|\eta|$, we reduce to an integral of form

$$\int_{\mathbb{R}} e^{is\tilde{\Phi}(\alpha, \beta)} m(\xi, \eta) \widehat{F}(\xi - \eta) \widehat{G}(\eta) d\beta,$$

where ξ and η are functions of α and β respectively, and

$$\tilde{\Phi}(\alpha, \beta) = \Phi^+(\alpha, \beta) + O(2^{-(1-o)m});$$

for notations see Section 4.8.1. Now we choose a parameter $\rho \leq (1 - o)m$, and consider the region where $|\Phi^+(\alpha, \beta)| \sim 2^{-\rho}$, or when $|\Phi^+(\alpha, \beta)| \lesssim 2^{-(1-o)m}$; in the first situation we integrate by parts in s , producing the I_0 , I_1 and I_2 terms, in the second situation we will estimate the integral I directly. In estimating all these terms we shall use Schur's lemma.

For the term with $|\Phi_+(\alpha, \beta)| \lesssim 2^{-(1-o)m}$, we will Propositions 4.4.1 and 4.8.2 to bound

$$|\widehat{F}| \lesssim \varepsilon_1; \quad \sup_{\eta_e} \|\widehat{G}\|_{L_\beta^2} \lesssim 2^{l_2} \|\widehat{G}\|_{L_\eta^2} \lesssim 2^{-(5/18-2\delta')m} \varepsilon_1,$$

then use Propositions 4.8.1 and to bound

$$\sup_{\alpha} \int_{\mathbb{R}} \mathbf{1}_E(\alpha, \beta) d\beta \lesssim 2^{-(1/3-o)m}; \quad \sup_{\beta} \int_{\mathbb{R}} \mathbf{1}_E(\alpha, \beta) d\alpha \lesssim 2^{-(1-o)m} 2^r,$$

using the fact that $|\partial_\alpha \Phi_+| \sim 2^{-r}$, where E is the set where $|\Phi_+(\alpha, \beta)| \lesssim 2^{-(1-o)m}$. Putting these into Schur's lemma, and considering that η_e ranges in a ball centered at ξ_e with radius $2^{-(1-o)m/2}$, we obtain

$$\varepsilon_1^{-2} \|I\|_{L^2} \lesssim 2^m 2^{-(1-o)m} 2^{-(5/18-2\delta')m} (2^{-(1/3-o)m})^{1/2} (2^{-(1-o)m} 2^r)^{1/2} \lesssim 2^{-(17/18-2\delta'-o)m} 2^{r/2},$$

which is bounded above by $2^{-(17/18-2\delta'-o)j}$.

For the term I_0 , the estimate is the same as above; simply replace $2^{-(1-o)m}$ by $2^{-\rho}$ and repeat the argument above. For I_1 and I_2 , we shall estimate them in the same way as I_0 , but with the additional time integral which counts as 2^m ; however, for the term with ∂_s derivative, we have from (4.86) and Remark 4.4.2 that

$$\|\partial_t G\|_{L^2} \lesssim 2^{-(3/2-o)m} \varepsilon_1^2, \quad \|\widehat{\partial_t F}\|_{L^\infty} \lesssim 2^{-(1-o)m} \varepsilon_1^2,$$

which is almost 2^m better than the corresponding G and F factors, so this cancels the time integration.

Finally, considering the Fourier L^1 bound, we may assume $j_2 \leq m/2$, otherwise it would directly follow from the L^2 bound (where for $\|G\|_{L^2}$ we have a bound of $2^{-5m/12} \varepsilon_1$ instead of $2^{-5m/18} \varepsilon_1$). Now since $j_2 \leq m/2$, we can assume $|\nabla_\eta \Phi| \lesssim 2^{-(1/4-o)m}$, since otherwise we can integrate by parts in η (or β) to get 2^{-20m} decay; also we can assume $|\Phi(\xi, \eta)| \lesssim 2^{-m/3}$, since otherwise we can integrate by parts in s , using the trick in (4.118) and the bound $\|\partial_t G\|_{L^2} \lesssim 2^{-(3/2-o)m} \varepsilon_1^2$ to close. Now using (4.124) and (4.125), we can restrict ξ to a region of volume $\lesssim 2^{-m/8}$, so we use the L^2 bound obtained above and Hölder to obtain a good L^1 bound.

The case of small j 's

Assume $j' = \max(j_1, j_2) \prec m/3$. In particular we know that $|\nabla_\eta \Phi| \ll 1$, so we are close to one of the spacetime resonance spheres described in Proposition 4.8.1, say (α_0, β_0) .

Recall that $|\sin \angle(\xi, \eta)| \lesssim 2^{-(1/2-o)m}$; in the discussion below we will further assume $|\partial_\beta^2 \Phi^+| \ll 1$, where $\alpha = \pm|\xi|$ and $\beta = \pm|\eta|$. In fact, when $|\partial_\beta^2 \Phi^+| \gtrsim 1$ we have a non-degenerate oscillatory integral (which is exactly the case in [32]), so for example we have $|\beta - R_i(\alpha)| \lesssim 2^{-(1/2-o)m}$ (instead of $2^{-m/3} \langle m \rangle^A$), and the proof will be completed in the same way as below, and every estimate will be strictly better.

Now we will fix a time s , and restrict $||\xi| - \alpha_0| \sim 2^{-2l}$ with $0 \leq l \leq m/2$, or $||\xi| - \alpha_0| \lesssim 2^{-m}$, using suitable cutoff functions. We are thus considering the integral

$$H(s, \xi) = \int_{\mathbb{R}^3} e^{is\Phi(\xi, \eta)} m(\xi, \eta) \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\eta.$$

As the first step, we can use Lemma 4.6.1 to restrict $|\sin \angle(\xi, \eta)| \lesssim 2^{-m/2} \langle m \rangle^A$. We then fix the direction vectors ξ_e and η_e , noticing that η_e stays in a set with volume $2^{-m} \langle m \rangle^A$ with fixed ξ_e , and

let $\alpha = \pm|\xi|$ and $\beta = \pm|\eta|$. We then have $\Phi(\xi, \eta) = \Phi^+(\alpha, \beta) + O(2^{-m}\langle m \rangle^A)$; with suitable choice of A , this remainder will be safely ignored. Below we will consider the integral

$$\int_{\mathbb{R}} e^{is\Phi^+(\alpha, \beta)} m(\xi, \eta) \widehat{F}(s, \xi - \eta) \widehat{G}(s, \eta) d\beta.$$

Next, recall that

$$\partial_\beta \Phi^+(\alpha, \beta) = P(\alpha, \beta) \cdot (\beta - R_1(\alpha)) \cdot [(\beta - R_2(\alpha))^2 - Q(\alpha)],$$

by our assumptions, we may assume that β is to $R_3(\alpha)$ (the case of R_4 is similar).

If we restrict $|\beta - R_3(\alpha)| \gtrsim 2^{-\mu}$ by inserting some cutoff function, where $\mu \prec \max(\frac{m}{3}, \frac{m-l}{2})$, then we have, by elementary calculus, that $|\partial_{\eta_1} \Phi| \gtrsim 2^{-\min(2\mu', \mu'+l)}$ and $|\partial_{\eta_1}^2 \Phi| \lesssim 2^{-\min(\mu', l)}$. We then set in Proposition 4.2.5 that

$$K = 2^m, \quad n = 2, \quad \epsilon_1 \sim 2^{-\min(2\mu', \mu'+l)}, \quad \epsilon_2 \sim 2^{-\min(\mu', l)}, \quad \lambda \sim 2^{\max(\mu, j')},$$

and check that the choice of parameters (in particular the choice that $\mu' \prec \max(m/3, (m-l)/2)$) guarantees sufficient decay.

Now, suppose $|\beta - R_3(\alpha)| \lesssim 2^{-\mu_0} \langle m \rangle^A$, where $\mu_0 = \max(\frac{m}{3}, \frac{m-l}{2})$, then we have

$$|\Phi_+(\alpha, \beta) - \Phi_+(\alpha, R_3(\alpha))| \lesssim 2^{-m} \langle m \rangle^A,$$

which can be checked using the expressions of Φ_+ . Using the support of β and η_e , we can already bound $\|\widehat{I}\|_{L^\infty} \lesssim 2^{-m/3} \langle m \rangle^A \varepsilon_1^2$. To bound the L^2 and Fourier L^1 norms we let

$$\lambda = (\partial_\alpha \Phi)(\alpha_0, R_2(\alpha_0))$$

as in Proposition 4.8.1, we will consider the case when $\lambda \neq 0$ and when $\lambda = 0$.

(1) Suppose $\lambda \neq 0$. If we assume $|\alpha - \alpha_0| \lesssim 2^{-2l}$ with $l \geq m/2$, then we directly use the Fourier L^∞ bound obtained above and the smallness of support to conclude; therefore we may assume $||\xi| - \alpha_0| \sim 2^{-2l}$ with $l \prec m/2$. From part (5) of Proposition 4.8.1, we have that $|\Phi_+(\alpha, R_3(\alpha))| \sim 2^{-2l}$, while

$$|\Phi(\xi, \eta) - \Phi_+(\xi, R_3(|\xi|))| \lesssim 2^{-m} \langle m \rangle^A,$$

so we have that $|\Phi(\xi, \eta)| \sim 2^{-2l}$, thus we will integrate by parts in s . The boundary term I_0 can be

estimated in L^2 norm (using Hölder) by $2^\kappa \varepsilon_1^2$, where

$$\kappa \preceq A \log m - l + 2l - \frac{4m}{3} \preceq -\frac{5j}{6}$$

(notice that the $2^{-4m/3}$ factor is due to the support of β and η_e together), and the Fourier L^1 norm would be bounded by $2^{\kappa'}$, where

$$\kappa' \preceq A \log m - 2l + 2l - \frac{4m}{3} \preceq -\frac{5j}{6} \leq -5j/4,$$

which will be acceptable. As for I_1 (I_2 is the same), we will have one more time integration, but the Fourier L^∞ norm of $\partial_s F$ (or $\partial_s G$) will be 2^m better than the corresponding function themselves (F or G) due to (4.86) and Remark 4.4.2, so the proof will go exactly the same way.

(2) Suppose $\lambda = 0$. We may again assume $||\xi| - \alpha_0| \sim 2^{-2l}$ with $l \prec m/2$; we next consider the case when

$$j \succ \max(2l, m - l, m - \mu_0, l + \mu_0) = \max(2l, m - l)$$

where we recall $\mu_0 = \max(m/3, (m - l)/2)$. We then fix η and s , and repeat the argument of integrating by parts in ξ as before; for a fixed point x with $|x| \sim 2^j$, we analyze the integral

$$\int_{\mathbb{R}^3} e^{i(s\Phi + x \cdot \xi)} \chi \left(2^{n_1} \left(\eta - \frac{\eta \cdot \xi}{|\xi|^2} \xi \right) \right) \chi \left(2^\mu \left(\frac{\eta \cdot \xi}{|\xi|} - R_3(|\xi|) \right) \right) \widehat{f}(\xi - p_3(\xi)) \widehat{g}(p_3(\xi)) d\xi.$$

Note that with the cutoff functions, we have

$$|\nabla_\xi \Phi(\xi, \eta)| \lesssim 2^{-\mu} + |\nabla_\xi \Phi(\xi, p_3(\xi))| \lesssim 2^{-\mu} + 2^{-l},$$

the last inequality following easily from the proof of Proposition 4.8.1. Now we can set in Proposition 4.2.5 that $K = 2^m$, $\epsilon = 2^{j_0 - m}$, $k = 1$, $L = 2^{100M}$ and $\lambda = \max(j', l + \mu, n_1)$ to obtain a decay of $2^{-100M} \varepsilon_1^2$. Note that taking each derivative of the cutoff function

$$\chi \left(2^\mu \left(\frac{\eta \cdot \xi}{|\xi|} - R_3(|\xi|) \right) \right)$$

may cost us up to $2^{\max(2l, l + \mu)}$; this can be proved using the same argument as the proof of Proposition 4.2.5.

Now we may assume $j \preceq \max(2l, m - l)$. Using this and the bound for $\varepsilon_1^{-2} \|\widehat{I}\|_{L^\infty}$ we already

have, we can bound $\varepsilon_1^{-2}\|I\|_{L^2}$ *without integrating by parts in s* by 2^κ , where

$$\kappa \preceq m - l - 4m/3 \preceq -5l/3 \preceq -5j/6$$

when $l \geq m/3$ (the Fourier L^1 bound follows from Hölder), and

$$\kappa \preceq m - l - 4m/3 \preceq -5(m-l)/6 \preceq -5j/6$$

when $3m/11 \leq l \leq m/3$ (the Fourier L^1 bound also follows from Hölder). Finally when $l \leq 3m/11$, we note that $|\Phi| \sim 2^{-3l}$ so we can integrate by parts in s to bound $\varepsilon_1^{-2}\|I\|_{L^2}$ by $2^{\max(\kappa_1, \kappa_2)}$, where

$$\kappa_1 \preceq 3l - l - 4m/3 \preceq -(m-l) - m/10 \preceq -j - m/10,$$

and

$$\kappa_2 \preceq m + 3l - l - (4m/3) - m \preceq -j - m/10.$$

4.6.3 Low frequency output

In this section we assume $k \leq -K_0^2$. As before, we can now exclude the cases when $j \succ m$, or when $\max(j_1, j_2) \succeq m$, which allows us to assume $|\sin \angle(\xi, \eta)| \lesssim 2^{-(m+k)/2}$ due to Lemma 4.6.1 (we ignore the powers $2^{2\bar{k}}$ since they are unimportant here). The rest of the proof then goes in the same way as in Section 4.6.2, making necessary changes due to the fact that $|\xi| \ll 1$.

To be precise, if $j_1 \geq (1/2 - o)m$, we can first fix the angle $\angle(\xi, \eta)$, then reduce to a one-dimensional integral as before (note that $|\partial_\alpha \Phi_+| \gtrsim 1$ here), so that we use Schur's test to obtain $\|I\|_{L^2} \lesssim 2^{-\kappa} \varepsilon_1^2$, where

$$\kappa \leq k + \rho - (m + k) - \rho/2 - (\rho/2)/2 - 5/6(1/2 - o)m \leq -(1 + o)m,$$

where ρ is the one defined in Section 4.6.2, the first 2^k factor is due to the fact that

$$\|\widehat{I}\|_{L^2} \lesssim 2^k \sup_{\theta \in \mathbb{S}^2} \|\widehat{I}(\alpha\theta)\|_{L_\alpha^2},$$

when $\widehat{I}$ is restricted to $|\xi| \sim 2^k$.

Now let us assume $\max(j_1, j_2) \leq (1/2 - o)m$. Again we must have $|\sin \angle(\xi, \eta)| \lesssim 2^{-(m+k)/2}$, and

again the relation

$$\Phi(\xi, \eta) = \Phi^+(\alpha, \beta) + O(2^{-(1-o)m})$$

holds. After fixing the directions of ξ and η and reducing to the one-dimensional integral, notice that when $|\Phi^+| + |\partial_\beta \Phi^+| \ll 1$ we have $|\partial_\beta^2 \Phi^+| \gtrsim 1$ by Proposition 4.8.1, thus under the condition $\max(j_1, j_2) \leq (1/2 - o)m$ we can restrict $|\beta - p(\alpha)| \lesssim 2^{-(1/2-o)m}$ where $\beta = p(\alpha)$ is the unique solution to $\partial_\beta \Phi_+(\alpha, \beta) = 0$, thus obtaining $\|\widehat{I}\|_{L^\infty} \lesssim 2^{-(1/2-o)m} \varepsilon_1^2$ in exactly the same way as Section 4.6.2, and again we have

$$\Phi_+(\alpha, \beta) = \Phi^+(\alpha, p(\alpha)) + O(2^{-(1-o)m});$$

notice that $|\partial_\alpha \Phi^+(\alpha, p(\alpha))| \gtrsim 1$, we can then analyze the size of $|\Phi^+(\alpha, p(\alpha))|$ and integrate by parts in s (and use (4.86) and Remark 4.4.2 to bound the L^∞ norm of $\widehat{\partial_s F}$ and $\widehat{\partial_s G}$) when necessary, as we did in Section 4.6.2 above. Compared with that case, the L^∞ bound here is better, and the volume bound for $\{\alpha : |\Phi^+(\alpha, p(\alpha))| \sim 2^{-\rho}\}$ is also better with fixed ρ . This completes the proof.

4.7 High frequency case

In this section we assume $\bar{k} := \max(k, k_1, k_2) \geq K_0$. By Proposition 4.8.1, we have either $|\Phi| \gtrsim 2^{-2\bar{k}}$, or $\min(k, k_1, k_2) \geq -D_0$. In the latter case we also have that, either $c_\sigma = c_\mu = c_\nu$ and $b_\sigma - b_\mu - b_\nu = 0$, or $|\nabla_\eta \Phi| \gtrsim 2^{-4\bar{k}}$.

Now, if $|\Phi| \gtrsim 2^{-2\bar{k}}$, we simply integrate by parts in s and estimate the corresponding I_0 , I_1 and I_2 terms as before; for example by a version of (4.88) we have

$$\|I_0\|_{L^2} \lesssim 2^{3\bar{k}} \cdot 2^{6\bar{k}} 2^{-9m/8} \varepsilon_1^2 \lesssim 2^{-(1+o)m} \varepsilon_1^2$$

and

$$\|I_1\|_{L^2} \lesssim 2^{m+3\bar{k}} \cdot 2^{-(1-o)m} 2^{-9m/8} \varepsilon_1^2 \lesssim 2^{-(1+o)m} \varepsilon_1^2.$$

If $|\Phi| \lesssim 2^{-2\bar{k}}$ and $|\nabla_\eta \Phi| \gtrsim 2^{-4\bar{k}}$, then we may assume $j_1 \geq (1-o)m - 4\bar{k}$, thus we can use Lemma 4.6.1 to restrict $|\sin \angle(\xi, \eta)| \lesssim 2^{-(1/2-o)m}$; we then restrict $|\Phi| \sim 2^{-\gamma}$, integrate by parts in s , and use Schur's lemma as before, to obtain $\|I\|_{L^2} \lesssim 2^{\max(\kappa_0, \kappa_1)} \varepsilon_1^2$, where

$$\kappa_0 = \gamma - (1-o)m + 8\bar{k} - \gamma - 5m/6 \leq -(1+o)m$$

and

$$\kappa_1 = \gamma - (1 - o)m + 8\bar{k} - \gamma - 9m/8 \leq -(1 + o)m.$$

Finally, when $c_\sigma = c_\mu = c_\nu = 1$ (say) and $b_\sigma - b_\mu - b_\nu = 0$, then we have $|\Phi| \gtrsim 2^{-2\bar{k}}$ so we can analyze as above, unless $|\xi|, |\eta|, |\xi - \eta| \sim 2^k$, in which case we have

$$|\nabla_\xi \Phi| \sim |\nabla_\eta \Phi| \sim 2^{-3k} |\eta - \rho\xi|$$

as in Proposition 4.8.1. Then, using the same integration by parts argument (in ξ or η) as above, we conclude that $j \preceq m - 3k - l \preceq \max(j_1, j_2)$, where $|\eta - \rho\xi| \sim 2^l$; moreover we can assume $|\sin \angle(\xi, \eta)| \lesssim 2^{-(1/2-o)m}$ by Lemma 4.6.1, so we assume $|\Phi| \sim 2^{-\gamma}$ and reduce to one-dimensional integral, and integrate by parts in s so get that $\|I\|_{L^2} \lesssim 2^{\max(\kappa_0, \kappa_1)} \varepsilon_1^2$ where

$$\kappa_0 = \gamma - (1 - o)m + (3k + l) - \gamma - \max(j_1, j_2) \leq -3j/2,$$

$$\kappa_1 = \gamma + m - (1 - o)m + (3k + l) - \gamma - 9m/8 \leq -(1 + 1/20)j,$$

so in any case we get the desired estimate.

4.8 Auxiliary results

4.8.1 Properties of phase functions

Let $\Phi = \Phi_{\sigma\mu\nu}$ and Φ^+ be defined as in Section 4.1.1. Define the spacetime resonance set

$$\mathcal{R} = \{(\xi, \eta) : \Phi(\xi, \eta) = \nabla_\eta \Phi(\xi, \eta) = 0\}, \quad (4.120)$$

and also assume

$$|\xi| \sim 2^k, \quad |\xi - \eta| \sim 2^{k_1}, \quad |\eta| \sim 2^{k_2}; \quad \max(k, k_1, k_2) := \bar{k}.$$

Proposition 4.8.1. *(1) Suppose $|\bar{k}| \leq K_0^2$ and $(\xi, \eta) = (\alpha e, \beta e)$ for some $e \in \mathbb{S}^2$, then we have*

$$\sup_{\mu \leq 3} |\partial_\beta^\mu \Phi^+(\alpha, \beta)| \gtrsim 1, \quad (4.121)$$

and the same holds for ∂_α . If moreover $\min(k, k_1, k_2) \leq -D_0$, then the range $\mu \leq 3$ above can be

improved to $\mu \leq 2$. Note that (4.121) implies the measure bound

$$\sup_{\alpha} |\{\beta : |\Phi(\alpha, \beta)| \leq \epsilon\}| \lesssim \epsilon^{1/3} \quad (4.122)$$

for bounded α, β by well-known results; see for example [18], Section 8.

(2) If c_σ, c_μ and c_ν are not all equal, then the set

$$\mathcal{R} = \{(\alpha\theta, \beta\theta) : \theta \in \mathbb{S}^2, (\alpha, \beta) \in R\}, \quad (4.123)$$

where $R \subset \mathbb{R}^2$ is a finite set. If $c_\sigma = c_\mu = c_\nu$, then if $b_\sigma - b_\mu - b_\nu \neq 0$ then $\mathcal{R} = \emptyset$; otherwise $\mathcal{R} = \{(\xi, \rho\xi)\}$, where $\rho = b_\nu/b_\sigma$. In this case we have

$$|\nabla_\eta \Phi| \sim (1 + 2^{5k_1})^{-1} |\eta - \rho\xi|$$

if $|k_1 - k_2| = O(1)$, and $|\nabla_\eta \Phi| \gtrsim 2^{-2\bar{k}}$ otherwise.

(3) Suppose $k \geq \max(k_1 + D_0, D_0^2/2)$, then we have $|\Phi| + |\nabla_\eta \Phi| \gtrsim 2^{-2k_1}$; if $|k_1 - k_2| \leq 2D_0$ and $k_1 \geq D_0^2/2$, then either $|\Phi| + |\nabla_\eta \Phi| \gtrsim 2^{-4k_1}$, or $c_\sigma = c_\mu = c_\nu$ and $b_\sigma - b_\mu - b_\nu = 0$.

(4) For $\alpha \neq 0$, there exist smooth functions $R_1(\alpha)$ and $R_2(\alpha)$, strictly positive functions $P(\alpha, \beta)$, and a function $Q(\alpha)$ which has only simple zeros, such that

$$\partial_\beta \Phi^+(\alpha, \beta) = \pm P(\alpha, \beta) \cdot (\beta - R_1(\alpha)) \cdot [(\beta - R_2(\alpha))^2 - Q(\alpha)]. \quad (4.124)$$

Moreover, near each point where $Q(\alpha) = 0$, we have that $R_1(\alpha) - R_2(\alpha)$ is bounded away from zero. We will also define

$$R_3(\alpha) = R_2(\alpha) + \sqrt{|Q(\alpha)|}, \quad R_4(\alpha) = R_2(\alpha) - \sqrt{|Q(\alpha)|}.$$

(5) Suppose at some point α_0 we have $Q(\alpha_0) = 0 = \Phi_+(\alpha_0, R_2(\alpha_0))$. Let also $\lambda = (\partial_\alpha \Phi)(\alpha_0, R_2(\alpha_0))$, then we have

$$\Phi^+(\alpha, R_3(\alpha)) = \lambda(\alpha - \alpha_0) + \lambda'|\alpha - \alpha_0|^{\frac{3}{2}} + O(|\alpha - \alpha_0|^2), \quad (4.125)$$

where λ' may take different values depending on whether $\alpha > \alpha_0$ or $\alpha < \alpha_0$, but is always nonzero; the same thing happens for R_4 . Moreover, when $\lambda = 0$, we also have

$$|\partial_\alpha(\Phi^+(\alpha, R_3(\alpha)))| \sim |\alpha - \alpha_0|^{1/2}. \quad (4.126)$$

Proof. (1) we only need to prove that there is no (α, β) where $\partial_\beta^j \Phi^+ = 0$ for all $j \leq 3$, unless $\alpha = \beta = 0$. In fact, if $\partial_\beta^j \partial_\beta \Phi^+(\alpha, \beta) = 0$ for $j \leq 2$, we must have $\mu\nu < 0$, since the function $\sqrt{ax^2 + b}$ is strictly convex; if we write $\beta - \alpha = \gamma$, we will obtain

$$\frac{c_\nu^2 \beta}{\sqrt{c_\nu^2 \beta^2 + b_\nu^2}} = \frac{c_\mu^2 \mu}{\sqrt{c_\mu^2 \gamma^2 + b_\mu^2}}, \quad \frac{c_\nu^2 b_\nu^2}{(c_\nu^2 \beta^2 + b_\nu^2)^{3/2}} = \frac{c_\mu^2 b_\mu^2}{(c_\mu^2 \gamma^2 + b_\mu^2)^{3/2}}, \quad (4.127)$$

and

$$\frac{c_\nu^2 \beta}{(c_\nu^2 \beta^2 + b_\nu^2)^{5/2}} = \frac{c_\mu^2 \mu}{(c_\mu^2 \gamma^2 + b_\mu^2)^{5/2}}. \quad (4.128)$$

By elementary algebra, these three equations will force $c_\nu = c_\mu$, $b_\tau + b_\sigma = 0$ and $\beta = \gamma$, but in this case we clearly have $\Phi^+(\alpha, \beta) \neq 0$.

For the case when $\min(k, k_1, k_2) \leq -D_0$, we again only need to show that there is no (α, β) where $\partial_\beta^j \Phi^+ = 0$ for all $j \leq 2$, and $\min(|\alpha|, |\beta|, |\alpha - \beta|) \leq 2^{-D_0}$. In fact, using part (2) below, we only need to consider spacetime resonance other than $(0, 0)$ with $\alpha\beta(\alpha - \beta) = 0$, unless $c_\sigma = c_\mu = c_\nu$ and $b_\sigma - b_\mu - b_\nu = 0$ (in this latter case we must have $\beta = \rho\alpha$ using the notation in (2), in which case we can directly compute $\partial_\beta^2 \Phi^+ \neq 0$); clearly we cannot have $\beta = 0$ or $\alpha - \beta = 0$, and when $\alpha = 0$ then $\beta = \gamma$ as above, so we can repeat the arguments to show that now $\partial_\beta \Phi^+ = \partial_\beta^2 \Phi_+ = 0$ implies $c_\nu = c_\mu$ and $b_\nu + b_\mu = 0$, so that $\Phi^+(0, \beta) \neq 0$.

(2) If $\nabla_\eta \Phi = 0$, then η and $\xi - \eta$ must be collinear, so we have $\xi = \alpha\theta$ and $\eta = \beta\theta$, where $\theta \in \mathbb{S}^2$, and $\Phi^+(\alpha, \beta) = \partial_\beta \Phi^+(\alpha, \beta) = 0$.

First assume $c_\mu \neq c_\nu$, then we have $|\partial_\beta \Lambda_\mu(\alpha - \beta)| = |\partial_\beta \Lambda_\nu(\beta)| := k$, and hence

$$\alpha - \beta = \frac{\pm b_\mu k}{c_\mu \sqrt{c_\mu^2 - k^2}}, \quad \Lambda_\mu(\alpha - \beta) = \frac{b_\mu c_\mu}{\sqrt{c_\mu^2 - k^2}}$$

and similarly for β , so we get

$$\left(\frac{b_\mu c_\mu}{\sqrt{c_\mu^2 - k^2}} + \frac{b_\nu c_\nu}{\sqrt{c_\nu^2 - k^2}} \right)^2 = c_\sigma^2 \left(\frac{b_\mu k}{c_\mu \sqrt{c_\mu^2 - k^2}} + \frac{b_\nu k}{c_\nu \sqrt{c_\nu^2 - k^2}} \right)^2 + b_\sigma^2,$$

which reduces to a polynomial equations in k that is not an identity (since $c_\mu \neq c_\nu$), so we only have a finite number of k 's.

Now assume $c_\mu = c_\nu = 1$, then with $\rho = b_\nu/b_\sigma$, it is easy to check that $\partial_\beta \Phi^+ = 0$ implies $\beta = \rho\alpha$. Plugging this into $\Phi^+ = 0$, we obtain an equation for α that is an identity when $c_\sigma = 1$ and $b_\sigma - b_\mu - b_\nu = 0$; thus this equation has at most two solutions when $c_\sigma \neq 1$, and no solution

when $c_\sigma = 1$ and $b_\sigma - b_\mu - b_\nu \neq 0$. Finally the bound on $|\nabla_\eta \Phi|$ when $c_\mu = c_\nu$ follows from the fact that $\xi \mapsto \nabla \Lambda_\mu(\xi)$ is diffeomorphism whose Jacobian matrix has an inverse bounded by $(1 + |\xi|)^{-3}$ pointwise.

(3) In the first case, we may assume $c_\sigma = c_\nu$, otherwise we would have $|\Phi| \gtrsim 2^k$. Now if $c_\sigma = c_\nu = 1$ and $c_\mu < 1$, we would have

$$|\nabla_\eta \Phi| \geq \frac{|\eta|}{\sqrt{|\eta|^2 + b_{n\nu}^2}} - \frac{c_\mu^2 |\xi - \eta|}{\sqrt{c_\mu^2 |\xi - \eta|^2 + b_\mu^2}} \geq \frac{1 - c_\mu}{2};$$

if $c_\mu > 1$ we would have

$$|\Phi| \geq \max(c_\mu |\xi - \eta|, |b_\mu|) - ||\xi| - |\eta|| + O(2^{-k}) \gtrsim 1.$$

If $c_\mu = 1$, we may directly compute that $|\nabla_\eta \Phi| \gtrsim 2^{-2k_1}$.

In the second case, assume $|\Phi| + |\nabla_\eta \Phi| \ll 2^{-4k_1}$, then we must have $c_\mu = c_\nu = 1$, since when $c_\mu \neq c_\nu$ we have $|\nabla_\eta \Phi| \gtrsim 1$; this in turn implies $|\eta - \rho\xi| \lesssim 2^{-k_1}$ where $\rho = b_\nu/(b_\mu + b_\nu)$ (or $|\xi| \lesssim 2^{-k_1}$ if $b_\mu + b_\nu = 0$, which is easily seen to be impossible). Plugging into $|\Phi| \ll 2^{-4k_1}$, we see that the only possibility is $c_\sigma = 1$ and $b_\sigma - b_\mu - b_\nu = 0$.

(4) When $c_\mu = c_\nu$, the unique solution of $\partial_\beta \Phi^+(\alpha, \beta) = 0$ is $\beta = \rho\alpha$ with $\rho = b_\nu/(b_\mu + b_\nu)$, and $\partial_\beta^2 \Phi^+(\alpha, \beta) \neq 0$; below we will assume $c_\mu \neq c_\nu$.

After squaring, the equation $\partial_\beta \Phi_+(\alpha, \beta) = 0$ becomes

$$\frac{c_\mu^4 (\beta - \alpha)^2}{c_\mu^2 (\beta - \alpha)^2 + b_\mu^2} = \frac{c_\nu^4 \beta^2}{c_\nu^2 \beta^2 + b_\nu^2}. \quad (4.129)$$

Assume $\alpha > 0$, note that (4.129) has at least two roots β : a unique solution exists in $(0, \alpha)$, and when $c_1 > c_2$, a unique solution exists in $(\alpha, +\infty)$, and when $c_1 < c_2$, a unique solution exists in $(-\infty, 0)$. These two solutions are both simple, and depends smoothly on α , and exactly one of them actually solves the equation $\partial_\beta \Phi^+(\alpha, \beta) = 0$, depending on the signs of $b_\mu b_\nu$.

On the other hand, (4.129) simplifies to a polynomial equation of β with degree four, so it has at most four roots. If we quotient out the two roots mentioned above, we are left with a quadratic equation

$$P(\alpha)\beta^2 + Q(\alpha)\beta + R(\alpha) = 0,$$

where $P(\alpha) > 0$. After completing the square, we can then write $\partial_\beta \Phi^+(\alpha, \beta)$ in the form of (4.124). It is also clear from above that R_1 and R_2 are always separated. Next, note that $\partial_\alpha \partial_\beta \Phi^+(\alpha, \beta) \neq 0$;

if we choose $\beta = R_2(\alpha)$, it then follows that $Q(\alpha) = 0$ and $Q'(\alpha) = 0$ cannot happen at the same point.

(5) First, note that

$$\partial_\alpha(\Phi^+(\alpha, R_2(\alpha)))_{\alpha=\alpha_0} = \lambda + (\partial_\beta \Phi^+)(\alpha_0, R_2(\alpha_0)) = \lambda,$$

we thus have $\Phi_+(\alpha, R_2(\alpha)) = \sigma(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)$. Next, since we are close to the point $(\alpha_0, R_2(\alpha_0))$, we will have

$$\Phi^+(\alpha, R_3(\alpha)) - \Phi^+(\alpha, R_2(\alpha)) = \int_0^{\sqrt{|Q(\alpha)|}} P(\alpha, \beta + R_2(\alpha)) \cdot (\beta^2 - Q(\alpha)) d\beta,$$

where P is (say) smooth and strictly positive. We then compute, with α close to α_0 , that this integral equals some $c|Q(\alpha)|^{3/2}$ plus an error of $O(|Q(\alpha)|^2)$, where c is nonzero but may depend on the sign of $Q(\alpha)$. Since $Q(\alpha)$ has a simple zero at α_0 , this proves (4.125). The proof of (4.126) is similar, and will be omitted here. $\square$

4.8.2 Functions with low angular momentum

Proposition 4.8.2. *For each function f on $\mathbb{R}^3$, let its spherical harmonics decomposition be as in (4.12) and define S_l as in (4.13). Then we have*

$$\|S_l f\|_{L^p} \lesssim \|f\|_{L^p} \tag{4.130}$$

uniformly in l , for $1 \leq p \leq \infty$, and also

$$\|S_l f\|_{L^\infty} \lesssim 2^l \left(\sup_r r^{-2} \int_{|\xi|=r} |f|^2 d\omega \right)^{1/2}. \tag{4.131}$$

Proof. In what follows we will assume f is a function on $\mathbb{S}^2$, because we can always use polar coordinates $(\rho, \theta) \in \mathbb{R}^+ \times \mathbb{S}^2$ and then fix ρ . For the standard identities about zonal harmonics, the reader may consult [47], Chapter 4.

(1) Assume $l \geq 1$. Recall the zonal harmonics $Z_\theta^q(\theta')$ of degree q , and define

$$K(\theta, \theta') = \sum_{q \geq 0} \varphi(2^{-l} q) Z_\theta^q(\theta'),$$

then we have

$$S_l f(\theta) = \int_{\mathbb{S}^2} f(\theta') K(\theta, \theta') d\omega(\theta').$$

Therefore (4.130) would follow from a bound of $\|K(\theta, \cdot)\|_{L^1}$ uniform in θ , which would be a consequence of the pointwise inequality

$$|K(\theta, \theta')| \lesssim \min(2^{2l}, 2^{-l}|\theta - \theta'|^{-3}). \quad (4.132)$$

When $|\theta - \theta'| \lesssim 2^{-l}$ the bound (4.132) will be trivial, since we have $|Z_\theta^q(\theta')| \lesssim 2q + 1$ for each q . Now we assume $\epsilon = |\theta - \theta'| \gg 2^{-l}$, let $\lambda = \theta \cdot \theta' = 1 - \epsilon^2/2$, and write $\alpha_r = Z_\theta^q(\theta')$. Then we have the generating function identity

$$\sum_{q=0}^{\infty} \alpha_q \rho^q = \frac{1 - \rho^2}{(\rho^2 - 2\lambda\rho + 1)^{3/2}} \quad (4.133)$$

for $0 \leq \rho < 1$. Since the function $\rho^2 - 2\lambda\rho + 1$ never vanishes when $\rho \in \mathbb{C}$ and $|\rho| < 1$, the right hand side of (4.133) will have a unique holomorphic continuation to the unit disc in $\mathbb{C}$, and it must equal the left hand side. If we now choose $\rho = \exp(\frac{iy-1}{N})$ for each $y \in \mathbb{R}$, and let $h(y)$ be the Fourier transform of $\varphi(x)e^x$ (which is Schwartz because φ has compact support), we will get

$$\begin{aligned} K(\theta, \theta') &= \sum_{q=0}^{\infty} e^{-\frac{q}{N}} \alpha_q \int_{\mathbb{R}} h(y) e^{i\frac{q}{N}y} dy \\ &= \int_{\mathbb{R}} h(y) dy \cdot \sum_{r=0}^{\infty} \alpha_q (e^{\frac{iy-1}{N}})^q \\ &= \int_{\mathbb{R}} h(y) \frac{1 - e^{\frac{2(iy-1)}{N}}}{(1 - 2\lambda e^{\frac{iy-1}{N}} + e^{\frac{2(iy-1)}{N}})^{3/2}} dy. \end{aligned}$$

Now, if $\rho = \exp(\frac{iy-1}{N})$, we will have $|1 - \rho^2| \lesssim N^{-1}\langle y \rangle$. For the denominator we have

$$|1 - 2\lambda\rho + \rho^2| = |\rho - \rho_+| \cdot |\rho - \rho_-| \geq (1 - |\rho|)^2 \sim N^{-2}$$

where $\rho_{\pm} = \lambda \pm i\sqrt{1 - \lambda^2}$, so the integral for $|y| \gtrsim N\epsilon$ will be bounded by

$$N^{-1}N^3(N\epsilon)^{-3} \int_{|y| \gtrsim N\epsilon} \langle y \rangle^{-4} \lesssim N^{-1}\epsilon^{-3}.$$

In the region $|y| \ll N\epsilon$, we will have $|\arg(\rho)| \ll \epsilon$ and $|\arg(\rho_{\pm})| = \arccos(\lambda) \sim \epsilon$, which implies

$|1 - 2\lambda\rho + \rho^2| \sim \epsilon^2$ (note also that $1 \ll N\epsilon$), which allows us to bound the integral by

$$N^{-1}\epsilon^{-3} \int_{|y| \lesssim N\epsilon} \langle y \rangle^{-4} \lesssim N^{-1}\epsilon^{-3},$$

and the proof is complete.

(2) Recall that $\{Y_q^m\}$ form an orthonormal basis for the space of spherical harmonics of degree r (with L^2 inner product), where $-q \leq m \leq q$, then we have

$$Z_\theta^q(\theta') = \sum_{m=-q}^q Y_q^m(\theta) \overline{Y_q^m(\theta')}. \quad (4.134)$$

Assume $l \geq 1$, since

$$S_l f(\theta) = \int_{\mathbb{S}^2} \left(\sum_q \varphi_1(2^{-l}q) Z_\theta^q \right) (\theta') f(\theta') d\omega(\theta'),$$

we only need to bound the $L_{\theta'}^2$ norm of $\sum_q \varphi_1(2^{-l}q) Z_\theta^q$ for each θ . Since Z_θ^q and $Z_\theta^{q'}$ are orthogonal for $q \neq q'$, from (4.134) and the properties of Z_θ^r we can deduce

$$\left\| \sum_q \varphi_1(2^{-l}q) Z_\theta^q \right\|_{L^2}^2 \lesssim \sum_{q \lesssim 2^l} \sum_{m=-q}^q |Y_q^m(\theta)|^2 \sim \sum_{q \lesssim 2^l} |Z_\theta^q(\theta)| \sim \sum_{q \lesssim 2^l} (2q+1) \sim 2^{2l},$$

which is exactly what we need. □

Chapter 5

The 2D Euler-Maxwell system

In this chapter we shall describe the methods used in [18] to prove small data scattering for the 2D irrotational Euler-Maxwell system, as an application of the ideas introduced in Chapter 4. This work is joint with A. Ionescu and B. Pausader.

5.1 The one-fluid Euler-Maxwell model

The Euler-Maxwell system models the motion of electrons and positive ions in plasma physics; at high temperature and speed, they are treated as two fluids that satisfy a system of compressible Euler equations coupled with Maxwell equations of their self-consistent electromagnetic fields. In this chapter we consider a simplified model where the motion of the ions is neglected; this gives the so-called Euler-Maxwell electron system, which in 2D is written as the evolution system

$$\left\{ \begin{array}{l} \partial_t n_e + \operatorname{div}(n_e v_e) = 0, \\ n_e m_e (\partial_t v_e + v_e \cdot \nabla v_e) + \nabla p_e = -n_e e \left(E - \frac{b v_e^\perp}{c} \right), \\ \partial_t b + c \cdot \operatorname{curl}(E) = 0, \\ \partial_t E + c \nabla^\perp b = 4\pi e n_e v_e, \end{array} \right. \quad (5.1)$$

and an elliptic equation

$$\operatorname{div}(E) = 4\pi e(n_0 - n_e), \quad (5.2)$$

where n_0 is a constant, the unknowns $n_e \in \mathbb{R}$ and $v_e \in \mathbb{R}^2$ are density and velocity of the electrons, $E \in \mathbb{R}^2$ and $b \in \mathbb{R}$ are electric and magnetic fields respectively; also p_e is the pressure which we

assume to satisfy the quadratic law $p_e = Tn_e^2/2$.

Note that (5.2) is propagated by (5.1), since we have

$$\partial_t(\operatorname{div}(E) - 4\pi e(n_0 - n_e)) = 0$$

under (5.1). Moreover, we shall make the *irrotationality* assumption

$$\operatorname{curl}(v_e) = \frac{e}{m_e c} b, \quad (5.3)$$

which is also propagated by (5.1) due to a similar reason.

The system (5.1) has a constant equilibrium

$$(n_e, v_e, E, b) = (n_0, 0, 0, 0), \quad (5.4)$$

and we shall study small perturbations of this equilibrium under (5.2) and (5.3).

The difficulty of the analysis is due to the fact that we are in $2D$; the corresponding $3D$ problem has been solved in Germain-Masmoudi [21] and then Ionescu-Pausader [32]. In fact, even the full, two-fluid problem has been settled in [26]; see also [27] for a relativistic version of Euler-Maxwell, and [25], [28], [31] for results about the simpler Euler-Poisson model.

5.1.1 Reduction to a Klein-Gordon system

Under the irrotationality assumption (5.3), the system (5.1) and (5.2) can be reduced, via nondimensionalization and diagonalization, to a multi-speed, quasilinear Klein-Gordon system. More precisely, as in [18], let

$$\begin{aligned} n_e(x, t) &:= n_0 \cdot (1 + \rho(\lambda x, \beta t)), & v_e(x, t) &:= c \cdot u(\lambda x, \beta t), \\ b(x, t) &:= c\sqrt{4\pi n_0 m_e} \cdot \tilde{b}(\lambda x, \beta t), & E(x, t) &:= c\sqrt{4\pi n_0 m_e} \cdot \tilde{E}(\lambda x, \beta t), \\ \lambda &:= c^{-1}\sqrt{4\pi e^2 n_0 / m_e}, & \beta &:= \sqrt{4\pi e^2 n_0 / m_e}, \end{aligned} \quad (5.5)$$

and further define

$$U_e := |\nabla|^{-1} \operatorname{div}(u) - i\Lambda_e |\nabla|^{-1} \operatorname{div}(\tilde{E}), \quad U_b := \Lambda_b |\nabla|^{-1} \operatorname{curl}(u) - i|\nabla|^{-1} \operatorname{curl}(\tilde{E}), \quad (5.6)$$

then we can reduce (5.1), (5.2) and (5.3) to the following system

$$\begin{aligned} (\partial_t - i\Lambda_e)U_e &= \frac{1}{2}|\nabla| \left| \frac{\nabla}{|\nabla|} \Re U_e + \frac{\nabla^\perp}{|\nabla|} \Lambda_b^{-1} \Re U_b \right|^2 + i\Lambda_e |\nabla|^{-1} \operatorname{div} \left\{ \frac{|\nabla|}{\Lambda_e} \Im U_e \cdot \left(\frac{\nabla}{|\nabla|} \Re U_e + \frac{\nabla^\perp}{|\nabla|} \Lambda_b^{-1} \Re U_b \right) \right\}, \\ (\partial_t - i\Lambda_b)U_b &= i|\nabla|^{-1} \operatorname{curl} \left\{ \frac{|\nabla|}{\Lambda_e} \Im U_e \cdot \left(\frac{\nabla}{|\nabla|} \Re U_e + \frac{\nabla^\perp}{|\nabla|} \Lambda_b^{-1} \Re U_b \right) \right\}. \end{aligned} \quad (5.7)$$

Here

$$\Lambda_e = \sqrt{1 - \theta\Delta}, \quad \Lambda_b = \sqrt{1 - \Delta}$$

are Klein-Gordon phases with different speeds, and $\theta = (Tn_0)/(m_e c^2)$ is a physical constant (the speed of sound over the speed of light) such that $0 < \theta < 1$. From now on we will only consider the system (5.7). The main theorem of [18] is then stated in terms of this reduced system, namely

Theorem 5.1.1. *Assume that $\theta \in (0, 1)$, and let $N \gg N_1$ be sufficiently large, and assume that*

$$\|(U_e^0, U_b^0)\|_{H^N} + \|(U_e^0, U_b^0)\|_{H_\Omega^{N_1}} + \|(U_e^0, U_b^0)\|_Z = \epsilon_0 \leq \bar{\epsilon} \quad (5.8)$$

where $\bar{\epsilon} = \bar{\epsilon}(\theta) > 0$ is sufficiently small, $H_\Omega^{N_1}$ is defined by

$$\|f\|_{H_\Omega^{N_1}} = \sup_{k \leq N_1} \|\Omega^k f\|_{L^2}, \quad \Omega = x_1 \partial_2 - x_2 \partial_1,$$

and Z is a certain localization norm. Then there exists a unique global solution $(U_e, U_b) \in C(\mathbb{R}, H^N \cap H_\Omega^{N_1})$ of the system (5.7) with initial data $(U_e(0), U_b(0)) = (U_e^0, U_b^0)$. Moreover, for any $t \in [0, \infty)$

$$\|(U_e(t), U_b(t))\|_{H^N} + \|(U_e(t), U_b(t))\|_{H_\Omega^{N_1}} + \sup_{|\alpha| \leq 4} (1+t)^{0.99} \|D_x^\alpha (U_e(t), U_b(t))\|_{L^\infty} \lesssim \epsilon_0. \quad (5.9)$$

5.2 Main ingredients of the proof

In this section we will explain the main ideas involved in the proof of Theorem 5.1.1; the reader is referred to [18] for details. The starting point is still the basic scheme described in Section 4.1.2, but here we cannot directly carry it out due to inadequate decay in $2D$.

In fact, for this scheme to work, one must have a decay bound $\|(U_e(t), U_b(t))\|_{L^\infty} \lesssim |t|^{-1}$ in order to close the energy estimate in Step 1. In $3D$ linear Klein-Gordon waves decay like $t^{-3/2}$; although actual solutions decay slower due to presence of spacetime resonances, one still have enough room in Chapter 4 to prove t^{-1} decay.

However, in $2D$ the linear decay for Klein-Gordon is t^{-1} , which is just barely enough; moreover,

Bernicot-Germain [7] showed that the decay for the second iteration is at most $t^{-1} \log t$, in the presence of spacetime resonance, and in our proof we will only assume the decay of $t^{-1+\delta}$ for some small δ . Therefore, the simple energy estimate in Step 1 does not close, and we have to add correction terms to the basic energy $\mathcal{E}$, hoping to eliminate the cubic terms on the right hand side of (4.16), which is the first new ingredient of the proof.

5.2.1 The modified energy

We shall use the I -method as in [12], see also the arguments in Section 3.3.1; there is also the normal form method of Shatah [46], which serves the same purpose but works through a change of variables.

The basic energy functional

We start with the quadratic energy functional $\mathcal{E}^2 = \|U_e\|_{H^N}^2 + \|U_b\|_{H^N}^2$, and compute that $\partial_t \mathcal{E}^2 = \mathcal{C}^0 + \mathcal{C}^1 + \mathcal{C}^2$, where $\mathcal{C}^1$ and $\mathcal{C}^2$ does not lose derivatives, and in Fourier space

$$\mathcal{C}^0 \approx \int_{\xi_1 + \xi_2 + \xi_3 = 0} (\mathcal{F}|\nabla|^{N+1}U_e(\xi_1)\mathcal{F}|\nabla|^N U_e(\xi_2) - \mathcal{F}|\nabla|^{N+1}\overline{U_e}(\xi_1)\mathcal{F}|\nabla|^N \overline{U_e}(\xi_2))\widehat{U}_\sigma(\xi_3)$$

which, under the assumption $|\xi_3| \ll |\xi_1|$, is basically

$$\mathcal{C}^0 \approx \int_{\xi_1 + \xi_2 + \xi_3 = 0} |\xi_1|^{N+1}|\xi_2|^N (\widehat{U_e}(\xi_1)\widehat{U_e}(\xi_2) - \widehat{\overline{U_e}}(\xi_1)\widehat{\overline{U_e}}(\xi_2))\widehat{U}_\sigma(\xi_3). \quad (5.10)$$

To handle the loss of derivative in $\mathcal{C}_0$, noticing that U_b satisfies a semilinear equation, and that

$$(\partial_t - i\Lambda_e)U_e = -\nabla U_e \cdot u - i\left(\Lambda_e \frac{U_e + \overline{U_e}}{2}\right) \cdot \frac{|\nabla|}{\Lambda_e} \Im U_e + l.o.t., \quad (5.11)$$

we may add to $\mathcal{E}^2$ a correction term

$$\mathcal{E}_0^3 := \int_{\xi_1 + \xi_2 + \xi_3 = 0} |\xi_1|^N |\xi_2|^N [c_1 (\widehat{U_e}(\xi_1)\widehat{U_e} + \widehat{\overline{U_e}}(\xi_1)\widehat{\overline{U_e}}) + c_2 \widehat{U_e}(\xi_1)\widehat{\overline{U_e}}(\xi_2)]\widehat{U}_\sigma(\xi_3),$$

which is suggested by the form of the *exact* conservation law of (5.7), where c_1 and c_2 are two appropriate constants. Plugging in (5.11), after suitable symmetrization, one can cancel the term $\mathcal{C}^0$ and obtains that $\partial_t(\mathcal{E}^2 + \mathcal{E}_0^3) = \mathcal{C}^1 + \mathcal{C}^2 + \mathcal{Q}^0$, where $\mathcal{C}^1$ and $\mathcal{C}^2$ are cubic terms and $\mathcal{Q}^0$ is a quartic term, all without loss of derivatives.

The I -method and correction terms

Note that $\mathcal{Q}^1$ is already acceptable; we next use the I -method to eliminate $\mathcal{C}^1$.

A typical term in $\mathcal{C}^1$ is

$$\mathcal{C}^1 \approx \int_{\xi_1+\xi_2+\xi_3=0} |\xi_1|^N |\xi_2|^N \widehat{U}_e(\xi_1) \widehat{U}_e(\xi_2) \widehat{U}_e(\xi_3).$$

To eliminate it, we need to add the correction term

$$\mathcal{E}_1^3 \approx \int_{\xi_1+\xi_2+\xi_3=0} \frac{|\xi_1|^N |\xi_2|^N}{\Lambda_e(\xi_1) - \Lambda_e(\xi_2) + \Lambda_e(\xi_3)} \widehat{U}_e(\xi_1) \widehat{U}_e(\xi_2) \widehat{U}_e(\xi_3).$$

Note that we can choose the terms in $\mathcal{C}^1$ such that for each of them, the denominator is bounded below; thus, let the multiplier in $\mathcal{E}_1^3$ be m , then we have $|m| \lesssim |\xi_1|^N |\xi_2|^N$ (where we ignore powers of $|\xi_3|$). Plugging in (5.11), we obtain an additional quartic term with one derivate loss (in contrast with the semilinear situation as in [12]). However, this quartic term turns out to have the form

$$\mathcal{Q}^1 = \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} |\xi_1|^{N+1} |\xi_2|^N (\widehat{U}_e(\xi_1) \widehat{U}_e(\xi_2) - \widehat{U}_e(\xi_1) \widehat{U}_e(\xi_2)) \widehat{U}_\mu(\xi_3) \widehat{U}_\nu(\xi_4), \quad (5.12)$$

which has *exactly* the same structure as (5.10). Therefore, we can use the same argument as in the construction of $\mathcal{E}_0^3$ above to construct a correction term $\mathcal{E}_0^4$ to eliminate this additional term with derivative loss.

Now we have constructed a modified energy functional

$$\mathcal{E} = \mathcal{E}^2 + \mathcal{E}_0^2 + \mathcal{E}_1^3 + \mathcal{E}_0^4,$$

such that

$$\partial_t \mathcal{E} = \mathcal{C}^2 + (\text{quartic}) + (\text{quintic}),$$

where the quartic and quintic terms do not lose derivatives. Moreover, each term in $\mathcal{C}^2$ has the form

$$\mathcal{C}^2 \approx \int_{\xi_1+\xi_2+\xi_3=0} |\xi_1|^{N-1} |\xi_2|^N \widehat{U}_b(\xi_1) \widehat{U}_b(\xi_2) \widehat{U}_e(\xi_3),$$

where the “divisor” $\Lambda_b(\xi_1) - \Lambda_b(\xi_2) + \Lambda_e(\xi_3)$ can vanish, but we have the crucial *smoothing* effect in the multiplier which can be checked by direct computations.

This $\mathcal{C}^2$ term cannot be eliminated by adding correction terms without producing singularity;

we thus have the control them using another new ingredient, which is discussed below.

5.2.2 The L^2 lemma

Suppose we want to analyze a cubic bulk term coming from $\mathcal{C}^2$, namely

$$\int_0^t \int_{\mathbb{R}} m(\xi, \eta) \widehat{U_b}(\xi) \widehat{U_b}(\eta) \widehat{U_e}(\xi - \eta) d\eta ds, \quad |m| \approx |\xi|^{N-1} |\eta|^N, |\xi| \gg |\xi - \eta|.$$

Let, as in (4.17),

$$f_e(t) = e^{it\Lambda_e} U_e(t), \quad f_b(t) = e^{it\Lambda_b} U_b(t),$$

then we only need to bound

$$\int_0^t \int_{\mathbb{R}} e^{is\Phi(\xi, \eta)} m(\xi, \eta) \widehat{f_b}(\xi) \widehat{f_b}(\eta) \widehat{f_e}(\xi - \eta) d\eta ds$$

with some $\Phi = \Phi_{\sigma\mu\nu}$, using the notations in Section 4.1.1. An easier situation would be when $|\Phi(\xi, \eta)| \gtrsim |s|^{-0.999}$; in fact, in this case we can gain at least $|s|^{0.001}$ from integration by parts in time, which is already enough to cancel the t^δ loss coming from the insufficient decay.

Now let us consider the case when $|\Phi| \ll |s|^{-0.999}$. Here we shall use a crucial L^2 lemma, which states that

$$\left| \int_{\mathbb{R}} e^{is\Phi(\xi, \eta)} \chi(|s|^{0.999} \Phi(\xi, \eta)) \widehat{F}(\xi) \widehat{G}(\eta) \widehat{H}(\xi - \eta) d\eta \right| \lesssim |s|^{-1.001} \|F\|_{L^2} \|G\|_{L^2} \|H\|_Z \quad (5.13)$$

under suitable assumptions, with a Schwarz function χ ; the $|s|^{-1.001}$ power allows us to trump the $|s|^{O(\delta)}$ loss and close the estimate for $\mathcal{C}^2$.

The proof of (5.13) relies on integrating by parts in η using one particular vector field, $V := (\nabla_\eta \Phi)^\perp \cdot \nabla_\eta$, which annihilates $\Phi(\xi, \eta)$, and in particular the function

$$\Upsilon := \nabla_{\xi, \eta}^2(\xi, \eta) [\nabla_\xi^\perp \Phi(\xi, \eta), \nabla_\eta^\perp \Phi(\xi, \eta)]$$

will play an important role. Roughly speaking, if $|\Upsilon| \geq |s|^{-1/10}$, we shall use the TT^* method and reduce to the control of

$$K(\xi, \xi') := \int_{\mathbb{R}^2} e^{is(\Phi(\xi, \eta) - \Phi(\xi', \eta))} \chi(R\Phi(\xi, \eta)) (R\Phi(\xi', \eta)) m(\xi, \eta) m(\xi', \eta) H(\xi - \eta) H(\xi' - \eta) d\eta,$$

where $R = |s|^{0.999}$. For simplicity let us assume H and m are Schwartz; also we may assume $|\xi - \xi'|$

is small enough by partition of unity, thus by geometry and the assumption that

$$|\Phi(\xi, \eta)| \lesssim R^{-1}, \quad |\Phi(\xi', \eta)| \lesssim R^{-1},$$

we know that $\xi - \xi'$ is almost perpendicular to the vector $\nabla_\xi \Phi(\xi, \eta)$.

Therefore, for the vector V we will have

$$(V \cdot \partial_\eta)(\Phi(\xi, \eta) - \Phi(\xi', \eta)) \approx |\xi - \xi'| \cdot \Upsilon(\xi, \eta).$$

Since $(V \cdot \partial_\eta)\Phi(\xi, \eta) = 0$, $|\Upsilon| \geq |s|^{-1/10}$ and $R \ll |s|$, we know by integrating by parts using $V \cdot \partial_\eta$ that $|K(\xi, \xi')| \lesssim |s|^{-100}$ unless $|\xi - \xi'| \lesssim |s|^{-0.9}$. On the other hand, $|\xi - \xi'| \lesssim |s|^{-0.9}$ gives

$$\int_{\mathbb{R}^2} |K(\xi, \xi')| d\xi' \lesssim (|s|^{-0.9})^2 |s|^{-1} \log |s|,$$

due to the volume bound

$$\sup_\xi |\{\Phi(\xi, \eta) \leq \epsilon\}| \lesssim \epsilon \log(1/\epsilon),$$

which implies (5.13) by Schur's lemma.

If instead $|\Upsilon| \leq |s|^{-1/10}$, we shall bound the L^2 norm of the integral directly. In fact, using Schur's lemma and the improved volume bound

$$\sup_\xi |\{\Phi(\xi, \eta) \leq \epsilon, |\Upsilon(\xi, \eta)| \leq \epsilon'\}| \lesssim \epsilon(\epsilon')^{1/12} \log(1/\epsilon),$$

we can clearly gain a small power and close the estimate.

5.2.3 The remaining parts

The rotation vector field

In order to work out the Z norm estimate, as in the $3D$ case in Chapter 4 above, we will include the rotation vector field $\Omega = x_1 \partial_2 - x_2 \partial_1$ in the energy norm, in order to exploit the advantage of the improved linear dispersion estimate (see Proposition 4.2.7) and the bilinear integration by parts lemma (see Lemma 4.6.1). There is nevertheless one difficulty, namely that the vector field Ω is not consistent with the modified energy estimate described in Section 5.2.1 above.

This can be settled by noticing that Ω commutes with the linear flow $e^{it\Lambda}$ (since the linear phase

Λ is radial), thus the Duhamal formula (4.18) gives

$$\Omega^k(\widehat{f_\sigma}(t, \xi) - \widehat{f_\sigma}(0, \xi)) = \sum_{\mu, \nu} \sum_{k_1 + k_2 \leq k} \int_0^t \int_{\mathbb{R}^2} e^{is\Phi(\xi, \eta)} m_{k_1 k_2}(\xi, \eta) \widehat{\Omega^{k_1} f_\mu}(\xi - \eta) \widehat{\Omega^{k_2} f_\nu}(\eta) d\eta ds.$$

Thus if we assume in the worst case $k_1 = k$ and $k_2 = 0$, we will be able to bound the L^2 norm $\Omega^k f$ using the L^2 lemma in Section 5.2.2 above, provided that the standard energy control already allows us to exclude the case of very high frequencies.

Decomposition of $\partial_t f$

In controlling the Z norm, in particular when integrating by parts in time, it is important to have a good control of $\partial_t f$. In three dimensions, we have $\|\partial_t f\|_{L^2} \lesssim |t|^{-3/2}$ at medium frequencies due to strong dispersion, which allows us to treat this as a remainder; in $2D$ however, the bounds for $\partial_t f$ is much weaker and we cannot rely solely on the size of $\partial_t f$. Instead we have to exploit the *oscillation* of $\partial_t f$ also, and this is captured in the following decomposition (the actual decomposition is more complicated, see [18]):

$$\partial_t f_\sigma = \sum_{i=1}^3 \mathcal{B}_i,$$

where

$$\widehat{\mathcal{B}}_i(\xi) = \sum_{\mu, \nu} \int_{\mathbb{R}^2} \varphi_{E_i} e^{it\Phi(\xi, \eta)} m(\xi, \eta) \widehat{f_\mu}(\xi - \eta) \widehat{f_\nu}(\eta) d\eta,$$

where φ_{E_i} is some cutoff function supported in a neighborhood of E_i . Moreover, we have that $|\Phi(\xi, \eta)| \gtrsim 1$ on E_1 , that $\widehat{\mathcal{B}}_2$ is supported near some specific frequency, and that $\|\mathcal{B}_3\|_{L^2} \lesssim |t|^{-3/2}$.

In the actual analysis, $\mathcal{B}_3$ will be treated as a remainder, $\mathcal{B}_2$ will not enter the worst spacetime resonance interaction, and for the contribution of $\mathcal{B}_1$ we can always integrate by parts in time *once more*, so that the corresponding terms become quartic, and are thus acceptable.

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