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Doctoral Dissertations, 2011-2026

Mathematics

2012

Finite-Sheeted Covering Spaces and Solenoids over 3-manifolds

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Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp01gx41mh898>

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FINITE-SHEETED COVERING SPACES AND SOLENOIDS
OVER 3-MANIFOLDS

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: DAVE GABAI

JUNE 2012

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Abstract

This thesis develops techniques for studying towers of finite-sheeted covering spaces of 3-manifolds. The central question we seek to address is the following: given a π_1 -injective continuous map $f : S \rightarrow M$ of a 2-manifold S into a 3-manifold M , when does there exist a non-trivial connected finite-sheeted covering space M' of M such that f lifts to M' ? We approach this problem by reformulating it in terms of isometric actions of $\pi_1(M)$ on compact metric spaces. We then study regular solenoids over M , which give natural examples of compact metric spaces with isometric $\pi_1(M)$ -actions. We conclude by introducing a construction that we call the *mapping solenoid* of a map $f : S \rightarrow M$, which can be used to derive cohomological criteria that guarantee the existence of a lift of f to a non-trivial connected finite-sheeted covering space of M .

Acknowledgements

I would like to thank Dave Gabai for his encouragement and guidance, and for all that he has taught me during my time at Princeton. Thanks also to Bill Browder and Alan Reid, both for their comments on this thesis and for many mathematical discussions that have helped shape this project. I am also very grateful to Jill LeClair, who, as graduate administrator, has been extremely helpful at every stage of my graduate studies.

I would also like to thank Alexis Cohen, Henry Cowels and Cotton Seed for their support and friendship. Finally, I would like to thank Sam Lewallen. In addition to being a great friend who has listened eagerly to each new “proof” of the virtual Haken conjecture, trying to emulate Sam’s enthusiastic, inventive and open-minded approach to mathematics has taught me how enjoyable mathematical research can be.

The author was partially supported by grants NSF DMS-0504110 and NSF DMS-0854969 during the writing of this thesis.

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Chapter 1

An Introduction to 3-manifolds and Their Finite-sheeted Covering Spaces

1.1 Virtual Conjectures for 3-manifolds

Finite-sheeted covering spaces of 3-manifolds have been an active area of research since Friedhelm Waldhausen's work in the late 1960's on the Borel conjecture. Recall that a topological space X is called *aspherical* if $\pi_k(X) = 0$ for $k > 1$. The Borel conjecture asserts that the homeomorphism type of an aspherical 3-manifold is determined by its fundamental group. Waldhausen was successful in resolving this conjecture for 3-dimensional manifolds that are *Haken*, i.e. aspherical 3-manifolds satisfying the following definition:

Definition 1.1.1. *Let M be a closed aspherical 3-manifold. M is Haken if there exists an embedding $f : S \rightarrow M$ such that S is a connected orientable surface other than the 2-sphere, and $f_* : \pi_1(S) \rightarrow \pi_1(M)$ is injective.*

In the course of his work on the Borel conjecture, Waldhausen showed in [Wal] that the universal covering space of a Haken manifold is homeomorphic to $\mathbb{R}^3$. Waldhausen concluded his paper by remarking that this result immediately extended to any 3-manifold that is finitely covered by a

Haken manifold, and that he did not know of any examples of aspherical 3-manifolds that did not have such a cover. Waldhausen's remarks gave rise to the following conjecture:

Conjecture 1.1.1 (The Virtually Haken Conjecture). *Every closed aspherical 3-manifold has a finite-sheeted cover that is Haken.*

A decade later, Thurston formulated his geometrization conjecture, which gave a vast generalization of the 3-dimensional Borel conjecture. Thurston's work established the geometrization conjecture for Haken 3-manifolds, and the virtually Haken conjecture was thought to be a promising approach to extending this result to the case of general aspherical 3-manifolds. Thurston's work showed that the most prevalent geometric 3-manifolds are hyperbolic, i.e. they are homeomorphic to the quotient of hyperbolic 3-space $\mathbb{H}^3$ by a discrete fixed-point free group of isometries. An important special case of the geometrization conjecture for Haken manifolds dealt with 3-manifolds that are homeomorphic to surface bundles over the circle. Thurston's proof of geometrization for this class of manifolds required different techniques from those Thurston employed to resolve the non-fibered case of the geometrization theorem for Haken manifolds, and generated a great deal of research on fibered 3-manifolds. Though fibered 3-manifolds appear to be a very special class of Haken manifolds, Thurston pointed out that an arbitrary finite volume hyperbolic 3-manifold does not carry any obvious obstruction to having a finite-sheeted covering space that fibers over the circle. The assertion that every hyperbolic 3-manifold has such a finite-sheeted cover came to be known by the following name:

Conjecture 1.1.2 (The Virtually Fibered Conjecture). *Every closed hyperbolic 3-manifold has a finite-sheeted cover that fibers over the circle.*

A 3-manifold M that fibers over the circle with fiber S is homeomorphic to the mapping torus of a homeomorphism $\phi : S \rightarrow S$, i.e. $M \cong S \times [0, 1] / (x, 1) \sim (\phi(x), 0)$. The mapping torus construction provides one of the simplest ways of combining a 2-manifold and a 1-manifold to produce a 3-manifold, so if true the virtual fibered conjecture would show that every hyperbolic 3-manifold can "virtually" be constructed using this simple method.

In addition to the virtually fibered conjecture and the virtually Haken conjecture, there are

several other well-known conjectures about covering spaces of 3-manifolds. Recall that the first Betti number of a 3-manifold M , denoted $b_1(M)$, is the rank of the abelian group $H_1(M)$.

Conjecture 1.1.3 (The Virtually Positive First Betti Number Conjecture). *Every closed aspherical 3-manifold has a finite-sheeted cover with positive first Betti number.*

We will refer to the supremum of the first Betti numbers of the set of finite sheeted covers of M as the *virtual Betti number* of M , and we will denote this quantity by $vb_1(M)$. The example of the 3-torus, or more generally any torus bundles over the circle, shows that there exist aspherical 3-manifolds all of whose covering spaces have bounded first Betti number. Indeed, if M is a torus bundle over the circle then any finite-sheeted covering space M' of M is also a torus bundle over the circle. A simple Mayer-Vietoris argument shows that the first Betti number of such a manifold is bounded above by 3, so it follows that $vb_1(M') \leq 3$. For hyperbolic 3-manifolds, however, virtual Betti numbers are expected to be unbounded.

Conjecture 1.1.4 (The Virtually Infinite First Betti Number Conjecture). *If M is a closed hyperbolic 3-manifold then $vb_1(M) = \infty$.*

A group G is said to be *large* if there is a surjective homomorphism from a finite-index subgroup of G onto a non-abelian free group. It is not hard to show that if $\pi_1(M)$ is large then M has finite-sheeted covering spaces with arbitrarily large first Betti number. The following conjecture is therefore stronger than the virtually infinite first Betti number conjecture.

Conjecture 1.1.5 (Largeness Conjecture). *The fundamental group of a closed hyperbolic 3-manifold is large.*

It is clear that there are a number of relationships between the above conjectures. As mentioned above, the largeness conjecture implies the virtually infinite first Betti number conjecture, and the virtually infinite first Betti number conjecture is clearly stronger than the virtually positive first Betti number conjecture. A simple argument using Dehn's lemma and Poincaré duality shows that an aspherical 3-manifold with positive Betti number is Haken, so we therefore have the following string of implications:

$$\pi_1(M) \text{ Large} \implies vb_1(M) = \infty \implies vb_1(M) > 0 \implies M \text{ virtually Haken}$$

The fundamental group of a 3-manifold M that fibers over the circle surjects onto $\mathbb{Z}$, from which it follows that $b_1(M) > 0$, so there is one other elementary relationship between these conjectures:

$$M \text{ virtually fibered} \implies vb_1(M) > 0.$$

The virtually Haken conjecture is therefore the weakest of the virtual conjectures listed above. Note that if $p : M' \rightarrow M$ is a connected covering space of M then p is π_1 -injective, i.e. the induced homomorphism $p_* : \pi_1(M') \rightarrow \pi_1(M)$ is 1-1. It follows that if M is virtually Haken, a π_1 -injective embedding $f : S \rightarrow M'$ into a Haken cover of M yields a π_1 -injective *immersion* $p \circ f : S \rightarrow M$. The group $(p \circ f)_*(\pi_1(S))$ is therefore isomorphic to the fundamental group of a closed surface of positive genus. Such groups are known as surface groups, and the statement that the fundamental group of every aspherical 3-manifold contains a surface subgroup was known as the surface subgroup conjecture until its resolution in 2009 by Kahn and Markovic in [KM1].

Theorem 1.1.1 (The Surface Subgroup Theorem). *Let M be an aspherical 3-manifold. Then $\pi_1(M)$ contains a surface subgroup.*

One of the central concerns of this thesis is to study conditions under which such π_1 -injective immersions into a 3-manifold M can be used to produce an embedded π_1 -injective surface in a finite-sheeted covering space of M .

1.2 Recent Developments in the Covering Space Theory of 3-manifolds

Many further relationships between these conjectures have come to light over the past decade as a result of several significant advances in 3-manifold theory. The most important such advance has been Perelman's resolution of Thurston's geometrization conjecture. Indeed, as will be discussed in chapter 2, without this result we would be unable to show that a 3-manifold with infinite fundamental group has *any* finite-sheeted covers at all. The resolution of the geometrization conjecture

has also established the virtually Haken conjecture for any aspherical manifold M that is *not* hyperbolic. In studying the virtually Haken conjecture one may therefore assume that the 3-manifolds in question have hyperbolic structures, allowing techniques from hyperbolic geometry to be brought to bear on the question.

A second important development has been the resolution of Marden's tameness conjecture, by Agol in [?] and Calegari-Gabai in [CG]. This theorem shows that every non-compact complete hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold with boundary. This theorem has important implications for non-compact 3-manifolds that arise as infinite-sheeted covering spaces of closed hyperbolic 3-manifolds. To describe these implications, we will need to recall a few facts from hyperbolic 3-manifold theory. Recall that the isometry group of hyperbolic 3-space $\mathbb{H}^3$ may be identified with $PSL_2(\mathbb{C})$, the group of conformal automorphisms of the Riemann sphere $\mathbb{CP}^1$, and hence the fundamental group of a closed hyperbolic 3-manifold may be identified with a discrete subgroup Γ of $PSL_2(\mathbb{C})$. The *limit set* $\Lambda(H)$ of a subgroup $H < PSL_2(\mathbb{C})$ is defined to be the accumulation points of the set $\{h \cdot z \mid h \in H\}$, where $z \in \mathbb{CP}^1$ is an arbitrary point. It is not difficult to check that this set does not depend on the point $z \in \mathbb{CP}^1$. The Riemann sphere $\mathbb{CP}^1$ gives a compactification of $\mathbb{H}^3$, so that any complete geodesic γ in $\mathbb{H}^3$ has exactly two endpoints γ^+ and γ^- in $\mathbb{CP}^1$. The *convex hull* $CH(H)$ of a subgroup $H < PSL_2(\mathbb{C})$ is defined to be the smallest geodesically convex subset of $\mathbb{H}^3$ that contains any bi-infinite geodesic γ such that $\{\gamma^+, \gamma^-\} \subset \Lambda(H)$. It is easy to see that $CH(H)$ is invariant under that action of H on $\mathbb{CP}^1$, and that $CH(H)/H$ gives a geodesically convex subset of $\mathbb{H}^3/H$. The subgroup H is said to be *geometrically finite* if an ϵ -neighborhood of $CH(H)/H$ has finite volume. The resolution of the tameness conjecture together with the Thurston-Canary covering lemma (see [Can]) gives the following theorem.

Theorem 1.2.1. *Let M be a 3-manifold, and let $H < \pi_1(M)$ be a finitely generated subgroup. Either H is geometrically finite, or there exists a finite sheeted covering space M' of M that fibers over the circle with fiber Σ such that $f_*(\pi_1(\Sigma))$ has finite index in H .*

A subgroup $H < \pi_1(M)$ that comes from a virtual fibration in the manner described in the

above theorem is known as a *virtual fiber subgroup*. Up to finite index, such subgroups are *surface groups*, i.e. they are homeomorphic to fundamental groups of closed orientable surfaces of positive genus. Virtual fiber subgroups form one side of an important dichotomy for surface subgroups. Recall that a Jordan curve in $\mathbb{CP}^1$ is the image of an injective continuous map from the circle into $\mathbb{CP}^1$.

Definition 1.2.1. *Let $H < PSL_2(\mathbb{C})$ be a surface group. H is quasi-Fuchsian if the limit set $\Lambda(H) < \mathbb{CP}^1$ is a Jordan curve.*

A result of Bonahon [Bon] shows that geometrically finite surface groups are quasi-Fuchsian, so combining this result with the tameness theorem we obtain the following dichotomy for surface groups:

Corollary 1.2.1. *Let M be a closed hyperbolic 3-manifold, and let $H < \pi_1(M)$ be a surface subgroup. Then H is either a virtual fiber subgroup or H is quasi-Fuchsian.*

While it remains unknown whether virtual fiber subgroups of 3-manifold groups exist in general, Kahn and Markovic showed in their proof of the surface subgroup conjecture that the fundamental group of every closed hyperbolic 3-manifold M contains quasi-Fuchsian surface subgroups. The surfaces constructed by Kahn and Markovic are carried by π_1 -injective immersions $f : S \rightarrow M$ that have many self intersection, however, and are therefore far from being embedded.

While it is not yet known whether the fundamental group of every hyperbolic 3-manifold contains a virtual fiber subgroup, recent work of Agol has shown that a broad class of 3-manifolds do virtually fiber over the circle. The statement of Agol's theorem requires the following definition:

Definition 1.2.2. *A group G is residually finite rationally solvable (or RFRS) if there exists a nested sequence $G = G_0 > G_1 > G_2 > \dots$ such that*

- G_i is normal in G ,
- $\cap_{i=1}^{\infty} G_i = \{e\}$,
- $[G : G_i] < \infty$, and

- The kernel of the natural map $G_i \rightarrow H_1(G_i, \mathbb{Q})$ is contained in G_{i+1} .

Agol's result is the following:

Theorem 1.2.2. *Let M be an aspherical 3-manifold with boundary a union of tori such that $\pi_1(M)$ is RFRS. Then M is virtually fibered.*

Agol showed that a wide class of groups are virtually RFRS, including a class of groups known as right-angled Artin groups. Wise has recently announced in [Wis] that if M is a hyperbolic 3-manifold admitting an embedding $f : S \rightarrow M$ such that $f_*(\pi_1(S))$ is quasi-Fuchsian, then $\pi_1(M)$ has a finite-index subgroup that embeds in a right angled Artin group and is hence RFRS. This result implies the virtual fibered conjecture for Haken hyperbolic 3-manifolds, since if $f : S \rightarrow M$ is a π_1 -injective embedding of a surface into a hyperbolic 3-manifold M then $f_*(\pi_1(S))$ is either quasi-Fuchsian or a virtual fiber subgroup by the dichotomy for surface groups given in Corollary 1.2.1, so either the surface subgroup itself is already a virtual fiber or Wise's result applies. Wise's result also implies if M is a hyperbolic 3-manifold that contains an embedded quasi-Fuchsian surface then $\pi_1(M)$ satisfies a very strong group theoretic condition known as subgroup separability. It is also known that if M is a hyperbolic 3-manifold such that $\pi_1(M)$ is subgroup separable then $\pi_1(M)$ is large. It therefore appears that a positive resolution of following strong form of the virtually Haken conjecture for hyperbolic 3-manifolds would resolve all of the virtual conjectures listed above.

Conjecture 1.2.1. *Let M be a closed hyperbolic 3-manifold. Then M has a finite sheeted covering space M' admitting an embedding $f : S \rightarrow M'$ of a closed surface such that $f_*(\pi_1(S))$ is quasi-Fuchsian.*

The resolution of this conjecture is therefore one of the central aims of current research on finite-sheeted covering spaces of 3-manifolds.

1.3 Very Recent Developments in the Covering Space Theory of 3-manifolds

On March 12, 2012, Ian Agol announced in a lecture at the Institut Henri Poincaré that he has proven a theorem that has Conjecture 1.2.1 as a corollary. This remarkable theorem, a preprint of which is now available on the arxiv (see [Ago2]), resolves all of the conjectures presented in the previous section.

1.4 Thesis Summary

This thesis studies towers of finite-sheeted covering spaces of 3-manifolds, taking as a point of departure the following unpublished theorem of Jaco. Recall that given a map $f : X \rightarrow Y$ and a covering space $\pi : Y' \rightarrow Y$, f *lifts* to Y' if there exists a continuous map $\tilde{f} : X \rightarrow Y'$ such that $\pi \circ \tilde{f} = f$.

Theorem 1.4.1 (Jaco's Virtually Haken Criterion). *Let M be an aspherical 3-manifold, and let $f : S \rightarrow M$ be a π_1 -injective map. Suppose that there exist infinitely many finite-sheeted covering spaces $\pi : M' \rightarrow M$ such that f lifts to M' . Then M is virtually Haken.*

This theorem allows one to prove that a manifold M is virtually Haken under a variety of assumptions on the structure of the profinite topology on $\pi_1(M)$. Chapter 2 gives background information on the profinite topology on a group G and its profinite completion $\widehat{G}$, and introduces the notions of subgroup separability and engulfing and their topological applications. The main result of this chapter is following proposition, which generalizes a theorem of A.V. Egorov to show that subgroup can be understood in terms of isometric actions of $\pi_1(M)$ on compact metric spaces.

Proposition 2.4.1. *A subgroup H of a finitely generated group G is separable if and only if there is a compact metric space X with an isometric G -action and a point $x \in X$ such that $H = \text{Stab}(x)$.*

Chapter 3 discusses a natural compact metric space $\widehat{M}$ on which the fundamental group of CW

complex M acts, called the *universal solenoid* over M . This space is defined to be the inverse limit of the set of all finite-sheeted covering spaces of M , and gives a principal $\widehat{\Gamma}$ -bundle over M , where Γ denotes the fundamental group of M and $\widehat{\Gamma}$ denotes its profinite completion. This chapter discusses some the basic topological structure of general solenoids, and then turns to a more detailed study of solenoids over 3-manifolds. We show in this chapter that if finite volume hyperbolic 3-manifolds are virtually fibered, then for any aspherical 3-manifold M the Čech cohomology groups $H^i(\widehat{M}, A)$ vanish for any finite coefficient module A and any $i > 0$. The fundamental group of a manifold whose universal solenoid satisfies this condition is said to be *good in the sense of Serre*, a result which we show can be extended to the fundamental group of any 3-manifold (aspherical or not with or without boundary) under the assumption that finite volume hyperbolic 3-manifolds are virtually fibered. We conclude by discuss the relevance of this result to a question known as Grothendieck's problem for 3-manifold groups, and show that the fundamental groups of closed prime 3-manifolds have a property known as Grothendieck rigidity. This extends work of Long and Reid in [LR1], who showed that fundamental groups of closed geometric 3-manifolds are Grothendieck rigid.

Chapter 4 introduces a construction that we call the *mapping solenoid* of a map $f : X \rightarrow Y$. This object can be used under favorable circumstances to show that f can be lifted to some finite-sheeted covering space of Y . The central tool in this section is the Cartan-Leray spectral sequence for a fiber bundle, which we show can be used to derive spectral sequences for mapping solenoids. These spectral sequences are then used to derive an engulfing criterion, which provides a new (though rather complicated) proof of the well-known fact that a π_1 -injective map of a torus into closed 3-manifold M can be lifted to infinitely many finite-sheeted covering spaces of M . We conclude by studying the mapping solenoids associated to non-engulfed surface subgroups of 3-manifold groups and π_1 -injective maps of surfaces into towers of non-Haken 3-manifolds.

Chapter 2

The Profinite Topology on a 3-manifold Group

2.1 An introduction to the profinite topology

It is a basic result in covering space theory that given a connected manifold M there is a one-to-one correspondence, known as the Galois correspondence, between subgroups of $\pi_1(M)$ and connected covering spaces of M . This correspondence holds more generally for any topological space M admitting a universal covering space $p : \widetilde{M} \rightarrow M$. In this setting, $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations of p and any connected covering space of M can be shown to be homeomorphic to a covering space of the form $\widetilde{M}/N$ for some subgroup $N < \pi_1(M)$. The number of sheets of the covering space $\widetilde{M}/N \rightarrow M$ is equal to the index $[\pi_1(M) : N]$, so it follows that studying *finite-sheeted* covering spaces of M is equivalent to studying *finite index* subgroups of $\pi_1(M)$. This section gives an introduction to the profinite topology on a group G , which provides a useful way of organizing information about G 's finite-index subgroups.

Given a group G , the *profinite topology* on G is defined to be the smallest topology in which cosets of finite index subgroups of G are open. Alternatively, one can define the profinite topology

on G to be the smallest topology in which every homomorphism $\rho : G \rightarrow F$, where F is a finite group equipped with the discrete topology, is continuous. Intuitively speaking, this topology considers two elements h and g of G to be “near” if whenever F is a finite group of small order, any homomorphism from G to F maps h and g to the same element.

The profinite topology on a general finitely generated group need not be very rich. Indeed, there are many finitely presented groups G that have no finite quotients at all (see [Hig]), and hence the profinite topology on G is the trivial topology. When the profinite topology on G is Hausdorff, however, it can serve as a powerful tool for studying G . It is easily seen that the profinite topology on G is Hausdorff if and only if every element of $\pi_1(M)$ has non-trivial image in a finite quotient of $\pi_1(M)$, a condition which is known as *residual finiteness*. When G is the fundamental group of a 3-manifold, the following theorem, which is consequence of the geometrization conjecture together with work of Hempel and Malcev (see [Th2]), shows that the profinite topology has this property,

Theorem 2.1.1. *The fundamental group of a compact 3-manifold is residually finite.*

Given a group G , we will denote the set of finite index normal subgroup of G by $\mathcal{N}(G)$. This set comes equipped with a natural partial ordering given by inclusion, which yields a partial ordering on the set of finite quotients of G , given by declaring $G/N_1 \geq G/N_2$ if $N_1 \subset N_2$. Note that if $G/N_1 \geq G/N_2$ then there is a natural quotient map

$$\phi_{21} : G/N_1 \rightarrow (G/N_1)/(N_2/N_1) \cong G/N_2,$$

and that if $G/N_1 \geq G/N_2 \geq G/N_3$, then $\phi_{32} \circ \phi_{21} = \phi_{31}$. If G is a finitely generated group, then the following theorem of Marshall Hall (see [Hal1]) guarantees that the collection of finite index subgroups is closed under intersection. In fact, Hall’s theorem show that if G is finitely generated, then finite index *characteristic* subgroups of G , i.e. those subgroups of G that are fixed under any automorphism of G , give a neighborhood basis of the identity for the profinite topology on G .

Theorem 2.1.2. *Let G be a finitely generated group. Given any two finite index subgroups H_1 and H_2 of G , there exists a finite index characteristic subgroup $K < G$ such that $K < H_1 \cap H_2$.*

This shows that the collection of finite quotients of G satisfy the axioms of an inverse system. The *profinite completion* of G is defined to be the inverse limit of this inverse system, which can be defined as follows:

$$\widehat{G} := \{(x_N) \in \prod_{N \in \mathcal{N}(G)} G/N \mid \text{if } G/N_1 \geq G/N_2 \text{ then } \phi_{21}(x_{N_1}) = x_{N_2}\}.$$

By equipping each finite quotient of G with the discrete topology, equipping $\prod_{N \in \mathcal{N}(G)} G/N$ with the product topology, and equipping $\widehat{G}$ with the subspace topology, $\widehat{G}$ obtains the structure of a topological group. By Tychonoff's theorem, $\prod_{N \in \mathcal{N}(G)} G/N$ is compact, and it is not hard to check that $\widehat{G}$ is a closed subspace of $\prod_{N \in \mathcal{N}(G)} G/N$ so $\widehat{G}$ is compact.

There is a natural homomorphism $\eta : G \rightarrow \prod_{N \in \mathcal{N}(G)} G/N$, where the N -th entry of $\eta(g)$ in the product is given by the reduction of g modulo N , and it is clear that $\eta(G) \subset \widehat{G}$. Another way of defining the profinite topology on G is as the pull-back of the topology on $\widehat{G}$ by η , i.e. the topology in which a subset of G is open if and only if it is the preimage under η of an open set in $\widehat{G}$. If G is residually finite, then for any $g \in G$ there exists a normal subgroup N such the image of g in G/N is non-trivial, and hence the map η is injective. Conversely if the map η is injective, then for any $g \in G$ the N -th entry of $\eta(g)$ is non-trivial for some N , so g has nontrivial image in G/N . It follows that η is injective if and only if G is residually finite.

In addition to the profinite topology on G , we will also be interested in potentially coarser topologies on G coming from inverse systems of finite-index subgroups of G other than $\mathcal{N}(G)$. Given an inverse system $\mathcal{F}$ of finite index normal subgroups of G that is closed under intersection, we will denote by $\widehat{G}_{\mathcal{F}}$ the inverse limit of the inverse system $\{G/N \mid N \in \mathcal{F}\}$. Just like the profinite completion, $\widehat{G}_{\mathcal{F}}$ is a profinite group that comes equipped with a natural homomorphism $\eta_{\mathcal{F}} : G \rightarrow \widehat{G}_{\mathcal{F}}$. We will call the topology given by pulling back the topology on $\widehat{G}_{\mathcal{F}}$ to G by $\eta_{\mathcal{F}}$ the $\mathcal{F}$ -topology on G .

2.2 Subgroup Separability and Scott's Lemma

Given a subgroup H of a group G , we will denote the closure of H in the profinite topology on G by H^* , i.e.

$$H^* := \bigcap_{N \in \mathcal{N}(G)} N = \eta^{-1} \left(\overline{\eta(H)} \right).$$

A subgroup H of a group G is said to be *separable* if $H = H^*$, and is said to be *engulfed* if $H^* \neq G$. Note that H is engulfed if and only if H is not dense in the profinite topology on G .

As we will see below, the notion of separability has interesting topological applications, and many of the conjectures discussed in the introduction can be addressed for a 3-manifold M whose fundamental group has suitable separability properties. There are many different commonly studied separability conditions that one can impose on a group G . The most basic of these is residual finiteness, which was discussed in the previous section. Using the above notation, residual finiteness holds if and only if $e^* = e$, where e denotes the identity subgroup of G . A group G is said to be *extended residually finite* if given any subgroup $H < G$, $H = H^*$, i.e. H is closed in the profinite topology. This is an extremely restrictive property for a group to have, and fails to be satisfied even by non-abelian free groups. A much less restrictive property is that of being *locally extended residually finite*, or *LERF*, which means that every *finitely generated* subgroup of G is closed in the profinite topology. A group satisfying this condition is also said to be *subgroup separable*. This property was shown by Hall in [Hal2] to be satisfied for free groups, and by Scott in [Sco] to hold for fundamental groups of closed surfaces.

Many of the topological applications of subgroup separability are derived from the following lemma, which is a straightforward generalization of Lemma 1.4 from Scott's paper [Sco] that we will need in the forthcoming sections.

Lemma 2.2.1. *Let $\mathcal{F}$ be a inverse system of finite index subgroups of G , let X be a connected metric space, and let $p : X' \rightarrow X$ be a regular cover of X with covering group G . A subgroup $H < G$ is closed in the $\mathcal{F}$ -topology if and only if given any compact subset C of X'/H , there exists a $K \in \mathcal{F}$ such that $H < K$ and the covering map $p_{KH} : X'/H \rightarrow X'/K$ is 1-1 when restricted to*

C .

Proof. We will denote the identity subgroup of G by e , and given two subgroups $H_1 < H_2 < G$, we will denote the natural covering map $X'/H_1 \rightarrow X'/H_2$ by $p_{H_2H_1}$. Suppose that H is closed in the $\mathcal{F}$ -topology, let $C \subset X'/H$ be a compact set. Let $x \in X$ be a basepoint lying in $p_{GH}(C)$ and let $\tilde{x} \in p_{Ge}^{-1}(x)$. Since C is compact, there exists $R > 0$ such that the R -ball around $p_{He}(x)$, $B_R(p_{He}(x))$, contains C . Let C' denote the closure of $B_R(\tilde{x})$, and note that $C \subset p_{He}(B_R(\tilde{x}))$. Since C' is compact, the collection of elements

$$\{g \in G \mid (g \cdot C') \cap C' \neq \emptyset\}$$

is a finite set $\{g_1, \dots, g_n\} \subset G$ since the action of a deck group of a covering space is proper. By reordering if necessary, we may assume that $\{g_1, \dots, g_k\}$ are those elements of $\{g_1, \dots, g_n\}$ that do not lie in H . Since H is closed in the $\mathcal{F}$ -topology, there exist subgroups $K_1, \dots, K_k \in \mathcal{F}$ such that $H < K_i$ and $g_i \notin K_i \cap H$. Since $\mathcal{F}$ is closed under intersections, $K := \bigcap_{i=1}^k K_i$ is an element of $\mathcal{F}$ that contains H but does not contain g_i for $1 \leq i \leq k$. Let $a, b \in C$, and let $\tilde{a}, \tilde{b} \in C'$ be points in $p_{He}^{-1}(a)$ and $p_{He}^{-1}(b)$ respectively. Since $p_{KH}(a) = p_{KH} \circ p_{He}(\tilde{a}) = p_{Ke}(\tilde{a})$ and likewise $p_{KH}(b) = p_{Ke}(\tilde{b})$, if $p_{KH}(a) = p_{KH}(b)$ then $p_{Ke}(\tilde{a}) = p_{Ke}(\tilde{b})$ and thus there exists $k \in K$ such that $k \cdot \tilde{a} = \tilde{b}$. Since $\tilde{a}$ and $\tilde{b}$ both lie in C' , it follows that $k \in \{g_1, \dots, g_n\}$. Since $K \cap \{g_1, \dots, g_k\} = \emptyset$, $k \in \{g_{k+1}, \dots, g_n\} \subset H$. It follows that

$$a = p_{He}(\tilde{a}) = p_{He}(k \cdot \tilde{a}) = p_{He}(\tilde{b}) = b,$$

so $p_{KH}|_C$ is injective as claimed. □

In the case that the $\mathcal{F}$ -topology is equal to the full profinite topology, this gives Scott's well-known geometric interpretation of the LERF condition.

Lemma 2.2.2. *Let X be a Hausdorff topological space with a regular covering X' and covering group G . Then G is LERF if and only if given any finitely generated subgroup of $H < \Gamma$ and any*

compact subset C of X'/H , there exists a finite-sheeted covering X'' of X such that the covering map $X'/H \rightarrow X$ factors through a map $\pi : X'/S \rightarrow X''$ that restricts to an embedding on C .

The above lemma can be used to force desirable features of an infinite-sheeted covering space of a space X with LERF fundamental group to appear in finite-sheeted covering spaces of X . The proof of the following well-known corollary of Scott's lemma gives a model example of how this type of argument works.

Corollary 2.2.1. *Let M be an aspherical 3-manifold, and let $f : S \rightarrow M$ be a π_1 -injective map from a closed surface into M . If $f_*(\pi_1(S))$ is closed in the profinite topology on $\pi_1(M)$, then there exists a finite-sheeted covering space $p : M' \rightarrow M$ such that f lifts to a map $\tilde{f} : S \rightarrow M'$, and such that $\tilde{f}$ is homotopic to an embedding. Consequently, M is virtually Haken.*

Proof. Let S be a closed surface with fundamental group H . Since S is a $K(H, 1)$ space, there exists a continuous map $f : S \rightarrow M$ such that $f_*(\pi_1(S)) = H$. Let $p : M_H \rightarrow M$ be the connected covering space of M corresponding to H . By the lifting criterion in covering space theory, there exists a lift $\tilde{f} : S \rightarrow M_H$. It is a well known fact in 3-manifold topology (see [Hem]) that there is a homeomorphism $\Phi : M_H \rightarrow S \times \mathbb{R}$, so M_H deformation retracts onto the surface $S_0 := \Phi^{-1}(S \times \{0\})$. It follows that $\tilde{f}$ is homotopic to a map $\tilde{f}' : S \rightarrow S_0$. Since $\tilde{f}'$ gives a homotopy equivalence between the surfaces S and S_0 , the Baer-Nielsen theorem (see [FM]) shows that $\tilde{f}'$ is homotopic to a homeomorphism $\tilde{f}'' : S \rightarrow S_0$. Since S_0 is compact, there exists a finite sheeted covering $r : M' \rightarrow M$ and a map $\pi : M_H \rightarrow M'$ such that $\pi|_{S_0}$ is an embedding. Since π_* is a π_1 -injective map, it follows that $\pi|_{S'} : S' \rightarrow M'$ is a π_1 -injective embedding, so M' is Haken. $\square$

This shows that 3-manifolds containing separable surface subgroups are virtually Haken. By Kahn and Markovic's resolution of the surface subgroup conjecture, it follows that 3-manifolds with LERF fundamental groups are virtually Haken. While it is conjectured that *hyperbolic* 3-manifolds have LERF fundamental groups, fundamental groups of general 3-manifolds are known not to be LERF. The first examples of non-LERF 3-manifold groups were constructed by Burns, Karass and Solitar [BKS]. Such examples were subsequently studied by Niblo and Wise in [NW2], who showed that the fundamental group of the complement of a very simple 4-component link in S^3 is a

subgroup of every known non-LERF 3-manifold group. This link is given by a 4-component chain, i.e. a link with components C_1, C_2, C_3 and C_4 such that $C_1 \cup C_2, C_2 \cup C_3$ and $C_3 \cup C_4$ are Hopf links, and $C_i \cup C_j$ is a 2 component unlink for all other pairs i, j .

Theorem 2.2.1. *Any 3-manifold M such that $\pi_1(S^3 \setminus L) < \pi_1(M)$ is not LERF.*

Niblo and Wise also showed that $\pi_1(S^3 \setminus L)$ is a subgroup of every known non-LERF 3-manifold group. The existence of such a subgroup may therefore be the only obstruction to being LERF for 3-manifold groups.

2.3 Jaco's Virtually Haken Criterion

As was discussed in the previous section, one approach to proving that a manifold M is virtually Haken is to establish the existence of a surface subgroup $H < \pi_1(M)$ that is closed in the profinite topology. In order for H to be closed in $\pi_1(M)$, for every element $g \in \pi_1(M) \setminus H$ there must exist a finite index subgroup $K_g < \pi_1(M)$ containing H such that $g \notin K$. In this section, we discuss a result of Jaco that shows that the virtually Haken conjecture can also be proved under the much weaker hypothesis that there exist infinitely many finite index subgroups $K < \pi_1(M)$ such that $H < K$, or in the notation of the previous section, that $[\pi_1(M) : H^*] = \infty$. We will call such a surface subgroup *strongly engulfed*. It is a simple exercise to show that a subgroup $H < G$ is strongly engulfed if and only if it is nowhere dense in the profinite topology on G . In this section we present Jaco's argument (see [SW] and [Nib]), and observe that when combined with the tameness theorem it can be used to show that a hyperbolic 3-manifold containing a strongly engulfed quasi-Fuchsian surface subgroup virtually contains an embedded quasi-Fuchsian surface.

Lemma 2.3.1. *A non-compact aspherical 3-manifold N admitting a π_1 -injective map $f : \Sigma \rightarrow N$ from a closed surface contains an embedded π_1 -injective surface S . If N is hyperbolic and f is quasi-Fuchsian, then S is quasi-Fuchsian.*

Proof. Let d be the distance function for any complete Riemannian metric on N , let $r > 0$ be an arbitrary constant, and let $N^1 := \{x \in N \mid d(x, f(\Sigma)) \leq r\}$. Since Σ is compact, N^1 is a

compact submanifold of N . Let $S_1, \dots, S_n$ be the components of ∂N^1 . If any component S_i is non-separating in N , then N has positive first Betti number and hence N contains an embedded π_1 -injective surface. We may therefore assume that each component of ∂N^1 is separating. If any component C of ∂N^1 is a sphere, since C bounds embedded 3-ball B in N by the sphere theorem we will replace N^1 by $N^1 \cup B$ to obtain a 3-manifold with one fewer boundary component. By performing this operation on each spherical boundary component of N^1 , we may assume that each component of N^1 has positive genus. We may also assume by enlarging N^1 if necessary that given any component S_i of ∂N^1 , the component of $N \setminus S_i$ which does not contain N^1 has infinite diameter, since if the other component of $N \setminus S_i$ is a compact manifold K we can replace N^1 with $N^1 \cup K$.

Suppose that the inclusion map $i : S_1 \rightarrow N^1$ does not induce an injective map $i_* : \pi_1(S_1) \rightarrow \pi_1(N^1)$. By Dehn's lemma, we may find an embedded disk D in N^1 whose boundary gives a homotopically non-trivial closed curve on S_1 . Let $N^2 \subset N^1$ be the submanifold given by the complement of a regular neighborhood of D , and S_1^1 the boundary component of N^2 intersecting S_1 . Note that S_1^1 is given by cutting S_1 along the closed curve $D \cap S_1$ and gluing in two disks, so the components of S_1^1 have genus strictly less than that of S_1 . We can repeat this process if the restriction of $i^1 : S_1^1 \rightarrow N^2$ to some component of S_1^1 is not π_1 -injective. Proceeding inductively, we obtain a sequence $S_1, S_1^1, \dots, S_1^k$ of surfaces bounding 3-manifolds $N^1 \supset N^2 \supset \dots \supset N^k$, until $i^k : S_1^k \rightarrow N^k$ restricts to a π_1 -injective map on each component of S_1^k . Note that the submanifold N^{i-1} is homeomorphic to the manifold obtained from N^i by identifying two embedded disks in ∂N^i .

Suppose that S_1^k is a union of spheres. Since N is aspherical, each component of S_1^k must bound an embedded 3-ball in N to one side or the other. Let A_i denote the union of the components of $N \setminus S_1^i$ that do not contain N^i . Since A_0 has a component with infinite diameter and $A_{i-1} \subset A_i$, it follows that there exists a component of $N \setminus S_1^k$ that has infinite diameter and does not contain N^k . Let C be a component of S_1^k that border such a component. The embedded 3-ball B that C bounds is compact, and therefore cannot contain a subset of N of infinite diameter. It follows that B lies to the side of C containing N^k , and since $\partial B = C$, $N^k \subset B$. This implies, in particular, that S_1 was the only boundary component of N^1 , since by assumption each component of ∂N^1

bounded an infinite diameter subsets of N^1 to the outside. Since N^k is a subset of 3-ball with spherical boundary, N^k is homeomorphic to a punctured 3-ball and hence has trivial fundamental group. Since N^i is given by identifying disjoint disks in the boundary of N^{i+1} , it follows by a simple induction argument and Van Kampen's theorem that N^i has free fundamental group for all i . In particular, $\pi_1(N^1)$ is free. Since the map $f : \Sigma \rightarrow \pi_1(N)$ factors through a map $f' : \Sigma \rightarrow N^1$ and $f_* : \pi_1(\Sigma) \rightarrow \pi_1(N)$ is injective, it follows that $f'_* : \pi_1(\Sigma) \rightarrow \pi_1(N^1)$ is injective. This shows that $f'_*(\pi_1(\Sigma))$ gives a surface subgroup of the free group $\pi_1(N^1)$, which is a contradiction since subgroups of a free group are free. It follows that the component C must have positive genus.

The component $C \subset S_1^k$ therefore gives a surface of positive genus such that the map $i^k|_C : C \rightarrow N^k$ is π_1 -injective. We would now like to compress C along embedded disks in $N \setminus N^k$ as above to produce a new surface that is π_1 -injective in N^1 . Let A denote the infinite diameter component of $N \setminus C$ that does not contain N^k as above. If the inclusion of C into A is not π_1 -injective, then we can compress C in A inductively as above until we obtain a surface C^l bounding a subset A^l of A such that the inclusion $i : C^l \rightarrow A_k$ is in π_1 -injective. As above, C^l cannot consist entirely of sphere, so it has a component S of positive genus whose inclusion into A is π_1 -injective.

If the inclusion of S into all of N is not π_1 -injective, then S bounds an embedded disk D in N to the side of S containing N^k . We may assume that this disk does not intersect C , since the inclusion of C into N^k is π_1 -injective. It therefore follows that D lies in the submanifold of N lying between S and C . This submanifold is given by inductively adding one-handles to C^l , however, so the inclusion of S into this submanifold is π_1 -injective. It follows that the inclusion of S into N is π_1 -injective.

We now consider the geometric version of this lemma. Note that since N^1 is a compact manifold with boundary, $\pi_1(N^1)$ is finitely generated. Let $i : N^1 \rightarrow N$ denote the inclusion map. The subgroup $i_*(\pi_1(N^1))$ cannot be a virtual fiber subgroup, since it contains a quasi-Fuchsian surface subgroup. It therefore follows from the tameness theorem that the limit set of $i_*(\pi_1(N^1))$ is not all of $\partial\mathbb{H}^3$. Since the surface S constructed above is given by compressing a boundary component of N^1 , $\pi_1(S) < i_*(\pi_1(N^1))$, and therefore $\pi_1(S)$ cannot be a virtual fiber. By the dichotomy for surface subgroups of hyperbolic three manifold groups given in Theorem 1.2.1 above, $\pi_1(S)$ is

quasi-Fuchsian.

□

Given the above lemma, the following geometric version of Jaco's virtually Haken criterion follows from Lemma 2.2.2:

Theorem 2.3.1 (Jaco's Criterion). *Let M be an irreducible 3-manifold. Suppose there exists a π_1 -injective map $f : \Sigma \rightarrow M$, where Σ is a closed surface with $\chi(\Sigma) \leq 0$, and suppose that f lifts to infinitely many finite-sheeted covers of M . The M is virtually Haken. If M admits a π_1 -injective quasi-Fuchsian immersion $f : S \rightarrow M$ that lifts to infinitely many finite-sheeted covering spaces of M , then M virtually contains an embedded quasi-Fuchsian surface.*

Proof. Let $\{p_i : M_i \rightarrow M \mid 1 \leq i \leq \infty\}$ be an infinite collection of finite sheeted covering spaces to which f lifts. Let K denote the subgroup of $\pi_1(M)$ given by $\cap_{i=1}^{\infty} (p_i)_*(\pi_1(M_i))$. Note that since K is an intersection of finite index subgroups, $K^* = K$ and hence K is closed in the profinite topology. Let $p_K : M_K \rightarrow M$ be the covering space of M corresponding to K . Note that $[\pi_1(M) : K]$ is infinite, and therefore M_K is non-compact. By the lifting criterion, there exists a lift $\tilde{f} : \Sigma \rightarrow M_K$ of f . It therefore follows by Lemma 2.3.1 that there exists a π_1 -injective embedding $i : S \rightarrow M_K$ which is quasi-Fuchsian if f is quasi-Fuchsian. Since S is compact and K is closed, by Lemma 2.2.2 there exists a finite-sheeted covering space $p : M' \rightarrow M$ such that $p_K : M_K \rightarrow M$ factors through a covering map $\pi : M_K \rightarrow M'$ such that $\pi|_{i(S)}$ is an embedding. It follows that $\pi \circ i : S \rightarrow M'$ is an embedding, and since π and i are π_1 -injective $\pi \circ i$ is π_1 -injective as well.

□

Jaco's criterion has the following corollary:

Corollary 2.3.1. *Let M be a 3-manifold such that $\pi_1(M)$ contains a strongly engulfed surface subgroup. Then M is virtually Haken.*

While it may seem reasonable to conjecture that surface subgroups of 3-manifolds groups should always be strongly engulfed, the following theorem of Niblo and Wise [NW1], which builds on examples of Rubinstein and Wang in [RW], shows that this is not always the case:

Theorem 2.3.2. *There exist 3-manifolds M and surface subgroups $H < \pi_1(M)$ such that H is dense in the profinite topology on $\pi_1(M)$.*

The manifolds M in these examples are not hyperbolic, and it is conjectured that this phenomenon does not occur in hyperbolic 3-manifolds.

2.4 Indirect Methods for Studying the Profinite Topology on a Group

If G is a finitely generated group, then many features of the profinite topology on G can be easily understood in terms of actions of G on finite sets. A subgroup $H \subset G$ is engulfed, for example, if and only if there is a finite G -set X and a point $x \in X$ such that $H \cdot x \neq G \cdot x$. To see this, note that since X is finite, the power set $\mathcal{P}(X)$ is also a finite G -set. The orbit $H \cdot x \in \mathcal{P}(X)$ is stabilized by H , however it is not stabilized by all of G . It follows that $\text{Stab}(H \cdot x)$ is a proper subgroup of G , and since $\mathcal{P}(X)$ is finite this subgroup has finite index. To prove the converse, note that if H is engulfed by a finite index subgroup K , H stabilizes the coset K under the left action of G on the coset space G/K .

The goal of this section is generalize the argument presented in the above paragraph by showing that the finite set in the above discussion can be replaced by an arbitrary compact metric space X with an isometric G -action. We will call such a metric space a *G -compactum*. The main result in this section is the following criterion for closedness of the subgroup $H < G$:

Proposition 2.4.1. *A subgroup H of a finitely generated group G is closed if and only if there is a G -compactum X and a point $x \in X$ such that $H = \text{Stab}(x)$.*

This generalizes a result of A.V. Egorov from [Ego], who showed that a group G is residually finite if and only if G acts by isometries on a compact metric space X so that the induced homomorphism $\rho : G \rightarrow \text{Isom}(X)$ is injective. We point out the following corollary of Proposition 2.4.1:

Corollary 2.4.1. *A subgroup H of a finitely generated group G is engulfed if and only if there is a G -compactum X and a point $x \in X$ such that $\overline{H \cdot x} \neq \overline{G \cdot x}$.*

Proof. Suppose that $H < G$ is engulfed, and let $K < G$ be a proper finite index subgroup containing H . By endowing G/K with a metric d where $d(g_1K, g_2K) = 1$ for any two distinct cosets g_1K and g_2K , G/K becomes G -compactum. The H orbit of the coset K is simply K , whereas the G orbit of the coset K is all of G/K , so $\overline{H \cdot K} \neq \overline{G \cdot K}$.

To prove the converse direction, recall that the space $\mathcal{C}(X)$ of closed subsets of a compact metric space X is itself a compact metric space when equipped with the Hausdorff metric d_H , defined by

$$d_H(X, Y) = \max\left\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\right\}.$$

Note that if X is a G -compactum then the induced action of G on $\mathcal{C}(X)$ is also isometric, and that H stabilizes the point $\overline{H \cdot x} \in \mathcal{C}(X)$. Since $\overline{H \cdot x} \neq \overline{G \cdot x}$, there exists $g \in G$ such that $g \cdot \overline{H \cdot x} \neq \overline{H \cdot x}$, so $\text{Stab}(\overline{H \cdot x}) \neq G$. Since $\text{Stab}(\overline{H \cdot x})$ is closed in the profinite topology on G by Proposition 2.4.1, there exists a proper finite index subgroup K such that $\text{Stab}(\overline{H \cdot x}) < K$. Since $H < \text{Stab}(\overline{H \cdot x})$, H is engulfed.

□

One direction of Proposition 2.4.1 can be easily shown by metrizing the profinite completion of G .

Lemma 2.4.1. *Let G be a finitely generated group, and let $H < G$ be closed. Then there exists a G -compactum X such that $H = \text{Stab}(x)$ for some $x \in X$.*

Proof. Let $\eta : G \rightarrow \widehat{G}$ denote the natural map from G to its profinite completion. Since G is finitely generated, $\widehat{G}$ admits a left-invariant metric $d : \widehat{G} \times \widehat{G} \rightarrow \mathbb{R}$ (see Chapter I.1 of [Ser]). Thus $\widehat{G}$ is a compact metric space on which G acts by isometries, via the action $g \cdot h = \eta(g)h$. Let $\mathcal{C}(\widehat{G})$ denote the space of closed sets in $\widehat{G}$ equipped with the Hausdorff metric. Suppose that H is closed in the profinite topology on G , and let C_H denote the closure in $\widehat{G}$ of $\eta(H)$. Since C_H is a subgroup of $\widehat{G}$ it contains the identity element $\eta(e)$. Let $g \in G \setminus H$. Since $H = \eta^{-1}(C_H)$, $\eta(g) \notin C_H$, so $\eta(g) \cdot \eta(e) \notin C_H$. Thus $g \cdot C_H \neq C_H$, and g does not stabilize the point $C_H \in \mathcal{C}(\widehat{G})$. On the other hand, $h \cdot C_H = \eta(h)C_H = C_H$ since C_H is a subgroup of $\widehat{G}$, so $H = \text{Stab}(C_H)$. □

The other implication in Proposition 2.4.1 is less elementary, and relies on the Peter-Weyl theorem to translate information about the action of G on X into information about representations of G into $Gl_n(\mathbb{C})$. Before presenting the proof of Proposition 2.4.1, we will first outline some well-known arguments that show how linear representations relate to the profinite topology on G . The first of these arguments, originally due independently to Selberg [Sel] and Malcev [Mal], shows that finitely generated linear groups are residually finite.

Theorem 2.4.1 (Selberg's Lemma). *Let F be a field, and let G be a finitely generated subgroup of $Gl_n(F)$. Then G is residually finite.*

Proof. Recall that the *Jacobson radical* of a ring R , denoted $\mathcal{J}(R)$ is defined to be the intersection of the collections of all maximal ideals in R . It is a basic fact in commutative algebra (a proof of which may be found in [Eis]) that if R is a finitely generated integral domain with multiplicative identity, then $\mathcal{J}(R) = 0$, and $R/\mathcal{M}$ is a finite field for any maximal ideal $\mathcal{M} \subset R$.

Let $g_1, \dots, g_n$ be a set of generators for G , and let $\alpha_1, \dots, \alpha_l$ be the collection of all matrix coefficients appearing in the matrices $g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}$. Let R denote the subring of F generated by the elements $\alpha_1, \dots, \alpha_l$. Since any element of G is given by multiplying matrices in the set $\{g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}\}$ together, the matrix entries of any element of G lie in the ring R , i.e. $G \leq Gl_n(R)$. Note that since the identity matrix is in G , the multiplicative identity in F lies in R , and since R is a subring of a field R is an integral domain.

Let g be a nontrivial element of G , and let $I \in Gl_n(F)$ denote the identity matrix. Since $g - I$ is not the identity matrix, there exists a matrix entry $(g - I)_{ij}$ that is non-zero. Since $(g - I)_{ij} \in R$ and $\mathcal{J}(R) = 0$, there exists a maximal ideal $\mathcal{M} \subset R$ such that $(g - I)_{ij} \notin \mathcal{M}$. Consider the homomorphism $\rho_{\mathcal{M}} : G \rightarrow Gl_n(R/\mathcal{M})$ given by composing the inclusion map $G \hookrightarrow Gl_n(R)$ with the map $Gl_n(R) \rightarrow Gl_n(R/\mathcal{M})$ given by reducing matrix coefficients modulo $\mathcal{M}$. Since $(\rho_{\mathcal{M}}(g) - \rho_{\mathcal{M}}(I))_{ij} = (g - I)_{ij} + \mathcal{M}$, it follows that $\rho_{\mathcal{M}}(g) \neq \rho_{\mathcal{M}}(I)$. Since $Gl_n(R/\mathcal{M})$ is a finite group and $g \in G$ was arbitrary, it follows that G is residually finite.

□

An important observation of Long from [Lon2] shows that the above argument can be generalized

to show that any subgroup H of a residually finite group G is closed in the profinite topology provided that it is maximal with respect to vanishing of so-called abstract polynomials on G . In the presence of a linear representation, the following lemma of Bergeron from [Ber] shows that this can be phrased in terms of the Zariski topology:

Theorem 2.4.2. *Let G be a finitely generated group, let $H < G$, and let $\rho : G \rightarrow \mathrm{GL}_n(F)$ be a faithful representation. If $\rho(H)$ is not Zariski dense in $\rho(G)$, then $H < G$ is engulfed. If $H = \rho^{-1}(C)$ for some Zariski closed subset of G , then H is closed in profinite topology on G .*

To see that this is a generalization of Selberg's lemma, note that for a finitely generated matrix group G , the identity subgroup I is the collection of all elements that give the trivial matrix under the map $g \mapsto g - I$. Since the matrix entries of this function are polynomial (indeed linear) maps, I satisfies the hypotheses of the above lemma. We now discuss the proof of this Lemma, following Long in [Lon2].

Proof. Consider $\mathrm{GL}_n(F)$ as a subset of F^{n^2} , and suppose that there is a polynomial function $f : F^{n^2} \rightarrow F$ such that f vanishes on $\rho(H)$ and does not vanish on $\rho(g)$ for some $g \in G \setminus H$. Let R be the ring of coefficients of ρ as in the proof of Selberg's lemma above. Since f is a polynomial map, f sends $\mathcal{I}^{n^2}$ into $\mathcal{I}$ for any ideal $\mathcal{I} \subset R$. It follows that f gives a well-defined map $f_{\mathcal{I}} : (R/\mathcal{I})^{n^2} \rightarrow R/\mathcal{I}$ such that $f_{\mathcal{I}}(x + \mathcal{I}^{n^2}) = f(x) + \mathcal{I}$. Since $\mathcal{J}(R) = 0$ as above and $f(g) \neq 0$, it follows that there exists a maximal ideal $\mathcal{M}$ such that $f(g) + \mathcal{M} \neq \mathcal{M}$. It follows that $f_{\mathcal{M}}(x + \mathcal{M}^{n^2}) \neq 0$. On the other hand, if $h \in H$ then

$$f_{\mathcal{M}}(\rho(h) + \mathcal{M}^{n^2}) = f(\rho(h)) + \mathcal{M} = 0 + \mathcal{M},$$

since f vanishes on $\rho(H)$. This shows that $\rho_{\mathcal{M}}(g) \notin \rho_{\mathcal{M}}(H)$, and since the image of $\rho_{\mathcal{M}}$ is a finite group, $\rho_{\mathcal{M}}^{-1}(\rho_{\mathcal{M}}(H))$ gives a finite index subgroup of G that contains H and does not contain g .

If H is the inverse image under ρ of a Zariski closed subset of $\mathrm{GL}_n(F)$, then given any $g \notin H$, there exists a polynomial function $f : \mathrm{GL}_n(F) \rightarrow F$ that vanishes on $\rho(H)$ and does not vanish on $\rho(g)$. The above argument shows that g can be separated from H , so H is closed in the profinite

topology.

□

We are now ready to present the proof of Proposition 2.4.1. As was remarked above, the main step in the proof of this proposition is the application of the Peter-Weyl theorem, background on which can be found in Chapter 4 of [Bum].

Proof. Let X be a G -compactum and $H = \text{Stab}(x)$ for some $x \in X$. This action gives rise to a homomorphism $\phi : G \rightarrow \text{Isom}(X)$. Note that $\text{Isom}(X)$ is a compact topological group with the topology given by the uniform metric. Let $\overline{G}$ be the closure of $\phi(G)$ in $\text{Isom}(X)$. $\overline{G}$ is a compact Hausdorff topological group, so $\overline{G}$ admits a left-invariant Haar measure μ .

Let V be the Hilbert space $L^2(\overline{G}, \mu)$. Given a square integrable function f , we will denote by $[f]$ the L^2 class of f . There is a natural representation $\overline{G} \rightarrow \text{GL}(V)$ coming from the unitary action of $\overline{G}$ on the Hilbert space $L^2(\overline{G}, \mu)$ by $g \cdot [f] = [f \circ g^{-1}]$. By the Peter-Weyl theorem, there exists an orthogonal decomposition $V = \bigoplus_{i=1}^{\infty} V_i$, where $g \cdot V_i = V_i$ for all $i \in \mathbb{N}$ and $g \in \overline{G}$, and $V_i \cong \mathbb{C}^{n_i}$. Note that each V_i is a finite dimensional vector space.

Let $f_x : \overline{G} \rightarrow \mathbb{R}$ be the map $g \mapsto d(x, g \cdot x)$. f_x is a continuous function on a compact space, so f_x is bounded and square integrable. Let $h \in H$ so that $h \cdot x = x$. Then $f_x(g) = d(x, g \cdot x) = d(h \cdot x, g \cdot x) = d(x, h^{-1}g \cdot x) = (f_x \circ h^{-1})(g) = (h \cdot f_x)(g)$, so h fixes f_x and hence fixes $[f_x]$. Thus $H \subset \text{Stab}([f_x])$.

Suppose that $g \notin \text{Stab}(x)$, and let $l = d(x, g \cdot x)$. We would like to show that g does not preserve the L^2 -class of f_x . To see this, note that if y is an element of $B_{l/4}(x)$, the ball of radius $l/4$ about x , then the distance from y to $g \cdot x$ is at least $3l/4$. Let $B = \{\gamma \in \overline{G} \mid d(\gamma \cdot x, x) \leq l/4\}$

$$\begin{aligned} \|g \cdot f_x - f_x\|_{L^2} &= \int_{\overline{G}} |d(g \cdot x, \gamma \cdot x) - d(x, h \cdot x)|^2 d\mu(\gamma) \geq \int_B |d(g \cdot x, \gamma \cdot x) - d(x, h \cdot x)|^2 d\mu(\gamma) \\ &\geq \int_B \left| \frac{3}{4}l - \frac{1}{4}l \right|^2 d\mu(\gamma) = \frac{1}{4}l^2 \mu(B) \end{aligned}$$

We claim that $\mu(B) \neq 0$. Since G is finitely generated, G is a countable group (though $\overline{G}$ may not be). G is dense in $\overline{G}$, so for all $g \in \overline{G}$ there exists a sequence $h_i \in G$ such that $h_i^{-1}g \rightarrow e \in \overline{G}$.

Thus $d(h_i^{-1}g \cdot x, x) \rightarrow 0$, so there exists an $h \in G$ such that $h^{-1}g$ moves x less than $l/4$, hence $g \in h \cdot B$. Thus $\cup_{h \in G} h \cdot B = \overline{G}$. $\mu(\overline{G}) = \mu(\cup_{h \in G} h \cdot B) \leq \sum_{h \in G} \mu(h \cdot B) = \sum_{h \in G} \mu(B)$, so if $\mu(B) = 0$ then $\mu(\overline{G}) = 0$. This is a contradiction, since Haar measure is non-trivial. We have therefore proved that $g \cdot [f_x] \neq [f_x]$, so $\text{Stab}([f_x]) \cap G = H$.

We now show that H is separable. Let $g \in G \setminus H$, and let π_i be the projection map from V to V_i . Since $\overline{G}$ preserves the decomposition $V = \bigoplus_{i=1}^{\infty} V_i$, we have a linear action of $\overline{G}$ on V_i by $g \cdot v = \pi_i(g \cdot \tilde{v})$, where $\tilde{v}$ is the unique element of V such that $\tilde{v}$ is orthogonal to V_j for all $j \neq i$ and $\pi_i(\tilde{v}) = v$. This action of $\overline{G}$ on V_i yields a finite dimensional representation $\bar{\rho}_i : \overline{G} \rightarrow \text{GL}(\mathbb{C}^{n_i})$. This gives us a finite dimensional representation ρ_i of the original group G given by $\rho_i = \bar{\rho}_i \circ \bar{\phi}$.

Since $g \cdot [f_x] - [f_x] \neq 0$, $\pi_i(g \cdot [f_x] - [f_x])$ must be non-trivial for some $i = i(g)$. $\pi_i(g \cdot [f_x] - [f_x]) = \pi_i(g \cdot [f_x]) - \pi_i([f_x]) = g \cdot \pi_i([f_x]) - \pi_i([f_x]) \neq 0$. Let $u = \pi_i([f_x])$. We now have that under the linear action of G on $\mathbb{C}^{n_i}$ given by the representation ρ_i , $H \subset \text{Stab}(u)$ and $g \notin \text{Stab}(u)$.

To see that stabilizers of a vector v in a finitely generated subgroup of $Gl_n(\mathbb{C})$ are separable, note that the function $\phi_i : Gl_n(\mathbb{C}) \rightarrow \mathbb{C}$ given by $g \mapsto \pi_i(g - g \cdot v)$, where $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}$ is the i -th coordinate function, is a polynomial map. The collection of points for which ϕ_i vanishes for all i is exactly $\text{Stab}(v)$, so $\text{Stab}(v)$ is closed in the Zariski topology. By Long's verbal lemma, there exists a finite index subgroup $K < \rho_{i(g)}(G)$ such that $g \notin K$ and $\text{Stab}(u) \subset K$. Since g was arbitrary, K is a separable subgroup.

□

Chapter 3

The Topology of Solenoids over 3-manifolds

As we saw in the last chapter, many important conjectures about finite-sheeted covering spaces of a 3-manifold M can be rephrased as questions about the profinite completion of the fundamental group of M . Just as much of the structure of $\pi_1(M)$ can be understood geometrically by looking at the action of $\pi_1(M)$ on the universal covering space $\widetilde{M}$ of M , many features of the profinite completion of $\pi_1(M)$ can be understood by viewing the profinite completion as the set of deck transformations of a “profinite covering space” of M given by the inverse limit of the set of all finite-sheeted covering spaces of M . We call this profinite covering space the *universal solenoid* over M .

This chapter begins with an introduction to solenoids and their Čech cohomology. We then go on to study the universal solenoid over a 3-manifold M , and show that such objects have trivial Čech cohomology over any finite coefficient module under either of the following assumptions:

1. $vb_1(M) \leq 1$, or
2. Every hyperbolic piece of the JSJ-decomposition of M is virtually fibered.

We remark that Wise’s announcement [Wis] implies that either condition 1 or 2 holds for every

3-manifold M , so it is expected that this result holds unconditionally. After proving this result, we discuss its relevance to the study of a property known as Grothendieck rigidity. Given a group G and a subgroup $H < G$, (G, H) is said to be a *Grothendieck pair* if the inclusion map $i : H \hookrightarrow G$ induces an isomorphism $\hat{i} : \hat{H} \rightarrow \hat{G}$. The existence of finitely generated Grothendieck pairs was first established by Platonov and Tavgen in [PT], and finitely presented such examples were given by Bridson and Grunewald in [BG]. A group G is said to be *Grothendieck rigid* if given any finitely generated subgroup $H < G$, (G, H) is *not* a Grothendieck pair. Many classes of groups are known to be Grothendieck rigid, including free groups, surfaces groups, and fundamental groups of geometric 3-manifolds (see [LR1]). We will show in this chapter that it follows from Wise's announcement that fundamental groups of closed, prime 3-manifolds are Grothendieck rigid as well.

3.1 Solenoids over CW complexes

Let X be a connected CW complex¹, and let $\mathcal{T} := \{\dots X_2 \rightarrow X_1 \rightarrow X_0 \cong X\}$ be a (finite or infinite) tower of distinct finite-sheeted covering spaces of X , and let $p_{i,j} : X_j \rightarrow X_i$ be the covering map given by composing the covering maps $X_j \rightarrow X_{j-1} \rightarrow \dots \rightarrow X_i$. The *solenoid over X associated to $\mathcal{T}$* , denoted $\hat{X}_{\mathcal{T}}$ is defined to be the inverse limit of this tower, i.e.

$$\hat{X}_{\mathcal{T}} := \lim_{\leftarrow} X_i = \{(x_i) \in \prod_{i=0}^{\infty} X_i \mid p_{i,j}(x_j) = x_i \text{ for all } i < j\}$$

endowed with the topology given by equipping $\prod_{i=0}^{\infty} X_i$ with the product topology and equipping $\hat{X}_{\mathcal{T}} \subset \prod_{\alpha} X_{\alpha}$ with the subspace topology. The space $\hat{X}_{\mathcal{T}}$ comes equipped with a natural continuous map $p_{\mathcal{T},0} : \hat{X}_{\mathcal{T}} \rightarrow X$, given by composing the inclusion map $\hat{X}_{\mathcal{T}} \hookrightarrow \prod_{i=0}^{\infty} X_i$ with the projection map $\prod_{i=0}^{\infty} X_i \rightarrow X_0$.

This definition is a special case of Sullivan's definition of a solenoid in [Sul], with the exception that we will not always require X to be a manifold. In this section we give the definition of a solenoid over a CW complex and describe the point-set topological structure of these spaces. The

¹The only feature of X that we will need is that X admits a universal covering space. It would therefore be sufficient to assume that X is a path-connected, locally path-connected, and semilocally simply-connected space (see [Hat1])

results in this section are simple adaptations of results in covering space theory, most of which are proved in McCord's paper *Inverse Limit Sequences for Covering Maps* [McC]. The definition of solenoid above differs slightly from McCord's definition of a "solenoidal space," however, in that the covering maps in the tower $\mathcal{T}$ are not required to be regular or connected, but as we will see below most of the point-set topological structure theorems from [McC] can be proved without any added difficulty in this setting.

If one of the spaces X_i in the tower is homeomorphic to a disjoint union $X_i^0 \sqcup X_i^1$, then $\widehat{X}_{\mathcal{T}} \cong p_{\mathcal{T}}^{-1}(X_i^0) \sqcup p_{\mathcal{T}}^{-1}(X_i^1)$. We may therefore decompose a general solenoid into a disjoint union of components that are inverse limits of towers of connected covering spaces. Given such a connected tower $\mathcal{T}$, it follows easily from the definition of the inverse limit topology that the solenoid $\widehat{X}_{\mathcal{T}}$ is connected, though in general it will not be path-connected. Given such a connected tower $\mathcal{T}$ and a basepoint $(x_n) \in \widehat{X}_{\mathcal{T}}$, one can associate a nested sequence of subgroups $\mathcal{S} = \{\dots > N_2 > N_1 > N_0 = \pi_1(X, x_0)\}$ to $\mathcal{T}$ by setting $N_i = (p_{i,0})_*(\pi_1(X, x_i))$. Conversely, given any nested sequence $\mathcal{S} = \{\dots N_3 > N_2 > N_1 > N_0 = \pi_1(X)\}$ of finite index subgroups of $\pi_1(X)$, we obtain a tower

$$\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 = X,$$

where X_i denotes the connected covering space of X corresponding to N_i . We will denote the solenoid associated to such a sequence $\mathcal{S}$ by $\widehat{X}_{\mathcal{S}}$. The following lemma describes the basic topological structure of $\widehat{X}_{\mathcal{S}}$.

Lemma 3.1.1. *Let X be a compact connected CW complex, and let $\mathcal{S}$ be an infinite nested sequence of finite-index subgroups of $\pi_1(X)$. Then $\widehat{X}_{\mathcal{S}}$ is a compact topological space, and the projection $p_{\mathcal{S},0} : \widehat{X}_{\mathcal{S}} \rightarrow X$ gives $\widehat{X}_{\mathcal{S}}$ the structure of a Cantor set bundle over X .*

Proof. Since X and all its finite sheeted covering spaces are compact, $\prod_{i=0}^{\infty} X_i$ equipped with the product topology is compact by Tychonoff's theorem. To show that $\widehat{X}$ is compact, it therefore suffices to show that $\widehat{X}$ is a closed subset of $\prod_{i=0}^{\infty} X_i$. Let $(x_j^1), (x_j^2), \dots$ be a sequence of points in $\widehat{X}_{\mathcal{S}}$. Since X_0 is compact, we can pass to a subsequence $\{i^{(1)}\} \subset \mathbb{N}$ such that $x_0^{i^{(1)}}$ converges to a point y . Let $p_{\infty,i} : \widetilde{X} \rightarrow X_i$ denote the universal covering map of X_i . Let U be a neighborhood

of y such that the map $p_{\infty,0}$ uniformly covers U , i.e. each component V of $p_{\infty,0}^{-1}(U)$ is mapped homeomorphically onto U by $\tilde{p}$. Note that given any i and any component W of $p_{i,0}^{-1}(U)$, $p_{i,0}$ also maps W homeomorphically onto U , since $W = p_{\infty,i}(V)$ for some component $V \subset \tilde{p}^{-1}(U)$ and $p_{i,0} \circ \tilde{p}_i = \tilde{p}$.

Since the degree of $p_{1,0} : X_1 \rightarrow X_0$ is finite, there are finitely many components $U_1^{(1)}, \dots, U_k^{(1)}$ of $p_{1,0}^{-1}(U)$. After passing to a further subsequence $\{i^{(2)}\} \subset \{i^{(1)}\}$, we can assume that each element $x_1^{i^{(2)}}$ lies in some component $U_l^{(1)}$. Proceeding by induction, given $U_l^{(m)} \subset X_m$ and a subsequence $x_m^{i^{(m)}}$ such that $x_m^{i^{(m)}} \subset U_l^{(m)}$, there exist finitely many components $U_1^{(m+1)}, \dots, U_k^{(m+1)}$ of $p_{m+1,m}^{-1}(U_l^{(m)})$, so we may find a subsequence $\{i^{(m+1)}\} \subset \{i^{(m)}\}$ such that $x_{m+1}^{i^{(m+1)}} \subset U_{l'}^{(m+1)}$ for some l' . The diagonal sequence $(x_j^{i^{(j)}})$ has the property that $x_j^{i^{(j)}}$ eventually lies in a single component V_j of $p_{j,0}^{-1}(U)$. Since $p_{1,0}(x_j^{i^{(j)}}) = x_0^{i^{(j)}}$ converges to y and $p_{1,0}|_{V_1}$ is a homeomorphism, it follows that $x_1^{i^{(j)}}$ converges to the unique point y_1 of $p_{1,0}^{-1}(y)$ lying in V . By the same argument, if for all $k \leq j$ $x_j^{i^{(j)}}$ converges to a point y_j such that $p_{i,j}(y_j) = y_k$, $x_{j+1}^{i^{(j+1)}}$ converges to the unique point of $p_{j+1,j}^{-1}(y_j)$ lying in V_{j+1} . It follows by induction that $x_j^{i^{(j)}}$ converges for all j to a point y_j such that for all $k \leq j$, $p_{k,j}(y_j) = y_k$. Note that the point $(y_j) \in \prod_{j=0}^{\infty} X_i$ lies in $\widehat{X}_S$. Since $x_j^{i^{(j)}}$ converges to y_j for all j , it follows by the definition of the product topology that $(x_j^{i^{(j)}})$ converges to (y_j) . This shows that $\widehat{X}_S$ is closed.

To see that $\widehat{X}_{(S)}$ has the structure of a Cantor set bundle, let $y \in X$ and U any connected neighborhood of y as above whose closure $\overline{U}$ is regularly covered. Let F denote the preimage of y under the map $p_S : \widehat{X}_S \rightarrow X$. Note that F is equal to the inverse limit of finite sets

$$\lim_{\leftarrow} p_{0,i}^{-1}(y) = \{(y_i) \in p_{0,i}^{-1}(x_0) \mid p_{i,j}(x_j) = x_i \text{ for all } i < j\}.$$

Any inverse limit of finite sets with unbounded cardinality is homeomorphic to the Cantor set, so since $\mathcal{T}$ is an infinite tower by assumption F is homeomorphic to a Cantor set. Given a point $z \in \overline{U}$, and $y_i \in p_{i0}^{-1}(y)$, let z_i denote the unique point of $p_{i0}^{-1}(z)$ that lies in the same component of $p_{i0}^{-1}(\overline{U})$ as y_i . Consider the map $\Phi : F \times \overline{U} \rightarrow p_S^{-1}(\overline{U})$ given by sending $((y_i), z)$ to the sequence (z_i) . Note that Φ is equal to the inverse limit of the homeomorphisms $\Phi_i : p_{i,0}^{-1}(y) \times \overline{U} \rightarrow p_{i,0}^{-1}(U)$. This

shows that Φ is a continuous bijection, and since both $p_S^{-1}(\bar{U})$ and $F \times \bar{U}$ are compact Hausdorff spaces Φ is a homeomorphism.

□

The fact that the Cantor set is disconnected allows many arguments from covering space theory to be applied directly to solenoids, since, as the following lemma shows, this is enough to guarantee that solenoids have the unique path-lifting property.

Lemma 3.1.2. *Let X be a CW complex, and let $p : E \rightarrow X$ be a fiber bundle with disconnected fiber over X . Then p is a fibration with the unique path lifting property.*

Proof. A theorem of Hurewicz and Huebsch shows that fiber bundles over paracompact base spaces are fibrations (see [Spa]). It follows that p has the homotopy lifting property with respect, and in particular one can lift paths (though of as homotopies of maps of points into X) from X to E . Let $\eta : [0, 1] \rightarrow X$ be a path and $\tilde{\eta}_i : [0, 1] \rightarrow E$ a pair of lifts such that $\tilde{\eta}_1(0) = \tilde{\eta}_2(0)$. There exists a neighborhood U of X such that $p^{-1}(U) \cong U \times F$ where F is a Cantor set, and $p : U \times F \rightarrow U$ is given by projection onto the first factor. Since F is totally disconnected, it follows that $\tilde{\eta}_1$ and $\tilde{\eta}_2$ must agree on $p^{-1}(U)$. Covering the path η by such neighborhoods, we obtain the desired result. □

While in many respects solenoids over X behave similarly to covering spaces of X , many arguments in covering space theory cannot be adapted to the solenoidal setting due to the fact that the solenoid $X_{\mathcal{T}}$ is not path-connected unless the tower $\mathcal{T}$ has finitely many elements. The following lemma shows, however, that solenoids contain canonical dense path components, which for many purposes is all that one needs.

Lemma 3.1.3. *Let X be a CW complex, and let $\mathcal{S} := \{\dots < N_2 < N_1 < \pi_1(X)\}$ be a nested sequence of finite-index subgroups of $\pi_1(X)$. The universal covering map $\tilde{p} : \tilde{X} \rightarrow X$ factors through a continuous map $\rho : \tilde{X} \rightarrow \hat{X}_{\mathcal{S}}$ with dense image, and $\rho(\tilde{X})$ is a path component of $\hat{X}_{\mathcal{S}}$. The map ρ is injective if and only if $\bigcap_{i=1}^{\infty} N_i = \{e\}$.*

Proof. Note that the diagonal map $\prod_{i=1}^{\infty} p_{\infty,i} : \tilde{X} \rightarrow \prod_{i=1}^{\infty} \tilde{X}/N_i$ is continuous and has image in $\hat{X}_{\mathcal{S}}$. Given any open set $U \subset \hat{X}_{\mathcal{S}}$, $U = O \cap \hat{X}_{\mathcal{S}}$ for some open set $O \subset \prod_{i=1}^{\infty} \tilde{X}/N_i$ since $\hat{X}_{\mathcal{S}}$ is

equipped with the subspace topology. By the definition of the product topology, O is a union of subsets of the form $\prod_{i=1}^n O_i \times \prod_{i=n+1}^{\infty} X/N_i$ where $O_i \subset X_i$ is an open set. It therefore follows that U contains a non-empty open set U' of the form

$$\left(\prod_{i=1}^n O_i \times \prod_{i=n+1}^{\infty} X/N_i \right) \cap \widehat{X}_S.$$

Let $(x_i) \in \widehat{X}_S$ be an element of U' . Since the map $\tilde{p}_n : \tilde{X} \rightarrow \tilde{X}/N_n$ is surjective, we can find a point $\tilde{x}_n \in \tilde{X}$ such that $\tilde{p}_n(\tilde{x}_n) \in O_n \subset \tilde{X}/N_n$. Since

$$\tilde{p}_i(\tilde{x}_n) = p_{i,n} \circ \tilde{p}_n(\tilde{x}_n) = p_{i,n}(x_n) = x_i,$$

it follows that $(\tilde{p}_i(\tilde{x}_n))$ agrees with (x_i) for the first n -entries, and therefore lies in U' .

Let C denote the path component of $\widehat{X}_S$ containing $\rho(\tilde{X})$. Let $y \in C$, and let $\eta : [0, 1] \rightarrow C$ satisfy $\eta(0) = \rho(x)$ for some $x \in \tilde{X}$ and $\eta(1) = y$. $\hat{p} \circ \eta$ gives a path in X from

$$\hat{p} \circ \eta(0) = \hat{p} \circ \rho(x) = \tilde{p}(x)$$

to $\hat{p}(y)$. Note that $\hat{p} \circ \eta$ has a lift $\tilde{\eta} : [0, 1] \rightarrow \tilde{X}$ such that $\tilde{\eta}(0) = x$. It follows by the unique path-lifting property that $\rho \circ \tilde{\eta} = \eta$, so $x_1 = \eta(1) = \rho(\tilde{\eta}(1))$. This shows that $\rho(\tilde{X}) = C$.

To verify the last claim, suppose that $\cap_{i=1}^{\infty} N_i = \{e\}$. Suppose that $\rho(x_1) = \rho(x_2)$, and let $\gamma : [0, 1] \rightarrow \tilde{X}$ be a path from x_1 to x_2 . Note that

$$\tilde{p}(x_1) = \hat{p} \circ \rho(x_1) = \hat{p} \circ \rho(x_2) = \tilde{p}(x_2),$$

so $\tilde{p}(\gamma)$ gives an element of $\pi_1(X, \tilde{p}(x_1))$. Suppose $\tilde{p} \circ \gamma$ gives a trivial element of $\pi_1(X, \tilde{p}(x_1))$. Since $\cap_{i=1}^{\infty} N_i = \{e\}$, there exists N_i such that the covering map $\tilde{p}_i : \tilde{X} \rightarrow \tilde{X}/N_i$ sends x_1 and x_2 to distinct elements. By the definition of the map ρ , however, this implies that $\rho(x_1) \neq \rho(x_2)$, which gives a contradiction. It follows that $\tilde{p} \circ \gamma$ must be a contractible loop, and since γ is a lift of this loop to $\tilde{X}$, $x_0 = \gamma(0) = \gamma(1) = x_1$. This shows that ρ is injective.

Conversely if $\cap_{i=1}^{\infty} N_i \neq \{e\}$, let g be a nontrivial element of $\cap_{i=1}^{\infty} N_i$. Given $x \in \tilde{X}$, $p_i(g \cdot x) = p_i(x)$ since $g \in N_i$, so $\rho(g \cdot x) = \rho(x)$. Since g is nontrivial $g \cdot x \neq x$, so ρ is not injective.

□

Given a connected solenoid $\hat{X}_{\mathcal{S}}$ as above, we will refer to the path component $\rho(\tilde{X}) \subset \hat{X}_{\mathcal{S}}$ as the *baseleaf* of $\hat{X}$.

We will say that the solenoid $\hat{X}_{\mathcal{S}}$ is *regular* if each N_i in the nested sequence of finite-index subgroups $\mathcal{S}$ is a normal subgroup of $\pi_1(X)$. In this setting, each covering space $p_{0,j} : X_j \rightarrow X$ is a regular covering space. It is easy to show that the diagonal action of $\pi_1(X)$ on $\prod X_i$ whose action on each factor is by deck transformations of $p_{0,j}$ preserves $\hat{X}_{\mathcal{S}}$, and that this gives $\hat{X}_{\mathcal{S}}$ the structure of a $\pi_1(X)$ -space. Moreover, one can easily show that this action extends to a fixed-point free action of $\varprojlim \pi_1(M)/N_i$ on $\hat{X}_{\mathcal{S}}$ that acts transitively on the fibers of the map $p_{\mathcal{S}} : \hat{X}_{\mathcal{S}} \rightarrow X$ (see [McC]), which yields the following lemma:

Lemma 3.1.4. *Let $\mathcal{S} = \{\dots N_3 > N_2 > N_1 > N_0 = \pi_1(X)\}$ be a nested sequence of finite-index normal subgroups of $\pi_1(X)$, and let $\hat{G}$ denote the profinite group $\varprojlim \pi_1(X)/N_i$. The regular solenoid $\hat{X}_{\mathcal{S}}$ is a principal $\hat{G}$ -bundle over X .*

Regular solenoids therefore give a natural family of compact metric spaces with $\pi_1(M)$ -actions.

3.2 Universal Solenoids

Given a compact CW complex X , there is a “largest” connected solenoid over X , which we will call the *universal solenoid* over X and will denote by $\hat{p} : \hat{X} \rightarrow X$. The universal solenoid over X is defined to be the inverse limit of the set of all finite-sheeted covering spaces of X , i.e.

$$\hat{X} := \{(x_N) \in \prod_{\substack{N \leq \pi_1(X) \\ f.i.}} \tilde{X}/N \mid p_{KH}(x_H) = x_K \text{ for all } K < H\},$$

where $\tilde{X}$ denotes the universal covering space of X and $p_{KH} : \tilde{X}/H \rightarrow \tilde{X}/K$ is the natural covering map.

Though the definition of the universal solenoid involves every finite index subgroup of $\pi_1(X)$, the universal solenoid can also be defined using a much sparser collection of subgroups. Given a nested sequence of finite-index subgroups $\mathcal{S} = \{\cdots \rightarrow N_2 \rightarrow N_1 > \pi_1(X)\}$ we will say that $\mathcal{S}$ *induces the full profinite topology on $\pi_1(X)$* if for every finite index subgroup $H < \pi_1(X)$, there exists an i such that $N_i < H$. As an example of such a tower, consider the sequence $\{\cdots < K_2 < K_1 < \pi_1(X)\}$ where K_n is equal to the intersection of all subgroups of $\pi_1(X)$ of index at most n . Since $\pi_1(X)$ is finitely generated, it follows from Hall's Theorem, Theorem 2.1.2 above, that the subgroups K_n each have finite index. Given any finite index subgroup $H < \pi_1(X)$, note that $H < K_{[\pi_1(X):H]}$, so this sequence of subgroups induces the full profinite topology on $\pi_1(M)$. Given a tower $\mathcal{T}$ of finite sheeted covering spaces of M , we will say that $\mathcal{T}$ *induces the full profinite topology on $\pi_1(X)$* if the sequence of subgroups of $\pi_1(X)$ given by each subtower of $\mathcal{T}$ consisting of connected covering spaces $\{\dots X_2 \rightarrow X_1 \rightarrow X\}$ has the property that $\{\dots \pi_1(X_2) < \pi_1(X_1) < \pi_1(X)\}$ induces the full profinite topology on $\pi_1(X)$. The following lemma shows that a linear tower of covering spaces can be used to define the universal solenoid over X , so $\widehat{X}$ fits the definition of a solenoid given in the previous section.

Lemma 3.2.1. *If $\mathcal{S} := \{\cdots < K_2 < K_1 < \pi_1(X)\}$ is a nested sequence of finite index subgroups of $\pi_1(X)$ that induces the full profinite topology on $\pi_1(X)$, then $\hat{p} : \widehat{X} \rightarrow X$ is isomorphic as a bundle to $p_{\mathcal{S}} : \widehat{X}_{\mathcal{S}} \rightarrow X$.*

Proof. Consider the map

$$\Phi : \prod_{\substack{N < \pi_1(X) \\ \text{f.i.}}} \widetilde{X}/N \rightarrow \prod_{i=1}^{\infty} \widetilde{X}/K_i$$

given by projecting onto the factors indexed by K_i . This map is continuous, and clearly maps $\widehat{X} \rightarrow \widehat{X}_{\mathcal{S}}$. If H is any finite index subgroup of $\pi_1(X)$, then since $\mathcal{S}$ induces the full profinite topology on $\pi_1(M)$, $K_n < H$ for some n . It follows that given any point in $(x_N) \in \widehat{X}$, the entry x_H is determined by the entry x_{K_n} . It follows that quotient map $\Phi|_{\widehat{X}} : \widehat{X} \rightarrow \widehat{X}_{\mathcal{S}}$ is bijective. Since $\widehat{X}$ and $\widehat{X}_{\mathcal{S}}$ are compact Hausdorff spaces, this map is a homeomorphism. □

If $\{\dots K_2 < K_1 < \pi_1(X)\}$ induces the full profinite topology on X , then given any finite-sheeted covering space $p : X' \rightarrow X$, there exists an i such that $K_i < p_*(\pi_1(X'))$. It follows that the cover $\tilde{X}/K_i \rightarrow X$ factors through p . Conversely, if a tower of covering spaces $\{\dots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0\}$ has the property that given any finite-sheeted cover $p : X' \rightarrow X$, there exists an i such that $p_{i,0} : X_i \rightarrow X$ factors through p , then the groups $(p_{i,0})_*(\pi_1(X_i))$ must induce the full profinite topology on $\pi_1(X)$. This yields the following characterization of the universal solenoid:

Lemma 3.2.2. *Let $p : \tilde{S} \rightarrow S$ be a connected solenoid that factors through every finite sheeted covering space of S . Then $\hat{S}$ is isomorphic to the universal hyperbolic solenoid.*

As was discussed in the previous chapter, the profinite group $\varprojlim \pi_1(M)/K_n$ is isomorphic to the profinite completion $\widehat{\pi_1(M)}$ of $\pi_1(M)$, so Lemma 3.2.1 together with Lemma 3.1.4 yield the following:

Lemma 3.2.3. *Let X be a compact CW complex, and let Γ denote $\pi_1(M)$. $\hat{X}$ is a principal $\hat{\Gamma}$ -bundle over X .*

The following lemma justifies the use of the term “universal” to describe $\hat{X}$.

Lemma 3.2.4. *Let X be a compact CW complex, and let $\mathcal{S}$ be any nested sequence of finite-index subgroup of $\pi_1(X)$. Then there exists a map $\hat{p}_{\mathcal{S}} : \hat{X} \rightarrow \hat{X}_{\mathcal{S}}$ yielding the following commutative diagram:*

$$\begin{array}{ccc} \hat{X} & & \\ \downarrow \hat{p} & \searrow \hat{p}_{\mathcal{S}} & \\ & \hat{X}_{\mathcal{S}} & \\ & \swarrow p_{\mathcal{S}} & \\ & X & \end{array}$$

Proof. Let $\mathcal{S} = \{\dots > N_2 > N_1 > N_0 = \pi_1(X)\}$, and consider the map

$$\Phi : \prod_{\substack{N < \pi_1(X) \\ f.i.}} \tilde{X}/N \rightarrow \prod_{i=1}^{\infty} \tilde{X}/N_i.$$

This map induces a continuous map from $\widehat{X} \rightarrow \widehat{X}_{\mathcal{S}}$. To check the surjectivity of this map, let (x_i) be a point in $\widehat{X}_{\mathcal{S}}$. For each n , let $\tilde{p}_n : \tilde{X} \rightarrow \tilde{X}/N_n$ be the natural covering map and let $\tilde{x}_n$ be an arbitrary point in $\tilde{p}_n^{-1}(x_n)$. Let $\rho : \tilde{X} \rightarrow \widehat{X}$ be the map given by Lemma 3.1.3. Since the first n entries of $\Phi(\rho(\tilde{x}_n))$ agrees with (x_i) , it follows that the sequence $\Phi(\rho(\tilde{x}_n))$ converges to (x_i) . Since the image of Φ is dense and the domain of Φ is compact, it follows that Φ is surjective. $\square$

Recall that two spaces X_1 and X_2 are said to be *commensurable* if X_1 and X_2 share a finite sheeted covering space. The following lemma follows immediately from the definition of the universal solenoid.

Lemma 3.2.5. *Let X_1 and X_2 be commensurable spaces. Then $\widehat{X}_1$ is homeomorphic to $\widehat{X}_2$.*

We remark that it is not known whether or not the converse to this statement holds when X is a 3-manifold.

3.3 The Algebraic Topology of Solenoids

In this section we discuss the algebraic topology of solenoids. The following lemma shows that the homotopy groups of the baseleaf of a solenoid have simple interpretations.

Lemma 3.3.1. *Let X be a connected CW complex, and let $\mathcal{S} := \cdots < N_2 < N_1 < \pi_1(X)$ be a nested sequence of finite index subgroups of $\pi_1(X)$. Let C be the baseleaf of $\widehat{X}_{\mathcal{S}}$. Then $\pi_1(C) \cong \cap_{i=1}^{\infty} N_i$, and $\pi_k(C) \cong \pi_k(X)$ for $k > 1$.*

Proof. Let $x \in X$, let $F := p_{\mathcal{S}}^{-1}(x)$, and let $\hat{x} \in C \cap F$. Since $p : \widehat{X}_{\mathcal{S}} \rightarrow X$ has the homotopy lifting property, it follows (see e.g. [Hat1] Theorem 4.41) that there is a long exact sequence

$$\cdots \rightarrow \pi_k(F, \hat{x}) \rightarrow \pi_k(E, \hat{x}) \rightarrow \pi_k(X, x) \rightarrow \pi_{k-1}(F, \hat{x}) \rightarrow \cdots \rightarrow \pi_0(E, \hat{x}) \rightarrow 0$$

This induces isomorphisms $\pi_k(C, \hat{x}) \cong \pi_k(E, \hat{x}) \cong \pi_k(X, x)$ for all $k > 1$, since $\pi_i(F, \hat{x})$ is trivial for

$i > 0$. To compute $\pi_1(E, \hat{x})$ note that the exact sequence

$$0 \rightarrow \pi_1(E, \hat{x}) \rightarrow \pi_1(X, x) \rightarrow \pi_0(F, \hat{x}) \rightarrow \pi_0(E, \hat{x}) \rightarrow 0$$

(where the π_0 terms are sets and not groups) has the interpretation that $\pi_1(E, \hat{x})$ corresponds to the elements of $\pi_1(X, x)$ that lift to closed loops based at $\hat{x}$. Let X_i be the covering space of X corresponding to N_i , let $\hat{p}_i : \hat{X} \rightarrow X_i$ be the natural map to X_i and let $x_i = \hat{p}_i(\hat{x})$. It is easy to see from the definition of $\hat{X}_{\mathcal{S}}$ that the loops in X based at x that lift to a loop in $\hat{X}_{\mathcal{S}}$ based at $\hat{x}$ are exactly those that lift to a closed loop in X_i based at x_i for all i . Since these loops yield elements of $\pi_1(X, x)$ that are contained in N_i , the claim follows. $\square$

We point out the following special case of the above lemma:

Lemma 3.3.2. *Let X be a compact aspherical CW complex, and let $x \in X$. Each component of $\hat{X}$ has trivial homotopy groups if and only if $\pi_1(X)$ is residually finite.*

Proof. Since $\hat{X}$ is a regular solenoid and regular solenoids are principal bundles, each component of $\hat{X}$ is homeomorphic. Since $\pi_1(X)$ is residually finite, the collection $\mathcal{S} := \{\dots < K_2 < K_1 < \pi_1(X)\}$, where K_n is the intersection of all finite-index subgroups of index at most n , has trivial intersection. Since $\hat{X} \cong \hat{X}_{\mathcal{S}}$ by Lemma 3.2.1, the result follows from Lemma 3.3.1. $\square$

Since most of the spaces X we will be interested in are aspherical with residually finite fundamental group, Lemma 3.3.1 shows that homotopy groups will not provide us with interesting invariants of universal solenoids. As we will see in what follows, however, cohomology groups do provide useful algebraic invariants of solenoids in this setting. Unlike in the case of CW complexes, however, the various constructions of standard cohomology theory for CW complexes do not give equivalent theories on solenoids. To see this, note that by Lemma 3.3.2, the path components of universal solenoid over an aspherical manifold X with residually finite fundamental group all have trivial homotopy groups. By the Hurewicz theorem, it follows that the simplicial homology and

cohomology groups of $\widehat{X}$ vanish above dimension 0. The k -th Čech cohomology groups of $\widehat{X}$, on the other hand, can be easily show to be non-vanishing whenever $\mathbb{Z} < H^k(X, \mathbb{Z})$. This is due to the following “continuity” property of Čech cohomology (see [Bre] for a proof):

Lemma 3.3.3. *Let $\{X_\alpha\}$, be an inverse system of CW complexes, with maps $p_{\alpha\beta} : X_\beta \rightarrow X_\alpha$ for $\alpha < \beta$. Let R_α be a $\pi_1(X_\alpha)$ -module for all α , and for $\beta > \alpha$ let $\psi_{\alpha\beta} : R_\alpha \rightarrow R_\beta$ be homomorphisms making $\{R_\alpha\}$ into a directed system of abelian groups. Suppose the module structures on $\{R_\alpha\}$ are compatible, i.e. given $\alpha < \beta$, $a \in R_\alpha$ and $g \in \pi_1(X_\beta)$, $(p_{\alpha\beta})_*(g) \cdot a = g \cdot \psi_{\alpha\beta}(a)$. Then $\check{H}(\lim_{\leftarrow} X_\alpha, \lim_{\rightarrow} R_\alpha) \cong \lim_{\rightarrow} \check{H}(X_\alpha, R_\alpha)$.*

We will therefore use Čech cohomology exclusively in what follows, and for simplicity of notation we will denote the Čech cohomology functor by “ H ” rather than “ $\check{H}$.”

Note that when R_α is a fixed abelian group R with a trivial $\pi_1(X_\alpha)$ action for all α , the coefficient compatibility condition in the above lemma is automatically satisfied. More generally, given an inverse system of spaces $(\{X_\alpha\}, p_{\alpha\beta})$ with terminal space X_0 and R a $\pi_1(X_0)$ -module, there is a unique $\pi_1(X_\alpha)$ -module $p_{0,\alpha}^*(R)$ with underlying abelian group R that satisfies the coefficient compatibility conditions. This module is simply R itself equipped with the $\pi_1(X_\alpha)$ -action given by $g \cdot x = (p_{0,\alpha})_*(g) \cdot x$. For simplicity of notation, we will denote $p_{0,\alpha}^*(R)$ simply by R . This allows us to consider R as a coefficient system on each of the spaces X_α simultaneously. Using this convention, Lemma 3.3.3 immediately yields the following:

Lemma 3.3.4. *Let $\mathcal{T} = \{\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \cong X\}$ be a tower of covering spaces of X . Given any $\pi_1(X)$ -module R ,*

$$H^k(\widehat{X}_{\mathcal{T}}, R) \cong \lim_{\rightarrow} H^k(X_i, R).$$

Recall that the direct limit A_∞ of a directed system of abelian groups (A_i, ϕ_{ij}) can be defined by

$$A_\infty := \left(\bigoplus_{i=1}^{\infty} A_i \right) / \sim,$$

where $\sim$ is the equivalence relation generated by declaring $x \sim y$ for $x \in A_i$ and $y \in A_j$ if there exists $k > i, j$ such that $\phi_{ik}(x) = \phi_{jk}(y)$. There are natural maps $\phi_{i\infty} : A_i \rightarrow A_\infty$ given composing

the inclusion $A_i \hookrightarrow \bigoplus_i^\infty A_i$ with the quotient $\bigoplus_i^\infty A_i \rightarrow A_\infty$, and one can easily check that

$$\ker(\phi_{i\infty}) = \{x \in A_i \mid \exists k > i \text{ s.t. } \phi_{ik}(x) = 0\}.$$

It therefore follows that kernel of the map $\hat{p}_{\mathcal{T}} : X \rightarrow \hat{X}_{\mathcal{T}}$ is the cohomology classes on X that pull back trivially to some covering space in the tower $\mathcal{T}$. We will call such cohomology classes $\mathcal{T}$ -trivial, and when $\mathcal{T}$ is the tower of all finite sheeted covering spaces of X we will call such classes *virtually trivial*. The above discussion gives the following lemma:

Lemma 3.3.5. *Let R be a $\pi_1(X)$ -module. The group $H^k(\hat{X}, R) = 0$ if and only if given any finite-sheeted covering space X' of X , every element of $H^k(X', R)$ is virtually trivial.*

We also remark for future reference that when each of the spaces X_α in Lemma 3.3.3 is homeomorphic to a fixed space X and the maps $p_{\alpha\beta}$ are all identity maps, then Lemma 3.3.3 reduces to the following:

Lemma 3.3.6. *Let X be a space and let $(R_\alpha, \psi_{\beta\alpha})$ be a directed system of $\pi_1(X)$ -modules. Then $\varinjlim H^k(X, R_\alpha) \cong H^k(X, \varinjlim R_\alpha)$.*

3.4 Cohomology over Finite Coefficient Modules and Goodness in the Sense of Serre

A simple argument involving the transfer homomorphism shows that an infinite order cohomology class defined on X cannot be virtually trivial. For this reason the cohomology groups of the universal solenoid $\hat{X}$ over a space X with coefficients in a finite $\pi_1(X)$ -module are often much simpler than the cohomology of $\hat{X}$ with coefficients in $\mathbb{Z}$. This is especially true in dimension 1 as the following lemma shows.

Lemma 3.4.1. *Let X be a space, and let R be any finite $\pi_1(X)$ -module. $H^1(\hat{X}, R) = 0$.*

Proof. The action of $\pi_1(X)$ on R yields a homomorphism $\rho : \pi_1(X) \rightarrow \text{Aut}(R)$. Since R is finite, $\text{Aut}(R)$ is also finite thus $\ker(\rho)$ is a finite index subgroup. After passing to a finite sheeted

covering space of X we may therefore assume that R is equipped with the trivial $\pi_1(X)$ -action. In this setting, there is a natural homomorphism $\Phi : H^1(X, R) \rightarrow \text{Hom}(\pi_1(X), R)$. Given an element $\omega \in H^1(X, R)$, the kernel of $\Phi(\omega)$ corresponds to a finite sheeted covering space of $p : X' \rightarrow X$. By naturality of Φ ,

$$\Phi(p^*(\omega)) = p^*(\Phi(\omega)) = \Phi(\omega)|_{\pi_1(X')} = \Phi(\omega)|_{\ker(\Phi(\omega))} = 0.$$

Since every element of $H^1(X, R)$ is virtually trivial, it follows by Lemma 3.3.5 that $H^1(\widehat{X}, R) = 0$. $\square$

When X is an n -manifold, $H^n(\widehat{X}, R)$ can also be shown to be trivial under a simple hypothesis about the orders of the finite quotients of $\pi_1(X)$.

Lemma 3.4.2. *Let X be an n -manifold. Suppose that for all $k > 0$, there exists a finite index subgroup $N < \pi_1(X)$ such that k divides $[\pi_1(X) : N]$. Then given any finite $\pi_1(X)$ -module R , $H^n(\widehat{X}, R) = 0$.*

Proof. As in the proof of the previous lemma, we may assume after passing to a finite sheeted covering space that R is a trivial module. Since X is an n -manifold, for any finite sheeted covering space $p : X' \rightarrow X$ of X , $H^n(X', R) \cong R$, and given any finite sheeted covering space $\rho : X'' \rightarrow X'$ the map $\rho^* : H^n(X', R) \rightarrow H^n(X'', R)$ is given by multiplication by $\deg(\rho)$. Suppose that $\pi_1(X)$ has finite index subgroups of arbitrarily divisible indices. Given $\omega \in H^n(X', R)$, there exists $N < \pi_1(X)$ such that $|R| \cdot [\pi_1(X) : \pi_1(X')]$ divides $[\pi_1(X), N]$. Let $N' = \pi_1(X') \cap N$, and let $\rho : X'' \rightarrow X'$ be the covering space of X' corresponding to $N' < \pi_1(X')$. Since $|R| \cdot [\pi_1(X) : \pi_1(X')]$ divides $[\pi_1(X), N]$ and $[\pi_1(X), N]$ divides $[\pi_1(X), N'] = [\pi_1(X) : \pi_1(X')] \cdot [\pi_1(X') : N']$, it follows that $|R|$ divides $[\pi_1(X') : N'] = \deg(\rho)$. This shows that induces the map $\rho^* : H^n(X', R) \rightarrow H^n(X'', R)$ is trivial, so any element of $H^n(X', R)$ is virtually trivial. $\square$

The virtual triviality of cohomology classes on a space X with coefficient in a finite module R can be used to show that $\pi_1(X)$ has a property known as *goodness in the sense of Serre*, or simply *goodness* for short.

Definition 3.4.1. *A discrete group G is good if given any finite G -module R and $k \geq 0$, the natural map $\eta : G \rightarrow \widehat{G}$ induces an isomorphism $\eta^* : H^k(\widehat{G}, R) \rightarrow H^k(G, R)$.*

The following lemma from [Ser] gives an alternative characterization of goodness:

Lemma 3.4.3. *A group G is good if and only if for any finite index subgroup $N < G$ and any cohomology class $\omega \in H^k(N, R)$, there exists a finite index subgroup $N' < N$ such that ω is in the kernel of the restriction map $H^k(N, R) \rightarrow H^k(N', R)$.*

This formulation of goodness has the following immediate consequence:

Lemma 3.4.4. *Let G be a group. If G has a finite index good subgroup then G is good.*

Given Lemma 3.3.5 from the previous section, this result has the following interpretation in terms of universal solenoids:

Lemma 3.4.5. *Let G be a group, and let X be a $K(G, 1)$ -space. G is good in the sense of Serre if and only if given any G -module R and $k > 0$, $H^k(\widehat{X}, R)$ is trivial.*

A number of low-dimensional groups are well-known to be good.

Lemma 3.4.6. *Finite groups, free groups, surface groups and fundamental groups of 2-orbifolds are good.*

Proof. Since the trivial subgroup has finite index in any finite group, goodness of finite groups is clear. Goodness of free groups follows immediately from Lemma 3.4.1, since the higher cohomology groups of free groups vanish. The goodness of surface groups follows from Lemmas 3.4.2, since surface groups have non-trivial maps onto $\mathbb{Z}$ and hence have finite-index subgroups of any index. It is a consequence of Selberg's lemma that the fundamental group of a 2-orbifold has a finite-index subgroup isomorphic to the fundamental group of a surface, so the result for orbifolds follows as well. □

The following lemma from [Ser] gives a useful extension theorem for good groups.

Lemma 3.4.7. *Let F and N be finitely generated discrete groups, and assume that $H^k(N, R)$ is finite for any finite N -module R . Let G be a group fitting into the exact sequence*

$$1 \rightarrow N \rightarrow G \rightarrow F \rightarrow 1.$$

If N and F are good, then G is good.

This result can be used to show that the fundamental groups of many 3-manifolds are good:

Lemma 3.4.8. *Let M be the fundamental group of a 3-manifold that is either Seifert fibered or fibered over the circle. Then $\pi_1(M)$ is good.*

Proof. If M is Seifert fibered, then there is an exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow \pi_1^{\mathcal{O}}(B) \rightarrow 1,$$

where B is the base orbifold of the Seifert fibration of M and $\pi_1^{\mathcal{O}}(B)$ denotes its orbifold fundamental group. If M fibers over the circle, then the long exact sequence of homotopy groups for the fibration yields an exact sequence

$$1 \rightarrow \pi_1(S) \rightarrow \pi_1(M) \rightarrow \mathbb{Z} \rightarrow 1.$$

In both cases, goodness of $\pi_1(M)$ then follows from Lemma 3.4.6 and Lemma 3.4.7.

□

As we will see in the next section, this result can be extended to show that the fundamental group of any 3-manifold is good, provided that the virtually fibered conjecture holds for Haken hyperbolic 3-manifolds.

3.5 Goodness of 3-manifold groups

In this section we study goodness for fundamental groups of general 3-manifolds. Goodness was proved for fundamental groups of graph manifolds by Wilton and Zalesski in [WZ], and was proved to

hold for 3-dimensional Poincaré duality groups with trivial virtual first Betti number independently by Weigel [Weig] and Kochloukova-Zalesski [KZ]. We observe in this section that, provided that fundamental groups of finite volume hyperbolic 3-manifolds are virtually fibered, these results can be combined to show that the fundamental groups of all closed 3-manifolds are good. We will also show that Weigel and Kochloukova-Zalesski's result can be extended to 3-manifolds M with $vb_1(M) \leq 1$. This yields the following:

Proposition 3.5.1. *Let M be a closed 3-manifold such that either*

- *Every hyperbolic piece of the JSJ decomposition of every prime factor M is virtually fibered,*
- or*
- *$vb_1(M) \leq 1$.*

Then $\pi_1(M)$ is good.

We remark that this has the following corollary:

Corollary 3.5.1. *If finite-volume hyperbolic 3-manifolds M with $vb_1(M) > 1$ are virtually fibered, then fundamental groups of 3-manifolds are good.*

We begin by showing that the top-dimensional cohomology of universal solenoids over 3-manifolds always vanishes.

Lemma 3.5.1. *Given any 3-manifold M with infinite fundamental group and any finite $\pi_1(M)$ -module R , $H^3(\widehat{M}, R)$ is trivial.*

Proof. If $\pi_1(M)$ virtually surjects onto $\mathbb{Z}$, then this result is clear by Lemma 3.4.2. It is well known that toroidal manifolds have virtually infinite first Betti number, so if $\pi_1(M)$ does not virtually surject onto $\mathbb{Z}$ then M is hyperbolic, and $\pi_1(M)$ is isomorphic to a finitely generated subgroup of $SL(2, \mathbb{C})$. The main theorem of [Lub] gives that a finitely generated subgroup of $GL(n, \mathbb{C})$ has a subgroup of index divisible by n for any n , so Lemma 3.4.2 can be applied in the hyperbolic setting as well. □

The following lemma shows that to prove goodness it suffices to check that cohomology classes with coefficients in the group $\mathbb{Z}/n$ are virtually trivial.

Lemma 3.5.2. *Let M be an aspherical 3-manifold. $\pi_1(M)$ is good if and only if $H^2(\widehat{M}, \mathbb{Z}/n) = 0$ for every n .*

Proof. Given any finite coefficient module R , by passing to a finite sheeted covering space of M if necessary we may assume that the action of $\pi_1(M)$ on R is trivial. Since R is a finite abelian group, $R \cong \bigoplus_{i=1}^k \mathbb{Z}/n_i$ by the fundamental theorem of modules over a PID. Given any finite sheeted covering space M' , $H^k(M', R) \cong H^k(M', \bigoplus_{i=1}^k \mathbb{Z}/n_i) \cong \bigoplus_{i=1}^k H^k(M', \mathbb{Z}/n_i)$. It therefore follows that elements of $H^k(M', R)$ are virtually trivial if and only if elements of $H^k(M', \mathbb{Z}/n_i)$ are virtually trivial. By Lemmas 3.4.1, 3.5.1, the only dimension in which virtual triviality is not guaranteed is dimension 2. □

The remainder of the proof of Proposition 3.5.1 in the case of 3-manifolds with virtually positive first Betti number follows from the decomposition theory of 3-manifolds together with results of Grunewald, Jaikin-Zapirain and Zalesskii in [GJZ], and Wilton Zalesskii in [WZ]. Recall from the previous chapter that the profinite topology on Γ is the smallest topology on Γ in which finite index subgroups of Γ are closed. Given $H < \Gamma$, a finite index subgroup of Γ intersects H in a finite index subgroup of H , so open sets in the subspace topology on H are open in the full profinite topology in H . If these two topologies agree, Γ is said to *induce the full profinite topology on H* . We will say that a subgroup $H < \Gamma$ is *efficiently embedded* in Γ if H is closed in Γ and Γ induces the full profinite topology on H .

Given groups A and B together with C a subgroup of A , and $f : C \hookrightarrow B$ an injective homomorphism, recall that an *amalgamated free product* $A *_C B$ is defined to be the quotient of the free product $A * B$ by the group normally generated by the elements $\{f(c)c^{-1} \mid c \in C\}$. Given a single group A , a subgroup C , and an injective homomorphism $f : C \hookrightarrow A$, the *HNN-extension of A by f* is defined to be the quotient of the group $A * \mathbb{Z}$ by the group normally generated by $\{tct^{-1}f(c)^{-1} \mid c \in C\}$, where t is a generator for the $\mathbb{Z}$ factor of $A * \mathbb{Z}$. Following [GJZ], we say that

an amalgamated free product decomposition $G \cong A *_C B$ is *efficient* if A, B and C are efficiently embedded in G , and that an *HNN* decomposition $G \cong A *_f C$ is efficient if C and $f(C)$ are efficiently embedded in G . The following lemma from [GJZ] shows that a group is good if it has an efficient decomposition into good groups.

Lemma 3.5.3. *An efficient amalgamated product or an efficient HNN extension of good groups is good.*

We now show how standard geometric decomposition theorems for 3-manifolds can be used to produce efficient decompositions of $\pi_1(M)$.

Lemma 3.5.4. *The fundamental group of a closed 3-manifold is good if the fundamental group of each of the prime factors of its orientable double cover are good.*

Proof. Let M be a closed 3-manifold. By Lemma 3.4.4, the orientable double cover of M has good fundamental group if and only if M has good fundamental group, so we may assume without loss of generality that M is orientable. By the prime decomposition theorem for 3-manifolds, M may be decomposed as a connected sum $M \cong M_1 \# M_2 \# \dots \# M_k$, where M_i is a prime 3-manifold. This yields a free product decomposition $\pi_1(M) \cong \ast_{i=1}^k \pi_1(M_i)$. Note that a free product of groups induces the full profinite topology on its free factors, since given any normal finite index normal subgroup $N < \pi_1(M_i)$, the quotient map $\pi_1(M_i) \rightarrow \pi_1(M_i)/N$ factors through the inclusion of $\pi_1(M_i)$ into $\ast_{i=1}^k \pi_1(M_i)$. It therefore suffices to check that fundamental groups of prime 3-manifolds are good. With the exception of $S^2 \times S^1$, all such manifolds are either aspherical or have finite fundamental group. Since $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$ is a good group and finite groups are good, $\pi_1(M)$ is good if $\pi_1(M_i)$ is good for each aspherical prime summand of M . □

Recall that a closed, orientable, irreducible 3-manifold M admits a canonical decomposition along tori, called the JSJ decomposition of M , into pieces $M_1, M_2, \dots, M_n$, each of which is either Seifert fibered or atoroidal. The following theorem of Wilton and Zalesskii from [WZ] reduces the question of whether the fundamental group of an arbitrary 3-manifold is good to the question of whether these JSJ pieces have good fundamental groups.

Theorem 3.5.1. *Let M be a closed, irreducible, orientable 3-manifold and suppose that the fundamental groups of the pieces of the JSJ decomposition of M are good. Then the fundamental group of M is good.*

Combining this lemma with Lemma 3.4.8 from the previous section, we obtain lemma 3.5.1 for any manifold with non-trivial JSJ decomposition.

Lemma 3.5.5. *If M is a 3-manifold with non-trivial JSJ decomposition and the hyperbolic pieces of the JSJ of M are virtually fibered, then $\pi_1(M)$ is good.*

Proof. By Thurston's hyperbolization theorem for Haken manifolds, the pieces of the JSJ decomposition of M are either hyperbolic or Seifert fibered. Since Seifert fibered manifolds and virtually fibered manifolds have good fundamental group by Lemma 3.4.8, the result follows from Lemma 3.5.1 □

In order to prove Proposition 3.5.1, it therefore remains to deal with 3-manifolds with $vb_1(M) < 2$. We remark that a result proved independently by Weigel [Weig] and Kochloukova-Zalesskii [KZ] shows that an arbitrary Poincaré duality group Γ of dimension 3 with trivial virtual first Betti number is good. This proves the desired result when $vb_1(M) = 0$. The following lemma strengthens Weigel and Kochloukova-Zalesskii's result in the case that Γ arises as the fundamental group of a closed 3-manifold.

Proposition 3.5.2. *Let M be a 3-manifold with $vb_1(M) < 2$. Then $\pi_1(M)$ is good.*

This proposition follows easily from the following lemma.

Lemma 3.5.6. *Let M be a 3-manifold with $vb_1(M) < 2$. Then any element $\omega \in H^2(M, \mathbb{Z}/n)$ is virtually trivial.*

Proof. We claim that it is enough to show that there exists a finite-sheeted covering space $\pi : M' \rightarrow M$ such that $\pi_*(H_2(M', \mathbb{Z})) \subset nH_2(M, \mathbb{Z})$. This is a simple consequence of the universal coefficients theorem, which gives the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ext}(H_1(M), \mathbb{Z}/n) & \xrightarrow{\phi} & H^2(M, \mathbb{Z}/n) & \xrightarrow{\psi} & \text{Hom}(H_2(M), \mathbb{Z}/n) \longrightarrow 0 \\
& & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* \\
0 & \longrightarrow & \text{Ext}(H_1(M'), \mathbb{Z}/n) & \xrightarrow{\phi} & H^2(M', \mathbb{Z}/n) & \xrightarrow{\psi} & \text{Hom}(H_2(M'), \mathbb{Z}/n) \longrightarrow 0.
\end{array}$$

If $\pi_*(H_2(M', \mathbb{Z})) \subset nH_2(M, \mathbb{Z})$, then clearly $\pi^* : \text{Hom}(H_1(M), \mathbb{Z}/n) \rightarrow \text{Hom}(H_1(M'), \mathbb{Z}/n)$ is the trivial map. It follows that $\pi^*(\omega) = \phi(\eta)$ for some $\eta \in \text{Ext}(H_1(M'), \mathbb{Z}/n)$. It follows from basic properties of the Ext functor that $\text{Ext}(H_1(M'), \mathbb{Z}/n) \cong \text{Ext}(\text{Tor}(H_1(M')), \mathbb{Z}/n)$. By passing to the further cover of M' given by the kernel of the natural map $\pi_1(M') \rightarrow \text{Tor}(H_1(M'))$, we obtain a covering space M' to which η pulls back trivially. It follows that ω pulls back trivially to M' as well.

It therefore remains to show that there exists a finite-sheeted covering space $\pi : M' \rightarrow M$ such that $\pi_*(H_2(M', \mathbb{Z})) \subset nH_2(M, \mathbb{Z})$. If $vb_1(M) = 0$, then $H_2(M, \mathbb{Z}) = 0$ so this inclusion holds trivially for any covering space.

Suppose $vb_1(M) = 1$, and let Σ be a surface such that $[\Sigma]$ generates $H_2(M)$. By a standard application of Poincaré duality and Dehn's lemma, we may assume that Σ is an embedded π_1 -injective surface. Given a finite-sheeted covering space $\pi : M' \rightarrow M$, let Σ' be a connected component of the preimage of Σ . Σ' is an embedded non-separating surface, so there exists an embedded loop C in M intersecting Σ' exactly once. The intersection pairing between the homology class $[C]$ and $[\Sigma'] = \pm 1$, so it follows that $[\Sigma']$ is a primitive homology class. Since $H_2(M') \cong \mathbb{Z}$, $[\Sigma']$ generates $H_2(M')$. Let d be the degree of the covering map $\pi|_{\Sigma'} : \Sigma' \rightarrow \Sigma$. Since $\pi_*([\Sigma']) = d[\Sigma]$, it follows that $p_*(H_2(M')) \subset dH_2(M)$.

It therefore suffices to find a finite-sheeted covering space $\pi : M' \rightarrow M$ such that n divides the degree of $\pi|_{\Sigma'}$ for some (and hence every) component Σ' of $\pi^{-1}(\Sigma)$. Let x be a point on Σ , and let γ be an oriented closed curve based at x giving a non-trivial element of $\pi_1(\Sigma, x)$. Since Σ is π_1 -injective, γ gives a non-trivial element $[\gamma] \in \pi_1(M, x)$. It follows from work of Hamilton [Ham] that abelian subgroups of 3-manifold groups are closed in the profinite topology, so there exists a

subgroup $K < \pi_1(M, x)$ such that $K \cap \langle \gamma \rangle = \langle \gamma^n \rangle$. Let $\pi_K : (M_K, \tilde{x}) \rightarrow (M, x)$ be the finite-sheeted covering space of M corresponding to K , and let $\tilde{\gamma}$ and Σ' be the components of $\pi_K^{-1}(\gamma)$ and $\pi_K^{-1}(\Sigma)$ respectively that pass through $\tilde{x}$. Since the loop $\tilde{\gamma}$ generates $K \cap \langle \gamma \rangle = \langle \gamma^n \rangle$, it follows that $\pi_K|_{\tilde{\gamma}}$ is an n -fold covering map. Since $\tilde{\gamma} \subset \Sigma'$, it follows that n must divide the degree of $\pi_K|_{\Sigma'}$. $\square$

We are now ready to show Proposition 3.5.2.

Proof of Proposition 3.5.2. Given any finite sheeted covering space M' of M , M' satisfies the hypothesis of Lemma 3.5.6. It follows that any element of $H^2(M', \mathbb{Z}/n)$ is virtually trivial. By Lemma 3.3.5, this shows that $H^2(\widehat{M}, \mathbb{Z}/n)$ is trivial, which together with Lemma 3.5.2 implies Proposition 3.5.2. $\square$

3.6 The Cohomology of non-aspherical 3-dimensional Solenoids over Finite Coefficient Modules

When M is an aspherical 3-manifold, the cohomology of M is identical to the cohomology of $\pi_1(M)$, so the goodness of $\pi_1(M)$ allows us to trivialize any cohomology class on M with finite coefficients by passing to a finite sheeted cover. When M is not aspherical, however, the following lemma shows that there always exist cohomology classes on M valued in finite coefficient modules that are not virtually trivial.

Lemma 3.6.1. *Let M be a 3-manifold such that M is not aspherical. Then there exists p such that $H^2(\widehat{M}, \mathbb{Z}/p)$ is not trivial.*

Given this fact, the proof of Lemma 3.6.1 follows from well known arguments in 3-manifold topology:

Proof. If $\pi_2(M)$ is trivial and M is not aspherical, then M is finitely covered by the 3-sphere S^3 by the classification of 3-manifolds. In this case $\widehat{M} \cong S^3$, so $H^2(\widehat{M}, \mathbb{Z}/p) \neq 0$.

If $\pi_2(M)$ is non-trivial, then by the sphere theorem of Papakyriakopoulos (see [Hem]), there is an embedding $f : S^2 \rightarrow M$ such that $f(S^2)$ is essential in $\pi_2(M)$. Suppose that $f(S^2)$ is non-separating in M . It follows that $f_*([S^2])$ gives a non-trivial element of $H_2(M)$. By naturality of the universal coefficients exact sequence, the non-triviality of the map $f^* : \text{Hom}(H_2(M), \mathbb{Z}/p) \rightarrow \text{Hom}(H_2(S^2), \mathbb{Z}/p)$ implies that the map $f^* : H^2(M, \mathbb{Z}/p) \rightarrow H^2(S^2, \mathbb{Z}/p)$ is non-trivial for every p . Let $\omega \in H^2(M, \mathbb{Z}/p)$ be an element such that $f^*(\omega) \neq 0$. Given any finite-sheeted covering space $\rho : M' \rightarrow M$, the map f lifts to a map $\tilde{f} : S^2 \rightarrow M'$. Since $\tilde{f}^* \circ \rho^*(\omega) = (\rho \circ \tilde{f})^*(\omega) = f^*(\omega) \neq 0$, it follows that $\rho^*(\omega)$ is non-trivial. Since ρ was an arbitrary covering space, it follows that ω is not virtually trivial.

If $f(S^2)$ separates M into pieces M_1 and M_2 , then since each of M_1 and M_2 is not a 3-ball, there exist non-trivial connected finite-sheeted covering spaces $p_1 : M'_1 \rightarrow M$ and $p_2 : M'_2 \rightarrow M$ by the residual finiteness of compact 3-manifold groups. It is a simple exercise to check that one can glue several copies of M'_1 and M'_2 together to form a finite-sheeted covering space of $p : M' \rightarrow M$, and that the resulting space contains a non-separating lift $\tilde{f} : S^2 \rightarrow M$ of f . It follows from the argument of the previous paragraph that there exists an element $\omega \in H^2(M', \mathbb{Z}/p)$ that is not virtually trivial. By Lemma 3.3.5, $H^2(\widehat{M}, \mathbb{Z}/p)$ is nontrivial. □

We will see below that essential 2-spheres in M for all of account for all the cohomology classes of $\widehat{M}$ that do not arise from the failure of goodness of $\pi_1(M)$. The following lemma is a geometric version of Hopf's exact sequence. Our proof of this fact uses the Cartan-Leray spectral sequence. The basic properties of this spectral sequence will be reviewed in section 4.3 below.

Lemma 3.6.2. *There is a natural map $\Phi : H^2(\pi_1(M), R) \rightarrow H^2(M, R)$ mapping onto the set of elements $\omega \in H^2(M, R)$ that restrict trivially to every embedded 2-sphere in M .*

Proof. Let $\pi : \widetilde{M} \rightarrow M$ be the universal covering space of M . The Cartan-Leray spectral sequence for the regular covering $\widetilde{M} \rightarrow M$ has E_2 -page given by $E_2^{p,q} = H^p(\pi_1(M), H^q(\widetilde{M}, R))$, and converges to $H^{p+q}(M, R)$.

Since $\widetilde{M}$ is simply connected, R gives a trivial coefficient system over $\widetilde{M}$. It therefore follows

that $H^k(\widetilde{M}, R) = 0$ unless $k = 0$ or $k = 2$. This shows that the first row of the E_2 -page vanishes, and therefore $E_\infty^{2,0} = E_2^{2,0} = H^2(\pi_1(M), R)$. Convergence of the spectral sequence yields the exact sequence

$$0 \rightarrow H^2(\pi_1(M), R) \xrightarrow{\Phi} H^2(M, R) \rightarrow E_\infty^{0,2} \rightarrow 0.$$

$E_\infty^{0,2}$ is a subspace of $E_2^{0,2} = H^0(\Gamma, H^2(\widetilde{M}, R))$, and it follows from naturality that the third map in the above sequence is induced by $\pi^* : H^2(M, R) \rightarrow H^2(\widetilde{M}, R)$.

Let M' denote the aspherical manifold given by cutting M along a maximal disjoint collection of non-isotopic embedded essential spheres in M . It is a consequence of the geometrization theorem that after gluing 3-balls into the boundary of M' to form a manifold $\overline{M}'$, the universal covering space of each component of $\overline{M}'$ is homeomorphic to $\mathbb{R}^3$. It follows that the universal covering space of each component of M' is homeomorphic to the complement of a collection of disjoint balls in $\mathbb{R}^3$, and hence $\widetilde{M}$ consists of a simply connected union of such spaces glued together along their boundary. A simple Mayer-Vietoris argument shows that an element of $H^2(\widetilde{M}, R)$ is non-trivial if and only if it is supported non-trivially on some embedded sphere. This shows that if ω is a class in $H^2(M, R)$ that vanishes on every 2-sphere, then $\pi^*(\omega)$ is trivial and therefore ω must be in the image of Φ . $\square$

Note that naturality of the map Φ means that if $K < \pi_1(M)$, $\pi : M_K \rightarrow M$ is the covering space corresponding to K , and $\omega \in H^k(\pi_1(M), R)$, then $\pi^*(\Phi(\omega)) = \Phi(\omega|_K)$ where $\omega|_K$ denotes the restriction of ω to K . Goodness of 3-manifold groups, together with the above lemma yields the following proposition.

Proposition 3.6.1. *Let M be a 3-manifold, and suppose that finite volume hyperbolic 3-manifolds N with $b_1(N) \geq 2$ are virtually fibered. Let R be a finite coefficient module and let $\omega \in H^k(M, R)$. ω is virtually trivial if and only if $f^*(\omega) = 0$ for every continuous map $f : S^2 \rightarrow M$.*

Proof. Let $\pi : \widetilde{M} \rightarrow M$ be a covering space and $f : S^2 \rightarrow M$ a continuous map. f lifts to a map $\tilde{f} : S^2 \rightarrow \widetilde{M}$. Since $\pi \circ \tilde{f} = f$ and $\tilde{f}^* \circ \pi^* = f^*$, $f^*(\omega) \neq 0$ implies $\pi^*(\omega) \neq 0$. This establishes one direction of the proposition.

Assume that $\omega \in H^k(M, R)$ is such that $f^*(\omega) = 0$ for each continuous map $f : S^2 \rightarrow M$. By Lemma 3.5.2, it suffices to establish the proposition in the case $k = 2$. Since ω restricts trivially to each embedded 2-sphere, it follows from Lemma 3.6.2 that $\omega = \Phi(\eta)$, where $\Phi : H^2(\pi_1(M), R) \rightarrow H^2(M, R)$ is a natural homomorphism. Since $\pi_1(M)$ is good there exists a finite index subgroup K of $\pi_1(M)$ such that η restricts trivially to K , so it follows by the naturality of Φ that if $\pi_K : \widetilde{M} \rightarrow M$ is the finite sheeted covering space corresponding to K then

$$\pi_K^*(\omega) = \pi_K^*(\Phi(\eta)) = \Phi(\eta|_K) = 0.$$

□

3.7 Grothendieck rigidity of closed, prime 3-manifold groups

We now turn to Grothendieck's question for 3-manifold groups. This requires proving goodness for fundamental groups of compact 3-manifolds with boundary. We achieve this by a doubling trick that requires the following simple lemma. Recall that the *double* of a compact 3-manifold with boundary M is the closed manifold obtained by gluing together the boundaries of two copies of M via the identity map.

Lemma 3.7.1. *Let M be a compact 3-manifold with boundary. If the fundamental group of the double of M has good fundamental group, then $\pi_1(M)$ is good.*

Proof. As in the previous sections, it suffices to show that 2-dimensional cohomology classes on M with coefficients in a finite module are virtually trivial.

Let DM denote the double of M . Since DM retracts onto M via a map $r : DM \rightarrow M$ given by identifying the two copies of M contained in DM , the map $r^* : H^2(M, R) \rightarrow H^2(DM, R)$ is injective.

Let $\omega \in H^2(M, R)$ be an element such that $f^*(\omega) = 0$ for any continuous map $f : S^2 \rightarrow M$. Given a continuous map $f : S^2 \rightarrow DM$, $r \circ f$ is a continuous map from S^2 to M , so $f^*(r^*(\omega)) = (r \circ f)^*(\omega)$ is trivial. This shows that $r^*(\omega)$ pulls back trivially under any map from a two sphere

to DM , so $r^*(\omega)$ is in the image of the map $\Phi : H^2(\pi_1(DM), R) \rightarrow H^2(DM, R)$ given by Lemma 3.6.2.

Let η be such that $\Phi(\eta) = r^*(\omega)$. Since $\pi_1(DM)$ is good, there exists a finite index subgroup $K < \pi_1(DM)$ such that $\eta|_K = 0$. Let $\pi : DM_K \rightarrow DM$ denote the cover of DM corresponding to K . By the naturality of Φ , $0 = \Phi(\eta|_K) = \pi^*(r^*(\omega))$.

Let $\pi^i : M_K \rightarrow M$ denote the pullback of the covering space $\pi : DM_K \rightarrow DM$ to M via the inclusion $i : M \hookrightarrow DM$. We then have the following commutative diagram:

$$\begin{array}{ccccc} M_K & \xrightarrow{i^\pi} & DM_K & & \\ \downarrow \pi^i & & \downarrow \pi & & \\ M & \xrightarrow{i} & DM & \xrightarrow{r} & M. \end{array}$$

Since

$$0 = (i^\pi)^*(\pi^*(r^*(\omega))) = (\pi \circ i^\pi)^*(r^*(\omega)) = (i \circ \pi^i)^*(r^*(\omega)) = (\pi^i)^*(i^*(r^*(\omega))) = (\pi^i)^*(\omega),$$

ω pulls back trivially under the map π^i .

□

Proposition 3.7.1. *If finite volume hyperbolic 3-manifolds M with $vb_1(M) > 1$ are virtually fibered, then fundamental groups of closed prime 3-manifolds are Grothendieck rigid.*

Proof. Since $S^2 \times S^1$ is the only three manifold with infinite fundamental group that is prime but is not aspherical and $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$ is clearly Grothendieck rigid, we may assume that M is aspherical.

Let Γ denote $\pi_1(M)$ and let $i : H \hookrightarrow \Gamma$ be the inclusion of a proper finitely generated subgroup H of Γ into Γ . If H is finite index in Γ , H is a proper closed subset of Γ so $\hat{i} : \hat{H} \rightarrow \hat{\Gamma}$ cannot be surjective. We may therefore conclude that H is infinite index in Γ . Let M_H be the covering space of M corresponding to H . By the Scott core theorem, there is a retraction $r : M_H \rightarrow N_H$ onto a compact 3-manifold with boundary $N_H \subset M_H$ such that r is a homotopy equivalence.

Since M is aspherical, N_H must be aspherical as well, so the group cohomology of $\pi_1(N_H)$ is identical to the cohomology of the space N_H . Thus $\mathbb{Z}/2 \cong H^3(M, \mathbb{Z}/2) \cong H^3(\Gamma, \mathbb{Z}/2)$, while $H^3(H, \mathbb{Z}/2) \cong H^3(N_H, \mathbb{Z}/2) = 0$. By goodness, $H^3(\widehat{\Gamma}, \mathbb{Z}/2) \cong H^3(\Gamma, \mathbb{Z}/2)$ and $H^3(\widehat{H}, \mathbb{Z}/2) \cong 0$. This shows that $\widehat{H}$ and $\widehat{\Gamma}$ are not isomorphic, so $\hat{i} : \widehat{H} \rightarrow \widehat{\Gamma}$ cannot be an isomorphism.

□

3.8 The Cohomology of 3-dimensional Solenoids over $\mathbb{Z}$

In this section we study the cohomology of solenoids over $\mathbb{Z}$. This will require a few basic results about direct limits and divisible groups.

Lemma 3.8.1. *A torsion-free divisible abelian group is a $\mathbb{Q}$ -vector space.*

Proof. Let A be a torsion free divisible group, and let $x \in A$. Given $\frac{a}{b} \in \mathbb{Q}$, define $\frac{a}{b} \cdot x$ to be $a \cdot x'$, where x' is an element in A such that $b \cdot x' = x$. If x'' is another element such that $b \cdot x'' = x$, then $b \cdot (x'' - x') = 0$. Since A is torsion free, this implies $x'' = x'$. □

The following lemma follows immediately from the fact that the Tor functor commutes with direct limits:

Lemma 3.8.2. *A direct limit of torsion free groups is torsion free, and a direct limit of torsion groups is torsion.*

These lemmas can be used to compute the top dimensional cohomology groups of solenoids over n -manifolds.

Lemma 3.8.3. *Let $\mathcal{T} = \{\cdots \rightarrow M_2 \rightarrow M_1 \rightarrow M\}$ be a tower of connected covering spaces of a closed orientable n -manifold M , and suppose that for all $N > 0$, there exists k such that $\deg(M_k \rightarrow M)$ is divisible by N . Then $H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z}) \cong \mathbb{Q}$.*

Proof. Since $H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z})$ is the direct limit of the torsion free groups $H^n(M_i, \mathbb{Z}) \cong \mathbb{Z}$, it follows from the previous lemma that $H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z})$ is torsion free. Let $\hat{p}_i : \widehat{M} \rightarrow M_i$ and $p_{ji} : M_j \rightarrow M_i$ for $j > i$ denote the natural covering maps. Let ω_i denote a generator for $H^n(M_i, \mathbb{Z})$, and recall that

after choosing orientations compatibly $p_{ij}^*(\omega_i) = \deg(p_{ij})\omega_j$. Given $\alpha \in H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z})$, there exists i such that $\alpha = \hat{p}_i^*(k\omega_i)$ for some $k \in \mathbb{Z}$. Given $m \in \mathbb{N}$, there exists $j > i$ such that $m \mid \deg(p_{ij})$, so $p_{ij}^*(k\omega_i) = k \deg(p_{ij})\omega_j = \ell m\omega_j$ for some $\ell \in \mathbb{Z}$. Since

$$m\hat{p}_j^*(\ell\omega_j) = \hat{p}_j^*(\ell m\omega_j) = \hat{p}_j^* \circ p_{ji}^*(k\omega_i) = (p_{ji} \circ \hat{p}_j)^*(k\omega_i) = \hat{p}_i^*(k\omega_i) = \alpha,$$

α is divisible by m . It follows from Lemma 3.8.1 that $H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z})$ is a $\mathbb{Q}$ -vector space. Given α and β in $\mathbb{Q}$, α and β are both in the image of $\hat{p}_k^* : H^n(M_k, \mathbb{Z}) \rightarrow H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z})$ for some k . Since $H^n(M_k, \mathbb{Z}) \cong \mathbb{Z}$, it follows that α and β are linearly dependent, so $\dim(H^n(\widehat{M}_{\mathcal{T}}, \mathbb{Z})) = 1$. $\square$

These lemmas can also be used to study $H^1(\widehat{X}, \mathbb{Z})$ for an arbitrary compact CW complex X .

Lemma 3.8.4. *Let X be a compact CW complex. Then $H^1(\widehat{X}, \mathbb{Z})$ is a $\mathbb{Q}$ -vector space of rank $vb_1(X)$.*

Proof. Given a finite-sheeted covering spaces $X_i \rightarrow X$, let $\hat{p}_i : \widehat{X} \rightarrow X_i$ denote the natural covering map. Given $\hat{\alpha} \in H^1(\widehat{X}, \mathbb{Z})$, there exists an element $\alpha \in H^1(X_i, \mathbb{Z})$ such that $\hat{\alpha} = \hat{p}_i^*(\alpha)$. There exists a map $\rho : X_i \rightarrow S^1$ such that $\alpha = \rho^*(\omega)$ where ω denotes an appropriately chosen generator of $H^1(S^1, \mathbb{Z})$. Let $r : S^1 \rightarrow S^1$ be an n -fold covering map, and let $r^\rho : (X_i)_\rho^r \rightarrow X_i$ be the pullback of this n -fold covering map by ρ , so that we have the commutative diagram

$$\begin{array}{ccc} (X_i)_\rho^r & \xrightarrow{\rho^r} & S^1 \\ r^\rho \downarrow & & \downarrow r \\ X_i & \xrightarrow{\rho} & S^1 \end{array}$$

Since $\hat{p}_i : \widehat{X} \rightarrow X_i$ factors $r^\rho \circ q$ for some map $q : \widehat{X} \rightarrow (X_i)_\rho^r$,

$$\hat{\alpha} = \hat{p}_i^*(\alpha) = (r^\rho \circ q)^*(\alpha) = q^* \circ (r^\rho)^* \circ \rho^*(\omega) =$$

$$q^*((\rho \circ r^\rho)^*(\omega)) = q^*((r \circ \rho^r)^*(\omega)) = q^* \circ (\rho^r)^*(n\omega) = n(\rho^r \circ q)^*(\omega)$$

so $\hat{\alpha}$ is divisible by n . This shows that $H^1(\hat{X}, \mathbb{Z})$ is divisible. Since it is a limit of the torsion free groups, it follows that it is a $\mathbb{Q}$ -vector space. The claim on the rank is easily checked, since if α and β are linearly independent elements of $H^1(X_i, \mathbb{Z})$, then $\hat{p}_i^*(\alpha)$ and $\hat{p}_i^*(\beta)$ are linearly independent in $H^1(\hat{X}, \mathbb{Z})$, and if $\hat{\alpha}$ and $\hat{\beta}$ are linearly independent elements of $H^1(\hat{X}, \mathbb{Z})$, then they are of the form $\hat{p}_i^*(\alpha)$ and $\hat{p}_i^*(\beta)$ for some linearly independent elements $\alpha, \beta \in H^1(X_i, \mathbb{Z})$.

□

When M is a 3-manifold, the above results suffice to compute $H^k(\widehat{M}, \mathbb{Z})$ for every k except for $k = 2$. We now turn to the case $k = 2$.

Lemma 3.8.5. *Let M be a 3-manifold. $H^k(\widehat{M}, \mathbb{Z})$ is torsion free for all k .*

Proof. By Lemmas 3.8.3 and 3.8.4, it suffices to check that $H^2(\widehat{M}, \mathbb{Z})$ is torsion free. The universal coefficients theorem yields a natural exact sequence of the form

$$0 \longrightarrow \text{Ext}(H_1(M'), \mathbb{Z}) \xrightarrow{\phi} H^2(M', \mathbb{Z}) \xrightarrow{\psi} \text{Hom}(H_2(M'), \mathbb{Z}) \longrightarrow 0$$

for each finite-sheeted covering space M' of M . Since $\text{Ext}(H_1(M), \mathbb{Z})$ is naturally isomorphic to $\text{Ext}(\text{Tor}(H_1(M), \mathbb{Z}), \mathbb{Z})$, $\lim_{\rightarrow} \text{Ext}(H_1(M), \mathbb{Z}) = 0$ as in the proof of Lemma 3.5.2. Since taking direct limits preserves exact sequences, the limit of the above exact sequence show that $H^2(\widehat{M}, \mathbb{Z})$ is isomorphic to $\lim_{\rightarrow} \text{Hom}(H_2(M'), \mathbb{Z})$. Since $\text{Hom}(H_2(M'), \mathbb{Z})$ is torsion free, the result follows from Lemma 3.8.2.

□

The next lemma shows that divisibility of $H^2(\widehat{M}, \mathbb{Z})$ is equivalent to goodness.

Lemma 3.8.6. *Let M be an aspherical 3-manifold. $H^2(\widehat{M}, \mathbb{Z}) \cong \mathbb{Q}^{vb_1(M)}$ if and only if $\pi_1(M)$ is good.*

Proof. Suppose that $H^k(\widehat{M}, \mathbb{Z}) \cong \mathbb{Q}^{vb_1(M)}$. For each finite sheeted covering space M' of M , the

Bockstein sequence gives a natural long exact sequence of the form

$$\cdots \rightarrow H^1(M', \mathbb{Z}/p) \rightarrow H^2(M', \mathbb{Z}) \rightarrow H^2(M', \mathbb{Z}) \rightarrow H^2(M', \mathbb{Z}/p) \rightarrow 0,$$

since the tail of the Bostein sequence given by

$$H^3(M', \mathbb{Z}) \rightarrow H^3(M', \mathbb{Z}) \rightarrow H^3(M', \mathbb{Z}/p)$$

is exact. Since the direct limit functor is exact and $H^1(\widehat{M}, \mathbb{Z}/p)$ is trivial, taking the direct limit gives an exact sequence

$$0 \rightarrow H^2(\widehat{M}, \mathbb{Z}) \xrightarrow{\phi} H^2(\widehat{M}, \mathbb{Z}) \rightarrow H^2(\widehat{M}, \mathbb{Z}/p) \rightarrow 0.$$

If $H^2(\widehat{M}, \mathbb{Z})$ is a $\mathbb{Q}$ -vector space, then $H^2(\widehat{M}, \mathbb{Z})/\phi(H^2(\widehat{M}, \mathbb{Z})) \cong H^2(\widehat{M}, \mathbb{Z}/p)$ is a $\mathbb{Q}$ -vector space, and hence must be trivial. By Lemma 3.5.2, this shows that $\pi_1(M)$ is good.

We now prove the converse. As was shown in the proof of the previous lemma, $H^2(\widehat{M}, \mathbb{Z}) \cong \varinjlim \text{Hom}(H_2(M'), \mathbb{Z})$ as M' ranges over the set of finite-sheeted covering spaces of M . Let M' be a fixed finite-sheeted covering space of M , let $x_1, \dots, x_k$ be generators for $H_2(M')$, and let $\psi_1, \psi_2, \dots, \psi_k$ be dual elements so that $\psi_i(x_j) = \delta_{ij}$. Since there is a surjective map from $H^2(M', \mathbb{Z}/n)$ onto $\text{Hom}(H_2(M'), \mathbb{Z}/n)$, since M is good it follows that we may find a finite-sheeted covering space $p : M'' \rightarrow M$ such that $p^*(\overline{\psi_i})$ is trivial, where $\overline{\psi_i}$ denotes the reduction of ψ_i modulo n . It follows that $p_*(H_2(M'')) \subset nH_2(M')$, and hence $p^*(\text{Hom}(H_2(M'), \mathbb{Z})) \subset n\text{Hom}(H_2(M'), \mathbb{Z})$.

This shows that $\varinjlim \text{Hom}(H_2(M'), \mathbb{Z})$ is divisible, so by Lemmas 3.8.5 and Lemma 3.8.1, $H^2(\widehat{M}, \mathbb{Z})$ is a $\mathbb{Q}$ -vector space. Since the rank of $H^2(M', \mathbb{Z})$ is identical to the rank of $H^1(M', \mathbb{Z})$, the same argument as was given in the proof of Lemma 3.8.4 shows that the rank of $H^2(\widehat{M}, \mathbb{Z})$ is $vb_1(M)$.

□

Combining the above results, we have established the following proposition describing the inte-

gral cohomology of the universal solenoid over a 3-manifold with good fundamental group.

Proposition 3.8.1. *Let M be an aspherical 3-manifold. M has good fundamental group if and only if*

$$H^*(\widehat{M}, \mathbb{Z}) \cong \mathbb{Z}, \mathbb{Q}^{vb_1(M)}, \mathbb{Q}^{vb_1(M)}, \mathbb{Q}.$$

It is interesting to observe that the conjectures about 3-manifolds outlined in the introduction imply that given any 3-manifold M , either $vb_1(M) \leq 3$ or $vb_1(M) = \infty$.

Chapter 4

Mapping Solenoids

Given a continuous map $f : S \rightarrow M$ between compact spaces and a finite-sheeted covering space $\pi : M' \rightarrow M$, there exists a covering space $\pi^f : S_f^\pi \rightarrow S$, called the *pull-back of π* , and a map $f^\pi : S_f^\pi \rightarrow M'$ called an *elevation* of S into M' making the following diagram commute:

$$\begin{array}{ccc} S_f^\pi & \xrightarrow{f^\pi} & M' \\ \pi^f \downarrow & & \downarrow \pi \\ S & \xrightarrow{f} & M. \end{array}$$

The covering space S_f^π is given by the set $\{(x, y) \in S \times M' \mid f(x) = \pi(y)\}$, the map $f^* : S_f^\pi \rightarrow S$ is given by the composition of the inclusion $i : S_f^\pi \hookrightarrow S \times M'$ with the projection map $S \times M' \rightarrow S$, and the map $\tilde{f}^\pi : S_f^\pi \rightarrow M'$ is given by the composition of the inclusion i with the projection map $S \times M' \rightarrow M'$. In this chapter we address the following question: given such a map $f : S \rightarrow M$, which finite-sheeted covering spaces of S are induced by finite-sheeted covering spaces of M ?

What sort of answer could we hope to have for this question? On one extreme, if the map $f : S \rightarrow M$ is a constant map it is easy to see that for any cover $\pi : M' \rightarrow M$, S_f^π has $\deg(\pi)$ components each of which is homeomorphic to S . On the other extreme, if $f : S \rightarrow M$ is π_1 -injective and $\pi_1(M)$ is LERF then it is straight-forward to show that given any finite-sheeted connected covering space $\rho : S' \rightarrow S$ there exists a covering space $\pi : M' \rightarrow M$ such that the

covering space $f^*(\pi) : S_f^\pi \rightarrow S$ factors through ρ .

One way of describing the collection of covering spaces of S induced by M is to study the solenoid over S given by pulling-back the universal solenoid $\hat{\pi} : \widehat{M} \rightarrow M$ via f . We will denote this solenoid by $\widehat{S}_f$, i.e.

$$\widehat{S}_f := \{(x, y) \in S \times \widehat{M} \mid f(x) = \hat{\pi}(y)\}.$$

Each of the situations described in the previous paragraph may be succinctly described in terms of the topology of $\widehat{S}_f$. In f is a constant map, for instance, each connected component of $\widehat{S}_f$ is homeomorphic to S . If f is π_1 -injective and $\pi_1(M)$ is LERF, each connected component of $\widehat{S}_f$ is homeomorphic to the universal solenoid over S . It is also easy to show that $f_*(\pi_1(S))$ is engulfed if and only if $\widehat{S}_f$ is disconnected.

In this chapter we will study the topology of $\widehat{S}_f$ by studying an auxiliary space $\widehat{B}_f$ that we call the *mapping solenoid* of f . The mapping solenoid of f can be used to derive a spectral sequence that relates the cohomology of $\widehat{S}_f$ and $\widehat{M}$. Under certain circumstances, this spectral sequence can be used to show that $\widehat{S}_f$ is disconnected, and that the group $f_*(\pi_1(S))$ is therefore engulfed. We then study π_1 -injective maps from surfaces into 3-manifold groups. We show that if M is *not* virtually Haken, then for any π_1 -injective map $f : S \rightarrow M$ the terms of the E_2 -pages of the relevant spectral sequences can be computed exactly.

4.1 Basic Results on Pullback Covering Spaces and Solenoids

In this section we gather some basic facts about pull-back covering spaces and solenoids that we will use in what follows.

Lemma 4.1.1. *If $f : S \rightarrow M$ is a continuous map and $\pi : M' \rightarrow M$ is a regular solenoid over M , then $\pi^f : S_f^\pi \rightarrow S$ is a regular solenoid over S , and $f^\pi : S_f^\pi \rightarrow M'$ is $\pi_1(M)$ -equivariant.*

Proof. This follows immediately from the definition of the pull-back when π is a finite-sheeted covering space, since if the action of $\pi_1(M)$ on M' by deck transformations is transitive on the fibers of π , then the $\pi_1(M)$ -action on $S \times M'$ given by extending the action of M' to act by the

identity on S acts transitively on the fibers of the map $\pi^f : S_f^\pi \rightarrow S$. The result follows easily for all solenoids by taking limits. □

Lemma 4.1.2. *Let $f : S \rightarrow M$ be a continuous map, and let $\pi : M' \rightarrow M$ be a finite sheeted covering space. Let $x \in S_f^\pi$, let S' be the component of S_f^π containing x , and let K be the finite index subgroup of $\pi_1(M)$ given by $\pi_*(\pi_1(M', f^\pi(x)))$. The connected covering space $\pi^f|_{S'} : S' \rightarrow S$ is the covering space of S corresponding to the subgroup $f_*^{-1}(K)$.*

Proof. Since $\pi \circ f^\pi = f \circ \pi^f$, it follows that

$$(\pi^f)_*(\pi_1(S', x)) \subset f_*^{-1}((\pi \circ f^\pi)_*(\pi_1(S', x))) \subset f_*^{-1}(\pi_*(\pi_1(M', f^\pi(x)))) = f_*^{-1}(K).$$

To show the reverse containment, let $\gamma : [0, 1] \rightarrow S$ represent an element of $f_*^{-1}(K)$, and let $\eta : [0, 1] \rightarrow M'$ be a lift of $f \circ \gamma$ to M' such that $\eta(0) = f^\pi(x)$. The path $\tilde{\gamma} : [0, 1] \rightarrow S \times M'$ defined by $\tilde{\gamma}(t) = (\gamma(t), \eta(t))$ clearly lies in S_f^π and is closed. Since $f^\pi((\gamma(0), \eta(0))) = \eta(0) = f^\pi(x)$ and $\pi^f(\gamma(0), \eta(0)) = \gamma(0) = f(x)$, it follows that $(\gamma(0), \eta(0)) = x$, so $\tilde{\gamma}(t)$ is a closed loop based at x . This shows that $[\gamma]$ is in the image under π_*^f of $\pi_1(S', x)$. □

Lemma 4.1.3. *Let $f : S \rightarrow M$ be a continuous map, let $\rho : S' \rightarrow S$ be a finite-sheeted covering space of S and let $H = \rho_*(\pi_1(S'))$. There exists a covering space $\pi : M' \rightarrow M$ such that $\pi^f : S_f^\pi \rightarrow S$ factors through ρ if and only if $f_*(H)$ is separable in $\pi_1(M)$ from $f_*(gH)$ for each $g \notin H$.*

Proof. Let $g_1, \dots, g_n$ be coset representatives for $\pi_1(S)/H$. Suppose $f_*(H)$ is separable in $\pi_1(M)$ from $f_*(g_i H)$ for all i . Then for each i there exists a finite index subgroup $K_i < \pi_1(M)$ such that $f_*(H) < K_i$ and $K_i \cap f_*(g_i H) = \emptyset$. Let $K = \cap_{i=1}^n K_i$. Then $f_*^{-1}(K) = H = \rho_*(\pi_1(S'))$. Let $\pi : M' \rightarrow M$ be the covering space corresponding to K . By Lemma 4.1.2, $\rho : S' \rightarrow S$ is isomorphic to a component of the covering space $\pi^f : S_f^\pi \rightarrow S$. By pulling back a further covering space of M' that is regular, we obtain a regular covering space $p : S'' \rightarrow S$ that factors through S_f^π . Since every component of S'' is isomorphic, p restricted to each component factors through S' .

Conversely, suppose that there exists a covering space $\pi : M' \rightarrow M$ such that $\pi^f : S_f^\pi \rightarrow S$ factors through $\rho : S' \rightarrow S$. Let H' be the image under π_*^f of the fundamental group of a component of S_f^π . By the previous lemma, H' is given by $f_*^{-1}(K)$, where $K = \pi_*(\pi_1(M', y))$ for an appropriately chosen basepoint y . Given $g \notin H'$, $f_*(gH') = f_*(g)f_*(H') \subset f_*(g)K$. Since $f_*(g)$ does not lift to M' , $(f_*(g) \cdot K) \cap K = \emptyset$. It follows that if $g_1, g_2, \dots, g_n$ are distinct cosets of H' , then $f_*(g_1) \cdot K, \dots, f_*(g_n) \cdot K$ are distinct cosets of K . The set

$$\bigcup_{\{i \mid g_i \in H\}} f_*(g_i)K$$

is an open set containing $f_*(H)$ and is disjoint from the open set

$$\bigcup_{\{i \mid g_i \notin H\}} f_*(g_i)K.$$

Since $H' < H$, the non-trivial coset representatives for H' contain a set of non-trivial coset representatives for H , so it follows that $f_*(H)$ is separable from $f_*(gH)$ for each $g \notin H$.

□

The previous lemma has the following immediate corollary.

Lemma 4.1.4. *Let $f : S \rightarrow M$ be a continuous map and suppose that $\pi_1(M)$ induces the full profinite topology on $\pi_1(S)$. Then each component of the pullback solenoid $\widehat{S}_f$ is isomorphic to the universal solenoid over S .*

Proof. If $\pi_1(M)$ induces the full profinite topology on S , it follows from the previous lemma that for any finite-sheeted cover $\rho : S' \rightarrow S$, there exists a regular covering $\pi : M' \rightarrow M$ such that π^f factors through ρ . It follows that the inverse limit of these coverings $\widehat{\pi}^f : \widehat{S}_f \rightarrow S$ factors through every finite-sheeted covering space. By Lemma 3.2.2, any connected solenoid factoring through every finite-sheeted covering space of S is isomorphic to the universal solenoids over S , so each component of $\widehat{S}_f$ is isomorphic to $\widehat{S}$. □

Lemma 4.1.5. *Let $f : S \rightarrow M$ be a π_1 -injective map. $\widehat{S}_f$ is disconnected if and only if $f_*(\pi_1(S))$ is engulfed.*

Proof. Since $H^0(\widehat{S}_f, \mathbb{Z}) \cong H^0(\varprojlim S_f^\pi, \mathbb{Z}) \cong \varinjlim H^0(S_f^\pi, \mathbb{Z})$, it follows that $\widehat{S}_f$ is disconnected if and only if one of the pullback covers S_f^π is disconnected.

Suppose that S_f^π is disconnected for some finite-sheeted covering space $\pi : M' \rightarrow M$. Let x be a point in S and let y denote $f(x)$. Consider the (right) monodromy action of $\pi_1(S, x)$ on $(\pi^f)^{-1}(x)$. Since this action is given by path lifting, for any component $C \subset S_f^\pi$, the image of $\pi_1(S, x)$ in $\text{Sym}((\pi^f)^{-1}(x))$, where $\text{Sym}((\pi^f)^{-1}(x))$ denotes the permutation group on the set $(\pi^f)^{-1}(x)$, lies in $\text{Stab}(C \cap (\pi^f)^{-1}(x))$. On the other hand, the monodromy action of $\pi_1(M, f(x))$ on $\pi^{-1}(y)$ is transitive, since M' is a connected covering space of M . Given $g \in \pi_1(S, x)$ and $\tilde{x} \in C \cap (\pi^f)^{-1}(x)$,

$$f^\pi(\tilde{x}) \cdot f_*(g) = f^\pi(\tilde{x} \cdot g) \in f^\pi(C \cap (\pi^f)^{-1}(x)).$$

It follows that under the map $\rho : \pi_1(M) \rightarrow \text{Sym}(\pi^{-1}(y))$, $f_*(\pi_1(S))$ lies in the proper subgroup $\text{Stab}(f^\pi(C \cap (\pi^f)^{-1}(x)))$, so $f_*(\pi_1(S, x))$ is engulfed by the finite index subgroup

$$\rho^{-1}(\text{Stab}(f^\pi(C \cap (\pi^f)^{-1}(x)))).$$

Conversely, suppose that $f_*(\pi_1(S, x))$ is engulfed by a subgroup $K < \pi_1(M, y)$, and let $\pi : (M', \tilde{y}) \rightarrow (M, y)$ be the finite-sheeted cover of M such that $\pi_*(\pi_1(M', \tilde{y})) = K$. Note that $(x, \tilde{y})$ lies in $S_f^\pi \subset S \times M'$. Let $g \notin K$, and suppose that there exists a path $\eta : [0, 1] \rightarrow S_f^\pi$ such that $\eta(0) = (x, \tilde{y})$ and $\eta(1) = (x, \tilde{y} \cdot g)$. Then $f \circ \pi^f \circ \eta$ gives a path in M based at y , and $f^\pi \circ \eta$ gives a lift of this path to M' based at $\tilde{y}$. Since $f_*(\pi_1(S, x)) < K$, however, any map $\gamma : [0, 1] \rightarrow M$ representing an element of $f_*(\pi_1(S, x))$ has a lift $\tilde{\gamma} : [0, 1] \rightarrow M'$ such that $\tilde{\gamma}(0) = \tilde{y}$ and $\tilde{\gamma}$ is closed. This yields a contradiction, so it follows that no such path η exists, and hence S_f^π is disconnected. $\square$

The final lemma that we will need follows from the basic naturality properties of pull-backs of bundles:

Lemma 4.1.6. *Let $f : S \rightarrow M$ be a map, let $\pi : M' \rightarrow M$ be a finite-sheeted covering space, and let $f^\pi : S_f^\pi \rightarrow M$ be the elevation of f to M' . The solenoid $\widehat{S}'$ given by pulling back the universal solenoid over M' to S_f^π via $\tilde{f}^\pi$ is isomorphic as a $\pi_1(M)$ -space to $\widehat{S}_f$.*

4.2 The Mapping Solenoid

In this section we introduce the mapping solenoid associated to a continuous map $f : S \rightarrow M$ and a tower $\mathcal{T} = \{\dots M_2 \rightarrow M_1 \rightarrow M\}$ of regular finite-sheeted covering spaces of M . Throughout this section we will denote $\pi_1(M)$ by Γ , and we will denote the inverse limit of the finite groups $\pi_1(M)/\pi_1(M_i)$ by $\widehat{\Gamma}_{\mathcal{T}}$. Note that since $\widehat{M}_{\mathcal{T}}$ is a regular solenoid, $\widehat{S}_f^{\mathcal{T}}$ is a regular solenoid over S by Lemma 4.1.1. The *mapping solenoid* for the map $f : S \rightarrow M$ with respect to the tower $\mathcal{T}$, which we will denote by $\widehat{B}_f^{\mathcal{T}}$, is defined to be

$$\left(\widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}}\right) / \widehat{\Gamma}_{\mathcal{T}},$$

where $\widehat{\Gamma}_{\mathcal{T}}$ acts diagonally on the product $\widehat{S}_f \times \widehat{M}_{\mathcal{T}}$. The following lemma gives a useful alternative construction of this object:

Lemma 4.2.1. *Let $f : S \rightarrow M$ be a continuous map, let $\mathcal{T} := \dots \rightarrow M_2 \rightarrow M_1 \rightarrow M$ be a tower of connected regular covering spaces of M , and let $\dots \rightarrow S_2 \rightarrow S_1 \rightarrow S$ be the corresponding pullback covering spaces of S . Let G_n denote the quotient $\pi_1(M)/\pi_1(M_n)$. Then*

$$\widehat{B}_f^{\mathcal{T}} \cong \varprojlim (S_n \times M_n)/G_n.$$

Proof. Let N_n be the normal subgroup corresponding to $\pi_1(M_n)$. Since $\widehat{N}_n$ acts trivially on S_n , $(S_n \times M_n)/G_n \cong (S_n \times \widehat{M}_{\mathcal{T}})/\widehat{\Gamma}_{\mathcal{T}}$. Consider the quotient map

$$\Phi : \prod_{n=1}^{\infty} S_n \times \widehat{M}_{\mathcal{T}} \rightarrow \prod_{n=1}^{\infty} \left((S_n \times \widehat{M}_{\mathcal{T}})/\widehat{\Gamma}_{\mathcal{T}}\right).$$

The map Φ is clearly invariant under the diagonal action of $\widehat{\Gamma}_{\mathcal{T}}$ on $\prod_{n=1}^{\infty} S_n \times \widehat{M}_{\mathcal{T}}$, so it factors

through a continuous map to $(\prod_{n=1}^{\infty} S_n \times \widehat{M}_{\mathcal{T}})/\widehat{\Gamma}_{\mathcal{T}}$. Since the $\widehat{\Gamma}_{\mathcal{T}}$ -action preserves the subspace $\lim_{\leftarrow} S_n \times \widehat{M}_{\mathcal{T}}$, we obtain a continuous map

$$\phi : \widehat{B}_f^{\mathcal{T}} \cong \left(\lim_{\leftarrow} S_n \times \widehat{M}_{\mathcal{T}} \right) / \widehat{\Gamma}_{\mathcal{T}} \rightarrow \lim_{\leftarrow} \left((S_n \times \widehat{M}_{\mathcal{T}}) / \widehat{\Gamma}_{\mathcal{T}} \right).$$

If $\phi(\overline{(x_n)}) = \phi(\overline{(y_n)})$, then $\overline{x_n} = \overline{y_n}$ for all n . There therefore exists a unique element $g_n \in G_n$ sending $\overline{x_n}$ to $\overline{y_n}$, so (g_n) defines an element of $\lim_{\leftarrow} G_n$, so since $(g_n) \cdot (x_n) = (y_n)$, $\overline{(x_n)} = \overline{(y_n)}$ it follows that ϕ is injective. Since ϕ is also surjective and continuous, it follows that ϕ is a homeomorphism since its domain and image are compact Hausdorff spaces. $\square$

We remark that when M is aspherical, then the universal covering space $\widetilde{M}$ is a classifying space for $\pi_1(M)$ so the cohomology of the bundle $(S_n \times M_n)/G_n \cong (S_n \times \widetilde{M})/\pi_1(M)$ is equal to the Γ -equivariant cohomology of the $\pi_1(M)$ -space S_n . The cohomology groups $H^k(\widehat{B}_f, R)$ may therefore be viewed as the direct limit of the equivariant cohomology groups $H_{\Gamma}^k(S_n, R)$.

Note that the bundle $(S_n \times M_n)/G_n$ from the previous lemma is both an S_n bundle over M and a M_n bundle over S . By taking the inverse limits of these bundles, we obtain the following lemma:

Lemma 4.2.2. *There exists a continuous surjective map $\rho_1 : \widehat{B}_f^{\mathcal{T}} \rightarrow S$ giving $\widehat{B}_f^{\mathcal{T}}$ the structure of a $\widehat{M}_{\mathcal{T}}$ -bundle over S , and a continuous surjective map $\rho_2 : \widehat{B}_f^{\mathcal{T}} \rightarrow M$ giving $\widehat{B}_f^{\mathcal{T}}$ the structure of a $\widehat{S}_f^{\mathcal{T}}$ -bundle over M .*

A second important feature of the mapping solenoid is the existence of a natural section of the map $\widehat{B}_f \rightarrow S$.

Lemma 4.2.3. *Let $\rho_1 : \widehat{B}_f^{\mathcal{T}} \rightarrow S$ and $\rho_2 : \widehat{B}_f^{\mathcal{T}} \rightarrow M$ denote the natural projection maps. There exists a section $\sigma : S \rightarrow \widehat{B}_f^{\mathcal{T}}$ of ρ_1 such that $\rho_2 \circ \sigma = f$.*

Proof. Let $\hat{f} : \widehat{S}_f^{\mathcal{T}} \rightarrow \widehat{M}_{\mathcal{T}}$ denote the elevation of f to $\widehat{S}_f^{\mathcal{T}}$. Let $\mathcal{G} \subset \widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}}$ be the graph of $\hat{f}$, i.e.

$$\mathcal{G} := \{(x, y) \in \widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}} \mid \hat{f}(x) = y\}.$$

Since $\hat{f}$ is $\widehat{\Gamma}_{\mathcal{T}}$ -equivariant, $\mathcal{G}$ is a $\widehat{\Gamma}_{\mathcal{T}}$ -invariant subset of $\widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}}$, so there is an inclusion map

$$i : \mathcal{G}/\widehat{\Gamma}_{\mathcal{T}} \hookrightarrow (\widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}})/\widehat{\Gamma}_{\mathcal{T}} = \widehat{B}_f^{\mathcal{T}}.$$

We claim that $\rho_1 \circ i$ is a homeomorphism. To see this, suppose (x_1, y_1) and (x_2, y_2) are two points in $\mathcal{G}$ whose images in $\mathcal{G}/\widehat{\Gamma}_{\mathcal{T}}$ map to the same point under ρ_1 . Since $\rho_1(\overline{(x_1, y_1)}) = \rho_1(\overline{(x_2, y_2)})$, it follows that there exists $g \in \widehat{\Gamma}_{\mathcal{T}}$ such that $x_1 = g \cdot x_2$. Since $\hat{f}$ is $\widehat{\Gamma}_{\mathcal{T}}$ -equivariant,

$$y_1 = \hat{f}(x_1) = \hat{f}(g \cdot x_2) = g \cdot \hat{f}(x_2) = g \cdot y_2.$$

It follows that $\overline{(x_2, y_2)} = \overline{(x_1, y_1)}$, so $\rho_1 \circ i$ is injective. $\rho_1 \circ i$ is also surjective and continuous, so it follows since $\mathcal{G}/\widehat{\Gamma}_{\mathcal{T}}$ and S are compact Hausdorff spaces that $\rho_1 \circ i$ is a homeomorphism. Let $\sigma : S \rightarrow \widehat{B}_f^{\mathcal{T}}$ be given by $i \circ (\rho_1 \circ i)^{-1}$. The map σ is clearly a section of ρ_1 , so it remains to check that $\rho_2 \circ \sigma = f$. Consider the map $q : \widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}} \rightarrow \widehat{B}_f^{\mathcal{T}}$ given by taking the quotient by $\widehat{\Gamma}_{\mathcal{T}}$. Let $\tilde{\rho}_1$ be given by the composition of the projection $\widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}} \rightarrow \widehat{S}_f^{\mathcal{T}}$ with the map $\hat{p}^f : \widehat{S}_f^{\mathcal{T}} \rightarrow S$, and let $\tilde{\rho}_2$ be given by the composition of the projection $\widehat{S}_f^{\mathcal{T}} \times \widehat{M}_{\mathcal{T}} \rightarrow \widehat{M}_{\mathcal{T}}$ with the map $\hat{p} : \widehat{M}_{\mathcal{T}} \rightarrow M$. Note that $\tilde{\rho}_1 = \rho_1 \circ q$ and $\tilde{\rho}_2 = \rho_2 \circ q$. Given any $x \in S$, since $\sigma(x) \in \mathcal{G}/\widehat{\Gamma}_{\mathcal{T}}$, there exists $(\tilde{x}, \tilde{y}) \in \mathcal{G}$ such that $q((\tilde{x}, \tilde{y})) = \sigma(x)$. Since $(\tilde{x}, \tilde{y}) \in \mathcal{G}$, $\tilde{y} = \hat{f}(\tilde{x})$. It follows that

$$\rho_2(\sigma(x)) = \rho_2 \circ q(\tilde{x}, \tilde{y}) = \tilde{\rho}_2((\tilde{x}, \tilde{y})) = \tilde{\rho}_2((\tilde{x}, \hat{f}(\tilde{x}))) = \hat{p}(\hat{f}(\tilde{x})) = f \circ \hat{p}^f(\tilde{x}) = f(x).$$

□

4.3 Spectral Sequences for Mapping Solenoids

In this section we review some basic facts about spectral sequences for fiber bundles, which will provide us with the basic tool we need in the following sections to compute cohomology groups of mapping solenoids.

We begin this section by describing the basic properties of the Cartan-Leray spectral sequence

that we will need in what follows. Recall that given a fiber bundle $\pi : X \rightarrow M$ with fiber F , there is a well-defined action of $\pi_1(M)$ on the cohomology groups $H^k(F, A)$ for any coefficient module A . This gives $H^k(F, A)$ the structure of a $\pi_1(M)$ -module, yielding a local coefficient system on M . The cohomology groups of M with respect to this local coefficient system form a bigraded group E_2 , such that the (r, s) graded piece of E_2 is given by

$$E_2^{r,s} := H^r(M, H^s(F, A)),$$

when r and s are both non-negative, and $E_2^{r,s} = 0$ otherwise. This bigraded group comprises what is known as the E_2 -page of the Cartan-Leray spectral sequence. There is a homomorphism $d_2 : E_2 \rightarrow E_2$, known as the d_2 -differential that has bidegree $(2, -1)$, i.e. it maps the group $E_2^{r,s}$ to the group $E_2^{r+2,s-1}$.

$$\begin{array}{ccccccc}
0 & H^0(M, H^2(F, A)) & H^1(M, H^2(F, A)) & H^2(M, H^2(F, A)) & H^3(M, H^2(F, A)) & & \\
& \searrow & & \searrow & & & \\
0 & H^0(M, H^1(F, A)) & H^1(M, H^1(F, A)) & H^2(M, H^1(F, A)) & H^3(M, H^1(F, A)) & & \\
& \searrow & & \searrow & & & \\
0 & H^0(M, H^0(F, A)) & H^1(M, H^0(F, A)) & H^2(M, H^0(F, A)) & H^3(M, H^0(F, A)) & & \\
& \searrow & & \searrow & & & \\
0 & 0 & 0 & 0 & 0 & &
\end{array}$$

The E_2 -page together with the d_2 -differential.

The differential d_2 satisfies $d_2 \circ d_2 = 0$, so we can take the homology of this map to form the groups

$$E_3^{r,s} := (\ker(d_2) \cap E_2^{r,s}) / d_2(E_2^{r-2,s+1}),$$

which comprise the E_3 -page of the spectral sequence. Similar to the E_2 page, the E_3 -page comes

equipped with a d_3 -differential with bidegree $(3, -2)$, and we form the E_4 -page of the spectral sequence by taking the homology of this differential. More generally, the E_k page comes equipped with a differential d_k of bidegree $(k, -k + 1)$, and the E_{k+1} page is given by taking the homology of d_k . The collection of groups and differentials $\{E_r, d_r\}$ therefore satisfies the following definition:

Definition 4.3.1. *A spectral sequence $\{E_r, d_r\}$ is a collection of bigraded groups E_r indexed by $\{r \in \mathbb{N} \mid r \geq r_0\}$ together with differentials $d_r : E_r \rightarrow E_r$ such that $d_r^2 = 0$, and E_{r+1} is given by $\ker(d_r)/d_r(E_r)$.*

Note that in the Cartan-Leray spectral sequence, the groups $E_k^{r,s}$ are trivial for all k if either r or s is negative. As a consequence, when $k > s$ the differential $d_k|_{E_k^{r,s}}$ has image in group $E_k^{r+k, s-k+1} = 0$, so for $k > s$, $\ker(d_k) \cap E_k^{r,s} \cong E_k^{r,s}$. Similarly, for $k > r+1$, the group $E_k^{r-k, s+k-1} \cong 0$ so $d_2(E_k^{r-k, s+k-1}) = 0$. It therefore follows that for large enough k , (namely $k > \max\{r+1, s\}$), $E_k^{r,s} \cong E_{k+1}^{r,s}$. These stable terms are denoted by $E_\infty^{r,s}$, and they comprise a bigraded group known as the E_∞ -page of the spectral sequence.

The terms of the E_∞ -page of the spectral sequence give a great deal of information about the cohomology groups $H^n(X, R)$. This is due to the fact that $E_\infty^{k, n-k} \cong F_k^n / F_{k+1}^n$ for a filtration

$$0 \subset F_n^n \subset F_{n-1}^n \subset \cdots \subset F_0^n = H^n(X, R).$$

In many cases, for example when R is a field, this allows one to reconstruct the group $H^n(X, R)$ from the E_∞ -page of the Cartan-Leray spectral sequence. It is in this sense that the Cartan-Leray spectral sequence is said to *converge* to the cohomology of the total space X , and this is often denoted

$$H^r(M, H^s(F, A)) \implies H^{r+s}(X, A).$$

The Cartan-Leray spectral sequence has many nice naturality properties. Let $F_i \rightarrow X_i \rightarrow B_i$ be fiber bundles, for $i \in \{1, 2\}$, let $(E_r^{p,q})_i$ and $(d_r)_i$ be the terms and differentials of the E_r -page of the Cartan-Leray spectral sequence for $F_i \rightarrow X_i \rightarrow B_i$, and let $0 \subset (F_n^n)_i \subset \cdots \subset (F_0^n)_i = H^n(X_i, R)$ be the corresponding filtration of $H^n(X_i, R)$. Given a bundle map $f_T : X_1 \rightarrow X_2$, inducing maps

$f_B : B_1 \rightarrow B_2$ and $f_F : F_1 \rightarrow F_2$, we have the following naturality properties (see [Hat2] p. 18):

1. There are induced maps $f_r^* : (E_r^{p,q})_2 \rightarrow (E_r^{p,q})_1$ such that $(d_r)_1 \circ f_r^* = f_r^* \circ (d_r)_2$ and f_{r+1} is the map induced on homology by f_r .
2. The map $f_T^* : H^n(X_2, R) \rightarrow H^n(X_1, R)$ sends the filtration $0 \subset (F_n^n)_2 \subset \cdots \subset (F_0^n)_2 = H^n(X_2, R)$ to the filtration $0 \subset (F_n^n)_1 \subset \cdots \subset (F_0^n)_1 = H^n(X_1, R)$, and the map $f_\infty^* : (E_\infty^{p,q})_2 \rightarrow (E_\infty^{p,q})_1$ is given by the induced map $\overline{f_T^*} : (F_k^n)_2 / (F_{k+1}^n)_2 \rightarrow (F_k^n)_1 / (F_{k+1}^n)_1$.
3. The maps $f_2^* : H^p(B_2, H^q(F_2, R)) \rightarrow H^p(B_1, H^q(F_1, R))$ are induced by the maps f_F and f_B .

The following lemma shows that the Cartan-Leray spectral sequence can easily be generalized to inverse limits of bundles.

Lemma 4.3.1. *Let I be a partially ordered set, let $\{S_i \rightarrow X_i \rightarrow M_i \mid i \in \mathcal{I}\}$ be a set of fiber bundles indexed by $\mathcal{I}$ and let $\{\phi_{ji} : X_i \rightarrow X_j \mid j, i \in \mathcal{I}, j > i\}$ be an inverse system of bundle maps. Let $(E_r^{p,q})_i$ and $(d_r)_i$ denote the terms and differentials on the E_r -page of the Cartan-Leray spectral sequence for the bundle $S_i \rightarrow X_i \rightarrow M_i$. There exists a spectral sequence $\{(E_r^{p,q})_\infty, (d_r)_\infty\}$ such that $(E_r^{p,q})_\infty = \varinjlim (E_r^{p,q})_i$ and $\{(E_r^{p,q})_\infty, (d_r)_\infty\}$ converges to $\varinjlim H^*(X_i, R)$.*

Proof. By the naturality properties for the Cartan-Leray spectral sequence listed above, we have maps $(\phi_r)_{ji} : (E_r^{p,q})_i \rightarrow (E_r^{p,q})_j$ for $j > i$ satisfying $(\phi_r)_{ji} \circ (d_r)_i = (d_r)_j \circ (\phi_r)_{ji}$. We will denote the natural map $(E_r^{p,q})_i \rightarrow (E_r^{p,q})_\infty$ by $(\phi_r)_{\infty i}$. We define the differentials $(d_r)_\infty$ as follows. Given any $x \in (E_r^{p,q})_\infty := \varinjlim (E_r^{p,q})_i$, there exists i and $a \in (E_r^{p,q})_i$ such that $x = (\phi_r)_{\infty i}(a)$. We define $(d_r)_\infty(x)$ to be equal to $(\phi_r)_{\infty i} \circ (d_r)_i(a)$. To see that this is well defined, note that if $x = (\phi_r)_{\infty j}(b)$ for some j and $b \in (E_r^{p,q})_j$ then by the definition of the direct limit there exists $k \geq \max(i, j)$ such that $(\phi_r)_{ki}(a) = (\phi_r)_{kj}(b)$. We therefore have

$$\begin{aligned} (\phi_r)_{\infty i} \circ (d_r)_i(a) &= (\phi_r)_{\infty k} \circ (\phi_r)_{ki} \circ (d_r)_i(a) = (\phi_r)_{\infty k} \circ (d_r)_k \circ (\phi_r)_{ki}(a) \\ &= (\phi_r)_{\infty k} \circ (d_r)_k \circ (\phi_r)_{kj}(b) = (\phi_r)_{\infty k} \circ (\phi_r)_{kj} \circ (d_r)_j(b) = (\phi_r)_{\infty j} \circ (d_r)_j(b). \end{aligned}$$

This shows that $(d_r)_\infty$ is well defined. To check that $(d_r)_\infty \circ (d_r)_\infty$ is the trivial map, note that given $x \in (E_r^{p,q})_\infty$, and $a \in (E_r^{p,q})_i$ such that $x = (\phi_r)_{\infty i}(a)$ we have

$$(d_r)_\infty \circ (d_r)_\infty(x) = (d_r)_\infty \circ (d_r)_\infty \circ (\phi_r)_{\infty i}(a) = (\phi_r)_{\infty i} \circ (d_r)_i^2(a) = 0.$$

We now check that $\{(E_r^{p,q})_\infty, (d_r)_\infty\}$ is a spectral sequence, i.e. that $(E_{r+1}^{p,q})_\infty$ is given by taking the homology of the map $(d_r)_\infty : (E_r)_\infty \rightarrow (E_r)_\infty$. For each i , we have the exact sequence

$$0 \rightarrow \text{Im}((d_r)_i) \rightarrow \ker((d_r)_i) \rightarrow (E_{r+1})_i \rightarrow 0.$$

Note that $(\phi_r)_{ji} \circ (d_r)_i = (d_r)_j \circ (\phi_r)_{ji}$ implies that $(\phi_r)_{ji}$ sends $\text{Im}((d_r)_i)$ into $\text{Im}((d_r)_j)$. Likewise if $(d_r)_i(x) = 0$, then $(d_r)_j \circ (\phi_r)_{ji}(x) = (\phi_r)_{ji} \circ (d_r)_i(x) = 0$, so $(\phi_r)_{ji}$ maps the $\ker((d_r)_i)$ to $\ker((d_r)_j)$. This shows that $(\phi_r)_{ji}$ preserves each of the above exact sequences. Since the direct limit functor is exact on the category of modules (see [Weig]), it follows that we have an exact sequence

$$0 \rightarrow \varinjlim \text{Im}((d_r)_i) \rightarrow \varinjlim \ker((d_r)_i) \rightarrow (E_{r+1})_\infty \rightarrow 0.$$

We now show that $\varinjlim \text{Im}((d_r)_i) = \text{Im}((d_r)_\infty)$ and $\ker((d_r)_\infty) = \varinjlim \ker((d_r)_i)$. Note that if $x \in \varinjlim \text{Im}((d_r)_i)$, then there exists i and $a \in (E_r^{p,q})_i$ such that $x = (\phi_r)_{\infty i}((d_r)_i(a)) = (d_r)_\infty((\phi_r)_{\infty i}(a))$, so $x \in \text{Im}((d_r)_\infty)$ and hence $\varinjlim \text{Im}((d_r)_i) \subset \text{Im}((d_r)_\infty)$. Conversely, if $x \in \text{Im}((d_r)_\infty)$ then $x = (d_r)_\infty((\phi_r)_{\infty i}(a)) = (\phi_r)_{\infty i}((d_r)_i(a))$ for some i and $a \in (E_r^{p,q})_i$, so $x \in \varinjlim \text{Im}((d_r)_i)$. This shows that $\varinjlim \text{Im}((d_r)_i) = \text{Im}((d_r)_\infty)$.

Similarly, if $x \in \varinjlim \ker((d_r)_i)$, then there exists i and $a \in \ker(d_r)_i$ such that $x = (\phi_r)_{\infty i}(a)$, so $(d_r)_\infty(x) = (\phi_r)_{\infty i} \circ (d_r)_i(a) = 0$. Thus $\varinjlim \ker((d_r)_i) \subset \ker((d_r)_\infty)$. If $x \in \ker((d_r)_\infty)$, then given i and $a \in (E_r^{p,q})_i$ such that $x = (\phi_r)_{\infty i}(a)$, $0 = (d_r)_\infty(x) = (\phi_r)_{\infty i} \circ (d_r)_i(a)$. It follows that there exists some $k > i$ such that $\phi_{ki} \circ (d_r)_i(a) = 0$, so $(d_r)_k \circ (\phi_r)_{ki}(a) = 0$. Since $(\phi_r)_{ki}(a) \in \ker((d_r)_k)$ and $x = (\phi_r)_{\infty i}(a) = (\phi_r)_{\infty k} \circ (\phi_r)_{ik}(a)$, it follows that $x \in \varinjlim \ker((E_r^{p,q})_i)$. This shows that $\ker((d_r)_\infty) = \varinjlim \ker((d_r)_i)$.

This shows that the previous exact sequence may be rewritten as

$$0 \rightarrow \text{Im}((d_r)_\infty) \rightarrow \ker((d_r)_\infty) \rightarrow (E_{r+1})_\infty \rightarrow 0,$$

and hence $(E_{r+1})_\infty$ is given by taking the homology of $(E_r)_\infty$ with respect to the map $(d_r)_\infty$. This establishes the claim that $\{(E_r)_\infty, (d_r)_\infty\}$ is a spectral sequence.

Finally, we check convergence. By the second naturality property for the Cartan-Leray sequence listed above, the map $(\phi_r)_{ji}$ sends $(F_k^n)_i$ to $(F_k^n)_j$. Let $(F_k^n)_\infty$ denote the direct limit of these groups. Taking the direct limit of the family of filtrations

$$0 \subset (F_n^n)_i \subset (F_{n-1}^n)_i \subset \cdots \subset (F_0^n)_i \cong H^k(X_i, A)$$

gives a filtration

$$0 \subset (F_n^n)_\infty \subset (F_{n-1}^n)_\infty \subset \cdots \subset (F_0^n)_\infty \cong \lim_{\rightarrow} H^k(X_i, A).$$

The groups $(F_k^n)_i$ fit into short exact sequences of the form

$$0 \rightarrow (F_{k+1}^n)_i \rightarrow (F_k^n)_i \rightarrow (E_\infty^{k,n-k})_i \rightarrow 0,$$

so using the exactness of the direct limit functor again we obtain exact sequences

$$0 \rightarrow (F_{k+1}^n)_\infty \rightarrow (F_k^n)_\infty \rightarrow (E_\infty^{k,n-k})_\infty \rightarrow 0.$$

This shows that the groups $(E_\infty^{k,n-k})_\infty$ are isomorphic to the successive quotients in a filtration of $\lim_{\rightarrow} H^k(X_i, A)$, i.e. the spectral sequence $\{(E_r^{p,q})_\infty, (d_r)_\infty\}$ converges to $\lim_{\rightarrow} H^k(X_i, A)$. □

We now apply this lemma to the cohomology of a mapping solenoid $\widehat{B}_f^{\mathcal{T}}$ for a mapping $f : S \rightarrow M$ where $\mathcal{T} = \{\dots M_2 \rightarrow M_1 \rightarrow M\}$ is a tower of regular covering spaces of M . Recall from the previous section that $\widehat{B}_f^{\mathcal{T}}$ is homeomorphic to the inverse limit $\lim_{\leftarrow} (S_i \times M_i)/(\Gamma/N_i)$, and that

$(S_i \times M_i)/(\Gamma/N_i)$ is a fiber bundle in two different ways, one given by

$$S_i \rightarrow (S_i \times M_i)/(\Gamma/N_i) \rightarrow M$$

and the other given by

$$M_i \rightarrow (S_i \times M_i)/(\Gamma/N_i) \rightarrow S.$$

Applying the previous lemma, for any coefficient module A , we obtain a pair of spectral sequences converging to the cohomology of $\widehat{B}_f^{\mathcal{T}}$. The first of these has E_2 -page given by $\varinjlim H^p(M, H^q(S_i, A))$. By Lemma 3.3.4 and 3.3.6, $\varinjlim H^p(M, H^q(S_i, A)) \cong H^p(M, \varinjlim H^q(S_i, A))$ and $\varinjlim H^q(S_i, A) \cong H^q(\varprojlim S_i, A) = H^q(\widehat{S}_f, A)$. We therefore have that the E_2 -page of this spectral sequence is given by $H^p(M, H^q(\widehat{S}_f, A))$. Applying similar reasoning to the second fibration gives a spectral sequence with E_2 -page given by $H^p(S, H^q(\widehat{M}_{\mathcal{T}}, A))$. We therefore obtain the following lemma.

Lemma 4.3.2. *Let $\mathcal{T}$ be a tower of regular covering spaces over M , and let $\widehat{B}_f^{\mathcal{T}}$ denote the mapping solenoid for a map $f : S \rightarrow M$. For any coefficient module R , there exist a pair of spectral sequences whose E_2 -pages are given by $H^r(S, H^s(\widehat{M}_{\mathcal{T}}, R))$ and $H^r(M, H^s(\widehat{S}_f^{\mathcal{T}}, R))$ that both converge to the cohomology of $\widehat{B}_f^{\mathcal{T}}$.*

The next two lemmas provide basic information about the maps that the fibrations ρ_1 and ρ_2 induce on cohomology. The first follows immediately from the existence of the section $\sigma : S \rightarrow \widehat{B}_f^{\mathcal{T}}$ given by Lemma 4.2.3.

Lemma 4.3.3. *The map $\rho_1^* : H^k(S, R) \rightarrow H^k(\widehat{B}_f^{\mathcal{T}}, R)$ induced by the natural projection $\rho_1 : \widehat{B}_f^{\mathcal{T}} \rightarrow S$ is injective.*

The second lemma follows from the naturality properties of the Cartan-Leray spectral sequence outlined above.

Lemma 4.3.4. *The map $\rho_1^* : H^k(M, R) \rightarrow H^k(\widehat{B}_f^{\mathcal{T}}, R)$ factors through the group $E_{\infty}^{k,0}$ in the spectral sequence $H^r(M, H^s(\widehat{S}_f^{\mathcal{T}}, R)) \implies H^{r+s}(\widehat{B}_f^{\mathcal{T}}, R)$. Likewise, the map $\rho_2^* : H^k(S, R) \rightarrow H^k(\widehat{B}_f^{\mathcal{T}}, R)$ factors through the group $E_{\infty}^{k,0}$ in the spectral sequence $H^r(S, H^s(\widehat{M}_{\mathcal{T}}, R)) \implies H^{r+s}(\widehat{B}_f^{\mathcal{T}}, R)$.*

Proof. Let B_n denote the bundle $(S_n \times M_n)/(\Gamma/N_n)$. We can view the projection $q_n : B_n \rightarrow M$ as a bundle map between the bundle $S_n \rightarrow B_n \rightarrow M$ and the trivial bundle $\{pt\} \rightarrow M \xrightarrow{id} M$. We will denote the terms of the Cartan-Leray spectral sequences for these bundles by $(E_k^{p,q})_1$ and $(E_k^{p,q})_2$, respectively. Note that the Cartan-Leray spectral sequence for the second bundle is trivial, so it follows that $(E_\infty^{k,0})_2 \cong H^k(M, R)$. By the third naturality statement listed above, q_i^* maps $(E_\infty^{k,0})_2$ to $(E_\infty^{k,0})_1$, so the map $q_n^* : H^k(M, R) \rightarrow H^k((S_n \times M_n)/(\Gamma/N_n), R)$ factors through the E_∞ -page of the spectral sequence $H^r(M, H^s(S_n, R)) \implies H^{r+s}(B_n, R)$. Since ρ_1^* is the direct limit of the maps q_n^* , and since the terms of the E_∞ -page for the spectral sequence $H^r(M, H^s(\widehat{S}_f, R)) \implies H^{r+s}(\widehat{B}_f, R)$ are the direct limits of the terms of the E_∞ -page for spectral sequences $H^r(M, H^s(S_n, R)) \implies H^{r+s}(B_n, R)$, ρ_1^* factors as required. The proof of the analogous result for ρ_2 is identical. $\square$

As we will see in the following sections, the previous lemmas can be used under certain circumstances to show that $H^0(\widehat{S}_f, R) \neq R$, so by Lemma 4.1.5 S is not engulfed. For now, we remark the following simple pair of lemmas on the cohomology of mapping solenoids.

Lemma 4.3.5. *Let M be an aspherical space, let $f : S \rightarrow M$ be a continuous map, and let $\widehat{B}_f$ be the mapping solenoid for f . If $\pi_1(M)$ is good, then $H^k(\widehat{B}_f, \mathbb{Z}/p) \cong H^k(S, \mathbb{Z}/p)$ for all k .*

Proof. Since $\pi_1(M)$ is good and aspherical, it follows from Lemma 3.4.5 that $H^k(\widehat{M}, \mathbb{Z}/p)$ is trivial for all $k > 0$. The E_2 -page of the spectral sequence $H^r(S, H^s(\widehat{M}, \mathbb{Z}/p))$ therefore vanishes above the first row and is equal to $H^r(S, \mathbb{Z}/p)$ on the first row since $\widehat{M}$ is connected. Since all differentials in the spectral sequence are trivial, it follows that

$$H^k(\widehat{B}_f, \mathbb{Z}/p) \cong E_\infty^{k,0} \cong E_2^{k,0} \cong H^k(S, \mathbb{Z}/p).$$

$\square$

Lemma 4.3.6. *Let S be an aspherical space with good fundamental group, let $f : S \rightarrow M$ be a continuous map, and let $\widehat{B}_f$ be the mapping solenoid for f . If $\pi_1(M)$ induces the full profinite*

topology on $\pi_1(S)$ and $f_*(\pi_1(S))$ is dense in the profinite topology on $\pi_1(M)$, then $H^k(\widehat{B}_f, \mathbb{Z}/p) \cong H^k(M, \mathbb{Z}/p)$ for all k .

Proof. Since $f_*(\pi_1(S))$ is dense, it follows that from Lemma 4.1.5 that $\widehat{S}_f$ is connected. Since S is aspherical and good, the universal solenoid $\widehat{S}$ over S has trivial cohomology, and since $\pi_1(M)$ induces the full profinite topology on $\pi_1(S)$, $\widehat{S}_f$ is homeomorphic to $\widehat{S}$ by Lemma 3.2.1. Using the spectral sequence $H^r(\pi_1(M), H^s(\widehat{S}_f, \mathbb{Z}/p))$ as in the proof of the previous lemma, the result follows. $\square$

4.4 Solenoids with finite $\mathbb{Z}/p$ -cohomology rings

In this section we apply Lemma 4.3.2 from the previous section to prove the following theorem. Given a space X , we will let $cd_p(X)$ denote the p -cohomological dimension of X , i.e. the largest integer n such that there exists a $\pi_1(X)$ -module A that is annihilated by p^k for some k such that $H^n(X, A) \neq 0$.

Theorem 4.4.1. *Let M be an aspherical space with good fundamental group, and let S be an aspherical space such that $cd_p(S) < cd_p(M)$. Let $f : S \rightarrow M$ be a π_1 -injective map. Suppose that there exists a prime p such that the cohomology groups of $\widehat{S}_f$ over $\mathbb{Z}/p$ have finite dimension. Then $f_*(\pi_1(S))$ is strongly engulfed.*

Proof. Suppose that $f_*(\pi_1(S))$ is not engulfed and the hypotheses of the above theorem hold. It follows by Lemma 4.1.5 that $\widehat{S}_f$ is connected. Since M is an aspherical n -manifold, $\pi_1(M)$ has $\mathbb{Z}/p$ cohomological dimension n , and since $\pi_1(S) < \pi_1(M)$ and S is aspherical, $cd_p(S) = cd_p(\pi_1(S)) \leq cd_p(M) = n$.

Let k be the largest dimension for which $H^k(\widehat{S}_f, \mathbb{Z}/p)$ is non-trivial, and note that $k \leq cd_p(S) < cd_p(M)$. Since $H^k(\widehat{S}_f, \mathbb{Z}/p)$ is finite dimensional, there exists a finite-index subgroup $K < \pi_1(M)$ such that K acts trivially on $H^k(\widehat{S}_f, \mathbb{Z}/p)$. Let $\pi : M_K \rightarrow M$ denote the connected covering space of M corresponding to K . By Lemma 4.1.6, the solenoid given by pulling back the universal solenoid over M_K to S_f^π via f^π is isomorphic as $\pi_1(M_K)$ -space to $\widehat{S}_f$. Note that M_k has good fundamental

group since $\pi_1(M_k) < \pi_1(M)$, so by replacing M by M_K , S by S_f^π and f by f^π if necessary, we can assume that $\pi_1(M)$ acts trivially on $H^k(\widehat{S}_f, \mathbb{Z}/p)$.

Since the $\pi_1(M)$ -action on $H^k(\widehat{S}_f, \mathbb{Z}/p)$ is trivial, the group $H^n(M, H^k(\widehat{S}_f, \mathbb{Z}/p))$ is isomorphic to $H^n(M, \mathbb{Z}/p)^l \cong (\mathbb{Z}/p)^l$, where l is the rank of $H^k(\widehat{S}_f, \mathbb{Z}/p)$. Since $cd_p(M) = n > k$, this group is the upper-rightmost non-trivial entry in the E_2 -page of the spectral sequence $H^r(M, H^s(\widehat{S}_f, \mathbb{Z}/p)) \implies H^{r+s}(\widehat{B}_f, \mathbb{Z}/p)$. It follows that this entry survives to the E_∞ -page of the spectral sequence, so $H^{n+k}(\widehat{B}_f, \mathbb{Z}/p)$ is non-trivial. Since $\pi_1(M)$ is good and $f_*(\pi_1(S))$ is not engulfed, we have by Lemma 4.3.5 that $H^{n+k}(S, \mathbb{Z}/p) \cong H^{n+k}(\widehat{B}_f, \mathbb{Z}/p)$. Since $n + k > cd_p(S)$, however, $H^{n+k}(S, \mathbb{Z}/p)$ is trivial. This gives the desired contradiction.

It follows that $f_*(\pi_1(S))$ is engulfed, so there exists a lift $\tilde{f} : S \rightarrow M'$ for some finite sheeted covering space M' of M . Repeating the above argument, $\tilde{f}_*(\pi_1(S))$ is also engulfed, so by induction we obtain that $\tilde{f}$ is strongly engulfed.

□

Theorem 4.4.1 also has the following immediate corollary:

Corollary 4.4.1. *Let M be an aspherical 3-manifold with infinite fundamental group, and let $f : S \rightarrow M$ be any π_1 -injective immersion of a closed surface into M such that $f_*(\pi_1(S))$ is not strongly engulfed in the profinite topology on $\pi_1(M)$. Then $H^1(\widehat{S}_f, \mathbb{Z}/p)$ is infinite dimensional.*

We remark that if M is a 3-manifold that is not virtually Haken, then this holds for any π_1 -injective immersion of a surface into M by Jaco's virtually Haken criterion.

As a corollary, we obtain a proof of the following well-known fact in 3-manifold topology.

Corollary 4.4.2. *Let T be a torus or Klein bottle, let M be a closed 3-manifold with good fundamental group, and let $f : T \rightarrow M$ be a continuous map. Then f lifts to infinitely many finite sheeted covering spaces of M .*

Proof. Since any finite-sheeted connected covering space of a torus or Klein bottle is itself a torus or Klein bottle, it follows that for any finite-sheeted covering map $\pi : M' \rightarrow M$, the components of T_f^π are Klein bottles or tori. It follows that each component of $\widehat{T}_f$ has finite rank cohomology

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to an n -manifold, since iterated torus bundles have virtually bounded $\mathbb{Z}/p$ -Betti numbers.

4.5 Mapping solenoids associated to non-engulfed surfaces

$\pi_1(M)$. The existence of such maps was first established by Niblo and Wise in [NW1].

$H^r(\pi_1(M), H^s(S_f, \mathbb{Z}/p))$ that are non-trivial are as follows:

$s = 1$	$(\mathbb{Z}/p)^{2g-\epsilon}$	$(\mathbb{Z}/p)^{2-\epsilon}$	0	0
$s = 0$	$H^0(M, \mathbb{Z}/p)$	$H^1(M, \mathbb{Z}/p)$	$H^2(M, \mathbb{Z}/p)$	$H^3(M, \mathbb{Z}/p)$
	$r = 0$	$r = 1$	$\dots$	

where $\epsilon = 0$ if the differential $d_2^{0,1} : H^0(M, H^1(\widehat{S}_f, \mathbb{Z}/p)) \rightarrow H^2(M, \mathbb{Z}/p)$ in the spectral sequence $H^r(M, H^s(\widehat{S}_f, \mathbb{Z}/p)) \Rightarrow H^{r+s}(\widehat{B}_f, \mathbb{Z}/p)$ is surjective, and $\epsilon = 1$ otherwise.

Proof. Since M is aspherical with good fundamental group, it follows from Lemma 3.4.5 that $H^k(\widehat{M}, \mathbb{Z}/p)$ is trivial for all $k > 0$. By Lemma 4.3.5, it follows that $H^k(\widehat{B}_f, \mathbb{Z}/p) \cong H^k(S, \mathbb{Z}/p)$ for all k . By Lemma 4.1.5 and Proposition 4.4.1, $H^0(\widehat{S}_f, \mathbb{Z}/p) \cong \mathbb{Z}/p$ and $H^1(\widehat{S}_f, \mathbb{Z}/p)$ is infinite dimensional. To see that $H^2(\widehat{S}_f, \mathbb{Z}/p)$ is trivial, note that since S_f^π is connected for each finite-sheeted covering space $\pi : M' \rightarrow M$, it follows that $\deg(\pi^f) = \deg(\pi)$ so the result follows from Lemma 3.4.2.

Consider the map $d_2^{0,1} : H^0(M, H^1(\widehat{S}_f, \mathbb{Z}/p)) \rightarrow H^2(M, \mathbb{Z}/p)$ coming from the spectral sequence $H^r(M, H^s(\widehat{S}_f, \mathbb{Z}/p)) \implies H^{r+s}(\widehat{B}_f, \mathbb{Z}/p)$. Since $H^2(M, \mathbb{Z}/p)/\text{Im}(d_2)$ injects in $H^2(\widehat{B}_f, \mathbb{Z}/p) \cong \mathbb{Z}/p$, it follows that either $d_2^{0,1}$ is surjective or $d_2^{0,1}$ has rank $\dim(H^2(M, \mathbb{Z}/p)) - 1$. Since

$$H^1(\widehat{B}_f, \mathbb{Z}) \cong \ker(d_2^{0,1}) \oplus H^1(M, \mathbb{Z}/p),$$

it follows that

$$\dim(\ker(d_2^{0,1})) = \dim(H^1(\widehat{B}_f, \mathbb{Z}/p)) - \dim(H^1(M, \mathbb{Z}/p)) = 2g - \dim(H^2(\mathbb{Z}/p)) = 2g - rk(d_2^{0,1}) - \epsilon,$$

where $\epsilon = 0$ if $d_2^{0,1}$ is surjective and $\epsilon = 1$ otherwise. Thus

$$\dim(H^0(M, H^1(\widehat{S}_f, \mathbb{Z}/p))) = rk(d_2^{0,1}) + \dim(\ker(d_2^{0,1})) = 2g - \epsilon.$$

The map $d_2^{1,1} : H^1(M, H^1(\widehat{S}_f, \mathbb{Z}/p)) \rightarrow H^3(M, \mathbb{Z}/p)$ must be surjective, so $H^1(M, H^1(\widehat{S}_f, \mathbb{Z}/p))$ must have dimension at least 1. If $\epsilon = 0$, then $E_\infty^{2,0} = E_3^{2,0} = 0$, so $\ker(d_2^{1,1}) \cong E_3^{1,1} \cong E_\infty^{1,1} \cong H^2(\widehat{B}_f, \mathbb{Z}/p) \cong \mathbb{Z}/p$. In this case it follows that

$$\dim(H^1(M, H^1(\widehat{S}_f, \mathbb{Z}/p))) = \dim(\ker(d_2^{1,1})) + rk(d_2^{1,1}) = 2 = 2 - \epsilon.$$

If $\epsilon = 1$, then since

$$\mathbb{Z}/p \cong H^2(\widehat{B}_f, \mathbb{Z}/p) \cong E_3^{2,0} \oplus \ker(d_2^{1,1})$$

and $E_3^{2,0} \neq 0$, $\ker(d_2^{1,1})$ is trivial. In this case $d_2^{1,1}$ is an isomorphism, so

$$\dim(H^1(M, H^1(\widehat{S}_f, \mathbb{Z}/p))) = 1 = 2 - \epsilon.$$

The vanishing of the other terms on the E_2 -page are clear, so the proposition follows. $\square$

We remark that we do not know whether the term ϵ in the above proposition can take both the values 0 and 1 in the above theorem. The following lemma shows, however, that if the genus of the dense surface is small enough then the $\epsilon = 1$ case must occur.

Proposition 4.5.2. *Let M be an aspherical 3-manifold, let $f : S \rightarrow M$ be a continuous map of a surface S of genus g into M such that $f_*(\pi_1(S))$ is dense in $\pi_1(M)$. If $\dim(H^1(M, \mathbb{Z}/p)) > g$, then $\epsilon = 1$ in the above theorem.*

Proof. Suppose that $\dim(H^1(M, \mathbb{Z}/p)) > g$. Since $f_*(\pi_1(S))$ is dense in $\pi_1(M)$, it follows that $f^* : H^1(M, \mathbb{Z}/p) \rightarrow H^1(S, \mathbb{Z}/p)$ is injective. Given $0 \neq \alpha \in H^1(M, \mathbb{Z}/p)$, Poincaré duality in S implies that the kernel of the map $H^1(S, \mathbb{Z}/p) \rightarrow H^2(S, \mathbb{Z}/p) \cong \mathbb{Z}/p$ given by $x \mapsto x \cup f^*(\alpha)$ has dimension g . Since $\dim(f^*(H^1(M, \mathbb{Z}/p))) > g$, it follows that there exists an element $\beta \in H^1(M, \mathbb{Z}/p)$ such that $f^*(\beta) \cup f^*(\alpha) \neq 0$. Since $f^*(\beta) \cup f^*(\alpha) = f^*(\beta \cup \alpha)$, it follows that the map $f^* : H^2(M, \mathbb{Z}/p) \rightarrow H^2(S, \mathbb{Z}/p)$ is non-trivial.

Let ω be an element of $H^2(M, \mathbb{Z}/p)$ such that $f^*(\omega) \neq 0$, and let $\sigma : S \rightarrow \widehat{B}_f$ be the section of $\rho_1 : \widehat{B}_f \rightarrow S$ given by Lemma 4.2.3. Since $\rho_2 \circ \sigma = f$, it follows that $\rho_2^*(\omega) \neq 0$. By Lemma 4.3.4, the map $H^2(M, \mathbb{Z}/p) \rightarrow H^2(\widehat{B}_f, \mathbb{Z}/p)$ factors through the group $E_\infty^{2,0}$ on the E_∞ -page of the spectral sequence $H^r(S, H^s(M, \mathbb{Z}/p)) \implies H^{r+s}(\widehat{B}_f, \mathbb{Z}/p)$, so it follows that $\omega \in H^2(M, \mathbb{Z}/p) = E_2^{2,0}$ must survive to the E_∞ -page. This implies that $d_2^{0,1}$ is non-surjective, so $\epsilon = 1$. $\square$

4.6 Mapping solenoids for maps of surfaces into towers of non-Haken 3-manifolds

Our goal in this section is to prove the following proposition:

Proposition 4.6.1. *Let M be an aspherical 3-manifold, and let $f : S \rightarrow M$ be a π_1 -injective immersion of a closed surface of genus g into M . Let $\mathcal{T} = \{\cdots \rightarrow M_2 \rightarrow M_1 \rightarrow M\}$ be a tower of regular covering spaces of M such that*

- *For all $n \in \mathbb{Z}$, there exists k such that n divides the degree of the covering map $M_k \rightarrow M$,*
- *$\pi_1(M_k) < \ker(\pi_1(M) \rightarrow H_1(M))$*
- *M_k is not Haken for all k .*

There exists a finite abelian group T (depending on f) such that the cohomology groups $H^r(\pi_1(M), H^s(\widehat{S}_f^{\mathcal{T}}, \mathbb{Z}))$ satisfy

$s = 2$	$\mathbb{Q}$	0	0	$\mathbb{Q}$
$s = 1$	$\mathbb{Z}^{2g}$	0	$(\mathbb{Q}/\mathbb{Z})^2$	$\mathbb{Q}^{2g}$
$s = 0$	$\mathbb{Z}$	0	T	$\mathbb{Z}$
	$r = 0$	$r = 1$	$\dots$	

We remark that there are many towers $\mathcal{T}$ satisfying the first two conditions in the hypotheses of this proposition, however there are no known infinite towers of covering spaces of non-Haken manifolds (see [CD]).

The proof of Proposition 4.6.1 relies on the following consequence of Jaco's virtually Haken criterion.

Lemma 4.6.1. *Let $\mathcal{T} := \cdots \rightarrow M_2 \rightarrow M_1 \rightarrow M_0 \cong M$ be a regular tower of aspherical 3-manifolds, and let $f : S \rightarrow M$ be a π_1 -injective map of a closed surface into M . If $\widehat{S}_f^{\mathcal{T}}$ has infinitely many components, then M_i is Haken for some i .*

Proof. Let $\pi : \widetilde{M} \rightarrow M$ denote the universal covering space of M , and $\tilde{y} \in \widetilde{M}$ be a basepoint defining the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations, so that $g \cdot y$ is the image of y under the unique deck transformation that takes $\tilde{y}$ to $\tilde{y} \cdot g$, where $\tilde{y} \cdot g$ denotes the monodromy action of $\pi_1(X)$ on $\pi^{-1}(\pi(\tilde{y}))$ by path lifting. Let y_n denote the image of $\tilde{y}$ in M_n , let $x_0 \in f^{-1}(\pi(\tilde{y}))$ and let $x_n \in S_f^{p_0, n} \subset S \times M_n$ denote the point (x_0, y_n) .

Let $G_k < \pi_1(M)$ denote the stabilizer of the component of $S_f^{p_0, k}$ containing x_k under the action of $\pi_1(M)$ on $S_f^{p_0, k}$ by deck transformations. Note that $\pi_1(M_k) < G_k$ for all k since $\pi_1(M_k)$ acts trivially on $S_f^{p_0, k}$. Furthermore, since the component of $S_f^{p_0, k+1}$ containing x_{k+1} maps to the component of $S_f^{p_0, k}$ containing x_k under the covering map $p_{k, k+1}^f$, $G_{k+1} < G_k$.

Suppose that $H^0(S_f^{p_0, k+1}, \mathbb{Z})$ has larger rank than $H^0(S_f^{p_0, k}, \mathbb{Z})$. Since the preimage of some component of $S_f^{p_0, k}$ is disconnected and $p_{k, k+1}^f$ is regular, it follows that the preimage of every component of $S_f^{p_0, k}$ is disconnected. Let y be a point in $(p_{k, k+1}^f)^{-1}(x_k)$ that lies in a different component of $S_f^{p_0, k+1}$ from x_{k+1} , and let $g \in \pi_1(M)$ be an element such that $g \cdot x_{k+1} = y$. By the definition of G_{k+1} , $g \notin G_{k+1}$, however since

$$x_k = p_{k, k+1}^f(y) = p_{k, k+1}^f(g \cdot x_{k+1}) = g \cdot p_{k, k+1}^f(x_{k+1}) = g \cdot x_k,$$

it follows that $g \in G_k$. It follows that the containment $G_{k+1} < G_k$ is proper. If $\widehat{S}_f^{\mathcal{T}}$ has infinitely many components, there must be infinitely many k for which the rank of $H^0(S_f^{p_0, k+1}, \mathbb{Z})$ is strictly larger than the rank of $H^0(S_f^{p_0, k}, \mathbb{Z})$. It follows that $G_{k+1} \neq G_k$ for infinitely many k .

Let M'_k denote the covering space of M corresponding to G_k . Since the action of $\pi_1(S)$ on $S_f^{p_k, 0}$ preserves the component of $S_f^{p_k, 0}$ containing x_k , $\pi_1(S) < G_k$. It follows that $f : S \rightarrow M$ lifts to M'_k for all k by the lifting criterion, so by Jaco's criterion M'_n is Haken for some n . Since M_n covers

M'_n, M_n is Haken as well so the result follows. □

The proof of Proposition 4.6.1 also requires the following lemma from group cohomology, which is a consequence of Shapiro's lemma.

Lemma 4.6.2. *Let G be a group, let N be a G module that decomposes as a direct sum $\bigoplus_{i=1}^n N_i$. Suppose that G acts transitively on the summands N_i , and let $H < G$ be the stabilizer of N_i for some i . Then for all k , $H^k(G, N) \cong H^k(H, N_i)$.*

Proof. This lemma is a direct consequence of material in Chapter III of [Bro], which we cite here. By Proposition III.5.3 in [Bro], the module N is isomorphic to the induced module $\text{Ind}_H^G M_i$. Since the number of indices are finite, H is a finite index subgroup of G . It follows by Proposition III.5.9 in [Bro] that $\text{Ind}_H^G M_i \cong \text{Coind}_H^G M_i$. By Shapiro's lemma, $H^*(H, M_i) \cong H^*(G, \text{Coind}_H^G M_i)$, so the result follows. □

We are now ready to give the proof of Proposition 4.6.1.

Proof of Proposition 4.6.1. Let $\mathcal{T}$ be a tower of covering spaces satisfying the hypotheses of Proposition 4.6.1. Note that since each M_k in the tower $\mathcal{T}$ is non-Haken, M_k is rational homology sphere for all k . The results of Chapter 3 show that

$$H^*(\widehat{M}_{\mathcal{T}}, \mathbb{Z}) \cong \mathbb{Z}, 0, 0, \mathbb{Q}.$$

For simplicity of notation, we will omit the character " $\mathcal{T}$ " and denote $\widehat{M}_{\mathcal{T}}, \widehat{S}_f^{\mathcal{T}}$ and $\widehat{B}_f^{\mathcal{T}}$ by $\widehat{M}, \widehat{S}_f$ and $\widehat{B}_f$ respectively.

Lemma 4.3.2 gives that there is a spectral sequence computing $H^k(\widehat{B}_f, \mathbb{Z})$ with E_2 -page given by $H^r(S, H^s(\widehat{M}, \mathbb{Z}))$. The non-zero groups in this spectral sequence are sufficiently far apart that all differentials in the spectral sequence must be trivial. It follows that

$$H^*(\widehat{B}_f, \mathbb{Z}) \cong \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^{2g}, \mathbb{Q}.$$

Since M is not virtually Haken by assumption, $\widehat{S}_f$ has finitely many components $\widehat{S}_f^1, \widehat{S}_f^2, \dots, \widehat{S}_f^n$ by Lemma 4.6.1. This gives a decomposition of the modules $H^k(\widehat{S}_f, \mathbb{Z})$ of the form

$$H^k(\widehat{S}_f, \mathbb{Z}) \cong \bigoplus_{i=1}^n H^k(\widehat{S}_f^i, \mathbb{Z}).$$

Let $H < \pi_1(M)$ be the stabilizer of $\widehat{S}_f^1$. Since $\pi_1(M)$ acts transitively on these components, we can apply Lemma 4.6.2, which gives

$$H^r(\pi_1(M), H^s(\widehat{S}_f, \mathbb{Z})) \cong H^r(H, H^s(\widehat{S}_f^1, \mathbb{Z})).$$

Let $p_H : M_H \rightarrow M$ be the covering space of M corresponding to H . Note that for large enough i , $\pi_1(M_i)$ acts trivially on the components of $\widehat{S}_f$, so $\pi_1(M_i) < H$. It follows that the map $\widehat{p} : \widehat{M} \rightarrow M$ factors through a map. After choosing basepoints appropriately, we may assume that $\pi_1(S)$ stabilizes $\widehat{S}_f^1$, and hence $f_*(\pi_1(S)) < H$. It follows that $f : S \rightarrow M$ lifts to a map $\tilde{f} : S \rightarrow M_H$.

We claim that $\widehat{S}_f^1 \cong \widehat{S}_{\tilde{f}}$ as an H -module. To see this, note that we have the commutative diagram

$$\begin{array}{ccc} \widehat{S}_f & \longrightarrow & \widehat{M} \\ \downarrow & & \downarrow \widehat{p}_H \\ S_f^{p_H} & \xrightarrow{f^{p_H}} & M_H \\ \downarrow p_H^f & & \downarrow p_H \\ S & \xrightarrow{f} & M \end{array}$$

The existence of the lift $\tilde{f} : S \rightarrow M_H$ gives a section $\sigma : S \rightarrow S_f^{p_H}$ of the map p_H^f such that $f^{p_H} \circ \sigma = \tilde{f}$, and it is easy to check that $\sigma(S)$ is the image of $\widehat{S}_f^1$ in $S_f^{p_H}$. It follows that the set $\widehat{S}_{\tilde{f}} = \{(x, y) \in S \times \widehat{M} \mid \tilde{f}(x) = \widehat{p}_H(y)\}$ is exactly $\widehat{S}_f^1$.

We now consider the mapping solenoid $\widehat{B}_{\tilde{f}}$ for the map $\tilde{f} : S \rightarrow M_H$. Note that since for all n there exists an i such that n divides $[\pi_1(M) : \pi_1(M_i)]$, it follows that for all n there exists a k such

that n divides $[H, \pi_1(M_k)]$. Since M_H is not virtually Haken, by the same argument given above to compute the cohomology of $\widehat{B}_f$,

$$H^*(\widehat{B}_{\tilde{f}}, \mathbb{Z}) \cong \mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^{2g}, \mathbb{Q}.$$

Since $\widehat{S}_{\tilde{f}}$ is connected, $H^0(\widehat{S}_{\tilde{f}}, \mathbb{Z}) \cong \mathbb{Z}$. It follows that the bottom row of the E_2 -page of the spectral sequence $H^r(H, H^s(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \implies H^*(\widehat{B}_{\tilde{f}}, \mathbb{Z})$ is given by $E_2^{k,0} \cong H^k(H, \mathbb{Z})$. Since M is not virtually Haken, the first Betti number of H is trivial, and hence $H^2(H, \mathbb{Z})$ is a torsion abelian group that we will denote by T .

Note that since $\widehat{S}_{\tilde{f}}$ is connected, it follows that for every $M_i \in \mathcal{T}$ such that $\pi_1(M_i) < H$, the covering space $p_{H_i} : M_i \rightarrow M_H$ induces a connected covering space $p_{H_i}^{\tilde{f}} : S_{\tilde{f}}^{p_{H_i}} \rightarrow S$ that has degree $[H : \pi_1(M_i)]$. It follows by Lemma 3.8.3 that $H^2(\widehat{S}_{\tilde{f}}, \mathbb{Z}) \cong \mathbb{Q}$. Since M_H is a rational homology sphere, we have that

$$H^*(H, H^2(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \cong \mathbb{Q}, 0, 0, \mathbb{Q}.$$

We therefore have that the E_2 -page of the spectral sequence $H^r(H, H^s(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \implies H^*(\widehat{B}_{\tilde{f}}, \mathbb{Z})$ has the following form:

$\mathbb{Q}$	0	0	$\mathbb{Q}$
$H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$	$H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$	$H^2(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$	$H^3(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$
$\mathbb{Z}$	0	T	$\mathbb{Z}$

We now compute the groups $H^k(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ for each k .

- $H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \cong \mathbb{Z}^{2g}$:

Since T is a finite group and $E_3^{2,0}$ is the quotient of T by the image of d_2 , $E_3^{2,0}$ is also finite. The group $E_3^{2,0}$ is a stable term of the spectral sequence, however, so $E_3^{2,0} \cong E_\infty^{2,0}$. Since $E_\infty^{2,0} \subset H^2(\widehat{B}_{\tilde{f}}, \mathbb{Z}) \cong \mathbb{Z}$ and $\mathbb{Z}$ is torsion free, it follows that $E_3^{2,0}$ is trivial and hence d_2 surjects onto T . We therefore have the following exact sequence:

$$0 \rightarrow H^1(\widehat{B}_{\tilde{f}}, \mathbb{Z}) \rightarrow H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \xrightarrow{d_3} T \rightarrow 0.$$

Since $H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})$ is a direct limit of torsion free groups, any submodule of $H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})$ is torsion free. It follows that $H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is torsion free. Since $H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is a finitely generated torsion free abelian group with $\mathbb{Z}^{2g}$ as a finite index subgroup, it follows that $H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \cong \mathbb{Z}^{2g}$.

- $H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \cong 0$:

We first claim that $H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is torsion. To see this, note that $H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is the direct limit of groups of the form $H^1(H, H^1(S_i, \mathbb{Z}))$ where S_i denotes the pull-back by $\tilde{f}$ of the cover $M_i \rightarrow M_H$, so it suffices by Lemma 3.8.2 to show that each of these groups is torsion.

Given a finite-sheeted regular covering map $\pi : M' \rightarrow M_H$, let $N \triangleleft H$ denote the finite index subgroup of H given by $\pi_*(\pi_1(M'))$. Note that N acts trivially on $S_{\tilde{f}}^\pi$, so it follows that the action of N on $H^1(S_{\tilde{f}}^\pi, \mathbb{Z})$ is trivial. We therefore have that

$$H^1(N, H^1(S_{\tilde{f}}^\pi, \mathbb{Z})) \cong H^1(N, \mathbb{Z}^{2g(S_{\tilde{f}}^\pi)}) \cong H^1(N, \mathbb{Z})^{2g(S_{\tilde{f}}^\pi)},$$

which is trivial, since M_H has virtually trivial first Betti number. It follows that the restriction map.

$$\text{res}_N^H : H^1(H, (S_{\tilde{f}}^\pi, \mathbb{Z})) \rightarrow H^1(N, (S_{\tilde{f}}^\pi, \mathbb{Z}))$$

is trivial, so the map $\text{cor}_N^H \circ \text{res}_N^H : H^1(H, (S_{\tilde{f}}^\pi, \mathbb{Z})) \rightarrow H^1(H, (S_{\tilde{f}}^\pi, \mathbb{Z}))$ must also be trivial. Since the composition of the corestriction and restriction maps is given by multiplication by $[H : N]$, it follows that any element of $H^1(H, (S_{\tilde{f}}^\pi, \mathbb{Z}))$ has order at most $[H : N]$.

Since $H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is torsion, it follows that the map $d_2 : H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \rightarrow E_2^{3,0} \cong \mathbb{Z}$ must be trivial, hence $E_\infty^{1,1} \cong E_3^{1,1} \cong H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is a torsion group. Since the spectral sequence converges to $H^k(\widehat{B}_{\tilde{f}}, \mathbb{Z})$, we have that a filtration

$$E_\infty^{2,0} \cong F_2^2 \subset F_1^2 \subset F_0^2 \cong H^2(\widehat{B}_{\tilde{f}}, \mathbb{Z})$$

such that $E_\infty^{1,1} \cong F_1^2/F_2^2$. In the computation of $H^0(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ above, however, we showed that $E_\infty^{2,0}$ is trivial, and hence $E_\infty^{1,1} \cong F_1^2 \subset H^2(\widehat{B}_{\tilde{f}}, \mathbb{Z})$. Since $H^2(\widehat{B}_{\tilde{f}}, \mathbb{Z})$ is torsion free, it follows that $E_\infty^{1,1}$ must be trivial. The claim follows.

- $H^2(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \cong (\mathbb{Q}/\mathbb{Z})^2$:

We now have that $E_\infty^{2,0} \cong E_\infty^{1,1} \cong 0$. It follows that $E_\infty^{0,2} \cong H^2(\widehat{B}_{\tilde{f}}, \mathbb{Z}) \cong \mathbb{Z}$. Since $E_2^{2,0} \cong H^1(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z}))$ is trivial, it follows that $E_3^{3,0} \cong E_2^{3,0} \cong \mathbb{Z}$. The map $d_3 : E_3^{0,2} \rightarrow E_3^{3,0} \cong \mathbb{Z}$ has kernel isomorphic to $E_\infty^{0,2} \cong \mathbb{Z}$, so we have the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow E_3^{0,2} \rightarrow \text{Im}(d_3) \rightarrow 0.$$

Since $\mathbb{Z}$ and $\text{Im}(d_3)$ are finitely generated, $E_3^{0,2}$ is finitely generated as well. Since finitely generated subgroups of $\mathbb{Q}$ are cyclic it follows that $E_3^{0,2} \cong \mathbb{Z}$. We note for future reference that this implies that $\text{Im}(d_3)$ is torsion, and hence trivial since $\text{Im}(d_3) \subset E_3^{3,0} \cong \mathbb{Z}$. It follows that $E_4^{3,0} \cong E_\infty^{3,0} \cong \mathbb{Z}$. Since $E_3^{0,2} \cong \mathbb{Z}$ and $E_2^{0,2} \cong \mathbb{Q}$, the exact sequence

$$0 \rightarrow E_3^{0,2} \rightarrow E_2^{0,2} \xrightarrow{d_3} \text{Im}(d_2) \rightarrow 0,$$

shows that $\text{Im}(d_2) \cong \mathbb{Q}/\mathbb{Z}$. This gives us the exact sequence

$$0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow H^2(H, H^1(\widehat{S}_{\tilde{f}}, \mathbb{Z})) \rightarrow E_3^{2,1} \rightarrow 0.$$

Note that $E_3^{2,1} \cong E_\infty^{2,1}$. Since $E_\infty^{1,2}$ and $E_\infty^{0,3}$ are trivial, the terms F_0^3, F_1^3 , and F_2^3 in the filtration

$$E_\infty^{3,0} \cong F_3^3 \subset F_2^3 \subset F_1^3 \subset F_0^3 \cong H^3(\widehat{B}_{\tilde{f}}, \mathbb{Z})$$

are identical. It follows that we have an exact sequence

$$0 \rightarrow E_{\infty}^{3,0} \rightarrow H^3(\widehat{B}_{\widehat{f}}, \mathbb{Z}) \rightarrow E_{\infty}^{2,1} \rightarrow 0.$$

Since $E_{\infty}^{3,0} \cong \mathbb{Z}$ and $H^3(\widehat{B}_{\widehat{f}}, \mathbb{Z}) \cong \mathbb{Q}$, it follows that

$$E_3^{2,1} \cong E_{\infty}^{2,1} \cong \mathbb{Q}/\mathbb{Z}.$$

Plugging this into the exact sequence involving $E_3^{2,1}$ above, we obtain

$$0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow H^2(H, H^1(\widehat{S}_{\widehat{f}}, \mathbb{Z})) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$ -module, this sequence splits, and therefore $H^2(H, H^1(\widehat{S}_{\widehat{f}}, \mathbb{Z})) \cong (\mathbb{Q}/\mathbb{Z})^2$.

• $H^3(H, H^1(\widehat{S}_{\widehat{f}}, \mathbb{Z})) \cong \mathbb{Q}^{2g}$:

Since all differentials entering and leaving the $(3, 1)$ entry on all pages of the spectral sequence are trivial, $E_2^{3,1} \cong E_{\infty}^{3,1}$. Since $E_{\infty}^{k,4-k}$ is trivial for all $k \neq 3$, it follows that $E_2^{3,1} \cong H^4(\widehat{B}_{\widehat{f}}, \mathbb{Z}) \cong \mathbb{Q}^{2g}$.

Since $H^k(H, H^1(\widehat{S}_{\widehat{f}}, \mathbb{Z}))$ is trivial for $k > 3$, we have established that

$$H^r(H, H^1(\widehat{S}_{\widehat{f}}, \mathbb{Z})) \cong \mathbb{Z}^{2g}, 0, (\mathbb{Q}/\mathbb{Z})^2, \mathbb{Q}^{2g}.$$

Since $H^k(H, H^1(\widehat{S}_{\widehat{f}}, \mathbb{Z})) \cong H^k(M, H^1(\widehat{S}_f, \mathbb{Z}))$, the proof is complete. □

We remark that it is not known whether such towers of non-Haken 3-manifolds exists. It is therefore interesting to ask whether the existence of a $\pi_1(M)$ -module such as $H^1(\widehat{S}_f, \mathbb{Z})$ with cohomology groups $\mathbb{Z}^{2g}, 0, (\mathbb{Q}/\mathbb{Z})^2, \mathbb{Q}^{2g}$ can be ruled out on algebraic grounds.

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