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Ergodic theory of the geodesic flow and entropy at infinity

Velozo, Anibal

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Ergodic theory of the geodesic flow and entropy at infinity

Anibal Velozo

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

In this thesis we study the geodesic flow on a non-compact pinched negatively curved manifold. We prove the upper semicontinuity of the entropy map and relate the escape of mass phenomenon with the topological entropy at infinity of the geodesic flow. We also study the thermodynamic formalism of the geodesic flow. We obtain a complete description of the pressure map of potentials that vanish at infinity, and construct Hölder potentials that exhibit phase transitions. We remark that phase transitions for regular potentials is a feature that can only occur in the non-compact situation. We introduce the family of strongly positive recurrent potentials and prove some important properties of such potentials. We also obtain large deviation bounds for the geodesic flow on geometrically finite manifolds and very strongly positive recurrent potentials.

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Chapter 1

Introduction

In this thesis we will study the ergodic theory and the thermodynamic formalism of the geodesic flow on non-compact pinched negatively curved manifolds. In order to put our results into context we will start with a brief discussion on what is known in the compact case, which is better understood than the non-compact case and motivates some of the central topics in this thesis.

By the work of Bowen [B5] and Ratner [Rat], we know that the geodesic flow on a compact negatively curved manifold can be modelled as a suspension flow over a shift of finite type. This symbolic representation has many important consequences to the thermodynamic formalism of Hölder potentials: the topological pressure can be computed using a weighted sum over the periodic orbits, the equilibrium states are unique and satisfy the Gibbs property, the central limit theorem holds, and large deviation estimates are available, just to mention a few (for a more complete discussion we refer the reader to [B7]). Another important property of the geodesic flow on a compact negatively curved manifold is the upper semicontinuity of the entropy map [B1]. In the compact case the upper semicontinuity of the entropy map implies the existence of a measure of maximal entropy. For the geodesic flow, the measure of maximal entropy was constructed independently by Margulis [Mar] and Bowen [B5],

and it is usually called the Bowen-Margulis measure.

Unfortunately, if the ambient manifold is non-compact then most of this discussion does not apply: we do not have at our disposal a symbolic representation, and the proof of the upper semicontinuity of the entropy map does not work. Let me be a bit more precise about these two technical difficulties. First, in the study of non-compact manifolds we are often interested in situations where the injectivity radius is not bounded below (for instance in the presence of 'cusps'); this is a very problematic issue when trying to extend the symbolic model of Bowen and Ratner to the non-compact case. Second, the proof of the upper semicontinuity of the entropy map is based on two hypotheses: the compactness of the ambient space, and that the dynamical system is expansive (or some weaker assumption like h-expansive, or asymptotically h-expansive). If one removes the compactness assumption then the upper semicontinuity of the entropy map does not necessarily hold, even when the dynamical system is expansive. As an example, consider the full shift on a countable (infinite) alphabet. In this case, the entropy map is not upper semicontinuous at any measure of finite entropy. In this thesis we will see that the dynamics of the geodesic flow on a non-compact pinched negatively curved manifold is, in many respects, similar to the one of a countable Markov shift. For this reason, if we want to prove the upper semicontinuity of the entropy map on non-compact manifolds, then the poor behaviour of the entropy map exhibited by the full shift on a countable alphabet is not very encouraging. Fortunately it is not all bad news: in a recent work Paulin, Pollicott and Schapira [PPS] were able to extend many of the results known to hold for Hölder potentials on compact negatively curved manifolds to the non-compact case. More precisely, they proved the uniqueness of equilibrium states and the equivalence between the topological pressure, the Gurevich pressure, and the critical exponent for Hölder potentials (for precise statements see Theorem 2.3.12 and Theorem 2.3.14). In this work we will continue the study of the thermodynamic formalism of Hölder

potentials, but we will also be interested in continuous potentials. In the continuous category the techniques developed in [PPS] do no apply.

Before stating some of the main results of this thesis we will introduce some notation. The dynamical system of interest will always be the geodesic flow on a pinched negatively curved manifold. In particular, whenever we say 'invariant measure', we really mean 'non-negative Borel measure invariant by the geodesic flow'. The measure theoretic entropy of an invariant probability measure μ is denoted by $h_\mu(g)$. We say that $(\mu_n)_n$ converges vaguely to μ if for every compactly supported continuous function f we have that $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$. Similarly, we say that $(\mu_n)_n$ converges in the weak-* topology to μ if for every bounded continuous function f we have that $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$. Since M is non-compact, the vague limit of a sequence of probability measures might not be a probability measure. In other words, a sequence of probability measures might lose mass. We recall that the mass of a measure μ is the number $\mu(T^1M)$, and it is denoted by $|\mu|$. A fundamental quantity in this thesis will be the topological entropy at infinity of the geodesic flow (see Definition 5.2.1), which we denote by δ_∞. The following result is proved in Section 5.

Theorem 1.0.1 (=Theorem 5.2.3)**.** *Let (M, g) be a pinched negatively curved manifold. Let $(\mu_n)_n$ be a sequence of invariant probability measures converging to μ in the vague topology. Then*

$$\limsup_{n\to\infty} h_{\mu_n}(g) \leqslant |\mu| h_{\mu/|\mu|}(g) + (1 - |\mu|)\delta_\infty.$$

If the sequence converges vaguely to the zero measure, then the right hand side is understood as δ_∞.

A simple consequence of Theorem 1.0.1 is the upper semicontinuity of the entropy map.

Theorem 1.0.2 (=Theorem 5.2.6)**.** *Let (M, g) be a pinched negatively curved manifold. Let $(\mu_n)_n$ be a sequence of invariant probability measures converging to μ in the weak-* topology. Then*

$$\limsup_{n\to\infty} h_{\mu_n}(g) \leqslant h_\mu(g).$$

A result similar to Theorem 1.0.1 was previously obtained by Einsiedler, Kadyrov and Pohl in the context of homogeneous dynamics (see [EKP]). Later on, Kadyrov and Pohl proved that the formula obtained in [EKP] is sharp (see [KP]). In a joint work with F. Riquelme, we adapted the methods used in [EKP] to obtain Theorem 1.0.1 for geometrically finite manifolds (see [RV]). In the paper [V2] we extended the results from [RV] to arbitrary pinched negatively curved manifolds.

The topological entropy at infinity has a measure theoretic counterpart; this is what we call the measure theoretic entropy at infinity. Since its definition does not require to introduce a lot of notation we will provide it here. The measure theoretic entropy at infinity of the geodesic flow is defined as

$$h_\infty = \sup_{(\mu_n)_n} \limsup_{n\to\infty} h_{\mu_n}(g),$$

where the supremum runs over sequences of invariant probability measures converging vaguely to the zero measure. We will prove a type of variational principle for the entropy at infinity.

Theorem 1.0.3 (=Theorem 5.3.2)**.** *The topological entropy at infinity is equal to the measure theoretic entropy at infinity. In other words $\delta_\infty = h_\infty$.*

The measure theoretic entropy at infinity was introduced in [IRV] and proved to be equal to the topological entropy at infinity for the geodesic flow on extended Schottky manifolds (see Definition 7.4.1) via symbolic methods. This result was later extended to cover all geometrically finite manifolds in [RV]. In this thesis we will constantly refer to real valued functions with domain T^1M as 'potentials'; this is a standard

convention in thermodynamic formalism. The topological pressure of a potential F is the quantity

$$P(F) = \sup_{\mu \in \mathcal{M}(g)} \{h_\mu(g) + \int F d\mu\},$$

where the supremum runs over the space of invariant probability measures $\mathcal{M}(g)$. An invariant probability measure satisfying $P(F) = h_\mu(g) + \int F d\mu$, is called an equilibrium state for the potential F. We will constantly refer to the map $t \mapsto P(tF)$ as the pressure map of the potential F. We denote by $C_0(T^1M)$ the space of continuous potentials vanishing at infinity (see Definition 3.1.1). Among potentials vanishing at infinity we will be particularly interested in strongly positive recurrent potentials (SPR for short); these are potentials satisfying the critical gap condition $P(F) > \delta_\infty$ (for the general definition of SPR potentials see Definition 2.3.20). One of the main features of SPR potentials is part (1) in our next result; in the compact case this is an immediate consequence of the upper semicontinuity of the entropy map. Parts (2) and (3) exhibit some big differences between the compact and the non-compact cases.

Theorem 1.0.4 (=Theorem 6.1.8)**.** *Let (M, g) be a pinched negatively curved manifold anf $F \in C_0(T^1M)$. Let $(\mu_n)_n$ be a sequence of invariant probability measures such that*

$$\lim_{n\to\infty} \left(h_{\mu_n}(g) + \int F d\mu_n\right) = P(F).$$

Then the following statements hold.

1. *If F is SPR, then $(\mu_n)_n$ converges in the weak-* topology to an equilibrium state of F.*

2. *Suppose that F does not admit any equilibrium state. Then $(\mu_n)_n$ converges vaguely to the zero measure. In this case we have $P(F) = \delta_\infty$.*

3. *Suppose that F does admit an equilibrium state. Then the accumulation points*

of $(\mu_n)_n$ *lies in the set*

$$\{t\mu : t \in [0,1] \text{ and } \mu \text{ is an equilibrium state of } F\}.$$

We emphasize that we require no higher regularity (on F) than continuity. In particular the theory developed in [PPS] does not apply. A big difference with respect to more regular potentials is the lack of uniqueness of equilibrium states for continuous potentials. It is proven in [IV] that one can slightly C^0-perturb any bounded continuous potential $F \in C_b(T^1M)$ into a potential with uncountably many equilibrium states. A crucial ingredient for that result is Theorem 1.0.2, which allows us to identify subderivatives of the pressure at F to its equilibrium states. For Hölder SPR potentials we can compute the first derivative of the topological pressure.

Theorem 1.0.5 (=Theorem 6.1.9)**.** *Let* M *be a pinched negatively curved manifold and* $F \in C_0(T^1M)$ *be a SPR Hölder potential. For every* $G \in C_b(T^1M)$ *the following holds*

$$\frac{d}{dt}_{|t=0} P(F+tG) = \int G d\mu_F,$$

where μ_F *is the equilibrium state of* F*.*

We will prove in Section 7.2 that SPR potentials are open and dense in $C_0(T^1M)$. Theorem 1.0.1 and Theorem 1.0.5 are important ingredients in order to describe the pressure map of positive potentials in $C_0(T^1M)$, as the following result establishes.

Theorem 1.0.6 (=Theorem 7.1.2)**.** *The pressure map,* $t \mapsto P(tF)$*, of a positive Hölder potential* F *that vanishes at infinity verifies the following properties:*

(1) for every $t \in \mathbb{R}$ *we have that* $P(tF) \geqslant \delta_\infty$

(2) the function $t \mapsto P(tF)$ *has a horizontal asymptote at* $-\infty$*, that is*

$$\lim_{t \to -\infty} P(tF) = \delta_\infty.$$

Moreover, if $t_F := \sup\{t \leqslant 0 : P(tF) = \delta_\infty\}$*, then*

(3) for every $t > t_F$ *the potential* tF *has an equilibrium state, and*

(4) the pressure map of F *is differentiable in* (t_F, ∞)*, and it verifies*

$$P(tF) = \begin{cases} \delta_\infty & \text{if } t < t_F \\ \text{strictly increasing} & \text{if } t > t_F, \end{cases}$$

(5) If $t < t_F$ *then the potential* tF *has not equilibrium measure.*

We remark that in Theorem 1.0.6 the number t_F can be $-\infty$; this is the case if $P(tF) > \delta_\infty$, for all $t \in \mathbb{R}$. In the non-compact case the pressure map is not always regular ([S2] and [S3] are good references in the context of countable Markov shifts). Sometimes the pressure map can develop singularities: these singularities are usually called phase transitions. Phase transitions usually detect a significant change in the dynamics of our system (from the point of view of the potential). We will be interested in finding conditions that guarantee the existence and the lack of phase transitions (see Section 7.4 and Section 7.3 resp.). In general it is a difficult problem to determine when a potential will (or not) exhibit phase transitions. For this reason we will restrict our attention to some particular classes of geometrically finite manifolds (see Definition 2.3.6). Since for the geodesic flow on non-compact manifolds we do not know how regular is the pressure map (our best result on the regularity of the pressure is Theorem 1.0.5), we will use the following definition for our phase transitions.

Definition 1.0.7 (Phase transition)**.** *We say that a potential* F *exhibits* a phase transition at t_0 *if there exists* $\epsilon > 0$ *such that* $P(tF)$ *has an equilibrium state for* $t \in (t_0, t_0 + \epsilon)$*, but it has not for* $t \in (t_0 - \epsilon, t_0)$ *(or vice versa). A potential* F *exhibits a phase transition if it exhibits a phase transition for some* $t_0 \in \mathbb{R}$*.*

We remark that in the compact case Hölder potentials cannot develop phase transitions. This follows from [BR] and the symbolic model available for those systems. The philosophy behind our construction of phase transitions is very simple: if a potential decays very slowly to zero through the cusps, then it is likely to develop a phase transition. After the paper [V1] was uploaded to arxiv, G. Iommi pointed out to us that the same principle was used in the construction of phase transitions for Pomeu-Manneville maps. As in Theorem 1.0.6 we use the notation $t_F = \sup\{t : P(tF) = \delta_\infty\}$.

Theorem 1.0.8 (=Theorem 7.4.1)**.** *There exists a geometrically finite manifold M such that the following holds. For every non-negative Hölder continuous potential F going* slowly to zero *through the cusp of M we have that*

1. *$t_F \in [-1, 0)$,*
2. *The potential tF has equilibrium measure for $t > t_F$,*
3. *The potential tF has not equilibrium measure for $t < t_F$.*

In other words, the pressure map of F exhibits a phase transition at $t = t_F$. Moreover, the pressure map is differentiable in $(-\infty, t_F) \cup (t_F, \infty)$. With respect to the behaviour at $t = t_F$ we have two possibilities:

(4) If the potential $t_F F$ does not have an equilibrium state, then the pressure map is differentiable everywhere.

(5) If $t_F F$ has an equilibrium state, then the pressure map is not differentiable at $t = t_F$.

Theorem 1.0.9 (=Theorem 7.4.2)**.** *There exists an extended Schottky manifold for which the geodesic flow has a measure of maximal entropy and the geometric potential exhibits a phase transition.*

We remark that the manifolds in Theorem 1.0.8 can be hyperbolic, and that the class of potentials going slowly to zero will be introduced in Section 7.4.3 (Definition 7.4.11). The family of extended Schottky manifolds is defined in Section 7.4.1. We would like to emphasize that phase transitions for the geodesic flow were also constructed in [IJ] and [IRV] by symbolic methods (for the modular surface and extended Schottky manifolds resp.). Our method has the advantage of being more geometric (we do not use any symbolic representation) and that we can incorporate the geometric potential to the family of potentials exhibiting phase transitions.

In Section 8 we will obtain large deviation estimates for the geodesic flow. In order to state our results we need to introduce some notation. Each periodic orbit of the geodesic flow defines an invariant probability measure, this is just Lebesgue measure on the orbit divided by the length of the orbit. More precisely, to each periodic orbit τ we associate an invariant probability measure μ_τ. We refer to the measures of the form μ_τ as periodic measures. The set of periodic measures is denoted by $\mathcal{M}_p(g)$. Each periodic measure η has associated the length of its original periodic orbit, we refer to this number as the length of the periodic measure and it is denoted by $l(\eta)$. Given $W \subset T^1M$ and real numbers r, s we define

$$\mathcal{M}_p(W, r, s) = \{\eta \in \mathcal{M}_p(g) : l(\eta) \in [s, r] \text{ and } \eta(W) > 0\}.$$

From now on we will fix a relatively compact open subset W of T^1M intersecting the non-wandering set of the geodesic flow and c a sufficiently large constant (as in Proposition 8.0.1). We denote by $\mathcal{M}_{\leqslant 1}(g)$ the space of invariant sub-probability measures of the geodesic flow, that is, invariant measures such that $|\mu| \in [0, 1]$. We remark that $\mathcal{M}_{\leqslant 1}(g)$ endowed with the vague topology is a compact convex metric space (see Section 2.1). Given a potential $G \in C_0(T^1M)$ we define the G-pressure

$Q_G : C_0(T^1M) \to \mathbb{R}$, as

$$Q_G(f) := P(G+f) - P(G),$$

and the rate function

$$I_G(\mu) =: \sup_{f \in C_0(T^1M)} \{ \int f d\mu - Q_G(f) \},$$

whose domain is $\mathcal{M}_{\leqslant 1}(g)$. Our next result follows easily from [Pol] (see also [Kif]).

Theorem 1.0.10 (=Theorem 8.1.1)**.** *Let M be a pinched negatively curved manifold, G a Hölder potential in $C_0(T^1M)$, and $\mathcal{K}$ a closed subset of $\mathcal{M}_{\leqslant 1}(g)$. Then we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c) \cap \mathcal{K}} \exp(l(\tau) \int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau) \int G d\mu_\tau)} \right) \leqslant -\beta,$$

where $\beta = \inf_{\mu \in \mathcal{K}} I_G(\mu)$.

If we assume that G is SPR, then $\beta = \inf_{\mu \in \mathcal{K}} I_G(\mu)$ is positive if and only if $\mathcal{K}$ does not contain the equilibrium state of G (see Lemma 8.2.5). For our large deviation lower bound we will restrict to geometrically finite manifolds. This is not an optimal assumption, and we expect that Theorem 1.0.11 can be extended to the entire class of negatively curved manifolds. We remark that for the lower bound we use 'very strongly positive recurrent potentials' (vSPR for short), which is a sub-class of SPR potentials (for a precise definition see Definition 7.3.2). This is done in order to ensure that any compact Hölder perturbation of our potential is still SPR. In the general case there is still some work to do (we need a more general version than Theorem 2.3.15), and this is the main reason why we restrict to geometrically finite manifolds.

Theorem 1.0.11 (=Theorem 8.2.2)**.** *Let M be a geometrically finite manifold with cusps and G a vSPR potential. Let $\mathcal{U}$ be an open subset of $\mathcal{M}_{\leqslant 1}(g)$. Then we have*

$$\liminf_{t\to\infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{U} \cap \mathcal{M}_p(W,t,t-c)} \exp(l(\tau) \int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau) \int G d\mu_\tau)} \right) \geqslant -\beta,$$

where $\beta = \inf_{\mu \in \mathcal{U}} I_G(\mu)$.

We remark that for hyperbolic geometrically finite manifolds the zero potential is vSPR, in particular we obtain the following result.

Theorem 1.0.12 (=Theorem 8.2.7)**.** *Let M be a hyperbolic geometrically finite manifold with cusps. Then the following are true.*

1. *Let $\mathcal{K}$ a closed subset of $\mathcal{M}_{\leqslant 1}(g)$ not containing the measure of maximal entropy. Then we have*

$$\limsup_{t \to \infty} \frac{1}{t} \log \left(\frac{\#(\mathcal{M}_p(W, t, t-c) \cap \mathcal{K})}{\#\mathcal{M}_p(W, t, t-c)} \right) \leqslant -\beta,$$

   where $\beta = \inf_{\mu \in \mathcal{K}} I_0(\mu)$ is a positive number.

2. *Let $\mathcal{U}$ be an open subset of $\mathcal{M}_{\leqslant 1}(g)$ whose closure does not contain the measure of maximal entropy. Then we have*

$$\liminf_{t \to \infty} \frac{1}{t} \log \left(\frac{\#(\mathcal{M}_p(W, t, t-c) \cap \mathcal{K})}{\#\mathcal{M}_p(W, t, t-c)} \right) \geqslant -\gamma,$$

   where $\gamma = \inf_{\mu \in \mathcal{U}} I_0(\mu)$ is a positive number.

Chapter 2

Preliminaries

In this section we will collect some important facts about the geodesic flow on a pinched negatively curved manifolds. The definitions and facts given in this section will be constantly used in following sections. Since the spaces of interest for this thesis are non-compact we need to be careful with the spaces of invariant probability measures that we use (it is a technically unpleasant to restrict all our attention to the space of invariant probability measures, which is non-compact). We start by making explicit the spaces of measures that we will use, and their topologies.

2.1 Measure theory

Let (X, d) be a locally compact metric space and $T : X \to X$, a continuous map. In this thesis whenever we say *measure* we mean a non-negative countably additive measure. The mass of a measure is the number $\mu(X)$ and it is denoted by $|\mu|$. We say that a Borel measure μ is T-invariant if $\mu(T^{-1}A) = \mu(A)$, for every Borel set A. We denote by $\mathcal{M}(X, T)$ the space of Borel T-invariant probability measures on X and $\mathcal{M}_{\leqslant 1}(X, T)$ the space of Borel T-invariant measures such that $|\mu| \in [0, 1]$ (we also refer to them as sub-probability measures). Clearly $\mathcal{M}(X, T) \subset \mathcal{M}_{\leqslant 1}(X, T)$. We endow the space $C_b(X)$ (resp. $C_c(X)$) of bounded (resp. compactly supported) continuous

functions with the uniform norm $||f||_0 = \sup_{x\in X} |f(x)|$. We endow $\mathcal{M}(X,T)$ with the *weak-* topology*: a sequence $(\mu_n)_n$ converges weakly to μ if for every $f \in C_b(X)$ we have

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu.$$

In a similar way we endow $\mathcal{M}_{\leqslant 1}(X,T)$ with the *vague topology*: a sequence $(\mu_n)_n$ converges vaguely to μ if for every $f \in C_c(X)$ we have

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu.$$

If there exists a compact exhaustion of X, i.e. an increasing sequence $(K_n)_n$ of compact sets such that $X = \bigcup_{n\geqslant 1} K_n$, then the space $C_c(X)$ is separable. In this case consider a dense subset $(f_n)_n$ of the unit ball of $C_c(X)$. We define a metric d on $\mathcal{M}_{\leqslant 1}(X,T)$ by the formula

$$d(\mu_1,\mu_2) = \sum_{n\geqslant 1} \frac{1}{2^n} |\int f d\mu_1 - \int f d\mu_2|.$$

This metric is compatible with the vague topology. By Banach-Alaoglu theorem we know that $\mathcal{M}_{\leqslant 1}(X,T)$ is a compact metric space. If X is a non-compact manifold, then X admits a compact exhaustion, and therefore this discussion applies to that case. The connection between this two topologies is given by the following simple fact.

Theorem 2.1.1. *Let X be a locally compact metric space. Then for a sequence $(\mu_n)_n \subset \mathcal{M}(X,T)$ the following statements are equivalent.*

1. *$(\mu_n)_n$ converges to μ in the weak-* topology.*

2. *$(\mu_n)_n$ converges to μ in the vague topology and μ is a probability measure.*

Theorem 2.1.1 tells us that the only obstruction to converge in the weak-* topology is the possible escape of mass. In Section 3 we will explain the topological structure

of the space of invariant sub-probability measures for the geodesic flow, which also leads to a description of the space of invariant probability measures.

2.2 Entropy theory

This theory provides tools to measure the complexity of a dynamical system. In this thesis we will be mainly interested in the following two notions of entropy.

2.2.1 Topological entropy

Let (X, d) will be a metric space and $T : X \to X$, a continuous map. We define the dynamical metrics $(d_n)_n$ given by the formula:

$$d_n(x, y) = \max_{i \in \{0,\dots,n-1\}} d(T^i x, T^i y).$$

We denote by $B_n(x, r)$ to the ball centered at x of radius r in the metric d_n. A ball $B_n(x, r)$ is also called a (n, r)-*dynamical ball*. Given a compact subset K of X we denote by $N(K, n, r)$ to the minimum number of (n, r)-dynamical balls needed to cover K. The *topological entropy* of (X, d, T) is defined as

$$h_d(T) = \sup_{K \subset X} \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{n} \log N(K, n, r),$$

where the last limit runs over compact subsets of X. This definition was first introduced by R. Bowen in [B1] as a way to extend the classical definition of topological entropy on compact spaces to the noncompact setting.

2.2.2 Measure theoretic entropy

Given a countable measurable partition $\mathcal{P} = \{P_i\}_{i\in I}$ of X we define the entropy of $\mathcal{P}$ as

$$H_\mu(\mathcal{P}) = -\sum_{i\in I} \mu(P_i)\log\mu(P_i).$$

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ we can construct the smallest common refinement of $\mathcal{P}$ and $\mathcal{Q}$, this is denoted by $\mathcal{P}\vee\mathcal{Q}$. Define

$$h_\mu(T,\mathcal{P}) = \lim_{n\to\infty}\frac{1}{n}H(\bigvee_{i=0}^{n} T^{-i}\mathcal{P}).$$

By taking the supremum over all countable partitions of finite entropy we obtain the *entropy of T with respect to μ*, this is denoted by $h_\mu(T)$. In other words

$$h_\mu(T) = \sup_{\mathcal{P}} h_\mu(T,\mathcal{P}),$$

where the supremum runs over all countable partitions of finite entropy.

Definition 2.2.1 (Entropy map)**.** *Given a dynamical system (X,d,T) we refer to the map $\mu\mapsto h_\mu(T)$, as the* entropy map*.*

Katok's entropy formula

The topological entropy and the measure theoretic entropy seem, at first, a bit unrelated. The following theorem provides a very strong connection between them, it helps to understand the measure theoretic entropy in the spirit of the topological entropy.

Theorem 2.2.2. *[A. Katok] Let (X,d) be a compact metric space and $T: X\to X$, a continuous transformation. Let μ be an ergodic T-invariant probability measure and $\delta\in(0,1)$. Then*

$$h_\mu(T) = \lim_{r\to\infty}\liminf_{n\to\infty}\frac{1}{n}\log N_\mu(n,r,\delta),$$

where $N_\mu(n, r, \delta)$ is the minimum number of (n, r)-dynamical balls needed to cover a set of μ-measure at least $1 - \delta$. In particular the limit above is independient of $\delta \in (0, 1)$.

If one follows the proof of Theorem 2.2.2 (see Section 1 in [Kat]), it is clear that the inequality

$$h_\mu(T) \leqslant \lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log N_\mu(n, r; \delta),$$

holds regardless of the compactness of X. For most of our purposes this inequality will be enough. Recently F. Riquelme [Riq] proved the equality when X is assumed to be a manifold and T a Lipschitz map. More precisely we have the following result.

Theorem 2.2.3. *Let (X, d) be a metric space and $T : X \to X$ a continuous transformation. Then for every ergodic T-invariant probability measure μ we have*

$$h_\mu(T) \leqslant \lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log N_\mu(n, r, \delta),$$

where $N_\mu(n, r, \delta)$ as in Theorem 2.2.2. If we moreover assume that (X, d) is a topological manifold and T is Lipschitz, then the equality holds.

The following two definitions will be of great importance in this thesis.

Definition 2.2.4 (Simplified entropy formula)**.** *We say that (X, d, T) satisfies a* simplified entropy formula *if there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, for every ergodic probability measure μ and $\delta \in (0, 1)$ we have*

$$h_\mu(T) = \limsup_{n\to\infty} \frac{1}{n} \log N_\mu(n, \epsilon, \delta).$$

We say that (X, d, T) satisfies a simplified entropy inequality *if there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, for every ergodic probability measure μ and $\delta \in (0, 1)$*

we have

$$h_\mu(T) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log N_\mu(n, \epsilon, \delta).$$

Definition 2.2.5 (Weak entropy dense)**.** *We say that the space of ergodic measures is* weak entropy dense *in* $\mathcal{M}(X,T)$ *if the following holds. For every* $\mu \in \mathcal{M}(X,T)$ *and* $\eta > 0$ *there exists a sequence* $(\mu_n)_n$ *of ergodic measures satisfying*

1. $(\mu_n)_n$ *converges in the weak-* topology to* μ
2. *For every* $n \in \mathbb{N}$ *we have* $h_{\mu_n}(T) > h_\mu(T) - \eta$.

We also refer to this property by saying that (X, d, T) *is* weak entropy dense.

We will prove in Section 4 that the geodesic flow on a pinched negatively curved manifold satisfies a mild modification of the simplified entropy inequality and that its ergodic measures are weak entropy dense.

2.3 Geodesic flows

Let (M, g) be a complete Riemannian manifold. We define the unit tangent bundle of M as $T^1M = \{v \in TM : ||v||_g = 1\}$. Since (M, g) is complete, the geodesic flow on T^1M is well defined for all times, we denote it by $(g_t)_{t\in\mathbb{R}}$. The Riemannian metric g makes M into a metric space, the induced distance function (shortest path distance) is denoted by d. Let $\pi : T^1M \to M$ be the canonical projection. We define a metric on T^1M, which we still denote by d, in the following way

$$d(x, y) = \max_{t\in[0,1]} d(\pi g_t(x), \pi g_t(y)), \tag{2.1}$$

for every $x, y \in T^1M$. We emphasize that this is the metric used in all the statement about the geodesic flow (but other possible candidates of metrics are usually uniformly equivalent to d, at least in the pinched negatively curved case). The universal cover

of M will be denoted by $\widetilde{M}$. For simplicity we will use the same notation used for M to denote the metric and distance function on $\widetilde{M}$. Similarly to the construction of the metric on T^1M we define a metric on $T^1\widetilde{M}$. We remark that (n,ϵ)-dynamical balls in $T^1\widetilde{M}$ will play an important role in Section 4.2. We will restrict the discussion given in Section 2.1 to the geodesic flow. In this case we will use the following notation.

Definition 2.3.1. *The space of invariant probability measures of the geodesic flow is denoted by $\mathcal{M}(g)$. The space of invariant sub-probability measures of the geodesic flow is denoted by $\mathcal{M}_{\leqslant 1}(g)$.*

Similarly, we restrict the discussion given in subsection 2.2 to the geodesic flow. We will use the following notation.

Definition 2.3.2. *Given $\mu \in \mathcal{M}(g)$, we denote by $h_\mu(g)$ to the measure theoretic entropy of the measure μ under the map g_1 (the time one map of the geodesic flow). We denote by $h_{top}(g)$ to the topological entropy of the geodesic flow (the entropy of g_1), where the metric used to compute the topological entropy is given by equation (2.1).*

2.3.1 Structure of negatively curved manifolds

From now on we will assume that (M,g) is a complete Riemannian manifold of negative sectional curvature. We will moreover assume that $K_g \in [-a,-b]$, for some $a,b>0$, where K_g is the sectional curvature of M. We refer to a manifold satisfying those properties as a *pinched negatively curved manifold*. As before, the universal cover of M is denoted by $\widetilde{M}$.

The space $\widetilde{M}$ has a natural compactification, the so-called Gromov compactification. As with any compactification, we will add a 'boundary' to $\widetilde{M}$ in order to make it compact, the visual boundary $\partial_\infty\widetilde{M}$. The visual boundary, informally speaking, is the set of geodesic rays in $\widetilde{M}$ up to bounded distance, in other words we identify two

geodesic rays $\alpha, \beta : [0, \infty) \to \widetilde{M}$, if $t \mapsto d(\alpha(t), \beta(t))$ is a bounded function. We emphasize that $\widetilde{M} \cup \partial_\infty \widetilde{M}$ is homeomorphic to a closed ball. We remark that using the ball model of hyperbolic space we can identify this compactification with the round unit ball; this is also the picture we will have in mind even if the curvature is not constant.

We denote by $\mathrm{Iso}(\widetilde{M})$ to the group of isometries on $\widetilde{M}$. It is well known that every isometry of $\widetilde{M}$ extends to a homeomorphism of $\partial_\infty \widetilde{M}$ and that the fundamental group of M acts (via Deck transformations) isometrically, freely and discontinuously on $\widetilde{M}$. We identify $\pi_1(M)$ with a subgroup $\Gamma < \mathrm{Iso}(\widetilde{M})$. In this context there is a complete classification of the elements of $\mathrm{Iso}(\widetilde{M})$: an isometry can be elliptic, hyperbolic or parabolic. An *elliptic isometry* fixes a point in $\widetilde{M}$, this type of isometry will not be presented in Γ since M is a manifold. A *hyperbolic isometry* fixes (pointwise) exactly two points at infinity and the geodesic connecting these two points (setwise), this geodesic is called the *axis* of the hyperbolic isometry. The axis of a hyperbolic isometry in Γ will descend to a closed geodesic on M. A *parabolic isometry* fixes exactly one point at infinity. Parabolic subgroups will be important when studying geometrically finite manifolds, they are defined as follows

Definition 2.3.3. *A* parabolic subgroup *of* $Iso(\widetilde{M})$ *is a group of isometries where each element is parabolic, and they all fix the same point at infinity. A* maximal parabolic subgroup *of* Γ *is a parabolic subgroup that is not strictly contained in another parabolic subgroup of* Γ.

Definition 2.3.4. *The* limit set *of* Γ, *which we denote by* $L(\Gamma)$, *is the set of accumulation points in* $\partial_\infty \widetilde{M}$ *of the orbit of a point* $x \in \widetilde{M}$ *under* Γ. *The* conical limit set, *denoted by* $L_c(\Gamma)$, *is the set of points* $\xi \in \partial_\infty \widetilde{M}$ *such that there exists a sequence of translates of* $x \in \widetilde{M}$ *under* Γ *which converges to* ξ *while staying at bounded distance from a geodesic ray ending at* ξ.

Through this thesis, $\pi_1(M) = \Gamma$ will be a *non-elementary group*, in other words it

is not generated by one hyperbolic element, nor a parabolic subgroup (see for instance [Bow]). In this case we can relate the non-wandering set of the geodesic flow with the limit set of Γ. To make this precise we denote by $\Omega \subset T^1M$ to the *non-wandering set* of the geodesic flow. We start by recalling the Hopf's parametrization of $T^1\widetilde{M}$. The unit tangent bundle of $\widetilde{M}$ is identified with

$$T^1\widetilde{M} = (\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M})\backslash(\text{Diagonal}) \times \mathbb{R},$$

via Hopf's coordinates by sending each vector $\tilde{v} \in T^1\widetilde{M}$ to $(v_-, v_+, b_{v_+}(o, \pi(\tilde{v})))$. Here v_- and v_+ are respectively the negative and positive ends at infinity of the oriented geodesic line determined by $\tilde{v}$ in $\widetilde{M}$, and $b_{v_+}(o, \pi(\tilde{v}))$ is the Busemann function defined for all $x, y \in \widetilde{M}$ and all $\xi \in \partial_\infty\widetilde{M}$ as

$$b_\xi(x, y) = \lim_{t\to\infty} d(x, \xi_t) - d(y, \xi_t),$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ. Under this identification the geodesic flow acts by translation in the third coordinate. The non-wandering set of T^1M corresponds to the projection of $(L(\Gamma) \times L(\Gamma)\backslash(Diagonal)) \times \mathbb{R}$ to T^1M (see [Ebe]). Fix a point $x_0 \in \widetilde{M}$, a set of the form

$$H_\xi(r) = \{x \in \widetilde{M} : b_\xi(x_0, x) \geqslant r\},$$

is called a (closed) *horoball center at* ξ. We say that ξ is the base point of the horoball $H_\xi(r)$. We now proceed to define the important class of geometrically finite manifolds.

Definition 2.3.5. *A point $p \in \partial_\infty\widetilde{M}$ is a* bounded parabolic fixed point *of Γ if the maximal parabolic subgroup of Γ fixing p acts cocompactly in $L(\Gamma)\backslash\{p\}$.*

Definition 2.3.6 (Geometrically finite manifolds)**.** *We say that $\Gamma < Iso(\widetilde{M})$ is geometrically finite if every point in $L(\Gamma)$ is a conical point or bounded parabolic. We*

say that M is geometrically finite if $\pi_1(M)$ is geometrically finite.

It follows from [Bow] that the non-compact part of the non-wandering set of the geodesic flow go through the cusps of T^1M and that each cusp is standard, i.e. the quotient of a horoball by the action of the maximal parabolic subgroup of $\pi_1(M)$ fixing the base point of the horoball. Moreover, $\pi_1(M)$ has a finite number of non-conjugate maximal parabolic subgroups (finite number of cusps).

2.3.2 Thermodynamic formalism

In this thesis we will constantly refer to real valued functions whose domain is T^1M as *potentials*. We follow this convention to be consistent with the existing literature on the subject, but it also helps to simplify some notation. The following definition is not standard, despite this, is the notation used in [PPS] and we find it convenient.

Definition 2.3.7 (Topological pressure)**.** *Let $F : T^1M \to \mathbb{R}$ be a continuous potential. We define the* topological pressure *of F as*

$$P(F) = \sup_{\mu \in \mathcal{M}(g)} \{h_\mu(g) + \int F d\mu\}.$$

Definition 2.3.8 (Equilibrium state)**.** *We say that a measure $\mu \in \mathcal{M}(g)$ is an* equilibrium state *for F if it realizes the supremum in the definition of topological pressure. In other words*

$$P(F) = h_\mu(g) + \int F d\mu.$$

In the compact case there is a well known connection between the topological pressure of a continuous potential and a weighted version of the topological entropy [Wal]. If $F = 0$ one would expect to recover the topological entropy of the geodesic (see Definition 2.3.2). This is in fact the case (even for non-compact manifolds) as the following result states.

Theorem 2.3.9 (Variational principle)**.** *[OP] Let M be a pinched negatively curved manifold. Then*

$$h_{top}(g) = \sup_{\mu \in \mathcal{M}(g)} h_\mu(g).$$

It turns out that the topological pressure of Hölder potential have a nice characterization in terms of some critical exponent. More importantly, with Hölder regularity there exists at most one equilibrium state. Before making precise those results we start with some notation. Given two points $x, y \in \widetilde{M}$, we denote by $[x, y]$ to the oriented geodesic segment starting at x and ending at y. For a function $G : T^1\widetilde{M} \to \mathbb{R}$, we use the notation $\int_x^y G$ to represent the integral of G over the vectors tangent to the path $[x, y]$ (in direction from x to y). Given a function $F : T^1M \to \mathbb{R}$, we denote by $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ to the function $\widetilde{F} = F \circ p$, where p is the canonical projection $p : T^1\widetilde{M} \to T^1M$. The following definition was first introduced in [PPS] (a similar definition was used earlier in [Cou]).

Definition 2.3.10. *Let $F : T^1M \to \mathbb{R}$, be a continuous function and $\widetilde{F}$ its lift to $T^1\widetilde{M}$. Define the* Poincaré series associated to (Γ, F) *based at $z \in \widetilde{M}$ as*

$$P(s, F) = \sum_{\gamma \in \Gamma} \exp\left(\int_z^{\gamma z} (\widetilde{F} - s) \right).$$

The critical exponent of (Γ, F) *is*

$$\delta_\Gamma^F = \inf\{s \mid P(s, F) \text{ is finite}\}.$$

We say that the pair (Γ, F) is of convergence type *if $P(\delta_\Gamma^F, F) < \infty$, in other words the Poincaré series converges at its critical exponent. Otherwise we say (Γ, F) is of* divergence type*.*

We use the notation δ_Γ when refering to the critical exponent of $(\Gamma, 0)$, where 0 is the zero function. If F is a Hölder potential, then the critical exponent does not

depend on the base point z. We also remark that if F is bounded, then δ_Γ^F is finite.

A general procedure due to S. Patterson [Pat] and D. Sullivan [Sul] associates to Γ a family of conformal measures on $\partial_\infty \widetilde{M}$ of exponent δ_Γ, the so called *Patterson-Sullivan conformal measures* of Γ. Using this family of confomal measures one can canonically construct a (g_t)-invariant measure, we refer to this measure as the *Bowen-Margulis measure* (for its precise construction we refer the reader to [OP]). Similar to the construction of the Patterson-Sullivan conformal measures, we can associate to the pair (Γ, F) a family of conformal measures. We briefly recall this procedure (which generalizes the Patterson-Sullivan construction to include Hölder potentials). A *Patterson density* of dimension δ for (Γ, F) is a family of finite Borel measures $(\sigma_x)_{x\in\widetilde{M}}$ on $\partial_\infty \widetilde{M}$, such that, for every $\gamma \in \Gamma$, for all $x, y \in \widetilde{M}$ and for every $\xi \in \partial_\infty \widetilde{M}$ we have

$$\gamma_* \sigma_x = \sigma_{\gamma x} \quad \text{and} \quad \frac{d\sigma_x}{d\sigma_y}(\xi) = e^{-C_{F-\delta,\xi}(x,y)},$$

where $C_{F,\xi}(x, y)$ is the *Gibbs cocycle* defined as

$$C_{F,\xi}(x, y) = \lim_{t\to\infty} \int_y^{\xi_t} \widetilde{F} - \int_x^{\xi_t} \widetilde{F},$$

for any geodesic ray $t \mapsto \xi_t$ ending at ξ. The limit in the definition of the Gibbs cocycle always exists because the manifold has negative curvature and the potential is Hölder continuous. If $\delta_\Gamma^F < \infty$, then there exists at least one Patterson density of dimension δ_Γ^F for (Γ, F), which support lies in the limit set $L(\Gamma)$ of Γ [PPS, Proposition 3.9]. If (Γ, F) is of divergence type then there is an unique Patterson-Sullivan density of dimension δ_Γ^F [PPS, Corollary 5.12]. For now on assume that $\delta_\Gamma^F < \infty$ and let (σ_x) be a Patterson density of dimension δ_Γ^F. Denote by (σ_x^ι) the Patterson density of dimension δ_Γ^F for $(\Gamma, F \circ \iota)$, where ι is the flip isometry map $v \mapsto -v$ on $T^1\widetilde{M}$. Using the Hopf parametrisation $v \mapsto (v_-, v_+, t)$ with respect to a base point $o \in \widetilde{M}$, the

measure

$$d\tilde{m}(v) = e^{C_{F\circ\iota-\delta_\Gamma^F,v_-}(x_0,\pi(v))+C_{F-\delta_\Gamma^F,v_+}(x_0,\pi(v))} d\sigma_o^\iota(v_-)d\sigma_o(v_+)dt,$$

is independent of $o \in \widetilde{M}$, Γ-invariant and (g_t)-invariant. This induces a measure m on T^1M called the *Gibbs measure associated to the Patterson density* σ_x. If (Γ, F) is of divergence type this construction is unique and we refer to the resulting measure as the *Gibbs measure associated to* F. A fundamental property of m_F is that, whenever finite, is the unique equilibrium state for the potential F. For $F = 0$ this construction gives us the so-called Bowen-Margulis measure, which we denote by m_{BM}. The convergence/divergence type of the pair $(\Gamma, 0)$ has strong ergodic implications for the geodesic flow with respect to the Bowen-Margulis measure as we can see from the theorem stated below (see [Yue]).

Theorem 2.3.11. *Let Γ be a discrete non-elementary torsion free subgroup of isometries of $\widetilde{M}$. Then, we have*

(a) the group Γ is of divergence type if and only if the geodesic flow (g_t) is ergodic and completely conservative with respect to m_{BM}, and

(b) the group Γ is of convergence type if and only if the geodesic flow (g_t) is non-ergodic and completely dissipative with respect to m_{BM}.

Finally we state one of the main results in [PPS], which is a crucial input in this work.

Theorem 2.3.12. *[PPS, Theorem 2.3] Let F be a bounded Hölder potential. Then*

$$P(F) = \delta_\Gamma^F.$$

Moreover, if there exists a finite Gibbs measure m_F for (Γ, F), then $m_F/||m_F||$ is the unique equilibrium state of F. Otherwise there is not equilibrium measure.

We remark that this result was obtained by Otal and Peigne in the case $F = 0$ (see [OP]). The proof of Theorem 2.3.12 follows very closely the proof in [OP]. If $F = 0$ this is simply saying that if the Bowen-Margulis measure is finite, then its normalization $m_{BM}/||m_{BM}||$ is the measure of maximal entropy of the geodesic flow, otherwise there is not measure of maximal entropy. It worth mentioning that in the compact case this measure was constructed in two different ways by Bowen [B2],[B3] and Margulis [Mar] (and so the reason for its name). Later on Bowen [B6] proved the uniqueness of the measure of maximal entropy, so both constructions must coincide.

Following [PPS] we will now introduce the Gurevich pressure of a potential. It worth mentioning that the Gurevich pressure is defined in the spirit of the work of Gurevich and Sarig on countable Markov shifts [G1], [S1]. We start with some notations. Let $Per(t)$ be the set of periodic orbits of the geodesic flow of length less or equal than t. Each periodic orbit defines an invariant probability measure, this is just Lebesgue measure on the orbit divided by its length. More precisely to each periodic orbit τ we associate an invariant probability measure μ_τ and we refer to the measures of the form μ_τ as *periodic measures*. The set of periodic measures is denoted by $\mathcal{M}_p(g)$. Observe that each periodic measure η has associated the length of its original closed geodesic, we refer to this number as the *length of the periodic measure* and it is denoted by $l(\eta)$. Given $W \subset T^1M$ and real numbers r, s we define

$$\mathcal{M}_p(W, r, s) = \{\eta \in \mathcal{M}_p(g) : l(\eta) \in [s, r] \text{ and } \eta(W) > 0\}.$$

Definition 2.3.13 (Gurevich pressure)**.** *Let F be a continuous potential, W an open relatively compact subset of T^1M intersecting the non-wandering set of the geodesic flow and $c > 0$. The* Gurevich pressure *of F with respect to W and c is defined as*

$$P_{Gur}(F, W, c) = \limsup_{t\to\infty} \sum_{\mu\in\mathcal{M}_p(W,t,t-c)} \exp(l(\mu) \int F d\mu).$$

It is proven in [PPS] that the Gurevich pressure is independent of W and c, in particular we can write it as

$$P_{Gur}(F) = \limsup_{t\to\infty} \sum_{\mu\in\mathcal{M}_p(W,t,t-c)} \exp(l(\mu)\int F d\mu).$$

The connection between the Gurevich pressure and the topological pressure is given by the following result.

Theorem 2.3.14. *[PPS, Theorem 1.1] Let F be a Hölder potential. Then*

$$P(F) = P_{Gur}(F).$$

We will now briefly discuss under what assumptions a Hölder potential admits an equilibrium state, in other words, when we can ensure that the Gibbs measure m_F is finite. For the first part of this discussion we will assume that M is geometrically finite. In this case we have a finite number of non-conjugate maximal parabolic subgroups. Each maximal parabolic $\mathcal{P}$ defines a critical exponent $\delta_{\mathcal{P}}^F$ (see Definition 2.3.10). The following result was proved in [DOP] for the case $F = 0$, and later extended by Coudene in [Cou] for his definition of critical exponent. For a proof of the version stated here we refer the reader to [RV, Theorem 2.12].

Theorem 2.3.15. *Let M be a geometrically finite manifold and F a Hölder potential. Assume that $(\mathcal{P}, F)$ is of divergence type for a parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$. Then $\delta_{\mathcal{P}}^F < \delta_{\Gamma}^F$.*

This result will be essential in Section 7, where we define the notion of 'very strongly positive recurrent potentials' motivated by Theorem 2.3.15. The next result was also initially proved in [DOP] for $F = 0$, but the proof also extend to include Hölder potentials [PPS, Corollary 8.6].

Theorem 2.3.16. *Let M be a geometrically finite manifold. Let F be a bounded Hölder potential. If $\delta_{\mathcal{P}}^F < \delta_\Gamma^F$ for every parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$, then the Gibbs measure of F is finite.*

We will now explain the results in [PS], where a characterization for the existence of equilibrium states is provided. Given $\widetilde{U} \subset \widetilde{M}$ we define $\Gamma_{\widetilde{U}}$ as the set of elements in Γ such that there exists a geodesic starting at $\widetilde{U}$ and finishing at $\gamma\widetilde{U}$ that only meet $\Gamma\widetilde{U}$ at the beginning and at the end of its trajectory. Recall that if F is a potential in T^1M we denote by $\widetilde{F}$ to its lift to $T^1\widetilde{M}$ and that Ω is the non-wandering set of the geodesic flow. Let $\mathcal{P}$ be the set of closed geodesics and $n_U(p)$ the number of times a geodesic $p \in \mathcal{P}$ crosses U.

Definition 2.3.17. *A potential $F : T^1M \to \mathbb{R}$ is said to be* recurrent *if there exists an open relatively compact subset $U \subset M$, such that $T^1U \cap \Omega \neq \varnothing$, and*

$$\sum_{p\in\mathcal{P}} n_U(p) \exp(\int_p F - P(F)) = \infty.$$

Definition 2.3.18. *We say that the pair $(\Gamma, \widetilde{F})$ is* positive recurrent *with respect to $\widetilde{U} \subset \widetilde{M}$ if the following properties hold.*

1. *$T^1\widetilde{U}$ has non-empty intersection with the lift of Ω to the universal cover.*
2. *F is a recurrent potential.*
3. *There exists $x \in \widetilde{M}$ such that $\sum_{\gamma\in\Gamma_{\widetilde{U}}} d(x, \gamma x) \exp(\int_x^{\gamma x} \widetilde{F} - P(F))$, is finite.*

Theorem 2.3.19. *[PS, Theorem 2] Let F be a Hölder potential with finite pressure. Then*

1. *If F is recurrent and $(\Gamma, \widetilde{F})$ is positive recurrent with respect to some open relatively compact set $\widetilde{U} \subset \widetilde{M}$, then m_F is finite.*

2. *If m_F is finite, then F is recurrent and $(\Gamma, \widetilde{F})$ is positive recurrent with respect to any open relatively compact set $\widetilde{U} \subset \widetilde{M}$ meeting the projection to $\widetilde{M}$ of the lift of Ω to $T^1\widetilde{M}$.*

In this thesis we will be particularly interested in the following family of potentials.

Definition 2.3.20 (Strongly positive recurrent potentials)**.** *Let $\widetilde{U}$ be an open relatively compact subset of $\widetilde{M}$, such that $T^1\widetilde{U}$ has non-empty intersection with the lift of Ω to $T^1\widetilde{M}$. We define $\delta_\Gamma^F(\widetilde{U})$ as the critical exponent of the Poincaré series*

$$P(F, \widetilde{U}, s) = \sum_{\gamma \in \Gamma_{\widetilde{U}}} \exp(\int_x^{\gamma x} \widetilde{F} - s).$$

Then define

$$P_\infty(F) = \inf_{\{\widetilde{U}_n\}} \liminf_{n\to\infty} \delta_\Gamma^F(\widetilde{U}_n),$$

where the infimum runs over collections $(\widetilde{U}_n)_n$ such that $\widetilde{U}_n \subset \widetilde{U}_{n+1}$, and $\Omega \subset \bigcup_n \widetilde{U}_n$. The quantity $P_\infty(F)$ is called the topological pressure at infinity *of the potential F. We say that F is* strongly positive recurrent *(SPR for short) if $P_\infty(F) < P(F)$.*

In the geometrically finite case it is easy to verify that

$$P_\infty(F) = \sup_{\mathcal{P}} \delta_{\mathcal{P}}^F,$$

where the supremum runs over the parabolic subgroups of $\pi_1(M)$. In particular a potential F is SPR if and only if $\sup_{\mathcal{P}} \delta_{\mathcal{P}}^F < P(F)$. We will mainly focus on the case where F vanishes at infinity (see Definition 3.1.1). In this case we will prove many important properties of the pressure of SPR potentials; this should be compared with the results of Sarig on SPR potentials for countable Markov shifts (see [S2]). In some sense potentials such that $P_\infty(F) < P(F)$ behave very similar to those on compact manifolds. In Section 6 we will prove the main properties of the family of

SPR potentials. It worth mentioning that a SPR potential is positive recurrent, in particular by Theorem 2.3.19 it admits a unique equilibrium state.

We finish this section with a remarkable property of the Gibbs measure associated to F. We use the notation $B_n^p(v,r)$ to denote the projection of the dynamical ball $B_n(v,r)$ in the universal cover of M (for a precise definition see Definition 4.2.1). We say that $B_n^p(v,r)$ is a $p(n,r)$-dynamical ball. We remark that this modification of dynamical ball will be very relevant for our definition of the topological entropy at infinity (see Definition 5.2.1).

Definition 2.3.21 (Gibbs property)**.** *Let m be an invariant measure on T^1M. We say that m verifies the* Gibbs property for the potential $F : T^1M \to \mathbb{R}$ *if for every compact set $K \subset T^1M$ and $r > 0$ there exists a constant $C_{K,r} \geqslant 1$ such that for every $v \in K$ and every $n \geqslant 1$ such that $g^n(v) \in K$, we have*

$$C_{K,r}^{-1} \leqslant \frac{m(B_n^p(v,r))}{e^{\int_0^n F(g_t(v))-P(F)dt}} \leqslant C_{K,r}.$$

This definition resemble the usual Gibbs property in symbolic dynamics, but it is not identical (the compact subset has some role here) due to the non-compactness of M. For shifts of finite type, every equilibrium state of a Hölder potential satisfies the Gibbs property [B7]. For countable Markov shifts Gibbs measures exist only if the shift space satisfies the BIP property (see [S1]). It is proven in [PPS] that the Gibbs measures of Hölder potentials satisfy the Gibbs property. This fact will be important in order to prove our modification of the simplified entropy inequality (Section 4).

Chapter 3

The spaces $\mathcal{M}_{\leqslant 1}(g)$ and $C_0(T^1M)$

In this section we will describe some properties of the space of continuous functions vanishing at infinity and the topological structure of the space of invariant sub-probability measures $\mathcal{M}_{\leqslant 1}(g)$.

3.1 The space $C_0(T^1M)$

The space of continous functions vanishing at infinity is the most relevant function space in this thesis. For this class of potentials we will obtain a fairly good understanding of the pressure map (Theorem 7.1.2). We will also prove that Hölder potentials in this class can develop phase transitions, which is a novel feature in comparison with what is known in the compact case (see Section 7.4).

Definition 3.1.1. *We say that a continuous function F* vanishes at infinity *if for every $\epsilon > 0$, there exists a compact set K such that $\sup_{x\in K^c} |F(x)| < \epsilon$. The space of continuous functions vanishing at infinity is denoted by $C_0(T^1M)$.*

The following easy lemma is one of the main reasons why $C_0(T^1M)$ is so useful. We will see in Section 6 that properties of the entropy extend to properties of the pressure because of this lemma.

Lemma 3.1.2. *Let $F \in C_0(T^1M)$, then the map*

$$\mu \mapsto \int F d\mu,$$

is continuous in $\mathcal{M}_{\leqslant 1}(g)$.

Proof. Suppose we have a sequence $(\mu_n)_n \subset \mathcal{M}_{\leqslant 1}(g)$ converging in the vague topology to μ. Fix $\epsilon > 0$, and let $K = K(\epsilon)$ be a compact subset of T^1M such that $\mu(\partial K) = 0$, and $\sup_{x \in K^c} |F(x)| < \epsilon$. Since $\mu(\partial K) = 0$ we know that $\lim_{n\to\infty} \int_K F d\mu_n = \int_K F d\mu$. Finally

$$\limsup_{n\to\infty} |\int F d\mu - \int F d\mu_n| \leqslant \limsup_{n\to\infty} |\int_K F d\mu - \int_K F d\mu_n| + 2\epsilon \leqslant 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary we are done. □

The following result is well known and will be used very often in this work. For convenience we state it here.

Lemma 3.1.3. *Let F and G in $C_b(T^1M)$. Then $|P(F) - P(G)| \leqslant ||F - G||_0$.*

Proof. Observe that

$$h_\mu(g) + \int F d\mu \leqslant h_\mu(g) + \int G d\mu + ||F - G||_0,$$

therefore $P(F) \leqslant P(G) + ||F - G||_0$. The other inequality is analogous. □

The space of Hölder potentials is dense in $C_0(T^1M)$ (in the C^0-topology). This fact follows from the local compactness of T^1M and Stone-Weierstrass theorem. Here we state a few simple, but useful consequences of this fact.

Proposition 3.1.4. *Let $F \in C_0(T^1M)$. Then the critical exponent of (Γ, F) corresponds to the topological pressure of F. More precisely*

$$\delta_\Gamma^F = \sup_{\mu \in \mathcal{M}(g)} \{h_\mu(g) + \int F d\mu\}.$$

In particular the critical exponent is independent of the base point used in the definition of the Poincaré series of (Γ, F).

Proof. By Theorem 2.3.12 we know that this property holds for bounded Hölder potentials. Take a sequence $(F_n)_{n\in\mathbb{N}}$ of Hölder potentials converging to F in the C^0-topology. There exists a subsequence $(n_k)_k$ such that $||F_{n_k} - F||_0 \leqslant \frac{1}{k}$. Observe that

$$\sum_{\gamma\in\Gamma} \exp\big(\int_x^{\gamma x} \widetilde{F_{n_k}} - (s + \frac{1}{k})\big) \leqslant \sum_{\gamma\in\Gamma} \exp\big(\int_x^{\gamma x} \widetilde{F} - s\big) \leqslant \sum_{\gamma\in\Gamma} \exp\big(\int_x^{\gamma x} \widetilde{F_{n_k}} - (s - \frac{1}{k})\big).$$

From this we obtain the inequality

$$\delta_\Gamma^{F_{n_k}} - \frac{2}{k} \leqslant \delta_\Gamma^F \leqslant \delta_\Gamma^{F_{n_k}} + \frac{2}{k}.$$

By assumption we had $||F_{n_k} - F||_0 \leqslant \frac{1}{k}$, in particular $|P(F_{n_k}) - P(F)| \leqslant \frac{1}{k}$ (Lemma 3.1.3). Using that $P(F_{n_k}) = \delta_\Gamma^{F_{n_k}}$, we conclude that $\delta_\Gamma^F = P(F)$. □

Definition 3.1.5 (Topological pressure of a subset)**.** *Let $A \subset T^1M$ an invariant subset of the geodesic flow, we define the topological pressure of F restricted to A as*

$$P_A(F) = \sup_\mu \{h_\mu(g) + \int F d\mu\},$$

where the supremum runs over the invariant probability measures satisfying $\mu(A) = 1$.

The following result is obtained in the proof of [PPS, Lemma 6.7].

Theorem 3.1.6 (Aproximation by compact subsets)**.** *Let F be a Hölder potential with finite topological pressure. Then $P(F) = \sup_K P_K(F)$, where the supremum runs over the compact invariant subsets of the geodesic flow.*

Proposition 3.1.7. *Let $F \in C_0(T^1M)$. Then the topological pressure of F is approximated by compact subsets.*

Proof. Let $(F_n)_n$ be a sequence of Hölder continuous potentials converging to F in the C^0 topology. Similar to Lemma 3.1.3 we have $|P_K(F) - P_K(F_n)| \leqslant ||F - F_n||_0$, for every invariant subset K. Then

$$\begin{aligned} |P(F) - P_K(F)| &\leqslant |P(F) - P(F_n)| + |P(F_n) - P_K(F_n)| + |P_K(F_n) - P_K(F)| \\ &\leqslant 2||F - F_n||_0 + |P(F_n) - P_K(F_n)|. \end{aligned}$$

Taking supremum over the invariant compact subsets and sending n to infinity we obtain that $P(F) = \sup_K P_K(F)$. □

In Section 6 we will give a criterion for the existence of equilibrium states for potentials in $C_0(T^1M)$ satisfying $P(F) > \delta_\infty$, where δ_∞ is the topological entropy at infinity of the geodesic flow (see Definition 5.2.1). In this case the theory developed in [PPS] does not apply, our method has the advantage of being flexible enough to include continuous potentials. In contrast to Theorem 2.3.12 in $C_0(T^1M)$ we can not ensure the uniqueness of equilibrium states; this is also the case for continuous potentials on compact negatively curved manifolds.

3.2 The space $\mathcal{M}_{\leqslant 1}(g)$

The results contained in this section are joint work with G. Iommi [IV]. Since the paper [IV] is still in preparation we will provide proofs of some statements concerning the geodesic flow. As mentioned in Section 2.1 the space $\mathcal{M}_{\leqslant 1}(g)$ endowed with the

vague topology is compact metrizable, in particular sequentially compact. Moreover, $\mathcal{M}_{\leqslant 1}(g)$ is convex, i.e. a convex combination of sub-probability invariant measures is still a sub-probability invariant measure. Recall that a point in a convex set is an *extreme point* if it is not the convex combination of at least two different elements in the convex set. The following lemma characterize the extreme points of $\mathcal{M}_{\leqslant 1}(g)$.

Lemma 3.2.1. *The extreme points of $\mathcal{M}_{\leqslant 1}(g)$ are the ergodic probability measures and the zero measure.*

Proof. Observe that any measure $\mu \in \mathcal{M}_{\leqslant 1}(g)$ such that $\mu(T^1X) \in (0,1)$ is the convex combination between a probability measure and the zero measure, in particular it can not be an extreme point. The zero measure is clearly an extreme point. Finally, by [Wal, Theorem 6.10] we know that the ergodic measures are the extreme points of $\mathcal{M}(g)$, but the same argument implies that they are extreme points of $\mathcal{M}_{\leqslant 1}(g)$. □

Definition 3.2.2 (Poulsen simplex)**.** *A metrizable convex Choquet simplex with at least two points K is a* Poulsen simplex *if its extreme points are dense in K.*

The first example of such a simplex was constructed by Poulsen [Pou] in 1961. It was later shown by Lindenstrauss, Olsen and Sternfeld [LOS, Theorem 2.3] that the Poulsen simplex is unique up to affine homemorphism. Note that a Poulsen simplex is always infinite dimensional. A remarkable feature observed in [LOS, Section 3] is that,

Proposition 3.2.3. *The set of extreme points in the Poulsen simplex is path connected.*

We will now prove that $\mathcal{M}_{\leqslant 1}(g)$ is affine homeomorphic the Poulsen simplex. Since $\mathcal{M}_{\leqslant 1}(g)$ is a metrizable convex choquet simplex with at least two points K we only need to check that its extreme points are dense. We will, in fact, prove that the periodic measures are dense in $\mathcal{M}_{\leqslant 1}(g)$. We will need the following result from [CS].

Theorem 3.2.4. *Let M be a pinched negatively curved manifold. The periodic measures are dense in $\mathcal{M}(g)$ with respet to the weak-* topology.*

The main inputs to obtain this result are the closing lemma and the local product structure of the geodesic flow (for precise definitions we refer the reader to Section 4.1). The strategy of the proof is very simple: the closing lemma allows us to approximate any ergodic measure by a periodic measure, and the local product structure together with the closing lemma allows us to approximate any average of periodic measures by a single periodic measure. It is a standard fact that any invariant probability measure can be approximated by the average of ergodic measures (consequence of ergodic decomposition), and therefore one obtains the density statement. To obtain the density of periodic measures in $\mathcal{M}_{\leqslant 1}(g)$ we will need the following result, but its proof will be postponed to Section 7 (see Lemma 7.1.3).

Lemma 3.2.5. *Let M be a non-compact pinched negatively curved manifold. Then there exists a sequence of periodic measures that converges vaguely to the zero measure.*

The proof we will give to this fact is a direct consequences of our results on the pressure map of functions in $C_0(T^1M)$. We will now outline a different proof of Lemma 3.2.5 (this is not really necessary, but allows us to state a recent result that is interesting by itself). The following result recently obtained in [KL] and extends to higher dimensions (and variable curvature) a well known result of Bonahon for hyperbolic 3-manifolds [Bon].

Theorem 3.2.6 (Characterization of geometrically infinite)**.** *Let M be a pinched negatively curved manifold. Assume that M is not geometrically finite, then there exists a sequence of closed geodesics that escape from any compact subset of M.*

It worth mentioning that in the geometrically finite case closed geodesics can not escape from every compact set. The sequence of periodic measures associated to the geodesics provided by Theorem 3.2.6 converges vaguely to the zero measure. To finish

the proof of Lemma 3.2.5 we can now assume that M is geometrically finite. This part of the proof can be made rigurous, but we will just sketch the ideas involve (these ideas were very useful in [IRV]). Pick a maximal parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$ and h a hyperbolic isometry such that $\mathcal{P}$ and $\langle h \rangle$ are in Schottky position (see Definition 7.4.1). The covering N of M induced by the subgroup $\mathcal{P} * \langle h \rangle \subset \pi_1(M)$ is also geometrically finite. Moreover the geodesic flow of N can be coded as a suspension flow over a countable Markov shift (see [BP]). It is important to observe that the covering map $q : N \to M$ is one to one nearby the cusp associated to $\mathcal{P}$. In the symbolic model it is easy to construct periodic orbits that converge to the zero measure through the cusp associate to $\mathcal{P}$ (their mass concentrate in the cusp). The image of this closed geodesics will have the same property in M (since q send the cusp of N into the cusp in M associated to $\mathcal{P}$) and we obtain Lemma 3.2.5.

Theorem 3.2.7 (Topological structure of the space of measures)**.** *Let M be a non-compact pinched negatively curved manifold. Then $\mathcal{M}_{\leqslant 1}(g)$ is affine homeomorphic to the Poulsen simplex and $\mathcal{M}(g)$ is affine homeomorphic to the Poulsen simplex minus a vertex and all of its convex combinations.*

Proof. In light of Theorem 3.2.4 and Lemma 3.2.5 we need to prove that every measure $\mu \in \mathcal{M}_{\leqslant 1}(g)$ such that $\mu(T^1M) \in (0,1)$ can be approximated by periodic measures. Since periodic measures are dense in $\mathcal{M}(g)$ it is enough to approximate every measure of the form $\lambda\mu$, where $\lambda \in (0,1)$ is a rational number and μ is a periodic measure. Let $\lambda = p/(p+q)$, where $p, q \in \mathbb{N}$, and $(\mu_n)_n$ be a sequence of periodic measures converging vaguely to the zero measure. Define

$$\nu_n = \frac{1}{p+q}(p\mu + q\mu_n).$$

Let d be a metric on $\mathcal{M}_{\leqslant 1}(g)$. By Theorem 3.2.4 we know there exists a sequence of periodic measures $(\eta_n)_n$ such that $d(\nu_n, \eta_n) < \frac{1}{n}$. Since $\lim_{n\to\infty} \nu_n = \lambda\mu$, we get that

$\lim_{n\to\infty} \eta_n = \lambda\mu$. □

Remark 3.2.8. *Theorem 3.2.7 describes completely the topology of our spaces of invariant measures. We emphasize that in the compact case Sigmund proved that $\mathcal{M}(g)$ is affine homeomorphic to a Poulsen simplex [Sig]. As a consequences of the results in this section we can conclude that the only topological property of M that is detected by the topology of $\mathcal{M}(g)$ is its compactness (or lack of compactness). If we wanted to recover some geometric property of M from the space of invariant measures one needs to add some extra data to this set.*

Chapter 4

Entropy density and simplified entropy formula

In this section we will prove that the geodesic flow on a pinched negatively curved manifold satisfies a mild modification of the simplified entropy inequality and that its ergodic measures are weak entropy dense. This two properties will be used to prove the upper semicontinuity of the entropy map.

4.1 Weak entropy density

In this section we will check that the proof of [EKW, Theorem B] extends to the non-compact case for the geodesic flow on a negatively curved manifold. We will need the following definitions.

Definition 4.1.1 (Closing lemma)**.** *We say that a flow $(\varphi_t)_{t\in\mathbb{R}}$ on a metric space (X, d) satisfy the closing lemma if for all $x \in X$, there exists a neighborhood W_x of x such that the following holds. Given $\epsilon > 0$, there exists δ and t_0 such that for all $y \in W_x$ and $t \geqslant t_0$, if $d(y, \varphi_t y) < \delta$ and $\varphi_t y \in W_x$, then there exists y' and $s > 0$ such that $|t - s| < \epsilon$, $\varphi_s y' = y'$ and $d(\varphi_h y, \varphi_h y') < \epsilon$, for $h \in (0, \min\{t, s\})$.*

Define the sets

$$W_\epsilon^{ss}(x) = \{y \in X : d(\varphi_t(x), \varphi_t(y)) \leqslant \epsilon, \text{ for all } t \geqslant 0\},$$

$$W_\epsilon^{su}(x) = \{y \in X : d(\varphi_t(x), \varphi_t(y)) \leqslant \epsilon, \text{ for all } t \leqslant 0\}.$$

Definition 4.1.2 (Local product structure)**.** *We say that the flow $(\varphi_t)_{t\in\mathbb{R}}$ admits a local product structure if for all $x \in X$, there exists a neighborhood V_x of x such that the following holds. Given $\epsilon > 0$, there exists $\delta > 0$ such that for all $y, z \in V_x$ satisfying $d(x, y) < \delta$, there exists a point $w = \langle y, z\rangle \in X$ and a real number $t \in (-\epsilon, \epsilon)$ so that $\langle y, z\rangle \in W_\epsilon^{su}(\varphi_t(x)) \cap W_\epsilon^{ss}(y)$.*

Remark 4.1.3. *It is a well known fact (see for instance [Bal]) that the geodesic flow on a pinched negatively curved manifold is transitive, satisfies the closing lemma and admits local product structure. These properties are important for us because of the following result.*

Proposition 4.1.4. *Let (X, d) be a metric space where closed balls are compact. Let $(\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow which is transitive, admits local product structure and satisfies the closing lemma. Then for every measure $\mu \in \mathcal{M}(X, \varphi_1)$ and $\epsilon > 0$, there exists an ergodic measure μ_e arbitrarily close to μ (in the weak-* topology) such that $h_{\mu_e}(\varphi_1) > h_\mu(\varphi_1) - \epsilon$. We can moreover assume that* $\mathrm{supp}\mu_e$ *is compact.*

Proof. As in the proof of the entropy density in the compact case we start with the following general fact.

Lemma 4.1.5. *[EKW, Proposition 6.1] Let (X, d) be a metric space and T a continuous transformation. Given an ergodic measure μ, $\alpha > 0$, $\beta > 0$ and $f_1, ..., f_l \in C_b(X)$, there exists n_0 and $\gamma > 0$ such that for all $n \geqslant n_0$ there exists a (n, γ) separated set $S \subset X$ such that*

1. $|S| \geqslant \exp(n(h_\mu(T) - \alpha))$.

2. $|\frac{1}{n}\sum_{k=0}^{n-1} f_j(T^k x) - \int f_j d\mu| < \beta$*, for all* $x \in S$ *and* $j \in \{1, ..., l\}$*.*

Let $K \subset X$ *be a measurable set satisfying* $\mu(K) > 3/4$*. Then we can moreover assume that* $S \subset K \cap T^{-n}K$*. We can choose* γ *so that it does not depend on* K*, or* n*.*

We only need to justify the last part of the proposition since points (1) and (2) are taken without modification from [EKW]. We follow the notation in the proof of [EKW, Proposition 6.1] and modify the definition of F_n by the formula

$$F_n = E_n \cap K \cap T^{-n}K \cap \{x \in X : \frac{1}{n}\sum_{k=0}^{n-1} \chi_V(T^k x) \leqslant 2\delta\},$$

where $E_n = \{x \in X : |\frac{1}{n}\sum_{k=0}^{n-1} f_j(T^k x) - \int f_j d\eta| < \beta, \forall j \in \{1, ..., l\}\}$. Since $\mu(K \cap T^{-n}K) > \frac{1}{2}$, and the measure of the other two sets involved in the definition of F_n, by Birkhoff ergodic theorem, tends to 1 as n goes to infinity, we conclude that for n sufficiently large we have $\mu(F_n) > \frac{1}{2}$. With this definition of F_n the rest of the proof follows without modification the proof of [EKW, Proposition 6.1].

We will use the notation $T = \varphi_1$. We want to prove that given $\mu \in \mathcal{M}(X, T)$, $\epsilon > 0$, $\eta > 0$, and $f_1, ..., f_l \in C_c(X)$, there exists an ergodic measure $\mu_e \in V(f_1, ..., f_l; \mu, \epsilon)$ such that $h_{\mu_e}(T) > h_\mu - \eta$, where

$$V(f_1, ..., f_l; \mu, \epsilon) = \{\nu \in \mathcal{M}(X, T) : |\int f_i d\nu - \int f_i d\mu| < \epsilon, \forall i = \{1, ..., l\}\}.$$

As in the proof of [EKW, Theorem B] we can reduce the problem to the case $\mu = \frac{1}{N}\sum_{k=1}^{N} \mu_k$, where $\{\mu_k\}_{k=1}^N$ is a collection of ergodic measures. Fix a compact set K such that $\mu_i(K) > 3/4$, $\forall i \in \{1, ..., l\}$. By the uniform continuity of the functions $(f_i)_i$, there exists $\epsilon_0 > 0$ such that if $d(x, y) < \epsilon_0$, then $|f_i(x) - f_i(y)| < \frac{\epsilon}{4}$. Using Lemma 4.1.5 we can find n_0 and $\gamma > 0$ such that for every $n \geqslant n_0$, there exist an (n, γ)-separated set $S_i \subset K \cap T^{-n}K$ satisfying the conclusions of Lemma 4.1.5 for the measure μ_i, $\beta = \epsilon/4$ and $\alpha = \eta/2$. By the definition of the local product structure and

the compactness of K, given $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that if $x, y \in K$ satisfy $d(x, y) < \delta_0$, then there exists a point $\langle x, y\rangle$ such that $\langle x, y\rangle \in W^{su}_{\epsilon_0}(\varphi_t(x)) \cap W^{ss}_{\epsilon_0}(y)$ and $|t| < \epsilon_0$. We will moreover assume that ϵ_0 satisfy $\gamma > \epsilon_0/4$. By the transitivity of the flow and the compactness of K, given $\delta_0 > 0$ there exists a constant $R = R(\delta_0)$ such that for every $x, y \in K$, there exists $z \in X$ such that $d(x, z) < \delta_0$, and $d(\varphi_p z, y) < \delta_0$ for some $p \in [0, R]$. In particular if $(x, y) \in \left(\bigcup_{k=1}^N S_k\right)^2$, there exists $z = z(x, y) \in X$ such that $d(\varphi_n x, z) < \delta_0$, and $d(\varphi_p z, y) < \delta_0$, for some $z = z(x, y) \in [0, R]$. By choosing n sufficiently large we can assume $R(\delta_0)/n$ is very small to be determined later. Choose $(x_1, x_2, ..., x_{MN}) \in (\prod_{i=1}^N S_i)^M$. As in the proof of [CS, Proposition 3.2], using the closing lemma and the local product structure, we can construct a periodic orbit that ϵ_0-shadows the broken orbit

$$W = O_0^n(x_1) \cup O_0^{p(x_1,x_2)} z(x_1, x_2) \cup O_0^n(x_2) \cup ... \cup O_0^n(x_n) \cup O_0^{p(x_{MN},x_1)} z(x_{MN}, x_1).$$

We think of W as a parametrized map which represent the sequence of segments above. We denote the periodic orbit shadowing W by $w(x) = w(x_1, ..., x_{MN})$. The period of w is approximately the domain of W. By the choice of ϵ_0 (in terms of the sequence $(f_i)_i$) and because $R(\delta_0)/n$ is sufficiently small, one easily verifies that the periodic measure μ_w associated to w belongs to $V(f_1, ..., f_l; \mu, \frac{\epsilon}{2})$. Define $S = \bigcup_{M \geqslant n_0} (\prod_{i=1}^N S_i)^M$. As before, to each element $x \in S$ we associate a periodic orbit $w(x)$. Fix some reference point $q \in K$. There exists a constant $L = L(n, R(\delta_0))$ such that $w(x) \subset B(q, L)$, for every $x \in S$. In particular the set

$$\Psi = \bigcup_{x \in S} w(x),$$

is relatively compact in X and (φ_t)-invariant. This implies that $\Psi_0 = \overline{\Psi}$ is compact

and (φ_t)-invariant. Define

$$A_n = \{x \in X : |\frac{1}{nN}\int_0^{nN} f_i(\varphi_t x)dt - \int f_i d\mu| \leqslant \epsilon, \forall i \in \{1, ..., l\}\}.$$

It is easy to see that because of the choice of ϵ_0, δ_0 and the sets $(S_i)_{i=1}^N$ we have $\Psi_0 \subset \bigcup_{n\geqslant 1} A_n$. Since Ψ_0 is (φ_t)-invariant we know that $x \in \Psi_0$ implies that for every $s \in \mathbb{R}$ we have $\varphi_s x \in \bigcup_{n\geqslant 1} A_n$. Let ν be an ergodic measure supported in Ψ_0 and $x \in \Psi_0$ a point such that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n f_i(\varphi_t x)dt = \int f_i d\nu,$$

for every $i \in \{1, ..., l\}$. Note that $x \in A_n$, for some n, and that $\varphi_{nN} x \in A_m$, for some m. In particular we have

$$|\frac{1}{nN}\int_0^{nN} f_i(\varphi_t x)dt - \int f_i d\mu| \leqslant \epsilon, \text{ and } |\frac{1}{mN}\int_0^{mN} f_i(\varphi_{nN+t} x)dt - \int f_i d\mu| \leqslant \epsilon.$$

These inequalities imply that

$$|\frac{1}{(n+m)N}\int_0^{(n+m)N} f_i(\varphi_t x)dt - \int f_i d\mu| \leqslant \epsilon.$$

We can generalize this argument to conclude that there exists a sequence $(m_i)_i$ such that $\lim_{t\to\infty} m_i = \infty$, and

$$|\frac{1}{m_i}\int_0^{m_i} f_i(\varphi_t x)dt - \int f d\mu| \leqslant \epsilon.$$

Since by assumption we know that $\lim_{n\to\infty} \frac{1}{n}\int_0^n f_i(\varphi_t x)dt = \int f_i d\nu$, we conclude that

$$|\int f_i d\nu - \int f_i d\mu| \leqslant \epsilon.$$

In particular every ergodic measure supported in Ψ_0 belongs to $V(f_1, ..., f_l; \mu, \epsilon)$. Recall that $\gamma > \epsilon_0/4$. By construction if $x, y \in (\prod_{i=1}^{N} S_i)^M$, and $x \neq y$, then

$$d_{M(n+N(\delta_0))}(w(x), w(y)) > \epsilon_0/2.$$

In other words Ψ_0 contains a $(M(n + N(\delta_0)), \epsilon_0/2)$-separated set of cardinality

$$\exp\left(nM\left(\frac{1}{n}\sum_{k=1}^{N} h_{\mu_k}(\varphi_1) - \frac{\eta}{2}\right)\right).$$

Then

$$h_{top}(\Psi_0) \geqslant \limsup_{M\to\infty} \frac{nM(h_\mu(\varphi_1) - \frac{\eta}{2})}{M(n + N(\delta_0))}.$$

By choosing $N(\delta_0)/n$ sufficiently small we can make the left hand side in the last inequality strictly bigger than $h_\mu(\varphi_1) - \eta$. Finally, take an ergodic measure μ_e supported in Ψ_0 with entropy at least $h_\mu(\varphi_1) - \eta$. Since we had already proved that $\mu_e \in V(f_1, ..., f_l; \mu, \epsilon)$, this finishes the proof. □

Combining Remark 4.1.3 and Proposition 4.1.4 we obtain the main result of this section.

Theorem 4.1.6 (Weak entropy density of geodesic flow)**.** *Let M be a pinched negatively curved manifold. Then the ergodic measures are weak entropy dense in the space of invariant probability measures.*

As a corollary of Theorem 4.1.6 we obtain a new version of Theorem 3.1.6, where we can weaken the Hölder assumption, instead we require the potential to be bounded.

Theorem 4.1.7. *Let $F \in C_b(T^1M)$, then*

$$P(F) = \sup_K P_K(F),$$

where the supremum runs over the set of compact invariant subsets.

Proof. Fix $\epsilon > 0$ and choose $\mu_\epsilon \in \mathcal{M}(g)$ such that

$$h_{\mu_\epsilon}(g) + \int F d\mu_\epsilon > P(F) - \epsilon.$$

Using Proposition 4.1.4 we can approximate μ_ϵ with an invariant probability measure of compact support ν_ϵ such that $h_{\nu_e}(g) > h_{\mu_\epsilon}(g) - \epsilon$, and $\int F d\nu_\epsilon > \int F d\mu_\epsilon - \epsilon$. This implies that

$$h_{\nu_\epsilon}(g) + \int F d\nu_\epsilon > P(F) - 3\epsilon,$$

where ν_ϵ is compactly supported. In particular $\sup_K P_K(F) \geqslant h_{\nu_\epsilon}(g) + \int F d\nu_\epsilon$. It is clear that $P(F) \geqslant \sup_K P_K(F)$. Therefore

$$P(F) \geqslant \sup_K P_K(F) \geqslant P(F) - 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary we conclude that $P(F) = \sup_K P_K(F)$. □

4.2 Simplified entropy inequality for geodesic flows

We will now proceed to prove that the geodesic flow on a pinched negatively curved manifold satisfies a simplified entropy inequality after a mild modification in the definition of the dynamical balls. As before $p : T^1\widetilde{M} \to T^1M$ is the canonical projection.

Definition 4.2.1. *Let $y \in T^1\widetilde{M}$ and $x = p(y)$, we define $B_n^p(x, r)$ as the image under p of the (n, r)-dynamical ball in $T^1\widetilde{M}$ centered at y. We say that $B_n^p(x, r)$ is the $p(n, \epsilon)$-dynamical ball centered at x, where p stands for projection.*

We recall that $N_\mu(n, \epsilon, \delta)$ is the minimum number of (n, ϵ)-dynamical balls needed to cover a set of measure strictly bigger than $1 - \delta$. We define $N_\mu^p(n, \epsilon, \delta)$ as the minimum number of $p(n, \epsilon)$-dynamical balls needed to cover a set of measure strictly bigger than $1 - \delta$. We also use the notation $N^p(C, n, \epsilon)$ to denote the minimum

number of $p(n,\epsilon)$-dynamical balls needed to cover the set $C \subset T^1M$. To simplify notation we define $X = T^1M$. To prove the main result of this subsection we will need the following lemma, that follows easily from the criterion given in [PS].

Lemma 4.2.2. *Let (M,g) be a pinched negatively curved manifold. Then there exists a Hölder continuous function $\varphi : T^1M \to \mathbb{R}$ which admits an equilibrium state.*

Theorem 4.2.3 (Simplified entropy inequality)**.** *Let (M,g) be a pinched negatively curved manifold. Then for every ergodic measure μ have*

$$h_\mu(g) \leqslant \liminf_{n\to\infty} \frac{1}{n} \log N^p_\mu(n,r,\delta),$$

where $\delta \in (0,1)$ and $r > 0$.

Proof. Fix $\delta \in (0,1)$, $r > 0$ and $r' \in (0,r)$. Choose $m \in \mathbb{N}$ such that $1-\delta > \frac{1}{m}$. Let $F_n \subset X$ be a set satisfying $N^p_\mu(n,r,1-\frac{1}{m}) = N^p(F_n,r,1-\frac{1}{m})$ and $\mu(F_n) > \frac{1}{m}$. By Birkhoff ergodic theorem there exists $F' \subset X$ and $N_0 > 0$ such that $\mu(F') > 1 - \frac{1}{8m}$, and $|\frac{1}{n}\int_0^{n-1} \varphi(g_t x)dt - \int \varphi d\mu| < \epsilon$, for every $x \in F'$ and $n \geqslant N_0$. From now on we will assume $n \geqslant N_0$. Let K be a compact subset such that $\mu(K) > 1 - \frac{1}{8m}$. We will need the following fact, which follows directly from the formula

$$\mu(A_1 \cup ... \cup A_t) = \sum_{i=1}^{n-1} (-1)^{i+1} \sum \mu(A_{t_1} \cap ... \cap A_{t_i}).$$

Lemma 4.2.4. *Let F be a measurable set satisfying $\mu(F) > \frac{1}{s}$. Then for every $h \in \mathbb{Z}$ there exists $k \in [h, h+2s)$ such that $\mu(F \cap g_{-k}F) > \frac{1}{2^{2s}}$.*

Define $S_n = F_n \cap K \cap F'$ and observe that by construction $\mu(S_n) > \frac{1}{2m}$. Then there exists $k_n \in (-4m, 0]$ such that

$$\mu(S_n \cap g_{-(n-1+k_n)} S_n) > \frac{1}{2^{4m}}.$$

Define $A_n = S_n \cap g_{-(n-1+k_n)} S_n$. In particular we get

$$N_\mu^p(n, r', 1 - \frac{1}{2^{4m}}) \leqslant N^p(A_n, n, r').$$

Consider a $p(n,r)$-covering with minimal cardinality of A_n and denote by R the set of centers of such dynamical balls. For each $x \in R$, let E_x be a $p(n,r')$-separated set of maximal cardinality in $B_n^p(x,r)$. We can moreover assume that $E_x \subset A_n$. By definition, the $p(n,r'/2)$-dynamical balls with centers in E_x are disjoint. Moreover, since $\#E_x$ is maximal, the collection of $p(n,r')$-dynamical balls having centers in E_x is a $p(n,r')$-covering of $B_n^p(x,r)$. Define $Y = \bigcup_{l=0}^{4m} g_l K$. Since K is compact, the same holds for Y. Let $\varphi : T^1M \to \mathbb{R}$ be a Hölder continuous function admitting an equilibrium state. Let m be its (unique) equilibrium state. Therefore

$$\sum_{y \in E_x} m(B_n^p(y, r'/2)) = m\left(\bigcup_{y \in E_x} B_n^p(y, r'/2) \right) \leqslant m(B_n^p(x, r + r')),$$

and so

$$\#E_x \leqslant \frac{m(B_n^p(x, r + r'))}{\min_{y \in E_x} m(B_n^p(y, r'/2))}.$$

Observe that by construction if $x \in A_n$, then $g_{n-1+k_n}(x) \in S_n$, which implies $g_n(x) \in Y$. In particular $A_n \subset Y \cap g_{-n}Y$. Recall that m satisfies the Gibbs property, i.e. there exists a constant $C = C(Y, r + r', r'/2)$ such that

$$C^{-1} \leqslant \frac{m(B_n^p(y, r_0))}{\exp(\int_0^{n-1} \varphi(g_t y)dt - nP(\varphi))} \leqslant C,$$

for every $y \in Y \cap g_{-n}Y$ and $r_0 \in \{r + r', r'/2\}$. Using the notation above this implies the bound

$$\#E_x \leqslant C^2 \exp\left(\int_0^{n-1} \varphi(g_t x)dt - \min_{y \in E_x} \int_0^{n-1} \varphi(g_t y)dt \right).$$

Therefore, by the definition of F', we have

$$\#E_x \leqslant C' \exp(2n\epsilon).$$

Observe that

$$\begin{aligned} N_\mu^p(n, r', 1 - \frac{1}{2^{4m}}) &\leqslant N^p(A_n, n, r') \leqslant C' \exp(2n\epsilon) \#R = C' \exp(2n\epsilon) N^p(A_n, n, r) \\ &\leqslant C' \exp(2n\epsilon) N^p(F_n, n, r) = C' \exp(2n\epsilon) N_\mu^p(n, r, 1 - \frac{1}{m}) \\ &\leqslant C' \exp(2n\epsilon) N_\mu^p(n, r, \delta). \end{aligned}$$

We remark that C' is independent of n; it only depends on r, m and K. Then

$$\liminf_{n\to\infty} \frac{1}{n} \log N_\mu^p(n, r', 1 - \frac{1}{2^{4m}}) \leqslant \liminf_{n\to\infty} \frac{1}{n} \log N_\mu^p(n, r, \delta) + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary we get

$$\liminf_{n\to\infty} \frac{1}{n} \log N_\mu^p(n, r', 1 - \frac{1}{2^{4m}}) \leqslant \liminf_{n\to\infty} \frac{1}{n} \log N_\mu^p(n, r, \delta).$$

By definition $B_n^p(x, r) \subset B_n(x, r)$. This implies that $N_\mu(n, s, q) \leqslant N_\mu^p(n, s, q)$, for every $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $q \in (0, 1)$. Using Theorem 2.2.3 we get

$$h_\mu(g) = \lim_{r'\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N_\mu(n, r', 1 - \frac{1}{2^{4m}}) \leqslant \lim_{r'\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N_\mu^p(n, r', 1 - \frac{1}{2^{4m}}).$$

Combining the inequalities above we obtain

$$h_\mu(g) \leqslant \liminf_{n\to\infty} \frac{1}{n} \log N_\mu^p(n, r, \delta).$$

□

Chapter 5

Semicontinuity of the entropy map

In this section we will prove the upper semicontinuity of the entropy map. More precisely we will prove that if $(\mu_n)_n$ converges in the weak-* topology to μ, then

$$\limsup_{n\to\infty} h_{\mu_n}(g) \leqslant h_\mu(g).$$

In order to do so we will prove a more general inequality that involves the escape of mass and the topological entropy at infinity of the geodesic flow (Section 5.2). We will also prove that the topological entropy at infinity coincides with a measure theoretic counterpart, this is what we call the variational principle for the entropy at infinity. The results in Section 5.2 will have many consequences to the thermodynamic formalism of potentials vanishing at infinity, but we will discuss those applications in Section 7.

5.1 A more general result

In this section we will prove a more general version than the one we need for the geodesic flow. This has the advantage of emphasizing the key ingredients to obtain similar results.

We start with our definition of topological entropy at infinity. Let (X, d) be a non-compact topological manifold, and $T : X \to X$ be an L-Lipschitz homeomorphism, i.e. a homeomorphism satisfying $d(Tx, Ty) \leqslant Ld(x, y)$, for every $(x, y) \in X \times X$. Let K be a compact subset of X. Define

$$K(n) = K \cap \bigcap_{i=1}^{n-2} T^{-i}K^c \cap T^{-(n-1)}K.$$

Given a point $x \in K$ and $r > 0$, we define $C(x, n, r)$ as the number of (n, r)-dynamical balls needed to cover $B(x, r) \cap K(n)$. We define the *topological entropy outside K at scale r* by the formula

$$\delta_\infty(K, r) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in K} C(x, n, r),$$

and the *topological entropy outside K* by $\delta_\infty(K) = \inf_{r>0} \delta_\infty(K, r)$.

Definition 5.1.1 (Topological entropy at infinity)**.** *The* topological entropy at infinity *of (X, d, T) is the quantity*

$$\delta_\infty = \inf_{\{K_n\}} \liminf_{n\to\infty} \delta_\infty(K_n),$$

where the infimum in front runs over sequences $\{K_n\}_{n\in\mathbb{N}}$ where each K_n is compact, $K_n \subset K_{n+1}$ and $X = \bigcup_{n\geqslant 1} K_n$.

Our goal is to prove the following result.

Theorem 5.1.2. *Let (X, d) be a non-compact topological manifold and T an L-Lipschitz homeomorphism with finite topological entropy at infinity. Assume that (X, d, T) satisfies a simplified entropy inequality and that its ergodic measures are weak entropy dense. Let $(\mu_n)_n$ be a sequence of T-invariant probability measures con-*

verging to μ in the vague topology. Then

$$\limsup_{n\to\infty} h_{\mu_n}(T) \leqslant |\mu| h_{\mu/|\mu|}(T) + (1-|\mu|)\delta_\infty.$$

If the sequence $(\mu_n)_n$ converges to the zero measure the right hand side of the inequality is understood as δ_∞.

We start with the following preliminary version of Theorem 5.1.2.

Proposition 5.1.3. *Let (X,d) be a non-compact topological manifold and T a L-Lipschitz homeomorphism with finite topological entropy at infinity. Assume that (X,d,T) satisfies a simplified entropy inequality. Let $(\mu_n)_n$ be a sequence of ergodic measures converging to μ in the vague topology. Suppose there exists a compact set K such that $\mu(\partial K) = 0$, and $\mu_n(K) > 0$, for every $n \in \mathbb{N}$. Then*

$$\limsup_{n\to\infty} h_{\mu_n}(T) \leqslant |\mu| h_{\mu/|\mu|}(T) + (1-\mu(A(K)))\delta_\infty(K,r),$$

where $A(K) = \{x \in X : T^k x \in K, \text{ for some } k \geqslant 0 \text{ and some } k \leqslant 0\}$.

Proof. Define $Y = A(K)$ as the set of points in X that enter to K under positive and negative iterates of T. Let A_k be the set of points in K that have their first return to K at time k. Given $x \in Y$ define $n_2(x)$ as the smallest non-negative number such that $T^{n_2(x)}x \in K$ and $n_1(x)$ as the smallest non-negative number such that there exists $y \in K$ satisfying $T^{n_1(x)}(y) = x$. For $x \in Y$ define $n(x) = n_1(x) + n_2(x)$, and declare $n(x) = \infty$ whenever $x \in Y^c$. For $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ define

$$C_n = \{x \in X : n(x) = n\}.$$

By the ergodicity of μ_m and the hypothesis $\mu_m(K) > 0$ we have that $\mu_m(C_\infty) = 0$. Moreover, the fact $x \in C_n$ means that x is in the orbit of a point in A_n. Since T

is Lipschitz and $\sup_{x\in K} d(x,Tx)$ is finite, we conclude that $\bigcup_{n=0}^{M} C_n$ is bounded, and therefore relatively compact. Define

$$\alpha_{N,M} = \bigcup_{n>N}^{M} C_n, \text{ and } \alpha_N = \bigcup_{n>N} C_n.$$

It worth mentioning that $\partial\alpha_N \subset \bigcup_{k\in\mathbb{Z}} T^{-k}\partial K$, and that the same holds for $\alpha_{N,M}$ and C_∞. This implies that $\mu(\partial\alpha_{N,M}) = \mu(\partial\alpha_N) = \mu(\partial C_\infty) = 0$. By the definition of $\delta_\infty(K,r)$ we know that given $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that the following holds. For every $n \geqslant N_0$ and $x \in K(n) = K \cap \bigcap_{s=1}^{n-2} T^{-s}K^c \cap T^{-(n-1)}K$, we have that $B(x,r) \cap K(n)$ can be covered by at most $e^{n(\delta_\infty(K,r)+\epsilon)}$ (n,r)-dynamical balls.

Choose natural numbers $k \geqslant 2$ and $N \geqslant N_0(\epsilon)$. Define the partition

$$\beta_{k,N} = \{\alpha_{kN}, \alpha_{N,kN}, Q_N^1, ..., Q_N^s, C_\infty\},$$

where Q_N^i are balls of diameter less than r/L^{kN+2} covering $\bigcup_{n=0}^{N} C_n$ (which is relatively compact) and let $\beta'_{k,N} = \{Q_N^1, ..., Q_N^s\}$. We choose this covering such that $\mu(\partial Q_N^i) = 0$ for every i. In particular we know $\mu(\partial\beta_{k,N}) = 0$.

Recall that for a partition $\mathcal{P}$ we denote $\mathcal{P}^n$ to the partition $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$. Let $Q \in \beta_{k,N}^n$ be such that $Q \subset \alpha_N^c$. We say that $[r,s) \subset [0,n)$ is an *excursion* of Q into α_N if $T^tQ \subset \alpha_N$ for every $t \in [r,s)$, $T^{r-1}Q \subset \alpha_N^c$ and $T^sQ \subset \alpha_N^c$. Define $m_{k,N,n}(Q)$ as the number of excursions of Q into α_N that contain at least one excursion into α_{kN} and let $|E_{N,n}(Q)| = \#\{k \in [0,n) : T^kQ \subset \alpha_N\}$.

Remark 5.1.4.

1. *Observe that if $x \in C_n$, then $Tx \in C_n$, or $Tx \in K$. In other words if $x \in K^c$, then $n(Tx) \leqslant n(x)$.*

2. *Let $[r,s)$ be an excursion of Q into α_N. Suppose there exists $x \in T^{r-1}Q \cap K^c$, then by (1) we have $n(x) \geqslant n(Tx) > N$. In particular $x \in \alpha_N$, which contradicts*

that $T^{r-1}Q \subset \alpha_N^c$. *We conclude that* $T^{r-1}Q \subset K$.

3. *If* $x \in Q$ *and* $Q \subset \alpha_N^c$, *then* $T^t x \in K$, *for some* $0 \leqslant t \leqslant N$. *If* $x \in \bigcup_{n=0}^{N} C_n$, *then by definition* $n_2(x) \leqslant N$ *and the conclusion follows.*

We claim that an atom $Q \in \beta_{k,N}^n$ such that $Q \subset K \cap T^{-(n-1)}K$, can be covered with an appropriate number of (n, r)-dynamical balls. This estimate is very important in order to prove Proposition 5.1.3.

Proposition 5.1.5. *Let* $\beta_{k,N}$ *as above. Then an atom* $Q \in \beta_{k,N}^n$ *such that* $Q \subset K \cap T^{-(n-1)}K$ *can be covered by no more than*

$$C_0 e^{|E_{N,n}(Q)|(\delta_\infty(K,r)+\epsilon)} e^{m_{k,N,n}(Q)N(\delta_\infty(K,r)+\epsilon)},$$

(n, r)*-dynamical balls, where* $C_0 = C_0(m, q, N, k)$.

To simplify notation we will forget the subindex N and k. We remark that since C_∞ satisfies $TC_\infty \subset C_\infty$, the assumption $T^{n-1}Q \subset K$ rules out the possibility that Q entered to C_∞ before the $(n-1)$th iterate. The proof is inductive. First decompose $[0, n-1]$ as

$$[0, n-1] = W_1 \cup V_1 \cup ... \cup V_s \cup W_{s+1},$$

according to the excursions into α_N that contain at least one excursion into α_{kN}. It worth mentioning that by Remark 5.1.4 each excursion into α_N can contain at most one excursion into α_{kN}. More precisely, let $V_i = [n_i, n_i + h_i)$ and $W_i = [l_i, l_i + L_i)$ with $l_i + L_i = n_i$ and $n_i + h_i = l_{i+1}$. The segment V_i denotes an excursion into α_N that contains an excursion into α_{kN} and $(W_i)_i$ are the complementary intervals.

Step 0: Cover $\bigcup_{n=0}^{N} C_n$ by balls of diameter r/L^{kN+2}. We do this covering such that the boundary of each ball has zero measure under μ. We denote the number of balls required for this covering as $C_0 = C_0(K, r, N, k)$.

Step 1: Assume we have covered Q by

$$C_0 C_1^{i-1} e^{(\delta_\infty(K,r)+\epsilon)(|V_1|+\ldots+|V_{i-1}|)} e^{(i-1)N(\delta_\infty(K,r)+\epsilon)},$$

(l_i+1, r)-dynamical balls. We claim the same number of balls suffices to cover Q with (l_i+L_i, r)-dynamical balls. Observe that by hypothesis $T^{l_i}Q \subset \alpha_N^c$, therefore diam $T^{l_i}Q \leqslant r/L^{kN+2}$. Since the balls used to cover β' have all diameter smaller than r/L^{kN+2} the same hold if Q spends some extra time in β'. If Q have an excursion into α_N that does not enter to α_{kN}, then by definition it must come back to β' before kN iterates (see Remark 5.1.4 (3)). In particular if the excursion into α_N is $[p_i, p_i+q_i)$, then $q_i \leqslant kN$. Observe that diam $T^{p_i-1}Q \leqslant r/L^{kN+2}$ implies that diam $T^{p_i+t}Q \leqslant r$ for every $t \in [0, kN)$. In particular the same holds for $t \in [0, q_i]$. Now we have entered to β' again and we can repeat this process until we find an excursion into α_{kN}, in that case we proceed to Step 2.

Step 2: Assume we have covered Q by

$$C_0 e^{(\delta_\infty(K,r)+\epsilon)(|V_1|+\ldots+|V_{i-1}|)} e^{(i-1)N(\delta_\infty(K,r)+\epsilon)},$$

(n_i, r)-dynamical balls. To get a covering with (n_i+h_i+1, r)-dynamical balls we will cover each (n_i, r)-dynamical ball in the given covering by (n_i+h_i+1, r)-dynamical balls. Let $x \in Q$ be the center of one of the (n_i, r)-dynamical balls (if the center of the ball is not in Q one takes a point in the ball that do belong to Q and change r by $2r$ in our next argument, for simplicity we assume that $x \in Q$ is the center of the dynamical ball). By definition $T^t x \in \alpha_N$ for $t \in [n_i, n_i+h_i)$, $T^{n_i-1}x \in \alpha_N^c$ and $T^{n_i+h_i}x \in \alpha_N^c$. Let $s(x) \geqslant 0$ be the smallest number such that $T^{n_i+h_i+s(x)}(x) \in K$. Notice that by Remark 5.1.4(2) we have $T^{n_i-1}(x) \in K$, and by Remark 5.1.4(3) we

know that $s(x) \leqslant N$. Since $T^{n_i-1}B_{n_i}(x,r) \subset B(T^{n_i-1}x,r)$, we can just focus on covering $B = B(T^{n_i-1}x,r) \cap K(h_i + s(x) + 1)$ with (h_i+1,r) dynamical balls. By the definition of $\delta_\infty(K,r)$ we know that B can be covered with at most $e^{(\delta_\infty(K,r)+\epsilon)(h_i+s(x))}$ $(h_i + s(x) + 1, r)$-dynamical balls. We conclude that B can be covered by at most $e^{(\delta_\infty(K,r)+\epsilon)(h_i+N)}$ (h_i+1,r)-dynamical balls. This proves that the number of (n_i+h_i+1,r)-dynamical balls needed to cover Q is at most the number of balls we had at the beginning of Step 2 times $e^{(\delta_\infty(K,r)+\epsilon)h_i}e^{(\delta_\infty(K,r)+\epsilon)N}$.

We conclude that Q can be covered with at most

$$C_0 e^{|E_{N,n}(Q)|(\delta_\infty(K,r)+\epsilon)} e^{m_{k,N,n}(Q)N(\delta_\infty(K,r)+\epsilon)},$$

(n,r)-dynamical balls, where $C_0 = C_0(m,q,N,k)$. We remark that C_0 is a constant independent of n. We also remark that the term $|E_{N,n}(Q)|$ is a very rough bound, we can actually use the time spent in excursions into α_N containing excursions into α_{kN}.

Proposition 5.1.6. *Let $\beta_{k,N}$ the partition defined in Proposition 5.1.5. Let μ be an ergodic invariant probability measure satisfying $\mu(K) > 0$. Then*

$$h_\mu(T) \leqslant h_\mu(T,\beta_{k,N}) + \mu(\alpha_N)(\delta_\infty(K,r)+\epsilon) + \frac{1}{k}(\delta_\infty(K,r)+\epsilon).$$

Proof. Recall that by Definition 2.2.4 we know that for every ergodic measures μ we have

$$h_\mu(T) \leqslant \lim_{n\to\infty} \frac{1}{n} \log N_\mu(n,r,\delta).$$

Using the ergodicity of μ and the assumption $\mu(K) > 0$ we can find an increasing sequence $(n_i)_i$ such that

$$\mu(K \cap T^{-n_i}K) > \delta_1,$$

for every $i \in \mathbb{N}$, where δ_1 is sufficiently small but positive (and independent of n_i).

By Shannon-McMillan-Breiman theorem the set

$$A_{\epsilon_1,N} = \{x \in X : \forall n \geqslant N, \mu(\beta^n(x)) \geqslant \exp(-n(h_\mu(T,\beta) + \epsilon_1))\},$$

has measure converging to 1 as N goes to ∞, for every $\epsilon_1 > 0$. Fix $\epsilon_1 > 0$ small. By Birkhoff ergodic theorem there exists a set W_{ϵ_1} such that

$$\frac{1}{n}\sum_{i=0}^{n} 1_{\alpha_N}(T^n x) < \mu(\alpha_N) + \epsilon_1, \text{ and } \mu(W_{\epsilon_1}) > 1 - \frac{\delta_1}{4},$$

for all $x \in W_{\epsilon_1}$ and $n \geqslant n(\epsilon_1)$. We finally define

$$X_i = W_{\epsilon_1} \cap A_{\epsilon_1,n_i} \cap K \cap T^{-n_i}K.$$

By construction, for i sufficiently large, we have $\mu(X_i) > \frac{\delta_1}{2}$. From now on we will always assume i is sufficiently large. Our goal is to cover X_i by (n_i, r)-dynamical balls. By definition of A_{ϵ_1,n_i} we know X_i can be covered with $\exp(n_i(h_\mu(T,\beta) + \epsilon_1))$ many elements of β^{n_i}. We will use Proposition 5.1.5 to cover efficiently each of those atoms by dynamical balls. Let $Q \in \beta^{n_i}$ be an atom intersecting X_i, in particular $Q \in K \cap T^{-(n-1)}K$. By the choice of W_{ϵ_1} we have

$$|E_{N,n_i}(Q)| < (\mu(\alpha_N) + \epsilon_1)n_i.$$

We claim that $m_{k,N,n_i}(Q) \leqslant \frac{1}{Nk}n_i$. This follows from Remark 5.1.4. Let $[p, p+q)$ be is an excursion of Q into α_N that contain an excursion into α_{kN}. There exists a smallest $h \geqslant 0$ such that $T^{p+q+h}Q \subset K$. By definition of α_{kN} we have $q+h+1 \geqslant kN$. Moreover $T^kQ \subset \alpha_N^c$ for every $k \in [p+q+1, p+q+h]$. In particular each excursion into α_{kN} generates an interval of length at least kN where no other excursion into α_N can occur. Putting all together, we get that $N(n_i, r, 1 - \frac{\delta_1}{2})$ is bounded from above

by

$$e^{n_i(h_\mu(T,\beta)+\epsilon_1)}C_0 e^{n_i(\delta_\infty(K,r)+\epsilon)(\mu(\alpha_N)+\epsilon_1)}e^{\frac{1}{kN}n_i N(\delta_\infty(K,r)+\epsilon)}.$$

Finally we obtained

$$h_\mu(T) \leqslant h_\mu(T,\beta_{k,N}) + \epsilon_1 + (\delta_\infty(K,r)+\epsilon)(\mu(\alpha_N)+\epsilon_1) + \frac{1}{k}(\delta_\infty(K,r)+\epsilon).$$

Since $\epsilon_1 > 0$ was arbitrary we are done. □

We now explain how to get Proposition 5.1.3 from Proposition 5.1.6. First assume $\mu(X) > 0$, and fix $\varepsilon_0 > 0$. We remark that by construction $\mu(\partial\beta_{k,N}) = 0$. To simplify notation we use β instead of $\beta_{k,N}$ to denote our partition. Choose m sufficiently large such that

$$h_{\frac{\mu}{|\mu|}}(T) + \varepsilon_0 > \frac{1}{m}H_{\frac{\mu}{\|\mu\|}}(\beta^m), \quad 2\frac{e^{-1}}{m} < \frac{\varepsilon_0}{2},$$

and $-(1/m)\log\mu(X) < \varepsilon_0$. Then

$$|\mu| h_{\frac{\mu}{|\mu|}}(T) + 2\varepsilon_0 > \frac{1}{m}\sum_{P\in\beta^m}\mu(P)\log\mu(P).$$

Define $A = \bigcap_{i=0}^{m-1} T^{-i}\alpha_{kN}$ and observe that by the definition of the vague convergence

$$\lim_{n\to\infty}\sum_{Q\in\beta^m\setminus\{A\}}\mu_n(Q)\log\mu_n(Q) = \sum_{Q\in\beta^m\setminus\{A\}}\mu(Q)\log\mu(Q).$$

Choosing n sufficiently large we get the inequality

$$|\mu| h_{\frac{\mu}{|\mu|}}(T) + 3\varepsilon_0 \geqslant \frac{1}{m}H_{\mu_n}(\beta^m).$$

Finally using Proposition 5.1.6 we get

$$\begin{aligned}\mu(X)h_{\frac{\mu}{\|\mu\|}}(T)+3\varepsilon >&\frac{1}{m}H_{\mu_n}(\beta^m)\geqslant h_{\mu_n}(T,\beta)\\ \geqslant&h_{\mu_n}(T)-(\delta_\infty(K,r)+\epsilon)\mu_n(\alpha_N)-\frac{1}{k}(\delta_\infty(K,r)+\epsilon).\end{aligned}$$

Observe that $\bigcup_{n=0}^{N} C_n$ is relatively compact and that $\mu_n(\alpha_N)=1-\mu_n(\bigcup_{n=0}^{N} C_n)$. We remark that by construction $\mu(\partial\alpha_N)=0$. Therefore

$$\begin{aligned}\limsup_{n\to\infty} h_{\mu_n}(T)\leqslant&\mu(X)h_{\frac{\mu}{\|\mu\|}}(T)+(\delta_\infty(K,r)+\epsilon)(1-\mu(\bigcup_{n=0}^{N} C_n))\\ &+\frac{1}{k}(\delta_\infty(K,r)+\epsilon).\end{aligned}$$

Observe that by construction we can send ϵ to zero as N goes to infinity. Finally take $k\to\infty$ and $N\to\infty$. We obtained the desired inequality

$$\limsup_{n\to\infty} h_{\mu_n}(T)\leqslant|\mu|h_{\mu/|\mu|}+(1-\mu(A(K)))\delta_\infty(K,r).$$

The case when $\mu(X)=0$ follows directly from Proposition 5.1.6 since $h_{\mu_n}(g,\beta)\to 0$ and $\mu_n(\alpha_N)=1-\mu_n(\bigcup_{s=1}^{N} C_s)\to 1$ as n tends to ∞. □

We have finally all the ingredients to prove Theorem 5.1.2.

Proof of Theorem 5.1.2. We will prove that

$$\limsup_{n\to\infty} h_{\mu_n}(T)\leqslant|\mu|h_{\mu/|\mu|}+(1-\mu(A(K)))\delta_\infty(K,r), \tag{5.1}$$

for every sufficiently large compact set K and $r>0$. Let μ_0 be a T-invariant measure that gives positive measure to K (which exists because K is sufficiently large) and define $\mu_n'=(1-\frac{1}{n})\mu_n+\frac{1}{n}\mu_0$. By hypothesis, the space of ergodic measures is weak entropy dense (see Definition 4.1.6), therefore we can find an ergodic measure ν_n

arbitrarily close to μ_n' (in the weak-* topology) such that $h_{\nu_n}(T) > h_{\mu_n'}(T) - \frac{1}{n}$. In particular we can assume $\nu_n(K) > 0$ and that $(\nu_n)_n$ converges to μ in the weak-* topology. We can now use Proposition 5.1.3 to the sequence $(\nu_n)_n$ and get

$$\limsup_{n\to\infty} h_{\nu_n}(T) \leqslant |\mu| h_{\mu/|\mu|} + (1 - \mu(A(K)))\delta_\infty(K, r).$$

By construction we have

$$h_{\nu_n}(T) > h_{\mu_n'}(T) - \frac{1}{n} = \big(1 - \frac{1}{n}\big) h_{\mu_n}(T) + \frac{1}{n} h_{\mu_0}(T) - \frac{1}{n},$$

and therefore

$$\limsup_{n\to\infty} h_{\nu_n}(T) \geqslant \limsup_{n\to\infty} h_{\mu_n}(T),$$

which implies the inequality (5.1). Take an increasing sequence $(K_i)_i$ of compact sets such that $\lim_{i\to\infty} \delta_\infty(K_i) = \delta_\infty$. Now observe that $A(K_i) \subset A(K_{i+1})$ and $\bigcup_{i\geqslant 1} A(K_i) = X$. This implies that $\lim_{i\to\infty} \mu(A(K_i)) = \mu(X)$, and therefore inequality (5.1) finishes the proof. □

5.2 Upper semicontinuity for the geodesic flow

As explained in Section 4.2 the geodesic flow satisfies a mild modification of the simplified entropy inequality. This fact does not affect the proof of Theorem 5.1.2, but modifies the definition of the topological entropy at infinity (instead of using (n, ϵ)-dynamical balls we use $p(n, \epsilon)$-dynamical balls). In this case the topological entropy at infinity (using $p(n, \epsilon)$-dynamical balls) takes a much more concrete form in terms of the action of the fundamental group on $\widetilde{M}$. Let $Q \subset \widetilde{M}$ be a compact subset, and let $\Gamma = \pi_1(M)$. Given $n \in \mathbb{N}$ we define

$$\Gamma_Q(n) = \{\gamma \in \Gamma : \exists x \in T^1Q \text{ such that } g_k(x) \in T^1Q^c, \text{ for } k \in [1, n-2] \text{ and } g_{n-1}(x) \in T^1\gamma Q\}.$$

Then define

$$\delta_\infty(Q) = \limsup_{n\to\infty} \frac{1}{n} \log \#\Gamma_Q(n).$$

Let P be a Dirichlet fundamental domain of M in $\widetilde{M}$.

Definition 5.2.1 (Topological entropy at infinity)**.** *The* topological entropy at infinity *of M is the quantity*

$$\delta_\infty = \inf_{(P_k)} \liminf_{k\to\infty} \delta_\infty(P_k),$$

where the infimum runs over compact exhaustions of P, in other words, increasing sequences $(P_k)_{k\geqslant 1}$ such that each P_k is compact and $\bigcup_{n\geqslant 1} P_k = P$.

Remark 5.2.2. *Shortly after the paper [V2] was finished, B. Schapira told me that in a joint work with S. Tapie [ST] they also defined the topological entropy at infinity. The critical gap $\delta_\infty < h_{top}(g)$, has important consequence in the study of the regularity of the topological entropy under C^1 perturbations of the metric [ST].*

It is not hard to verify that the topological entropy at infinity computed with $p(n,\epsilon)$-dynamical balls coincides with the one in Definition 5.2.1. As mentioned above, the proof of Theorem 5.1.2 implies one of the main results of this thesis.

Theorem 5.2.3. *Let (M,g) be a pinched negatively curved manifold. Let $(\mu_n)_n$ be a sequence of invariant probability measures converging to μ in the vague topology. Then*

$$\limsup_{n\to\infty} h_{\mu_n}(g) \leqslant |\mu| h_{\mu/|\mu|}(g) + (1-|\mu|)\delta_\infty.$$

If the sequence converges vaguely to zero, then the right hand side is understood as δ_∞.

It worth mentioning that if M is geometrically finite, then

$$\delta_\infty = \max_{\mathcal{P}} \delta_{\mathcal{P}},$$

where the maximum runs over the parabolic subgroups of $\pi_1(M)$. In particular we recover one of the main results of [RV].

Remark 5.2.4. *We will prove in Section 5.3 that the quantity δ_∞ is sharp. It is not clear to the authors that the topological entropy at infinity computed with the $p(n,\epsilon)$-dynamical balls coincides with the one using (n,ϵ)-dynamical balls (although in geometrically finite manifolds they seem to coincide). For some reason, in this theory, the role of the $p(n,\epsilon)$-dynamical balls is more significant than the usual dynamical balls.*

Definition 5.2.5 (Strongly positive recurrent manifolds)**.** *We say a pinched negatively curved manifold M is* strongly positive recurrent *(SPR for short) if $\delta_\infty < h_{top}(g)$.*

Among the consequences of Theorem 5.2.3 we will get that SPR manifolds have a measure of maximal entropy (Theorem 6.1.1). This definition should also be compared with Definition 2.3.20, where an analogous critical gap condition was required. Theorem 5.2.3 implies that the entropy map, $\mu \mapsto h_\mu(g)$, and the map $\mu \mapsto h_\mu(g) + \int F d\mu$, are upper semicontinuous .

Theorem 5.2.6 (Upper semicontinuity of the entropy map)**.** *Let (M,g) be a pinched negatively curved manifold. Let $(\mu_n)_n$ be a sequence of invariant probability measures converging to μ in the weak-* topology. Then*

$$\limsup_{n\to\infty} h_{\mu_n}(g) \leqslant h_\mu(g).$$

Theorem 5.2.7. *Let (M,g) be a pinched negatively curved manifold. Let $(\mu_n)_n$ be a sequence of invariant probability measures converging to μ in the weak-* topology and $F \in C_b(T^1M)$. Then*

$$\limsup_{n\to\infty} \big(h_{\mu_n}(g) + \int F d\mu_n\big) \leqslant \big(h_\mu(g) + \int F d\mu\big).$$

Remark 5.2.8. *After the work of Yomdin [Yom] and Newhouse [New] we know that on a compact manifold the entropy map is upper semicontinuous if the dynamics is of class C^∞. Later on, Buzzi [Buz] refined Newhouse's result and proved that in this context the dynamical system is asymptotically h-expansive, which it is known to imply the upper semicontinuity of the entropy map in the compact setting. We remark that their methods use, in an essential way, the compactness of the manifold.*

5.3 Variational principle for the entropy at infinity

In Section 5.2 we introduced the topological entropy at infinity of the geodesic flow. We will now define a measure theoretic counterpart and prove that this two a priori different notions of entropy at infinity coincide.

Definition 5.3.1 (Measure theoretic entropy at infinity)**.** *The* measure theoretic entropy at infinity *of the geodesic flow is defined as*

$$h_\infty = \sup_{(\mu_n)} \limsup_{\mu_n \to 0} h_{\mu_n}(g),$$

where the supremum runs over sequences of invariant probability measures converging vaguely to the zero measure.

This invariant was defined in [IRV] and proved to be equal to the topological entropy at infinity for the geodesic flow on extended Schottky manifolds via symbolic methods. This result was later extended to cover all geometrically finite manifolds in [RV]. In this thesis we generalize those results to any pinched negatively curved manifold.

Theorem 5.3.2 (Variational principle for the entropy at infinity)**.** *The topological entropy at infinity is equal to the measure theoretic entropy at infinity. Equivalently,* $\delta_\infty = h_\infty$.

To prove the variational principle we will need the following lemma.

Lemma 5.3.3. *Let φ be a Hölder potential in $C_0(T^1M)$. Then*

$$P(\varphi) \geqslant \delta_\infty.$$

Proof. Let N be a Dirichlet fundamental domain of M in $\widetilde{M}$. Given $\epsilon > 0$, there exists a compact subset $K = K(\epsilon)$ of M such that $\varphi(x) \in [-\epsilon, \epsilon]$, for every $x \in T^1K^c$. Let $Q = Q(\epsilon)$ be the subset of N corresponding to K and $M = \sup_{x\in T^1M} |\varphi(x)|$. Choose a reference point $z \in T^1Q$. Then

$$\begin{aligned}\sum_{\gamma\in\Gamma} \exp(\int_z^{\gamma z} \widetilde{\varphi} - s) &\geqslant \sum_{n\geqslant 1} \sum_{\gamma\in\Gamma_Q(n)} \exp(\int_z^{\gamma z} \widetilde{\varphi} - s) \\ &\geqslant e^{-2M} \sum_{n\geqslant 1} \sum_{\gamma\in\Gamma_Q(n)} e^{-\epsilon(n-3)} \exp(-sd(z,\gamma z)).\end{aligned}$$

Let $C = C(Q)$ be the diameter of Q. By the definition of $\Gamma_Q(n)$ we have that $d(z,\gamma z) < n + 2C$, for every $\gamma \in \Gamma_Q(n)$. This implies that

$$\begin{aligned}\sum_{\gamma\in\Gamma} \exp(\int_z^{\gamma z} \widetilde{\varphi} - s) &\geqslant e^{-2M} \sum_{n\geqslant 1} \sum_{\gamma\in\Gamma_Q(n)} e^{-\epsilon(n-3)} \exp(-sd(z,\gamma z)) \\ &\geqslant e^{-2M} e^{-2sC} e^{3\epsilon} \sum_{n\geqslant 1} \#\Gamma_Q(n) \exp(-n(s+\epsilon)).\end{aligned}$$

By definition if $\delta_\infty(Q)$ there exists an increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that $\#\Gamma_Q(n_i) \geqslant \exp(n_i(\delta_\infty(Q) - \epsilon))$. In particular we get

$$\sum_{\gamma\in\Gamma} \exp(\int_z^{\gamma z} \widetilde{\varphi} - s) \geqslant e^{-2M} e^{-2sC} e^{3\epsilon} \sum_{i\geqslant 1} \exp(n_i(-s + \delta_\infty(Q) - 2\epsilon)).$$

Observe that if $s < \delta_\infty(Q) - 2\epsilon$, then the right hand side diverges, so does the left

hand side. In particular we obtain that

$$P(\varphi) \geqslant \delta_\infty(Q) - 2\epsilon.$$

Taking an increasing sequence $(Q_k)_k$ such that $\delta_\infty(Q_k)$ converges to δ_∞ (and such that Q_k contains Q) we obtain $P(\varphi) \geqslant \delta_\infty - \epsilon$, but ϵ was arbitrary.

□

Proof of Theorem 5.3.2. If follows from Theorem 5.2.3 that if a sequence $(\mu_n)_n$ converges vaguely to the zero measure, then

$$\limsup_{n\to\infty} h_{\mu_n}(g_1) \leqslant \delta_\infty.$$

This implies that $h_\infty \leqslant \delta_\infty$. To prove the converse inequality we start with a strictly positive Hölder continuous function $\varphi \in C_0(T^1M)$. By definition of the topological pressure we can find a sequence of invariant probability measures $(\mu_n)_n$ such that

$$P(-n\varphi) - \frac{1}{n} \leqslant h_{\mu_n}(g) - n\int \varphi d\mu_n. \tag{5.2}$$

Using Lemma 5.3.3 we get

$$n\int \varphi d\mu_n \leqslant h_{top}(g) + \frac{1}{n} - \delta_\infty. \tag{5.3}$$

Let K be a compact set in T^1M and set $m = \min_{x\in K} |\varphi(x)|$. Therefore using (5.3) we get

$$\mu_n(K) \leqslant \frac{C}{nm},$$

for some constant C. This immediately implies that $(\mu_n)_n$ converges vaguely to zero.

Finally observe that inequality (5.2) and Lemma 5.3.3 implies that

$$\delta_\infty \leqslant h_{\mu_n}(g) + \frac{1}{n},$$

and therefore

$$\delta_\infty \leqslant \limsup_{n\to\infty} h_{\mu_n}(g).$$

We conclude that $h_\infty \geqslant \delta_\infty$, which finishes the proof of the equality $h_\infty = \delta_\infty$. □

Chapter 6

Applications of the semicontinuity

In this section we will obtain some applications of the main results from Section 5.2 to the existence of equilibrium states and maximizing measures. We will also prove that the Gibbs measures are dense in $\mathcal{M}_{\leqslant 1}(g)$, generalizing a result from [Bel].

6.1 Existence of equilibrium states

We start with a criterion for the existence of measures of maximal entropy. This result was also obtained in [ST] by different methods.

Theorem 6.1.1 (Criterion for existence of measure of maximal entropy)**.** *Let (M, g) be a pinched negatively curved manifold. Assume that M is SPR, i.e. $\delta_\infty < h_{top}(g)$. Then the geodesic flow on M admits a measure of maximal entropy.*

Proof. Let $(\mu_n)_n$ be a sequence of invariant probability measures such that

$$h_{\mu_n}(g) > h_{top}(g) - \frac{1}{n}.$$

We will assume that $(\mu_n)_n$ converges in the vague topology (otherwise take a subsequence). Let μ be the vague limit of the sequence. By Theorem 5.2.3 we have

that

$$h_{top}(g) = \limsup_{n\to\infty} h_{\mu_n}(g) \leqslant |\mu| h_{\mu/|\mu|}(g) + (1-|\mu|)\delta_\infty$$
$$\leqslant |\mu| h_{top}(g) + (1-|\mu|)\delta_\infty.$$

Assume for a second that μ is not a probability measure, then we get

$$|\mu| h_{top}(g) + (1-|\mu|)\delta_\infty < h_{top}(g),$$

which leads to a contradiction. We conclude that μ is a probability measure. By Theorem 5.2.6 we obtain that

$$h_{top}(g) = \limsup_{n\to\infty} h_{\mu_n}(g) \leqslant h_\mu(g),$$

which proves the statement. □

We remark that by [OP] the measure of maximal entropy is unique (see Theorem 2.3.12). The same argument used in the proof of Theorem 6.1.1 gives us a slightly more general result.

Theorem 6.1.2. *Let (M, g) be a pinched negatively curved manifold. Let $(\mu_n)_n$ be a sequence of invariant probability measures such that*

$$\lim_{n\to\infty} h_{\mu_n}(g) = h_{top}(g).$$

Then the following statements hold.

1. *Suppose that M is SPR. Then $(\mu_n)_{n\in\mathbb{N}}$ converges to the measure of maximal entropy.*

2. *Suppose that the geodesic flow on M does not admit a measure of maximal*

entropy. Then $(\mu_n)_{n\in\mathbb{N}}$ *converges vaguely to zero. In this case we have* $\delta_\infty = h_{top}(g)$.

3. *Suppose that the geodesic flow on* M *admits a measure of maximal entropy. Then the accumulation points of* $(\mu_n)_{n\in\mathbb{N}}$ *lies in the set* $\{t\mu_{max} : t \in [0,1]\}$, *where* μ_{max} *is the measure of maximal entropy.*

It worth mentioning that an analogous result to Theorem 6.1.2 holds in the context of countable Markov shifts [GS]. We will now state similar results to Theorem 6.1.1 and Theorem 6.1.2 in connection to the pressure of continuous potentials. We start with a definition.

Definition 6.1.3. *We say that a function* F *is* negative at infinity *if there exists a compact set* $K \subset T^1M$ *such that* F *is negative in the complement of* K. *The set of continuous functions negative at infinity is denoted by* $C_{<0}(T^1M)$.

It is easy to check that if $F \in C_{<0}(T^1M)$, then the map

$$\mu \mapsto \int F d\mu,$$

is upper semicontinuous in the vague topology (the proof is basically the same as the one of Lemma 3.1.2). Our next result is the basic tool to obtain the existence of equilibrium states.

Theorem 6.1.4. *Let* (M,g) *be a pinched negatively curved manifold. Suppose we have a sequence* $(\mu_n)_n$ *of invariant probability measures converging to* μ *in the vague topology and a potential* F *in* $C_0(T^1M)$ *or* $C_{<0}(T^1M)$. *Then*

$$\limsup_{n\to} \left(h_{\mu_n}(g) + \int F d\mu_n\right) \leqslant |\mu|\left(h_{\mu/|\mu|}(g) + \int F d\mu/|\mu|\right) + (1-|\mu|)\delta_\infty.$$

Proof. This result follows directly fom Theorem 5.2.3 and the upper semicontinuity of the map $\mu \mapsto \int F d\mu$, if F belongs to $C_0(T^1M)$, or $C_{<0}(T^1M)$. □

Using Theorem 6.1.4 and the strategy of the proof of Theorem 6.1.1 we obtain the following two results.

Theorem 6.1.5 (Criterion for the existence of equilibrium states)**.** *Let (M, g) be a pinched negatively curved manifold. Suppose that F is a potential in $C_0(T^1M)$ or $C_{<0}(T^1M)$, such that $P(F) > \delta_\infty$. Then F admits at least one equilibrium state.*

Remark 6.1.6. *We would like to emphasize that even though Theorem 6.1.5 gives a criterion for the existence of equilibrium states for potentials that are negative at infinity, the critical gap condition $\delta_\infty < P(F)$ is not optimal. In the geometrically finite case we know that the inequality*

$$P_\infty(F) = \max_{\mathcal{P}} \delta_{\mathcal{P}}^F < P(F),$$

implies the existence of an equilibrium state (Theorem 2.3.16). If F is negative at infinity then $\max_{\mathcal{P}} \delta_{\mathcal{P}}^F \leqslant \delta_\infty$ (with strict inequality if, for instance, F tends to a negative constant at infinity). For this reason we will mainly focus on potentials that vanish at infinity, where our bound is sharp. In Section 9 we will explain what we expect to be true, but we have not been able to prove so far. This would provide a sharp statement for arbitrary bounded potentials.

The following lemma shows that SPR potentials in $C_0(T^1M)$ are exactly those satisfying the critical gap $P(F) > \delta_\infty$.

Lemma 6.1.7. *Let $F \in C_0(T^1M)$. Then $P_\infty(F) = \delta_\infty$.*

Proof. Since F vanishes at infinity we know that for each $\epsilon > 0$, there exists a compact subset $K = K(\epsilon) \subset M$ such that $\sup_{x \in T^1K^c} |F(x)| < \epsilon$. Let $\widetilde{U} \subset \widetilde{M}$ be an open relatively compact subset of $\widetilde{M}$, such that $T^1\widetilde{U}$ has non-empty intersection with the lift of Ω to $T^1\widetilde{M}$. Assume that $K \subset p(\widetilde{U})$. It follows from the definition of the Poincaré series $P(F, \widetilde{U}, s)$ (see Definition 2.3.20) and our choice of $\widetilde{U}$ that $P(F, \widetilde{U}, s)$

can be bounded below by $\sum_{\gamma\in\Gamma_{\tilde{U}}}\exp(-(s+\epsilon)d(x,\gamma x))$, and above by $\sum_{\gamma\in\Gamma_{\tilde{U}}}\exp(-(s-\epsilon)d(x,\gamma x))$ (up to multiplicative constants). From this we can conclude that $|\delta_\Gamma^F(\widetilde{U})-\delta_\infty|<2\epsilon$. Since this computation works for every $\widetilde{U}$ as in Definition 2.3.20 which also satisfies $K\subset p(\widetilde{U})$, we obtain that $|P_\infty(F)-\delta_\infty|<2\epsilon$. Since $\epsilon>0$ was arbitrary we obtain that $P_\infty(F)=\delta_\infty$.

□

In particular Theorem 6.1.5 proves that any SPR potential in $C_0(T^1M)$ admits an equilibrium state.

Theorem 6.1.8. *Let (M,g) be a pinched negatively curved manifold anf $F\in C_0(T^1M)$. Let $(\mu_n)_n$ be a sequence of invariant probability measures such that*

$$\lim_{n\to\infty}\left(h_{\mu_n}(g)+\int F d\mu_n\right)=P(F).$$

Then the following statements hold.

1. *If F is SPR, then $(\mu_n)_n$ converges to an equilibrium state of F.*
2. *Suppose that F does not admit any equilibrium state. Then $(\mu_n)_n$ converges vaguely to zero. In this case we have $P(F)=\delta_\infty$.*
3. *Suppose that F does admit an equilibrium state. Then the accumulation points of $(\mu_n)_{n\in\mathbb{N}}$ lies in the set*

$$\{t\mu : t\in[0,1] \text{ and } \mu \text{ is an equilibrium state of } F\}.$$

We emphasize that we require no higher regularity (on F) than continuity. In particular the theory developed in [PPS] does not apply. A big difference with respect to more regular potentials is the lack of uniqueness of equilibrium states for continuous potentials. It is proven in [IV] that one can slightly C^0-perturb any potential

$F \in C_b(T^1M)$ into a potential with uncountably many equilibrium states. A crucial ingredient for that result is Theorem 5.2.6, which allows us to identify subderivatives of the pressure at F to its equilibrium states. We finish this section with the following result (that will be important in Section 7 and 8).

Theorem 6.1.9 (First derivative of the pressure)**.** *Let M be a pinched negatively curved manifold and $F \in C_0(T^1M)$ a SPR Hölder potential. For every $G \in C_b(T^1M)$ the following holds*

$$\frac{d}{dt}_{|t=0} P(F+tG) = \int G d\mu_F,$$

where μ_F is the equilibrium state of F.

Proof. Let μ_0 be the equilibrium state of F and for every $t \neq 0$ we choose an invariant probability measure μ_t such that

$$h_{\mu_t}(g) + \int (F+tG) d\mu_t \geqslant P(F+tG) - t^2.$$

Observe that

$$\begin{aligned} P(F+tG) - P(F) &\leqslant \big(h_{\mu_t}(g) + \int (F+tG)d\mu_t + t^2\big) - \big(h_{\mu_t}(g) + \int F d\mu_t\big) \\ &= t\int G d\mu_t + t^2. \end{aligned}$$

Similarly

$$\begin{aligned} P(F+tG) - P(F) &\geqslant \big(h_{\mu_0}(g) + \int (F+tG)d\mu_0\big) - \big(h_{\mu_0}(g) + \int F d\mu_0\big) \\ &= t\int G d\mu_0. \end{aligned}$$

In particular for $t > 0$ we get

$$\int G d\mu_0 \leqslant \frac{P(F+tG) - P(F)}{t} \leqslant \int G d\mu_t + t,$$

and the reversed inequality for $t < 0$. We now claim that $(\mu_t)_t$ converges in the weak-* topology to μ_0 as t goes to zero. First observe that

$$P(F + tG) - t^2 \leqslant h_{\mu_t}(g) + \int (F + tG) d\mu_t \leqslant P(F + tG),$$

therefore

$$\lim_{t\to 0} h_{\mu_t}(g) + \int F d\mu_t = \lim_{t\to 0} h_{\mu_t}(g) + \int (F + tG) d\mu_t = \lim_{t\to 0} P(F + tG) = P(F).$$

Since F is Hölder we know that μ_0 is the unique equilibrium state of F. We now use Theorem 6.1.8 to conclude that $\lim_{t\to 0} \mu_t = \mu_0$. As a consequence we obtain that $\lim_{t\to 0} \int G d\mu_t = \int G d\mu_0$. This together with the inequalities above give us that

$$\frac{d}{dt}_{|t=0} P(F + tG) = \int G d\mu_0.$$

□

Remark 6.1.10. *The same proof of Theorem 6.1.9 implies the following more general fact. Suppose we have a continuous family of potentials $(F_t)_{t\in(-\epsilon,\epsilon)}$, and F_0 is a strongly positive recurrent Hölder potential in $C_0(T^1M)$. Moreover assume that $\frac{d}{dt}_{|t=0} F_t$ is a bounded continuous function. Then*

$$\frac{d}{dt}_{|t=0} P(F_t) = \int \big(\frac{d}{dt}_{|t=0} F_t\big) d\mu_0,$$

where μ_0 is the unique equilibrium state of F_0.

As a corollary of the proof of Theorem 6.1.9 we get the continuity of equilibrium states in the SPR range.

Theorem 6.1.11 (Continuity of equilibrium states)**.** *Suppose we have Hölder potentials F and G in $C_0(T^1M)$. Moreover assume that $F + tG$ is SPR for $t \in (-\epsilon, \epsilon)$, and*

denote by μ_t to the equilibrium state of $F + tG$. Then the map $t \mapsto \mu_t$, is continuous in the weak- topology.*

6.2 Maximizing measures as zero temperature limits

The results in this sections are part of an ongoing work with F. Riquelme. In this section we will prove the existence of maximizing measures as zero temperature limits (under certain assumptions). We remark that the parameter t in the pressure map $t \mapsto P(tF)$ is interpreted as the inverse of the temperature. In particular, a zero temperature limit is the vague limit of a sequence of equilibrium states for potentials $t_n F$, while t_n goes to infinity. In some sense the maximizing measures that appear as zero temperature limits are the most natural from the thermodynamic point of view, and that is one reason why they have been extensively studied (see [JMU], [CGU], [CH]). In the non-compact case we need to be a bit careful with the escape of mass phenomenon, but we will see in Theorem 6.2.2 that this is possible to do with a result like Theorem 5.2.3.

Definition 6.2.1 (Maximizing measure)**.** *We say that a measures $\mu \in \mathcal{M}(g)$ is a* maximal measure *for the potential F if*

$$\int F d\mu = \sup_{\nu \in \mathcal{M}(g)} \int F d\nu.$$

We denote by $M_F = \sup_{\nu \in \mathcal{M}(g)} \int F d\nu$, and $m_F = \inf_{\nu \in \mathcal{M}(g)} \int F d\nu$.

Observe that for $t > 0$ we have that $P(tF) > tM_F$. If $M_F > 0$, then for t sufficiently large we get $P(tF) \geqslant tM_F > \delta_\infty$. The potential tF is SPR if $F \in C_0(T^1M)$ and t is large enough. We conclude that for t sufficiently large the potential tF has an equilibrium state.

Theorem 6.2.2 (Existence of maximizing measures as zero temperature limits)**.** *Let F be a Hölder potential in $C_0(T^1M)$ such that $M_F > 0$. Denote by μ_t to the equilibrium state of tF if exists. Then the accumulation points of $(\mu_t)_t$ as t goes to infinity are maximizing measures (in particular probability measures).*

Proof. It follows from the definition of the topological pressure that for $t > 0$ we have

$$tM_F \leqslant P(tF) \leqslant h_{top}(g) + tM_F. \tag{6.1}$$

If t is sufficiently large we will have that $P(tF) \geqslant tM_F > \delta_\infty$, and by Theorem 6.1.5 we conclude that tF admits an equilibrium state. Let μ_t be the unique equilibrium state for tF. Pick an accumulation point μ_∞ of the sequence $(\mu_t)_t$ as t goes to infinity with respect to the vague topology. By definition there exists a sequence $(t_n)_n$ such that $(\mu_{t_n})_n$ converges vaguely to μ_∞ and n goes to infinity. We claim that μ_∞ has positive mass. Indeed, if $\mu_\infty(T^1M) = 0$, then by the continuity of $\mu \mapsto \int F d\mu$ (Lemma 3.1.2) we would have that

$$\lim_{n\to\infty} \int F d\mu_{t_n} = 0. \tag{6.2}$$

On the other hand, by Theorem 6.1.9 we know that for t sufficiently large

$$\left.\frac{d}{ds}\right|_{s=t} P(sF) = \int F d\mu_t.$$

Inequality (6.1) implies that for positive t we have

$$\lim_{t\to\infty} \frac{d}{dt}_{s=t} P(sF) = M_F > 0. \tag{6.3}$$

This contradicts equation (6.2), we conclude that $\mu_\infty(T^1M) > 0$. Let ν be the

normalization of μ_∞. Using Theorem 5.2.3 we get

$$\limsup_{n\to\infty} h_{\mu_{t_n}}(g) \leqslant \|\mu_\infty\| h_\nu(g) + (1 - \|\mu_\infty\|)\delta_\infty.$$

In particular, for any $\epsilon > 0$, and large enough $n \geqslant 1$, we have

$$h_{\mu_{t_n}}(g) - \varepsilon \leqslant \|\mu_\infty\| h_\nu(g) + (1 - \|\mu_\infty\|)\delta_\infty.$$

On the other hand, the definition of the topological pressure implies that

$$h_\nu(g) + t_n \int F d\nu \leqslant h_{\mu_{t_n}}(g) + t_n \int F d\mu_{t_n}.$$

Putting all this together

$$h_{\mu_{t_n}}(g) - \varepsilon \leqslant \|\mu_\infty\| \left(h_{\mu_{t_n}}(g) + t_n \int F d\mu_{t_n} - t_n \int F d\nu \right) + (1 - \|\mu_\infty\|)\delta_\infty,$$

and therefore

$$\frac{1}{t_n \|\mu_\infty\|} [(1 - \|\mu_\infty\|)(h_{\mu_{t_n F}}(g) - \delta_\infty) - \epsilon] \leqslant \int F d\mu_{t_n F} - \int F d\nu.$$

Taking limit as n goes to infinity

$$\int F d\nu \leqslant \int F d\mu_\infty,$$

but since $\nu = \frac{1}{\|\mu_\infty\|}\mu_\infty$, we necessarily have $\mu_\infty(X) = 1$. Using (6.3) we get that $\int F d\mu_\infty = \lim_{n\to\infty} \int F d\mu_{t_n} = M_F$. We conclude that μ_∞ is a maximizing measure for F. □

Remark 6.2.3. *In the hypothesis of Theorem 6.2.2 we could have assumed that F is negative at infinity and the same result would follow. As before, what we really*

need is the upper semicontinuity of the map $\mu \mapsto \int F d\mu$*, which is known to hold for potentials in* $C_0(T^1M)$ *and* $C_{<0}(T^1M)$*.*

6.3 Density of Gibbs measures

We will use the results in Section 6.2 to prove the density of Gibbs measures in $\mathcal{M}_{\leqslant 1}(g)$. We remark that this result appeared earlier in [Bel] for geometrically finite manifolds, and that our strategy is similar, but more straighforward (after all the work we have done). We know from the proof of Theorem 3.2.7 that the periodic measures are dense in $\mathcal{M}_{\leqslant 1}(g)$, so it is enough to prove that we can approximate any periodic measures by Gibbs measures. Let τ be a periodic orbit and μ_τ the periodic measure associated to τ. Consider a compactly supported Hölder potential F_τ such that

$$F_\tau(x) = \begin{cases} 1, \text{ if x} \in \tau \\ < 1, \text{ otherwise.} \end{cases}$$

Observe that $M_{F_\tau} = 1$, and that the only maximizing measure of F_τ is the periodic measure μ_τ. Using the results from Section 6.2 we get that the equilibrium states of the family $(tF_\tau)_t$ converges to μ_τ as t goes to infinity. Summarizing we obtained

Theorem 6.3.1. *Let* M *be a pinched negatively curved manifold. The space of Gibbs measures is dense in* $\mathcal{M}_{\leqslant 1}(g)$*.*

Chapter 7

Thermodynamic formalism

In this section we will study the thermodynamic formalism of potentials that vanishes at infinity. More precisely, we will describe the behaviour (regularity, limits, etc) of the map $t \mapsto P(tF)$.

Definition 7.0.1 (Pressure map)**.** *Given a potential F, we refer to the map*

$$t \mapsto P(tF),$$

as the pressure map *of F.*

We will give a fairly good description of the pressure map for positive potentials in $C_0(T^1M)$ (Theorem 7.1.2). In the non-compact case the pressure map of Hölder potentials is not always regular, sometimes the pressure map can develop singularities: these singularities are usually called phase transitions. Phase transitions usually detect a significant change in the dynamics of our system (from the point of view of the potential). We will be interested in finding conditions that guarantee the existence, and the lack of phase transition (Section 7.4 and Section 7.3 resp.). In general it is a difficult problem to determine when a potential will (or not) exhibit phase transitions, for this reason we will restrict our attention to some particular

class of geometrically finite manifolds. Since for the geodesic flow on non-compact manifolds we do not know how regular is the pressure map (our best result on the regularity of the pressure is Theorem 6.1.9), we will use the following definition.

Definition 7.0.2 (Phase transition)**.** *We say that a potential F exhibits* a phase transition at t_0 *if there exists $\epsilon > 0$ such that $P(tF)$ has an equilibrium state for $t \in (t_0, t_0 + \epsilon)$, but it has not for $t \in (t_0 - \epsilon, t_0)$ (or vice versa). A potential F exhibits a phase transition if it exhibits a phase transition for some $t_0 \in \mathbb{R}$.*

We emphasize that the potentials for which we will describe phase transitions are Hölder continuous. We remark that in the compact case Hölder potentials can not develop phase transitions. By the work of Bowen [B5] and Ratner [Rat], we know that the geodesic flow on a compact negatively curved manifold can be modelled as a suspension flow over a shift of finite type. As a consequence we obtain that the pressure map of a Hölder potential is real analytic, and that every Hölder potential has a unique equilibrium state (for a complete discussion we refer the reader to [B5], [B7] and [Rue]). In the context of symbolic dynamics the study of phase transitions has a long story. In this situation a phase transition is understood as a time t where the pressure map has discontinuous first, or some higher derivative. For shifts of finite type and non-Hölder potentials phase transitions were constructed in [Hof] and [Lop]. For countable Markov shifts phase transitions have been extensively studied by Sarig (see [S2], [S3] and [S5]). Based on the work of Sarig, phase transitions have also been studied for some interval maps (see for instance [PZ], [BI], [IT]). For suspension flows over countable Markov shifts phase transitions have been studied in [IJ]. It worth pointing out that for suspension flows over countable Markov shifts it is possible to construct phase transitions even when the shift map satisfies the BIP property (see [IJ, Theorem 4.1]). Sarig proved that on a countable Markov shift satisfying the BIP property, the pressure map of a locally Hölder potential is real analytic whenever finite (see [S4, Corollary 4]). This is an significant difference

between the thermodynamic formalism of the suspension flow and the shift map. The phase transitions constructed in this paper exhibit a similar behaviour to those from [IJ]. Phase transitions for quadratic-like maps have been extensively studied in [CR1], [CR2], [CR3] and [CR4]. In this body of work Coronel and Rivera-Letelier also constructed quadratic-like maps having sensitive dependence of the geometric potential at zero and positive temperature. Sensitive dependence is an interesting phenomenon that has not been studied for the geodesic flow.

The following notation will be constantly used in this section: if G is a group, we denote by G^* to $G\backslash\{id\}$. We finish this discussion with a definition.

Definition 7.0.3 (Groups in Schottky position)**.** *Let F_1 and F_2 be discrete, torsion free subgroups of Iso($\widetilde{M}$). We say that F_1 and F_2 are in* Schottky position *if there exist disjoint closed subsets U_{F_1} and U_{F_2} of $\partial_\infty \widetilde{M}$ such that $F_1^*(\partial_\infty \widetilde{M}\backslash U_{F_1}) \subset U_{F_1}$ and $F_2^*(\partial_\infty \widetilde{M}\backslash U_{F_2}) \subset U_{F_2}$.*

7.1 Description of the pressure map

In this section we will describe the pressure map $t \mapsto P(tF)$, where F is a positive potential vanishing at infinity. Recall that by Lemma 5.3.3 we know that if $G \in C_0(T^1M)$, then

$$P(G) \geqslant \delta_\infty.$$

In the proof of our next theorem we will need the following simple lemma.

Lemma 7.1.1. *Let F be a continuous positive potential. If (μ_n) is a sequence of invariant probability measures such that*

$$\lim_{n\to\infty} \int F d\mu_n = 0.$$

Then $(\mu_n)_n$ converges vaguely to the zero measure.

Proof. Let $K \subset X$ be a compact set. Then

$$\begin{aligned} \int F d\mu_n &\geqslant \int_K F d\mu_n \\ &\geqslant \min\{F(v) : v \in K\}\mu_n(K). \end{aligned}$$

Observe that $\min\{F(v) : v \in K\} > 0$, because of the positivity of F, so

$$\lim_{n\to\infty} \int F d\mu_n = 0 \;\Rightarrow\; \lim_{n\to\infty} \mu_n(K) = 0,$$

which is equivalent to say that $(\mu_n)_n$ converges vaguely to the zero measure. □

Our next result follows closely the strategy of the proofs of [IRV, Theorem 1.3] and [RV, Theorem 5.7] (also see [IJ]), where the same properties were obtained for some particular cases of negatively curved manifolds. We are now able to generalize those results to any pinched negatively curved manifold.

Theorem 7.1.2 (Pressure map for positive potentials in $C_0(T^1M)$)**.** *The pressure map of a positive Hölder potential F that vanishes at infinity verifies the following properties.*

(1) for every $t \in \mathbb{R}$ we have that $P(tF) \geqslant \delta_\infty$

(2) the function $t \mapsto P(tF)$ has a horizontal asymptote at $-\infty$, that is

$$\lim_{t\to-\infty} P(tF) = \delta_\infty.$$

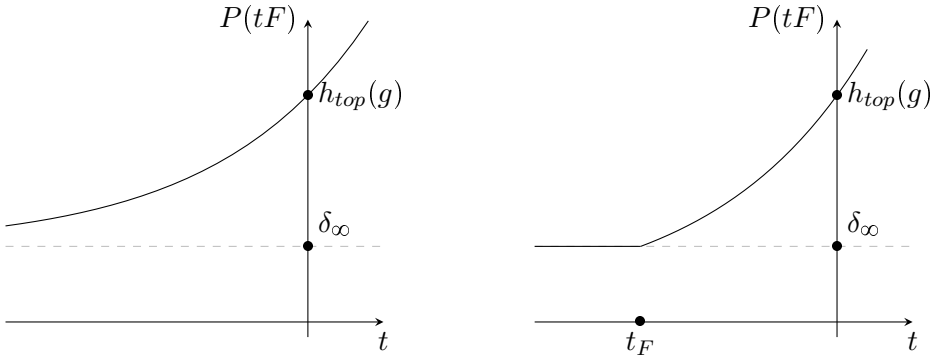
Moreover, if $t_F := \sup\{t \leqslant 0 : P(tF) = \delta_\infty\}$, then

(3) for every $t > t_F$ the potential tF has an equilibrium state, and

(4) the pressure function $t \mapsto P(tF)$ is differentiable in (t_F, ∞), and it verifies

$$P(tF) = \begin{cases} \delta_\infty & \text{if } t < t_F \\ \text{strictly increasing} & \text{if } t > t_F, \end{cases}$$

(5) If $t < t_F$ then the potential tF has not equilibrium measure.



No phase transitions Phase transition at $t = t_F$

Figure 1

We emphasize that if $P(tF) > \delta_\infty$, for every $t \in \mathbb{R}$, then t_F is equal to $-\infty$.

Proof. We already proved in Lemma 5.3.3 that $P(tF) \geqslant \delta_\infty$ (observe that tF is a potential vanishing at infinity). This fact gives part (1). In order to prove (2) recall that the pressure map, $t \mapsto P(tF)$, is convex and non-decreasing (since F is positive). From the lower bound $P(tF) \geqslant \delta_\infty$, we obtain that the limit $\lim_{t\to-\infty} P(tF) =: A$ exists. It follows from part (1) that $A \geqslant \delta_\infty$. By definition of the topological pressure we know there exists a sequence of invariant probability measures $(\mu_n)_n$ such that

$$\lim_{n\to\infty} \left(h_{\mu_n}(g) - n \int F d\mu_n\right) = A. \tag{7.1}$$

Since A is a fixed number we necessarily have that

$$\lim_{n\to\infty} \int F d\mu_n = 0.$$

By hypothesis F is positive, we can conclude by Lemma 7.1.1 that $(\mu_n)_n$ converges vaguely to the zero measure. In particular $\limsup_{n\to\infty} h_{\mu_n}(g) \leqslant \delta_\infty$ (Theorem 5.3.2). Equation (7.1) implies $A \leqslant \limsup_{n\to\infty} h_{\mu_n}(g)$, therefore $A \leqslant \delta_\infty$. We conclude that $A = \delta_\infty$, as required. It follows from Theorem 6.1.5 that if $P(tF) > \delta_\infty$, then tF has an equilibrium state, and therefore we obtain (3). The differentiability of the pressure map in the interval (t_F, ∞) follows from Theorem 6.1.9, and because F is positive we also get that $P(tF)$ is strictly increasing in (t_F, ∞). This concludes the proof of (4). To prove (5) assume μ is an equilibrium state for tF, where $t < t_F$. If $t < t'' < t_F$, then

$$P(tF) = h_\mu(g) + t\int F d\mu < h_\mu(g) + t''\int F d\mu \leqslant P(t''F),$$

but $P(t''F) = P(tF)$, which is a contradiction. □

As a corollary of the proof of Theorem 7.1.2 we obtain a property claimed in Section 3.

Lemma 7.1.3. *Let M be a non-compact pinched negatively curved manifold. Then there exists a sequence of periodic measures that converges vaguely to the zero measure.*

Proof. Take a positive potential F in $C_0(T^1M)$ and consider a sequence $(\mu_n)_n$ such that

$$\lim_{n\to\infty} \left(h_{\mu_n}(g) - n\int F d\mu_n\right) = \delta_\infty.$$

This is granted by the definition of the topological pressure and part (2) of Theorem 7.1.2. This immediately implies that $\lim_{n\to\infty} \int F d\mu_n = 0$, and by Lemma 7.1.1 we get that $(\mu_n)_n$ converges to the zero measure. Now we use Theorem 3.2.4 to approximate μ_n by a periodic measure η_n, we can assume that $d(\mu_n, \eta_n) < \frac{1}{n}$. We conclude that $(\eta_n)_n$ converges vaguely to the zero measure, as desired. □

If the potential F is just non-positive, then a similar description of the pressure map can be obtained. In this case we can not immediately conclude that

$\lim_{t\to-\infty} P(tF) = \delta_\infty$ (it might happen that the limit is strictly bigger than δ_∞). If, for instance, F vanishes (exactly) on an invariant subset Y of T^1M with topological entropy less or equal than δ_∞, then we can also conclude that $\lim_{t\to-\infty} P(tF) = \delta_\infty$. We will briefly explain why. As in the proof of Theorem 7.1.2 define A such that $\lim_{t\to-\infty} P(tF) = A$ (which exists because the pressure map is non-decreasing). By definition of the topological pressure we can find a sequence $(\mu_n)_n$ such that

$$\lim_{n\to\infty} \left(h_{\mu_n}(g) - n\int F d\mu_n\right) = A.$$

We take a convergent subsequence $(\mu_{n_k})_k$ with limit μ. As before in Theorem 7.1.2 we can conclude that $\lim_{k\to\infty}\int F d\mu_{n_k} = 0$, which implies that μ is supported in Y. Finally we get

$$A \leqslant \limsup_{k\to\infty} h_{\mu_{n_k}}(g) \leqslant |\mu| h_{\mu/|\mu|}(g) + (1-|\mu|)\delta_\infty \leqslant |\mu| h_{top}(Y) + (1-|\mu|)\delta_\infty.$$

It follows from Lemma 5.3.3 that $A \geqslant \delta_\infty$. By assumption $h_{top}(Y) \leqslant \delta_\infty$, so

$$\delta_\infty \leqslant A \leqslant |\mu| h_{top}(Y) + (1-|\mu|)\delta_\infty \leqslant \delta_\infty,$$

and therefore $A = \delta_\infty$. Observe that if $h_{top}(Y) < \delta_\infty$, then we can conclude that μ has zero mass, in other words, that the sequence $(\mu_n)_n$ converges vaguely to the zero measure. The rest of the properties in Theorem 7.1.2 follow without modification. It is likely that if F vanishes on an invariant set Y such that $h_{top}(Y) \in (\delta_\infty, h_{top}(g))$, then

$$\lim_{t\to-\infty} P(tF) = h_{top}(Y),$$

but we will not try to prove that here. We also remark that if Y is not invariant, and if we understand $h_{top}(Y)$ as the supremum of the entropy of measures supported on Y, then the same conclusions will hold.

7.2 Strongly positive recurrent potentials in $C_0(T^1M)$

In this section we will prove that SPR potentials are open and dense in $C_0(T^1M)$. This property should be compared with what is known for SPR potentials in countable Markov shifts [CSa]. We recall the definition of strongly positive recurrent potentials in $C_0(T^1M)$ (for the general definition see Definition 2.3.20).

Definition 7.2.1 (Strongly positive recurrent)**.** *We say that a potential $F \in C_0(T^1M)$ is* strongly positive recurrent *(SPR for short) if $P(F) > \delta_\infty$. The family of SPR potentials in $C_0(T^1M)$ will be denoted by $\mathcal{S}$.*

Lemma 7.2.2. *$\mathcal{S}$ is an open subset of $C_0(T^1M)$.*

Proof. Let $F \in \mathcal{S}$ and define $r = P(F) - \delta_\infty$. Observe that if $||G - F||_0 < r$, then $P(G) > P(F) - r = \delta_\infty$. □

Lemma 7.2.3. *$\mathcal{S}$ is dense in $C_0(T^1M)$.*

Proof. Let $F \in C_0(T^1M)$, we will prove that we can approximate F by SPR potentials. If $F \in \mathcal{S}$ there is nothing to prove. We will assume that F is not strongly positive recurrent, in other words, that $P(F) = \delta_\infty$. By Proposition 3.1.7 we know that the pressure of F can be approximated by compact subsets. In particular, given $\epsilon > 0$, there exists a compact invariant subset K and a measure μ supported on K such that

$$h_\mu(g) + \int F d\mu > P(F) - \frac{\epsilon}{2}.$$

Choose a compactly supported continuous function G such that $G(x) = \epsilon$, for every $x \in K$, and $||G||_0 = \epsilon$. Observe that

$$h_\mu(g) + \int (F + G) d\mu = \epsilon + h_\mu(g) + \int F d\mu \geqslant P(F) + \frac{\epsilon}{2} > \delta_\infty.$$

In particular we obtain that $H = (F + G) \in C_0(T^1M)$ is strongly positive recurrent and $||H - F||_0 \leqslant \epsilon$. Since ϵ was arbitrary we obtain that $\mathcal{S}$ is dense in $C_0(T^1M)$. $\square$

Combining Lemma 7.2.2 and Lemma 7.2.3 we obtain the following result.

Theorem 7.2.4. *The space of SPR potentials $\mathcal{S}$ is open and dense in $C_0(T^1M)$.*

The two most important properties of strongly positive recurrent potentials are described in Theorem 6.1.8 and Theorem 6.1.9, for convenience we will state them again here.

Theorem 7.2.5. *Let F be a Hölder potential in $\mathcal{S}$ with equilibrium state μ_F. Then the following facts hold.*

1. *Let $G \in C_b(T^1M)$. Then*

$$\frac{d}{ds}_{|s=0} P(F + sG) = \int G d\mu_F.$$

2. *Let $(\mu_n)_n$ be a sequence of invariant probability measures satisfying*

$$\lim_{n\to\infty} \left(h_{\mu_n}(g) + \int F d\mu_n\right) = P(F),$$

then $(\mu_n)_n$ converges to μ_F.

We emphasize that in the compact case any Hölder potential satisfies the conclusions of Theorem 7.2.5, in the non-compact situation we need to be much more careful.

7.3 No phase transition

In this section we will restrict our attention to geometrically finite manifolds. This does not seem to be an essential assumption, we believe the results in this section can

be extended to any pinched negatively curved manifolds, but we are still missing an important ingredient (an extension of Theorem 2.3.15 to arbitrary negatively curved manifolds). From now on we will assume that M is geometrically finite. We remark that in this case

$$\delta_\infty = \sup_{\mathcal{P}} \delta_{\mathcal{P}},$$

where the supremum runs over the parabolic subgroups of $\pi_1(M)$. For convenience we will fix a maximal collection of non-conjugate maximal parabolic subgroups (one for each cusps of M), say $\{\mathcal{P}_1, ..., \mathcal{P}_n\}$. With this notation we have that

$$\delta_\infty = \max_{k\in\{1,...,n\}} \delta_{\mathcal{P}_k}.$$

Definition 7.3.1. *Using the notation above, we say that a maximal parabolic $\mathcal{P}_k$ is* maximizing *if*

$$\delta_\infty = \delta_{\mathcal{P}_k}.$$

We start defining a sub-class of SPR potentials that are stable under compact perturbations. For convenience (it is more suitable for our purposes) we will define this sub-class for potentials vanishing at infinity, but this is not strictly necessary. This class of potentials will also be important in Section 8, since for those potentials we obtain large deviations lower bounds.

Definition 7.3.2 (Very strongly positive recurrent)**.** *We say that a Hölder potential $F \in C_0(T^1M)$ is* very strongly positive recurrent *(vSPR for short) if for some maximizing maximal parabolic $\mathcal{P}_k$, we have that $(\mathcal{P}_k, F)$ is of divergence type.*

We emphasize that in the definition of vSPR potential we assumed that our potential is Hölder. This is an important assumption in order to use Theorem 2.3.15.

Lemma 7.3.3. *Suppose that M is geometrically finite, and $F \in C_0(T^1M)$. Then for every maximal parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$ we have $\delta_{\mathcal{P}}^F = \delta_{\mathcal{P}}$.*

Proof. Pick a reference point $x \in M$ that belongs to the region where the cusp is standard, i.e. where it looks like the quotient of a horoball by the parabolic subgroup $\mathcal{P}$. We will moreover assume that x belongs to the region of the cusp where $|F| < \epsilon$. Using the convexity of the horoballs we get that the Poincaré series of $\mathcal{P}$ based at x can be bounded below by $\sum_{p\in\mathcal{P}} \exp\left(-(s+\epsilon)d(x,px)\right)$, and above by $\sum_{p\in\mathcal{P}} \exp\left(-(s-\epsilon)d(x,px)\right)$. This implies that $|\delta_{\mathcal{P}}^F - \delta_{\mathcal{P}}| < 2\epsilon$. Since ϵ was arbitrary we conclude the lemma. We remark that for potentials in $C_0(T^1M)$ we have freedom to choose the reference point of the Poincaré series (Lemma 3.1.4) □

Lemma 7.3.4. *A vSPR potential is SPR.*

Proof. Let F be a vSPR potential. By definition $(\mathcal{P}_k, F)$ is of divergence type for some maximal parabolic $\mathcal{P}_k$ such that $\delta_\infty = \delta_{\mathcal{P}_k}$. Using Theorem 2.3.15 we get that $\delta_{\mathcal{P}_k}^F < P(F)$. Since F vanishes at infinity we can use Lemma 7.3.3 to get that $\delta_{\mathcal{P}_k}^F = \delta_{\mathcal{P}_k} = \delta_\infty$, and therefore $\delta_\infty < P(F)$. □

As mentioned before, vSPR potentials behave nicely under compact perturbations. More precisely, we have the following result.

Lemma 7.3.5. *Let F be a vSPR potential. If G is a compactly supported Hölder potential, then $F+G$ is vSPR.*

Proof. By definition of vSPR potentials $(\mathcal{P}_k, F)$ is of divergence type for some maximal parabolic $\mathcal{P}_k$ such that $\delta_\infty = \delta_{\mathcal{P}_k}$. Since G is compactly supported, we can pick a point x in a sufficiently small neighborhood of the cusp associated to $\mathcal{P}_k$ such that the whole horoball defining the cusp is in the region where G vanishes. Using the convexity of the horoballs we can conclude that the Poincaré series associate to $(\mathcal{P}_k, F)$ based at x is equal to the Poincaré series associated to $(\mathcal{P}_k, F+G)$ based at x. Since for Hölder potentials the behaviour of the Poincaré series is independent of the base point (see Proposition 3.1.4) we conclude that $(\mathcal{P}_k, F+G)$ is of divergence type. In particular we obtain that $F+G$ is vSPR. □

Remark 7.3.6. *Every geometrically finite manifold with cusps admit strongly positive recurrent potentials. This construction was first done by Coudene in [Cou]. For a very similar construction we refer the reader to the example constructed in [RV, Section 5]. If a potential decays fast enough to zero at infinity, then it will be very strongly positive recurrent. On the contrary, as it will be explained in Section 7.4, if the potential decays very slowly to zero at infinity, then it could even not be positive recurrent (not even divergence type).*

Proposition 7.3.7. *Let F be a vSPR potential and G a compactly supported Hölder potential. Then for every $t \in \mathbb{R}$ the potential $F + tG$ is SPR. In particular the line $t \mapsto (F + tG)$, does not exhibits a phase transition.*

Proof. Since $F + tG$ is a compact perturbation of F, by Lemma 7.3.5 we get that $F + tG$ is vSPR. Using Lemma 7.3.4 we conclude the result. □

An immediate consequence of this result is that if the zero potential is vSPR, then a compactly supported potential can not exhibit a phase transition. We remark that the examples of potentials exhibiting phase transitions (constructed in Section 7.4) satisfy that the zero potential is vSPR.

Corollary 7.3.8 (No phase transition)**.** *Suppose that the zero potential is vSPR, more precisely, that $\mathcal{P}_k$ is of divergence type for some maximizing maximal parabolic subgroup. Let G be a compactly supported continuous potential. Then the pressure map of G does not exhibit a phase transition.*

In [RV, Section 5] it was constructed a potential F (not compactly supported) such that the pressure map of F does not exhibit a phase transition (under the assumption that the zero potential is vSPR). In light of Corollary 7.3.8, the same conclusion hold for any compactly supported potentials (and therefore the construction in [RV, Section 5] becomes unnecessarily complicated).

7.4 Phase transitions

In this section we will study some geometrically finite manifolds on which it is possible to construct potentials that exhibit phase transitions. We will also construct a geometrically finite manifold for which the geometric potential exhibits a phase transition. The philosophy behind our constructions is very simple: if a potential decays very slowly to zero through the cusps, then it is likely to develop a phase transition. After the paper [V1] was uploaded to arxiv, G. Iommi pointed out to us that the same principle was used in the construction of phase transitions for Pomeu-Manneville maps. Recall that for $F \in C_0(T^1M)$ we use the notation $t_F = \sup\{t : P(tF) = \delta_\infty\}$. The main goal of this section is to prove the following two results.

Theorem 7.4.1. *There exists a geometrically finite manifold M such that the following holds. For every non-negative Hölder continuous potential F going* slowly to zero *through the cusp of M we have that*

1. $t_F \in [-1, 0)$,
2. *The potential tF has equilibrium measure for $t > t_F$,*
3. *The potential tF has not equilibrium measure for $t < t_F$.*

In other words, the map $t \mapsto P(tF)$ exhibits a phase transition at t_F. Moreover $t \mapsto P(tF)$ is differentiable for $t \neq t_F$.

Theorem 7.4.2. *There exists an extended Schottky manifold for which the geodesic flow has a measure of maximal entropy and the geometric potential exhibits a phase transition.*

We remark that the manifolds in Theorem 7.4.1 can be hyperbolic, and that the class of potentials going *slowly to zero* will be introduced in Section 7.4.3 (Definition 7.4.11). The family of extended Schottky manifolds is defined in Section 7.4.1.

7.4.1 Extended Schottky groups

Let N_1, N_2 be two non-negative integers such that $N_1 + N_2 \geqslant 2$ and $N_2 \geqslant 1$. Consider N_1 hyperbolic isometries $h_1, ..., h_{N_1}$ and N_2 parabolic ones $p_1, ..., p_{N_2}$ satisfying the following conditions:

1. For $1 \leqslant i \leqslant N_1$ there exists a compact neighbourhood C_{h_i} of the attracting point ξ_{h_i} of h_i and a compact neighbourhood $C_{h_i^{-1}}$ of the repelling point $\xi_{h_i^-}$ of h_i, such that
$$h_i(\partial\widetilde{M}\backslash C_{h_i^{-1}}) \subset C_{h_i}.$$

2. For $1 \leqslant i \leqslant N_2$ there exists a compact neighbourhood C_{p_i} of the unique fixed point ξ_{p_i} of p_i, such that
$$\forall n \in \mathbb{Z}^* \quad p_i^n(\partial\widetilde{M}\backslash C_{p_i}) \subset C_{p_i}.$$

3. The $2N_1 + N_2$ neighbourhoods introduced in (1) and (2) are pairwise disjoint.

4. The elementary parabolic groups $\langle p_i \rangle$, for $1 \leqslant i \leqslant N_2$, are of divergence type.

The group $\Gamma =< h_1, ..., h_{N_1}, p_1, ..., p_{N_2} >$ is a non-elementary free group which acts properly discontinuously and freely on M (see [DP, Corollary II.2]). Such a group Γ is called an *extended Schottky group*. It is proven in [DP] that it is a geometrically finite group. Note that if $N_2 = 0$, then the group Γ only contains hyperbolic elements, then Γ is a classical Schottky group and its geometric and dynamical properties are well understood.

7.4.2 The Geometric potential

We briefly recall the construction of the geometric potential of T^1M (there is some

overlap with notation that was already introduced, but for completeness we define everything here). For $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$ we define the *Busemann function* as

$$b_\xi(x,y) = \lim_{t\to\infty} d(x,\xi_t) - d(y,\xi_t),$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ. Pick a reference point $o \in \widetilde{M}$. For every $\xi \in \partial_\infty M$ and $s > 0$, denote by $B_\xi(s)$ the *horoball* centered at ξ of height s relative to o, that is

$$B_\xi(s) = \{y \in \widetilde{M} : b_\xi(o,y) \geqslant s\},$$

where $b_\xi(o,\cdot)$ is the Busemann function at ξ relative to o. The family $\{\partial B_\xi(s)\}_{s\in\mathbb{R}}$ foliates $\widetilde{M}$ by codimension one hypersurfaces. This family corresponds to the level sets of $b_\xi(o,\cdot)$, each leaf is called a *horosphere* centered at ξ. Suppose we are given a unit vector v, we are going to define the stable and unstable manifolds passing through v. Let $x \in \widetilde{M}$ be the base of v and $s_0 \in \mathbb{R}$ such that $x \in \partial B_\xi(s_0)$. Each vector $\{\nabla_y b_\xi(o,y)\}_{y\in\partial B_\xi(s_0)}$ points to ξ and is perpendicular to $\partial B_\xi(s_0)$. This defines a Hölder-submanifold of $T^1\widetilde{M}$ passing through v, the so called *strong stable submanifold* at v, this will be denoted by $W^{ss}(v)$. A similar construction defines the *strong unstable submanifold* at v; it will be denoted by $W^{su}(v)$. The strong (un)stable foliation is the foliation whose leaves are the strong (un)stable manifolds. One can characterize the points lying in $W^{ss}(v)$ and $W^{su}(v)$ by the following condition:

$$W^{ss}(v) = \{w \in T^1\widetilde{M} : \lim_{t\to\infty} d(g_t v, g_t w) = 0\}.$$

$$W^{su}(v) = \{w \in T^1\widetilde{M} : \lim_{t\to-\infty} d(g_t v, g_t w) = 0\}.$$

Observe that for every $\gamma \in Iso(\widetilde{M})$ we have $W^{ss}(\gamma v) = \gamma W^{ss}(v)$, and $W^{su}(\gamma v) = \gamma W^{su}(v)$; it follows from this that both foliations descend to T^1M. It also follows from the definition that $W^{ss}(g_t(v)) = g_t(W^{ss}(v))$ and $W^{su}(g_t(v)) = g_t(W^{su}(v))$, i.e.

the geodesic flow preserves the foliations.

Definition 7.4.3 (Geometric potential)**.** *We define the* geometric potential *or* unstable jacobian *by the formula*

$$F^{su}(\xi) = -\left.\frac{d}{dt}\right|_{t=0} \log \det dg_t|_{W^{su}}(\xi),$$

where the determinant of dg_t is computed with respect to orthonormal basis of the unstable subspaces with respect to Riemannian metric g.

The study of the geometric potential for (transitive) Anosov diffeomorphisms and flows is a classical subject. In those cases the equilibrium state of this potential corresponds to the SRB measure. We will need the following theorem.

Theorem 7.4.4. *[PPS, Theorem 7.2] Let $\widetilde{M}$ be a simply connected pinched negatively curved manifold. Moreover assume that the derivatives of the sectional curvature are uniformly bounded. Then the geometric potential F^{su} is Hölder continuous.*

It is proven in [Kli, Theorem 3.9.1] that if (M, g) satisfies the pinching condition $-a^2 \leqslant K_g \leqslant -b^2 < 0$, then we have

$$-(N-1)a \leqslant F^{su}(v) \leqslant -(N-1)b, \tag{7.2}$$

where N is the real dimension of M and $v \in T^1M$ (here K_g stands for the sectional curvature of (M, g)). We emphasize that inequality (7.2) follows from a local computation: it is also true that if the sectional curvature of an open set $Z \subset M$ lies in $[-a_1^2, -b_1^2]$, then for every $v \in T^1Z$ we have

$$-(N-1)a_1 \leqslant F^{su}(v) \leqslant -(N-1)b_1.$$

Inequality (7.2) also implies that for every invariant probability measure μ, and $t \geqslant 0$

$$h_\mu(g) - t(N-1)a \leqslant h_\mu(g) + t\int F^{su}(v)d\mu(v) \leqslant h_\mu(g) - t(N-1)b.$$

By the definition of the topological pressure it follows that

$$h_{top}(g) - t(N-1)a \leqslant P(tF^{su}) \leqslant h_{top}(g) - t(N-1)b.$$

The following remark is a consequence of the local nature of inequality (7.2), and inspires the computations done in Section 7.5.

Remark 7.4.5. *Let $U := F^{su} + (N-1)$. As mentioned above if*

$$-(1+\frac{1}{N-1}M)^2 \leqslant K_g \leqslant -(1+\frac{1}{N-1}L)^2,$$

on an open set W of M, then

$$-(N-1)(1+\frac{1}{N-1}M) \leqslant F^{su}(v) \leqslant -(N-1)(1+\frac{1}{N-1}L),$$

for every vectors $v \in T^1W$. Equivalently $-M \leqslant U \leqslant -L$.

It worth mentioning that under the assumptions of Theorem 7.4.4 Ruelle's inequality holds [Riq]. More precisely we have that

$$h_\mu(g) \leqslant -\int F^{su}d\mu,$$

for every $\mu \in \mathcal{M}(g)$. In other words $P(F^{su}) \leqslant 0$.

7.4.3 Construction of phase transitions

In this section we will prove Theorem 7.4.1. We start with a modification of [DOP, Theorem C].

Proposition 7.4.6. *Suppose we have a parabolic subgroup $\mathcal{P}$ and a hyperbolic isometry h such that $\mathcal{P}$ and $H = \langle h \rangle$ are in Schottky position. Denote by Γ_k the group generated by $\mathcal{P}$ and $\langle h^k \rangle$, and define $M_k = \widetilde{M}/\Gamma_k$. Let $F : T^1M_1 \to \mathbb{R}$ be a bounded Hölder potential for which $(\mathcal{P}, F)$ is of convergence type and $\delta^F_{\mathcal{P}} > 0$. Suppose that $\int_\gamma F = 0$, where γ is the periodic orbit associated to h. Denote by F_n the lift of F to T^1M_n. Then there exists N_0 such that if $n \geqslant N_0$, then (Γ_n, F_n) is of convergence type, and $\delta^F_{\mathcal{P}} = \delta^{F_n}_{\Gamma_n}$.*

The heart of the proof of [DOP, Theorem C] is the following elementary fact about Hadamard manifolds. Given $D > 0$, there exists $C = C(D) > 0$ such that for every geodesic triangle with vertices x, y, z, and angle at z bigger than D, then

$$d(x, y) \geqslant d(x, z) + d(z, y) - C.$$

We begin with the following analogous inequality that the Gibbs cocycle $C_{F,\xi}(x, y)$ satisfies.

Lemma 7.4.7. *[PPS, Lemma 3.4] Let F be a Hölder potential such that $||F||_0 < B$. Then for every $r > 0$, for all $x, y \in \widetilde{M}$ and $\xi \in \mathcal{O}_x B(y, r)$ we have*

$$|C_{F,\xi}(x, y) + \int_x^y \widetilde{F}| \leqslant L(r, B),$$

for certain uniform constant $L(r, B)$. Here $\mathcal{O}_x B(y, r) \subset \partial_\infty \widetilde{M}$ denotes the set of end points of geodesic rays emanating from x that intersect $B(y, r)$.

In the inequality above we are using the Gibbs cocycle defined in Section 2. In the proof of Proposition 7.4.6 we will use that $C_{F,\xi}(x, y)$ is an additive cocycle, more

precisely that

$$C_{F,\xi}(x,y) + C_{F,\xi}(y,z) = C_{F,\xi}(x,z).$$

Proof of Proposition 7.4.6. Since H and $\mathcal{P}$ are in Schottky position we can find $U_{\mathcal{P}}, U_H \subset \widetilde{M} \cup \partial_\infty \widetilde{M}$ so that

1. $\mathcal{P}^*(\partial_\infty \widetilde{M} \backslash U_{\mathcal{P}}) \subset U_{\mathcal{P}}$.
2. $H^*(\partial_\infty \widetilde{M} \backslash U_H) \subset U_H$.
3. $U_H \cap U_{\mathcal{P}} = \varnothing$.

Fix $x \in X$ over the axis of h so that $x \notin U_H \cup U_{\mathcal{P}}$. As a consequence of the Ping Pong Lemma we have that Γ is isomorphic to the free product $H * \mathcal{P}$. By the comments above Lemma 8.2 and the fact that $U_H \cap U_{\mathcal{P}} = \varnothing$, we know that there exists a positive constant C such that for every $y \in U_H$ and $z \in U_{\mathcal{P}}$ we have

$$d(y,z) \geqslant d(x,y) + d(x,z) - C.$$

Applying inequality (7.4.3) and the inclusion properties described above we obtain

$$d(x, p_1 h^{kn_1} ... p_j h^{kn_j} x) \geqslant \sum_i d(x, p_i x) + \sum_i d(x, h^{kn_i} x) - 2jC,$$

where $n_i \in \mathbb{Z}^*$, $k \neq 0$ and $p_i \in \mathcal{P}^*$. Let B be a bound for F, i.e. $||F||_0 < B$. Choose ξ outside $U_{\mathcal{P}} \cup U_H$ and r big enough we can apply Lemma 8.2. Then

$$\begin{aligned}\int_x^{h^n x} \widetilde{F} + \int_x^{px} \widetilde{F} = \int_{px}^{ph^n x} \widetilde{F} + \int_x^{px} \widetilde{F} &\geqslant -2L(r,B) - C_{F,\xi}(x, ph^n x) \\ &\geqslant -3L(r,B) + \int_x^{ph^n x} \widetilde{F}.\end{aligned}$$

This immediately generalize to

$$\int_x^{p_1 h^{kn_1} \ldots p_j h^{kn_j} x} \widetilde{F} \leqslant (2j+1)L(r,B) + \sum_i \int_x^{h^{kn_i} x} \widetilde{F} + \sum_i \int_x^{p_i x} \widetilde{F}.$$

Let $l := d(x, hx)$. By the choice of x we have that $d(x, h^N x) = |N|l$, and since $\int_x^{hx} \widetilde{F} = 0$, we also have $\int_x^{h^N x} \widetilde{F} = 0$. Finally

$$\sum_{n \in \mathbb{Z}^*} \exp\left(\int_x^{h^{kn} x} (\widetilde{F} - s) \right) = \sum_{n \in \mathbb{Z}^*} \exp(-s|n|lk) = 2\frac{\exp(-slk)}{1 - \exp(-slk)}.$$

For simplicity we will bound the expression

$$\tilde{P}(s) = \sum_{j \geqslant 1} \sum_{p_i \in \mathcal{P}^*, m_i \in \mathbb{Z}^*} \exp(\int_x^{p_1 h^{kn_1} \ldots p_j h^{kn_j} x} (\widetilde{F} - s)).$$

A bound for $P(s)$ follows identically, but here we have more symmetry. Using the inequalities above we obtain

$$\tilde{P}(s) \leqslant \sum_{j \geqslant 1} \left(e^{2(C + L(r,B)+1)} \sum_{n \in \mathbb{Z}^*} \exp\left(\int_x^{h^{kn} x} (\widetilde{F} - s) \right) \sum_{p \in \mathcal{P}^*} \exp\left(\int_x^{px} (\widetilde{F} - s) \right) \right)^j. \tag{7.3}$$

By taking k big enough, we can make the right hand side of (7.3) to be convergent at $s = \delta_{\mathcal{P}}^F$. In particular $\delta_{\mathcal{P}}^F \geqslant \delta_{\Gamma_k}^{F_k}$, which immediately implies $\delta_{\Gamma_k}^{F_k} = \delta_{\mathcal{P}}^F$. Inequality (7.3) also implies the convergence property of the pair (Γ_k, F_k). □

Remark 7.4.8. *Let $\{h_1, ..., h_l\}$ be a collection of hyperbolic isometries and denote by H_i the group generated by h_i. Suppose that the subgroups $\{H_1, ..., H_l, \mathcal{P}\}$ are pairwise in Schottky position (as in the definition of extended Schottky but allowing $\mathcal{P}$ to have bigger rank). Moreover assume that the integral of F over the closed geodesics associated to each h_i vanishes. Define Γ_n as the group generated by $\mathcal{P}$ and the elements $\{h_1^n, ..., h_l^n\}$. The proof of Proposition 7.4.6 can be modified to conclude that for big*

enough k we have

$$\delta_{\mathcal{P}}^{F} = \delta_{\Gamma_k}^{F_k},$$

and that (Γ_k, F_k) is of convergence type. For simplicity we will state our results only for the case treated in Proposition 7.4.6, but everything works identically under the hypothesis of this remark.

The next proposition is very important for us, it provides the family of potentials for which we will have phase transitions. In Section 7.5 we will also use ideas from the proof of Proposition 7.4.9 to modify the metric at the cusp. Recall that $(\mathcal{P}, -F)$ is of convergence type if the sum

$$\sum_{p\in\mathcal{P}} \exp\Big(-\int_x^{px} \widetilde{F} - \delta_{\mathcal{P}}^{-F} d(x, px)\Big),$$

is finite.

Proposition 7.4.9. *Let M be a geometrically finite manifold. There exists a bounded Hölder potential F_0 satisfying the following properties:*

1. *F_0 is positive near the cusps,*
2. *F_0 goes to zero through the cusps of M,*
3. *$(\mathcal{P}, -F_0)$ is of convergence type for every maximal parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$.*

Proof. We will define F_0 in a neighborhood of each cusp, and then we will extend it to the rest of the manifold in a Hölder continuous way (making sure that F_0 is a bounded function). Pick a maximal parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$, and denote by $\xi \in \partial_\infty \widetilde{M}$ to its fixed point. There exists a neighborhood $\mathcal{U}$ of the cusp associated to $\mathcal{P}$ which is isometric to $B_\xi(q_0)/\mathcal{P}$, for big enough q_0. Pick a reference point $x \in \partial\mathcal{U}$.

For $w \in \mathcal{U}$ we define $d(w) = q$, if $\pi(w) \in \partial B_\xi(q + q_0)/\mathcal{P}$. We say that the geodesic $\gamma : [a, b] \to T^1M$ has *height* H if

$$\max_{t\in[a,b]} d(\gamma(t)) = H.$$

For $l < L$, define

$$S(l, L) := \{p \in \mathcal{P} : l < d(x, px) \leqslant L\}.$$

By the definition of critical exponent, for every $\epsilon > 0$, there exists a natural number $C(\epsilon)$ so that

$$\sum_{p\in S(C(\epsilon),\infty)} \exp(-(\delta_\mathcal{P} + \epsilon)d(x, px)) < \epsilon^2.$$

We define a sequence $(A_n)_n$ inductively as follows: let $A_1 = C(1)$, and $A_{n+1} = \max(A_n + 1, C(1/n))$. By construction the sequence of real numbers $(A_n)_n$ is strictly increasing and satisfies

$$\sum_{p\in S(A_n,\infty)} \exp(-(\delta_\mathcal{P} + 1/n)d(x, px)) < \frac{1}{n^2}.$$

We define H_n as the maximum height of the geodesic segments $[x, px]$, where p runs in $S(A_n, A_{n+1})$. With the heights $(H_n)_n$ we construct a sequence $(B_n)_n$ by declaring $B_1 = H_1$, and inductively define $B_{n+1} = \max(B_n + 1, H_{n+1})$. Define a function f on $\mathcal{V} = d^{-1}([B_1, \infty)) \subset \mathcal{U}$, by the following expression

$$f(x) = \begin{cases} -d(x) + 1/n - B_n & \text{if } x \in d^{-1}([B_n, B_n + 1/n - 1/(n + 1)]) \\ 1/(n + 1) & \text{if } x \in d^{-1}([B_n + 1/n - 1/(n + 1), B_{n+1}]). \end{cases}$$

Let $F : T^1\mathcal{V} \to \mathbb{R}$, be the composition of the projection from $T^1\mathcal{V}$ to $\mathcal{V}$ and f. We do the same construction for every cusp in M. Using this functions (at the cusps) and any Hölder continuous extension to the rest of the manifold, we obtain our bounded

Hölder potential F_0. We will now check that F_0 satisfies the properties described in Proposition 7.4.9. It follows from the construction that F_0 goes to zero through the cusps. By Lemma 7.3.3 to get that $\delta_{\mathcal{P}}^{-F_0} = \delta_{\mathcal{P}}$. It only remains to check that $(\mathcal{P}, -F_0)$ is of convergence type for every maximal parabolic subgroup of $\pi_1(M)$. Observe that if $p \in S(A_n, A_{n+1})$, then $[x, px]$ has height at most H_n, in particular at most B_n. Because of the way we defined the function f we obtain that if $p \in S(A_n, A_{n+1})$ and $v \in T^1[x, px]$, then $F_0(v) \geqslant \frac{1}{n}$. In other words, for $p \in S(A_n, A_{n+1})$ we have

$$\int_x^{px} F_0 \geqslant \frac{d(x, px)}{n}.$$

Finally

$$\sum_{p \in S(A_1, \infty)} \exp\left(-\int_x^{px} F_0 - \delta_{\mathcal{P}} d(x, px) \right) = \sum_{n=1}^{\infty} \sum_{p \in S(A_n, A_{n+1})} \exp\left(-\int_x^{px} F_0 - \delta_{\mathcal{P}} d(x, px) \right)$$

$$\begin{aligned}
&\leqslant \sum_{n=1}^{\infty} \sum_{p \in S(A_n, A_{n+1})} \exp\left(-\frac{1}{n} d(x, px) - \delta_{\mathcal{P}} d(x, px) \right) \\
&\leqslant \sum_{n=1}^{\infty} \sum_{p \in S(A_n, \infty)} \exp\left(-\left(\frac{1}{n} + \delta_{\mathcal{P}} \right) d(x, px) \right) \\
&\leqslant \sum_{n=1}^{\infty} \frac{1}{n^2},
\end{aligned}$$

which is finite.

□

Remark 7.4.10. *Since the numbers $(B_n)_n$ can be arbitrarity far apart, we interpret the decay of F through the cusp associated to $\mathcal{P}$ as 'very slow'.*

Definition 7.4.11. *A potential G belongs to the family $\mathcal{F}_s$ if the following conditions are satisfied.*

1. *G is a bounded Hölder potential,*

2. *G is positive in a neighborhood of the cusps of M and vanishes at infinity,*

3. *$(\mathcal{P}, -G)$ is of convergence type for every maximal parabolic subgroup $\mathcal{P}$ of $\pi_1(M)$.*

The elements in $\mathcal{F}_s$ are called potentials going slowly to zero *through the cusps of M. The class of non-negative potentials in $\mathcal{F}_s$ is denoted by $\mathcal{F}_s^+$.*

The family $\mathcal{F}_s$ is not empty because of Lemma 7.3.3. We remark that the family $\mathcal{F}_s$ is quite big. Let $F \in \mathcal{F}_s$, if G is a bounded Hölder potential going to zero through the cusps of M satisfying $G \geqslant F$ in a neighborhood of the cusps, then $G \in \mathcal{F}_s$. We now proceed to prove Theorem 7.4.1. The statement of our next result looks a bit more complicated that Theorem 7.4.1, but it has the advantage of being more precise. Our manifolds are geometrically finite with one cusp, in particular $\delta_\infty = \delta_\mathcal{P}$. As before, we use the notation

$$t_F = \sup\{t : P(tF) = \delta_\mathcal{P}\}.$$

Theorem 7.4.12. *Let $\mathcal{P}$ be a divergence type parabolic subgroup and h a hyperbolic isometry such that $\langle h \rangle$ and $\mathcal{P}$ are in Schottky position. Define Γ_n as the group generated by $\mathcal{P}$ and $\langle h^n \rangle$. Let $M_k = \widetilde{M}/\Gamma_k$, and $F_1 : T^1M_1 \to \mathbb{R}$ a potential in the class $\mathcal{F}_s^+$. Assume that $\int_\gamma F_1 = 0$, where γ is the periodic orbit associated to h. Let F_n be the lift of F_1 to T^1M_n, and N_0 be the constant provided by Proposition 7.4.6 for $F = -F_1$. Then for $n \geqslant N_0$ we have:*

1. *$t_{F_n} \in [-1, 0)$.*

2. *The potential tF_n has an equilibrium state for $t > t_{F_n}$.*

3. *The potential tF_n does not have an equilibrium state for $t < t_{F_n}$.*

In conclusion, the pressure map of F_n exhibits a phase transition at $t = t_{F_n}$. Moreover, the pressure map is differentiable in $(-\infty, t_{F_n}) \cup (t_{F_n}, \infty)$. With respect to the behaviour at $t = t_{F_n}$ we have two possibilities:

(4) If the potential $t_{F_n}F_n$ does not have an equilibrium state, then the pressure map is differentiable everywhere.

(5) If $t_{F_n}F_n$ has an equilibrium state, then the pressure map is not differentiable at $t = t_{F_n}$.

We remark that since $\mathcal{P}$ is of divergence type, then the geodesic flow on M_n has a measure of maximal entropy (a consequence of Theorem 2.3.15 and Theorem 2.3.16).

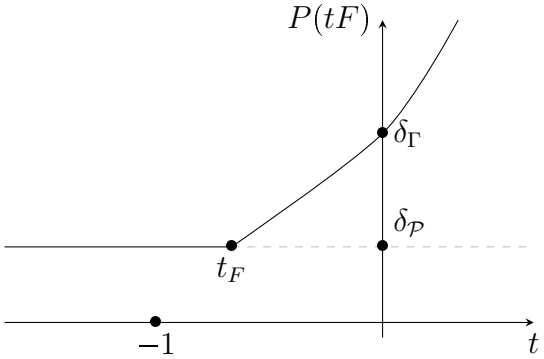


Figure 2: Phase transition for $F \in \mathcal{F}_s^+$

Proof. Since the covering map $M_k \to M_1$, is one to one in a neighborhood of the cusp associated to $\mathcal{P}$ we get that the potential F_n goes to zero through the cusp of T^1M_n. This implies that $\delta_\mathcal{P} = \delta_\mathcal{P}^{tF_n}$. Since F_n is non-negative, the pressure map $t \mapsto P(tF_n)$ is non-decreasing. It follows from Theorem 6.1.5 that for $t > t_{F_n}$ there exists an unique equilibrium state for tF_n, and from Theorem 6.1.9 that the pressure map is differentiable in (t_{F_n}, ∞). By assumption $\mathcal{P}$ is of divergence type, then Theorem 2.3.15 gives us $\delta_\infty = \delta_\mathcal{P} < \delta_\Gamma = P(0)$. We conclude that $t_F < 0$. Since $F_1 \in \mathcal{F}_s^+$ we know that $(\mathcal{P}, -F_1)$ is of convergence type. Notice that the potentials $(F_n)_n$ lift to the same potential on $T^1\widetilde{M}$, in particular the Poincaré series of $(\mathcal{P}, -F_n)$ is independient of n. It follows that $(\mathcal{P}, -F_n)$ is of convergence type for every $n \geqslant 1$. By Proposition 7.4.6

we know that $\delta_{\mathcal{P}}^{-F_n} = P(-F_n)$, for every $n \geqslant N_0$. We can conclude that $\delta_\infty = P(-F_n)$, which also implies that $t_{F_n} \geqslant -1$. Moreover, by Hopf-Tsuji-Sullivan-Roblin theorem there is not equilibrium state for the potential $-F_n$ (since $(\Gamma_n, -F_n)$ is of convergence type). As in the proof of Theorem 7.1.2 we can conclude that the pressure map is constant in $(-\infty, t_F)$. We will now check part (3). Suppose there exists $t \in (-\infty, t_{F_n})$ such that tF_n has an equilibrium state, say μ_t. Recall that by definition of the family $\mathcal{F}_s^+$, the potential F_n is non-negative and positive in a neighborhood of the cusp associated to $\mathcal{P}$. Since the support of μ_t contains some portion of the cusp we conclude that $\int F_n d\mu_t > 0$. Observe that for every $t' > t$ we have

$$\delta_\infty = h_{\mu_t}(g) + t \int F_n d\mu_t < h_{\mu_t}(g) + t' \int F_n d\mu_t.$$

This implies that for $t' \in (t, t_{F_n})$ we have $P(t'F_n) > \delta_\infty$, which is a contradiction. We conclude that the potential tF_n does not have an equilibrium state for $t < t_{F_n}$. This proves part (3). We now proceed to prove part (4) and part (5). First suppose that t_nF_n does not have an equilibrium state. Let μ_t be the equilibrium state of tF_n, for $t > t_{F_n}$. Observe that

$$\begin{aligned}\lim_{t \to t_{F_n}^+} \left(h_{\mu_t}(g) + t_{F_n} \int F_n d\mu_t \right) &= \lim_{t \to t_{F_n}^+} \left(h_{\mu_t}(g) + t \int F_n d\mu_t \right) \\ &= \lim_{t \to t_{F_n}^+} P(tF_n) = P(t_nF_n).\end{aligned}$$

By Theorem 6.1.8 we conclude that $(\mu_t)_t$ converges vaguely to the zero measure as t goes to t_{F_n}. In particular

$$\lim_{t \to t_{F_n}^+} \int F_n d\mu_t = 0.$$

By Theorem 6.1.9 we know that the derivative of the pressure map of F_n at t, for $t > t_{F_n}$, is given by $\int F_n d\mu_t$. We conclude that the pressure map is differentiable at $t = t_{F_n}$, and that the derivative at that point is equal to zero. This proves part (4).

Now assume that $t_n F_n$ has an equilibrium state, and denote it by $\mu_{t_{F_n}}$. As in the proof of part (4) we will denote by μ_t the equilibrium state of tF_n, for $t > t_{F_n}$. The same argument used in the proof of part (4) allows us to conclude that $\int F_n d\mu_{t_{F_n}} > 0$. The left hand side derivative of the pressure map at $t = t_{F_n}$ is clearly equal to zero. We will prove that the right hand side derivative of the pressure map at $t = t_n$ is equal to $\int F_n d\mu_{t_{F_n}}$. As observed in part (3) we know that

$$\lim_{t \to t_{F_n}^+} \left(h_{\mu_t}(g) + t_{F_n} \int F_n d\mu_t \right) = P(t_n F_n).$$

Theorem 6.1.8 allows us to conclude that every vague limit point of $(\mu_t)_t$ as t goes to t_{F_n}, must be of the form $\lambda \mu_{t_{F_n}}$ (for some $\lambda \in [0,1]$). We now claim that $(\mu_t)_t$ actually converges to $\lambda \mu_{t_{F_n}}$ (for a fixed λ). The convexity and differentiability of the pressure map in (t_{F_n}, ∞) implies that the limit

$$\lim_{t \to t_{F_n}^+} \int F_n d\mu_t = A,$$

exists. If $\lambda \mu_{t_{F_n}}$ is a limit point of $(\mu_t)_t$ as t goes to t_{F_n}, then we must have $A = \lambda \int F_n d\mu_{t_{F_n}}$. We conclude that there is at most one possible choice for λ, and that the sequence of measures is convergent. Observe that the right hand side derivative of the pressure at $t = t_{F_n}$ is equal to $A = \lambda \int F_n d\mu_{t_{F_n}}$. In particular, it is less than or equal to $\int F_n d\mu_{t_{F_n}}$. Define

$$L(t) = h_{\mu_{t_{F_n}}}(g) + t \int F_n d\mu_{t_{F_n}}.$$

Observe that $P(tF) \geqslant L(t)$, and that $P(t_n F) = L(t_n)$. The convexity of the pressure map implies that the right hand side derivative at $t = t_{F_n}$ is at least $\int F_n d\mu_{t_{F_n}}$. We conclude that the right hand side derivative of the pressure at $t = t_{F_n}$ must be equal to $\int F_n d\mu_{t_{F_n}}$. We remark that since $A = \int F_n d\mu_{t_{F_n}}$, we necessarily have $\lambda = 1$. In

other words, the sequence $(\mu_t)_t$ converges to $\mu_{t_{F_n}}$ as t goes to t_{F_n}.

□

Remark 7.4.13. *Suppose the potential F satisfies the following additional property: $(\mathcal{P}, -F)$ is of convergence type and $(\mathcal{P}, tF)$ is of divergence type for $t > -1$. In this case we can ensure that for every $t > -1$ the potential tF_n is SPR; this implies that $t_{F_n} = -1$. Moreover, at $t = -1$ the potential tF_n does not have an equilibrium state. By part (4) of Theorem 7.4.12 we get that the pressure map is differentiable everywhere.*

Remark 7.4.14. *If the potential t_nF_n has an equilibrium state, then the pressure map exhibits a first order phase transition (using Ehrenfest classification). In this case the first derivative of the pressure develops a singularity. If the potential t_nF_n does not have an equilibrium state, then it is reasonable to expect that some higher order derivative of the pressure map should develop a singularity. In the context of countable Markov shifts, Sarig investigated the relation between critical exponents and abnormal fluctuations (see [S5]). It is a very interesting project to try to prove analogous result to those in [S5] for the geodesic flow on a pinched negatively curved manifold.*

We will briefly discuss what happen if we only assume $F \in \mathcal{F}_s$, i.e. we allow F to take negative values. As before we will assume that M has only one cusp. First suppose that for every $\mu \in \mathcal{M}(g)$ we have $\int F_n d\mu \geqslant 0$. Then Theorem 6.1.9 and the convexity of the pressure map implies that $t \mapsto P(tF_n)$, has the same description as the one of a potential in $\mathcal{F}_s^+$. As explain in the paragraph after Theorem 7.1.2 we need to be a bit careful here, in general we can not immediately conclude that $\lim_{t\to-\infty} P(tF) = \delta_\infty$. In our case this issue does not apply: our construction ensures that for some t we have $P(tF) = \delta_\infty$. If there exists $\mu \in \mathcal{M}(g)$ such that $\int F_n d\mu < 0$, then the set

$$J = \{t \in \mathbb{R} : P(tF_n) = \delta_\infty\},$$

is a compact interval. Observe that for $t \in \mathbb{R}\backslash J$ the potential tF_n is strongly positive recurrent and therefore admits an equilibrium state. We claim that if $t \in int(J)$, then tF_n does not have an equilibrium state. Suppose that for some $t \in int(J)$ there exists an equilibrium state for tF, say μ_t. If $\int F_n d\mu_t > 0$, then for $t' > t$ we have

$$\delta_\infty = h_{\mu_t}(g) + t\int F_n d\mu_t < h_{\mu_t}(g) + t'\int F_n d\mu_t.$$

In particular if $t' \in J$ and $t' > t$, then $P(t'F) > \delta_\infty$. Similarly, if $\int F_n d\mu_t < 0$, then for $t' < t$ we have

$$\delta_\infty = h_{\mu_t}(g) + t\int F_n d\mu_t < h_{\mu_t}(g) + t'\int F_n d\mu_t.$$

In particular if $t' \in J$ and $t' < t$, then $P(t'F) > \delta_\infty$. We conclude that $\int F_n d\mu_t = 0$ (otherwise we contradict the definition of J). Since μ_t is an equilibrium state for tF_n we have

$$\delta_\infty = P(tF_n) = h_{\mu_t}(g) + t\int F_n d\mu_t = h_{\mu_t}(g).$$

Observe that $\int F_n d\mu_t = 0$ and $h_{\mu_t}(g) = \delta_\infty$ implies that μ_t is an equilibrium state for sF_n, for every $s \in J$. Since by construction $-F_n$ does not have an equilibrium state, and $-1 \in J$, we conclude that the measure μ_t can not exist. We remark that potentials with this description of the pressure map (see Figure 2) can be constructed in a similar fashion to those in Theorem 7.4.12. For instance suppose that F_1 is negative in the complement of a neighborhood of the cusp and identically zero on γ (the geodesic associated to the hyperbolic generator h). If we take a closed geodesic that wrap around the lift of γ to M_n a large number of times (the geodesic represented by ph^{nk} for big k works) we get an invariant measure with negative integral against F_n. As mentioned in the paragraph below Definition 7.4.11, if $F \in \mathcal{F}_s$ and $t > 1$, then $tF \in \mathcal{F}_s$. In particular, given $M > 1$, there exists $N_1 = N_1(M, F)$ such that the following holds: for every $n \geqslant N_1$, the pairs $(\Gamma_n, -MF_n)$ and $(\Gamma_n, -F_n)$ are of convergence type and have critical exponent $\delta_\infty = \delta_\mathcal{P}$. In particular (by the convexity

of the pressure map) J contains the interval $(-M,-1)$. A similar argument allows us to construct potentials such that J contains any compact subset of $(-\infty,0)$.

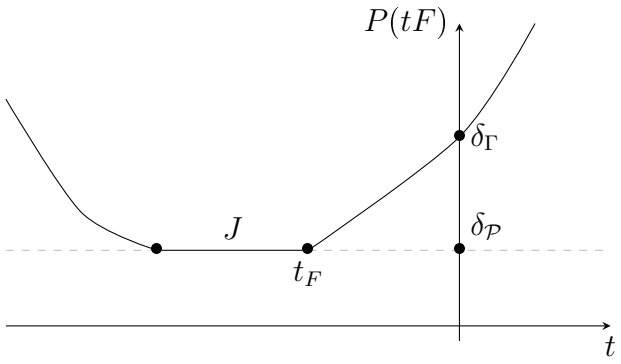


Figure 3: Phase transition for $F \in \mathcal{F}_s$

In light of this discussion we have the following definition.

Definition 7.4.15 (Types of phase transition)**.** *A Hölder potential $F \in C_0(T^1M)$ exhibits a phase transition of* type A *if the graph of $t \mapsto P(tF)$, looks like Figure 2. A potential F exhibits a phase transition of* type B *if the graph of $t \mapsto P(tF)$, looks like Figure 3.*

Type A and B phase transitions represent basically all types of phase transitions for Hölder potentials in $C_0(T^1M)$.

Remark 7.4.16. *The manifolds $(M_n)_n$ constructed in Theorem 7.4.12 are all diffeomorphic to M_1. It is a well known fact that every parabolic subgroup of $Iso(\mathbb{H}^N)$ is of divergence type: we can verify the hypothesis of Theorem 7.4.12 if $(\widetilde{M},g)$ is isometric to $\mathbb{H}^N$.*

A concrete situation where Theorem 7.4.12 applies is given in the following corollary.

Corollary 7.4.17. *Let M be a thrice-punctured sphere. We can endow M with a complete hyperbolic metric for which it is possible to construct Hölder potentials exhibiting phase transitions (as described in Theorem 7.4.12). By Remark 7.4.8 the same holds if M is a k-punctured sphere and $k \geqslant 3$.*

7.5 Phase transitions for the geometric potential

In this section we will modify the metric at the cusp of a hyperbolic manifold in such a way that the geometric potential exhibits a phase transition. Since F^{su} is not a potential that goes to zero through the cusps of the manifold, to apply the previous techniques we need to consider a normalization. From now on we will assume that our manifold has real dimension N.

Definition 7.5.1. *We define the* normalized unstable jacobian *as the function*

$$U = F^{su} + (N-1).$$

Our goal is to prove that under certain conditions U exhibits a phase transition, just as the potentials in Theorem 7.4.12. The construction starts with the hyperbolic space $\mathbb{H}^N$. It is convenient to think in the half space model, i.e. the space $\mathbb{R}^{N-1} \times \mathbb{R}^+$ with coordinates $(x_1, ..., x_{N-1}, x_0)$ and metric

$$ds^2 = \frac{1}{x_0^2}(dx_1^2 + ... + dx_{N-1}^2 + dx_0^2).$$

To simplify notation we denote $(x_1, ..., x_{N-1}, x_0) = (\mathbf{x}, x_0)$. In this model we have a preferent point at infinity, we denote this by $\xi_\infty \in \partial_\infty \widetilde{M}$. We will modify the hyperbolic metric in a neighborhood of ξ_∞. It will be convenient to consider the diffeomorphism $\mathbb{H}^N \to \mathbb{R}^N$ taking $(\mathbf{x}, x_0)$ to $(\mathbf{x}, \log(x_0))$. In this model the hyperbolic metric takes the form $g = e^{-2t}d\mathbf{x}^2 + dt^2$. For a positive function $T : \mathbb{R} \to \mathbb{R}$, we define the Riemannian metric

$$g_T = T(t)^2 d\mathbf{x}^2 + dt^2.$$

The sectional curvature of g_T has value $-(T''(t)/T(t))$ for the planes generated by

$\langle \partial/\partial x_i, \partial/\partial x_j \rangle$ and value $-(T''(t)/T(t))^2$ for those generated by $\langle \partial/\partial x_i, \partial/\partial t \rangle$. Define

$$K(t) = -\frac{T''(t)}{T(t)}.$$

Bounds on $K(t)$ clearly imply bounds on the curvature of g_T. The lines $t \mapsto (\mathbf{x}, t)$ are still geodesics and any isometry of $\mathbb{R}^{N-1}$ acts isometrically on $(\mathbb{H}^N, g_T)$, where the action is given by $A.(\mathbf{x}, t) = (A(\mathbf{x}), t)$. Observe that translations act transitively in $H_t = \{(\mathbf{x}, t) : \mathbf{x} \in \mathbb{R}^{N-1}\}$. This two basic observations and the definition of the Busemann function are enough to conclude that H_t are the horospheres associated to ξ_∞. For a function constant on horospheres we will use the notation $F(t) = F(\mathbf{x}, t)$. In this section the *height of a segment* will be the maximum value of the t coordinate over the segment. Suppose we have a surjective, strictly increasing function $u : (0, \infty) \to \mathbb{R}$. We can define $T = T(u)$ by the equation

$$T(u(t)) = 1/t.$$

In this context we will use $o = (\mathbf{0}, u(1))$ as reference point. Let p be a translation in $\mathbb{R}^N$ such that $d(o, po) = 1$. It is proven in [DOP, Section 3] that there exists a uniform constant C such that

$$|d_T(o, p^n o) - 2u(|n|)| \leqslant C.$$

It will be important for us the fact that C only depends on the pinching of the sectional curvature of $(\mathbb{H}^N, g_T)$. Using the symmetry of our metric we can also conclude that

$$|t_n - u(|n|))| \leqslant D,$$

where t_n is the maximum height of the geodesic segment $[o, p^n o]$. Similarly to what

happened with C, D only depends on the pinching of the metric. From now on D and C will be the constants associated to a metric with pinching $-(1/3)^2 \geqslant K_g \geqslant -2^2$. Observe that

$$\begin{aligned} K(u(t)) &= -\frac{2tu'(t)+t^2u''(t)}{(u't)^3}, \\ &= -\frac{1+2t\varphi'(t)+t^2\varphi''(t)}{(1+t\varphi'(t))^3}, \\ &= -\frac{g_1(t)+tg_1'(t)}{g_1(t)^3}, \\ &= -(g_2(t)-\frac{t}{2}g_2'(t)), \end{aligned}$$

where we have made the substitutions

$$u(t) = \log(t)+\varphi(t); g_1(t) = 1+t\varphi'(t) \text{ and } g_2(t) = 1/g_1(t)^2.$$

7.5.1 Construction of a special metric at the cusp

We will start by constructing a function g_2 satisfying several properties. Using the equalities above, this will give us a function u and therefore a metric on $\mathbb{R}^N$.

Let $a_{n+1} := (1+\frac{1}{(N-1)n})^2$, $b_n := 1-(2a_n-1)^{-\frac{1}{2}}$ and $c_n := (a_{n-1}-a_n)$. For $n \geqslant 2$ we choose $k(n) \in \mathbb{N}$ such that

$$\exp\left(\left(\frac{1}{n}+\frac{1}{2}\right)D\right)\sum_{|k|\geqslant k(n)}\exp\left(-\left(\frac{1}{2}+\frac{1}{n}\right)2\log(|k|)\right) \leqslant \frac{1}{n^2}.$$

Without loss of generality we can assume that $(k(n))_n$ is strictly increasing sequence. We will define a sequence $(p_n)_n$ so that the conditions below are satisfied. We do this by induction, i.e. the choice of $p_1, ..., p_n$ will determine p_{n+1}. For $n \geqslant 2$ define $\Delta_n = p_n - p_{n-1}$.

1. The sequence $\left(\frac{c_n}{\Delta_n}\right)$ is a strictly decreasing.

2. $b_n \log(p_{n+1}) \geqslant -b_n \log(p_n) + \sum_{i=2}^{n-1} b_i \log(p_{i+1}/p_i)$, for $n \geqslant 3$.

3. $(1 - 2b_n) \log(p_{n+1}) \geqslant \log(k(n+1))$, whenever $1 - 2b_n > 0$.

4. $\lim_{n\to 0} \dfrac{p_n c_n}{\Delta_n} = 0$.

For $n \geqslant 2$ define the line connecting the points (p_n, a_n) and (p_{n+1}, a_{n+1}) as J_n. We could have assumed that p_2 is big enough compared to p_1 so that $J_2(0) \leqslant 2$, we will do assume that. Define the intervals $I_n = [p_{n-1}, p_n]$. We will construct a $\mathcal{C}^\infty$ function $g_2 : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties

1. For $t \in I_n$ and $n \geqslant 3$ we have $2a_{n-1} - 1 \geqslant g_2(t) \geqslant a_n$.

2. $g_2''(t) \geqslant 0 \geqslant g_2'(t)$ for $t \geqslant 2$.

3. $2 \geqslant g_2(t) - \frac{t}{2} g_2'(t) \geqslant 1/3$ for all $t \in \mathbb{R}^+$.

4. For $t \in I_n$ and $n \geqslant 3$ we have $g_2(t) - \frac{t}{2} g_2'(t) \geqslant a_n$.

5. $g_2(t) = 1$ for $t < 1$.

Observe that $(g_2 - (t/2)g_2')' = g_2'/2 - (t/2)g_2''$, so the condition $g_2'' \geqslant 0 \geqslant g_2'$ implies that $g_2(t) - (t/2)g_2'(t)$, is non-increasing. Notice that if $0 \geqslant g_2'$, then $g_2(t) - \frac{1}{2} t g_2'(t) \geqslant g_2(t)$, in particular if $g_2 \geqslant a_n$, then the same holds for $g_2(t) - \frac{1}{2} t g_2'(t)$. We now explain how to construct g_2. First define $J_0 : \mathbb{R}^+ \to \mathbb{R}$ as $J_0(t) := \sup_{n \geqslant 2} J_n(t)$. Now define

$$J(t) := \begin{cases} 1 & \text{if } t \in (0, 1] \\ \min\{J_0(t), t\} & \text{if } t \geqslant 1 \end{cases}$$

The function J is not smooth, we can smooth J in a neighborhood of the nodes to obtain a smooth function J^* as close to J (in the $\mathcal{C}^\infty$ topology) as needed. When $t \geqslant 2$ we can assume that $J^* \geqslant J$ and that J^* is convex decreasing. The fact that J^* can be taken convex on that region comes from the condition (1) in the definition

of $(p_n)_n$. We remark that the choice of $(2a_{n-1}-1)$ as the upper bound is done just to get room for this perturbation (notice $2a_{n-1}-1 > a_{n-1}$). We finally set $g_2 = J^*$. Define a function φ (up to additive constant) by the equation

$$\varphi'(t) = (g_2^{-1/2} - 1)/t.$$

First observe that $g_2 \geqslant 1$, implies that $0 \geqslant \varphi'$. For $n \geqslant 3$ and $t \in I_n$ we have $2a_{n-1} - 1 \geqslant g_2(t)$, therefore $\varphi'(t) \geqslant -b_{n-1}/t$. Since $(b_n)_n$ is a positive decreasing sequence we actually have that $\varphi'(t) \geqslant -b_{n-1}/t$, for every $t \geqslant p_{n-1}$. Then for $n \geqslant 3$, and $t \geqslant p_{n+1}$ we get

$$\begin{aligned}
\varphi(t) - \varphi(p_2) &= \sum_{i=2}^{n-1} \int_{p_i}^{p_{i+1}} \varphi'(s)ds + \int_{p_n}^{t} \varphi'(s)ds \\
&\geqslant \sum_{i=2}^{n-1} \int_{p_i}^{p_{i+1}} -\frac{b_i}{s}ds + \int_{p_n}^{t} -\frac{b_n}{t}ds \\
&= -\sum_{i=2}^{n-1} b_i \log(p_{i+1}/p_i) - b_n \log(t) + b_n \log(p_n) \\
&\geqslant -b_n \log(p_{n+1}) - b_n \log(t) \\
&\geqslant -2b_n \log(t).
\end{aligned}$$

We will normalize φ so that $\varphi(p_2) = 0$. We observe that by making the p_i's even bigger we obtain that $\lim_{t\to\infty} \varphi(t) = -\infty$. We will make that assumption. Finally define $u : \mathbb{R}^+ \to \mathbb{R}$, by the equation

$$u(t) = \log(t) + \varphi(t).$$

By definition of φ' we have that $u' = \frac{1}{t} + \varphi'(t) = \frac{g_2^{-1/2}}{t}$, therefore u is surjective and strictly increasing (recall $g_2 \leqslant 2$). As commented at the beginning of this section,

a function with the properties of u determine a function $T = T(u)$ and therefore a metric g_T. We now pick the reference point $o = (\mathbf{0}, u(1))$ and a parabolic isometry such that $d(o, po) = 1$. We will now state a number of observations which will lead to the proof of Theorem 7.4.2.

Observation 1: The formula $K(u(t)) = -(g_2(t) - (t/2)g_2'(t))$, and property (3) in the definition of g_2 implies that $-2 \leqslant K \leqslant -1/3$. In particular the curvature of the metric g_T satisfies the pinching

$$-\frac{1}{9} \geqslant K_{g_T} \geqslant -4.$$

Observation 2: The hypothesis (4) in the definition of the sequence $(p_n)_n$, and the calculation of $J^*(t) - \frac{1}{2}t(J^*)'(t)$ on the intervals I_n gives us that

$$\lim_{t\to\infty} K(t) = -1.$$

For big enough t, $K(t)$ increases to -1. In particular the function U is going to zero as t goes to infinity (see Remark 7.4.5).

Observation 3: By property (4) in the definition of g_2 we know that

$$K(u(t)) \leqslant -a_{n+1},$$

for $t \leqslant p_{n+1}$. Combining this and Remark 7.4.5 we get that for every $t < u(p_{n+1})$, we have that $U(t) \leqslant -\frac{1}{n}$.

Observation 4: Using hypothesis (3) in the definition of the sequence $(p_n)_n$ and

the lower bound for $\varphi(t)$ we get

$$\begin{aligned}\log(k(n+1)) &< (1-2b_n)\log(p_{n+1}) \\ &< \log(p_{n+1}) + \varphi(p_{n+1}) = u(p_{n+1}).\end{aligned}$$

Lemma 7.5.2. *There exists $m \in \mathbb{N}$ such that for every $n \geqslant k(m)$ and $|k| \leqslant k(n+1)$, the function U is at most $-\frac{1}{n}$ on the geodesic segment $[o, p^k o]$.*

Proof. It follows from **Obs. 1** that if $|k| \leqslant k(n+1)$, then the height of $[o, p^k o]$ is at most $u(k(n+1)) + D$. For n big enough this is less than $\log(k(n+1))$. Using **Obs. 4** we conclude that if $|k| \leqslant k(n+1)$, then the height of $[o, p^k o]$ is at most $u(p_{n+1})$. Finally **Obs. 3** gives us that $U \leqslant -\frac{1}{n}$ on the geodesic segment $[o, p^k o]$ if $|k| \leqslant k(n+1)$. □

Lemma 7.5.3. *The critical exponent of $\mathcal{P} = \langle p \rangle$ is equal to $1/2$. Moreover, $\mathcal{P}$ is of divergence type.*

Proof. As explained in [DOP, Section 3], the Poincaré series of $\mathcal{P}$ for the metric g_T is equivalent to the series

$$\sum_{n \in \mathbb{Z}} \exp(-2su(|n|)).$$

Observe that for every $\epsilon > 0$, there exists a natural number N such that if $n \geqslant N$, then we have $\varphi(n) > -\epsilon \log(t)$. In particular

$$\sum_{n \geqslant N} \exp(-2su(|n|)) \leqslant \sum_{n \geqslant N} \exp(-2s(1-\epsilon)\log(|n|)).$$

If $s(1-\epsilon) > 1/2$, then the right hand side converges. This implies that for every $\epsilon > 0$ the following inequality holds: $\delta_{\mathcal{P}} \leqslant \frac{1}{2(1-\epsilon)}$. On the other hand for big n we

have $u(n) < \log(n)$. Then

$$\sum_{n>N'} \exp(-2s\log(|n|)) < \sum_{n>N'} \exp(-2su(|n|)).$$

Since the critical exponent of the LHS is $1/2$ (and of divergence type) we get the inequality $\delta_{\mathcal{P}} \geqslant 1/2$. We conclude that $\delta_{\mathcal{P}} = 1/2$, and $\mathcal{P}$ is of divergence type.

□

Lemma 7.5.4. *The pair $(\mathcal{P}, U)$ is of convergence type with respect to the metric g_T.*

Proof. We denote by d_T the distance function induced by g_T. Combining Lemma 7.5.2 and the definition of $k(n)$ we get the bounds

$$\begin{aligned}
&\sum_{|k|\geqslant k(m)} \exp\left(\int_o^{p^k o} U - \frac{1}{2} d_T(o, p^k o)\right) \\
&= \sum_{n\geqslant m} \sum_{k=k(n)}^{k(n+1)-1} \exp\left(\int_o^{p^k o} U - \frac{1}{2} d_T(o, p^k o)\right) \\
&\leqslant \sum_{n\geqslant m} \sum_{k=k(n)}^{k(n+1)-1} \exp\left(-\left(\frac{1}{n} + \frac{1}{2}\right) d_T(o, p^k o)\right) \\
&\leqslant \sum_{n\geqslant m} \sum_{k=k(n)}^{k(n+1)-1} \exp\left(-\left(\frac{1}{n} + \frac{1}{2}\right) (2u(|k|))\right) \exp\left(\left(\frac{1}{n} + \frac{1}{2}\right) D\right) \\
&\leqslant \sum_{k=1} \frac{1}{k^2}.
\end{aligned}$$

Obs. 2 and Lemma 7.3.3 implies that $\delta_{\mathcal{P}} = \delta_{\mathcal{P}}^U$ and then by Lemma 7.5.3 we know that the series above is exactly the Poincaré series associated to $(\mathcal{P}, U)$. This finishes the proof of the lemma. □

7.5.2 Construction of the family $\{M_{n,m}\}$

We have now all the ingredients to construct the Riemannian manifold announced in Theorem 1.0.2. We will use the notation introduced at the beginning of this section.

We start with $(\mathbb{R}^N, g_{hyp})$, where g_{hyp} is the hyperbolic metric, and the function u constructed in Section 7.5.1. We choose a hyperbolic isometry h (for the hyperbolic metric) so that $H = \langle h \rangle$ is in Schottky position with respect to $\mathcal{P} = \langle p \rangle$. We moreover assume that the axis of h has height smaller that $u(1/2)$. Define Γ as the group generated by p and h and let $M = \mathbb{R}^N/\Gamma$. The closed geodesic associated to h is denoted by $\gamma = \gamma_h$. We can 'cut' the cusp associated to $\mathcal{P}$ above height $u(1/2)$ and replace it by the cusp endowed with the metric g_T. This is possible because g_T is the hyperbolic metric on the region $\{(\mathbf{x}, t) : t < u(1)\}$. We have constructed a new Riemannian metric g on M. We lift the metric to the universal cover; this is our new Hadamard manifold $(\widetilde{M}, g)$. We will check that the Riemannian manifold (M, g) satisfies the properties announced in Theorem 1.0.2. Since the geometric structure has change we will be careful with our notation. The generator of the parabolic subgroup of $Iso(\widetilde{M}, g)$ corresponding to the cusp is denoted by p_* and h_* is the hyperbolic isometry associated to the closed geodesic γ in M. The group generated by p_* is denoted by $\mathcal{P}_*$ and the group generated by h_* is denoted by H_*. The geometric potential of M is denoted by F^{su}, and its normalization by U (see Definition 7.5.1). As before, we will organize the relevant information in a couple of observations and lemmas. It will be convenient to define $Q = -U$. We start with the following definition.

Definition 7.5.5. *We denote by $\Gamma_{n,m}$ the group generated by h_*^n and p_*^m. Let $M_{n,m}$ be the covering of M associated to the subgroup $\Gamma_{n,m}$ of Γ. The lift of a potential G on T^1M to $T^1M_{n,m}$ is denoted by $G_{n,m}$.*

Observation 5: The closed geodesic γ lies in the region where g is hyperbolic. This implies that U and Q vanish along γ.

Observation 6: Since F^{su} is locally defined in terms of g and every local structure

is preserved under taking coverings, we conclude that $F^{su}_{n,m}$ is the geometric potential of $M_{n,m}$.

Observation 7: The potential Q vanishes at infinity and it is positive in a neighborhood of the cusp associated to $\mathcal{P}_*$. This follows directly from **Obs. 2**.

Lemma 7.5.6. *The pair $(\mathcal{P}_*, -tQ)$ is of convergence type for every $t \geqslant 1$.*

Proof. The reference point used in the proof of Lemma 7.5.4 lies in the piece of M coming from $(\mathbb{R}^N, g_T)$. Since horoballs are convex it follows that $d_T = d$ on that region. This implies that the series estimated in Lemma 7.5.4 is exactly the Poincaré series associated to $\mathcal{P}_*$, which implies that $(\mathcal{P}_*, -Q)$ is of convergence type. By the construction of g_2 we know that Q is positive above height $u(2)$ (see **Obs. 7**). Change the reference point to o', a point with height $u(2)$. By convexity of this region and the definition of g_2 we get that $-t\int_{o'}^{p^k o'} Q \leqslant -\int_{o'}^{p^k o'} Q$, for every $t \geqslant 1$. Plugging this into the Poincaré series of $-tF$ and $-F$ implies the lemma. □

The idea now is to apply Theorem 7.4.12 to the potential $Q = -U$. Lemma 7.5.6 and **Obs. 7** implies that Q belongs to the class $\mathcal{F}_s$. Despite that Q is not non-negative, by **Obs. 5** and the discussion at the end of Section 7.4.3, we know that F_n exhibits a phase transition for sufficiently big n. This is not a very satisfactory answer, one would like to know which type of phase transition we encountered, either type A or type B phase transition. By the construction of the modified metric at the cusp, a phase transition of type B seems unlikely to occur. Since we are not able to completely rule out that case, we present an argument to justify that a type A phase transition is always possible to construct. As before, we denote by $\widetilde{Q}$ to the lift of Q to the universal cover of M. Denote by $\xi \in \partial_\infty \widetilde{M}$ to the parabolic fixed point of $\mathcal{P}_*$. By construction of g_2 we know that there exist real numbers s and L such that $\widetilde{Q}$ is negative precisely at vectors whose base lies in the interior of $B_\xi(s)\backslash B_\xi(s+L)$ and it

is zero for vectors with base in $M \backslash B_\xi(s)$. It follows easily from this fact that there exists m_0 such that if $m \geqslant m_0$, then

$$\int_o^{g.o} \widetilde{Q} > 0, \tag{7.4}$$

for every $g \in \Gamma_{1,m}$, in particular for every $g \in \Gamma_{n,m}$. It follows from the definition of the Poincaré series and the fact that Q goes to zero through the cusp of M (see **Obs. 2**), that if the pair $(\Gamma_{n,1}, -Q_{n,1})$ is of convergence type with $\delta_{\Gamma_{n,1}}^{-Q_{n,1}} = \delta_{\mathcal{P}}$, then the same holds for $(\Gamma_{n,m}, -Q_{n,m})$. Moreover, if $m \geqslant m_0$, then inequality (7.4) and the definition of the Poincaré series implies that if $(\Gamma_{n,m}, -Q_{n,m})$ is of convergence type with $\delta_{\Gamma_{n,m}}^{-Q_{n,m}} = \delta_{\mathcal{P}}$, then the same holds for $(\Gamma_{n,m}, -tQ_{n,m})$, for every $t \geqslant 1$. Combining **Obs. 6** and the discussion above we conclude that $Q_{n,m}$ admits a phase transition of type A for big enough n and m.

7.5.3 Conclusions

In Section 7.5.2 we constructed a family of geometrically finite negatively curved manifolds $M_{n,m}$ (see Definition 7.5.5) for which $Q_{n,m}$ exhibits a phase transition of type A if n and m are big enough. Since

$$tQ_{n,m} = -tF_{n,m}^{su} - t(N-1),$$

the existence of an equilibrium state for $tQ_{n,m}$ is equivalent to the existence of one for $-tF_{n,m}^{su}$. Just as in Theorem 7.4.12, there exists $t_{n,m} \in [-1, 0)$ such that $tQ_{n,m}$ has an equilibrium state for all $t > t_{n,m}$, and there is not equilibrium state for $t < t_{n,m}$. We remark that $F_{n,m}^{su}$ does not admit an equilibrium state since $(\Gamma_{n,m}, -Q_{n,m}) = (\Gamma, F_{n,m}^{su} + (N-1))$ is of convergence type. It is clear from the construction that $M_{n,m}$ are extended Schottky manifolds. By Lemma 7.5.3 we know that $\mathcal{P}_*$ and $\langle p_*^n \rangle$ are of divergence type, then using Theorem 2.3.15 we obtain $\delta_{\langle p_*^n \rangle} < \delta_{\Gamma_{n,m}}$. We conclude

that the geodesic flow on $M_{n,m}$ has a measure of maximal entropy. All this together gives us the following result.

Theorem 7.5.7. *Each $M_{n,m}$ is an extended Schottky manifold and $F^{su}_{n,m}$ is the geometric potential of $M_{n,m}$. Suppose that n and m are sufficiently large. Then there exists $s_{n,m} \in (0,1]$ such that the following holds:*

1. *$tF^{su}_{n,m}$ has an equilibrium state for $t < s_{n,m}$.*
2. *$tF^{su}_{n,m}$ does not has an equilibrium state for $t > s_{n,m}$.*
3. *The pressure map is linear in $(s_{n,m}, \infty)$.*

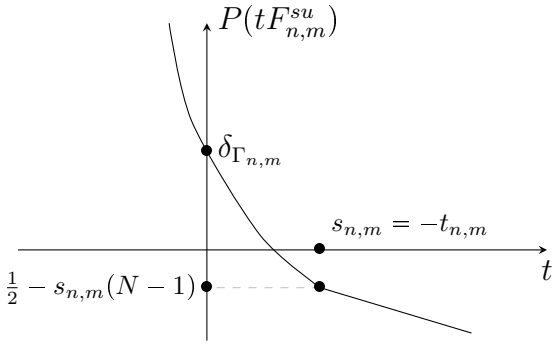


Figure 4: Phase transition for $P(tF^{su}_{n,m})$

As mentioned in Remark 7.4.8 we could have done the same construction for an extended Schottky manifold with one parabolic and arbitrary number of hyperbolic generators (modulo taking big powers of the parabolic and hyperbolic generators). Combining Remark 7.4.16 and the proof of Theorem 7.5.7 we obtain the following result.

Corollary 7.5.8. *Let M be a k-punctured sphere with $k \geqslant 3$. We can endow M with a complete Riemannian metric with pinched negative sectional curvature, such that the geometric potential exhibits a phase transition of type A. Moreover, the geodesic flow of M has an unique measure of maximal entropy, and the Riemannian metric is hyperbolic outside a neighborhood of one of the punctures.*

Chapter 8

Large deviations

In this section we will obtain large deviation estimates for the geodesic flow on negatively curved manifolds. Although this section is initially motivated by the work of Pollicott in [Pol], our goal will be to follow closely the paper of Kifer [Kif] (which is also the main input in [Pol]), and to check that his strategy extends to the non-compact case when suitable assumptions are satisfied. We will see that for the large deviation upper bound we do not need strong assumptions on the manifold, nor the potential. In contrast, to obtain large deviation lower bounds we will restrict to geometrically finite manifolds. This is not an optimal assumption, we expect that the results in that Section can be extended to the entire class of negatively curved manifolds (with analog assumptions to the ones we will require in the geometrically finite case). We would also like to point out that for our lower bound we use 'very strongly positive recurrent potentials' (vSPR for short), which is a sub-class of SPR potentials (for a precise definition see Definition 7.3.2). This is necessary in order to ensure that any compact Hölder perturbation of our potential is still SPR. For geometrically finite manifolds a convenient definition of vSPR was already familiar to the authors. In the general case there is still some work to do (we need a more general version than Theorem 2.3.15), this is the main reason why we restrict to geometrically finite

manifolds.

We will start by collecting some results that will be relevant for this section. Recall that the *Gurevich pressure* of a Hölder potential F is defined as

$$P_{Gur}(F) = \limsup_{t\to\infty} \frac{1}{t} \log \sum_{g\in\mathcal{M}_p(W,t,t-c)} \exp(l(g) \int F dg),$$

where W is a relatively compact open subset of T^1M intersecting the non-wandering set of the geodesic flow and c any positive real number (for a more precise description see Definition 2.3.13). It is proven in [PPS, Section 4.1, Section 4.3] that this quantity is independent of W and the real number c. Moreover, it is proven that if c is large enough, then

$$P_{Gur}(F) = \lim_{t\to\infty} \frac{1}{t} \log \sum_{g\in\mathcal{M}_p(W,t,t-c)} \exp(l(g) \int F dg).$$

If one follows the proof of this last property one can verifies that the constant c depends on the geometry of M and W, but it is independent of the Hölder potential F. As mentioned in Section 2, for every Hölder potential F we have

$$P_{Gur}(F) = P(F).$$

We will summarize all this information in the following proposition.

Proposition 8.0.1. *Let W be a relatively compact open subset of T^1M intersecting the non-wandering set of the geodesic flow. Then there exists a constant $c(W) > 0$ sufficiently large such that*

$$P(F) = P_{Gur}(F) = \lim_{t\to\infty} \frac{1}{t} \log \sum_{g\in\mathcal{M}_p(W,t,t-c)} \exp(l(g) \int F dg),$$

for every Hölder potential F.

The following two definitions are the starting point of all the results in this section.

We remark that in the non-compact case the choice of domain for the G-pressure and the rate function might not be canonical. For convenience we defined the rate function in the space of sub-probability measures $\mathcal{M}_{\leqslant 1}(g)$, once this decision is made the domain of Q_G must be contained in $C_0(T^1M)$ (to ensure duality). There are a couple of reasons that justify this choice. We will see that the compactness of the domain of I_G is important to prove the lower semicontinuity of the rate function I_n (to be defined in the proof of Theorem 8.2.2), which is a fundamental fact in order to obtain the lower bound large deviation estimate. Another reason is that in the upper bound estimate we want to use arbitrary closed subsets, instead of compact subsets (by a standard Contraction Principle we could bypassed this issue if the sequence $(\mathcal{M}(W, n, n-c))_n$ is exponentially tight, but we prefer not to use this approach, see [DZ, Theorem 4.2.1]). The upper bound for arbitrary closed subsets is also used to derive the lower bound.

Definition 8.0.2 (G-pressure)**.** *Given $G \in C_0(T^1M)$ we define $Q_G : C_0(T^1M) \to \mathbb{R}$, by the formula*

$$Q_G(f) = P(G + f) - P(G).$$

Definition 8.0.3 (Rate function)**.** *Given $\mu \in \mathcal{M}_{\leqslant 1}(g)$ we define*

$$I_G(\mu) = \sup_{f \in C_0(T^1M)} \big\{ \int f d\mu - Q_G(f) \big\}.$$

Observe that by definition (take $f = 0$) we know $I_G(\mu) \geqslant 0$. If the choice of function G is clear we will use the notation Q (resp. I) instead of Q_G (resp. I_G).

Lemma 8.0.4. *The fuction I_G is lower semicontinuous.*

Proof. We want to prove that the set $I_G^{-1}((t, \infty])$ is open. Pick a measure $\mu \in \mathcal{M}_{\leqslant 1}(g)$ such that $I_G(\mu) > t$. This implies that there exists $f \in C_0(T^1M)$ such that $\int f d\mu - Q_G(f) > t$. By the definition of the vague topology the map $\mu \mapsto \int f d\mu - Q_G(f)$, is

continuous. In particular the inequality $\int f d\nu - Q_G(f) > t$, holds for measures ν in a neighorhood of μ. This implies that $I_G(\nu) > t$, for every ν in this neighborhood. $\square$

The following proposition follows from the upper semicontinuity of the entropy map (Theorem 5.2.6).

Proposition 8.0.5. *For every $\mu \in \mathcal{M}(g)$ the following formula holds*

$$h_\mu(g) = \inf_{f \in C_c(T^1M)} \{P(f) - \int f d\mu\}.$$

Remark 8.0.6. *Observe that by definition of the topological pressure we have $h_\mu(g) + \int f d\mu \leqslant P(f)$. In particular*

$$h_\mu(g) \leqslant \inf_{f \in C(T^1M)} \{P(f) - \int f d\mu\}.$$

Proposition 8.0.5 gives the lower bound $h_\mu(g) \geqslant \inf_{f \in C_c(T^1M)} \{P(f) - \int f d\mu\}$. In particular $h_\mu(g) \geqslant \inf_{f \in C_0(T^1M)} \{P(f) - \int f d\mu\}$. We conclude that the formula in Proposition 8.0.5 holds when replacing $C_c(T^1M)$ by $C_0(T^1M)$.

Lemma 8.0.7. *Let $G \in C_0(T^1M)$. Then for every $\mu \in \mathcal{M}(g)$ the following formula holds*

$$I_G(\mu) = P(G) - (h_\mu(g_1) + \int G d\mu).$$

Proof. Observe that

$$\begin{aligned} I_G(\mu) = & \sup_{f \in C_0(T^1M)} \{ \int (G+f) d\mu - P(G+f)\} + P(G) - \int G d\mu \\ = & - \inf_{f \in C_0(T^1M)} \{P(G+f) - \int (G+f) d\mu\} + P(G) - \int G d\mu \\ = & - \inf_{g \in C_0(T^1M)} \{P(g) - \int g d\mu\} + P(G) - \int G d\mu \\ = & P(G) - (h_\mu(g) + \int G d\mu). \end{aligned}$$

□

8.1 Upper bound for closed sets

The following result follows from [Pol, Theorem 1 (i)] without modification (also see [Kif]). For completeness we include its proof.

Theorem 8.1.1. *Let G be a Hölder potential in $C_0(T^1M)$. Let $\mathcal{K}$ be a closed subset of $\mathcal{M}_{\leqslant 1}(g)$. Then we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)\cap\mathcal{K}} \exp(l(\tau)\int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)} \right) \leqslant -\beta,$$

where $\beta = \inf_{\mu\in\mathcal{K}} I_G(\mu)$.

We emphasize that in Theorem 8.1.1 the constant c is the one from Proposition 8.0.1.

Proof. From now on we will use the notation $I = I_G$, and $Q = Q_G$. We remark that since $K \subset \mathcal{M}_{\leqslant 1}(g)$ is closed, it is immediately compact. For every $\epsilon > 0$ we have

$$\mathcal{K} \subset \{\mu \in \mathcal{M}_{\leqslant 1}(g) : I(\mu) > \beta - \epsilon\}.$$

By the definition of $I(\mu)$ we have that

$$\{\mu \in \mathcal{M}_{\leqslant 1}(g) : I(\mu) > \beta - \epsilon\} = \bigcup_{f\in C_0(T^1M)} \{\mu \in \mathcal{M}_{\leqslant 1}(g) : \int f d\mu - Q(f) > \beta - \epsilon\}.$$

Define $\mathcal{V}_f = \{\mu \in \mathcal{M}_{\leqslant 1}(g) : \int f d\mu - Q(f) > \beta - \epsilon\}$, and observe that $\mathcal{V}_f$ is open in the vague topology (since $f \in C_0(T^1M)$). By the compactness of $\mathcal{K}$ we obtain a finite subcover $K \subset \bigcup_{i=1}^N \mathcal{V}_{f_i}$. Define $\mathcal{U}_{f_i} = \mathcal{V}_{f_i} \cap \mathcal{M}_p(W,t,t-c) \cap \mathcal{K}$. We have the following inequality

$$\frac{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)\cap\mathcal{K}} \exp(l(\tau)\int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)} \leqslant \sum_{i=1}^{N} \frac{\sum_{\mu_\tau \in \mathcal{U}_{f_i}} \exp(l(\tau)\int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)}$$
$$= \sum_{i=1}^{N} \frac{\sum_{\mu_\tau \in \mathcal{U}_{f_i}} \exp(l(\tau)\int (G+f_i) d\mu_\tau) e^{-l(\tau)\int f_i d\mu_\tau}}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)}$$
$$\leqslant \sum_{i=1}^{N} \frac{\sum_{\mu_\tau \in \mathcal{U}_{f_i}} \exp(l(\tau)\int (G+f_i) d\mu_\tau) e^{-l(\tau)(Q(f_i)+\beta-\epsilon)}}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)}$$

We will bound the RHS according to the sign of the term $Q(f_i)+\beta-\epsilon$. If $Q(f_i)+\beta-\epsilon \geqslant 0$, we use the inequality $e^{-l(\tau)(Q(f_i)+\beta-\epsilon)} \leqslant e^{-(t-c)(Q(f_i)+\beta-\epsilon)}$, and if $Q(f_i)+\beta-\epsilon < 0$, we use the inequality $e^{-l(\tau)(Q(f_i)+\beta-\epsilon)} \leqslant e^{-t(Q(f_i)+\beta-\epsilon)}$. In any case we get that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{U}_{f_i}} \exp(l(\tau)\int (G+f_i) d\mu_\tau) e^{-l(\tau)(Q(f_i)+\beta-\epsilon)}}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)} \right)$$
$$\leqslant -(Q(f_i)+\beta-\epsilon) + P(G+f_i) - P(G) = -\beta+\epsilon.$$

Here we used the fact that $\lim_{t\to\infty} \frac{1}{t} \log \sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)$, is well defined and equal to $P(G)$. The inequality

$$\limsup_{t\to\infty} \frac{1}{t} \log(\sum_{i=1}^{N} a_i(t)) \leqslant \max_{i\in\{1,..,N\}} \limsup_{t\to\infty} \frac{1}{t} \log a_i(t),$$

implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)\cap\mathcal{K}} \exp(l(\tau)\int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)} \right) \leqslant -\beta+\epsilon,$$

but $\epsilon > 0$ was arbitrary.

□

8.2 Lower bound for open sets

For the lower bound in our large deviation estimate we will restrict to the class of geometrically finite manifolds. As mentioned in the introduction of this section this is a technical assumption that is likely to be removable. The ingredient that would allows us to push this result to general negatively curved manifolds is a more general version of Theorem 2.3.15.

Remark 8.2.1. *When we refer to geometrically finite manifolds, we essentially mean geometrically finite manifolds with cusps. If M is geometrically finite without cusps, then the non-wandering set of the geodesic flow is compact (M is convex cocompact). In this case most of our results are well known (the geodesic flow can be modelled as a suspension flow over a shift of finite type).*

For our large deviation lower bound we will use vSPR potentials. The main feature of this class of potentials is that any compact Hölder perturbation will remain strongly positive recurrent (Lemma 7.3.5). We know that the strongly positive recurrent condition is open in the $\mathcal{C}^0$ topology (Lemma 7.2.2), but this is apparently not enough to obtain Theorem 8.2.2.

Theorem 8.2.2. *Let M be a geometrically finite manifold with cusps and G a vSPR potential. Let $\mathcal{U}$ be an open subset of $\mathcal{M}_{\leqslant 1}(g)$. Then we have*

$$\liminf_{t\to\infty} \frac{1}{t} \log\left(\frac{\sum_{\mu_\tau \in \mathcal{U}\cap \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau)\int G d\mu_\tau)} \right) \geqslant -\beta,$$

where $\beta = \inf_{\mu\in\mathcal{U}} I_G(\mu)$.

Proof. We will use the simplified notation $I = I_G$, and $Q = Q_G$. Let $(V_k)_k$ be a sequence of Hölder continuous functions in $C_c(T^1M)$ such that $||V_i||_0 = 1$, and whose

linear combinations form a dense subset of $C_0(T^1M)$. The metric

$$\rho(\mu,\nu)=\sum_{k\geqslant 1}2^{-k}|\int V_k d\mu-\int V_k d\nu|, \tag{8.1}$$

generates de vague topology of $\mathcal{M}_{\leqslant 1}(g)$. Define $f_n:\mathcal{M}(g)\to\mathbb{R}^n$, given by

$$f_n(\mu)=(\int V_1 d\mu,...,\int V_n d\mu).$$

We also define functions $Q_n, I_n:\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$, by the formulas

$$Q_n(\beta)=Q(\sum_{i=1}^{n}\beta_i V_i),$$

and

$$I_n(\alpha)=\begin{cases}\inf\{I(\mu):f_n\mu=\alpha\},\ \text{if } \alpha\in f_n\mathcal{M}_{\leqslant 1}(g)\\ \infty,\ \text{otherwise.}\end{cases}$$

We claim that $I_n(\alpha)$ is lower semicontinuous. More precisely, we want to prove that $I_n^{-1}((t,\infty])$ is open. Observe that

$$I_n^{-1}((t,\infty])=\{\alpha:t<I_n(\alpha)\}=\{\alpha:I(\mu)>t,\forall\mu \text{ such that } f_n\mu=\alpha\}.$$

Suppose by contradiction that $I_n^{-1}((t,\infty))$ is not open. Take $\alpha\in I_n^{-1}((t,\infty])$ that is not an interior point. There exists a sequence $(\mu_n)_n$ of measures in $\mathcal{M}_{\leqslant 1}(g)$ such that $I(\mu_n)\leqslant t$, and $f_n\mu\to\alpha$. After passing to a subsequence (because of the compactess of $\mathcal{M}_{\leqslant 1}(g)$) we can assume that μ_n converges to $\mu\in\mathcal{M}_{\leqslant 1}(g)$. Since I is lower semicontinuous we know that $I(\mu)\leqslant t$. The continuity of f_n implies that $f_n\mu=\alpha$. This contradicts the fact that $\alpha\in I_n^{-1}((t,\infty])$.

Remark 8.2.3. *In the proof of the semicontinuity of I_n we used the compactness of $\mathcal{M}_{\leqslant 1}(g)$. This is one of the reasons why we needed to define I in a non-trivial way*

for the invariant sub-probability measures.

It follows from the definition of Q_n that

$$Q_n(\beta) = \sup_{\alpha \in \mathbb{R}^n} (\langle \beta, \alpha \rangle - I_n(\alpha)).$$

It is also easy to check the convexity of I_n ([Kif, Theorem 2.1]). The convexity and lower semicontinuity of I_n implies by a standard duality principle ([Ro, Theorem 12.2]) that

$$I_n(\alpha) = \sup_{\beta \in \mathbb{R}^n} (\langle \alpha, \beta \rangle - Q_n(\beta)).$$

Let $J_n(A) = \inf_{\alpha \in A} I_n(\alpha)$. As in [Kif] we will prove the following lemma.

Lemma 8.2.4. *Let U be an open subset of $\mathbb{R}^n$. Then*

$$\liminf_{t \to \infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c) \cap f_n^{-1} U} \exp(l(\tau) \int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau) \int G d\mu_\tau)} \right) \geqslant -J_n(U). \tag{8.2}$$

Proof. We will assume that $J_n(U) < \infty$, otherwise there is nothing to prove. Fix $\epsilon > 0$, and let $\alpha_\epsilon \in U \subset \mathbb{R}^n$ such that $I_n(\alpha_\epsilon) < J_n(U) + \epsilon$. As in [Kif] we can ensure the existence of $\beta_\epsilon \in \mathbb{R}^n$ such that

$$Q_n(\beta_\epsilon) = \langle \beta_\epsilon, \alpha_\epsilon \rangle - I_n(\alpha_\epsilon).$$

Choose r sufficiently small such that $B_r(\alpha_\epsilon) = \{\alpha : |\alpha - \alpha_\epsilon| < r\}$ is contained in U. Observe that $f_n^{-1} U \supset f_n^{-1} B_r(\alpha_\epsilon)$, therefore the LHS in inequality (8.2) is bounded below by

$$\liminf_{t \to \infty} \frac{1}{t} \log \left(\frac{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c) \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int G d\mu_\tau)}{\sum_{\mu_\tau \in \mathcal{M}_p(W,t,t-c)} \exp(l(\tau) \int G d\mu_\tau)} \right).$$

To simplify notation define $A_t = \mathcal{M}_p(W, t, t-c)$, and $B_t = \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int G d\mu_\tau)}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int G d\mu_\tau)}$.

Observe that

$$B_t = C_t \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau) e^{-l(\tau)\langle \beta_\epsilon, f_n \mu_\tau - \alpha_\epsilon \rangle} e^{-l(\tau)\langle \beta_\epsilon, \alpha_\epsilon \rangle}}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau},$$

where $V^n = (V_1, ..., V_n)$ and

$$C_t = \frac{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int G d\mu_\tau}.$$

It follows from the definition of the Gurevich pressure that

$$\lim_{t \to \infty} \frac{1}{t} \log C_t = Q_G(\langle \beta_\epsilon, V^n \rangle) = Q_n(\beta_\epsilon).$$

Using the definition of A_t and that for $\mu_\tau \in f_n^{-1} B_r(\alpha_\epsilon)$ we have $|f_n \mu_\tau - \alpha_\epsilon| < r$, we obtain

$$B_t \geqslant C_t \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau)}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau} e^{-tr|\beta_\epsilon|} e^{(-t\langle \beta_\epsilon, \alpha_\epsilon \rangle},$$

here we assumed that $\langle \beta_\epsilon, \alpha_\epsilon \rangle \geqslant 0$, otherwise the term $e^{-t\langle \beta_\epsilon, \alpha_\epsilon \rangle}$ should be replaced by $e^{-(t-c)\langle \beta_\epsilon, \alpha_\epsilon \rangle}$. In any case we derive the inequality

$$\begin{aligned} \liminf_{t \to \infty} \frac{1}{t} \log B_t &\geqslant \liminf_{t \to \infty} \frac{1}{t} \log \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau)}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau} \\ &\qquad + Q_n(\beta_\epsilon) - r|\beta_\epsilon| - \langle \beta_\epsilon, \alpha_\epsilon \rangle \\ &\geqslant \liminf_{t \to \infty} \frac{1}{t} \log \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau)}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau} \\ &\qquad - I_n(\alpha_\epsilon) - r|\beta_\epsilon|. \end{aligned}$$

In the last inequality we used that by definition of β_ϵ we have $Q_n(\beta_\epsilon) + I_n(\alpha_n) =$

$\langle \beta_\epsilon, \alpha_\epsilon \rangle$. Define

$$D_t = \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} B_r(\alpha_\epsilon)} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau)}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau},$$

and

$$E_t = \frac{\sum_{\mu_\tau \in A_t \cap f_n^{-1} (B_r(\alpha_\epsilon)^c)} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau)}{\sum_{\mu_\tau \in A_t} \exp(l(\tau) \int (G + \langle \beta_\epsilon, V^n \rangle) d\mu_\tau}.$$

By definition $D_t + E_t = 1$. We claim that $\lim_{t\to\infty} D_t = 1$, or equivalently that $\lim_{t\to\infty} E_t = 0$. This will immediately imply that $\liminf_{t\to\infty} \frac{1}{t} \log D_t = 0$, and therefore the inequality

$$\liminf_{t\to\infty} \frac{1}{t} \log B_t \geqslant -I_n(\alpha) - r|\beta_\epsilon|.$$

Since r was arbitrary we are done. It remains to prove that $\lim_{t\to\infty} E_t = 0$. First observe that $f_n^{-1}(B_n(\alpha_\epsilon)^c)$ is a closed subset of $\mathcal{M}_{\leqslant 1}(g)$. We can use Theorem 8.1.1 to obtain the inequality

$$\limsup_{t\to\infty} \frac{1}{t} \log E_t \leqslant - \inf_{\mu \in f_n^{-1}(B_r(\alpha_\epsilon)^c)} I_{G+\langle \beta_\epsilon, V^n \rangle}(\mu).$$

It is enough to prove that $\inf_{\mu \in f_n^{-1}(B_r(\alpha_\epsilon)^c)} I_{G+\langle \beta_\epsilon, V^n \rangle}(\mu) > 0$.

Lemma 8.2.5. *Let $H \in C_0(T^1M)$ be a Hölder SPR potential. Then the only measure $\mu \in \mathcal{M}_{\leqslant 1}(g)$ satisfying the equation*

$$I_H(\mu) = 0,$$

is the equilibrium state of H.

Proof. Recall that $I_H(\mu) \geqslant 0$, for every $\mu \in \mathcal{M}_{\leqslant 1}(g)$. If $I_H(\mu) = 0$, we must have

$$\int f d\mu - P(H + f) + P(H) \leqslant 0,$$

for every $f \in C_0(T^1M)$. In particular we have $t\int f d\mu \leqslant P(H+tf) - P(H)$. Suppose $t > 0$, we divide by t and take $t \to 0$. By Theorem 6.1.9 we obtain $\int f d\mu \leqslant \int f dm_H$, where m_H is the equilibrium state of H. Since f was an arbitrary function in $C_0(T^1M)$ this concludes that necessarily $\mu = m_H$. □

Remark 8.2.6. *If we assume that $H \in C_0(T^1M)$ is not strongly positive recurrent, then*

$$I_H(\mu_0) = \sup_{f \in C_0(T^1M)} \{P(H) - P(H+f)\} = P(H) - \inf_{f \in C_0(T^1M)} P(H+f),$$

where μ_0 is the zero measure. As an application of Theorem 7.1.2 we know that $\inf_{f \in C_0(T^1M)} P(H+f) = \inf_{g \in C_0(T^1M)} P(g) = \delta_\infty$. Since H is not strongly positive recurrent we get that $I_H(\mu_0) = 0$. In this case the conclusion of Lemma 8.2.5 does not hold.

Suppose by contradiction that $\inf_{\mu \in f_n^{-1}(B_r(\alpha_\epsilon)^c)} I_{G+\langle \beta_\epsilon, V^n \rangle}(\mu) = 0$. This would imply the existence of a sequence $(\mu_k)_k \subset \mathcal{M}_{\leqslant 1}(g)$ such that $|f_n \mu_k - \alpha_\epsilon| \geqslant r$, and that $\lim_{k\to\infty} I_{G+\langle \beta_\epsilon, V^n \rangle}(\mu_k) = 0$. By compactness of $\mathcal{M}_{\leqslant 1}(g)$ we conclude that there exists a measure $\mu \in \mathcal{M}_{\leqslant 1}(g)$ such that $I(\mu) = 0$ (by the lower semicontinuity of I), and $|f_n \mu - \alpha_\epsilon| \geqslant r$. By Lemma 8.2.5 we know that μ must be the equilibrium state of $G + \langle \beta_\epsilon, V^n \rangle$. We will now prove that $f_n m_{\beta_\epsilon} = \alpha_\epsilon$, where $\mu = m_{\beta_\epsilon}$ is the equilibrium state of $G + \langle \beta_\epsilon, V^n \rangle$. This would imply that the inequality $|f_n \mu - \alpha_\epsilon| \geqslant r$, is not possible and therefore that the assumption $\inf_{\mu \in f_n^{-1}(B_r(\alpha_\epsilon)^c)} I_{G+\langle \beta_\epsilon, V^n \rangle}(\mu) = 0$, is false.

We start with the following observation

$$\begin{aligned} I_{G+\langle\beta_\epsilon,V^n\rangle}(\mu) &= \sup_{f\in C_0(T^1M)} \Big\{ \int f d\mu - P(G+\langle\beta_\epsilon,V^n\rangle + f)\Big\} + P(G+\langle\beta_\epsilon,V^n\rangle) \\ &= \sup_{g\in C_0(T^1M)} \Big\{ \int g d\mu - P(G+g)\Big\} + P(G+\langle\beta_\epsilon,V^n\rangle) - \int \langle\beta_\epsilon,V^n\rangle d\mu \\ &= I_G(\mu) - P(G) + P(G+\langle\beta_\epsilon,V^n\rangle) - \int \langle\beta_\epsilon,V^n\rangle d\mu \\ &= I_G(\mu) + Q_n(\beta_\epsilon) - \int \langle\beta_\epsilon,V^n\rangle d\mu. \end{aligned}$$

We conclude that

$$Q_n(\beta_\epsilon) = \int \langle\beta_\epsilon,V^n\rangle dm_{\beta_\epsilon} - I_G(m_{\beta_\epsilon}).$$

Define $\widehat{\alpha} = (\int V_1 dm_{\beta_\epsilon}, ..., \int V_n dm_{\beta_\epsilon})$. By definition $f_n m_{\beta_\epsilon} = \widehat{\alpha}$, and therefore

$$Q_n(\beta_\epsilon) = \langle\widehat{\alpha},\beta_\epsilon\rangle - I_G(m_{\beta_\epsilon}) \leqslant \langle\widehat{\alpha},\beta_\epsilon\rangle - I_n(\widehat{\alpha}) \leqslant Q_n(\beta_\epsilon). \tag{8.3}$$

We conclude that $Q_n(\beta_\epsilon) = \langle\widehat{\alpha},\beta_\epsilon\rangle - I_n(\widehat{\alpha})$. Recall that by construction we had $Q_n(\beta_\epsilon) = \langle\alpha_\epsilon,\beta_\epsilon\rangle - I_n(\alpha_\epsilon)$. By standard convex analysis we know that the set

$$\{\alpha \in \mathbb{R}^n : I_n(\alpha) = \langle\alpha,\beta_\epsilon\rangle - Q_n(\beta_\epsilon)\},$$

corresponds to the set $\partial Q_n(\beta_\epsilon)$ of subdifferentials of Q_n at β_ϵ. Recall that a vector $v \in \mathbb{R}^n$ is a subdifferential of Q_n at β, if for every $w \in \mathbb{R}^n$ the following inequality holds

$$\langle v, w\rangle \leqslant Q_n(\beta + w) - Q_n(\beta).$$

Note that if $v \in \partial Q_n(\beta_\epsilon)$, then

$$t\langle v, w\rangle \leqslant P(G+\langle\beta + tw, V^n\rangle) - P(G+\langle\beta, V^n\rangle).$$

In particular we get that for every $w \in \mathbb{R}^n$ we have

$$\langle v, w \rangle \leqslant \frac{d}{dt}_{|t=0} P(G + \langle \beta + tw, V^n \rangle) = \int \langle w, V^n \rangle dm_{\beta_\epsilon} = \langle w, \widehat{\alpha} \rangle.$$

This implies that $v = \widehat{\alpha}$. We conclude that $\alpha_\epsilon = \widehat{\alpha} = f_n m_{\beta_\epsilon}$, as desired. This finishes the proof of Lemma 8.2.4. □

We now proceed to finish the proof of Theorem 8.2.2 using Lemma 8.2.4. Fix $\epsilon > 0$, and choose $\mu_\epsilon \in \mathcal{M}_{\leqslant 1}(g)$ such that $I(\mu_\epsilon) \leqslant \beta + \epsilon$. Choose $\delta > 0$ small enough such that $B_\delta(\mu_\epsilon) = \{\mu : \rho(\mu, \mu_\epsilon) < \delta\}$ is contained in $\mathcal{U}$, where ρ is the distance defined by equation (8.1). Define

$$\rho_n(\mu, \nu) = \sum_{k=1}^{n} 2^{-k} \left| \int V_k d\mu - \int V_k d\nu \right|.$$

Choose n sufficiently large such that $B_{n,\delta/2}(\mu_\epsilon) = \{\mu \in \mathcal{M}_{\leqslant 1}(g) : \rho_n(\mu, \mu_\epsilon) < \delta/2\}$ is contained in $B_\delta(\mu_\epsilon)$. Set $\alpha_\epsilon = f_n \mu_\epsilon$, and

$$C_{n,\delta/2}(\alpha_\epsilon) = \{\alpha \in \mathbb{R}^n : \sum_{k=1}^{n} 2^{-k} |\alpha_k - (\alpha_\epsilon)_k| < \delta/2\},$$

where $\alpha = (\alpha_1, ..., \alpha_k)$ and $\alpha_\epsilon = ((\alpha_\epsilon)_1, ..., (\alpha_\epsilon)_k)$. Observe that $C_{n,\delta/2}(\alpha_\epsilon)$ is open and that $B_{n,\delta/2}(\mu_\epsilon) = f_n^{-1} C_{n,\delta/2}(\alpha_\epsilon)$. Using the inclusion

$$\mathcal{U} \supset B_\delta(\mu_\epsilon) \supset B_{n,\delta/2}(\mu_\epsilon) = f_n^{-1} C_n(\alpha_\epsilon),$$

and Lemma 8.2.4 we obtain

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log\left(\frac{\sum_{\mu_\tau\in\mathcal{U}\cap\mathcal{M}_p(W,t,t-c)}\exp(l(\tau)\int G d\mu_\tau)}{\sum_{\mu_\tau\in\mathcal{M}_p(W,t,t-c)}\exp(l(\tau)\int G d\mu_\tau)}\right) &\geqslant -J_n(B_{n,\delta}(\alpha_\epsilon))\\ &\geqslant -I_n(\alpha_\epsilon)\\ &\geqslant -I(\mu_\epsilon)\\ &\geqslant -\beta-\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary we are done. □

In hyperbolic space every parabolic subgroup is of divergence type, in particular the zero potential is a vSPR potential. Combining Theorem 8.1.1 and Theorem 8.2.2 we obtain

Theorem 8.2.7. *Let M be a hyperbolic geometrically finite manifold with cusps. Then*

1. *Let $\mathcal{K}$ a closed subset of $\mathcal{M}_{\leqslant 1}(g)$ not containing the measure of maximal entropy. Then we have*

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\frac{\#(\mathcal{M}_p(W,t,t-c)\cap\mathcal{K})}{\#\mathcal{M}_p(W,t,t-c)}\right)\leqslant -\beta,$$

   where $\beta=\inf_{\mu\in\mathcal{K}} I_0(\mu)$ is a positive number.

2. *Let $\mathcal{U}$ be an open subset of $\mathcal{M}_{\leqslant 1}(g)$ whose closure does not contain the measure of maximal entropy. Then we have*

$$\liminf_{t\to\infty}\frac{1}{t}\log\left(\frac{\#(\mathcal{M}_p(W,t,t-c)\cap\mathcal{K})}{\#\mathcal{M}_p(W,t,t-c)}\right)\geqslant -\gamma,$$

   where $\gamma=\inf_{\mu\in\mathcal{U}} I_0(\mu)$ is a positive number.

8.3 Application

Before jumping into our next application (which is a direct application of the proof of Theorem 8.2.2), we will explain some concepts from convex analysis. This is necessary to make sense of the hypothesis in our result, but it also worth explaining some facts that we took for granted from [Kif] for the proof of Theorem 8.2.2. As for the lower bound large deviation estimate, the geometrically finite assumption does not seem to be crucial (once we have the right definition of very strongly positive recurrent potentials everything will follow analogously).

Definition 8.3.1 (Affine hull)**.** *Given $S \subset \mathbb{R}^n$, the* affine hull *of S is the set*

$$Aff\ S = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{n} \lambda_i x_i,\ \textit{where } x_i \in S \textit{ and } \sum_{i=1}^{n} \lambda_i = 1\}.$$

Observe that the difference between the convex hull and the affine hull is that in the former we require the coefficients λ_i to be non-negative. The affine hull of S is also the smallest affine subset (translation of linear subspaces) containing S.

Definition 8.3.2 (Relative interior)**.** *Given $S \subset \mathbb{R}^n$, the* relative interior *of S is defined as*

$$ri(S) = \{x \in S : \exists \epsilon > 0, B(x, \epsilon) \cap Aff\ S \subset S\}.$$

The relative interior is a meaningful notion of interior when the convex subset is not 'full dimensional' and we care about the linear structure. The following definition was used in the proof of Theorem 8.2.2 and will be needed again.

Definition 8.3.3. *To a sequence $V^n = (V_1, ..., V_n)$ of potentials we associate the map $f_n^V : \mathcal{M}_{\leqslant 1}(g) \to \mathbb{R}^n$, given by the formula*

$$f_n^V(\mu) = (\int V_1 d\mu, ..., \int V_n d\mu).$$

Proposition 8.3.4. *Let M be a geometrically finite manifold and G a very strongly positive recurrent potential. Suppose we have a sequence of compactly supported Hölder potentials $V^n = (V_1, ..., V_n)$, and $\alpha \in ri(f_n^V(\mathcal{M}_{\leqslant 1}(g)))$. Then there exists $\beta \in \mathbb{R}^n$ such that the equilibrium state of $G + \langle \beta, V^n \rangle$, say m_β, satisfies*

$$f_n^V(m_\beta) = \alpha.$$

Proof. We will follow the notation used in the proof of Theorem 8.2.2. Since I_n and Q_n are convex duals and α lies in the relative interior of the domain of I_n, we know the existence of $\beta \in \mathbb{R}^n$ (see [Ro, Theorem 23.4]) such that

$$Q_n(\beta) = \langle \beta, \alpha \rangle - I_n(\alpha).$$

As mentioned before, this is equivalent to say that $\beta \in \partial I_n(\alpha)$. Let m_β be the equilibrium state of $G + \langle \beta, V^n \rangle$ (which exists because G is very strongly positive recurrent and $\langle \beta, V^n \rangle$ has compact support). Define $\alpha_0 = f_n^V m_\beta$. As before (see equation 8.3) we obtain

$$Q_n(\beta) = \langle \alpha_0, \beta \rangle - I_G(m_\beta) \leqslant \langle \alpha_0, \beta \rangle - I_n(\alpha_0) \leqslant Q_n(\beta).$$

We conclude that $Q_n(\beta) = \langle \alpha_0, \beta \rangle - I_n(\alpha_0)$. The argument used in the proof of Lemma 8.2.4 applies without modification here and we get that $\alpha = \alpha_0$. Moreover $I_n(\alpha) = I(\mu_\beta)$. In particular we also obtain that the function $P(G) - (h_\mu(g) + \int G d\mu)$ is minimized at m_β among all probability measures μ such that $f_n^V \mu = \alpha$. □

Remark 8.3.5. *Fix a sequence $V = (V_1, ..., V_n)$ of Hölder potentials and suppose that*

$$dim\ Aff\ f_n^V(\mathcal{M}_{\leqslant 1}(g)) < n.$$

In this case there exists a plane $P = \{x : a_1x_1+...+a_nx_n = b\}$ *containing* $f_n^V(\mathcal{M}_{\leqslant 1}(g))$. *In particular for every* $\mu \in \mathcal{M}_{\leqslant 1}(g)$ *we have* $\sum_{k=1}^n a_k \int V_k d\mu = b$. *First observe that since the zero measure belongs to* $\mathcal{M}_{\leqslant 1}(g)$ *we must have that* $b = 0$. *Define* $\mathcal{W} = C(T^1M)/\sim$, *where* $f \sim g$ *if they are cohomologous through a Hölder function. More precisely* $f \sim g$ *if there exists a Hölder function* H, *differentiable along flow lines, such that*

$$f(v) - g(v) = \frac{d}{dt}_{|t=0} H(g_t v),$$

for every $v \in \Omega$, *where* Ω *is the non-wandering set of the geodesic flow. Let* $h = \sum_{k=1} a_k V_k$. *Observe that* $\int h d\mu = 0$ *for every* $\mu \in \mathcal{M}_{\leqslant 1}(g)$. *It is proven in [PPS, Remark 3.1] that* h *is then cohomologous to zero in restriction to the nonwandering set of the geodesic flow. In particular* $h = 0$ *in* $\mathcal{W}$. *We conclude that dim Aff* $f_n^V(\mathcal{M}_{\leqslant}(g))$ *is equal to the dimension of the span of* $\{V_1, ..., V_n\}$ *in* $\mathcal{W}$. *In particular if* $\{V_1, ..., V_n\}$ *are linearly independent in* $\mathcal{W}$, *then the relative interior of* $f_n^V \mathcal{M}_{\leqslant 1}(g)$ *is just its topological interior.*

As mentioned in Remark 7.3.6, every geometrically finite manifold admits very strongly positive recurrent potentials. An immediate consequence of Proposition 8.3.4 is the following result.

Corollary 8.3.6. *Let* M *be a geometrically finite manifold. Suppose we have a sequence of compactly supported Hölder potentials* $V^n = (V_1, ..., V_n)$, *and* $\alpha \in ri(f_n^V(\mathcal{M}_{\leqslant 1}(g)))$. *Then there exists a Gibbs measure* m *such that* $f_n^V m = \alpha$.

To end this section we will explain some details for $n = 1$ (even though in this case the result is not very surprising and can be recovered from the argument used in Theorem 6.2.2). In this case we have one compactly supported Hölder potential V_1. Observe that Aff $f_1^V(\mathcal{M}_{\leqslant 1}(g)) = \mathbb{R}$, and that

$$f_1^V(\mathcal{M}_{\leqslant 1}(g)) = [\min\{0, m\}, \sup\{0, M\}],$$

where $m = \inf_{\mu \in \mathcal{M}(g)} \int f d\mu$, and $M = \sup_{\mu \in \mathcal{M}(g)} \int f d\mu$. Clearly

$$ri(f_1^V(\mathcal{M}_{\leqslant 1}(g)) = (\min\{0, m\}, \sup\{0, M\}).$$

Corollary 8.3.6 tells us that for $\alpha \in (\min\{0, m\}, \sup\{0, M\})$, there exists a Gibbs measure m such that $\int V_1 dm = \alpha$.

Chapter 9

Final comments

In this thesis we have mainly focused on the thermodynamic formalism of potentials in $C_0(T^1M)$. One important reason for this is that if $F \in C_0(T^1M)$, then the map

$$\mu \mapsto \int F d\mu,$$

is continuous in the vague topology (see Lemma 3.1.2). We will now briefly discuss what happens if we weaken this hypothesis.

Definition 9.0.1. *We say that a potential* F converges to D at infinity *if for each* $\epsilon > 0$*, there exists a compact subset* $K \subset T^1M$ *such that* $\sup_{x \in K^c} |F(x) - D| < \epsilon$*. The space of continuous potentials converging to* D *at infinity is denoted by* $C(T^1M, D)$*.*

It follows from the definition that $C(T^1M, 0) = C_0(T^1M)$. We use the notation $C(T^1M, D)$ to avoid any confusion with the space of bounded continuous potentials $C_b(T^1M)$. The following lemma gives some insight on the type of behaviour that we will encounter when considering potentials that do not vanish at infinity.

Lemma 9.0.2. *Let* $(\mu_n)_n$ *be a sequence of invariant probability measures converging*

vaguely to μ. Then for every $F \in C(T^1M, D)$ we have

$$\lim_{n\to\infty} \int F d\mu_n = \int F d\mu + (1 - |\mu|)D.$$

Proof. Let $(K_m)_m$ be a compact exhaustion of T^1M such that the following hold:

1. $\sup_{x\in K_m^c} |F(x) - D| < \frac{1}{m}$.
2. $\mu(\partial K_m) = 0$.

It follows from this two assumptions that

$$\lim_{n\to\infty} \int_{K_m} F d\mu_n = \int_{K_m} F d\mu,$$

and that

$$|\int_{K_m^c} F d\mu_n - D\mu_n(K_m^c)| \leqslant \frac{1}{m}.$$

Similarly $\lim_{n\to\infty} \mu_n(K_m) = \mu(K_m)$, and therefore $\lim_{n\to\infty} \mu_n(K_m^c) = 1 - \mu(K_m)$. Observe that

$$\begin{aligned} |\int F d\mu_n - \int_{K_m} F d\mu - D(1 - \mu(K_m))| \leqslant \\ |\int_{K_m} F d\mu_n - \int_{K_m} F d\mu| + |\int_{K_m^c} F d\mu_n - D\mu_n(K_m^c)| \\ + |D\mu_n(K_m^c) - D(1 - \mu(K_m))|. \end{aligned}$$

Taking n to infinity we get

$$\limsup_{n\to\infty} \int F d\mu_n - \left(\int_{K_m} F d\mu + D(1 - \mu(K_m)) \right) \leqslant \frac{1}{m},$$

and

$$\liminf_{n\to\infty} \int F d\mu_n - \left(\int_{K_m} F d\mu + D(1 - \mu(K_m)) \right) \geqslant -\frac{1}{m}.$$

Finally taking m to infinity we get that

$$\lim_{n\to\infty}\int F d\mu_n = \int F d\mu + D(1-|\mu|).$$

□

Definition 9.0.3. *Given $F \in C_b(T^1M)$ we define* essential upper bound of F *as*

$$c(F) = \inf_{n\in\mathbb{N}} \sup_{x\in K_n^c} F(x),$$

where $(K_n)_n$ is a compact exhaustion of T^1M.

It is easy to verify that $c(F)$ is independent of the choice of compact exhaustion.

Lemma 9.0.4. *Let $(\mu_n)_n$ be a sequence of invariant probability measures converging vaguely to μ. Then for every $F \in C_b(T^1M)$ we have*

$$\limsup_{n\to\infty}\int F d\mu_n \leqslant \int F d\mu + (1-|\mu|)c(F).$$

Proof. Let $(K_m)_m$ be a compact exhaustion of T^1M such that the following hold:

1. $\sup_{x\in K_m^c} F(x) < c(F) + \frac{1}{m}$.
2. $\mu(\partial K_m) = 0$.

Observe that $\int_{K_m^c} F d\mu_n \leqslant (c(F)+\frac{1}{m})\mu_n(K_n^c)$, and that

$$\begin{aligned}\left(\int F d\mu_n - \int_{K_m} F d\mu - (1-\mu(K_m))c(F)\right) &\\ \leqslant \left(\int_{K_m} F d\mu_n - \int_{K_m} F d\mu\right) &+ \left(\int_{K_m^c} F d\mu_n - c(F)\mu_n(K_m^c)\right)\\ &+ (c(F)\mu_n(K_m^c) - c(F)(1-\mu(K_m))).\end{aligned}$$

As in Lemma 9.0.2 we obtain that

$$\limsup_{n\to\infty} \int F d\mu_n - \left(\int_{K_m} F d\mu + c(F)(1-\mu(K_m)) \right) \leqslant \frac{1}{m}.$$

Taking m to infinity proves the lemma. □

As a direct consequence of Lemma 9.0.4 and Theorem 5.2.3 we obtain

Theorem 9.0.5. *Let $(\mu_n)_n$ be a sequence of invariant probability measures converging vaguely to μ and $F \in C_b(T^1M)$. Then*

$$\limsup_{n\to\infty} \left(h_{\mu_n}(g) + \int F d\mu_n \right) \leqslant |\mu| \left(h_{\mu/|\mu|}(g) + \int F d\mu/|\mu| \right) + (1-|\mu|)(\delta_\infty + c(F)).$$

In particular, if $P(F) > \delta_\infty + c(F)$, then F has an equilibrium state (see proof of Theorem 6.1.5). We believe that Theorem 9.0.5 is not an optimal result, and we suggest the following conjecture.

Conjecture 9.0.6. *Let $(\mu_n)_n$ be a sequence of invariant probability measures converging vaguely to μ and $F \in C_b(T^1M)$. Then*

$$\limsup_{n\to\infty} \left(h_{\mu_n}(g) + \int F d\mu_n \right) \leqslant |\mu| \left(h_{\mu/|\mu|}(g) + \int F d\mu/|\mu| \right) + (1-|\mu|)P_\infty(F).$$

We emphasize that if $F \in C(T^1M, D)$, then

$$P_\infty(F) = \delta_\infty + D = \delta_\infty + c(F). \tag{9.1}$$

This follows from the proof of Lemma 6.1.7. In particular Conjecture 9.0.6 holds for potentials in $C(T^1M, D)$. It worth pointing out that potentials in $C(T^1M, D)$ are uniformly continuous. The uniform continuity of the potential could play a role in Conjecture 9.0.6 (since we do not have strong evidence to justify that the uniform continuity is relevant we state the conjecture in the general case). If Conjecture 9.0.6

holds then one can obtain Theorem 6.1.8 and Theorem 6.1.9 for arbitrary SPR Hölder potentials (we can remove the assumption that our potential belongs to $C_0(T^1M)$). We remark that Theorem 9.0.5 and equation (9.1) implies that Theorem 6.1.8 and Theorem 6.1.9 hold for potentials in $C(T^1M, D)$, for every $D \in \mathbb{R}$.

In Section 8 we obtained large deviation lower bounds for vSPR potentials on geometrically finite manifolds. As mentioned in Section 8 we do not have a good definition of vSPR potentials for arbitrary pinched negatively curved manifolds. We expect that the following is true (we use the notation from Definition 2.3.20):

Conjecture 9.0.7. *Let $\widetilde{U}$ be an open relatively compact subset of $\widetilde{M}$ such that $T^1\widetilde{U}$ has non-empty intersection with the lift of Ω to $T^1\widetilde{M}$. Assume that $P(F, \widetilde{U}, s)$ is of divergence type. Then $\delta_\Gamma^F(\widetilde{U}) < P(F)$.*

We recall that a Poincaré series is of divergence type if it is divergent at its critical exponent. Conjecture 9.0.7 should be considered as a generalization of Theorem 2.3.15. Assume for a second that Conjecture 9.0.7 holds. In this case we say that F is vSPR if there exists an increasing sequence $(\widetilde{U}_n)_n$ such that $\bigcup_{n\geqslant 1} \widetilde{U}_n = \widetilde{M}$, and that $P(F, \widetilde{U}_n, s)$ is of divergence type for every $n \geqslant 1$. If F is vSPR, then Conjecture 9.0.7 and Definition 2.3.20 imply that $P_\infty(F) < P(F)$. In other words vSPR potentials are SPR potentials. Let G be a compactly supported continuous function on T^1M. Choose $\widetilde{U} \subset \widetilde{M}$ big enough such that $\mathrm{supp}(G) \subset T^1U$, where $U = p(\widetilde{U})$. Choose n sufficiently large such that $\widetilde{U} \subset \widetilde{U}_n$. In this case the Poincaré series $P(F + G, \widetilde{U}_n, s)$ is identical to the Poincaré series $P(F, \widetilde{U}_n, s)$. Since by assumption $P(F, \widetilde{U}_n, s)$ is of divergence type, we get that $P(F+G, \widetilde{U}_n, s)$ is of divergence type as well. This implies that $F + G$ is vSPR, and therefore SPR. We conclude that any compact perturbation of a vSPR potential is SPR. For the family of vSPR potentials we can obtain large deviation lower bounds (without any modification in the proof of Theorem 8.2.2).

We finish this section with a few comments about the thermodynamic formalism of the geodesic flow on non-positively curved manifolds. In the paper [V2] we obtained

the upper semicontinuity of the entropy map for the geodesic flow on rank 1 manifolds when restricting to the regular part of the geodesic flow (for details we refer the reader to [V2]). For the geodesic flow on non-compact rank 1 manifolds it is not known the uniqueness of the measure of maximal entropy (in the compact setting the uniqueness of the measure of maximal entropy was proven by Knieper [Kni]), nor the uniqueness of equilibrium states for regular potentials (in the compact case this problem was recently treated in [BCFT]). We remark that in this setting the relevant critical gap conditions (e.g. SPR condition) must also consider the singular part of the geodesic flow, and not just the behaviour at infinity. The importance of controlling the singular part of the geodesic flow is clear from the works [BCFT] and [V2].

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