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Rational Curves on Algebraic Spaces and Projectivity Criteria

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Princeton, NJ : Princeton University

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Rational Curves on Algebraic Spaces and Projectivity Criteria

David E. Villalobos Paz

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
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Abstract

In Chapter 1, we provide background material that is relevant to the remainder of the thesis. The aim is to make the thesis accessible to readers whose background in algebraic geometry is at the level of one of the standard introductory graduate textbooks. The topics summarised here include group actions on schemes and the quotients of such actions, positivity conditions of divisors, the Kleiman-Mori cone, and the Minimal Model Program. We also give a brief and informal introduction to algebraic spaces; while not completely rigorous, it should be enough to understand the role of algebraic spaces in Chapter 3.

In Chapter 2, we give a short proof of a Grothendieck-Lefschetz Theorem for equivariant Picard groups of nonsingular varieties with the action of an affine algebraic group.

Chapter 3 is the core of the thesis. We show that a smooth Moishezon space Y is non-projective if and only if it contains a rational curve such that $-[C] \in \overline{\mathrm{NE}}(Y)$. More generally, this holds if Y has $\mathbb{Q}$-factorial, log terminal singularities. We derive this as a consequence of our main technical result: that we can run the relative minimal model program when the base is an algebraic space Y of finite type over a field of characteristic 0. As a second application, we show that every log canonical pair (Y, Δ), where Y is an algebraic space of finite type over a field of characteristic 0 admits a dlt modification that is projective over Y.

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Chapter 1

Background

1.1 Group Actions and Quotients

In this section, we describe actions of groups on schemes as well as quotients of such actions. We adopt Mumford's Geometric Invariant Theory point of view. For details and proofs, see for example [41] or [8]. Throughout this section, k will denote a field; for simplicity, we will assume that k has characteristic 0.

Definition 1.1.1. An algebraic group G over k is a scheme of finite type over k together with three morphisms $\mu : G \times G \to G$, $i : G \to G$, and $e : \mathrm{Spec}(k) \to G$ that satisfy the usual properties of group multiplication, inverses, and identity. We say that an algebraic group G is affine if G is an affine scheme.

Example 1.1.2. Every finite group is algebraic. The additive group $\mathbb{G}_a = \mathrm{Spec}(k[x])$ and the multiplicative group $\mathbb{G}_m = \mathrm{Spec}(k[x, x^{-1}])$ are algebraic groups.

Definition 1.1.3. An action of an algebraic group G on a k-scheme X is a morphism $\sigma : G \times X \to X$ that satisfies the usual properties of a group action.
We will write $g \cdot x$ for $\sigma(g, x)$.

Definition 1.1.4. Suppose a reduced algebraic group G acts on two k-schemes X and Y. We say that a morphism $\phi : X \to Y$ is G-equivariant if $\phi(g \cdot x) = g \cdot \phi(x)$ for

all $g \in G$ and $x \in X$.

When the action of G on Y is trivial, a G-equivariant morphism is called G-invariant.

Ideally, we would like to say that the quotient of a scheme by a group action is the scheme whose underlying set consists of the orbits of this action. Unfortunately, in many cases such a space does not exist.

Example 1.1.5. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ given by $t \cdot (x, y) = (tx, t^{-1}y)$. The orbits of this action are the origin $\{(0, 0)\}$, the punctured x-axis $\{(x, 0) : x \in k^\times\}$, the punctured y-axis $\{(0, y) : y \in k^\times\}$, and for each $a \in k^\times$ the curve $\{(x, y) : xy = a\}$. Note that the closures of the first three orbits intersect, so it is impossible for a regular function to separate these orbits. This then means that we cannot construct a quotient in which each point corresponds to an orbit of this action.

Consider, on the other hand, the action of a group G on a set X. In this case, the quotient G/X is the set of orbits of the action, and G acts trivially on this quotient. The quotient map $\phi : X \to X/G$ is then clearly G-invariant. Moreover, given any G-invariant function $f : X \to Y$, we can find a unique map $\overline{f} : X/G \to Y$ satisfying $\overline{f} \circ \phi = f$. This universal property for the G-quotient of a set motivates the following definition for the quotient of a scheme:

Definition 1.1.6. A categorical quotient of the action of an algebraic group G acting on a scheme X over k is a scheme Y over k together with a G-invariant morphism $\phi : X \to Y$, such that for every G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$, such that $\overline{f} \circ \phi = f$.

It follows that if a categorical quotient exists, then it is unique up to unique isomorphism.

Example 1.1.7. For the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ given by $t \cdot (x, y) = (tx, t^{-1}y)$, the morphism $\phi : \mathbb{A}^2 \to \mathbb{A}^1 : (x, y) \to xy$ is a categorical quotient. Observe that the given action induces an action of the coordinate ring $k[x, y] = k[\mathbb{A}^2]$. The invariant

subring of this action is $k[x,y]^{\mathbb{G}_m} = k[xy]$, and the categorical quotient of the action is precisely the morphism on schemes induced by the inclusion $k[x,y]^{\mathbb{G}_m} \to k[x,y]$. This is no coincidence.

Proposition 1.1.8. *Suppose an algebraic group G acts on an affine k-scheme $X = \mathrm{Spec}(A)$. Let $A^G = \{a \in A : g \cdot a = a \text{ for all } g \in G\}$ be the invariant subring. Then the morphism $X \to \mathrm{Spec}(A^G)$ induced by the inclusion of rings $A^G \to A$ is a categorical quotient.*

Remark. Suppose that $X = \mathrm{Spec}(A)$ is of finite type over k. Hilbert's Fourteenth Problem asks if the ring of invariants A^G is finitely generated. In 1958, Nagata produced counterexamples showing that this is not true in general; see [10] for a survey of counterexamples to this problem. This means that the categorical quotient of an affine variety need not be a variety. However, the ring of invariants is finitely generated when G is a reductive group; this is sometimes called Hilbert's Finiteness Theorem. By reductive, we mean that G contains no closed, normal, unipotent subgroups. For example, finite groups as well as $\mathbb{G}_m$ are reductive.

We would still like to have a notion of a quotient in which every point corresponds to an orbit of the action.

Definition 1.1.9. Suppose G is an algebraic group acting on a k-scheme X. A geometric quotient of X by G is a morphism of schemes $\phi : X \to X/G$, such that

1. ϕ is surjective and every fibre of ϕ is an orbit of the action,
2. A subset $U \subset X/G$ is open if and only if $\phi^{-1}(U) \subset X$ is open, and
3. $\mathcal{O}_{X/G} = (\phi_\star \mathcal{O}_X)^G$.

Proposition 1.1.10. *If $\phi : X \to Y$ is a geometric quotient, then it is a categorical quotient. Therefore, if a geometric quotient exists, then it is unique up to unique isomorphism.*

Example 1.1.11. Going back to the example action of $\mathbb{G}_m$ on $\mathbb{A}^2$, note that if we remove the three problematic fibres (the origin and the punctured coordinates axes) to obtain the variety $Y = \mathbb{A}^2 \setminus (\{x = 0\} \cup \{y = 0\})$, then the categorical quotient $Y \to Y/\mathbb{G}_m \cong k[t, t^{-1}]$ is in fact a geometric quotient.

This example suggests that given a group action on a scheme, even if a geometric quotient does not exist, one might be able to find an open subscheme where it does exist.

Definition 1.1.12. Let G be an algebraic group acting on an affine k-scheme X. A point $x \in X$ is said to be stable if its orbit $G \cdot x$ is closed and the stabiliser group G_x is finite. The set of stable points is denoted by X^s.

Proposition 1.1.13. *X^s is an open subset of X. If $\phi : X \to X//G$ denotes the categorical quotient, then the restriction $\phi_{X^s} : X^s \to \phi(X^s)$ is a geometric quotient.*

Remark. The stable locus X^s may be empty.

The next step is to construct quotients of schemes that are not necessarily affine. This is done by lifting the G-action on X to a line bundle over X, and defining stability in terms of that line bundle.

Definition 1.1.14. Suppose $\pi : L \to X$ is a line bundle on a scheme X with an action of G. A G-linearisation of the G-action is an action of G on L such that π is G-equivariant and the induced map on fibres $L_x \to L_{g \cdot x}$ is linear for all $g \in G$ and $x \in X$.

Example 1.1.15. Take $X = \mathrm{Spec}(k)$ and consider the line bundle $\mathbb{A}^1 \to X$ (which corresponds to the trivial invertible sheaf $\mathcal{O}_x$). Let G be any algebraic group over k. Then for each character $\chi : G \to \mathbb{G}_m$, we get a G-linearisation of the G-action on X, for which G acts on $\mathbb{A}^1$ by $g \cdot x = \chi(g)x$.

The tensor product of two G-linearised line bundles and the inverse of a G-linearised line bundle both carry canonical G-linearisations, so the set of isomorphism

classes of G-linearised line bundles over X forms a group, called the G-equivariant Picard group of X and denoted by $\mathrm{Pic}^G(X)$. This is functorial: If G acts on X and Y, and $f : X \to Y$ is G-equivariant, then there is an induced homomorphism $f^\star : \mathrm{Pic}^G(Y) \to \mathrm{Pic}^G(X)$.

Definition 1.1.16. Suppose G acts on X and L is a line bundle on X. L is called G-invariant if $g^\star L \cong L$ for all $g \in G$, where we think of each g as inducing an automorphism of X. The subgroup of $\mathrm{Pic}(X)$ consisting of the isomorphism classes of G-invariant line bundles on X is denoted $\mathrm{Pic}(X)^G$.

Definition 1.1.17. Suppose G acts on a quasi-projective k-scheme X and $L \to X$ is a G-linearised ample line bundle on X. A point $x \in X$ is called semistable if there exist $n \geq 1$ and $s \in \Gamma(X, L^d)^G$ such that X_s is affine and $x \in X_s$. The locus of semistable points is denoted by $X^{ss}(L)$. A point X^s is called stable if it is semistable, the orbit $G \cdot x$ is closed in $X^{ss}(L)$, and the stabiliser G_x is finite. The locus of stable points is denoted by $X^s(L)$.

Example 1.1.18. If G is a finite group acting on a projective variety X, then we can find a G-linearisable very ample line bundle L over X, along with a G-linearisation for L. Since X is projective, given any point $x \in X$, we may find a G-invariant hyperplane that excludes any given x. Since G is finite, we see that $X^s(L) = X$. This remains true even if X is only assumed to be quasi-projective.

Theorem 1.1.19. *(Mumford) Suppose G is a reductive group acting on a quasi-projective k-scheme X and L is a G-linearised line bundle over X. Then a categorical quotient $\phi : X^{ss}(L) \to X^{ss}(L)//G$ exists and there is an open set $U \subset X^{ss}(L)//G$ for which $\phi^{-1}(U) = X^s(L)$ and the restriction $\phi|_{X^s(L)} : X^s(L) \to U$ is a geometric quotient. Moreover, the quotients $X^{ss}(L)//G$ and U are quasi-projective varieties.*

Remark. It can happen that $X^{ss}(L) = \emptyset$.

Remark. If a finite group G acts on a quasi-projective variety X, then we have

$X^s(L) = X$ and the geometric quotient X/G exists as a quasi-projective variety.

1.2 Algebraic Spaces

In the previous section, we saw that we can form certain quotients of schemes by group actions. In particular, if we have a finite group G acting on a quasi-projective variety X, then we can form the quotient X/G and this turns out to be a quasi-projective variety as well.

We can then ask the following question: Suppose that G is a finite group acting on a scheme X of finite type over k. Does the quotient X/G exist as a scheme? The answer here turns out to be no. In fact, Hironaka has an example of a proper, but not projective, variety with an action of the finite group $\mathbb{Z}/2\mathbb{Z}$ for which the quotient is not a scheme. The quotient exists, however, as a more general object: an algebraic space. See the appendix to [16] for more details on this example.

In this section, we provide some background on the theory of algebraic spaces. For a rigorous and detailed treatment, see for example [23]. Artin introduced the notion of an algebraic space to deal with these more general quotients of schemes by finite group actions. In fact, informally, we can think of algebraic spaces as quotients of schemes by finite groups. While not completely correct in general, it turns out that this characterisation is accurate for Noetherian, normal, separated algebraic spaces.

For the purposes of this thesis, we think of algebraic spaces as follows: On the one hand, a scheme X has a Zariski open cover $\pi_i : U_i \to X$, where each U_i is an affine scheme, and these open immersions are compatible on the overlaps. An algebraic space X, on the other hand, can be covered by the images of étale morphisms $\pi_i : U_i \to X$, where each U_i is an affine scheme, and these morphisms are again compatible on the overlaps.

Just as we did for schemes, we can develop the theory of algebraic group actions

on algebraic spaces and the quotients of such actions. For details, see for example [28]. Most of the theory remains the same, but now geometric quotients need not be categorical quotients, and thus geometric quotients need not be unique. This can be fixed by only considering proper group actions.

Definition 1.2.1. Suppose $\sigma : G \times_S X \to X$ is the action of an algebraic group on an algebraic space (over a base S, say). We say that this action is proper if the morphism $(\sigma, \mathrm{pr}_2) : G \times_S X \to X \times_S X$ is proper, where $\mathrm{pr}_2 : G \times_S X \to X$ is the second projection.

If we look at these proper actions, we obtain that in many cases the geometric quotient of an algebraic space exists as an algebraic space. In particular, as a Corollary to the Keel-Mori Theorem, we have:

Theorem 1.2.2. *(Keel-Mori; see [21]) Let L be a locally Noetherian base scheme. Let G be a flat group scheme over L and let X be an algebraic space of finite type over L. Suppose that G acts properly on X with finite stabilisers. Then a geometric and uniform categorial quotient X/G exists as a separated algebraic space. Moreover, if X is normal, then so is X/G.*

1.3 Positivity and Cones in Birational Geometry

In this section, we describe positivity properties of divisors as well as various cones that can be associated to an algebraic variety or, more generally, to a proper algebraic space. For more details and proofs, see for example [32] or [35]. Throughout this section, all schemes and algebraic spaces will be assumed to be proper over a field k of characteristic 0.

We first recall the basic notions of Weil and Cartier divisors.

Definition 1.3.1. A prime Weil divisor on an irreducible normal algebraic space X is an integral codimension 1 closed subset. A Weil divisor on X is a finite linear

combination of prime divisors, that is, an expression of the form $D = \sum d_i D_i$, where the D_i are prime divisors, the d_i are integers, and only finitely many of the d_i are nonzero. We say that D is effective if all $d_i \geq 0$.

Remark. If the d_i are allowed to have rational (resp. real) coefficients, then we refer to D as a Weil $\mathbb{Q}$-divisor (resp. a Weil $\mathbb{R}$-divisor).

Definition 1.3.2. If $f \in k(X)^\times$ is a nonzero rational function on X, then we may define the divisor of zeroes and poles of f by $\text{div } f = \sum \text{ord}_Y(f)Y$, where the sum runs over all prime divisors Y of X and $\text{ord}_Y(f)$ is the order of vanishing of f along Y.

Definition 1.3.3. Suppose that D is a Weil divisor on X. We define a sheaf $\mathcal{O}_X(D)$ on X by $\Gamma(U, \mathcal{O}_X(D)) = \{0\} \cup \{f \in k(X)^\times : \text{div } f + D|_U \geq 0\}$ whenever $U \subset X$ is open. It is a fact that $\mathcal{O}_X(D)$ is a coherent sheaf.

Definition 1.3.4. We say that a Weil divisor D on X is a Cartier divisor if $\mathcal{O}_X(D)$ is an invertible sheaf. A Weil $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-Cartier if some nonzero integer multiple mD is Cartier. A Weil $\mathbb{R}$-divisor D is called $\mathbb{R}$-Cartier if it is an $\mathbb{R}$-linear combination of Cartier divisors. We say that X is $\mathbb{Q}$-factorial if every Weil $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.

Definition 1.3.5. If X is smooth, we can define the canonical sheaf $\omega_X = \Omega_X^{\dim(X)}$. This is, in fact, an invertible sheaf. If K_X is a divisor for which $\mathcal{O}_X(K_X) = \omega_X$, then we say that K_X is a canonical divisor.

We recall that if X is normal, then the complement of the smooth locus of X has codimension at least 2 in X.

Definition 1.3.6. Suppose X is normal and let U denote the smooth locus of X. A canonical divisor K_X on X is the closure in X of a canonical divisor K_U on U. There is a unique divisor class on X with this property.

Since X is proper, we can define intersection products of Cartier divisors with closed subsets of X. There are various ways of doing this; for example, if Z is a closed subset with $\dim(Z) = d$ and $L_1, \ldots, L_d$ are Cartier divisors on X, then we can prove that $\chi(Z, \mathcal{O}_Z(m_1 L_1 + \cdots m_d L_d)) = C \cdot m_1 \cdots m_d + \cdots$, and define the intersection product as $L_1 \cdot \cdots \cdot L_d \cdot Z = C$.

When $L = L_1 = \cdots = L_d$, we write $L^d \cdot Z$ for the intersection product. In the particular case where Z is a curve on X, the intersection product is $L \cdot Z = \deg_Z(L|_Z)$.

Extending by linearity we can make sense of intersection products when the divisors are $\mathbb{R}$-Cartier.

For a detailed treatment of intersection theory, see for example [12].

Definition 1.3.7. We say that two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors D_1 and D_2 on X are numerically equivalent if $D_1 \cdot C = D_2 \cdot C$ for all integral curves C on X. We say that D is numerically trivial if D is numerically equivalent to 0.

Definition 1.3.8. Let $\mathrm{Div}(X)$ denote the (abelian) group of Cartier divisors on X. Then the set of numerically trivial divisors on X forms a subgroup $\mathrm{Num}(X)$ of $\mathrm{Div}(X)$. The quotient group $\mathrm{Div}(X)/\mathrm{Num}(X)$ is called the Néron-Severi group of X, and we denote it by $N^1(X)$.

Remark. There are different ways to define the Néron-Severi group that result in different groups. However, all of these agree up to torsion.

Theorem 1.3.9. *(Néron-Severi Theorem; see Theorem 5.1 in Expose XIII of [4]) Let X be proper over k. Then the Néron-Severi group $N^1(X)$ is finitely generated.*

Definition 1.3.10. The rank of $N^1(X)$ is called the Picard number of X and denoted by $\rho(X)$.

We will now describe various cones in the finite-dimensional real vector space $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 1.3.11. Recall that an invertible sheaf $\mathcal{L}$ on X is said to be very ample

if it induces a closed embedding $\phi_{|\mathcal{L}|} : X \hookrightarrow \mathbb{P}^N$ into a projective space. A Cartier divisor A is said to be ample if there exists a positive integer m for which the invertible sheaf $\mathcal{O}_X(mA)$ is very ample. In fact, this definition of ampleness makes sense for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

Ampleness can be characterised numerically in terms of intersection products. The result for proper varieties is classical. For proper algebraic spaces, it was formulated by Kollár; see [25].

Theorem 1.3.12. *(Nakai, Moishezon, Kollár) [Nakai-Moishezon Criterion for Ampleness] Let X be a proper algebraic space and let H be a line bundle on X. Then H is ample on X if an only if for every irreducible closed subspace $Z \subset X$, the intersection product $H^{\dim(Z)} \cdot Z$ is positive.*

A shortcoming of ampleness is that the pullback of an ample divisor by a birational morphism need not be ample.

Example 1.3.13. Let $\pi : X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at a point. Let L be a line on $\mathbb{P}^2$; this is an ample divisor. However, $\pi^\star L$ has degree 0 on the exceptional divisor of the blow-up, so by the Nakai-Moishezon criterion it cannot be ample.

One can weaken ampleness to obtain a notion that is preserved under pullbacks by birational morphisms.

Definition 1.3.14. A Cartier divisor D on X is said to be nef if $D \cdot C \geq 0$ for every integral curve $C \subset X$.

By linearity of the intersection product, this definition still makes sense if D is assumed to be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. In view of the Nakai-Moishezon criterion, it is clear that ample divisors are nef. Moreover:

Proposition 1.3.15. *Suppose $f : X \to Y$ is a morphism and D is a nef divisor on Y. Then $f^\star D$ is nef on X.*

Recall that a cone in a vector space is a subset closed under multiplication by positive scalars. The numerical classes of ample $\mathbb{R}$-divisors on X form a convex cone in $N^1(X)_{\mathbb{R}}$, known as the ample cone and denoted by $\mathrm{Amp}(X)$. Similarly, the numerical classes of nef $\mathbb{R}$-divisors form a convex cone in $N^1(X)_{\mathbb{R}}$, known as the nef cone and denoted by $\mathrm{Nef}(X)$. We have seen that every ample divisor is nef, and so $\mathrm{Amp}(X) \subset \mathrm{Nef}(X)$. When X is projective, we can say more:

Theorem 1.3.16. *(Kleiman) Suppose that X is projective. Then* $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$ *and* $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$.

We now give another positivity property that weakens ampleness and is preserved under pullbacks by birational morphisms:

Definition 1.3.17. A Cartier divisor D on X is said to be big if for some $m > 0$, the rational map $\phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$ is birational onto its image. This definition makes sense even if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. When D is $\mathbb{R}$-Cartier, we say that it is big if it can be written as $D = \sum d_i D_i$, where each $d_i > 0$ and each D_i is a big Cartier divisor.

From the definition it is immediate that ample divisors are big. Moreover, we have:

Proposition 1.3.18. *Suppose $f : X \to Y$ is a birational morphism and D is a divisor on Y. Then D is big if and only if $f^\star D$ is big.*

Bigness is a numerical inavariant: If D_1 and D_2 are numerically equivalent divisors, and D_1 is big, then D_2 is also big. The numerical classes of big $\mathbb{R}$-divisors also form a convex cone $\mathrm{Big}(X)$ in $N^1(X)_{\mathbb{R}}$.

We now turn our view to curves in X.

Definition 1.3.19. A real 1-cycle on X is a linear combination $\gamma = \sum c_i C_i$, where the C_i are integral curves on X and the c_i are real numbers, only finitely many of the which are nonzero. The group of real 1-cycles is denoted by $Z_1(X)_{\mathbb{R}}$.

We know how to compute the intersection number of a Cartier divisor and an integral curve on X. Extending by linearity, we can compute the intersection number of a Cartier divisor and a real 1-cycle.

Definition 1.3.20. Two real 1-cycles γ_1 and γ_2 are said to be numerically equivalent if $D \cdot \gamma_1 = D \cdot \gamma_2$. A real 1-cycle is numerically trivial if it is numerically equivalent to 0.

Note that the set consisting of the numerically trivial 1-cycles on X is a vector subspace of $Z_1(X)_{\mathbb{R}}$. We can thus take the quotient $N_1(X)_{\mathbb{R}} = Z_1(X)_{\mathbb{R}}/\text{num.equiv.}$ By construction, we have a perfect pairing $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ given by the intersection product $(D, \gamma) \mapsto D \cdot \gamma$. Since we know that $N^1(X)_{\mathbb{R}}$ is finite-dimensional, this then implies that $N_1(X)_{\mathbb{R}}$ is also a finite-dimensional vector space.

Definition 1.3.21. The convex cone of effective curve classes on X is $\mathrm{NE}(X) = \{\sum c_i[C_i] \in N_1(X)_{\mathbb{R}} : c_1 \geq 0\}$. Here, by $[C_i]$ we mean the numerical class of the curve C_i. The Kleiman-Mori cone of X, or the closed cone of curves of X, denoted by $\overline{\mathrm{NE}}(X)$ is the closure of $\mathrm{NE}(X)$ in $N_1(X)_{\mathbb{R}}$, where $N_1(X)_{\mathbb{R}}$ is given the Euclidean topology.

Recall that $N_1(X)_{\mathbb{R}}$ and $N^1(X)_{\mathbb{R}}$ are dual vector spaces. We have:

Proposition 1.3.22. $\overline{\mathrm{NE}}(X) \subset N_1(X)_{\mathbb{R}}$ *is the dual cone of* $\mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$.

Example 1.3.23. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let $p \in \mathbb{P}^1$, and let $\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for $i = 1, 2$ be the projections. Let $L_i = \mathrm{pr}_i^\star p$. We have that $N^1(X) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$, and also $L_1^2 = L_2^2 = 0$ and $L_1 \cdot L_2 = 1$. In particular, L_1, L_2 are nef divisors that are not ample. The ample cone is the first quadrant in the (L_1, L_2) plane (without the axes), and the nef cone is the first quadrant together with the positive coordinate axes and the origin. The big cone is the same as the ample cone. The Kleiman-Mori cone is the same as the nef cone.

In general, the structure of the Kleiman-Mori cone can be quite complicated. For

smooth projective varieties, Mori proved a partial structure theorem in terms of the canonical divisor; see [39].

Definition 1.3.24. Suppose D is an $\mathbb{R}$-Cartier divisor. We define $D_{>0} = \{\gamma \in N_1(X)_{\mathbb{R}} : D \cdot \gamma > 0\}$. This is an open half-space in $N_1(X)_{\mathbb{R}}$. We also define $\overline{\mathrm{NE}}(X)_{D>0} = \overline{\mathrm{NE}}(X) \cap D_{>0}$.

Remark. These definitions still make sense if $>$ is replaced with $\geq, <$, or $\leq$.

Theorem 1.3.25 (Mori's Cone Theorem)**.** *Suppose X is a non-singular projective variety. Then*

1. *There are countably many rational curves $C_i \subset X$ such that $0 < -(C_i \cdot K_X) \leq \dim(X) + 1$, and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

2. *For any $\epsilon > 0$ and ample divisor H,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X + \epsilon H) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i],$$

where the sum is finite.

Remark. With contributions from many authors, it was eventually proved that the Cone Theorem still holds if X is allowed to have "mild" singularities. What we mean by mild here will be summarised in Section 1.4.1. For some of the contributions to the proof of the singular version of the Cone Theorem, see [26], [39], [19], and [46].

Remark. In the Cone Theorem, the $\mathbb{R}_{\geq 0}[C_i]$ are extremal rays of the closed cone of curves.

We recall that an extremal face, or simply a face, of a cone is a subset F such that $v + w \in F$ implies that $v \in F$ and $w \in F$. A 1-dimensional extremal face is

called an extremal ray. If v is a vector in a face F, then v can be written as a linear combination of vectors in the extremal rays of F.

Remark. It is sometimes stated that the Cone Theorem implies that the $(K_X+\epsilon H)$-negative part of the closed cone of curves is polyhedral. This is a misconception. What the Cone Theorem says is that the closed cone of curves is generated by the curve classes on which $(K_X + \epsilon H)$ is non-negative together with finitely many more rays.

When the canonical divisor K_X satisfies certain properties, we can say more about $\overline{\mathrm{NE}}(X)$:

Definition 1.3.26. Suppose that K_X is $\mathbb{R}$-Cartier. X is said to be Fano if $-K_X$ is ample; X is said to be Calabi-Yau if K_X is numerically trivial; and X is said to be of general type if K_X is big.

Example 1.3.27. The Cone Theorem implies that $\overline{\mathrm{NE}}(X)$ is a convex polyhedral cone when X is Fano.

Already for Calabi-Yau spaces, the picture is much more complicated even in the case of surfaces:

Theorem 1.3.28. *(Kovács; see [34]) If X is a smooth algebraic surface with trivial K_X and whose first Betti number is zero (such an X is called a smooth K3 surface), then $\overline{\mathrm{NE}}(X)$ is completely circular or it has no circular part at all. In other words, either every extremal ray is a limit of extremal rays or all of the extremal rays are isolated.*

We can characterise ampleness numerically in terms on the cone of curves. The result is due to Kleiman, and typically stated for projective varieties. In fact, it holds more generally; we discuss this more thoroughly in Chapter 3.

Theorem 1.3.29 (Kleiman's Ampleness Criterion)**.** *Suppose X is a projective variety and D is a divisor on X. Then D is ample if and only if $\overline{\mathrm{NE}}(X) \setminus \{0\} \subset D_{>0}$.*

We end this section by remarking that all the notions that we have introduced have analogues in the relative setting, where instead of considering a proper variety or algebraic space X over some field, we look at a morphism $f : X \to Y$ of proper varieties or algebraic spaces over k.

For example, given $f : X \to Y$ and a divisor D on X, we say that D is relatively nef, or f-nef, if $D \cdot C \geq 0$ for all integral curves C on X for which $f(C)$ is a point. Similarly, the group of relative real 1-cycles $Z_1(X/Y)_{\mathbb{R}}$ consists of linear combinations $\sum c_i C_i$, where only finitely many of the c_i are nonzero and the C_i are integral curves on X for which $f(C_i)$ is a point. The relative version of Kleiman's ampleness criterion then says that a divisor D on X is relatively ample if and only if $\overline{\mathrm{NE}}(X/Y) \setminus \{0\} \subset D_{>0}$.

1.4 The Minimal Model Program

A natural question in algebraic geometry is that of the classification of say, smooth varieties over k. While one can attempt to classify these objects up to isomorphism, the point of view taken in birational geometry is to perform this classification up to a weaker equivalence relation:

Definition 1.4.1. We say that two varieties X and Y over k are birational if they contain isomorphic open dense subvarieties.

We note that by Chow's Lemma every variety is birational to a projective variety. Tentatively, the question then becomes: Can we classify smooth projective varieties up to birational equivalence?

Consider the case of curves: If two smooth projective curves are birational, then they are already isomorphic, so we get a unique smooth projective representative in each birational class. We can try to understand curves by looking at their genus $g = g(C)$. If $g = 0$, then $C \cong \mathbb{P}^1$ and $-K_C$ is ample, so C is Fano. If $g = 1$, then C

is an elliptic curve and K_C is trivial, so C is Calabi-Yau. Finally, if $g \geq 2$, then K_C is ample, so C is of general type.

The hope is that the positivity of K_X can help us understand higher-dimensional varieties X. Already when X is a surface, the classification is much more complicated. We outline some of the ideas behind the classification here; for more details see for example [3] or [2].

Definition 1.4.2. Suppose X is a smooth projective surface and $C \subset X$ is a curve with self-intersection $C^2 = -1$ and such that $C \cong \mathbb{P}^1$. We say that C is a (-1)-curve.

Theorem 1.4.3. *(Castelnuovo's Contractibility Criterion) Let C be a (-1)-curve on a smooth projective surface. Then there exists a birational morphism $\pi : X \to Y$, such that Y is a smooth projective surface, $\pi(C)$ is a point, π is the blow-up of Y at $\pi(C)$, and C is the exceptional divisor of this blow-up.*

Definition 1.4.4. We say that a smooth projective surface X is minimal if every birational morphism $\pi : X \to Y$ to another smooth projective surface is an isomorphism.

We recall that if $\pi : Y' = \mathrm{Bl}_p Y \to Y$ is the blow-up of Y at a point and E is the exceptional divisor, then $\mathrm{NS}(Y') \cong \mathrm{NS}(Y) \oplus \mathbb{Z}[E]$. Therefore, every time we contract a (-1)-curve on a surface, we decrease the rank of the Néron-Severi group by 1.

Say that we start with a smooth projective surface X. We know that $\mathrm{NS}(X)$ is finitely-generated, so after finitely many contractions we must be left with another smooth projective surface X' that contains no (-1)-curves. This X' is then minimal. Therefore, if we want to classify the smooth projective surfaces up to birational equivalence, it is then enough to classify the minimal smooth projective surfaces. This classification was done by Enriques and Kodaira.

Definition 1.4.5. Let D be a Cartier divisor on a variety X. The Iitaka dimension of D, denoted $\kappa(D)$ is the maximal dimension of the image of the rational maps

$\phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$, where $m \geq 1$. This definition makes sense even when D is only $\mathbb{Q}$-Cartier. When the linear system $|mD|$ is empty for every $m \geq 1$, we say that $\kappa(D) = -\infty$. When $D = K_X$, we write $\kappa(X) = \kappa(K_X)$ and call this the Kodaira dimension of X.

Remark. If $\dim(X) = n$, then $\kappa(X) = n$ if and only if K_X is big.

Theorem 1.4.6. *Suppose X is a smooth projective surface with $\kappa(X) \geq 0$. Then there is a unique minimal smooth projective surface birational to X.*

Remark. The above result does not hold when $\kappa(X) = -\infty$. For example, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational and both are minimal.

Definition 1.4.7. A surface X is a ruled surface if it is birational to $C \times \mathbb{P}^1$ for some smooth projective curve C.

Definition 1.4.8. A surface X is elliptic if there is a smooth curve C and a surjective morphism $X \to C$ whose generic fibre is an elliptic curve.

Theorem 1.4.9. *(Enriques-Kodaira Classification of Minimal Surfaces) Suppose X is a minimal smooth projective surface over an algebraically closed field k of characteristic 0. Let $\rho_g = h^0(X, \Omega_X^2)$ be the geometric genus and $q = h^1(X, \mathcal{O}_X)$ be the irregularity. Then:*

1. *If $\kappa(X) = -\infty$, then $X \cong \mathbb{P}^2$ or X is a minimal ruled surface over a smooth curve C. In particular,*
   (a) *If C is rational, then X itself is rational. Moreover, $X \cong \mathbb{P}^2$ or X is a Hirzebruch surface $X \cong \mathbb{F}_n$ for some $n \neq 1$.*
   (b) *If C is irrational, then $X \cong \mathbb{P}_C(E)$ for some rank two vector bundle E over C.*
2. *If $\kappa(X) = 0$, then there are four possibilities:*

(a) $\rho_g = 0, q = 0$: *X is an Enriques surface.*

(b) $\rho_g = 0, q = 1$: *X is a hyperelliptic surface.*

(c) $\rho_g = 1, q = 0$: *X is a K3 surface.*

(d) $\rho_g = 1, q = 2$: *X is an abelian surface.*

3. *If $\kappa(X) = 1$, then X is an elliptic surface.*

4. *if $\kappa(X) = 2$, then X is of general type.*

Remark. Not all elliptic surfaces have $\kappa(X) = 1$. In fact, all Enriques surfaces and some K3 are elliptic.

Remark. There are some additional types of surfaces that appear if we work in characteristic 2 or 3.

We now explain a different perspective for finding minimal surfaces, which will generalise to higher dimensions. Recall that the adjunction formula for surfaces says that $K_X \cdot C + C^2 = 2g(C) - 2$, so if we know that $C^2 = -1$ and C is rational, then we obtain $K_X \cdot C = -1$. Therefore, if X contains a (-1)-curve, then the canonical divisor K_X is not nef.

The Cone Theorem says that $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$, where the $\mathbb{R}_{\geq 0}[C_i]$ are extremal rays. We can then attempt to perform the contraction of the (-1)-curves from the point of these K_X-negative extremal rays.

Definition 1.4.10. A contraction is a morphism $f : X \to Z$ such that $f_\star \mathcal{O}_X = \mathcal{O}_Z$. If X and Z are normal, then this says precisely that the fibres of f are connected and non-empty.

Definition 1.4.11. Given a projective variety X and an extremal face $F \subset \overline{\mathrm{NE}}(X)$, a morphism $f : X \to Z$ is called the contraction of F, and we write $f = \mathrm{cont}_F$ if the following conditions hold:

1. f is a contraction.

2. Let $C \subset X$ be an irreducible curve. Then $f(C)$ is a point if and only if $[C] \in F$.

Remark. Given an extremal face $F \subset \overline{\mathrm{NE}}(X)$, the contraction cont_F may or may not exist.

Theorem 1.4.12. *(Contraction Theorem for Smooth Projective Surfaces) Let X be a smooth projective surface and $R \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray. Then the contraction $\mathrm{cont}_R : X \to Z$ exists and satisfies one of the following:*

1. *Z is a smooth surface and X is obtained from Z by blowing-up a closed point.*
2. *Z is a smooth curve and X is a minimal ruled surface over Z.*
3. *Z is a point and $X \cong \mathbb{P}^2$.*

So if we start with a smooth projective surface X, we follow the following algorithm:

1. If K_X is nef, then X is already minimal, so we stop. If K_X is not nef, then contract a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$.
2. Consider the contraction $\mathrm{cont}_R : X \to Z$. If $\dim(Z) < 2$, then we stop. If $\dim(Z) = 2$, then we go back to Step 1.

Definition 1.4.13. We say that X has a Mori fibre space structure if there exists a morphism $\pi : X \to Z$ with connected fibres, such that $\dim(Z) < \dim(X)$, $-K_X$ is relatively ample, and the relative Picard rank $\rho(X/Z) = \rho(X) - \rho(Z)$ is equal to 1.

Thus, the algorithm for surfaces terminates in either a Mori fibre space or in a smooth minimal model. We might hope that these steps generalise to higher-dimensional varieties. Mori, see [39], described the possible contractions of K_X-negative extremal rays for a smooth projective 3-fold over $\mathbb{C}$:

Theorem 1.4.14. *Let X be a non-singular projective 3-fold over $\mathbb{C}$ and $\mathrm{cont}_R : X \to Z$ the contraction of a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$. The following is a list of all possibilities for cont_R:*

1. $\dim(Z) = 3$, cont_R *is birational, and there are five types of local behaviour near the contracted surface:*
   (a) cont_R *is the (inverse of the) blow-up of a smooth curve in the smooth 3-fold Z.*
   (b) cont_R *is the (inverse of the) blow-up of a smooth point in the smooth 3-fold Z.*
   (c) cont_R *is the (inverse of the) blow-up of an ordinary double point of Z.*
   (d) cont_R *is the (inverse of the) blow-up of a point of Z which is locally analytically given by the equation $x^2 + y^2 + z^2 + w^3 = 0$.*
   (e) cont_R *contracts a smooth $\mathbb{CP}^2$ with normal bundle $\mathcal{O}(2)$ to a point of multiplicity 4 on Z which is locally analytically the quotient of $\mathbb{C}^3$ by the involution $(x, y, z) \mapsto (-x, -y, -z)$.*
2. $\dim(Z) = 2$ *and* cont_R *is a fibration whose fibres are plane conics.*
3. $\dim(Z) = 1$ *and general fibres of of* cont_R *are del Pezzo surfaces.*
4. $\dim(Z) = 0$, $-K_X$ *is ample and hence X is a Fano 3-fold.*

We see that in situations $1(c)$, $1(d)$, and $1(e)$, the resulting 3-fold Z is no longer smooth, and so we cannot proceed in the same was as we did for surfaces. For one thing, it is not clear that the Cone Theorem holds for singular Z. For another, when we worked with surfaces and cont_R was birational, it always contracted a divisor, which implied that the process terminated; for 3-folds, the exceptional set of $\mathrm{cont_R}$ could be of higher codimension like in $1(e)$, and then it is not clear that this terminates.

The Minimal Model Program aims to nevertheless make the process that we have described into a workable algorithm. For the MMP to run successfully, we need to settle a few questions: What singularities can arise as we perform contractions of extremal rays? Does the Cone Theorem hold for varieties with those singularities? Do the steps in the algorithm eventually terminate with a Mori fibre space or with a minimal model? We will summarise the answers to these questions in the following two subsections. For details and proofs, see for example [32] or [31].

1.4.1 Singularities of the MMP

In this subsection, we will describe the singularities that arise in the Minimal Model Program. Instead of looking at varieties, we will look at pairs (X, Δ) consisting of a variety X and a divisor Δ. Then, rather than looking at the intersection of K_X with curves on X, we will need to consider intersection products of $K_X + \Delta$ with curves. Therefore, we will want to choose Δ so that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Typically, we want to think of $K_X + \Delta$ as a small perturbation of K_X, and so the coefficients of Δ will usually be small.

Definition 1.4.15. A divisor $\Delta = \sum d_i D_i$ is said to be a boundary divisor if $0 \leq d_i \leq 1$ for all i.

We also want to have a notion of smoothness not just for a variety, but for a pair.

Definition 1.4.16. A divisor $\Delta = \sum d_i D_i$ is said to be a simple normal crossings (snc) divisor if each component D_i is smooth and these components intersect everywhere transversally.

Example 1.4.17. If L_1, L_2 are two distinct lines on $\mathbb{P}^2$, then $\Delta = L_1 + L_2$ is an snc divisor. However, if L_3 is a line passing through the intersection point of L_1 and L_2, then $\Delta' = L_1 + L_2 + L_3$ is not snc.

Definition 1.4.18. We say that a pair (X, Δ) is log smooth if X is smooth and D

is an snc divisor.

Example 1.4.19. The pair $(\mathbb{P}^2, L_1 + L_2)$ is log smooth. Also, if X is any smooth variety, then the pair $(X, 0)$ is log smooth.

Definition 1.4.20. A log resolution of a pair (X, Δ) is a proper birational morphism $f : Y \to X$, such that Y is smooth, $\mathrm{Ex}(f)$ is a divisor, and the divisor $\mathrm{Ex}(f) \cup f^{-1}(\Delta)$ is snc.

That log resolutions exist for varieties over fields of characteristic 0 is a result of Hironaka's; see [17].

Theorem 1.4.21. *(Hironaka's Resolution of Singularities) Let X be an irreducible algebraic variety over $\mathbb{C}$ and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme Z. Then there exist a smooth variety Y and a projective morphism $f : Y \to X$, such that*

1. *f is an isomorphism over $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Supp}(Z))$,*
2. *$f^\star \mathcal{I} \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$, and*
3. *$\mathrm{Ex}(f) \cup D$ is an snc divisor.*

We can use resolutions to measure the singularities of the pair (X, Δ).

Definition 1.4.22. Suppose $(X, \Delta = \sum d_i D_i)$ is a pair such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Whenever $f : Y \to X$ is a birational morphism such that Y is normal, we can write $K_Y = f^\star(K_X + \Delta) + \sum a(E, X, D)E$, where the only non-exceptional divisors in the sum are the $f_\star^{-1} D_i$ with coefficient $-d_i$. A component of E that is f-exceptional is said to be an exceptional divisor over X. We define the discrepancy of (X, Δ) to be

$$\mathrm{discrep}(X, \Delta) = \inf_E \{a(E, X, D) : E \text{ is an exceptional divisor over } X\}.$$

Moreover, the total discrepancy of (X, Δ) id given by

$$\text{totaldiscrep}(X, \Delta) = \inf_E\{a(E, X, D) : E \text{ is a divisor over } X\}.$$

Example 1.4.23. Consider the pair $(X, 0)$, where X is a smooth variety. Let $Z \subset X$ be a codimension k smooth subvariety, and consider the blow-up $f : \text{Bl}_Z X \to X$ with exceptional divisor E. Then $K_{\text{Bl}_Z X} = f^\star K_X + (k-1)E$, so that $a(E, X, 0) = k - 1$. From this it follows that $\text{discrep}(X, 0) = 1$.

Proposition 1.4.24. *With the set-up from above, either* $\text{discrep}(X, \Delta) = -\infty$ *or* $-1 \leq \text{totaldiscrep}(X, \Delta) \leq \text{discrep}(X, \Delta) \leq 1$.

Definition 1.4.25. Let $(X, \Delta = \sum d_i D_i)$ be a pair such that X is normal and $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that

1. (X, Δ) is terminal if $\text{discrep}(X, \Delta) > 0$;
2. (X, Δ) is canonical if $\text{discrep}(X, \Delta) \geq 0$;
3. (X, Δ) is kawamata log terminal (klt) if $\text{discrep}(X, \Delta) > -1$ and $0 \leq d_i < 1$ for all i;
4. (X, Δ) is purely log terminal (plt) if $\text{discrep}(X, \Delta) > -1$;
5. (X, Δ) is log canonical if $\text{discrep}(X, \Delta) \geq -1$.

Remark. When $\Delta = 0$, the notions of klt and plt agree, and we simply say that X has log terminal (lt) singularities.

Remark. We can check whether a pair has klt or lc singularities by looking at the discrepancy coming from a single, arbitrary log resolution of the pair.

Each of these classes of singularities appears naturally in various settings of birational geometry. For example, when $\Delta = 0$, terminal singularities are the smallest

class that we need to make sense of minimal models of smooth varieties. At the other end of the spectrum, lc singularities are the largest class for which discrepancies make sense, though they are typically hard to work with.

Remark. If $(X, \Delta = \sum d_i D_i)$ has lc singularities, then by Proposition 1.4.24 we have $d_i \leq 1$ for all i.

Example 1.4.26. If X is smooth, then $(X, 0)$ has terminal singularities, and we typically just say that X has terminal singularities. More generally, if $(X, \Delta = \sum d_i D_i)$ is log smooth and $0 \leq d_i < 1$ for all i, then (X, Δ) has klt singularities.

Example 1.4.27. Suppose X is a surface. Then X is terminal if and only if X is smooth. X is canonical if and only if it is smooth or has Du Val singularities, and these are precisely those singularities that are analytically isomorphic to quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Log terminal surface singularities are precisely those that are analytically isomorphic to quotients of $\mathbb{C}^2$ by finite groups. Log canonical surface singularities are precisely those that are analytically isomorphic to a quotient of $\mathbb{C}^2$, an elliptic singularity, or a cusp singularity by a finite group.

We will use one more class of singularities that is better behaved than lc singularties but a bit more general than klt singularities.

Definition 1.4.28. Let (X, Δ) be a pair where X is normal, Δ is a boundary divisor, and $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is divisorially log terminal (dlt) if there is closed subset $Z \subset X$ such that

1. $X \setminus Z$ is smooth and $\Delta|_{X \setminus Z}$ is an snc divisor, and
2. If $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $\mathrm{center}_X E \subset Z$, then $a(E, X, \Delta) > -1$.

Proposition 1.4.29. *A dlt pair $(X, \Delta = \sum d_i D_i)$ is klt if and only if $\lfloor \Delta \rfloor = \sum \lfloor d_i \rfloor D_i = 0$.*

Informally, this says that a dlt pair (X, Δ) is lc, and wherever (X, Δ) is not klt, Δ is an snc divisor. It turns out that dlt pairs form the smallest class of singularities for which the MMP handles the case where X is smooth and Δ is an snc boundary divisor, and for which the steps of the MMP do not lead outside of the starting class. Therefore, when we start with a lc pair, it is very useful to be able to reduce to the dlt case.

Definition 1.4.30. A dlt modification of a pair (X, Δ) is a projective, birational morphism $g^{\mathrm{dlt}} : X^{\mathrm{dlt}} \to X$, such that

1. $(X^{\mathrm{dlt}}, \left(g^{\mathrm{dlt}}\right)_\star^{-1} \Delta + E^{\mathrm{dlt}})$ is dlt, where E^{dlt} is the sum of all g^{dlt}-exceptional divisors with coefficient 1, and

2. $K_{X^{\mathrm{dlt}}} + \left(g^{\mathrm{dlt}}\right)_\star^{-1} \Delta + E^{\mathrm{dlt}}$ is g^{dlt}-nef.

Theorem 1.4.31. *(Odaka-Xu; see [42]) A normal, quasi-projective pair (X, Δ) over a field of characteristic 0 for which $K_X + \Delta$ is $\mathbb{R}$-Cartier has a dlt modification.*

Remark. In general, dlt modifications are not unique. In fact, there often are infinitely many dlt modifications.

Finally, we will need the fact that if we start with pair with mild singularities, we can often perturb the divisor and the singularities of the resulting pair remain mild.

Theorem 1.4.32. *Let (X, Δ) be a pair with Δ effective, and let Δ' be an effective $\mathbb{Q}$-Cartier divisor. Then*

1. *If $(X, \Delta + \Delta')$ is terminal (resp. canonical, klt, plt, dlt, lc), then (X, Δ) is also terminal (resp. canonical, klt, plt, dlt, lc).*

2. *If (X, Δ) is terminal (resp. klt), then $(X, \Delta + \epsilon\Delta')$ is also terminal (resp. klt) for $0 \leq \epsilon \ll 1$.*

3. *If (X, Δ) is dlt, then $(X, \Delta + \epsilon\Delta')$ is also dlt for $0 \leq \epsilon \ll 1$, assuming that $\mathrm{Supp}(\Delta') \subset \mathrm{Supp}(\Delta - \lfloor \Delta \rfloor)$.*

1.4.2 Running the MMP

In this section, we will discuss the steps of the MMP. Rather than working in the absolute setting, we will work in the relative setting mentioned in Section 1.3. We will start with a projective morphism $f : (X, \Delta) \to Y$ of projective varieties, where (X, Δ) has $\mathbb{Q}$-factorial, dlt singularities. The first ingredient is that the Cone Theorem and the Contraction Theorem both hold in this level of generality.

Theorem 1.4.33. *(Relative Cone and Contraction Theorems) Let $f : X \to Y$ be a projective morphism of projective varieties over a field of characteristic 0, where X is $\mathbb{Q}$-factorial and (X, Δ) is dlt. Then:*

1. *There are countably many rational curves $C_j \subset X$ such that $f(C_j)$ is a point, $0 < -(K_X + \Delta) \cdot C_j \leq 2\dim(X)$, and*

$$\overline{\mathrm{NE}}(X/Y) = \overline{\mathrm{NE}}(X/Y)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

2. *For any $\epsilon > 0$ and f-ample divisor H,*

$$\overline{\mathrm{NE}}(X/Y) = \overline{\mathrm{NE}}(X/Y)_{(K_X+\Delta+\epsilon H)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j],$$

   where the sum is finite.

3. *Let $F \subset \overline{\mathrm{NE}}(X/Y)$ be a $(K_X + \Delta)$-negative extremal face. Then there is a unique morphism $\mathrm{cont}_F : X/Y \to Z/Y$ such that $(\mathrm{cont}_F)_\star \mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by cont_F if and only if $[C] \in F$. cont_F is called the contraction of F. Z is projective over Y.*

4. *Let F and cont_F be as in 3. Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there exists a line bundle L_Z on Z such that $L \cong (\mathrm{cont}_F)^{-1}_\star L_Z$.*

From now on, we will suppose that $F = R$ is an extremal ray. We have three possibilities for the contraction morphism $\mathrm{cont}_R : X \to Z$. First, it could happen that $\dim(Z) < \dim(X)$, and we say that cont_R is a Fano contraction. Otherwise, the contraction morphism is birational and: either the exceptional set is a divisor, in which case we call cont_R a divisorial contraction, or it is a higher codimension subset of X, in which case we call cont_R a small contraction. We explain how to deal with each one of these three cases.

Proposition 1.4.34. *Let (X, Δ) be a projective, $\mathbb{Q}$-factorial dlt pair and $\mathrm{cont}_R : X \to Z$ the contraction of a $(K_X + \Delta)$-negative extremal ray. Assume that cont_R is either a divisorial or a Fano contraction. Then*

1. *Z is $\mathbb{Q}$-factorial, and*
2. *$\rho(Z) = \rho(X) - 1$.*

Unfortunately, if $\mathrm{cont}_R : X \to Z$ is a small contraction and X is $\mathbb{Q}$-factorial, then Z is no longer $\mathbb{Q}$-factorial. Instead of keeping Z for the next step on the MMP, we will replace X by another variety X^+ as follows:

Definition 1.4.35. Let X be a normal scheme and Δ an $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A $(K_X + \Delta)$-flipping contraction is a proper birational morphism $f : X \to Z$ to a normal scheme Z such that $\mathrm{Ex}(f)$ has codimension at least 2 in X and $-(K_X + \Delta)$ is f-ample. A normal scheme X^+ together with a proper birational morphism $f^+ : X^+ \to Z$ is called a $(K_X + \Delta)$-flip if

1. $K_{X^+} + \Delta^+$ is $\mathbb{R}$-Cartier, where Δ^+ is the birational transform of Δ on X^+,
2. $K_{X^+} + \Delta^+$ is f^+-ample, and
3. $\mathrm{Ex}(f^+)$ has codimension at least 2 in X^+.

We also call the induced rational map $\phi : X \dashrightarrow X^+$ a $(K_X + \Delta)$-flip.

When flips exist they are unique. The existence of flips in dimension 3 was proved by Mori, see [40]; in dimension 4 by Shokurov, see [48]; and in arbitrary dimension by Birkar, Cascini, Hacon, and McKernan, see [6]:

Theorem 1.4.36. *(BCHM) Let (X, Δ) be a klt pair and let $\pi : X \to Z$ be a small $(K_X + \Delta)$-extremal contraction. Then the flip of π exists.*

Proposition 1.4.37. *Let (X, Δ) be a projective, $\mathbb{Q}$-factorial dlt pair and $\mathrm{cont}_R : X \to Z$ the flipping contraction of a $(K_X + \Delta)$-negative extremal ray with flip $\mathrm{cont}_R^+ : X^+ \to Z$. Then*

1. *X^+ is $\mathbb{Q}$-factorial, and*
2. *$\rho(X^+) = \rho(X)$.*

Proposition 1.4.38. *Let (X, Δ) be a dlt pair. Let $g : X \dashrightarrow X'$ be either a divisorial contraction of a $(K_X + \Delta)$-negative extremal ray or a $(K_X + \Delta)$-flip. Let $\Delta' = g_\star \Delta$. Then (X', Δ') is also dlt.*

Putting everything together, we can now run the MMP:

Algorithm 1.4.39. *Start with a dlt pair $(X, \Delta) = (X_0, \Delta_0)$, where X is $\mathbb{Q}$-factorial, and a projective morphism $f_0 : X_0 \to S$.*

1. *If $K_{X_0} + \Delta_0$ is f_0-nef, then we stop. If not, then by the Cone Theorem we can find a $(K_{X_0} + \Delta_0)$-negative extremal ray $R_0 \subset \overline{\mathrm{NE}}(X/Y)$, and by the Contraction Theorem we can contract this ray. Let $\mathrm{cont}_{R_0} : X_0/Y \to Z_0/Y$ be this contraction. We have a commutative diagram*

$$\begin{array}{ccc} X_0 & \xrightarrow{\mathrm{cont}_{R_0}} & Z_0 \\ & {\scriptstyle f_0}\searrow \quad \swarrow{\scriptstyle g_0} & \\ & Y & \end{array}$$

2. *If cont_{R_0} is a Fano contraction, then we stop.*

3. *If* cont_{R_0} *is a divisorial contraction, then we put* $X_1 = Z_0$, $f_1 = g_0$, *and* $\Delta_1 = (\mathrm{cont}_{R_0})_\star \Delta_0$. *Then* (X_1, Δ_1) *is dlt,* X_1 *is* $\mathbb{Q}$*-factorial, and* $f_1 : X_1 \to Y$ *is projective, so we go back to Step 1 with these new data.*

4. *If* cont_{R_0} *is a small contraction, then we set* $(X_1, \Delta_1) = (X_0^+, \Delta_0^+)$ *(the flip of* cont_{R_0}*) and* $f_1 = g_0 \circ f_0^+$. *Then* (X_1, Δ_1) *is dlt,* X_1 *is* $\mathbb{Q}$*-factorial, and* $f_1 : X_1 \to Y$ *is projective, so we go back to Step 1 with these new data.*

If this process stops, then we are left with a Mori fibre space or with a pair $(X^{\min}, \Delta^{\min})$ such that $K_{X^{\min}} + \Delta^{\min}$ is nef over Y, and we call this a relative minimal model. What remains to be established is whether this algorithm terminates. We note that there can only be a finite number of divisorial contraction steps: indeed, every time we contract a divisor, we Picard rank drops by 1. It is not clear, however, that infinite sequences of flips cannot arise. Termination of flips is not known in complete generality.

In [6], rather than considering the MMP as we have described it here, the authors run the so-called MMP with scaling. Informally, this means that they do not contract arbitrary $K_X + \Delta$-negative extremal rays in $\overline{\mathrm{NE}}(X/Y)$. Instead, they choose a divisor A on X that is ample over Y, so that for t large enough, $K_X + \Delta + tA$ is also ample over Y. Now the value of t can be decreased until at some t_0 we have for the first time some rational curve C with $(K_X + \Delta + t_0 A) \cdot C = 0$. Such a rational curve generates a $K_X + \Delta$-negative extremal ray, and then such a ray is contracted. This scaling gives enough structure that it can be shown that under certain conditions, any sequence of flips for the MMP with scaling terminates.

Theorem 1.4.40. *(BCHM) Let* $f : X \to Y$ *be a projective morphism of normal quasi-projective varieties. Let* (X, Δ) *be a* $\mathbb{Q}$*-factorial klt pair, where* $K_X + \Delta$ *is* $\mathbb{R}$*-Cartier and* Δ *is* f*-big. Let* $H \geq 0$ *be an* $\mathbb{R}$*-divisor. If* $K_X + \Delta + H$ *is klt and* f*-nef, then we may run the* $(K_X + \Delta)$*-MMP over* Y *with scaling of* H. *In particular, this*

MMP terminates.

We will also rely on the following result by Fujino; see [11].

Theorem 1.4.41. *Assume that $f : X \to Y$ is a projective morphism of normal quasi-projective varieties over a field of characteristic 0, (X, Δ) is $\mathbb{Q}$-factorial and dlt, and $K_X + \Delta \sim_{f,\mathbb{Q}} E$, where E is effective and f-exceptional. Then (X, Δ) has a dlt, $\mathbb{Q}$-factorial, minimal model over Y.*

Chapter 2

An Equivariant Grothendieck-Lefschetz Theorem

2.1 Introduction

The Grothendieck-Lefschetz Theorem for Picard Groups, see Corollaries XII.3.6 and XII.3.7 in [13] or Corollary IV.3.3 in [15], states that if X is a nonsingular projective variety over a field of characteristic zero and Y is a subvariety of X, such that $\dim(Y) \geq 3$ and Y is a complete intersection in X, then the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism. In positive characteristic, there are analogous results by Ji; see Theorem 1.1 in [18].

If G is an algebraic group acting on X, we let $\mathrm{Pic}^G(X)$ denote the equivariant Picard group of X, as defined in Chapter 1.3 of [41]; in other words, the elements of $\mathrm{Pic}^G(X)$ are line bundles over X with a G-linearisation, and this is a group via the tensor product. It was recently shown, see [44], that a version of Grothendieck-Lefshetz holds for the equivariant Picard group, provided that the group G is assumed to be finite. This was done by adapting the proof of Grothendieck-Lefschetz in [15] to deal with the group action.

Here, we drop the finiteness hypothesis and prove the result for any affine algebraic group. More specifically, we prove the following:

Theorem 2.1.1. *Let k be an algebraically closed field of characteristic zero. Let X be a nonsingular projective variety over k and let G be an affine algebraic group over k acting on X. Let Y be a G-invariant normal subvariety of X, such that $\dim(Y) \geq 3$ and Y is a complete intersection in X. Then the natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ is an isomorphism. If instead we assume that $\dim(Y) = 2$, then the natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ is injective.*

2.2 Equivariant Grothendieck-Lefschetz

Before proving Theorem 2.1.1, we need a technical lemma that will give us an exact sequence involving $\mathrm{Pic}^G(X)$. Throughout this section, we will assume that varieties are irreducible.

Lemma 2.2.1. *Let X be a normal proper variety over k and G be an affine algebraic group over k acting on X. Then there exists an exact sequence of abelian groups*

$$0 \to \mathrm{Hom}_{\mathrm{GpSch}}(G, \mathcal{O}_X(X)^\times) \to \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)^G \to \mathrm{Ext}^1_{\mathrm{GpSch}}(G, \mathcal{O}_X(X)^\times).$$

Proof. Let $\mathrm{Ext}^1_{\mathrm{Gp}}(G, \mathcal{O}_X(X)^\times)$ denote the set of (not necessarily central) extensions of abstract groups, where $\mathcal{O}_X(X)^\times$ is viewed as a G-module via the morphism $G \to \mathrm{Aut}(X)$. Since $\mathcal{O}_X(X)^\times$ is an abelian group, then the set $\mathrm{Ext}^1_{\mathrm{Gp}}(G, \mathcal{O}_X(X)^\times)$ is itself endowed with the structure of an abelian group, with the Baer sum as the group operation; see Ex. 7 on page 114 of [37]. We can then consider the subset $\mathrm{Ext}^1_{\mathrm{GpSch}}(G, \mathcal{O}_X(X)^\times)$ of extensions of group schemes. In fact, from the definition of the Baer sum, we can see that $\mathrm{Ext}^1_{\mathrm{GpSch}}(G, \mathcal{O}_X(X)^\times)$ is closed under the group operation, so that it is a subgroup of $\mathrm{Ext}^1_{\mathrm{Gp}}(G, \mathcal{O}_X(X)^\times)$.

We define the map $\epsilon : \mathrm{Pic}(X)^G \to \mathrm{Ext}^1_{\mathrm{GpSch}}(G, \mathcal{O}_X(X)^\times)$ exactly as in Section 3.4 of [7]. We will introduce some notation to spell out this definition. For a line bundle L over X, we define $\mathrm{Aut}(X)_L = \{\phi \in \mathrm{Aut}(X) : \phi^\star L \cong L\}$ to be the subgroup of $\mathrm{Aut}(X)$ that stabilises L, and we let $\mathrm{Aut}^{\mathbb{G}_m}(L)$ denote the group of automorphisms of L that commute with the action of $\mathbb{G}_m$ by multiplication on the fibres of L.

By Lemma 3.4.1 in [7], we have an extension

$$1 \to \mathbb{G}_m \to \mathrm{Aut}^{\mathbb{G}_m}(L) \to \mathrm{Aut}(X)_L \to 1.$$

If, furthermore, $L \in \mathrm{Pic}(X)^G$, then the image of the homomorphism $G \to \mathrm{Aut}(X)$ is contained in $\mathrm{Aut}(X)_L$. Thus, we can pull back the given group extension by the homomorphism $G \to \mathrm{Aut}(X)_L$, and we obtain another short exact sequence

$$1 \to \mathbb{G}_m \to \mathcal{G}(L) \to G \to 1,$$

where $\mathcal{G}(L)$ is called the lifting group associated to L. We then define

$$\epsilon(L) = (1 \to \mathbb{G}_m \to \mathcal{G}(L) \to G \to 1).$$

We observe that since the morphism $G \to \mathrm{Aut}(X)$ is algebraic, then $\epsilon(L)$ is an extension of group schemes.

We now let $\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)^G$ be the homomorphism that forgets the linearisation. By Corollary 7.1 of [8], we have that $\ker(\alpha) = \mathrm{Hom}_{\mathrm{GpSch}}(G, \mathcal{O}_X(X)^\times)$.

As in Remark 3.4.3(i) of [7], $L \in \mathrm{Pic}(X)^G$ is linearisable if and only if the extension $\epsilon(L)$ splits as an extension of abstract groups and the resulting action of G on L is algebraic. Note that this latter condition is automatic when G is finite; in general, what we then require is that the extension $\epsilon(L)$ splits when viewed as an extension of algebraic groups, rather than just abstract groups. Indeed, in this situation, the

induced group homomorphism $G \to \mathcal{G}(L) \to \mathrm{Aut}^{\mathbb{G}_m}(L)$ will be algebraic, and hence we obtain a linearisation of L. Thus, we have $\mathrm{im}(\alpha) = \ker(\epsilon)$, which gives our assertion. □

Theorem 2.2.2. *Let k be an algebraically closed field. Let $Y \subset X$ be normal proper k-varieties. Let G be an affine algebraic group over k acting on X, and suppose Y is G-invariant. If the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective (resp. an isomorphism), then the natural map on equivariant Picard groups $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ is also injective (resp. an isomorphism).*

Proof. Since k is algebraically closed, then $\mathcal{O}_X(X)^\times = k^\times$ and $\mathcal{O}_Y(Y)^\times = k^\times$. By Lemma 2.2.1, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(G, k^\times) & \longrightarrow & \mathrm{Pic}^G(X) & \longrightarrow & \mathrm{Pic}(X)^G & \longrightarrow & \mathrm{Ext}^1_{\mathrm{GpSch}}(G, k^\times) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(G, k^\times) & \longrightarrow & \mathrm{Pic}^G(Y) & \longrightarrow & \mathrm{Pic}(Y)^G & \longrightarrow & \mathrm{Ext}^1_{\mathrm{GpSch}}(G, k^\times)
\end{array}$$

The two leftmost and the rightmost vertical arrows are obviously isomorphisms. By hypothesis, the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective (resp. an isomorphism), and because it is G-equivariant, it induces an injection (resp. an isomorphism) of the G-invariant Picard groups $\mathrm{Pic}(X)^G \to \mathrm{Pic}(Y)^G$. In other words, the second vertical arrow (counting from the right) is injective (resp. an isomorphism). By the Five Lemma, the middle arrow is injective (resp. an isomorphism) as well. □

We now obtain the equivariant version of Grothendieck-Lefschetz as an immediate corollary:

Corollary 2.2.3 (cf. Theorem 2.1.1)**.** *Let k be an algebraically closed field of characteristic zero. Let X be a nonsingular projective variety over k and let G be an affine algebraic group over k acting on X. Let Y be a G-invariant normal subvariety of X, such that $\dim(Y) \geq 3$ and Y is a complete intersection in X. Then the natural map*

$\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ *is an isomorphism. If instead we assume that* $\dim(Y) = 2$*, then the natural map* $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ *is injective.*

Proof. In view of Theorem 2.2.2, we just need to show that $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective when $\dim(Y) = 2$ and an isomorphism when $\dim(Y) \geq 3$. This is exactly the conclusion of the usual Grothendieck-Lefschetz Theorem for Picard Groups. □

Remark. If $X = \mathbb{P}^n_k$, then Corollary 2.2.3 holds even if we drop the hypothesis that k has characteristic zero. This is due to Corollary XII.3.7 of [13].

Chapter 3

Rational Curves on Algebraic Spaces and Projectivity Criteria

3.1 Introduction

A proper, irreducible, reduced analytic space Y is called a Moishezon space if it is bimeromorphic to some projective variety. Any two such projective varieties are birational, and thus Y is endowed with a unique algebraic structure. In fact, Artin showed in [1] that the category of Moishezon spaces is equivalent to the category of proper algebraic spaces over $\mathbb{C}$. Starting in dimension 3, there exist Moishezon spaces that are not projective varieties, or even schemes. Hironaka constructed two such examples, which can be found in the Appendix to [16]. We note that the threefolds in Hironaka's examples contain rational curves whose sum is numerically trivial.

Peternell conjectured that any Moishezon manifold without rational curves is projective, and in [43] he proved this in dimension 3. In [47], Shokurov proved Peternell's conjecture in arbitrary dimension when Y has mild singularities, assuming the existence and termination of flips. In view of [6], the assumptions in Shokurov's proof are satisfied. The precise statement is that if $\psi : (Y, \Delta) \to U$ is a proper, but not pro-

jective, morphism of normal algebraic spaces, where (Y, Δ) is analytically $\mathbb{Q}$-factorial and dlt, then ψ contracts a rational curve in Y. However, this is clearly not enough to characterise the non-projectivity of ψ. In Theorem 5.1.4 of [24], Kollár shows that a proper, algebraic threefold Y with normal, $\mathbb{Q}$-factorial singularities is non-projective if and only if Y contains a curve C such that $-[C] \in \overline{\mathrm{NE}}(Y)$.

We extend Kollár's characterisation to arbitrary dimension and to the relative setting, but with an additional restriction on the singularities of Y. Our main result is:

Theorem 3.1.1. *Suppose that $\psi : (Y, \Delta) \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0 and that (Y, Δ) has klt singularities. If ψ is non-projective, then*

1. *either Y contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$,*
2. *or Y has a small, $\mathbb{Q}$-factorial modification Y^{qf} that is projective over U (more precisely, the composite morphism $Y^{\mathrm{qf}} \to Y \to U$ is projective).*

Corollary 3.1.2. *Suppose that $\psi : Y \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0, and that Δ is some divisor on Y such that the pair (Y, Δ) has klt singularities. Assume additionally that Y is $\mathbb{Q}$-factorial. Then ψ is non-projective if and only if Y contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$.*

In particular, when $k = \mathbb{C}$, $U = \mathrm{Spec}(k)$, we get:

Corollary 3.1.3. *Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities. Then Y is non-projective if and only if it contains a rational curve C such that $-[C] \in \overline{\mathrm{NE}}(Y)$.*

As a consequence of Corollary 3.1.2, we can prove a version of Kleiman's criterion for Moishezon spaces:

Corollary 3.1.4. *Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities and that L is a Cartier divisor on Y. Then L is ample if and only if L has positive degree on every irreducible curve on Y and L induces a strictly positive function on $\overline{\mathrm{NE}}(Y)$.*

In the works of Shokurov and Birkar-Cascini-Hacon-McKernan, they find a log resolution (X, Θ) of (Y, Δ) that is projective over U and run a minimal model program for (X, Θ) over (an étale-open affine patch of) U. The difference in our approach is that we run an MMP for (X, Θ) over Y, and this gives us a stronger result. Our main theorem will follow once we are able to run relative MMPs for $X \to Y$, where the base Y is an algebraic space of finite type over a field of characteristic 0.

Roughly speaking, we do this as follows: Y is an algebraic space, so it admits étale morphisms $Y_j \to Y$ whose images cover Y, where the Y_j are affine. We can then run a relative MMP for $X \times_Y Y_j$ over Y_j for each j, so it is enough to verify that the resulting minimal models patch together and descend to give a minimal model for X over Y. The right way to run the relative MMP for this application is to do as Kollár describes in [29], so that the steps of the MMP do not preserve $\mathbb{Q}$-factoriality; we need this precisely because étale base change does not preserve $\mathbb{Q}$-factoriality either.

The technical details are contained in Section 3.2. The precise MMP statement that we need for our applications is the following:

Theorem 3.1.5. *Suppose $g : (X, \Theta) \to (Y, \Delta)$ is a projective morphism of algebraic spaces of finite type over a field of characteristic 0, where (X, Θ) is $\mathbb{Q}$-factorial and dlt. Suppose that g has exceptional divisor $E = E_1 + \cdots + E_n$ and that $K_X + \Theta \sim_{g,\mathbb{R}} E_\Theta = \Sigma e_j E_j$ for some numbers $e_j \geq 0$. Finally, suppose that H is a divisor on X, such that $K_X + \Theta + cH$ is g-ample for some number c. Then we may run the g-relative $(K_X + \Theta)$-MMP with scaling of H.*

In our applications, we will start with an algebraic space Y of finite type over a field of characteristic 0. We will have some boundary divisor Δ on Y, such that the

pair (Y, Δ) has mild singularities, and $g : (X, \Theta) \to (Y, \Delta)$ will be a suitably chosen log resolution. We will need a divisor H on X, such that $K_X + \Theta + cH$ is ample over Y for some number c. If X is taken to be globally projective, we could, for example, choose H to be some ample divisor on X. In fact, to prove Theorem 3.1.1, we take H to be ample and general enough that each step in the resulting MMP arises from the contraction of an extremal ray in the relative cone of curves. See Section 3.3 for the proof of our projectivity criterion and its corollaries.

In addition to our characterisation of projectivity, Theorem 3.1.5 also allows us to prove:

Theorem 3.1.6. *Suppose (Y, Δ) is a log canonical pair, where Y is an algebraic space of finite type over a field of characteristic 0. Then (Y, Δ) admits a dlt modification.*

The proof of Theorem 3.1.6 is in Section 3.4. For this, we will want H to have support equal to $\mathrm{Ex}(g)$, which puts us in the setting of [29]; in this case, we choose our log resolution so that it carries a g-ample, g-exceptional divisor. This can be arranged by Theorem 1 in [33].

In [5], it is shown that, with certain assumptions, the minimal model program runs for threefolds in mixed characteristic, so our results extend to algebraic spaces of dimension 3 in mixed characteristic. There is also work in progress by Lyu and Murayama, [36], with the aim of showing that the MMP can be run over quasi-excellent schemes of characteristic 0. In view of this, our results would extend to the case where Y is an algebraic space of finite type over a quasi-excellent scheme of characteristic 0.

3.2 Technical Results on the Relative MMP

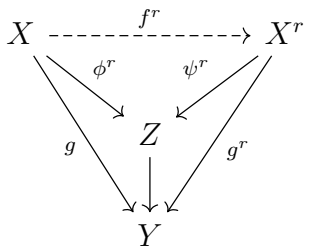
Our aim is to run the relative MMP over agebraic spaces, but we first develop our theory in a more abstract setting, where in place of $K_X + \Theta$ we consider an arbi-

trary $\mathbb{R}$-Cartier divisor D. Throughout this paper we will work under the following hypotheses:

(†) V will be assumed to be a quasi-excellent, normal scheme, Y a quasi-excellent, normal, irreducible algebraic space, and X a quasi-excellent, normal algebraic space. $\pi : V \to Y$ will be a universally open and quasi-finite morphism, and $g : X \to Y$ a projective morphism. D and H will denote $\mathbb{R}$-Cartier divisors on X, such that for some real number c, $D + cH$ is g-ample.

By decreasing the scaling constant c in (†), either we eventually reach a number $r > 0$, such that $D + rH$ is g-nef but not g-ample, and $D + (r + \epsilon)H$ is g-ample for all $\epsilon > 0$; or we eventually reach $r = 0$, in which case D is already g-nef, so we do not do anything.

We define $\phi^r : X \to Z$ to be the morphism that contracts all curves C in X, such that C is contracted by g and $(D + rH) \cdot C = 0$. These curves are those in some extremal face of the relative cone of curves $\overline{\mathrm{NE}}(X/Y)$. We will assume that the contraction ϕ^r and the corresponding step f^r of the relative D-MMP with scaling of H exist:



If the birational transforms $D^r = (f^r)_\star D$ and $H^r = (f^r)_\star H$ are $\mathbb{R}$-Cartier, then the divisor $D^r + (r - \epsilon)H^r$ on X^r is g^r-ample for sufficiently small $\epsilon > 0$. Then we obtain the next step of this relative MMP by decreasing $r - \epsilon$ and proceeding as above.

We are interested in how the steps of this MMP behave with respect to base change, so we consider the commutative diagram

$$\begin{array}{ccc} X_V = (X \times_Y V)^\nu & \xrightarrow{g_V} & V \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\quad g \quad} & Y \end{array}$$

where the superscript ν will always denote the normalisation. Note that, in view of our quasi-excellence hypothesis, the normalisation morphism is finite, and hence projective, and so π_V is quasi-finite and g_V is projective. Additionally, since π is universally open, every connected component of X_V dominates a component of X. We can then set $D_V = \pi_V^\star(D)$ and $H_V = \pi_V^\star(H)$. Observe that if $D + cH$ is g-ample, then $D_V + cH_V$ is g_V-ample, and so by decreasing c we can similarly define the steps of the relative D_V-MMP with scaling of H_V, assuming they exist.

We will index the outputs of these MMPs in a non-standard way. The steps of the MMP are typically indexed discretely, but this can lead to the steps of the MMP for $X \to Y$ and those of the MMP for $X_V \to V$ to be indexed differently: It could happen that as we decrease c in $D + cH$, we hit the first number r for which $D + rH$ stops being g-ample, forcing us to do an MMP step over Y, and yet $D_V + rH_V$ is still g_V-ample, so that for the moment no MMP step is performed over V. In this situation, the first step of the MMP over V would come later than the first step of the MMP over Y.

For this reason, we will index the outputs of the MMP continuously, rather than discretely, using the scaling constant as the index. More formally:

Notation. If it exists, the r^{th} output of the g-relative D-MMP with scaling of H, denoted by $f^r : X \dashrightarrow X^r$, will mean the composite $X \dashrightarrow X^{r_1} \dashrightarrow \cdots \dashrightarrow X^{r_n}$ for numbers $r_1 > \cdots > r_n \geq r$, where each r_i is such that $D^{r_{i-1}} + r_i H^{r_{i-1}}$ is nef, but not ample, over Y, $D^{r_{i-1}} + (r_i + \epsilon)H^{r_{i-1}}$ is ample over Y, and $D^{r_n} + (r - \epsilon)H^{r_n}$ is ample over Y (where $\epsilon > 0$ is sufficiently small). Each birational map in the composition

then corresponds to a relative MMP step as described above.

We note that when the relative MMP with scaling is run as in [6], we are allowed to have $r_i = r_{i+1}$, since each MMP step comes from the contraction of an extremal ray in the relative cone of curves. However, in [29], each MMP step is allowed to contract a higher-dimensional face of this cone, and we must have $r_i > r_{i+1}$. In particular, X^r does not vary as r decreases along the interval $(r_{i+1}, r_i]$.

Finally, we emphasise that when we talk about the r^{th} output, we do not mean the output after performing r MMP steps, but rather the output after the scaling constant has been decreased to r. For example, if r is such that $D + rH$ on X is still g-ample, then the r^{th} output $f^r : X \dashrightarrow X^r$ is simply the identity map on X.

Suppose we are in the setting of (†). We first give an abstract characterisation of the r^{th} output of our relative MMP.

Lemma 3.2.1. *Suppose the hypotheses in (†) are satisfied. The r^{th} output of the relative D-MMP over Y with scaling of H, if it exists, is characterised by the following properties:*

1. *$f^r : X \dashrightarrow X^r$ is a birational contraction to a normal variety over Y,*
2. *$f^r_\star(D + (r - \epsilon)H)$ is ample over Y for sufficiently small $\epsilon > 0$.*
3. *f^r only contracts divisors contained in the relative exceptional locus $\mathbb{E}(D + rH)$ of $D + rH$; that is, it only contracts divisors E for which the restriction $(D + rH)|_E$ is not big over Y.*

Proof. That the r^{th} output satisfies these three properties is clear. Suppose that $h : X \dashrightarrow X'$ is another birational map satisfying the same three properties. Consider the composite $f^r \circ h^{-1} : X' \dashrightarrow X^r$. We claim that this is an isomorphism in codimension 1. For this, it is enough to show that f^r and h contract the same divisors. Suppose E is some divisor contracted by f^r. Then $(D + rH)|_E$ is not big

over Y. Since $h_\star(D+(r-\epsilon)H)$ is ample, and in particular big, over Y, then h must also contract E: Otherwise, $h_\star$E would be a divisor on X^r on which $h_\star(D+(r-\epsilon)H)$ is not big. Reversing the roles of f and h, we see that X^r and X' have the same divisors.

We also know that $h_\star(D+(r-\epsilon)H)$ is ample over Y and its birational transform by $f^r \circ h^{-1}$, namely $f^r_\star(D+(r-\epsilon)H)$, is also ample over Y. We thus conclude that $f^r \circ h^{-1} : X' \dashrightarrow X^r$ is an isomorphism. This follows from the relative version of a result of Matsusaka and Mumford; see [38]. □

We are now ready to show that these MMP steps are compatible with base change as in (†):

Theorem 3.2.2. *Suppose the hypotheses in* (†) *are satisfied. If* X^r *exists, then* $(X_V)^r$ *exists and* $(X_V)^r \cong (X^r \times_Y V)^\nu$.

Proof. We assume the existence of $f^r : X \dashrightarrow X^r$. Consider the normalised base change $f^r_V : X_V \dashrightarrow (X^r \times_Y V)^\nu$. We claim that this satisfies the three conditions of Lemma 3.2.1. Indeed, we have a commutative diagram

$$\begin{array}{ccc} X_V = (X \times_Y V)^\nu & \overset{f^r_V}{\dashrightarrow} & (X^r \times_Y V)^\nu \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi^r_V} \\ X & \overset{f^r}{\dashrightarrow} & X^r \end{array}$$

By construction, $(X^r \times_Y V)^\nu$ is normal. Suppose that the birational inverse $(f^r_V)^{-1}$ contracts some divisor E. Since π^r_V is quasi-finite, then $\pi^r_V(E)$ is a divisor in X^r whose image in X by $(f^r)^{-1}$ has codimension greater than 1. This is a contradiction, because f^r is a birational contraction. Therefore, f^r_V is a birational contraction as well. This gives us Condition 1.

Similarly, we know that $(f^r)_\star(D+(r-\epsilon)H)$ is ample over Y. Since π^r_V is quasi-finite, then the pullback $(\pi^r_V)^\star(f^r)_\star(D+(r-\epsilon)H)$ is ample over Y. Because we are

dealing with divisors on normal schemes, it is clear that $(\pi_V^r)^\star(f^r)_\star(D+(r-\epsilon)H) = (f_V^r)_\star(D_V+(r-\epsilon)H_V)$, which shows Condition 2.

Finally, suppose that f_V^r contracts some divisor E. Since π_V is quasi-finite, then $\pi_V(E)$ is a divisor in X that must be contracted by f^r. Hence, $(D+rH)|_{\pi_V(E)}$ is not big, which implies that $(D_V+rH_V)|_E$ is not big, which is Condition 3. □

As a corollary, we show that, if they exist, we can glue MMP steps over an open cover:

Corollary 3.2.3. *Suppose the hypotheses in (†) are satisfied. Suppose $\{Y_j\}$ is an open cover for Y. For each index j, define $X_j = X \times_Y Y_j = g^{-1}(Y_j)$, $g_j = g|_{X_j} : X_j \to Y_j$, $D_j = D|_{Y_j}$, and $H_j = H|_{Y_j}$. Suppose that for each j we know the existence of the r^{th} output $X_j \dashrightarrow (X_j)^r$ of the D_j-MMP with scaling of H_j over Y_j. Then the r^{th} output $X \dashrightarrow X^r$ of the D-MMP with scaling of H over Y exists.*

Proof. For each pair of indices j and k, we will use Theorem 3.2.2 on the base change diagram

$$\begin{array}{ccc} (X_j)_V = X_j \cap X_k & \xrightarrow{(g_j)|_{X_j\cap X_k}} & V = Y_j \cap Y_k \\ \downarrow & & \downarrow \\ X_j & \xrightarrow{\quad g_j \quad} & Y_j \end{array}$$

Note that the inclusion $Y_j \cap Y_k \hookrightarrow Y_j$ is universally open and quasi-finite. Since the r^{th} output $X_j \dashrightarrow (X_j)^r$ is assumed to exist, then Theorem 3.2.2 implies that there exists an r^{th} output $X_j \cap X_k \dashrightarrow (X_j \cap X_k)^r$ and Lemma 3.2.1 says that this output is unique.

This means exactly that $(X_j)^r$ and $(X_k)^r$ are isomorphic over $Y_j \cap Y_k$. Therefore, we may glue these outputs together to obtain an algebraic space X^r together with a birational map $f^r : X \dashrightarrow X^r$ over Y. The three conditions of Lemma 3.2.1 are satisfied locally on the base of f^r, and so f^r itself satisfies them. □

Lemma 3.2.4. *Suppose that $\tilde{W}$ is a quasi-excellent, normal scheme and $\psi : \tilde{W} \to Y$*

is a Galois covering with Galois group G. We still assume that Y is a quasi-excellent, normal algebraic space and that $g : X \to Y$ is a projective morphism from a normal algebraic space. With $\mathbb{R}$-Cartier divisors D and H on X as before, suppose that we know that the r^{th} output $(f_{\tilde{W}})^r : X_{\tilde{W}} \dashrightarrow (X_{\tilde{W}})^r$ over $\tilde{W}$ exists and is G-equivariant. Then its quotient by G gives the r^{th} output of the MMP over Y.

Proof. We need to show that this quotient satisfies the three properties in Lemma 3.2.1. We have a commutative diagram:

$$\begin{array}{ccc} X_{\tilde{W}} & \xrightarrow{(f_{\tilde{W}})^r} & (X_{\tilde{W}})^r \\ \downarrow /G & & \downarrow /G \\ X & \xrightarrow{f^r} & (X_{\tilde{W}})^r/G \end{array}$$

First, we note that the quotient $(X_{\tilde{W}})^r/G$ exists as a normal algebraic space. Suppose that the birational inverse $(f^r)^{-1}$ contracted some divisor E. This pulls back to some divisor on $(X_{\tilde{W}})^r$ whose birational transform in $X_{\tilde{W}}$ must have higher codimension, because taking quotients by G gives finite morphisms. This is a contradiction, so f^r must be a birational contraction.

Also, $(f^r)_\star(D+(r-\epsilon)H)$ is obtained by taking the quotient of $((f_{\tilde{W}})^r)_\star(D_{\tilde{W}}+(r-\epsilon)H_{\tilde{W}})$ by G. The latter divisor is, by assumption, ample over $\tilde{W}$, and since taking a quotient by G gives a finite morphism, then our divisor over $(X_{\tilde{W}})^r/G$ is ample over Y.

Finally, suppose that f^r contracts some divisor E for which $(D + rH)|_E$ is big. Then taking pullbacks by the finite quotient map, we obtain that the divisor $\pi^\star_{\tilde{W}}E$ is contracted by $(f_{\tilde{W}})^r$ even though $(D_{\tilde{W}} + rH_{\tilde{W}})|_{\pi^\star_{\tilde{W}}E}$ is big. This is a contradiction.

We are done by Lemma 3.2.1. □

Theorem 3.2.5. *Suppose the hypotheses in (†) are satisfied and that $\pi : V \to Y$ is surjective. If $(X_V)^r$ exists, then X^r exists and $(X_V)^r \cong (X^r \times_Y V)^\nu$.*

Proof. Since the morphism $\pi : V \to Y$ is quasi-finite, then we can extend it to a finite

morphism; that is, there is some scheme W with an open immersion $i : V \to W$ and a surjective, finite morphism $\tilde{\pi} : W \to Y$, such that $\tilde{\pi} \circ i = \pi$.

This $\tilde{\pi}$ gives rise to a field extension $k(W)/k(Y)$. Note that W might be disconnected; in this case, by $k(W)$ we mean the compositum of the function fields of the connected components of W. We can take a Galois closure $L/k(Y)$ of this field extension. By taking integral closures in L, we get a Galois covering $\tilde{W} \to Y$ that dominates W. We will denote this by $\phi : \tilde{W} \to W$, such that $\tilde{\pi} \circ \phi : \tilde{W} \to Y$ is Galois. We also note that the intermediate field extension $L/k(W)$ is Galois, and so ϕ is itself a Galois covering.

Set $\tilde{V} = \phi^{-1}(V)$ and $\tilde{\phi} = \phi|_{\tilde{V}}$. Then $\tilde{\phi}$ is quasi-finite and universally open, so we can apply Theorem 3.2.2 to conclude the existence of the r^{th} output $(f_{\tilde{V}})^r : X_{\tilde{V}} \dashrightarrow (X_{\tilde{V}})^r \cong ((X_V)^r \times_V \tilde{V})^\nu$ of the MMP over $\tilde{V}$. We note that the universal openness of $\tilde{\phi}$ follows from TAG 0F32 in [49].

Put $G = \mathrm{Gal}(L/k(Y))$, so that G acts on $\tilde{W}$ and $\tilde{W}/G \cong Y$. This action is transitive on the fibres of $\tilde{W} \to Y$. Note that since π was assumed to be surjective, then the G-translates of $\tilde{V}$ cover $\tilde{W}$. In other words, $\{g\tilde{V}\}_{g \in G}$ is an open cover for $\tilde{W}$. Since $g\tilde{V} \cong \tilde{V}$ is an isomorphism for each $g \in G$, and we know that the r^{th} MMP output exists over $\tilde{V}$, then in fact the r^{th} MMP output exists over each open set of our cover. By Corollary 3.2.3, we deduce the existence of the r^{th} MMP output over $\tilde{W}$, which we will denote by $(f_{\tilde{W}})^r : X_{\tilde{W}} \dashrightarrow (X_{\tilde{W}})^r$.

Moreover, we observe that these schemes $X_{\tilde{W}}$ and $(X_{\tilde{W}})^r$ carry G-actions, lifted from the G-action on $\tilde{W}$, and $(f_{\tilde{W}})^r$ is G-equivariant. We now take quotients by G, resulting in a birational map $f^r : X \cong X_{\tilde{W}}/G \dashrightarrow (X_{\tilde{W}})^r/G$. By Lemma 3.2.4, this is the output of the MMP over Y that we wished to construct, so we can write $X^r = (X_{\tilde{W}})^r/G$.

Finally, if we base change this output to V and normalise, we obtain the scheme $(X^r \times_Y V)^\nu$. By Theorem 3.2.2, we know that this satisfies the three conditions of

Lemma 3.2.1, so by uniqueness, we conclude that $(X_V)^r \cong (X^r \times_Y V)^\nu$. $\square$

Now we focus on the case where $D = K_X + \Theta$.

Theorem 3.2.6 (cf. Theorem 3.1.5)**.** *Suppose $g : (X, \Theta) \to (Y, \Delta)$ is a projective morphism of algebraic spaces of finite type over a field of characteristic 0, where (X, Θ) is $\mathbb{Q}$-factorial and dlt. Suppose that g has exceptional divisor $E = E_1 + \cdots + E_n$ and that $K_X + \Theta \sim_{g,\mathbb{R}} E_\Theta = \Sigma e_j E_j$ for some numbers $e_j \geq 0$. Finally, suppose that H is a divisor on X, such that $K_X + \Theta + cH$ is g-ample for some number c. Then we may run the g-relative $(K_X + \Theta)$-MMP with scaling of H.*

Proof. Since Y is an algebraic space of finite type over k, we can find finitely many affine k-schemes of finite type Y_j along with étale morphisms $\pi_j : Y_j \to Y$, such that Y is covered by the images of the π_j. Let $X_j = X \times_Y Y_j$, which is already normal because π_j is étale. Let Θ_j and H_j be the pullbacks of Θ and H to X_j. Note that K_X pulls back to K_{X_j} because π_j is étale. Since X is $\mathbb{Q}$-factorial, then E_Θ is $\mathbb{R}$-Cartier. Now Y_j is affine and the pullback of E_Θ to X_j is effective, exceptional over Y_j, and $\mathbb{R}$-Cartier so we can run the relative $(K_{X_j} + \Theta_j)$-MMP over Y_j with scaling of H_j by [11].

For each j, the divisor $K_X + \Theta + cH$ pulls back to a divisor on X_j that is ample over Y_j, and so we may decrease c to get the MMP started. Let r be the first value of the scaling constant for which $K_X + \Theta + rH$ is no longer ample over Y_j for at least one index j. We know that the output $(X_j)^r$ over Y_j exists for each j. Let V be the disjoint union of the Y_j, so that X_V is the disjoint union of the X_j. Then the output $(X_V)^r$ over V exists: indeed, it is the disjoint union of the $(X_j)^r$. We observe that $V \to Y$ is quasi-finite and universally open by TAG 0F32 in [49]. Now by Theorem 3.2.5, $(X_V)^r$ descends to give the r^{th} output X^r of the relative $(K_X + \Theta)$-MMP over Y with scaling of H.

We now proceed by decreasing r further to obtain the subsequent MMP outputs.

Since the MMP over each of the Y_j eventually terminates, then the MMP over Y terminates, too. □

3.3 Projectivity Criteria

Lemma 3.3.1. *Suppose that, in addition to the hypotheses of Theorem 2.6, $H = \sum h_i H_i$ is such that the numerical classes of the H_i span $\mathrm{NS}(X)_{\mathbb{Q}}$ and the coefficients h_i are linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$. Then each step in the MMP of Theorem 2.6 arises from the contraction of an extremal ray in the relative cone of curves.*

Proof. We prove this for the first step $X \dashrightarrow X^r$, and the proof for the other steps is identical. Let $\phi^r : X \to Z$ be the associated contraction. We mimic the argument in [29]: If C, C' are any two curves contracted by ϕ^r, then we have $\sum h_i(H_i \cdot C) = -r^{-1}(E_\Theta \cdot C)$, and similarly for C'.

Eliminating r gives $\sum h_i((E_\Theta \cdot C')(H_i \cdot C) - (E_\Theta \cdot C)(H_i \cdot C')) = 0$. Since the h_i are linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$, then we have $(E_\Theta \cdot C')(H_i \cdot C) = (E_\Theta \cdot C)(H_i \cdot C')$ for every i. Therefore, C and C' are proportional as functions on $\langle H_i \rangle_{\mathbb{R}}$. Since the numerical classes of the H_i span $\mathrm{NS}(X)$, then C and C' are numerically equivalent. This then shows that $\overline{\mathrm{NE}}(X/Z)$ is 1-dimensional. □

Theorem 3.3.2 (cf. Theorem 3.1.1)**.** *Suppose that $\psi : (Y, \Delta) \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0 and that (Y, Δ) has klt singularities. If ψ is non-projective, then*

1. *either Y contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$,*
2. *or Y has a small, $\mathbb{Q}$-factorial modification that is projective over U.*

Proof. We first find a log resolution $g : X \to (Y, \Delta)$, such that X is projective over U. Write $E = E_1 + \cdots + E_n$ for the exceptional divisor of g. We can write $K_X + F_1 = g^\star(K_Y + \Delta) + F_2$, where F_1, F_2 are effective and F_2 is g-exceptional. Since (Y, Δ) is klt, then the coefficients of F_1 are all less than 1. For $0 < \eta \ll 1$, the coefficients of $F_1 + \eta E$ are still less than 1. Choose such a value of η, and let $\Theta = F_1 + \eta E$. Then we have $K_X + \Theta \sim_{g,\mathbb{R}} E_\Theta$, where $E_\Theta = F_2 + \eta E$ is effective and $\mathrm{Supp}(E_\Theta) = \mathrm{Ex}(g)$.

With this choice of Θ, the pair (X, Θ) is klt. Now choose a divisor A, sufficiently ample over U, such that $\mathrm{Supp}(A) \cup \mathrm{Supp}(\Theta)$ is an snc divisor, and such that $K_X + \Theta + cA$ is ample over Y, where $0 < c < 1$ is sufficiently general. We may pick such a divisor by Bertini's Theorem. Note that this choice of A implies that the pair $(X, \Theta + cA)$ is klt. Next, we perturb cA as follows: for every divisor class in some basis for $\mathrm{NS}(X)$, we pick a divisor representing that class and we add a sufficiently small, sufficiently general multiple of it to cA. Call the resulting divisor H. With these choices, we can arrange that $K_X + \Theta + H$ is ample over Y, $(X, \Theta + H)$ is klt, and the coefficients of H are linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$, where $e_1, \ldots, e_n$ are the coefficients of E_Θ.

We now run the relative $(K_X + \Theta)$-MMP with scaling of H over Y, whose steps exist and terminate by Theorem 2.6. Since (X, Θ) is klt, then this MMP terminates in a klt pair $(X^{\min}, \Theta^{\min})$ projective over Y. Since Y is klt, then by Theorem 3.52 in [32], $X^{\min}$ is a small modification of Y. Note that our choice of X was smooth, and hence $\mathbb{Q}$-factorial. Since each step of this MMP comes from the contraction of an extremal ray, then $X^{\min}$ is still $\mathbb{Q}$-factorial. If $X^{\min}$ is projective over U, then $X^{\min}$ is the claimed small, $\mathbb{Q}$-factorial modification of Y that is projective over U.

Otherwise, along the course of the MMP described above, there is a first step $X^r \dashrightarrow X^{r'}$, such that X^r is projective over U but $X^{r'}$ is not. We will focus on this MMP step; this is not necessarily the first step where the relative MMPs over Y and

over U deviate from each other. We emphasise that this is where our approach differs from that of [47] and [6].

Note first that since $K_X + \Theta + H$ is ample over U, then in particular it is relatively ample, and therefore the index r_1 corresponding to the first step in this MMP satisfies $r_1 < 1$. Since X is smooth (and hence $\mathbb{Q}$-factorial), then $(X, \Theta + r_1 H)$ is klt. Additionally, after we do the first MMP step, the output $(X^{r_1}, \Theta^{r_1} + r_1 H^{r_1})$ remains klt. This step arises from the contraction of an extremal ray in the cone of curves, so X^{r_1} is still $\mathbb{Q}$-factorial. Since the next index, say r_2, is smaller than r_1 and X^{r_1} is $\mathbb{Q}$-factorial, then in fact $(X^{r_1}, \Theta^{r_1} + r_2 H^{r_1})$ is klt.

Inductively, we see that right before we do the MMP step $X^r \dashrightarrow X^{r'}$ that loses projectivity over U, we have a klt pair $(X^r, \Theta^r + r' H^r)$ projective over U. Since X^r is $\mathbb{Q}$-factorial, we can actually conclude that $(X^r, \Theta^r + (r' - \epsilon) H^r)$ is klt for $0 < \epsilon \ll 1$. By Lemma 3.3.1, this MMP step arises from the contraction of some extremal ray R in $\overline{\mathrm{NE}}(X^r/Y)$. Let F be the minimal extremal face of the larger cone of curves $\overline{\mathrm{NE}}(X^r/U)$ that contains R. We have two possibilities to consider:

First, suppose that F is itself a ray. If F contains only curves whose images in Y are points, then our step of the relative MMP over Y is actually a step of the relative MMP over U. However, the steps of the MMP over U preserve projectivity over U. This then gives us a contradiction, because we assumed that $X^{r'}$ is not projective U. It may also happen that F contains some curves whose images in Y are curves; in other words, our step of the MMP over Y doesn't contract every curve in F. Then there are curves C, C', such that $[C'] = \lambda [C]$ in $\overline{\mathrm{NE}}(X^r/U)$ for some $\lambda > 0$, and such that $g^r(C)$ is a point and $g^r(C')$ is a curve in Y. In fact, by Corollary 1.4 in [14] applied to the contraction morphism $\mathrm{cont}_F : X^r \to Z$ over U, we can take the curves C, C' to be rational. This implies that $[g^r(C')] = 0$ in $\overline{\mathrm{NE}}(Y/U)$, so certainly $-[g^r(C')] \in \overline{\mathrm{NE}}(Y/U)$.

The second possibility is that F has dimension greater than 1. Since R is an

extremal ray contracted by a step of some MMP, then it is spanned by the class of some rational curve C, and $[C] \in F$. Then we can write $[C] = \sum \lambda_j v_j$, where each $v_j \in R_j$ is some vector contained in an extremal ray R_j of F, and $\lambda_j \neq 0$. Here we allow the extremal rays R_j to have non-negative intersection with the log canonical divisor, so at this point we do not know that any v_j is the class of some curve in X^r.

We have that $K_{X^r}+\Theta^r+(r'+\epsilon)H^r$ is relatively ample, and $(K_{X^r}+\Theta^r+r'H^r)\cdot C = 0$. This means that if we decrease r', we get a negative intersection product: $(K_{X^r}+\Theta^r+(r'-\epsilon)H^r)\cdot C < 0$, so there must exist some j_0, such that $(K_{X^r}+\Theta^r+(r'-\epsilon)H^r)\cdot v_{j_0} < 0$ for all sufficiently small $\epsilon > 0$. Since we know that $(X, \Theta^r + (r'-\epsilon)H^r)$ is klt and R_{j_0} is a negative extremal ray for this pair, then by [20] this means that $R_{j_0} = \mathbb{R}_{\geq 0}[C_{j_0}]$ for some rational curve C_{j_0} in X^r. Replacing λ_{j_0} if necessary, we can assume that $v_{j_0} = [C_{j_0}]$. Letting $\epsilon \to 0$, we see that actually $(K_{X^r} + \Theta^r + r'H^r) \cdot C_{j_0} \leq 0$. Suppose for a contradiction that $g^r(C_{j_0})$ is a point. Then $(K_{X^r} + \Theta^r + r'H^r) \cdot C_{j_0} \geq 0$ because the divisor is relatively nef. Combining our two inequalities, we deduce that $(K_{X^r} + \Theta^r + r'H^r) \cdot C_{j_0} = 0$, so that the ray $R_{j_0} \cap \overline{\mathrm{NE}}(X^r/Y)$ gets contracted by this MMP step. This is impossible because this step contracts only the ray R. Therefore, $g^r(C_{j_0})$ is a rational curve in Y, and $-[g^r(C_{j_0})] = \lambda_{j_0}^{-1} \sum_{j \neq j_0} \lambda_j [g^r(v_j)] \in \overline{\mathrm{NE}}(Y/U)$. □

Remark. We observe that it really could happen that, in our proof, F is an extremal ray, yet it contains curve classes that are not in R. For example, consider $U = \mathrm{Spec}(k)$ and let X be any threefold with an ordinary double point x. Consider the blow-up $\mathrm{Bl}_x X \to X$ with exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Contracting one of these factors of $\mathbb{P}^1$ gives a small resolution X' of $(x \in X)$, so overall we have morphisms $\mathrm{Bl}_x X \to X' \to X$. A step of the absolute MMP for $\mathrm{Bl}_x X$ would contract the entire exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. However, if we ran a relative MMP for $\mathrm{Bl}_x X \to X'$, we would contract only one of the two $\mathbb{P}^1$ factors.

Corollary 3.3.3 (cf. Corollary 3.1.2)**.** *Suppose that $\psi : Y \to U$ is a proper morphism*

of normal algebraic spaces of finite type over a field k of characteristic 0, and that Δ is some divisor on Y, such that the pair (Y, Δ) has klt singularities. Assume additionally that Y is $\mathbb{Q}$-factorial. Then ψ is non-projective if and only if Y contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$.

Proof. If $\psi : Y \to U$ is projective, then Y carries some ψ-ample divisor A. Then A is positive on every curve on Y that is contracted by ψ, so Y cannot contain a curve like the one in our statement.

Conversely, if ψ is non-projective, then we can run the argument in the proof of Theorem 3.3.2. Our relative MMP over Y terminates in a small modification $X^{\min}$ that is projective over U. However, we are now assuming that Y is $\mathbb{Q}$-factorial, so Y admits no non-trivial small modifications that are projective over it. This implies that $X^{\min} \cong Y$, so in particular $X^{\min}$ is not projective over U. Therefore, there must be some MMP step over Y where projectivity over U is lost. We then argue as in the proof of Theorem 3.3.2 to find the desired rational curve. □

We then deduce the following characterisation of projectivity. In dimension 3, without the need for klt singularities, this was done by Kollár; see Corollary 5.1.5 in [24].

Corollary 3.3.4. *Suppose that $\psi : (Y, \Delta) \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0 and that (Y, Δ) has $\mathbb{Q}$-factorial klt singularities. Then ψ is projective if and only if there is a Cartier divisor L on Y such that $L \cdot C > 0$ for every irreducible curve C in Y such that $\psi(C)$ is a point. (L need not be ample over U).*

Proof. If ψ is projective, then Y contains a Cartier divisor L ample over U, so we are done. Conversely, suppose that there is a Cartier divisor L on Y that is positive on every irreducible curve contracted by ψ. Assume for a contradiction that ψ is

not projective. By Corollary 3.3.3, Y contains a rational curve C such that $-[C] \in \overline{\mathrm{NE}}(Y)$, but then for this curve we have $L \cdot (-C) \geq 0$, which is impossible. $\square$

We recall Kleiman's numerical criterion for ampleness. The usual formulation is that if Y is projective over U, then a Cartier divisor L on Y is ample over U if and only if L induces a strictly positive function on $\overline{\mathrm{NE}}(Y/U) \setminus \{0\}$. In fact, Kleiman proved a more general version, where Y is only assumed to be quasi-divisorial over U; see Theorem IV.4.2 in [22]. There are, however, proper algebraic spaces, even smooth ones, that are not quasi-divisorial: see, for example, Hironaka's second example in the Appendix of [16]. As a consequence of Corollary 3.3.4, we prove a version of Kleiman's criterion for which we do not need quasi-divisoriality, but instead we require Y to have mild singularities. Note that this version of Kleiman's criterion also follows from Lemma 21 in [30].

Corollary 3.3.5 (Kleiman's Criterion, cf. Corollary 3.1.3)**.** *Suppose that $\psi : (Y, \Delta) \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0 and that (Y, Δ) has $\mathbb{Q}$-factorial klt singularities. Let L be a Cartier divisor on Y. Then L is ψ-ample if and only if L has positive degree on every curve C such that $\psi(C)$ is a point and L induces a strictly positive function on $\overline{\mathrm{NE}}(Y/U) \setminus \{0\}$.*

Proof. Suppose first that L is ψ-ample. Then L has positive degree on every curve and ψ is projective, so the usual form of Kleiman's Criterion applies. Conversely, if L is positive on every curve, then by Corollary 3.3.4, Y is projective, so the usual form of Kleiman's Criterion applies, and we deduce that L is ample. $\square$

Remark. For our version of Kleiman's Criterion, it is not enough to assume that L induces a strictly positive function on $\overline{\mathrm{NE}}(Y/U) \setminus \{0\}$: We really need the assumption that L has positive degree on every curve contracted by ψ, or equivalently that Y has no numerically trivial curves that are contracted by ψ. See Exercise VI.2.19.3 in [27]

for an example where the latter hypothesis is not satisfied and Kleiman's Criterion does not hold.

3.4 Existence of dlt Modifications

Our next applications concern the existence of dlt modifications of lc pairs.

Theorem 3.4.1 (cf. Theorem 3.1.6)**.** *Suppose* (Y, Δ) *is a log canonical pair, where* Y *is an algebraic space of finite type over a field of characteristic* 0*. Then* (Y, Δ) *admits a dlt modification that carries a relatively ample, exceptional divisor over* Y*.*

Proof. Let $g : (X, \Theta) \to (Y, \Delta)$ be a log resolution such that the morphism g is projective and such that $\mathrm{Ex}(g)$ supports a relatively ample divisor. This can be arranged by Theorem 1 in [33]. Here, Θ is defined by $K_X + \Theta = K_X + E + \widetilde{\Delta} = g^\star(K_Y + \Delta) + E_\Theta$, where $E = E_1 + \cdots + E_n$ is the exceptional divisor of g, $\widetilde{\Delta}$ is the strict transform of Δ, and $E_\Theta = \Sigma e_i E_i$ is some combination of the exceptional divisors. In particular, $e_i \geq 0$ for each i because (Y, Δ) is log canonical. Additionally, since (X, Θ) is log smooth, then it is dlt and each E_i is $\mathbb{Q}$-Cartier.

Since H is relatively ample, then there exists a number c, such that $K_X + \Theta + cH$ is g-ample. We can then wiggle the coefficients of H so that they become linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$ and relative ampleness is preserved.

Note that if Y were assumed to be a scheme, then we would be in the setting of Theorem 1 in [29], and our result would follow from Theorem 3.2.6. When Y is an algebraic space, we cannot directly apply Kollár's result. Additionally, the morphisms $\pi : V \to Y$ and $\pi_V : X_V \to X$ are no longer open immersions and are instead only known to be étale (here, V is as in the proof of Theorem 3.2.6). Thus, the pullback of an irreducible divisor can have multiple components, and hence the coefficients of $H_V = \pi_V^\star H$ are no longer linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$, so we also cannot directly apply Theorem 1 in [29] to the relative $(K_{X_V} + \Theta_V)$-MMP with scaling of

H_V.

Suppose that the first step of the MMP over V contracts some irreducible divisor F_0 contained in some $\pi_V^\star E_0$. We will show that actually all of $\pi_V^\star E_0$ is contracted by this same step.

Indeed, suppose C is some curve in $\pi_V^\star E_0$, so that $\pi_V(C)$ is contained in E_0. Let C' be the component of $\pi_V^\star(\pi_V(C))$ that is contained in F_0. Since F_0 is contracted, then $(K_{X_V} + \Theta_V + rH_V) \cdot C' = 0$. By the Projection Formula, we have that $(K_X + \Theta + rH) \cdot \pi_V(C) = 0$. But the morphism π_V restricts to a finite, étale morphism over $\pi_V^\star(\pi_V(C))$, say of degree d. Then we have $(K_{X_V} + \Theta_V + rH_V) \cdot \pi_V^\star(\pi_V(C)) = d(K_X + \Theta + rH) \cdot \pi_V(C) = 0$. Since C is a component of $\pi_V^\star(\pi_V(C))$ and $K_{X_V} + \Theta_V + rH_V$ is relatively nef, then $(K_{X_V} + \Theta_V + rH_V) \cdot C = 0$, so C is contracted.

In summary, we have shown that if, in the first MMP step over V, we contract an irreducible component of H appearing with coefficient h_i, then every component with that same coefficient is contracted as well. The same argument shows that is true for all subsequent steps. This is all we need to be in the setting of Theorem 1 in [29].

By Theorem 3.2.6, we may run the relative $(K_X + \Theta)$-MMP with scaling of H over Y. This MMP will now terminate in a relative minimal model $g^{\min} : (X^{\min}, \Theta^{\min}) \to (Y, \Delta)$, and this pair $(X^{\min}, \Theta^{\min})$ is the dlt modification we were seeking. Since the birational transform of each one of the exceptional divisors E_i to $X^{\min}$ is $\mathbb{Q}$-Cartier, then the divisor $K_{X^{\min}} + \Theta^{\min} + cH^{\min}$ on $X^{\min}$ is ample over Y for c sufficiently small. □

Remark. The dlt modification that we obtain in Theorem 3.4.1 has the property that it carries a relatively ample exceptional divisor. We could, instead, ask for dlt modifications for which the underlying space is $\mathbb{Q}$-factorial. We observe again that the steps of Kollár's MMP need not preserve $\mathbb{Q}$-factoriality because we are allowed to contract extremal faces of dimension ≥ 2. This means that we need to do some more work to obtain $\mathbb{Q}$-factorial dlt modifications.

Corollary 3.4.2. *Let (Y, Δ) be an lc pair, where Y is an algebraic space of finite type over a field k of characteristic 0. Then (Y, Δ) admits a $\mathbb{Q}$-factorial dlt modification that is projective over Y.*

Proof. We first take a log resolution $g : (X, \Theta) \to (Y, \Delta)$ and then run the relative MMP with scaling described in Theorem 3.4.1 to obtain a dlt modification $g^{\min} : (X^{\min}, \Theta^{\min}) \to (Y, \Delta)$. If $\dim(Y) \leq 2$, then $X^{\min}$ is already $\mathbb{Q}$-factorial, so we only need to look at the case where $\dim(Y) \geq 3$.

We claim that the abelian group $\mathrm{Cl}(X^{\min})/\mathrm{Pic}(X^{\min})$ is finitely generated. We show this by induction on the MMP steps. Observe that since X is smooth, then $\mathrm{Cl}(X)/\mathrm{Pic}(X)$ is trivial and so certainly finitely generated. Suppose next that after j MMP steps $\mathrm{Cl}(X^j)/\mathrm{Pic}(X^j)$ is finitely generated, and consider the next step:

$$\begin{array}{ccc} X^j & \overset{f}{\dashrightarrow} & X^{j+1} \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & Z & \end{array}$$

Here, ϕ is the contraction morphism of an extremal face V of $\overline{\mathrm{NE}}(X^j/Y)$, and ψ is a small modification. By the Contraction Theorem, we have a short exact sequence $0 \to \mathrm{Pic}(Z) \to \mathrm{Pic}(X^j) \to \mathbb{Z}^{\dim(V)}$. We also know that ψ induces an inclusion $\mathrm{Pic}(Z) \to \mathrm{Pic}(X^{j+1})$. This means that going from $\mathrm{Pic}(X^j)$ to $\mathrm{Pic}(Z)$ decreases the Picard rank by $\dim(V)$, and then going to $\mathrm{Pic}(X^{j+1})$ may increase the rank. Similarly, going from $\mathrm{Cl}(X^j)$ to $\mathrm{Cl}(X^{j+1})$ decreases the rank by 1 if ϕ contracts a divisor, and there is no change in the class group if ϕ does not contract any divisors. So at worst, going from $\mathrm{Cl}(X^j)/\mathrm{Pic}(X^j)$ to $\mathrm{Cl}(X^{j+1})/\mathrm{Pic}(X^{j+1})$ increases the rank by $\dim(V)$, and hence our quotient remains finitely generated.

By induction it follows that $\mathrm{Cl}(X^{\min})/\mathrm{Pic}(X^{\min}) \cong \mathbb{Z}^m \oplus A$, where m is a non-negative integer and A is a finite abelian group. Each $\mathbb{Z}$ summand is generated by some divisor class $[D_i], 1 \leq i \leq m$, and these divisors D_i are not $\mathbb{Q}$-Cartier. Blowing up along one of these divisors D_i gives a small morphism because $\dim(X^{\min}) \geq 3$,

and it decreases the rank of our abelian group by 1 because the strict transform of D_i is $\mathbb{Q}$-Cartier. To be more precise, we wish to take the symbolic blow-up by $-D_i$ (see e.g. Remark 6.33 in [32]).

After m blow ups, we have a small modification $h : X^{\mathrm{qf}} \to X^{\mathrm{min}}$, where $\mathrm{Cl}(X^{\mathrm{qf}})/\mathrm{Pic}(X^{\mathrm{qf}})$ is finite, and therefore X^{qf} is $\mathbb{Q}$-factorial. Now $g^{\mathrm{min}} \circ h : (X^{\mathrm{qf}}, h_\star^{-1}\Theta^{\mathrm{min}}) \to (Y, \Delta)$ is the desired $\mathbb{Q}$-factorial dlt modification. □

Finally, we look at an application of Theorem 3.2.2 where the morphism $\pi : V \to Y$ is quasi-étale.

Corollary 3.4.3. *Suppose $\pi : V \to Y$ is a universally open, quasi-finite and quasi-étale morphism of normal quasi-projective varieties over a field of characteristic 0. Let (Y, Δ) be lc, and take a projective log resolution $g : (X, \Theta) \to (Y, \Delta)$, such that there exists a relatively ample divisor H with $\mathrm{Supp}(H) = \mathrm{Ex}(g)$. Let $E = E_1 + \cdots E_n$ be the exceptional divisor of g. Then running the relative $(K_X + \Theta)$-MMP with scaling of H and base changing it to V is equivalent to running the relative $(K_{X_V} + \Theta_V)$-MMP with scaling of H_V.*

Proof. As before, we take Θ to be such that $K_X + \Theta \sim_{g,\mathbb{R}} E_\Theta$, where $E_\Theta = \sum e_i E_i$ is g-exceptional and effective, so that [11] applies. H is relatively ample, so there exists a number c for which $K_X + \Theta + cH$ is g-ample. We can wiggle the coefficiens of H to guarantee that they are linearly independent over $\mathbb{Q}(e_1, \ldots e_n)$. Since π is quasi-étale, then all the ramification of $\pi_V : X_V \to X$ is along the exceptional divisor E. We have a log resolution $g_V : (X_V, \Theta_V) \to (V, \pi^\star\Delta)$, where Θ_V is defined similarly to Θ. By the Riemann-Hurwitz formula, it follows that $\pi_V^\star(K_X + \Theta) = K_{X_V} + \Theta_V$.

Now we decrease c until we reach the first value r for which $K_X + \Theta + rH$ is no longer g-ample. Then we have $(X_V)^r \cong X^r \times_Y V$ by Theorem 3.2.2. We can keep on decreasing r until the MMP over Y terminates. □

Remark. We observe that $\pi_V^\star K_X \neq K_{X_V}$ and $\pi_V^\star \Theta \neq \Theta_V$, and that we need to be

in characteristic 0 to avoid wild ramification.

Remark. A particular example of Corollary 3.4.3 is when $\pi : V \to Y$ is the quotient by some group action on V. In this case, we may run the relative MMP on $g : (X, \Theta) \to (Y, \Delta)$ to obtain a dlt modification of (Y, Δ). However, the corresponding relative MMP on $g_V : X_V \to V$ will not yield a dlt modification of $(V, \pi^\star\Delta)$; this will only be a quotient-dlt modification. See Section 5 of [9] for more on quotient-dlt pairs.

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