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Etale covering between resolution of isolated singularities and Gaffney-Lazarsfeld Theorem for Homogeneous Spaces

Tu, Yu-Chao

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Étale covering between resolution of isolated singularities and Gaffney-Lazarsfeld Theorem for Homogeneous Spaces

Yu-Chao Tu

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of Princeton University
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Abstract

We discuss two topics in the thesis. One is the resolution of isolated singularities and its application in CR geometry, and the other one is extending the Gaffney-Lazarsfeld theorem to homogeneous spaces which is a theorem dealing with dimension bounds for branched loci. We explain the background of these problems and their significance in the beginning, including the minimal model program and branched coverings.

In the second chapter, first we provide some technical results in order to carry out the proofs of main theorems, then prove the multiplicativity of the canonical volume defined for isolated singularities. This gives that any étale covering map between resolutions of a variety having only finitely many isolated singularities is an isomorphism. We also explain some results of CR geometry and show that any non-constant CR morphisms between embeddable strongly pseudo-convex CR manifolds is a biholomorphism.

In the third chapter, we begin with the notions of local degrees and higher ramification loci in order to obtain the Gaffney-Lazarsfeld theorem on projective spaces and Grassmannians. After some fundamental properties of homogeneous spaces, we give a generalization of the Gaffney-Lazarsfeld theorem for homogeneous spaces of Picard number one.

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To my family.

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Chapter 1

Introduction

This thesis deals with two topics: resolution of isolated singularities and ramification of branched coverings. Singularities study is one of the central topics in algebraic geometry, and isolated singularity is a certain type of singularities which may have some good properties to characterize. On the other hand, the branched loci of coverings consist of coincided merged points in the pre-image of coverings. The two questions discussed here are:

1. What relation of two resolutions of singular varieties can have when there is a nice morphism between them.

2. The structure of the branched locus when the covering map has certain target.

Algebraic geometry deals with geometric objects consisting of the zero sets of polynomials called algebraic varieties. Because of the origin, the study combines methods of algebra and geometry. A guiding problem in algebraic geometry is classification problem. The ideal situation is to find good typical cases and any given variety can be classified in just one class. For example, a plane curve that is determined by an irreducible quadratic polynomial over $\mathbb{C}$ is isomorphic either to the parabola defined by $y = x^2$, or to the hyperbola defined by $xy = 1$. For curves and surfaces, there are already good results. We can classify curves by their genera. For example, smooth

projective curves with genus 0 are just projective lines $\mathbb{P}^1$, with genus 1 are elliptic curves, like $y^2 = x^3 + x + 1$, and there are other higher genus curves. In the case of surfaces, there are good classification theorems according to Kodaira dimensions $-\infty, 0, 1, 2$, which is the dimension of the image of a rational map determined by canonical divisors. Similar to curves case, a smooth projective surface with negative Kodaira dimension is birational to the projective plane $\mathbb{P}^2$, where two algebraic varieties are birational if each variety has a nonempty Zariski open subset and these two subsets are isomorphic. The notion "birationality" comes in the classification as an efficient equivalence to reduce the problem, and birational geometry has contributed lots of results in studying varieties.

The minimal model program provides a systematic way to reduce varieties within their birational equivalent classes and then people can go further from the reduced varieties. The spirit of running MMP is contracting a divisor or replace a small closed subvariety at each step until it comes out a minimal model, which can be thought as no more divisors to contract, or a fibration with special fibers: Fano varieties. There are some technical problems to deal with in MMP and one is the singularity. In surfaces, running MMP from a smooth surface would give smooth surfaces in the whole process by the Castelnuovo theorem, that is, contracting a rational curve with self-intersection -1 in a smooth projective surface gives a new smooth projective surface. In higher dimensions, however, the contraction would bring singular varieties [Mor82] and some bad singularities cause difficulty in continuing MMP. One may replace the former variety by another one to continue, and a central problem in MMP is to determine:

1. Does the replacement, called flip, exist?

2. If 1. is true, do the flips occur only finitely many times?

The first question is confirmed in [BCHM10]. The second question is still unknown, but [BCHM10] gives an affirmed answer by adding some extra condition. On the other hand, in order to run MMP, the singularities can not be arbitrary. In fact, only some mild singularities are allowed and their definitions involve resolution of singularities. One reason of the definition's choice is that every singular algebraic variety defined over a field of characteristic 0 can be resolved, using the method of blow-ups, to a nonsingular algebraic variety according to [Hir64]. Thus it is natural to study the properties of resolutions.

Note that the construction does not give a unique resolution. For example, if Y is a resolution of a singular variety X, we blow up Y to another nonsingular variety Y', then Y' is also a resolution of X. In dimension 2, we can have minimal resolution of a singular surface, that is, any resolution is a blow-up of this minimal resolution. We start from any resolution of a given surface, then we contract the rational curve with -1 self-intersection step by step until we can not do it anymore. The final surface is smooth by the Castelnuovo theorem. So we have some uniqueness of resolution in dimension 2. But the minimal resolution does not exist for higher dimensions. So a question comes in: given two resolutions of a singular algebraic variety, what can we say about their relation?

One answer to this question is that if there is an étale morphism between two resolutions of an algebraic variety with only finitely many isolated singularities, then the resolutions are actually the same. The result can be generalized that if there is an étale morphism between two resolutions of two varieties with finitely many isolated singularities, then there is some multiplicity relation between their "canonical volume" which is defined in the second chapter. The idea of the proof is introducing the construction of relative canonical model, a partial resolution of variety, and we can describe the canonical volume of isolated singularities on it. Then we reduce general resolutions back to the relative canonical models by Nakayama's theorem

where canonical singularities play an important role in the argument.

A major application of the multiplicities is in CR geometry. CR geometry studies CR manifolds which can be thought as differentiable manifolds together with preferred complex distribution, an integrable and almost Lagrangian subbundle of its complexified tangent bundle. Generally speaking, CR manifolds are abstract models of boundaries of complex manifolds. If the CR structure, i.e., the subbundle above, of a CR manifold has a definite local frame at every point, then such a CR manifold is called strongly pseudoconvex. In [Yau11], the conjecture 1.1 asks that for a compact strongly pseudoconvex embeddable CR manifold X of dimension at least 3, every non-constant CR morphism from X to itself must be a CR biholomorphism. It bases on some known results of compact strongly pseudoconvex embeddable CR manifolds and the most important one is that they are boundaries of varieties with only isolated normal singularities. It is proved in [Yau11] that the conjecture 1.1 is true for dimension just 3, and the method uses the existence and uniqueness of minimal resolution of complex surfaces. The difficulty of extending the proof to higher dimensions is then no minimal resolutions anymore. The main theorem in chapter two overcomes the problem and then give the affirmative answer to the conjecture.

If we consider the morphisms further, we know that étale morphism is a special case of finite coverings. This is a covering map such that no points merge together in the pre-image of a target point. For example, the map $\pi : \mathbb{R}^1 \to S^1$ by $t \mapsto e^{it}$ is étale, but the map $f : C = \{(z, x) \in \mathbb{P}^2 | z^2 = (x - a_1)...(x - a_6)\} \to \mathbb{P}^1$ by $(z, x) \mapsto x$ is not. f is a 2-to-1 covering map with branched locus $\{(0, a_1), ..., (0, a_6)\}$. In general, a finite covering map has branched locus, and it is important to study its behavior.

Take a branched covering $f : X \to Y$ between complex projective smooth varieties. A famous result is the purity of branched locus [Zar58], which says that for $f : X \to Y$ a finite separable covering map between non-singular algebraic varieties, the branch locus on X must be made up purely of codimension 1 subvarieties, i.e., a Weil divisor

R and

$$K_X = f^* K_Y + R$$

In curves case, Hurwitz formula relates the genera of two curves and the branched locus: for $f : C_1 \to C_2$ a separable covering map of degree d between two smooth complete curves, there is

$$2g(C_1) - 2 = d.(2g(C_2) - 2) + deg\ R$$

where $g(C_1), g(C_2)$ are the genus of C_1, C_2 respectively, and R is the branched locus.

Besides the general results, we can think of the situation of certain covering maps. An interesting result was obtained in 1980, where [GL80][Laz80] show that if one considers branched coverings from normal projective varieties to projective spaces and refine the definition of branched locus, we can obtain information of the hierarchy inside branched locus. The idea is that for branched locus having higher ramifying degree, its codimension is smaller but with a suitable upper bound, that is, the higher branched locus gets smaller and smaller but it would not shrink to empty too soon. To be more precise, after defining the notion of local degree which means the number of sheets coming together, in [GL80], the notion of higher ramification locus is defined as the set whose points have higher local degree. When X is an irreducible projective variety and Y is a projective space $\mathbb{P}^n$, it is proved by Gaffney and Lazarsfeld in [GL80] that the codimension of higher ramification locus is bounded above by the local degree of its points minus one. The method in [GL80] uses induction on the dimension of $\mathbb{P}^n$ and the Fulton-Hansen connectedness theorem [FL81][Laz04].

In order to generalize this result to other projective smooth variety Y, first one may need to find suitable subvarieties in Y to run the dimension induction, i.e., a subvariety $Z \subset Y$ with similar property as Y and the preimage $f^{-1}(Z)$ is also irreducible. In [Deb96], it is proved that there are natural subvarieties which are

also Grassmannians and their pull-backs are irreducible, e.g., for a branced covering $f : X \to Gr(r, n)$, the preimage $f^{-1}(Gr(r, n-1))$ is irreducible, so one may run the dimension induction for Grassmannians but there would be some gaps. The second thing is the Fulton-Hansen connectedness theorem. There are some generalized forms of Fulton-Hansen connectedness theorem: [Han83][Deb96] for Grassmannians and flagmanifolds, [Fal81] for homogenous spaces, [B$\breve{\text{9}}$6] for weighted projective spaces, and [BR08] for products of weighted projective spaces. On the other hand, [KM99] considers some certain homogeneous spaces of Picard number 1 and then show Barth-Lefschetz type theorems(cf. [Laz04], Theorem 3.2.1) which may be applied to Gaffney-Larzarsfeld theorem for smooth varieties. Another direction is that Debarre [Deb95] proves the Gaffney-Lazarsfeld theorem on abelian varieties by using the similar argument as [GL80](see also [Laz04]).

A reason to consider the situation of homogeneous spaces of Picard number 1 is the following conjecture due to Lazarsfeld [KM99]:

Conjecture. *Let $f : Y \to X = G/P$ be a branched covering where Y is a nonsingular connected complex projective variety, Pic $X = \mathbb{Z}$. Then the associated vector bundle $\mathcal{E}$ on X is ample.*

The associated vector bundle is defined by the splitting of vector bundles:

$$f_*\mathcal{O}_Y \cong \mathcal{O}_X \bigoplus \mathcal{E}^*$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$. Assume that we have this result, we can prove the Gaffney-Lazarsfeld theorem in this case of branched coverings as [Laz04], 6.3.56. Therefore, obtaining the GL theorem can be a sign of confirming the conjecture.

When it comes to a general homogeneous space with Picard number 1, however, there is no natural subvariety which is suitable for running the dimension induction, and the known forms of the Fulton-Hansen connectedness theorem is too weak to

apply in general cases. We attack these problems by using the results in [Kol13] to show that the connectedness theorem holds indeed, and we avoid the dimension induction on target variety by using induction on the higher ramification locus instead. The main theorem of this paper is that if $f : X \to Y$ is a branched covering from an irreducible projective variety to a homogeneous space Y with Picard number 1, then the higher ramification locus has anticipated codimension bound. As an application, we show that if a projective irreducible variety has a low degree branched covering to a projective homogeneous space with Picard number one, it is simply connected.

The thesis proceeds as the following: At the beginning of the second chapter, we provide some background of singularities and resolutions. Second, we state the main theorems and their proofs. Then we give an application of the theorems in differential geometry concerning CR manifolds and CR morphisms with some necessary terminologies. In the third chapter, we explain the preliminary of branched coverings and known results at first. We also provide some basic background of homogeneous spaces and some description about homogeneous spaces of Picard number one. Then we turn into the main theorem and its proof. Finally, we obtain an application to a sufficient condition of simply connectedness.

Chapter 2

Étale covering between resolution of isolated singularities

2.1 Resolution of singularities and canonical volume

We fix the terminologies which will be used below.

Definition 2.1.1. Let X be a scheme over an algebraic closed field k. X is an algebraic variety if X is reduced, irreducible, and finite type over k. X is complete (projective) if X is proper (projective) over $Spec\ k$ respectively.

We consider only the situation $k = \mathbb{C}$ in the whole article. At first, we give the definition of birationality.

Definition 2.1.2. [Har77] Let X, Y be two algebraic varieties. We say X and Y are birationally equivalent if there are open subsets $U \subset X, V \subset Y$ such that U is isomorphic to V, written as $U \cong V$.

Remark 1. This is also equivalent to say that the functional fields $K(X)$ and $K(Y)$ are isomorphic. For any open affine subset $U = Spec\ A \subset X$, $K(X)$ is isomorphic to

the quotient ring of A.

A birational map $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ defined on an open subset $U \subset X$ such that there are open subsets $U' \subset X$, $V' \subset Y$ with $f|_{U'} : U' \to V'$ an isomorphism. Note that the map f also gives a birational map $g : Y \dashrightarrow X$ as the inverse map of f. A typical case of birational map is blow-up [Har77]. For example, given an affine space $\mathbb{A}^n$, define a variety

$$Bl_0(\mathbb{A}^n) := \{(x_1, ..., x_n, y_1, ..., y_n) \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i, i, j = 1, ..., n\}.$$

The projection $p : Bl_0(\mathbb{A}^n) \to \mathbb{A}^n$ is a birational morphism and its inverse $q : \mathbb{A}^n \dashrightarrow Bl_0(\mathbb{A}^n)$ is a birational map defined on $\mathbb{A}^n - 0$. We call $p : Bl_0(\mathbb{A}^n) \to \mathbb{A}^n$ the blow-up of the origin of $\mathbb{A}^n$. In general, a blow-up can be defined on any closed subset, which corresponds to a coherent sheaf of ideals when the variety is noetherian.

General blow-ups can be constructed along subvarieties and defined by sheaves of ideals [Har77].

Definition 2.1.3. Let X be a scheme, $\mathcal{F}$ a sheaf of finitely generated graded $\mathcal{O}_X$-algebra. Then for any open affine subset $U = \text{Spec } A \subset X$, $\mathcal{F}(U)$ is a graded A-algebra and we have $\pi_U : \text{Proj } \mathcal{F}(U) \to U$. Gluing all the open affine subsets gives a scheme $\mathbf{Proj}\ \mathcal{F}$ with a morphism $\pi : \mathbf{Proj}\ \mathcal{F} \to X$. Now, let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent sheaf of ideals on X. Define the sheaf of graded algebras $\bigoplus_{d \geq 0} \mathcal{I}^d$, $\mathcal{I}^0 := \mathcal{O}_X$, then the blow-up of X at $\mathcal{I}$ is

$$Bl_{\mathcal{I}} := \mathbf{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d.$$

Example 2.1.1. The former example, $Bl_0(\mathbb{A}^n)$, can be restated in the general definition. For $X = \text{Spec } A$, $A = k[x_1, ..., x_n]$, X is $\mathbb{A}^n$ and the origin of X corresponds to the ideal $\mathcal{I} = (x_1, ..., x_n)$. Then the blow-up is $Bl_0(\mathbb{A}^n) = \text{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d$. Note that it is the closed subscheme of Proj $A[y_1, ..., y_n]$ defined by $\{x_i y_j = x_j y_i | i, j = 1, ..., n\}$ which is the same as before.

Blow-ups give a method of resolving singularities on varieties.

Example 2.1.2. Consider the curve C defined by $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the origin $(0,0)$ of $\mathbb{A}^2$, we get $\pi : Bl_{(0,0)}(\mathbb{A}^2) \to \mathbb{A}^2$ with

$$\pi^{-1}(C) = \{(x, y; u, v) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid y^2 = x^3, yu = xv\}$$

Assume $u \neq 0$, and we can set $u = 1$. Then $y = xv$, $x^2(v^2 - x) = 0$, and the blow-up of C is

$$\widetilde{C} = \{(x, y; u, v) \mid y = xv, v^2 = x\}.$$

In general, the singularities can be improved after blow-ups.

Definition 2.1.4. Let X be a variety. A resolution of X is a proper birational morphism $f : Y \to X$ such that Y is smooth. The exceptional set of f is the subset $E \subset Y$ where f^{-1} is not a morphism at $f(x)$ for every $x \in E$.

For example, the exceptional set of $p : Bl_0(\mathbb{A}^n) \to \mathbb{A}^n$ is $\{0\} \times \mathbb{P}^{n-1}$.

Blow-ups contribute to the study of resolution of singularities. A fundamental theorem in algebraic variety is that over the field of characteristic 0, any singular algebraic variety X has a resolution [Hir64]. That is, for any algebraic variety, we can find a birational model which is nonsingular. The idea of the proof is repeatedly blowing up along subvarieties and using induction on dimension.

One can see [Kol07] for more information about resolution of singularities.

In general, the process of resolution involves iterating blowing up and the situation can be really complicated. Moreover, we can have arbitrary resolution of a given variety. For instance, let $f : Y \to X$ be a resolution of X and $y \in Y$ is a point in Y. Then the composition $f \circ \pi : Bl_y(Y) \to X$ of the blow-up $\pi : Bl_y(Y) \to Y$ and f is also a resolution of X. Therefore, there is no "biggest" resolution. If we turn to find "minimal" resolution, it is the right thing to do when the dimension is 2. That

is, given any surface X, we have a minimal resolution $f : X' \to X$ such that for any other resolution $g : Y \to X$, there is a birational morphism $h : Y \to X'$ such that g factors through h, $g = f \circ h$. This comes from the fact that any birational map between two nonsingular surfaces can be factored as composition of blow-ups and blow-downs [Har77], and the minimal resolution of a surface is the nonsingular surface having no a rational curve with -1 self-intersection, i.e., a rational curve E such that $E \cdot E = -1$(see Definition 2.1.6 for definition of intersection).

However, minimal resolutions do not exist in higher dimensional cases.

Example 2.1.3. (Atiyah flop) Let $Y = \{(xy = zw)\} \subset \mathbb{A}^4$ and $V = Bl_0(Y)$. The exceptional locus is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. V can be blown down in two different ways to X_1 and X_2 which are not isomorphic. Both are resolutions of Y and can not be blown down further.

Although there is no uniquely determined resolution, it is still possible to study relations between them, especially there are nice morphisms between resolutions. An example is étale morphisms.

Definition 2.1.5. A morphism $f : X \to Y$ is called an étale morphism if

1. (*unramification*) For every $x \in X$, $y = f(x)$, we have $m_y.\mathcal{O}_x = m_x$ and $k(x)$ is a separable algebraic extension of $k(y)$.

2. (*flatness*) For every $x \in X$, $y = f(x)$, the stalk $\mathcal{O}_{x,X}$ is a flat $\mathcal{O}_{y,Y}$-module.

Topologically, if $f : X \to Y$ is an étale morphism, every point $y \in Y$ has a neighborhood $U \subset Y$ such that there are disjoint open subsets $U_i \subset X$ with $f_{U_i} : U_i \to V$ being an isomorphism, which means that f is locally an isomorphism. Étale morphisms give good results in connecting the canonical divisors of two morphisms. Recall that on a normal scheme X, a (Weil) divisor $D = \sum a_i D_i$ is a $\mathbb{Z}$-linear combination of irreducible and reduced codimensional one closed subvarieties $D_i \subset X$. D_i are

called prime divisors. Then for a smooth variety X, its canonical divisor K_X, up to linear equivalence, is a Weil divisor which corresponds to the top wedge-product of differentials. For general normal variety X, its singular locus has codimension ≥ 2, and its canonical divisor is the extension of the canonical divisor on the smooth locus $X^{reg} \subset X$. The definition of Weil divisors can be extended to $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors while the coefficients are numbers in $\mathbb{Q}$ and $\mathbb{R}$ respectively.

On the other hand, a Cartier divisor on X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ where $\mathcal{K}_X^*$ means the sheaf of invertible elements in the sheaf of total quotient ring of the structure sheaf $\mathcal{O}_X$. When X is normal, the Cartier divisors on X are the locally principal Weil divisors. Note that on algebraic varieties, all Cartier divisors modulo linear equivalence are isomorphic to line bundles. A $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-Cartier if mD is Cartier for some integer $m > 0$.

By higher dimensional Hurwitz formula and purity of branched locus, for any finite, separable morphism $f : X \to Y$ between two nonsingular algebraic varieties, we have

$$K_X = f^* K_Y + R$$

where K_X, K_Y are the canonical divisors of X, Y respectively and R is the ramification locus which is a divisor. If f is étale, then $R = 0$ and we have $K_X = f^* K_Y$, K_X is just the pull-back of K_Y. This is an important fact used in the proof of the main theorem.

An essential invariant involved in the argument is the self-intersection of canonical divisors on exceptional sets. We at first define the notion of intersection number.

Definition 2.1.6. [KM98] Let X be a proper scheme, $Z \subset X$ a closed subscheme of dimension d, $L_1, ..., L_d$ Cartier divisors on X. The intersection number $L_1 \cdot ... L_d \cdot Z$ is defined as the degree of the zero cycle determined by the intersection. In particular, if $L_1, ..., L_d, Z$ intersect at smooth points with independent tangent planes, the

intersection number is just the number of the intersection points.

We come to a special case of singularities, the isolated singularities. $(x \in X)$ is an isolated singularity if x is a singular point and there is a neighborhood $U \subset X$ such that $U - x$ is nonsingular. The example 2.1.2 above gives that $(0,0) \in C$ is an isolated singularity (the cusp). In [BFF11], the volume of an isolated singularity is defined and it can be used to characterize the singular type. We follow the idea of volume to define similar notion for isolated singularities which involves their resolutions. For an isolated singularity $(x \in X)$, since locally it is the only singular point, we can shrink X so that we assume X affine and $x \in X$ is the only singular point. Equivalently, we can also consider a germ $(x \in V)$ where $V \subset X$ is an open affine subset in X. Replace X by such V. Take a resolution $f : Y \to X$ of $(x \in X)$. The exceptional set $E \subset Y$ is actually the preimage of x. Note that E is proper and it is the largest proper closed subset in Y. This is another important fact used in the proof of main theorem.

To define the "canonical volume" of an isolated singularity, we need a "canonical resolution". We already know that there is no minimal resolution for dimension larger than 2, so another choice of resolution is necessary. Fortunately, there is a good candidate if we allow partial resolutions $f : Y \to X$ such that Y may be singular but its singularity is better than X. Before we can go further, we need to define types of some mild singularities.

To give description of singularities, we note that the canonical divisor K_X on a normal variety X may not be Cartier. However, after adding another $\mathbb{Q}$-divisor $\Delta = \sum a_i D_i, 0 \leq a_i \leq 1$ on X, we can assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. This invention brings the natural definition of pullback of Cartier divisors as the corresponding line bundle. Therefore, for resolution $f : Y \to X$ of X, we can compare the canonical divisors mK_Y and the pullback $f^*(m(K_X + \Delta))$ where $m(K_X + \Delta))$ is Cartier. We call (X, Δ) a pair. Resolution of singularities can be extended to log resolution of

pairs.

Definition 2.1.7. Let (X, Δ) be a pair, X a normal variety, $\Delta = \sum a_i D_i$ a $\mathbb{Q}$-divisor on X. A log resolution of (X, Δ) is a proper birational morphism $f : Y \to X$ such that Y is smooth, the exceptional set E of f is a divisor and $E \cup f^{-1}(\text{Supp } D)$ is simple normal crossing.

A $\mathbb{Q}$-divisor $D = \sum a_i D_i$ on a smooth variety X is a simple normal crossing divisor if each D_i is smooth and they intersect everywhere transversally.

The existence of log resolution is also prove in [Hir64] for varieties over a field of characteristic 0.

Theorem 1. [Hir64] *Let X be an irreducible algebraic variety over $\mathbb{C}$ and $\mathcal{I}_Z \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a subvariety $Z \subset X$. Then there are a smooth variety Y and a projective morphism $f : Y \to X$ such that*

1. *f is an isomorphism over $X - (\text{Sing}(X) \cup \text{Supp(Z)})$.*
2. *$f^*\mathcal{I}_Z \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$.*
3. *$E \cup D$ is simple normal crossing where E is the exceptional divisor of f.*

Remark 2. The theorem above has corresponding analytic version. That is, we can replace X by a small neighborhood of a compact subset of an irreducible reduced analytic space, $Z \subset X$ by a closed subspace, Y by an analytic space.

We now define discrepancy for divisors over X.

Definition 2.1.8. ([KM98], Definition 2.25, 2.28) Let (X, Δ) be a pair where X is a normal variety and $\Delta = \sum a_i D_i, 0 \leq a_i \leq 1$ is a sum of distinct prime divisors. Assume that $m(K_X + \Delta)$ is Cartier for some positive integer $m > 0$. Suppose that $f : Y \to X$ is a birational morphism from a normal variety Y and $E \subset Y$ is the

exceptional locus of f with $E_i \subset E$ the irreducible exceptional divisors.

$$f_*^{-1}\Delta := \sum a_i f_*^{-1} D_i$$

is the strictly transform of Δ by f. Since

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta))|_{Y-E} \cong f^*\mathcal{O}_X(m(K_X + \Delta))|_{Y-E},$$

there are some rational numbers $a(E_i, X, \Delta)$ such that $m \cdot a(E_i, X, \Delta) \in \mathbb{N}$ and

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta)) \cong f^*\mathcal{O}_X(m(K_X + \Delta))(\sum_i (m.a(E_i, X, \Delta))E_i),$$

which can be written as

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum_{E_i:\text{exceptional}} a(E_i, X, \Delta)E_i$$

where $\equiv$ means the numerically equivalence. Then the discrepancy of (X, Δ) is defined as

$$\text{discrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) : E \text{ is an exceptional divisor over } X.\}$$

Moreover, the total discrepancy of (X, Δ) is defined as

$$\text{totaldiscrep}(X, \Delta) := \inf_E \{a(E, X, \Delta) : E \text{ is a divisor over } X.\}$$

There are some basic properties of $a(E, X, \Delta)$.

Proposition 1. *([KM98], 2.27-31)*

1. *Assume that $\Delta' \geq 0$ is effective and $\mathbb{Q}$-Cartier. Then $a(E, X, \Delta) \geq a(E, X, \Delta + \Delta')$ for every divisor E over X,*

2. *Let X be a smooth variety and $\Delta = \sum a_i D_i$ a sum of distinct prime divisors.*

Let $Z \subset X$ be a closed subvariety of codimension k, $\pi : Bl_Z X \to X$ the blow-up of Z, and $E \subset Bl_Z X$ an irreducible component of the exceptional divisor which dominates Z. Then

$$a(E, X, \Delta) = k - 1 - \sum_i a_i \cdot mult_Z D_i.$$

$mult_Z D_i$ is the multiplicity of D_i along Z and $\pi^* D_i = (\pi_*)^{-1} D_i + (mult_Z D_i)E$ when Z is smooth, $(\pi_*)^{-1} D_i$ is the strict transform of D_i.

3. *Let $f : Y \to X$ be a proper birational morphism between normal varieties. Let $(Y, \Delta_Y), (X, \Delta_X)$ be two pairs such that*

$$K_Y + \Delta_Y \equiv f^*(K_X + \Delta_X) \text{and } f_* \Delta_Y = \Delta_X$$

Then

$$a(F, Y, \Delta_Y) = a(F, X, \Delta_X)$$

for any divisor F over X.

4. *Assume that $\Delta = \sum a_i D_i$ is a $\mathbb{Q}$-divisor.*

   (a) *Either discrep $(X, \Delta) = -\infty$, or $-1 \leq$ totaldiscrep $(X, \Delta) \leq$ discrep $(X, \Delta) \leq 1$.*

   (b) *If X is smooth, then* discrep $(\mathrm{X}, 0) = 1$.

   (c) *Assume that X is smooth, $\sum D_i$ is an snc divisor and $a_i \leq 1$ for all i. Then*

$$\text{discrep } (X, \Delta) = \min \left\{ \min_{i \neq j, D_i \cap D_j \neq \emptyset} \{1 - a_i - a_j\}, \min_i \{1 - a_i\}, 1 \right\}$$

   Recall that a $\mathbb{Q}$-divisor $D = \sum a_i D_i$ is simple normal crossing (snc) if each

D_i is smooth and they intersect everywhere transversally.

When $\Delta = 0$, we write X instead of $(X, 0)$.

Definition 2.1.9. [KM98] Let (X, Δ) be a pair where X is a normal variety and $\Delta = \sum a_i D_i$ is a sum of distinct prime divisors. Assume that $m(K_X + \Delta)$ is Cartier for some positive integer $m > 0$, we say that (X, Δ) is

$$\begin{array}{ccc} \textit{terminal} & \text{if} & \text{discrep}(X, \Delta) > 0 \\ \textit{canonical} & \text{if} & \text{discrep}(X, \Delta) \geq 0 \\ \textit{klt} & \text{if} & \text{discrep}(X, \Delta) > -1, a_i < 1 \forall i \\ \textit{plt} & \text{if} & \text{discrep}(X, \Delta) > -1 \\ \textit{lc} & \text{if} & \text{discrep}(X, \Delta) \geq -1 \end{array}$$

Remark 3. klt means "Kawamata log terminal", plt for "purely log terminal", and lc for "log canonical". Most of the theorems in MMP hold in klt case, e.g., the Kawamata-Viehweg vanishing theorem. lc is the largest class for meaningful discrepancy as the properties above. There are still many questions within this class.

Let X be a normal variety. By [BCHM10], there is a projective birational morphism $f^{can} : X^{can} \to X$ such that X^{can} has only canonical singularity and $K_{X^{can}}$ is f^{can}-ample. We recall the definition of relative ampleness.

Definition 2.1.10. Let $f : X \to Y$ be a proper morphism between two varieties X and Y. A line bundle L on X is f-very ample if the canonical map $f^* f_* L \to L$ is surjective and induces an embedding

$$\begin{array}{ccc} X & \xhookrightarrow{\phi_L} & \mathbb{P}_Y(f_* L) \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & Y & \end{array}$$

where $\mathbb{P}_Y(f_* L)$ is the projective vector bundle over Y associate to $f_* L$.

L is f-ample if $L^{\otimes m}$ is f-very ample for some $m > 0$.

There are some criteria for f-ampleness. The following is a famous one.

Proposition 2. *(Nakai-Moishezon criterion)*([KM98], 1.42); ([Laz04], 1.7.9) *Let $f : X \to Y$ be a proper morphism and D is a $\mathbb{Q}$-Cartier divisor on X. Then D is f-ample if and only if $(D^{\dim V} \cdot V) > 0$ for every irreducible subvariety $V \subset X$ of positive dimension with $f(V)$ being a point in Y.*

There is a similar criterion of nefness and relative nefness. We just consider the case of varieties.

Definition 2.1.11. Let X be a normal variety. A $\mathbb{Q}$-Cartier divisor D is nef if the intersection $(D \cdot C) \geq 0$ for every irreducible curve $C \subset X$. For a proper morphism $f : Y \to X$, a $\mathbb{Q}$-Cartier divisor D on Y is f-nef if $(D \cdot C) \geq 0$ for every irreducible curve contracted by f.

Proposition 3. ([KM98], 1.43) *Let $f : X \to Y$ be a proper morphism and D is a $\mathbb{Q}$-Cartier divisor on X. Then D is f-nef if and only if $(D^{\dim V} \cdot V) \geq 0$ for every irreducible subvariety $V \subset X$ of positive dimension with $f(V)$ being a point in Y.*

We also provide the definition of bigness which will be used in the main result.

Definition 2.1.12. Let X be a proper variety of dimension n. A Cartier divisor D is called big if

$$c \cdot k^n < h^0(X, \mathcal{O}_X(kD)) < C \cdot k^n$$

for some positive numbers c, C and $k >> 1$.

There is also a relative version of bigness. When Y is a point, it reduces to the original definition.

Definition 2.1.13. Let $f : X \to Y$ be a proper morphism between varieties. A Cartier divisor D is called f-big if rank $f_*\mathcal{O}_X(kD) > c \cdot k^n$ for some $c > 0$ and $k >> 1$. n is the dimension of the general fiber of f.

Proposition 4. ([KM98], 3.23) *Let $f : X \to Y$ be a proper morphism between varieties and D is a Cartier divisor. Then the following are equivalent:*

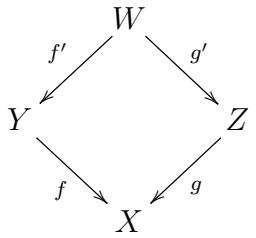
1. *D is f-big.*
2. *mD is linearly equivalent to $A+E$ where A is a f-ample divisor and E effective for some $m > 0$.*
3. *the rational map $\phi_{|mD|/Y}$ is birational for some $m > 0$.*
4. *$\phi_{|mD|/Y}$ is generically finite for some $m > 0$.*

Note that the rational map $\phi_{|mD|/Y} : X \dashrightarrow \mathbb{P}_Y(f_*\mathcal{O}_X(mD))$ over Y is induced by the natural homomorphism $f^*f_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$.

We denote X^{can} the relative canonical model of X. The construction of X^{can} is taking any resolution $f : Y \to X$ of X at first, then

$$X^{can} := Proj_X \bigoplus_{m\geq 0} f_*\mathcal{O}(mK_Y).$$

The key result proved in [BCHM10] is that the graded $\mathcal{O}_X$-algebra $\bigoplus_{m\geq 0} f_*\mathcal{O}(mK_Y)$ is always finitely generated, so $Proj_X \bigoplus_{m\geq 0} f_*\mathcal{O}(mK_Y)$ is defined. Note that this is independent of the choice of resolutions of X. In fact, for any two resolutions $f : Y \to X$ and $g : Z \to X$ of X, we can find a common resolution W of Y and Z



then we have

$$\begin{aligned} Proj_X \bigoplus_{m\geq 0} f_*\mathcal{O}(mK_Y) &= Proj_X \bigoplus_{m\geq 0} (f\circ f')_*\mathcal{O}(mK_W) \\ &= Proj_X \bigoplus_{m\geq 0} (g\circ g')_*\mathcal{O}(mK_W) \\ &= Proj_X \bigoplus_{m\geq 0} g_*\mathcal{O}(mK_Z). \end{aligned}$$

We have morphism $f^{can} : X^{can} \to X$ and birational map $\phi : Y \dashrightarrow X^{can}$, ϕ^{-1} has no exceptional divisors, such that $f = f^{can} \circ \phi$.

Example 2.1.4. For surface, the relative canonical model is obtained by contracting all (-2) self-intersection rational curves in the minimal resolution of V.

We show that X^{can} defined above has the described properties and it is the unique partial resolution having these properties.

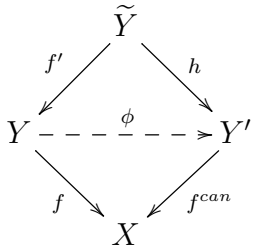
Proposition 5. *X^{can} defined above has at most canonical singularities with $K_{X^{can}}$ f^{can}-ample. Moreover, assume that there is a canonical variety Z with a commutative diagram*

$$\begin{array}{ccc} Y & \overset{\alpha}{\dashrightarrow} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

such that α is a birational map, α^{-1} has no exceptional set, g a proper morphism, K_Z g-ample, and $a(E, Y, 0) \leq a(E, Z, 0)$ for every α-exceptional divisor $E \subset Y$. Then $Z = Y' := Proj_X \bigoplus_{m\geq 0} f_*\mathcal{O}(mK_Y)$.

Proof. We follow the method in [KM98], (3.51-52). Given a resolution $f : Y \to X$ of X, We show that Y' has at most canonical singularities. We can find a common

resolution $\widetilde{Y}$ of Y and Y' as the following commutative diagram:



We have

$$\begin{aligned} K_{\widetilde{Y}} &\equiv f'^*(K_Y) + E_1 \\ &\equiv h^*(K_{Y'}) + E_2 \end{aligned}$$

where E_1 and E_2 are exceptional divisors. Note that $a(E, Y', 0) - a(E, Y, 0)$ is the coefficient of the prime divisor E in $E_2 - E_1 = f'^*(K_Y) - h^*(K_{Y'}) := D$. $-D$ is f'-nef since the intersection between D and any exceptional set is 0. Moreover, $f'_*D = (E_2 - E_1)$ is effective since this divisor is ϕ-exceptional. Then we have $E_2 - E_1$ effective by the following lemma. Since all coefficients of E_1 are positive, it gives that all coefficients of E_2 are non-negative, then Y' has at most canonical singularities. Note that $K_{Y'}$ is f^{can}-ample by the definition of f-ample.

Lemma 1. *(Negativity Lemma) [KM98] Let $f : X \to Y$ be a proper birational morphism between normal varieties, $-D$ a f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then*

1. *D is effective if and only if f_*D is.*
2. *Assume D effective. Then for every $y \in Y$, either $f^{-1}(y) \subset Supp(D)$ or $f^{-1}(y) \cap Supp(D) = \emptyset$.*

Idea of proof. The main part is to show D effective if f_*D is effective. By Chow's Lemma, we can assume f projective, and assume Y affine since it is a local problem. Write $D = \sum D^i$ where $f(D^i)$ has codimension i in Y. For surfaces, $D = D^2 + D^1$, D^1 is f-nef, and we need to show D^2 effective. This can be obtained by the Hodge Index Theorem. For the general case, let $S \subset Y$ be a complete intersection of $n - 2$

general hypersurfaces, $T = f^{-1}(S)$, then the results for surfaces can be applied to $D|_T = D^2|_T + D^1|_T$, and D^2 is effective. Take a general very ample divisor $H \subset X$ and consider the restriction $D|_H$. The proof is done by induction on dimension.

Now, we know Y and Y' are canonical, so E_1, E_2 are both effective. Take Z as described in the property and a common resolution W of Y and Z: $f' : W \to Y$ and $g' : W \to Z$, we have

$$\begin{aligned} Z &= Proj_X \bigoplus_{m\geq 0} g_*\mathcal{O}(mK_Z) \\ &= Proj_X \bigoplus_{m\geq 0} (f \circ f')_*\mathcal{O}(mK_W) \\ &= Proj_X \bigoplus_{m\geq 0} f_*\mathcal{O}(mK_Y) \end{aligned}$$

□

Now we define the canonical volume of isolated singularities.

Definition 2.1.14. Let $(x \in V)$ be a germ such that x is the only isolated singularity. Take the relative canonical model $f : V^{can} \to V$ of V and denote E the exceptional set. Define $cv_V(x) = (K_{V^{can}})^{dimE} \cdot E$, called the *canonical volume* of x.

If V^{can} is not isomorphic to V, we have E nonempty and $cv_V(x) = (K_{V^{can}})^{dimE} \cdot E > 0$ by the f-ampleness. If V^{can} is isomorphic to V and x is a singular point, we set $cv_V(x) = (K_{V^{can}})^{dimE} \cdot E = (K_{V^{can}})^0 \cdot x = 1$. Finally, if x is a smooth point, we set $cv_V(x) = 0$.

In general, if V has finitely many isolated normal singularities $x_i, i = 1, ..., m$, then we consider the sum of canonical volume

$$\sum_{i=1}^{m} cv_V(x_i) = \sum_{i=1}^{m} (K_{V^{can}})^{dimE_{x_i}} \cdot E_{x_i}$$

where $f : V^{can} \to V$ is the relative canonical model of V and E_{x_i} is the exceptional set over x_i. From the definition, we see that $\sum_{i=1}^{m} cv_V(x_i) > 0$ for nonempty isolated

normal singularities $x_i, i = 1, ..., m$, on V.

Example 2.1.5. We can compute volume for some surface singularities.

1. Let $(V, 0)$ be a surface singularity with a resolution such that the exceptional set is $\mathbb{P}^1$ with self-intersection $-k, k \geq 3$. Then the relative canonical model of $(V, 0)$ is the same as its minimal resolution Y since there are no (-1) and (-2)-rational curves on Y. According to the adjunction formula, denote the exceptional curve by E, we have

$$2g_E - 2 = E \cdot (E + K_Y),$$

thus $E \cdot K_Y = -2 - (-k) = k - 2$ which is the volume of $(V, 0)$.

2. Let $(V, 0) = \{z^2 = y(x^4 + y^6)\}$. The exceptional set of its minimal resolution consists of a curve of genus one with self-intersection -1 and two (-2)-rational curves. Denote the three exceptional curves by E_1, E_2, E_3 and $g_{E_1} = 1, g_{E_2} = g_{E_3} = 0$. The relative canonical model Y of $(V, 0)$ is derived by contracting E_2 and E_3 from the minimal resolution Y^m, and E_1' is the image of E_1 of the contraction $p : Y^m \to Y$. By the adjunction formula on Y^m and pull-back from Y to Y^m, we have

$$K_Y \cdot E_1' = p^* K_Y \cdot p^* E_1' = K_{Y^m} \cdot E_1 + a_2 K_{Y^m} \cdot E_2 + a_3 K_{Y^m} \cdot E_3 = K_{Y^m} \cdot E_1$$

since $K_{Y^m} \cdot E_2 = K_{Y^m} \cdot E_3 = 0$, and then $K_{Y^m} \cdot E_1 = 2g_{E_1} - 2 - E_1 \cdot E_1 = 1$. The volume of $(V, 0)$ is 1.

3. Let $(V, 0) = \{z^2 = y^3 + x^{9+6\ell}\}$. The exceptional set of its minimal resolution consists of $\ell + 4$ many (-2)-rational curves and one rational curve with self-intersection -3. Denote the (-3)-rational curve on Y^m by E and its image in

Y by E' through the contraction $\pi : Y^m \to Y$. Then by the adjunction formula and pull-back as above,

$$K_Y \cdot E' = K_{Y^m} \cdot E = 2g_E - 2 - E \cdot E = 1.$$

The volume of $(V, 0)$ is 1.

2.2 Multiplicity of étale morphism between resolutions of isolated singularities

We can start our main theorem for the multiplicity of canonical volume for étale covers of resolutions.

Theorem 2. *Let V be an algebraic variety with finitely many normal isolated singularities $x_i, i = 1, ..., m$, on V. Then the sum of canonical volume $\sum_{i=1}^{m} cv_V(x_i)$ is multiplicative in étale covering maps between resolutions. That is, if W is another algebraic variety or complex space with normal isolated singularities $y_j, j = 1, ..., s$, $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$ are resolutions of W and V respectively, $\Phi : \widetilde{W} \to \widetilde{V}$ is an étale covering map, then we have $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$ where d is the degree of Φ.*

Proof. Suppose we have resolutions $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$, and we consider relative canonical models $p_1' : W^{can} \to W$ and $p_2' : V^{can} \to V$. We start from the following claim:

Claim: If $\Phi : W^{can} \to V^{can}$ is an étale covering map, then $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$ where d is the degree of Φ.

Proof. Let E_1 be the exceptional set of p_1' and E_2 the exceptional set of p_2'. $E_1 = \sum_{j=1}^{s} E_{y_j}$ where E_{y_j} is the exceptional set over y_j, and similarly $E_2 = \sum_{i=1}^{m} E_{x_i}$ where E_{x_i} is the exceptional set over x_i. Since the canonical divisors $K_{W^{can}}$ and $K_{V^{can}}$ are p_1'-ample and p_2'-ample respectively, if E_1 is not empty, we have $\sum_{j=1}^{s} cv_W(y_j) = \sum_{j=1}^{s} (K_{W^{can}})^{dimE_{y_j}} \cdot E_{y_j} > 0$. Similarly, $\sum_{i=1}^{m} cv_V(x_i) = \sum_{i=1}^{m} (K_{V^{can}})^{dimE_{x_i}} \cdot E_{x_i} > 0$ if E_2 is not empty.

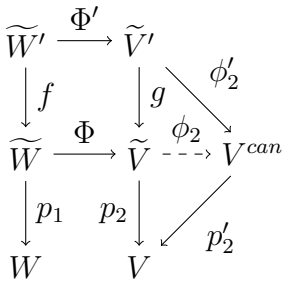
By Φ being an étale covering map, we have the pullback $\Phi^* K_{V^{can}} = K_{W^{can}}$. Also, $\Phi^* E_2 = E_1$ since E_1 and E_2 are the only proper sets in W^{can} and in V^{can} respectively if we shrink V and W. Therefore, from

$$\sum_{j=1}^{s} (K_{W^{can}})^{dimE_{y_j}} \cdot E_{y_j} = \sum_{i=1}^{m} (\Phi^* K_{V^{can}})^{dimE_{x_i}} \cdot \Phi^* E_{x_i} = d \sum_{i=1}^{m} (K_{V^{can}})^{dimE_{x_i}} \cdot E_{x_i}$$

where d is the degree of Φ, we have $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$.

If $V^{can} \cong V$, E_2 has dimension 0 and $E_2 = \sum_{i=1}^{m} x_i$. Since étale morphisms are locally isomorphisms, Φ sends singular points to singular points, and we have $E_1 = \sum_{j=1}^{s} y_j$ also 0-dimensional and $W^{can} \cong W$. The intersection $(K_{V^{can}})^0 \cdot x_i$ is just 1 by definition. By counting the singular points, we have $\sum_{j=1}^{s} cv_W(y_j) = s = dm = d \sum_{i=1}^{m} cv_V(x_i) > 0$. The claim is proved. □

Now, for two resolutions $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$, we have birational map $\phi_2 : \widetilde{V} \dashrightarrow V^{can}$ over V. (see the diagram below)



Take a common resolution $\widetilde{V}'$ of $\widetilde{V}$ and V^{can} with birational morphisms $g : \widetilde{V}' \to \widetilde{V}$ and $\phi_2' : \widetilde{V}' \to V^{can}$, $\Phi' : \widetilde{W}' \to \widetilde{V}'$ is the base-change of $\Phi : \widetilde{W} \to \widetilde{V}$. Then Φ' is also an étale covering map with the same degree of Φ. After replacing Φ and ϕ_2 by Φ' and ϕ_2' respectively, we can assume $\phi_2 : \widetilde{V} \to V^{can}$ a birational morphism.

By the property of relative canonical model, V^{can} and W^{can} have at worst canonical singularities. We use a theorem in [Tak03]:

Theorem 3. ([Tak03], Theorem 1.1) *Let V be a normal analytic space and let $f : \widetilde{V} \to V$ be a resolution of singularities. Then the induced homomorphism $f_* : \pi_1(\widetilde{V}) \to \pi_1(V)$ is an isomorphism if (V, Δ) is Kawamata log-terminal (klt) for some divisor Δ.*

Ramark. In [Kol93], it is proved that f induces isomorphism between algebraic fundamental groups when V is log terminal. The fundamental group $\pi_1(\widetilde{V})$ is independent of the choice of resolution of singularities, and we can denote $\pi_1(Res\ V)$ as the fundamental group of any resolution of V. In general, for any resolution $f : Y \to X$ of a normal variety X, the induced map $\pi_1(Y) \to \pi_1(X)$ is surjective and has nontrivial kernel. The theorem means that the klt singularity is mild in topology.

Remark 4. When the base field is characteristic 0, the results of [Tak03] and [Kol93] used in the following also hold when X is an algebraic variety and U can be the Henselization of the local ring $\mathcal{O}_{X,x}$ instead of a contractible neighborhood. A local ring R with maximal ideal m is Henselian if for any monic polynomial $P(x)$ in $R[x]$,

any factorization of the image of $P(x)$ in $R/m[x]$, $p(x) = f(x)g(x)$, into two coprime monic polynomials can be lifted to a factorization $P(x) = F(x)G(x)$ in $R[x]$. A Henselian local ring is called strictly Henselian if its residue field is separably closed. For a local ring A, its Henselization is a Henselian ring B such that any local homomorphism $h : A \to C$ to a Henselian ring can be extended to $h' : B \to C$. Similarly, the strict Henselization can be defined. Therefore, we abuse the terms of analytic space and contractible neighborhood to the situation of algebraic varieties.

We recall the definition of algebraic fundamental groups.

Definition 2.2.1. Let X be a connected scheme. For any finite étale cover $f : Y \to X$, we denote $Aut_X(Y)$ the Galois group of f. If there is a X-morphism $X_i \to X_j$ between two finite étale covers $X_i \to X$ and $X_j \to X$, it induces a homomorphism $Aut_X(X_i) \to Aut_X(X_j)$. The algebraic fundamental group $\hat{\pi}_1(X)$ of X is the inverse limit

$$\hat{\pi}_1(X) = \lim_{\leftarrow} \ Aut_X X_i$$

over all finite étale covers of X.

When $\pi_1(X)$ is finite, we have $\hat{\pi}_1(X) = \pi_1(X)$.

Proof. The theorem can be reduced to a local problem by the topological lemma below.

Lemma 2. ([Tak03], Lemma 3.1) *Let $f : Y \to X$ be a surjective continuous map of topological spaces. Assume that*

1. *f has arcwise connected fibers.*

2. *$\exists$ an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ such that U_λ is arcwise connected with $\pi_1(U_\lambda) = 1$, and every $f^{-1}(U_\lambda)$ is also arcwise connected with $\pi_1(f^{-1}(U_\lambda)) = 1$.*

Then the induced homomorphism $f_* : \pi_1(Y) \to \pi_1(X)$ is an isomorphism.

So we need to check that for a resolution $f : Y \to X$, every point $p \in X$ has an open neighborhood U such that $\pi_1(U_\lambda) = 1$ and $\pi_1(f^{-1}(U_\lambda)) = 1$. This is called locally simply connected in [Kol93], Definition 7.1. Condition 1 of the lemma is satisfied since the fibers of f are connected by Zariski Main Theorem, and connected analytic space is arcwise connected.

Theorem 4. *(Zariski Main Theorem)*([Har77], 11.4) *Let $f : X \to Y$ be a birational projective morphism of noetherian integral schemes, and assume that Y is normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.*

The most technical part is the following finiteness result.

Proposition 6. ([Tak03], Prop. 2.1) *Let $g : X \to S$ be a proper surjective morphism of normal varieties with connected fibres. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that the pair (X, Δ) is klt with $-(K_X + \Delta)$ g-nef and g-big. Let $f : Y \to X$ be a resolution of singularities. Then for every point $p \in S$, there exists a contractible open neighbourhood $p \in U \subset S$ such that $(g \circ f)^{-1}(U)$ is connected and $\pi_1((g \circ f)^{-1}(U))$ is finite.*

The idea of the proposition's proof is by contradiction. Assume that U is a contractible open subset with infinite fundamental group. Take a point $0 \in X$ and $Y_0 = f^{-1}(0)$. Since (X, Δ) is klt, we have

$$K_Y + \Delta' \equiv f^*(K_X + \Delta) + \sum e_i E_i$$

where Δ' is the strictly transform of Δ, $\sum E_i$ is the exceptional divisor of f, $\sum E_i \cup f^{-1}(\text{Supp }(\Delta))$ is simple normal crossing, and $e_i > -1$ for all i. Take

$$F = \sum \ulcorner e_i \urcorner E_i, \Delta'' = \sum (\ulcorner e_i \urcorner - e_i) E_i$$

where $\ulcorner e_i \urcorner = N$ if $N - 1 < e_i \leq N$, which means the round-up of e_i. We also have

$-f^*(K_X + \Delta) \equiv A + D$ where A is a $(g \circ f)$-ample $\mathbb{Q}$-divisor and D is an effective $\mathbb{Q}$-divisor. D can be taken very small such that $(Y, \Delta_Y := \Delta' + \Delta'' + D)$ is also klt and then

$$F \equiv K_Y + \Delta_Y + A.$$

The second main goal is to show $H^0_{(2)}(\widetilde{Y}, \widetilde{F}) \neq 0$, where $\pi : \widetilde{Y} \to Y$ is the universal cover of Y, $\widetilde{F} = \pi^* F$, and

$$H^0_{(2)}(\widetilde{Y}, \widetilde{F}) := H^0(Y, \pi_{*(2)} F)$$

is the 0th L^2-cohomology group, where for every open subset $W \subset Y$, $s \in \Gamma(W, \pi_{*(2)}F) \subset \Gamma(\pi^{-1}(W), F)$ has the property that for any $p \in W$, there is an open neighborhood $U \subset\subset V$ of $y \in Y$, V is a small coordinate neighborhood of y, such that $s|_{\pi^{-1}(U)}$ is L^2 with respect to the induced metrics on $\widetilde{Y}, \widetilde{F}$ from Y, F.

From the non-vanishing of the 0th L^2-cohomology, there is some $0 \neq g \in H^0_{(2)}(\widetilde{Y}, \widetilde{F})$. There are two possibilities:

1. $g|_{\widetilde{Y_0}} \neq 0$.

2. General cases.

Assume the first case. Note

$$\int_{\widetilde{Y}'} |g|^{2m} dv_{\widetilde{\omega}} < \infty, m \geq 1,$$

for every $\widetilde{Y}' \in \mathcal{U}_{\widetilde{Y}}$,

$\mathcal{U}_{\widetilde{Y}} := \{(g \circ f \circ \pi)^{-1}(S') | S'$ is a relatively compact contractible Stein open neighbourhood of $0 \in$

(Stein variety is defined in the next section.) $\widetilde{\omega}$ is the induced metric on $\widetilde{Y}$ from the

metric ω on Y. Then the Poincaré series

$$P(g^{\otimes 2m}) := \sum_{\gamma \in \pi_1(Y)} \gamma^* g^{\otimes 2m}$$

uniformly converges on every $\widetilde{Y}' \in \mathcal{U}_{\widetilde{Y}}$. Denote the set

$$P^{2m} := \left\{ \Pi P(g^{\otimes 2m_i}) | m_i \in \mathbb{N}, \sum m_i = 2m \right\}$$

By the assumption that $\pi_1(Y)$ is infinite and results in [Gro91], we can obtain $p_1, p_2 \in P^{2m}$ with $\frac{p_1}{p_2}$ non-constant. But this contradicts the fact that $H^0(Y, F^{\otimes k}) \cong H^0(X, \mathcal{O}_X)$. The general cases can also be reduced to the first one if we divide g by a suitable section which vanishes on an irreducible component of Y_0.

To obtain the non-vanishing, apply the Kawamata-Viehweg vanishing theorem to F, we have $H^q(Y, F) = 0$ for $q > 0$.

Theorem 5. (Kawamata-Viehweg vanishing)([KM98], Theorem 2.64) *Let X be a smooth complex projective variety, L a line bundle on X such that $L \equiv M + \sum a_i D_i$, where*

1. *M is a nef and big $\mathbb{Q}$-divisor.*
2. *$\sum D_i$ is simple normal crossing.*
3. *$0 \leq a_i < 1$ and $a_i \in \mathbb{Q}$ for all i.*

Then $H^q(X, \omega_X \otimes L) = 0$ for $q > 0$.

Moreover, consider the short exact sequence

$$0 \to F \otimes \mathcal{I}_W \to F \to F|_W \to 0,$$

we can get $H^q(Y, F \otimes \mathcal{I}_W) = 0, q > 0$, $H^0(W, F|_W) \neq 0$ where $W \subset Y$ is a subscheme

with Supp $W = Y_0$. Similarly, the short exact sequence

$$0 \to \widetilde{F} \otimes \mathcal{I}_{\widetilde{W}} \to \widetilde{F} \to \widetilde{F}|_{\widetilde{W}} \to 0,$$

$\widetilde{W} = \pi^{-1}(W)$, and the Demailly-Nadel vanishing theorem of L^2-cohomologies give $H^q_{(2)}(\widetilde{Y}, \widetilde{F}) = H^q_{(2)}(\widetilde{Y}, \widetilde{F} \otimes \mathcal{I}_{\widetilde{W}}) = 0, q > 0$. This implies that $H^0_{(2)}(\widetilde{Y}, \widetilde{F}) \to H^q_{(2)}(\widetilde{Y}, \widetilde{F} \otimes \mathcal{I}_{\widetilde{W}})$ is surjective. By the equality of Euler characters $\chi_{(2)}(\widetilde{W}, \widetilde{F}|_{\widetilde{W}}) = \chi(W, F|_W)$ which is derived from L^2-index theorem for $\pi|_{\widetilde{W}} : \widetilde{W} \to W$ and $F|_W$, we have $h^0_{(2)}(\widetilde{W}, \widetilde{F}|_{\widetilde{W}}) = h^0(W, F|_W)$, and then $H^0_{(2)}(\widetilde{Y}, \widetilde{F}) \neq 0$ from the results above. The proposition is then proved.

Apply the proposition of finiteness to $X = S$ in the theorem, there is a contractible open neighborhood $p \in U \subset X$ such that $f^{-1}(U)$ is connected and $\pi_1(f^{-1}(U))$ is finite. Then we apply the theorem in [Kol93].

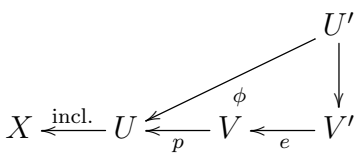
Theorem 6. ([Kol93], Theorem 7.5) *Let X be a normal variety, $f : Y \to X$ a resolution of singularities of X.*

1. *If (X, Δ) is klt for some $\mathbb{Q}$-divisor Δ, then Y is locally algebraic simply connected.*

2. *If X has quotient singularities, then Y is locally simply connected.*

Since $\pi_1(f^{-1}(U))$ is finite, the first part implies that $f^{-1}(U)$ has a non-trivial étale covering $\widetilde{U} \to f^{-1}(U)$ with trivial fundamental group $\pi_1(\widetilde{U}) = 1$. Note that $f^{-1}(U)$ is connected, so it means that $\pi_1(f^{-1}(U)) = 1$. Then we get the isomorphism $f_* : \pi_1(Y) \xrightarrow{\cong} \pi_1(X)$.

The idea of theorem's proof is considering the diagram



where $p : V \to U$ is a resolution, $e : V' \to V$ is an étale cover, U' is the normalization of U in V', and then $\phi : U' \to U$ is étale over the smooth part of U. $(U', \phi^*\Delta)$ is klt and U' has rational singularities by the following two propositions:

Proposition 7. ([Kol92], Prop. 20.3) *Let $h : U \to V$ be a finite and dominant morphism between irreducible normal schemes of characteristic 0. $B = \sum b_i B_i$ is a divisor on V such that $\sum B_i$ contains the branch locus of h. Let $h^*(K_V + B) = K_U + \bar{B}$, then $K_V + B$ is lc (resp. plt) if and only if $K_U + \bar{B}$ is lc (resp. plt).*

Proposition 8. ([KMM87], Theorem 1-3-6; [KM98], Theorem 5.22) *Let (X, Δ) be a klt pair. Then X has rational singularities.*

Definition 2.2.2. ([KM98], 5.8) Let X be a variety over a field of characteristic 0 and $f : Y \to X$ a resolution of singularities. $f : Y \to X$ is a rational resolution if

1. $f_*\mathcal{O}_Y = \mathcal{O}_X$.
2. $R^i f_*\mathcal{O}_X = 0$ for $i > 0$.

X has rational singularities if every resolution $f : Y \to X$ is rational.

When the base field is char. 0, we have good results in determining rational singularities.

Theorem 7. ([KM98], 5.10) *Let X be a variety over a field of characteristic 0. Then the following are equivalent.*

1. *X has a rational resolution.*
2. *Every resolution of Y is rational.*
3. *X is Cohen-Macaulay, i.e., the local ring $\mathcal{O}_{X,x}$ for every $x \in X$ is Cohen-Macaulay: depth $\mathcal{O}_{X,x}$ = dim $\mathcal{O}_{X,x}$, and $f_*\omega_Y = \omega_X$ for any resolution $f : Y \to X$.*

Apply the following lemma to $\phi : U' \to U$, then the proof is completed.

Lemma 3. ([Kol93], Lemma 7.4) *Let $p : X_1 \to X_2$ be a finite and dominant morphism between irreducible analytic spaces. Assume that*

1. *X_1 has rational singularities.*
2. *$\pi_1(Res\ X_1)$(resp. $\hat{\pi}_1(Res\ X_1)$) is trivial.*
3. *$\pi_1(X_2)$(resp. $\hat{\pi}_1(X_2)$) is trivial.*

Then $\pi_1(Res\ X_2)$(resp. $\hat{\pi}_1(Res\ X_2)$) is trivial.

To show this lemma, we can replace X_2 by a contractible neighborhood of a point $x \in X_2$. Take a resolution $f_2 : Y_2 \to X_2$ such that $D_2 := \text{red } f_2^{-1}(x)$ is a normal crossing divisor. The proposition above gives that $\pi_1(Res\ X_2)$(resp. $\hat{\pi}_1(Res\ X_2)$) is finite, thus there is an étale cover $e_2 : Y_2' \to Y_2$ which is a finite morphism. Let X_2' be the normalization of X_2 in Y_2' and $f : X_2' \to X_2$ the induced morphism. The factorization $r : X_1 \to X_2'$, $q : X_2' \to X_2$ of p are finite and dominant. Take $x' \in q^{-1}(x) \subset X_2'$ and note that $D' = \text{red } f^{-1}(x') \to D_2$ is a finite étale morphism, so D' is a normal crossing divisor and $\chi(D', \mathcal{O}_{D'}) = \deg q \cdot \chi(D_2, \mathcal{O}_{D_2})$. Since X' has rational singularities, by a result in [Ste83], it gives $\chi(D', \mathcal{O}_{D'}) = 1$, hence $\deg q = 1$.

Proposition 9. ([Ste83], 2.14; [Kol93], Lemma 5.2.3) *Let X be a complex space and let $D \subset X$ be a divisor being proper and normal crossing. Assume that D is topologically a retract of X. Then the restriction maps $h^i(X, \mathcal{O}_X) \to h^i(D, \mathcal{O}_D)$ are surjective for all i.*

□

We see that $\pi_1(V^{can}) \cong \pi_1(\widetilde{V})$ by the above theorem since V^{can} has canonical singularities, $(V^{can}, 0)$ is *klt*. Now we take the étale cover $\beta : W' \to V^{can}$ which gives the subgroup $\beta_*\pi_1(W') \subseteq \pi_1(V^{can})$ isomorphic to the subgroup $\Phi_*\pi_1(\widetilde{W}) \subseteq \pi_1(\widetilde{V})$.

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\Phi} & \widetilde{V} \\ h \downarrow & & \downarrow \phi_2 \\ W' & \xrightarrow[\beta]{} & V^{can} \end{array}$$

We see that β is an étale covering map with the same degree as Φ. Note that W' has canonical singularities since étale morphisms are locally isomorphisms. Because $\phi_{2*}\Phi_*\pi_1(\widetilde{W}) = \beta_*\pi_1(W')$, there is a morphism $h : \widetilde{W} \to W'$ coming from the morphism $\phi_2 : \widetilde{V} \to V^{can}$ extending to the étale covers $\widetilde{W}$ and W' of $\widetilde{V}$ and V^{can} respectively, and h is birational since ϕ_2 is. In fact, h is a resolution morphism from $\widetilde{W}$ to W'. Since $h^{-1} : W' \setminus \text{Sing } W' \to \widetilde{W}$ is a birational morphism, $p_1 \circ h^{-1} : W' \setminus \text{Sing } W' \to W$ is also a birational morphism. As the singular set of W' consists of isolated normal singularities, $p_1 \circ h^{-1}$ extends through singular set of W' and we get a birational morphism $q : W' \to W$. If $E \subseteq W'$ is an exceptional curve over W, by the projection formula, we have

$$K_{W'} \cdot E = \beta^* K_{V^{can}} \cdot E = K_{V^{can}} \cdot \beta_* E > 0$$

since $\beta_* E$ is a sum of exceptional curves in V^{can} over V and $K_{V^{can}}$ is relative ample over V. So $K_{W'}$ is relatively ample over W. Then, by the uniqueness of relative canonical model, we have W' isomorphic to W^{can}. Replace W' by W^{can}, we have an étale covering map $\beta : W^{can} \to V^{can}$ and this is the claim above, which gives $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$ where $d = \deg\beta = \deg\Phi$.

□

We now can prove the main theorem easily.

Proof. We take $W = V$ in Theorem C. Since the isolated singular points are nonempty, we have $\sum_{i=1}^{m} cv_V(x_i) > 0$. The equation $\sum_{i=1}^{m} cv_V(x_i) = d \sum_{i=1}^{m} cv_V(x_i)$, where d is the de-

gree of Φ, gives $d = 1$, hence Φ is an isomorphism. $\square$

2.3 Application in CR geometry

The prime result in [TYZ13] is about CR geometry. CR stands for Cauchy-Riemann or complex-real. CR geometry considers real manifolds having complex structure models on real hypersurfaces. [TYZ13] gives a condition when a nontrivial CR morphism is actually a CR biholomorphism. We provide some terminologies of CR geometry.

Definition 2.3.1. Let X be a connected orientable manifold of real dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $\mathbb{C}T(X)$ such that

(1) $S \cap \overline{S} = \{0\}$

(2) If L, L' are local sections of S, then so is $[L, L']$.

A manifold with a CR structure is called a CR manifold. There is a unique subbundle $\mathcal{H}$ of the tangent bundle $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{u - iu\colon u \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the CR structure.

The notion of a CR structure attempts to describe intrinsically the property of being a hypersurface in a complex space by studying the properties of holomorphic vector fields which are tangent to the hypersurface. Therefore, a standard class of CR manifolds is real submanifolds in complex spaces. These manifolds are called embedded CR manifolds and their CR structure can be defined more concretely. Let $M \subset \mathbb{C}^n$ be a real submanifold which is locally defined by the real value functions $F_1 = F_2 = ... = F_k = 0$. Suppose that the system satisfies the independence condition

$$\partial F_1 \wedge \bar{\partial} F_1 \wedge ... \wedge \partial F_k \wedge \bar{\partial} F_k \neq 0,$$

M is a manifold of real dimension $2n-k$ and it is an embedded CR manifold of CR codimension k. Let $S \subset T^{1,0}\mathbb{C}^n|_M$ be the subbundle of vectors annihilating all F_i. Note that S satisfies the CR structure in the definition and S has rank $n-k$ by the independence condition. Embedded CR manifolds have more structures than the abstractly defined CR manifolds, usually comes from Cauchy-Riemann equations.

Example 2.3.1. We see two examples of CR manifolds.

1. The classical example of a CR manifold is the real $2n+1$-dimensional sphere S^{2n+1} as a submanifold of $\mathbb{C}^{n+1}$. The subbundle S is given by

$$S = \mathbb{C}T(S^{2n+1}) \cap T^{1,0}\mathbb{C}^{n+1},$$

   $T^{1,0}\mathbb{C}^{n+1}$ is the holomorphic vector bundle. The real expression of the CR structure $(\mathcal{H}, J)$ is $\mathcal{H} = \mathbb{R}(S \oplus \overline{S})$,

$$\mathcal{H}_p = \{X \in T_pS^{2n+1} | JX \in T_pS^{2n+1} \subset T_p\mathbb{C}^{n+1}\}$$

   where $p \in S^{2n+1}$ is any point.

2. A more complicated example is the Heisenberg group $H_3(\mathbb{R})$. It is the group of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

   The fact is that the CR structure on S^3 can be viewed as the one-point compactification of $H_3(\mathbb{R})$. See [CDPT07] for more details.

Definition 2.3.2. Let $(X, \mathcal{H}_X, J_X), (Y, \mathcal{H}_Y, J_Y)$ be two CR manifolds with given CR structures. A differentiable map $f : X \to Y$ is called a CR morphism if f preserves

the CR structure, that is,

$$df_p(S_{p,X}) \subset S_{f(p),Y}$$

for any $p \in X$ and S_X, S_Y are the corresponding subbundles in $\mathbb{C}T(X), \mathbb{C}T(Y)$ respectively.

Definition 2.3.3. Let $\{L_1, \ldots, L_{n-1}\}$ be a local frame of S. Then $\{\overline{L}_1, \ldots, \overline{L}_{n-1}\}$ is a local frame of $\overline{S}$ and one may choose a local section N of $T(X)$ which is purely imaginary such that

$$L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, N$$

is a local frame of $\mathbb{C}T(X)$. The matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \Sigma a_{ij}^k L_k + \Sigma b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

is Hermitian and is called the Levi form of S.

Remark 5. For an embedded CR manifold $M \subset \mathbb{C}^n$ of CR codimension 1 with defining function $F = 0$, The Levi form is defined by the Hermitian form

$$h = i\partial\bar{\partial}F|_S.$$

Proposition 10. *The number of nonzero eigenvalues and the absolute value of the signature of the Levi form (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 2.3.4. The CR manifold X is called pseudoconvex if the Levi form is semi-definite at each point of X. X is strongly pseudoconvex if the form is actually definite everywhere.

The name "(strongly) pseudoconvex" comes from the study of of pseudoconvex domains.

Definition 2.3.5. Let $U \subset \mathbb{C}^n$ be a domain, i.e., an open connected subset. A function $f : U \to \mathbb{R} \cup \{-\infty\}$ is called a plurisubharmonic if it is upper semi-continuous and for any complex line $L := \{a + bz | a, b \in \mathbb{C}\}$, the function $z \to f(a + bz)$ is subharmonic on $L \cap U$, i.e., for any closed ball $\overline{B(y, r)} \subset L \cap U$ of center y and radius r, any real-valued continuous function h on $\overline{B(y, r)}$ and harmonic on $B(y, r)$, and $f(a + bz) \leq h(z)$ for all $z \in \partial B(y, r)$, then $f(a + bz) \leq h(z)$ for all $z \in B(y, r)$.

U is a pseudoconvex domain if there is a continuous plurisubharmonic function ϕ on U such that $\{z \in U | \phi(z) < x\}$ is a relatively compact subset of U for any real x.

Remark 6. If f is a C^2-function, f is plurisubharmonic if and only if the hermitian matrix $(\lambda)_{ij}$, called Levi matrix, $\lambda_{ij} = \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}$ is positive semi-definite.

Proposition 11. *Let $M \subset \mathbb{C}^n$ be a real submanifold. M is the boundary of a (strongly) pseudoconvex domain in $\mathbb{C}^n$ if and only if it is a (strongly) pseudoconvex as a CR manifold.*

Studies in strongly pesudoconvex CR manifolds show that they are embeddable in complex Euclidean space when their dimensions are larger than 3, and they are actually the boundaries of singular complex varieties.

Theorem 8 (Boutet De Monvel [Bou75])**.** *If X is a compact strongly pseudoconvex CR manifold of dimension $(2n - 1)$ and $n \geqslant 3$, then X is CR embeddable in $\mathbb{C}^N$ for some N, i.e., a real submanifold with CR structure in $\mathbb{C}^N$.*

Remark 7. There are non-embeddable compact 3-dimensional CR manifolds. So we need to emphasize the embeddable condition so that we can have the following theorem.

Theorem 9 (Harvey–Lawson [HL75],[HL])**.** *For any compact connected embeddable strongly pseudoconvex CR manifold X, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X and V has only normal isolated singularities.*

Because of the theorem above, isolated singularities relate to CR geometry.

Another concept used in the theorem is Stein manifold. The key fact is that if the boundary of a Stein domain in $\mathbb{C}^n$ is smooth, the boundary is actually a CR manifold. Moreover, we consider varieties with only finitely many isolated singularities. Locally, we can find an affine open subset containing only one singular point, and a complex affine variety is also a Stein variety.

Definition 2.3.6. A complex space X is a Stein variety if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and any analytic coherent sheaf $\mathcal{F}$. Equivalently, X is Stein if it satisfies:

1. (holomorphically convex) For every compact subset $K \subset X$, the set

$$\overline{K} := \{z \in X || f(z)| \leq \sup_K |f|, \forall f \in \Gamma(X, \mathcal{O}_X^{hol})\}$$

   is also a compact subset of X.

2. (holomorphically separable) For $x \neq y$ on X, $\exists f \in \Gamma(X, \mathcal{O}_X^{hol})$ such that $f(x) \neq f(y)$.

Example 2.3.2. 1. Any complex affine variety is a Stein variety.

2. (Behnke and Stein) A connected non-compact Riemann surface is Stein.

Note that the converse of the first example is not true. There are Stein varieties which are not affine varieties. For a quasi-affine complex normal variety X with X^{an} Stein, X is affine if and only if $\Gamma(X, \mathcal{O}_X)$ is finitely generated. Every closed submanifold of a Stein manifold is also Stein. Moreover, every Stein manifold X of complex dimension n can be embedded into $\mathbb{C}^{2n+1}$ by a proper holomorphic map. So a Stein manifold is a closed complex submanifold of $\mathbb{C}^n$ for some n.

Rigidity problems of CR-immersions into spheres and hyperquadrics were studied by Ebenfelt, Huang and Zaitsev [EHZ04] [EHZ05]. Let X be a strongly pseudoconvex

CR manifold of dimension $2n-1$. For $p \in X$, let f: $(X,p) \longrightarrow S^{2n+2d-1}$ be a local CR immersion of X near p into unit sphere $S^{2n+2d-1}$ in $\mathbb{C}^{n+d}$. The beautiful result of [EHZ04] states that if $d < \frac{n}{2} - 1$, then f is rigid in the sense that any other immersion of (X,p) into $S^{2n+2d-1}$ is of the form $\phi \circ f$, where ϕ is biholomorphic automorphism of the unit ball $B \subseteq \mathbb{C}^{n+d}$. As a striking corollary, they show that if X and X' are two strongly pseudoconvex CR manifolds of dimension $2n-1$ in $S^{2n+2d-1}$ with $d < \frac{n}{2} - 1$ and if X and X' are locally CR equivalent at some points $p \in X$ and $p' \in X'$, then there exists a unitary linear transformation which maps X to X'.

In a remarkable paper [Pin74], Pinchuk showed that a proper holomorphic mapping between strongly pseudoconvex domains in $\mathbb{C}^n$ is locally biholomorphic. In fact he proved that proper holomorphic self-maps of strongly pseudoconvex domains are necessarily biholomorphic. It was proved in [BC82] and [DF82] that proper holomorphic maps extend smoothly to the boundaries and hence induce CR morphisms between the boundaries.

In [Yau11], there is another point of view. In view of Fornaess theory on strongly pseudoconvex domains [For76], it investigates the rigidity property of CR morphisms between strongly pseudoconvex CR manifolds by means of the singularities theory.

Since we are in the range of complex geometry, we need to check if the resolution of singularities still hold. Locally, we shall assume that $x \in V$ is a germ of complex analytic space with only one isolated singularity x. By Hironaka's paper[Hir63], it's biholomorphic equivalent to a germ of a complex algebraic singularity. Since there are only finitely many isolated singularities, after the equivalence and the resolution theorems of algebraic varieties over field of characteristic 0, we can construct a resolution $\pi : \widetilde{V} \to V$ of V such that $\widetilde{V}$ is smooth and π is a bimeromorphic proper morphism.

The main result about resolution of isolated singularities then give sufficient conditions of CR biholomorphisms.

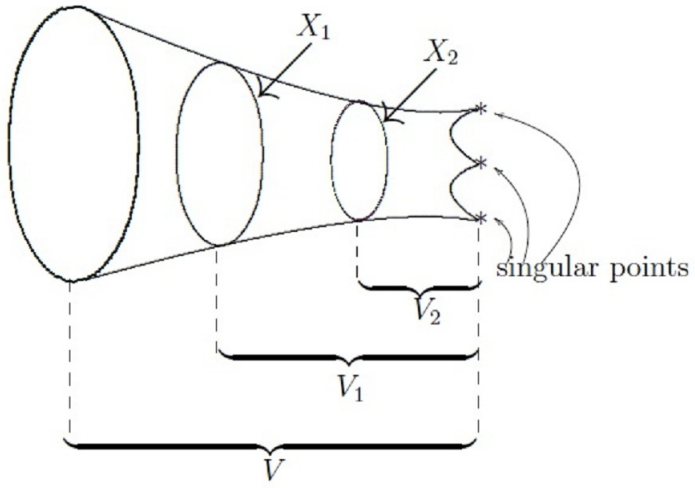


Figure 2.1: situation of the theorem [TYZ13]

Theorem 10. *Let X_1 and X_2 be two $(2n-1)$-dimensional compact strongly pseudoconvex CR manifolds lying in a Stein variety V of dimension n in $\mathbb{C}^N$. Let $V_1 \subseteq V$, $V_2 \subseteq V$ such that $\partial V_1 = X_1$ and $\partial V_2 = X_2$. Assume that the singular set S of V is nonempty and $S = Sing\ V = Sing\ V_1 = Sing\ V_2$(see Figure 2.1). Then non-constant CR morphisms from X_1 to X_2 are necessarily CR biholomorphisms.*

Corollary 1. *Let X be a compact strongly pseudoconvex embeddable CR manifold of dimension ≥ 3 lying in $\mathbb{C}^N$. Then any non-constant CR morphism from X to itself must be a CR automorphism.*

Proof. The following proof besides the application of the main theorem is due to Stephen S.-T. Yau and Huaiqing Zuo [TYZ13]. The first step to prove the theorem is extending the morphism $X_1 \to X_2$ to $V_1 \to V_2$. This is done in [Yau11].

Proposition 12. *([Yau11], Prop. 1.1) Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geq 3$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Suppose that the singular set $S_i \subset V_i, i = 1, 2$ is either empty or consisting of only isolated normal singularities. If $\Phi : X_1 \to X_2$ is a non-constant CR morphism, then Φ is surjective and Φ can be extended to a*

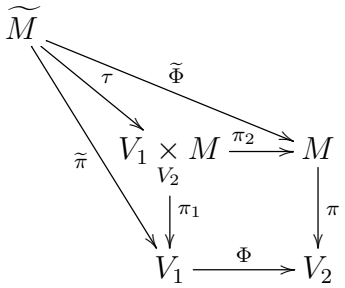
proper surjective holomorphic map $\Phi : V_1 \to V_2$ *(use the same notation) such that* $\Phi(S_1) \subseteq S_2, \Phi^{-1}(X_2) = X_1$, *and* $\Phi : V_1 - \Phi^{-1}(S_2) \to V_2 - S_2$ *is a covering map. Moreover, if* S_2 *does not have quotient singularities, then* $\Phi^{-1}(S_2) = S_1$.

So we start with a non-constant CR morphism $\Phi : X_1 \to X_2$. By the proposition above, Φ can be extended to a proper holomorphic map from V_1 to V_2 such that $\Phi\colon V_1 \setminus \Phi^{-1}(S) \longrightarrow V_2 \setminus S$ is a covering map of degree d and $\Phi(S) = S$. Let $S = \{q_1, \ldots, q_m\}$. Then $\Phi^{-1}(S) = \{q_1, \ldots, q_m, p_1, \ldots, p_k\}$. We shall prove that $\Phi^{-1}(S) = S$. Let $\pi\colon M \longrightarrow V_2$ be a resolution of singularities of V_2 such that the exceptional sets

$$E_1 = \pi^{-1}(q_1) = \bigcup_{i=1}^{\ell_1} A_i^1, \ldots, E_m = \pi^{-1}(q_m) = \bigcup_{i=1}^{\ell_m} A_i^m$$

are normal crossing divisors.

Consider the fibre product $V_1 \underset{V_2}{\times} M$ of the maps $\Phi\colon V_1 \longrightarrow V_2$ and $\pi\colon M \longrightarrow V_2$. Let $\tau\colon \widetilde{M} \longrightarrow V_1 \underset{V_2}{\times} M$ be the normalization map. Then we have the following commutative diagram where π_1 and π_2 are natural projections. Notice that $\pi_1\colon V_1 \underset{V_2}{\times} M \longrightarrow V_1$



is a biholomorphism outside $\pi_1^{-1}(\Phi^{-1}(S))$ and $\pi_2\colon V_1 \underset{V_2}{\times} M \to M$ is a covering map outside $\bigcup_{i=1}^{m} E_i$. Thus

$$\widetilde{\Phi} := \pi_2 \circ \tau\colon \widetilde{M} \longrightarrow M$$

is a d-fold branch covering. For each $A_i^j \subseteq E_j$, and any point $q_i^j \in A_i^j$ which is a smooth point in $\bigcup_{i=1}^{m} E_i$, we choose a germ of a curve Γ_i^j at the point q_i^j which intersects with $\bigcup_{i=1}^{m} E_i$ only at q_i^j and the intersection of A_i^j and Γ_i^j is transversal at q_i^j.

Let $\Gamma = \bigcup \Gamma_i^j$, $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant \ell_j$. Notice that $\widetilde{\pi} := \pi_1 \circ \tau$ is a proper map which is a biholomorphism outside $\widetilde{E} := \widetilde{\pi}^{-1}(\Phi^{-1}(S)) = \widetilde{\Phi}^{-1}(E)$ where $E = E_1 \cup \cdots \cup E_m$. Observe that $\widetilde{E}$ has exactly $m+k$ connected components $\widetilde{E} = \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_{m+k}$. Clearly

$$\widetilde{\Phi}_*(\widetilde{E}) = \sum_{i,j} d_i^j A_i^j, \text{ where } d_i^j \leqslant d.$$

By the projection formula (cf. p. 34 of [Ful98], or p. 426 of [Har77]),

$$\begin{aligned} \sum_{j=1}^{m} \sum_{i=1}^{\ell_j} d_i^j &= \Gamma \cdot \widetilde{\Phi}_*(\widetilde{E}) \\ &= \widetilde{\Phi}^*(\Gamma) \cdot \widetilde{E} \\ &\geqslant (\ell_1 + \cdots + \ell_m) d. \end{aligned}$$

The last inequality comes from the fact that $\widetilde{\Phi}^*(\Gamma_i^j)$ has d distinct branches because $\widetilde{\Phi} \colon \widetilde{M} \setminus \widetilde{\Phi}^{-1}(E) \longrightarrow M \setminus E$ is a d-fold covering space. Since $d_i^j \leqslant d$, we conclude that $d_i^j = d$ for all i, j. It follows that the branch locus of $\widetilde{\Phi}$ is contained in the singular locus of $\bigcup_{i=1}^{m} E_i$ which is of dimension $n-2$. As $\widetilde{M}$ is normal and M is smooth, $\widetilde{\Phi} \colon \widetilde{M} \longrightarrow M$ is a covering map by purity of branch locus. In particular, $\widetilde{M}$ is smooth.

Now we are ready to prove that $\Phi^{-1}(S) = S$, i.e., there are no $p_1, \ldots, p_k$ points in $\Phi^{-1}(S)$. Observe that $\widetilde{\pi}^{-1}(p_i)$, $1 \leqslant i \leqslant k$ and E_j, $1 \leqslant j \leqslant m$ are maximal compact connected analytic subsets in $\widetilde{M}$ and M respectively. Since $\widetilde{\Phi}$ is a covering map, there is a neighborhood $\widetilde{U}_i$ of $\widetilde{\pi}^{-1}(p_i)$ which maps biholomorphically to a neighborhood U_j of E_j for some j via $\widetilde{\Phi}$. As $\widetilde{\pi} \colon \widetilde{M} \longrightarrow V_1$ is a point modification in a neighborhood of p_i, there is a neighborhood D_i of p_i such that

$$\widetilde{\pi} \colon \widetilde{\pi}^{-1}(D_i) \setminus \widetilde{\pi}^{-1}(p_i) \longrightarrow D_i \setminus \{p_i\}$$

is a biholomorphism. Similarly, there is a neighborhood O_j of q_j such that

$$\pi\colon \pi^{-1}(O_j) \setminus E_j \longrightarrow O_j \setminus \{q_j\}$$

is a biholomorphism. Therefore

$$\pi \circ \widetilde{\Phi} \circ \widetilde{\pi}^{-1}\colon D_i \setminus \{p_i\} \longrightarrow O_j \setminus \{q_j\}$$

is a biholomorphism. Observe that p_i is a smooth point of D_i and q_j is an isolated normal singularity. It follows that $\pi \circ \widetilde{\Phi} \circ \widetilde{\pi}^{-1}$ extends to a biholomorphism from D_i to O_j. In particular q_j is not a singular point. This contradiction shows that $\Phi^{-1}(S) = S$ and hence $\widetilde{\pi}\colon \widetilde{M} \longrightarrow V_1$ is also a resolution of singularities of V_1.

If $\Phi(q_i) = q_j$, then (V, q_i) is isomorphic to (V, q_j) as germs of singularities. This is because the resolution of (V, q_j) is a resolution of (V, q_i).

Therefore, we have an étale covering map $\widetilde{\Phi} : \widetilde{M} \to M$ between two resolutions. By the main theorem in the former section and V_1, V_2 have the same singular set, we have deg $\widetilde{\Phi} = 1$ and the theorem is proved. □

Remark 8. When V is a complex surface, we can reduce the resolution M to be the minimal resolution [Yau11] and have another proof. Let A be an irreducible component of the exceptional divisor on M over V_2, $\widetilde{A} \subset \widetilde{\Phi}^{-1}(A)$ an irreducible component of $\widetilde{\Phi}^{-1}(A)$. Then the restriction map $\widetilde{\Phi}|_{\widetilde{A}} : \widetilde{A} \to A$ is an étale covering with degree d_A, and

$$\widetilde{A} \cdot \widetilde{A} = (\widetilde{\Phi}|^*_{\widetilde{A}} A) \cdot (\widetilde{\Phi}|^*_{\widetilde{A}} A) = A \cdot (\widetilde{\Phi}|_{\widetilde{A}_*} \widetilde{\Phi}|^*_{\widetilde{A}} A) = d_A (A \cdot A)$$

If M is minimal, there is no (-1)-curve on M and the Hurwitz formula gives $g(\widetilde{A}) > 0$ if $g(A) > 0$. The equation above shows that there is no (-1)-curve on $\widetilde{M}$, hence $\widetilde{M}$ is also minimal. By the uniqueness of minimal resolution, degree of $\widetilde{\Phi}$ is one. If

originally the resolution M is not minimal, we can contract all the (-1)-curves on M to get minimal resolution M', and we can also contract corresponding (-1)-curves on $\widetilde{M}$ and get a new resolution $\widetilde{M'}$ such that the étale covering map $\widetilde{\Phi} : \widetilde{M} \to M$ having degree d induces another étale covering map $\widetilde{\Phi}' : \widetilde{M'} \to M'$ with the same degree d. Then we conclude as above.

But in higher dimensional cases, there is no minimal resolution in general. Therefore we use the relative canonical model instead.

Chapter 3

Gaffney-Lazarsfeld Theorem

3.1 Theorems on projective spaces and Grassmannians

We still assume that all varieties are over $\mathbb{C}$.

Given a finite surjective mapping $f : X \to Y$ between irreducible varieties and Y is smooth. The local degree of a point is

$$e_f(x) := deg\ (V(x) \to U(f(x))),$$

where $x \in X$, $U(f(x))$ is a small neighborhood of $f(x)$, and

$$f^{-1}(U(f(x))) = \coprod_{f(x')=f(x)} V(x')$$

splits as disjoint union of connected neighborhoods of the inverse image. We have the basic formula

$$\sum_{f(x')=f(x)} e_f(x') = \deg\ f.$$

We can define the higher ramification loci as

$$R_\ell(f) := \{x \in X \mid e_f(x) > \ell\}.$$

(See [GL80], [Laz04] Definition 3.4.7., for more detailed description). Note that $R_1(f)$ is the usual ramification locus.

The main result in [GL80] is the following and we call it the Gaffney-Lazarsfeld Theorem.

Theorem 11. ([GL80], Theorem 1) *Let X be an irreducible projective variety of dimension n and $f : X \to \mathbb{P}^n$ a branched covering of degree d. Then $R_\ell(f)$ is nonempty and*

$$codim_X R_\ell(f) \leq \ell, \text{ provided } \ell \leq min\{d-1, n\}.$$

To begin the proof, we need some propositions of connectedness and behavior of local degrees. The first one is Bertini theorem.

Theorem 12. ([FL81], 1.1; [Laz04], 3.3.1) *Let X be an irreducible variety and $f : X \to \mathbb{P}^r$ a morphism. Fix an integer $d <$ dim $\overline{f(X)}$. If $L \subset \mathbb{P}^r$ is a general $(r-d)$-plane, then $f^{-1}(L)$ is irreducible.*

Remark 9. A plane $L \subseteq \mathbb{P}^r$ of dimension $(r-d)$ is general if it lies in a dense open subset of the Grassmannian $Gr(r-d, \mathbb{P}^r)$.

Proof. Assume that the preimage of f is generically finitely many. This is what we need in the proof of Gaffney-Lazarsfeld Theorem. It suffices to prove the case $d = n-1, n =$ dim $X =$ dim $\overline{f(X)}$. Fix a general $(r-d)$-plane $\Lambda \subseteq \mathbb{P}^r$, we assume that $f^{-1} \subseteq X$ consists of a non-empty finite set of smooth points of X and each point is reduced. The set of all $(r-n+1)$-planes $L \subseteq \mathbb{P}^r$ containing Λ is parameterized by

$T = \mathbb{P}^{n-1}$, i.e., $L_t \supseteq \Lambda$ corresponds to $t \in T$. Define the set

$$V := \{(x,t) | f(x) \in L_t\} \subseteq X \times T.$$

We need to show that a general fibre of the projection $p_2 : V \to T$ is irreducible.

Note that $p_1 : V \to X$ is the blow-up of X along $f^{-1}(\Lambda)$, so V is irreducible since X is. The map $s : T \to V, t \mapsto (O,t)$ where $O \in f^{-1}(\Lambda)$ is a fixed point, is a section of p with image intersecting to V^{sm}, the smooth locus of V. Then we use the following lemma to end the proof.

Lemma 4. *Let $p : V \to T$ be a dominating morphism between irreducible complex varieties. Assume that p admits a section $s : T \to V$ with $s(t) \in V^{sm}$ for general point $t \in T$. Then the fibre $V_t = p^{-1}(t)$ is irreducible for general $t \in T$.*

To prove the lemma, we can shrink both V and T, and assume that p is a smooth morphism by generically smoothness theorem in char. 0. Then it suffices to show fibres of p connected. We can assume p topolocially locally trivial by shrinking T again, i.e., every $t \in T$ has a neighborhood U such that $p|_{p^{-1}(U)}$ is topologically isomorphic to the trivial fibration $U \times B \to U$ for some B. Use the fact that a locally trivial fibration between path-connected spaces admitting sections has connected fibres, then the lemma is proved. □

The general irreducibility can be extended to arbitrary connectedness by compactness condition. We will use this fact in the proof of main theorem in section 3. Here we provide the case of Bertini theorem.

Theorem 13. ([FL81], 2.1; [Laz04], 3.3.3) *Let X be an irreducible variety and $f : X \to \mathbb{P}^r$ a morphism. $L \subseteq \mathbb{P}^r$ is an arbitrary linear space of codimension $d <$ dime $\overline{f(X)}$. If X is complete, then $f^{-1}(L)$ is connected. The same statement holds more generally if X is not complete provided that f is proper over a Zariski open subset*

$V \subseteq \mathbb{P}^r$ *and* $L \subseteq V$.

Proof. Denote $G = Gr(r-d, \mathbb{P}^r)$ the Grassmannian of all codimension d linear spaces of $\mathbb{P}^r$, $W \subseteq G$ the open subset consisting of linear spaces contained in a Zariski open subset $V \subseteq \mathbb{P}^r$ over which f is proper. Define

$$Z := \left\{(x, L) | x \in f^{-1}(L)\right\} X \times W.$$

This is a Zariski open subset of a Grassmannian bundle $X \times G \to X$ and then Z is irreducible. By base change, the projection $p_2 : Z \to W$ is proper since f is proper over V. Consider the Stein factorization

$$Z \xrightarrow{q} W' \xrightarrow{r} W,$$

q has connected fibres and r is finite. Bertini theorem says that general fibres of p_2 is irreducible, thus r is generically one-to-one. But r is a surjective branched covering map of the normal variety W, so r is one-to-one and the theorem is proved. □

If X is also locally irreducible, e.g., X is normal, the theorems above can be generalized that the induced morphism

$$\pi_1(f^{-1}(L)) \to \pi_1(X)$$

is surjective. It also implies that if $U \subseteq \mathbb{P}^r$ is a neighborhood of L, $\pi_1(f^{-1}(U)) \to \pi_1(X)$ is surjective.

Another important connectedness theorem is the Fulton-Hansen connectedness theorem.

Theorem 14. *([FL81], 3.1; [Laz04], 3.3.6) Let X be an irreducible projective (or complete) variety, and*

$$f : X \to \mathbb{P}^r \times \mathbb{P}^r$$

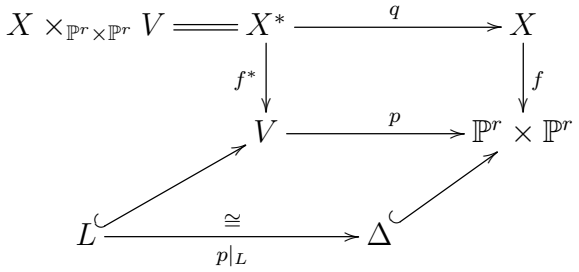
a morphism. Assume that dim $f(X) > r$*. Then the inverse image* $f^{-1}(\Delta) \subseteq X$ *of the diagonal* $\Delta \subseteq \mathbb{P}^r \times \mathbb{P}^r$ *is connected.*

Proof. Let $(x) = (x_0, ..., x_r)$ and $(y) = (y_0, ..., y_r)$ be the homogeneous coordinates on factors of $\mathbb{P}^r \times \mathbb{P}^r$, and $(x_0, ..., x_r, y_0, ..., y_k) \in \mathbb{P}^{2r+1}$. Let $V \subseteq \mathbb{P}^{2r+1}$ be the complement of $\{(0, y)\} \cup \{(x, 0)\}$. The morphism

$$p : V \to \mathbb{P}^r \times \mathbb{P}^r,\ (x, y) \mapsto ((x), (y))$$

makes V as a $\mathbb{C}^*$-bundle over $\mathbb{P}^r \times \mathbb{P}^r$. Let $L \subseteq V$ be the linear subspace defined by equations $x_i = y_i, 0 \leq i \leq r$, so L maps isomorphically to the diagonal $\Delta \subseteq \mathbb{P}^r \times \mathbb{P}^r$.

Consider the commutative diagram:



X^* is irreducible from X and f^* is proper from f. Note that $\dim \overline{f^*(X^*)} > r + 1 = \operatorname{codim}_V L$ and $f^{-1}(L)$ is connected by the connectedness theorem above. Since $f^{*-1}(L) \xrightarrow[q]{\cong} f^{-1}(\Delta)$, the theorem is proved. □

Remark 10. As before, if X is locally irreducible, the induced map

$$\pi_1(f^{-1}(\Delta)) \to \pi_1(X)$$

is surjective.

The Fulton-Hansen connectedness theorem is generated to many different cases including several product of projective spaces, Grassmanninas, and weighted projective

spaces.

On the other hand, we need to know the behavior of local degrees. The main properties we use are the semi-continuity and super-additivity.

Proposition 13. ([GL80], Lemma 1) *Let $f : X \to Y$ be a finite surjective morphism between irreducible varieties and Y smooth. Then the function $x \mapsto e_f(x)$ is upper semicontinuous on X.*

Proposition 14. ([GL80], Lemma 2) *Let T be an integral variety with generic point $t \in T$, X an irreducible projective variety of dimension n, $f : X \to Y$ a finite morphism where Y is integral and normal, and all varieties over an algebraically closed field k. Let $c_1, c_2 : T \to X$ be distinct morphisms with $f \circ c_1 = f \circ c_2 = h$ such that $c_1(t_0) = c_2(t_0) = x_0$ for some closed point $t_0 \in T$. Then*

$$e_f(x_0) \geq e_f(c_1(t)) + e_f(c_2(t)).$$

Proof. We check the first proposition. It suffices to show that given $x_0 \in X, e_f(x) \leq e_f(x_0)$ for all x in a neighborhood of x_0. Denote $y_0 = f(x_0)$, $\bar{X}(x_0) = \text{Spec } (\mathcal{O}^{sh}_{X,x_0})$ where $\mathcal{O}^{sh}_{X,x_0}$ is the strict Henselization of the local ring $\mathcal{O}_{X,x_0}$ ($\mathcal{O}^{sh}_{X,x_0}$ can be viewed as the local ring of x_0 in the étale topology). Then we consider the commutative diagram:

$$\begin{array}{ccccc} \bar{X}(x_0) & \xrightarrow{a} & W & \xrightarrow{p} & X \\ \downarrow{\scriptstyle f_{x_0}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \bar{Y}(y_0) & \xrightarrow[b]{} & V & \xrightarrow[q]{} & Y \end{array}$$

$\bar{X}(x_0)$ is a connected component of $X \times_Y \bar{Y}(y_0)$, $p : W \to X, q : V \to Y$ are étale neighborhoods of x_0, y_0 respectively, g is a finite morphism, W is a connected component of the fibre product $U := X \times_Y V$, and V is integral and normal. Since a

is dominant, deg $g = \deg f_{x_0} = \deg e_f(x_0)$ and $e_g(w) \leq \deg g$ for all $w \in W$ by

$$\sum_{w' \in W, g(w') = g(w)} e_g(w') = \deg g.$$

$e_g(w) = e_f(p(w))$ for all $w \in W$ since p, q are étale. Thus $e_f(x) \leq e_f(x_0)$ for all $x \in \mathrm{Im}(p)$ and $\mathrm{Im}(p)$ is an open neighborhood of x_0.

To see the second proposition, we keep the same notation in the paragraph above. Take a point $v_0 \in q^{-1}(y_0)$ and $u_0 = (x_0, v_0)$. Denote $g' : U \to V, p' : U \to X$ the projections. Assume that there is a map $j : T \to V$ such that $j(t_0) = v_0$ and $q \circ j = h$ if we replace T by its normalization and then a connected component of $T \times_Y V$. Then there are $d_i : T \to U$ such that $p' \circ d_i = c_i$, $g' \circ d_i = j$, and $d_1(t_0) = d_2(t_0) = u_0$. The hypothesis of distinct c_1, c_2 gives $c_1 \circ \bar{t} \neq c_2 \circ \bar{t}$ where $\bar{t} : \mathrm{Spec}\ K$ is a geometric point centered at t. Take $u_i = d_i \circ \bar{t}$. The rest is the same as the argument in the former paragraph. □

We can now proceed the proof of Gaffney-Lazarsfeld theorem of projective spaces.

Proof. ([Laz04]) The theorem is clear for dimension $n = 1$. Assume $n \geq 2$, for general hyperplane $L \subset \mathbb{P}^n$, the inverse image $f^{-1}(L)$ is irreducible and then by induction hypothesis the theorem is true for the restriction morphism $f' : f^{-1}(L) \to L = \mathbb{P}^{n-1}$. Note $e_{f'}(x) = e_f(x)$ for all $x \in f^{-1}(L)$ when L general, so $codim_X R_\ell(f) \leq \ell$, provided $\ell \leq min\{d-1, n-1\}$. Then it remains to show $R_n \neq \emptyset$ if $d \geq n+1$. By induction hypothesis, there is an irreducible component S of R_{n-1} with dim ≥ 1. Apply the Fulton-Hansen connectedness theorem to the mapping

$$F = f \times f|_S : X \times S \to \mathbb{P}^n \times \mathbb{P}^n.$$

There are two situations:

1. $\Delta_S = F^{-1}(\Delta_{\mathbb{P}^n})$.

2. $\Delta_S \neq F^{-1}(\Delta_{\mathbb{P}^n})$.

In 1., all the sheets of f come together and $e_f(x) = d \geq n+1 \forall s \in S$ by $\sum_{f(x')=f(x)} e_f(x') = \deg f$. In 2., the Fulton-Hansen connectedness theorem gives that there is a component $T \subseteq F^{-1}(\Delta_{\mathbb{P}^n})$, $T \neq \Delta_S$ and $T \cap \Delta_S \neq \emptyset$. So there is a sequence

$$(x_k, s_k) \in T \subseteq X \times S$$

with $x_k \neq s_k, f(x_k) = f(s_k)$, and it converges to $(s^*, s^*) \in \Delta_S$. Since $e_f(s_k) \geq n$ for all k, the super-additivity of e_f gives that $e_f(s^*) \geq n+1$. □

A direct application of the Gaffney-Lazarsfeld theorem is a sufficient condition of algebraically simply connectedness.

Corollary 2. ([Laz04], 3.4.10) *Let X be an irreducible normal projective variety of dimension n. Assume that there is a branched covering map $f : X \to \mathbb{P}^n$ of degree $d \leq n$. Then X is algebraically simply connected.*

Proof. Suppose that there is an étale covering map $p : Y \to X$ of degree ≥ 2 with Y connected. Y is irreducible since X is normal, which implies locally irreducible. Consider the composition

$$g = f \circ p : Y \to \mathbb{P}^n.$$

The G-L theorem implies that $\exists y \in Y$ with $e_g(y) \geq \min\{\deg g, n+1\}$. Note

$$\min\{\deg g, n+1\} > d = \deg f,$$

so $e_g(y) \geq d+1$. But $e_g(y) = e_f(p(y))$ since p is étale, and $e_f(x) \leq d$ for all $x \in X$, a contradiction. □

Remark 11. The corollary above can be refined such that X is actually simply connected.

Beyond the projective spaces, [Deb96] generalizes the above theorems to Grassmannians with some non-empty intersection condition as the following:

Theorem 15. ([Deb96], 6.2, 6.3) *Let X be an irreducible variety, $f : X \to Gr(d, \mathbb{P}^n)$ a morphism, and $d < l \leq n$. Suppose that for general $M \in Gr(l-1, \mathbb{P}^n)$, $f(X)$ intersects $Gr(d, M)$. Then for general $L \in Gr(l, \mathbb{P}^n)$,*

1. *$f^{-1}(Gr(d, L))$ is irreducible.*
2. *If f is proper over an open subset V of $Gr(d, \mathbb{P}^n)$ and $Gr(d, L)$ is contained in V, then $f^{-1}(Gr(d, L))$ is connected.*
3. *If X is locally irreducible, $\pi_1(f^{-1}(Gr(d, L))) \to \pi_1(X)$ is surjective.*

Theorem 16. ([Deb96], 7.1) *Let X be an irreducible variety and $f : X \to Gr(r, \mathbb{P}^n) \times Gr(r, \mathbb{P}^n)$ a morphism. Suppose that there are partitions λ and μ, either $\lambda < \overline{\mu}$ or $\lambda^* < \overline{\mu}^*$ (see Definition 3.1.1), such that $[\overline{f(X)}] \cdot p_1^* \sigma_{\bar{\lambda}} \cdot p_2^* \sigma_{\bar{\mu}} \neq 0$. For every $\gamma \in \mathrm{PGL}(n+1, k)$, denote ${}^{\gamma}\Delta$ the image of the diagonal Δ under the automorphism $(x, y) \mapsto (x, \gamma y)$ of $Gr(r, \mathbb{P}^n) \times Gr(r, \mathbb{P}^n)$. Then*

1. *For general γ, $f^{-1}({}^{\gamma}\Delta)$ is irreducible.*
2. *If X is complete, $f^{-1}(\Delta)$ is connected.*
3. *If X is locally irreducible and complete, $\pi_1(f^{-1}(\Delta)) \to \pi_1(X)$ is surjective.*

The proof of the general irreducibility and arbitrary connectedness is very similar to the case of projective spaces. It suffice to prove the case that L is a hyperplane of $\mathbb{P}^n$. Consider the set

$$Z = \{(x, u) \in X \times Gr(n-2, \mathbb{P}^n) | \Lambda_{f(x)} \subset \Lambda_u\}$$

where $\Lambda_{f(x)}, \Lambda_u$ are the corresponding planes for $f(x) \in Gr(d, \mathbb{P}^n)$ and $u \in Gr(n-2, \mathbb{P}^n)$. Z is a Zariski open subset of a Grassmannian bundle, hence Z is irreducible. Z

is locally irreducible if X is. The projection $p_2 : Z \to Gr(n-2, \mathbb{P}^n)$ is dominant, and $p_1(p_2^{-1}(Gr(n-2, L))) = f^{-1}(Gr(d, L))$ for all hyperplane L of $\mathbb{P}^n$. So the situation is reduced to $d = n-2$. Now, by dualization, there is a dominant morphism $f' : X \to Gr(1, \mathbb{P}^n)$. Then (1) is finished by the fact below whose proof is more technical.

Theorem 17. ([Deb96], 6.1) *Let X be an irreducible variety, $f : X \to Gr(1, \mathbb{P}^n)$ a dominant morphism, and $p \in \mathbb{P}^n$ a general point. Then*

1. *$f^{-1}(\Sigma_p)$ is irreducible, where Σ_p is the variety $\{z \in Gr(1, \mathbb{P}^n) | p \in \Lambda_z\}$.*
2. *If X is locally irreducible, $\pi_1(f^{-1}(\Sigma_p)) \to \pi_1(X)$ is surjective.*

To show (2) in Theorem 16, we similarly take

$$Z' = \{(x, u) \in X \times Gr(d, L) | \Lambda_{f(x)} \subset \Lambda_u\}$$

Z' is irreducible since it comes from a Grassmannian bundle over X. The projection $q : Z' \to Gr(d, L)$ is proper, and it has the Stein factorization $Z' \xrightarrow{q'} G' \xrightarrow{\rho} Gr(d, L)$ where q' has connected fibres and ρ is a finite morphism. By (1) in the theorem, ρ is surjective and one-to-one on a non-empty open subset. Thus ρ is bijective and then q has connected fibres, which proves (2).

We see that the Fulton-Hansen connectednesss-like theorem for Grassmannian needs some condition of intersection.

Definition 3.1.1. Denote $\lambda = (\lambda_0, ..., \lambda_d)$ a partition that $n - d \geq \lambda_0 \geq ... \geq \lambda_d \geq 0$ and $|\lambda| = \sum \lambda_i$. Given such a partition, we also denote $\overline{\lambda}$ the partition $(n - d - \lambda_d, ..., n - d - \lambda_0)$ and a partition $\lambda^* = (\lambda_0^*, ..., \lambda_{n-d-1}^*)$ where $\lambda_i^* = \max \{j | \lambda_j > i\}, i = 0, ..., n-d-1$. Note that $\overline{\lambda^*} = \overline{\lambda}^*$. For two partitions λ and μ, denote $\lambda \leq \mu$(resp. $\lambda < \mu$) if $\lambda_i \leq \mu_i$ (resp. $\lambda_i < \mu_i$) for all $i = 0, ..., d$.

The Schubert class σ_λ corresponding to partition λ is a cycle class in $A^{|\lambda|}(Gr(d, \mathbb{P}^n))$

which stands for the class of Schubert variety

$$\Sigma_L = \{u \in Gr(d, \mathbb{P}^n) | \dim\ (\Lambda_u \cap L^{(i)}) \geq i, i = 0, ..., d\}$$

where $L = (L^{(0)} \subset ... \subset L^{(d)} \subset \mathbb{P}^n)$ with $\dim\ L^{(i)} = n - d + i - \lambda_i$. In particular, for $u \in \{0, ..., n-d\}$, the class σ_u stands for the Schubert variety

$$\Sigma_L = \{u \in Gr(d, \mathbb{P}^n) | \Lambda_u \cap L \neq \emptyset\}$$

where L is a linear space in $\mathbb{P}^n$ of codimension $d + m$.

A basic property for Schubert classes are that they form a basis of the $\mathbb{Z}$-module $A^*(Gr(d, \mathbb{P}^n))$. We have $\sigma_\lambda \cdot \sigma_{\overline{\lambda}} = 1$ and for any subvariety of $Gr(d, \mathbb{P}^n)$, $[X] = \sum_\lambda [X]_\lambda \sigma_\lambda, [X]_\lambda = [X] \cdot \sigma_{\overline{\lambda}}$

The argument of Fulton-Hansen theorem on Grassmannians is similar to the projective spaces. Consider homogeneous coordinates (x, y) on $\mathbb{P}^{2n+1}$ where x, y are $(n+1)$-tuples of $\mathbb{C}$. Take $\tilde{\gamma} \in GL(n+1, \mathbb{C})$ and $\gamma \in PGL(n+1, \mathbb{C})$ is the projection of $\tilde{\gamma}$. Denote L_1, L_2 the linear subspaces of $\mathbb{P}^{2n+1}$ defined by $\{x = 0\}, \{y = 0\}$ respectively, and ${}^{\tilde{\lambda}}L$ defined by $\{x = \tilde{\gamma}(y)\}$. Consider $V := Gr(d, \mathbb{P}^{2n+1}) - \Sigma_{L_1} - \Sigma_{L_2}$, $\rho_i : \mathbb{P}^{2n+1} - L_i \to \mathbb{P}^n$ the projection $(x, y) \mapsto x, (x, y \mapsto y)$ respectively, $\phi_i : V \to Gr(d, \mathbb{P}^n)$ the induced morphisms, and $\phi := (\phi_1, \phi_2) : V \to Gr(d, \mathbb{P}^n) \times Gr(d, \mathbb{P}^n)$. Note that if $X' := X \times_{Gr(d,\mathbb{P}^n) \times Gr(d,\mathbb{P}^n)} V$ and $f' : X' \to V$ is the projection, there is $f'^{-1}(Gr(d, {}^{\tilde{\lambda}}L)) \cong f^{-1}({}^{\gamma}\Delta)$. Let M be a general linear subspace of $\mathbb{P}^{2n+1}$ with dimension $n-1$ such that $M \cap L_i = \emptyset$ and $\rho_i|_M : M \xrightarrow{\cong} H_i \subset \mathbb{P}^n$ where H_i is a hyperplane. $\gamma_i : Gr(d, M) \to Gr(d, H_i)$ are the induced isomorphisms. Moreover, ϕ induces an isomorphism of $Gr(d, M)$ on

$$\Omega_M = \{(u, \gamma_2(\gamma_1^{-1}(u))) | u \in Gr(d, H_1)\}$$

and $f'^{-1}(Gr(d,M)) \cong f^{-1}(\Omega_M)$. By the Künneth decomposition theorem of $A^*(Gr(d,\mathbb{P}^n) \times Gr(d,\mathbb{P}^n))$, the class of Ω_M can be written as

$$\sum p_1^*\sigma_\alpha \cdot p_2^*\sigma_\beta,$$

and the partitions α, β satisfy $\alpha_i + \beta_{d-i} = n-d+1, i = 0, ..., d$. By the hypothesis of the theorem, there are partitions λ, μ such that $\lambda < \overline{\mu}$ and $[\overline{f(X)}] \cdot p_1^*\sigma_{\bar{\lambda}} \cdot p_2^*\sigma_{\bar{\mu}} \neq 0$. Define a new partition λ' by $\lambda'_i = \overline{\mu}_i - 1$. Thus $\lambda \leq \lambda'$ and $[\overline{f(X)}] \cdot p_1^*\sigma_{\bar{\lambda'}} \cdot p_2^*\sigma_{\bar{\mu}} \neq 0$ by the fact below([Deb96], 4.1):

$$\sigma_\lambda \cdot \sigma_\mu \neq 0 \Longleftrightarrow \lambda \leq \overline{\mu}.$$

Thus, the decomposition of $[\Omega_M]$ gives $[\overline{f(M)}] \cdot [\Omega_M] \neq 0$. Therefore, $[\overline{f'(X')}] \cdot [Gr(d,M)] \neq 0$, and then [Deb96] gives the first result. If X is complete, then f' is proper, and [Deb96] gives the second result.

The condition of algebraically simply connectedness can also be implied.

Corollary 3. [Deb96], 7.4 *If X is an irreducible subvariety of $Gr(d,\mathbb{P}^n)$ such that $[X] \cdot [X] \cdot (\sigma_{1,...,1} + \sigma_{n-d}) \neq 0$, then $\pi_1^{\text{alg}}(X) = 0$.*

However, we can not use the same strategy to prove the Gaffney-Lazarsfeld Theorem on Grassmannians since the dimension induction in [GL80] seems not sufficient here. In fact, we can take $Gr(d,L)$ irreducible, but its codimension in $Gr(d,\mathbb{P}^n)$ is ≥ 2 if $d \geq 2$, and then we do not get further information on $R_\ell(f)$ when $d(n-2-d) < \ell < d(n-1-d)$. So avoiding the induction on dimension may be a good direction to generalize theorems.

3.2 Homogeneous spaces

We recall the definitions of algebraic groups and homogeneous spaces. The discussion below follows [Tev05].

Definition 3.2.1. An algebraic group G is an algebraic variety with a group structure such that multiplication and inverse maps

$$G \times G \to G, (x, y) \mapsto x \cdot y; G \to G, x \mapsto x^{-1}$$

are morphisms of algebraic varieties. A subgroup $H \subset G$ is algebraic if H is an algebraic subvariety of G. Algebraic homomorphisms between algebraic groups are morphisms of algebraic varieties which are also homomorphisms for the group structures. It is easy to see that images and kernels of algebraic homomorphisms are algebraic groups.

We see that the general linear group $GL(V)$, the group of invertible linear transformations on a vector space V, is an example of algebraic groups. An algebraic group G is called an linear algebraic group if G can be embedded into some $GL(V)$.

Remark 12. In this article, we only consider G as linear algebraic groups, that is, G can be embedded into some general linear group $GL(V)$.

Definition 3.2.2. An action of an algebraic group G on an algebraic variety X is algebraic if the map $G \times X \to X, (g, x) \mapsto g \cdot x$ is a morphism. If G acts transitively on X, then X is called a homogeneous space. For $H \subset G$ an algebraic subgroup, the cosets G/H is a smooth algebraic variety with natural action of G.

Moreover, G/H is a quasi-projective homogeneous space because there is an action of G on some projective space $\mathbb{P}^N$ such that G/H is Gp(Chevalley theorem).

Definition 3.2.3. An element $u \in G$ in a group is called unipotent if $(u - id)^k = id$ for some $k > 0$. An algebraic group G is called reductive if its unipotent radical

$R_u(G)$, i.e., the maximal normal unipotent subgroup of G, is trivial. Let G be a connected reductive group and $H \subset G$ an algebraic subgroup. If G/H is a projective variety, then H is called a parabolic subgroup of G. A maximal connected solvable subgroup of G is called a Borel subgroup.

A basic property is that a subgroup $H \subset G$ is parabolic if and only if it contains some Borel subgroup. In particular, any Borel subgroup is parabolic.

We can see that both projective spaces and Grassmannians are projective homogeneous spaces.

Example 3.2.1. [Tev05]

1. Let V be $\mathbb{C}^N$ and $G = \mathrm{SL}\ (V)$. The projective space $\mathbb{P}(V) = \mathbb{P}^{N-1}$ is G/P, where P is the normalizer in G of a line in V:

$$P = \{g \in G | g \cdot (e_1) \subseteq (e_1)\}.$$

2. The Grassmannian $Gr(d, V)$ of d-dimensional linear subspaces of vector space V is G/P', where P' is the normalizer in G of a d-dimensional linear subspace:

$$P' = \{g \in G | g \cdot (e_1, ..., e_d) \subseteq (e_1, ..., e_d)\}$$

   Note that $Gr(d, V) = Gr(d-1, \mathbb{P}(V))$ since d-dimensional subspaces in V correspond to $(d-1)$-dimensional subspaces in $\mathbb{P}(V)$.

We prepare to generalize the Gaffney-Lazarsfeld theorem to homogeneous spaces. The main theorem in next section is the case of projective homogeneous spaces of Picard number one, that is, the Picard group Pic $X = \mathbb{Z}$ for the homogeneous space X.

Example 3.2.2. 1. Projective spaces have Picard number one. In fact, all line bundles on $\mathbb{P}(V)$ have the form $\mathcal{O}_{\mathbb{P}}(d)$ for $d \in \mathbb{Z}$ so it is clear that Pic $(\mathbb{P}) = \mathbb{Z}$.

2. Grassmannians have Picard number one. Take the tautological vector bundle $\mathcal{S}$ over $Gr(d, V)$ which is a vector bundle such that the fibre of $\mathcal{S}$ over a d-dimensional linear subspace $U \subset V$ is U itself. Then the line bundle $\Lambda^d \mathcal{S}^\wedge$ is a very ample generator of Pic $(Gr(d, V))$, hence Pic $(Gr(d, V)) = \mathbb{Z}$.

We can construct more examples of homogeneous spaces of Picard number 1. We review the definition and fundamental facts about the corresponding Lie algebras of algebraic groups.

Definition 3.2.4. Let G be an algebraic group and $e \in G$ is the identity element of G. The tangent space $T_{e,G}$ of G at e has a natural Lie algebra structure, i.e., the tangent vector space has bracket $[,]$ defined by

$$[X, Y] = XY - YX$$

for $X, Y \in T_{e,G}$ and we denote the Lie algebra by $\mathfrak{g}$.

Example 3.2.3. If $G = GL(V)$, then $\mathfrak{g} = \mathfrak{gl}(V)$ is the Lie algebra of all linear operators of V.

A homomorphism between Lie algebras $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called algebraic if there exists an algebraic homomorphisms $F : G_1 \to G_2$ such that f is F's differential.

Give a connected reductive algebraic group G with Lie algebra $\mathfrak{g}$, consider a maximal subtorus T inside G, that is, a subgroup of G which is isomorphic to $\mathbb{C}^* \times ... \times \mathbb{C}^*$ and is maximal in the order of inclusion. Dim T is called the rank of G and the Lie algebra $\mathfrak{t}$ of T is called a Cartan algebra. The root decomposition of $\mathfrak{g}$ is

$$\mathfrak{f} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \alpha \neq 0, \mathfrak{g}_\alpha \neq 0$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | \mathrm{Ad}(t)x = \alpha(t)x \text{for any } t \in T\},$$

$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation, e.g., if $G = \mathrm{GL}(V)$ then $\mathrm{Ad}(X)M = XMX^{-1}$. Elements of $\Delta \subset \chi(T)$, $\chi(T)$ is the set of characters of T, are called roots. Roots can be divided into positive roots and negative roots $\Delta = \Delta^+ \cup \Delta^-$, and there is a set of simple roots $\prod = \{\alpha_1, ..., \alpha_r\} \subset \Delta^+$ such that any root $\alpha \in \Delta$ can be expressed as

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i, n_i \in \mathbb{Z},$$

with all $n_i \geq 0$ if $\alpha \in \Delta^+$, all $n_i \leq 0$ if $\alpha \in \Delta^-$.

Let $\prod_P \subset \prod$ be a subset of simple roots and $\Delta_P^+ \subset \Delta^+$ the positive roots generated by $\prod_P$. The parabolic subgroup P of G is determined by its Lie algebra

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta_P^+} \mathfrak{g}_{-\alpha},$$

$\mathfrak{b}$ is the Lie algabra of the Borel subgroup B that $P \supset B$. Denote $\prod_{G/P} = \prod \setminus \prod_P$ and $\Delta_{G/P}^+ = \Delta^+ \setminus \Delta_P^+$. P is a maximal proper parabolic subgroup if and only if $\prod_{G/P}$ consists of only one simple root. In this case, G/P is called a minimal flag variety and $\mathrm{Pic}\, G/P = \mathbb{Z}$ generated by a very ample line bundle.

To get some concrete examples, we use the classical results of classification of simple algebraic groups. They can be classified by their Dynkin diagrams. The Dynkin diagram of an algebraic group G is the graph whose vertices are indexed by G's simple roots and two vertices A, B are connected by $\frac{2(A,B)}{(B,B)}$ edges where $(,)$ is the inner product of $\mathfrak{t}^*$, $\mathfrak{t}^*$ is the dual of the Lie subalgebra $\mathfrak{t} \subset \mathfrak{g}$ of a maximal subtorus $T \subset G$. The Dynkin diagrams of simple algebraic groups have types $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, and G_2 [Tev05]. Then, to get homogeneous spaces of Picard number 1, we can take maximal parabolic subgroups as above and make $\prod_{G/P}$ only one simple root. There is a table in [Mok08] which gives examples of Fano homogeneous spaces of Picard

number 1 from this manner.

$\mathfrak{g}$	$\mathfrak{l}$
B_n	$A_1 \times B_{n-2}$
D_n	$A_1 \times D_{n-2}$
G_2	A_1
F_4	C_3
E_6	A_5
E_7	D_6
E_8	E_7

where $\mathfrak{g} \cong \mathfrak{s} \ltimes \mathfrak{l}$ is the semi-direct product of a solvable ideal $\mathfrak{s}$ and a semi-simple subalgebra $\mathfrak{l}$. $\mathfrak{l}$ is called a Levi factor of $\mathfrak{g}$ and we have $\mathfrak{l} \subset \mathfrak{p} \subset \mathfrak{g}$, $\mathfrak{p}$ is Lie subalgebra corresponding to the maximal parabolic subgroup $P \subset G$ and we have G/P a homogeneous space of Picard number 1.

We need an important notion in the next section which is the non-degeneracy defined in [Kol13] for subvarieties in a homogeneous space. Let $Y = G/H$ be a simply connected, quasi-projective, homogeneous space. An irreducible subvariety $Z \subset Y$ is called degenerate if there is a subgroup $H \subset K \subset G$ such that Z is contained in a fiber of the natural projection $p_k : G/H \to G/K$. Z is non-degenerate if it is not degenerate. We provide part of the main theorem in [Kol13] which is used in the argument of next section.

Theorem 18. *([Kol13], Theorem 2) Let G be a connected algebraic group over $\mathbb{C}$, $X = G/H$ a quasi-projective, simply connected, homogeneous space and $Z \subset X$ a compact, irreducible subvariety. Let $D \subset X$ denote a sufficiently small Euclidean open neighborhood of Z. Then the following are equivalent.*

1. *Z is nondegenerate in X.*
2. *$\pi_1(D \cap X^0) \to \pi_1(X^0)$ is surjective for every Zariski open subset $X^0 \subset X$.*

3. $\pi_1(u_D^{-1}(X^0)) \to \pi_1(X^0)$ *is surjective for every* X^0 *and every finite, étale cover* $u_D : \tilde{D} \to D$.

4. $\pi_1(\tau_g(Z) \cap X^0) \to \pi_1(X^0)$ *is surjective for every* X^0 *and general* $f \in G$.

5. $\pi_1((\tau_g \circ u_Z)^{-1}(X^0)) \to \pi_1(X^0)$ *is surjective for every* X^0, *every finite cover* $u_Z : \tilde{Z} \to Z$ *and general* $g \in G$.

Example 3.2.4. [Kol13] Let X be a projective homogeneous space of Picard number 1, then every positive dimensional subvariety is non-degenerate. If X_i are projective homogeneous spaces of Picard number 1, then a positive dimensional subvariety $Z \subset \prod X_i$ is non-degenerate if and only if the projections $Z \to X_i$ are constant.

3.3 Gaffney-Lazarsfeld Theorem for Homogeneous Spaces

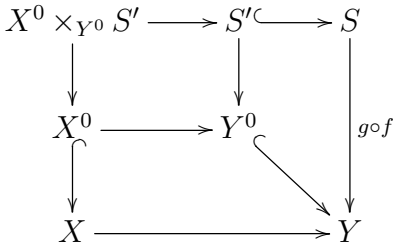
The main theorem is the following:

Theorem 19. *Let* X *be an irreducible projective variety.* Y *is a projective homogeneous space* G/P. G *is a connected linear algebraic group.* Y *has Picard number* 1 *and dimension* n. $f : X \to Y$ *is a branched covering of degree* d. *Then* $R_\ell(f)$ *is nonempty and*

$$codim_X R_\ell(f) \leq \ell, \text{ provided } \ell \leq min\{d-1, n\}.$$

Proof. We use induction on ℓ. For $\ell = 1$, assume that $f : X \to Y$ is unramified. By [Bor91], (11.16), every parabolic subgroup of G is connected and P is maximal, so G/P is simply connected. Therefore, Y is simply connected and f is actually an isomorphism. So if $d \geq 2$, the morphism f must ramify and *codim* $R_1(f) = 1$.

Now assume that the result holds for all $\ell \leq k-1$. Assume that $k \leq \min\{d-1, n\}$. We show that $R_k(f) \neq \emptyset$. By the hypothesis of $R_{k-1}(f)$, we can pick an irreducible component $S \subset R_{k-1}(f)$ which has dimension $\geq n-k+1 \geq 1$. Since Y is a projective homogeneous space of Picard number 1, $f(S)$ is nondegenerate as in the definition of [Kol13]. Take $Y^0 = Y \backslash f(R_1(f)), X^0 = f^{-1}(Y^0)$ the unramified part of f, and $g \in G$ a general element in the action group G of Y such that $(g \circ f(S)) \cap Y^0 \neq \emptyset$. By Theorem 2 in [Kol13] above, the map between fundamental groups $\pi_1((g \circ f|_S)^{-1}(Y^0)) \to \pi_1(Y^0)$ is surjective, and $S' := (g \circ f|_S)^{-1}(Y^0) = S \backslash$(closed subset in S) is irreducible. Then the fiber product $X^0 \times_{Y^0} S'$ is irreducible and $X \times_Y S_g$ is irreducible where S_g emphasizes the twisted morphism $g \circ f|_S : S \to Y$. Then $X \times_Y S$ is the limit of $X \times_Y S_g$, so it is connected.



If $X \times_Y S = \Delta_S = \{(s,s) \in X \times X \mid s \in S\}$, we have $e_f(x) = d \geq k+1$ and the result is true. So assume $X \times_Y S \neq \Delta_S$. We already know that $X \times_Y S$ is connected, so there is an irreducible component T of $X \times_Y S$ such that $T \neq \Delta_S$ and $T \cap \Delta_S \neq \emptyset$. We can pick a sequence $(t_h, s_h) \in T$ such that $t_h \neq s_h, f(t_h) = f(s_h)$ for every $h \geq 1$ and $lim_{h\to\infty} t_h = lim_{h\to\infty} s_h = s^*$. We have $e_f(t_h) \geq 1, e_f(s_h) \geq k$, hence

$$e_f(s^*) \geq k+1$$

which means $s^* \in R_k(f) \neq \emptyset$.

To show that $R_k(f)$ indeed has the required codimension count, we use the result obtained by Lazarsfeld. We first need the notion of connectedness in dimension k.

Definition 3.3.1. ([Laz04], 3.3.13) A Noetherian scheme X is connected in dimension k if $\dim X > k$ and $X \backslash T$ is connected for every closed subset $T \subset X$ of dimension $< k$.

We note that X is connected in dimension 0 is the same as usual connectedness. If X is irreducible, e.g., X is an algebraic variety, then X is connected in dimension $n-1$. There is a basic property of connectedness in dimension k.

Proposition 15. ([Laz04], 3.3.15) *X is connected in dimension k if and only if it satisfies the following:*

1. *Every irreducible component X' of X has dimension $> k$.*

2. *For any two irreducible components $X', X"$ of X, there is a sequence*

$$X' = X_0, X_1, ..., X_d = X"$$

of irreducible components such that $\dim (X_{i-1} \cap X_i) \geq k$ for $1 \leq i \leq d$.

Proposition 16. *([GL80][FL81][Gaf88]) Let $f : X \to Y$ be a branched covering of irreducible varieties with X normal and Y nonsingular. If $R_\ell(f)$ is non-empty, then every irreducible component of $R_\ell(f)$ has codimension $\leq \ell$ in X.*

Proof. We show this property by induction on ℓ. For $\ell = 0$, we have $R_0(f) = X$ and it is clear. Assume it is true for all $\ell < k$ and $R_k(f)$ is non-empty. Take $V \subset X$ an irreducible component of $R_{k-1}(f)$ containing $x \in R_k(f)$. V has dimension $\geq n-k+1$ by induction hypothesis. If V is also a component of $R_k(f)$, then we are done. So we assume that V is not a component of $R_k(F)$. Since f is a finite surjective and Y smooth, each fiber has the same cardinality counting the multiplicity where the sume is the degree of f ([GL80],[Laz04]). So we have $p \in V, q \in X, p \neq q$ with $f(p) = f(q)$.

Consider the map $F := f \times f \mid_V : X \times V \to Y \times Y$. Set Δ_Y as the diagonal set in $Y \times Y$. From the existence of p, q, we have a component $T \subset F^{-1}(\Delta_Y)$ with

$T \neq (V \times V)$, $T \cap (V \times V) \neq \emptyset$. We have $p_1(T \cap (V \times V)) \subset R_k(f)$ where p_1 is the projection map. By $F^{-1}(\Delta_Y)$ being connected in dimension $n-k$ (see [Gaf88], lemma 2.4), we get dim $R_k(f) \geq n-k$ which is the desired result. □

Remark 13. The proposition above is also proved in Lazarsfeld's thesis [Laz80].

Now we come back to the theorem. Let $\hat{n} : \hat{X} \to X$ be the normalization of X. Then $f \circ \hat{n} : \hat{X} \to Y$ is a branched covering satisfying the property above. We have $\hat{n}^{-1}(s*) \subset R_k(f \circ \hat{n})$, so $R_k(f \circ \hat{n})$ has codimension $\leq k$, and $\hat{n}(R_k(f \circ \hat{n}))$ has codimension $\leq k$, which gives that $R_k(f)$ has codimension $\leq k$. So the induction holds and we proved the theorem. □

According to the theorem, we have a corollary about simply connectedness of coverings of low degree as the one mentioned in [GL80][Laz04].

Corollary 4. *Let $f : X \to Y$ be a branched covering from an irreducible normal projective variety X to a homogeneous space Y with Picard number 1. If deg $f \leq$ dim X, X is algebraically simply connected.*

Proof. The proof is exactly the same as in [Laz04], Corollary 3.4.10.. We assume on the contrary that there is an étale covering $\pi : \widetilde{X} \to X$ with *deg* $\pi \geq 2$. $\widetilde{X}$ is irreducible since X is normal. We apply the main theorem to the composition

$$g := f \circ \pi : \widetilde{X} \to Y,$$

so there is at least a point $p \in \widetilde{X}$ such that $e_g(p) \geq \min\{deg\ g, n+1\} > deg\ f$. On the other hand, we have $e_g(p) = e_f(\pi(p))$ since π is étale. This is a contradiction. Thus X must be algebraically simply connected. □

In fact, we have the following result:

Corollary 5. *In the same setting of the corollary above, X is actually simply connected.*

Proof. The process is to show some surjectivity between fundamental groups and it would imply that $\pi_1(X) = 1$. We have the following lemma:

Lemma 5. *Let $f : X \to Y$ be a finite surjective morphism from an irreducible projective variety to a homogeneous space $Y = G/P$ with dimension n and Picard number one. If $d := deg f \leq n$, there is a subvariety $T \subset X$ of dimension $\geq n+1-d \geq 1$ such that f has only one preimage over T.*

Proof. By the main theorem, $R_{d-1}(f)$ is nonempty and codimension $R_{d-1}(f) \leq d-1$. Therefore, there is an irreducible component $T \subset R_{d-1}(f)$ which has dimension $\geq n+1-d \geq 1$. Since every point $x \in R_{d-1}(f)$ has local degree $e_f(x) = d$, f has only one preimage over T. □

From the lemma, we have a subvariety $T \subset Y$ with positive dimension in the corollary assumption. Take normalization $n : \widetilde{T} \to T$ and consider the map $F := (f\mid_T \circ n, f\mid_T \circ n) : \widetilde{T} \to Y \times Y$, note that $(X \times \widetilde{T}) \times_{Y\times Y} \widetilde{T}$ is homeomorphic to $\widetilde{T}$. The map

$$\pi_1(\widetilde{T}) = \pi_1((X \times \widetilde{T}) \times_{Y\times Y} \widetilde{T}) \to \pi_1(X \times \widetilde{T})$$

is surjective by using the main result in [Kol13] which appears in the proof of the main theorem above. More specifically, $Y \times Y$ is homogeneous and $F(\widetilde{T})$ is a nondegenerate irreducible subvariety in $Y \times Y$. For $g \in G$ a general element of group action on $Y \times Y$, $(g \circ F(\widetilde{T})) \cap (Y^0 \times Y^0) \neq \emptyset$ where $Y^0 \times Y^0$ is the unramified locus of $(f, f) : X \times X \to Y \times Y$. Denote $\widetilde{T}^0 := (g \circ F)^{-1}(Y^0 \times Y^0) \subset \widetilde{T}$ and $X^0 := f^{-1}(Y^0)$, we have the diagram

$$\begin{array}{ccccc}
(X^0 \times \widetilde{T}^0) \times_{Y^0\times Y^0} \widetilde{T}^0 & \longrightarrow & \widetilde{T}^0 & \hookrightarrow & \widetilde{T} \\
\downarrow & & \downarrow & & \\
X^0 \times \widetilde{T}^0 & \longrightarrow & Y^0 \times Y^0 & & \downarrow{\scriptstyle g\circ F} \\
\downarrow & & & \searrow & \\
X \times \widetilde{T} & & \longrightarrow & & Y \times Y
\end{array}$$

By [Kol13], Theorem 2.12, $\pi_1(\widetilde{T}^0) \to \pi_1(Y^0 \times Y^0)$ is surjective. Consider the the diagram

$$\begin{array}{ccccc} & & \pi_1((X^0 \times \widetilde{T}^0) \times_{Y^0 \times Y^0} \widetilde{T}^0) & \longrightarrow & \pi_1(X^0 \times \widetilde{T}^0) \\ & & \downarrow & & \downarrow \\ \pi_1(\widetilde{T}) & = & \pi_1((X \times \widetilde{T}) \times_{Y \times Y} \widetilde{T}) & \longrightarrow & \pi_1(X \times \widetilde{T}) \end{array}$$

The upper horizontal map is surjective since it is lifted by an étale morphism. The two vertical maps are also surjective because of the open inclusion and irreducibility of $(X \times \widetilde{T}) \times_{Y \times Y} \widetilde{T}$ and $X \times \widetilde{T}$. So we get the surjection $\pi_1(\widetilde{T}) \to \pi_1(X \times \widetilde{T})$. The surjectivity then implies that $\pi_1(X) = 1$, X is simply connected. □

Remark 14. The argument for $Y = \mathbb{P}^n$ can be found in [FL81](also see [Laz04], Example 3.4.11.) and the proof here is essentially the same way. In [Deb95], Cor. 7.4, Debarre shows the simply connectedness when Y is Grassmannian and $f : X \to Y$ satisfies certain nonempty intersection condition.

Therefore, any irreducible projective variety X with nontrivial fundamental group can only have finite surjective morphisms to a homogeneous space of Picard number one with degree lower bound $dim\ X + 1$.

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