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Doctoral Dissertations, 2011-2026

Mathematics

2018

Voevodsky Motives, Stable Homotopy Theory, and Integration

Zargar, Masoud

Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp018g84mp961>

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VOEVODSKY MOTIVES, STABLE HOMOTOPY THEORY, AND INTEGRATION

MASOUD ZARGAR

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS

CO-ADVISERS: DENIS-CHARLES CISINSKI, PETER OZSVATH, AND CHARLES
WEIBEL

JUNE 2018

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Abstract

In this thesis, we study applications and connections of Voevodsky's theory of motives to stable homotopy theory, birational geometry, and arithmetic. On the one hand, we show that we can use the stable ∞ -category of Voevodsky motives to develop a theory of integration similar to classical motivic integration. We show that this theory allows us to circumvent some of the complications of classical motivic integration that obstruct the path to concrete arithmetic and geometric results. One main application of this part is that up to a common direct sum, K-equivalent smooth projective k -varieties have the same integral Chow motives (once we invert the exponential characteristic of k), partially answering a conjecture of Chin-Lung Wang. In addition to generalizing a theorem of Kontsevich on the equality of Hodge numbers of K-equivalent smooth projective complex varieties, we show that such varieties have isomorphic *integral* singular cohomology groups and that K-equivalent complex varieties have isomorphic rational mixed Hodge structures. On the arithmetic side, we show that K-equivalent smooth $\mathbb{F}_q$ -varieties have isomorphic ℓ -adic Galois representations (up to semi-simplification). Furthermore, we connect this theory of integration of Voevodsky motives to the existence of motivic t -structures for geometric Voevodsky motives; we show that if the expected motivic t -structure on rational geometric Voevodsky motives exists, then K-equivalent smooth projective varieties, in particular birational Calabi-Yau smooth projective varieties over a field admitting resolution of singularities, have equivalent rational (Chow) motives. This implies that all cohomology theories (considering only the group structures) agree for K-equivalent varieties. We also connect this to a conjecture of Orlov concerning bounded derived categories of coherent sheaves. The other part of this thesis is concerned with if classical stable homotopy theory in algebraic topology sits inside stable motivic homotopy theory over an algebraically closed field. We show that up to inverting the exponential characteristic, the latter subsumes the former theory. In order to do this, we construct a

stable étale realization functor. As a consequence of this construction, we also prove a homotopy theoretic generalization of the étale version of the Suslin-Voevodsky comparison theorem comparing étale and motivic cohomology, both with suitable torsion coefficients.

Acknowledgements

I would like to thank my advisers Professors Denis-Charles Cisinski at the University of Regensburg, Peter Ozsvath at Princeton University, and Charles Weibel at Rutgers University for giving me the opportunity to freely pursue my interests in the past several years. I would also like to thank them for their trust, support, and encouragement.

I suppose that one should not just thank those who helped mathematically, but also those whose presence in my life changed me in other positive ways in the past few years. This thesis is a shadow of my life spent at Princeton and elsewhere in the past five years. Given that our mathematical lives are not independent of our lives beyond mathematics, the people around me have in some sense affected me mathematically as well, even if they may not realize this. I believe that the graduate students at Princeton are like my family. We ate together, partied together, complained to each other, traveled together, discussed mathematics together, and most important of all, found ourselves together. I will not try to encapsulate into words why the following people are valuable to me; words would not do justice to the strong bonds we have formed. Anibal Velozo, Naser Talebizadeh Sardari, Aurelien Gribinski, Michael Jemison, Artem Kotelskiy, Florian Sprung, Yuchen Liu, Daniel Kriz, Rafael von Känel, Nikita Lvov, Injee Jeong, Georgios Moschidis, Alexandros Eskenazis, Charlie Stibitz, Eric Naslund, and Mehdi Yazdi.

Outside of Princeton, I would like to thank my family for their support and encouragement. I would also like to thank Dr. Valentin Rodingher for always being a good friend, and last, but certainly not least, Ioana Tirziu whose patience and existence had a lasting impact on my psyche.

To all that is good, to all whose depth touched me in the last several years, to all who touched me badly locally but touched me well globally, to all the affection and care, to all the sleepless nights, to all the sad moments we shared together, to all the smiles sprung and tears shed around and among us.

To the sensation of witnessing beauty.

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Chapter 1

Introduction

In the past twenty years, the theory of Voevodsky motives and the homotopy theory of schemes have provided us with new tools for attacking problems in geometry, arithmetic, and topology. An example of such an achievement is the proof of the Bloch-Kato conjecture by Voevodsky ([80],[82],[84],[85],[86]). It gives a description of Galois cohomology groups in terms of Milnor K-theory, a very concrete object. These theories have also led to new results and techniques regarding ℓ -independence ([23],[73]), new results in algebraic K-theory ([22]), new results on vector bundles and their algebraicity ([4],[5],[6],[7]), and new methods in computations of stable stems ([32]). One reason Voevodsky constructed his triangulated categories of motives was to answer a wish of Beilinson and Deligne for a category that behaves like the derived category of the hypothetical abelian category of mixed motives. Afterall, this would suffice for defining cohomology. Though it is still unknown if the triangulated category of rational geometric Voevodsky motives has a motivic t -structure (and so contains the desired abelian category of mixed motives as the heart of the motivic t -structure), it is clear that this triangulated category and its variants provide us with tools to obtain new results in a large number of areas of mathematics.

In this thesis, we focus on two connections of Voevodsky's theory of motives to other areas of mathematics. The first is an application of the stable ∞ -category of Voevodsky motives to birational geometry, arithmetic, and questions on derived categories of coherent sheaves. On the other hand, we also study the connection of stable motivic homotopy theory to classical stable homotopy theory.

1.1 Arithmetic and Geometry of K-equivalent Varieties

Calabi-Yau varieties are of great importance to both mathematicians and physicists. They show up in string theory as objects of interest as well as in the classification of varieties in the minimal model program. In 1996, Batyrev used the Weil conjectures and p -adic integration to prove that two birational smooth projective Calabi-Yau complex varieties (an instance of K-equivalent varieties) have the same Betti numbers [13]. Later, Kontsevich generalized this by proving that they, in fact, have the same Hodge numbers [57]. Kontsevich circumvented the Weil conjectures by using his theory of motivic integration, developed shortly afterwards by Denef and Loeser [30]. We now describe the classical theory of motivic integration.

Let k be an algebraically closed field of characteristic zero. Consider the Grothendieck group of k -varieties $K_0(\mathrm{Var}_k)$, a group encapsulating the idea of cutting and pasting k -varieties. Precisely, it is defined as the free abelian group on isomorphism classes of k -varieties modulo the subgroup generated by relations of the form $[X] - [U] - [Z]$, where Z is a closed subvariety of a k -variety X with open complement U . In fact, fiber product of varieties over k induces a ring structure on this group. Because of

encapsulating the idea of cutting and pasting, this group or ring is called the group or ring of *virtual motives*, depending on whether we care about the ring structure or not. Let $\mathbb{L}$ be the class of the affine line. By construction, every additive function on the category of k -varieties factors through the group $K_0(\mathrm{Var}_k)$. Motivic integration allows us to understand the class of k -varieties X , $[X]$, inside a completion $\widehat{\mathcal{M}}_k$ of the ring $\mathcal{M}_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$. Given D an effective divisor of a smooth k -variety X , motivic integration spits out an element

$$\int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\mathrm{ord}_D} d\mu_X \in \widehat{\mathcal{M}}_k.$$

This assignment satisfies two important properties. The first is the transformation rule. Precisely, suppose X and Y are smooth k -varieties, $f : Y \rightarrow X$ is a proper birational morphism over k with relative canonical divisor $K_{Y/X}$, and suppose $D \subset X$ is an effective divisor. Then

$$\int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\mathrm{ord}_D} d\mu_X = \int_{\mathcal{J}_\infty(Y)} \mathbb{L}^{-\mathrm{ord}_{f^{-1}D + K_{Y/X}}} d\mu_Y.$$

The other is an easy computation that

$$\int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\mathrm{ord}_\emptyset} d\mu_X = [X].$$

These properties allow us to study the class of k -varieties in $\widehat{\mathcal{M}}_k$ that are related via sequences of resolutions of singularities. For example, suppose X and Y are K-equivalent smooth k -varieties, that is, they are smooth k -varieties such that there is a smooth k -variety Z and proper birational morphisms π_X and π_Y such that $\pi_X^* K_X = \pi_Y^* K_Y$:

$$\begin{array}{ccc} & Z & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & \cdots\cdots\cdots & Y. \end{array}$$

Examples of K-equivalent complex varieties are birational smooth projective complex varieties with nef canonical divisors. If k admits resolution of singularities, then birational smooth projective Calabi-Yau k -varieties are K-equivalent. The two important properties of motivic integration above imply that

$$\begin{aligned}
[X] &= \int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\text{ord}_\emptyset} d\mu_X \\
&= \int_{\mathcal{J}_\infty(Z)} \mathbb{L}^{-\text{ord}_{K_{Z/X}}} d\mu_Z \\
&= \int_{\mathcal{J}_\infty(Z)} \mathbb{L}^{-\text{ord}_{K_{Z/Y}}} d\mu_Z \\
&= \int_{\mathcal{J}_\infty(Y)} \mathbb{L}^{-\text{ord}_\emptyset} d\mu_Y = [Y]
\end{aligned}$$

in $\widehat{\mathcal{M}}_k$. Indeed, K-equivalence implies that $K_{Z/X} = K_Z - \pi_X^* K_X = K_Z - \pi_Y^* K_Y = K_{Z/Y}$. The Deligne-Hodge polynomial encodes the Hodge numbers of varieties, and factors through $\widehat{\mathcal{M}}_k$. Using this, one can show that the Hodge numbers of X and Y are equal. In summary, Kontsevich along with the theoretical joint work of Denef and Loeser on motivic integration proved the following theorem.

Theorem 1.1.1 (Kontsevich [57], Denef-Loeser [30]). *If X and Y are two K-equivalent complex varieties, then $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$. In particular, they have the same Hodge numbers.*

A few natural and important questions arise.

Question 1.1.2. *We know that $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, but do we have this equality in $\mathcal{M}_{\mathbb{C}}$?*

The answer to this question is not known. In fact, the following conjecture is of great importance to classical motivic integration.

Conjecture 1.1.3. *The completion map $\mathcal{M}_k \rightarrow \widehat{\mathcal{M}}_k$ is injective.*

This is a conjecture in classical motivic integration whose validity would allow us to deduce more refined information about classes of varieties. The reason this is the case is that not every map out of $\mathcal{M}_k$ factors through $\widehat{\mathcal{M}}_k$. For example, the counting measure $C_q : K_0(\mathrm{Var}_{\mathbb{F}_q})[\mathbb{L}^{-1}] \rightarrow \mathbb{Q}$ does not factor through the completion. Indeed, $q^n/\mathbb{L}^n \rightarrow 0$ in the topology of the completion while $C_q(q^n/\mathbb{L}^n) = 1$. This is one reason it is difficult to deduce arithmetic information using classical motivic integration. There has been some progress in this direction by Loeser and Cluckers [25], for example. They develop a different kind of motivic integration that allows them to refine Kontsevich's theorem above to saying that $[X] = [Y]$ in $K_0(\mathrm{Var}_{\mathbb{C}})$ localized with respect to the classes $\mathbb{L}$ and $1 - \mathbb{L}^n, n \geq 1$.

We no longer assume that k is algebraically closed or of characteristic zero. As a result, we can also ask about arithmetic data.

Question 1.1.4. *What if we want even more refined information? What if we were asking about the arithmetic relationship between X and Y that are K -equivalent? Do X and Y have the same ℓ -adic Galois representations (up to semi-simplification)? If k is a finite field, do X and Y have the same zeta functions?*

Through the work of Voevodsky, we now know that the triangulated category $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ of geometric effective Voevodsky motives is a rich category also encapsulating the idea of "cutting and pasting" k -varieties. As a result, we may ask the following question.

Question 1.1.5. *Suppose X and Y are K -equivalent smooth k -varieties. Is it true that $[M(X)] = [M(Y)]$ in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k))$?*

If this is true, we would have a motivic result using which we have an affirmative answer to the above arithmetic question. Indeed, there are realization functors from Voevodsky motives to (the derived category of) ℓ -adic sheaves, mixed Hodge structures, etc. whenever they make sense. Let us make the following convention from now on.

In the context of the second chapter, unless otherwise stated, whenever we speak of a pair of a field k and a commutative ring R , we assume that k is a field admitting resolution of singularities or whose exponential characteristic is invertible in the commutative ring R .

In the second chapter, I construct a theory of integration for Voevodsky motives with R -coefficients taking values in a group $\mathcal{M}(k; R)$ close to the group $K_0(\varprojlim_n \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n))$, where the limit is a limit of stable ∞ -categories. This theory has the feature that the natural map

$$K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R)$$

is injective! (See proposition 2.3.8.) As the analogous statement in motivic integration, conjecture 1.1.3, is open as of today, the injectivity of this map allows us to deduce new (concrete) results. The idea is that one main source of complications in classical motivic integration is the fact that the completion is taken at the ring-theoretic level. However, Voevodsky motives allow us to stay on the categorical level as much as possible, and eventually take advantage of the flexibility that categories have over rings in this case. Additionally, this theory of integration of Voevodsky motives comes equipped with a transformation rule that allows us to understand how our integrals change with respect to proper birational morphisms (see theorem 2.3.25).

Using this new theory of integration, we obtain the following answer to the last question above.

Theorem 1.1.6 (Theorem 2.4.1). *If X and Y are K -equivalent smooth k -varieties, then $[M(X)] = [M(Y)]$ in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$.*

A conjecture of Chin-Lung Wang [88] related to the above question is as follows.

Conjecture 1.1.7. *If X and Y are K -equivalent smooth projective complex varieties, then X and Y have the same integral Chow motives.*

Using our theorem 1.1.6 above along with Bondarko's weight structures [20], we give the following partial answer to his conjecture.

Theorem 1.1.8 (Theorem 2.4.2). *If X and Y are K -equivalent smooth projective k -varieties, then there is an effective Chow motive $P \in \text{Chow}^{\text{eff}}(k; R)$ such that $M(X) \oplus P \simeq M(Y) \oplus P$.*

Note that Chow motives sit inside geometric Voevodsky motives. As corollaries, we obtain the following consequences.

Corollary 1.1.9 (Corollary 2.4.4). *If X and Y are K -equivalent smooth projective k -varieties, then they have the same rational numerical motives.*

Corollary 1.1.10 (Corollary 2.4.5). *If k admits resolution of singularities and X and Y are birational smooth projective Calabi-Yau k -varieties, then there is an effective integral Chow motive $P \in \text{Chow}^{\text{eff}}(k; \mathbb{Z})$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k; \mathbb{Z}) \hookrightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z})$.*

An example of a concrete consequence is the following.

Theorem 1.1.11 (Theorem 2.4.7). *If X and Y are K -equivalent smooth projective k -varieties with k of characteristic zero with $k \hookrightarrow \mathbb{C}$ an embedding, then $H^*(X^{\text{an}}; \mathbb{Z}) \simeq H^*(Y^{\text{an}}; \mathbb{Z})$ as graded abelian groups.*

Remark 1.1.12. *Mark McLean has recently claimed a proof that birational Calabi-Yau complex manifolds admit Hamiltonians with isomorphic Hamiltonian Floer cohomology algebras after a certain change of Novikov rings. From this, he can deduce that birational Calabi-Yau manifolds have isomorphic integral singular cohomology groups.*

Theorem 1.1.13 (Theorem 2.4.9). *If X and Y are K -equivalent smooth k -varieties, then they have the same ℓ -adic Galois representations (up to semi-simplification). In particular, in the case $k = \mathbb{F}_q$, two such $\mathbb{F}_q$ -varieties have the same zeta functions.*

If $\mathbb{F}_q$ admits resolution of singularities, then all of this is true for X and Y that are birationally equivalent smooth projective Calabi-Yau $\mathbb{F}_q$ -varieties.

This last theorem answers our arithmetic questions. One last corollary to our theorem 1.1.6 is the following.

Corollary 1.1.14 (Corollary 2.4.10). *If X and Y are K -equivalent complex varieties, then they have the same classes of integral mixed Hodge structures in $K_0(\text{MHS}_{\mathbb{Z}})$. By the semisimplicity of polarizable rational mixed Hodge structures, two such varieties have the same rational mixed Hodge structures.*

This recovers the well-known result of Kontsevich mentioned earlier (and that of Batyrev) as a special case.

This work is also related to the very important question of the existence of a motivic t -structure. First, a direct consequence of theorem 1.1.8 is the following.

Theorem 1.1.15 (Theorem 2.4.12). *Suppose $\text{Chow}^{\text{eff}}(k; R)$ is a Krull-Schmidt category. Then for X and Y K -equivalent smooth projective k -varieties, the Chow motives of X and Y in $\text{Chow}(k; R)$ are equivalent.*

See definition 2.4.11 for the definition of Krull-Schmidt categories. Intuitively, it means that we can uniquely, up to permutation, decompose objects into a direct sum of indecomposables.

When working with rational coefficients, the previous theorem is related to the previous theorem and the existence of a motivic t -structure on rational geometric Voevodsky motives.

Theorem 1.1.16 (Theorem 2.4.18). *Suppose the motivic t -structure conjecture is true for $\text{DM}_{\text{gm}}(k; \mathbb{Q})$. Then if X and Y are K -equivalent smooth projective k -varieties, $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

Since it is expected that there is a motivic t -structure, it is reasonable to make the following conjecture.

Conjecture 1.1.17. *If X and Y are K -equivalent smooth projective k -varieties, then $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

Clearly, this is the rational version of the conjecture 1.1.7 above of Wang (generalized to arbitrary characteristics); however, even the above rational version is not known. Note that in the case of birational smooth projective Calabi-Yau complex varieties, we have the following conjecture due to Bondal and Orlov that is in the same spirit.

Conjecture 1.1.18 (Bondal-Orlov [17]). *If X and Y are birational smooth projective Calabi-Yau complex varieties, then $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y)$.*

There is also the following conjecture of Orlov.

Conjecture 1.1.19 (Orlov [74]). *If X and Y are smooth projective complex varieties such that $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y)$, then $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

Though conjecture 1.1.19 has been met with skepticism, it is still reasonable to expect the validity of conjecture 1.1.17. The hope is that by categorifying this new theory of integration of Voevodsky motives, one may be able to unconditionally prove the equivalence $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$ for K -equivalent smooth projective k -varieties. The difficulty is in setting up the theory so that the transformation rule becomes valid categorically, not just in my $\mathcal{M}(k; R)$ which is close to a Grothendieck group. Finally, a result of Kawamata combined with theorems 1.1.6 and 1.1.8 allows us to obtain new results on the connections between D-equivalence and Chow motives. For example, we show the following.

Theorem 1.1.20 (Theorem 2.4.26). *Then for X and Y smooth projective complex varieties such that $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$, $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y) \implies M(X) \oplus P \simeq M(Y) \oplus P$ for some integral Chow motive P .*

Theorem 1.1.21 (Theorem 2.4.27). *Suppose the motivic t -structure conjecture is true for $\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}; \mathbb{Q})$. Then for X and Y smooth projective complex varieties such that $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$, $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y) \implies M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

1.2 Comparison of stable homotopy categories

In [69], Morel and Voevodsky initiated the homotopy theory of schemes with the hope of introducing topological ideas into algebraic geometry. As mentioned in the introduction, this theory along with its stabilization with respect to smashing with $\mathbb{P}^1$ has had many applications in algebraic geometry and homotopy theory. Just as in algebraic topology where we have a triangulated category of *spectra* Spt whose objects roughly represent generalized cohomology theories in topology, in the motivic world over a base scheme S we have a triangulated category of motivic spectra $\mathrm{Spt}(S)$ whose objects roughly represent generalized motivic cohomology theories like *algebraic K-theory*. In the topological world, we impose that smashing with S^1 is an equivalence, while in the motivic world we impose that smashing with $\mathbb{P}^1$ is an equivalence. Each topological spectrum can be represented by a sequence of objects $(X_n)_n$ with weak equivalences

$$X_n \xrightarrow{\sim} \Omega X_{n+1}.$$

In the motivic world, the natural replacement of topological spaces is the notion of *motivic spaces*. We can take the ∞ -category of *pointed* sheaves on the site $(\mathrm{Sm}/S)_{\mathrm{Nis}}$, (where *Nis* indicates that we are working with respect to the Nisnevich topology) and localize with respect to the projections $X \times_S \mathbb{A}_S^1 \rightarrow X$. We obtain the ∞ -category $\mathrm{Spc}_*(S)$ of *pointed motivic spaces over S* . We then have the stable ∞ -category of

motivic spectra

$$\mathcal{Spt}(S) := \lim_n \left(\dots \xrightarrow{\Omega_{\mathbb{P}_S^1}} \mathcal{Spc}_*(S) \xrightarrow{\Omega_{\mathbb{P}_S^1}} \mathcal{Spc}_*(S) \right),$$

a limit in the ∞ -category of ∞ -categories with morphisms right adjoint functors. Its objects are sequences of motivic spaces $(X_n)_n$ with equivalences

$$X_n \xrightarrow{\sim} \Omega_{\mathbb{P}_S^1} X_{n+1}.$$

In $\mathcal{Spc}_*(S)$, we have the homotopy pushout square

$$\begin{array}{ccc} \mathbb{G}_{m,S} & \longrightarrow & \mathbb{A}_S^1 \simeq S \\ \downarrow & & \downarrow \\ S \simeq \mathbb{A}_S^1 & \longrightarrow & \mathbb{P}_S^1 \end{array}$$

Therefore, $\mathbb{P}_S^1 \simeq S^1 \wedge \mathbb{G}_{m,S}$ in $\mathcal{Spc}_*(S)$. In the category of topological spectra $\mathcal{Spt}$, we inverted smashing with S^1 , while in the motivic world we inverted both smashing with S^1 and $\mathbb{G}_{m,S}$. Since $\mathcal{Spt}(S)$ is motivic while $\mathcal{Spt}$ is topological, we ask the following natural question.

Question 1.2.1. *To what extent is the natural functor $c_S : \mathcal{Spt} \rightarrow \mathcal{Spt}(S)$ induced by the constant presheaf functor fully faithful? In other words, does motivic stable homotopy theory encapsulate classical stable homotopy theory? What happens as the base S varies?*

On a philosophical level, we are asking about the relation between the motivic world and the topological world. On a more practical level, such full faithfulness results could help in computing stable stems using stable motivic homotopy theory. Levine proved the following theorem.

Theorem 1.2.2 (Levine, [63]). *Suppose k is an algebraically closed field of characteristic zero. Then for $S = \mathrm{Spec} k$, c_S is fully faithful.*

In the second part of this thesis, we use étale homotopy theory to prove the following generalization of his theorem to arbitrary characteristics.

Theorem 1.2.3 (Theorem 3.6.1). *Let k be an algebraically closed field of exponential characteristic p . Then*

$$c[1/p] : \mathcal{Spt}[1/p] \rightarrow \mathcal{Spt}(k)[1/p]$$

induced by the constant presheaf functor is fully faithful.

By definition, the exponential characteristic is 1 if k has characteristic zero; otherwise it is equal to the characteristic. Suslin and Voevodsky have proved the following comparison theorem between motivic and étale cohomology.

Theorem 1.2.4 (Suslin-Voevodsky [81]). *If X is a finite-type k -scheme, and $N > 1$ is an integer coprime to the characteristic of k , then*

$$H_n^{\mathrm{sus}}(X; \mathbb{Z}/N) := H_{*,0}(X; \mathbb{Z}/N) \simeq H_{\mathrm{\acute{e}t}}^*(X; \mathbb{Z}/N)^*,$$

where $*$ denotes the Pontryagin dual.

Using the techniques for proving the above theorem on full-faithfulness, we generalize this comparison theorem of Suslin and Voevodsky to the following homotopy theoretic result.

Theorem 1.2.5 (Generalized Suslin-Voevodsky, theorem 3.7.1). *Suppose k is an algebraically closed field of exponential characteristic p , $E \in \mathcal{Spt}^{\mathrm{eff}}(k)_{\mathrm{tor}}$ is an effective torsion motivic spectrum, and $\ell \neq p$ is a prime. Then*

$$\underline{\mathrm{Et}}_{\ell*} : \pi_{n,0}(E)(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n \left(\underline{\mathrm{Et}}_{\ell}(E) \right)$$

is an isomorphism.

On the other hand, if k is not algebraically closed, then c need not be fully faithful.

For example, Heller and Ormsby have proved the following theorem.

Theorem 1.2.6 (Heller-Ormsby, [43],[44]). *If k is real closed, then there is a fully faithful functor*

$$\mathcal{Spt}^{C_2} \rightarrow \mathcal{Spt}(k)$$

from the ∞ -category $\mathcal{Spt}^{C_2}$ of C_2 -equivariant spectra.

In light of these results, we ask the following question.

Question 1.2.7. *If k is a field, does $\mathcal{Spt}(k)$ contain a category of $\mathrm{Gal}(k^{sep}|k)$ -equivariant spectra as a full subcategory?*

In order to precisely formulate this question, one should find the appropriate ∞ -category of equivariant spectra with *profinite* group actions. Barwick has developed an equivariant stable homotopy theory with profinite group actions [12], which may help in precisely formulating this question.

Chapter 2

Integration of Voevodsky motives

2.1 Introduction

Let k be a perfect field and R a commutative ring. Suppose throughout that k either admits resolution of singularities or its characteristic is invertible in the coefficient ring R . In this chapter, we take inspiration from classical motivic integration and its wealth of applications to birational geometry to develop a theory of integration for Voevodsky's mixed motives. Though the core goals of the theory are the same, we construct it so that it circumvents some of the complications of classical motivic integration.

After setting up the foundations, we use our theory to connect the theory of Voevodsky motives to birational geometry and classical questions on derived categories of coherent sheaves. The first main theorem proved in this chapter is the following.

Theorem 2.1.1. *If X and Y are K -equivalent smooth k -varieties, then $[M(X)] = [M(Y)]$ in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$.*

Recall that two smooth k -varieties X and Y are said to be *K-equivalent* if there is a smooth k -variety Z along with proper birational morphisms $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $f^*K_X = g^*K_Y$. Examples of K-equivalent complex varieties are birational smooth projective complex varieties with nef canonical divisors. If k admits resolution of singularities, then birational smooth projective Calabi-Yau k -varieties are K-equivalent.

As a consequence, we obtain the following stronger result using Bondarko's weight structures [20].

Theorem 2.1.2. *If X and Y are K-equivalent smooth projective k -varieties, then $M(X) \oplus P \simeq M(Y) \oplus P$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ for some Chow motive P .*

Note that (effective) Chow motives sit inside (effective) geometric Voevodsky motives, and so the above is also a statement about effective Chow motives with coefficients in R . This is of great philosophical importance because it gives a partial answer to the following conjecture of Chin-Lung Wang.

Conjecture 2.1.3 (Chin-Lung Wang [87]). *If X and Y are K-equivalent smooth projective complex varieties, then they have isomorphic integral Chow motives.*

What theorem 2.1.2 says is that at least up after direct summing with a common Chow motive, this conjecture is true. We will come back to this conjecture later in the introduction.

Note that theorem 2.1.1 has, as a particular consequence, the equality of Hodge numbers of K-equivalent complex varieties, which is a theorem due to Kontsevich [57]. We prove the following

Corollary 2.1.4. *If k is of characteristic zero and X and Y are K-equivalent smooth projective k -varieties, then $H^*(X^{an}; \mathbb{Z}) \simeq H^*(Y^{an}; \mathbb{Z})$.*

Of course, a special case of this corollary is when X and Y are birational Calabi-Yau complex varieties.

Remark 2.1.5. *Mark McLean has recently claimed a proof that birational Calabi-Yau complex manifolds admit Hamiltonians with isomorphic Hamiltonian Floer cohomology algebras after a certain change of Novikov rings. From this, he can deduce that birational Calabi-Yau manifolds have isomorphic integral singular cohomology groups.*

We also have the following new arithmetic result.

Corollary 2.1.6. *If X and Y are K -equivalent smooth projective k -varieties, then $H^*(X_{\bar{k}}; \mathbb{Z}_\ell)^{ss} \simeq H^*(Y_{\bar{k}}; \mathbb{Z}_\ell)^{ss}$ as graded Galois representations (up to semi-simplification).*

As a particular consequence, if $k = \mathbb{F}_q$, then X and Y have the same zeta functions. If $\mathbb{F}_q$ admits resolution of singularities, then two birational Calabi-Yau $\mathbb{F}_q$ -varieties have the same ℓ -adic Galois representations (up to semi-simplification), and so also the same zeta functions.

Recall the notion of a Krull-Schmidt category.

Definition 2.1.7 (Krull-Schmidt category). *An R -linear additive category $\mathcal{C}$ is said to be a Krull-Schmidt category if every object is a finite direct sum of objects with local endomorphism rings.*

The Krull-Schmidt theorem says that an object in a Krull-Schmidt category has a local endomorphism ring if and only if it is indecomposable. Furthermore, it also says that any object is uniquely, up to permutation, a direct sum of indecomposable objects. Many examples of Krull-Schmidt categories come from abelian categories in which every object has finite length. A concrete example is the category of finitely generated modules over a finite R -algebra, where R is a commutative Noetherian local complete ring (e.g. $\mathbb{Z}_\ell$). It is not known if the category of effective Chow

motives over a field k with R -coefficients $\mathrm{Chow}^{\mathrm{eff}}(k; R)$ is a Krull-Schmidt category, even if k is of characteristic zero and $R = \mathbb{Q}$.

Using theorem 2.1.2, we can prove the following.

Theorem 2.1.8. *Suppose $\mathrm{Chow}^{\mathrm{eff}}(k; R)$ is a Krull-Schmidt category. Then for X and Y K -equivalent smooth projective k -varieties, the Chow motives of X and Y in $\mathrm{Chow}(k; R)$ are equivalent.*

In particular, if $\mathrm{Chow}^{\mathrm{eff}}(k; \mathbb{Z}[1/p])$, where p is the exponential characteristic of k , is a Krull-Schmidt category, then conjecture 2.1.3 is true at least after inverting p .

If there is a motivic t -structure on rational geometric Voevodsky motives, then $\mathrm{Chow}(k; \mathbb{Q})$ is a Krull-Schmidt category. As we will see, a consequence of the existence of a motivic t -structure is the following theorem.

Theorem 2.1.9. *Suppose the motivic t -structure conjecture is true for $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$. If X and Y are two K -equivalent smooth projective k -varieties, then $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

The existence of such a motivic t -structure for rational geometric Voevodsky motives is a central conjecture in Voevodsky's theory of motives, and if it is not true then his theory is inadequate. For the statement of the motivic t -structure conjecture, see section 2.4.2. Note that the category of rational (effective) Chow motives embeds fully faithfully into the triangulated category of rational (effective) geometric Voevodsky motives (Voevodsky [83], corollary 4.2.6), and so the conclusion is equivalent to the equivalence of rational Chow motives. As particular consequences, this theorem implies that the existence of a suitable motivic t -structure on $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$ implies that K -equivalent smooth projective k -varieties have isomorphic rational noncommutative motives, rational ℓ -adic Galois representations, rational mixed Hodge structures (if

defined), rational Chow groups, and rational algebraic K-groups.

Recall the following standard definition.

Definition 2.1.10 (D-equivalence). *Two k -varieties X and Y are said to be D-equivalent if their bounded derived categories of coherent sheaves are equivalent as k -linear triangulated categories.*

Another consequence of the integration of Voevodsky motives is that we can prove a theorem pertaining to a conjecture of Orlov [74]. Indeed, we have the following.

Corollary 2.1.11. *If X and Y are D-equivalent smooth projective complex varieties such that either $\kappa(X) = \dim X$ (general type) or $\kappa(X, -K_X) = \dim X$, then $M(X) \oplus P \simeq M(Y) \oplus P$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; \mathbb{Z})$ for some Chow motive $P \in \mathrm{Chow}^{\mathrm{eff}}(k; \mathbb{Z})$.*

In particular, if integral (rational) Chow motives form a Krull-Schmidt category, then we have $M(X) \simeq M(Y)$ (respectively, $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$). In particular, this would be the case if there is a motivic t -structure.

Remark 2.1.12. *It is known that if X and Y are D-equivalent k -varieties with ample or anti-ample canonical bundles, then X and Y are isomorphic [18]. When they have ample canonical bundles for example, we are in the setting of varieties of general type.*

The proof of the above is an easy consequence of our main theorem. Indeed, for smooth projective complex varieties of general type, Kawamata [55] has proved that D-equivalence implies K-equivalence. See theorem 2.4.27. As far as the author knows, this is the first time the existence of a motivic t -structure or Chow motives forming a Krull-Schmidt category has been connected to this conjecture of Orlov.

One reason we obtain stronger concrete results inaccessible to classical motivic integration is that an advantage of our theory is that it comes equipped with a

natural *injective* map

$$c_R : K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R),$$

where $\mathcal{M}(k; R)$ is an abelian group in which our integrals will land. In classical motivic integration, the analogous injectivity statement is only conjectural, and so our path to more refined information encounters great complications. In our theory of integration, however, $\mathcal{M}(k; R)$ is constructed fine enough so that our theory comes equipped with a transformation rule analogous to that in classical motivic integration, but not too fine so that it would be difficult to have the injectivity of c_R above. As stated above, we use this to show that K-equivalent smooth k -varieties have motives whose classes are equal in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$. In particular, if k admits resolutions of singularities, then this is the case for classes of integral motives of birational smooth projective Calabi-Yau k -varieties. In the final section, using our main theorem, we deduce concrete information about the geometry and number theory of K-equivalent varieties. We note that the strongest result in this direction previously known is that in characteristic zero a theory of motivic integration due to Cluckers and Loeser gives the equality of classes of K-equivalent complex varieties in the localization of $K_0(\mathrm{Var}_k)$ with respect to the classes of the affine line and the classes of all projective spaces [25]. Consequently, in some sense, our theory not only gives stronger motivic results in characteristic zero, but it also works in positive characteristics. Another advantage of working with Voevodsky motives is that we are now able to use Bondarko's weight structures [20] to deduce results about Chow motives that were inaccessible to classical motivic integration. Furthermore, we are able to connect important questions in the theory of motives to K-equivalence.

Let us recall why the idea of motivic integration is important by recalling the

history behind the theory. To mathematicians as well as physicists, the classification of Calabi-Yau varieties is important. Batyrev proved that two birationally equivalent smooth projective Calabi-Yau complex varieties have the same Betti numbers [13]. This result dating to 1996 was used by Beauville to explain the Yau-Zaslow formula counting the number of rational curves on K3 surfaces [14]. Such results are also used to bound Hodge numbers of elliptic Calabi-Yau varieties [31], used to prove the log canonical threshold formula [33],[71], and prove transfer principles that allow the study of the Fundamental Lemma in the Langlands program [26], [27].

Roughly, Batyrev's proof of the equality of Betti numbers of birational smooth projective Calabi-Yau complex varieties goes as follows. Choose a lift of the two varieties to the maximal compact subring B of an appropriately chosen local field, and suppose its maximal ideal is $\mathfrak{q}$ and its residue field is isomorphic to the finite field $\mathbb{F}_q$ of characteristic p . Count the number of $\mathbb{F}_q$ -points on the varieties using p -adic integration with respect to canonical measures induced by gauge forms on the two varieties (gauge forms exist because the varieties are assumed to be Calabi-Yau). By showing that the p -adic integrals in the setup can be computed on dense open subsets of the varieties, he showed via the transformation rule for Haar integrals that the reduction modulo $\mathfrak{q}$ of the models have the same number of $\mathbb{F}_q$ -points. By doing the same process with cyclotomic extensions of A , he showed that they have the same number of $\mathbb{F}_{q^n}$ -points. Concisely, the zeta functions of the reductions modulo $\mathfrak{q}$ are the same. By using the Weil conjectures proved by Deligne [28], he concluded that they have the same Betti numbers. This method suggested to the author that something motivic is going on in the background. Indeed, it turns out to be the case.

On the other hand, it was conjectured that two such varieties in fact have the same Hodge numbers. This generalizes the result of Batyrev; indeed, given this

result, the decomposition theorem in Hodge theory implies the result of Batyrev. On December 7 1995, Kontsevich gave a lecture at Orsay envisioning a theory of motivic integration to prove the stronger statement that two such complex varieties have the same Hodge numbers [57]. In fact, he proved the following more general result.

Theorem 2.1.13 (Kontsevich [57], Denef-Loeser [30]). *If X and Y are K -equivalent smooth projective complex varieties, then they have the same Hodge numbers.*

Kontsevich's idea of using motivic integration in the proof of his theorem is based on the following observation. Hodge numbers are encoded by the Deligne-Hodge polynomial. In the case of a smooth projective complex variety X , the polynomial is given by

$$E(X) = \sum_{i,j} (-1)^{i+j} h^i(X; \Omega_X^j) u^i v^j \in \mathbb{Z}[u, v].$$

For general smooth complex varieties, this is defined using mixed Hodge structures. The Deligne-Hodge polynomial has the property that it can be viewed as a function on the Grothendieck ring of complex varieties $K_0(\text{Var}_{\mathbb{C}})$. In his lecture in Orsay in 1995, Kontsevich envisioned a theory of motivic integration that allows us to prove that the classes in a completion of $K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ ($\mathbb{L} := [\mathbb{A}^1]$ the Lefschetz motive) of two K -equivalent complex varieties are equal. The Deligne-Hodge polynomial extends to this completion, from which the result follows.

In 1996, a preprint of Denef and Loeser containing such a construction of motivic integration was circulating in the mathematical community. This construction was published in 1999 [30]. The advantage of this proof is that it not only circumvents the usage of the Weil conjectures proved by Deligne and used by Batyrev, but it gives a stronger result in the sense that two such complex varieties have the same value on any function on the aforementioned completion of the Grothendieck ring of varieties with the Lefschetz motive inverted. We remark that later on Chin-Lung

Wang in 2002 [87] and Tetsushi Ito in 2004 [53] independently proved this result on Hodge numbers by using p -adic Hodge theory to refine Batyrev's proof [53].

Motivic integration is by now a highly developed theory with many applications in birational geometry. In the classical theory of motivic integration, motivic integrals take values in a *completion* $\widehat{\mathcal{M}}_k$ of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ for suitable k . Since not all motivic measures factor through this completion, we cannot deduce all the results we want. For example, the counting measure $C_q : K_0(\mathrm{Var}_{\mathbb{F}_q})[\mathbb{L}^{-1}] \rightarrow \mathbb{Q}$ does not factor through the completion. Indeed, $q^n/\mathbb{L}^n \rightarrow 0$ in the topology of the completion while $C_q(q^n/\mathbb{L}^n) = 1$. This is one reason it is difficult to deduce arithmetic information using classical motivic integration. Therefore, often it is needed to know that the motivic integrals computed take values in a proper subring of $\widehat{\mathcal{M}}_k$. Unfortunately, it is unknown if the natural map from $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ to its completion $\widehat{\mathcal{M}}_k$ is injective. On the other hand, there have been improvements to classical motivic integration in characteristic zero in this direction; however, the integrals still take values in the localization of $K_0(\mathrm{Var}_k)$ with respect to the classes of the affine line and the classes of projective spaces [25]. Such problems have been central to the theories of motivic integration developed using the Grothendieck ring of varieties.

The elements of $K_0(\mathrm{Var}_k)$ are called *virtual motives* because this ring encapsulates the idea of cutting and pasting k -varieties. On the other hand, it is by now clear that the formalism of mixed motives à la Voevodsky is a better framework for thinking about motives because they are ∞ -*categories* carrying a very rich structure. For example, they are stable ∞ -categories that come with six functors formalisms. Moreover, there are realization functors to derived categories of ℓ -adic sheaves, mixed Hodge structures, etc. The fact that Voevodsky motives are categorical as opposed to ring-theoretic roughly means that they are richer in information and flexibility.

We will see that developing a theory of integration for Voevodsky motives has its advantages.

It should be pointed out that it is not clear to the author how the two theories of motivic integration relate to one another beyond both being the same in philosophy. One main goal of motivic integration theories based on $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ is proving equality of classes in this ring; however, as previously mentioned, to a great extent this is not achieved. Since Voevodsky motives are richer than virtual motives, contrary to classical motivic integration, we are able to show that our theory comes equipped with a natural *injective* map $c_R : K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R)$ into the target $\mathcal{M}(k; R)$ of our theory of integration (see proposition 2.3.8). We will be able to use this injectivity result to obtain new information about classes of geometric motives in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$, not just in $\mathcal{M}(k; R)$. If we want to study more concrete structures like ℓ -adic Galois representations and mixed Hodge structures, having equality in either $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$ (or $K_0(\mathrm{DM}_{\mathrm{gm}}(k; R))$) or $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ is good enough. We must note that however, the natural morphism

$$K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \rightarrow K_0(\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}; \mathbb{Q}))$$

induced by sending X to $\pi_!^X \mathbf{1}_X$, where $\pi^X : X \rightarrow \mathrm{Spec} \mathbb{C}$ is the structure morphism of X , has a nontrivial kernel. Indeed, for $g \geq 2$, there are abelian g -folds A such that $\mathrm{End}(A) \simeq \mathbb{Z}$ and $A \not\simeq \widehat{A} := \mathrm{Pic}^0(A)$. $[A] - [\widehat{A}]$ is a nonzero element of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ that maps to zero because A and $\widehat{A}$ are isogenous and so have equivalent rational Chow motives. (For details, see [52].) As a result, equality in $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ is more refined than equality in $K_0(\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}; \mathbb{Q}))$. That being said, what is clear is that using Voevodsky motives allows us to circumvent many of the deficiencies of classical motivic integration that prevent us from obtaining stronger concrete results

in geometry and arithmetic.

This part of the thesis grew out of an attempt to construct a categorified motivic integration taking values in some refinement of $\mathrm{DM}_{\mathrm{gm}}(k; R)$. This categorified version, though, requires a better understanding of the geometric side of motivic integration, and is an ongoing project. If we manage to develop a categorified motivic integration, then we will be able to make theorem 2.1.9 independent of the existence of a motivic t -structure. We may even be able to prove an integral version of this unconditional version, a conjecture of Chin-Lung Wang [88].

2.2 Conventions and Preliminaries

Throughout this chapter, the pair (k, R) will be such that k is a perfect field of exponential characteristic p that admits resolution of singularities or has characteristic invertible in the commutative ring R . When k admits resolution of singularities, we can take coefficients in $\mathbb{Z}$ as opposed $\mathbb{Z}[1/p]$. For us, a Calabi-Yau k -variety X is one whose canonical bundle ω_X is trivial. Henceforth, we will use the language of ∞ -categories as developed in [65] and [67]. In particular, homotopy (co)limits will be called (co)limits. Note that if $\mathcal{C}_0$ is an ordinary category, we abuse notation and still write $\mathcal{C}_0$ instead of its (classical) nerve $N(\mathcal{C}_0)$ which is an ∞ -category. Classical (co)limits in $\mathcal{C}_0$ correspond to ∞ -categorical (co)limits in $N(\mathcal{C}_0)$, and so this convention will not cause confusion when dealing with ordinary categories.

In the rest of this section, we recall the definition of Jet schemes and prove some of its properties. Furthermore, we discuss the essentials of the stable ∞ -category of mixed motives in the sense of Voevodsky, and discuss its properties that will be

important in this chapter. Everything in this section is known, and is discussed here only for the convenience of the reader and to fix the notation.

2.2.1 Jet Schemes

In this subsection, we give the definition and some of the properties of Jet schemes.

Proposition 2.2.1. *Suppose X is a k -scheme of finite type. For each $n \geq 0$, there is a k -scheme $\mathcal{J}_n(X)$ of finite type representing the functor*

$$Z \mapsto \text{Mor}_k(Z \times_k \text{Spec } k[[t]]/(t^{n+1}), X).$$

For a proof, see [39] and [40]. For $n \geq m$, $k[[t]]/(t^{n+1}) \rightarrow k[[t]]/(t^{m+1})$ induces a morphism of k -schemes

$$\pi_m^n : \mathcal{J}_n(X) \rightarrow \mathcal{J}_m(X).$$

These are all *affine* morphisms, and so we have a k -scheme

$$\mathcal{J}_\infty(X) := \varprojlim_n \mathcal{J}_n(X).$$

We sometimes use X_∞ as a substitute notation for $\mathcal{J}_\infty(X)$. We shall call this the *Jet scheme* of X , and we let

$$\pi_n^X : \mathcal{J}_\infty(X) \rightarrow \mathcal{J}_n(X)$$

be the natural projection. We know that $\mathcal{J}_\infty(X)$ has the property that for each

$$Z \mapsto \text{Mor}_k(Z \times_k \text{Spec } k[[t]], X),$$

and that π_n^X is induced by the ring homomorphism $k[[t]] \rightarrow k[[t]]/(t^{n+1})$.

Jet schemes are higher order versions of tangent spaces. In particular, if X is a k -variety, then $\mathcal{J}_1(X) \simeq TX$ is the tangent space of X .

We now recall some known properties of Jet schemes.

Proposition 2.2.2. *Let $X \rightarrow Y$ be an étale morphism of k -schemes of finite type. The following natural square is a pullback square.*

$$\begin{array}{ccc} \mathcal{J}_m(X) & \longrightarrow & \mathcal{J}_m(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y, \end{array}$$

that is, $\mathcal{J}_m(X) \simeq \mathcal{J}_m(Y) \times_Y X$.

Proof. We show the isomorphism on the level of the corresponding functor of points.

Precisely, we want to show that

$$\mathrm{Mor}_k(-, \mathcal{J}_m(X)) \simeq \mathrm{Mor}_k(- \times_k \mathrm{Spec} k[[t]]/(t^{m+1}), X)$$

and

$$\begin{aligned} \mathrm{Mor}_k(-, \mathcal{J}_m(Y) \times_Y X) &\simeq \mathrm{Mor}_k(-, \mathcal{J}_m(Y)) \times_{\mathrm{Mor}_k(-, Y)} \mathrm{Mor}_k(-, X) \\ &\simeq \mathrm{Mor}_k(- \times_k \mathrm{Spec} k[[t]]/(t^{m+1}), Y) \times_{\mathrm{Mor}_k(-, Y)} \mathrm{Mor}_k(-, X) \end{aligned}$$

are equivalent. Let Z be a k -scheme and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ p \uparrow & \searrow \gamma & \uparrow \theta \\ Z & \longrightarrow & Z \times_k \mathrm{Spec} k[[t]]/(t^{m+1}). \end{array}$$

Since $X \rightarrow Y$ is étale, and so formally étale, there is a one-to-one correspondence between $\gamma \in \text{Mor}_k(Z \times_k \text{Spec } k[[t]]/(t^{m+1}), X)$ as in the diagram above, and morphisms θ and p making the above diagram commute. The conclusion follows. $\square$

From this, we obtain the following important corollary.

Corollary 2.2.3. *Let X be a smooth k -scheme of finite type of pure dimension d . Then $\mathcal{J}_m(X)$ is an $\mathbb{A}^{dm}$ -bundle over X . In particular, $\mathcal{J}_m(X)$ is smooth of pure dimension $d(m+1)$. Consequently, $\pi_m^{m+1} : \mathcal{J}_{m+1}(X) \rightarrow \mathcal{J}_m(X)$ is an $\mathbb{A}^d$ -bundle.*

Proof. By the étale invariance above, it suffices to check this locally. In other words, it suffices to check this for $X = \mathbb{A}^d$, which is a simple computation. $\square$

2.2.2 Mixed Motives

In the mid-eighties, Beilinson and Deligne conjectured the existence of a triangulated category with a t -structure whose heart is the conjectural abelian category of mixed motives. Voevodsky constructed triangulated categories with the hope that (with rational coefficients) their full subcategory of constructible objects is the derived category of the hypothetical abelian category of mixed motives. Though this property is, as of today, conjectural, these triangulated categories have had many applications. For example, the Bloch-Kato conjecture about the relation of Galois cohomology to Milnor K-theory was proved by Voevodsky [80],[82],[84],[85],[86]. We now have a theory of motivic cohomology, a spectral sequence from motivic cohomology to algebraic K-theory, and a six functor formalism among many other things. This gives us the flexibility to study motivic phenomena within a very rich framework. In our case, the theory of Voevodsky motives is indispensable.

In this subsection, we briefly recall the definition of the main stable ∞ -category

of Voevodsky motives. We will then discuss a part of the six functor formalism, localization sequences, and purity, all of which will be essential to our work. Our exposition is terse, and so we recommend the reader to look at [21] for the basic definitions and theorems.

There are many variants to the construction to motives in the spirit of Voevodsky. One way the stable ∞ -category of Voevodsky motives is using smooth correspondences. It is defined using the Nisnevich topology, partly because algebraic K-theory satisfies Nisnevich descent. We say that a morphism $f : X \rightarrow Y$ is *completely decomposed* at $y \in Y$ if there is an $x \in X$ above y such that the residual field extension $k(y) \rightarrow k(x)$ is an isomorphism. A Nisnevich covering of an S -scheme Y in this topology is a finite family $\{f_i : U_i \rightarrow Y\}$ of étale S -morphisms with the property that for every point $y \in Y$, there is a j such that f_j is completely decomposed at y . This gives us the site $(\mathcal{S}m/S)_{Nis}$ of smooth S -schemes with the Nisnevich topology. In this setting, a Nisnevich sheaf of R -modules with transfers is a contravariant functor on the category of correspondences Cor_S to the category of R -modules that is a Nisnevich sheaf once we restrict it to $\mathcal{S}m/S$. Let us denote this category of Nisnevich sheaves of R -modules with transfers by $\text{Sh}^{\text{tr}}(S; R)$. Consider the derived ∞ -category of chain complexes of R -modules $D(\text{Sh}^{\text{tr}}(S; R))$ on the category of Nisnevich sheaves with transfers, and invert the chain complexes of the form

$$\dots \rightarrow 0 \rightarrow h_{X \times \mathbb{A}^1} \rightarrow h_X \rightarrow 0 \rightarrow \dots,$$

where h_T denotes the representable Nisnevich sheaf with transfers associated to T . This gives us the stable ∞ -category $\text{DM}^{\text{eff}}(S; R)$ of *effective Voevodsky motives*. The object in $\text{DM}^{\text{eff}}(S; R)$ associated to h_X will be denoted by $M_S(X)$, and will be called the motive of X . Voevodsky motives in general are obtained by stabilizing with

respect to the Tate twist. The structure morphism $\mathbb{P}_S^1 \rightarrow S$ induces the morphism of motives

$$M_S(\mathbb{P}_S^1) \rightarrow M_S(S).$$

Its fiber shifted by $[-2]$ is denoted by $\mathbf{1}_S(1)$ and called the *Tate twist*. Stabilizing with respect to the Tate twist gives us

$$\mathrm{DM}(S; R) := \operatorname{colim}_n \left(\mathrm{DM}^{\mathrm{eff}}(S; R) \xrightarrow{-\otimes \mathbf{1}_S(1)} \mathrm{DM}^{\mathrm{eff}}(S; R) \xrightarrow{-\otimes \mathbf{1}_S(1)} \dots \right),$$

where the colimit is taken in the ∞ -category of presentable ∞ -categories with morphisms left adjoint functors. This is the stable ∞ -category of *Voevodsky motives* (without the effectivity condition). The smallest stable subcategory of $\mathrm{DM}^{\mathrm{eff}}(S; R)$ containing $M_S(X)$ is denoted by $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S; R)$ and called the ∞ -category of *geometric effective* Voevodsky motives. $\mathrm{DM}_{\mathrm{gm}}(S; R)$, called the ∞ -category of *geometric* Voevodsky motives, is obtained by inverting $\mathbf{1}_S(1)$ in the category of geometric effective Voevodsky motives $\mathrm{DM}^{\mathrm{eff}}(S; R)$.

In our case, however, we will need to work with schemes of finite type. The construction above does not allow us to do this because of its restriction to *smooth* schemes. Therefore, we may consider another variant of the above construction. Instead of working with smooth correspondences, we may work with the category $\mathrm{Sch}^{\mathrm{cor}}/S$ consisting of separated finite type S -schemes with morphisms finite S -correspondences. Given a topology τ , we may consider the big site $(\mathrm{Sch}/S)_\tau$ of finite type S -schemes with the τ -topology. Considering presheaves of R -modules on $\mathrm{Sch}^{\mathrm{cor}}/S$ that restrict to τ -sheaves on $(\mathrm{Sch}/S)_\tau$ gives us the category $\underline{\mathrm{Sh}}_\tau^{\mathrm{tr}}(S; R)$ of τ -sheaves of R -modules with transfers. $\mathbb{A}^1$ -localization and inverting the Tate twist as before gives us the category $\underline{\mathrm{DM}}_\tau(S; R)$ of τ -motives. In this general context, any separated finite type S -scheme X defines an object $\underline{M}_S(X) \in \underline{\mathrm{DM}}_\tau(S; R)$. When we do

not use the symbol τ , we mean that $\tau = Nis$. Therefore, $\underline{DM}(S; R) := \underline{DM}_{Nis}(S; R)$. We denote by $DM_{gm}^{eff}(S; R)$ the smallest stable ∞ -category containing motives of the form $M_S(X)(n)$, $n \geq 0$ and X smooth S -scheme. Its stabilization with respect to the Tate twist will be denoted by $DM_{gm}(S; R)$. Considering the largest localizing full subcategory of $\underline{DM}_\tau(S; R)$ generated by motives $\underline{M}_S(X)(n)$ for X smooth S -schemes and $n \in \mathbb{Z}$ gives us the category $DM_\tau(S; R)$. From now on, we let S be Noetherian. Taking $\tau = cdh$ gives us the categories $DM_{cdh}(S; R)$ and $\underline{DM}_{cdh}(S; R)$.

For all such categories, we have basic functors $f^*, f_*, \otimes, \underline{Hom}(-, -)$. For smooth $f : X \rightarrow Y$, we also have a left adjoint $f_\#$ to f^* . If X is a smooth S -scheme with structure morphism $\pi^X : X \rightarrow S$, then $M_S(X) = \pi_\#^X \mathbf{1}_X$.

The functor that will be of greatest importance to us is $f_!$, which is, up to equivalence, given by $p_* j_\#$ for $f = p \circ j$ a factorization of f into an open embedding j followed by a proper morphism p (Nagata compactification).

Essential to our work will be localization sequences. By theorem 5.11 of Cisinski and Déglise in [23], since by assumption k has characteristic exponent invertible in R , for each open embedding $j : U \hookrightarrow X$ of k -varieties with complement the closed embedding $i : Z \hookrightarrow X$, there is a cofiber sequence

$$j_\# j^* \mathbf{1}_X \rightarrow \mathbf{1}_X \rightarrow i_* i^* \mathbf{1}_X \rightarrow j_\# j^* \mathbf{1}_X[1]$$

in $DM_{cdh}(k; R)$. Applying the exact functor $\pi_!^X$ to it, we obtain the cofiber sequence

$$\pi_!^U \mathbf{1}_U \rightarrow \pi_!^X \mathbf{1}_X \rightarrow \pi_!^Z \mathbf{1}_Z \rightarrow \pi_!^U \mathbf{1}_U[1]$$

in $\mathrm{DM}_{cdh}(k; R)$. Note that we are not assuming smoothness, which is an advantage of working with the cdh topology. Note, however, that by corollary 5.9 of Cisinski and Déglise [23], the natural morphism $\mathrm{DM}(X; R) \rightarrow \mathrm{DM}_{cdh}(X; R)$ is an equivalence of symmetric monoidal categories if X is a regular k -scheme. Furthermore, using proposition 8.1(c) of Cisinski and Déglise [23] for the field k we have that $\mathrm{DM}(k; R) \simeq \mathrm{DM}_{cdh}(k; R) = \underline{\mathrm{DM}}_{cdh}(k; R)$, and so we can work in the larger category $\underline{\mathrm{DM}}_{cdh}(k; R)$ and pass to $\mathrm{DM}(k; R)$.

Furthermore, we also have purity, a part of which says that if $f : X \rightarrow Y$ is a smooth separated morphism of finite type over k of relative dimension d , then $f_! \simeq f_{\#}(-d)[-2d]$ (see theorem 11.4.5 of [21]). We will take advantage of these properties when proving the well-definedness of our motivic measure.

2.3 Integration of Motives

In this section, we define integration in the setting of geometric Voevodsky motives. In order to do so, we first define the category of *completed* geometric effective motives $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$. This will be defined in subsection 2.3.1. We also define the notion of *convergent* motives that will be used in the definition of the group $\mathcal{M}(k; R)$ in which our integrals will land. We show that $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$ naturally injects into $\mathcal{M}(k; R)$. In order to define our integrals, we need a notion of measure; we define a measure on the Jet scheme $\mathcal{J}_{\infty}(X)$ taking values in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$. We do so by first defining the measure on the so-called *stable subschemes* of $\mathcal{J}_{\infty}(X)$, and then extending this by defining the notions of *good* and *measurable* subsets and their measures. Given this measure, we define the integration of Voevodsky motives, and specialize to a particular class of functions, called *measurable*, that will be of greatest interest to us. In the subsection on computations, we set the stage for the transformation rule

by doing some computations demonstrating the way integrals change with respect to blowups along smooth centers. In the final subsection, we prove the transformation rule.

2.3.1 Completed and Convergent Motives

Integration of Voevodsky motives will deal with the category of effective geometric Voevodsky motives $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ and will take values in an abelian group $\mathcal{M}(k; R)$ close to the Grothendieck group of some categorical limit $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$ of Verdier quotients of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ by $\otimes$ -ideals. In this subsection, we define this categorical completion and the notion of convergent motives, and discuss their relations to the usual categories of Voevodsky's motives.

Note that by theorem 1.1.4.4 of [67], the ∞ -category $\mathrm{Cat}_{\infty}^{Ex}$ of small stable ∞ -categories with exact functors is closed under small limits. The ∞ -category of commutative algebra objects in it, denoted by $\mathrm{Cat}_{\infty}^{Ex, \otimes}$, is also closed under small limits. In fact, limits in $\mathrm{Cat}_{\infty}^{Ex, \otimes}$ can be computed in $\mathrm{Cat}_{\infty}^{Ex}$. See section 3.2.2. of [67].

Definition 2.3.1. *Define the ∞ -category of completed effective (geometric) Voevodsky motives with R -coefficients as the following limit in $\mathrm{Cat}_{\infty}^{Ex, \otimes}$ of small stable symmetric monoidal ∞ -categories with exact functors:*

$$\mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}, \wedge}(k; R) := \lim_n \mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R) / \mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R)(n),$$

where $\mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R)(n)$ is the full sub- ∞ -category of $\mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R)$ generated by effective (geometric) motives Tate-twisted n times. The transition functors $\mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R) / \mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R)(n+1) \rightarrow \mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R) / \mathrm{DM}_{(\mathrm{gm})}^{\mathrm{eff}}(k; R)(n)$ are the localization functors.

Let us set some notation that will soon be useful to us. Let by

$$L_n : \mathrm{DM}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}}(k; R)/\mathrm{DM}^{\mathrm{eff}}(k; R)(n)$$

be the natural localization functor. Since $\mathrm{DM}^{\mathrm{eff}}(k; R)$ is presentable, this localization functor has a fully faithful right adjoint that we denote by

$$i_n : \mathrm{DM}^{\mathrm{eff}}(k; R)/\mathrm{DM}^{\mathrm{eff}}(k; R)(n) \hookrightarrow \mathrm{DM}^{\mathrm{eff}}(k; R).$$

By the universal property of limits, there is a natural functor $L_\infty : \mathrm{DM}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$. We also denote the restriction of L_∞ to $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ by L_∞ .

Let us prove a lemma that will be used in the proof of subsequent lemmas.

Lemma 2.3.2. *Suppose $N \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$, and $A \in \mathrm{DM}^{\mathrm{eff}}(k; R)$. Then for $n \gg 0$, $A(n) \rightarrow N$ is trivial.*

Proof. $\mathrm{DM}^{\mathrm{eff}}(k; R)$ is compactly generated by motives of the form $M(X)(n)$ for $n \geq 0$ and X smooth projective k -varieties. Since N is geometric it is generated by finitely many such $M(X_i)(n_i)[m_i]$, X_i smooth projective k -varieties of dimension d_i . Suppose A is a colimit of objects $A_j = M(Y_j)(k_j)[t_j]$, where Y_j are smooth projective k -varieties. Consider

$$\begin{aligned} & \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}}(k; R)}(A_j(n), N) \\ \simeq & \mathrm{colim}_i \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}}(k; R)}(M_j(Y_j)(k_j + n)[t_j], M(X_i)(n_i)[m_i]) \\ \simeq & \mathrm{colim}_i \mathrm{Map}_{\mathrm{DM}(k; R)}(M_j(Y_j)(k_j + n - n_i)[t_j - m_i], M(X_i)) \\ \simeq^{(1)} & \mathrm{colim}_i \mathrm{Map}_{\mathrm{DM}(k; R)}(M_j(Y_j)(k_j + n - n_i)[t_j - m_i], M^c(X_i)) \\ \simeq^{(2)} & \mathrm{colim}_i \mathrm{Map}_{\mathrm{DM}(k; R)}(M_j(Y_j)(k_j + n - n_i)[t_j - m_i], \mathrm{Map}_{\mathrm{DM}(k; R)}(M(X_i), \mathbf{1}_k)(d_i)[2d_i]) \\ \simeq & \mathrm{colim}_i \mathrm{Map}_{\mathrm{DM}(k; R)}(M_j(Y_j \times_k X_i)(k_j + n - n_i)[t_j - m_i], \mathbf{1}_k(d_i)[2d_i]), \end{aligned}$$

where (1) follows from the fact that the X_i are smooth *projective*, and (2) follows from $M^e(X) \simeq M(X)^\vee(d)[2d]$, where d is the dimension of smooth k -variety X and the $\vee$ denotes dualization. The last mapping space is contractible if $k_j + n - n_i > d_i$, that is, when $n > n_i + d_i - k_j$. We can take $n \geq \max_i \{n_i + d_i + 1\}$, which depends only on N and ranges over finitely many i by the fact that N is geometric. Consequently, $\mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}}(k;R)}(A(n), N) \simeq \lim_j \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}}(k;R)}(A_j(n), N) \simeq 0$ for $n \geq \max_i \{n_i + d_i + 1\}$, that is, for $n \gg 0$. The conclusion follows. $\square$

A lemma that will be of importance to us later is the following. In particular, it demonstrates that completed effective geometric Voevodsky motives contain usual effective geometric Voevodsky motives as a full subcategory.

Lemma 2.3.3. *The natural functors*

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$$

and

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$$

are fully faithful.

Proof. Consider the natural commutative diagram

$$\begin{array}{ccc} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) & \longrightarrow & \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R) \\ \downarrow & & \downarrow \\ \mathrm{DM}^{\mathrm{eff}}(k; R) & \longrightarrow & \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R) \end{array}$$

of functors. In order to show that $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$ is fully faithful, we show that $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$ and $L_\infty : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$ are fully faithful functors.

First, let us show that $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$ is fully faithful. Since the limit of fully faithful functors is fully faithful, it suffices to show that for each $n \in \mathbb{Z}$,

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n) \rightarrow \mathrm{DM}^{\mathrm{eff}}(k; R)/\mathrm{DM}^{\mathrm{eff}}(k; R)(n)$$

is fully faithful. However, this is a consequence of theorem 2.1 of [72]. It is also easy to see that the above commutative diagram exists.

We now show that $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}}(k; R) \rightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$ is fully faithful. Suppose M and N are geometric motives in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$. We are to show that

$$\mathrm{Map}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)}(M, N) \rightarrow \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)}(L_{\infty}M, L_{\infty}N)$$

is a weak equivalence. However,

$$\begin{aligned} \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)}(L_{\infty}M, L_{\infty}N) &\simeq \lim_n \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}}(k; R)/\mathrm{DM}^{\mathrm{eff}}(k; R)(n)}(L_nM, L_nN) \\ &\simeq \lim_n \mathrm{Map}_{\mathrm{DM}^{\mathrm{eff}}(k; R)}(M, i_nL_nN). \end{aligned}$$

We claim that since N is geometric, for $n \gg 0$, $N \xrightarrow{\varphi} i_nL_nN$ is an equivalence, from which the result will follow. The cokernel of φ lies in $\mathrm{DM}^{\mathrm{eff}}(k; R)(n)$. By lemma 2.3.2, for $n \gg 0$, all maps $A(n) \rightarrow N$, $A \in \mathrm{DM}^{\mathrm{eff}}(k; R)$ are trivial. Geometric motives are generated by objects of the form $M(X)(m)$, where X is a smooth projective k -variety and $m \in \mathbb{Z}$. Therefore, for $n \gg 0$, $N(-n)$ lies in the negative part of the homotopy t -structure on $\mathrm{DM}(k; R)$. Consider the distinguished triangle

$$N \rightarrow i_nL_nN \rightarrow \mathrm{coker} \varphi \rightarrow N[1]$$

for $n \gg 0$. Since $\text{coker} \varphi \rightarrow N[1]$ is 0, the sequence splits, and so $N \oplus \text{coker} \varphi \simeq i_n L_n N$. Since $\text{Map}(\text{coker} \varphi, i_n L_n N) \simeq \text{Map}(L_n \text{coker} \varphi, L_n N) \simeq 0$, $\text{coker} \varphi \rightarrow i_n L_n N$ is 0. Therefore, $\text{coker} \varphi$ must be 0. Therefore, φ is an equivalence for $n \gg 0$. We conclude that

$$\text{Map}_{\text{DM}_{\text{gm}}^{\text{eff}}(k;R)}(M, N) \rightarrow \text{Map}_{\text{DM}^{\text{eff}, \wedge}(k;R)}(L_\infty M, L_\infty N) \simeq \text{Map}_{\text{DM}^{\text{eff}}(k;R)}(M, N)$$

is the identity morphism, as required. $\square$

Remark 2.3.4. *Note that it is not true that $L_\infty : \text{DM}^{\text{eff}}(k; R) \rightarrow \text{DM}^{\text{eff}, \wedge}(k; R)$ is fully faithful. Indeed, Ayoub (lemma 2.4 of [9]) constructs a phantom motive $F = \text{hocolim}_{n \in \mathbb{N}} \mathbb{Q}(n)[n]$ in $\text{DM}^{\text{eff}}(k; \mathbb{Q})$, k a field of infinite transcendence degree over its prime field, that is not equivalent to 0. This maps to 0 under L_∞ . His construction is as follows. Note that $H^n(\text{Spec } k; \mathbb{Q}(n)) = K_n^M(k) \otimes \mathbb{Q}$, where $K_n^M(k)$ is the n -th Milnor K -group of k . Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements of $k^\times$ that are algebraically independent. Let $a_n : \mathbb{Q}(n)[n] \rightarrow \mathbb{Q}(n+1)[n+1]$ be the map corresponding to $a_n \in k^\times$ modulo the isomorphism $\text{Hom}(\mathbb{Q}(n)[n], \mathbb{Q}(n+1)[n+1]) \simeq \text{Hom}(\mathbb{Q}(0), \mathbb{Q}(1)[1]) = H^1(\text{Spec } k; \mathbb{Q}(1)) = k^\times \otimes \mathbb{Q}$. This gives an $\mathbb{N}$ -inductive system $\{\mathbb{Q}(n)[n]\}_{n \in \mathbb{N}}$, using which we obtain the object $F := \text{hocolim}_{n \in \mathbb{N}} \mathbb{Q}(n)[n]$. Ayoub shows that F is nonzero by showing that the natural map $\alpha_\infty : \mathbb{Q}(0) \rightarrow F$ is nonzero. Indeed, since $\mathbb{Q}(0)$ is compact, α_∞ is zero if and only if for some n the natural map $\alpha_n : \mathbb{Q}(0) \rightarrow \mathbb{Q}(n)[n]$ is zero. But this corresponds, under the identification $\text{Hom}(\mathbb{Q}(0), \mathbb{Q}(n)[n]) = K_n^M(k) \otimes \mathbb{Q}$, to the symbol $\{a_0, \dots, a_n\} \in K_n^M(k) \otimes \mathbb{Q}$. This is nonzero as a result of the assumption that the a_i are algebraically independent. As a side remark, it is called a phantom motive because its Betti realization is 0.*

2.3.2 $\mathcal{M}(k; R)$ and injectivity from $K_0(\mathbf{DM}_{\text{gm}}^{\text{eff}}(k; R))$

In this subsection, we define the abelian group $\mathcal{M}(k; R)$ in which our integrals will land. We also show that $K_0(\mathbf{DM}_{\text{gm}}^{\text{eff}}(k; R))$ naturally injects into this group. First, let us fix some notation for convenience.

Suppose $\mathcal{A}$ is a stable ∞ -category. Denote by $F(\mathcal{A})$ the free abelian group on equivalence classes $[X]$ of objects X of $\mathcal{A}$. Denote by $E(\mathcal{A})$ the subgroup of $F(\mathcal{A})$ generated by elements of the form $[X \oplus Y] - [X] - [Y]$. Let $T(\mathcal{A})$ be the subgroup of $F(\mathcal{A})$ generated by elements of the form $[A] - [B] + [C]$, where

$$A \rightarrow B \rightarrow C \rightarrow A[1]$$

is a cofiber sequence in $\mathcal{A}$. Define $G(\mathcal{A}) := F(\mathcal{A})/E(\mathcal{A})$ and $K_0(\mathcal{A}) := F(\mathcal{A})/T(\mathcal{A})$. Clearly, there is a surjection $G(\mathcal{A}) \rightarrow K_0(\mathcal{A})$. Let $I(\mathcal{A})$ be its kernel.

Definition 2.3.5. *An effective convergent motive X is an object of $\mathbf{DM}^{\text{eff}}(k; R)$ with the property that for $N \gg 0$, $i_N L_N X$ is in the essential image of $\mathbf{DM}_{\text{gm}}^{\text{eff}}(k; R) \hookrightarrow \mathbf{DM}^{\text{eff}}(k; R)$. Let $\mathbf{DM}_{\text{conv}}^{\text{eff}}(k; R)$ be the full subcategory of the stable ∞ -category $\mathbf{DM}^{\text{eff}}(k; R)$ consisting of effective convergent motives. We call this stable ∞ -category the ∞ -category of effective convergent motives.*

An example of an effective convergent motive is $M(B\mathbb{G}_m) = \bigoplus_{i=0}^{\infty} \mathbf{1}_k(i)[2i]$. We remark that there is a natural functor $\mathbf{DM}_{\text{conv}}^{\text{eff}}(k; R) \rightarrow \mathbf{DM}^{\text{eff}, \wedge}(k; R)$ with essential image in $\mathbf{DM}_{\text{gm}}^{\text{eff}, \wedge}(k; R)$. Note that the phantom motive F in remark 2.3.4 is convergent and effective. Furthermore, its image under $\mathbf{DM}_{\text{conv}}^{\text{eff}}(k; R) \rightarrow \mathbf{DM}^{\text{eff}, \wedge}(k; R)$ is 0, and so this functor is not fully faithful.

We define here the notion of virtual dimension; it will recur throughout this chapter.

Definition 2.3.6. *The virtual dimension of an object M of $\mathrm{DM}(k; R)$, denoted by $\mathrm{vdim} M$, is the largest $n \in \mathbb{Z} \cup \{\pm\infty\}$ such that $M \in \mathrm{DM}^{\mathrm{eff}}(k; R)(n)$.*

Remark 2.3.7. *Note that $\mathrm{vdim} 0 = +\infty$. Also, for F the phantom motive in remark 2.3.4, we have $\mathrm{vdim} F = \infty$, while $F \not\cong 0$. The value $-\infty$ can also be attained: $\mathrm{vdim} \bigoplus_{n=-\infty}^{\infty} \mathbf{1}_k(n)[2n] = -\infty$.*

Suppose for each $i \geq 0$,

$$X_0^{(i)} \rightarrow X_1^{(i)} \rightarrow \dots \rightarrow X_n^{(i)} \rightarrow \dots$$

is a tower of objects in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ such that the cofibers $X_n^{(i)}/X_{n-1}^{(i)}$ have virtual dimensions going to infinity as n goes to infinity. Furthermore, assume that the virtual dimensions of $X_n^{(i)}$ go to infinity as i goes to infinity. Denote by $T_{\infty}(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$ the image in $F(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R))$ of the smallest subgroup of $F(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$ containing $T(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$ and elements of the form

$$\left[\bigoplus_i \operatorname{colim}_n X_n^{(i)} \right] - \left[\bigoplus_i \bigoplus_n X_n^{(i)}/X_{n-1}^{(i)} \right]$$

coming from all towers above. Define the abelian groups

$$\mathcal{M}(k; R) := F(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R))/T_{\infty}(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$$

and

$$\mathcal{M}_0(k; R) := F(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R))/E(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R)),$$

where we are abusing notation and viewing $E(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$ as its image in $F(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R))$. Note that $\mathcal{M}(k; R)$ and $\mathcal{M}_0(k; R)$ are modules over the rings

$K_0(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$ and $G(\mathrm{DM}_{\mathrm{conv}}^{\mathrm{eff}}(k; R))$, respectively. There is a natural $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$ -linear morphism

$$c_R : K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R),$$

as well as a natural $G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$ -linear morphism

$$G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}_0(k; R).$$

Our integrals will take values in $\mathcal{M}(k; R)$. Consequently, if we want to prove results about classes of geometric motives in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$ using integration, it will be very useful to know the following injectivity result.

Proposition 2.3.8. *The natural homomorphism*

$$c_R : K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R)$$

is injective.

Proof. Prior to showing that c_R is injective, we show that

$$G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}_0(k; R)$$

is injective. Every object of $G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$ is of the form $[X]$ for some $X \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$. Suppose $[X] = 0$ in $\mathcal{M}_0(k; R)$. Then there is an effective convergent motive Z such that $X \oplus Z \simeq Z$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R) \hookrightarrow \mathrm{DM}^{\mathrm{eff}, \wedge}(k; R)$. Since X is geometric, by lemma 2.3.2, for $N \gg 0$, $X \rightarrow i_N L_N X$ is an equivalence (see proof of lemma 2.3.3). Furthermore, since Z is convergent, we may choose N so that it additionally satisfies $i_N L_N Z \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$. Applying the exact functor $i_N L_N$ to both sides, we obtain $X \oplus i_N L_N Z \simeq i_N L_N Z$ as objects in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$. This implies that

$[X] = 0$ in $G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$. This argument establishes the injectivity of

$$G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}_0(k; R).$$

We now show that $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R)$ is injective. Note that there is a natural quotient map $\mathcal{M}_0(k; R) \rightarrow \mathcal{M}(k; R)$. Denote its kernel by J . The injectivity of $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow \mathcal{M}(k; R)$ is equivalent to showing that $G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \cap J = I(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$. The inclusion $I(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \subseteq G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \cap J$ is clear. Now suppose $[X]$ is in $G(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \cap J$. Then $[X] = [A] + [C] - [B] + [\bigoplus_i \mathrm{colim}_n X_n^{(i)}] - \left[\bigoplus_i \bigoplus_n X_n^{(i)} / X_{n-1}^{(i)} \right]$ in $\mathcal{M}_0(k; R)$ for $A \rightarrow B \rightarrow C \rightarrow A[1]$ a cofiber sequence with A, B, C (in the image of) effective convergent motives in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$, and $X_0^{(i)} \rightarrow X_1^{(i)} \rightarrow \dots$ a sequence of towers as above. Therefore, there is an effective convergent motive U such that

$$X \oplus B \oplus U \oplus \bigoplus_i \bigoplus_n X_n^{(i)} / X_{n-1}^{(i)} \simeq A \oplus C \oplus U \oplus \bigoplus_i \mathrm{colim}_n X_n^{(i)}.$$

As in the previous argument, we may apply $i_N L_N$ for $N \gg 0$ such that $X \rightarrow i_N L_N X$ is an equivalence and $i_N L_N A$, $i_N L_N B$, $i_N L_N C$, and $i_N L_N U$ are geometric motives. The exactness of $i_N L_N$ gives us the equivalence

$$X \oplus i_N L_N B \oplus i_N L_N U \oplus \bigoplus_i \bigoplus_n i_N L_N (X_n^{(i)} / X_{n-1}^{(i)}) \simeq i_N L_N A \oplus i_N L_N C \oplus i_N L_N U \oplus \bigoplus_i \mathrm{colim}_n i_N L_N X_n^{(i)}$$

in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$. Let k be such that the virtual dimensions of $X_n^{(i)}$ and $X_n^{(i)} / X_{n-1}^{(i)}$ are at least N for $i, n \geq k$. Then, for $M \gg 0$, we have

$$L_M \left(X \oplus i_N L_N B \oplus i_N L_N U \oplus \bigoplus_{i, n \leq k} X_n^{(i)} / X_{n-1}^{(i)} \right) \simeq L_M \left(i_N L_N A \oplus i_N L_N C \oplus i_N L_N U \oplus \bigoplus_{i=0}^k X_k^{(i)} \right).$$

Using lemma 2.3.3, we obtain that

$$X \oplus i_N L_N B \oplus i_N L_N U \oplus \bigoplus_{i, n \leq k} X_n^{(i)} / X_{n-1}^{(i)} \simeq i_N L_N A \oplus i_N L_N C \oplus i_N L_N U \oplus \bigoplus_{i \leq k} X_k^{(i)}$$

with $i_N L_N A \rightarrow i_N L_N B \rightarrow i_N L_N C \rightarrow i_N L_N A[1]$ a cofiber sequence in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$.

It follows that

$$[X] = [i_N L_N A] + [i_N L_N C] - [i_N L_N B] + \left[\bigoplus_{i \leq k} X_k^{(i)} \right] - \left[\bigoplus_{i, n \leq k} X_n^{(i)} / X_{n-1}^{(i)} \right],$$

which is, as required, an element of $I(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$. $\square$

The injectivity of the analogous result for virtual motives is unknown and is a source of complications in classical motivic integration. Indeed, in classical motivic integration, integrals take values in a completion $\widehat{\mathcal{M}}_k$ of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$, and it is not known if the map from $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ to its completion is injective. Therefore, motivic integration has its complications when we try to show equality of classes in $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$. In our case, however, integrals take values in $\mathcal{M}(k; R)$, and so the above proposition will help us deduce results about classes in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$. Let me point out that I believe that at least for $R = \mathbb{Q}$, the natural morphism

$$K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R))$$

is injective. In fact, I believe the stronger result that the composition

$$K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)) \rightarrow K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)) \rightarrow \varprojlim_n K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) / \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n))$$

is injective. Not being able to prove this is the reason $\mathcal{M}(k; R)$ is considered instead of $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R))$ or $\varprojlim_n K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R) / \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n))$. One main reason for

developing the integration of Voevodsky motives is to have such an injective map so that we can deduce stronger concrete results in geometry and arithmetic.

2.3.3 Motivic measure and measurable subsets

In this section, we define the motivic measure on the Jet scheme $\mathcal{J}_\infty(X)$ of a smooth k -variety X . We first define *stable* subschemes of the Jet scheme and define a measure on such subschemes. We will then define *good* and *measurable* subsets of the Jet scheme and extend our measure to such subsets.

Definition 2.3.9. *Let X be a smooth k -scheme. A subscheme $A \subseteq \mathcal{J}_\infty(X)$ is said to be stable if there is an $m \in \mathbb{N}$ such that $A_m := \pi_m(A)$ is a locally closed subscheme of $\mathcal{J}_m(X)$ and $A = \pi_m^{-1}(A_m)$. We shall say that such an A is stable at least at the m th level.*

In order to define the motivic measure on stable subschemes, we need the following lemma.

Lemma 2.3.10. *Suppose X is a smooth k -variety of dimension d . If A is a stable subscheme of $\mathcal{J}_\infty(X)$ that is stable at least at the N th level, then for every $m \geq N$,*

$$\pi_!^{A_{m+1}} \mathbf{1}_{A_{m+1}}((m+1)d)[2(m+1)d] \simeq \pi_!^{A_m} \mathbf{1}_{A_m}(md)[2md]$$

in $\mathrm{DM}(k; R)$.

Proof. Note that by the smoothness of X , for every $m \geq N$

$$A_{m+1} \xrightarrow{\pi_m^{m+1}} A_m$$

is an $\mathbb{A}^d$ -bundle. By purity, we know that $\pi_{m,!}^{m+1} \mathbf{1}_{A_{m+1}} \simeq \pi_{m,\#}^{m+1} \mathbf{1}_{A_{m+1}}(-d)[-2d]$. Therefore, we obtain

$$\begin{aligned} \pi_{!}^{A_{m+1}} \mathbf{1}_{A_{m+1}}((m+1)d)[2(m+1)d] &\simeq \pi_{!}^{A_m} \pi_{m,!}^{m+1} \mathbf{1}_{A_{m+1}}((m+1)d)[2(m+1)d] \\ &\simeq \pi_{!}^{A_m} \pi_{m,\#}^{m+1} \mathbf{1}_{A_{m+1}}(md)[2md] \\ &\simeq \pi_{!}^{A_m} \mathbf{1}_{A_m}(md)[2md], \end{aligned}$$

where the last equivalence follows from $\mathbb{A}^1$ -homotopy invariance and the fact that π_m^{m+1} is an $\mathbb{A}^d$ -bundle implying that $\pi_{m,\#}^{m+1} \mathbf{1}_{A_{m+1}} \simeq \mathbf{1}_{A_m}$. The conclusion follows. $\square$

Therefore, on such stable subschemes, we can define a measure μ_X as follows.

Definition 2.3.11. *For A a stable subscheme of $\mathcal{J}_\infty(X)$, we define its volume by*

$$\mu_X(A) \simeq \pi_{!}^{A_m} \mathbf{1}_{A_m}((m+1)d)[2(m+1)d] \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$$

for sufficiently large m .

By lemma 2.3.10, this is independent of m for sufficiently large m . Note that it is clearly geometric as A_m is locally closed. We also used the fact that $\pi_{!}^Y \mathbf{1}_Y \in \mathrm{DM}^{\mathrm{eff}}(k; R)(-d_Y)$, where $d_Y := \dim Y$. We show this in the following lemma.

Lemma 2.3.12. *Suppose X is a k -variety of dimension d . Then $\mathrm{vdim} \pi_{!}^X \mathbf{1}_X = -d$.*

Proof. First assume that X is a smooth k -variety. By purity, $\pi_{!}^X \mathbf{1}_X \simeq M(X)(-d)[-2d]$, and so $\mathrm{vdim} \pi_{!}^X \mathbf{1}_X \geq -d$. If $\mathrm{vdim} \pi_{!}^X \mathbf{1}_X > -d$, then $\mathrm{vdim} M(X) \geq 1$, that is $M(X) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(1)$. From this we obtain $M(X_L) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(L; R)(1)$ for any finite extension $L|k$. Take a point $\mathrm{Spec} L \rightarrow X_L$ for some large enough L . From this, we see that $\mathbf{1}_L$ splits off $M(X_L)$ as a direct summand. As a result, $M(X_L)$ has virtual dimension 0. Consequently, $\mathrm{vdim} \pi_{!}^X \mathbf{1}_X = -d$.

We now prove the lemma when X need not be smooth. We do so by inducting on the dimension of X . The lemma is true if X is of dimension 0. Suppose the lemma is true for dimensions $< d$. Let $\text{Sing}X$ be the singular locus of X . Then $\dim \text{Sing}X < d$, and so by the inductive hypothesis, $\text{vdim } \pi_!^{\text{Sing}X} \mathbf{1}_{\text{Sing}X} = -\dim \text{Sing}X > -d$. By the smoothness of $X \setminus \text{Sing}X$, we have $\text{vdim } \pi_!^{X \setminus \text{Sing}X} \mathbf{1}_{X \setminus \text{Sing}X} = -d$. Consider the localization cofiber sequence

$$\pi_!^{X \setminus \text{Sing}X} \mathbf{1}_{X \setminus \text{Sing}X} \rightarrow \pi_!^X \mathbf{1}_X \rightarrow \pi_!^{\text{Sing}X} \mathbf{1}_{\text{Sing}X} \rightarrow \pi_!^{X \setminus \text{Sing}X} \mathbf{1}_{X \setminus \text{Sing}X}[1]$$

in $\underline{\text{DM}}_{cdh}(k; R)$. Twisting it d times, we obtain the cofiber sequence

$$\pi_!^{X \setminus \text{Sing}X} \mathbf{1}_{X \setminus \text{Sing}X}(d) \rightarrow \pi_!^X \mathbf{1}_X(d) \rightarrow \pi_!^{\text{Sing}X} \mathbf{1}_{\text{Sing}X}(d) \rightarrow \pi_!^{X \setminus \text{Sing}X} \mathbf{1}_{X \setminus \text{Sing}X}(d)[1]$$

in $\underline{\text{DM}}_{cdh}(k; R)$. In the above cofiber sequence, $\pi_!^X \mathbf{1}_X(d)$ is an extension of two effective motives, and so is itself an effective cdh motive. Consequently, $\text{vdim } \pi_!^X \mathbf{1}_X \geq -d$. Assume to the contrary that $\text{vdim } \pi_!^X \mathbf{1}_X > -d$. Then we have $\text{vdim } \pi_!^{X \setminus \text{Sing}X} \mathbf{1}_{X \setminus \text{Sing}X} > -d$ from the localization sequence above, a contradiction. As a result, $\text{vdim } \pi_!^X \mathbf{1}_X = -d$, as required. $\square$

Later, we will extend μ_X to a larger collection of subsets of $\mathcal{J}_\infty(X)$ that we shall call measurable subsets. The following corollary will be used in proving that our measure on measurable subsets is well-defined up to equivalence.

Corollary 2.3.13. *Suppose $A \subseteq \bigcup_{i=1}^N A_i \subseteq \mathcal{J}_\infty(X)$, where A and the A_i are stable subschemes. Then*

$$\text{vdim } \mu_X(A) \geq \min_{1 \leq i \leq N} \text{vdim } \mu_X(A_i).$$

Proof. This is an easy consequence of lemma 2.3.12, and the fact that if $X \subseteq \bigcup_{i=1}^N X_i$ is a covering of a subscheme by a finite collection of subschemes, then $\dim X \leq \max_{1 \leq i \leq N} \dim X_i$. $\square$

As a sanity check, we have additivity of μ_X in the following weak sense.

Lemma 2.3.14. *μ_X is additive on finite disjoint unions of stable subschemes of $\mathcal{J}_\infty(X)$. More precisely, if the subscheme $A = \sqcup_{i=1}^k A_i$ is a finite disjoint union of stable subschemes of $\mathcal{J}_\infty(X)$, then*

$$\mu_X(A) \simeq \bigoplus_{i=1}^k \mu_X(A_i).$$

Proof. By the stability of the A_i , there is an N such that every A_i stabilizes from the N th level onward. Then it is easy to see that $A_N = \sqcup_{i=1}^k A_{i,N}$, and so

$$\begin{aligned} \mu_X(A) &\simeq \pi_!^{A_N} \mathbf{1}_{A_N}((N+1)d)[2(N+1)d] \\ &\simeq \pi_!^{A_N} \mathbf{1}_{\sqcup_{i=1}^k A_{i,N}}((N+1)d)[2(N+1)d] \\ &\simeq \bigoplus_{i=1}^k \pi_!^{A_{i,N}} \mathbf{1}_{A_{i,N}}((N+1)d)[2(N+1)d] \\ &\simeq \bigoplus_{i=1}^k \mu_X(A_i). \end{aligned}$$

□

Another lemma that will be important in the proof of the well-definedness of the measure μ_X once we extend it to measurable subsets, to be defined later, is the following.

Lemma 2.3.15. *Suppose D and D_n , $n \in \mathbb{N}$, are stable subschemes of $\mathcal{J}_\infty(X)$ such that $D \subseteq \bigcup_{n \in \mathbb{N}} D_n$ and the virtual dimensions of $\mu_X(D_n)$ go to infinity. Then D is contained in the union of finitely many of the D_n .*

Proof. The proof is the same as that of lemma 2.3 of [64] but made scheme-theoretic. Suppose $D = \pi_n^{-1} \pi_n(D)$, where $\pi_n(D)$ is a locally closed k -subscheme of $\mathcal{J}_n(X)$. Assume to the contrary that D cannot be covered by a finite subcol-

lection. Choose $k \in \mathbb{N}$ so that $\text{vdim } \mu_X(D_i) > (n+2)d$ for $i > k$, and choose $x_{n+1} \in \pi_{n+1}^{-1}(D \setminus \cup_{i \leq k} D_i)$. Then the (scheme-theoretic) fiber $\pi_{n+1}^{-1}x_{n+1} \subset D$. This set is not covered by finitely many of the D_i . Indeed, $\pi_{n+1}^{-1}(x_{n+1})$ is not covered by the D_i when $i \leq k$. Furthermore, $\pi_{n+1}^{-1}(x_{n+1}) \cap D_i$ has positive codimension in D_i for $i > k$.

Consequently, we can inductively construct a sequence $(x_m)_{m>n}$ such that for every $m > n$ $x_m \in \mathcal{J}_m(X)$, x_{m+1} is above x_m , and $\pi_m^{-1}(x_m)$ is not coverable by finitely many of the D_i . This determines an element $x \in \mathcal{J}_\infty(X)$ such that $\pi_n(x) \in \pi_n(D) \implies x \in D$ because D is stable at least at level n . Since D is covered by the D_i , there is a j such that $x \in D_j$. D_j is stable, and so is stable at least at some level $m > n$. This implies that $\pi_m^{-1}(x_m) \subseteq D_j$, contradicting the fact that $\pi_m^{-1}(x_m)$ is not finitely coverable by the D_i . $\square$

The subsets of $\mathcal{J}_\infty(X)$ that will show up in the integration of Voevodsky motives will not necessarily be stable subschemes; they will be subsets of $\mathcal{J}_\infty(X)$, called measurable subsets, that can be approximated by stable subschemes in the following sense.

Definition 2.3.16. *A subset $C \subseteq \mathcal{J}_\infty(X)$ is good if there is a monotonic sequence (the inclusions are locally closed embeddings of k -schemes) of stable subschemes*

$$C_0 \supseteq \dots \supseteq C_n \supseteq C_{n+1} \supseteq \dots \quad (\text{or } C_0 \subseteq \dots \subseteq C_n \subseteq C_{n+1} \subseteq \dots)$$

of $\mathcal{J}_\infty(X)$ containing (resp. contained in) C , and stable $C_{n,i}$, $i, n \in \mathbb{N}$ such that

$$C_n \setminus C \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i} \quad \left(\text{resp. } C \setminus C_n \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i} \right),$$

and for every n, i ,

$$n \leq \text{vdim } \mu_X(C_{n,i})$$

and

$$\mathrm{vdim} \mu_X(C_{n,i}) \xrightarrow{i \rightarrow \infty} \infty.$$

We then define the volume of C as the object

$$\mu_X(C) := (\mu_X(C_n))_n \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R).$$

We call a pair $(C, \mathcal{S})$ consisting of a subset $C \subseteq \mathcal{J}_\infty(X)$ and a finite decomposition $\mathcal{S}$ ($C = \sqcup_{i \in \mathcal{S}} C_i$, $C_i \subseteq \mathcal{J}_\infty(X)$) measurable if each C_i is a good subset. We then let

$$\mu_X(C, \mathcal{S}) := \bigoplus_{i \in \mathcal{S}} \mu_X(C_i).$$

We view a good subset without a prescribed decomposition as a measurable subset with the trivial decomposition.

Just as in classical measure theory, the question arises as to whether this measure is well-defined (up to equivalence).

Proposition 2.3.17. *This measure μ_X is well-defined up to equivalence, that is, for a fixed measurable subset $(C, \mathcal{S})$, any two sets of data in the above definition give rise to volumes that are equivalent in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$.*

Proof. Since the decomposition is fixed, we may assume without loss of generality that our measurable subset $(C, \mathcal{S})$ is a good subset with the trivial decomposition. Suppose $(C_n)_n$ and $(D_n)_n$ are two monotonic sequences as in the definition above. We are to show that

$$(\mu_X(C_n))_n \simeq (\mu_X(D_n))_n$$

in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$. Suppose first that both are decreasing sequences. Note that $(C_n \cap D_n)_n$ is also a decreasing sequence satisfying the properties in the definition above. Indeed, $(C_n \cap D_n) \setminus C \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i}$. Therefore, we may assume without loss of

generality that for each n , $C_n \subseteq D_n$. If both sequences are increasing, then a similar argument shows that we may assume without loss of generality that $C_n \subseteq D_n$. If $(C_n)_n$ is increasing and $(D_n)_n$ is decreasing, it is then clear that $C_n \subseteq C \subseteq D_n$. Therefore, in all three cases, we may assume without loss of generality that $C_n \subseteq D_n$.

We show that there are equivalences $\mu_X(C_n) \simeq \mu_X(D_n)$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$ that are compatible with each other as n varies. This can be done by first writing $C_n \hookrightarrow D_n$ as the composition of an open embedding $C_n \hookrightarrow K_n$ followed by a closed embedding $K_n \hookrightarrow D_n$, where K_n is the inverse image under π_m of a locally closed k -subvariety of $\mathcal{J}_m(X)$ for some m . Then, we may use the localization sequences. Indeed, in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ we have cofiber sequences

$$\mu_X(C_n) \rightarrow \mu_X(K_n) \rightarrow \mu_X(K_n \setminus C_n) \rightarrow \mu_X(C_n)[1]$$

$$\mu_X(D_n \setminus K_n) \rightarrow \mu_X(D_n) \rightarrow \mu_X(K_n) \rightarrow \mu_X(D_n \setminus K_n)[1]$$

from localization sequences. Note that $K_n \setminus C_n \subseteq D_n \setminus C_n$. If (C_n) and (D_n) are both increasing, then $D_n \setminus C_n \subseteq C \setminus C_n \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i}$. By lemma 2.3.15, there are finitely many $C_{n,i}$ such that

$$K_n \setminus C_n \subseteq \bigcup_{i=1}^N C_{n,i}.$$

Note that $K_n \setminus C_n$ is a stable subscheme of $\mathcal{J}_{\infty}(X)$. Consequently, corollary 2.3.13 implies that the virtual dimension of $\mu_X(K_n \setminus C_n)$ is at least the minimum of the virtual dimensions of the $\mu_X(C_{n,i})$ which is at least n . Therefore, $\mu_X(K_n \setminus C_n) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$. As a result,

$$\mu_X(C_n) \rightarrow \mu_X(K_n)$$

is an equivalence in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$. Similarly, using $D_n \setminus K_n \subseteq C \setminus K_n \subseteq C \setminus C_n$, we deduce that

$$\mu_X(D_n) \rightarrow \mu_X(K_n)$$

is also an equivalence in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$. Consequently, we have equivalences

$$\begin{array}{c} \mu_X(C_n) \\ \downarrow \sim \\ \mu_X(K_n) \\ \uparrow \sim \\ \mu_X(D_n) \end{array}$$

in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$. We can similarly construct morphisms

$$\mu_X(C_n) \rightarrow \mu_X(C_{n+1})$$

that are equivalences in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$. As a result, we have a diagram

$$\begin{array}{ccc} \mu_X(C_n) & \xrightarrow{\sim} & \mu_X(C_{n+1}) \\ \downarrow \sim & & \downarrow \sim \\ \mu_X(K_n) & & \mu_X(K_{n+1}) \\ \uparrow \sim & & \uparrow \sim \\ \mu_X(D_n) & \cdots\cdots\cdots & \mu_X(D_{n+1}) \end{array}$$

in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$. We can fill in the dotted arrows (with necessarily an equivalence in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$) so that the diagram commutes. We have shown that if (C_n) and (D_n) are both increasing, then we can find equivalences $\mu_X(C_n) \xrightarrow{\sim} \mu_X(D_n)$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)/\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(n)$ that are compatible as n varies. Equivalently, we have established the existence of an equivalence $(\mu_X(C_n))_n \simeq (\mu_X(D_n))_n$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$, as required. The other two cases are proved in a similar manner. $\square$

Some examples of measurable subsets of $\mathcal{J}_\infty(X)$ are stable subschemes, subschemes of the form $\mathcal{J}_\infty(Y)$ for any $Y \subseteq X$ a k -subvariety, and pairs of the form $(\pi_n^{-1}Z, \pi_n^{-1}\mathcal{S})$, where $(Z, \mathcal{S})$ is a constructable subset of $\mathcal{J}_n(X)$ for some n with a prescribed finite decomposition $\mathcal{S}$ into locally closed subschemes. Here, if $\mathcal{S} = \{S_1, \dots, S_k\}$, then $\pi_n^{-1}\mathcal{S} := \{\pi_n^{-1}S_1, \dots, \pi_n^{-1}S_k\}$. Another example of a measurable subset of $\mathcal{J}_\infty(X)$ is the *scheme-theoretic* disjoint union of countably many stable subschemes whose virtual dimensions go to ∞ ; see lemma 2.3.19. In the following lemma, we compute the measure of $\mathcal{J}_\infty(Y)$ when Y is a locally closed k -subvariety of X of dimension less than that of X .

Lemma 2.3.18. *Suppose Y is a subvariety of a smooth k -scheme X of strictly positive codimension. Then $\mathcal{J}_\infty(Y)$ is good (and so measurable) with $\mu_X(\mathcal{J}_\infty(Y)) \simeq 0$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$.*

Proof. Note that

$$\mathcal{J}_\infty(Y) = \bigcap_{n=0}^{\infty} \pi_n^{-1}\mathcal{J}_n(Y).$$

which is the intersection of the following decreasing sequence

$$\pi_0^{-1}\mathcal{J}_0(Y) \supseteq \pi_1^{-1}\mathcal{J}_1(Y) \supseteq \dots$$

of stable subschemes of $\mathcal{J}_\infty(X)$ containing $\mathcal{J}_\infty(Y)$.

By the main result of [41], there is a positive integer e such that for all m sufficiently large,

$$\pi_{\lfloor m/e \rfloor}^{-1}\mathcal{J}_\infty(Y) = \pi_{\lfloor m/e \rfloor}^m \mathcal{J}_m(Y).$$

Choose the decreasing sequence

$$\pi_0^{-1}\mathcal{J}_0(Y) \supseteq \pi_e^{-1}\mathcal{J}_e(Y) \supseteq \pi_{2e}^{-1}\mathcal{J}_{2e}(Y) \supseteq \pi_{3e}^{-1}\mathcal{J}_{3e}(Y) \supseteq \dots$$

We show that for each n , $\mu_X(\pi_{ne}^{-1}\mathcal{J}_{ne}(Y)) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(nc)$, where c is the codimension of Y in X . We follow a well-known argument in classical motivic integration. By lemma 4.3 of [30], $\dim \pi_n(\mathcal{J}_\infty(Y)) \leq (n+1) \dim Y$. As a result, for $m \gg 0$

$$\begin{aligned}
\dim \mathcal{J}_m(Y) &\leq \dim \pi_{\lfloor m/e \rfloor}^m \mathcal{J}_m(Y) + (m - \lfloor m/e \rfloor) \dim X \\
&= \dim \pi_{\lfloor m/e \rfloor} \mathcal{J}_\infty(Y) + (m - \lfloor m/e \rfloor) \dim X \\
&\leq (\lfloor m/e \rfloor + 1) \dim Y + (m - \lfloor m/e \rfloor) \dim X \\
&= (m+1) \dim X - (\lfloor m/e \rfloor + 1)(\dim X - \dim Y)
\end{aligned}$$

Using this inequality, for $n \gg 0$

$$\begin{aligned}
\mathrm{vdim} \mu_X(\pi_{ne}^{-1}\mathcal{J}_{ne}(Y)) &= \mathrm{vdim} \pi_{\dagger}^{\mathcal{J}_{ne}(Y)} \mathbf{1}_{\mathcal{J}_{ne}(Y)}((ne+1) \dim X)[2(ne+1) \dim X] \\
&= (ne+1) \dim X - \dim \mathcal{J}_{ne}(Y) \\
&\geq (ne+1) \dim X - (ne+1) \dim X + (n+1)c \\
&= (n+1)c.
\end{aligned}$$

Consequently, $\mu_X(\pi_n^{-1}\mathcal{J}_n(Y)) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)(nc)$. Since by assumption $c \geq 1$, $\mu_X(\mathcal{J}_\infty(Y)) \simeq 0$ in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$. $\square$

Lemma 2.3.19. *Suppose $C = \bigsqcup_{n=0}^\infty C_n$ as subschemes of $\mathcal{J}_\infty(X)$, where C_n are stable subschemes of $\mathcal{J}_\infty(X)$ with $\mathrm{vdim} \mu_X(C_n) \rightarrow \infty$. Then C is measurable and*

$$\mu_X(C) \simeq \bigoplus_{n=0}^\infty \mu_X(C_n).$$

Proof. Choose the sequence $C_0 \subseteq C_0 \sqcup C_1 \subseteq \dots \subseteq C_0 \sqcup C_1 \sqcup \dots \sqcup C_n \subseteq \dots \subseteq C$ of stable subschemes of $\mathcal{J}_\infty(X)$, and use lemma 2.3.14. $\square$

We end this subsection by making the observation that if C is a stable subscheme with a finite (locally closed) decomposition $\mathcal{S}$ into stable subschemes C_i given by $C =$

$\sqcup_{i \in \mathcal{S}} C_i$, then $\mu_X(C)$ is given by a sequence of extensions of $\mu_X(C_i)$, while $\mu_X(C, \mathcal{S}) = \bigoplus_i \mu_X(C_i)$ is given by the trivial extensions. Therefore, *up to a sequence of extensions*, $\mu_X(C)$ and $\mu_X(C, \mathcal{S})$ are the same.

2.3.4 Measurable functions, integrals, and computations

After the previous preparatory sections, we give here the definition of measurable functions and define our integrals. We then do some computations that allow us to anticipate the transformation rule that is at the heart of the theory of integration of Voevodsky motives.

Definition 2.3.20. *A function $F : \mathcal{J}_\infty(X) \rightarrow \mathbb{N}_{\geq 0} \cup \{\infty\}$ is measurable if for each $s \in \mathbb{N}_{\geq 0}$, $F^{-1}(s)$ is measurable and $F^{-1}(\infty)$ has measure 0.*

An important example of a measurable function to us is ord_Y , where $i : Y \hookrightarrow X$ is a closed k -subvariety of X of positive codimension. This function is defined as follows. Given a closed k -subvariety Y of a k -variety X , the associated ideal sheaf $\mathcal{J}_{X/Y}$ fits into a short exact sequence

$$0 \rightarrow \mathcal{J}_{X/Y} \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y \rightarrow 0.$$

Given $\gamma \in \mathcal{J}_\infty(X)$, we may view it as a morphism $\gamma : \text{Spec } k(\gamma)[[t]] \rightarrow X$ or $\gamma^* : \mathcal{O}_X \rightarrow k(\gamma)[[t]]$. Then $\text{ord}_Y(\gamma)$ is defined as the largest number $e \in \mathbb{N}_{\geq 0} \cup \{\infty\}$ such that the composition

$$\mathcal{O}_X \xrightarrow{\gamma^*} k(\gamma)[[t]] \rightarrow k(\gamma)[[t]]/(t^e)$$

sends $\mathcal{J}_{X/Y}$ to zero. The fact that $\text{ord}_Y^{-1}(\infty)$ has measure zero when Y has strictly positive codimension in X immediately follows from lemma 2.3.18.

Definition 2.3.21. *Given a measurable function $F : \mathcal{J}_\infty(X) \rightarrow \mathbb{N}_{\geq 0} \cup \{\infty\}$, define the motivic integral*

$$\int_{X_\infty} \mathbf{1}_k(F)[2F] d\mu_X := \left[\bigoplus_{s=0}^{\infty} \mu_X(F^{-1}(s))(s)[2s] \right]$$

viewed as an element of $\mathcal{M}(k; R)$.

The reason it makes sense for (the image of the convergent motive) $\bigoplus_{s=0}^{\infty} \mu_X(F^{-1}(s))(s)[2s]$ (in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$) to be viewed as an object of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$ is that $\mu_X(F^{-1}(s))(s)[2s] \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)(s)$. Note that $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}, \wedge}(k; R)$ is not closed under arbitrary small colimits just as $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R)$ is not. Also, note that $\mathbf{1}_k(F)[2F]$ is the analogue of functions of the form $\mathbb{L}^{-F}$ in classical motivic integration, and that the classical motivic integral of $\mathbb{L}^{-F}$ is defined by

$$\int_{X_\infty} \mathbb{L}^{-F} d\mu_X := \sum_{s=0}^{\infty} \mu_X(F^{-1}(s)) \mathbb{L}^{-s}$$

as an element of $\widehat{\mathcal{M}}_k$, where μ_X here is the measure in classical motivic integration. Note that in classical motivic integration, one reason we must complete $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ is so that we can talk about infinite sums as above. In our case, we do not complete, but work categorically so that we have a notion of infinite direct sums. This infinite direct sum cannot be taken in a presentable category like $\mathrm{DM}^{\mathrm{eff}}(k; R)$ because passing to K_0 would give us a value group that is zero; this is the reason we consider completed Voevodsky motives. Finally, note that both definitions of integration above are completely analogous to the way Lebesgue integration is defined in real analysis.

Of great importance to us in this chapter is a formula that will allow us to understand how integrals change via resolution of singularities. In fact, in the next

section, we shall prove that if $f : X \rightarrow Y$ is a proper birational morphism of smooth k -varieties with $K_{X/Y}$ its relative canonical divisor, and $D \subseteq Y$ is an effective divisor, then

$$\int_{Y_\infty} \mathbf{1}_k(\text{ord}_D)[2\text{ord}_D]d\mu_Y = \int_{X_\infty} \mathbf{1}_k(\text{ord}_{f^{-1}D+K_{X/Y}})[2\text{ord}_{f^{-1}D+K_{X/Y}}]d\mu_X.$$

This is the analogue of the transformation rule in classical motivic integration:

$$\int_{Y_\infty} \mathbb{L}^{-\text{ord}_D}d\mu_Y = \int_{X_\infty} \mathbb{L}^{-\text{ord}_{f^{-1}D+K_{X/Y}}}d\mu_X.$$

Prior to proving the transformation rule in our setting in the next subsection, we do some computations and compare them to their analogues in classical motivic integration.

Example 2.3.22. *Set X to be a smooth k -variety and $F = \text{ord}_\emptyset$. Then*

$$\int_{X_\infty} \mathbf{1}_k(\text{ord}_\emptyset)[2\text{ord}_\emptyset]d\mu_X = [M(X)].$$

Indeed, $\text{ord}_\emptyset^{-1}(s) = 0$ for $s > 0$ and $\text{ord}_\emptyset^{-1}(0) = \mathcal{J}_\infty(X)$. Therefore,

$$\int_{X_\infty} \mathbf{1}_k(\text{ord}_\emptyset)[2\text{ord}_\emptyset]d\mu_X = \left[\bigoplus_{s=0}^{\infty} \mu_X(\text{ord}_\emptyset^{-1}(s))(s)[2s] \right] = [\mu_X(\mathcal{J}_\infty(X))] = [M(X)]$$

in $\mathcal{M}(k; R)$. This is analogous to the calculation in classical motivic integration that

$$\int_{X_\infty} \mathbb{L}^{-\text{ord}_\emptyset}d\mu_X = [X]$$

in $\widehat{\mathcal{M}}_k$.

Example 2.3.23. Suppose X is a smooth k -variety of dimension d and Y is a smooth divisor. Let n be a positive integer. We claim that

$$\int_{X_\infty} \mathbf{1}_k(\text{nord}_Y)[2\text{nord}_Y]d\mu_X = [M(X \setminus Y)] + [\pi_!^Y \mathbf{1}_Y(d+n)[2(d+n)]]/[M(\mathbb{P}^n)].$$

Indeed, we have $\text{ord}_Y^{-1}(0) = \mathcal{J}_\infty(X \setminus Y)$, and for $s > 0$ $\text{ord}_Y^{-1}(s) = \text{ord}_Y^{-1}([s, \infty]) \setminus \text{ord}_Y^{-1}([s+1, \infty]) = \pi_{s-1}^{-1}\mathcal{J}_{s-1}(Y) \setminus \pi_s^{-1}\mathcal{J}_s(Y) = \pi_s^{-1}((\pi_{s-1}^s)^{-1}(\mathcal{J}_{s-1}(Y)) \setminus \mathcal{J}_s(Y))$. As a result,

$$\begin{aligned} & \int_{X_\infty} \mathbf{1}_k(\text{nord}_Y)[2\text{nord}_Y]d\mu_X \\ &= \left[\bigoplus_{s=0}^{\infty} \mu_X(\text{ord}_Y^{-1}(s))(ns)[2ns] \right] \\ &= \left[\mu_X(\mathcal{J}_\infty(X \setminus Y)) \oplus \bigoplus_{s=1}^{\infty} \mu_X(\pi_s^{-1}((\pi_{s-1}^s)^{-1}(\mathcal{J}_{s-1}(Y)) \setminus \mathcal{J}_s(Y)))(ns)[2ns] \right] \\ &= [\pi_!^{X \setminus Y} \mathbf{1}_{X \setminus Y}(d)[2d]] \\ &+ \left[\bigoplus_{s=1}^{\infty} \pi_!^{(\pi_{s-1}^s)^{-1}(\mathcal{J}_{s-1}(Y)) \setminus \mathcal{J}_s(Y)} \mathbf{1}_{(\pi_{s-1}^s)^{-1}(\mathcal{J}_{s-1}(Y)) \setminus \mathcal{J}_s(Y)}((s+1)d)[2(s+1)d](ns)[2ns] \right] \\ &= [M(X \setminus Y)] + \left[\bigoplus_{s=1}^{\infty} \pi_!^{(\pi_{s-1}^s)^{-1}(\mathcal{J}_{s-1}(Y))} \mathbf{1}_{(\pi_{s-1}^s)^{-1}(\mathcal{J}_{s-1}(Y))}((s+1)d)[2(s+1)d](ns)[2ns] \right] \\ &- \left[\bigoplus_{s=1}^{\infty} \pi_!^{\mathcal{J}_s(Y)} \mathbf{1}_{\mathcal{J}_s(Y)}((s+1)d)[2(s+1)d](ns)[2ns] \right] \\ &= [M(X \setminus Y)] + \left[\pi_!^Y(d-1)[2(d-1)]\mathbf{1}_Y \otimes \bigoplus_{s=1}^{\infty} \mathbf{1}_k((n+1)s)[2(n+1)s] \right] \\ &- \left[\pi_!^Y \mathbf{1}_Y(d-1)[2(d-1)] \otimes \bigoplus_{s=1}^{\infty} \mathbf{1}_k((n+1)s)[2(n+1)s](1)[2] \right] \\ &= [M(X \setminus Y)] + \left[\pi_!^Y \mathbf{1}_Y(d-1)[2(d-1)] \otimes \pi_!^{\mathbb{G}^m} \mathbf{1}_{\mathbb{G}^m}(1)[2] \otimes \bigoplus_{s=1}^{\infty} \mathbf{1}_k((n+1)s)[2(n+1)s] \right] \\ &= [M(X \setminus Y)] + [\pi_!^Y \mathbf{1}_Y(d+n)[2(d+n)]]/[M(\mathbb{P}^n)]. \end{aligned}$$

If $n = 1$ and Y is a divisor, then this is analogous to the computation in classical motivic integration that

$$\int_{X_\infty} \mathbb{L}^{-\text{ord}_Y} d\mu_X = [X \setminus Y] + [Y]/[\mathbb{P}^1].$$

Example 2.3.24. Let $\pi : \tilde{X} \rightarrow X$ be the blowup morphism of X along a smooth center Y of codimension c . Let $K_{\tilde{X}/X}$ be the canonical divisor of π . We claim that

$$\int_{\tilde{X}_\infty} \mathbf{1}_k(\text{ord}_{K_{\tilde{X}/X}})[2\text{ord}_{K_{\tilde{X}/X}}] d\mu_{\tilde{X}} = [M(X)].$$

Indeed, let E be the exceptional divisor of the blowup. Then $K_{\tilde{X}/X} = (c-1)E$. Using the computations in the previous example, we obtain

$$\begin{aligned} & \int_{\tilde{X}_\infty} \mathbf{1}_k(\text{ord}_{K_{\tilde{X}/X}})[2\text{ord}_{K_{\tilde{X}/X}}] d\mu_{\tilde{X}} \\ &= \int_{\tilde{X}_\infty} \mathbf{1}_k(\text{ord}_{(c-1)E})[2\text{ord}_{(c-1)E}] d\mu_{\tilde{X}} \\ &= [M(\tilde{X} \setminus E)] + [\pi_!^E \mathbf{1}_E(d+c-1)[2(d+c-1)]]/[M(\mathbb{P}^{c-1})] \\ &= [M(\tilde{X} \setminus E)] + [\pi_!^{Y \times \mathbb{P}^{c-1}} \mathbf{1}_{Y \times \mathbb{P}^{c-1}}(d+c-1)[2(d+c-1)]]/[M(\mathbb{P}^{c-1})] \\ &= [M(\tilde{X} \setminus E)] + [\pi_!^Y \mathbf{1}_Y(d)[2d]][M(\mathbb{P}^{c-1})]/[M(\mathbb{P}^{c-1})] \\ &= [M(X \setminus Y)] + [\pi_!^Y \mathbf{1}_Y(d)[2d]] = [M(X)]. \end{aligned}$$

This is analogous to the formula in classical motivic integration that

$$\int_{\tilde{X}_\infty} \mathbb{L}^{-\text{ord}_{K_{\tilde{X}/X}}} d\mu_{\tilde{X}} = [X].$$

This example, in combination with example 2.3.22 gives us

$$\int_{\tilde{X}_\infty} \mathbf{1}_k(\text{ord}_{K_{\tilde{X}/X}})[2\text{ord}_{K_{\tilde{X}/X}}] d\mu_{\tilde{X}} = \int_{X_\infty} \mathbf{1}_k(\text{ord}_\emptyset)[2\text{ord}_\emptyset] d\mu_X,$$

which is analogous to the equality

$$\int_{\tilde{X}_\infty} \mathbb{L}^{-\text{ord}_{K_{\tilde{X}/X}}} d\mu_{\tilde{X}} = \int_{X_\infty} \mathbb{L}^{-\text{ord}_\emptyset} d\mu_X$$

in classical motivic integration. In the next subsection, we generalize this last example.

2.3.5 The transformation rule

The aim of this subsection is to prove the following transformation rule. Recall that we are assuming that either k admits resolution of singularities or the characteristic exponent of k is invertible in the coefficient ring R . The main reason we make this assumption is because as of today, we do not have resolution of singularities in positive characteristics. The recent work of Hironaka claims a proof which is to be verified [45].

Theorem 2.3.25. *Suppose $f : X \rightarrow Y$ is a proper birational morphism of smooth k -varieties with $K_{X/Y}$ its relative canonical divisor, and let $D \subseteq Y$ be an effective divisor. Then*

$$\int_{Y_\infty} \mathbf{1}_k(\text{ord}_D)[2\text{ord}_D] d\mu_Y = \int_{X_\infty} \mathbf{1}_k(\text{ord}_{f^{-1}D+K_{X/Y}})[2\text{ord}_{f^{-1}D+K_{X/Y}}] d\mu_X.$$

As we will shortly see, this is an analogue of the change of variables formula in calculus. In the next section, this formula will be crucial to our applications regarding K-equivalent varieties.

Prior to proving the transformation rule above, we recall the first fundamental exact sequence for Kähler differentials and related topics.

Given a morphism of smooth k -schemes $f : X' \rightarrow X$, we have an $\mathcal{O}_{X'}$ -linear morphism $f^*\Omega_X \xrightarrow{df} \Omega_{X'}$ (of locally free sheaves on X') that fits into the exact sequence

$$f^*\Omega_X \xrightarrow{df} \Omega_{X'} \rightarrow \Omega_{X'/X} \rightarrow 0$$

of locally free sheaves. Suppose now that f is *birational* and let $d = \dim X = \dim X'$ as before. Since f is birational, X and Y have the same function fields. The stalk of $\Omega_{X'/X}$ at the generic point is isomorphic to $\Omega_{K(X')/K(X)} = 0$. Therefore, taking stalks at the generic point for $f^*\Omega_X \xrightarrow{df} \Omega_{X'}$ gives a surjection of finite-dimensional vector spaces of the same dimension (X and X' are birational), and so an isomorphism. Therefore, when f is birational, we obtain the short exact sequence

$$0 \rightarrow f^*\Omega_X \xrightarrow{df} \Omega_{X'} \rightarrow \Omega_{X'/X} \rightarrow 0.$$

Taking the d th exterior power of df , we obtain the exact sequence

$$0 \rightarrow f^*\omega_X \rightarrow \omega_{X'}$$

of line bundles, where $\omega_X = \Omega_X^d$ and $\omega_{X'} = \Omega_{X'}^d$ denote the canonical bundles of X and X' (over k). Tensoring by $\omega_{X'}^{-1}$, we obtain the exact sequence

$$0 \rightarrow f^*\omega_X \otimes \omega_{X'}^{-1} \rightarrow \mathcal{O}_{X'}$$

of line bundles. Therefore, we may view $f^*\omega_X \otimes \omega_{X'}^{-1}$ as a locally principal ideal of $\mathcal{O}_{X'}$, which we denote by $J_{X'/X}$. Let $K_{X'/X}$ be the Cartier divisor given locally by the vanishing of $J_{X'/X}$. This is called the relative canonical divisor of f . Note that $J_{X'/X}$ is locally given by the vanishing of $\det df$.

Suppose now that $L|k$ is a field extension. Let $\gamma : \operatorname{Spec} L[[t]] \rightarrow X'$ be an L -

point of $\mathcal{J}_\infty(X')$ with $\text{ord}_{K_{X'/X}}(\gamma) = e$. This is equivalent to $\gamma^* J_{X'/X} = (t^e) \subseteq L[[t]]$. Since $J_{X'/X}$ is locally defined by the vanishing of $\det df$, this is equivalent to $\gamma^*(\det df) = (t^e)$, or equivalently, $\det(\gamma^*(df)) = (t^e)$.

The pillar on which the proof of the transformation rule rests is the following proposition due to Denef and Loeser. From context, it will be clear what π_m and π_m^n mean; we abuse notation in the following lemma.

Proposition 2.3.26. *(Lemma 3.4 of [30]) Suppose $f : X' \rightarrow X$ is a proper birational morphism of smooth k -varieties. Let $C'_e := \text{ord}_{K_{X'/X}}^{-1}(e)$, where $K_{X'/X}$ is the relative canonical divisor of f . Let $C_e := f_\infty C'_e$. Let $L|k$ be a field extension. Let $\gamma \in C'_e(L)$ be an L -point, that is, a map $\gamma^* : \mathcal{O}_{X'} \rightarrow L[[t]]$ such that $\gamma^*(J_{X'/X}) = (t^e)$. Then for $m \geq 2e$ one has:*

- (a) *For all $\xi \in \mathcal{J}_\infty(X)$ such that $\pi_m(\xi) = f_m(\pi_m(\gamma))$, there is $\gamma' \in \mathcal{J}_\infty(X')$ such that $f_\infty(\gamma') = \xi$ and $\pi_{m-e}(\gamma') = \pi_{m-e}(\gamma)$. In particular, the fiber of f_m over $f_m(\gamma_m)$ lies in the fiber of π_{m-e}^m over $\pi_{m-e}(\gamma)$.*
- (b) *$\pi_m(C'_e)$ is a union of fibers of f_m .*
- (c) *The map $f_m : \pi_m(C'_e) \rightarrow \pi_m(C_e)$ is a piecewise trivial $\mathbb{A}^e$ -fibration.*

For the proof of the transformation rule, we also need the following two lemmas.

Lemma 2.3.27. *If $f : X' \rightarrow X$ is a proper birational morphism of smooth k -varieties, then away from a measure zero subscheme, $f_\infty : \mathcal{J}_\infty(X') \rightarrow \mathcal{J}_\infty(X)$ is bijective.*

Proof. The proof is more or less that given in classical motivic integration. Let $Z \subset X'$ be a proper closed k -subvariety on whose complement f is an isomorphism. Given $\gamma \in \mathcal{J}_\infty(X)$, we denote by $k(\gamma)$ its residue field. Then γ can be viewed as a morphism $\gamma : \text{Spec } k(\gamma)[[t]] \rightarrow X$. We show that any γ which does not lie entirely in $f(Z)$ uniquely lifts to an arc in X' . We apply the valuative criterion for properness

to show this. Consider the following diagram.

$$\begin{array}{ccccc}
\mathrm{Spec} (k(\gamma)[[t]]_{(0)}) & \xrightarrow{\quad \cdot \quad} & X' & \xleftarrow{\quad} & X' \setminus Z \\
\downarrow & \nearrow & \downarrow & \searrow & \downarrow \simeq \\
\mathrm{Spec} (k(\gamma)[[t]]) & \xrightarrow{\quad \gamma \quad} & X & \xleftarrow{\quad} & X \setminus f(Z).
\end{array}$$

By assumption, the generic point $\mathrm{Spec} (k(\gamma)[[t]]_{(0)})$ is in $X \setminus f(Z)$. Since f is an isomorphism from $X' \setminus Z \rightarrow X \setminus f(Z)$, this generic point lifts to X' uniquely (the upper left horizontal dashed arrow). f is proper, and so the valuative criterion for properness yields the unique existence of the dotted arrow $\mathrm{Spec} (k(\gamma)[[t]]) \rightarrow X'$. Consequently, the morphism

$$f_\infty : \mathcal{J}_\infty(X') \setminus \mathcal{J}_\infty(Z) \rightarrow \mathcal{J}_\infty(X) \setminus \mathcal{J}_\infty(f(Z)),$$

of k -schemes is bijective. □

Lemma 2.3.28. *If $f : X' \rightarrow X$ is a proper birational morphism of smooth k -varieties, then for every m , $f_m : \mathcal{J}_m(X') \rightarrow \mathcal{J}_m(X)$ is surjective.*

Proof. Suppose $Z \subset X$ is a proper closed k -subvariety such that $f : X' \setminus f^{-1}(Z) \rightarrow X \setminus Z$ is an isomorphism of schemes. From the previous lemma 2.3.27, we know that if $\gamma \notin \mathcal{J}_\infty(Z)$, then it lies in the image of f_∞ . If $\gamma \in \mathcal{J}_m(X)$, then $\pi_m^{-1}(\gamma)$ cannot be contained entirely in $\mathcal{J}_\infty(Z)$ because the latter has measure 0. As a result, there is an element in $\pi_m^{-1}(\gamma)$ that is in the image of f_∞ . In particular, there is an element in $\mathcal{J}_m(X')$ that maps to $\gamma \in \mathcal{J}_m(X)$, and so f_m is surjective. □

Proof of the transformation rule. Using the above proposition and lemmas, we deduce the transformation rule. Let $C'_{\leq e} = \mathrm{ord}_{K_{X'/X}}^{-1}([0, e])$ and let $C'_e = \mathrm{ord}_{K_{X'/X}}^{-1}(e)$. Note that $\mathcal{J}_\infty(K_{X'/X})$ has measure 0 by lemma 2.3.18. Up to a measure 0 subscheme,

therefore,

$$C'_{\leq 0} \subseteq C'_{\leq 1} \subseteq \dots \subseteq \mathcal{J}_\infty(X')$$

is a filtration of $\mathcal{J}_\infty(X')$ by locally closed subschemes. We can refine the filtration into even smaller locally closed subschemes according to the order of contact along $f^{-1}D$: set

$$C'_{\leq e, \geq k} := C'_{\leq e} \cap \text{ord}_{f^{-1}D}^{-1}([k, \infty]) \text{ and } C_{\leq e, \geq k} := f_\infty(C'_{\leq e, \geq k}).$$

We can similarly define $C'_{e, \geq k}$, $C_{e, \geq k}$, $C'_{e, k}$, and $C_{e, k}$. We have the filtration (up to measure zero) of $\text{ord}_D^{-1}([k, \infty])$:

$$(C_{\leq 0, \geq k}, \mathcal{S}_{\leq 0, \geq k}) \subseteq (C_{\leq 1, \geq k}, \mathcal{S}_{\leq 1, \geq k}) \subseteq \dots \subseteq (C_{\leq n, \geq k}, \mathcal{S}_{\leq n, \geq k}) \subseteq \dots \subseteq \text{ord}_D^{-1}([k, \infty]),$$

where the $\mathcal{S}_{\leq e, \geq k}$, as $e \geq 0$ varies, form an increasing sequence of finite decompositions of the constructable subsets $C_{\leq e, \geq k}$ into stable subschemes. By this assumption, we can define $\mathcal{S}_{e, \geq k} := \mathcal{S}_{\leq e, \geq k} \setminus \mathcal{S}_{\leq e-1, \geq k}$. For each pair $(k, k+1)$, we can take refinements $\mathcal{S}_{e, \geq k}^{tr}$ of $\mathcal{S}_{e, \geq k}$ so that $\mathcal{S}_{e, \geq k+1}^{tr} \subseteq \mathcal{S}_{e, \geq k}^{tr}$. Define $\mathcal{S}_{e, k} := \mathcal{S}_{e, \geq k}^{tr} \setminus \mathcal{S}_{e, \geq k+1}^{tr}$. Though the notation may suggest otherwise, the refinements chosen for the pairs $(k-1, k)$ and $(k, k+1)$ need not be compatible. Define $\mathcal{S}_{\leq e, \geq k}^{tr}$ to be $\cup_{i \leq e} \mathcal{S}_{i, \geq k}^{tr}$.

Using proposition 2.3.26, we know that $C_{e, k}$ are constructable subsets of $\mathcal{J}_\infty(X)$. Note that $f_\infty : C'_{e, k} \rightarrow C_{e, k}$ is a piecewise trivial $\mathbb{A}^e$ -bundle by proposition 2.3.26. Since $C_{e, k}$ is constructable, choose a finite decomposition into stable subschemes, say $\mathcal{S} = \{C_i\}_i$, such that atop each C_i , $f_\infty : C'_{e, k} \rightarrow C_{e, k}$ is a trivial $\mathbb{A}^e$ -bundle. Furthermore, $\mathcal{S}' := \{f_\infty^{-1}(C_i)\}_i$ is a decomposition of $C'_{e, k}$ into stable subschemes. We may assume without loss of generality that above each object of $\mathcal{S}_{e, k}$, f_∞ is an $\mathbb{A}^e$ -bundle.

Consider the filtration

$$(C_{\leq 0, \geq k}, \mathcal{S}_{\leq 0, \geq k}^{tr}) \subseteq (C_{\leq 1, \geq k}, \mathcal{S}_{\leq 1, \geq k}^{tr}) \subseteq \dots \subseteq (C_{\leq n, \geq k}, \mathcal{S}_{\leq n, \geq k}^{tr}) \subseteq \dots \subseteq \text{ord}_D^{-1}([k, \infty]),$$

From this filtration, we obtain the tower

$$\bigoplus_k \mu_X(C_{\leq 0, \geq k}, \mathcal{S}_{\leq 0, \geq k}^{tr})(k)[2k] \rightarrow \bigoplus_k \mu_X(C_{\leq 1, \geq k}, \mathcal{S}_{\leq 1, \geq k}^{tr})(k)[2k] \rightarrow \dots$$

Note that $\mu_X(\text{ord}_D^{-1}[k, \infty])$ is a sequential colimit of measures of finite unions of objects in $\cup_e \mathcal{S}_{e, \geq k}^{tr}$. The reason this is so is because $\text{ord}_D^{-1}[k, \infty]$ is a closed subset of $\mathcal{J}_\infty(X)$, and the objects of $\cup_e \mathcal{S}_{e, \geq k}^{tr}$ are stable subschemes covering it. Indeed, the closure of each object of $\cup_e \mathcal{S}_{e, \geq k}^{tr}$ (which is locally closed) is stable and contained in $\text{ord}_D^{-1}([k, \infty])$ which is the union of the stable subschemes in $\cup_e \mathcal{S}_{e, \geq k}^{tr}$ whose virtual dimensions go to infinity. By lemma 2.3.15, the closure of each object of $\cup_e \mathcal{S}_{e, \geq k}^{tr}$ is contained in the union of finitely many of the objects of $\cup_e \mathcal{S}_{e, \geq k}^{tr}$. Furthermore, the measures of the objects of $\mathcal{S}_{e, \geq k}^{tr}$ go to 0 as e goes to infinity. This implies that we can assume without loss of generality that we have an ordering on the objects of $\cup_e \mathcal{S}_{e, \geq k}^{tr}$ so that $\text{ord}_D^{-1}([k, \infty])$ is a sequential colimit of measures of finite unions of the first N objects in $\cup_e \mathcal{S}_{e, \geq k}^{tr}$ as N varies.

The above tower has slices $\bigoplus_k \mu_X(C_{e, \geq k}, \mathcal{S}_{e, \geq k}^{tr})(k)[2k]$ with e varying. Therefore, in

$\mathcal{M}(k; R)$, we have

$$\begin{aligned}
\left[\bigoplus_k \mu_X(\text{ord}_D^{-1}(k))(k)[2k] \right] &= \left[\text{colim}_e \bigoplus_k \mu_X(C_{\leq e, \geq k}, \mathcal{S}_{\leq e, \geq k}^{tr})(k)[2k] \right] \\
&- \left[\text{colim}_e \bigoplus_k \mu_X(C_{\leq e, \geq k+1}, \mathcal{S}_{\leq e, \geq k+1}^{tr})(k)[2k] \right] \\
&= \left[\bigoplus_k \bigoplus_e \mu_X(C_{e, \geq k}, \mathcal{S}_{e, \geq k}^{tr})(k)[2k] \right] \\
&- \left[\bigoplus_k \bigoplus_e \mu_X(C_{e, \geq k+1}, \mathcal{S}_{e, \geq k+1}^{tr})(k)[2k] \right] \\
&= \left[\bigoplus_k \bigoplus_e \mu_X(C_{e, k}, \mathcal{S}_{e, \geq k}^{tr} \setminus \mathcal{S}_{e, \geq k+1}^{tr})(k)[2k] \right] \\
&= \left[\bigoplus_k \bigoplus_e \mu_X(C_{e, k}, \mathcal{S}_{e, k})(k)[2k] \right].
\end{aligned}$$

By assumption, atop each object of $\mathcal{S}_{e, k}$, f_∞ is an $\mathbb{A}^e$ -bundle. Consequently,

$$\mu_{X'}(C'_{e, k}, f_\infty^{-1}\mathcal{S}_{e, k})(e)[2e] = \bigoplus_{C \in \mathcal{S}_{e, k}} \mu_{X'}(f_\infty^{-1}(C))(e)[2e] \simeq \bigoplus_{C \in \mathcal{S}_{e, k}} \mu_X(C) = \mu_X(C_{e, k}, \mathcal{S}_{e, k}).$$

From this, we obtain

$$\begin{aligned}
\int_{X_\infty} \mathbf{1}_k(\text{ord}_D)[2\text{ord}_D] d\mu_X &= \left[\bigoplus_k \mu_X(\text{ord}_D^{-1}(k))(k)[2k] \right] \\
&= \left[\bigoplus_k \left(\bigoplus_e \mu_X(C_{e, k}, \mathcal{S}_{e, k}) \right) (k)[2k] \right] \\
&= \left[\bigoplus_{k, e} \mu_{X'}(C'_{e, k}, f_\infty^{-1}(\mathcal{S}_{e, k}))(k+e)[2(k+e)] \right] \\
&= \left[\bigoplus_t \left(\bigoplus_{k+e=t} \mu_{X'}(C'_{e, k}, f_\infty^{-1}(\mathcal{S}_{e, k})) \right) (t)[2t] \right] \\
&= \left[\bigoplus_t \mu_{X'}(\text{ord}_{f^{-1}D+K_{X'/X}}^{-1})(t)[2t] \right] \\
&= \int_{X'_\infty} \mathbf{1}_k(\text{ord}_{f^{-1}D+K_{X'/X}})[2\text{ord}_{f^{-1}D+K_{X'/X}}] d\mu_{X'},
\end{aligned}$$

as required. □

2.4 Applications

Now that we have laid the foundations for the integration of Voevodsky motives, we apply it to answer a few questions related to the number theory and geometry of K-equivalent varieties. In particular, we obtain new results in the case of birational Calabi-Yau varieties and birational minimal varieties.

2.4.1 Classes of motives of K-equivalent varieties

In this subsection, we prove the following theorem using our theory of motivic integration.

Theorem 2.4.1. *Suppose X and Y K-equivalent smooth k -varieties. Then in $K_0(\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; R))$, $[M(X)] = [M(Y)]$.*

Proof. Since X and Y are K-equivalent, there is a smooth k -variety Z as well as proper birational morphisms $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ such that $f^*K_X = g^*K_Y$:

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & \cdots\cdots\cdots & Y. \end{array}$$

Applying the transformation rule to each of f and g , we obtain

$$[M(X)] = \int_{X_\infty} \mathbf{1}_k(\mathrm{ord}_\emptyset)[2\mathrm{ord}_\emptyset] d\mu_X = \int_{Z_\infty} \mathbf{1}_k(\mathrm{ord}_{K_{Z/X}})[2\mathrm{ord}_{K_{Z/X}}] d\mu_Z$$

and

$$[M(X)] = \int_{Y_\infty} \mathbf{1}_k(\text{ord}_\emptyset)[2\text{ord}_\emptyset]d\mu_Y = \int_{Z_\infty} \mathbf{1}_k(\text{ord}_{K_{Z/Y}})[2\text{ord}_{K_{Z/Y}}]d\mu_Z.$$

Since X and Y are K -equivalent,

$$K_{Z/X} = K_Z - f^*K_X = K_Z - g^*K_Y = K_{Z/Y},$$

and so

$$[M(X)] = \int_{Z_\infty} \mathbf{1}_k(\text{ord}_{K_{Z/X}})[2\text{ord}_{K_{Z/X}}]d\mu_Z = \int_{Z_\infty} \mathbf{1}_k(\text{ord}_{K_{Z/Y}})[2\text{ord}_{K_{Z/Y}}]d\mu_Z = [M(Y)]$$

in $\mathcal{M}(k; R)$. However, by proposition 2.3.8 $[M(X)] = [M(Y)]$ in $K_0(\text{DM}_{\text{gm}}^{\text{eff}}(k; R))$, as required. $\square$

When X and Y are smooth projective k -varieties, we have the following stronger result.

Theorem 2.4.2. *Suppose X and Y are K -equivalent smooth projective k -varieties, then there is an effective Chow motive $P \in \text{Chow}^{\text{eff}}(k; R)$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k; R)^{op} \hookrightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k; R)$.*

Proof. By proposition 3.1.3 of Bondarko's paper [20], we know that the inclusion $\text{Chow}^{\text{eff}}(k; R)^{op} \hookrightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k; R)$ induces an isomorphism after taking K_0 . As a consequence of theorem 2.4.1 and the fact that X and Y are smooth projective, $[M(X)] = [M(Y)]$ in $\text{Chow}^{\text{eff}}(k; R)$, from which it follows that there is an effective Chow motive $P \in \text{Chow}^{\text{eff}}(k; R)$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k; R)^{op} \hookrightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k; R)$. $\square$

This gives a good partial answer to the following conjecture of Chin-Lung Wang [88]. Wang conjectured it for complex varieties, while we conjecture it in general.

Conjecture 2.4.3. *If X and Y are K -equivalent smooth projective k -varieties, then $M(X) \simeq M(Y)$, that is, they have equivalent integral Chow motives.*

Birational compact hyperkähler manifolds have isomorphic Hodge structures. In particular, they have the same Hodge and Betti numbers. Furthermore, they have isomorphic integral singular cohomology *rings*. However, the isomorphism between the integral singular cohomology *rings* is not only the result of them being K -equivalent smooth projective complex varieties. It is not in general true that K -equivalent smooth projective complex varieties have isomorphic integral cohomology *rings*. For example, Nam-Hoon Lee and Keiji Oguiso have jointly constructed birational Calabi-Yau complex threefolds with non-isomorphic integral singular cohomology *rings* [60]. It has been conjectured that if we take into account quantum corrections, then the (quantum) cohomology rings are isomorphic, that is, K -equivalent smooth projective complex varieties have isomorphic quantum cohomology rings in the extended Kähler moduli space. In the hyperkähler case, the usual cup product coincides with its quantum-corrected cup product which gives the isomorphism of singular cohomology rings. This begs the question of how can one bring in the data of quantum corrections into our theory of motivic integration?

In three dimensions, Reid and Mori's classification of three dimensional singularities has allowed Kollár and Mori to completely understand the relation between birational minimal models. Via a complete classification of three dimensional flops and flips, it can be shown that three dimensional birational minimal models have isomorphic integral singular (and intersection) cohomology groups and mixed (and intersection pure) Hodge structures. These integral results point to the possibility that the above conjecture may even be valid integrally.

As a small corollary to theorem 2.4.2, let us mention the following which is a shadow of the proof underlying Batyrev's proof of the equality of Betti numbers of birational Calabi-Yau complex varieties.

Corollary 2.4.4. *If X and Y are K -equivalent smooth projective k -varieties, then their rational numerical motives are isomorphic.*

Proof. The proof is a simple consequence of theorem 2.4.2 and the fact that rational numerical motives form a semisimple abelian category [54]. $\square$

As another corollary to theorem 2.4.2, we obtain the following.

Corollary 2.4.5. *If k admits resolution of singularities and X and Y are birational smooth projective Calabi-Yau k -varieties, then there is an effective integral Chow motive $P \in \text{Chow}^{\text{eff}}(k; \mathbb{Z})$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k; \mathbb{Z}) \hookrightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z})$.*

Proof. If X and Y are birational smooth projective Calabi-Yau k -varieties, choose a common resolution of singularities (possible since k admits resolution of singularities)

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & \cdots\cdots\cdots & Y. \end{array}$$

Since $K_X = 0$ and $K_Y = 0$, two such birational Calabi-Yau k -varieties are therefore K -equivalent. $\square$

Remark 2.4.6. *The same corollary is true more generally for birational smooth projective k -varieties with nef canonical bundles.*

An advantage of obtaining motivic results is that through realization functors we can deduce results of interest to geometers and number theorists. Before mentioning

some concrete implications of the above motivic results, we make a few comments regarding some important realization functors.

1. Let $\text{Vec}_{\mathbb{Q}_\ell}$ and $\text{Vec}_{\mathbb{Q}}$ be the category of finite dimensional $\mathbb{Q}_\ell$ - and $\mathbb{Q}$ -vector spaces. For each ℓ prime to the characteristic of the field k , there is an ℓ -adic realization functor $r_{\mathbb{Q}_\ell} : \text{DM}_{\text{gm}}(k; \mathbb{Q}) \rightarrow D^b(\text{Vec}_{\mathbb{Q}_\ell})$ induced by sending a scheme X to $H_{\text{ét}}^*(X_{\bar{k}}; \mathbb{Q}_\ell)^*$. If k is a field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$, we have the Betti realization functor $r_\sigma : \text{DM}_{\text{gm}}(k; \mathbb{Q}) \rightarrow D^b(\text{Vec}_{\mathbb{Q}})$ induced by sending scheme X to $H_*(X^{\text{an}}; \mathbb{Q})$. There is also an integral contravariant version of the Betti realization

$$B_\sigma : \text{DM}_{\text{gm}}^{\text{gm}}(k; \mathbb{Z})^{\text{op}} \rightarrow D^b(\mathbb{Z})$$

given by sending $M(X)$ to $H^*(X^{\text{an}}; \mathbb{Z})$, which we denote by B_ℓ . See, for example, Lecomte's work [58].

2. (ℓ -adic realization) Voevodsky's triangulated or stable ∞ -category of geometric effective motives $\text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z})$ is based on the Nisnevich topology, a topology coarser than the étale topology. For each n , reducing modulo ℓ^n and sheafifying with respect to the étale topology gives us functors

$$\text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z}) \rightarrow \text{DM}_{\text{gm}, \text{ét}}^{\text{eff}}(k; \mathbb{Z}/\ell^n) \hookrightarrow \text{DM}_{\text{ét}}^{\text{eff}}(k; \mathbb{Z}/\ell^n).$$

By rigidity (see Cisinski-Dégli's [24] for the case that k does not have finite cohomological dimension), the latter category is equivalent to the derived category of sheaves of $\mathbb{Z}/\ell^n$ -modules on the small étale site of k , that is $D(k_{\text{ét}}, \mathbb{Z}/\ell^n)$. These functors assemble into the ℓ -adic realization functor

$$\text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z}) \rightarrow \widehat{D}(k_{\text{ét}}; \mathbb{Z}_\ell),$$

where $\widehat{D}(k_{\text{ét}}; \mathbb{Z}_\ell)$ is the derived category of ℓ -adic sheaves. This latter category is equivalent to the derived category of continuous ℓ -adic Galois representations $D(\text{Rep}_{\text{cnt}}(G_k; \mathbb{Z}_\ell))$. We know by work of Cisinski and Déglise that the above functor factors through the bounded derived category of constructible ℓ -adic sheaves. We can also rationalize everything if we want to work over $\mathbb{Q}_\ell$.

3. (Hodge realization) Recall that an integral mixed Hodge structure consists of a finitely generated abelian group $V_{\mathbb{Z}}$ together with a finite increasing filtration W_i on $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes \mathbb{Q}$, called a weight filtration, and a finite decreasing filtration F^p on $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C}$, called a Hodge filtration, with some compatibility conditions. The importance is that this is an abstraction of the algebraic structure one obtains from Hodge theory on integral cohomology groups of compact Kähler manifolds. It is well-known that the integral homology of every complex variety carries a canonical integral Hodge structure [38]. There is an abelian category of integral mixed Hodge structures $\text{MHS}_{\mathbb{Z}}$. Consider the category $\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Z})$ of geometric Voevodsky motives over $\text{Spec } \mathbb{C}$. Lecomte and Wach [59] have constructed an integral Hodge realization functor

$$\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Z})^{\text{op}} \rightarrow D^b(\text{MHS}_{\mathbb{Z}})$$

to the derived category of integral mixed Hodge structures. It sends $M(X)$ to $H^*(X^{\text{an}}; \mathbb{Z})$ with its canonical integral mixed Hodge structure. We could also rationalize to pass to (polarizable) rational mixed Hodge structures.

As a corollary to theorem 2.4.2, we have the following result.

Theorem 2.4.7. *If X and Y are K -equivalent smooth projective k -varieties with k of characteristic zero with $\sigma : k \hookrightarrow \mathbb{C}$, then $H^*(X^{\text{an}}; \mathbb{Z}) \simeq H^*(Y^{\text{an}}; \mathbb{Z})$ as graded abelian groups.*

Proof. Indeed, there is the Betti realization functor $B_\sigma : \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k; \mathbb{Z}) \rightarrow D^b(\mathbb{Z})$ into the bounded derived category of finitely generated abelian groups. This is given by sending $M(X)$ to the graded abelian group $\bigoplus_n H^n(X^{an}; \mathbb{Z})[-n] \in D^b(\mathbb{Z})$. Applying theorem 2.4.2, we deduce that $H^*(X^{an}; \mathbb{Z}) \oplus H^*(B_\sigma(P)) \simeq H^*(Y^{an}; \mathbb{Z}) \oplus H^*(B_\sigma(P))$ as finitely generated abelian groups. Since we are landing in finitely generated abelian groups, the classification of finitely generated abelian groups allows us to cancel $H^*(B_\sigma(P))$ from both sides to obtain $H^*(X^{an}; \mathbb{Z}) \simeq H^*(Y^{an}; \mathbb{Z})$ as graded abelian groups. $\square$

Remark 2.4.8. *The cohomology rings need not be isomorphic. In fact, Nam-Hoon Lee and Keiji Ogusio have jointly constructed birational Calabi-Yau complex threefolds with non-isomorphic integral singular cohomology rings [60]. However, for birational smooth projective Calabi-Yau complex manifolds X and Y , Mark McLean has claimed a proof of this by identifying the quantum cohomologies of X and Y up to tensoring with a suitable Novikov ring. His proof uses Hamiltonian Floer cohomology.*

From theorem 2.4.1, X and Y that are K -equivalent smooth k -varieties satisfy $[M(X)] = [M(Y)]$ at least in $K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Z}[1/p]))$, where p is the exponent characteristic of k . Applying the ℓ -adic realization functor described above, we deduce that X and Y have the same ℓ -adic Galois representations (up to semi-simplification). In particular, if $k = \mathbb{F}_q$, then taking traces of the Frobenius we deduce that they have the same zeta functions: $\zeta_X(t) = \zeta_Y(t)$. Summarizing, we have

Theorem 2.4.9. *If X and Y are K -equivalent smooth k -varieties, then they have the same ℓ -adic Galois representations (up to semi-simplification). In particular, in the case $k = \mathbb{F}_q$, two such $\mathbb{F}_q$ -varieties have the same zeta functions. If $\mathbb{F}_q$ admits resolution of singularities, then all this is true for X and Y that are birationally equivalent smooth projective Calabi-Yau $\mathbb{F}_q$ -varieties.*

Using the Hodge realization functor, we deduce the following corollary of theorem 2.4.1.

Corollary 2.4.10. *If X and Y are K -equivalent complex varieties, then they have the same classes of integral mixed Hodge structures in $K_0(\text{MHS}_{\mathbb{Z}})$. By the semisimplicity of polarizable rational mixed Hodge structures, two such varieties have the same rational mixed Hodge structures.*

This recovers the well-known result of Kontsevich [57] mentioned earlier (and that of Batyrev [13]) as a special case.

2.4.2 Motivic t -structure, Krull-Schmidt categories, and rational motives

In this section, we show how the the notion of Krull-Schmidt categories could improve theorem 2.4.2 above, and so prove conjecture 2.4.3 of Wang. We also study how the existence of a motivic t -structure on $\text{DM}_{\text{gm}}(k; \mathbb{Q})$ with the expected properties implies the equivalence of rational Voevodsky motives of K -equivalent smooth projective k -varieties.

Recall the following definition.

Definition 2.4.11 (Krull-Schmidt category). *An R -linear additive category $\mathcal{C}$ is said to be a Krull-Schmidt category if every object is a finite direct sum of objects with local endomorphism rings.*

The Krull-Schmidt theorem says that an object in a Krull-Schmidt category has a local endomorphism ring if and only if it is indecomposable. Furthermore, it also say that any object is uniquely, up to permutation, a direct sum of indecomposable objects. Many examples of Krull-Schmidt categories come from abelian categories in

which every object has finite length. A concrete example is the category of finitely generated modules over a finite R -algebra, where R is a commutative Noetherian local complete ring (e.g. $\mathbb{Z}_\ell$). It is not known if the category of effective Chow motives over a field k with R -coefficients $\text{Chow}^{\text{eff}}(k; R)$ is a Krull-Schmidt category, even if k is of characteristic zero and $R = \mathbb{Q}$.

A direct consequence of theorem 2.4.2 is the following.

Theorem 2.4.12. *Suppose $\text{Chow}^{\text{eff}}(k; R)$ is a Krull-Schmidt category, k and R as in our conventions set at the beginning of this introduction. Then for X and Y K -equivalent smooth projective k -varieties, the Chow motives of X and Y in $\text{Chow}(k; R)$ are equivalent.*

When working with rational coefficients, can approach the rational version of conjecture 2.4.3 from the point of view of motivic t -structures, which is what we pursue now.

Let μ be a t -structure on $\text{DM}_{\text{gm}}(k; \mathbb{Q})$ with $\text{DM}_{\text{gm}}(k; \mathbb{Q})^{\leq 0}$ and $\text{DM}_{\text{gm}}(k; \mathbb{Q})^{\geq 0}$ its positive and negative parts, respectively. We denote by $\text{DM}_{\text{gm}}(k; \mathbb{Q})^\heartsuit := \text{DM}_{\text{gm}}(k; \mathbb{Q})^{\leq 0} \cap \text{DM}_{\text{gm}}(k; \mathbb{Q})^{\geq 0}$ the heart of μ . Furthermore, let ${}^\mu H^\bullet : \text{DM}_{\text{gm}}(k; \mathbb{Q}) \rightarrow \text{DM}_{\text{gm}}(k; \mathbb{Q})^\heartsuit$ be the cohomology functors with respect to the motivic t -structure μ . We say that μ is a *non-degenerate* t -structure if the functors $\{{}^\mu H^a\}_a$ form a conservative family of functors.

Definition 2.4.13. μ is said to be a *motivic t -structure* if it is non-degenerate and compatible with $\otimes$ and r , that is, $\otimes$ and r are t -exact. Here, r is either $r_{\mathbb{Q}_\ell}$ or r_σ as described above.

A notoriously difficult conjecture in the theory of Voevodsky motives states the following.

Conjecture 2.4.14. (*Motivic t -structure conjecture*) *There is a motivic t -structure on the stable ∞ -category $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$ whose heart $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})^\heartsuit$ has semisimple part the category $\mathrm{Num}(k; \mathbb{Q})$ of rational numerical motives, and such that every motive has a filtration by rational numerical motives. Additionally, for each smooth projective k -variety X , each ${}^\mu H^a M(X)_\mathbb{Q}$ is a semisimple object of the heart. In characteristic zero, this last condition follows from the existence of a motivic t -structure (proposition 1.5 of [15]).*

We now show a concrete consequence of the validity of this conjecture. First a proposition.

Proposition 2.4.15. *Suppose the motivic t -structure conjecture is true for $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$. Then for X and Y smooth projective k -varieties such that $[M(X)_\mathbb{Q}] = [M(Y)_\mathbb{Q}]$ in $K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q}))$, $M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q}$.*

Proof. Suppose the motivic t -structure conjecture is true for $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$. The inclusions $\mathrm{Num}(k; \mathbb{Q}) \hookrightarrow \mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})^\heartsuit$ and $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})^\heartsuit \hookrightarrow \mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$ induce isomorphisms

$$K_0(\mathrm{Num}(k; \mathbb{Q})) \rightarrow K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})^\heartsuit)$$

and

$$K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})^\heartsuit) \rightarrow K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})).$$

The latter isomorphism follows from the theorem of the heart [12]. The second morphism has inverse given by $[M] \mapsto \sum_a (-1)^a [{}^\mu H^a M]$. Since $[M(X)_\mathbb{Q}] = [M(Y)_\mathbb{Q}]$ in $K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q}))$, we obtain that

$$[(\oplus_{a \text{ even}} {}^\mu H^a M(X)_\mathbb{Q}) \oplus (\oplus_{a \text{ odd}} {}^\mu H^a M(Y)_\mathbb{Q})] = [(\oplus_{a \text{ odd}} {}^\mu H^a M(X)_\mathbb{Q}) \oplus (\oplus_{a \text{ even}} {}^\mu H^a M(Y)_\mathbb{Q})]$$

in $K_0(\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})^\heartsuit)$. By assumption, for each smooth projective k -variety Z , ${}^\mu H^a M(Z)$ is semisimple, and so a numerical motive by the assumption that the

semisimple part of the heart is the category of rational numerical motives. Therefore, the last equality holds in $K_0(\text{Num}(k; \mathbb{Q}))$. By Jannsen's theorem [54], $\text{Num}(k; \mathbb{Q})$ is a semisimple abelian category, and so

$$(\oplus_{a \text{ even}} {}^\mu H^a M(X)_{\mathbb{Q}}) \oplus (\oplus_{a \text{ odd}} {}^\mu H^a M(Y)_{\mathbb{Q}}) \simeq (\oplus_{a \text{ odd}} {}^\mu H^a M(X)_{\mathbb{Q}}) \oplus (\oplus_{a \text{ even}} {}^\mu H^a M(Y)_{\mathbb{Q}})$$

as numerical motives, and so also in $\text{DM}_{\text{gm}}(k; \mathbb{Q})^\vee$. By proposition 1.7 of [15], this forces us to have ${}^\mu H^a M(X)_{\mathbb{Q}} \simeq {}^\mu H^a M(Y)_{\mathbb{Q}}$ for every a . By proposition 1.4 of [15], for each smooth projective k -variety Z , $M(Z)_{\mathbb{Q}} \simeq \oplus_a {}^\mu H^a M(Z)_{\mathbb{Q}}[-a]$. Consequently, $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$. $\square$

Remark 2.4.16. *We do not need to actually identify the semisimple part of the heart with rational numerical motives to make the above proof work. Therefore, there is no need to invoke Jannsen's theorem.*

Remark 2.4.17. *A similar proof as in that of proposition 2.4.15 show that if the motivic t -structure conjecture is true for $\text{DM}_{\text{gm}}(k; \mathbb{Q})$, then $\text{Chow}(k; \mathbb{Q})$ is a Krull-Schmidt category.*

Consequently, we have the following theorem.

Theorem 2.4.18. *Suppose the motivic t -structure conjecture is true for $\text{DM}_{\text{gm}}(k; \mathbb{Q})$. Then if X and Y are K -equivalent smooth projective k -varieties, $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

Proof. Combine proposition 2.4.15 and theorem 2.4.1. $\square$

All of these suggest the following conjecture, which is a weaker version of conjecture 2.4.3 above.

Conjecture 2.4.19. *If X and Y are K -equivalent smooth projective k -varieties, then $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

For rational coefficients, since it is highly unexpected that the motivic t -structure conjecture above is false, and so theorem 2.4.18 suggests that it is very likely that

this last conjecture is true. The goal for the future is to refine the construction of motivic integration in order to unconditionally prove this last conjecture and possibly its more general integral version.

2.4.3 Relation to D-equivalence and Orlov's conjecture

In this section, we study the implications of our theorems for the following two important conjectures in noncommutative geometry due to Bondal-Orlov and Orlov. The first is as follows.

Conjecture 2.4.20 (Bondal-Orlov [17]). *If X and Y are birationally equivalent smooth projective Calabi-Yau complex varieties, then $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y)$.*

Though true for birational Calabi-Yau varieties of dimension at most 3 and birational symplectic 4-folds, this conjecture is vastly open. Note that there is a functor

$$\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q}) \rightarrow \mathrm{KMM}(k)_{\mathbb{Q}}$$

sending $M(X)_{\mathbb{Q}}$ to $NM(X) \in \mathrm{KMM}(k)_{\mathbb{Q}}$, where $\mathrm{KMM}(k)_{\mathbb{Q}}$ is Kontsevich's category of noncommutative motives. See [78] for details. We thus obtain the following corollary of theorem 2.4.18.

Corollary 2.4.21. *Suppose the motivic t -structure conjecture is true for $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Q})$. Then if X and Y are K -equivalent smooth projective k -varieties, $NM(X) \simeq NM(Y)$ in $\mathrm{KMM}(k)_{\mathbb{Q}}$.*

We know this to be true unconditionally if we replace K -equivalence with D -equivalence (combine proposition 1 of [17] with the functor in [77] or [78]). In light of conjecture 2.4.20, it is still open if K -equivalence, at least in the setting of birational smooth projective complex varieties, implies D -equivalence. What this last corollary states is that if the expected motivic t -structure on rational geometric Voevodsky mo-

tives exists, then K-equivalence implies the equivalence of noncommutative motives in the sense of Kontsevich. Consequently, many of the noncommutative cohomology theories agree for K-equivalent smooth projective varieties. The second conjecture is the relation between the bounded derived category of coherent sheaves on smooth projective complex varieties and their rational motives.

Conjecture 2.4.22 (Orlov [74]). *If X and Y are smooth projective complex varieties such that $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y)$ as k -linear triangulated categories, then $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$ in $\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}, \mathbb{Q})$.*

Equivalences of such bounded derived categories are given by Fourier-Mukai transforms, and under certain conditions on the kernel of this transform, the latter conjecture is true [74]. However, the unconditional conjecture is still vastly open.

Remark 2.4.23. *Conjecture 2.4.22 is false if we require the equivalence of integral motives in the conclusion. Indeed, there are D -equivalent Calabi-Yau threefolds with non-isomorphic third integral singular cohomology groups [1]. Furthermore, the converse is also false. Indeed, take X to be $\mathbb{P}_k^2$ blown-up at a point and Y to be $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$. Both have motives $\mathbf{1}_k \oplus \mathbf{1}_k(1)[2]^{\oplus 2} \oplus \mathbf{1}_k(2)[4]$ and are both Fano. If they were D -equivalent, then they would be isomorphic (use result of Bondal and Orlov [19]), which is not true.*

Remark 2.4.24. *It is a result of Balmer [10] in $\otimes$ -triangular geometry that quasi-compact and quasi-separated schemes X can be recovered from their $\otimes$ -triangulated categories of perfect complexes $D_{\mathrm{perf}}(X)$. This is not true if we forget the $\otimes$ -structure, as shown by Mukai that there are abelian varieties A such that $A \not\simeq \widehat{A} := \mathrm{Pic}^0(A)$ but are D -equivalent. Therefore, Orlov's conjecture states that if we forget the $\otimes$ -structure on $D_{\mathrm{perf}}(X) \simeq D^b\mathrm{Coh}(X)$, for X any smooth projective complex variety, we can at least recover the rational (Chow) motive of X . From another perspective, we have Gabriel's theorem stating that if $\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y)$ as abelian categories, then X and Y are isomorphic [36]. Therefore, another perhaps more enlightening*

interpretation of Orlov's conjecture is that considering derived equivalence on the left hand side forces us to consider equivalence in Voevodsky's category of rational geometric motives, conjecturally a bounded derived category of mixed motives.

In any case, the combination of the above two conjectures suggests that if X and Y are birational smooth projective Calabi-Yau complex varieties, then $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$, giving more evidence for the validity of the rational version of the conjecture of Wang.

A priori, there is no obvious connection between the existence of a motivic t -structure on $\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}; \mathbb{Q})$ and conjecture 2.4.22 of Orlov. However, a theorem of Kawamata states the following.

Theorem 2.4.25 (Part (2) of Theorem 2.3 of [55]). *Suppose X and Y are D -equivalent smooth projective complex varieties such that X is of general type ($\kappa(X) = \dim X$, i.e. maximal Kodaira dimension) or $\kappa(X, -K_X) = \dim X$. Then they are K -equivalent.*

In combination with theorem 2.4.2 and theorem 2.4.18, we obtain the following conditional theorems, respectively.

Theorem 2.4.26. *Then for X and Y smooth projective complex varieties such that $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$, $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y) \implies M(X) \oplus P \simeq M(Y) \oplus P$ for some integral Chow motive P .*

Theorem 2.4.27. *Suppose the motivic t -structure conjecture is true for $\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}; \mathbb{Q})$. Then for X and Y smooth projective complex varieties such that $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$, $D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y) \implies M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$.*

Remark 2.4.28. *We remark that when X and Y are smooth projective complex varieties such that X has ample or anti-ample canonical bundle, then an $\mathbb{C}$ -linear equivalence of bounded derived categories of coherent sheaves implies an isomorphism of X and Y . This is a theorem due to Bondal and Orlov [18].*

In a good sense, the vast majority of smooth projective complex varieties are of general type. For example, in the moduli space of curves, the space of curves of Kodaira dimension $-\infty$ (genus 0) is a point, the space of curves of Kodaira dimension 0 (genus 1) is 1-dimensional, while the space of curves of genus $g \geq 2$ (general type) has dimension $3g - 3$. Also, a hypersurface of degree d in $\mathbb{P}^n$ is of general type if and only if $d > n + 1$, and so *most* hypersurfaces are of general type. Consequently, conditional on the existence of the expected motivic t -structure on $\mathrm{DM}_{\mathrm{gm}}(\mathbb{C}; \mathbb{Q})$, we have established conjecture 2.4.22 for the vast majority of smooth projective complex varieties. Most likely, any possible counterexample to Orlov's conjecture can only be found among *exceptional* varieties, that is, those not of general type. As mentioned at the end of the previous subsection, if we develop a more refined theory of integration for Voevodsky motives, we may be able to unconditionally prove that K-equivalent smooth projective varieties have equivalent rational (Chow) motives, in which case we will make unconditional the theorems and corollaries above that are conditional on the existence of a motivic t -structure. This technique of passing through K-equivalence to show that D-equivalence implies equivalence of rational motives will not work for all smooth projective varieties since D-equivalence does not in general imply K-equivalence. Uehara has given an example of two birational smooth projective complex varieties that are D-equivalent but not K-equivalent [79].

Chapter 3

Comparison of stable homotopy categories

3.1 Introduction

Motivic homotopy theory is designed to be the homotopy theory of algebraic geometry. Just as spectra in stable homotopy theory represent generalized cohomology theories, motivic spectra in stable motivic homotopy theory represent *homotopy invariant* algebraic cohomology theories. There are comparison theorems that relate cohomology theories. For example, the Artin comparison theorem gives an isomorphism between the étale cohomology $H_{\text{ét}}^*(X; \mathbb{Z}/n\mathbb{Z})$ of a separated finite type $\mathbb{C}$ -scheme X and the singular cohomology $H^*(X^{\text{an}}; \mathbb{Z}/n\mathbb{Z})$ of its complex analytification X^{an} . Another such comparison theorem is the Suslin-Voevodsky theorem establishing an isomorphism between the (motivic) singular (co)homology of a separated finite type $\mathbb{C}$ -scheme with finite coefficients and the singular (co)homology of its associated complex analytic space X^{an} . Suslin and Voevodsky also show that if X is a separated finite type k -scheme, where k is an algebraically closed field of

exponential characteristic p *admitting resolution of singularities*, then its (motivic) singular/Suslin cohomology $H_{sus}^*(X; \mathbb{Z}/n\mathbb{Z})$ with n coprime to p agrees with the étale cohomology $H_{et}^*(X; \mathbb{Z}/n\mathbb{Z})$ [81]. The method of de Jong alterations allows the removal of the condition that k admits resolution of singularities [37]. In combination, these results say that for suitable finite torsion coefficients motivic cohomology is a generalization of both Betti and étale cohomology, whenever such cohomology theories are defined and well-behaved. In general, the study of realization functors is important because they, in particular, give us comparison theorems that allow us to understand complicated mathematical objects via their less complicated or concrete shadows. In particular, the existence of realization functors in the study of motives justifies the term *motivic*. In this chapter, we construct the stable ℓ -adic étale realization functor, a realization functor on the level of motivic spectra. Using this construction, we relate the classical stable homotopy category to the stable motivic homotopy category and generalize the Suslin-Voevodsky theorem to the homotopy theoretic level.

In the classical stable homotopy category SH , the (topological) circle S^1 is inverted, while in the stable motivic homotopy category $\mathrm{SH}(S)$, the projective line $\mathbb{P}_S^1$ is inverted. Since two different objects are inverted in the two theories, the question arises whether there is a fully faithful functor from SH to $\mathrm{SH}(S)$, i.e. whether stable motivic homotopy theory subsumes classical stable homotopy theory. There has been progress in this respect for various base fields. In [63], Levine used the Betti realization functor on the level of motivic spectra to prove that the functor

$$\mathrm{SH} \rightarrow \mathrm{SH}(k)$$

induced by the constant sheaf functor is fully faithful whenever k is algebraically closed of characteristic zero. The proof given by Levine is by comparing the slice spectral sequence to the Betti realization of the slice spectral sequence. On the other hand, Wilson and Østvær [89] jointly proved that for k algebraically closed of positive characteristic p , the functor $\mathrm{SH} \rightarrow \mathrm{SH}(k)$ induces an isomorphism $\pi_n(\mathbb{S})[1/p] \rightarrow \pi_{n,0}(\mathbf{1}_k)(k)[1/p]$. Their proof passes through Witt vectors to reduce the positive characteristic case to Levine's characteristic zero case. However, they use motivic Serre finiteness (see corollary 3.6.3 and remark 3.6.4), now a theorem due to Ananyevskiy-Levine-Panin [2], and the motivic Adams spectral sequence to establish their result.

On the other hand, we use étale homotopy theory to deduce this result and more. Artin and Mazur defined the étale homotopy type of a scheme X as a pro-homotopy type [3], and this was later rigidified to the notion of a pro-space $\Pi_\infty^{\mathrm{ét}} X$ by Friedlander [34]. The pro-space $\Pi_\infty^{\mathrm{ét}} X$ is defined such that its singular cohomology and the étale cohomology of X agree for constant coefficients. Over a non-separably closed field k , it would be better to define $\Pi_\infty^{\mathrm{ét}} X$ *relative* to $\mathrm{Spec} \ k$ so that the action of the absolute Galois group $G_k := \mathrm{Gal}(k^{\mathrm{sep}}|k)$ is also considered; this would give us a pro-sheaf of spaces on the small étale site of $\mathrm{Spec} \ k$. The étale topological type $X \mapsto \Pi_\infty^{\mathrm{ét}} X$ of smooth k -schemes, k an algebraically closed field, has been lifted to the level of motivic spaces by Isaksen [50]. In this chapter, we will construct a stable version of the ∞ -categorical incarnation of this construction, that is, for each prime $\ell \neq \mathrm{expchar} \ k$, we construct an exact small colimit-preserving functor

$$\underline{\mathrm{Ét}}_\ell : \mathcal{Spt}(k) \rightarrow \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}$$

between stable ∞ -categories. Note that this stable étale realization is different from that constructed by Quick in [76]. His construction is basically the formal stabilization of the unstable étale realization, whereas the one in this chapter is not.

The first application of our stable ℓ -adic étale realization functor is the generalization of Levine's result to algebraically closed fields that need not be of characteristic zero. Secondly, we generalize the étale Suslin-Voevodsky theorem to the homotopy-theoretic level. Indeed, we prove the following theorems.

Theorem 3.1.1 (Theorem 3.6.1). *Let k be an algebraically closed field of exponential characteristic p . Then*

$$c[1/p] : \mathcal{Spt}[1/p] \rightarrow \mathcal{Spt}(k)[1/p]$$

induced by the constant presheaf functor is fully faithful.

Theorem 3.1.2 (Generalized Suslin-Voevodsky, theorem 3.7.1). *Suppose k is an algebraically closed field of exponential characteristic p , $E \in \mathcal{Spt}^{\text{eff}}(k)_{\text{tor}}$ is an effective torsion motivic spectrum, and $\ell \neq p$ is a prime. Then*

$$\underline{\text{Ét}}_{\ell*} : \pi_{n,0}(E)(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n \left(\underline{\text{Ét}}_{\ell}(E) \right)$$

is an isomorphism.

Though the first theorem can be proved using a density argument in conjunction with the main result of [89], the generalized étale Suslin-Voevodsky theorem is completely new; even for its statement one needs a suitable stable étale realization functor. That being said, in this chapter we give a completely different proof of the first theorem; there is hope that this approach will lead to a comparison of equivariant and motivic stable homotopy theories at least when the base field is not algebraically closed. We deduce the full-faithfulness result by comparing the slice spectral sequence to the spectral sequence induced by the stable étale realization of the slice tower. In

particular, instead of using deformation theory, the motivic Adams spectral sequence, and motivic Serre finiteness (as in [89]) to reduce to the characteristic zero case proved by Levine, we replace the Betti realization in Levine’s proof with the stable étale realization functor. Furthermore, the techniques in this chapter give us the second theorem above which is a homotopy-theoretic generalization of the étale version of the Suslin-Voevodsky theorem. Furthermore, we avoid using motivic Serre finiteness, and so obtain a new proof of this theorem in the special case that k is algebraically closed. Though our proof uses the philosophy of Levine’s proof, there are a number of technicalities that arise due to the appearance of pro-homotopy theory. Consequently, we have to work harder to prove some of the necessary lemmas and propositions. In particular, the proofs of the above theorems are not immediate consequences of our construction of the stable étale realization functor.

There are a number of generalizations of the results in this chapter that could be pursued. The author expects that SH fully faithfully embeds in $\mathrm{SH}(k)$ without the inversion of the exponential characteristic; however, the techniques of this chapter do not work, at least with their current limitations. The inversion of the exponential characteristic occurs in a number of places in motivic homotopy theory. Most importantly, it occurs when dealing with resolution of singularities in the spirit of de Jong and Gabber (see Shane Kelly’s work [56]). Furthermore, étale cohomology does not behave well at the residue characteristics. The author knows of no $\mathbb{A}^1$ -homotopy invariant cohomology theory good enough to deal with this issue at the prime p . Another possible generalization that should be pursued is Galois equivariance, i.e. the dropping of the condition that k is algebraically closed. There has been some progress in this direction. Now that we know the validity of motivic Serre finiteness [2], Heller and Ormsby have proved that there is a fully faithful functor from the C_2 -equivariant stable homotopy category to the stable motivic homotopy

category of real closed fields [43],[44]. The ideas behind the construction of the stable étale realization functor constructed in this chapter may lead to Galois equivariant generalizations. Note, however, that the construction given here works well only when dealing with algebraically closed fields. In order to deal with non-algebraically closed fields, we need to construct a stable étale realization functor that takes care of continuous Galois actions. This will be pursued in a future paper. That being said, proving that stable motivic homotopy theory subsumes some sort of Galois equivariant stable homotopy theory for fields that are not real closed requires a good theory of equivariant stable homotopy theory with profinite group actions. Perhaps, the work of Barwick [12] is relevant here.

We warn the reader that the default language in this chapter is that of ∞ -categories as in [65] and [67]. In particular, throughout this chapter, $\mathcal{S}$ is the ∞ -category of spaces. For notational simplicity, throughout this chapter we view $\mathcal{S}\mathrm{pt}$ as the stable ∞ -category of S^2 -spectra. The unit object is $\Sigma_{S^2}^\infty S^0$, also designated by $\mathbb{S}$.

3.2 Motivic Homotopy Theory

In this section, we recall the foundations of motivic homotopy theory as well as its stable version. We will furthermore describe Voevodsky's slice tower. The bigraded slice tower can be found in [63] in the language of triangulated categories. However, since we work in the ∞ -categorical language, we reproduce here the basic constructions for completeness and for the benefit of the reader.

Let S be a Noetherian scheme of finite Krull dimension. Let $\mathcal{S}\mathrm{m}/S$ be the category of separated smooth schemes (of finite type) over S , which will henceforth be shortened to *smooth schemes*. Let $\mathcal{P}\mathrm{Sh}_\infty(\mathcal{S}\mathrm{m}/S)$ and $\mathcal{P}\mathrm{Sh}_\infty^*(\mathcal{S}\mathrm{m}/S)$ be the

∞ -categories $\mathrm{Fun}((\mathcal{S}\mathrm{m}/S)^{\mathrm{op}}, \mathcal{S})$ and $\mathrm{Fun}((\mathcal{S}\mathrm{m}/S)^{\mathrm{op}}, \mathcal{S}_*)$ of presheaves of spaces and pointed presheaves of spaces, respectively, on $\mathcal{S}\mathrm{m}/S$. Note that whenever $\mathcal{C}_0$ is an ordinary category, we identify it with its nerve $N(\mathcal{C}_0)$. Therefore, when we speak of ∞ -functors from $\mathcal{C}_0$ to an ∞ -category $\mathcal{D}$, we actually mean ∞ -functors from the nerve $N(\mathcal{C}_0)$ of $\mathcal{C}_0$ to $\mathcal{D}$. Note that $N(\mathcal{C}_0)$ is an ∞ -category, and that the nerve of an ordinary category coincides with the simplicial nerve of $\mathcal{C}_0$ viewed as a simplicial category with discrete spaces of morphisms. In particular, we abuse notation to identify $\mathcal{S}\mathrm{m}/S$ with its nerve $N(\mathcal{S}\mathrm{m}/S)$. Taking $\mathcal{P}\mathrm{Sh}_\infty(\mathcal{S}\mathrm{m}/S)$ and sheafifying with respect to the Nisnevich topology and localizing with respect to the S -projections $X \times_S \mathbb{A}_S^1 \rightarrow X$ gives us the ∞ -category $\mathrm{Spc}(S)$ of motivic spaces (since S is assumed to be Neotherian of finite Krull dimension, this is already hypercomplete). By localization, we mean the following. Call a Nisnevich sheaf E in the ∞ -category $\mathrm{Shv}_\infty^{\mathrm{Nis}}(\mathcal{S}\mathrm{m}/S)$ $\mathbb{A}^1$ -local if it has the property that for every S -projection $X \times_S \mathbb{A}_S^1 \rightarrow X$ viewed as a map of representable Nisnevich sheaves, $\mathrm{Map}(X, E) \rightarrow \mathrm{Map}(X \times_S \mathbb{A}_S^1, E)$ is a weak equivalence of spaces. Then $\mathrm{Spc}(S)$ is the full subcategory of $\mathrm{Shv}_\infty^{\mathrm{Nis}}(\mathcal{S}\mathrm{m}/S)$ consisting of the $\mathbb{A}^1$ -local objects. To make this precise, we proceed in the following well-known manner. The category $\mathcal{P}\mathrm{Sh}_\infty(\mathcal{S}\mathrm{m}/S)$ is the ∞ -category associated to the projective object-wise model structure on simplicial presheaves on $\mathcal{S}\mathrm{m}/S$. This model structure is a left proper combinatorial simplicial model structure, and so it can be left Bousfield localized along any small set of morphisms (see proposition A.3.7.3 of Lurie's [65]). We can localize to obtain $\mathrm{Shv}_\infty^{\mathrm{Nis}}(\mathcal{S}\mathrm{m}/S)$, and then obtain $\mathrm{Spc}(S)$ via the left Bousfield localization that corresponds to the $\mathbb{A}^1$ -equivalences, that is, maps of Nisnevich sheaves of spaces $X \rightarrow Y$ such that for every $\mathbb{A}^1$ -local object Z , $\mathrm{Map}(Y, Z) \rightarrow \mathrm{Map}(X, Z)$ is a weak equivalence. We can do this for any topology τ . We have the following lemma.

Lemma 3.2.1. *The inclusion $i : \mathcal{Spc}(S) \rightarrow \mathcal{Shv}_\infty^\tau(\mathcal{Sm}/S)$ has a left adjoint*

$$L_{\mathbb{A}^1} : \mathcal{Shv}_\infty^\tau(\mathcal{Sm}/S) \rightarrow \mathcal{Spc}(S).$$

Doing the same with $\mathcal{PSh}_\infty^*(\mathcal{Sm}/S)$ gives us the ∞ -category $\mathcal{Spc}_*(S)$ of pointed motivic spaces. We can see from the homotopy pushout square

$$\begin{array}{ccc} \mathbb{G}_{m,S} & \longrightarrow & \mathbb{A}_S^1 \simeq * \\ \downarrow & & \downarrow \\ * \simeq \mathbb{A}_S^1 & \longrightarrow & \mathbb{P}_S^1 \end{array}$$

in the ∞ -category $\mathcal{Spc}_*(S)$ that the S^1 -suspension of (pointed) $\mathbb{G}_{m,S}$ is in fact homotopy equivalent to $\mathbb{P}_S^1$ (pointed at ∞). Let $\Omega_{\mathbb{P}_S^1}(-)$ be the functor $\mathrm{Map}_{\mathcal{Spc}_*(S)}(\mathbb{P}_S^1, -)$. Let $\mathcal{Spt}(S)$ be the ∞ -category of $\mathbb{P}_S^1$ -spectra of motivic spaces over S :

$$\mathcal{Spt}(S) := \lim \left(\dots \xrightarrow{\Omega_{\mathbb{P}_S^1}} \mathcal{Spc}_*(S) \xrightarrow{\Omega_{\mathbb{P}_S^1}} \mathcal{Spc}_*(S) \right),$$

where the limit is taken in the ∞ -category of ∞ -categories. Let $\Sigma_{\mathbb{P}_S^1}^\infty : \mathcal{Spc}_*(S) \rightarrow \mathcal{Spt}(S)$ be the left adjoint of $\Omega_{\mathbb{P}_S^1}^\infty : \mathcal{Spt}(S) \rightarrow \mathcal{Spc}_*(S)$, and call it the suspension functor. Note that such a left adjoint exists because $\mathcal{Spc}_*(S)$ is a presentable ∞ -category; it is the localization of a category of presheaves. See proposition 1.4.4.4 of [67] for the existence of the left adjoint.

We remind the reader that an ∞ -category $\mathcal{C}$ is said to be *accessible* if for some regular cardinal κ , there is a small ∞ -category $\mathcal{C}_0$ such that $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{C}_0)$ [65]. Basically, all this is saying is that even though $\mathcal{C}$ may be large, it is generated under κ -small filtered colimits by a small category. An ∞ -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be an *accessible functor* if $\mathcal{C}$ is an accessible ∞ -category and there is a regular

cardinal κ such that F preserves κ -small filtered colimits [65]. Note that presentable ∞ -categories are, by definition, accessible ∞ -categories that admit all small colimits.

Note that that $\mathcal{Spt}(S)$ is a presentable stable ∞ -category. Indeed, the inclusion from $\mathcal{Pr}^R$, the ∞ -category of presentable ∞ -categories with morphisms those accessible functors preserving small limits, into the ∞ -category of ∞ -categories preserves limits (proposition 5.5.3.18 of [65]). Therefore, $\mathcal{Spt}(S)$ is a limit in $\mathcal{Pr}^R$, which is closed under small limits, again by Proposition 5.5.3.18 of [65]. The presentability of $\mathcal{Spt}(S)$ can also be seen by noting that $\mathcal{Spt}(S)$ is a colimit in $\mathcal{Pr}^L \simeq (\mathcal{Pr}^R)^{op}$, the ∞ -category of presentable ∞ -categories with morphisms those functors preserving small colimits.

$\mathbf{1}_S \in \mathcal{Spt}(S)$ is the motivic sphere spectrum $\Sigma_{\mathbb{P}^1}^\infty S_+$ over S . Smash product makes $\mathcal{Spt}(S)$ a symmetric monoidal ∞ -category with unit object $\mathbf{1}_S$. Throughout this chapter, we drop S in all the above notations whenever the context makes the choice of S clear.

Throughout this chapter, the motivic sphere $S^{p,q}$ will be defined as

$$(S^1)^{\wedge p} \wedge \mathbb{G}_{m,S}^{\wedge q}.$$

Note that $S^{p,q}$ is usually $(S^1)^{\wedge(p-q)} \wedge \mathbb{G}_{m,S}^{\wedge q}$; however, we find the above convention more convenient for our purposes.

Given an ∞ -category $\mathcal{C}$, we let $[X, Y]_{\mathcal{C}}$ be $\pi_0 \text{Map}_{\mathcal{C}}(X, Y)$. We will omit the subscript $\mathcal{C}$ whenever there is no possibility of confusion. We let $\Sigma^{p,q} : \mathcal{Spt}(S) \rightarrow \mathcal{Spt}(S)$ be the bigraded suspension functor $E \mapsto S^{p,q} \wedge E$. $\Omega^{p,q} : \mathcal{Spt}(S) \rightarrow \mathcal{Spt}(S)$ be the loop

functor $E \mapsto S^{-p,-q} \wedge E$ with the expected definition. It is the right adjoint of $\Sigma^{p,q}$. If $E \in \mathcal{Spt}(S)$ is a motivic spectrum, then $\pi_{p,q}(E)$ is the Nisnevich sheaf associated to the presheaf on $\mathcal{Sm}/S$ of abelian groups

$$X \mapsto [\Sigma^{p,q} \Sigma_{\mathbb{P}^1}^\infty X_+, E]$$

In the proof of the main theorem, Voevodsky's slice spectral sequence will play an essential role. We now set up the necessary notation and recall the construction of this spectral sequence. Let $\mathcal{Spt}(S)_{\geq(a,b)}$ be the full subcategory of $\mathcal{Spt}(S)$ generated under colimits and extensions by $\{\Sigma^{p,q} \Sigma_{\mathbb{P}^1}^\infty X_+ | X \in \mathcal{Sm}/S, p \geq a \text{ and } q \geq b\}$. $\mathcal{Spt}(S)_{\geq(a,-\infty)}$ denotes the full subcategory of $\mathcal{Spt}(S)$ generated under colimits and extensions by $\{\Sigma^{p,q} \Sigma^\infty X_+ | X \in \mathcal{Sm}/S, p \geq a \text{ and } q \in \mathbb{Z}\}$. Similarly, $\mathcal{Spt}(S)_{\geq(-\infty,b)}$ denotes the full subcategory of $\mathcal{Spt}(S)$ generated under colimits and extensions by $\{\Sigma^{p,q} \Sigma^\infty X_+ | X \in \mathcal{Sm}/S, p \in \mathbb{Z} \text{ and } q \geq b\}$. By construction, $\mathcal{Spt}(S)$ is a *stable* ∞ -category. As a result, $\mathcal{Spt}(S)_{\geq(-\infty,b)}$ is also a stable ∞ -category. For each fixed $b \geq -\infty$, from proposition 1.4.4.11 of [67] and the stability of $\mathcal{Spt}(S)_{\geq(-\infty,b)}$ we deduce the existence of a t -structure on $\mathcal{Spt}(S)_{\geq(-\infty,b)}$ with non-negative part $\mathcal{Spt}(S)_{\geq(0,b)}$. For each (a,b) (where a, b can be $-\infty$), we have a truncation functor $f_{a,b} : \mathcal{Spt}(S) \rightarrow \mathcal{Spt}(S)$ given by the truncation $\tau_{\geq(a,b)} : \mathcal{Spt}(S) \rightarrow \mathcal{Spt}(S)_{\geq(a,b)}$ followed by the inclusion $i_{a,b} : \mathcal{Spt}(S)_{\geq(a,b)} \rightarrow \mathcal{Spt}(S)$. $(i_{a,b}, \tau_{\geq(a,b)})$ is an adjoint pair. Note that the reason we consider t -structures is because we want the inclusion to be a left adjoint to truncation; this allows us to construct maps $f_{a,b}E \rightarrow E$. For convenience, $E \mapsto E_{\geq(a,b)}$ will be the truncation $f_{a,b}E$ of E . Given a fixed b , this t -structure on $\mathcal{Spt}(S)_{\geq(-\infty,b)}$ with non-negative part $\mathcal{Spt}(S)_{\geq(0,b)}$ gives us maps $E_{\geq(a,b)} \rightarrow E$, and so the cofiber sequences

$$E_{\geq(a,b)} \rightarrow E \rightarrow E_{\leq(a-1,b)} \rightarrow \Sigma^{1,0} E_{\geq(a,b)}.$$

We write $f_n := f_{-\infty, n}$. We call $\mathrm{Spt}^{\mathrm{eff}}(S) := \mathrm{Spt}(S)_{\geq(-\infty, 0)}$ the category of *effective spectra*. For each spectrum $E \in \mathrm{Spt}(S)$ and each n , the adjunction $(i_{-\infty, n}, \tau_{\geq(-\infty, n)})$ allows us to produce the natural morphism $f_n E \rightarrow E$, and the n -th *slice* $s_n E$ is defined by the cofiber sequence

$$f_{n+1} E \rightarrow f_n E \rightarrow s_n E \rightarrow \Sigma^{1,0} f_{n+1} E.$$

Note that $f_{n+1} f_n = f_{n+1}$. For every effective spectrum E , these cofiber sequences fit together to produce Voevodsky's *slice tower*

$$\dots \rightarrow f_{t+1} E \rightarrow f_t E \rightarrow \dots \rightarrow f_0 E = E$$

with n^{th} layer $s_n E$. This gives us an exact couple of graded abelian groups

$$\begin{array}{ccc} \pi_{*,0}(f_* E)(S) & \xrightarrow{\quad} & \pi_{*,0}(f_* E)(S) \\ & \swarrow \partial \quad \searrow & \\ & \pi_{*,0}(s_* E)(S), & \end{array}$$

where ∂ is of degree -1 and the unlabeled morphisms are of degree 0 . From this exact couple, we get Voevodsky's *slice spectral sequence*

$$E_{p,q}^2 = \pi_{p+q}(s_q E)(S) \implies \pi_{p+q}(E)(S),$$

where the differentials on the r -th page are given by

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r.$$

In section 3.5, we recall conditions under which the convergence of this spectral sequence is guaranteed. We will also study the spectral sequence obtained via the stable

ℓ -adic étale realization of the above slice tower. By comparing these two spectral sequences for $S = \operatorname{Spec} k$, k an algebraically closed field, we will deduce our comparison theorems.

3.3 ℓ -adic Homotopy Theory

The stable ℓ -adic étale realization functor will take values in the ∞ -category of $(H\mathbb{Z}/\ell)^*$ -local pro-spectra (those that are *cohomologically* Bousfield localized with respect to the Eilenberg-MacLane spectrum $H\mathbb{Z}/\ell$). In this section, we recall pro-categories in the spirit of Grothendieck (and Lurie in the ∞ -categorical world) and give its universal property. This will be essential to our construction of the stable étale realization functor. We then discuss Bousfield localization of pro-spaces and pro-spectra using the language of ∞ -categories. We now recall the central notion of pro-categories in the world of ∞ -categories.

Definition 3.3.1 (Definition 3.1.1 of [66]). *Given an accessible ∞ -category $\mathcal{C}$ admitting finite limits, its pro-category $\operatorname{Pro}(\mathcal{C})$ is the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{S})^{\operatorname{op}}$ spanned by the accessible functors $F : \mathcal{C} \rightarrow \mathcal{S}$ that preserve finite limits.*

The notion of a pro-category as defined above is very close to the more familiar classical notion. Indeed, if $\mathcal{C}$ is the nerve of an ordinary accessible category $\mathcal{C}_0$, then the ∞ -category $\operatorname{Pro}(\mathcal{C})$ is the nerve of the (ordinary) pro-category $\operatorname{Pro}(\mathcal{C}_0)$ (example 3.1.3 of [66]).

As an example, the ∞ -category $\mathcal{S}\operatorname{pt}$ of spectra is a presentable, hence accessible, ∞ -category admitting finite limits. Therefore, we may construct $\operatorname{Pro}(\mathcal{S}\operatorname{pt})$ as above. Similarly, we have $\operatorname{Pro}(\mathcal{S})$.

Remark 3.3.2. If $\mathcal{C}$ is an accessible ∞ -category admitting finite limits, then the collection of left-exact accessible functors $\mathcal{C} \rightarrow \mathcal{S}$ is closed under small filtered colimits in $\text{Fun}(\mathcal{C}, \mathcal{S})$. Therefore, $\text{Pro}(\mathcal{C})$, which is a full subcategory of the opposite category of $\text{Fun}(\mathcal{C}, \mathcal{S})$, is closed under small cofiltered limits. This suggests the following universal proposition.

Proposition 3.3.3 (Proposition 3.1.6 of [66]). *Suppose $\mathcal{C}$ is an accessible ∞ -category admitting finite limits, and suppose $\mathcal{D}$ is an ∞ -category admitting small cofiltered limits. Let $\text{Fun}'(\text{Pro}(\mathcal{C}), \mathcal{D})$ be the full subcategory of $\text{Fun}(\text{Pro}(\mathcal{C}), \mathcal{D})$ spanned by those functors that preserve small cofiltered limits. Then the Yoneda embedding $\mathcal{C} \rightarrow \text{Pro}(\mathcal{C})$ induces an adjunction*

$$\text{Fun}'(\text{Pro}(\mathcal{C}), \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}).$$

Suppose $f : \mathcal{C} \rightarrow \mathcal{D}$ is a left exact functor between accessible categories admitting finite limits. Composing with the Yoneda embedding $\mathcal{D} \hookrightarrow \text{Pro}(\mathcal{D})$ gives us a functor $\mathcal{C} \rightarrow \text{Pro}(\mathcal{D})$. By the above universal property of pro-categories, this can be extended to $\text{Pro}(f) : \text{Pro}(\mathcal{C}) \rightarrow \text{Pro}(\mathcal{D})$ that preserves small cofiltered limits. On the other hand, there is a functor $G : \text{Pro}(\mathcal{D}) \rightarrow \text{Pro}(\mathcal{C})$ given by composition with f . These maps have the following important property.

Lemma 3.3.4. *If $f : \mathcal{C} \rightarrow \mathcal{D}$ is a left exact functor between accessible ∞ -categories admitting finite limits, then*

$$G : \text{Pro}(\mathcal{D}) \rightleftarrows \text{Pro}(\mathcal{C}) : \text{Pro}(f)$$

is an adjunction.

Proof. Given a function $H : \mathcal{X} \rightarrow \mathcal{Y}$, there is a composition of functors

$$\mathcal{Y} \rightarrow \text{Fun}(\mathcal{Y}, \mathcal{S})^{\text{op}} \rightarrow \text{Fun}(\mathcal{X}, \mathcal{S})^{\text{op}}$$

given by the Yoneda embedding followed by pre-composition with H . We denote this by $\tilde{H}$. This is called the *formal left adjoint* of H . Indeed, it has the following property. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then we have the following functorial equivalence:

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{X}, \mathcal{S})^{\mathrm{op}}}(\tilde{H}(y), x) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{X}, \mathcal{S})^{\mathrm{op}}}(\mathrm{Map}_{\mathcal{Y}}(y, H(-)), x) \simeq \mathrm{Map}_{\mathcal{Y}}(y, H(x)),$$

where the last equivalence follows from Yoneda's lemma. Note that if the image of $\tilde{H}$ lies in the full subcategory $\mathcal{E}$ of $\mathrm{Fun}(\mathcal{X}, \mathcal{S})^{\mathrm{op}}$, then we can write the equivalence as

$$\mathrm{Map}_{\mathcal{E}}(\tilde{H}(y), x) \simeq \mathrm{Map}_{\mathcal{Y}}(y, H(x)).$$

Let us apply this to the functor $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{C}) \rightarrow \mathrm{Pro}(\mathcal{D})$. Since f is a left exact and accessible functor between accessible categories admitting finite limits, such $\mathrm{Pro}(f)$ exists. Let us take $H := \mathrm{Pro}(f)$. Composing $\tilde{H}$ with $R : \mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{S})^{\mathrm{op}} \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ induced by restriction. We show that $R \circ \tilde{H} : \mathrm{Pro}(\mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ is equal to $G : \mathrm{Pro}(\mathcal{D}) \rightarrow \mathrm{Pro}(\mathcal{C})$ composed with $\mathrm{Pro}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$. Indeed, the former is given by sending $d \in \mathrm{Pro}(\mathcal{D})$ to $\mathrm{Map}_{\mathrm{Pro}(\mathcal{D})}(d, f(-)) = d \circ f$, while the latter is given by sending $d : \mathcal{D} \rightarrow \mathcal{C}$ to $d \circ f \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$. The conclusion follows. $\square$

Definition 3.3.5. $g : \mathcal{D} \rightarrow \mathrm{Pro}(\mathcal{C})$ is called the *pro-left adjoint* of $f : \mathcal{C} \rightarrow \mathcal{D}$.

The lemma above says that it exists whenever f a left exact functor between accessible ∞ -categories admitting finite limits.

Given a prime ℓ , $\mathcal{S}^{\ell\text{-fc}}$ is defined as the full subcategory of $\mathcal{S}$ of ℓ -finite spaces, that is, those spaces X with finitely many nontrivial homotopy groups with the additional conditions that $\pi_0 X$ is finite and $\pi_i X$, $i \geq 1$, are ℓ -finite groups. We then let $\mathcal{S}^{\mathrm{Pro}(\ell)} := \mathrm{Pro}(\mathcal{S}^{\ell\text{-fc}})$, and call it the ∞ -category of ℓ -profinite spaces.

There is a notion of $(H\mathbb{Z}/\ell)^*$ -localization of $\mathrm{Pro}(\mathcal{S})$. A morphism $X \rightarrow Y$ in $\mathrm{Pro}(\mathcal{S})$ is called an $(H\mathbb{Z}/\ell)^*$ -equivalence if it induces an isomorphism $H^*(Y; \mathbb{Z}/\ell) \rightarrow H^*(X; \mathbb{Z}/\ell)$. A pro-space Z is $(H\mathbb{Z}/\ell)^*$ -local if for every $(H\mathbb{Z}/\ell)^*$ -equivalence $X \rightarrow Y$, $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}(Y, Z) \rightarrow \mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}(X, Z)$ is a weak equivalence in $\mathcal{S}$. $\mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$ is the full ∞ -subcategory of $(H\mathbb{Z}/\ell)^*$ -local pro-spaces in $\mathrm{Pro}(\mathcal{S})$. It is not hard to show that $\mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} = \mathrm{Pro}(\mathcal{S}^{\ell\text{-fc}})$ (see remark 3.8 of [46]). We construct localization functors by using the universal property of pro-categories. The inclusion $i : \mathcal{S}^{\ell\text{-fc}} \hookrightarrow \mathcal{S}$ is accessible and left exact. Therefore, lemma 3.3.4 implies that the inclusion $i : \mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \rightarrow \mathrm{Pro}(\mathcal{S})$ admits a left adjoint $L^{\mathbb{Z}/\ell} : \mathrm{Pro}(\mathcal{S}) \rightarrow \mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$.

Since the ∞ -categories $\mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$ and $\mathrm{Pro}(\mathcal{S}^{\ell\text{-fc}})$ are equivalent, we will henceforth also call $(H\mathbb{Z}/\ell)^*$ -local profinite spaces ℓ -profinite spaces. Similarly, there is also a pointed version of this construction.

Similarly, we may localize $\mathrm{Pro}(\mathcal{S}\mathrm{pt})$ with respect to the Eilenberg-MacLane spectrum $H\mathbb{Z}/\ell$ as follows. Call a morphism $E \rightarrow F$ of pro-spectra an $(H\mathbb{Z}/\ell)^*$ -equivalence if $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{pt})}(F, H\mathbb{Z}/\ell) \rightarrow \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{pt})}(E, H\mathbb{Z}/\ell)$ is a weak equivalence. Note that we are abusing terminology here; when we speak of $(H\mathbb{Z}/\ell)^*$ -local pro-spaces, we will mean the definition given in the previous paragraph, while if we speak of $(H\mathbb{Z}/\ell)^*$ -local pro-spectra, we mean this definition. It should be clear from the context which we mean. Let $\mathrm{Pro}(\mathcal{S}\mathrm{pt})^{H\mathbb{Z}/\ell}$ be the full subcategory of $\mathrm{Pro}(\mathcal{S}\mathrm{pt})$ spanned by the $(H\mathbb{Z}/\ell)^*$ -local objects, that is, the full subcategory of $\mathrm{Pro}(\mathcal{S}\mathrm{pt})$ consisting of spectra G such that for every $(H\mathbb{Z}/\ell)^*$ -equivalence $E \rightarrow F$, $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{pt})}(F, G) \rightarrow \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}\mathrm{pt})}(E, G)$ is an equivalence. Similar to the case of $\mathrm{Pro}(\mathcal{S})$, we produce an adjoint pair of functors

$$L^{H\mathbb{Z}/\ell} : \mathrm{Pro}(\mathcal{S}\mathrm{pt}) \rightleftarrows \mathrm{Pro}(\mathcal{S}\mathrm{pt})^{H\mathbb{Z}/\ell} : i_{st},$$

where i_{st} is the inclusion functor. The existence of the Bousfield localization functors could have also been shown by using the strict model structures of Isaksen [51].

As we will see in our construction of the étale realization functors, the categories $\mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$ and $\mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}$ will play an important role for us.

In classical stable homotopy theory, we have the adjunction between infinite suspension and infinite loop space functors

$$\Sigma_+^\infty : \mathcal{S} \rightleftarrows \mathcal{Spt} : \Omega^\infty.$$

This induces the adjunction

$$\mathrm{Pro}(\Sigma_+^\infty) : \mathrm{Pro}(\mathcal{S}) \rightleftarrows \mathrm{Pro}(\mathcal{Spt}) : \mathrm{Pro}(\Omega^\infty),$$

from which we naturally obtain the adjunction

$$\Sigma_{S_\ell^2+}^\infty : \mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \rightleftarrows \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell} : \Omega_{S_\ell^2}^\infty$$

of the Bousfield localized categories.

3.4 Shape Theory and Étale Realization Functors

In [63], Levine used the Betti realization of motivic spectra to prove his full-faithfulness result for algebraically closed fields of characteristic zero. The stable Betti realization is an easy extension of the unstable Betti realization of motivic *spaces*. In positive characteristic, however, the extension of the étale realization of motivic spaces due to Isaksen [50] is not immediate because we need a good stable

homotopy theory of pro-spaces. The main purpose of this section is to show that for k algebraically closed, there is a stable étale realization of motivic spectra that behaves well enough for us.

In this section, we will first describe shape theory and recall the étale realization of schemes. We will then describe Hoyois' (see [46]) ∞ -categorical incarnation of Isaksen's étale realization of motivic spaces that gives the unstable ℓ -adic étale realization functor

$$\mathcal{S}pc(S) \rightarrow \mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}.$$

For our construction of the stable ℓ -adic étale realization, we first construct the étale realization functor for Nisnevich sheaves of E -modules on $\mathcal{S}m/S$, E an E_∞ -ring spectrum. Roughly, specializing this construction to $S = \mathrm{Spec} \ k$, k algebraically closed, and $E = \mathbb{S}$, and performing $\mathbb{A}^1$ -localization and Bousfield localization, we obtain the stable ℓ -adic étale realization functor of motivic spectra

$$\mathcal{S}pt(k) \rightarrow \mathrm{Pro}(\mathcal{S}pt)^{H\mathbb{Z}/\ell}.$$

Also taking $S = \mathrm{Spec} \ k$, k an algebraically closed field, in Hoyois' unstable case, we will be able to compare the unstable and stable étale realization functors. Before doing so, here are some preliminaries on shape theory and Grothendieck sites.

If τ is a Grothendieck topology on the category of schemes, and X is a scheme, the small τ -site will be the full subcategory of $\mathcal{S}ch/X$, the category of separated finite type X -schemes, generated by members of the τ -coverings of X and with the Grothendieck topology induced by τ . X_τ denotes the ∞ -topos of sheaves of spaces on the small τ -site of X . This assignment of X_τ to a scheme X is functorial in

the sense that a morphism of schemes $f : X \rightarrow Y$ induces a morphism of ∞ -topoi $f_* : X_\tau \rightarrow Y_\tau$ given by $f_*(\mathcal{F})(-) := \mathcal{F}(- \times_Y X)$. In this chapter, the étale topology will be of greatest importance to us.

Let $\mathcal{T}\text{opoi}^R$ be the ∞ -category of ∞ -topoi with morphisms the geometric morphisms (Lurie [65] uses the notation $\mathcal{T}po^R$ instead). Recall that a geometric morphism $f_* : \mathcal{X} \rightarrow \mathcal{Y}$ between ∞ -topoi is a functor that admits a left exact left adjoint, designated by $f^* : \mathcal{Y} \rightarrow \mathcal{X}$. Given a geometric morphism $f_* : \mathcal{X} \rightarrow \mathcal{Y}$ with left adjoint $f^* : \mathcal{Y} \rightarrow \mathcal{X}$, we have a pro-left adjoint $f_! : \mathcal{X} \rightarrow \text{Pro}(\mathcal{Y})$ to f^* given by $f_!(X)(Y) := \text{Map}_{\mathcal{X}}(X, f^*Y)$. Indeed, note that by the Yoneda embedding, $f_!(X)(Y) \simeq \text{Map}_{\text{Pro}(\mathcal{S})}(f_!(X), Y)$. Therefore, we see that $f_!$ is a pro-left adjoint to f^* . In particular, we have a geometric morphism $\pi_* : \mathcal{X} \rightarrow \mathcal{S}$ that is unique up to homotopy ($\mathcal{S}$ is final in $\mathcal{T}\text{opoi}^R$ by proposition 6.3.4.1 of [65]). This has the constant sheaf functor $\pi^* : \mathcal{S} \rightarrow \mathcal{X}$ as its left adjoint. The pro-left adjoint $\pi_! : \mathcal{X} \rightarrow \text{Pro}(\mathcal{S})$ is given by

$$\pi_!(F)(K) = \text{Map}_{\mathcal{X}}(F, \pi^*K).$$

The *shape* $\Pi_\infty \mathcal{X}$ of the ∞ -topos $\mathcal{X}$ is defined as the pro-space $\pi_!(\mathbf{1}_{\mathcal{X}})$, where $\mathbf{1}_{\mathcal{X}}$ is the final object of $\mathcal{X}$. $\Pi_\infty : \mathcal{T}\text{opoi}^R \rightarrow \text{Pro}(\mathcal{S})$ is in fact a pro-left adjoint to the functor $\mathcal{S} \rightarrow \mathcal{T}\text{opoi}^R$ sending a space to the ∞ -topos of ∞ -sheaves on the space. See [47] for details regarding this functor Π_∞ .

Important examples come from Grothendieck sites. For example, if X is a scheme, then $\Pi_\infty^{\text{ét}} X := \Pi_\infty X_{\text{ét}}$ is the *étale topological type* of X . Note that this construction is functorial in the sense that a morphism of schemes $X \rightarrow Y$ functorially gives a morphism $\Pi_\infty^{\text{ét}} X \rightarrow \Pi_\infty^{\text{ét}} Y$ of étale topological types. We denote the functor by

$$\Pi_\infty^{\text{ét}} : \mathcal{S}\text{m}/S \rightarrow \text{Pro}(\mathcal{S}).$$

Remark 3.4.1. By the construction of $\Pi_\infty^{\text{ét}}X$, for constant coefficients A , $H^*(\Pi_\infty^{\text{ét}}X; A) \cong H_{\text{ét}}^*(X; A)$. This feature of the étale topological type allows us to prove theorems about étale cohomology by using algebraic topology. Artin-Mazur [3] produced a pro object in the homotopy category of spaces representing the étale topological type while Friedlander [34] rigidified the Artin-Mazur construction to the level of pro-spaces, again giving an explicit construction.

The ℓ -adic unstable étale realization functor on motivic spaces is constructed as follows. We first lift $\Pi_\infty^{\text{ét}} : \mathcal{S}m/S \rightarrow \text{Pro}(\mathcal{S})$ to a functor

$$\hat{\Pi}_\infty^{\text{ét}} : \text{Shv}_\infty^\tau(\mathcal{S}m/S) \rightarrow \text{Pro}(\mathcal{S})$$

that preserves small homotopy colimits. Here τ is any topology coarser than the étale topology, for example the Nisnevich topology. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{S}m/S & \xrightarrow{\Pi_\infty^{\text{ét}}} & \text{Pro}(\mathcal{S}) \\ \downarrow & \nearrow \hat{\Pi}_\infty^{\text{ét}} & \\ \mathcal{P}Sh_\infty(\mathcal{S}m/S) & & \\ \downarrow & \nearrow \hat{\Pi}_\infty^{\text{ét}} & \\ \text{Shv}_\infty^\tau(\mathcal{S}m/S) & & \end{array}$$

The middle dotted arrow can be constructed uniquely from $\Pi_\infty^{\text{ét}}$ because $\text{Pro}(\mathcal{S})$ is co-complete and every presheaf of spaces on $\mathcal{S}m/S$ is a colimit of representable presheaves of spaces. In order to lift to all of $\text{Shv}_\infty^\tau(\mathcal{S}m/S)$, we need the following lemma of Hoyois [46].

Lemma 3.4.2. *If U is an étale covering diagram in the small étale site of X , then $\Pi_\infty^{\text{ét}}X$ is the colimit of the diagram of pro-spaces $\Pi_\infty^{\text{ét}}U$.*

Proof. This is lemma 1.5 of Hoyois [46]. The proof is that the ∞ -topos $X_{\text{ét}}$ is the colimit in $\mathcal{T}\text{opoi}^R$ of the diagram of ∞ -topoi $U_{\text{ét}}$. Since $\Pi_\infty : \mathcal{T}\text{opoi}^R \rightarrow \text{Pro}(\mathcal{S})$ is a pro-left adjoint by construction, it preserves colimits. The conclusion follows. $\square$

Using this, we lift $\Pi_\infty^{\text{ét}}$ to a left adjoint $\hat{\Pi}_\infty^{\text{ét}} : \text{Shv}_\infty^\tau(\mathcal{S}\mathfrak{m}/S) \rightarrow \text{Pro}(\mathcal{S})$.

Remark 3.4.3. *Note that $\hat{\Pi}_\infty^{\text{ét}}$ must be the composition of $\text{Shv}_\infty^\tau(\mathcal{S}\mathfrak{m}/S) \hookrightarrow \mathcal{P}\text{Sh}_\infty(\mathcal{S}\mathfrak{m}/S)$ and $\Pi_\infty^{\text{ét}}$. However, the above lemma comes in because we want to show that sheafifying a presheaf of spaces in $\mathcal{P}\text{Sh}_\infty(\mathcal{S}\mathfrak{m}/S)$ and then applying $\Pi_\infty^{\text{ét}}$ is the same as applying $\Pi_\infty^{\text{ét}}$ to the presheaf itself. This requires more than just existence.*

Alternatively, $\Pi_\infty^{\text{ét}}$ can be globalized by instead working with the *big* étale site $(\mathcal{S}\mathfrak{m}/S)_{\text{ét}}$. Indeed, consider the global sections functor

$$\Gamma_* : \text{Shv}_\infty^{\text{ét}}(\mathcal{S}\mathfrak{m}/S) \rightarrow \mathcal{S}$$

with left adjoint

$$\Gamma^* : \mathcal{S} \rightarrow \text{Shv}_\infty^{\text{ét}}(\mathcal{S}\mathfrak{m}/S)$$

given by sending a space to the constant sheaf on the big étale site over S . This has a pro-left adjoint

$$\Gamma_! : \text{Shv}_\infty^{\text{ét}}(\mathcal{S}\mathfrak{m}/S) \rightarrow \text{Pro}(\mathcal{S}).$$

Note that $\Gamma_!(X) = \Pi_\infty(\text{Shv}_\infty^{\text{ét}}(\mathcal{S}\mathfrak{m}/S)/X)$. If X is representable, then this is the same as $\Pi_\infty X_{\text{ét}}$ (see remark 5.3 of Hoyois [46]). Therefore, $\hat{\Pi}_\infty^{\text{ét}}$ is given by the composition

$$\text{Shv}_\infty^\tau(\mathcal{S}\mathfrak{m}/S) \xrightarrow{a_{\text{ét}}} \text{Shv}_\infty^{\text{ét}}(\mathcal{S}\mathfrak{m}/S) \xrightarrow{\Gamma_!} \text{Pro}(\mathcal{S}).$$

This preserves colimits as it is the composition of (restrictions of) two left adjoint functors.

For any prime ℓ not equal to the residue characteristics of S , and X a separated finite type S -scheme, we have that the map $H_{\text{ét}}^*(X; \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H_{\text{ét}}^*(X \times_S \mathbb{A}_S^1; \mathbb{Z}/\ell\mathbb{Z})$ induced by the projection $X \times_S \mathbb{A}_S^1 \rightarrow X$ is an isomorphism (see VII, Cor. 1.2 of [42]). By lemma 3.2.1, there is a *Bousfield* localization functor $L_{\mathbb{A}^1} : \mathcal{S}h\mathbf{v}_{\infty}^{\tau}(\mathcal{S}m/S) \rightleftarrows \mathcal{S}pc_{\tau}(S)$. Therefore, the composition

$$\mathcal{S}h\mathbf{v}_{\infty}^{\tau}(\mathcal{S}m/S) \xrightarrow{\hat{\Pi}_{\infty}^{\text{ét}}} \text{Pro}(\mathcal{S}) \xrightarrow{L^{\mathbb{Z}/\ell}} \text{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$$

factors through $\mathcal{S}pc(S)$ to give us *unstable ℓ -adic étale realization*

$$\acute{\text{Et}}_{\ell} : \mathcal{S}pc_{\tau}(S) \rightarrow \text{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$$

of τ -motivic spaces (instead of the Nisnevich topology in the construction of motivic spaces, use the topology τ). In our future computations, we will need the following lemma.

Lemma 3.4.4. *For $p, q \geq 0$, $\acute{\text{Et}}_{\ell} S^{p,q} \simeq K(\mathbb{Z}_{\ell}, 1)^{\wedge p} \wedge K(T\mu_{\ell^{\infty}}, 1)^{\wedge q}$, where $T\mu_{\ell^{\infty}}$ is the Tate module of $\mu_{\ell^{\infty}}$.*

For a proof of this, see Proposition 6.2 of Hoyois [46]. Since k is assumed to be algebraically closed, we can choose an isomorphism $T\mu_{\ell^{\infty}} \simeq \mathbb{Z}_{\ell}$ and identify $\acute{\text{Et}}_{\ell} S^{p,q} \simeq K(\mathbb{Z}_{\ell}, 1)^{\wedge p+q}$.

We now construct the stable étale realization functor using analogues of this alternative description. Instead of considering sheaves of spaces, we can consider sheaves of E -modules for some E_{∞} -ring spectrum E . In this case, consider the global sections functor

$$\Gamma_*^E : \mathcal{S}h\mathbf{v}_{\infty}^{\text{ét}}(\mathcal{S}m/S, \text{Mod}_E) \rightarrow \text{Mod}_E$$

with left adjoint

$$\Gamma_E^* : \mathcal{M}od_E \rightarrow \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S, \mathcal{M}od_E)$$

given by sending an E -module M to its associated constant sheaf of E -modules on the big étale site over S . This functor preserves finite limits, and so by lemma 3.3.4, this has a pro-left adjoint

$$\Gamma_!^E : \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S, \mathcal{M}od_E) \rightarrow \text{Pro}(\mathcal{M}od_E).$$

We then consider the morphism

$$\Pi_\infty^E : \mathcal{S}hv_\infty^\tau(\mathcal{S}m/S, \mathcal{M}od_E) \xrightarrow{a_{\acute{e}t}} \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S, \mathcal{M}od_E) \xrightarrow{\Gamma_!^E} \text{Pro}(\mathcal{M}od_E).$$

Specializing to $E = \mathbb{S}$, we obtain the commutative diagram

$$\begin{array}{ccccc} \hat{\Pi}_\infty^{\acute{e}t} : \mathcal{S}hv_\infty^\tau(\mathcal{S}m/S) & \xrightarrow{a_{\acute{e}t}} & \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S) & \xrightarrow{\Gamma_!} & \text{Pro}(\mathcal{S}) \\ \Sigma_+^\infty \downarrow & & \downarrow \Sigma_+^\infty & & \downarrow \text{Pro}(\Sigma_+^\infty) \\ \Pi_\infty^{\mathbb{S}} : \mathcal{S}hv_\infty^\tau(\mathcal{S}m/S; \mathcal{S}pt) & \xrightarrow{a_{\acute{e}t}} & \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S; \mathcal{S}pt) & \xrightarrow{\Gamma_!^{\mathbb{S}}} & \text{Pro}(\mathcal{S}pt). \end{array}$$

Furthermore, we also have the commutative diagram

$$\begin{array}{ccc} \text{Pro}(\mathcal{S}) & \xrightarrow{L^{\mathbb{Z}/\ell}} & \text{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \\ \text{Pro}(\Sigma_+^\infty) \downarrow & & \downarrow \Sigma_{S_\ell^1}^\infty + \\ \text{Pro}(\mathcal{S}pt) & \xrightarrow{L^{H\mathbb{Z}/\ell}} & \text{Pro}(\mathcal{S}pt)^{H\mathbb{Z}/\ell}. \end{array}$$

The previous two diagrams fit together to form

$$\begin{array}{ccccccc} \mathcal{S}hv_\infty^\tau(\mathcal{S}m/S) & \xrightarrow{a_{\acute{e}t}} & \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S) & \xrightarrow{\Gamma_!} & \text{Pro}(\mathcal{S}) & \xrightarrow{L^{\mathbb{Z}/\ell}} & \text{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \\ \Sigma_+^\infty \downarrow & & \downarrow \Sigma_+^\infty & & \downarrow \text{Pro}(\Sigma_+^\infty) & & \downarrow \Sigma_{S_\ell^1}^\infty + \\ \mathcal{S}hv_\infty^\tau(\mathcal{S}m/S; \mathcal{S}pt) & \xrightarrow{a_{\acute{e}t}} & \mathcal{S}hv_\infty^{\acute{e}t}(\mathcal{S}m/S; \mathcal{S}pt) & \xrightarrow{\Gamma_!^{\mathbb{S}}} & \text{Pro}(\mathcal{S}pt) & \xrightarrow{L^{H\mathbb{Z}/\ell}} & \text{Pro}(\mathcal{S}pt)^{H\mathbb{Z}/\ell}. \end{array}$$

As in the construction of the unstable étale realization, we can see that $L^{H\mathbb{Z}/\ell}\Pi_\infty^E$ factors through the $\mathbb{A}^1$ -localized τ -sheaves of E -modules. This gives us the S^1 -stable étale realization

$$\acute{\text{Et}}_\ell^{\text{Spt}} : \text{Spt}_\tau^{S^1}(S) \rightarrow \text{Pro}(\text{Spt})^{H\mathbb{Z}/\ell}.$$

These give us the following commutative diagram of étale realizations:

$$\begin{array}{ccccc} \acute{\text{Et}}_\ell : \text{Spc}_\tau(S) & \xrightarrow{a_{\acute{\text{et}}}} & \text{Spc}_{\acute{\text{et}}}(S) & \xrightarrow{L^{\mathbb{Z}/\ell}\Gamma_!} & \text{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \\ \Sigma_+^\infty \downarrow & & \Sigma_+^\infty \downarrow & & \downarrow \Sigma_{S_\ell^1}^\infty \\ \acute{\text{Et}}_\ell^{\text{Spt}} : \text{Spt}_\tau^{S^1}(S) & \xrightarrow{a_{\acute{\text{et}}}} & \text{Spt}_{\acute{\text{et}}}^{S^1}(S) & \xrightarrow{L^{H\mathbb{Z}/\ell}\Gamma_!^{\mathcal{S}}} & \text{Pro}(\text{Spt})^{H\mathbb{Z}/\ell} \end{array}$$

From now on, we restrict to $S = \text{Spec } k$, where k is a separably closed field. The final step in our construction is to invert $\mathbb{G}_m$ in the bottom row of the above diagram specialized to $S = \text{Spec } k$. We do so by showing that there is a commutative diagram of the following form.

$$\begin{array}{ccccc} \acute{\text{Et}}_\ell : \text{Spc}_\tau(k) & \xrightarrow{a_{\acute{\text{et}}}} & \text{Spc}_{\acute{\text{et}}}(k) & \xrightarrow{L^{\mathbb{Z}/\ell}\Gamma_!} & \text{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \\ \Sigma_+^\infty \downarrow & & \Sigma_+^\infty \downarrow & & \downarrow \Sigma_{S_\ell^1}^\infty \\ \acute{\text{Et}}_\ell^{\text{Spt}} : \text{Spt}_\tau^{S^1}(k) & \xrightarrow{a_{\acute{\text{et}}}} & \text{Spt}_{\acute{\text{et}}}^{S^1}(k) & \xrightarrow{L^{H\mathbb{Z}/\ell}\Gamma_!^{\mathcal{S}}} & \text{Pro}(\text{Spt})^{H\mathbb{Z}/\ell} \\ \Sigma_{\mathbb{G}_m}^\infty \downarrow & & \Sigma_{\mathbb{G}_m}^\infty \downarrow & & \downarrow \Sigma_{S_\ell^1}^\infty \\ \text{Spt}_\tau(k) & \xrightarrow{a_{\acute{\text{et}}}} & \text{Spt}_{\acute{\text{et}}}(k) & \cdots \cdots \cdots & \text{Pro}(\text{Spt})^{H\mathbb{Z}/\ell} \end{array}$$

We need to construct the dotted functor

$$\text{Spt}_{\acute{\text{et}}}(k) \rightarrow \text{Pro}(\text{Spt})^{H\mathbb{Z}/\ell}$$

so that the lower right square commutes. We do so by first checking that the square

$$\begin{array}{ccc} \mathcal{Spt}_{\acute{e}t}^{S^1}(k)^\omega & \xrightarrow{F} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell} \\ \Sigma_{\mathbb{G}_m} \downarrow & & \Sigma_{S^1_\ell} \downarrow \\ \mathcal{Spt}_{\acute{e}t}^{S^1}(k)^\omega & \xrightarrow{F} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}, \end{array}$$

where $F := L^{H\mathbb{Z}/\ell}\Gamma_!^{\mathbb{S}}$ and $\mathcal{Spt}_{\acute{e}t}^{S^1}(k)^\omega$ is the full subcategory of $\mathcal{Spt}_{\acute{e}t}^{S^1}(k)$ spanned by the ω -compact objects, commutes. Indeed, take $\Sigma_{S^1}^{a+\infty}X_+$, X a smooth k -scheme and $a \in \mathbb{Z}$. Such objects compactly generate $\mathcal{Spt}_{\acute{e}t}^{S^1}(k)$ under colimits and extensions (k is algebraically closed, and so a descent spectral sequence shows that $\mathcal{Spt}_{\acute{e}t}^{S^1}(k)$ is compactly generated), and our functors are exact and preserve colimits. It therefore suffices to check commutativity of the above square on such objects. Note that if $\mathcal{Spc}_{\mathrm{fin}}(k)$ is the full subcategory of $\mathcal{Spc}(k)$ generated by $\mathcal{S}m/k$ under finite colimits, then $\acute{E}t_\ell|_{\mathcal{Spc}_{\mathrm{fin}}(k)} : \mathcal{Spc}_{\mathrm{fin}}(\mathcal{S}m/k) \rightarrow \mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell}$ preserves finite products. This is lemma 6.1 of [46]. We remark that it is not necessarily true that $\acute{E}t_\ell$ preserves finite products on all of $\mathcal{Spc}(k)$ because the Kunneth formula requires some finiteness conditions. Using this, we deduce that

$$\begin{aligned} F(\Sigma_{\mathbb{G}_m} \Sigma_{S^1}^{a+\infty} X_+) &\simeq L^{H\mathbb{Z}/\ell}\Gamma_!^{\mathbb{S}}(\Sigma_{S^1}^\infty(\Sigma_{\mathbb{G}_m} \Sigma_{S^1}^a X_+)) \\ &\simeq \Sigma_{S^1_\ell}^\infty L^{\mathbb{Z}/\ell}\Gamma_!(\Sigma_{\mathbb{G}_m} \Sigma_{S^1}^a X_+) \\ &\simeq \Sigma_{S^1_\ell}^{a+1+\infty} L^{\mathbb{Z}/\ell}\Gamma_!(X_+) \\ &\simeq \Sigma_{S^1_\ell} L^{H\mathbb{Z}/\ell}\Gamma_!^{\mathbb{S}}(\Sigma_{S^1}^{a+\infty} X_+) \\ &\simeq \Sigma_{S^1_\ell} F(\Sigma_{S^1}^{a+\infty} X_+). \end{aligned}$$

By proposition 5.5.7.8 of [65], $\mathcal{Spt}_{\acute{e}t}(k)^\omega$ is the colimit

$$\mathrm{colim} \left(\mathcal{Spt}_{\acute{e}t}^{S^1}(k)^\omega \xrightarrow{\Sigma_{\mathbb{G}_m}} \mathcal{Spt}_{\acute{e}t}^{S^1}(k)^\omega \xrightarrow{\Sigma_{\mathbb{G}_m}} \dots \right)$$

in $\mathcal{Cat}_\infty(\omega)$, the ∞ -category of small ∞ -categories admitting ω -filtered colimits and with morphisms those functors that preserve ω -filtered colimits. We can similarly define $\widehat{\mathcal{Cat}}_\infty(\omega)$, but with big ∞ -categories instead. $\mathcal{Cat}_\infty(\omega) \subset \widehat{\mathcal{Cat}}_\infty(\omega)$ preserves colimits. Taking colimits of the above square in the category $\widehat{\mathcal{Cat}}_\infty(\omega)$, we obtain the commutative square

$$\begin{array}{ccc} \mathcal{Spt}_{\acute{e}t}^{S^1}(k)^\omega & \xrightarrow{F} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell} \\ \Sigma_{\mathbb{G}_m}^\infty \downarrow & & \Sigma_{S_\ell^1}^\infty \downarrow \\ \mathcal{Spt}_{\acute{e}t}(k)^\omega & \xrightarrow{F} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}, \end{array}$$

from which we get the commutative square

$$\begin{array}{ccc} \mathcal{Spt}_{\acute{e}t}^{S^1}(k) & \xrightarrow{F} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell} \\ \Sigma_{\mathbb{G}_m}^\infty \downarrow & & \Sigma_{S_\ell^1}^\infty \downarrow \\ \mathcal{Spt}_{\acute{e}t}(k) & \xrightarrow{F} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell} \end{array}$$

by left Kan extensions. We thus obtain the commutative diagram

$$\begin{array}{ccccc} \acute{E}t_\ell : \mathcal{Spc}_\tau(k) & \xrightarrow{a_{\acute{e}t}} & \mathcal{Spc}_{\acute{e}t}(k) & \xrightarrow{L^{\mathbb{Z}/\ell}\Gamma!} & \mathrm{Pro}(\mathcal{S})^{\mathbb{Z}/\ell} \\ \Sigma_{\mathbb{P}^1+}^\infty \downarrow & & \Sigma_{\mathbb{P}^1+}^\infty \downarrow & & \downarrow \Sigma_{S_\ell^2+}^\infty \\ \underline{\acute{E}t}_\ell : \mathcal{Spt}_\tau(k) & \xrightarrow{a_{\acute{e}t}} & \mathcal{Spt}_{\acute{e}t}(k) & \xrightarrow{L^{H\mathbb{Z}/\ell}\Gamma^{\mathcal{S}}} & \mathrm{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}. \end{array}$$

The lower row corresponds to an exact functor $\underline{\acute{E}t}_\ell$, which we call the *stable ℓ -adic étale realization functor*. Furthermore, it is not hard to see that this functor is oplax monoidal as it is constructed from the pro-left adjoint of a monoidal functor.

3.5 Two Spectral Sequences

In this section, we discuss the convergence of Voevodsky's slice spectral sequence and that of the spectral sequence obtained from the stable ℓ -adic étale realization of

the slice tower. We show that both spectral sequences are strongly convergent under suitable conditions. It is by comparing these two spectral sequences that we will prove our main theorems.

Let $\mathcal{Spt}_{\text{fin}}(k)$ to be the smallest full subcategory of $\mathcal{Spt}(k)$ containing $\Sigma_{\mathbb{P}^1}^n \Sigma_+^\infty X$, $n \in \mathbb{Z}$, $X \in \mathcal{Sm}/k$, and closed under finite colimits and extensions. Call such motivic spectra *finite motivic spectra*. In [62], Levine calls them finite spectra as apposed to finite *motivic* spectra. Given a finite effective motivic spectrum $E \in \mathcal{Spt}_{\text{fin}}^{\text{eff}}(k)$, we obtain the slice tower

$$\dots \rightarrow f_{t+1}(E) \rightarrow f_t(E) \rightarrow \dots \rightarrow f_1(E) \rightarrow f_0(E) = E$$

as discussed in section 3.2. This gives rise to the following exact couple:

$$\begin{array}{ccccc} \dots \pi_{*,0}(f_{q+1}(E))(k) & \xrightarrow{i} & \pi_{*,0}(f_q(E))(k) & \longrightarrow & \dots & \pi_{*,0}(f_1(E))(k) & \xrightarrow{i} & \pi_{*,0}(E)(k) \\ & \nwarrow k & \downarrow j & & & \nwarrow k & \downarrow j & \\ \dots & & \pi_{*,0}(s_q(E))(k) & & & \dots & & \pi_{*,0}(s_1(E))(k). \end{array}$$

Since $\mathbb{Z}_\ell$ is a flat $\mathbb{Z}$ -module, tensoring with $\mathbb{Z}_\ell$ gives the exact couple

$$\begin{array}{ccc} \pi_{*,0}(f_\bullet E)(S) \otimes \mathbb{Z}_\ell & \xrightarrow{i \otimes \mathbb{Z}_\ell} & \pi_{*,0}(f_\bullet E)(S) \otimes \mathbb{Z}_\ell \\ & \nwarrow k \otimes \mathbb{Z}_\ell \quad \swarrow j \otimes \mathbb{Z}_\ell & \\ & \pi_{*,0}(s_\bullet E)(S) \otimes \mathbb{Z}_\ell, & \end{array}$$

where the $\bullet$ denotes the indices of the tower and the $*$ denotes the grading of the homotopy groups. i decreases $\bullet$ by 1, while k has degree -1 (in $*$).

For each q , consider the filtration

$$F^{n,q} := \text{Filt}^n \pi_{*,0}(f_q(E))(k)[1/p] := \text{Im}(i^{(n)} : \pi_{*,0}(f_{n+q}(E))(k) \rightarrow \pi_{*,0}(f_q(E))(k))[1/p].$$

By theorem 7.3 of Levine [62], since E finite, we have that $F^{n,q} = 0$ for $n \gg 0$, where this lower bound is not universal and depends on $*$. Note that $F^{n,q} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} = \text{Im}(i^{(n)}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \cong \text{Im}(i^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell})$. The claim is that this last exact couple induces a *strongly convergent* spectral sequence

$$E_{p,q}^2 = \pi_{p+q,0}(s_q E)(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \implies \pi_{p+q,0}(E)(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$$

where the differentials on the r -th page are given by

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r.$$

Let $Q^q := \bigcap_n (F^{n,q} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell})$ and $RQ^s := R\lim_n F^{n,q} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$. In our case, $Q^q = 0$ for every q by what was said above. It is also easy to see that since $F^{n,q} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} = 0$ for $n \gg 0$, $RQ^s = 0$. Note that $\lim_q \pi_{*,0}(f_q(E))(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} = \lim_q Q^q = 0$, and so the spectral sequence is *Hausdorff*. By the Mittag-Leffler condition,

$$0 \longrightarrow R\lim_q Q^q \longrightarrow R\lim_q (\pi_{*,0}(f_q(E))(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}) \longrightarrow \lim_q RQ^q \longrightarrow 0,$$

and these conditions, we deduce that $R\lim_q (\pi_{*,0}(f_q(E))(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}) = 0$, i.e. the associated spectral sequence is *complete*. We conclude that the spectral sequence

$$E_{p,q}^2 = \pi_{p+q,0}(s_q(E))(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \implies \pi_{p+q,0}(E)(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$$

is *at least conditionally convergent*. Consider the exact sequence

$$0 \longrightarrow \frac{F^{n,0} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}}{F^{n+1,0} \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}} \longrightarrow E_{\infty}^n \longrightarrow Q^{n+1} \longrightarrow Q^n \longrightarrow RE_{\infty}^n \longrightarrow RQ^{n+1} \longrightarrow RQ^n \longrightarrow 0$$

of lemma 5.6 of Boardman [16]. Since $Q^n = 0$ for all n , we conclude from the exact sequence that $RE_{\infty}^n = 0$. As a result of conditional convergence, $RE_{\infty}^n = 0$, and the

fact that $E_s^r = 0$ for $s < 0$, we can conclude the following proposition via Theorem 7.1 of Boardman [16].

Proposition 3.5.1. *If $E \in \mathcal{Spt}_{\text{fin}}^{\text{eff}}(k)$ is a finite effective motivic spectrum over an algebraically closed field k , we have a strongly convergent spectral sequence*

$$E_{p,q}^2 = \pi_{p+q,0}(s_q(E))(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \implies \pi_{p+q,0}(E)(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell},$$

where the differentials on the r -th page are given by

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r.$$

We will now study the stable ℓ -adic étale analogue of the slice tower above. Applying the stable ℓ -adic étale realization $\underline{\mathcal{E}t}_{\ell}$, we obtain the tower

$$\dots \rightarrow \underline{\mathcal{E}t}_{\ell}(f_{t+1}(E)) \rightarrow \underline{\mathcal{E}t}_{\ell}(f_t(E)) \rightarrow \dots \rightarrow \underline{\mathcal{E}t}_{\ell}(E).$$

Since $\underline{\mathcal{E}t}_{\ell}$ is an exact functor, we obtain a spectral sequence

$$E_{p,q}^2 = \pi_{p+q} \left(\underline{\mathcal{E}t}_{\ell}(s_q(E)) \right) \implies \pi_{p+q} \left(\underline{\mathcal{E}t}_{\ell}(E) \right),$$

where the differentials on the r -th page are given by

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r.$$

In order to establish the strong convergence of this spectral sequence, we need the following connectivity result. Recall that a motivic spectrum $E \in \mathcal{Spt}(k)$ is *topologically N -connected* if for all $a \leq N$ and $b \in \mathbb{Z}$, $\pi_{a,b}(E) = 0$.

Proposition 3.5.2. *If $E \in \mathcal{Spt}(k)$ is topologically $(N-1)$ -connected, then $\underline{\mathcal{E}t}_{\ell}(f_q(E))$ is $(q + N - 1)$ -connected.*

Proof. Note that $\Sigma^{m,n}\Sigma_+^\infty X$ has stable ℓ -adic étale realization $\Sigma_{S_\ell^1}^{m+n}\Sigma_{S_\ell^2}^\infty \acute{\text{Et}}_\ell X_+$ which is $(m+n-1)$ -connected. $\mathcal{Spt}(k)_{\geq(N,q)}$ is generated under homotopy colimits and extensions by $\{\Sigma^{m,n}\Sigma_+^\infty Y | Y \in \mathcal{S}m/k, m \geq N \text{ and } n \geq q\}$, and so every $E \in \mathcal{Spt}(k)_{\geq(N,q)}$ has étale realization that is $(N+q-1)$ -connected. Therefore, by proposition 4.7 of Levine [63], if E is topologically $(N-1)$ -connected, then $\underline{\acute{\text{Et}}}_\ell(f_q(E))$ is $(q+N-1)$ -connected, as required. $\square$

By proposition 3.5.2, $\underline{\acute{\text{Et}}}_\ell(f_q(E))$ is $(q-1)$ -connected in $\text{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}$, and so this spectral sequence is also strongly convergent. As a result, we conclude that

Proposition 3.5.3. *If $E \in \mathcal{Spt}_{\text{fin}}^{\text{eff}}(k)$ is a finite effective motivic spectrum over an algebraically closed field k , the stable ℓ -adic étale realization of the slice tower gives a strongly convergent spectral sequence*

$$E_{p,q}^2 = \pi_{p+q} \left(\underline{\acute{\text{Et}}}_\ell(s_q(E)) \right) \implies \pi_{p+q} \left(\underline{\acute{\text{Et}}}_\ell(E) \right),$$

where the differentials on the r -th page are given by

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r.$$

3.6 Proof of the Main Theorem

In this section, we prove our main result saying that up to inversion of the exponential characteristic p , classical stable homotopy theory is subsumed by stable motivic homotopy theory.

Before stating our theorem, we need to state some generalities about inverting primes in stable ∞ -categories. Suppose $\mathcal{C}$ is a presentable stable ∞ -category. Let $\mathcal{C}[1/p]$ be the full subcategory of p -local objects, that is, those objects $E \in \mathcal{C}$ such

that $p \cdot \text{id}_E$ is an isomorphism. It can be shown that if $\mathcal{C}$ is compactly generated presentable ∞ -category, for example $\mathcal{S}\text{pt}$, the inclusion $\mathcal{C}[1/p] \hookrightarrow \mathcal{C}$ admits a left adjoint which we denote by $(-)[1/p]$, and that $\mathcal{C}[1/p]$ is the Verdier quotient by the smallest localizing subcategory containing the cofibers of $p \cdot \text{id}_E$ for each compact E in $\mathcal{C}$. Furthermore, if E is compact in $\mathcal{C}$ and F is also in $\mathcal{C}$, then $\text{Map}_{\mathcal{C}}(E, F)[1/p] \rightarrow \text{Map}_{\mathcal{C}[1/p]}(E[1/p], F[1/p])$ is an equivalence. See proposition A.2.8 and corollary A.2.12 of Kelly [56]. Both categories $\mathcal{S}\text{pt}$ and $\mathcal{S}\text{pt}(k)$ have this kind of localization.

Let $\Sigma_{\mathbb{G}_m} : \mathcal{S}_* \rightarrow \mathcal{S}\text{pc}_*(k)$ be the functor that suspends pointed spaces by $\mathbb{G}_m$. We can stabilize this functor to obtain the functor $c := \Sigma_{\mathbb{G}_m}^\infty : \mathcal{S}\text{pt} \rightarrow \mathcal{S}\text{pt}(k)$, which is a left adjoint. Localizing, we obtain $c[1/p] : \mathcal{S}\text{pt}[1/p] \rightarrow \mathcal{S}\text{pt}(k)[1/p]$. One of the two main theorems of this chapter is that away from the exponential characteristic of k , k an algebraically closed field, c is fully faithful.

Theorem 3.6.1. *Let k be an algebraically closed field of exponential characteristic p . Then*

$$c[1/p] : \mathcal{S}\text{pt}[1/p] \rightarrow \mathcal{S}\text{pt}(k)[1/p]$$

is a fully faithful functor of stable ∞ -categories.

We deduce this theorem as a corollary of the following special case.

Theorem 3.6.2. *Let k be an algebraically closed field of exponential characteristic p . Then for every prime $\ell \neq p$, the stable ℓ -adic étale realization induces an isomorphism*

$$\underline{\text{Ét}}_{\ell*} : \underline{\pi}_{*,0}(\mathbf{1}_k)(k) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \rightarrow \pi_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}.$$

Indeed, assuming theorem 3.6.2, we prove theorem 3.6.1 as follows.

Proof of theorem 3.6.1. Note that for any space $X \in \mathcal{S}$,

$$\underline{\mathcal{E}t}_\ell \circ c(\Sigma_{S^1}^\infty X_+) \simeq \underline{\mathcal{E}t}_\ell(\Sigma_{\mathbb{P}^1}^\infty X_+) \simeq \Sigma_{S_\ell^2}^\infty \underline{\mathcal{E}t}_\ell X_+$$

the latter spectrum being in $\text{Pro}(\mathcal{S}\text{pt})^{H\mathbb{Z}/\ell}$. This follows from lemma 3.4.4 and, by construction, the compatibility of the stable ℓ -adic realisation functor with its unstable version. For each $\ell \neq p$ any prime, the composition

$$[\Sigma_{S^1}^n \mathbb{S}, \mathbb{S}]_{\mathcal{S}\text{pt}} \otimes \mathbb{Z}_\ell \xrightarrow{c_* \otimes \mathbb{Z}_\ell} [\Sigma_{S^1}^n \mathbf{1}_k, \mathbf{1}_k]_{\mathcal{S}\text{pt}(k)} \otimes \mathbb{Z}_\ell \xrightarrow{\underline{\mathcal{E}t}_{\ell*}} [\Sigma_{S_\ell^1}^n L^{H\mathbb{Z}/\ell} \mathbb{S}, L^{H\mathbb{Z}/\ell} \mathbb{S}]_{\text{Pro}(\mathcal{S}\text{pt})^{H\mathbb{Z}/\ell}}$$

is an isomorphism. Since theorem 3.6.2 implies that the second map is an isomorphism as well, we deduce that $c_* \otimes \mathbb{Z}_\ell$ is an isomorphism. Since $\mathbb{Z}_{(\ell)}$ is a local Noetherian ring, its completion $\mathbb{Z}_\ell$ is a faithfully flat $\mathbb{Z}_{(\ell)}$ -module. Therefore, $c_* \otimes \mathbb{Z}_{(\ell)}$ is an isomorphism for each $\ell \neq p$, which, in turn, implies that $c_*[1/p]$ is an isomorphism. Let $\mathcal{R}$ be the full subcategory of $\mathcal{S}\text{pt}[1/p]$ having all objects $E \in \mathcal{S}\text{pt}[1/p]$ such that

$$c[1/p]_* : [\Sigma_{S^1}^n \mathbb{S}[1/p], E]_{\mathcal{S}\text{pt}[1/p]} \rightarrow [c(\Sigma_{S^1}^n \mathbb{S}[1/p]), c(E)]_{\mathcal{S}\text{pt}(k)[1/p]}$$

is an isomorphism for every n . $c[1/p]_*$ preserves small colimits and both $\Sigma_{S^1}^n \mathbb{S}[1/p]$ and $c(\Sigma_{S^1}^n \mathbb{S}[1/p]) \simeq \Sigma_{S^1}^n \mathbf{1}_k[1/p]$ are compact. By the above calculations, $\mathbb{S}[1/p] \in \mathcal{R}$. Indeed, as stated above, by corollary A.2.12 of Kelly [56], the isomorphism above is implied by theorem 3.6.2 above and the calculations at the beginning of this proof. Since the stable ∞ -category $\mathcal{S}\text{pt}[1/p]$ is compactly generated by $\mathbb{S}[1/p]$, we conclude that $\mathcal{R} = \mathcal{S}\text{pt}[1/p]$, that is

$$c[1/p]_* : [\Sigma_{S^1}^n \mathbb{S}, E]_{\mathcal{S}\text{pt}[1/p]} \rightarrow [c(\Sigma_{S^1}^n \mathbb{S}), c(E)]_{\mathcal{S}\text{pt}(k)[1/p]}$$

is an isomorphism for every $E \in \mathcal{Spt}[1/p]$. We can define $\mathcal{L}$ as the full subcategory of $\mathcal{Spt}[1/p]$ of p -local spectra E such that

$$c[1/p]_* : [E, F]_{\mathcal{Spt}[1/p]} \rightarrow [c(E), c(F)]_{\mathcal{Spt}(k)[1/p]}$$

is an isomorphism for every $F \in \mathcal{Spt}[1/p]$. Indeed, fix an $F \in \mathcal{Spt}[1/p]$. By the argument above, we know that the map is an isomorphism for $E = \Sigma_{\mathbb{S}^1}^n \mathbb{S}[1/p]$. Note that if $E = \text{colim } E_i$ is a small colimit of compact objects in the stable ∞ -category of p -local spectra, then $[E, F]_{\mathcal{Spt}[1/p]} = \lim [E_i, F]_{\mathcal{Spt}[1/p]}$ by the universal property of colimits. Furthermore, $c[1/p]$ preserves colimits as it is a (colimit of) left adjoint(s). Therefore, $[c(E), c(F)]_{\mathcal{Spt}(k)[1/p]} = \lim [c(E_i), c(F)]_{\mathcal{Spt}(k)[1/p]}$. In fact, the map above is the limit of the maps $[E_i, F]_{\mathcal{Spt}[1/p]} \rightarrow [c(E_i), c(F)]_{\mathcal{Spt}(k)[1/p]}$. Since $\mathbb{S}[1/p]$ compactly generates $\mathcal{Spt}[1/p]$, the same argument as above shows that $\mathcal{L} = \mathcal{Spt}[1/p]$. We conclude that $c[1/p]$ is a fully faithful functor. $\square$

A direct corollary of theorem 3.6.2 is motivic Serre finiteness for algebraically closed fields.

Corollary 3.6.3 (Special Case of Motivic Serre Finiteness). *For k an algebraically closed field and $n > 0$ an integer, $\pi_{n,0}(\mathbf{1}_k)(k) \otimes \mathbb{Q} = 0$.*

Remark 3.6.4. It is now a theorem of Ananyevskiy, Levine, and Panin that motivic Serre finiteness is valid for all fields k [2].

In order to prove theorem 3.6.2, we compare the two spectral sequences of the previous section to prove a number of comparison theorems. In combination, these comparison theorems will give us the above theorem. Given a triangulated category $\mathcal{T}$, we define $\mathcal{T}_{\text{tor}}$ to be the full subcategory with objects E such that $\text{Hom}_{\mathcal{T}}(A, E) \otimes \mathbb{Q} = 0$ for every compact object A in $\mathcal{T}$. These objects will be called *torsion* objects. Given a stable ∞ -category $\mathcal{C}$, $\mathcal{C}_{\text{tor}}$ is the full subcategory consisting of the torsion objects in its homotopy category $\text{h}\mathcal{C}$ which is a triangulated category.

Definition 3.6.5. *For each commutative ring R , MR is the Eilenberg-MacLane spectrum representing motivic cohomology with coefficients in R . It can be defined as the motivic spectrum*

$$MR := \operatorname{colim}_{n \rightarrow \infty} \Sigma_{\mathbb{P}^1}^{-n} \Sigma_{\mathbb{P}^1}^{\infty} K(R(n), 2n).$$

We need the following result of Hoyois, Kelly, and Østvær.

Lemma 3.6.6. *(Theorem 5.8 of [48]) Let k be a perfect field of exponential characteristic p , and let R be a commutative ring in which p is invertible. There is a symmetric monoidal Quillen equivalence*

$$\Phi : \operatorname{Mod}_{MR} \rightleftarrows \operatorname{DM}(k; R) : \Psi.$$

Taking $R = \mathbb{Z}[1/p]$, we obtain an equivalence $\Psi : \operatorname{DM}(k; \mathbb{Z}[1/p]) \xrightarrow{\sim} \operatorname{Mod}_{M\mathbb{Z}[1/p]}$.

Definition 3.6.7. *Let*

$$\operatorname{EM} : \operatorname{DM}(k; \mathbb{Z}[1/p]) \rightarrow \operatorname{Spt}(k)$$

be the functor given by the equivalence $\operatorname{DM}(k; \mathbb{Z}[1/p]) \simeq \operatorname{Mod}_{M\mathbb{Z}[1/p]}$ followed by the forgetful functor $\operatorname{Mod}_{M\mathbb{Z}[1/p]} \rightarrow \operatorname{Spt}(k)$. Furthermore, let

$$\mathbb{Z}[1/p]^{\operatorname{tr}} : \operatorname{Spt}(k) \rightarrow \operatorname{DM}(k; \mathbb{Z}[1/p])$$

be the left adjoint of EM .

The functors in definition 3.6.7 have the property that $(\operatorname{EM} \circ \mathbb{Z}[1/p]^{\operatorname{tr}})(X) = M\mathbb{Z}[1/p] \wedge X_+$ for every $X \in \operatorname{Sm}/k$. We claim that EM preserves small (homotopy) colimits.

Lemma 3.6.8. *EM preserves small (homotopy) colimits.*

Proof. Indeed, it is easy to show that an ∞ -functor $R : \mathcal{A} \rightarrow \mathcal{B}$ between cocomplete compactly generated stable ∞ -categories preserves all small colimits if it has a left adjoint $L : \mathcal{B} \rightarrow \mathcal{A}$ with the property that $L(B)$ is compact for B in a set of compact generators of $\mathcal{B}$. Applying this to $R = \text{EM}$ and $L = \mathbb{Z}[1/p]^{\text{tr}}$, choosing the compact generators $\Sigma^{p,q}\Sigma_{\mathbb{P}^1}^\infty X_+$, $p, q \in \mathbb{Z}$, $X \in \mathcal{S}\text{m}/k$, and noting that $\mathbb{Z}[1/p]^{\text{tr}}(\Sigma^{p,q}\Sigma_{\mathbb{P}^1}^\infty X_+) \simeq \mathbb{Z}[1/p]^{\text{tr}}(X)(q)[p+q]$ is compact in $\text{DM}(k; \mathbb{Z}[1/p])$, we deduce that EM preserves all small (homotopy) colimits. $\square$

By theorem 3.6.22 of Palaez [75], for each motivic spectrum E , $s_q E$ has the natural structure of an $M\mathbb{Z}$ -module, and so $s_q(E)[1/p]$ is an $M\mathbb{Z}[1/p]$ -module. By the above discussion, $s_q(E)[1/p] = \text{EM}(\pi_q^\mu(E)(q)[2q])$ for some motive $\pi_q^\mu(E) \in \text{DM}^{\text{eff}}(k; \mathbb{Z}[1/p])$.

Lemma 3.6.9. *(Lemmas 6.1 and 6.2 of [63]) Suppose k is a field of finite cohomological dimension. Then for $X \in \mathcal{S}\text{m}/k$ of dimension d over k , and for $q \geq d+1$, $f_q(\Sigma_{\mathbb{P}^1}^\infty X_+)$ and $s_q(\Sigma_{\mathbb{P}^1}^\infty X_+)$ are in $\mathcal{S}\text{pt}(k)_{\text{tor}}$, and $\pi_q^\mu(\Sigma_{\mathbb{P}^1}^\infty X_+)$ is in $\text{DM}^{\text{eff}}(k)_{\text{tor}}$. In particular, for $q \geq 1$, $f_q(\mathbf{1})$ and $s_q(\mathbf{1})$ are in $\mathcal{S}\text{pt}(k)_{\text{tor}}$, and $\pi_q^\mu(\mathbf{1})$ is in $\text{DM}^{\text{eff}}(k)_{\text{tor}}$.*

Lemma 3.6.10. *(Lemma 6.3 of [63]) If $E \in \mathcal{S}\text{pt}(k)_{\text{tor}}$, then for all q , $f_q(E), s_q(E) \in \mathcal{S}\text{pt}(k)_{\text{tor}}$ and $\pi_q^\mu(E) \in \text{DM}^{\text{eff}}(k)_{\text{tor}}$.*

In order to prove the next crucial proposition, we need the following lemma on motivic cohomology. Though this is well-known, we provide its short proof for the convenience of the reader. Suppose $\tau \in \{\text{ét}, \text{Nis}\}$ is a choice of a Grothendieck topology, R a commutative ring, and k any field of finite cohomological dimension. Designate by $D_\tau := D(\text{Shv}_\tau^{\text{tr}}(k; R))$ the unbounded derived category associated to the abelian category of τ -sheaves of R -modules with transfers $\text{Shv}_\tau^{\text{tr}}(k; R)$.

Lemma 3.6.11. *For every smooth k -scheme X , every complex $\underline{K} \in D_\tau$, and every $i \in \mathbb{Z}$, we have*

$$\text{Ext}_{\text{Shv}_\tau^{\text{tr}}(k; R)}^i(R_{\text{tr}}(X), \underline{K}) = \mathbb{H}_\tau^i(X; \underline{K}).$$

Proof. First, we reproduce the proof of this when the complex $\underline{K}$ is a single τ -sheaf of R -modules with transfers (concentrated in degree 0), i.e. we show that if F is a τ -sheaf of R -modules with transfer, then for each smooth k -scheme X and each $i \in \mathbb{Z}$ we have

$$\mathrm{Ext}_{\mathrm{Shv}_\tau^{tr}(k;R)}^i(R_{\mathrm{tr}}(X), F) = H_\tau^i(X; F).$$

This is proved in lemma 6.23 of [68] for $\tau = \text{ét}$ and in lemma 13.4 of [68] for $\tau = \text{Nis}$. For $i = 0$, $\mathrm{Hom}(R_{\mathrm{tr}}(X), F) = F(X)$ by the Yoneda lemma. For $i > 0$, it suffices to prove that if F is an injective τ -sheaf of R -modules with transfers then $H^i(X; F) = 0$. For the canonical flasque resolution $E^*(F)$ of F by étale sheaves, we have that the inclusion $F \rightarrow E^0$ splits by the injectivity of F . Therefore, F is a direct summand of E^0 in $\mathrm{Shv}_\tau^{tr}(k; R)$. Since $H_\tau^i(X; F)$ is a direct summand of $H_\tau^i(X; E^0)$, it must vanish for $i > 0$.

We prove the general theorem for the complex $\underline{K}$ using the hypercohomology spectral sequence. Indeed, we have the following morphism of spectral sequences

$$\begin{array}{ccc} E_2^{p,q} := \mathrm{Ext}_{\mathrm{Shv}_\tau^{tr}(k;R)}^p(R_{\mathrm{tr}}(X), \mathcal{H}^q(\underline{K})) & \Longrightarrow & \mathrm{Ext}_{\mathrm{Shv}_\tau^{tr}(k;R)}^{p+q}(R_{\mathrm{tr}}(X), \underline{K}) \\ \downarrow & & \downarrow \\ E_2^{p,q} := H_\tau^p(X, \mathcal{H}^q(\underline{K})) & \Longrightarrow & \mathbb{H}_\tau^{p+q}(X, \underline{K}). \end{array}$$

By the result for sheaves, we have an isomorphism on the E_2 page, and so an isomorphism on the abutment. The conclusion follows. $\square$

Remark 3.6.12. This is not optimal. See proposition 2.2.3 of [24] in which the étale version is proved for all S -schemes where S is any fixed Noetherian base scheme *without* cohomological finiteness assumptions. Also, if S is a Noetherian scheme of finite Krull dimension d , then its Nisnevich cohomological dimension is bounded

above by d . Therefore, the lemma is valid at least for Noetherian base schemes of finite Krull dimension.

Here, we prove an important proposition pertaining to the function EM defined in definition 3.6.7. Though an analogue of Levine's corollary 5.12 in [63], its proof in our étale setting is different.

Proposition 3.6.13. *Suppose k is algebraically closed of exponential characteristic p , and suppose $M \in \mathrm{DM}^{\mathrm{eff}}(k; \mathbb{Z}[1/p])_{\mathrm{tor}}$. For every prime $\ell \neq p$, we have isomorphisms*

$$\underline{\mathrm{Et}}_{\ell*} : \pi_{n,0}(\mathrm{EM}(M))(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n(\underline{\mathrm{Et}}_{\ell}(\mathrm{EM}(M))).$$

Proof. Let $\pi : \mathrm{Shv}_{\infty}^{\mathrm{Nis}}(\mathrm{Sm}/k) \rightarrow \mathrm{Shv}_{\infty}^{\mathrm{ét}}(\mathrm{Sm}/k)$ be the morphism of topoi. We can then write $\underline{\mathrm{Et}}_{\ell} = L^{H\mathbb{Z}/\ell} \Gamma_!^{\mathbb{S}} \pi^*$. For simplicity of notation, let F be the functor $L^{H\mathbb{Z}/\ell} \Gamma_!^{\mathbb{S}}$. Whenever π^* appears, it is the natural map induced by étale sheafification, unless otherwise stated. We show that

$$\underline{\mathrm{Et}}_{\ell*} : \pi_{n,0}(\mathrm{EM}(M))(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n(\underline{\mathrm{Et}}_{\ell}(\mathrm{EM}(M)))$$

is an isomorphism by showing that

$$\pi^* : \pi_{n,0}(\mathrm{EM}(M))(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_{n,0}^{\mathrm{ét}}(\pi^* \mathrm{EM}(M))(k) \otimes \mathbb{Z}_{\ell}$$

and

$$F : \pi_{n,0}^{\mathrm{ét}}(\pi^* \mathrm{EM}(M))(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n(\underline{\mathrm{Et}}_{\ell}(\mathrm{EM}(M)))$$

are isomorphisms. Here, $\pi_{a,b}^{\mathrm{ét}}$ is defined as in $\pi_{a,b}$ but with the Nisnevich topology replaced by the étale topology. We show that the two morphisms above are isomorphisms by showing that the morphisms

$$\mathrm{Map}_{\mathrm{Spt}(k)}(\mathbf{1}_k, \mathrm{EM}(M)) \xrightarrow{\pi^*} \mathrm{Map}_{\mathrm{Spt}_{\mathrm{ét}}(k)}(\pi^* \mathbf{1}_k, \pi^* \mathrm{EM}(M))$$

and

$$\mathrm{Map}_{\mathrm{Spt}_{\mathrm{\acute{e}t}}(k)}(\pi^* \mathbf{1}_k, \pi^* \mathrm{EM}(M)) \xrightarrow{F} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Spt})^{H\mathbb{Z}/\ell}}(F\pi^* \mathbf{1}_k, F\pi^* \mathrm{EM}(M))$$

are weak equivalences. By a density argument, the weak equivalence of π^* need only be checked for $M = \mathbb{Z}[1/p]^{tr}(X)[s]/N$ for all X smooth projective k -scheme, and all $N > 1$ coprime to p . Indeed, $\mathbb{Z}[1/p]^{tr}(X)[n]/N$, where $n \in \mathbb{Z}, N > 1$ coprime to p , and X smooth projective k -scheme, compactly generate $\mathrm{DM}^{\mathrm{eff}}(k; \mathbb{Z}[1/p])_{\mathrm{tor}}$ (see proposition 5.5.3 of [56]). Remember that by lemma 3.6.8 EM preserves small colimits.

Note that via the adjunction given in definition 3.6.7, π^* is equivalent to the morphism

$$\mathrm{Map}_{\mathrm{Mod}_{M\mathbb{Z}/N}}(M\mathbb{Z}/N, M\mathbb{Z}/N \wedge X_+) \xrightarrow{\pi^*} \mathrm{Map}_{\mathrm{Mod}_{M_{\mathrm{\acute{e}t}}\mathbb{Z}/N}}(\pi^* M\mathbb{Z}/N, \pi^*(M\mathbb{Z}/N \wedge X_+))$$

which we abuse notation and also write π^* . The equivalences of tensor triangulated categories $\mathrm{DM}(k; \mathbb{Z}/N) \simeq \mathrm{Mod}_{M\mathbb{Z}/N}$ (see lemma 3.6.6) and $\mathrm{DM}_{\mathrm{\acute{e}t}}(k; \mathbb{Z}/N) \simeq \mathrm{Mod}_{M_{\mathrm{\acute{e}t}}\mathbb{Z}/N}$ induce the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Mod}_{M\mathbb{Z}/N}}(M\mathbb{Z}/N, M\mathbb{Z}/N \wedge X_+) & \xrightarrow{\pi^*} & \mathrm{Map}_{\mathrm{Mod}_{M_{\mathrm{\acute{e}t}}\mathbb{Z}/N}}(\pi^* M\mathbb{Z}/N, \pi^*(M\mathbb{Z}/N \wedge X_+)) \\ \sim \downarrow & & \downarrow \sim \\ \mathrm{Map}_{\mathrm{DM}(k; \mathbb{Z}/N)}((\mathbb{Z}/N)^{tr}, (\mathbb{Z}/N)^{tr}(X)) & \xrightarrow{SV} & \mathrm{Map}_{\mathrm{DM}_{\mathrm{\acute{e}t}}(k; \mathbb{Z}/N)}((\mathbb{Z}/N)^{tr}, (\mathbb{Z}/N)^{tr}(X)). \end{array}$$

Note that we have labeled the lower horizontal map, which is induced by étale sheafification, as SV . For a Grothendieck topology τ , recall that $D_\tau := D(\mathrm{Shv}_\tau^{tr}(k; \mathbb{Z}/N))$. Let $C_* := C_*(\mathbb{Z}/N)^{tr}(X)$. Consider the following commutative diagram

$$\begin{array}{ccccc} \mathbb{H}_{Nis}^n(\mathrm{Spec} \ k, C_*) & \xrightarrow{\sim} & \mathrm{Ext}_{D_{Nis}}^n((\mathbb{Z}/N)^{tr}, C_*) & \longrightarrow & \tilde{\pi}_{-n} \mathrm{Map}_{\mathrm{DM}(k; \mathbb{Z}/N)}((\mathbb{Z}/N)^{tr}, (\mathbb{Z}/N)^{tr}(X)) \\ \downarrow a_{\mathrm{\acute{e}t}} = & & \downarrow = & & \downarrow \pi_{-n} SV \\ \mathbb{H}_{\mathrm{\acute{e}t}}^n(\mathrm{Spec} \ k, C_*) & \xrightarrow{\sim} & \mathrm{Ext}_{D_{\mathrm{\acute{e}t}}}^n((\mathbb{Z}/N)^{tr}, C_*) & \longrightarrow & \tilde{\pi}_{-n} \mathrm{Map}_{\mathrm{DM}_{\mathrm{\acute{e}t}}(k; \mathbb{Z}/N)}((\mathbb{Z}/N)^{tr}, (\mathbb{Z}/N)^{tr}(X)) \end{array}$$

Since C_* is $\mathbb{A}^1$ -local and $\mathrm{DM}^{\mathrm{eff}}(k; \mathbb{Z}/N)$ and $\mathrm{DM}_{\mathrm{\acute{e}t}}^{\mathrm{eff}}(k; \mathbb{Z}/N)$ embed fully faithfully into $\mathrm{DM}(k; \mathbb{Z}/N)$ and $\mathrm{DM}_{\mathrm{\acute{e}t}}(k; \mathbb{Z}/N)$, respectively, the upper and lower right morphisms are isomorphisms. Furthermore, both upper and lower left morphisms are also isomorphism by lemma 3.6.11. Since k is algebraically closed, $H_{\tau}^i(\mathrm{Spec} k; -)$ vanishes for $i > 0$. Consequently $H_{\tau}^0(\mathrm{Spec} k; -)$ is an exact functor. Therefore, $\mathbb{H}_{\tau}^*(\mathrm{Spec} k; \underline{K}) = H^*(K(\mathrm{Spec} k))$. As a result, the left vertical map is also an isomorphism. We conclude that SV is a weak equivalence. By the commutative square above relating π^* to SV , we deduce that π^* is a weak equivalence.

We use another density argument to prove that

$$F : \underline{\pi}_{n,0}^{\acute{e}t}(\pi^* \mathrm{EM}(M))(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n(\underline{\acute{E}t}_{\ell}(\mathrm{EM}(M)))$$

is an isomorphism. For $M = \mathbb{Z}[1/p]^{tr}(X)/N$ as before, $\pi^* M \in \mathrm{DM}_{\mathrm{\acute{e}t}}^{\mathrm{eff}}(k; \mathbb{Z}/N) \simeq \mathrm{Mod}_{H\mathbb{Z}/N}$. Since this category is compactly generated by the constant sheaf $\mathbb{Z}/N$, and since our functors are exact functors preserving colimits, it suffices to check that

$$\mathrm{Map}_{\mathrm{Spt}_{\mathrm{\acute{e}t}}(k)}(\pi^* \mathbf{1}_k, \pi^* M\mathbb{Z}/\ell^m) \xrightarrow{F} \mathrm{Map}_{\mathrm{Pro}(\mathrm{Spt})^{H\mathbb{Z}/\ell}}(F\pi^* \mathbf{1}_k, F\pi^*(M\mathbb{Z}/\ell^m))$$

is a weak equivalence. However, note that $\pi^*(M\mathbb{Z}/\ell^m) \simeq H\mathbb{Z}/\ell^m$ is in the image of the constant functor $c : \mathrm{Spt} \rightarrow \mathrm{Shv}_{\infty}^{\acute{e}t}(\mathrm{Sm}/k, \mathrm{Spt})$. Since F is a pro-left adjoint to c , the last map above is formally a weak equivalence. $\square$

In order to prove the next proposition, we need the following lemma.

Lemma 3.6.14.

$$\underline{\acute{E}t}_{\ell}(M\mathbb{Z}[1/p]) \simeq H\mathbb{Z}_{\ell}$$

is a weak equivalence in $\mathrm{Pro}(\mathrm{Spt})^{H\mathbb{Z}/\ell}$.

Proof. Recall that from definition 3.6.5,

$$M\mathbb{Z}[1/p] = \operatorname{colim}_{n \rightarrow \infty} \Sigma_{\mathbb{P}^1}^{-n} \Sigma_{\mathbb{P}^1}^{\infty} K(\mathbb{Z}[1/p](n), 2n).$$

Since $\underline{\mathcal{E}t}_{\ell}$ preserves colimits, we obtain

$$\begin{aligned} \underline{\mathcal{E}t}_{\ell}(M\mathbb{Z}[1/p]) &\simeq \operatorname{colim}_{n \rightarrow \infty} \underline{\mathcal{E}t}_{\ell}(\Sigma_{\mathbb{P}^1}^{-n} \Sigma_{\mathbb{P}^1}^{\infty} K(\mathbb{Z}[1/p](n), 2n)) \\ &\simeq \operatorname{colim}_{n \rightarrow \infty} \Sigma_{S_{\ell}^2}^{-n} \Sigma_{S_{\ell}^2}^{\infty} \underline{\mathcal{E}t}_{\ell}(K(\mathbb{Z}[1/p](n), 2n)) \\ &\simeq^* \operatorname{colim}_{n \rightarrow \infty} \Sigma_{S_{\ell}^2}^{-n} \Sigma_{S_{\ell}^2}^{\infty} K(\mathbb{Z}_{\ell}, 2n) \\ &\simeq L^{H\mathbb{Z}/\ell} \operatorname{colim}_{n \rightarrow \infty} \Sigma_{S^2}^{-n} \Sigma_{S^2}^{\infty} K(\mathbb{Z}, 2n) \\ &\simeq L^{H\mathbb{Z}/\ell} H\mathbb{Z} \simeq H\mathbb{Z}_{\ell}, \end{aligned}$$

where the equivalence $\simeq^*$ follows from corollary 8.5 of [46]. □

The proof of the following proposition is more or less a variant of its Betti analogue as proved by Levine in proposition 6.4 of [63].

Proposition 3.6.15. *For every q and n , the map*

$$\underline{\mathcal{E}t}_{\ell*} : \pi_{n,0}(s_q(\mathbf{1}_k))(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n(\underline{\mathcal{E}t}_{\ell}(s_q(\mathbf{1}_k)))$$

is an isomorphism.

Proof. Note that $\mathbf{1}_k \in \mathcal{Spt}^{\text{eff}}(k)$, hence for $q < 0$, $s_q(\mathbf{1}) = 0$. For $q = 0$, $s_0(\mathbf{1}) \simeq M\mathbb{Z}$ by [61]. As a result of this and the previous lemma, $\underline{\mathcal{E}t}_{\ell}(s_0(\mathbf{1})) = \underline{\mathcal{E}t}_{\ell}(M\mathbb{Z}[1/p]) \simeq H\mathbb{Z}_{\ell}$ in $\operatorname{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}$. Therefore,

$$\pi_{n,0}(s_0(\mathbf{1}))(k) = \pi_{n,0}(M\mathbb{Z}) = H_n^{\text{Sus}}(k; \mathbb{Z}) = \begin{cases} 0 & \text{for } n \neq 0 \\ \mathbb{Z} & \text{for } n = 0. \end{cases}$$

On the other hand,

$$\pi_n(H\mathbb{Z}_\ell) = \begin{cases} 0 & \text{for } n \neq 0 \\ \mathbb{Z}_\ell & \text{for } n = 0. \end{cases}$$

The unit in $\pi_{0,0}(M\mathbb{Z}[1/p])(k) \otimes \mathbb{Z}_\ell$ is induced by $\mathbf{1} \rightarrow M\mathbb{Z}[1/p]$ which goes to the unit $L^{H\mathbb{Z}/\ell}\mathbb{S} \rightarrow H\mathbb{Z}_\ell$ in $\pi_0(H\mathbb{Z}_\ell)$ under the stable ℓ -adic étale realization. Therefore, we have the isomorphism when $q \leq 0$. For $q > 0$, $\pi_q^\mu(\mathbf{1})$ is in $\mathrm{DM}^{\mathrm{eff}}(k)_{\mathrm{tor}}$ by lemma 3.6.9. Since $s_q(\mathbf{1}) = \mathrm{EM}(\pi_q^\mu(q)[2q])$ (see discussion immediately before lemma 3.6.9), we have the isomorphism by proposition 3.6.13. The conclusion follows. $\square$

Using this proposition, a comparison of the two spectral sequences of the previous section gives us our main theorem. This is an étale analogue of theorem 6.7 of Levine in [63]. The idea of our proof is the same as that of Levine.

Theorem 3.6.16. *Let k be an algebraically closed field of exponential characteristic p . Then for every prime $\ell \neq p$, the stable ℓ -adic étale realization induces an isomorphism*

$$\underline{\mathrm{Et}}_{\ell*} : \pi_{*,0}(\mathbf{1}_k)(k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow \pi_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

Proof. We prove this via a comparison of spectral sequences. Recall that we have a morphism

$$\begin{array}{ccc} {}_I E_{p,q}^2 := \pi_{p+q,0}(s_q(\mathbf{1}))(k) \otimes \mathbb{Z}_\ell & \Longrightarrow & \pi_{p+q,0}(\mathbf{1})(k) \otimes \mathbb{Z}_\ell \\ \underline{\mathrm{Et}}_{\ell*} \downarrow & & \downarrow \underline{\mathrm{Et}}_{\ell*} \\ {}_{II} E_{p,q}^2 := \pi_{p+q}(\underline{\mathrm{Et}}_\ell(s_q(\mathbf{1}))) & \Longrightarrow & \pi_{p+q}(\underline{\mathrm{Et}}_\ell(\mathbf{1})) \end{array}$$

of strongly convergent (using propositions 3.5.1 and 3.5.3) spectral sequences. By proposition 3.6.15, $\underline{\mathrm{Et}}_{\ell*}$ is an isomorphism on the E^2 -page, and so it is an isomorphism on the abutment. The conclusion follows. $\square$

3.7 Generalized Étale Suslin-Voevodsky Theorem

The Suslin-Voevodsky theorem gives an isomorphism between Suslin (co)homology and singular or étale (co)homology, all with $\mathbb{Z}/N$ coefficient with $(N, p) = 1$ (p the exponential characteristic of k), of a finite type separated k -scheme, k algebraically closed. In theorem 7.1 of [63], Levine proved a homotopy theoretic generalization of the Suslin-Voevodsky theorem relating motivic cohomology to singular cohomology. Here, we prove a homotopy theoretic generalization of the étale variant of this theorem.

Theorem 3.7.1. *Suppose k is an algebraically closed field of exponential characteristic p , $E \in \mathcal{Spt}^{\text{eff}}(k)_{\text{tor}}$ is a effective torsion motivic spectrum, and $\ell \neq p$ is a prime. Then*

$$\underline{\text{Ét}}_{\ell*} : \pi_{n,0}(E)(k) \otimes \mathbb{Z}_{\ell} \rightarrow \pi_n(\underline{\text{Ét}}_{\ell}(E))$$

is an isomorphism.

Before proving this lemma, let us justify why this is a generalization of the usual Suslin-Voevodsky theorem. We need the following lemma.

Lemma 3.7.2. *Given a prime number ℓ different from the exponential characteristic p of k , and given a smooth k -scheme X ,*

$$\underline{\text{Ét}}_{\ell}(M\mathbb{Z}/N \wedge X_+) \rightarrow \underline{\text{Ét}}_{\ell}(M\mathbb{Z}/N) \wedge \underline{\text{Ét}}_{\ell}X_+ \simeq H\mathbb{Z}/\ell^m \wedge \underline{\text{Ét}}_{\ell}X_+$$

is a weak equivalence in $\text{Pro}(\mathcal{Spt})^{H\mathbb{Z}/\ell}$ for every positive integer $N > 1$ coprime to p with ℓ -adic valuation m .

Proof. Note that

$$M\mathbb{Z}/N \wedge X_+ \simeq \text{colim}_{n \rightarrow \infty} \Sigma_{\mathbb{P}^1}^{-n} \Sigma_{\mathbb{P}^1}^{\infty} K(\mathbb{Z}/N(n), 2n) \wedge X_+.$$

Since $\underline{\text{Ét}}_\ell$ preserves colimits, we obtain

$$\begin{aligned}
& \text{Map}_{\text{Pro}(\mathcal{S}\text{pt})}^{H\mathbb{Z}/\ell}(\underline{\text{Ét}}_\ell(M\mathbb{Z}/N \wedge X_+), H\mathbb{Z}/\ell) \\
& \simeq \text{Map}_{\text{Pro}(\mathcal{S}\text{pt})}(\text{colim}_{n \rightarrow \infty} \underline{\text{Ét}}_\ell(\Sigma_{\mathbb{P}^1}^{-n} \Sigma_{\mathbb{P}^1}^\infty K(\mathbb{Z}/\ell^m(n), 2n) \wedge (\Sigma_{\mathbb{P}^1}^\infty X_+)), H\mathbb{Z}/\ell) \\
& \stackrel{(*)}{\simeq} \text{Map}_{\text{Pro}(\mathcal{S}\text{pt})}(\text{colim}_{n \rightarrow \infty} (\Sigma_{S_\ell^2}^{-n} \Sigma_{S_\ell^2}^\infty \underline{\text{Ét}}_\ell K(\mathbb{Z}/\ell^m(n), 2n) \wedge \underline{\text{Ét}}_\ell X_+), H\mathbb{Z}/\ell) \\
& \simeq \text{Map}_{\text{Pro}(\text{Mod}_{H\mathbb{Z}/\ell})}(\text{colim}_{n \rightarrow \infty} (\Sigma_{S_\ell^2}^{-n} \Sigma_{S_\ell^2}^\infty K(\mathbb{Z}/\ell^m, 2n) \wedge \underline{\text{Ét}}_\ell X_+ \wedge H\mathbb{Z}/\ell), H\mathbb{Z}/\ell) \\
& \stackrel{(**)}{\simeq} \text{Map}_{\text{Pro}(\text{Mod}_{H\mathbb{Z}/\ell})}((\text{colim}_{n \rightarrow \infty} \Sigma_{S_\ell^2}^{-n} \Sigma_{S_\ell^2}^\infty K(\mathbb{Z}/\ell^m, 2n)) \wedge (\underline{\text{Ét}}_\ell X_+ \wedge H\mathbb{Z}/\ell), H\mathbb{Z}/\ell) \\
& \simeq \text{Map}_{\text{Pro}(\text{Mod}_{H\mathbb{Z}/\ell})}(H\mathbb{Z}/\ell^m \wedge \underline{\text{Ét}}_\ell X_+ \wedge H\mathbb{Z}/\ell, H\mathbb{Z}/\ell) \\
& \simeq \text{Map}_{\text{Pro}(\mathcal{S}\text{pt})}(H\mathbb{Z}/\ell^m \wedge \underline{\text{Ét}}_\ell X_+, H\mathbb{Z}/\ell),
\end{aligned}$$

where the equivalence $(*)$ follows from the construction of the stable ℓ -adic realization functor (its compatibility with the unstable ℓ -adic realization functor and from corollary 8.5 of [46], $(**)$ follows from $\Sigma_{S_\ell^2}^\infty \underline{\text{Ét}}_\ell X_+ \wedge H\mathbb{Z}/\ell$ being homotopy equivalent to a space (by rigidity, it corresponds to a simplicial object in the category of finite dimensional $\mathbb{Z}/\ell$ -vector spaces), not just a pro-space. We conclude that $\underline{\text{Ét}}_\ell(M\mathbb{Z}/N \wedge X_+) \rightarrow H\mathbb{Z}/\ell^m \wedge \underline{\text{Ét}}_\ell X_+$ is a weak equivalence in $\text{Pro}(\mathcal{S}\text{pt})^{H\mathbb{Z}/\ell}$, as required. $\square$

In particular, if X is a smooth k -scheme and $E = M\mathbb{Z}/\ell \wedge X_+$ with $\ell \neq p$ a prime, then $\underline{\text{Ét}}_\ell(E) \simeq H\mathbb{Z}/\ell \wedge \underline{\text{Ét}}_\ell X_+$ by lemma 3.7.2. Note that $\pi_{n,0}(M\mathbb{Z}/\ell \wedge X_+)(k) = [\Sigma_{S^1}^n \mathbf{1}_k, M\mathbb{Z}/\ell \wedge X_+]_{\mathcal{S}\text{pt}(k)} = [\Sigma_{S^1}^n M\mathbb{Z}/\ell, M\mathbb{Z}/\ell \wedge X_+]_{\text{Mod}_{\mathbb{Z}/\ell}} = \text{Hom}_{DM(k; \mathbb{Z}/\ell)}(\mathbb{Z}/\ell^{tr}(k)[n], M(X)) = H_{n,0}(X; \mathbb{Z}/\ell)$ which is the n th motivic homology of X with coefficients in $\mathbb{Z}/\ell$. Note that Suslin homology $H_n^{sus}(-; \mathbb{Z}/\ell)$, defined as $H_*(C_*(\mathbb{Z}/\ell)^{tr}(X)(k))$ is isomorphic to motivic homology (see proposition 14.18 of [68]). We define Suslin cohomology with $\mathbb{Z}/\ell$ coefficients as the dual of Suslin homology with $\mathbb{Z}/\ell$ coefficients. On the other hand, $\pi_n(H\mathbb{Z}/\ell \wedge \underline{\text{Ét}}_\ell X_+) = [\Sigma_{S^1}^n \mathbb{S}, H\mathbb{Z}/\ell \wedge \underline{\text{Ét}}_\ell X_+]_{\mathcal{S}\text{pt}} = H_n(\underline{\text{Ét}}_\ell X; \mathbb{Z}/\ell) \cong H^n(\underline{\text{Ét}}_\ell X; \mathbb{Z}/\ell)^* \cong H_{\text{ét}}^n(X; \mathbb{Z}/\ell)$, where the second to last equivalence is a result of

the universal coefficient theorem, and the last equivalence is by the fact that $\dot{\text{Et}}X$ is the étale homotopy type of X . Applying theorem 3.7.1, we obtain the isomorphism

$$H_n^{\text{sus}}(X; \mathbb{Z}/\ell) \simeq H_n^{\text{ét}}(X; \mathbb{Z}/\ell),$$

from which we get the Suslin-Voevodsky comparison theorem

$$H_{\text{sus}}^n(X; \mathbb{Z}/\ell) \simeq H_{\text{ét}}^n(X; \mathbb{Z}/\ell)$$

via Pontryagin dualization (see the above argument for the identification of the dualization of the right hand side with étale cohomology). We now prove the above theorem 3.7.1.

Proof. The stable ∞ -category $\text{Spt}^{\text{eff}}(k)_{\text{tor}}$ is compactly generated by the objects $\{\Sigma^{p,q}\Sigma_{\mathbb{P}^1}^\infty X_+/N \mid q \geq 0, p \in \mathbb{Z}, (N, p) = 1, X \in \text{Sm}/k\}$. Therefore, it suffices to prove the theorem for $E \in \text{Spt}_{\text{fin}}^{\text{eff}}(k)_{\text{tor}}$. By proposition 3.5.1, the slice tower of E gives a strongly convergent spectral sequence

$$E_{p,q}^2 = \pi_{p+q,0}(s_q E)(k) \otimes \mathbb{Z}_\ell \implies \pi_{p+q,0}(E)(k) \otimes \mathbb{Z}_\ell.$$

By proposition 3.5.3, the étale realization of the slice tower gives the strongly convergent spectral sequence

$$E_{p,q}^2 = \pi_{p+q}(\dot{\text{Et}}_\ell(s_q E)) \implies \pi_{p+q}(\dot{\text{Et}}_\ell(E)).$$

In fact, it is clear that we have a morphism of spectral sequences from the former to the latter spectral sequence induced by stable ℓ -adic étale realization.

Since E is a effective torsion motivic spectrum, we know by lemma 3.6.10 that

$s_q E \simeq EM(\pi_q^\mu(E)(q)[2q])$ for some $\pi_q^\mu(E) \in \mathrm{DM}^{\mathrm{eff}}(k)_{\mathrm{tor}}$, and so proposition 3.6.13 implies that we have an isomorphism of the E^2 -pages, and so an isomorphism on the abutment. The conclusion follows. $\square$

Let k be an algebraically closed field of characteristic zero endowed with an inclusion $\sigma : k \hookrightarrow \mathbb{C}$. Let Re_B^σ be the stable Betti realization functor (see Levine's paper [63]). We can then use our main theorem 3.7.1 to deduce the following.

Corollary 3.7.3. *Suppose k is an algebraically closed field of characteristic zero endowed with an inclusion $\sigma : k \hookrightarrow \mathbb{C}$. Let E be an effective torsion motivic spectrum, and let ℓ be a prime. Then for each n , there is an isomorphism*

$$\pi_n(\underline{\mathrm{Et}}_\ell(E)) \cong \pi_n(\mathrm{Re}_B^\sigma(E)) \otimes \mathbb{Z}_\ell.$$

Proof. Combine theorem 3.7.1 above, and theorem 7.1 of [63]. $\square$

Remark 3.7.4. If our algebraically closed field k is of characteristic zero, we can replace ℓ -adic Bousfield localization in our construction of the stable ℓ -adic étale realization with profinite completion. Going through the same proofs and constructions tells us that when k has characteristic zero, there is a refinement of the results in this chapter. More precisely, we may replace all ℓ -completions with pro-finite completions, and all $- \otimes \mathbb{Z}_\ell$ with $- \otimes \hat{\mathbb{Z}}$. In particular, we may replace the isomorphism of corollary 3.7.3 with the isomorphism

$$\pi_n(\hat{\underline{\mathrm{Et}}}(E)) \cong \pi_n(\mathrm{Re}_B^\sigma(E))^\wedge,$$

where $\hat{\underline{\mathrm{Et}}}$ is our pro-finitely completed as opposed to the ℓ -adic stable étale realization functor we have constructed in this thesis.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ
 Email address: `mzargar@math.princeton.edu`