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BORDERED INVARIANTS IN LOW-DIMENSIONAL TOPOLOGY

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Abstract

In this thesis we present two projects. In the first project, which covers Chapters 2 and 3, we construct an algebraic version of Lagrangian Floer homology for immersed curves in a surface with boundary. We first associate to the surface an algebra $\mathcal{A}_1$. Then to an immersed curve L inside the surface we associate an A_∞ module $M(L)$ over $\mathcal{A}_1$. Then we prove that Lagrangian Floer homology $HF_*(L_0, L_1)$ is isomorphic to a suitable algebraic pairing of modules $M(L_0)$ and $M(L_1)$. We apply this theory to the pillowcase homology construction; namely we enhance it by extending the construction from knots to tangles: given a 4-ended tangle inside a 3-ball, after associating to it an immersed unobstructed curve inside the pillowcase, one can further associate an A_∞ module to that curve.

In the second project, which is described in Chapter 4, we compare two different types of mapping class invariants: the Hochschild homology of A_∞ bimodules coming from bordered Heegaard Floer homology, and fixed point Floer cohomology. We first develop effective methods to compute bimodule invariants and their Hochschild homology in the genus two case. We then compare the resulting computations to fixed point Floer cohomology, and make a conjecture that the two invariants are isomorphic. We also discuss a construction of a map potentially giving the isomorphism. It comes as an open-closed map in the context of a surface being viewed as a 0-dimensional Lefschetz fibration over $\mathbb{C}$.

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Chapter 1

Introduction

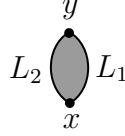
In the introduction we will first describe broadly the area of homological invariants in low-dimensional topology. Then we will outline the alternative Fukaya categorical approach of Auroux to bordered Heegaard Floer homology, which is the basis for both of our projects. After that we will present our results.

1.1 Modern homological invariants in low-dimensional topology

1.1.1 Symplectic geometry

Here I will outline important constructions in symplectic geometry, which are used in low-dimensional topology. Omitting a large amount of details and certain technical conditions, Lagrangian Floer homology [Flo88b] is a homological invariant $HF_*(L_1, L_2)$ of two Lagrangians inside a symplectic manifold $L_1, L_2 \rightarrow M^{2n}$, which is invariant under Hamiltonian isotopies of each Lagrangian. The underlying chain complex is generated by intersections: $CF_*(L_1, L_2) = \langle L_1 \cap L_2 \rangle_{\mathbb{F}_2}$. After picking an almost complex structure, the differential $d : CF_*(L_1, L_2) \rightarrow CF_*(L_1, L_2)$ comes from counting rigid pseudo-holomorphic discs between the intersection points, with Lagrangian

boundary conditions on L_1 and L_2 : a pseudo-holomorphic disc schematically drawn below contributes 1 to the coefficient c_{xy} in $d(x) = \sum_y c_{xy} \cdot y$.



Another invariant, which is relevant to low-dimensional topology, is the Fukaya category of a symplectic manifold $\mathcal{F}(M^{2n})$ [FukOhOhtOno09a], [Sei08], [Aur14]. It is a unified structure, which captures how all Lagrangians intersect with each other. The objects in this category are Lagrangians L_i , and morphism spaces are Lagrangian Floer complexes $CF_*(L_i, L_j)$. The product $CF_*(L_1, L_2) \otimes CF_*(L_2, L_3) \rightarrow CF_*(L_1, L_3)$ is given by counting pseudo-holomorphic triangles. It is not associative, and only is associative up to homotopy, given by counting pseudo-holomorphic rectangles. Thus $\mathcal{F}(M^{2n})$ is not a regular category, but rather is an A_∞ category (higher operations are given by counting pseudo-holomorphic polygons).

In the case $2n = 2$, where Lagrangians are curves on the surface, counting rigid pseudo-holomorphic discs is equivalent to counting immersed discs with convex angles at intersections. The Fukaya category in this case is similar to a curve complex, only it captures more information: minimal intersection numbers between the curves, and also all the immersed convex polygons with boundary sides on multiple curves.

1.1.2 Heegaard Floer theory

One of the main 3-manifold techniques in the field is called Heegaard Floer homology, developed by Ozsváth and Szabó in [OzsSza04d], [OzsSza04c]. The construction works as follows: first one picks a Heegaard splitting of a 3-manifold along a surface: $Y^3 = U_1 \cup_{\Sigma_g} U_2$. Second, one constructs a symplectic manifold: $Sym^g(\Sigma_g) = (\Sigma_g)^g / S_g$. Then, having picked a Heegaard diagram with a basepoint for

the Heegaard splitting, one obtains two Lagrangian tori: $\mathbb{T}(U_1), \mathbb{T}(U_2) \subset \text{Sym}^g(\Sigma_g)$. At last, Heegaard Floer homology of Y^3 is Lagrangian Floer homology of these two tori: $HF_*(\mathbb{T}(U_1), \mathbb{T}(U_2))$. There are different versions of Heegaard Floer homology depending on how one incorporates the basepoint in the diagram: they are denoted by $\widehat{HF}(Y), HF^-(Y), HF^+(Y), HF^\infty(Y)$. Among various applications, this theory can be used to effectively study surgery problems (for example, one can reprove property P [OzsSza04a]), homology cobordism group [OzsSza03a], and contact structures [OzsSza05]. One can also obtain such foundational 4-dimensional results as Donaldson's diagonalizability theorem and symplectic Thom conjecture for $\mathbb{C}P^2$ [OzsSza03a].

An analogous theory for knots, discovered independently in [OzsSza04b] and [Ras03], is called knot Floer homology $\widehat{HFK}(K)$. It categorifies the Alexander polynomial in the same way Khovanov homology categorifies the Jones polynomial. $\widehat{HFK}(K)$ also detects the genus of the knot [OzsSza04a], detects if the knot is fibered [Ghi08], [Ni07], and provides lower bounds for the 4-dimensional genus of the knot [OzsSza03b]. Another feature of knot Floer homology is that it contains essential information for studying Heegaard Floer homology $\widehat{HF}(S_{p/q}(K))$ of surgeries on a knot [OzsSza11], [ManOzs10].

One of the advantages of Heegaard Floer homology is that it can be computed for a large number of 3-manifolds (with small Heegaard diagrams) directly from its definition and properties. However, the construction of Heegaard Floer homology involves a choice of almost complex structure and subsequent counting of pseudo-holomorphic discs, and therefore a difficult and important question is how computable these invariants are in general. The first general algorithm to compute $\widehat{HF}(Y)$ was discovered by Sarkar and Wang in [SarWan10]. An effective algorithm to compute $\widehat{HF}(Y)$ was developed in [LipOzsThu14]. Other versions of Heegaard Floer homology are theoretically algorithmically computable [ManOzsThu09], but the methods are not practical.

Importantly, Heegaard Floer homology fits into 3+1 TQFT framework. It means that for 3-manifolds it assigns vector spaces, and for 4-dimensional cobordisms it assigns maps between the vector spaces, which satisfy the composition law.

1.1.3 Gauge theory and Instantons

There exists a 3-manifold theory called monopole Floer homology, which is parallel to Heegaard Floer theory. It is constructed using gauge theory by Kronheimer and Mrowka in [KroMro07], and is isomorphic to Heegaard Floer theory [KutLeeTau10a], [ColGhiHon11]. This means that Heegaard Floer homology, at its core, allows one to solve in a geometric way Seiberg-Witten differential equations on a 3-manifold.

There is another pair of parallel Floer homology theories for 3-manifolds. The one coming from gauge theory is instanton homology $I(Y^3)$, developed in [Flo88a] for integer homology sphere by Floer using anti-self-dual Yang-Mills equations. The following is an original formulation of the Atiyah-Floer conjecture, which describes what should be the parallel symplectic geometric theory. Having a Heegaard splitting $Y^3 = U_1 \cup_{\Sigma_g} U_2$, associate to U_i and Σ_g their $SU(2)$ representation varieties $R(U_i)$, $R(\Sigma_g)$. One then has maps $R(U_i) \rightarrow R(\Sigma_g)$. It was conjectured in [Ati88] that instanton Floer homology $I(Y^3)$ should be equal to Lagrangian Floer homology $HF_*(R(U_1), R(U_2))$. The spaces $R(U_1)$, $R(U_2)$, $R(\Sigma_g)$ are singular, and thus symplectic instanton Floer homology $HF_*(R(U_1), R(U_2))$ was not possible to define at the moment. The symplectic side of the isomorphism, as well as the proof of Atiyah-Floer conjecture, are still under development [DaeFuk17], [DosSal94].

1.1.4 Bordered invariants

Heegaard Floer homology was extended down to 2+1 TQFT by bordered Heegaard Floer homology theory, which was developed by Lipshitz, Ozsváth and Thurston in [LipOzsThu08], [LipOzsThu15]. This theory assigns a dg -algebra $\mathcal{A}$ to the parame-

terized surface $\Sigma = F(\mathcal{Z})$ (see Construction 4.1.2), and an A_∞ module $\widehat{CFA}(Y_1)_\mathcal{A}$ or a D structure ${}^{\mathcal{A}^{op}}\widehat{CFD}(Y_1)$ to a 3-manifold $(Y_1, \partial Y_1 = \Sigma)$. A gluing theorem, needed for TQFT structure, states that if $Y^3 = Y_1 \cup_\Sigma Y_2$, then $\widehat{HF}(Y^3) \cong \widehat{CFA}(Y_1)_\mathcal{A} \boxtimes^{\mathcal{A}} \widehat{CFD}(-Y_2)$, where $\boxtimes$ is an appropriate version of a tensor product. Appropriate objects called bimodules are assigned to 3-dimensional cobordisms between the surfaces, and the corresponding gluing theorems hold true.

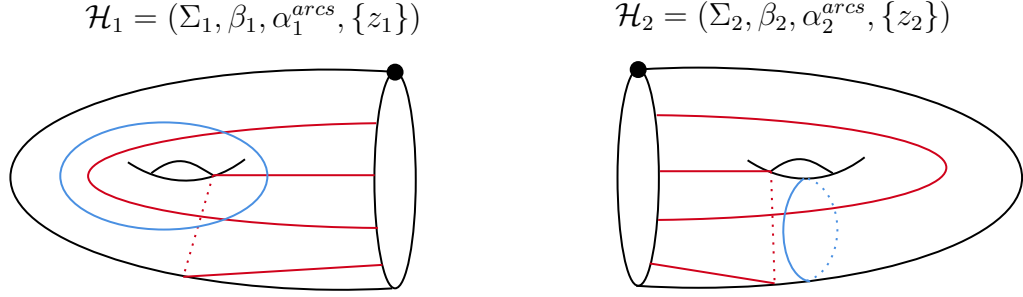
1.2 Symplectic geometric approach to bordered invariants

The starting point for our work is a construction of Auroux [Aur10b], which relates the partially wrapped Fukaya category $\mathcal{F}(Sym^g(\Sigma_g \setminus 1pt), \{z\})$ to bordered Heegaard Floer theory. This construction assigns the category $\mathcal{F}(Sym^g(\Sigma_g \setminus 1pt), \{z\})$ (see Section 2.3.2 for the definition of the category $\mathcal{F}(\Sigma, Z^{bps})$ for surfaces) to a surface Σ_g , and an object $\mathbb{T}_{Y_1}$ inside $\mathcal{F}(Sym^g(\Sigma_g \setminus 1pt), \{z\})$ to a 3-manifold $(Y_1, \partial Y_1 = \Sigma_g)$. A gluing theorem states that if $Y^3 = Y_1 \cup_{\Sigma_g} Y_2$, then $\widehat{HF}(Y^3) \cong \text{hom}_{\mathcal{F}(Sym^g(\Sigma_g \setminus 1pt), \{z\})}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2})$ (which is equal to $CF_*(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2})$ in case $\mathbb{T}_{Y_i}$ are embedded compact Lagrangians). A striking result¹ is the equivalence of this approach and bordered Heegaard Floer theory, which is addressed in [Aur10b, Theorems 1.2-1.4], where it is proved that $\widehat{CFA}(Y_1) = \mathcal{Y}(\mathbb{T}_{Y_1})$, the image of $\mathbb{T}_{Y_1}$ under Yoneda embedding w.r.t. a certain generating set of Lagrangians in $Sym^g(\Sigma_g \setminus 1pt)$, see [Aur10b, Theorem 1.3] for the description of the generating set. See also Section 4.4, and in particular Figure 4.22, where we explain the connection between the algebra from bordered Heegaard Floer theory and the partially wrapped Fukaya category.

Let us explain on a very simple example the difference between Auroux's and bordered Heegaard Floer theory approaches.

¹The full proof of this result was not written up, see [Aur10b, p.5, Caveat].

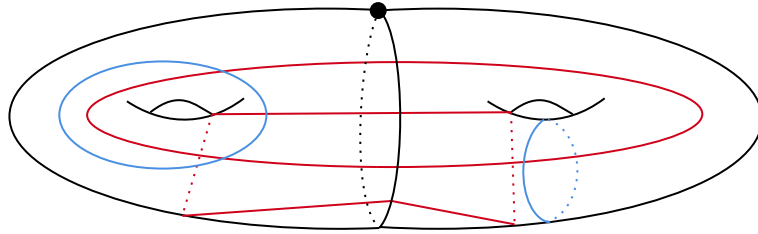
Suppose Y_1 and Y_2 are given by the following bordered Heegaard diagrams (see [LipOzsThu08, Section 4.1] for the definition of such diagrams):



Suppose $Y^3 = Y_1 \cup_{\Sigma} Y_2$ where the gluing is specified by the parameterizations of $\partial Y_i = \Sigma_i = \Sigma$ by arcs. And suppose we are interested in computing $\widehat{HF}(Y^3)$ in the “divide and conquer” fashion, i.e. we want to associate certain algebraic objects to Y_1 and Y_2 and then obtain $\widehat{HF}(Y^3)$ by pairing those objects. Then we have the following two possibilities:

- (1) Notice that the Heegaard diagram

$$\mathcal{H} = \mathcal{H}_1 \cup_{\partial} \mathcal{H}_2 = (\Sigma_{g=2} = \Sigma_1 \cup_{S^1} \Sigma_2, \beta^{curves}, \alpha^{curves}, \{z\})$$

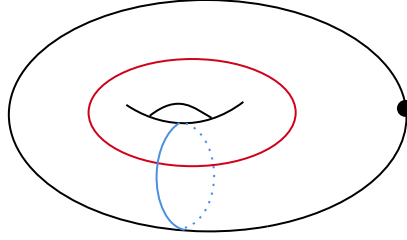


represents the manifold $Y^3 = Y_1 \cup_{\Sigma} Y_2$, see [LipOzsThu08, Lemma 4.30]. Because $\widehat{HF}(Y^3) = CF_*(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta})$, a Lagrangian Floer complex inside $Sym^2(\Sigma_{g=2} \setminus \{z\})$, the task is to compute the differential on this complex, which is given by counting pseudo-holomorphic curves in $Sym^2(\Sigma_{g=2} \setminus \{z\})$. One can try to understand these curves by cutting $\Sigma_{g=2} = \Sigma_1 \cup_{S^1} \Sigma_2$ along S^1 (and possibly degenerating the complex structure near it), and then studying pseudo-holomorphic curves in the two halves. The great achievement of bordered Heegaard Floer homology

[LipOzsThu08] is the realization of this approach. There the bordered algebra $\mathcal{A}$ comes as a convenient algebraic gadget to keep track of those pseudo-holomorphic curves inside symmetric products of $[0, 1] \times S^1$ with Lagrangian boundary conditions on $2g$ parallel different red arcs $[0, 1] \times pt$, which come off in the degeneration process of stretching along S^1 in $\Sigma_1 \cup_{S^1} \Sigma_2$.

(2) Notice that a different Heegaard diagram

$$\mathcal{H}' = (\Sigma_{closed} = \Sigma_1 \cup D^2 = \Sigma_2 \cup D^2, \beta_1, \beta_2, \{z\})$$



also represents the manifold $Y^3 = Y_1 \cup_{\Sigma} Y_2$. Thus $\widehat{HF}(Y^3) = CF_*(\beta_1, \beta_2)$, Lagrangian Floer complex inside $\Sigma_{closed} \setminus Nbd(z) = \Sigma = \Sigma_i$. One could try to understand the Lagrangian Floer complex $CF_*(\beta_1, \beta_2)$ by understanding β_1 and β_2 as objects in the partially wrapped Fukaya category $\mathcal{F}(\Sigma, \{z\})$. Having a parameterization of Σ (which in our case is given by arcs α_1, α_2), there are two convenient ways to algebraically represent an object β inside $\mathcal{F}(\Sigma, \{z\})$:

(a) To try to represent β as an iterated mapping cone of objects α_1 and α_2 . The algebraic object one gets at the end is a twisted complex, or equivalently a bounded D structure², see [HaiKatKon17, Section 4.1] for the details in the case of the partially wrapped Fukaya category of a surface.

(b) To understand how an object β intersects with the basic objects α_1, α_2 .

The algebraic object one gets at the end is an A_{∞} module

$$\mathcal{Y}(\beta) = \bigoplus_{i=1,2} hom_{\mathcal{F}(\Sigma, \{z\})}(\beta, \alpha_i) \text{ over the algebra}$$

²See [LipOzsThu15, Remark 2.2.37] for the explanation of why D structures and twisted complexes are essentially equivalent objects.

$\mathcal{A}_1(\Sigma) = \bigoplus_{1 \leq i, j \leq 2} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_j, \alpha_i)$. Then the Heegaard Floer homology is recovered algebraically: $\widehat{HF}(Y^3) = CF_*(\beta_1, \beta_2) = \text{Mor}(\mathcal{Y}(\beta_2), \mathcal{Y}(\beta_1))$.

Auroux systematically develops approach (2b) in [Aur10b], and we adopt it in this thesis. In the first project (Chapters 2 and 3) we translate approach (2b) to the case of 4-ended tangles in the pillowcase homology setting. In the second project (Chapter 4) approach (2) allows us to connect the Hochschild homology of an A_∞ bimodule $N(\phi)$ associated to a surface automorphism ϕ to the fixed point Floer cohomology $HF^*(\phi)$.

1.3 Summary of results

1.3.1 Recovering Lagrangian Floer homology algebraically

In Chapter 2 we construct an algebraic version of Lagrangian Floer homology for two immersed curves inside a surface with any non-zero number of basepoints $\{z_i\} = Z^{bps}$ on each of the components of its boundary $(\Sigma, Z^{bps} \subset \partial\Sigma \neq \emptyset)$. We first parameterize the surface by arcs α^{arcs} (see Definition 2.4.1). Then to a parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$ we associate an algebra $\mathcal{A}_1 = \mathcal{A}_1(\Sigma, Z^{bps}, \alpha^{arcs})$ (see Definition 2.4.3). To an immersed curve L (circle or arc with ends on $\partial\Sigma \setminus Z^{bps}$) inside Σ we associate an A_∞ module $M(L)_{\mathcal{A}_1}$ (see Section 2.5.2). Then, we prove the following pairing result:

Theorem 1.3.1. *Let L_0, L_1 be two admissible unobstructed immersed curves in the parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$. Then their Lagrangian Floer complex is chain homotopy equivalent to the algebraic pairing of curves:*

$$CF_*(L_0, L_1) \simeq M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{\text{bar}_r}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}.$$

Terms “admissible” and “unobstructed” are defined in Section 2.2, $_{\mathcal{A}_1} \overline{M(L_0)}$ de-

notes a dual module, and ${}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1}$ is a specific type DD structure constructed in such a way, that the above homotopy equivalence is true. From this homotopy equivalence it follows that

$$HF_*(L_0, L_1) \cong H_*(M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r{}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}).$$

Theorem 1.3.1 above proves that the Yoneda embedding functor

$$L \mapsto \bigoplus_{\alpha_i \in \alpha^{arcs}} \text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L, \alpha_i)$$

of the partially wrapped Fukaya category of a surface $\mathcal{F}(\Sigma, Z^{bps})$ into the category of A_∞ modules $Mod_{\mathcal{A}_1}$ is fully-faithful (see Section 2.3 and Remark 2.5.3 for more on this). The result was known before, at least for embedded curves it follows from the concept of split-generation of A_∞ categories (see [Sei08]), and for immersed curves it follows once one realizes that they are objects of the Fukaya category (see [HaiKatKon17, Theorem 4.3]). Our proof is simple and geometric, and doesn't rely on algebraic machinery and generation results.

Also, the algebraic pairing $H_*(M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r{}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)})$ gives an algorithm for computing geometric (i.e. minimal) intersection number of two curves, and geometric self-intersection number of one curve, on a surface with boundary³.

Our main motivation behind the algebraic pairing theorem was to construct algebraic invariants of tangles in the context of pillowcase homology. We now describe this construction.

³One should treat the case of $Per(L_0, L_1) = \mathbb{Z}$ separately, and subtract 2 from $rk(H_*)$ in order to obtain the geometric intersection number.

1.3.2 Bordered theory for pillowcase homology

Floer in [Flo90] and Kronheimer and Mrowka in [KroMro10] constructed a knot invariant called sutured instanton knot homology $KHI(K)$. It has properties similar to knot Floer homology, like detecting the genus of a knot (non-vanishing result). In fact, $KHI(K)$ is conjectured to be isomorphic to knot Floer homology $\widehat{HFK}(K)$ (both of them categorify the Alexander polynomial of a knot). In [KroMro11b], [KroMro11a] Kronheimer and Mrowka constructed another knot invariant called singular instanton knot homology, which is denoted by $I^\natural(K)$. In [KroMro11a] they proved that there is a spectral sequence from Khovanov homology $Kh(K)$ to $I^\natural(K)$, and that $I^\natural(K)$ is, in fact, isomorphic to $KHI(K)$. This, together with the non-vanishing result for $KHI(K)$, proved that Khovanov homology detects the unknot.

On the instanton side only gauge theoretic constructions of knot invariants are fully developed. We are interested in the potential symplectic counter-parts. The construction called *pillowcase homology* was developed by Hedden, Herald, and Kirk in [HedHerKir14a] and [HedHerKir14b], in order to better understand and compute $I^\natural(K)$. This geometric construction potentially gives a knot invariant, which we denote by $H_{pil}(K)$. It should be the symplectic side of Atiyah-Floer conjecture for singular instanton knot homology $I^\natural(K)$, see [HedHerKir14b, Conjecture 6.5].

The bordered construction

Based on the constructions from Chapter 2, in Chapter 3 we will enhance the construction of pillowcase homology, extending algebraic invariants from knots to tangles.

A quick pictorial plan of the construction is on the Figure 1.1. Having a knot in $K \subset S^3$, find a 2-sphere that intersects the knot in 4 points and thus divides it into two tangles $Q \cup T = K$, such that one of the two resulting tangles Q is trivial. Then proceed by applying applying traceless representation variety functor to the spaces $(D^3, Q) \leftarrow (S^2, 4pts) \rightarrow (D^3, T)$. The variety for the 2-sphere turns out to be the

pillowcase, i.e. a torus factorized by the elliptic involution, which is a 2-sphere with 4 singular conical points: $P = R(S^2, 4) = \{h \in \text{hom}(\pi_1(S^2 \setminus 4pt), SU(2)) \mid \text{tr}(h(\gamma_i)) = 0\} / \text{conj.} = T^2 / \tau$. The other ones are slightly trickier, because they involve holonomy perturbations π and a certain condition regarding the green earring (circle H + arc W) inside $(D^3, Q = A_1 + A_2)$:

$$L_Q = R_\pi^h(D^3, A_1 \cup A_2),$$

$$L_T = R_\pi(D^3, T = T_1 + T_2),$$

$$R(D^3, T=T_1+T_2) = \{h \in \text{hom}(\pi_1(D^3 \setminus T), SU(2)) \mid \text{tr}(\mu_{T_i})=0\} / \text{conj.},$$

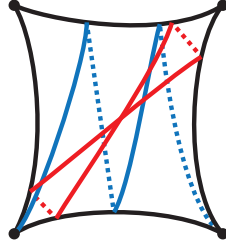
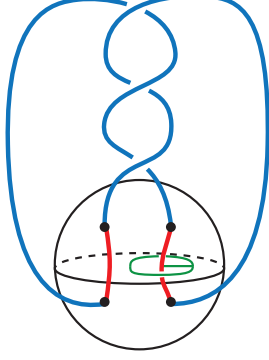
$$R^h(D^3, A_1 \cup A_2) = \{h \in \text{hom}(\pi_1(D^3 \setminus (A_1 \cup A_2 \cup H \cup W)), SU(2)) \mid \text{tr}(h(\mu_{A_i})) = \text{tr}(h(\mu_H)) = 0, h(\mu_W) = -I\} / \text{conj.}.$$

After having done all this, one obtains pillowcase homology of a knot as Lagrangian Floer homology $H_{pil}(K) = HF_*(L_Q, L_T)$ inside $P^* = P \setminus 4pts$. In spite of the fact that it was not proved that $H_{pil}(K)$ is a knot invariant, it was calculated in many examples and verified to be isomorphic to $I^h(K)$.

Our construction in Chapter 3 is an algebraic enhancement of pillowcase homology. It answers the following question: what algebraic structures should one associate to L_Q and L_T , in order to be able to recover $H_{pil}(K) = HF_*(L_Q, L_T)$ algebraically, without looking at the intersection picture on the pillowcase. The relevant objects can be seen in the third column of Figure 1.1. Using constructions from Chapter 2, we associate to the pillowcase with four small discs (containing singular points) removed $\overline{P} = P \setminus D_1 \cup D_2 \cup D_3 \cup D_4$ an algebra $\mathcal{A}_1(\overline{P}) = \mathcal{A}_p$, and to an immersed curve (circle or arc with ends on the boundary) L_T, L_Q inside P^* an A_∞ modules $M(L_T)_{\mathcal{A}_p}, M(L_Q)_{\mathcal{A}_p}$. Then Theorem 1.3.1 implies

$$H_{pil}(K) = HF_*(L_Q, L_T) \cong H_*(M(L_T)_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{\text{bar}}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L_Q)}).$$

• 2-sphere ($S^2, 4\text{pts}$)	→	• pillowcase P	→	• algebra A_p
• triv. tangle (D^3, Q)	→	• curve L_Q in P	→	• A_∞ module $M(L_Q)_{A_p}$
• tangle (D^3, T)	→	• curve L_T in P	→	• A_∞ module $M(L_T)_{A_p}$
• knot $Q+T=K$	→	• $H_{\text{pil}}(K)=HF_*(L_Q, L_T)$	$\cong$	• $H_*(M(L_T)_{A_p} \boxtimes \overline{A_p} \text{bar}_r \boxtimes \overline{M(L_Q)})$
(Lagrangian Floer homology in $P^*=P \setminus \{4\text{pts}\}$)				



$$L_T \rightarrow P \leftarrow L_Q$$

$$(D^3, T) \leftarrow (S^2, 4\text{pts}) \rightarrow (D^3, Q)$$

Figure 1.1: The pillowcase homology construction (1st and 2nd columns), and its algebraic extension (3rd column).

Potential applications of the bordered construction

Let us now describe the motivation behind the bordered construction. First, it provides a natural candidate for an algebraic invariant of a 2-stranded tangle T inside a ball D^3 . To such a tangle one can associate an immersed Lagrangian $L(T) : R_\pi(D^3, T) \looparrowright R(\partial D^3, 4) = P$, and then an A_∞ module $M(L(T))_{A_p}$ ⁴. As with pillowcase homology, there are missing ingredients in this construction: it needs to be proved that $L(T)$ is unobstructed, and the homotopy type of $M(L(T))_{A_p}$ does not depend on the perturbation π .

Building on this idea, one can isolate the part of $H_{\text{pil}}(K)$ which depends on L_K . I.e., if one changes L_K in some way, it is more natural to understand how $M(L_K)_{A_p}$ changes, rather than $H_{\text{pil}}(K) = H_*(M(L_K)_{A_p} \boxtimes \overline{A_p} \text{bar}_r \boxtimes \overline{M(L^\natural)})$.

⁴Here one must be careful. The definition of $M(L(T))_{A_p}$ requires a parameterization of the pillowcase $R(\partial D^3, 4)$. Thus there needs to be additional information, for this parameterization to be fixed. Namely, the boundary of the tangle $(\partial D^3, 4)$ must be bordered, i.e. parameterized by a standard fixed $(S^2, 4)$.

A very interesting direction of research is to further develop bordered theory for pillowcase homology $H_{pil}(K)$ into full bordered theory. Let us briefly describe the way such a theory would work. The strategy is the following:

- Step (1) To understand what algebra should be associated to $2n$ punctured sphere $(S^2, 2n)$.
- Step (2) To understand what bimodules (over the algebras from the previous step) correspond to tangles inside $S^2 \times I$, which connect $(S^2, 2k)$ to $(S^2, 2(k+1))$.
- Step (3) To build up a chain complex $C_{alg}(K)$, and prove that its homology $H_{alg}(K)$ is a knot invariant. The construction of $C_{alg}(K)$ should involve composing (via derived tensor product, or morphism space pairing) bimodules from the second step, and modules that correspond to trivial tangles $M(L_U), M(L^\natural)$ (examples 3.3.1, 3.3.2).
- Step (4) To prove that this construction, in fact, computes singular instanton knot homology: $H_{alg}(K) \cong I^\natural(K)$.

This is a difficult project. Even completing the step (1) is hard. The desired algebra should be the algebra of the Fukaya category of the smooth stratum of representation variety $R(S^2, 2n)$. After the pillowcase $R(S^2, 4)$, the next space of interest is $R(S^2, 6)$. It is already a complicated singular 6-dimensional manifold, see [Kir17]. See also [HerKir15] for the study of $R(S^2, 2n)$. Let us note that additional structures on representation spaces could help to compute their Fukaya category. For example, in the case of Heegaard Floer homology, the Fukaya category of $Sym^g(\Sigma_g \setminus 1pt)$ was computed in [Aur10b] using the structure of Lefschetz fibration over $\mathbb{C}$.

Nevertheless, if one manages to guess the algebras and bimodules, one can dismiss the underlying geometry and try to prove that the knot invariant is well defined algebraically (step (3)).

Examples of analogous bordered theories developed for other invariants are: bordered Heegaard Floer homology [LipOzsThu08], [LipOzsThu15]; bordered theory for

knot Floer homology [OzsSza17b], [OzsSza16], [OzsSza17a]; bordered theories for Khovanov homology [Rob16a], [Rob16b], [Man17]. Step (3) for Heegaard Floer homology was done in [Zha16], and for knot Floer homology in [OzsSza16], [OzsSza17a].

Let us mention that there is a related work by Zibrowius [Zib17]: by a different construction he associates an algebra to a 4-punctured 2-sphere and modules to 2-stranded tangles, and has a pairing theorem resulting in knot Floer homology.

1.3.3 Comparing homological invariants of mapping classes of surfaces

Overview

Consider a strongly based mapping class group $MCG_0(\Sigma_g, \partial\Sigma_g = S^1)$, which consists of all orientation preserving self-diffeomorphisms $\phi : (\Sigma_g, \partial\Sigma_g) \rightarrow (\Sigma_g, \partial\Sigma_g)$ fixing the boundary, up to isotopy. Suppose we are given a mapping class $\phi \in MCG_0(\Sigma_g, \partial\Sigma_g = S^1)$. In Chapter 4 to such an object we assign two homological invariants:

- (1) An A_∞ bimodule $N(\phi)$ (or, more precisely, its A_∞ homotopy equivalence class), and its Hochschild homology $HH_*(N(\phi))$, which is $\mathbb{Z}_2$ -graded. The bimodule $N(\phi)$ comes from bordered Heegaard Floer theory: one constructs a Heegaard diagram for a mapping cylinder of ϕ , then considers certain intersections between α and β curves as generators, and then obtains a differential via pseudo-holomorphic curve theory. For the definitions see [LipOzsThu13], where our bimodule is also denoted by $N(\phi)$, and the original paper [LipOzsThu15], where the corresponding bimodule is denoted by $\widehat{CFDA}(\phi, -g + 1)$. In Section 4.4.2, we also give an equivalent construction of this bimodule using the partially wrapped Fukaya category of the surface. This construction will be useful in understanding the connection with the next invariant.
- (2) Suppose for a moment that ϕ is a mapping class of a closed surface, and pick

a generic area-preserving representative ϕ in that mapping class. Then we consider a cohomology theory $HF^*(\phi)$, where the generators are non-degenerate constant sections of the mapping torus $T_\phi \rightarrow S^1$ (i.e. non-degenerate fixed points), and the differentials are pseudo-holomorphic cylinder sections of $T_\phi \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$. Sometimes the same theory can be set up using fixed points as generators and pseudo-holomorphic discs in Lagrangian Floer cohomology of graphs of id and ϕ as differentials, but we will use the sections and cylinders approach. This invariant $HF^*(\phi)$ is called *fixed point Floer cohomology*, or *symplectic Floer cohomology*. It is $\mathbb{Z}_2$ -graded by the sign of $\det(d\phi - Id)$ at the fixed points of ϕ .

In order to generalize this construction to mapping classes fixing the boundary, one has to specify in which direction to twist the boundary slightly to eliminate degenerate fixed points. There are two choices (we call them $+$ and $-$) for each boundary, see Figure 4.17 for the conventions.

In our case of a mapping class $\phi \in MCG_0(\Sigma_g, \partial\Sigma_g = S^1 = U_1)$, we actually consider the induced mapping class $\tilde{\phi} : (\tilde{\Sigma}_g, \partial\tilde{\Sigma}_g = U_1 \cup U_2) \rightarrow (\tilde{\Sigma}_g, \partial\tilde{\Sigma}_g = U_1 \cup U_2)$, where the surface $\tilde{\Sigma}_g = \Sigma_g \setminus D^2$ is obtained by removing a disc in the small enough neighborhood of the boundary U_1 , such that representative ϕ is identity on that neighborhood. We then consider fixed point Floer cohomology $HF^*(\tilde{\phi}; U_2+, U_1-)$ with two different perturbation twists on the two boundaries.

The bimodule invariant $N(\phi)$ was computed for mapping classes of the genus 1 surface in [LipOzsThu15, Section 10]. In Section 4.1.5, we compute $N(\phi)$ in the case of the genus 2 surface; we do it by explicitly describing the bimodules associated to Dehn twists τ_l , which generate the mapping class group. For that we write down the bimodules based on a holomorphic curve count, and then use the description of arc-slide DD bimodules from [LipOzsThu14] to prove that the bimodules $N(\tau_l)$ are

the correct ones.

We also describe how to compute the Hochschild homology of the bimodule. There is a problem one faces in computing Hochschild homology: none of the smallest models of bimodules $N(\phi)$ for the Dehn twists are bounded, and thus their Hochschild complex is infinitely generated. We write down a certain bounded identity bimodule $[\mathbb{I}]^b$ in the genus 2 case, so that the bimodule $[\mathbb{I}]^b \boxtimes N(\phi) \boxtimes [\mathbb{I}]^b$ is bounded and belongs to the same A_∞ homotopy equivalence class as $N(\phi)$. Thus by replacing $N(\phi)$ with $[\mathbb{I}]^b \boxtimes N(\phi) \boxtimes [\mathbb{I}]^b$ we solve the problem of $N(\phi)$ being not bounded.

In Section 4.3, based on our computations of the Hochschild homology $HH_*(N(\phi))$ in the genus 2 case, and the corresponding computations of the fixed point Floer cohomology, we make the following conjecture.

Conjecture 1.3.2. *For every mapping class $\phi \in MCG_0(\Sigma_g, \partial\Sigma_g = S^1 = U_1)$ there is an isomorphism of $\mathbb{Z}_2$ -graded vector spaces*

$$HH_*(N(\phi^{-1})) \cong HF^{*+1}(\tilde{\phi}; U_{2+}, U_{1-}).$$

We explain where such an isomorphism may come from. In Section 4.4, we sketch a symplectic geometric interpretation of bordered Heegaard Floer homology by Denis Auroux, which is based on the partially wrapped Fukaya categories of punctured surfaces and their symmetric products. In the framework of Fukaya categories of symplectic manifolds, if one wants to compare the Hochschild homology of the graph bimodule to fixed point Floer cohomology via open-closed map, one needs to consider the same kind of Hamiltonian perturbations for both invariants. Specifically, in partially wrapped theory, one needs to consider perturbations at infinity coming from non-constant Hamiltonian on the boundary. We denote the corresponding version of fixed point Floer cohomology by $HF^{1bp}(\phi)$. This kind of fixed point Floer cohomology was not considered before, but could be defined analogously to other versions.

Instead we choose to work with the version $HF^*(\tilde{\phi}; U_2+, U_1-)$, defined in terms of existing invariants. We then explain our choice, i.e. why $HF^*(\tilde{\phi}; U_2+, U_1-)$ is naturally isomorphic to $HF^{1bp}(\phi)$. After this, in Section 4.5, we show how our conjecture (the double basepoint version of it, to be precise) can be viewed as an instance of a more general conjecture in symplectic geometry. It states that the open-closed map in the Fukaya-Seidel category of a Lefschetz fibration is an isomorphism.

Notably, assuming the conjecture is true, our computational methods allow one to effectively compute the number of fixed points of a mapping class $\tilde{\phi}$ by simply running a program, even in the pseudo-Anosov case. For example, if one takes a mapping class $\psi = \tau_A \tau_B \tau_C \tau_D \tau_C \tau_A \tau_B \tau_E$ (see Figure 4.6), then the program [Kot18] finds:

Automorphism	Number of fixed points	Time in seconds
ψ	5	0.2
ψ^2	5	0.5
ψ^3	11	4.7
ψ^4	23	16
ψ^5	52	59.1
ψ^6	103	271.6

As a byproduct of the simplicity of the computations, we can determine if the mapping class is periodic, reducible with all components periodic, or pseudo-Anosov: the rank of $HH_*(N(\phi^n))$ is respectively bounded, grows linearly, or grows exponentially (see [LipOzsThu13, Corollary 4.2], [Cot09, Corollary 1.7]).

Context

There is another isomorphism of invariants of mapping classes of surfaces, which is directly related to our work. Consider a 3-manifold Y_ϕ^3 , where the subscript ϕ indicates that the manifold fibers over the circle $Y_\phi^3 \rightarrow S^1$ with a monodromy $\phi :$

$\Sigma_{closed} \rightarrow \Sigma_{closed}$. The statement says that the Heegaard Floer homology in the second lowest $spin^c$ structures (evaluating to $-2g+4$ on the fiber) is equal to the fixed point Floer cohomology of the corresponding monodromy: $HF_*(Y_\phi^3; -2g+4) \cong HF^*(\phi)$. Assuming the genus is greater than 2, this isomorphism follows from the following chain of isomorphisms:

- $HF^*(\phi) \cong HP_{degree=1}^*(\phi)$,

where $HF^*(\phi)$ is fixed point Floer cohomology of mapping class ϕ of a closed surface, see [Sei02], and $HP_{degree=1}^*(\phi)$ is periodic Floer cohomology, see [HutSul05]. This isomorphism is addressed in [LeeTau12, Appendix B].

- $HP_{degree=1}^*(\phi) \cong \widetilde{HM}_*(Y_\phi^3, c_+; -2g+4)$,

where Y_ϕ^3 is a 3-manifold fibered over the circle with a monodromy ϕ , and $\widetilde{HM}_*(Y_\phi^3, c_+; -2g+4)$ denotes an invariant of a 3-manifold called monopole Floer homology, defined in [KroMro07]. The c_+ indicates the version of $\widetilde{HM}_*$ where one uses a monotone positive perturbation. The isomorphism was proved in [LeeTau12]. The positivity of the perturbation comes from the genus being greater than 2.

- $\widetilde{HM}_*(Y_\phi^3, c_+; -2g+4) \cong \widetilde{HM}_\bullet(Y_\phi^3, c_+; -2g+4)$,

where $\bullet$ indicates the negative completion of the coefficient ring. The definition of $\widetilde{CM}_*$ does not depend on the negative completion, so the above groups are isomorphic, see [KroMro07, p. 606].

- $\widetilde{HM}_\bullet(Y_\phi^3, c_+; -2g+4) \cong \widetilde{HM}_\bullet(Y_\phi^3; -2g+4)$,

where the absence of c_+ indicates exactness of the perturbation in the definition of $\widetilde{HM}_\bullet$. The isomorphism is proved in [KroMro07, Theorems 31.1.2].

- $\widetilde{HM}_\bullet(Y_\phi^3; -2g+4) \cong \widetilde{HM}_\bullet(Y_\phi^3, c_b; -2g+4)$,

where c_b indicates the balanced perturbation in the definition of $\widetilde{HM}_\bullet$. The isomorphism is proved in [KroMro07, Theorems 31.1.1].

- $\widetilde{HM}_\bullet(Y_\phi^3, c_b; -2g+4) \cong \widetilde{HM}_*(Y_\phi^3, c_b; -2g+4),$

which again follows from the fact that the negative completion of the coefficient ring does not affect $\widetilde{HM}_*$.

- $\widetilde{HM}_*(Y_\phi^3, c_b; -2g+4) \cong HF_*(Y_\phi^3; -2g+4),$

which is a deep and very difficult theorem, despite the fact that the definition of Heegaard Floer homology was inspired by monopole Floer homology type constructions. It was proved via passing through another invariant, called embedded contact homology (ECH), in [KutLeeTau10a], [KutLeeTau10b], [KutLeeTau10c], [KutLeeTau11], [KutLeeTau12], and [ColGhiHon11], [ColGhiHon12a], [ColGhiHon12b], [ColGhiHon12c].

Our Conjecture 1.3.2 is analogous to the proved isomorphism $HF^+(Y_\phi^3; -2g+4) \cong HF^*(\phi)$. We work in a slightly different 3-manifold. Suppose we fix a lift of ϕ from the mapping class group of the closed surface $MCG(\Sigma_{closed})$ to the strongly based mapping class group $MCG_0(\Sigma_g, \partial\Sigma_g = U_1)$. Then, instead of the fibered manifold Y_ϕ^3 , we consider the open book corresponding to ϕ , which we denote by M_ϕ° . From Y_ϕ^3 one can obtain M_ϕ° by 0-surgery on the constant section of $Y_\phi^3 \rightarrow S^1$, which comes from the lift of ϕ . From M_ϕ° one can obtain Y_ϕ^3 by 0-surgery on the binding K of M_ϕ° . Instead of $HF^+(Y_\phi^3; -2g+4)$ we are working with the knot Floer homology of the binding (in the second lowest Alexander grading) $\widehat{HFK}(M_\phi^\circ, K; -g+1)$. It is equal to the Hochschild homology $HH_*(N(\phi))$ (see [LipOzsThu15, Theorem 14]) with which we are actually working in this thesis. The relevant version of fixed point Floer cohomology turns out to be $HF^*(\tilde{\phi}; U_{2+}, U_{1-})$. It is possible that the Conjecture 1.3.2 can be deduced from $HF^+(Y_\phi^3; -2g+4) \cong HF^*(\phi)$.

It is also interesting to compare our results to the work of Spano [Spa17]. He develops the full version of embedded contact knot homology $ECK(Y, K, \alpha)$, and conjectures it to be isomorphic to $HFK^-(Y, K)$. In [Spa17, Section 3.3.1], the connection to symplectic Floer homology is explained. Namely, in case of the knot

being a binding of an open book, the embedded contact knot homology is equal to a certain periodic Floer homology, see [Spa17, Theorem 3.19]. In the degree 1 case, it follows that if $ECK(Y, K, \alpha) \cong HFK^-(Y, K)$, then one has $HF^*(\phi, U_1+) \cong HFK^-(M_\phi^\circ, K; -g+1)$. Thus one can view our Conjecture 1.3.2 as the “hat” version of the conjecture of Spano.

Working in the “hat” version allows us to consider the Hochschild homology of the bimodule $HH(N(\phi))$ instead of the knot Floer homology $\widehat{HFK}(M_\phi^\circ, K; -g+1)$. This transition is quite powerful, because two things become possible: computations using bordered Floer theory, and the connection to the Fukaya category of a surface (in the “hat” version: to the partially wrapped Fukaya category of a surface), specifically to twisted open-closed maps there. The latter connection provides hope that the Conjecture 1.3.2 can be proved by more algebraic methods, using the structure of the Fukaya category. In this direction see [Gan12], where it is proved that the untwisted open-closed map is an isomorphism for the non-degenerate wrapped Fukaya category, in the exact setting.

1.4 Assumptions, conventions

- We will be using the $\mathbb{F}_2$ coefficient field.
- By differential we will mean not only a map of vector spaces $d : C \rightarrow C$ satisfying $d^2 = 0$, but also the following. Having a distinguished basis $\{y_j\}$ of C , if, for example, $d(y_7) = y_1 + y_2 + y_3$, then we will say that there is a differential from y_7 to y_1 (and from y_7 to y_2 , and from y_7 to y_3). We will denote these differentials by arrows: $y_7 \rightarrow y_1$. We will always have a distinguished basis of chain complexes, because chain complexes will be generated by geometric objects, such as intersections of curves or fixed points of maps.
- We will work with Lagrangian Floer homology, as opposed to cohomology, ex-

cept in Sections 2.3.2 and 4.4 where we adopt Auroux's cohomological convention.

- We will work with fixed point Floer cohomology, as opposed to homology.
- We will usually denote by ϕ not only the diffeomorphism of a surface, but also the mapping class which it represents.
- We will use the convention $\omega(X_H, \cdot) = -dH$ for Hamiltonian vector fields.

Chapter 2

From geometry to algebra

2.1 A_∞ algebraic notions

An overall broad goal in this section is to introduce algebraic machinery for cut and paste techniques in low-dimensional topology. Because we mainly work with Floer-theoretic *homological* invariants of 3-manifolds and links, we would like to be able to “decompose” chain complexes. I.e., suppose we have two 3-manifolds with one boundary component and with (possibly empty) tangles inside:

$$\begin{aligned} &\left(Y_1, T_1 \subset Y_1, \partial(Y_1, T_1) = (\Sigma, 2k \text{ points}) \right), \\ &\left(Y_2, T_2 \subset Y_2, \partial(Y_2, T_2) = (\Sigma, 2k \text{ points}) \right). \end{aligned}$$

Then we are looking for algebraic structures on $\mathbb{F}_2$ -vector spaces $M(Y_1, T_1)$ and $K(Y_2, T_2)$, such that $M(Y_1, T_1) \otimes K(Y_2, T_2)$ is naturally a chain complex, which will correspond to a 3-manifold with a link obtained from gluing:

$$(Y, L) = (Y_1, T_1) \cup_{(\Sigma, 2k)} (Y_2, T_2).$$

Definition 2.1.1. A *chain complex* is an $\mathbb{F}_2$ -vector space C equipped with a differ-

ential $d : C \rightarrow C$, satisfying $d^2 = 0$.

Definition 2.1.2. A *dg-algebra*¹ $\mathcal{A}$ is chain complex $(\mathcal{A}, d)$ equipped with a product $\mu_2 : A \otimes A \rightarrow A$, satisfying the associativity law $(ab)c = a(bc)$ and the Leibniz rule $d(ab) = d(a)b + ad(b)$.

Definition 2.1.3. A *right dg-module* $M_{\mathcal{A}}$ over *dg-algebra* $\mathcal{A}$ is a chain complex (M, d) together with an action of the algebra $m_{1,1} : M \otimes \mathcal{A} \rightarrow M$, satisfying $d_M(m_{1,1}(x, a)) = m_{1,1}(d_M x, a) + m_{1,1}(x, d_{\mathcal{A}} a)$.

The notion of *left dg-module* is defined analogously.

Definition 2.1.4. A *tensor product* of left and right *dg-modules* $(M_1)_{\mathcal{A}}$ and ${}_{\mathcal{A}}(M_2)$ is a quotient chain complex $M_1 \otimes_{\mathcal{A}} M_2 = M_1 \otimes_{\mathbb{F}_2} M_2 / (xa \otimes y = x \otimes ay)$ with a differential $d(x \otimes y) = dx \otimes y + x \otimes dy$.

Although the above algebraic structures satisfy our needs of being able to “decompose” chain complexes, it turns out that they are not rich enough. For example, if one wants to recover the structure of a *dg-algebra* $\mathcal{A}$ from its homology $H_*(\mathcal{A})$, it is not enough to remember the product on $H_*(\mathcal{A})$, see [Kel01]. Analogously, if one wants to recover the structure of *dg-module* M from its homology $H_*(M)$, it is not enough to remember the action of the algebra. Exactly the same kind of problems arise in symplectic geometry, when one wants to study Lagrangians in symplectic manifolds via their intersections with other Lagrangians, see [Smi15, Section 3.8] for the motivation in that context. The following are generalizations of *dg-algebraic* notions, which allow one to have more flexibility.

Definition 2.1.5. An *A_{∞} algebra* is an $\mathbb{F}_2$ -vector space $\mathcal{A}$ equipped with a family of

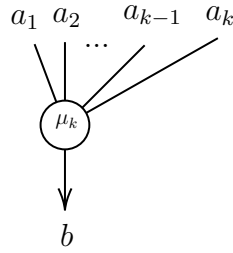
¹We will not introduce gradings on our objects, but to avoid confusion we will keep the standard name “*dg-algebra*”, which is derived from “*differential graded algebra*”.

operations $\mu_k : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}$ for each $k \geq 1$, satisfying A_∞ relations

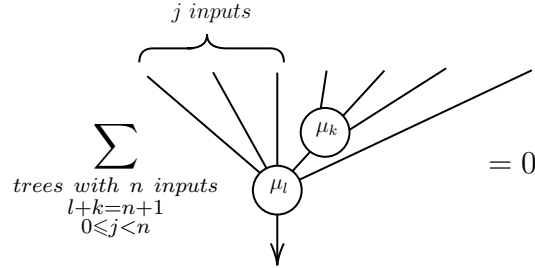
$$\sum_{l+k=n+1} \mu_k(a_1, \dots, a_j, \mu_l(a_{j+1}, \dots, a_{j+l}), a_{j+l+1}, \dots, a_n) = 0$$

for any $n \geq 1$ elements $a_1, \dots, a_n \in \mathcal{A}$.

It is convenient to think about the multiplication geometrically, in terms of trees. Namely, if $\mu_k(a_1, \dots, a_k) = b$, we will denote it like this:



The A_∞ relations, for each $n \geq 1$, can then be rewritten like so:



In the future we will often utilize such pictorial descriptions of maps.

If one thinks of μ_1 as a differential and μ_2 as a product, then the first two A_∞ relations are the conditions for being a dg -algebra, with the third relation implying that the associativity holds up to homotopy. Thus an A_∞ algebra with $\mu_k = 0$ for $k \geq 3$ is the same notion as a dg -algebra.

Because we will be working almost exclusively with dg -algebras, rather than A_∞ algebras, in what follows we will assume that $\mathcal{A}$ is a dg -algebra, despite the fact that it is possible to define the structures below over A_∞ algebras.

Definition 2.1.6. A *right* A_∞ module $M_{\mathcal{A}}$ over a dg -algebra $\mathcal{A}$ is an $\mathbb{F}_2$ -vector space M equipped with a family of actions $m_{1,k} : M \otimes \mathcal{A}^{\otimes k} \rightarrow M$ for each $k \geq 0$, satisfying A_∞ relations (for each $n \geq 0$):

$$\sum_{\substack{\text{trees with} \\ n+1 \text{ inputs} \\ l+k=n}} m_{1,k} \circ m_{1,l} + \sum_{\substack{\text{trees with} \\ n+1 \text{ inputs} \\ 1 \leq j \leq n-1}} m_{1,n-1} \circ \mu_2 + \sum_{\substack{\text{trees with} \\ n+1 \text{ inputs} \\ 1 \leq j \leq n}} m_{1,n} \circ \mu_1 = 0$$

Sometimes A_∞ modules are also called *A modules*. The notion of *left A module* can be defined analogously.

Definition 2.1.7. A *left type D structure* ${}^{\mathcal{A}}K$ over a dg -algebra $\mathcal{A}$ is an $\mathbb{F}_2$ -vector space N equipped with a map $\delta^1 : \mathcal{A} \otimes N \rightarrow \mathcal{A}$, satisfying equation

$$\delta^1 \circ \mu_2 + \delta^1 \circ \mu_1 = 0$$

The notion of *right type D structure* can be defined analogously. We will often omit the word “type”.

Remark 2.1.8. A type D structure over a dg -algebra has a natural dg -module associated to it. Namely, a type D structure ${}^{\mathcal{A}}K$ gives rise to an A module ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes^{\mathcal{A}} K$, where the algebra $\mathcal{A}$ is viewed as an $\mathcal{A}$ -bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ over itself, see [LipOzsThu08, Lemma 2.20] and the discussion below on bimodules and box tensor products. This A module ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes^{\mathcal{A}} K$ is in fact a dg -module, i.e. actions $m_{k,1}$ vanish for $k > 1$.

Before we proceed, we need to introduce an additional structure on our algebraic objects. In bordered Floer theory, we often have a distinguished subalgebra $\mathcal{I} \subset \mathcal{A}$, which is a vector space over $\mathbb{F}_2$ generated by the idempotent elements of $\mathcal{A}$. Each distinguished generator x of $\mathcal{A}$ will have its own left and right idempotent $i_{x,left}, i_{x,right} \in \mathcal{I}$. This defines actions $\mathcal{A} \curvearrowright \mathcal{I}$ and $\mathcal{I} \curvearrowright \mathcal{A}$ by setting $i_{x,left} \cdot x = x \cdot i_{x,right} = x$, and other actions to 0. Notice that the sum of all the idempotents is a unit in $\mathcal{A}$, and thus we denote it by 1. Analogous idempotent action structure $\mathcal{I} \curvearrowright N$ and $M \curvearrowright \mathcal{I}$ will hold for D structures ${}^{\mathcal{A}}K$ and A modules $M_{\mathcal{A}}$, i.e. each distinguished generator will have its unique left (or right) idempotent which preserves it. In our notation in the future ${}_i t$ will mean that unique left idempotent of the generator t is i .

We call the structure described above *idempotent subalgebra structure* on $\mathcal{A}$. It comes naturally if one obtains the algebra as the endomorphism algebra $\bigoplus_{L_i, L_j \in \mathcal{L}} \text{hom}(L_i, L_j)$ of a set of objects $\mathcal{L} = \{L_i\}$ in a strictly unital A_{∞} or dg category. The idempotents are the units of the objects.

Idempotents will be required to “match” in all algebra actions. This can be conveniently rephrased by saying that all tensor products over $\mathbb{F}_2$ are replaced by tensor products over $\mathcal{I}$ in the previous 3 definitions. In more details, in those definitions we should say that our dg or A_{∞} algebras are $\mathcal{I}$ -bimodules, the A modules and D structures are $\mathcal{I}$ -modules, and that all structure maps $\mu_k : \mathcal{A}^{(\otimes_{\mathcal{I}})k} \rightarrow \mathcal{A}$, $m_{1,k} : M \otimes_{\mathcal{I}} \mathcal{A}^{(\otimes_{\mathcal{I}})k} \rightarrow M$ and $\delta^1 : A \otimes_{\mathcal{I}} N \rightarrow A$ are $\mathcal{I}$ -linear.

Definition 2.1.9. Given a left D structure ${}^{\mathcal{A}}K$ and a right A module $M_{\mathcal{A}}$ over a dg -algebra $\mathcal{A}$, the *box tensor product* $M \boxtimes N$ is a chain complex $M \otimes_{\mathcal{I}} N$, with the differential defined as follows:

$$d(x_i \otimes_i y) =$$

The sum above is well-defined in case where the D structure or the A module satisfy certain boundedness conditions, see [LipOzsThu15, Section 2.2]. In bordered Heegaard Floer theory this corresponds to admissibility of Heegaard diagrams.

The A_∞ relations for the type D structure and the A module imply that the differential squares to zero, i.e. $M_{\mathcal{A}} \boxtimes^A K$ is indeed a chain complex, see [LipOzsThu08, Lemma 2.30] for the proof.

The box tensor product is a model for the A_∞ (or *derived*) tensor product, see [LipOzsThu08, Sections 2.2, 2.4] for the explanation of that.

We will be gluing 3-manifolds (possibly with a tangle inside) not only with one boundary component, but also with two. Thus we need to introduce bimodules.

Definition 2.1.10. An *opposite dg* or A_∞ algebra $\mathcal{A}^{op}$ is an algebra with the same underlying $\mathbb{F}_2$ -vector space, the same differential, but different higher operations: $\mu_k^{new}(a_1, \dots, a_k) = \mu_k(a_k, \dots, a_1)$.

A left type D structure ${}^{\mathcal{A}}K$ as an object is equivalent to a right type D structure $K^{\mathcal{A}^{op}}$. Analogously, a left A module ${}_{\mathcal{A}}M$ as an object is equivalent to a right A module $M_{\mathcal{A}^{op}}$.

Definition 2.1.11. A *left-right type DD structure* ${}^{\mathcal{A}}V^{\mathcal{B}}$ over two dg -algebras $\mathcal{A}$ and $\mathcal{B}$ is a left type D structure over a dg -algebra $\mathcal{A} \otimes_{\mathbb{F}_2} \mathcal{B}^{op}$, where the product is $a_1 \otimes b_1 \cdot a_2 \otimes b_2 = a_1 \cdot a_2 \otimes b_1 \cdot b_2$ and the differential is defined using the Leibniz rule.

Left-left and right-right type DD structures are defined analogously.

Definition 2.1.12. A *left-right A_∞ bimodule*² ${}^{\mathcal{A}}P_{\mathcal{B}}$ over two *dg*-algebras $\mathcal{A}$ and $\mathcal{B}$ (with the idempotent subalgebras $\mathcal{I}$ and $\mathcal{J}$ respectively) is a left-right $(\mathcal{I}, \mathcal{J})$ -bimodule P together with a family of $(\mathcal{I}, \mathcal{J})$ -bilinear A_∞ actions

$$m_{l,1,k} : \mathcal{A}^{(\otimes_{\mathcal{I}})^l} \otimes_{\mathcal{I}} P \otimes_{\mathcal{J}} \mathcal{B}^{(\otimes_{\mathcal{J}})^k} \rightarrow P$$

for each $l, k \geq 0$, satisfying the following relations (for each $n, s \geq 0$):

$$\begin{aligned}
& \sum_{\substack{n_1+n_2=n \\ s_1+s_2=s}} \text{Diagram 1} + \sum_{1 \leq j \leq s-1} \text{Diagram 2} + \sum_{1 \leq j \leq s} \text{Diagram 3} \\
& + \sum_{1 \leq j \leq n-1} \text{Diagram 4} + \sum_{1 \leq j \leq n} \text{Diagram 5} = 0
\end{aligned}$$

Definition 2.1.13. A *left-right DA bimodule* ${}^{\mathcal{A}}N_{\mathcal{B}}$ over two *dg*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a left-right $(\mathcal{I}, \mathcal{J})$ -bimodule N together with a family of $(\mathcal{I}, \mathcal{J})$ -bilinear A_∞ actions

$$\delta_{1+k}^1 : N \otimes_{\mathcal{J}} \mathcal{B}^{(\otimes_{\mathcal{J}})^k} \rightarrow \mathcal{A} \otimes_{\mathcal{I}} N$$

satisfying the following relations (for each $n \geq 0$):

²Sometimes also called *AA* bimodule.

$$\begin{aligned}
& \text{Diagram 1} + \sum_{k+l=n} \text{Diagram 2} + \sum_{1 \leq j \leq n-1} \text{Diagram 3} + \sum_{1 \leq j \leq n} \text{Diagram 4} = 0
\end{aligned}$$

The notions of *left-left* and *right-right type DA structures* are defined analogously.

The box tensor product operation allows to pair not only an A module to a D structure, but also allows to pair any two bimodules, modules or D structures as long as one pairs the right (left) A side to the left (right) D side, and certain boundedness conditions are met, see [LipOzsThu15, Section 2.3.2].

Suppose we are given two chain complexes C_1 and C_2 . The space $\text{Hom}_{\mathbb{F}_2}(C_1, C_2)$ has a natural chain complex structure: $d(f) = d \circ f + f \circ d$. The cycles in this space are called *chain maps*, and when two cycles are homologous $f_1 - f_2 = d(g)$ we say that f_1 is *chain homotopic* to f_2 . Then, instead of studying chain complexes up to *isomorphisms*, i.e. isomorphisms of vector spaces preserving the differential structure, we can study them up to *chain homotopy equivalence*. Two chain complexes C_1 and C_2 are called *chain homotopy equivalent* if there are chain maps $f : C_1 \rightarrow C_2$ and $g : C_2 \rightarrow C_1$ such that both of their compositions are homotopic to identities. It is very often the case that we assign a chain homotopy equivalence class of chain complexes as an algebraic invariant to a topological object. Homology of a chain complex is invariant w.r.t. chain homotopy equivalence, and this is why we often have homological invariants of topological objects.

A very similar situation happens with the other structures we have defined. Suppose S_1 and S_2 are either two type D structures, or two A modules, or two bimodules of the same type. There is an appropriate notion of morphisms between them, which form a chain complex $(\text{Mor}(S_1, S_2), d)$. Cycles in this morphism space are sometimes

called A_∞ *homomorphisms*, and two cycles are called *homotopic* if they are homologous. A corresponding notion of A_∞ *homotopy equivalence* between S_1 and S_2 can be introduced. We refer the reader to [LipOzsThu15, Section 2.2] for the definitions. Later in the thesis invariants of 3-manifolds with boundary, possibly with a tangle inside, will be A_∞ homotopy equivalence classes of either type D structures, or A modules, or bimodules.

2.2 Lagrangian Floer theory on a surface

2.2.1 Setup

Fix a compact smooth oriented surface with non-empty boundary, along with a finite set of basepoints $\{z_l\} = Z^{bps}$ on its boundary:

$$(\Sigma, Z^{bps} \subset \partial\Sigma \neq \emptyset).$$

Assumption 2.2.1. Throughout all of the chapters of this thesis we require that each boundary component of Σ contains at least one basepoint.

By a curve L we will mean an immersed circle or arc in the surface: $L : S^1 \looparrowright \Sigma$ or $L : [0, 1] \looparrowright \Sigma$. We will often denote by L both the map and the image $Im(L)$ inside Σ . Such a curve must satisfy the following properties:

- L is smoothly immersed, i.e. the differential of the map is injective. This implies that locally L is an embedding.
- If L is a circle, it is contained in the interior $int(\Sigma)$. If L is an arc then the endpoints of it are mapped to $\partial\Sigma \setminus Z^{bps}$, and the interior points of it are mapped to $int(\Sigma)$. Also the endpoints of L should be distinct points on $\partial\Sigma$, and transverse to the boundary $\partial\Sigma$.

- All self-intersections of L are transverse, and there are no triple self-intersections.
- L is *unobstructed*, i.e. it lifts to an embedded arc (if L is an arc) or properly embedded line (if L is a circle) in the universal cover of Σ . This is equivalent (see [Abo08, Lemma 2.2]) to saying that there is no fishtail (see Figure 2.1), and L is not null-homotopic. In other words, there should be no discs with boundary on L with 0 or 1 “switches” at self-intersections of L .

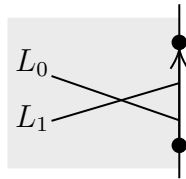


Figure 2.1: Fishtail.

Further in the text we will assume that these properties are satisfied, and call such immersed curves either “unobstructed curves”, or simply “curves”.

Lagrangian Floer homology is a homology theory for an ordered pair of curves, denoted by $HF_*(L_0, L_1)$. Often we will need pairs of curves to satisfy the following properties:

- All intersection points are transverse.
- There are no triple intersection points.
- If both curves L_0 and L_1 are arcs then the following condition should be satisfied. Suppose $l_0 \in \partial L_0$, $l_1 \in \partial L_1$, and l_0 and l_1 lie in the same connected component of $\partial\Sigma \setminus Z^{bts}$. Then, w.r.t. the induced orientation of ∂P , the order of the endpoints should be first l_0 , then l_1 .



- There is no essential immersed annulus with boundary on L_0 and L_1

$$A : (S^1 \times [0, 1], S^1 \times \{0\}, S^1 \times \{1\}) \rightarrow (\Sigma, L_0, L_1)$$

with no “switches” allowed. The latter means that boundaries of the annulus factor through L_0 and L_1 , i.e. one has $A|_{S^1 \times \{i\}} = L_i \circ f : S^1 \rightarrow \Sigma$ for some $f : S^1 \rightarrow S^1$ (or $f : S^1 \rightarrow [0, 1]$ if L_i is an arc).

We will call pairs of curves satisfying the above properties *admissible*.

Assumption 2.2.2. Throughout Chapter 2, when we state different Lagrangian boundary conditions, we will always assume that the maps on the boundaries of discs and polygons can be factored through the corresponding L_i , or equivalently (and intuitively more clear), that there are no switches allowed.

Convention. We will use notation $HF_*(L_0, L_1)$ for Lagrangian Floer homology of two curves, which are required to be admissible. We will use notation $HF_*^{pw}(L_0, L_1)$ for Lagrangian Floer homology of curves which are not necessarily admissible. Namely, we denote by $HF_*^{pw}(L_0, L_1)$ the homology $HF_*(L'_0, L_1)$ one gets after the curve L_0 is perturbed to L'_0 , in order to make (L_0, L_1) fully admissible. The superscript *pw* stands for partial wrapping, and indicates that the perturbation we use is the same kind of perturbation one does in the definition of the partially wrapped Fukaya category, see Section 2.3.2.

Our first goal is to define Lagrangian Floer homology $HF_*^{pw}(L_0, L_1)$ for a pair of immersed unobstructed L_0, L_1 in the surface Σ . We will describe the construction following closely [Abo08], [SilRobSal14], [HedHerKir14b], [HanRasWat16].

Step (1) For the homology to be well-defined one needs to restrict the class of curves they consider — the appropriate class for us are admissible pairs of unobstructed curves (the same setup as in [HanRasWat16]). Thus, having two unobstructed

curves (L_0, L_1) , one needs to know how to isotope L_0 to L'_0 so that (L'_0, L_1) is admissible.

Step (2) By definition, the chain complex $CF_*(L'_0, L_1)$ is generated over $\mathbb{F}_2$ by the intersection points $L'_0 \cap L_1$. The differential $\partial : CF_*(L'_0, L_1) \rightarrow CF_*(L'_0, L_1)$ is defined on the generators as mod 2 sum

$$\partial x = \sum_{y \in L'_0 \cap L_1} \mathcal{M}(x, y) \cdot y,$$

where $\mathcal{M}(x, y)$ counts the number of immersed discs from x to y in the surface. An immersed disc from x to y is an immersion of a *lune*, i.e. a disc with two convex corners, with right boundary on L'_0 and left boundary on L_1 (no switches allowed), see Figure 2.2. One requires these maps to be orientation preserving, and counts them up to reparameterizations, which are orientation preserving self-diffeomorphisms of the lune. Notice that the map is required to be an immersion on the boundary of the lune too, which guarantees that the angles of the disc at x and y are convex. The main difficulty in this step is to prove that $\mathcal{M}(x, y)$ is finite for any two generators $x, y \in L'_0 \cap L_1$.

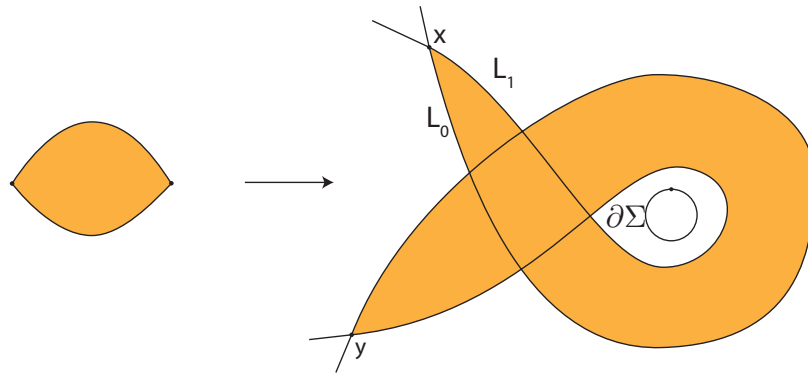


Figure 2.2: Immersed disc from x to y . Note that there are no fishtails due to the presence of $\partial\Sigma$.

Step (3) One proves that $\partial^2 = 0$, and, more generally, that A_∞ relations hold. Then

Lagrangian Floer homology is defined to be the homology of the chain complex:

$$HF_*^{pw}(L_0, L_1) = H_*(CF_*^{pw}(L_0, L_1)) := H_*(CF_*(L'_0, L_1), \partial).$$

The fact that Lagrangian Floer homology is well-defined follows from the following two steps.

Step (4) Suppose two curves are homotopic as maps $L_0 \sim L'_0$, where the only restriction is that the ends of immersed arcs are required to stay in $\partial\Sigma \setminus Z^{bts}$. Then if (L_0, L_1) and (L'_0, L_1) are both admissible pairs, then they can be connected through elementary isotopies (of both L_0 and L_1) called finger moves (see Figure 2.3) such that admissibility is preserved at each step.

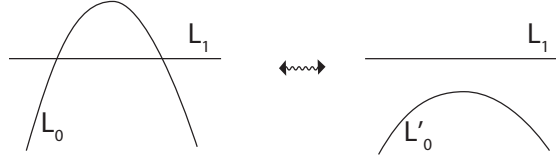


Figure 2.3: Finger move isotopy.

Step (5) If an admissible pair (L_0, L_1) is connected to an admissible pair (L'_0, L_1) by one finger move, then $HF_*(L_0, L_1) = HF_*(L'_0, L_1)$.

The statements from the last two steps also imply that Lagrangian Floer homology is invariant with respect to isotopies.

2.2.2 Lagrangian Floer homology is well-defined

We will follow the plan outlined in the previous section, giving more attention to the admissibility condition — the only place where our setup is different from [HedHerKir14b, Sections 2, 3].

Step 1. Isotoping (L_0, L_1) to an admissible pair (L'_0, L_1) .

The only problematic part in making (L'_0, L_1) admissible is making essential immersed annuli disappear. Let us first understand when essential annuli exist at all.

Definition 2.2.3. A *periodic map* is a smooth annulus $A : (S^1 \times [0, 1], S^1 \times \{0\}, S^1 \times \{1\}) \rightarrow (\Sigma, L_0, L_1)$ (as always, we assume that no switches are allowed).

Denote by $Per(L_0, L_1)$ the set of homotopy classes of periodic maps. Having a closed curve $L : S^1 \looparrowright \Sigma$ and an integer p denote by pL the curve $L \circ (\cdot)^p : S^1 \rightarrow S^1 \looparrowright \Sigma$ with a slight perturbation so that all the self-intersections are transverse. The map $(\cdot)^p : S^1 \rightarrow S^1$ is the p -power map of the complex plane where S^1 is the unit circle.

Lemma 2.2.4. *$Per(L_0, L_1) = \mathbb{Z}$ or $\{0\}$. Essential annuli can exist only if $Per(L_0, L_1) = \mathbb{Z}$. The condition $Per(L_0, L_1) = \mathbb{Z}$ is equivalent to the condition $pL_0 \sim qL_1$ for some co-prime integers p and q (in particular, both curves should be closed immersed circles).*

Proof. Assume $L_i : S_i \looparrowright \Sigma$. No switches on the boundaries of annulus A allows one to compose boundaries with L_i , i.e. $A : S^1 \times \{i\} \rightarrow S_i \xrightarrow{L_i} \Sigma$. Thus, after introducing an intersection point between L_0 and L_1 via isotopy if necessary, one gets the following sequence of maps:

$$Per(L_0, L_1) \rightarrow \pi_1(S_0 \times S_1) = \pi_1(S_0) \times \pi_1(S_1) \xrightarrow{L_{0*} + L_{1*}} \pi_1(\Sigma).$$

The statement of the lemma follows from the fact that

$$Per(L_0, L_1) \cong Ker(L_{0*} + L_{1*}).$$

Surjectivity is straightforward, while injectivity follows from the fact that $\pi_2(\Sigma) = 0$.

The latter is true because $\partial\Sigma \neq \emptyset$, and thus Σ is homotopy equivalent to a wedge

of circles. Wedges of circles always has a contractible universal cover, and because covering maps preserve higher (≥ 2) homotopy groups, one concludes that the wedge of circles has trivial second homotopy group, and so does Σ . $\square$

Definition 2.2.5. A *shadow* of a periodic annulus A is a two-chain

$$Sh(A) = \sum_{\text{open } D_i \subset \Sigma \setminus (L_0 \cup L_1)} \deg(A|_{D_i}) \cdot \overline{D_i}.$$

inside $C_2(\Sigma, (L_0 \cup L_1))$.

It is clear that $Sh(A)$ depends only on the homotopy class $[A] \in Per(L_0, L_1)$. The key observation is that for $[A]$ to have immersed orientation preserving representative requires $Sh([A])$ to have all coefficients positive. We call such shadows *positive*. In fact, we have:

Lemma 2.2.6. *The following four statements are equivalent:*

- a) *There exists an essential immersed periodic map $A : (S^1 \times [0, 1], S^1 \times \{0\}, S^1 \times \{1\}) \rightarrow (\Sigma, L_0, L_1)$ (in particular (L_0, L_1) is not admissible).*
- b) *There exists a periodic map A with positive shadow for a pair (L_0, L_1) in Σ .*
- c) *There exists a periodic map A , such that $\tilde{L}_0 \cap \tilde{L}_1 = \emptyset$, where $\tilde{L}_i$ is a lift of L_i to a covering $\tilde{\Sigma}^A$ corresponding to the subgroup $Im(A_*) \subset \pi_1(\Sigma)$.*
- d) *There exists a periodic map A , such that a pair $(\tilde{L}_0, \tilde{L}_1)$ in $\tilde{\Sigma}^A$ has a periodic map with positive shadow.*

Proof. Suppose, after possibly choosing a connecting path between L_0 and L_1 if $L_0 \cap L_1 = \emptyset$, we have $\pi_1(\Sigma) \supset Im(A_*) = \langle pL_0 \rangle = \langle qL_1 \rangle = \langle \gamma \rangle \neq 0$. Then $\tilde{\Sigma}^A := \tilde{\Sigma} / \langle \Gamma \rangle$, where $\tilde{\Sigma}$ is the universal cover, $\langle \Gamma \rangle = \mathbb{Z}$, and Γ acts on $\tilde{\Sigma}$ via action $\tilde{\alpha} \mapsto \widetilde{\gamma \cdot \alpha}$, where $\tilde{\alpha}$ is a lift of a path $\alpha : [0, 1] \rightarrow \Sigma$ to $\tilde{\Sigma}$. It follows that $\tilde{\Sigma}^A$ is

a space obtained by gluing a band (2 dimensional index 1 handle) to a contractible subset $H \subset \tilde{\Sigma}$, which is the fundamental domain of the $\mathbb{Z}$ -action by $\langle \Gamma \rangle$. The fact that H is contractible follows from the standard description of $\tilde{\Sigma}$ as a thickened universal cover of a wedge of circles.

This implies that $\text{int}(\tilde{\Sigma}^A)$ is homeomorphic to a cylinder $S^1 \times \mathbb{R}$. It is now straightforward to see that d) is equivalent to all other statements, because $\tilde{L}_0$ and $\tilde{L}_1$ are not self-intersecting curves on a cylinder.

□

Now we are prepared to make any pair (L_0, L_1) admissible. Suppose there is an essential immersed periodic annulus A . Then isotope one of the curves (say $\tilde{L}_0$) in the covering $\tilde{\Sigma}^A$ to introduce an intersection with another curve, and then push the isotopy down to the surface. Note that if $\text{Im}(A_*) = \langle pL_0 \rangle = \langle qL_1 \rangle$, then one needs to do isotopies of $\tilde{L}_0$ in p different points, so that it projects down to an isotopy of L_0 .

Step 2. Finiteness of immersed discs.

Here we need to show that $\mathcal{M}(x, y)$ is finite, assuming (L_0, L_1) is admissible. Denote by $\pi_2(x, y)$ the space of homotopy classes of smooth discs from x to y . The shadow of an element $\phi \in \pi_2(x, y)$ is defined in the same way as for $A \in \text{Per}(L_0, L_1)$. We first show that there are finite number of elements $\phi \in \pi_2(x, y)$, which can possibly have immersed representatives (the relevant condition is shadow $Sh(\phi)$ being positive). Then we argue that every such class ϕ has exactly one immersed representative from $\mathcal{M}(x, y)$.

Lemma 2.2.7. *In case $\pi_2(x, y) \neq \emptyset$, we have a free and transitive action*

$$\text{Per}(L_0, L_1) \cong \pi_2(x, x) \curvearrowright \pi_2(x, y).$$

Proof. The definition of multiplication

$$\pi_2(x, y) \times \pi_2(y, z) \rightarrow \pi_2(x, z), (\phi, \psi) \mapsto \phi * \psi$$

is given by pinching an arc in the middle of the disc and considering maps ϕ and ψ on the resulting two discs (which are connected by one point). The action $\pi_2(x, x) \curvearrowright \pi_2(x, y)$ is given by multiplying the elements, and the statement now becomes clear. $\square$

This implies that $\pi_2(x, y) = \{\phi\}$, $\mathbb{Z}$ or $\emptyset$. Next, let us prove that in case $\pi_2(x, y) = \mathbb{Z}$ we have only finite number of elements with immersed representatives.

Proposition 2.2.8. *Only a finite number of elements in $\pi_2(x, y)$ have positive shadow, and thus can have an immersed representative from $\mathcal{M}(x, y)$.*

Proof. Every $0 \neq \phi \in \pi_2(x, x)$ has a shadow with both negative and positive coefficients because (L_0, L_1) is admissible (see Lemma 2.2.6). For $\psi \in \pi_2(x, x) \cong \text{Per}(L_0, L_1)$ and $\phi \in \pi_2(x, y)$ one has $Sh(\psi * \phi) = Sh(\psi) + Sh(\phi)$. This, along with Lemma 2.2.7, implies the statement of the proposition. $\square$

Proposition 2.2.9. *Element $\phi \in \pi_2(x, y)$ with a positive shadow $Sh(\phi)$ can have at most one immersed representative, up to smooth reparameterizations.*

Proof. This follows from the fact that $\phi \in M(x, y)$ can be reconstructed from its positive shadow, see the proof of [SilRobSal14, Theorem 6.8], which applies in our case after passing to the universal cover and considering its compact simply connected subset containing immersed discs in question. Alternatively, one can argue as in [HedHerKir14b, Remark 3.14]. $\square$

Step 3. $\partial^2 = 0$ and A_∞ relations.

Suppose $L_0, L_1, \dots, L_k$ are pairwise admissible unobstructed immersed curves with no triple intersections inside (Σ, Z^{bps}) . Then we define *higher operations*

$$\mu_k : CF_*(L_0, L_1) \otimes \dots \otimes CF_*(L_{k-1}, L_k) \rightarrow CF_*(L_0, L_k)$$

$$\mu_k(p_0, p_1, \dots, p_{k-1}) = \sum_{q_k \in L_0 \cap L_k} \mathcal{M}(p_0, p_1, \dots, p_{k-1}, q_k) \cdot q_k,$$

where $\mathcal{M}(p_0, p_1, \dots, p_{k-1}, q_k)$ counts mod 2 the number of orientation preserving immersed polygons in Σ with $k + 1$ convex angles up to smooth reparameterizations, see the picture below.

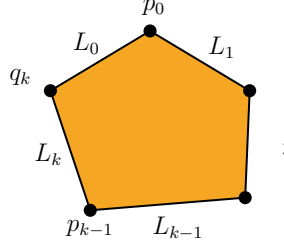


Figure 2.4

The finiteness of $\mathcal{M}(p_0, p_1, \dots, p_{k-1}, q_k)$ can be proved in the same way as finiteness of the space of immersed discs $\mathcal{M}(x, y)$; the difference is that one has to define an analog of periodic maps for multiple curves $Per(L_0, L_1, \dots, L_k)$, and give an algorithm of how to make all the elements of $Per(L_0, L_1, \dots, L_k)$ have non-positive shadows (i.e. one has to introduce a new notion of *admissible tuples* of curves). We do not give this argument here, because we do not need the higher products structure for our work.

Remark 2.2.10. In the definition of the Fukaya category in the exact setting the finiteness of moduli space of pseudo-holomorphic polygons $\mathcal{M}(p_0, p_1, \dots, p_{k-1}, q_k)$ follows from Gromov compactness and the uniform area bound, where the bound comes from Stokes theorem, see the top of [Aur14, Page 6]. In fact, we are using the same kind of mechanism: admissibility of $(L_0, L_1, \dots, L_k)$ corresponds to making it possible for

all $L_0, L_1, \dots, L_k$ to be exact w.r.t. some area form ω on Σ . The fact that all elements of $Per(L_0, L_1, \dots, L_k)$ have non-positive shadow corresponds to the fact that all periodic maps have area 0 w.r.t. ω (see [OzsSza04d, Lemma 4.12]). In the light of this correspondence the easiest way to make $L_0, L_1, \dots, L_k$ admissible would be to isotope each L_l to an exact representative w.r.t. some exact area form on a surface $\hat{\Sigma}$, which is the initial surface Σ completed by cylindrical ends at the boundaries, see Section 2.3.2 for the construction of such completion.

Proposition 2.2.11. *Suppose $L_0, L_1, \dots, L_n$ are pairwise admissible unobstructed immersed curves with no triple intersections inside (Σ, Z^{bps}) . Then, assuming the finiteness of $\mathcal{M}(p_0, p_1, \dots, p_{k-1}, q_k)$, higher operations satisfy A_∞ relations*

$$\sum_{l+k=n+1} \mu_k(a_1, \dots, a_j, \mu_l(a_{j+1}, \dots, a_{j+l}), a_{j+l+1}, \dots, a_n) = 0$$

for any $n \geq 1$ elements $a_1 \in CF_*(L_0, L_1), \dots, a_n \in CF_*(L_{n-1}, L_n)$.

Remark 2.2.12. The definition of μ_1 coincides with the definition of ∂ , and the first A_∞ relation states that $\mu_1^2 = \partial^2 = 0$. The second A_∞ relation states that the product μ_2 satisfies the Leibniz rule. The third A_∞ relation states that the product μ_2 is associative up to homotopy $H = \mu_3$.

The main idea behind proving $\mu_1^2 = \partial^2 = 0$ is that tuples of two consecutive immersed discs with convex angles come in pairs (here one uses the absence of fishtails), and so they cancel each other. The higher A_∞ relations are proved analogously, with the difference that one has to consider tuples of two consecutive immersed polygons. For the proofs here we refer to [Abo08, Lemmas 2.11, 3.6].

Step 4. How to keep the isotopies admissible.

Suppose we are given an isotopy L_0^t from an admissible pair $(L_0 = L_0^{t=0}, L_1)$ to an admissible pair $(L_0^{t=1}, L_1)$. One has problems with keeping this isotopy admissible at

each step only if $Per(L_0, L_1) = \mathbb{Z}$. Suppose A is a generator of that group. Then, passing to the covering $\tilde{\Sigma}^A$ corresponding to the subgroup $Per(L_0, L_1) \cong Im(A_*) \subset \pi_1(\Sigma)$, just like in Lemma 2.2.6, we can:

- 1) Isotope $\tilde{L}_1 = \tilde{L}_1^{s=0}$ to $\tilde{L}_1^{s=1}$ preserving $\tilde{L}_0 \cap \tilde{L}_1^s \neq \emptyset$ and $\tilde{L}_0^{t=1} \cap \tilde{L}_1^s \neq \emptyset$ for each s , but so that at the end point $s = 1$ all isotopies of the 0th curve $\tilde{L}_0^t$ intersect $\tilde{L}_1^{s=1}$.
- 2) Make an isotopy L_0^t from L_0 to $L_0^{t=1}$.
- 3) Isotope $\tilde{L}_1^{s=1}$ back to $\tilde{L}_1^{s=0}$.

Both steps 1) and 3) can be done in such a way that the isotopies can be projected to Σ . This sequence of isotopies keeps the pair admissible all the time.

Step 5. Finger move isotopy does not change Lagrangian Floer homology.

One possibility here is to use A_∞ relations to define chain maps between $CF_*(L_0, L_1)$ and $CF_*(L'_0, L_1)$ and prove that their compositions are homotopic to the identity, see [HedHerKir14b, Lemma 4.2] (with an appropriate change of argument because of the weaker notion of admissibility in our case).

Another approach is to note that a finger move (see Figure 2.3) on the level of Lagrangian Floer chain complex corresponds to a cancellation of the differential (see [SilRobSal14, Appendix C]). Here one needs to prove that there is exactly one immersed disc between the two points on the left of Figure 2.3. There is only one other possibility, which is a disc covering lower left and lower right domains on the left of Figure 2.3. But if such immersed disc exists, one would have an immersed annulus on the right of Figure 2.3, and this would contradict admissibility.

2.3 Partially wrapped Fukaya category of a surface

2.3.1 Different approaches

Definition 2.3.1. An A_∞ category $\mathcal{C}$ consists of a set of objects $\text{Ob}(\mathcal{C})$, a vector space $\text{hom}_{\mathcal{C}}(X_0, X_1)$ for any pair of objects, and *composition maps (higher operations)* for each $k \geq 1$

$$\mu_k : \text{hom}_{\mathcal{C}}(X_0, X_1) \otimes \cdots \otimes \text{hom}_{\mathcal{C}}(X_{k-1}, X_k) \rightarrow \text{hom}_{\mathcal{C}}(X_0, X_k),$$

satisfying A_∞ relations $\sum_{l+k=n+1} \mu_k(a_1, \dots, a_j, \mu_l(a_{j+1}, \dots, a_{j+l}), a_{j+l+1}, \dots, a_n) = 0$.

All unobstructed curves on a surface $(\Sigma, Z^{bps} \subset \partial\Sigma)$ comprise an A_∞ category called *partially wrapped Fukaya category*, which we denote by $\mathcal{F}(\Sigma, Z^{bps})$. Following the construction in the previous section, one might try to define it in the following way: for transverse L_0, L_1 define $\text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_1) := CF_*(L_0, L_1)$, define μ_k maps via counting immersed polygons with convex angles, and A_∞ relations follow from Proposition 2.2.11. However, there is a big problem one immediately encounters — since the curve is not transverse to itself $\text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_0)$ is not defined. One can remedy that by perturbing the first Lagrangian $L_0 \rightsquigarrow L'_0$ as we did in the previous section and define $\text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_1) := CF_*^{pw}(L_0, L_1)$, but then one encounters another problem — higher operations were defined only for admissible curves. For example it is not clear how to define $CF_*^{pw}(L_0, L_0) \otimes CF_*^{pw}(L_0, L_1) \rightarrow CF_*^{pw}(L_0, L_1)$.

There are different ways to resolve these problems, which correspond to different ways of defining the Fukaya category:

- (1) To simply work with the structure at hand, i.e. to work with A_∞ categories where hom spaces are defined only partially, for transverse pairs (these are called A_∞ pre-categories). This approach was developed in [KonSoi01] and implemented in the case of curves on a surface in [Abo08].

- (2) To first pick for each pair (L_0, L_1) a perturbation data, i.e. a certain Hamiltonian $H_{0,1}$, and define $hom_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_1) = CF_*(\phi_{X_{H_{0,1}}}^1(L_0), L_1) \cong CF_*^{pw}(L_0, L_1)$ (this is essentially what we did in the previous section). Then it turns out that one can choose additional perturbation data for each set of Lagrangians $\{L_0, L_1, \dots, L_k\}$ in Σ , so that counting solutions of perturbed Cauchy-Riemann equation for polygons gives higher operations μ_k satisfying A_∞ relations. This approach was realized in details in [Sei08], and was implemented in the relevant for us partially wrapped case in [Aur10b], [Aur10a].
- (3) To define an A_∞ category on a certain partially ordered countable set of Lagrangians³, and then identify Hamiltonian isotopic curves by inverting certain morphisms. This process is called localization of A_∞ categories, and was implemented in the relevant for us partially wrapped case (which is a special case of Liouville sectors) in [GanParShe17].
- (4) To define the partially wrapped Fukaya category combinatorially. For that one needs to parameterize (see Definition 2.4.1) the surface by a set of arcs α^{arcs} . From that one obtains an algebra $\mathcal{A}_1 = \mathcal{A}_1(\Sigma, Z^{bps}, \alpha^{arcs})$ (see Definition 2.4.3 and the remark afterwards explaining the connection to approach (2)). Then one can define $\mathcal{F}(\Sigma, Z^{bps})$ as
- (a) A dg category⁴ $\mathcal{A}_1 Mod$ of left D structures over the algebra $\mathcal{A}_1$. This approach was implemented in [HaiKatKon17], where $\mathcal{A}_1 Mod$ is denoted by $Tw\mathcal{F}_A(S, M)$. The equivalence of approaches (4a) and (2) on the level of homology is proved in [Zib17].

- (b) A dg category $Mod_{\mathcal{A}_1}$ of right A modules over the algebra $\mathcal{A}_1$. This is

³Having a poset of Lagrangians satisfying $L_i > L_j \implies L_i \pitchfork L_j$ one can define an A_∞ Fukaya category by $L_i > L_j \implies hom_*(L_i, L_j) = CF_*(L_i, L_j)$, $L_i = L_j \implies hom_*(L_i, L_j) = \langle id_{L_i} \rangle_{\mathbb{F}_2}$, $L_i < L_j \implies hom_*(L_i, L_j) = 0$.

⁴The category $\mathcal{A} Mod$ is an $\mathcal{A}_\infty$ category if the algebra $\mathcal{A}$ is an A_∞ algebra, see [LipOzsThu15, Lemma 2.2.27].

the approach we take in this thesis. Our main result in this chapter is Theorem 2.6.10, which proves the equivalence of approaches (4b) and (2) on the level of homology.

Remark 2.3.2. All of the approaches above work in the general case of high dimensional Liouville manifold instead of a surface, see [AboSei10], [Aur10b], [Syl15] for approach (2), and [GanParShe17] for approach (3). It should be noted that finding an algebra $\mathcal{A}_1$ for approach (4) is the same task as what is called “computing” the Fukaya category; it is highly non-trivial and is one of the main problems in symplectic topology, see [Smi15, Section 4.5]. In the case where M^{2n} is compact approach (3) does not work (because it uses the directive nature of wrapping around infinity), but there is another approach (5): to develop “Morse-Bott” Floer theory, as in [FukOhOhtOno09a][FukOhOhtOno09b]. There one defines $hom_{\mathcal{F}(M^{2n}, \omega)}(L_0, L_1) := C_*(L_0 \cap L_1)$ in case of “clean intersection”, i.e. $L_0 \cap L_1$ is smooth. The higher operations are defined by pulling back and intersecting the chains inside the moduli spaces of pseudo-holomorphic polygons, see the end of <http://www-math.mit.edu/~auroux/18.969/mirrorsymm-lect14.pdf>.

Remark 2.3.3. One can mix the approaches (4a) and (4b) and define Lagrangian Floer homology $CF_*(L_0, L_1)$ as a box tensor product of a module $M(L_0)_{\mathcal{A}_1}$ and a D structure ${}^{\mathcal{A}_1}K(L_1)$. An approach very close to this (a certain hyperelliptic involution bimodule has to be inserted there) was taken in [HanRasWat16].

We will outline below the setup of the approach (2), as it is the motivation behind the approach (4b), to which we devote the rest of the chapter. In the discussion below the most important part for us is the behavior of the Hamiltonian perturbation near the boundary, which motivates the definition of the algebra $\mathcal{A}_1$ in the next section.

2.3.2 Auroux's construction of the partially wrapped Fukaya category

This construction was developed in [Aur10b], [Aur10a] by Auroux in order to reinterpret bordered Heegaard Floer theory using Fukaya categories. Notice that in this subsection we follow Auroux's conventions and work with *cohomology*.

Having $(\Sigma, Z^{bps} \subset \partial\Sigma)$, fix a Liouville domain structure on Σ , i.e. an exact symplectic form $\omega = d\theta$, such that the Liouville vector field X_θ dual to θ points outwards the boundary. Then one associates to that data the partially wrapped Fukaya category $\mathcal{F}(\Sigma, Z^{bps})$ (Auroux also considers Fukaya categories of symmetric products of higher powers, but we only need the first one, i.e. the surface itself). Denote by $\hat{\Sigma}$ a Liouville manifold, which is obtained by a completion of the surface by a cylindrical end. I.e. consider a symplectization of the boundary $([0, +\infty) \times \partial\Sigma, d(r \cdot \theta))$, and glue its negative part to the Liouville flow collar neighborhood of $\partial\Sigma$ by $i : ((0, 1] \times \partial\Sigma, d(r \cdot \theta)) \rightarrow ((-\infty, 0] \times \partial\Sigma, \omega) \subset \Sigma$, s.t. $i((r, x)) = (e^r, x)$.

Objects of the partially wrapped Fukaya category $\mathcal{F}(\Sigma, Z^{bps})$ consist of closed exact embedded curves in $\hat{\Sigma}$, as well as non-compact properly embedded ones such that the ends stabilize to be rays in a cylindrical end, see [Aur10b] for the details. Morphism spaces are Lagrangian Floer cochain complexes $hom_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_1) = CF^*(\tilde{L}_0, L_1)$, where $\tilde{L}_0$ is a Lagrangian submanifold perturbed by a generic Hamiltonian. Because of non-compact Lagrangians, the behavior of Hamiltonian perturbation at infinity of $\hat{\Sigma}$ needs to be specified. Specifically the constructed Hamiltonian wraps the ray of the arc around the cylindrical end until it reaches the “stop”, i.e. one of the rays in $Z^{bps} \times [1, +\infty)$. See [Aur10b] for the details, and Figure 2.5 for a schematic picture. Note that this specific behavior of perturbation ensures that $hom_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_1) = CF^*(\tilde{L}_0, L_1) \simeq CF_*^{pw}(L_1, L_0)$, where the latter group was

defined in Section 2.2. The A_∞ operations

$$\text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_1) \otimes \dots \otimes \text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_{d-1}, L_d) \rightarrow \text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_0, L_d)$$

are given by counting holomorphic discs with $d + 1$ marked points on the boundary.

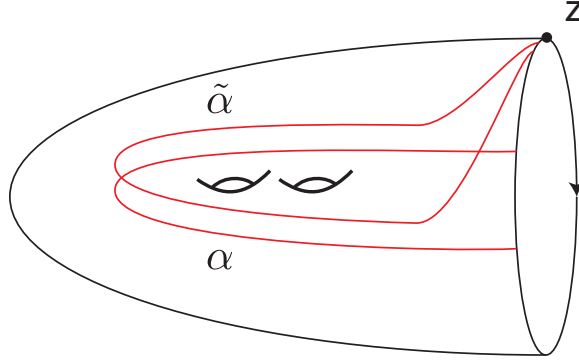


Figure 2.5: Perturbation near infinity for the partially wrapped Fukaya category of a surface.

2.4 From a surface to an algebra

As in the previous section, we fix a compact smooth oriented surface with non-empty boundary, along with a finite set of basepoints $\{z_l\} = Z^{bts}$ on its boundary: $(\Sigma, Z^{bps} \subset \partial\Sigma \neq \emptyset)$. We will also continue to assume that each boundary component of Σ contains at least one basepoint.

In this section we will first parameterize the surface $(\Sigma, Z^{bps} \subset \partial\Sigma)$ by arcs. Then we will associate to this parameterization a dg -algebra $\mathcal{A}_1$.

Definition 2.4.1. A *parameterization* of the surface Σ consists of a set $\{\alpha_k\} = \alpha^{arcs}$ of non-intersecting embedded arcs with ends on $\partial\Sigma \setminus Z^{bts}$, such that cutting along the arcs results into a set of discs each having exactly one basepoint on the boundary. Sometimes we will refer to the parameterizing arcs as *red arcs*.

A parameterization of Σ specifies a graph $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ — the vertices are the

arcs in the parameterization, and the edges are chords between the arcs on $\partial\Sigma$ which do not pass through basepoints. The orientation on the graph is induced by the orientation of chords, which in turn is induced by the orientation of Σ via the usual rule “outward normal first”.

Example 2.4.2. This will be our running example. Consider a surface $\overline{P}$ (*pillowcase*) in the figure below:

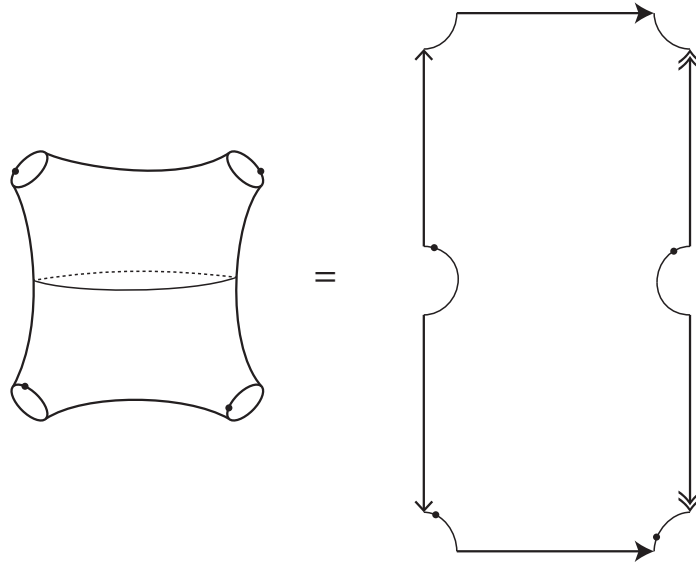


Figure 2.6: Pillowcase $\overline{P}$.

An example of a parameterization of this surface is on the left of Figure 2.7. The corresponding graph is on the right.

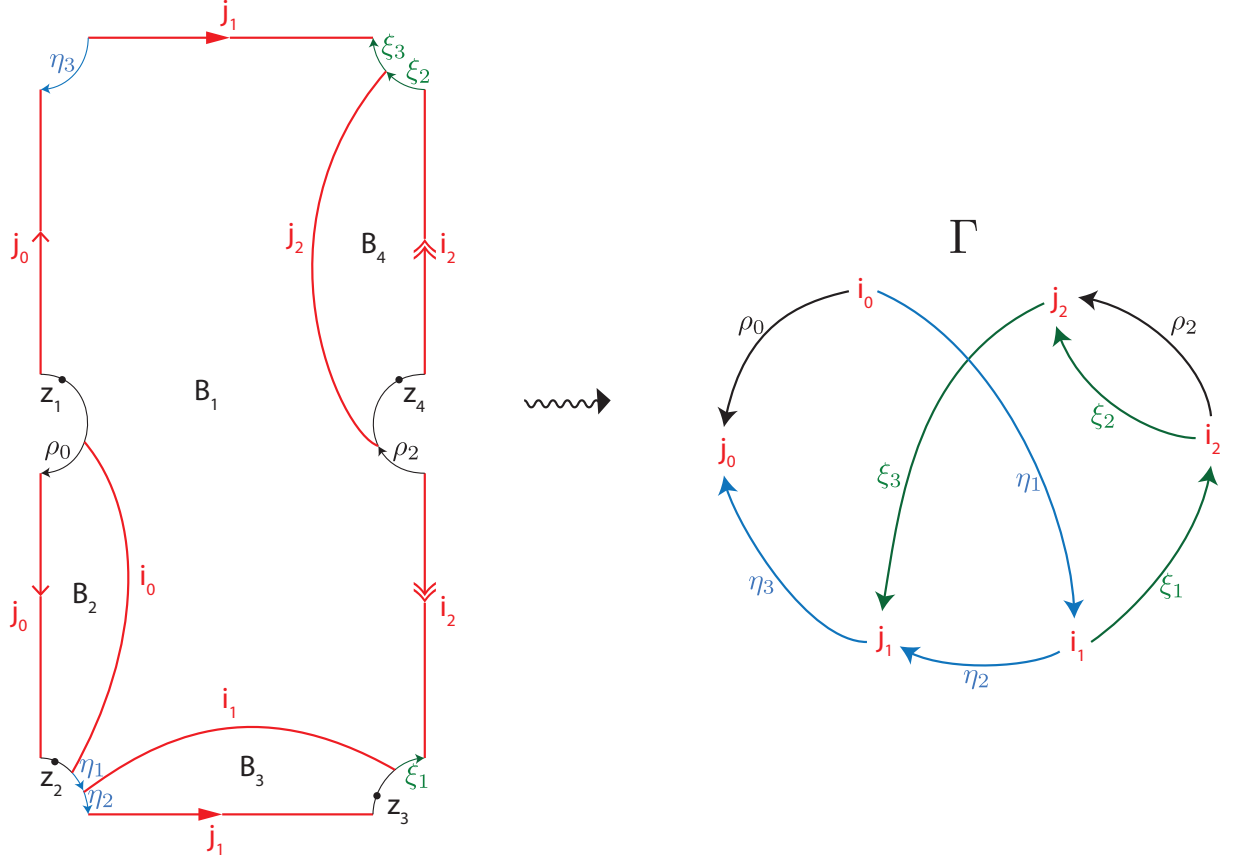


Figure 2.7: Parameterization of the pillowcase, and the corresponding graph.

Definition 2.4.3. *The Algebra $\mathcal{A}_1$ associated to a parameterized surface is a path algebra $\mathcal{A}_1(\Sigma, Z^{bps}, \alpha^{arcs}) = \mathcal{A}_1$ of the graph $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ with certain relations. *Path algebra* means that it is generated over $\mathbb{F}_2$ by paths in $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$, and concatenating of paths corresponds to multiplication. When concatenating is not possible, the multiplication is defined to be zero. Additional relations are defined locally for every arc α_k in the figure below:*

$$\begin{array}{ccc}
 \begin{array}{c} \downarrow p_1 \\ \downarrow p_2 \end{array} & \begin{array}{c} q_2 \uparrow \\ q_1 \uparrow \end{array} \\
 \hline
 \end{array} \implies p_1 q_2 = q_1 p_2 = 0$$

We can promote the algebra $\mathcal{A}_1$ to a *dg*-algebra by defining its differential to be 0. Also notice that the subalgebra generated by vertices of the graph $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ is

naturally an idempotent subalgebra, and we denote it by $\mathcal{I}$.

Remark 2.4.4. Consider the partially wrapped Fukaya category $\mathcal{F}(\Sigma, Z^{bps})$. The parameterization of Σ by α^{arcs} corresponds to picking a set of objects $\alpha^{arcs} \subset \mathcal{F}(\Sigma, Z^{bps})$. The algebra $\mathcal{A}_1$ is in fact quasi-isomorphic (isomorphic if one picks the simplest Hamiltonian perturbations) to the *hom*-algebra

$$\bigoplus_{\alpha_i, \alpha_j \in \alpha^{arcs}} \text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(\alpha_i, \alpha_j) = \bigoplus_{\alpha_i, \alpha_j \in \alpha^{arcs}} CF_*^{pw}(\alpha_j, \alpha_i),$$

see Example 4.4.4 for an illustration. Note that, if one makes sure to apply consistent perturbations to the curves, the A_∞ structure on $\bigoplus_{1 \leq i, j \leq k} CF_*^{pw}(\alpha_i, \alpha_j)$ can be seen by counting immersed polygons. One also should be careful with orientations, and in this case, due to cohomological conventions in [Aur10b], the higher operations are defined by counting not the polygons in Figure 2.4, but rather their mirror images.

Example 2.4.5 (Explicit description of $\mathcal{A}_1$ for the pillowcase).

Algebra $\mathcal{A}_p = \mathcal{A}_1(\overline{P}, Z^{bps}, \{i_0, i_1, i_2, j_0, j_1, j_2\})$ is a path algebra of the graph $\Gamma(\overline{P}, Z^{bps}, \{i_0, i_1, i_2, j_0, j_1, j_2\})$ in the Figure 2.7, with relations specified in the definition of $\mathcal{A}_1$. The algebra is generated by the following elements (we specify here only those non-trivial multiplications which do not involve vertices):

$$\mathcal{A}_p = \mathcal{A}_1(\overline{P}) = \langle i_0, i_1, i_2, j_0, j_1, j_2, \quad (2.1)$$

$$\rho_0, \rho_2, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3,$$

$$\xi_{12} = \xi_1 \xi_2, \xi_{23} = \xi_2 \xi_3, \xi_{123} = \xi_1 \xi_2 \xi_3 = \xi_{12} \xi_3 = \xi_1 \xi_{23},$$

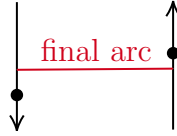
$$\eta_{12} = \eta_1 \eta_2, \eta_{23} = \eta_2 \eta_3, \eta_{123} = \eta_1 \eta_2 \eta_3 = \eta_1 \eta_{23} = \eta_{12} \eta_3 >_{\mathbb{F}_2}.$$

Regarding multiplications which involve vertices: constant paths (i.e. the vertices) are idempotents, and every path in $\mathcal{A}_p$ has its own left and right idempotent. These idempotents correspond to vertices of the start and the end of the path. All other

vertices annihilate the path. For example, for the path ξ_{12} we have $i_1 \xi_{12} j_2 = \xi_{12}$, and multiplication by other idempotents results in zero.

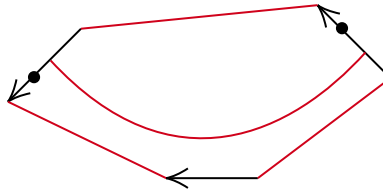
We will call a parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$ *directed* if $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ has no cycles (*acyclic*). We will call an algebra $\mathcal{A}$ with an idempotent subalgebra structure *directed*⁵ if there is an order on all of the idempotents $\{i_1, i_2, \dots\} = \mathcal{I}$ s.t. $i_k \cdot \mathcal{A} \cdot i_l = 0$ if $k > l$. It is clear that if $(\Sigma, Z^{bps}, \alpha^{arcs})$ is directed then so is $\mathcal{A}_1(\Sigma, Z^{bps}, \alpha^{arcs})$.

One can make any parameterized surface directed by adding some basepoints and arcs to the surface. Let us explain how this works. We will call an arc *final* if there are no chords coming out of it, i.e. if there are basepoints right after them on both of its ends (w.r.t. orientation of $\partial\Sigma$), i.e. if there are no edges going out of the corresponding vertex in Γ .



Suppose now that $(\Sigma, Z^{bps}, \alpha^{arcs})$ is not a directed parameterization. Then pick a cycle in Γ , and pick any edge in that cycle. Put a basepoint on the corresponding chord in $\partial\Sigma$. By this we reduced the number of cycles in the graph, but our parameterization is no longer valid, because the arcs now cut the surface into discs, among which one has two basepoints. We then add one additional arc to make the parameterization valid, and this arc will be the final arc for those two basepoints.

Adding a final arc to make Γ acyclic:



Because adding the final arc doesn't introduce any new cycles in Γ , we conclude that

⁵This definition is analogous to the definition of directedness in [Sei08, Page 75].

by adding one basepoint and one arc one can reduce the number of cycles in Γ . This implies that we can make Γ acyclic, i.e. $(\Sigma, Z^{bps}, \alpha^{arcs})$ directed.

2.5 From curves to A_∞ modules

Assumption 2.5.1. Throughout Chapter 2 we assume that the parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$ is directed. This implies that $\Gamma = \Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ is acyclic, and $\mathcal{A}_1 = \mathcal{A}_1(\Sigma, Z^{bps}, \alpha^{arcs})$ is directed.

Remark 2.5.2. Strictly speaking we do not need this assumption for our theory. But we find this assumption convenient, because then the module $M(L)_{\mathcal{A}_1}$, which we define below, has finitely many A_∞ actions. Moreover, this assumption is not very restrictive, as one can make the parameterized surface directed by adding basepoints and arcs.

To an immersed curve L in $(\Sigma, Z^{bps}, \alpha^{arcs})$ we associate a right A_∞ module $M(L)_{\mathcal{A}_1}$ over the algebra $\mathcal{A}_1$. Before defining the module, one needs to isotope L appropriately.

2.5.1 Preliminary isotopies of L

First, if L is an arc, one makes a perturbation of L in the small neighborhood of $\partial\Sigma$ by applying a twist along the orientation of $\partial\Sigma$, such that the end comes close to its closest basepoint, passing all the parameterizing arcs on its way, see Figure 2.8. This ensures that all the parameterizing arcs i are admissible with L as a pair (i, L) .

In fact, this perturbation is enough to define $M(L)$ up to A_∞ homotopy equivalence of A_∞ modules, but for further simplification of $M(L)$, and for defining a concrete $M(L)$ rather than its A_∞ homotopy equivalence class, we will do the following extra isotopy.

Denote the discs cut out by arcs by $\{B_k\} = \Sigma \setminus (\alpha^{arcs} \cup \partial\Sigma)$, see the left of Figure 2.7 for the pillowcase example. We call these discs *big domains*. Consider

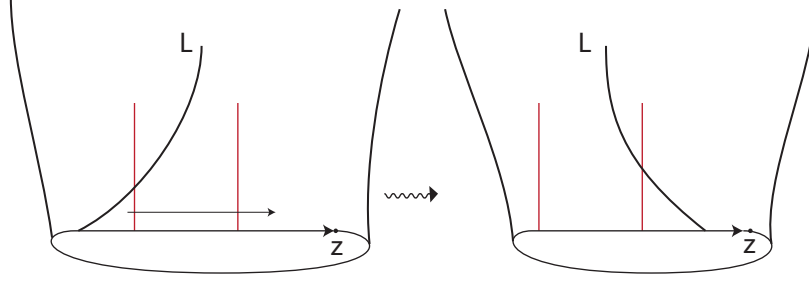


Figure 2.8: Perturbation near the boundary.

a process of traversing along L — this traversing (up to isotopies of L which do not change the intersections with arcs) is encoded in the cyclic sequence $S(L)$ of big domains B_k , which are visited by L , as well as the connecting arcs between them. For example for the curve $L^\natural$ in Figure 2.11 we have a cyclic sequence $S(L^\natural) = B_1j_2B_4i_2B_1i_0B_2j_0B_1j_1B_3i_1$. If L is an arc then the sequence is not cyclic.

We isotope a curve L further, so that the sequence $S(L)$ it gives does not have the same arcs around one big domain, i.e. it does not have a pattern iB_li . Such an isotopy exists because if one has such a pattern, there is a finger move isotopy of L removing iB_li from the sequence $S(L)$, see Figure 2.3. The length of the sequence decreases, so the process of doing such finger move isotopies has to stop.

2.5.2 Definition of $M(L)$

We now assume that L is isotoped according to the previous section. A right A_∞ module $M(L)_{\mathcal{A}_1}$ is defined as follows. Over $\mathbb{F}_2$ it is generated by all intersection of the curve L with the red arcs:

$$M(L) = \langle L \cap \alpha^{arcs} \rangle_{\mathbb{F}_2}$$

For example, for the curve $L^\natural$ in Figure 2.11 we have $M(L^\natural) = \langle z, w, s, t, y, x \rangle_{\mathbb{F}_2}$.

The idempotent subalgebra action $M(L) \curvearrowright \mathcal{I} \subset \mathcal{A}_1$ is defined as follows. Every generator has a unique right idempotent which preserves that generator. This idem-

potent corresponds to the arc on which the generator is sitting. Other idempotents annihilate the generator. For example for $M(L^\natural)$ in Figure 2.11 one gets the following idempotents (in the subscripts) for the generators: $z_{i_0}, w_{j_0}, s_{j_1}, t_{i_1}, y_{j_2}, x_{i_2}$.

The final piece of the A_∞ module structure, i.e. $\mathcal{I}$ -linear A_∞ actions

$$M(L) \otimes_{\mathcal{I}} \mathcal{A}_1^{\otimes n} \rightarrow M(L)$$

are defined by counting immersions (in fact they are all embeddings, because (1) we did the isotopies above and (2) graph Γ does not have cycles) of polygons, missing basepoints, from one generator to another generator, such that one side of the polygon is mapped to L , other sides are mapped to $\alpha^{arcs} \cup \partial\Sigma$, and all the angles are convex — see Figure 2.9. Such maps are counted up to orientation preserving smooth reparameterizations of the domain polygon. Non-idempotent elements of the algebra $a_1, \dots, a_n$ which such map picks up on $\partial\Sigma$ give an A_∞ action $x \otimes_{\mathcal{I}} a_1 \otimes_{\mathcal{I}} a_2 \otimes_{\mathcal{I}} \dots \otimes_{\mathcal{I}} a_n \rightarrow y$.

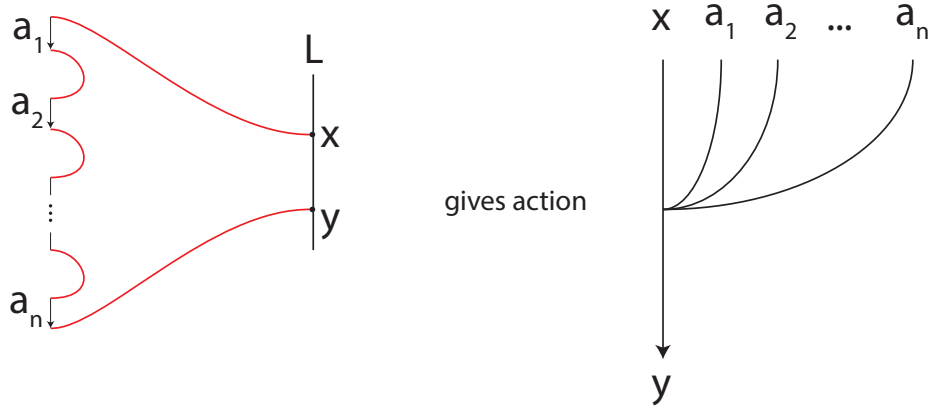


Figure 2.9: Immersed polygons of this type define A_∞ actions of the algebra $\mathcal{A}_1$ on the module $M(L)$.

Remark 2.5.3. The module $M(L)_{\mathcal{A}_1}$ is A_∞ homotopy equivalent to a module

$$\bigoplus_{\alpha_i \in \alpha^{arcs}} \text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L, \alpha_i) = \bigoplus_{\alpha_i \in \alpha^{arcs}} CF_*^{pw}(\alpha_i, L),$$

the image of L under Yoneda embedding $\mathcal{F}(\Sigma, Z^{bps}) \rightarrow \text{Mod}_{\mathcal{A}_1}$ w.r.t. the set α^{arcs}

(see [Aur14, Section 3.4]). We do not define it this way, because the partially wrapped Fukaya category was not defined for immersed Lagrangians.

Now, let us explain how $M(L)$ can be recovered combinatorially, by doing it on the example curve $L^\natural$ in Figure 2.11. First, let us count all the “basic” discs, which are contained entirely in one of the big domains B_l . Because there is one basepoint in each big domain B_l , there is exactly one basic disc between two consecutive generators. For the curve $L^\natural$ from Figure 2.11 here are all the basic discs: $z \xrightarrow{D_2} x$, $y \xrightarrow{D_0} x$, $t \xrightarrow{D_6} y$, $t \xrightarrow{D_4+D_3} s$, $w \xrightarrow{D_1} s$, $z \xrightarrow{D_4+D_5} w$. Thus first we get a circle (if L is an arc one gets a sequence) of generators and actions between them, see Figure 2.10. Notice that in addition to the basic actions that form a circle there are two extra actions: $z \xrightarrow{D_1+D_4+D_5} s$, $t \xrightarrow{D_0+D_6} x$. These are the actions which correspond to discs which are formed by juxtaposing basic discs along the arcs. These discs ensure that A_∞ relations are satisfied in our A_∞ module. Note that every immersed disc can be decomposed into basic discs. Also every basic disc is contained in the finite number of discs — otherwise one would have a cycle in the the graph Γ . Thus we can combinatorially recover all the finite number of actions by adding all the possible juxtapositions of the basic discs.

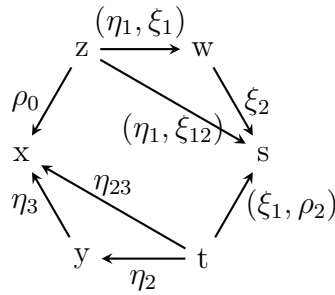


Figure 2.10: $M(L^\natural)_{\mathcal{A}_1}$, where $L^\natural$ is from Figure 2.11.

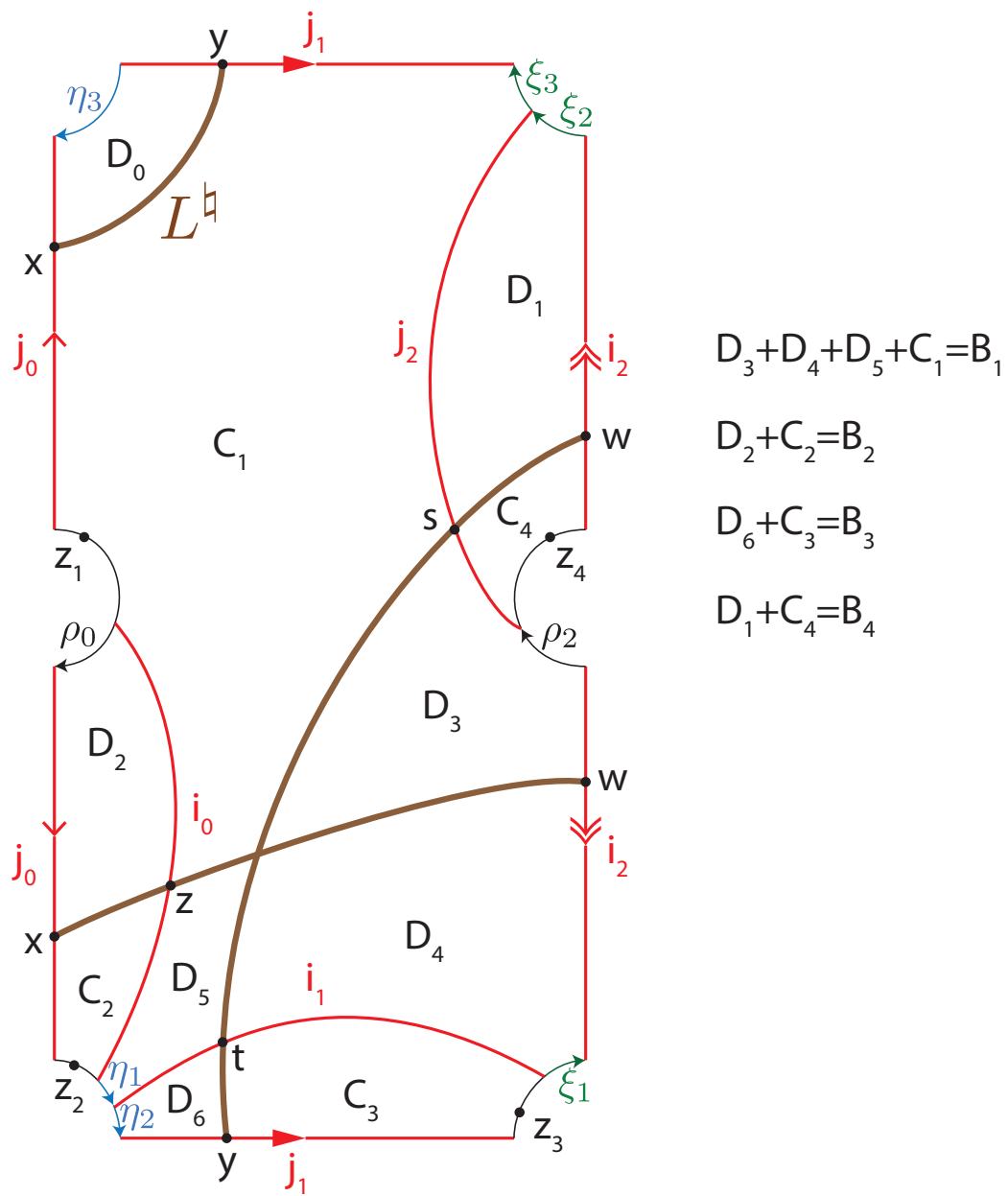


Figure 2.11: Curve $L^{\natural}$ on the pillowcase.

2.6 Recovering Lagrangian Floer homology algebraically

2.6.1 Algebra $\mathcal{A}_1^{\dagger h}$ and bimodule ${}_{\mathcal{A}_1}\overline{bar}_r\mathcal{A}_1$

Definition 2.6.1. The algebra $\mathcal{A}_1^{\dagger h}(\Sigma, Z^{bps}, \alpha^{arcs}) = \mathcal{A}_1^{\dagger h}$ ⁶ associated to a parameterized surface is a path algebra of the graph $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ with local relations:

$$\begin{array}{ccc}
 \downarrow p_1 & & q_2 \uparrow \\
 \text{---} & \text{---} & \text{---} \\
 \downarrow p_2 & & q_1 \uparrow
 \end{array}
 \implies p_1 p_2 = q_1 q_2 = 0$$

We can promote algebra $\mathcal{A}_1^{\dagger h}$ to a dg -algebra by defining its differential to be 0. Also notice that subalgebra generated by vertices of the graph $\Gamma(\Sigma, Z^{bps}, \alpha^{arcs})$ is naturally an idempotent subalgebra, and we denote it by $\mathcal{I}$.

Remark 2.6.2. Algebras $\mathcal{A}_1$ and $\mathcal{A}_1^{\dagger h}$ are related. First, denote by $\mathcal{A}_1^\dagger$ the $(|\alpha^{arcs}| - 1)$ -strands moving algebra associated to the parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$, see [Zar09, Section 2] for the definition. Then, analogously to [LipOzsThu11, Proposition 8.17], one can prove that $\mathcal{A}_1^\dagger$ is Koszul dual to $\mathcal{A}_1$. One can also prove that $\mathcal{A}_1^\dagger$, analogously to the algebra $C(\mathcal{Z})$ in [LipOzsThu13, Section 2.3], is formal, and $H_*(\mathcal{A}_1^\dagger) = \mathcal{A}_1^{\dagger h}$. This implies that algebras $\mathcal{A}_1$ and $\mathcal{A}_1^{\dagger h}$ are Koszul dual. Alternatively, one can prove Koszul duality by writing down the dualizing rank-1 DD bimodule, and proving that it is invertible. This bimodule comes from a certain $\alpha - \beta$ -bordered Heegaard diagram analogous to the diagram $\mathcal{D}(id)$ from the left of [LipOzsThu11, Figure 6]. Moreover, there is a dual parameterized surface $\Sigma^\dagger$, s.t. $\mathcal{A}_1(\Sigma^\dagger) = \mathcal{A}_1^{\dagger h}(\Sigma)$.

By e_k we denote “short” chords, i.e. those chords which correspond to paths of length 1 in Γ . Denote by $a^{\dagger h}(e_1, \dots, e_l)$ an element of $\mathcal{A}_1^{\dagger h}$ (possibly equal to 0) which

⁶The author thanks Claudius Zibrowius for pointing him to this definition, which simplified the exposition.

corresponds to the path $(e_1, \dots, e_l)$ in Γ . Analogously $a(e_1, \dots, e_l)$ will be used for the elements of the algebra $\mathcal{A}_1$.

Example 2.6.3 (Explicit description of elements of $\mathcal{A}_1^{\dagger h}$ for the pillowcase). Algebra $\mathcal{A}_1^{\dagger h}(\overline{P}, Z^{bps}, \{i_0, i_1, i_2, j_0, j_1, j_2\})$ is constructed as a path algebra of the graph $\Gamma(\overline{P}, Z^{bps}, \{i_0, i_1, i_2, j_0, j_1, j_2\})$ in the Figure 2.7, with relations specified in the definition of $\mathcal{A}_1^{\dagger h}$. Over $\mathbb{F}_2$ the algebra is generated by the following elements:

$$\begin{aligned} \mathcal{A}_1^{\dagger h}(\overline{P}) = & \langle a^{\dagger h}(i_0), a^{\dagger h}(i_1), a^{\dagger h}(i_2), a^{\dagger h}(j_0), a^{\dagger h}(j_1), a^{\dagger h}(j_2), \\ & a^{\dagger h}(\rho_0), a^{\dagger h}(\xi_2), a^{\dagger h}(\eta_2), \\ & a^{\dagger h}(\eta_1), a^{\dagger h}(\xi_1), a^{\dagger h}(\rho_2), a^{\dagger h}(\xi_3), a^{\dagger h}(\eta_3), \\ & a^{\dagger h}(\eta_1, \xi_1), a^{\dagger h}(\xi_1, \rho_2), a^{\dagger h}(\rho_2, \xi_3), a^{\dagger h}(\xi_3, \eta_3), \\ & a^{\dagger h}(\eta_1, \xi_1, \rho_2), a^{\dagger h}(\xi_1, \rho_2, \xi_3), a^{\dagger h}(\rho_2, \xi_3, \eta_3), \\ & a^{\dagger h}(\eta_1, \xi_1, \rho_2, \xi_3), a^{\dagger h}(\xi_1, \rho_2, \xi_3, \eta_3), \\ & a^{\dagger h}(\eta_1, \xi_1, \rho_2, \xi_3, \eta_3) \rangle_{\mathbb{F}_2}. \end{aligned}$$

The multiplication corresponds to concatenation of paths.

Definition 2.6.4. The dual reduced small bar resolution ${}^{\mathcal{A}_1}\overline{bar}_r^{\mathcal{A}_1}$ of the algebra $\mathcal{A}_1 = \mathcal{A}_1(\Sigma, Z^{bps}, \alpha^{arcs})$ is a type DD structure, elements of which are in 1-1 correspondence with elements of $\mathcal{A}_1^{\dagger h} = \mathcal{A}_1^{\dagger h}(\Sigma, Z^{bps}, \alpha^{arcs})$:

$${}_j a^{\dagger h}(e_1, \dots, e_l)_i \mapsto_i b(-e_l, \dots, -e_1)_j,$$

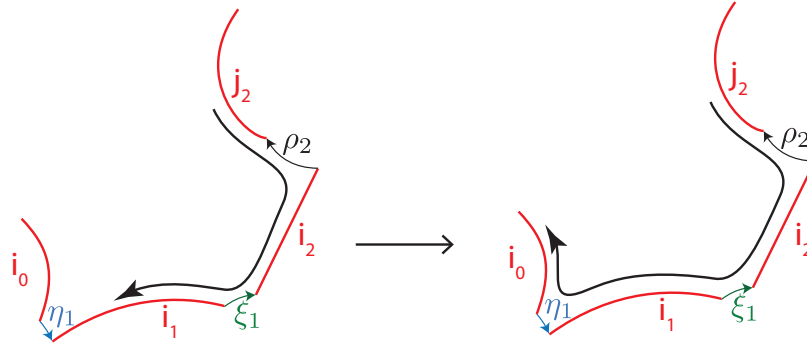
with a type DD structure map δ_1 defined by:

$$\delta^1 : \overline{bar}_r \rightarrow \mathcal{A}_1 \otimes_{\mathcal{I}} \overline{bar}_r \otimes_{\mathcal{I}} \mathcal{A}_1,$$

$$\begin{aligned} \delta^1({}_i b(-e_1, \dots, -e_l)_j) &= \sum_{e \in \text{Edges}(\Gamma), \text{end}(e)=j} 1 \otimes_i b(-e_1, \dots, -e_l, -e)_k \otimes_k a(e)_j + \\ &+ \sum_{e \in \text{Edges}(\Gamma), \text{start}(e)=i} {}_i a(e)_k \otimes_k b(-e, -e_1, \dots, -e_l)_j \otimes 1. \end{aligned}$$

It is easy to check that the type DD structure relation is satisfied for ${}^{\mathcal{A}_1} \overline{bar}_r {}^{\mathcal{A}_1}$.

We can interpret the generators of ${}^{\mathcal{A}_1} \overline{bar}_r {}^{\mathcal{A}_1}$ as paths (against orientation) on the boundaries of big domains B_k and encoding algebra elements one encounters on that path. The δ^1 map then corresponds to prolonging paths by one chord. See Figure 2.12 for an example of this geometric interpretation of the differential.



prolongation of the path
corresponds to differential

$$b(-\rho_2, -\xi_1) \longrightarrow 1 \otimes b(-\rho_2, -\xi_1, -\eta_1) \otimes \eta_1$$

Figure 2.12: Geometric interpretation of the differential in ${}^{\mathcal{A}_1} \overline{bar}_r {}^{\mathcal{A}_1}$.

Example 2.6.5 (Explicit description of ${}^{\mathcal{A}_1} \overline{bar}_r {}^{\mathcal{A}_1}$ for the pillowcase algebra). If $\mathcal{A}_1 = \mathcal{A}_1(\overline{P})$ is the pillowcase algebra described in the examples above, then ${}^{\mathcal{A}_1} \overline{bar}_r {}^{\mathcal{A}_1}$

consists of the following 24 generators (we list them with their idempotents):

$$\begin{aligned}
\mathcal{A}_1 \overline{bar}_r \mathcal{A}_1 = & \leq_{i_2} b(i_2)_{i_2, i_0} b(i_0)_{i_0, j_1} b(j_1)_{j_1, j_2} b(j_2)_{j_2, j_0} b(j_0)_{j_0, i_1} b(i_1)_{i_1,} \\
& j_0 b(-\rho_0)_{i_0, j_1} b(-\eta_2)_{i_1, j_2} b(-\xi_2)_{i_2,} \\
& j_0 b(-\eta_3)_{j_1, j_1} b(-\xi_3)_{j_2, j_2} b(-\rho_2)_{i_2, i_2} b(-\xi_1)_{i_1, i_1} b(-\eta_1)_{i_0,} \\
& j_2 b(-\rho_2, -\xi_1)_{i_1, j_1} b(-\xi_3, -\rho_2)_{i_2, j_0} b(-\eta_3, -\xi_3)_{j_2, i_2} b(-\xi_1, -\eta_1)_{i_0,} \\
& j_0 b(-\eta_3, -\xi_3, -\rho_2)_{i_2, j_1} b(-\xi_3, -\rho_2, -\xi_1)_{i_1, j_2} b(-\rho_2, -\xi_1, -\eta_1)_{i_0,} \\
& j_1 b(-\xi_3, -\rho_2, -\xi_1, -\eta_1)_{i_0, j_0} b(-\eta_3, -\xi_3, -\rho_2, -\xi_1)_{i_1,} \\
& j_0 b(-\eta_3, -\xi_3, -\rho_2, -\xi_1, -\eta_1)_{i_0} >_{\mathbb{F}_2} .
\end{aligned}$$

The differential $\delta^1 : \overline{bar}_r \rightarrow \mathcal{A}_1 \otimes_{\mathcal{I}} \overline{bar}_r \otimes_{\mathcal{I}} \mathcal{A}_1$ is described below by arrows:

$$\begin{aligned}
b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) & \longrightarrow 1 \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1, -\eta_1) \otimes \eta_1, \quad b(-\rho_2, -\xi_1, -\eta_1) \longrightarrow \\
\xi_3 \otimes b(-\xi_3, -\rho_2, -\xi_1, -\eta_1) \otimes 1, \quad b(-\eta_3) & \longrightarrow 1 \otimes b(-\eta_3, -\xi_3) \otimes \xi_3, \quad b(i_2) \longrightarrow \xi_2 \otimes b(-\xi_2) \otimes 1, \\
b(i_2) & \longrightarrow 1 \otimes b(-\xi_1) \otimes \xi_1, \quad b(i_2) \longrightarrow \rho_2 \otimes b(-\rho_2) \otimes 1, \quad b(j_1) \longrightarrow \eta_3 \otimes b(-\eta_3) \otimes 1, \\
b(j_1) & \longrightarrow 1 \otimes b(-\xi_3) \otimes \xi_3, \quad b(j_1) \longrightarrow 1 \otimes b(-\eta_2) \otimes \eta_2, \quad b(j_2) \longrightarrow 1 \otimes b(-\xi_2) \otimes \xi_2, \\
b(j_2) & \longrightarrow \xi_3 \otimes b(-\xi_3) \otimes 1, \quad b(j_2) \longrightarrow 1 \otimes b(-\rho_2) \otimes \rho_2, \quad b(j_0) \longrightarrow 1 \otimes b(-\rho_0) \otimes \rho_0, \\
b(j_0) & \longrightarrow 1 \otimes b(-\eta_3) \otimes \eta_3, \quad b(-\xi_3, -\rho_2, -\xi_1, -\eta_1) \longrightarrow \eta_3 \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1, -\eta_1) \otimes 1, \\
b(i_1) & \longrightarrow \xi_1 \otimes b(-\xi_1) \otimes 1, \quad b(i_1) \longrightarrow 1 \otimes b(-\eta_1) \otimes \eta_1, \quad b(i_1) \longrightarrow \eta_2 \otimes b(-\eta_2) \otimes 1, \\
b(-\rho_2, -\xi_1) & \longrightarrow 1 \otimes b(-\rho_2, -\xi_1, -\eta_1) \otimes \eta_1, \quad b(-\rho_2, -\xi_1) \longrightarrow \xi_3 \otimes b(-\xi_3, -\rho_2, -\xi_1) \otimes 1, \\
b(i_0) & \longrightarrow \rho_0 \otimes b(-\rho_0) \otimes 1, \quad b(i_0) \longrightarrow \eta_1 \otimes b(-\eta_1) \otimes 1, \quad b(-\xi_1) \longrightarrow \rho_2 \otimes b(-\rho_2, -\xi_1) \otimes 1, \\
b(-\xi_1) & \longrightarrow 1 \otimes b(-\xi_1, -\eta_1) \otimes \eta_1, \quad b(-\eta_1) \longrightarrow \xi_1 \otimes b(-\xi_1, -\eta_1) \otimes 1, \quad b(-\xi_3) \longrightarrow 1 \otimes \\
b(-\xi_3, -\rho_2) & \otimes \rho_2, \quad b(-\xi_3) \longrightarrow \eta_3 \otimes b(-\eta_3, -\xi_3) \otimes 1, \quad b(-\rho_2) \longrightarrow 1 \otimes b(-\rho_2, -\xi_1) \otimes \xi_1, \\
b(-\rho_2) & \longrightarrow \xi_3 \otimes b(-\xi_3, -\rho_2) \otimes 1, \quad b(-\eta_3, -\xi_3, -\rho_2) \longrightarrow 1 \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \otimes \\
\xi_1, \quad b(-\xi_3, -\rho_2, -\xi_1) & \longrightarrow \eta_3 \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \otimes 1, \quad b(-\xi_3, -\rho_2, -\xi_1) \longrightarrow 1 \otimes \\
b(-\xi_3, -\rho_2, -\xi_1, -\eta_1) & \otimes \eta_1, \quad b(-\xi_3, -\rho_2) \longrightarrow \eta_3 \otimes b(-\eta_3, -\xi_3, -\rho_2) \otimes 1, \quad b(-\xi_3, -\rho_2) \longrightarrow
\end{aligned}$$

$$1 \otimes b(-\xi_3, -\rho_2, -\xi_1) \otimes \xi_1, b(-\eta_3, -\xi_3) \longrightarrow 1 \otimes b(-\eta_3, -\xi_3, -\rho_2) \otimes \rho_2, b(-\xi_1, -\eta_1) \longrightarrow \rho_2 \otimes b(-\rho_2, -\xi_1, -\eta_1) \otimes 1.$$

2.6.2 Algebraic pairing recovers Lagrangian Floer homology

Let us refer to [LipOzsThu11, Definition 2.8] for the definitions of dual A_∞ modules. Having an A_∞ module $M_{\mathcal{A}_1}$, we denote its dual by ${}_{\mathcal{A}_1}\overline{M}$. Now we are ready to define an algebraic pairing of curves on the surface.

Definition 2.6.6. An *algebraic pairing* of two curves L_0, L_1 in the parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$ is given by a chain complex

$$M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}.$$

Remark 2.6.7. Note that ${}^{\mathcal{A}_1}\overline{bar}_r^{\mathcal{A}_1}$ does not have δ^1 differentials $\blacktriangleleft$ with non-idempotent algebra elements outgoing on both sides. Such type DD structures are called “separated”. This property implies that the chain complex structure of algebraic pairing does not depend on the brackets placement:

$$\begin{aligned} (M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1}) \boxtimes_{\mathcal{A}_1} \overline{M(L_0)} &= M(L_1)_{\mathcal{A}_1} \boxtimes ({}^{\mathcal{A}_1}\overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}) = \\ &= M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}. \end{aligned}$$

Usually only the chain homotopy equivalence class of the box tensor product does not depend on the brackets placement.

Remark 2.6.8. The fact that the differential on $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ is well-defined follows from the fact that ${}^{\mathcal{A}_1}\overline{bar}_r^{\mathcal{A}_1}$ is a bounded type DD structure (see [LipOzsThu15, Definition 2.2.56] for the definition of being bounded). This in turn follows from the geometric interpretation of its differential. Each time one applies the differential to an element, the path that element represents gets longer. Because there

is an upper bound on the length of possible paths, eventually $(\delta^1)^k$ has to become 0.

Remark 2.6.9. Moreover the modules $M(L_0), M(L_1)$ are also bounded, because they have finite number of A_∞ actions, which in turn follows from our assumption that $(\Sigma, Z^{bps}, \alpha^{arcs})$ is directed.

Theorem 2.6.10. *Let L_0, L_1 be two admissible unobstructed immersed curves in the parameterized surface $(\Sigma, Z^{bps}, \alpha^{arcs})$. Then their Lagrangian Floer complex is chain homotopy equivalent to the algebraic pairing of curves:*

$$CF_*(L_0, L_1) \simeq M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}.$$

Remark 2.6.11. Here we describe the motivation behind this theorem. By [Aur10a, Theorem 1], one knows that α^{arcs} generate⁷ the category $\mathcal{F}(\Sigma, Z^{bps})$, which consists of *embedded* Lagrangians. This follows from the fact that Lefschetz thimbles generate the Fukaya-Seidel category associated to a Lefschetz fibration, see [Sei08, Theorem 18.24]. The Fukaya-Seidel category is closely related to the partially wrapped Fukaya category, see Section 4.5.

Having a finite set of generating objects α^{arcs} is very useful, because the Yoneda embedding construction (see [Aur14, Section 3.4]) then gives a fully faithful embedding of the A_∞ category $\mathcal{F}(\Sigma, Z^{bps})$ into the dg -category of A_∞ modules over the *hom*-algebra $\bigoplus_{\alpha_i, \alpha_j \in \alpha^{arcs}} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \alpha_j) = \mathcal{A}_1$ (see [Sei08, Corollary 2.13]). I.e. if L_0, L_1 are embedded Lagrangians, then one has

$$HF_*^{pw}(L_0, L_1) = H_*(\text{hom}_{\mathcal{F}(\Sigma, Z^{bps})}(L_1, L_0)) \cong H_*(\text{Mor}_{\text{Mod}_{\mathcal{A}_1}}(M(L_0), M(L_1))).$$

In this theorem we extend this result to *immersed* Lagrangians⁸ in a surface, and

⁷This means that in certain enlarged category of twisted complexes $Tw\mathcal{F}(\Sigma, Z^{bps})$ every object in $\mathcal{F}(\Sigma, Z^{bps})$ is quasi-isomorphic to a direct summand of iterated mapping cones between the generating objects, see [Aur14, Section 3] for the details.

⁸One could actually derive this from the previous results because one could argue that the im-

more importantly, we prove the result in a geometric way, as opposed to proving generation of the Fukaya category by the parameterizing arcs.

The following is the precise underlying reasoning for the statement of the theorem. We first note that the morphism complex can be described in the following way via bar resolution, see [LipOzsThu11, Proposition 2.10]: $Mor_{Mod_{\mathcal{A}_1}}(M(L_0), M(L_1)) \cong M(L_1) \boxtimes_{\mathcal{A}_1} \overline{Bar_r(\mathcal{A}_1)} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$. It can be proved that ${}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1} \simeq {}^{\mathcal{A}_1}\overline{Bar_r(\mathcal{A}_1)}{}^{\mathcal{A}_1}$ and so $Mor(M(L_0)_{\mathcal{A}_1}, M(L_1)_{\mathcal{A}_1}) \simeq M(L_1)_{\mathcal{A}_1} \boxtimes {}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$. Now, suppose we have two immersed curves L_0, L_1 in the surface Σ . [Aur10a, Theorem 1] suggests that $HF_*^{pw}(L_0, L_1) = hom_{\mathcal{F}(\Sigma, Z^{bps})}(L_1, L_0) \cong H_*(Mor(M(L_0), M(L_1)))$. The way we constructed ${}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1}$ suggests that

$H_*(Mor(M(L_0), M(L_1))) \cong H_*(M(L_1)_{\mathcal{A}_1} \boxtimes {}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)})$. We now dismiss the morphism complex, and prove that $HF_*^{pw}(L_0, L_1) \cong H_*(M(L_1)_{\mathcal{A}_1} \boxtimes {}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)})$ using the geometric interpretation of ${}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1}$.

Proof. The plan for the proof is the following:

Step (1) We first isotope curves L_0, L_1 in a certain way.

Step (2) We prove that the isotoped pair (L_0, L_1) is admissible.

Step (3) We then prove that the two chain complexes we consider are isomorphic: $CF_*(L_0, L_1) \cong M(L_1)_{\mathcal{A}_1} \boxtimes {}^{\mathcal{A}_1}\overline{bar}_r{}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$, in two steps:

- a) We first see that the generators of complexes are in 1-1 correspondence.
- b) We then prove that the differentials coincide. This is done by “localizing” the differential in the geometric pairing, i.e. by noting that every disc contributing to the differential is contained almost entirely in one of the big domains B_k .

mersed Lagrangians are inside the derived partially wrapped Fukaya category, see [HaiKatKon17, Theorem 4.3]

We will be illustrating each step on our running example of curves: $L_0 = L^{\natural}$ — the curve in Figure 2.11, and $L_1 = L^b$ — the horizontal belt around the pillowcase on the left of Figure 2.6. For A_{∞} actions on the dual module ${}_{\mathcal{A}_1}\overline{M}(L^{\natural})$ see Figure 2.13, and A_{∞} actions on $M(L^b)_{\mathcal{A}_1} = \langle x, s, z, w \rangle_{\mathbb{F}_2}$ (see Figure 2.16 where the intersection of blue curve with red arcs $L^b \cap \alpha^{arcs} = \{x, s, z, w\}$ are depicted) are as follows: $z \otimes (\xi_3, \eta_3) \rightarrow x$, $z \otimes \rho_0 \rightarrow x$, $z \otimes (\eta_1, \xi_1) \rightarrow w$, $w \otimes \xi_2 \rightarrow s$.

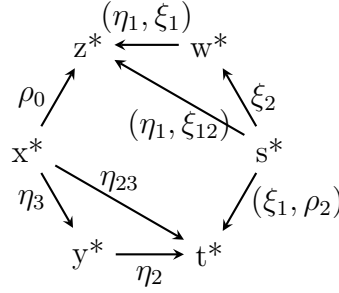


Figure 2.13: ${}_{\mathcal{A}_1}\overline{M}(L^{\natural})$

Step (1). For a short visual description of the required isotopy one may look at Figure 2.16. Let us describe it now.

First and foremost, one needs to isotope both curves L_0, L_1 in the way required for the definition of modules $M(L_0)_{\mathcal{A}_1}, M(L_1)_{\mathcal{A}_1}$, see Section 2.5.1.

Let us describe further isotopies of L_0 . Mark points b_k in the big domains B_k sufficiently far away from ∂B_k , like in Figure 2.16, and call them centers of the big domains. Also let us choose a center point for each red arc α which is sufficiently far away from $\partial \alpha$. Then isotope the curve L_0 in the following way: first make it intersect every red arc near its center (as in Figure 2.16). Then isotope L_0 so that in big domains it goes from the centers of big domains straight to the centers of red arcs (or, if L_0 is an arc, to the point on $\partial \Sigma$, see Figure 2.15). See Figure 2.16 for how the isotoped L_0 looks like in the case of $L_0 = L^{\natural} : S^1 \looparrowright \overline{P}$.

Concerning the curve L_1 , we also make it intersect the red arcs near their centers. But the rest of the isotopy is different from L_0 . First, tilt the angle in which it

intersects the centers of the red arcs, so that the following is true. 1) L_1 is almost parallel to the red arcs and intersects each nearby piece of L_0 exactly once. 2) Going clockwise around the center of the red arc, one encounters the rays in the following order: red arc, all the pieces of L_1 , all the pieces of L_0 . See Figure 2.14.

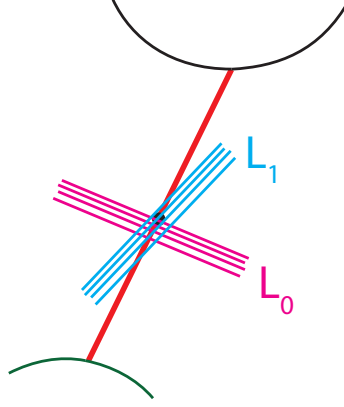


Figure 2.14: Perturbation near the centers of red arcs.

Next, we make the final isotopy of the L_1 curve, which has to do with the way it behaves inside the big domains. Divide L_1 into segments by intersections with red arcs. We already specified how L_1 looks near those intersections. Now we will describe how each segment between those intersections is isotoped, by traversing L_1 . First, it is important to note, the whole L_1 will not leave a small neighborhood of $\partial\Sigma \cup \{\text{red arcs}\}$. One starts at the center of the red arc, enters one of the big domains, and then goes near $\partial\Sigma \cup \{\text{red arcs}\}$ in that domain until it reaches a basepoint. If this is the end segment of L_1 being an arc, then L_1 is connected to ∂P near that basepoint, such that (L_0, L_1) is admissible, see Figure 2.15. Otherwise L_1 turns by 180° (in the direction towards the other end of the segment, i.e. such that it does not introduce a fishtail), and goes backwards until it reaches the other end of the segment. See Figure 2.16 for how the isotoped L_1 looks like in the case of $L_1 = L^b : S^1 \looparrowright \overline{P}$.

Step (2). Let us prove admissibility of the pair (L_0, L_1) after the isotopies above. All the non-smooth corners, triple intersections, and non-transverse intersections are eliminated by introducing a slight perturbation. There are no immersed annuli because

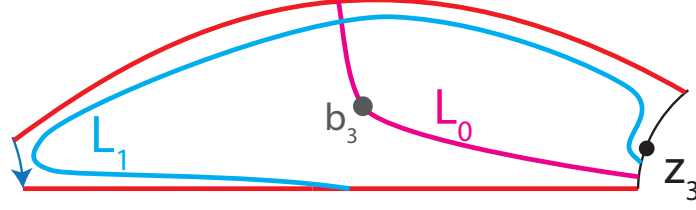


Figure 2.15: Perturbation of the end segments of the arcs.

of intersections near centers of red arcs introduced in Figure 2.14: if $pL_0 \sim qL_1$, and so $Per(L_0, L_1) = \mathbb{Z}$, those intersections lift to the covering from the Lemma 2.2.6(d). If L_0, L_1 are arcs, they are in admissible position relative to basepoints because we ensured it while isotoping L_1 , see Figure 2.15.

Step (3a). Here we assume that the required isotopies of L_0, L_1 were already made. Generators of $CF_*(L_0, L_1)$, as well as generators of $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$, are in 1-1 correspondence with the set of paths along the boundaries of big domains ∂B_k , from intersections $L_1 \cap \alpha^{arcs} = \{\text{generators of } M(L_1)\}$ to intersections $L_0 \cap \alpha^{arcs} = \{\text{generators of } \overline{M(L_0)}\}$. The paths are against the natural orientations of ∂B_k , and consist of chords $-e_l$ of length 1 (which are also elements of $\mathcal{A}_1$). Below we explain how to see the correspondence $\text{paths} \leftrightarrow \text{generators}$ for both chain complexes.

Remember that elements of ${}^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1}$ naturally correspond to such paths, see Figure 2.12. Each generator $u \otimes b \otimes v \in M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ has the corresponding element $b \in {}^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1}$, and that describes the path from the generator $u \in M(L_1)$ to the generator $v \in M(L_0)$. For the generators of $CF_*(L_0, L_1)$, the desired path can be traversed along the L_1 , see Figure 2.16. Notice that “0 length” paths are included, which correspond to intersections introduced when both L_0, L_1 cross the same red arc, see Figure 2.14.

Considering the example in Figure 2.16, we have:

$$M(L_1) = \langle x, s, z, w \rangle_{\mathbb{F}_2}, \quad \overline{M(L_0)} = \langle x^*, s^*, z^*, w^*, t^*, y^* \rangle_{\mathbb{F}_2}.$$

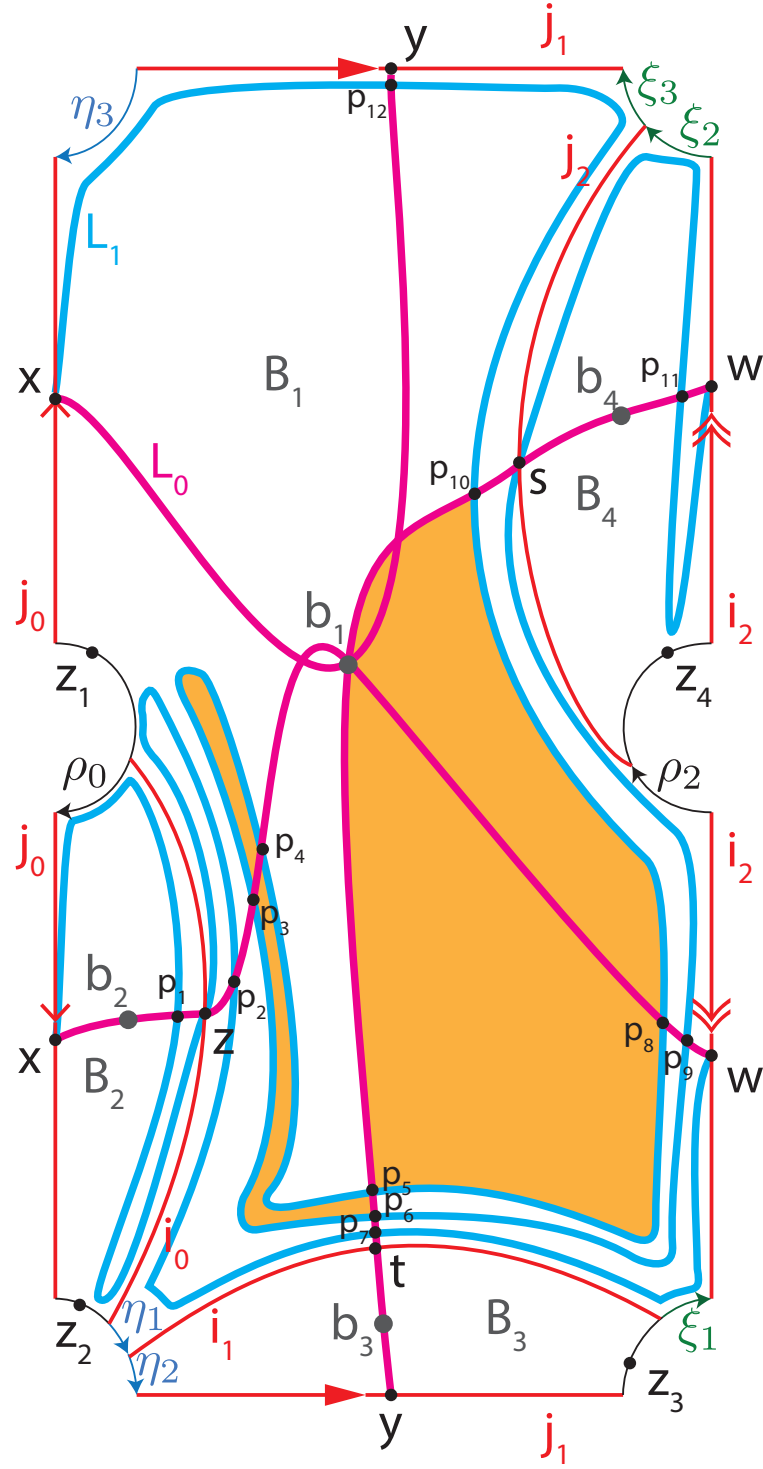


Figure 2.16: Isotopies of $L_0 = L^\natural$ and $L_1 = L^b$ so that the chain complexes of geometric and algebraic pairings become isomorphic: $CF_*(L_0, L_1) \cong M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M}(L_0)$.

The intersection points $L_0 \cap L_1$ in Figure 2.16 correspond to the generators of $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{\text{bar}}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ as follows:

$$\begin{aligned}
w \otimes b(i_2) \otimes w^* &\leftrightarrow w, \\
s \otimes b(j_2) \otimes s^* &\leftrightarrow s, \\
x \otimes b(j_0) \otimes x^* &\leftrightarrow x, \\
z \otimes b(i_0) \otimes z^* &\leftrightarrow z, \\
s \otimes b(-\rho_2, -\xi_1, -\eta_1) \otimes z^* &\leftrightarrow p_3, \\
s \otimes b(-\rho_2) \otimes w^* &\leftrightarrow p_9, \\
s \otimes b(-\xi_2) \otimes w^* &\leftrightarrow p_{11}, \\
s \otimes b(-\rho_2, -\xi_1) \otimes t^* &\leftrightarrow p_6, \\
w \otimes b(-\xi_1, -\eta_1) \otimes z^* &\leftrightarrow p_2, \\
w \otimes b(-\xi_1) \otimes t^* &\leftrightarrow p_7, \\
x \otimes b(-\eta_3) \otimes y^* &\leftrightarrow p_{12}, \\
x \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1, -\eta_1) \otimes z^* &\leftrightarrow p_4, \\
x \otimes b(-\eta_3, -\xi_3) \otimes s^* &\leftrightarrow p_{10}, \\
x \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \otimes t^* &\leftrightarrow p_5, \\
x \otimes b(-\eta_3, -\xi_3, -\rho_2) \otimes w^* &\leftrightarrow p_8, \\
x \otimes b(-\rho_0) \otimes z^* &\leftrightarrow p_1.
\end{aligned}$$

Step (3b). Here we want to show that the differentials in $CF_*(L_0, L_1)$ and $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{\text{bar}}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ coincide. We will do it by partitioning both differentials into smaller groups, and showing how the smaller groups correspond to each other.

Lemma 2.6.12. *Every immersed disc contributing to the differential in $CF_*(L_0, L_1)$ is contained inside a small neighborhood of one of the big domains B_k .*

Proof. For an immersed disc to enter two consecutive big domains it must pass through an intersection of type $u \otimes b(i) \otimes v$ (where $i \in \text{Vertices}(\Gamma)$), because the disc is not allowed to touch the $\partial\Sigma$. Here, in the notation, we use the 1-1 correspondence between generators of $CF_*(L_0, L_1)$ and generators of $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{\text{bar}}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$. Such intersections happen when both L_0 and L_1 cross the red arc, see Figure 2.14. Also only two opposite parts  of the intersection $u \otimes b(i) \otimes v \in L_0 \cap L_1$ are allowed to be filled by the disc. Thus the disc cannot pass through such an intersection point, as such a disc cannot be immersed. $\square$

Remark 2.6.13. We need to consider small neighborhoods of the big domains, as intersections of type $u \otimes b(i) \otimes v$ are not happening exactly on the red arc, but rather somewhere close to its center, see Figure 2.14.

Lemma 2.6.14. *Every differential in $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{\text{bar}}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ contains A_∞ action either on the $M(L_1)$ side (Figure 2.17), or on the $\overline{M(L_0)}$ side (Figure 2.18), but not on both sides. Moreover, every such A_∞ action comes from a “basic” disc in the definition of $M(L_i)_{\mathcal{A}_1}$, i.e. a disc contained entirely in one of the big domains.*

Proof. The fact that type DD structure ${}^{\mathcal{A}_1}\overline{\text{bar}}_r^{\mathcal{A}_1}$ is separated implies that the differential in $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{\text{bar}}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ is either on the $M(L_1)$ side (Figure 2.17), or on the $\overline{M(L_0)}$ side (Figure 2.18). We call them 1st and 2nd types of differentials.

Note that the δ^1 arrows in ${}^{\mathcal{A}_1}\overline{\text{bar}}_r^{\mathcal{A}_1}$ do not contain outgoing algebra elements of chord length more than 1. This observation implies the last statement of the lemma. $\square$

Now let us take one connected segment l_0 of L_0 , cut out by a small neighborhood of a big domain $N(B_k)$. And also take one connected segment l_1 of L_1 cut out by the same neighborhood $N(B_k)$. These segments almost coincide with two of the segments from the division of L_0 and L_1 by the intersections with red arcs. The behavior of l_0, l_1 inside $N(B_k)$ is completely described by our isotopy in step (1). Also note, that

such segments correspond to basic discs in the definition of $M(L_i)_{\mathcal{A}_1}$. These basic discs have a chance to contribute to differential in $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$.

Due to the first lemma above, the differential in $CF_*(L_0, L_1)$ is partitioned into differentials with boundaries on segments l_0, l_1 . We will denote such groups of differentials by $CF_*(l_0, l_1)$. Due to the second lemma above, the differential in $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$ is partitioned into differentials using different basic discs. We will denote such groups of differentials by $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$, as basic discs correspond to the segments.

We are left to show how differentials in $CF_*(l_0, l_1)$ correspond to differentials in $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$. We will do it by considering the 1st and 2nd type of differentials in $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$ separately.

The 1st type. In this case the differentials in $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$ have outgoing algebra elements on the left, see Figure 2.17. This corresponds to prolongation of paths in the backward direction, i.e. when new length one chords are concatenated to paths from the back. We use the following notation in Figure 2.17: i, k are the red arcs intersecting l_1 at (u_1) and (u_2) , j is the red arc intersecting l_0 at (v) , and $\gamma_1, \dots, \gamma_m$ represent chord length one elements of $\mathcal{A}_1$.

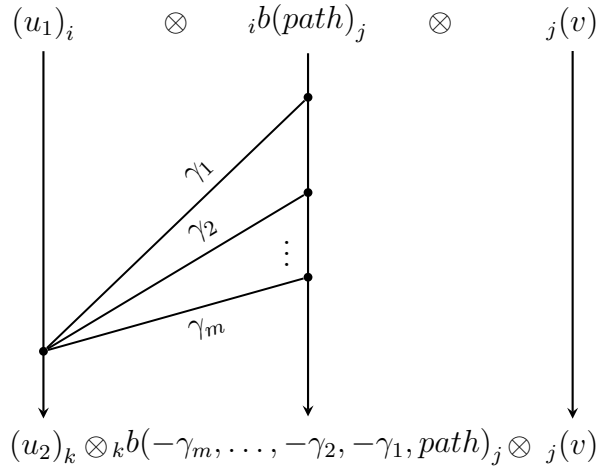


Figure 2.17: 1st type of differentials in $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$.

Let us describe the corresponding disc differentials in $CF_*(l_0, l_1)$. Suppose the disc goes from p to q . First, note that all intersections $l_0 \cap l_1$ are happening near one of two ends of segment l_0 . Points p and q can be on one end of the segment l_0 , or on the different ends. Let us consider those pairs which are on one end of the segment l_0 . For this to happen, traversing the l_1 boundary of the disc, the l_1 must pass the 180° rotation point and come back. See, for example, the disc from p_6 to p_5 in Figure 2.16. We say that such differentials are of the 1st type in $CF_*(l_0, l_1)$.

These are precisely the discs that correspond to the 1st type of differentials in $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$. The reason is that both 1st type of differentials in $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$, and 1st type of differentials in $CF_*(l_0, l_1)$ exist if and only if $(u_2)_k$ is before $(u_1)_i$, which is before $j(v)$ w.r.t. the direction against natural orientation of ∂B_k . For example, in Figure 2.16, the disc from p_6 to p_5 corresponds to the differential $s \otimes b(-\rho_2, -\xi_1) \otimes t^* \rightarrow x \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \otimes t^*$.

The 2nd type. In this case the differentials $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$ have outgoing algebra elements on the right, see Figure 2.18. This corresponds to prolongation of paths in the forward direction, i.e. when new length one chords are concatenated to the front of paths. We use the following notation in Figure 2.18: i is the red arc intersecting l_1 at (u) , j , o are the red arcs intersecting l_0 at (v_1) and (v_2) , and γ_m represent chord length one elements of $\mathcal{A}_1$.

The corresponding 2nd type of disc differentials in $CF_*(l_0, l_1)$ are those, which have their corners on two different ends of segment l_0 . They do not pass through the 180° rotation point of l_1 , but instead they pass through the center of the big domain, see the disc from p_{10} to p_5 in Figure 2.16.

Both 2nd type differentials in $M(l_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar}_r^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(l_0)}$, and 2nd type differentials in $CF_*(l_0, l_1)$ exist if and only if $(u)_i$ is before $j(v_1)$, which is before $o(v_2)$ w.r.t. the direction against the natural orientation of ∂B_k . For example, in Figure

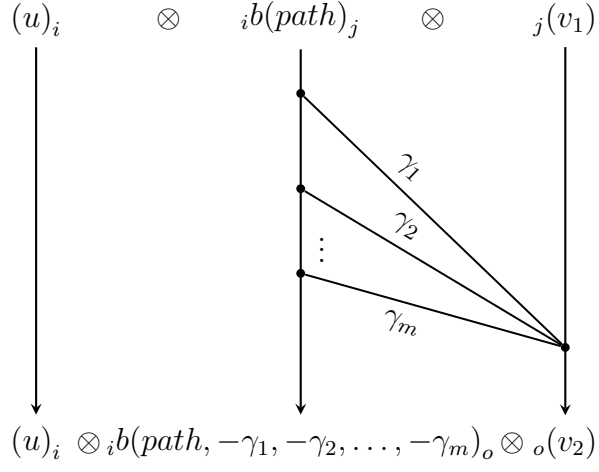


Figure 2.18: 2nd type of differentials in $M(L_1)_{\mathcal{A}_1} \boxtimes^{\mathcal{A}_1} \overline{bar_r}^{\mathcal{A}_1} \boxtimes_{\mathcal{A}_1} \overline{M(L_0)}$.

2.16, the disc from p_{10} to p_5 corresponds to the differential $x \otimes b(-\eta_3, -\xi_3) \otimes s^* \rightarrow x \otimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \otimes t^*$.

□

Chapter 3

Bordered theory for pillowcase homology

3.1 Pillowcase homology

We sketch how the pillowcase homology construction (from [HedHerKir14a] and [HedHerKir14b]) works in the first two columns of Figure 3.1. Having a knot in $K \subset S^3$, find a Conway sphere, i.e. a 2-sphere that intersects the knot in 4 points (denote it by $(S^2, 4)$). The knot is decomposed into two tangles by this 2-sphere, and we require one of the tangles to be a trivial tangle that consists of two arcs. Then proceed as follows:

- (1) To that Conway sphere with four marked points (denote by γ_i , $i = 1, 2, 3, 4$, loops around those points) associate a traceless character variety:

$$R(S^2, 4) = \{h \in \text{hom}(\pi_1(S^2 \setminus 4pt), SU(2)) \mid \text{tr}(h(\gamma_i)) = 0\} / \text{conj}.$$

It happens to be homeomorphic to the pillowcase — a torus factorized by the

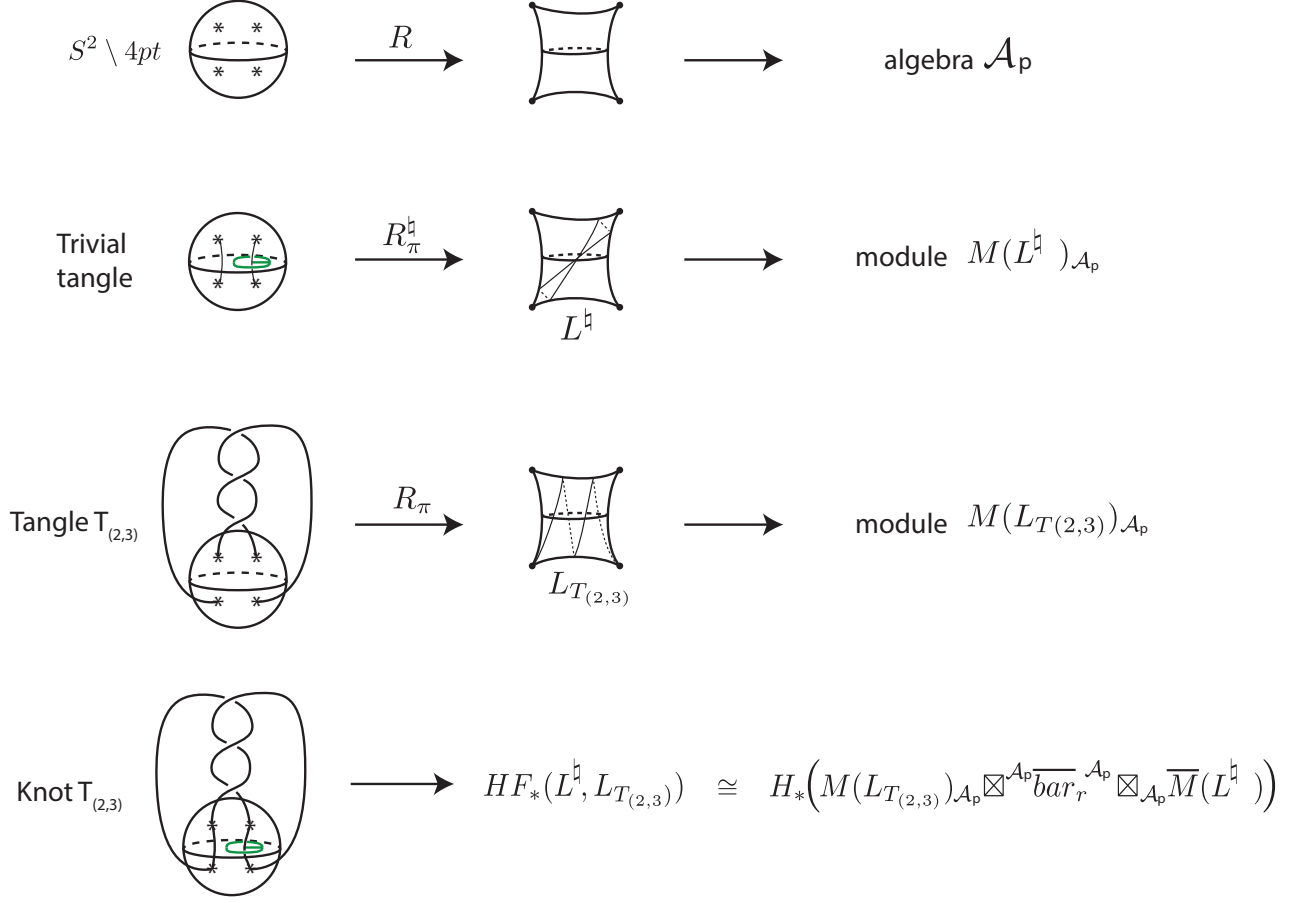


Figure 3.1: The pillowcase homology construction (1st and 2nd columns), and its algebraic extension (3rd column).

elliptic involution

$$R(S^2, 4) \cong P = S^1 \times S^1 / ((\gamma, \theta) \sim (-\gamma, -\theta)),$$

see [HedHerKir14a, Proposition 3.1] for the proof, and the 1st row 2nd column of Figure 3.1 for the picture of the pillowcase.

- (2) To a trivial tangle, which consists of two arcs A_1, A_2 , associate an immersed curve $L^\natural$ in the pillowcase by the following procedure. First, add to the tangle a circle H with an arc W as shown on the left of the second row of Figure 3.1.

Then form a space of traceless representations:

$$R^\natural(D^3, A_1 \cup A_2) = \{h \in \text{hom}(\pi_1(D^3 \setminus (A_1 \cup A_2 \cup H \cup W)), SU(2)) \mid \\ \mid \text{tr}(h(\mu_{A_i})) = \text{tr}(h(\mu_H)) = 0, h(\mu_W) = -I \} / \text{conj}.$$

Because $S \setminus 4pt \subset D^3 \setminus (A_1 \cup A_2 \cup H \cup W)$, there is a map in the reversed direction $R^\natural(D^3, A_1 \cup A_2) \rightarrow R(S^2, 4)$. Because this map is singular, and $R^\natural(D^3, A_1 \cup A_2)$ is not 1-dimensional, one needs to do a holonomy perturbation of the space. After a specifically defined perturbation (see [HedHerKir14a, Section 7]), one gets an immersed circle $L^\natural : R_\pi^\natural(D^3, A_1 \cup A_2) \looparrowright P$, which forms a “figure eight” depicted in the 2nd row 2nd column of Figure 3.1.

- (3) With the tangle $K \setminus (A_1 \cup A_2)$ from the other side one does almost the same procedure. The only difference is that the circle H and the arc W are not added (this is why here the image will often pass through a singular point). One still needs to perturb $R(D^3, K \setminus (A_1 \cup A_2))$ in this case (see, for example, [HedHerKir14b, Section 11.6] for the case of (3,4) torus knot). This results in the immersion $L_K : R_\pi(D^3, K \setminus (A_1 \cup A_2)) \looparrowright P^1$. An example of such an immersion for a tangle $T_{(2,3)} \setminus (A_1 \cup A_2)$ (coming from the (2,3)-torus knot) is in the 3rd row 2nd column of Figure 3.1. Examples of such immersions for other torus knots (with two arcs removed) are depicted in Figure 3.2.

- (4) Having done all that, one associates to the initial knot K a vector space called pillowcase homology. It is equal to Lagrangian Floer homology $H_{pil}(K) = HF_*(L^\natural, L_K)$ inside $\overline{P}$, where $\overline{P}$ is the pillowcase with small neighborhoods of 4 singular points deleted², see Figure 2.6.

¹Let us stress that the map L_K depends on the tangle decomposition of a knot. We chose this misleading notation for simplicity.

²One actually obtains Lagrangians in P and should consider Floer homology where discs do not cross singular points. But one can delete small neighborhoods of singular points to get $\overline{P}$, and the corresponding Lagrangian Floer complex will be unchanged.

On the level of chain complexes, the vector space isomorphism between the pillowcase chain complex and the singular instanton chain complex

$$C_{pil}(K) = CF_*(L^\natural, L_K) \cong CI^\natural(K)$$

is true by construction. In [HedHerKir14b] the authors provided lots of examples where the homologies of these chain complexes are indeed isomorphic.

Let us list the missing ingredients for $H_{pil}(K)$ to be a knot invariant. In the construction one makes certain choices. First, there is a tangle decomposition of a knot along the Conway sphere, and the map $R(D^3, K \setminus (A_1 \cup A_2)) \looparrowright \overline{P}$ depends on this decomposition. Second, there is a generic small perturbation of $R(D^3, K \setminus (A_1 \cup A_2))$ in order to obtain a non-singular immersed Lagrangian $L_K : R_\pi(D^3, K \setminus (A_1 \cup A_2)) \looparrowright \overline{P}$. Thus, in order to obtain a knot invariant, one needs to prove that $H_{pil}(K)$ does not depend on those two choices. Moreover, one needs to prove that, after those choices, the resulting immersion L_K is unobstructed, and admissible w.r.t. to $L^\natural$, so that $HF_*(L^\natural, L_K)$ is defined without difficulties.

3.2 The bordered construction

By applying constructions from Chapter 2, we obtain an algebraic enhancement of pillowcase homology. This answers the following question: what algebraic structures should one associate to $L^\natural$ and L_K in order to be able to recover $H_{pil}(K) = HF_*(L^\natural, L_K)$ algebraically, without looking at the intersection picture on the pillowcase. The relevant objects can be seen in the third column of Figure 3.1. Let us describe them:

- (1) To a 4-punctured 2-sphere we have associated a pillowcase $\overline{P}$, and now further associate an algebra $\mathcal{A}_p$ via the construction in Section 2.4. This is done by picking four basepoints on $\partial \overline{P}$ as in Figure 2.6, parameterizing the pillowcase

as in Example 2.4.2, and associating to this parameterization an algebra $\mathcal{A}_p = \mathcal{A}_1(\overline{P})$ as in Example 2.4.5.

- (2) To a trivial 2-stranded tangle we have associated an immersed circle $L^\natural$ in $\overline{P}$, and now further associate a specific module $M(L^\natural)_{\mathcal{A}_p}$, see Figure 2.10, via the construction in Section 2.5.
- (3) To a tangle from the other side $K \setminus (A_1 \cup A_2)$, similarly already having L_K , we associate a module $M(L_K)_{\mathcal{A}_p}$.
- (4) To a union of these two tangles, i.e. to a knot K , we associate a homology $H_*(M(L_K)_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar_r^{\mathcal{A}_p}} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)})$. The fact that this algebraic pairing is equal to pillowcase homology $H_{pil}(K) = HF_*(L^\natural, L_K)$ follows from Theorem 2.6.10, assuming L_K is unobstructed and $(L^\natural, L_K)$ is admissible.

3.3 Examples of modules associated to tangles

Here we give examples of modules $M(L_K)_{\mathcal{A}_p}$ associated to tangles $K - A_1 - A_2$. We use calculations from [HedHerKir14b][Sections 7,11] to get immersed curves L_K in the pillowcase, see Figure 3.2.

Example 3.3.1 (Trivial tangle to pair with). First we consider a trivial tangle $A_1 \cup A_2$ inside the Conway sphere, see the 2nd row 1st column in Figure 3.1. For that tangle (decorated with an additional arc and circle to avoid reducibles) one associates a curve $L^\natural$ in Figure 2.11, and to that curve one associates a module drawn in Figure 2.10.

In the algebraic pairing $M(L_U)_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar_r^{\mathcal{A}_p}} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}$ there is a dual module ${}_{\mathcal{A}_p} \overline{M(L^\natural)}$ involved. This module was described in Figure 2.13.

In the next examples we also compute pillowcase homology via algebraic pairing, using a computer program [Kot18] for box tensor product of modules. Another way

to see the chain complex and differentials is to isotope L_K as in the proof of Theorem 2.6.10, and then use Lagrangian Floer homology $CF_*(L^\natural, L_K)$.

Example 3.3.2 (The unknot). The next example is a trivial knot tangle $U - A_1 - A_2$. Depending on how one picks the second tangle for the trivial knot (the first tangle is a vertical smoothing $\gamma = \pi$), the resulting curve on the pillowcase can be different. It is either an arc $\{\gamma = \pi\}$ (in case the second tangle looks like a crossing $\times$), or an arc $\{\theta = 0\}$ (in case the second tangle is horizontal smoothing of that crossing). Note that one cannot pick a vertical smoothing $\gamma = \pi$ of a crossing, as it results in two circles if paired with $\gamma = \pi$. On the left of Figure 3.2 we depicted an arc $\{\theta = 0\} = L_U$ for a crossing $\times$.

The corresponding module $M(L_U)_{\mathcal{A}_p}$ is $q_{j_1} \xrightarrow{\eta_3} p_{j_0}$. The algebraic pairing chain complex $M(L_U)_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}$ has 13 generators and 12 differentials. Pillowcase homology then has rank one: $HF_*(L^\natural, L_U) = H_*(M(L_U)_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}) = \mathbb{F}_2$, which coincides with singular instanton knot homology $I^\natural(U)$.

Example 3.3.3 ($T_{(2,3)}$). An immersed curve for the right-handed trefoil is depicted on the left of Figure 3.2.

The corresponding module $M(L_{T_{(2,3)}})_{\mathcal{A}_p}$ has generators:

$$u_{i_0}, e_{j_1}, v_{j_1}, q_{i_1},$$

and actions:

$$u \otimes (\eta_1, \xi_1, \rho_2, \xi_3) \longrightarrow e, \quad q \otimes (\eta_2) \longrightarrow e, \quad q \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow v.$$

The algebraic pairing chain complex $M(L_U)_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}$ has 15 generators and 10 differentials. Pillowcase homology then has rank three: $HF_*(L^\natural, L_{T_{(2,3)}}) = H_*(M(L_{T_{(2,3)}})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}) = (\mathbb{F}_2)^3$, which coincides with singular in-

stanton knot homology $I^{\natural}(T_{(2,3)})$.

In the next three examples immersed curves are different unions of curves R_0, R_1, R_3, R_4 , see the right of Figure 3.2. Notice that R_3 differs from R_0 by a twist around the boundary, and thus their pairings with $L^{\natural}$ are the same (because $L^{\natural}$ is not an arc). We describe the modules $M(L_{R_i})_{\mathcal{A}_p}$ for $i = 0, 1, 4$ below.

Module $M(L_{R_0})_{\mathcal{A}_p}$.

4 generators with their idempotents: $a_{j_1}, c_{j_1}, b_{i_1}, d_{j_0}$.

Actions: $a \otimes \eta_3 \longrightarrow d, b \otimes \eta_{23} \longrightarrow d, b \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow c, b \otimes \eta_2 \longrightarrow a$.

Algebraic pairing with the trivial tangle module:

$$H_*(M(L_{R_0})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^{\natural})}) = \mathbb{F}_2.$$

Module $M(L_{R_1})_{\mathcal{A}_p}$.

4 generators with their idempotents: $x_{j_1}, y_{j_1}, z_{i_1}, t_{i_1}$

Actions: $z \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow y, t \otimes \eta_2 \longrightarrow y, z \otimes \eta_2 \longrightarrow x, t \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow x$.

Algebraic pairing with the trivial tangle module:

$$H_*(M(L_{R_1})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^{\natural})}) = (\mathbb{F}_2)^4.$$

Module $M(L_{R_4})_{\mathcal{A}_p}$.

4 generators with their idempotents: $a_{i_0}, c_{i_1}, b_{i_0}, e_{i_1}, d_{i_1}, g_{i_1}, h_{i_1}, m_{j_1}, l_{j_1}, q_{j_2}, p_{j_2}, s_{i_2}, r_{i_1}, u_{j_1}, t_{i_2}, w_{j_0}, v_{j_1}, y_{j_1}, x_{j_1}, z_{j_0}$.

Actions: $a \otimes \eta_3 \longrightarrow d, b \otimes \eta_{23} \longrightarrow d, b \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow c, b \otimes \eta_2 \longrightarrow a, p \otimes \xi_3 \longrightarrow u, t \otimes \xi_2 \longrightarrow q, d \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow l, y \otimes \eta_3 \longrightarrow z, a \otimes \eta_1 \longrightarrow g, s \otimes \xi_2 \longrightarrow p, c \otimes \eta_2 \longrightarrow x, r \otimes \xi_{123} \longrightarrow u, e \otimes \eta_2 \longrightarrow m, d \otimes \eta_2 \longrightarrow y, s \otimes \xi_{23} \longrightarrow u, h \otimes \eta_2 \longrightarrow v, c \otimes \eta_{23} \longrightarrow w, r \otimes \xi_{12} \longrightarrow p, q \otimes \xi_3 \longrightarrow v, r \otimes \xi_1 \longrightarrow s, a \otimes \rho_0 \longrightarrow w, t \otimes \xi_{23} \longrightarrow v, b \otimes \eta_1 \longrightarrow h, b \otimes \eta_{12} \longrightarrow v, d \otimes \eta_{23} \longrightarrow z, r \otimes \eta_2 \longrightarrow l, x \otimes \eta_3 \longrightarrow w, c \otimes (\xi_1, \rho_2, \xi_3) \longrightarrow m,$

$$e \otimes \xi_1 2 \longrightarrow q, a \otimes \eta_1 2 \longrightarrow u, b \otimes \rho_0 \longrightarrow z, e \otimes \xi_1 23 \longrightarrow v, g \otimes \eta_2 \longrightarrow u, e \otimes \xi_1 \longrightarrow t.$$

Algebraic pairing with the trivial tangle module:

$$H_*(M(L_{R_4})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}) = (\mathbb{F}_2)^4.$$

Using these modules we compute three more examples for tangles:

Example 3.3.4 ($T_{(3,7)}$). The corresponding immersed curve is depicted on the right of Figure 3.2.

The corresponding module is $M(L_{T_{(3,7)}}) = M(L_{R_0}) \oplus M(L_{R_1}) \oplus M(L_{R_1})$. Pillowcase homology then has rank 9: $HF_*(L^\natural, L_{T_{(3,7)}}) = H_*(M(L_{T_{(3,7)}})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}) = (\mathbb{F}_2) \oplus (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 = (\mathbb{F}_2)^9$, which coincides with singular instanton knot homology $I^\natural(T_{(3,7)})$.

Example 3.3.5 ($T_{(5,11)}$). The corresponding immersed curve is depicted on the right of Figure 3.2.

The corresponding module is $M(L_{T_{(5,11)}}) = M(L_{R_0}) \oplus M(L_{R_1}) \oplus M(L_{R_1}) \oplus M(L_{R_4}) \oplus M(L_{R_4})$. Pillowcase homology then has rank 17: $HF_*(L^\natural, L_{T_{(5,11)}}) = H_*(M(L_{T_{(5,11)}})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}) = (\mathbb{F}_2) \oplus (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 = (\mathbb{F}_2)^{17}$.

Singular instanton Floer homology is not known for $T_{(5,11)}$.

Example 3.3.6 ($T_{(3,4)}$). The corresponding immersed curve is depicted on the right of Figure 3.2. This is an example where one actually needs to perturb $L_{T_{(3,4)}}$ in order to get an immersed 1-manifold.

The corresponding module is $M(L_{T_{(3,4)}}) = M(L_{R_1}) \oplus M(L_{R_3})$. Pillowcase homology then has rank 5: $HF_*(L^\natural, L_{T_{(3,4)}}) = H_*(M(L_{T_{(3,4)}})_{\mathcal{A}_p} \boxtimes^{\mathcal{A}_p} \overline{bar}_r^{\mathcal{A}_p} \boxtimes_{\mathcal{A}_p} \overline{M(L^\natural)}) = (\mathbb{F}_2)^4 \oplus \mathbb{F}_2 = (\mathbb{F}_2)^5$, which coincides with singular instanton knot homology $I^\natural(T_{(3,4)})$.

See [HedHerKir14b] and [FukKirPin17] for other examples of immersed curves associated to tangles.

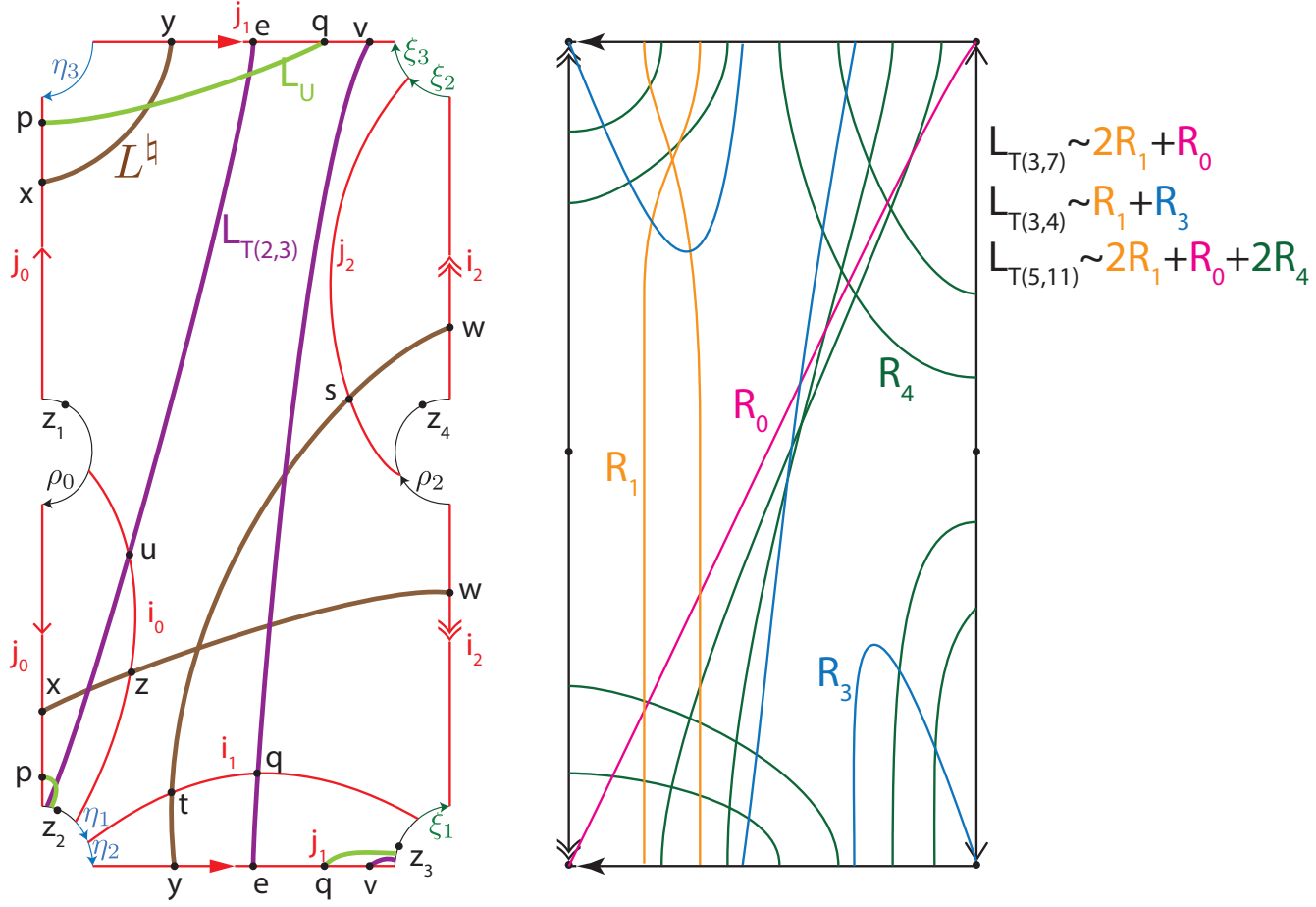


Figure 3.2: Different immersions associated to tangles. L^h denotes an immersed curve associated to a trivial tangle consisting of two arcs A_1, A_2 , see Figure 3.1. L_K denotes an immersed curve associated to a tangle $K \setminus (A_1 \cup A_2)$.

Chapter 4

Comparing homological invariants of mapping classes of surfaces

4.1 Bimodule invariant coming from bordered Heegaard Floer homology

Everything in this section is based on bordered Heegaard Floer homology theory. It was developed by Lipshitz, Ozsváth, and Thurston in [LipOzsThu08] and [LipOzsThu15].

4.1.1 Pointed matched circles

We will be considering parameterized surfaces, with one boundary component, with one basepoint on that component. In this chapter, though, we will take a slightly different approach to this object: the basic building block will be the combinatorics of the parameterization, instead of the parameterized surface itself.

Definition 4.1.1. A *pointed matched circle* is an oriented circle $\mathcal{Z}$, equipped with a basepoint z on it, and additional $4g$ points coming in pairs (distinct from each other and z) such that performing surgery on all $2g$ pairs results in one circle.

Construction 4.1.2 (surface associated to a pointed matched circle). Given a pointed matched circle $\mathcal{Z}$, we can associate a surface whose boundary is a circle $\mathcal{Z}$, viewing $2g$ pairs of points as feet of 1-handles. Specifically, one has to thicken $\mathcal{Z}$ into a band $\mathcal{Z} \times [0, 1]$, then glue the 1-handles to $\mathcal{Z} \times \{1\}$, and then cap off the boundary component which is not $\mathcal{Z} \times \{0\}$ (see below Figure 4.1 and [LipOzsThu08, Figure 1.1]). We denote this surface by $F^\circ(\mathcal{Z})$, and the orientation on it is induced from the boundary via the usual rule “outward normal first”. The surface $F^\circ(\mathcal{Z})$ is naturally parameterized by the 1-handles, and has a basepoint on the boundary $z \in \partial F^\circ(\mathcal{Z}) = \mathcal{Z}$. Let $F(\mathcal{Z})$ denote the result of filling in a disc $D_{\mathcal{Z}}$ to the boundary component of $F^\circ(\mathcal{Z})$ (so mapping classes of $F^\circ(\mathcal{Z})$ fixing the boundary naturally correspond to mapping classes of $F(\mathcal{Z})$ fixing the disc $D_{\mathcal{Z}}$). Note that any two surfaces specified by the same pointed matched circle are homeomorphic, via a homeomorphism which is uniquely determined up to isotopy.

Example 4.1.3. Below in Figure 4.1 one can see an example of a pointed matched circle in the $g = 2$ case, and its corresponding surface of genus 2. In our computations of mapping class invariant we will be using this pointed matched circle, which we denote by $\mathcal{Z}_{g=2}$. Notice that there are other pointed matched circles for a genus 2 surface, not isomorphic to $\mathcal{Z}_{g=2}$. We could have used them. Thus here we make a particular choice which can be understood as a choice of a parameterization of a surface by specifying the 0-handle (the preferred disc) and the 1-handles.

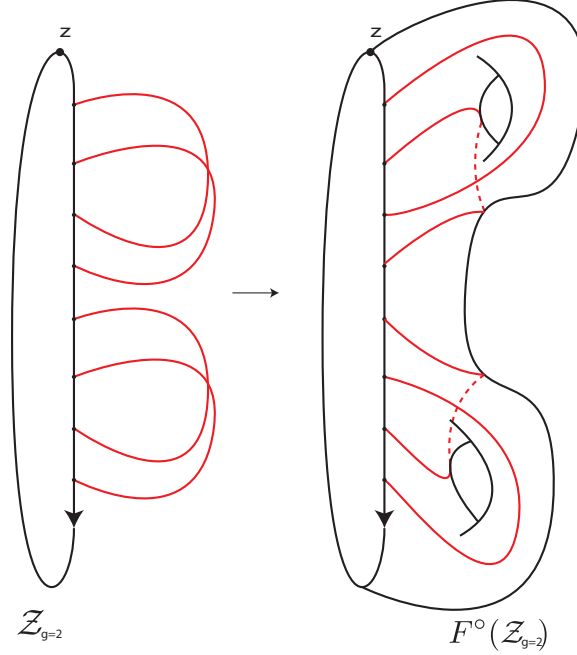


Figure 4.1: An example of a pointed matched circle for $g = 2$, and its corresponding genus 2 surface.

Having a pointed matched circle $\mathcal{Z}$, by $-\mathcal{Z}$ we denote the same circle but with the reversed orientation. The corresponding surface will also be the previous one, but with the reversed orientation: $F^\circ(-\mathcal{Z}) = -F^\circ(\mathcal{Z})$.

Consider now *the genus g strongly based mapping class groupoid*, which is a category where objects are pointed matched circles with $4g$ points, and morphism sets are

$$MCG_0(\mathcal{Z}_L, \mathcal{Z}_R) = \{\phi : F^\circ(\mathcal{Z}_L) \xrightarrow{\cong} F^\circ(\mathcal{Z}_R) \mid \phi(z_L) = z_R\} / \text{isotopy},$$

i.e. isotopy classes of orientation preserving diffeomorphisms respecting the boundary and the basepoint. For any pointed matched circle $\mathcal{Z}$ with $4g$ points the corresponding group of self-diffeomorphisms $MCG_0(\mathcal{Z}, \mathcal{Z}) \cong MCG_0(\Sigma_g, \partial\Sigma_g)$ is the mapping class group of the genus g surface with one boundary component.

Below in Figure 4.2 there is a plan for the rest of the section. If we take $\mathcal{Z}_L = \mathcal{Z}_R$, then we will produce an invariant of a mapping class of a surface. Now we explain the different pieces of this diagram.

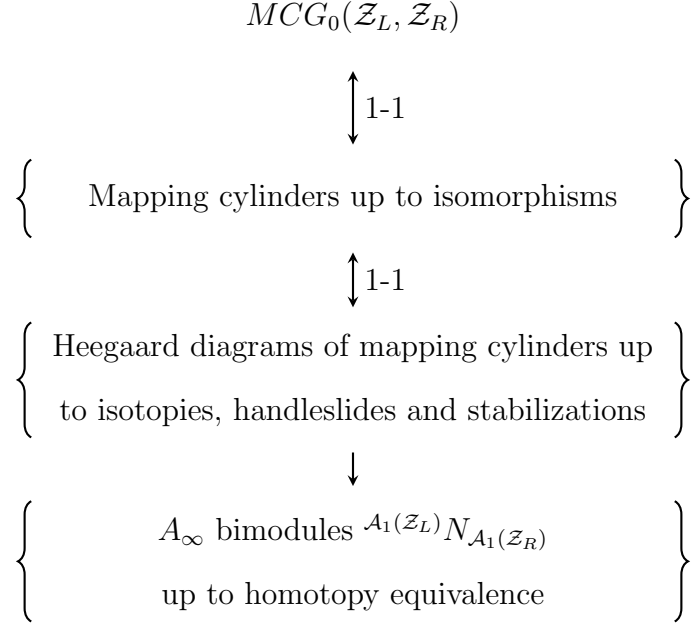


Figure 4.2: Plan for Section 4.1.

4.1.2 Mapping cylinders

We need a notion of *strongly bordered 3-manifold* Y with two boundary components $\partial_1 Y$ and $\partial_2 Y$. It consists of the following data (following [LipOzsThu15, Definition 5.1]):

- (1) a preferred disc and a basepoint (on the boundary of that disc) in each boundary component;
- (2) parameterizations by some fixed surfaces $\psi_i : (F_i, D_i, z_i) \rightarrow \partial_i Y$ of boundaries respecting distinguished discs and basepoints;
- (3) a framed arc connecting basepoints such that the framing on boundaries points into the distinguished discs.

If we fix surfaces F_1 and F_2 by which we parameterize the boundaries of Y , then there is a natural notion of an isomorphism of strongly bordered 3-manifolds — it is a diffeomorphism of the corresponding 3-manifolds, which respects every piece of the

additional data, i.e. parameterizations of boundaries, arcs connecting the basepoints, and their framings.

Having a strongly based mapping class we want to form a corresponding strongly bordered 3-manifold, which we call a mapping cylinder.

Construction 4.1.4 (mapping cylinder). Fix pointed matched circles $\mathcal{Z}_L$, $\mathcal{Z}_R$ and a mapping class $\phi : (F(\mathcal{Z}_L), D_L, z_L) \rightarrow (F(\mathcal{Z}_R), D_R, z_R)$. We can form a *mapping cylinder* $M_\phi =_\phi ([0, 1] \times F(\mathcal{Z}_R))_{Id}$, which is a strongly bordered 3-manifold $Y = [0, 1] \times F(\mathcal{Z}_R)$ with the following data:

- (1) a parametrization of its boundary given by

$$\psi_L = -\phi : -F(\mathcal{Z}_L) \rightarrow \partial_L Y = -F(\mathcal{Z}_R) = \{0\} \times -F(\mathcal{Z}_R) \text{ (note the twisting by } \phi)$$

$$\text{and } \psi_R = Id : F(\mathcal{Z}_R) \rightarrow \partial_R Y = F(\mathcal{Z}_R) = \{1\} \times F(\mathcal{Z}_R);$$

- (2) two distinguished discs $\{0\} \times D_R$ in $\partial_L Y$ and $\{1\} \times D_R$ in $\partial_R Y$;

- (3) a framed path $\gamma_z = [0, 1] \times \{z_R\}$ between $z_L \in \partial_L Y$ and $z_R \in \partial_R Y$ such that the framing points into the distinguished discs D_R at every fiber $\{t\} \times F(\mathcal{Z}_R)$.

See Figure 4.3.

The following lemma, the proof of which can be found in [LipOzsThu15, Lemma 5.29], allows us to talk about mapping cylinders instead of mapping classes (and vice versa), i.e. it explains the first correspondence in Figure 4.2.

Lemma 4.1.5. *Fix pointed matched circles $\mathcal{Z}_L$ and $\mathcal{Z}_R$. Then any strongly bordered 3-manifold Y , whose boundary is parameterized by $F(\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$, and whose underlying space can be identified with a product of a surface with an interval (so that arc γ_z is identified with the product of a point with the interval, respecting the framing) is of the form M_ϕ for some choice of strongly based mapping class $\phi : F^\circ(\mathcal{Z}_L) \rightarrow F^\circ(\mathcal{Z}_R)$. Moreover, two such strongly bordered three-manifolds are isomorphic if and only if they represent the same strongly based mapping class.*

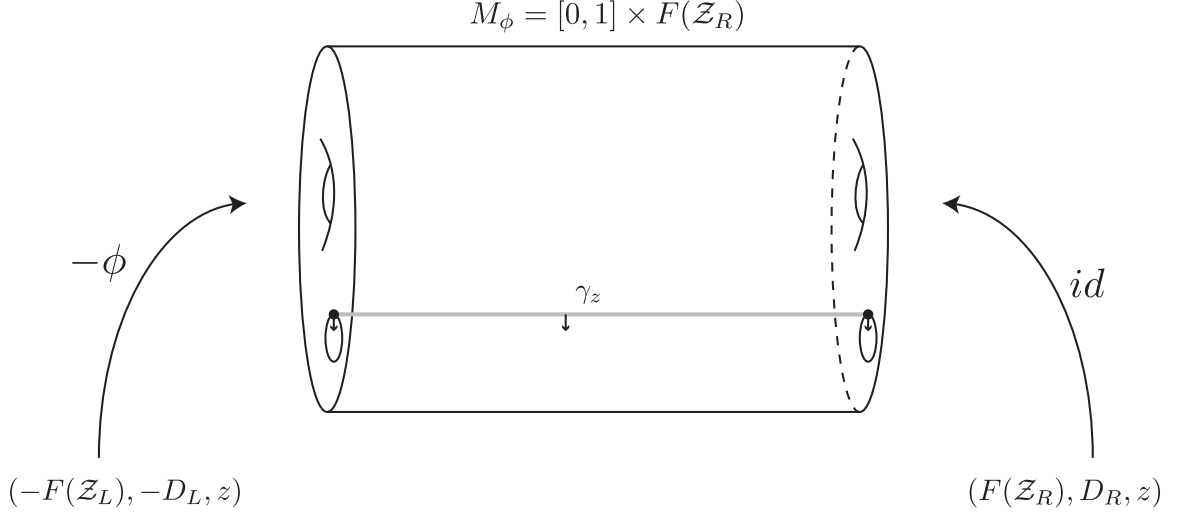


Figure 4.3: Mapping cylinder of $\phi : (F(\mathcal{Z}_L), D_L, z_L) \rightarrow (F(\mathcal{Z}_R), D_R, z_R)$.

4.1.3 Heegaard diagrams

Now, having constructed the mapping cylinder M_ϕ , we would like to have a 2-dimensional presentation of it. Following [LipOzsThu15, Definition 5.4], we introduce the following object.

Definition 4.1.6. An *arc bordered Heegaard diagram with two boundary components* is a quadruple $(\bar{\Sigma}, \bar{\alpha}, \beta, \mathbf{z})$ where

- $\bar{\Sigma}_g$ is an oriented compact surface of genus g with two boundary components, $\partial_L \bar{\Sigma}$ and $\partial_R \bar{\Sigma}$;
- $\bar{\alpha} = \{\bar{\alpha}_1^{arc, left}, \dots, \bar{\alpha}_{2l}^{arc, left}, \bar{\alpha}_1^{arc, right}, \dots, \bar{\alpha}_{2r}^{arc, right}, \dots, \alpha_1^{curve}, \dots, \alpha_{g-l-r}^{curve}\}$ is a collection of pairwise disjoint $2l$ embedded arcs with boundaries on $\partial_L \bar{\Sigma}$, $2r$ embedded arcs with boundaries on $\partial_R \bar{\Sigma}$, and $g - l - r$ circles in the interior (in particular $g \geq l + r$);
- $\beta = \{\beta_1, \dots, \beta_g\}$ is a g -tuple of pairwise disjoint curves in the interior of $\bar{\Sigma}$;
- $\mathbf{z}$ is a path in $\bar{\Sigma} \setminus (\bar{\alpha} \cup \beta)$ between $\partial_L \bar{\Sigma}$ and $\partial_R \bar{\Sigma}$;

These are required to satisfy:

- $\bar{\Sigma} \setminus \bar{\alpha}$ and $\bar{\Sigma} \setminus \beta$ are connected;
- $\bar{\alpha}$ intersect β transversely.

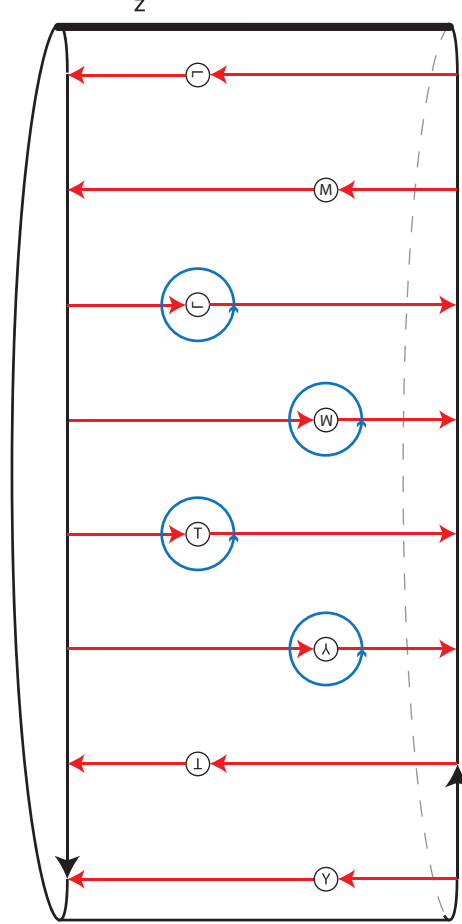


Figure 4.4: A Heegaard diagram for M_{id} , where $id : F^\circ(\mathcal{Z}_{g=2}) \rightarrow F^\circ(\mathcal{Z}_{g=2})$ is the identity mapping class of the genus 2 surface. The pointed matched circle $\mathcal{Z}_{g=2}$ is the one from Figure 4.1. We also indicate here the orientations of the α and β curves, because later they will give $\mathbb{Z}_2$ -grading on Hochschild homology.

Notice that two boundaries of any Heegaard diagram specify two pointed matched circles. In Figure 4.4 one can see an example of a Heegaard diagram of the mapping cylinder of $id : \Sigma_{g=2} \rightarrow \Sigma_{g=2}$.

The following proposition, which follows from [LipOzsThu15, Propositions 5.10, 5.11], provides the second correspondence in Figure 4.2.

Proposition 4.1.7. *Any arced bordered Heegaard diagram with two boundary components gives rise to a strongly bordered 3-manifold. For the other direction, suppose a strongly bordered 3-manifold has a boundary parameterized by $F(\mathcal{Z}_1)$ and $F(\mathcal{Z}_2)$. Then this 3-manifold has an arced bordered Heegaard diagram, boundary pointed matched circles of which are $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Diffeomorphism type of this Heegaard diagram is unique up to the following moves:*

- *Isotopies of α - and β - curves and arcs;*
- *Handleslides among the α -circles and among the β -circles;*
- *Handleslides of an α -arc over an α -circle;*
- *Stabilization of the diagram.*

The procedure of getting a strongly bordered 3-manifold from an arced bordered Heegaard diagram consists of thickening the surface, attaching 2-handles along the circles (along the β -circles from one side, and along the α -circles from the other side), and then carefully analyzing what happens on the α side of the boundary. There one has two surfaces of genera l and r with parameterizations coming from the Heegaard diagram, which are connected by an annulus, with the path $\mathbf{z}$ in it. This annulus, along with the path $\mathbf{z}$ in it, specify the framed arc $\gamma_{\mathbf{z}}$ by which the two boundary surfaces are connected in the definition of a strongly bordered 3-manifold. The existence of a Heegaard diagram (and the uniqueness up to the set of moves) follows from Morse theory. See [LipOzsThu08, Proposition 4.10] for the proof.

4.1.4 Bimodules

The invariant which we are going to investigate was defined in [LipOzsThu15] and then subsequently studied in [LipOzsThu13]. Here we give a classical definition of the bimodule from the original paper [LipOzsThu15], whereas in [LipOzsThu13] the

authors took another approach, which is similar in spirit to the definition we give in Section 4.4.2. See also [Sie] for the combinatorial geometric proof of correctness of the bimodule definition from [LipOzsThu13].

First, we need to specify the algebra.

Definition 4.1.8 (*dg-algebra associated to a pointed matched circle*). To a pointed matched circle $\mathcal{Z}$ one can associate a *dg-algebra* $\mathcal{A}_1(\mathcal{Z}) := \mathcal{A}_1(F^\circ(\mathcal{Z}))$ via Definition 2.4.3, keeping in mind that the surface $F^\circ(\mathcal{Z})$ is naturally marked with a basepoint and parameterized by 1-handles. The algebra $\mathcal{A}_1(\mathcal{Z})$ is denoted by $\mathcal{A}(\mathcal{Z}, -g + 1)$ in [LipOzsThu15] (i.e. we have only one strand moving). Below in Figure 4.5 there is an example of a pointed matched circle and the corresponding algebra in the genus 2 case.

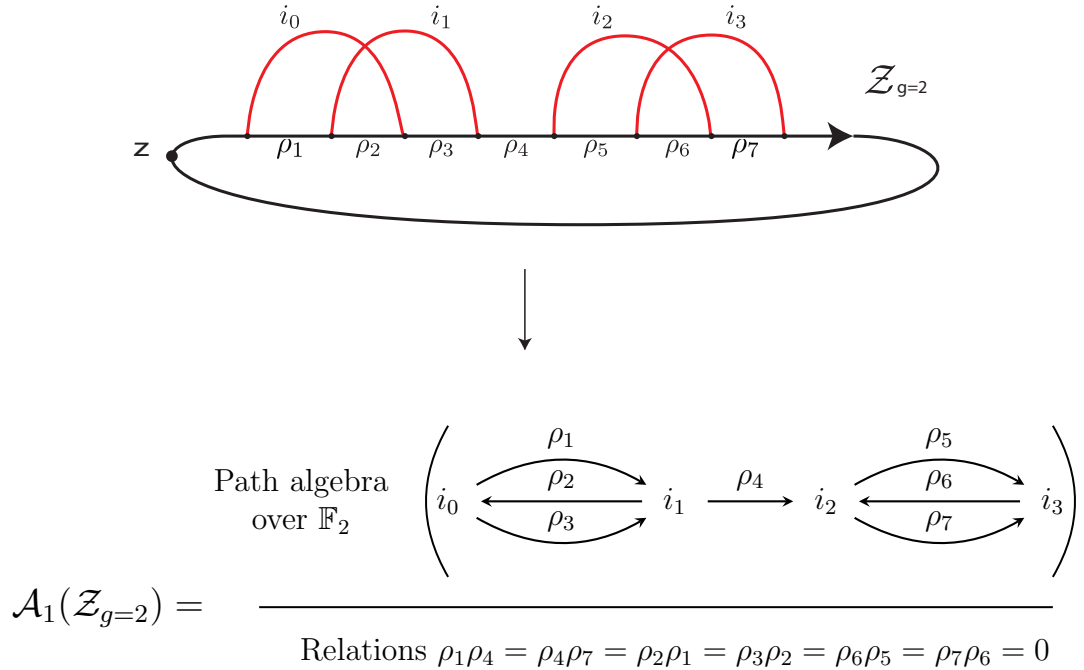


Figure 4.5: A genus 2 example of how to get a *dg-algebra* from a pointed matched circle.

Now we turn to the construction of the bimodule, following [LipOzsThu15, Section 6.3]. The notation $N(\phi)$ is taken from [LipOzsThu13], and in [LipOzsThu15] this

bimodule was denote by $\widehat{CFDA}(\phi, -g+1) \subset \widehat{CFDA}$, see the subsection “Other type of bimodules” below for the explanation of the notation.

Construction 4.1.9 (A_∞ bimodule associated to a mapping class). Fix a surface with one boundary component $(\Sigma, \partial\Sigma = S^1) = F^\circ(\mathcal{Z})$. A mapping class $\phi \in MCG_0(\Sigma, \partial\Sigma) = MCG_0(\mathcal{Z}, \mathcal{Z})$ gives rise to a strongly bordered 3-manifold M_ϕ , the mapping cylinder of ϕ . After this we can consider a Heegaard diagram $\mathcal{H}(M_\phi)$ representing M_ϕ , with pointed matched circles $\mathcal{Z}$ and $\mathcal{Z}$ on the boundaries. To such a Heegaard diagram one can associate an A_∞ bimodule ${}^{\mathcal{A}_1(\mathcal{Z})}N(\phi)_{\mathcal{A}_1(\mathcal{Z})}$ of DA type, over the algebra $\mathcal{A}_1(\mathcal{Z})$ from both sides.

The generators of the underlying $\mathbb{F}_2$ -vector space of the bimodule consist of tuples $\mathbf{x}$ of intersections between α - and β - curves such that every α and β circle gets 1 point, only one α -arc on the right boundary gets 1 point (denote this α -arc by $\alpha_{\mathbf{x}}^{arc, right}$), and all except one α -arc on the left boundary get 1 point (denote this α -arc by $\alpha_{\mathbf{x}}^{arc, left}$). See an example in Figure 4.11, where we marked all the generators on a Heegaard diagram.

The idempotent subalgebra $\mathcal{I}$ of $\mathcal{A}_1(\mathcal{Z})$ acts on these generators in the following way. Each α -arc in $\mathcal{H}(M_\phi)$ is comes from a matched pair in $\mathcal{Z}$, and thus has the corresponding idempotent in $\mathcal{A}_1(\mathcal{Z})$, which we denote by $i(\alpha^{arc})$. For a generator $\mathbf{x}$, we have actions $i(\alpha_{\mathbf{x}}^{arc, left}) \cdot \mathbf{x} = \mathbf{x}$, $\mathbf{x} \cdot i(\alpha_{\mathbf{x}}^{arc, right}) = \mathbf{x}$, while other idempotent actions are zero.

Other $\mathcal{I}$ -linear A_∞ actions of DA type $\delta_{1+k}^1 : N \otimes_{\mathcal{I}} \mathcal{A}_1(\mathcal{Z})^{\otimes k} \rightarrow \mathcal{A}_1(\mathcal{Z}) \otimes_{\mathcal{I}} N$ are defined using counting of pseudo-holomorphic curves. Along the way of the definition, one has to make an analytic choice of a family of almost complex structures on some space. If one makes a different analytic choice, or chooses a different Heegaard diagram for M_ϕ , the resulting bimodule will be the same up to homotopy equivalence. For the definitions and the proof of invariance see [LipOzsThu15, Section 6.3] and references there.

This finishes the explanation of how to a mapping class $\phi \in MCG_0(\Sigma, \partial\Sigma = S^1) = MCG_0(\mathcal{Z}, \mathcal{Z})$ one can associate an A_∞ homotopy equivalence class of bimodules ${}^{\mathcal{A}_1(\mathcal{Z})}N(\phi)_{\mathcal{A}_1(\mathcal{Z})}$. Along the way one makes a choice of a particular pointed matched circle $\mathcal{Z}$, and an identification $(\Sigma, \partial\Sigma)$ with a parameterized surface $F^\circ(\mathcal{Z})$. It turns out that for us this choice is not important. Mapping class groups with different parameterizations of the surface $(\Sigma, \partial\Sigma) \cong_1 F^\circ(\mathcal{Z})$ and $(\Sigma, \partial\Sigma) \cong_2 F^\circ(\mathcal{Z}')$ (note that if pointed matched circles are equal, it does not mean that the identifications are the same, for example they can differ by a Dehn twist) can be bijectively identified via conjugation by some element $a \in MCG_0(F^\circ(\mathcal{Z}), F^\circ(\mathcal{Z}'))$. Conveniently, the bimodules are also related: $N(\phi) \simeq N(a^{-1}) \boxtimes N(\psi) \boxtimes N(a)$ for $\phi = a\psi a^{-1} \in MCG_0(\mathcal{Z}, \mathcal{Z})$ and $\psi \in MCG_0(\mathcal{Z}', \mathcal{Z}')$, see the next paragraph for the explanation of this formula. Moreover, the main invariant for us is going to be the Hochschild homology of a bimodule, which is invariant w.r.t. conjugation of a mapping class.

An important feature of this mapping class invariant is that there is an operation which corresponds to multiplication in the mapping class group. Namely, if we have a mapping class $\phi = \phi_1 \circ \phi_2 = \phi_1 \phi_2$, then $M_{\phi_1 \phi_2} \cong M_{\phi_2 \partial_R} \cup_{\partial_L} M_{\phi_1}$, and for Heegaard diagrams we also have $\mathcal{H}_{\phi_1 \phi_2} \cong \mathcal{H}_{\phi_2 \partial_R} \cup_{\partial_L} \mathcal{H}_{\phi_1}$. The main theorem in bordered Heegaard Floer theory, the pairing theorem [LipOzsThu15, Theorem 12], implies that the corresponding operation for the bimodules is the box tensor product, and we have the following A_∞ homotopy equivalence of bimodules:

$${}^{\mathcal{A}_1(\mathcal{Z})}N(\phi_1 \phi_2)_{\mathcal{A}_1(\mathcal{Z})} \simeq {}^{\mathcal{A}_1(\mathcal{Z})}N(\phi_2)_{\mathcal{A}_1(\mathcal{Z})} \boxtimes {}^{\mathcal{A}_1(\mathcal{Z})}N(\phi_1)_{\mathcal{A}_1(\mathcal{Z})}. \quad (4.1)$$

Other types of bimodules

Our algebra is a direct summand of a bigger algebra $\mathcal{A}(\mathcal{Z}) = \bigoplus_{-g \leq k \leq g} \mathcal{A}(\mathcal{Z}, k)$, namely $\mathcal{A}_1(\mathcal{Z}) = \mathcal{A}(\mathcal{Z}, -g + 1)$. See [LipOzsThu08, Section 3] for the definition of $\mathcal{A}(\mathcal{Z})$. Denote the algebra $\mathcal{A}(\mathcal{Z}, g - 1)$ by $\mathcal{A}_1^\dagger(\mathcal{Z})$. It is Koszul to $\mathcal{A}_1(\mathcal{Z})$, and in general

we have Koszul duality of $\mathcal{A}^\dagger(\mathcal{Z}, -l) := \mathcal{A}(\mathcal{Z}, l)$ and $\mathcal{A}(\mathcal{Z}, -l)$, see [LipOzsThu11, Section 8] for that.

To a Heegaard diagram $\mathcal{H}(M_\phi)$ one can associate not only a DA type A_∞ bimodule ${}^{\mathcal{A}_1(\mathcal{Z})}N(\phi)_{\mathcal{A}_1(\mathcal{Z})} = {}^{\mathcal{A}_1(\mathcal{Z})}\widehat{CFDA}(\phi, -g+1)_{\mathcal{A}_1(\mathcal{Z})}$ (we will be dropping the index $-g+1$ later for this bimodule), but also DD and AA type bimodules ${}^{\mathcal{A}_1(\mathcal{Z})}\widehat{CFDD}(\phi)_{\mathcal{A}_1^\dagger(\mathcal{Z})}$ and ${}_{\mathcal{A}_1^\dagger(\mathcal{Z})}\widehat{CFAA}(\phi)_{\mathcal{A}_1(\mathcal{Z})}$ (note the changes of the algebra to its Koszul dual), see [LipOzsThu15, Section 6] for the definitions. The sets of generators for these three bimodules are the same, but the A_∞ actions are different. Note that all these three bimodules are direct summands of bigger bimodules ${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z})}$, ${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}(\phi)_{\mathcal{A}(\mathcal{Z})}$, and ${}_{\mathcal{A}(\mathcal{Z})}\widehat{CFAA}(\phi)_{\mathcal{A}(\mathcal{Z})}$, over the algebras $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}^\dagger(\mathcal{Z}) = \mathcal{A}(\mathcal{Z})$. These summands are characterized by the number of arcs generators occupy on the left and right side of a Heegaard diagram — for us it is $2g-1$ on the left, and 1 on the right. Equivalently, one can say that these generators are in such $spin^c$ structures of M_ϕ , that its Chern class evaluates to $-2g+2$ on the boundary surfaces of M_ϕ .

Hochschild homology

In order to relate the bimodule $N(\phi)$ to fixed point Floer cohomology, one needs to apply an algebraic operation to the bimodule, which is called Hochschild homology. It is a homology theory for A_∞ bimodules over the same algebra from both sides, which is obtained by a self-tensoring procedure. We refer to [LipOzsThu15, Section 2.3.5] for the algebraic definitions, basic properties, and a way to compute the Hochschild homology for bounded DA type bimodules.

There are two important points about this algebraic structure. First, Hochschild homology depends only on the A_∞ homotopy equivalence class of the bimodule. Thus, taking $HH_*({}^{\mathcal{A}_1(\mathcal{Z})}N(\phi)_{\mathcal{A}_1(\mathcal{Z})})$ would give us an invariant of a mapping class. Second, $HH_*({}^{\mathcal{A}_1(\mathcal{Z})}N(\phi)_{\mathcal{A}_1(\mathcal{Z})})$ is in fact naturally identified with a knot Floer homology of the binding of the open book M_ϕ° with a monodromy $\phi : (\Sigma, \partial\Sigma) \rightarrow (\Sigma, \partial\Sigma)$:

$$HH_*(N(\phi)) \cong \widehat{HFK}(M_\phi^\circ, K ; -g + 1).$$

Following [LipOzsThu15, Section 5], this open book is obtained from M_ϕ by identifying the two boundaries (thus the path γ_z becomes a circle) and then doing a surgery on γ_z with respect to the framing we had in Construction 4.1.4 of M_ϕ .

The construction of a Heegaard diagram for this open book is as follows. First take an arced bordered Heegaard diagram $\mathcal{H}(M_\phi)$ for the mapping cylinder M_ϕ , and glue the left boundary to the right one (on the algebraic level this corresponds to self-tensoring the bimodule). Then do a surgery on the arc z (which became a circle after self-gluing). Note that in order to block those discs which we did not count, one needs to place two basepoints on the two sides after surgery — these two basepoints specify a knot, which is the binding of the open book.

Up to the self-pairing procedure, this explains why $\widehat{HFK}(M_\phi^\circ, K ; -g + 1) \cong HH_*(N(\phi))$, see [LipOzsThu15, Theorem 14] for the details. Because we work in the second lowest $spin^c$ structure, we only get the Alexander grading $-g + 1$ summand.

Remark 4.1.10. This implies that $HH_*(N(\phi))$ is invariant w.r.t. conjugation of ϕ .

Using this interpretation of the Hochschild homology, we can endow it with a $\mathbb{Z}_2$ -grading — it corresponds to the sign of intersections of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in the definition of generators of knot Floer homology. Because the Heegaard diagram for the open book is constructed via self-gluing, the $\mathbb{Z}_2$ -grading, i.e. the choice of orientations on $\mathbb{T}_\alpha, \mathbb{T}_\beta$, amounts to the choice of orientations on β and α curves on the Heegaard diagram for M_ϕ (s.t. they are consistent, i.e. after gluing the orientations of the left arcs match with the orientations of the right arcs). Let us choose these orientations as in Figure 4.4. To obtain a consistent $\mathbb{Z}_2$ -grading on $HH_*(N(\phi))$ in general case for $\phi : MCG_0(\Sigma_g, \partial\Sigma_g)$, one needs to make such choices of orientations in the standard Heegaard diagram for M_ϕ (see [LipOzsThu15, Section 5.3] or the next subsection 4.1.5

for a construction of the standard $\mathcal{H}(M_\phi)$, so that the gradings of all $2g$ generators of $HH_*(N(id))$ are zero.

Cancellation

Let us finish this section by describing what cancellation is. Suppose there are two generators ${}_i x_j$ and ${}_i y_j$ (the subscripts indicate their left and right idempotents) in a DA bimodule P such that the only action between them is $\delta_1^1({}_i x_j) = i \otimes_i y_j$ (an example of such two generators is x_2 and t_{12} in Figure 4.14). Then one can cancel these two generators, i.e. erase x and y and the arrows involving them from the bimodule, and then add some other arrows between the generators left in the bimodule, guided by a certain cancellation rule. The outcome is a bimodule P' with less generators, but which is homotopy equivalent to the previous one, $P' \simeq P$. See [Zha16, Section 3.1] for the details of how cancellation works.

4.1.5 Computations

In this section we compute the DA bimodules for mapping classes of a genus 2 surface. We also describe an algorithm for computing the Hochschild homology.

First, fix a genus 2 surface $\Sigma_{g=2}$ with one boundary component, and a set of curves on it as in Figure 4.6.

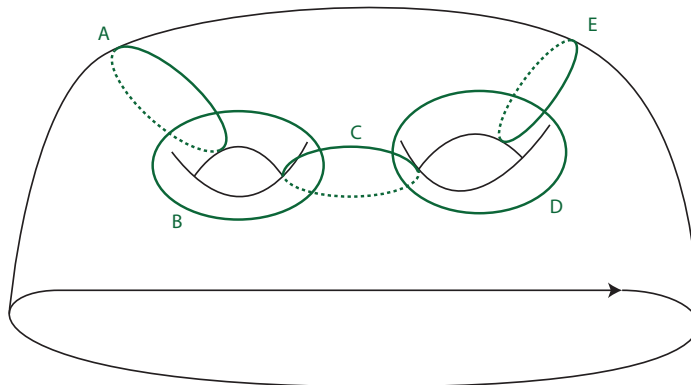


Figure 4.6: Dehn twists along these curves generate $MCG_0(\Sigma_{g=2})$.

The following is a presentation of the mapping class group of the genus 2 surface with one boundary component (see [Waj99, Theorem 2]):

$$\begin{aligned} MCG_0(\Sigma_{g=2}, \partial\Sigma_{g=2}) = < \tau_A, \tau_B, \tau_C, \tau_D, \tau_E \mid \text{commuting relations, braid relations,} \\ (\tau_E \tau_D \tau_C \tau_B)^5 = \tau_A \tau_B \tau_C \tau_D \tau_E \tau_E \tau_D \tau_C \tau_B \tau_A >, \end{aligned}$$

where τ_l is a right handed Dehn twist around the curve l . By commuting relations we mean that if curves l_1 and l_2 do not intersect, then $\tau_{l_1} \tau_{l_2} = \tau_{l_2} \tau_{l_1}$. By braid relations we mean that if curves l_1 and l_2 intersect at a single point transversely, then $\tau_{l_2} \tau_{l_1} \tau_{l_2} = \tau_{l_1} \tau_{l_2} \tau_{l_1}$.

Now we explain how to compute the bimodule invariant for a mapping class of the genus 2 surface. Every mapping class can be represented by a product of Dehn twists $\tau_A, \tau_B, \tau_C, \tau_D, \tau_E$ (or their inverses). We also know how the bimodule invariants behave with respect to composition of mapping classes: the corresponding operation is the box tensor product. Thus it is enough to compute $N(\phi)$ only for these Dehn twists and their inverses (10 bimodules in total).

Remark 4.1.11. Notice, that after this computation is done, one can redefine the bimodule invariant for ϕ in a combinatorial way. Namely, for a factorization of ϕ into Dehn twists above, let us assume it is $\phi = \tau_A \tau_C \tau_E^{-1}$ for example, one associates a tensor product of the corresponding bimodules $N(\tau_E^{-1}) \boxtimes N(\tau_C) \boxtimes N(\tau_A)$. To make sure this definition is correct, one needs to check the mapping class group relations. For example for a relation $\tau_A \tau_B \tau_A = \tau_B \tau_A \tau_B$ one needs to check the following homotopy equivalence: $N(\tau_A) \boxtimes N(\tau_B) \boxtimes N(\tau_A) \simeq N(\tau_B) \boxtimes N(\tau_A) \boxtimes N(\tau_B)$. After we wrote down the 10 bimodules, we checked the relations using a computer program [Kot18], and indeed they are satisfied. The homotopy equivalence actually always came as an isomorphism of bimodules after all possible cancellations were made.

This is a good strategy to assign Floer theoretic invariants to topological objects

without referring to pseudo-holomorphic theory. See the paper of Zhan [Zha16] for a combinatorial definition of $\widehat{CFDA}(\phi, 0)$ (it is an analogue of our invariant, where generators occupy g arcs on the left and g arcs on the right boundary of the Heegaard diagram). There he uses arc-slides (as opposed to Dehn twists) as generators of the mapping class groupoid. Moreover, using this combinatorial definition for $\widehat{CFDA}(\phi)$, he then defines combinatorially the “hat” version of Heegaard Floer homology of a 3-manifold $\widehat{HF}(Y^3)$.

First we need to fix a parameterization of our surface $(\Sigma_{g=2}, \partial\Sigma_{g=2}) \cong F^\circ(\mathcal{Z}_{g=2})$. This will specify a dq -algebra. We use a pointed matched circle $\mathcal{Z}_{g=2}$ and its corresponding algebra $\mathcal{A}_1(\mathcal{Z}_{g=2})$ from Figure 4.5. For an identification see Figure 4.7.

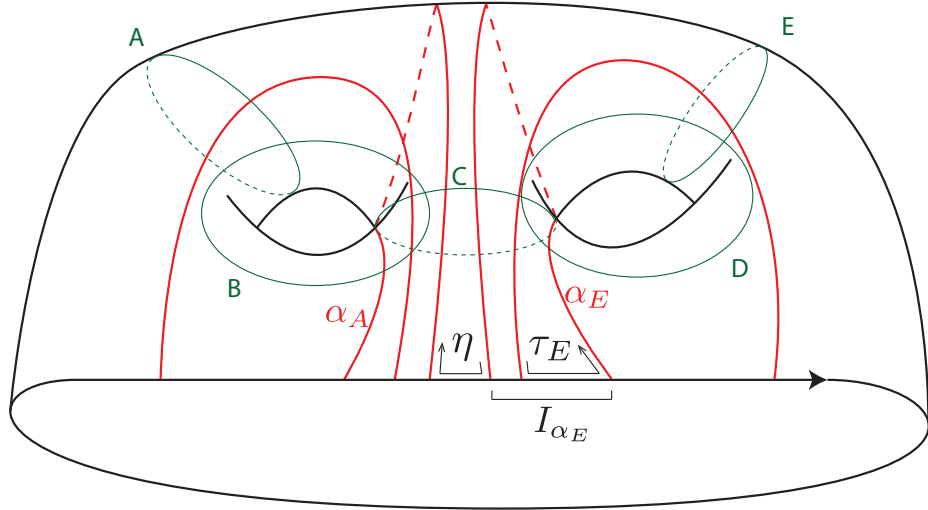


Figure 4.7: Parameterization of the surface $\Sigma_{g=2} \cong F^\circ(\mathcal{Z}_{g=2})$.

For every Dehn twist $\tau_l \in MCG_0(\Sigma_{g=2}, \Sigma_{g=2})$ we need to specify a Heegaard diagram for a mapping cylinder M_{τ_l} . Following [LipOzsThu15, Section 5.3], first of all, consider the standard Heegaard diagram $\mathcal{H}(M_{id})$ for $id : F^\circ(\mathcal{Z}_{g=2}) \rightarrow F^\circ(\mathcal{Z}_{g=2})$, see Figure 4.8. There is a shaded region of the diagram on the right that is identified with the right boundary $F^\circ(\mathcal{Z}_{g=2}) \setminus D^2$ of the mapping cylinder. Analogously there is a shaded region on the left part of the diagram which is identified with $-(F^\circ(\mathcal{Z}_{g=2}) \setminus D^2)$. There are also curves A, B, C, D and E on both of these surfaces, via the specified

above identification $F^\circ(\mathcal{Z}_{g=2}) \cong \Sigma_{g=2}$.

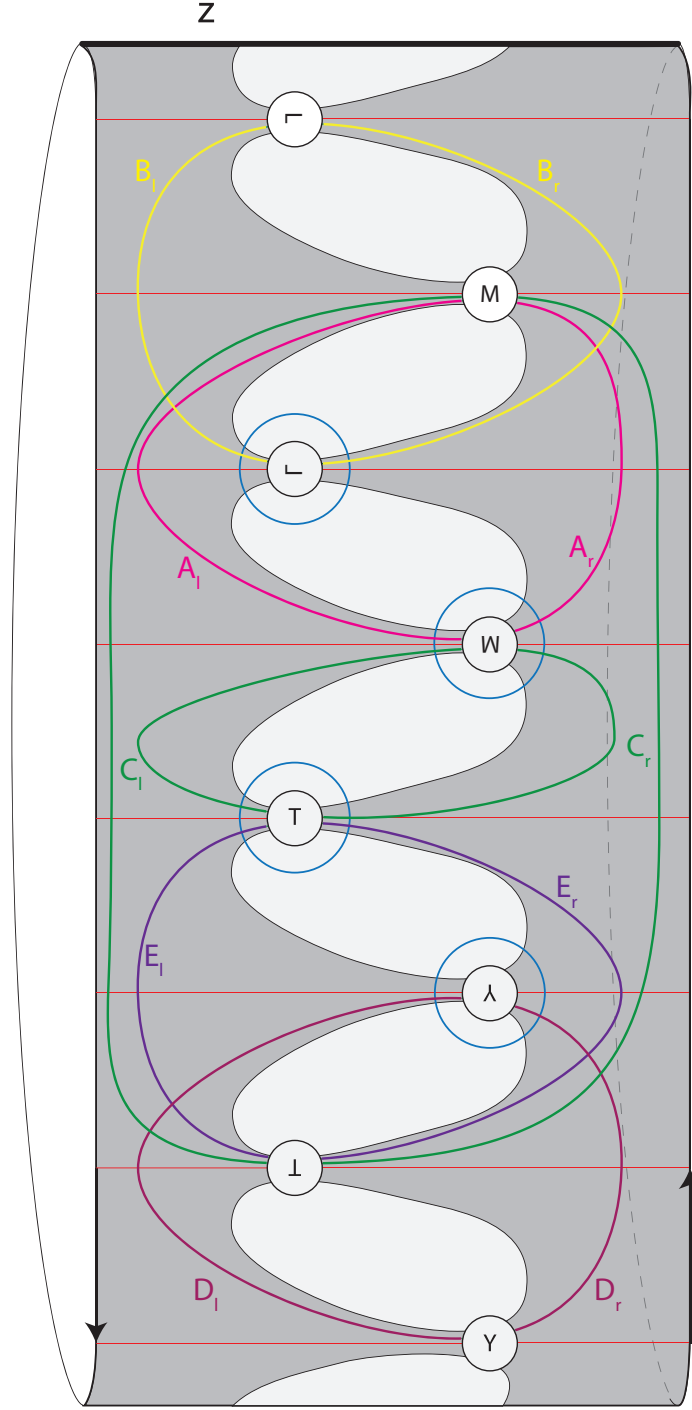


Figure 4.8: Heegaard diagram $\mathcal{H}(M_{id})$ of identity mapping class with curves over which we do the Dehn twists.

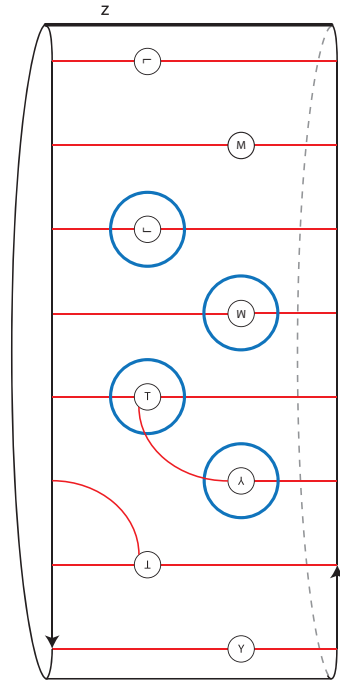
Now, suppose we want to draw a Heegaard diagram for M_{τ_E} . Then it is enough to change all the alpha arcs of the left side of $\mathcal{H}(M_{id})$ by applying τ_E . This corresponds to

a parameterization $-\tau_E : -F^\circ(\mathcal{Z}_{g=2}) \rightarrow \partial_L M_{\tau_E}$. So we apply the right handed Dehn twist τ_{E_l} to the alpha arcs. Alternatively one can apply $\tau_{E_l}^{-1}$ to all the beta curves (this corresponds to applying self-diffeomorphism $\tau_{E_l}^{-1}$ to the Heegaard diagram). We also could have applied τ_{E_r} to all the alpha arcs and curves, or $\tau_{E_r}^{-1}$ to all the beta curves. All these possibilities are depicted in Figure 4.9. One can see that all these 4 diagrams are equivalent up to the equivalence moves from Proposition 4.1.7 and self-diffeomorphisms applied to the diagrams. The resulting Heegaard diagrams here are analogous to the ones for the genus 1 case in [LipOzsThu15, Section 10.2].

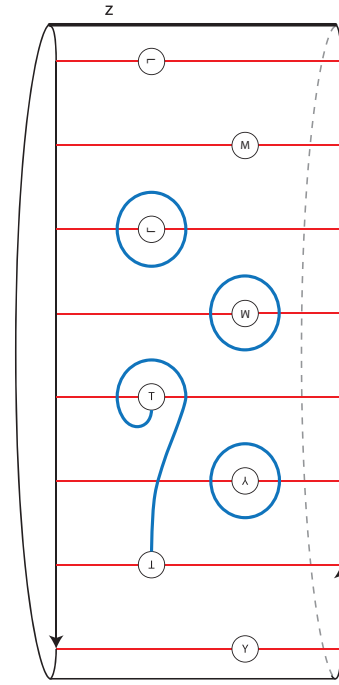
Remark 4.1.12. The orientation convention (essentially the signs of the Dehn twists on the diagrams) is chosen so that the map ϕ goes “from left to right” on the mapping cylinder and the Heegaard diagram, see [LipOzsThu11, Appendix A]. This ensures the desired behavior with respect to gluing: $\mathcal{H}_{\phi_1\phi_2} \cong \mathcal{H}_{\phi_2\partial_R} \cup_{\partial_L} \mathcal{H}_{\phi_1}$.

We will compute the bimodule $N(\tau_E)$ via the second type of Heegaard diagram for τ_E , and we will compute $N(\tau_C^{-1})$ via the third type of diagram for τ_C^{-1} . All other 8 bimodules can be computed analogously: for $\tau_E^{-1}, \tau_D, \tau_D^{-1}$ the bimodules can be computed very similarly to the τ_E case, and the other five bimodule invariants for $\tau_A^{-1}, \tau_A, \tau_B^{-1}, \tau_B, \tau_C$ can be deduced from the previous five by using a reflection of the diagrams about x -axis. We list all ten bimodules at the end in the appendix. For a general mapping class one first factorizes it into Dehn twists, and then box tensor multiplies all the bimodules for these Dehn twists.

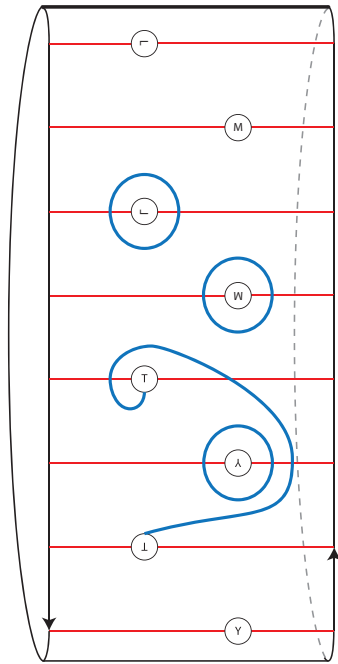
Computation 4.1.13 ($N(\tau_E)$). In Figure 4.11 we draw the Heegaard diagram $\mathcal{H}(M_{\tau_E})$ along with the marked generators of the bimodule, and also the idempotents corresponding to the arcs. Notice that we reversed the orientation of the boundary on the left, because it corresponds to the D side of the bimodule, which is over $\mathcal{A}_1(-(-\mathcal{Z}_{g=2})) = \mathcal{A}_1(\mathcal{Z}_{g=2})$. In Figure 4.12 we draw the candidate DA bimodule. The subscripts of generators of the DA bimodule represent the underlying left and right idempotents. The arrow between generators $\mathbf{x}$ and $\mathbf{y}$ with the label $a \otimes (b, c)$



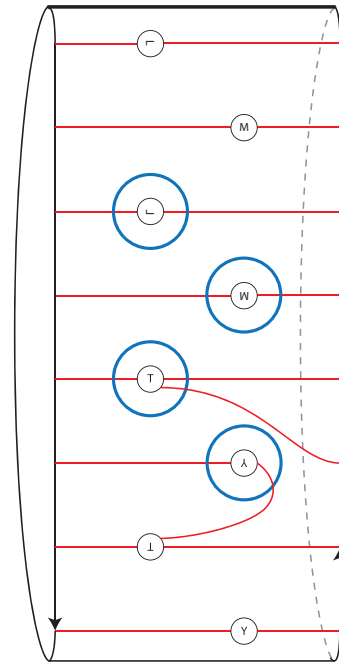
(a) 1st type.



(b) 2nd type.



(c) 3rd type.



(d) 4th type.

Figure 4.9: Four Heegaard diagrams for a Dehn twist along the curve E.

means that there is a DA type action $\mathbf{x} \otimes b \otimes c \rightarrow a \otimes \mathbf{y}$. If there is 1 in the label on the right, it means that there is no incoming algebra elements in that action. If there is 1 in the label on the left, it means that the corresponding action's outgoing algebra element is an idempotent. The notation refers to the fact that the sum of the idempotents in the algebra is a unit.

Let us denote for a moment our candidate bimodule by $N'(\tau_E)$, and the bimodule $N(\tau_E)$ will be the one which corresponds to the Heegaard diagram in Figure 4.11. Thus we want to prove that $N'(\tau_E) \simeq N(\tau_E)$. There are two ways to do it. One is to use directly the definition of A_∞ actions via pseudo-holomorphic curves. In the genus 1 case such computations were done in [LipOzsThu15, Section 10.2], and it is possible to generalize them to compute $N(\tau_E)$. However, it is more difficult to do this for $N(\tau_C^{-1})$, which is our next computation. Thus we choose another approach, which we will also use to compute $N(\tau_C^{-1})$.

Before proceeding to the proof of $N'(\tau_E) \simeq N(\tau_E)$, let us describe the necessary background material.

Just as the behavior of DA bimodules with respect to composition of mapping classes (4.1), the following homotopy equivalences follow from the pairing theorem [LipOzsThu15, Theorem 12] (we use here the notation $\widehat{CFDA}(\phi)$ instead of $N(\phi)$ just to emphasize that D sides are paired with A sides) :

$$\begin{aligned} \mathcal{A}_1(\mathcal{Z}) \widehat{CFDD}(\phi)^{\mathcal{A}_1^\dagger(\mathcal{Z})} \boxtimes_{\mathcal{A}_1^\dagger(\mathcal{Z})} \widehat{CFAA}(\psi)_{\mathcal{A}_1(\mathcal{Z})} &\simeq^{\mathcal{A}_1(\mathcal{Z})} \widehat{CFDA}(\psi\phi)_{\mathcal{A}_1(\mathcal{Z})}, \\ \mathcal{A}_1(\mathcal{Z}) \widehat{CFDA}(\phi)_{\mathcal{A}_1(\mathcal{Z})} \boxtimes^{\mathcal{A}_1(\mathcal{Z})} \widehat{CFDD}(\psi)^{\mathcal{A}_1^\dagger(\mathcal{Z})} &\simeq^{\mathcal{A}_1(\mathcal{Z})} \widehat{CFDD}(\psi\phi)^{\mathcal{A}_1^\dagger(\mathcal{Z})}. \end{aligned}$$

Arc-slides are generators of the mapping class groupoid, see [LipOzsThu14, Figure 3] for a definition of an arc-slide. In particular, they generate Dehn twists, and there is a standard way to decompose a Dehn twist $\tau_l : (\Sigma, \partial\Sigma) \rightarrow (\Sigma, \partial\Sigma)$ into a product of arc-slides. First, we pick a parameterization of a surface $(\Sigma, \partial\Sigma) \cong F^\circ(\mathcal{Z})$ such that

l is isotopic to an arc $\alpha \subset F^\circ(\mathcal{Z})$, whose ends are connected along the part of the boundary which does not contain a basepoint — we denote this part by I_α . Then we consider the composition of arc-slides, where each slide moves a point on I_α , in turn, once along the α . This will be the desired Dehn twist. For example, see Figure 4.7, where the curve E is in the correct position with the arc α_E . Thus the Dehn twist τ_E is equal to the single arc-slide over that arc, which is indicated on the picture. There is also a standard Heegaard diagram for an arc-slide. In our example, this is the 3rd type of the diagram in Figure 4.9.

The DD type bimodules for arc-slides were computed in [LipOzsThu14]. Let us mention that DD type bimodules are dg -bimodules (as opposed to A_∞ bimodules), and there are less moduli spaces of pseudo-holomorphic curves involved in their definition, compared to DA and AA type bimodules.

Lastly, we quote [LipOzsThu15, Theorem 4], which says that ${}^{\mathcal{A}_1(\mathcal{Z})}N(id)_{\mathcal{A}_1(\mathcal{Z})} \simeq {}^{\mathcal{A}_1(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z})}$ is the *identity DA bimodule*, i.e. it has generators ${}_{i_k}i_{ki_k}$ for every idempotent of $\mathcal{A}_1(\mathcal{Z})$, and actions $({}_{i_k}i_{ki_k}, a) \rightarrow a \otimes_{i_l} i_{li_l}$ for every element $i_k a i_l = a$ in algebra $\mathcal{A}_1(\mathcal{Z})$ (see [LipOzsThu15, Definition 2.2.48]). From the definition it follows that the box tensor product operation with ${}^{\mathcal{A}_1(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z})}$ does not change a bimodule.

Let us return to the proof of $N'(\tau_E) \simeq N(\tau_E)$. We first prove the following homotopy equivalence (which is, in fact, isomorphism):

$${}^{\mathcal{A}_1(\mathcal{Z}_{g=2})}N'(\tau_E)_{\mathcal{A}_1(\mathcal{Z}_{g=2})} \boxtimes {}^{\mathcal{A}_1(\mathcal{Z}_{g=2})}\widehat{CFDD}(id)^{\mathcal{A}_1^\dagger(\mathcal{Z}_{g=2})} \simeq {}^{\mathcal{A}_1(\mathcal{Z}_{g=2})}\widehat{CFDD}(\tau_E)^{\mathcal{A}_1^\dagger(\mathcal{Z}_{g=2})}.$$

All the terms are known: $N'(\tau_E)$ is our candidate bimodule, ${}^{\mathcal{A}_1(\mathcal{Z}_{g=2})}\widehat{CFDD}(id)^{\mathcal{A}_1^\dagger(\mathcal{Z}_{g=2})}$ is the identity DD bimodule described in [LipOzsThu14, Section 3], and ${}^{\mathcal{A}_1(\mathcal{Z}_{g=2})}\widehat{CFDD}(\tau_E)^{\mathcal{A}_1^\dagger(\mathcal{Z}_{g=2})}$ is an arc-slide DD type bimodule described in the same paper. See [LipOzsThu14, Definition 1.7] for the criterion for checking if a bimodule is an arc-slide bimodule, and [LipOzsThu14, Section 4] for a complete description of

arc-slide bimodules. Both sides of the homotopy equivalence above are equal to the DD bimodule 4.6.11, which we describe in appendix (including the types of differentials involved in the arc-slide bimodule). There, for elements of $\mathcal{A}_1^\dagger(\mathcal{Z}_{g=2})$ we use the strand diagram notation: strands are on the horizontal lines numbered by 0 to 7, and, as an example, $-(0, 2), (1, 3), (4 \rightarrow 5)-$ means an element of the algebra corresponding to the chord from the 4th to the 5th line supplemented with idempotents $(0, 2)$ and $(1, 3)$ (this makes this an element of three-strands-moving algebra). See [LipOzsThu08, Section 3] for a description of $\mathcal{A}_1^\dagger(\mathcal{Z}) = \mathcal{A}(\mathcal{Z}, g - 1)$ (and the whole $\mathcal{A}(\mathcal{Z})$) in terms of strand diagrams.

Now, using the pairing theorem [LipOzsThu15, Theorem 12], we finish the proof:

$$\begin{aligned}
N'(\tau_E) \boxtimes \widehat{CFDD}(id) &\simeq \widehat{CFDD}(\tau_E) \Rightarrow \\
N'(\tau_E) \boxtimes \widehat{CFDD}(id) \boxtimes \widehat{CFAA}(id) &\simeq \widehat{CFDD}(\tau_E) \boxtimes \widehat{CFAA}(id) \Rightarrow \\
N'(\tau_E) \boxtimes \widehat{CFDA}(id) &\simeq \widehat{CFDA}(\tau_E) \Rightarrow \\
N'(\tau_E) &\simeq \widehat{CFDA}(\tau_E) \simeq N(\tau_E).
\end{aligned}$$

Let us finish this computation by describing the way we wrote down the candidate bimodule $N'(\tau_E)$. Following closely [LipOzsThu15, Section 10.2], first we spotted the

following actions:

$$x_0 \otimes \rho_1 \rightarrow \rho_1 \otimes x_1,$$

$$r \rightarrow \rho_2 \otimes x_0,$$

$$r \otimes \rho_3 \rightarrow x_1,$$

$$x_1 \otimes \rho_4 \rightarrow \rho_4 \otimes x_2,$$

$$x_2 \otimes \rho_5 \rightarrow \rho_5 \otimes x_3,$$

$$x_3 \otimes \rho_6 \rightarrow \rho_6 \otimes x_3,$$

$$x_2 \otimes \rho_7 \rightarrow \rho_7 \otimes x_3,$$

$$x_1 \otimes \rho_2 \rightarrow \rho_{23} \otimes r,$$

$$x_0 \otimes (\rho_3, \rho_2) \rightarrow \rho_3 \otimes r.$$

All but one corresponding domains are polygons, and thus have a unique holomorphic representative. The domain for the action $x_0 \otimes (\rho_3, \rho_2) \rightarrow \rho_3 \otimes r$ has a cut from x_0 to the right boundary, and so it is an annulus with cut parameters at r going to the opposite boundaries, and so it also has a unique holomorphic representative. These are the actions which correspond to *short near-chords*, see [LipOzsThu14, Definition 1.6] (notice that, in the notation of that paper, our Dehn twist is a degenerate under-slide, and so one has to add two extra short near-chords). Then we filled out the other actions, so that A_∞ relations are satisfied (always first looking to add an action which contributes to the d^2 by factorizing the algebra element on the A side).

Remark 4.1.14. In the case of the arc-slide DD bimodules (see [LipOzsThu14, Definition 1.7]), all actions in the bimodule follow from the short near-chord actions, A_∞ relations, and the fact that domains which contribute to the actions should have Maslov index 1 (this corresponds to having an appropriate grading on the bimodule). This is not the case for the arc-slide DA bimodules. In our case above we have a

degenerate under-slide, and in this particular case short near-chord domains imply all other actions. However, in other cases of arc-slides (which we will need below), one should also add “by hand” one more action, which corresponds to the domain containing σ and touching both boundaries by chords of length 2, see [LipOzsThu14, Figure 16] for the notation. On that figure this domain is $\sigma + \sigma_-$ on the left and $\sigma + \sigma_+$ on the right.

Computation 4.1.15 ($N(\tau_C^{-1})$). In Figure 4.13 we draw the Heegaard diagram $\mathcal{H}(M_{\tau_C^{-1}})$ along with the marked generators of the bimodule. Because there are many generators, for the generators t_i we only mark an intersection point on the right side of the diagram; its corresponding $2g - 1 = 3$ intersections on the left are uniquely determined. In Figure 4.14 we draw the candidate DA bimodule. On some arrows (which are a little lighter on the picture) we did not write the actions; those actions correspond to all the rectangles in the rectangle area with vertices t_0, t_{12} and the right edge ρ_{23456} .

Let us denote for a moment our candidate bimodule by $N'(\tau_C^{-1})$, and the bimodule $N(\tau_C^{-1})$ will be the one which corresponds to the Heegaard diagram in Figure 4.13. Thus we want to prove that $N'(\tau_C^{-1}) \simeq N(\tau_C^{-1})$.

First, we factorize the Dehn twist τ_C^{-1} into the product of arc-slides. In Figure 4.7, consider a slide of the arc α_E over the arc α_A , and let us call this arc-slide η . Then one gets a new parameterization of the surface, where instead of the arc α_E one has a new arc α'_E which is isotopic to the curve $C' \sim \eta(C)$, if one connects the ends of the arc α'_E . So the Dehn twist $\tau_{C'}^{-1}$ can be factorized into four arc-slides $\mu_1, \mu_2, \mu_3, \mu_4$ along the arc α'_E , which we picture in Figure 4.10. Now, by a mapping class group relation $f\tau_L f^{-1} = \tau_{f(l)}$ we get the desired factorization:

$$\tau_C^{-1} = \tau_{\eta^{-1}(C')}^{-1} = \eta^{-1} \tau_{C'}^{-1} \eta = \eta^{-1} \mu_4 \mu_3 \mu_2 \mu_1 \eta.$$

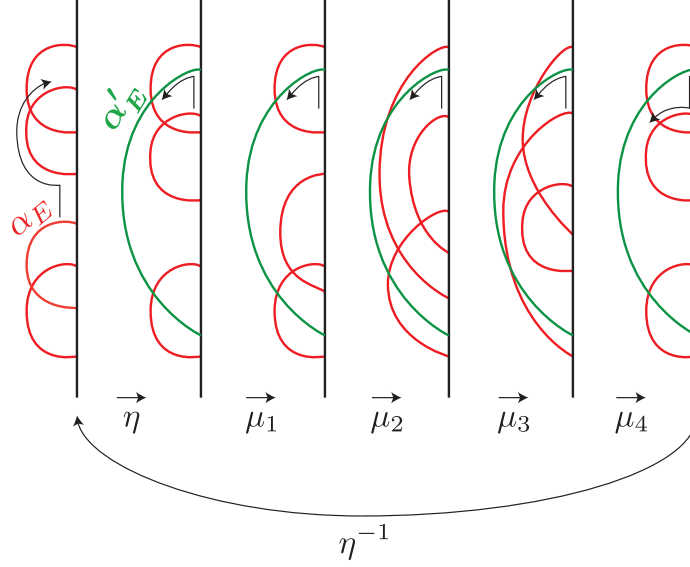


Figure 4.10: Composition of these 6 arc-slides gives a left handed Dehn twist around the curve C in Figure 4.7, i.e. $\tau_C^{-1} = \eta^{-1}\mu_4\mu_3\mu_2\mu_1\eta$.

From this factorization we get

$$N(\eta) \boxtimes N(\mu_1) \boxtimes N(\mu_2) \boxtimes N(\mu_3) \boxtimes N(\mu_4) \boxtimes N(\eta^{-1}) \simeq N(\tau_{C'}^{-1}),$$

and so to compute $N(\tau_{C'}^{-1})$ it is left to compute the bimodules for 6 arc-slides.

We computed them using exactly the same method we used in the previous computation of $N(\tau_E)$ (with the exception which is described in the remark at the end of Computation 4.1.13). We list them all at the end of the appendix.

For computing $N(\eta) \boxtimes N(\mu_1) \boxtimes N(\mu_2) \boxtimes N(\mu_3) \boxtimes N(\mu_4) \boxtimes N(\eta^{-1})$ we wrote a computer program [Kot18]. After tensoring all the six arc-slide bimodules, and then doing all possible cancellations, we get a bimodule, which is isomorphic to the one in Figure 4.14 with canceled differential $x_2 \rightarrow t_{12}$. This proves the desired homotopy equivalence:

$$N'(\tau_{C'}^{-1}) \simeq N(\eta) \boxtimes N(\mu_1) \boxtimes N(\mu_2) \boxtimes N(\mu_3) \boxtimes N(\mu_4) \boxtimes N(\eta^{-1}) \simeq N(\tau_{C'}^{-1}).$$

Computation 4.1.16 (Hochschild homology). It is the Hochschild homology of a

bimodule that we are going to equate with a version of fixed point Floer cohomology. Thus we would like to have an algorithm for computing it. The method from [LipOzsThu15, Section 2.3.5] for computing the Hochschild homology for type DA bimodules works well, as long as the DA bimodule is bounded (see [LipOzsThu15, Definition 2.2.46]). All bimodules in the genus 2 case that we computed previously are not bounded. To fix this problem, we multiply a bimodule by a certain bounded bimodule from the left and from the right such that the A_∞ homotopy equivalence class does not change: $[\mathbb{I}]^b \boxtimes N(\phi) \boxtimes [\mathbb{I}]^b \simeq N(\phi)$. Now we describe the bimodule $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}^b$.

We know that tensoring with the identity bimodule $\mathcal{A}_1(\mathcal{Z}_{g=2})N(id)_{\mathcal{A}_1(\mathcal{Z}_{g=2})} \simeq \mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}$ does not change the A_∞ homotopy equivalence class. Thus it is left to make $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}^b$ bounded, while not changing its A_∞ homotopy equivalence class.

In the genus 2 case, we claim that the needed bimodule $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}^b$ is depicted in Figure 4.16. The graph on that figure does not have any cycles, thus the bimodule is bounded. Canceling four differentials $c_1 \rightarrow c_2, t_1 \rightarrow t_2, z_1 \rightarrow z_2, w_1 \rightarrow w_2$ in $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}^b$ gives $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}$, hence $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}^b \simeq \mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}$.

We wrote down this bimodule using the diagram in Figure 4.15 (three intersections on the left side of the diagram are omitted for generators $z_1, z_2, c_1, c_2, t_1, t_2, w_1, w_2$), which is essentially the Heegaard diagram from Figure 4.4 but with perturbed β curves so that there are no periodic domains.

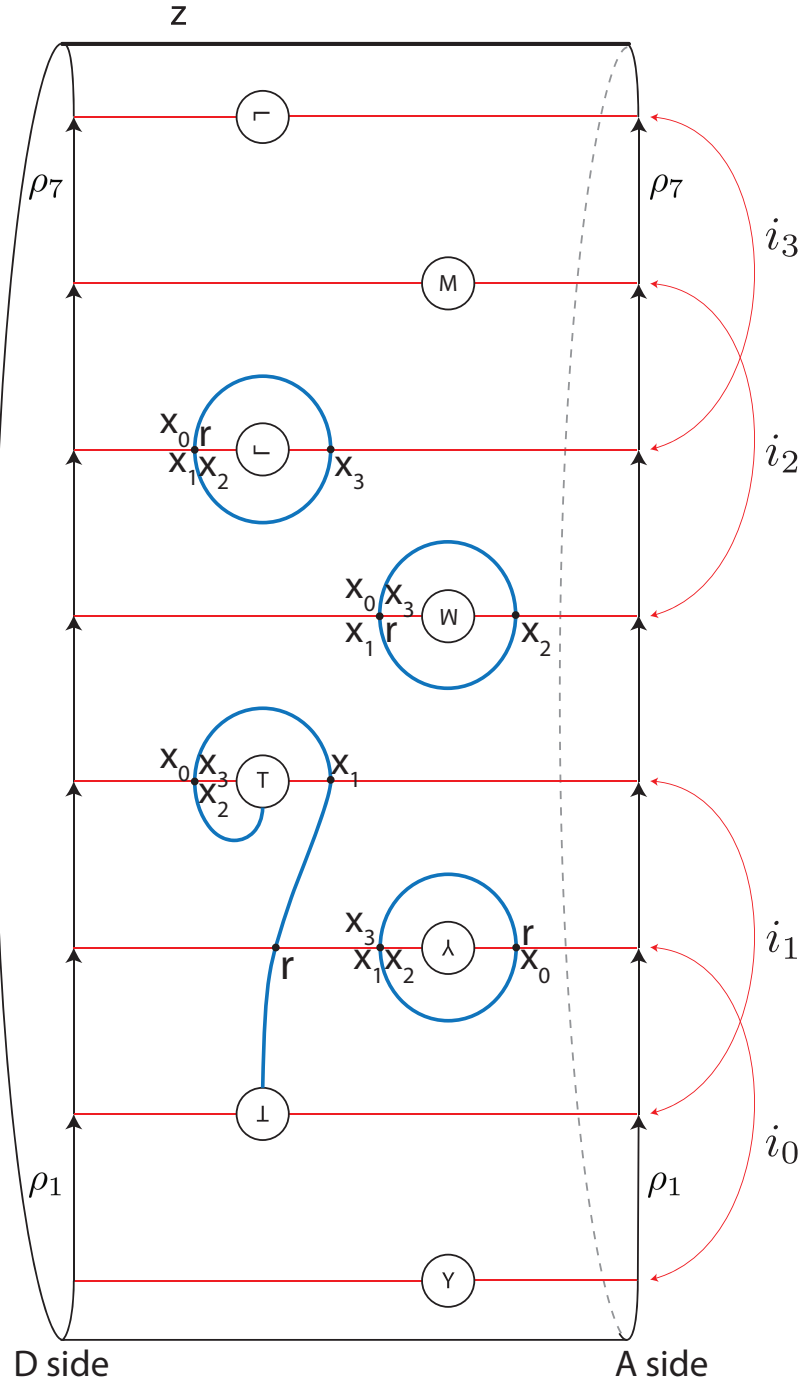


Figure 4.11: Heegaard diagram $\mathcal{H}(M_{\tau_E})$ for the right handed Dehn twist along the curve E.

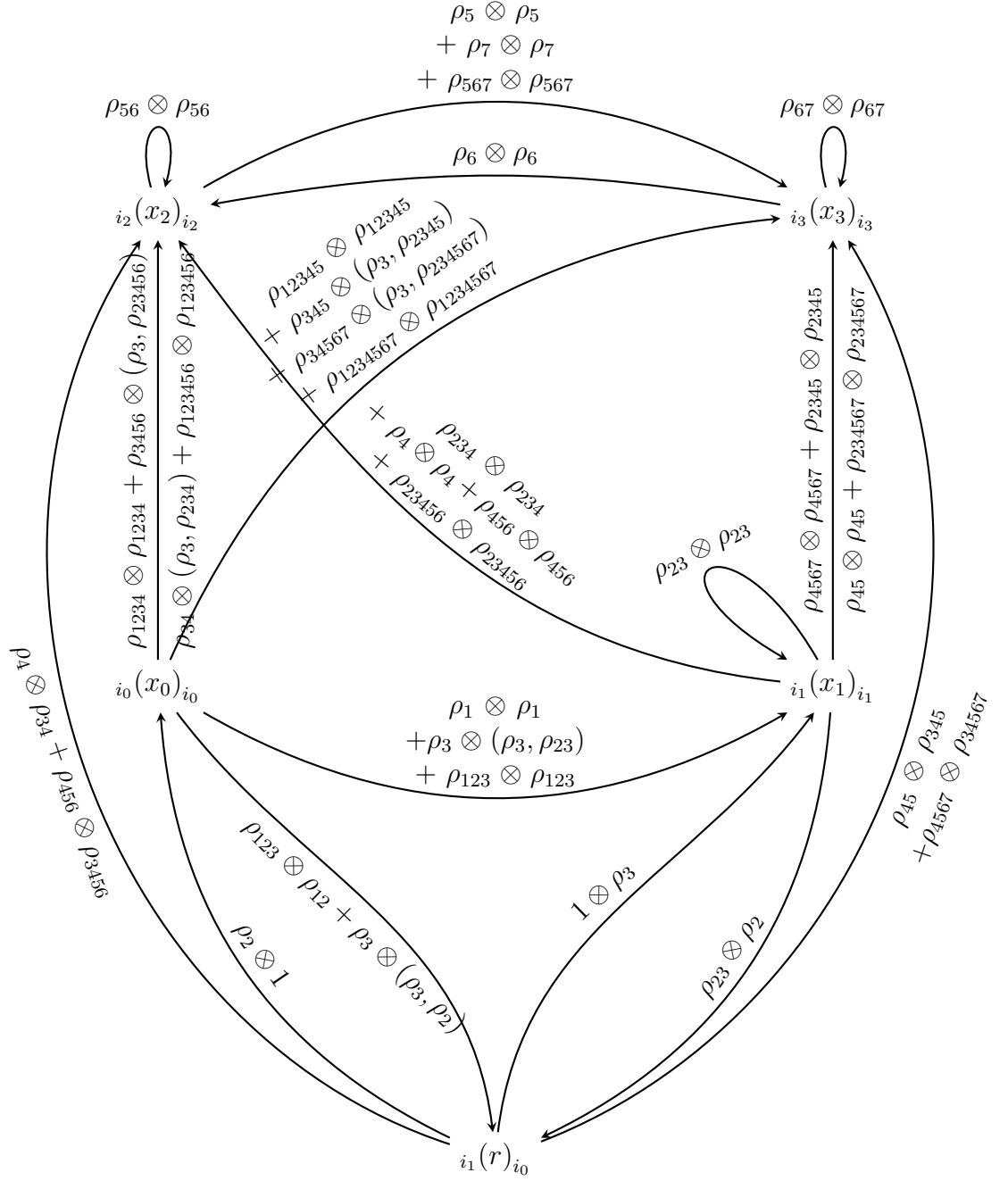


Figure 4.12: Bimodule $\mathcal{A}_1(\mathcal{Z}_{g=2})N(\tau_E)\mathcal{A}_1(\mathcal{Z}_{g=2})$.

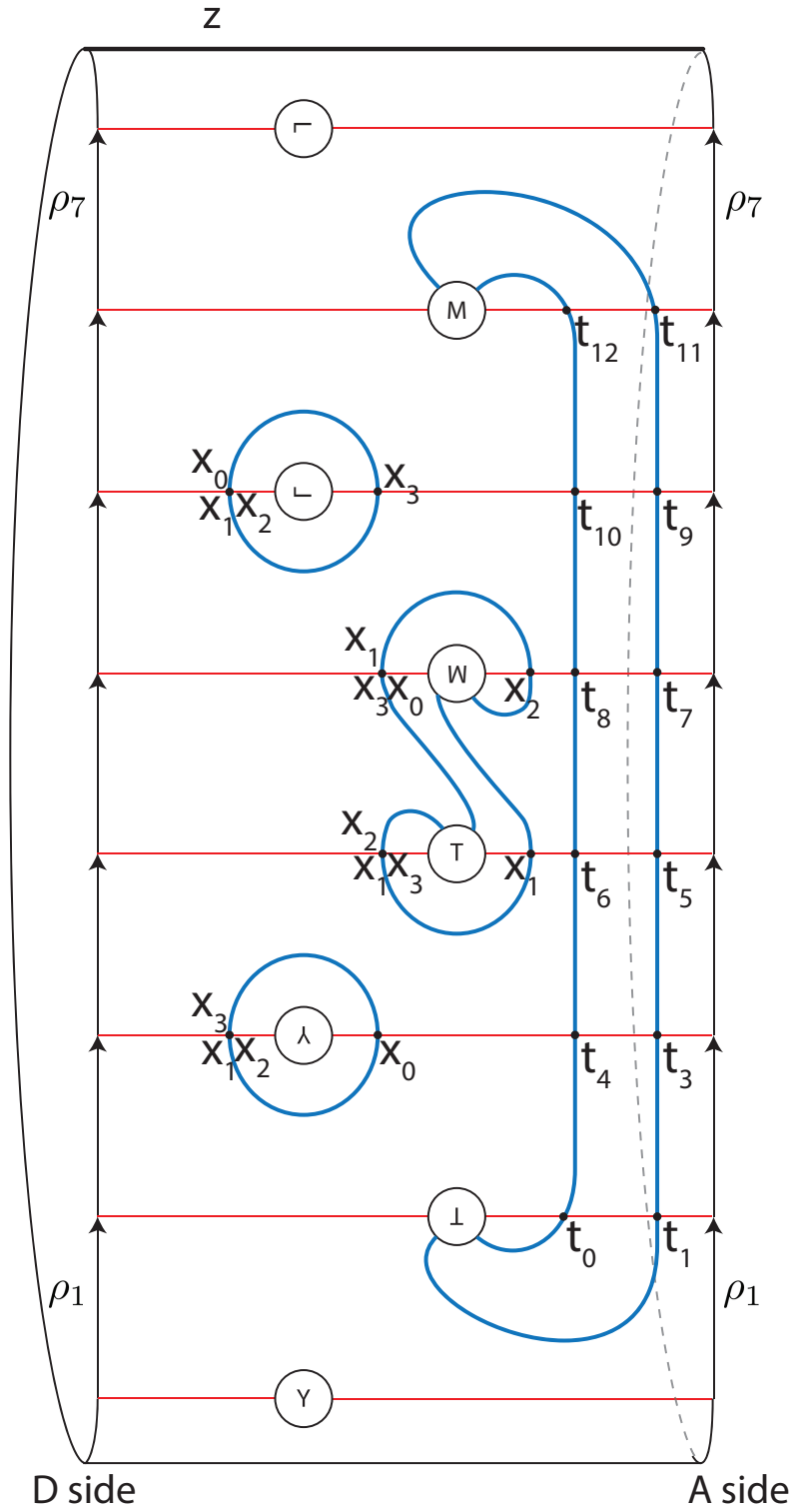


Figure 4.13: Heegaard diagram $\mathcal{H}(M_{\tau_C^{-1}})$ for the left handed Dehn twist along the curve C .

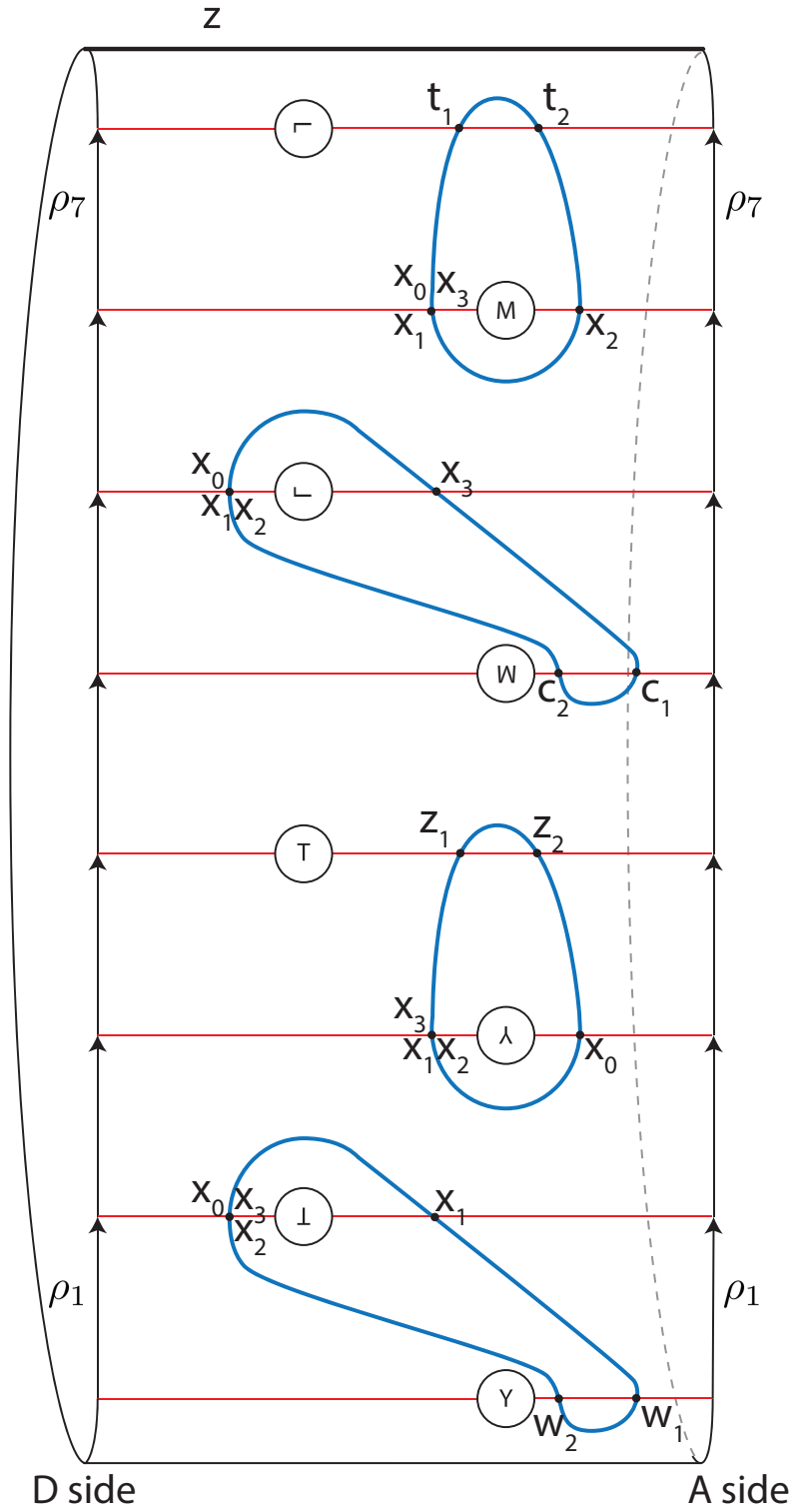


Figure 4.15: Heegaard diagram for M_{id} with no periodic domains, where $id : F^\circ(\mathcal{Z}_{g=2}) \rightarrow F^\circ(\mathcal{Z}_{g=2})$ is the identity mapping class.

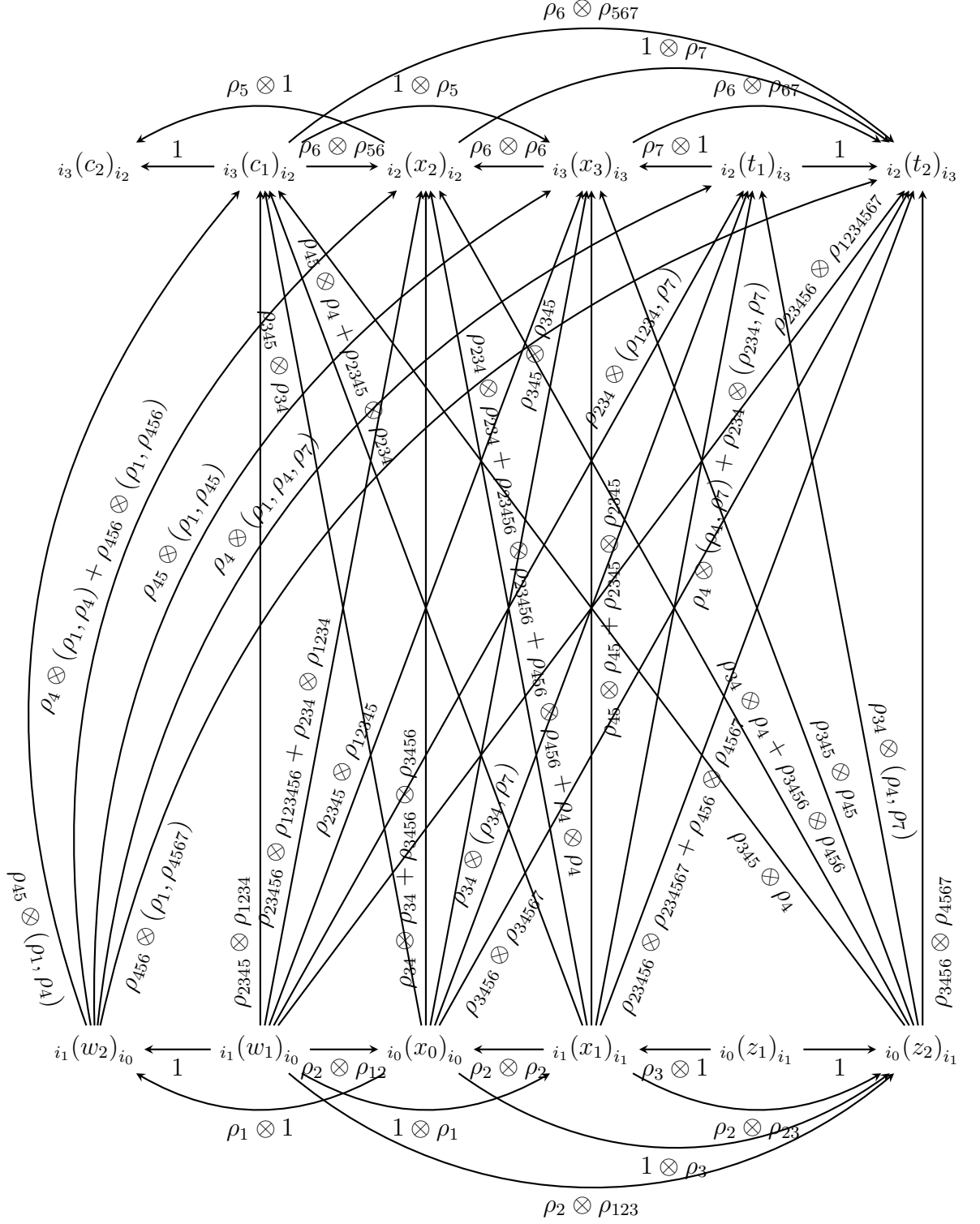


Figure 4.16: Bimodule $\mathcal{A}_1(\mathcal{Z}_{g=2})[\mathbb{I}]_{\mathcal{A}_1(\mathcal{Z}_{g=2})}^b$.

4.2 Fixed point Floer cohomology

At first fixed point Floer cohomology was defined for symplectomorphisms which are Hamiltonian isotopic to the identity, by Floer in [Flo89]. It was extended to other symplectomorphisms by Dostoglou and Salamon in [DosSal94]. Seidel in [Sei96] studied fixed point Floer cohomology of Dehn twists on surfaces, and then defined it for other mapping classes in [Sei02]. For the construction of fixed point Floer cohomology we refer the reader to articles [Gau03], [Cot09], [Sei17], [Ulj17]. Here we summarize how it works. We will be working with fixed point Floer cohomology, rather than homology.

4.2.1 Setup

Construction 4.2.1 (for closed surfaces). Consider a closed oriented surface Σ_g with genus $g > 1$. There is a classical version of fixed point Floer cohomology for orientation preserving mapping classes ϕ , which we denote by $HF^*(\phi)$. The construction is as follows. Choose a symplectic area form ω , and an area-preserving monotone representative ϕ of a mapping class (see [Sei02] for the definition of monotone, and particularly for why Floer cohomology does not depend on this choice). Then the chain complex $CF_*(\phi)$ is generated over $\mathbb{F}_2$ by the non-degenerate fixed points of ϕ . These are the fixed points where $\det(d\phi - Id) \neq 0$. For generic ϕ all fixed points are non-degenerate (hence are isolated). In the definition one can assume non-degeneracy, but in practice to achieve it one perturbs ϕ by the time 1 isotopy $\psi_{X_{H_t}}^1$ along the Hamiltonian vector field X_{H_t} , where $H_t : \Sigma \rightarrow \mathbb{R}$ is a time-dependent generic Hamiltonian. So by considering $\phi \circ \psi_{X_{H_t}}^1$ if necessary, we assume now that fixed points of ϕ are non-degenerate. Note that the fixed points of ϕ correspond to the constant sections of the mapping torus $T_\phi = \Sigma \times [0, 1] / (\phi(x), 0) \sim (x, 1)$. The differential $\partial : CF_*(\phi) \rightarrow CF_{*-1}(\phi)$ is defined by counting pseudo-holomorphic cylinder sections of $T_\phi \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$, which

limit to constant section at $\pm\infty$. One counts only index 1 cylinders (i.e. those which come in 1-dimensional family), up to translation along $\mathbb{R}$. The differential goes from $+$ to $-$, as in Morse homology. An almost complex structure on $T_\phi \times \mathbb{R}$ comes by picking a generic time-dependent almost complex structure on Σ and extending it to the rest of the tangent space naturally, i.e. the direction of the circle inside T_ϕ and the direction of $\mathbb{R}$ are interchanged. This differential satisfies $\partial^2 = 0$, and passing to the homology of the dual complex $HF^*(\phi) = H(CF^*(\phi), d)$ gives *fixed point Floer cohomology* — an invariant, which depends only on the mapping class ϕ . The $\mathbb{Z}_2$ -grading on this invariant is provided by the sign of $\det(d\phi - Id)$ at fixed points.

Construction 4.2.2 (for surfaces with boundary). There is a natural generalization of the above invariant to surfaces with boundary. Suppose $\Sigma = (\Sigma, \partial\Sigma = U_1 \cup U_2 \cup \dots \cup U_n)$ is an oriented surface with boundary, of any genus. We will consider orientation preserving mapping classes $\phi \in MCG_0(\Sigma)$ fixing the boundary. Choose an exact area-preserving representative (see [Gau03, Appendix C], or [Ul17, Lemma 3.3] for an explanation of why the definition does not depend on this choice). Because we want every fixed point to be non-degenerate, we will need to perturb ϕ near the boundary. Thus, in order to specify the perturbation, as an input we will also take decorations of every boundary component with a sign. These decorations tell us how the perturbation behaves near the boundary components. If U_i is decorated by $(+)$, then the perturbation in the neighborhood of U_i should be a twist in the direction of the orientation of U_i (the same direction as the Reeb flow on the boundary). This corresponds to perturbation along a Hamiltonian vector field with the Hamiltonian H having a time-independent local maximum on U_i , see Figure 4.17. If U_j is decorated by $(-)$, then the perturbation should be a twist in the opposite direction of the induced orientation, i.e. the Hamiltonian should have a local minimum on C_j . These twists near the boundary should be small enough, i.e. $\leq 2\pi$ if one full twist is 2π . We denote the resulting fixed point Floer cohomology by $HF^*(\phi; U_1\pm, U_2\pm, \dots, U_n\pm) =$

$HF^*(\phi ; U+, U-)$, where we denote $U+$ (respectively $U-$) to be a union of the positively (respectively negatively) decorated components.

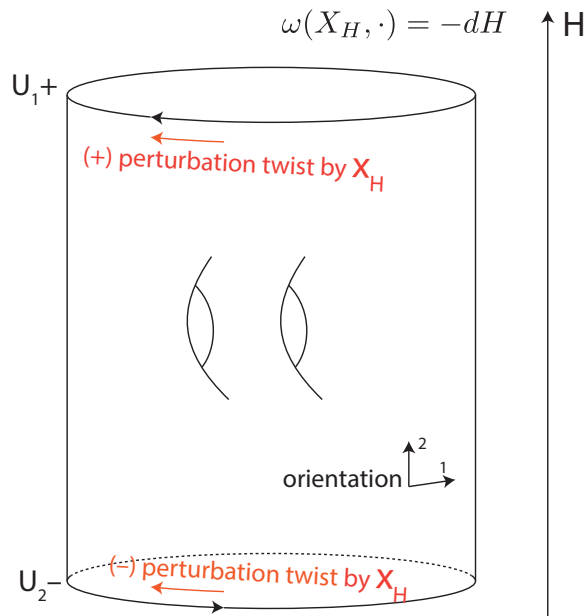


Figure 4.17: Perturbation twists near the boundary.

Remark 4.2.3. Notice that the naming of the twists comes from comparing the direction of the twist with the orientation on the boundary. It is not related to positive or negative Dehn twists. In fact, positive (+) direction of twisting corresponds to the left handed twisting, which appears in the negative (left handed) Dehn twists. (−) direction of twisting corresponds to the right handed twisting, which appears in positive (right handed) Dehn twist.

Remark 4.2.4. Note that paths of exact symplectomorphisms are Hamiltonian isotopies, see [Ulj17, Lemma 2.33].

4.2.2 Existing computational methods

First of all, in the case of the identity mapping class the Floer cohomology is the same as Morse cohomology with respect to the Hamiltonian we use for perturbing id . See [Sei17, Lemma 3.9] for a proof and references to the original works. Thus we have $HF^*(id; U+, U-) = H^*(\Sigma, U-)$, because Hamiltonian is local minimum on the curves in $U-$ and local maximum for the curves in $U+$.

First computations for non-trivial mapping classes were done by Seidel in [Sei96]. Suppose ϕ is a composition of right handed Dehn twists along curves $R = \{R_1, R_2, \dots, R_l\}$ and left handed Dehn twists along curves $L = \{L_1, L_2, \dots, L_k\}$. Suppose all the curves are disjoint and their complement has no disc components, and that no L_i is homotopic to R_j . Then $HF^*(\phi; U+, U-) = H^*(\Sigma - L, R \cup U-)$. This result is achieved again via reducing the computation to the Morse cohomology, where the Hamiltonian is local minimum on the curves R_i and local maximum on the curves L_j .

Then Gautschi in [Gau03] computed fixed point Floer cohomology for algebraically finite mapping classes (those consist of periodic mapping classes, and also reducible ones where the map on each component is periodic). Eftekhary in [Eft04] then generalized Seidel's work to Dehn twists along the curves which form a forest, see below. The last computations were done by Cotton-Clay in [Cot09], where he showed how to compute fixed point Floer cohomology for all pseudo-Anosov mapping classes and for all reducible ones (including those with Pseudo-Anosov components). Thus there is a way to compute fixed point Floer cohomology for any mapping class.

For our purposes we will need the following theorem (which is a generalization of Eftekhary's work to the case with boundary):

Theorem 4.2.5 (Eftekhary). *Suppose Σ is a surface with boundary, and $\phi : MCG_0(\Sigma)$ is a mapping class fixing the boundary. Suppose ϕ is a composition of right handed Dehn twists along the forest of transversely intersecting curves $R = \{R_1, R_2, \dots, R_l\}$ (i.e. there are no cycles in the intersection graph of the curves), along with left handed*

Dehn twists along the other forest of curves $L = \{L_1, L_2, \dots, L_m\}$. We assume that $L \cap R = \emptyset$, that no L_i is homotopic to R_j , and all the curves are homologically essential. Then

$$HF^*(\phi; U+, U-) = H^*(\Sigma - L, R \cup U-).$$

4.3 Conjectural isomorphism

4.3.1 Statement

As in Section 4.1 we consider the strongly based mapping class group $MCG_0(\Sigma)$ of the genus g surface with one boundary component. Having a mapping class $\phi : (\Sigma, \partial\Sigma = U_1 = S^1) \rightarrow (\Sigma, \partial\Sigma = U_1 = S^1)$, let us consider the induced $\tilde{\phi} : (\tilde{\Sigma}, \partial\tilde{\Sigma} = U_1 \cup U_2) \rightarrow (\tilde{\Sigma}, \partial\tilde{\Sigma} = U_1 \cup U_2)$, where $\tilde{\Sigma} = \Sigma \setminus D^2$ is obtained by removing a disc in the small enough neighborhood of the boundary such that ϕ is identity on that neighborhood.

Conjecture 4.3.1. *For every mapping class $\phi \in MCG_0(\Sigma_g, \partial\Sigma_g = S^1 = U_1)$ there is an isomorphism of $\mathbb{Z}_2$ -graded vector spaces*

$$HH_*(N(\phi^{-1})) \cong HF^{*+1}(\tilde{\phi}; U_2+, U_1-).$$

We support this conjecture by computations in the genus 2 case below. Then, in Section 4.4, we describe the symplectic geometric interpretation of bordered Heegaard Floer homology. Based on that, in Section 4.5, we outline a construction of the map potentially giving an isomorphism (in the double basepoint version of it). It comes from an open-closed map in the context of the partially wrapped Fukaya category, associated to a Lefschetz fibration.

Remark 4.3.2. The reason that on the left side of the isomorphism we have ϕ^{-1}

is because the bimodule coming from the bordered theory is homotopy equivalent to $\bigoplus_{i,j} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \phi(\alpha_j))$ (see Section 4.4 for the explanation of this), and the bimodule defined in [Sei17] is $\bigoplus_{i,j} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\phi(\alpha_i), \alpha_j) \simeq \bigoplus_{i,j} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \phi^{-1}(\alpha_j))$, see section 4.5.4 for more on this.

4.3.2 Computations

Here we perform computations in the genus 2 case. As in Section 4.1, we fix a set of curves generating the mapping class group as in Figure 4.6, and use a parameterization $\Sigma_{g=2} \cong F^\circ(\mathcal{Z}_{g=2})$ as in Figure 4.7.

For tensoring DA bimodules and for computing the Hochschild homology we used a computer program [Kot18]. For computing fixed point Floer cohomology we will use Theorem 4.2.5.

Computation 4.3.3 ($\phi = id$). For $\phi = id$ we have that the Hochschild homology is $HH_*(N(id)) = (\mathbb{F}_2)^4$, which is generated by all the four generators of $N(id)$ (note that all of them have identical left and right idempotents, which means that they contribute to $HC_*(N(id))$). All the four generators have grading 0. Fixed point Floer cohomology in this case is $HF^*(\tilde{id}; U_2+, U_1-) = H^*(\tilde{\Sigma}_{g=2}, U_1) = (\mathbb{F}_2)^4$, all concentrated in the grading 1, see Figure 4.18 for an illustration (by MH^* we denote Morse cohomology).

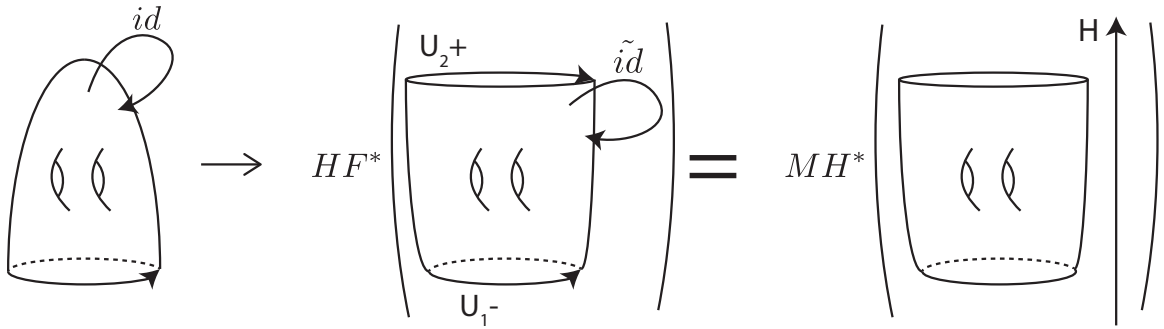


Figure 4.18: Computation for $id : (\Sigma_{g=2}, \partial\Sigma_{g=2}) \rightarrow (\Sigma_{g=2}, \partial\Sigma_{g=2})$.

Computation 4.3.4 ($\phi = \tau_l$). Suppose $\phi = \tau_l$ is a right handed Dehn twist around any of the curves A, B, C, D, or E. Then we have the same ranks as for the identity: the Hochschild homology is $HH_*(N(\tau_l^{-1})) = (\mathbb{F}_2)^4$. Fixed point Floer cohomology in this case is $HF^*(\tilde{\tau}_l; U_2+, U_1-) = H^*(\tilde{\Sigma}_{g=2}, l \cup U_1) = (\mathbb{F}_2)^4$, which corresponds to cutting $\tilde{\Sigma}_{g=2}$ along l and computing Morse cohomology w.r.t. Hamiltonian which looks like in Figure 4.19. Gradings for both invariants are the same as in the previous computation.

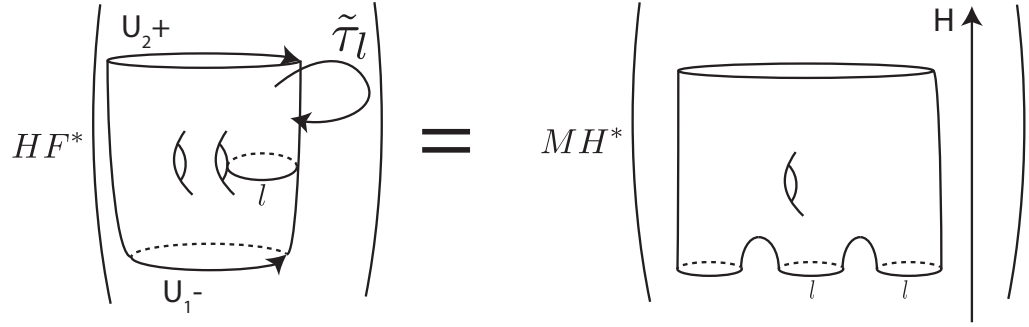


Figure 4.19: Computation for one right handed Dehn twist $\tau_l : (\Sigma_{g=2}, \partial\Sigma_{g=2}) \rightarrow (\Sigma_{g=2}, \partial\Sigma_{g=2})$.

Computation 4.3.5 ($\phi = \tau_l^{-1}$). For a left handed Dehn twist we have the same answers, $HH_*(N(\tau_l)) = (\mathbb{F}_2)^4$, and $HF^*(\tilde{\tau}_l^{-1}; U_2+, U_1-) = H^*(\tilde{\Sigma}_{g=2} - l, U_1) = (\mathbb{F}_2)^4$, which corresponds to cutting $\tilde{\Sigma}_{g=2}$ along l and computing Morse cohomology w.r.t. Hamiltonian which looks like in Figure 4.20. Gradings of both invariants are the same as in the previous examples.

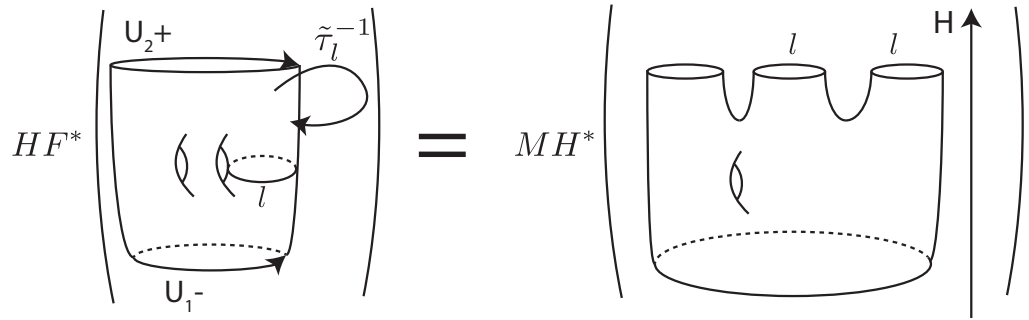


Figure 4.20: Computation for one left handed Dehn twist $\tau_l^{-1} : (\Sigma_{g=2}, \partial\Sigma_{g=2}) \rightarrow (\Sigma_{g=2}, \partial\Sigma_{g=2})$.

Experimenting with mapping classes arising from two disjoint forests of curves (i.e. those where one can use Eftekhary's result), we always got equal ranks of homologies. Let us highlight two more examples.

Computation 4.3.6 ($\phi = \tau_A \tau_B \tau_C \tau_D$). This mapping class is a monodromy of an open book on S^3 with the binding a torus (5,2) knot, and a page being a genus 2 surface with boundary. We get $\widehat{HFK}(S^3, T_{(5,2)}; 1 - g) = HH_*(N(\tau_D^{-1} \tau_C^{-1} \tau_B^{-1} \tau_A^{-1})) = \mathbb{F}_2$, and $HF^*(\tau_A \tau_B \tau_C \tau_D; U_2+, U_1-) = H^*(\tilde{\Sigma}_{g=2}, U_1 \cup A \cup B \cup C \cup D) = \mathbb{F}_2$. It is the lowest rank that we observed in our computations. Interestingly, it is known that $rk(HH_*(N(\phi))) > 0$ always, see [BalVel18]. It is natural to pose the following question.

Question. Is it true that all mapping classes $\phi \in MCG_0(\Sigma_{g=2}, \partial \Sigma_{g=2} = S^1)$ satisfy $rk(HH_*(N(\phi))) > 1$, except the products of permutations of $(\tau_A, \tau_B, \tau_C, \tau_D)$, and their inverses?

If yes, this would almost prove that $\widehat{HFK}(K)$ detects the torus (5,2) knot in S^3 , using the fact that $\widehat{HFK}(K)$ detects the genus of the knot [OzsSza04a], and detects if the knot is fibered [Ghi08], [Ni07]. One would only be left with the cases of products of permutations of $(\tau_A, \tau_B, \tau_C, \tau_D)$, and their inverses (the first thing to check in these case is if the ambient manifold M_ϕ° is in fact S^3).

Provided Conjecture 4.3.1 is true, one can try to answer the above question on the fixed point Floer cohomology side, i.e. to prove that $rk(HF^*(\tilde{\phi}; U_2+, U_1-)) > 1$ using the results of Cotton-Clay [Cot09].

Computation 4.3.7 ($\phi = \tau_A^5 \tau_B \tau_C \tau_D \tau_E^5$). This mapping class is pseudo-Anosov if viewed as a mapping class of a closed genus 2 surface (see [Eft04]). In this case $HH_*(N(\phi^{-1})) = (\mathbb{F}_2)^{10}$, and $HF^*(\tilde{\phi}; U_2+, U_1-) = H^*(\tilde{\Sigma}_{g=2}, U_1 \cup 5A \cup B \cup C \cup D \cup 5E) = (\mathbb{F}_2)^{10}$.

4.4 Alternative bimodule construction via the partially wrapped Fukaya category

Material in this section serves as a preparation for the subsequent Section 4.5.

Let us repeat the way we associate a bimodule to a mapping class in Section 4.1. To a surface one associates an algebraic structure, a dg -algebra in our case. To a mapping class one associates an A_∞ bimodule over that algebra. To a composition of mapping classes one associates the tensor product operation of bimodules. Here we describe another geometric way to arrive at the same mapping class invariant, which is based on the partially wrapped Fukaya category associated to a surface with one boundary component. Let us repeat that we follow Auroux's conventions and work with cohomology, i.e. $hom_{\mathcal{F}(\Sigma, \{z\})}(L_0, L_1) = CF_*^{pw}(L_1, L_0)$.

4.4.1 Relationship between the partially wrapped Fukaya category and bordered Heegaard Floer theory

The surface $(F^\circ(\mathcal{Z}), \{z\})$, associated to a genus g pointed matched circle $\mathcal{Z}$, has a distinguished set of generators of $\mathcal{F}(F^\circ(\mathcal{Z}), \{z\})$, which corresponds to the 1-handles coming from the matched pairs of points in $\mathcal{Z}$, see the surface on the right of Figure 4.1.

Theorem 4.4.1 (Auroux). *Suppose $\alpha_1, \dots, \alpha_{2g}$ are the arcs in $F^\circ(\mathcal{Z}) = \Sigma$ corresponding to the matched pairs in $\mathcal{Z}$. Then the bordered one-strand-moving dg -algebra $\mathcal{A}_1(F^\circ(\mathcal{Z})) = \mathcal{A}(\mathcal{Z}, -g+1)$ is quasi-isomorphic to the A_∞ hom-algebra of the partially wrapped Fukaya category w.r.t. the generating set $\alpha_1, \dots, \alpha_{2g}$, i.e.*

$$\mathcal{A}_1(F^\circ(\mathcal{Z})) = \mathcal{A}(\mathcal{Z}, -g+1) \simeq \bigoplus_{1 \leq i, j \leq 2g} hom_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \alpha_j).$$

Remark 4.4.2. The full statement of Auroux's theorem involves all the summands of

the bordered algebra and Fukaya categories of symmetric products: for $0 \leq k \leq 2g$ one has $\mathcal{A}(\mathcal{Z}, -g + k) \simeq \bigoplus_{1 \leq i, j \leq C_{2g}^k} \text{hom}_{\mathcal{F}(\text{Sym}^k(F^\circ(\mathcal{Z})), z)}(\lambda_i, \lambda_j)$, where $\{\lambda_i\}$ is a set of generators coming from the products of k arcs in $\alpha_1, \dots, \alpha_{2g}$.

Remark 4.4.3. Having a bordered Heegaard diagram for a bordered 3-manifold, Auroux identified not only the algebras, but also the type A bordered Heegaard Floer module with the module coming from the Fukaya category via the Yoneda embedding construction, see [Aur10b, Theorem 1.4].

Example 4.4.4 (torus algebra). Let us illustrate the above theorem on a torus. The way to get the torus bordered algebra from a pointed matched circle is pictured in Figure 4.21. One can see the corresponding elements of the algebra in Figure 4.22. They appear as generators of Lagrangian Floer complexes. One cannot see the product structure (i.e. holomorphic triangles) on this picture, because for that one needs to pick consistent perturbations for the three Lagrangians involved in the product operation.

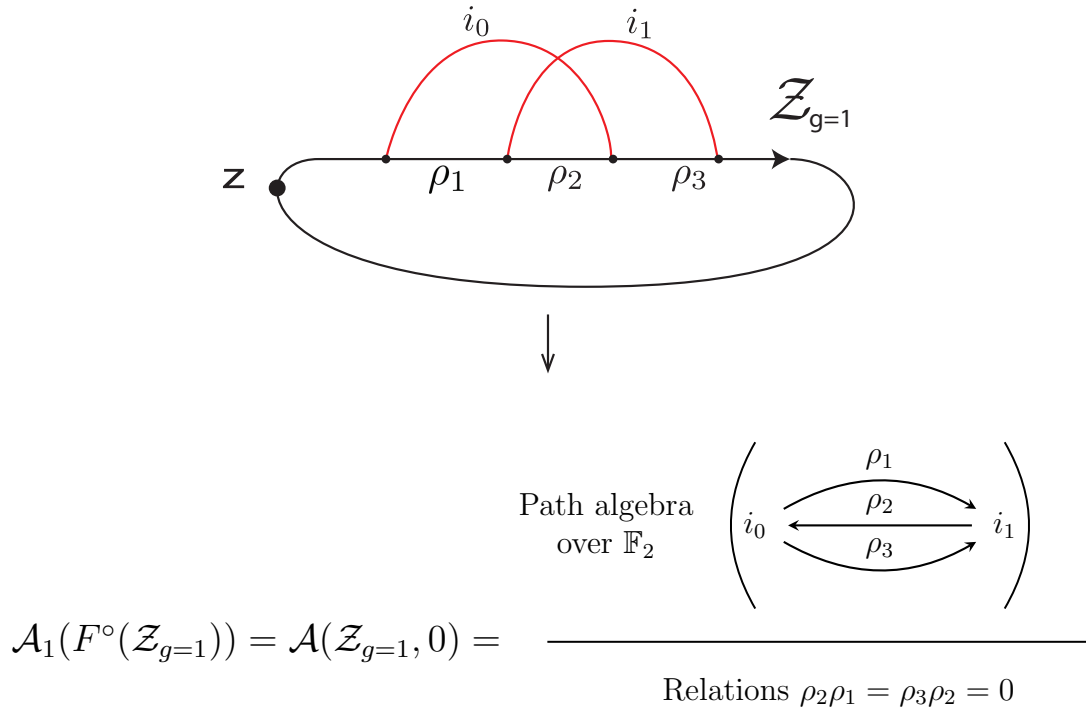


Figure 4.21: Torus bordered algebra, constructed from a pointed matched circle.

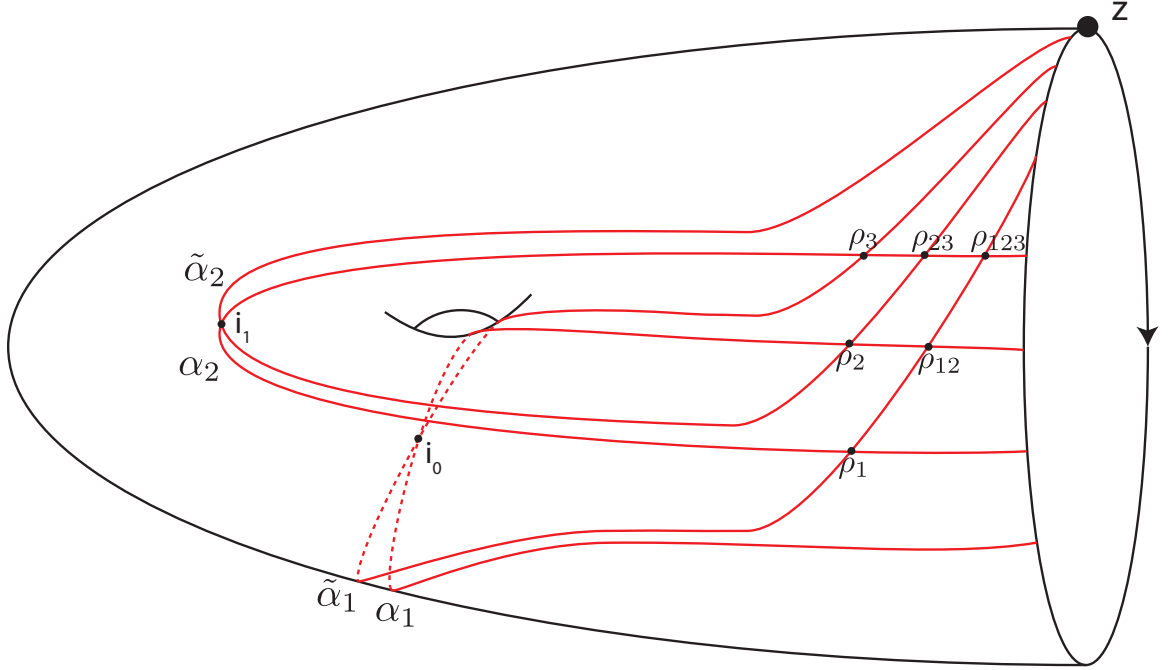


Figure 4.22: Bordered algebra elements in the genus 1 case, viewed as elements of morphism spaces between generating objects of the partially wrapped Fukaya category of a torus.

4.4.2 Bimodule construction

An exact self-diffeomorphism of $(\Sigma, \partial\Sigma = S^1)$ fixing the boundary induces an exact compactly supported self-diffeomorphism of a surface $\hat{\Sigma}$ (Σ completed with a cylindrical end). There is a standard way to associate to it an A_∞ bimodule over the hom -algebra of the Fukaya category $F(\Sigma, \{z\})$ (which is the same as the bordered algebra in our case). Namely, if $\alpha_1, \dots, \alpha_k$ generate the Fukaya category, then the bimodule as a vector space is equal to

$$N_F(\phi) = \bigoplus_{1 \leq i, j \leq k} hom_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \phi(\alpha_j)), \quad (4.2)$$

and the higher actions are given using A_∞ operations. For example the action $m^{1|1|1} : \mathcal{A}(\mathcal{F}(\Sigma, \{z\})) \otimes N_F(\phi) \otimes \mathcal{A}(\mathcal{F}(\Sigma, \{z\})) \rightarrow N_F(\phi)$ is given via the following operation counting holomorphic discs with 4 marked points (“rectangles”):

$$\begin{aligned}
& \left(\bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \alpha_j) \right) \otimes \left(\bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \phi(\alpha_j)) \right) \otimes \left(\bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\phi(\alpha_i), \phi(\alpha_j)) \right) \rightarrow \\
& \rightarrow \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z\})}(\alpha_i, \phi(\alpha_j)).
\end{aligned}$$

Note that this bimodule is of AA type in the bordered theory terminology, whereas $N(\phi)$ was of DA type. Now we unify the two constructions of bimodules over the same algebra.

Proposition 4.4.5. *Suppose $\alpha_1, \dots, \alpha_{2g}$ are arcs in $F^\circ(\mathcal{Z}) = \Sigma$ corresponding to the matched pairs in the pointed matched circle $\mathcal{Z}$. Then the two bimodules associated to a mapping class $\phi \in MCG_0(F^\circ(\mathcal{Z}), F^\circ(\mathcal{Z}))$ are A_∞ homotopy equivalent:*

$$N(\phi) \simeq N_F(\phi).$$

For the proof we refer the reader to [AurGriWeh14, Lemma 4.2]. The main idea is to use α - β -bordered Heegaard diagrams introduced in [LipOzsThu11].

4.5 Construction of a map in the double basepoint version of isomorphism

In this section we are going to explain how we arrived to Conjecture 4.3.1. We will state the double basepoint version of it, and will show that it is a special case of a more general conjecture.

4.5.1 From one basepoint to two: bimodule

Let us explain how to modify our previous constructions of bimodules if we want to have two basepoints instead of one. First of all, the partially wrapped Fukaya category was defined for any number of basepoints. In Figure 4.23 we draw a generating set of Lagrangians (red curves) in the genus 2 case, together with their perturbations (purple curves). Now we have five Lagrangian arcs as generators, instead of four in the one basepoint case (Figure 4.1). In general the number of generating arcs will be $2g+1$ and $2g$ for 2 and 1 basepoint cases, respectively.

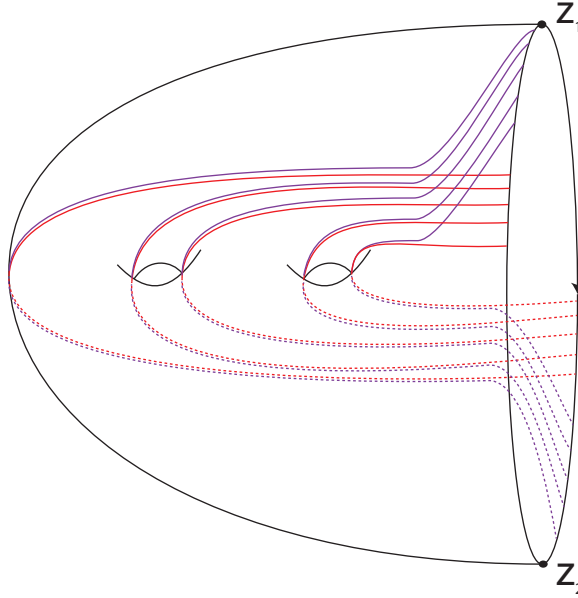


Figure 4.23: Generators of the partially wrapped Fukaya category $\mathcal{F}(\Sigma_{g=2}, \{z_1, z_2\})$.

After choosing this set of generators it is now clear what the corresponding pointed matched circle looks like, as well as the corresponding algebra, see Figure 4.24.

It is now possible to define an A_∞ bimodule (AA type) for a mapping class $\phi \in MCG_0(\Sigma)$ via the partially wrapped Fukaya category:

$$N_F^{2bp}(\phi) = \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z_1, z_2\})}(\alpha_i, \phi(\alpha_j)).$$

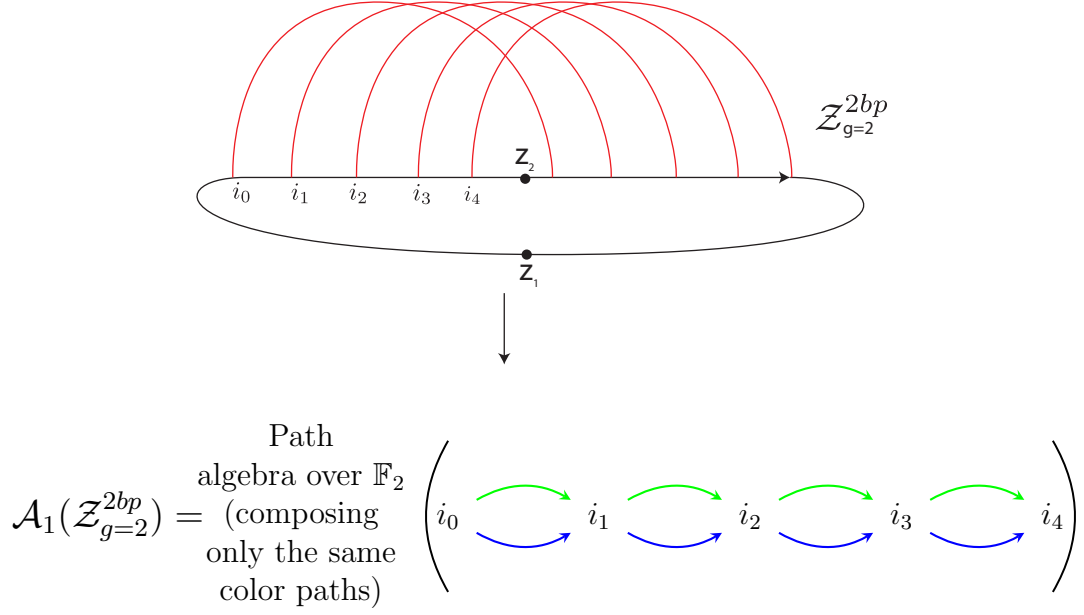


Figure 4.24: A genus 2 double basepoint example of how to get a dg -algebra out of a pointed matched circle. Paths consisting of different color arrows are prohibited.

Though for computations one would prefer to have a DA type bimodule $N^{2bp}(\phi)$. Such a bimodule, as in the one basepoint case, comes from a Heegaard diagram for the mapping cylinder, but equipped with two basepoints on each boundary, and two arcs connecting them. Bordered Heegaard diagrams with multiple basepoints were defined by Zarev in [Zar09]. In Figure 4.25 we draw a diagram for the identity mapping class in the genus two case. The generalization to other mapping classes is analogous to the 1 basepoint case. Two diagrams in Figure 4.25 are equivalent. On the right there is Zarev's bordered sutured Heegaard diagram, and on the left we drew a double basepoint Heegaard diagram, which would be a natural generalization of the one basepoint diagram. The two diagrams carry the same holomorphic information, and the bimodules coming from them are the same. The reason why they are different is that Zarev works with sutured manifolds and their bordered versions. In order to go from the left diagram to the right, instead of drawing basepoints and basepoint arcs one essentially deletes their neighborhoods and then says that the boundary coming from these neighborhoods (drawn in green) is forbidden for holomorphic discs.

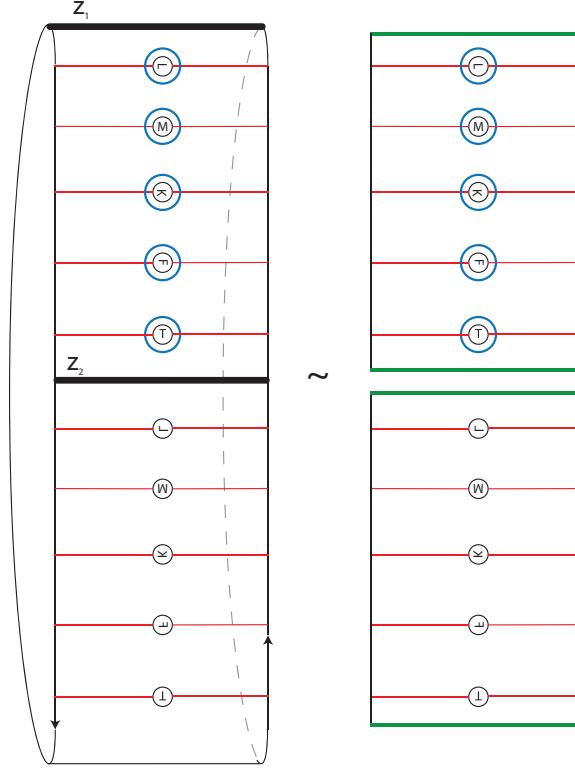


Figure 4.25: Double basepoint diagrams for mapping cylinder of $id : \Sigma_{g=2} \rightarrow \Sigma_{g=2}$.

The Hochschild homologies of one and two basepoint bimodules are related, and one actually has

$$rk(HH_*(N^{2bp}(\phi))) = rk(HH_*(N(\phi))) + 1.$$

The reason is that the Hochschild homology of the double basepoint bimodule is equal to knot Floer homology of the binding of the corresponding open book in the second lowest Alexander grading, where knot in a Heegaard diagram is specified by four basepoints, instead of two: $HH_*(N^{2bp}(\phi)) = \widehat{HFK}^{4bp}(M_\phi^\circ, K; -g)$. And the difference between the four basepoint and the usual two basepoint knot Floer homologies is known: $\widehat{HFK}^{4bp}(M_\phi^\circ, K) = (\mathbb{F}_2)^2 \otimes \widehat{HFK}(M_\phi^\circ, K)$, where the Alexander

gradings of two generators of $(\mathbb{F}_2)^2$ are 0 and -1. Thus one has

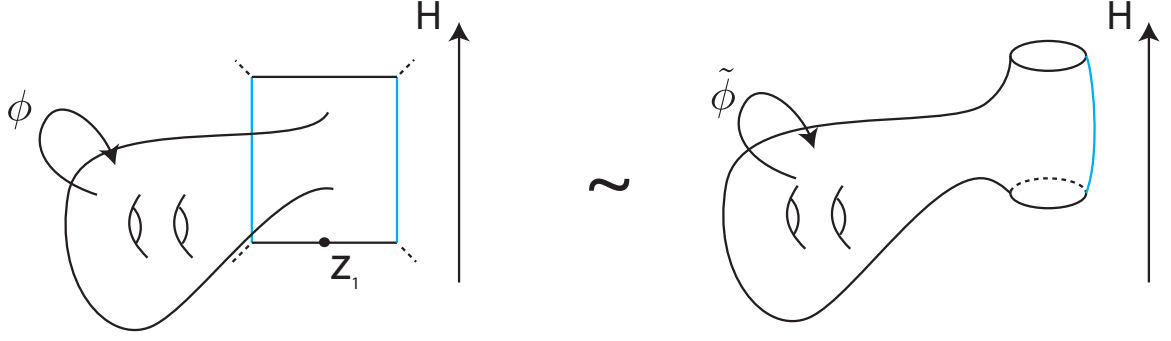
$$\begin{aligned} rk(HH_*(N^{2bp}(\phi))) &= rk(\widehat{HFK}^{4bp}(M_\phi^\circ, K; -g)) = \\ &= rk(\widehat{HFK}(M_\phi^\circ, K; -g+1) \oplus \widehat{HFK}(M_\phi^\circ, K; -g)) = rk(HH_*(N(\phi))) + 1, \end{aligned}$$

because the lowest $-g$ Alexander grading of knot Floer homology of a fibered knot (i.e. the binding of an open book) is always one.

4.5.2 From one basepoint to two: fixed point Floer cohomology

Let us first explain the choice of fixed point Floer cohomology for the one basepoint case.

There is a natural version of fixed point Floer cohomology $HF^{1bp}(\phi)$ which is equal to $HF^*(\tilde{\phi}; U_2+, U_1-)$, but defined without deleting a second disc from the surface. In Section 4.2, we decided not to give a rigorous definition of it (which would be analogous to [Sei17, Section 6]), and rather use existing methods and work with $HF^*(\tilde{\phi}; U_2+, U_1-)$. But let us indicate the setup — $HF^{1bp}(\phi)$ can be defined only for infinite area surfaces with a cylindrical end, rather than compact surfaces with boundary, and so one has to work with the induced compactly supported exact self-diffeomorphisms on the completion $\hat{\Sigma}$. Behavior of the Hamiltonian used for a perturbation near infinity should be a very specific one, see Figure 4.26, the left side. Upwards and downwards the Hamiltonian is linear w.r.t. radial coordinate. Comparing the left and the right side of the figure (on the right we glued the blue boundaries together), one can also see why such Hamiltonian perturbation is equivalent to considering $HF^*(\tilde{\phi}; U_2+, U_1-)$ — the generators (fixed points) and the differentials in Floer cohomology with perturbation on the left side and on the right side are 1-1 correspondence.



$$HF^{1bp}(\phi) \cong HF(\tilde{\phi}; U_2+, U_1-)$$

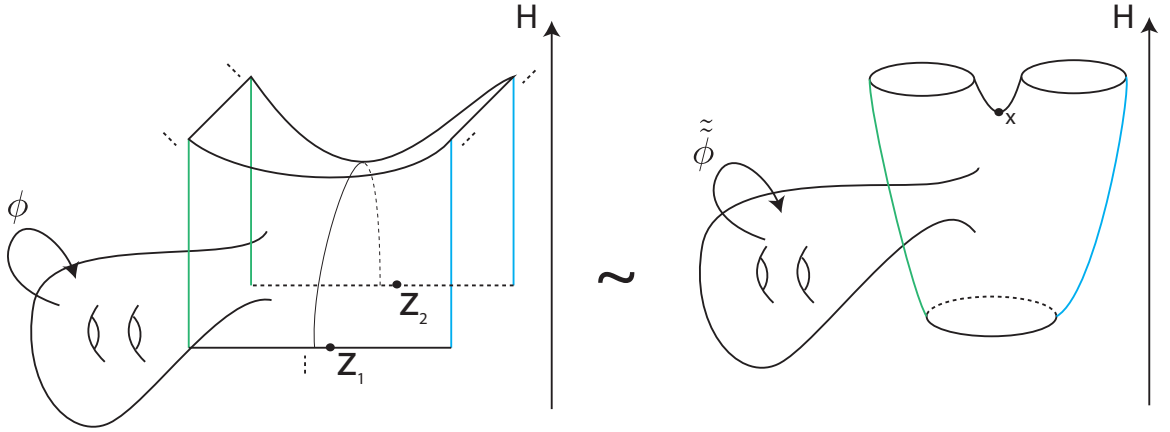
Figure 4.26: On the left: the behavior of Hamiltonian perturbation one actually needs to consider in one basepoint case, and an equivalent to it theory on the right in terms of classical version of fixed point Floer cohomology, where the Hamiltonian is constant on the boundaries.

Let us explain why we call it a one basepoint case. The Hamiltonian on the left side of the diagram could be used to define perturbations in the partially wrapped Fukaya category with one basepoint. One should imagine that the basepoint is in the bottom part of infinity, and Lagrangian arcs are allowed to go only to the upper part of infinity. Then the perturbation will send all the arcs to the left, i.e. to the basepoint.

Remark 4.5.1. When defining perturbations in the partially wrapped Fukaya category, one has to make sure every Lagrangian arc will be wrapped enough to intersect the other ones to the left of it at infinity (see Figure 4.22). Thus one actually cannot consider a linear Hamiltonian w.r.t. radial coordinate $H = r$ on the upper half of infinity, as it is pictured in Figure 4.26. Instead, in order to make sure that Hamiltonian has big enough derivative for every pair of Lagrangian arcs, one needs to take a limit $H = \delta r, \delta \rightarrow +\infty$, as it was done in [Aur10b, Definition 4.1]. But because we always consider only finite number of generating arcs going to infinity, it is enough to just consider linear Hamiltonian with big enough derivative $H = \delta r, \delta \gg 0$.

Now we consider the double basepoint counterpart of the above construction. The

corresponding behavior of Hamiltonian near infinity is pictured on the left of Figure 4.27, and we denote the resulting fixed point Floer cohomology by $HF^{2bp}(\phi)$. This cohomology theory was developed in [Sei17, Section 6], where one considers a surface as a total space of a 0-dimensional Lefschetz fibration over $\mathbb{C}$. The corresponding version of classical Floer cohomology (where Hamiltonian is constant on boundary components) is depicted on the right — instead of one disc one needs to take out two discs this time, and we denote the resulting Floer cohomology by $HF^*(\tilde{\phi}; U_2+, U_3+, U_1-)$.



$$HF^{2bp}(\phi) \cong HF(\tilde{\phi}; U_2+, U_3+, U_1-)$$

Figure 4.27: On the left: behavior of Hamiltonian perturbation one actually needs to consider in the two basepoints case (see [Sei17, Section 6] for a rigorous setup), and an equivalent to it theory on the right in terms of classical version of fixed point Floer cohomology, where Hamiltonian is constant on boundaries.

The one and two basepoint versions of fixed point Floer cohomology should be related, and for all the cases we considered, as in the case of the Hochschild homologies, one has

$$rk(HF^*(\tilde{\phi}; U_2+, U_3+, U_1-)) = rk(HF^*(\tilde{\phi}; U_2+, U_1-)) + 1.$$

We did not find a general explanation for this. The reason might be that if one compares cochain complexes, then they are identical except $CF^*(\tilde{\phi}; U_2+, U_3+, U_1-)$

has one more generator x depicted on the right of Figure 4.27. This generator does not have any differential going out of it, as they are gradient lines going up from x for suitable Hamiltonian, and there are no generators above x . Most likely it also does not have any differentials going in (or, rather, one can arrange the Hamiltonian in such a way).

Now we are ready to state a double basepoint version of Conjecture 4.3.1. Namely, the following should be true:

$$HH_*(N^{2bp}(\phi^{-1})) \cong HF^{*+1}(\tilde{\phi}; U_2+, U_3+, U_1-).$$

4.5.3 Lefschetz fibration structure on the surface

Take an area preserving double branched cover of an exact surface with cylindrical end over the complex numbers $f : \hat{\Sigma}_g \rightarrow \mathbb{C}$ (as an example one might take a quotient by the hyperelliptic involution, which we drew below). One can view this cover as an exact symplectic fibration with singularities, as in [Sei17, Setup 5.1]. This fibration is in fact a 0-dimensional Lefschetz fibration, with $2g+1$ critical points. Being a Lefschetz fibration corresponds to critical points not having more than order two branching (i.e. triple and more branch covers also can be Lefschetz fibrations). We assume that critical values $p_1, \dots, p_{2g+1}$ all satisfy $Re(p_i) = 0$ and $Im(p_1) < \dots < Im(p_{2g+1})$. The genus 2 case is drawn in Figure 4.28.

First, let us repeat that the structure of the exact symplectic fibration can be used to define a particular version of fixed point Floer cohomology. Following [Sei17, Section 6], having an exact compactly supported self-diffeomorphism $\phi : \hat{\Sigma} \rightarrow \hat{\Sigma}$ of 0-dimensional Lefschetz fibration, one can consider fixed point Floer cohomology $HF^*(\phi, \delta > 0, \epsilon)$, where ϵ is not important to us because fibers of $f : \hat{\Sigma} \rightarrow \mathbb{C}$ do not have boundary, and δ is responsible for perturbation at infinity by the Hamiltonian

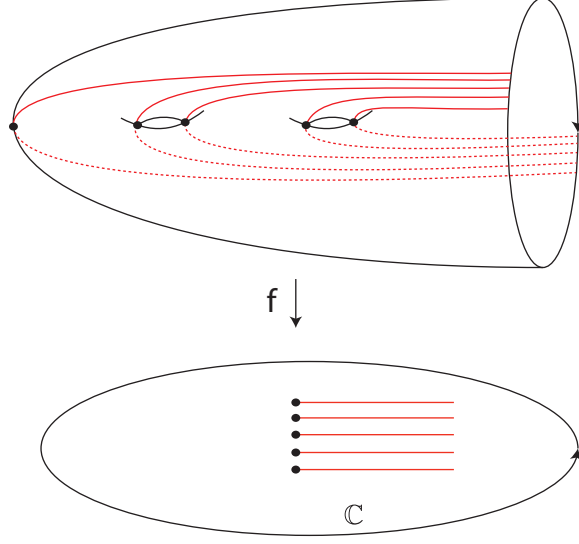


Figure 4.28: 0-dimensional Lefschetz fibration structure on the genus 2 surface.

$H(x) = \delta \operatorname{Re}(f(x))$. This theory depends only on the sign of δ , and for $\delta > 0$ is the cohomology theory from the previous subsection:

$$HF^{2bp}(\phi) = HF^*(\phi, \delta > 0).$$

The Lefschetz fibration structure over the complex plane can be also used to define a special type of A_∞ category $\mathcal{F}_f(\hat{\Sigma})$, which is called the Fukaya-Seidel category (see [Sei01], [Sei08], and more recent articles [Sei12], [Sei17] for a more relevant setup for us). In our double branched cover case the objects of the category are compact exact Lagrangians in $\hat{\Sigma}$ and also non-compact ones which are Lefschetz thimbles (in our case they are just preimages) associated to admissible arcs in $\mathbb{C}$. Admissible arcs in $\mathbb{C}$ are proper rays which start at the critical value of f , do not pass over other critical values, and at some point stabilize to be horizontal, oriented to the right rays. Perturbation at infinity is defined by pulling back to $\hat{\Sigma}$ the Hamiltonian $H = \delta \operatorname{Re}(z)$, $\delta \gg 0$ on $\mathbb{C}$. It is exactly the same type of Hamiltonian perturbation as for $\mathcal{F}(\Sigma, \{z_1, z_2\})$, see the left side of Figure 4.27.

The Fukaya-Seidel category $\mathcal{F}_f(\hat{\Sigma})$ is closely related (quasi-equivalent) to the par-

tially wrapped Fukaya category $\mathcal{F}(\Sigma, \{z_1, z_2\})$, despite the fact that the non-compact objects allowed are different. For a setup which mediates between the partially wrapped category with two basepoints and the Fukaya-Seidel category see [Aur10b, Section 3.2], where the Lefschetz fibration structure is used to define $\mathcal{F}(\Sigma, \{z_1, z_2\})$. For a generalization of that setup to the partially wrapped Fukaya category (which does not use fibration structure) see [Aur10b, Section 4.1].

It was proved in [Sei08] that Lefschetz thimbles (one for each critical points) generate the Fukaya-Seidel category. If one chooses a generating set of thimbles for the category $\mathcal{F}_f(\hat{\Sigma})$, then these thimbles also generate $\mathcal{F}(\Sigma, \{z_1, z_2\})$. Moreover, the hom -algebras are the same: $\bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}_f(\hat{\Sigma})}(\alpha_i, \alpha_j) \simeq \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z_1, z_2\})}(\alpha_i, \alpha_j) \simeq \mathcal{A}_1(\mathcal{Z}^{2bp})$, and the bimodules corresponding to exact automorphisms $\phi : \hat{\Sigma} \rightarrow \hat{\Sigma}$ are also the same $\bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}_f(\hat{\Sigma})}(\alpha_i, \phi(\alpha_j)) \simeq \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z_1, z_2\})}(\alpha_i, \phi(\alpha_j)) \simeq N^{2bp}(\phi)$.

An example of a generating set of Lefschetz thimbles in the genus two case is drawn in Figure 4.28, and the same set of generators for $\mathcal{F}(\Sigma, \{z_1, z_2\})$ was drawn in Figure 4.23.

4.5.4 Open-closed map

The open-closed map is a map between the Hochschild homology of a bimodule corresponding to an automorphism of a symplectic manifold, and a fixed point Floer cohomology. We refer the reader to [Sei17, Section 7] for the definition of this map in the case where the symplectic manifold is an exact symplectic fibration with singularities over $\mathbb{C}$ (which includes Lefschetz fibrations). This map counts isolated points in the moduli space of holomorphic maps from a Riemann surface drawn in Figure 4.29 to $\hat{\Sigma}$, with a twist ϕ along the gray line (compare with [Sei17, Figure 3]). These maps have the following boundary conditions: a twisted orbit of Hamiltonian vector field X_H on one end, which is equivalent to a constant section of $T_{\phi \circ \psi_{X_H}^1}$, and a

chain of Lagrangians on the other, with consistent perturbations. Along the gray line the map has a twist ϕ . So the strip end with the gray line limits to an intersection point of $\phi \circ \psi_{X_H}^1(L_3) \cap L_1$ to the left of the gray line, and to the intersection point $L_3 \cap (\phi \circ \psi_{X_H}^1)^{-1}(L_1)$ to the right of the gray line.

In the Lefschetz fibration setting Seidel in [Sei17, Equation 7.15] defines a bi-module $\mathcal{P}(\phi, \delta, \epsilon) = \bigoplus_{1 \leq i, j \leq k} \text{hom}(\psi_{X_H}^1(\phi(\alpha_i)), \alpha_j)$. The ϵ does not play any role for us, because in our case of 0-dimensional Lefschetz fibration there is no boundary in a fiber. The δ is responsible for the Hamiltonian $H(x) = \delta \text{Re}(f(x))$ which is used to perturb ϕ at infinity. If one assumes $\delta \gg 0$ (so that generating Lagrangians are wrapped enough to intersect each other at infinity), then, in our case of a 0-dimensional Lefschetz fibration, this bimodule is

$$\begin{aligned} \mathcal{P}(\phi, \delta \gg 0) &= \bigoplus_{1 \leq i, j \leq k} \text{hom}(\psi_{X_H}^1(\phi(\alpha_i)), \alpha_j) = \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}_f(\hat{\Sigma})}(\phi(\alpha_i), \alpha_j) = \\ &= \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z_1, z_2\})}(\phi(\alpha_i), \alpha_j) = \bigoplus_{1 \leq i, j \leq k} \text{hom}_{\mathcal{F}(\Sigma, \{z_1, z_2\})}(\alpha_i, \phi^{-1}(\alpha_j)) = N^{2bp}(\phi^{-1}). \end{aligned}$$

Now we turn our attention to [Sei17, Conjecture 7.18], which is stated for Lefschetz fibrations of any rank. In our case this amounts to the following — there is an open-closed map which gives an isomorphism:

$$HH_*(\mathcal{P}(\phi, \delta \gg 0)) \xrightarrow{\sim} HF_{2bp}^{*+1}(\phi).$$

As a consequence, our double basepoint conjecture stating that

$$HH_*(N^{2bp}(\phi^{-1})) \cong HF^{*+1}(\tilde{\phi}; U_2+, U_3+, U_1-)$$

is a special case of Seidel's conjecture. The one basepoint version, i.e. Conjecture 4.3.1, most likely fits in a similar framework, where instead of the Fukaya-Seidel

category (which is quasi-equivalent to the partially wrapped Fukaya category with two basepoints) one should work with the one basepoint partially wrapped Fukaya category, and construct there an appropriate version of a twisted open-closed map.

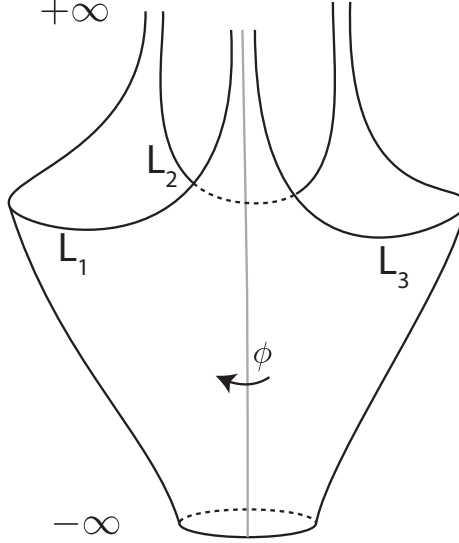


Figure 4.29: The open-closed map counts such holomorphic objects inside $\hat{\Sigma}$. Compare this to [Sei17, Figure 3].

4.6 Appendix. Bimodules for Dehn twists in the genus two surface

The set of curves that we consider in the genus two surface is pictured in Figure 4.6. We list ten DA type bimodules corresponding to the right and left handed Dehn twists along the curves A,B,C,D and E. Then we describe a DD bimodule for the Dehn twist τ_E , and also 6 arc-slide DA bimodules, which we used to compute $N(\tau_C^{-1})$ via factorization $\tau_C^{-1} = \eta^{-1} \mu_4 \mu_3 \mu_2 \mu_1 \eta$.

Bimodule 4.6.1. $N(\tau_A)$

5 generators with their idempotents: $i_1(x_2)_{i_1}$, $i_0(x_3)_{i_0}$, $i_3(x_0)_{i_3}$, $i_2(x_1)_{i_2}$, $i_2(r)_{i_3}$.

Actions: $x_2 \otimes (r23) \longrightarrow r23 \otimes x_2$, $x_2 \otimes (r2) \longrightarrow r2 \otimes x_3$, $x_2 \otimes (r234567) \longrightarrow$

$r234567 \otimes x0, x2 \otimes (r4567) \longrightarrow r4567 \otimes x0, x2 \otimes (r456) \longrightarrow r456 \otimes x1, x2 \otimes (r23456) \longrightarrow$
 $r23456 \otimes x1, x2 \otimes (r234) \longrightarrow r234 \otimes x1, x2 \otimes (r4) \longrightarrow r4 \otimes x1, x2 \otimes (r45) \longrightarrow r456 \otimes r,$
 $x2 \otimes (r2345) \longrightarrow r23456 \otimes r, x3 \otimes (r123) \longrightarrow r123 \otimes x2, x3 \otimes (r1) \longrightarrow r1 \otimes x2,$
 $x3 \otimes (r3) \longrightarrow r3 \otimes x2, x3 \otimes (r12) \longrightarrow r12 \otimes x3, x3 \otimes (r34567) \longrightarrow r34567 \otimes x0, x3 \otimes$
 $(r1234567) \longrightarrow r1234567 \otimes x0, x3 \otimes (r1234) \longrightarrow r1234 \otimes x1, x3 \otimes (r3456) \longrightarrow r3456 \otimes$
 $x1, x3 \otimes (r34) \longrightarrow r34 \otimes x1, x3 \otimes (r123456) \longrightarrow r123456 \otimes x1, x3 \otimes (r345) \longrightarrow r3456 \otimes r,$
 $x3 \otimes (r12345) \longrightarrow r123456 \otimes r, x0 \otimes (r6, r567) \longrightarrow r67 \otimes x0, x0 \otimes (r6, r56) \longrightarrow$
 $r6 \otimes x1, x0 \otimes (r6, r5) \longrightarrow r6 \otimes r, x1 \otimes (r567) \longrightarrow r567 \otimes x0, x1 \otimes (r7) \longrightarrow r7 \otimes x0,$
 $x1 \otimes (r56) \longrightarrow r56 \otimes x1, x1 \otimes (r5) \longrightarrow r56 \otimes r, r \otimes (r67) \longrightarrow r7 \otimes x0, r \otimes () \longrightarrow r5 \otimes x0,$
 $r \otimes (r6) \longrightarrow 1 \otimes x1.$

Bimodule 4.6.2. $N(\tau_A^{-1})$

5 generators with their idempotents: $i_1(x2)_{i1}, i_0(x3)_{i0}, i_3(x0)_{i3}, i_2(x1)_{i2}, i_2(r)_{i3}.$

Actions: $x2 \otimes (r23) \longrightarrow r23 \otimes x2, x2 \otimes (r2) \longrightarrow r2 \otimes x3, x2 \otimes (r234567) \longrightarrow$
 $r234567 \otimes x0, x2 \otimes (r4567) \longrightarrow r4567 \otimes x0, x2 \otimes (r23456, r5) \longrightarrow r2345 \otimes x0,$
 $x2 \otimes (r456, r5) \longrightarrow r45 \otimes x0, x2 \otimes (r4) \longrightarrow r4 \otimes x1, x2 \otimes (r234) \longrightarrow r234 \otimes x1,$
 $x2 \otimes (r456) \longrightarrow r456 \otimes x1, x2 \otimes (r23456) \longrightarrow r23456 \otimes x1, x2 \otimes (r45) \longrightarrow r4 \otimes r,$
 $x2 \otimes (r2345) \longrightarrow r234 \otimes r, x3 \otimes (r1) \longrightarrow r1 \otimes x2, x3 \otimes (r3) \longrightarrow r3 \otimes x2, x3 \otimes$
 $(r123) \longrightarrow r123 \otimes x2, x3 \otimes (r12) \longrightarrow r12 \otimes x3, x3 \otimes (r1234567) \longrightarrow r1234567 \otimes x0,$
 $x3 \otimes (r123456, r5) \longrightarrow r12345 \otimes x0, x3 \otimes (r3456, r5) \longrightarrow r345 \otimes x0, x3 \otimes (r34567) \longrightarrow$
 $r34567 \otimes x0, x3 \otimes (r3456) \longrightarrow r3456 \otimes x1, x3 \otimes (r1234) \longrightarrow r1234 \otimes x1, x3 \otimes$
 $(r123456) \longrightarrow r123456 \otimes x1, x3 \otimes (r34) \longrightarrow r34 \otimes x1, x3 \otimes (r345) \longrightarrow r34 \otimes r,$
 $x3 \otimes (r12345) \longrightarrow r1234 \otimes r, x0 \otimes () \longrightarrow r6 \otimes r, x1 \otimes (r567) \longrightarrow r567 \otimes x0, x1 \otimes (r7) \longrightarrow$
 $r7 \otimes x0, x1 \otimes (r56, r5) \longrightarrow r5 \otimes x0, x1 \otimes (r56) \longrightarrow r56 \otimes x1, x1 \otimes (r5) \longrightarrow 1 \otimes r,$
 $r \otimes (r67) \longrightarrow r567 \otimes x0, r \otimes (r6, r5) \longrightarrow r5 \otimes x0, r \otimes (r6) \longrightarrow r56 \otimes x1.$

Bimodule 4.6.3. $N(\tau_B)$

5 generators with their idempotents: $i_1(x2)_{i1}, i_0(x3)_{i0}, i_3(x0)_{i3}, i_2(x1)_{i2}, i_3(s)_{i2}.$

Actions: $x2 \otimes (r23) \longrightarrow r23 \otimes x2$, $x2 \otimes (r2) \longrightarrow r2 \otimes x3$, $x2 \otimes (r2345) \longrightarrow r2345 \otimes x0$,
 $x2 \otimes (r234567) \longrightarrow r234567 \otimes x0$, $x2 \otimes (r45) \longrightarrow r45 \otimes x0$, $x2 \otimes (r4567) \longrightarrow r4567 \otimes x0$,
 $x2 \otimes (r234) \longrightarrow r234 \otimes x1$, $x2 \otimes (r4) \longrightarrow r4 \otimes x1$, $x2 \otimes (r456) \longrightarrow r4567 \otimes s$,
 $x2 \otimes (r23456) \longrightarrow r234567 \otimes s$, $x3 \otimes (r1) \longrightarrow r1 \otimes x2$, $x3 \otimes (r3) \longrightarrow r3 \otimes x2$,
 $x3 \otimes (r123) \longrightarrow r123 \otimes x2$, $x3 \otimes (r12) \longrightarrow r12 \otimes x3$, $x3 \otimes (r345) \longrightarrow r345 \otimes x0$,
 $x3 \otimes (r12345) \longrightarrow r12345 \otimes x0$, $x3 \otimes (r34567) \longrightarrow r34567 \otimes x0$, $x3 \otimes (r1234567) \longrightarrow$
 $r1234567 \otimes x0$, $x3 \otimes (r34) \longrightarrow r34 \otimes x1$, $x3 \otimes (r1234) \longrightarrow r1234 \otimes x1$, $x3 \otimes (r3456) \longrightarrow$
 $r34567 \otimes s$, $x3 \otimes (r123456) \longrightarrow r1234567 \otimes s$, $x0 \otimes (r67) \longrightarrow r67 \otimes x0$, $x0 \otimes (r6) \longrightarrow$
 $r67 \otimes s$, $x1 \otimes (r7, r67) \longrightarrow r7 \otimes x0$, $x1 \otimes (r567) \longrightarrow r567 \otimes x0$, $x1 \otimes (r5) \longrightarrow r5 \otimes x0$,
 $x1 \otimes (r7, r6) \longrightarrow r7 \otimes s$, $x1 \otimes (r56) \longrightarrow r567 \otimes s$, $s \otimes (r7) \longrightarrow 1 \otimes x0$, $s \otimes () \longrightarrow r6 \otimes x1$.

Bimodule 4.6.4. $N(\tau_B^{-1})$

5 generators with their idempotents: $i1(x2)_{i1}$, $i0(x3)_{i0}$, $i3(x0)_{i3}$, $i2(x1)_{i2}$, $i3(s)_{i2}$.

Actions: $x2 \otimes (r23) \longrightarrow r23 \otimes x2$, $x2 \otimes (r2) \longrightarrow r2 \otimes x3$, $x2 \otimes (r4567) \longrightarrow r4567 \otimes x0$,
 $x2 \otimes (r234567) \longrightarrow r234567 \otimes x0$, $x2 \otimes (r45) \longrightarrow r45 \otimes x0$, $x2 \otimes (r2345) \longrightarrow r2345 \otimes x0$,
 $x2 \otimes (r234) \longrightarrow r234 \otimes x1$, $x2 \otimes (r4) \longrightarrow r4 \otimes x1$, $x2 \otimes (r4567, r6) \longrightarrow r456 \otimes x1$,
 $x2 \otimes (r234567, r6) \longrightarrow r23456 \otimes x1$, $x2 \otimes (r23456) \longrightarrow r2345 \otimes s$, $x2 \otimes (r456) \longrightarrow$
 $r45 \otimes s$, $x3 \otimes (r1) \longrightarrow r1 \otimes x2$, $x3 \otimes (r3) \longrightarrow r3 \otimes x2$, $x3 \otimes (r123) \longrightarrow r123 \otimes x2$,
 $x3 \otimes (r12) \longrightarrow r12 \otimes x3$, $x3 \otimes (r34567) \longrightarrow r34567 \otimes x0$, $x3 \otimes (r345) \longrightarrow r345 \otimes x0$,
 $x3 \otimes (r12345) \longrightarrow r12345 \otimes x0$, $x3 \otimes (r1234567) \longrightarrow r1234567 \otimes x0$, $x3 \otimes (r34) \longrightarrow$
 $r34 \otimes x1$, $x3 \otimes (r1234567, r6) \longrightarrow r123456 \otimes x1$, $x3 \otimes (r1234) \longrightarrow r1234 \otimes x1$,
 $x3 \otimes (r34567, r6) \longrightarrow r3456 \otimes x1$, $x3 \otimes (r3456) \longrightarrow r345 \otimes s$, $x3 \otimes (r123456) \longrightarrow$
 $r12345 \otimes s$, $x0 \otimes (r67) \longrightarrow r67 \otimes x0$, $x0 \otimes (r67, r6) \longrightarrow r6 \otimes x1$, $x0 \otimes (r6) \longrightarrow 1 \otimes s$,
 $x1 \otimes (r567) \longrightarrow r567 \otimes x0$, $x1 \otimes (r5) \longrightarrow r5 \otimes x0$, $x1 \otimes (r567, r6) \longrightarrow r56 \otimes x1$,
 $x1 \otimes (r56) \longrightarrow r5 \otimes s$, $x1 \otimes () \longrightarrow r7 \otimes s$, $s \otimes (r7) \longrightarrow r67 \otimes x0$, $s \otimes (r7, r6) \longrightarrow r6 \otimes x1$.

Bimodule 4.6.5. $N(\tau_C)$

16 generators with their idempotents: $i1(x2)_{i1}$, $i0(x3)_{i0}$, $i3(x0)_{i3}$, $i2(x1)_{i2}$, $i1(t8)_{i1}$,

$$i_2(t_9)_{i_0}, i_1(t_6)_{i_2}, i_2(t_7)_{i_1}, i_1(t_4)_{i_3}, i_2(t_5)_{i_2}, i_1(t_2)_{i_2}, i_2(t_3)_{i_3}, i_2(t_1)_{i_2}, i_1(t_{10})_{i_0}, i_2(t_{11})_{i_1}, i_1(t_{12})_{i_1}.$$

$$\begin{aligned} \text{Actions: } & x_2 \otimes (r_{23}) \longrightarrow r_{23} \otimes x_2, x_2 \otimes (r_2) \longrightarrow r_2 \otimes x_3, x_2 \otimes (r_{2345}) \longrightarrow r_{2345} \otimes x_0, \\ & x_2 \otimes (r_{45}) \longrightarrow r_{45} \otimes x_0, x_2 \otimes (r_{23456}) \longrightarrow r_{23456} \otimes x_1, x_2 \otimes (r_4) \longrightarrow r_4 \otimes x_1, \\ & x_2 \otimes (r_{234}) \longrightarrow r_{234} \otimes x_1, x_2 \otimes (r_{456}) \longrightarrow r_{456} \otimes x_1, x_3 \otimes (r_3) \longrightarrow r_3 \otimes x_2, \\ & x_3 \otimes (r_{345}) \longrightarrow r_{345} \otimes x_0, x_3 \otimes (r_{1234567}) \longrightarrow r_{1234567} \otimes x_0, x_3 \otimes (r_{34}) \longrightarrow r_{34} \otimes x_1, \\ & x_3 \otimes (r_{3456}) \longrightarrow r_{3456} \otimes x_1, x_3 \otimes (r_{12}) \longrightarrow r_{123456} \otimes t_9, x_3 \otimes (r_{123}) \longrightarrow r_{123456} \otimes t_7, \\ & x_3 \otimes (r_{1234}) \longrightarrow r_{123456} \otimes t_5, x_3 \otimes (r_1, r_4) \longrightarrow r_1 \otimes t_2, x_3 \otimes (r_{12345}) \longrightarrow r_{123456} \otimes t_3, \\ & x_3 \otimes (r_{123456}) \longrightarrow r_{123456} \otimes t_1, x_3 \otimes (r_1) \longrightarrow r_{123456} \otimes t_{11}, x_0 \otimes (r_6) \longrightarrow r_6 \otimes x_1, \\ & x_1 \otimes (r_5) \longrightarrow r_5 \otimes x_0, x_1 \otimes (r_{56}) \longrightarrow r_{56} \otimes x_1, t_8 \otimes () \longrightarrow r_{23} \otimes x_2, t_8 \otimes (r_4) \longrightarrow 1 \otimes t_6, \\ & t_8 \otimes () \longrightarrow r_4 \otimes t_7, t_8 \otimes (r_{45}) \longrightarrow 1 \otimes t_4, t_8 \otimes (r_{456}) \longrightarrow 1 \otimes t_2, t_9 \otimes (r_{34567}) \longrightarrow r_7 \otimes x_0, \\ & t_9 \otimes (r_3) \longrightarrow 1 \otimes t_7, t_9 \otimes (r_{34}) \longrightarrow 1 \otimes t_5, t_9 \otimes (r_{345}) \longrightarrow 1 \otimes t_3, t_9 \otimes (r_{3456}) \longrightarrow 1 \otimes t_1, \\ & t_6 \otimes () \longrightarrow r_{234} \otimes x_1, t_6 \otimes (r_5) \longrightarrow 1 \otimes t_4, t_6 \otimes () \longrightarrow r_4 \otimes t_5, t_6 \otimes (r_{56}) \longrightarrow 1 \otimes t_2, \\ & t_7 \otimes (r_{4567}) \longrightarrow r_7 \otimes x_0, t_7 \otimes (r_4) \longrightarrow 1 \otimes t_5, t_7 \otimes (r_{45}) \longrightarrow 1 \otimes t_3, t_7 \otimes (r_{456}) \longrightarrow 1 \otimes t_1, \\ & t_4 \otimes () \longrightarrow r_{2345} \otimes x_0, t_4 \otimes (r_6) \longrightarrow 1 \otimes t_2, t_4 \otimes () \longrightarrow r_4 \otimes t_3, t_5 \otimes (r_{567}) \longrightarrow r_7 \otimes x_0, \\ & t_5 \otimes (r_5) \longrightarrow 1 \otimes t_3, t_5 \otimes (r_{56}) \longrightarrow 1 \otimes t_1, t_2 \otimes () \longrightarrow r_{23456} \otimes x_1, t_2 \otimes () \longrightarrow r_4 \otimes t_1, \\ & t_3 \otimes (r_{67}) \longrightarrow r_7 \otimes x_0, t_3 \otimes (r_6) \longrightarrow 1 \otimes t_1, t_1 \otimes (r_7) \longrightarrow r_7 \otimes x_0, t_{10} \otimes () \longrightarrow r_2 \otimes x_3, \\ & t_{10} \otimes (r_3) \longrightarrow 1 \otimes t_8, t_{10} \otimes () \longrightarrow r_4 \otimes t_9, t_{10} \otimes (r_{34}) \longrightarrow 1 \otimes t_6, t_{10} \otimes (r_{345}) \longrightarrow 1 \otimes t_4, \\ & t_{10} \otimes (r_{3456}) \longrightarrow 1 \otimes t_2, t_{11} \otimes (r_{45}) \longrightarrow r_5 \otimes x_0, t_{11} \otimes (r_{234567}) \longrightarrow r_7 \otimes x_0, \\ & t_{11} \otimes (r_4) \longrightarrow 1 \otimes x_1, t_{11} \otimes (r_{456}) \longrightarrow r_{56} \otimes x_1, t_{11} \otimes (r_2) \longrightarrow 1 \otimes t_9, t_{11} \otimes (r_{23}) \longrightarrow \\ & 1 \otimes t_7, t_{11} \otimes (r_{234}) \longrightarrow 1 \otimes t_5, t_{11} \otimes (r_{2345}) \longrightarrow 1 \otimes t_3, t_{11} \otimes (r_{23456}) \longrightarrow 1 \otimes t_1, \\ & t_{12} \otimes () \longrightarrow 1 \otimes x_2, t_{12} \otimes (r_{23}) \longrightarrow 1 \otimes t_8, t_{12} \otimes (r_{234}) \longrightarrow 1 \otimes t_6, t_{12} \otimes (r_{2345}) \longrightarrow \\ & 1 \otimes t_4, t_{12} \otimes (r_{23456}) \longrightarrow 1 \otimes t_2, t_{12} \otimes (r_2) \longrightarrow 1 \otimes t_{10}, t_{12} \otimes () \longrightarrow r_4 \otimes t_{11}. \end{aligned}$$

Bimodule 4.6.6. $N(\tau_C^{-1})$

$$\begin{aligned} & 16 \text{ generators with their idempotents: } i_2(x_2)_{i_2}, i_3(x_3)_{i_3}, i_0(x_0)_{i_0}, i_1(x_1)_{i_1}, i_2(t_8)_{i_2}, \\ & i_1(t_9)_{i_3}, i_2(t_6)_{i_1}, i_1(t_7)_{i_2}, i_2(t_4)_{i_0}, i_1(t_5)_{i_1}, i_2(t_2)_{i_1}, i_1(t_3)_{i_0}, i_1(t_1)_{i_1}, i_2(t_{10})_{i_3}, i_1(t_{11})_{i_2}, \end{aligned}$$

$i_2(t12)_{i_2}$.

Actions: $x2 \otimes (r56) \longrightarrow r56 \otimes x2$, $x2 \otimes (r5) \longrightarrow r5 \otimes x3$, $x2 \otimes () \longrightarrow r56 \otimes t8$,
 $x2 \otimes () \longrightarrow 1 \otimes t12$, $x3 \otimes (r6) \longrightarrow r6 \otimes x2$, $x3 \otimes () \longrightarrow r6 \otimes t10$, $x0 \otimes (r34) \longrightarrow r34 \otimes x2$,
 $x0 \otimes (r3456) \longrightarrow r3456 \otimes x2$, $x0 \otimes (r345) \longrightarrow r345 \otimes x3$, $x0 \otimes (r1234567) \longrightarrow$
 $r1234567 \otimes x3$, $x0 \otimes (r3) \longrightarrow r3 \otimes x1$, $x0 \otimes (r12345) \longrightarrow r1 \otimes t9$, $x0 \otimes (r1234) \longrightarrow$
 $r1 \otimes t7$, $x0 \otimes () \longrightarrow r3456 \otimes t4$, $x0 \otimes (r123) \longrightarrow r1 \otimes t5$, $x0 \otimes (r12) \longrightarrow r1 \otimes t3$,
 $x0 \otimes (r1) \longrightarrow r1 \otimes t1$, $x0 \otimes (r123456) \longrightarrow r1 \otimes t11$, $x0 \otimes (r34) \longrightarrow r3 \otimes t11$,
 $x1 \otimes (r4) \longrightarrow r4 \otimes x2$, $x1 \otimes (r23456) \longrightarrow r23456 \otimes x2$, $x1 \otimes (r456) \longrightarrow r456 \otimes x2$,
 $x1 \otimes (r234) \longrightarrow r234 \otimes x2$, $x1 \otimes (r45) \longrightarrow r45 \otimes x3$, $x1 \otimes (r2345) \longrightarrow r2345 \otimes x3$,
 $x1 \otimes (r2) \longrightarrow r2 \otimes x0$, $x1 \otimes (r23) \longrightarrow r23 \otimes x1$, $x1 \otimes () \longrightarrow r456 \otimes t6$, $x1 \otimes () \longrightarrow$
 $r23456 \otimes t2$, $x1 \otimes (r234) \longrightarrow r23 \otimes t11$, $x1 \otimes (r4) \longrightarrow 1 \otimes t11$, $t8 \otimes (r5) \longrightarrow 1 \otimes t10$,
 $t8 \otimes (r56) \longrightarrow 1 \otimes t12$, $t9 \otimes (r67) \longrightarrow r234567 \otimes x3$, $t9 \otimes () \longrightarrow r4 \otimes t10$, $t9 \otimes (r6) \longrightarrow$
 $1 \otimes t11$, $t6 \otimes (r4) \longrightarrow 1 \otimes t8$, $t6 \otimes (r45) \longrightarrow 1 \otimes t10$, $t6 \otimes (r456) \longrightarrow 1 \otimes t12$,
 $t7 \otimes (r567) \longrightarrow r234567 \otimes x3$, $t7 \otimes () \longrightarrow r4 \otimes t8$, $t7 \otimes (r5) \longrightarrow 1 \otimes t9$, $t7 \otimes (r56) \longrightarrow$
 $1 \otimes t11$, $t4 \otimes (r34) \longrightarrow 1 \otimes t8$, $t4 \otimes (r3) \longrightarrow 1 \otimes t6$, $t4 \otimes (r345) \longrightarrow 1 \otimes t10$,
 $t4 \otimes (r3456) \longrightarrow 1 \otimes t12$, $t5 \otimes (r4567) \longrightarrow r234567 \otimes x3$, $t5 \otimes (r45) \longrightarrow 1 \otimes t9$,
 $t5 \otimes () \longrightarrow r4 \otimes t6$, $t5 \otimes (r4) \longrightarrow 1 \otimes t7$, $t5 \otimes (r456) \longrightarrow 1 \otimes t11$, $t2 \otimes (r4, r7) \longrightarrow r7 \otimes x3$,
 $t2 \otimes (r234) \longrightarrow 1 \otimes t8$, $t2 \otimes (r23) \longrightarrow 1 \otimes t6$, $t2 \otimes (r2) \longrightarrow 1 \otimes t4$, $t2 \otimes (r2345) \longrightarrow 1 \otimes t10$,
 $t2 \otimes (r23456) \longrightarrow 1 \otimes t12$, $t3 \otimes (r34567) \longrightarrow r234567 \otimes x3$, $t3 \otimes (r345) \longrightarrow 1 \otimes t9$,
 $t3 \otimes (r34) \longrightarrow 1 \otimes t7$, $t3 \otimes () \longrightarrow r4 \otimes t4$, $t3 \otimes (r3) \longrightarrow 1 \otimes t5$, $t3 \otimes (r3456) \longrightarrow 1 \otimes t11$,
 $t1 \otimes (r234567) \longrightarrow r234567 \otimes x3$, $t1 \otimes (r2345) \longrightarrow 1 \otimes t9$, $t1 \otimes (r234) \longrightarrow 1 \otimes t7$,
 $t1 \otimes (r23) \longrightarrow 1 \otimes t5$, $t1 \otimes () \longrightarrow r4 \otimes t2$, $t1 \otimes (r2) \longrightarrow 1 \otimes t3$, $t1 \otimes (r23456) \longrightarrow 1 \otimes t11$,
 $t10 \otimes (r6) \longrightarrow 1 \otimes t12$, $t11 \otimes (r7) \longrightarrow r234567 \otimes x3$, $t11 \otimes () \longrightarrow r4 \otimes t12$.

Bimodule 4.6.7. $N(\tau_D)$

5 generators with their idempotents: $i_2(x2)_{i_2}$, $i_3(x3)_{i_3}$, $i_0(x0)_{i_0}$, $i_1(x1)_{i_1}$, $i_0(s)_{i_1}$.

Actions: $x2 \otimes (r56) \longrightarrow r56 \otimes x2$, $x2 \otimes (r7) \longrightarrow r7 \otimes x3$, $x2 \otimes (r5) \longrightarrow r5 \otimes x3$,

$x2 \otimes (r567) \longrightarrow r567 \otimes x3, x3 \otimes (r6) \longrightarrow r6 \otimes x2, x3 \otimes (r67) \longrightarrow r67 \otimes x3, x0 \otimes (r1234) \longrightarrow r1234 \otimes x2, x0 \otimes (r123456) \longrightarrow r123456 \otimes x2, x0 \otimes (r34) \longrightarrow r34 \otimes x2, x0 \otimes (r3456) \longrightarrow r3456 \otimes x2, x0 \otimes (r12345) \longrightarrow r12345 \otimes x3, x0 \otimes (r1234567) \longrightarrow r1234567 \otimes x3, x0 \otimes (r345) \longrightarrow r345 \otimes x3, x0 \otimes (r34567) \longrightarrow r34567 \otimes x3, x0 \otimes (r12) \longrightarrow r12 \otimes x0, x0 \otimes (r123) \longrightarrow r123 \otimes x1, x0 \otimes (r3) \longrightarrow r3 \otimes x1, x0 \otimes (r1) \longrightarrow r12 \otimes s, x1 \otimes (r2, r1234) \longrightarrow r234 \otimes x2, x1 \otimes (r2, r123456) \longrightarrow r23456 \otimes x2, x1 \otimes (r4) \longrightarrow r4 \otimes x2, x1 \otimes (r456) \longrightarrow r456 \otimes x2, x1 \otimes (r2, r12345) \longrightarrow r2345 \otimes x3, x1 \otimes (r45) \longrightarrow r45 \otimes x3, x1 \otimes (r2, r1234567) \longrightarrow r234567 \otimes x3, x1 \otimes (r4567) \longrightarrow r4567 \otimes x3, x1 \otimes (r2, r12) \longrightarrow r2 \otimes x0, x1 \otimes (r2, r123) \longrightarrow r23 \otimes x1, x1 \otimes (r2, r1) \longrightarrow r2 \otimes s, s \otimes (r23456) \longrightarrow r3456 \otimes x2, s \otimes (r234) \longrightarrow r34 \otimes x2, s \otimes (r234567) \longrightarrow r34567 \otimes x3, s \otimes (r2345) \longrightarrow r345 \otimes x3, s \otimes (r2) \longrightarrow 1 \otimes x0, s \otimes (r23) \longrightarrow r3 \otimes x1, s \otimes () \longrightarrow r1 \otimes x1.$

Bimodule 4.6.8. $N(\tau_D^{-1})$

5 generators with their idempotents: $i_2(x2)_{i_2}, i_3(x3)_{i_3}, i_0(x0)_{i_0}, i_1(x1)_{i_1}, i_0(s)_{i_1}$.

Actions: $x2 \otimes (r56) \longrightarrow r56 \otimes x2, x2 \otimes (r7) \longrightarrow r7 \otimes x3, x2 \otimes (r5) \longrightarrow r5 \otimes x3, x2 \otimes (r567) \longrightarrow r567 \otimes x3, x3 \otimes (r6) \longrightarrow r6 \otimes x2, x3 \otimes (r67) \longrightarrow r67 \otimes x3, x0 \otimes (r123456) \longrightarrow r123456 \otimes x2, x0 \otimes (r34) \longrightarrow r34 \otimes x2, x0 \otimes (r1234) \longrightarrow r1234 \otimes x2, x0 \otimes (r3456) \longrightarrow r3456 \otimes x2, x0 \otimes (r12345) \longrightarrow r12345 \otimes x3, x0 \otimes (r345) \longrightarrow r345 \otimes x3, x0 \otimes (r1234567) \longrightarrow r1234567 \otimes x3, x0 \otimes (r34567) \longrightarrow r34567 \otimes x3, x0 \otimes (r12) \longrightarrow r12 \otimes x0, x0 \otimes (r12, r1) \longrightarrow r1 \otimes x1, x0 \otimes (r123) \longrightarrow r123 \otimes x1, x0 \otimes (r3) \longrightarrow r3 \otimes x1, x0 \otimes (r1) \longrightarrow 1 \otimes s, x1 \otimes (r456) \longrightarrow r456 \otimes x2, x1 \otimes (r4) \longrightarrow r4 \otimes x2, x1 \otimes (r4567) \longrightarrow r4567 \otimes x3, x1 \otimes (r45) \longrightarrow r45 \otimes x3, x1 \otimes () \longrightarrow r2 \otimes s, s \otimes (r234) \longrightarrow r1234 \otimes x2, s \otimes (r23456) \longrightarrow r123456 \otimes x2, s \otimes (r234567) \longrightarrow r1234567 \otimes x3, s \otimes (r2345) \longrightarrow r12345 \otimes x3, s \otimes (r2) \longrightarrow r12 \otimes x0, s \otimes (r23) \longrightarrow r123 \otimes x1, s \otimes (r2, r1) \longrightarrow r1 \otimes x1.$

Bimodule 4.6.9. $N(\tau_E)$

5 generators with their idempotents: $i_2(x2)_{i_2}, i_3(x3)_{i_3}, i_0(x0)_{i_0}, i_1(x1)_{i_1}, i_1(r)_{i_0}$.

Actions: $x2 \otimes (r56) \longrightarrow r56 \otimes x2, x2 \otimes (r7) \longrightarrow r7 \otimes x3, x2 \otimes (r567) \longrightarrow r567 \otimes x3,$

$x2 \otimes (r5) \longrightarrow r5 \otimes x3, x3 \otimes (r6) \longrightarrow r6 \otimes x2, x3 \otimes (r67) \longrightarrow r67 \otimes x3, x0 \otimes (r123456) \longrightarrow$
 $r123456 \otimes x2, x0 \otimes (r3, r23456) \longrightarrow r3456 \otimes x2, x0 \otimes (r1234) \longrightarrow r1234 \otimes x2,$
 $x0 \otimes (r3, r234) \longrightarrow r34 \otimes x2, x0 \otimes (r1234567) \longrightarrow r1234567 \otimes x3, x0 \otimes (r12345) \longrightarrow$
 $r12345 \otimes x3, x0 \otimes (r3, r2345) \longrightarrow r345 \otimes x3, x0 \otimes (r3, r234567) \longrightarrow r34567 \otimes x3,$
 $x0 \otimes (r123) \longrightarrow r123 \otimes x1, x0 \otimes (r3, r23) \longrightarrow r3 \otimes x1, x0 \otimes (r1) \longrightarrow r1 \otimes x1,$
 $x0 \otimes (r3, r2) \longrightarrow r3 \otimes r, x0 \otimes (r12) \longrightarrow r123 \otimes r, x1 \otimes (r4) \longrightarrow r4 \otimes x2, x1 \otimes (r234) \longrightarrow$
 $r234 \otimes x2, x1 \otimes (r456) \longrightarrow r456 \otimes x2, x1 \otimes (r23456) \longrightarrow r23456 \otimes x2, x1 \otimes (r2345) \longrightarrow$
 $r2345 \otimes x3, x1 \otimes (r4567) \longrightarrow r4567 \otimes x3, x1 \otimes (r45) \longrightarrow r45 \otimes x3, x1 \otimes (r234567) \longrightarrow$
 $r234567 \otimes x3, x1 \otimes (r23) \longrightarrow r23 \otimes x1, x1 \otimes (r2) \longrightarrow r23 \otimes r, r \otimes (r3456) \longrightarrow r456 \otimes x2,$
 $r \otimes (r34) \longrightarrow r4 \otimes x2, r \otimes (r345) \longrightarrow r45 \otimes x3, r \otimes (r34567) \longrightarrow r4567 \otimes x3,$
 $r \otimes () \longrightarrow r2 \otimes x0, r \otimes (r3) \longrightarrow 1 \otimes x1.$

Bimodule 4.6.10. $N(\tau_E^{-1})$

5 generators with their idempotents: $i_2(x2)_{i_2}, i_3(x3)_{i_3}, i_0(x0)_{i_0}, i_1(x1)_{i_1}, i_1(r)_{i_0}.$

Actions: $x2 \otimes (r56) \longrightarrow r56 \otimes x2, x2 \otimes (r7) \longrightarrow r7 \otimes x3, x2 \otimes (r567) \longrightarrow r567 \otimes x3,$
 $x2 \otimes (r5) \longrightarrow r5 \otimes x3, x3 \otimes (r6) \longrightarrow r6 \otimes x2, x3 \otimes (r67) \longrightarrow r67 \otimes x3, x0 \otimes (r1234) \longrightarrow$
 $r1234 \otimes x2, x0 \otimes (r123456) \longrightarrow r123456 \otimes x2, x0 \otimes (r12345) \longrightarrow r12345 \otimes x3,$
 $x0 \otimes (r1234567) \longrightarrow r1234567 \otimes x3, x0 \otimes (r123, r2) \longrightarrow r12 \otimes x0, x0 \otimes (r123) \longrightarrow$
 $r123 \otimes x1, x0 \otimes (r1) \longrightarrow r1 \otimes x1, x0 \otimes (r12) \longrightarrow r1 \otimes r, x0 \otimes () \longrightarrow r3 \otimes r,$
 $x1 \otimes (r4) \longrightarrow r4 \otimes x2, x1 \otimes (r234) \longrightarrow r234 \otimes x2, x1 \otimes (r456) \longrightarrow r456 \otimes x2,$
 $x1 \otimes (r23456) \longrightarrow r23456 \otimes x2, x1 \otimes (r234567) \longrightarrow r234567 \otimes x3, x1 \otimes (r2345) \longrightarrow$
 $r2345 \otimes x3, x1 \otimes (r4567) \longrightarrow r4567 \otimes x3, x1 \otimes (r45) \longrightarrow r45 \otimes x3, x1 \otimes (r23, r2) \longrightarrow$
 $r2 \otimes x0, x1 \otimes (r23) \longrightarrow r23 \otimes x1, x1 \otimes (r2) \longrightarrow 1 \otimes r, r \otimes (r3456) \longrightarrow r23456 \otimes x2,$
 $r \otimes (r34) \longrightarrow r234 \otimes x2, r \otimes (r34567) \longrightarrow r234567 \otimes x3, r \otimes (r345) \longrightarrow r2345 \otimes x3,$
 $r \otimes (r3, r2) \longrightarrow r2 \otimes x0, r \otimes (r3) \longrightarrow r23 \otimes x1.$

Bimodule 4.6.11. $\mathcal{A}_1(\mathcal{Z}_{g=2})DD(\tau_E)\mathcal{A}_1^\dagger(\mathcal{Z}_{g=2})$

5 generators with their idempotents:

$$\begin{aligned}
& i_2(x_2)_{|(0,2),(1,3),(5,7)|}, \\
& i_3(x_3)_{|(0,2),(1,3),(4,6)|}, \\
& i_0(x_0)_{|(1,3),(4,6),(5,7)|}, \\
& i_1(x_1)_{|(0,2),(4,6),(5,7)|}, \\
& i_1(r)_{|(1,3),(4,6),(5,7)|}.
\end{aligned}$$

Every action in arc-slide bimodules has its type, see [LipOzsThu14, Definition 4.19] for the relevant here case of an under-slide. We list all the actions grouped by their types:

U-1 type:

$$\begin{aligned}
& x_2 \longrightarrow r567 \otimes x_3 \otimes |(0,2), (1,3), (4 \rightarrow 7)|, \quad x_2 \longrightarrow r5 \otimes x_3 \otimes |(0,2), (1,3), (4 \rightarrow 5)|, \\
& x_2 \longrightarrow r7 \otimes x_3 \otimes |(0,2), (1,3), (6 \rightarrow 7)|, \quad x_3 \longrightarrow r6 \otimes x_2 \otimes |(0,2), (1,3), (5 \rightarrow 6)|, \\
& x_0 \longrightarrow r1234 \otimes x_2 \otimes |(1,3), (5,7), (0 \rightarrow 4)|, \quad x_0 \longrightarrow r123456 \otimes x_2 \otimes |(1,3), (5,7), (0 \rightarrow 6)|, \\
& x_0 \longrightarrow r1234567 \otimes x_3 \otimes |(1,3), (4,6), (0 \rightarrow 7)|, \quad x_0 \longrightarrow r12345 \otimes x_3 \otimes |(1,3), (4,6), (0 \rightarrow 5)|, \\
& x_0 \longrightarrow r1 \otimes x_1 \otimes |(4,6), (5,7), (0 \rightarrow 1)|, \quad x_0 \longrightarrow r123 \otimes x_1 \otimes |(4,6), (5,7), (0 \rightarrow 3)|, \\
& x_1 \longrightarrow r4 \otimes x_2 \otimes |(0,2), (5,7), (3 \rightarrow 4)|, \quad x_1 \longrightarrow r456 \otimes x_2 \otimes |(0,2), (5,7), (3 \rightarrow 6)|, \\
& x_1 \longrightarrow r23456 \otimes x_2 \otimes |(0,2), (5,7), (1 \rightarrow 6)|, \quad x_1 \longrightarrow r234 \otimes x_2 \otimes |(0,2), (5,7), (1 \rightarrow 4)|, \\
& x_1 \longrightarrow r4567 \otimes x_3 \otimes |(0,2), (4,6), (3 \rightarrow 7)|, \quad x_1 \longrightarrow r234567 \otimes x_3 \otimes |(0,2), (4,6), (1 \rightarrow 7)|, \\
& x_1 \longrightarrow r45 \otimes x_3 \otimes |(0,2), (4,6), (3 \rightarrow 5)|, \quad x_1 \longrightarrow r2345 \otimes x_3 \otimes |(0,2), (4,6), (1 \rightarrow 5)|.
\end{aligned}$$

U-2 type:

$$r \longrightarrow r2 \otimes x_0 \otimes 1, \quad r \longrightarrow 1 \otimes x_1 \otimes |(4,6), (5,7), (2 \rightarrow 3)|.$$

U-3 type:

$$\begin{aligned}
& r \longrightarrow r4 \otimes x_2 \otimes |(1,3), (5,7), (2 \rightarrow 4)|, \quad r \longrightarrow r456 \otimes x_2 \otimes |(1,3), (5,7), (2 \rightarrow 6)|, \\
& r \longrightarrow r45 \otimes x_3 \otimes |(1,3), (4,6), (2 \rightarrow 5)|, \quad r \longrightarrow r4567 \otimes x_3 \otimes |(1,3), (4,6), (2 \rightarrow 7)|.
\end{aligned}$$

U-4 type:

$$x_1 \longrightarrow r23 \otimes r \otimes |(4,6), (5,7), (1 \rightarrow 2)|, \quad x_0 \longrightarrow r3 \otimes r \otimes |(4,6), (5,7), (1 \rightarrow 3)|.$$

U-6 type:

$x0 \longrightarrow r34 \otimes x2 \otimes |(5, 7), (1 \rightarrow 4), (2 \rightarrow 3)|$, $x0 \longrightarrow r3456 \otimes x2 \otimes |(5, 7), (1 \rightarrow 6), (2 \rightarrow 3)|$,
 $x0 \longrightarrow r345 \otimes x3 \otimes |(4, 6), (1 \rightarrow 5), (2 \rightarrow 3)|$, $x0 \longrightarrow r34567 \otimes x3 \otimes |(4, 6), (1 \rightarrow 7), (2 \rightarrow 3)|$.

Bimodule 4.6.12. $N(\eta)$

5 generators with their idempotents: $i_2(x2)_{|(3,5)|}$, $i_3(x3)_{|(4,7)|}$, $i_0(x0)_{|(0,2)|}$, $i_1(x1)_{|(1,6)|}$,
 $i_2(r)_{|(1,6)|}$.

Actions: $x2 \otimes (|(3 \rightarrow 5)|) \longrightarrow r56 \otimes x2$, $x2 \otimes (|(3 \rightarrow 7)|) \longrightarrow r567 \otimes x3$, $x2 \otimes (|(5 \rightarrow 7)|) \longrightarrow r7 \otimes x3$, $x2 \otimes (|(3 \rightarrow 4)|) \longrightarrow r5 \otimes x3$, $x2 \otimes (|(5 \rightarrow 6)|) \longrightarrow 1 \otimes r$,
 $x2 \otimes (|(3 \rightarrow 6)|) \longrightarrow r56 \otimes r$, $x3 \otimes (|(4 \rightarrow 5)|) \longrightarrow r6 \otimes x2$, $x3 \otimes (|(4 \rightarrow 7)|) \longrightarrow r67 \otimes x3$,
 $x3 \otimes (|(4 \rightarrow 6)|) \longrightarrow r6 \otimes r$, $x0 \otimes (|(2 \rightarrow 3)|) \longrightarrow r34 \otimes x2$, $x0 \otimes (|(2 \rightarrow 5)|) \longrightarrow r3456 \otimes x2$,
 $x0 \otimes (|(0 \rightarrow 5)|) \longrightarrow r123456 \otimes x2$, $x0 \otimes (|(0 \rightarrow 3)|) \longrightarrow r1234 \otimes x2$, $x0 \otimes (|(0 \rightarrow 4)|) \longrightarrow r12345 \otimes x3$,
 $x0 \otimes (|(2 \rightarrow 7)|) \longrightarrow r34567 \otimes x3$, $x0 \otimes (|(0 \rightarrow 7)|) \longrightarrow r1234567 \otimes x3$,
 $x0 \otimes (|(2 \rightarrow 4)|) \longrightarrow r345 \otimes x3$, $x0 \otimes (|(0 \rightarrow 2)|) \longrightarrow r12 \otimes x0$, $x0 \otimes (|(2 \rightarrow 3)|, |(5 \rightarrow 6)|) \longrightarrow r3 \otimes x1$,
 $x0 \otimes (|(0 \rightarrow 1)|) \longrightarrow r1 \otimes x1$, $x0 \otimes (|(0 \rightarrow 3)|, |(5 \rightarrow 6)|) \longrightarrow r123 \otimes x1$,
 $x0 \otimes (|(0 \rightarrow 6)|) \longrightarrow r123456 \otimes r$, $x0 \otimes (|(2 \rightarrow 6)|) \longrightarrow r3456 \otimes r$, $x1 \otimes (|(1 \rightarrow 3)|) \longrightarrow r234 \otimes x2$,
 $x1 \otimes (|(1 \rightarrow 5)|) \longrightarrow r23456 \otimes x2$, $x1 \otimes (|(1 \rightarrow 7)|) \longrightarrow r234567 \otimes x3$,
 $x1 \otimes (|(1 \rightarrow 4)|) \longrightarrow r2345 \otimes x3$, $x1 \otimes (|(1 \rightarrow 2)|) \longrightarrow r2 \otimes x0$, $x1 \otimes (|(1 \rightarrow 3)|, |(5 \rightarrow 6)|) \longrightarrow r23 \otimes x1$,
 $x1 \otimes (|(1 \rightarrow 6)|) \longrightarrow r23456 \otimes r$, $x1 \otimes () \longrightarrow r4 \otimes r$, $r \otimes (|(6 \rightarrow 7)|) \longrightarrow r7 \otimes x3$.

Bimodule 4.6.13. $N(\mu_1)$

5 generators with their idempotents: $|(0,2)|(y1)_{|(0,3)|}$, $|(3,5)|(y0)_{|(2,4)|}$, $|(4,7)|(y3)_{|(5,7)|}$, $|(1,6)|(y2)_{|(1,6)|}$,
 $|(1,6)|(r)_{|(2,4)|}$.

Actions: $y1 \otimes (|(0 \rightarrow 3)|) \longrightarrow |(0 \rightarrow 2)| \otimes y1$, $y1 \otimes (|(3 \rightarrow 4)|) \longrightarrow |(2 \rightarrow 3)| \otimes y0$,
 $y1 \otimes (|(3 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(2 \rightarrow 5)| \otimes y0$, $y1 \otimes (|(0 \rightarrow 4)|) \longrightarrow |(0 \rightarrow 3)| \otimes y0$,
 $y1 \otimes (|(0 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(0 \rightarrow 5)| \otimes y0$, $y1 \otimes (|(0 \rightarrow 5)|) \longrightarrow |(0 \rightarrow 4)| \otimes y3$,
 $y1 \otimes (|(0 \rightarrow 7)|) \longrightarrow |(0 \rightarrow 7)| \otimes y3$, $y1 \otimes (|(3 \rightarrow 5)|) \longrightarrow |(2 \rightarrow 4)| \otimes y3$, $y1 \otimes (|(3 \rightarrow$

$$\begin{aligned}
& 7)| \longrightarrow |(2 \rightarrow 7)| \otimes y3, y1 \otimes (|(3 \rightarrow 6)|) \longrightarrow |(2 \rightarrow 6)| \otimes y2, y1 \otimes (|(0 \rightarrow 6)|) \longrightarrow \\
& |(0 \rightarrow 6)| \otimes y2, y1 \otimes (|(0 \rightarrow 1)|) \longrightarrow |(0 \rightarrow 1)| \otimes y2, y1 \otimes (|(0 \rightarrow 2)|) \longrightarrow |(0 \rightarrow 1)| \otimes r, \\
& y0 \otimes (|(4 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(3 \rightarrow 5)| \otimes y0, y0 \otimes (|(4 \rightarrow 7)|) \longrightarrow |(3 \rightarrow 7)| \otimes y3, \\
& y0 \otimes (|(4 \rightarrow 5)|) \longrightarrow |(3 \rightarrow 4)| \otimes y3, y0 \otimes (|(4 \rightarrow 6)|) \longrightarrow |(3 \rightarrow 6)| \otimes y2, y0 \otimes () \longrightarrow \\
& |(5 \rightarrow 6)| \otimes r, y3 \otimes (|(5 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(4 \rightarrow 5)| \otimes y0, y3 \otimes (|(5 \rightarrow 7)|) \longrightarrow |(4 \rightarrow \\
& 7)| \otimes y3, y3 \otimes (|(5 \rightarrow 6)|) \longrightarrow |(4 \rightarrow 6)| \otimes y2, y2 \otimes (|(1 \rightarrow 3)|) \longrightarrow |(1 \rightarrow 2)| \otimes y1, \\
& y2 \otimes (|(1 \rightarrow 4)|) \longrightarrow |(1 \rightarrow 3)| \otimes y0, y2 \otimes (|(1 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(1 \rightarrow 5)| \otimes y0, \\
& y2 \otimes (|(1 \rightarrow 5)|) \longrightarrow |(1 \rightarrow 4)| \otimes y3, y2 \otimes (|(1 \rightarrow 7)|) \longrightarrow |(1 \rightarrow 7)| \otimes y3, y2 \otimes (|(6 \rightarrow \\
& 7)|) \longrightarrow |(6 \rightarrow 7)| \otimes y3, y2 \otimes (|(1 \rightarrow 6)|) \longrightarrow |(1 \rightarrow 6)| \otimes y2, y2 \otimes (|(1 \rightarrow 2)|) \longrightarrow \\
& 1 \otimes r, r \otimes (|(2 \rightarrow 3)|) \longrightarrow |(1 \rightarrow 2)| \otimes y1, r \otimes (|(2 \rightarrow 4)|) \longrightarrow |(1 \rightarrow 3)| \otimes y0, \\
& r \otimes (|(2 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(1 \rightarrow 5)| \otimes y0, r \otimes (|(2 \rightarrow 7)|) \longrightarrow |(1 \rightarrow 7)| \otimes y3, \\
& r \otimes (|(2 \rightarrow 5)|) \longrightarrow |(1 \rightarrow 4)| \otimes y3, r \otimes (|(2 \rightarrow 6)|) \longrightarrow |(1 \rightarrow 6)| \otimes y2.
\end{aligned}$$

Bimodule 4.6.14. $N(\mu_2)$

5 generators with their idempotents: $_{|(1,6)|}(y1)_{|(1,6)|}, _{(5,7)}(y0)_{|(2,7)|}, _{(0,3)}(y3)_{|(0,4)|}, _{(2,4)}(y2)_{|(3,5)|},$
 $_{|(1,6)|}(r)_{|(2,7)|}.$

$$\begin{aligned}
& \text{Actions: } y1 \otimes (|(1 \rightarrow 6)|) \longrightarrow |(1 \rightarrow 6)| \otimes y1, y1 \otimes (|(6 \rightarrow 7)|) \longrightarrow |(6 \rightarrow 7)| \otimes y0, \\
& y1 \otimes (|(1 \rightarrow 7)|) \longrightarrow |(1 \rightarrow 7)| \otimes y0, y1 \otimes (|(1 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(1 \rightarrow 5)| \otimes y0, \\
& y1 \otimes (|(1 \rightarrow 4)|) \longrightarrow |(1 \rightarrow 3)| \otimes y3, y1 \otimes (|(1 \rightarrow 5)|) \longrightarrow |(1 \rightarrow 4)| \otimes y2, y1 \otimes (|(1 \rightarrow 3)|) \longrightarrow \\
& |(1 \rightarrow 2)| \otimes y2, y1 \otimes (|(1 \rightarrow 2)|) \longrightarrow 1 \otimes r, y0 \otimes () \longrightarrow |(5 \rightarrow 6)| \otimes r, y3 \otimes (|(0 \rightarrow 1)|) \longrightarrow \\
& |(0 \rightarrow 1)| \otimes y1, y3 \otimes (|(4 \rightarrow 6)|) \longrightarrow |(3 \rightarrow 6)| \otimes y1, y3 \otimes (|(0 \rightarrow 6)|) \longrightarrow |(0 \rightarrow 6)| \otimes y1, \\
& y3 \otimes (|(4 \rightarrow 7)|) \longrightarrow |(3 \rightarrow 7)| \otimes y0, y3 \otimes (|(4 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(3 \rightarrow 5)| \otimes y0, \\
& y3 \otimes (|(0 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(0 \rightarrow 5)| \otimes y0, y3 \otimes (|(0 \rightarrow 7)|) \longrightarrow |(0 \rightarrow 7)| \otimes y0, \\
& y3 \otimes (|(0 \rightarrow 4)|) \longrightarrow |(0 \rightarrow 3)| \otimes y3, y3 \otimes (|(0 \rightarrow 5)|) \longrightarrow |(0 \rightarrow 4)| \otimes y2, y3 \otimes (|(0 \rightarrow \\
& 3)|) \longrightarrow |(0 \rightarrow 2)| \otimes y2, y3 \otimes (|(4 \rightarrow 5)|) \longrightarrow |(3 \rightarrow 4)| \otimes y2, y3 \otimes (|(0 \rightarrow 2)|) \longrightarrow \\
& |(0 \rightarrow 1)| \otimes r, y2 \otimes (|(5 \rightarrow 6)|) \longrightarrow |(4 \rightarrow 6)| \otimes y1, y2 \otimes (|(3 \rightarrow 6)|) \longrightarrow |(2 \rightarrow 6)| \otimes y1, \\
& y2 \otimes (|(5 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(4 \rightarrow 5)| \otimes y0, y2 \otimes (|(3 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(2 \rightarrow 5)| \otimes y0,
\end{aligned}$$

$$\begin{aligned}
& y2 \otimes (|(3 \rightarrow 7)|) \longrightarrow |(2 \rightarrow 7)| \otimes y0, y2 \otimes (|(5 \rightarrow 7)|) \longrightarrow |(4 \rightarrow 7)| \otimes y0, y2 \otimes (|(3 \rightarrow 4)|) \longrightarrow \\
& |(2 \rightarrow 3)| \otimes y3, y2 \otimes (|(3 \rightarrow 5)|) \longrightarrow |(2 \rightarrow 4)| \otimes y2, r \otimes (|(2 \rightarrow 6)|) \longrightarrow |(1 \rightarrow 6)| \otimes y1, \\
& r \otimes (|(2 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(1 \rightarrow 5)| \otimes y0, r \otimes (|(2 \rightarrow 7)|) \longrightarrow |(1 \rightarrow 7)| \otimes y0, \\
& r \otimes (|(2 \rightarrow 4)|) \longrightarrow |(1 \rightarrow 3)| \otimes y3, r \otimes (|(2 \rightarrow 5)|) \longrightarrow |(1 \rightarrow 4)| \otimes y2, r \otimes (|(2 \rightarrow 3)|) \longrightarrow \\
& |(1 \rightarrow 2)| \otimes y2.
\end{aligned}$$

Bimodule 4.6.15. $N(\mu_3)$

5 generators with their idempotents: $_{|(1,6)|}(y1)_{|(1,6)|}, _{(3,5)|}(y0)_{|(2,4)|}, _{(0,4)|}(y3)_{|(0,5)|}, _{(2,7)|}(y2)_{|(3,7)|},$
 $_{|(1,6)|}(r)_{|(2,4)|}.$

$$\begin{aligned}
& \text{Actions: } y1 \otimes (|(1 \rightarrow 6)|) \longrightarrow |(1 \rightarrow 6)| \otimes y1, y1 \otimes (|(1 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(1 \rightarrow \\
& 5)| \otimes y0, y1 \otimes (|(1 \rightarrow 4)|) \longrightarrow |(1 \rightarrow 3)| \otimes y0, y1 \otimes (|(1 \rightarrow 5)|) \longrightarrow |(1 \rightarrow 4)| \otimes y3, \\
& y1 \otimes (|(6 \rightarrow 7)|) \longrightarrow |(6 \rightarrow 7)| \otimes y2, y1 \otimes (|(1 \rightarrow 3)|) \longrightarrow |(1 \rightarrow 2)| \otimes y2, y1 \otimes (|(1 \rightarrow \\
& 7)|) \longrightarrow |(1 \rightarrow 7)| \otimes y2, y1 \otimes (|(1 \rightarrow 2)|) \longrightarrow 1 \otimes r, y0 \otimes (|(4 \rightarrow 6)|) \longrightarrow |(3 \rightarrow 6)| \otimes y1, \\
& y0 \otimes (|(4 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(3 \rightarrow 5)| \otimes y0, y0 \otimes (|(4 \rightarrow 5)|) \longrightarrow |(3 \rightarrow 4)| \otimes y3, \\
& y0 \otimes (|(4 \rightarrow 7)|) \longrightarrow |(3 \rightarrow 7)| \otimes y2, y0 \otimes () \longrightarrow |(5 \rightarrow 6)| \otimes r, y3 \otimes (|(0 \rightarrow 6)|) \longrightarrow \\
& |(0 \rightarrow 6)| \otimes y1, y3 \otimes (|(0 \rightarrow 1)|) \longrightarrow |(0 \rightarrow 1)| \otimes y1, y3 \otimes (|(5 \rightarrow 6)|) \longrightarrow |(4 \rightarrow 6)| \otimes y1, \\
& y3 \otimes (|(0 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(0 \rightarrow 5)| \otimes y0, y3 \otimes (|(5 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(4 \rightarrow 5)| \otimes y0, \\
& y3 \otimes (|(0 \rightarrow 4)|) \longrightarrow |(0 \rightarrow 3)| \otimes y0, y3 \otimes (|(0 \rightarrow 5)|) \longrightarrow |(0 \rightarrow 4)| \otimes y3, y3 \otimes (|(5 \rightarrow \\
& 7)|) \longrightarrow |(4 \rightarrow 7)| \otimes y2, y3 \otimes (|(0 \rightarrow 7)|) \longrightarrow |(0 \rightarrow 7)| \otimes y2, y3 \otimes (|(0 \rightarrow 3)|) \longrightarrow \\
& |(0 \rightarrow 2)| \otimes y2, y3 \otimes (|(0 \rightarrow 2)|) \longrightarrow |(0 \rightarrow 1)| \otimes r, y2 \otimes (|(3 \rightarrow 6)|) \longrightarrow |(2 \rightarrow 6)| \otimes y1, \\
& y2 \otimes (|(3 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(2 \rightarrow 5)| \otimes y0, y2 \otimes (|(3 \rightarrow 4)|) \longrightarrow |(2 \rightarrow 3)| \otimes y0, \\
& y2 \otimes (|(3 \rightarrow 5)|) \longrightarrow |(2 \rightarrow 4)| \otimes y3, y2 \otimes (|(3 \rightarrow 7)|) \longrightarrow |(2 \rightarrow 7)| \otimes y2, r \otimes (|(2 \rightarrow \\
& 6)|) \longrightarrow |(1 \rightarrow 6)| \otimes y1, r \otimes (|(2 \rightarrow 6)|, |(1 \rightarrow 2)|) \longrightarrow |(1 \rightarrow 5)| \otimes y0, r \otimes (|(2 \rightarrow 4)|) \longrightarrow \\
& |(1 \rightarrow 3)| \otimes y0, r \otimes (|(2 \rightarrow 5)|) \longrightarrow |(1 \rightarrow 4)| \otimes y3, r \otimes (|(2 \rightarrow 7)|) \longrightarrow |(1 \rightarrow 7)| \otimes y2, \\
& r \otimes (|(2 \rightarrow 3)|) \longrightarrow |(1 \rightarrow 2)| \otimes y2.
\end{aligned}$$

Bimodule 4.6.16. $N(\mu_4)$

5 generators with their idempotents: $_{|(1,6)|}(y1)_{|(1,6)|}, _{(0,5)|}(y0)_{|(0,2)|}, _{(3,7)|}(y3)_{|(4,7)|}, _{(2,4)|}(y2)_{|(3,5)|},$

$$|(1,6)|^{(r)}_{|(0,2)|}.$$

Actions: $y1 \otimes (|(1 \rightarrow 6)|) \rightarrow |(1 \rightarrow 6)| \otimes y1$, $y1 \otimes (|(1 \rightarrow 6)|, |(1 \rightarrow 2)|) \rightarrow |(1 \rightarrow 5)| \otimes y0$, $y1 \otimes (|(1 \rightarrow 7)|) \rightarrow |(1 \rightarrow 7)| \otimes y3$, $y1 \otimes (|(1 \rightarrow 4)|) \rightarrow |(1 \rightarrow 3)| \otimes y3$, $y1 \otimes (|(6 \rightarrow 7)|) \rightarrow |(6 \rightarrow 7)| \otimes y3$, $y1 \otimes (|(1 \rightarrow 5)|) \rightarrow |(1 \rightarrow 4)| \otimes y2$, $y1 \otimes (|(1 \rightarrow 3)|) \rightarrow |(1 \rightarrow 2)| \otimes y2$, $y1 \otimes (|(1 \rightarrow 2)|) \rightarrow 1 \otimes r$, $y0 \otimes (|(0 \rightarrow 6)|) \rightarrow |(0 \rightarrow 6)| \otimes y1$, $y0 \otimes (|(0 \rightarrow 1)|) \rightarrow |(0 \rightarrow 1)| \otimes y1$, $y0 \otimes (|(0 \rightarrow 6)|, |(1 \rightarrow 2)|) \rightarrow |(0 \rightarrow 5)| \otimes y0$, $y0 \otimes (|(0 \rightarrow 7)|) \rightarrow |(0 \rightarrow 7)| \otimes y3$, $y0 \otimes (|(0 \rightarrow 4)|) \rightarrow |(0 \rightarrow 3)| \otimes y3$, $y0 \otimes (|(0 \rightarrow 5)|) \rightarrow |(0 \rightarrow 4)| \otimes y2$, $y0 \otimes (|(0 \rightarrow 3)|) \rightarrow |(0 \rightarrow 2)| \otimes y2$, $y0 \otimes () \rightarrow |(5 \rightarrow 6)| \otimes r$, $y0 \otimes (|(0 \rightarrow 2)|) \rightarrow |(0 \rightarrow 1)| \otimes r$, $y3 \otimes (|(4 \rightarrow 6)|) \rightarrow |(3 \rightarrow 6)| \otimes y1$, $y3 \otimes (|(4 \rightarrow 6)|, |(1 \rightarrow 2)|) \rightarrow |(3 \rightarrow 5)| \otimes y0$, $y3 \otimes (|(4 \rightarrow 7)|) \rightarrow |(3 \rightarrow 7)| \otimes y3$, $y3 \otimes (|(4 \rightarrow 5)|) \rightarrow |(3 \rightarrow 4)| \otimes y2$, $y2 \otimes (|(5 \rightarrow 6)|) \rightarrow |(4 \rightarrow 6)| \otimes y1$, $y2 \otimes (|(3 \rightarrow 6)|) \rightarrow |(2 \rightarrow 6)| \otimes y1$, $y2 \otimes (|(5 \rightarrow 6)|, |(1 \rightarrow 2)|) \rightarrow |(4 \rightarrow 5)| \otimes y0$, $y2 \otimes (|(3 \rightarrow 6)|, |(1 \rightarrow 2)|) \rightarrow |(2 \rightarrow 5)| \otimes y0$, $y2 \otimes (|(5 \rightarrow 7)|) \rightarrow |(4 \rightarrow 7)| \otimes y3$, $y2 \otimes (|(3 \rightarrow 7)|) \rightarrow |(2 \rightarrow 7)| \otimes y3$, $y2 \otimes (|(3 \rightarrow 4)|) \rightarrow |(2 \rightarrow 3)| \otimes y3$, $y2 \otimes (|(3 \rightarrow 5)|) \rightarrow |(2 \rightarrow 4)| \otimes y2$, $r \otimes (|(2 \rightarrow 6)|) \rightarrow |(1 \rightarrow 6)| \otimes y1$, $r \otimes (|(2 \rightarrow 6)|, |(1 \rightarrow 2)|) \rightarrow |(1 \rightarrow 5)| \otimes y0$, $r \otimes (|(2 \rightarrow 7)|) \rightarrow |(1 \rightarrow 7)| \otimes y3$, $r \otimes (|(2 \rightarrow 4)|) \rightarrow |(1 \rightarrow 3)| \otimes y3$, $r \otimes (|(2 \rightarrow 3)|) \rightarrow |(1 \rightarrow 2)| \otimes y2$, $r \otimes (|(2 \rightarrow 5)|) \rightarrow |(1 \rightarrow 4)| \otimes y2$.

Bimodule 4.6.17. $N(\eta^{-1})$

5 generators with their idempotents: $|(3,5)|^{(x2)}_{i2}$, $|(4,7)|^{(x3)}_{i3}$, $|(0,2)|^{(x0)}_{i0}$, $|(1,6)|^{(x1)}_{i1}$, $|(3,5)|^{(r)}_{i1}$.

Actions: $x2 \otimes (r56) \rightarrow |(3 \rightarrow 5)| \otimes x2$, $x2 \otimes (r7) \rightarrow |(5 \rightarrow 7)| \otimes x3$, $x2 \otimes (r5) \rightarrow |(3 \rightarrow 4)| \otimes x3$, $x2 \otimes (r567) \rightarrow |(3 \rightarrow 7)| \otimes x3$, $x3 \otimes (r6) \rightarrow |(4 \rightarrow 5)| \otimes x2$, $x3 \otimes (r67) \rightarrow |(4 \rightarrow 7)| \otimes x3$, $x0 \otimes (r1234) \rightarrow |(0 \rightarrow 3)| \otimes x2$, $x0 \otimes (r3456) \rightarrow |(2 \rightarrow 5)| \otimes x2$, $x0 \otimes (r123456) \rightarrow |(0 \rightarrow 5)| \otimes x2$, $x0 \otimes (r34) \rightarrow |(2 \rightarrow 3)| \otimes x2$, $x0 \otimes (r1234567) \rightarrow |(0 \rightarrow 7)| \otimes x3$, $x0 \otimes (r34567) \rightarrow |(2 \rightarrow 7)| \otimes x3$, $x0 \otimes (r345) \rightarrow |(2 \rightarrow 4)| \otimes x3$, $x0 \otimes (r12345) \rightarrow |(0 \rightarrow 4)| \otimes x3$, $x0 \otimes (r12) \rightarrow |(0 \rightarrow 2)| \otimes x0$,

$$\begin{aligned}
x0 \otimes (r1) &\longrightarrow |(0 \rightarrow 1)| \otimes x1, x0 \otimes (r123) \longrightarrow |(0 \rightarrow 3)| \otimes r, x0 \otimes (r3) \longrightarrow |(2 \rightarrow 3)| \otimes r, \\
x1 \otimes (r23456) &\longrightarrow |(1 \rightarrow 5)| \otimes x2, x1 \otimes (r234) \longrightarrow |(1 \rightarrow 3)| \otimes x2, x1 \otimes (r234567) \longrightarrow \\
&|(1 \rightarrow 7)| \otimes x3, x1 \otimes (r4, r7) \longrightarrow |(6 \rightarrow 7)| \otimes x3, x1 \otimes (r2345) \longrightarrow |(1 \rightarrow 4)| \otimes x3, \\
x1 \otimes (r2) &\longrightarrow |(1 \rightarrow 2)| \otimes x0, x1 \otimes (r23) \longrightarrow |(1 \rightarrow 3)| \otimes r, r \otimes (r4) \longrightarrow 1 \otimes x2, \\
r \otimes (r456) &\longrightarrow |(3 \rightarrow 5)| \otimes x2, r \otimes (r45) \longrightarrow |(3 \rightarrow 4)| \otimes x3, r \otimes (r4567) \longrightarrow |(3 \rightarrow 7)| \otimes x3, \\
r \otimes () &\longrightarrow |(5 \rightarrow 6)| \otimes x1.
\end{aligned}$$

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