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Level Set Shape For Ground State Eigenfunctions On Convex Domains

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Level Set Shape For Ground State Eigenfunctions On Convex Domains

Thomas Beck

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Charles Fefferman

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Abstract

In this thesis, we study the ground state Dirichlet eigenfunction of two classes of Schrödinger operators on convex domains. The assumptions on these operators ensure that the level sets of the eigenfunction are convex sets, and we find length scales and an orientation of the domain which determine the shape of the level sets. An an intermediate step we establish bounds on the ground state eigenvalue.

We first consider a class of two dimensional Schrödinger operators with a convex potential on a convex domain. In this case, the length scales and orientation determining the shape of the level sets are defined in terms of the geometry of the domain, the properties of the potential, and an associated family of one dimensional Schrödinger operators. Once we have established the shape of the level sets, we begin to analyse the behaviour of the eigenfunction near to where it achieves its maximum.

In the second part of the thesis we study the ground state eigenfunction of the Dirichlet Laplacian for a class of three dimensional convex domains. For each of these domains we use an approximate separation of variables to find an associated two dimensional Schrödinger operator of the form above. This then allows us to obtain sufficiently precise bounds on the first eigenvalue and also to use the same length scales and orientation as for the two dimensional operator in order to determine the shape of the level sets of the three dimensional eigenfunction.

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My family have been a constant source of support and inspiration my entire life. I want to take this opportunity to thank them for everything they have done, and continue to do for me. I know that I would not have been in a position to write this thesis without them.

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To my family.

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Chapter 1

Schrödinger operators on a two dimensional convex domain

1.1 Introduction and statement of results

We are interested in studying a class of Schrödinger operators

$$\mathcal{L} = -\Delta_{x,y} + V(x,y),$$

with Dirichlet boundary conditions. This operator acts on functions defined on the bounded, convex domain $\Omega \subset \mathbb{R}^2$, and $V(x,y)$ is a convex potential.

The operator $\mathcal{L}$ has an increasing sequence of Dirichlet eigenvalues

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \nearrow \infty,$$

with corresponding eigenfunctions $u_j(x,y)$ satisfying

$$\begin{cases} (-\Delta_{x,y} + V(x,y))u_j(x,y) &= \lambda_j u_j(x,y) \qquad \text{in } \Omega \\ u_j(x,y) &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Our main focus will be to study the first eigenvalue $\lambda = \lambda_1$ and eigenfunction $u(x,y) = u_1(x,y)$. The first eigenfunction $u(x,y)$ does not change sign inside Ω and so we normalise $u(x,y)$ so that it is positive inside Ω, and attains a maximum of 1. In Definitions 1.1.1 and 1.1.3 below, we will define

the class of convex domains Ω and potentials $V(x,y)$ that we are interested in. We will see that one consequence of the assumptions on Ω and $V(x,y)$ is that it ensures that the superlevel sets of $u(x,y)$,

$$W_c \coloneqq \{(x,y) \in \Omega : u(x,y) \geq c\},$$

are convex subsets of Ω for all $0 \leq c \leq 1$.

A theorem of John, [Jo], therefore implies that for each c we can find an ellipse E_c contained within this superlevel set W_c, such that a dilate of E_c, with scaling factor bounded by an absolute constant contains W_c. We are interested in determining the shape of the level sets of $u(x,y)$, and to do this we will study the lengths and orientation of the axes of the ellipse E_c. One of the main steps in establishing the shape of the level sets of $u(x,y)$ will be to prove sufficiently precise bounds on the first eigenvalue λ.

We know that the level set $\{(x,y) \in \Omega : u(x,y) = 0\}$ is equal to the boundary, $\partial\Omega$, and so in particular the shape of this level set is determined solely by the geometry of Ω. However, we will see that, in general, for the intermediate level sets, for example $\{(x,y) \in \Omega : u(x,y) = \frac{1}{2}\}$, it is not solely the shape of $\partial\Omega$ that governs its shape, but instead the two length scales L_1 and L_2. These length scales L_1 and L_2 will be given in Definitions 1.1.4 and 1.1.7, but the key feature of their definitions is the following: The length scale L_1 will be defined purely in terms of the geometry of Ω and properties of the potential $V(x,y)$, but the length scale L_2 will also depend on a family of associated one dimensional Schrödinger operators. Moreover, the definition of L_2 will also describe the orientation of these level sets of $u(x,y)$.

Our motivation for studying this problem is as follows: First, λ and $\Psi(t,x,y) = e^{\lambda t}u(x,y)$ are the lowest energy and ground state eigenfunction of the quantum system governed by the Schrödinger operator

$$\partial_t \Psi(t,x,y) + \mathcal{L}\Psi(t,x,y) = 0.$$

The main motivation comes from the series of papers [J1], [GJ1], [GJ2]. There, the authors study the first two Dirichlet eigenfunctions on two dimensional convex domains Ω, normalised so that the inner radius is comparable to 1, and the diameter is equal to the large parameter N. We will describe their results and techniques in more detail below, but for now we will briefly describe one of the techniques used that is most relevant for us: Using their normalisation of the domain Ω, they

write it as

$$\Omega = \{(x, y) : f_1(x) < y < f_2(x), a < x < b\},$$

for functions $f_1(x)$ and $f_2(x)$, which are convex and concave respectively, and they consider the concave *height function* $h(x)$,

$$h(x) = f_2(x) - f_1(x),$$

with $\max_{x\in[a,b]} h(x) = 1$. This allows us to define a large parameter L, purely in terms of the function $h(x)$ (and hence just depending on the geometry of the domain). This number L is the largest value such that

$$h(x) \geq 1 - L^{-2} \tag{1.1}$$

on an interval I of length at least L. Rather than the length of the diameter N, this parameter L is the relevant length scale to study the low energy eigenfunctions. Since the inner radius of their domain is comparable to 1, while the projection of the domain onto the x-axis is large compared to 1, it is natural to study the two dimensional problem via an approximate separation of variables. For each fixed x, the domain Ω consists of the interval $[f_1(x), f_2(x)]$ of length $h(x)$, which has first eigenvalue $\pi^2 h(x)^{-2}$. Thus, the ordinary differential operator on the interval $[a, b]$, which is naturally associated with this separation of variables is

$$-\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2}, \tag{1.2}$$

with zero boundary conditions. In [J1] the eigenvalues and eigenfunctions of this operator are used to generate appropriate test functions to provide bounds on the first eigenvalue in terms of L, and to estimate the location and width of the nodal line of the second eigenfunction. In [GJ1], they give a sharper estimate on the nodal line, and in [GJ2] they study the location of the maximum of the first eigenfunction of Ω, and its behaviour near this maximum where they use this approximate separation of variables to relate it to the first eigenfunction of the one dimensional operator. As a straightforward consequence of their work, it is this length scale L and orientation of the domain Ω given above, which determines the shape of the level sets of the eigenfunction $u(x, y)$ in this special case.

The papers [J1], [GJ1], [GJ2] also provide more motivation for studying the operators $\mathcal{L}$. In the same way that the one dimensional Schrödinger operator in (1.2) is used in a crucial way to study the eigenfunctions of two dimensional convex domains, it will be important to understand the properties of the eigenfunctions of $\mathcal{L}$ when considering the eigenfunctions of three (and higher) dimensional convex domains. In Chapter 2, we will give an application where we can use the properties of the eigenvalue λ and eigenfunction $u(x, y)$ to deduce results about the first eigenvalue and eigenfunction of a class of three dimensional convex domains.

Before stating our results, let us define precisely the class of domains Ω and potentials $V(x, y)$ that we will be considering here.

Definition 1.1.1 (The Domain Ω)**.** *The domain Ω is a bounded, convex two dimensional domain with inner radius N_1, and diameter N_2. We assume that the diameter is large compared to an absolute constant, while the inner radius is bounded below by an absolute constant.*

Remark 1.1.2. *Throughout, the constants that appear will depend on these absolute constants, but the dependence of any bounds on the diameter and inner radius themselves (and the other parameters introduced below) will be explicitly stated.*

We now state the class of potentials of interest.

Definition 1.1.3 (The Potential $V(x, y)$)**.** *The potential $V(x, y)$ on the domain Ω satisfies*

$$V(x, y) = \frac{1}{h(x, y)^2},$$

where $h(x, y)$ is a concave function with $0 \leq h(x, y) \leq 1$ and $\max_\Omega h(x, y) = 1$. In other words, $V(x, y)^{-1/2}$ is concave on Ω and

$$\min_\Omega V(x, y) = 1.$$

In particular, this also ensures that $V(x, y)$ is convex.

We see that this ensures that the first derivatives of V are bounded almost everywhere, and that the second derivatives of V are positive measures. However, we do not impose any further regularity assumptions on the potential. Before continuing, let us briefly discuss the motivation behind Definition 1.1.3.

1. One allowed potential is the constant potential $V(x, y) = 1$. In this case, our operator is analogous to the purely two dimensional operator studied in [J1]. In particular, we can renormalise

our domain Ω to ensure that the inner radius is comparable to 1. Note in general, our potential $V(x,y)$ is not scale invariant, and so this is not as useful a normalisation for us.

2. The assumption that $V(x,y)^{-1/2}$ is concave is a natural one when we recall the motivation for studying this class of Schrödinger operators. In the same way that the operator in (1.2) has been used to study the eigenfunctions of two dimensional domains, the potential $V(x,y)$ that we are considering is naturally related to the three dimensional domain with height function proportional to $h(x,y)$. This assumption that $V(x,y)^{-1/2}$ is concave also appears in the work of Borell, [B1], [B2], when studying the concavity properties of the Green's functions associated to these Schrödinger operators.

3. We do not claim that this is the only class of potentials for which the results below will be valid. In fact, many of the results can be restated to hold for a more general class of convex potentials (including those related to the harmonic oscillator). However, at times we will see that it is convenient to restrict to those potentials given in Definition 1.1.3, and so we will only state the results for this class of potentials.

We can now introduce the crucial parameters L_1 and L_2 that will appear as important length scales in our study of the first eigenfunction $u(x,y)$. For each $c \geq 0$, let us define the sublevel sets of $V(x,y)$ by

$$\Omega_c := \{(x,y) \in \Omega : V(x,y) \leq 1 + c\}.$$

Since $V(x,y)$ is convex, these sublevel sets Ω_c are convex subsets of Ω.

Definition 1.1.4 (The Parameter L_1)**.** *Let L_1 be the largest value such that the sublevel set $\Omega_{L_1^{-2}}$ has inner radius at least equal to L_1.*

Remark 1.1.5. *This definition is analogous to the definition of the parameter L from [J1] described above, and roughly speaking is equal to the largest length scale L_1 on which the potential increases by at most L_1^{-2} from its minimum.*

With L_1 fixed, we let $\tilde{L}_1$ be the diameter of the set $\Omega_{L_1^{-2}}$. If L_1 and $\tilde{L}_1$ are comparable in size, then we define L_2 to be equal to L_1, but if

$$\tilde{L}_1 \gg L_1,$$

then we now describe how to find L_2.

Remark 1.1.6. *Throughout, the notation $A \gg B$ denotes $A \geq \tilde{C}B$, for some large fixed absolute constant $\tilde{C} > 0$, and if this, and the converse $B \gg A$, do not hold then we say that A and B are comparable. In particular, we are not interested in the exact values of L_1 and L_2, but instead are interested in knowing whether any length scale is, or is not, comparable to L_1 and L_2. We will use the notation C to represent an absolute constant, that is small compared to $\tilde{C}$, which may change from line to line.*

To obtain a value for L_2, we first rotate our domain Ω, so that the projection of $\Omega_{L_1^{-2}}$ onto the y-axis is of the smallest length amongst the projections onto any line. In particular, this means that the projection of $\Omega_{L_1^{-2}}$ onto the x-axis is comparable to $\tilde{L}_1$, while the projection of $\Omega_{L_1^{-2}}$ onto the y-axis is comparable to L_1. This also fixes the orientation of Ω.

For each fixed x, let the interval $\Omega(x)$ be the cross-section of Ω at x, and consider the ordinary differential operator

$$\mathcal{L}(x) \coloneqq -\frac{d^2}{dy^2} + V(x, y), \tag{1.3}$$

with zero boundary conditions on $\Omega(x)$. We let $\mu(x)$ be the first eigenvalue of $\mathcal{L}(x)$, and define the minimum of these eigenvalues,

$$\mu^* \coloneqq \min_x \mu(x).$$

We can now define the parameter L_2.

Definition 1.1.7 (The Parameter L_2)**.** *We define L_2 to be the largest value such that*

$$\mu^* \leq \mu(x) \leq \mu^* + L_2^{-2},$$

for all x in an interval I of length at least L_2.

Remark 1.1.8. *Note that in this definition of L_2, we have used the orientation of $\Omega_{L_1^{-2}}$ fixed above. Therefore, from now on, whenever we consider any property of the eigenvalue or eigenfunction that depends on the value of L_2, we will have to use this orientation of $\Omega_{L_1^{-2}}$. In contrast, the definition of L_1 does not depend on the orientation of $\Omega_{L_1^{-2}}$.*

Our main aim in the study of the first eigenfunction is to give precise information about the shape of the level sets $\{(x, y) \in \Omega : u(x, y) = c\}$ which are near to the point where $u(x, y)$ attains

its maximum of 1. Since the potential $V(x, y)$ is a convex function and Ω is a convex set, Theorem 6.1 in [BL2] tells us that $u(x, y)$ is log concave. Alternative proofs of this result have also been given in [CF], [K], [KL]. In particular, this tells us that the superlevel sets are all convex. Since $\{(x, y) \in \Omega : u(x, y) \geq 0\} = \Omega$, one way of viewing this result is that

$$\{(x, y) \in \Omega : u(x, y) \geq 0\} \text{ convex } \Rightarrow \{(x, y) \in \Omega : u(x, y) \geq c\} \text{ convex}$$

for all $0 \leq c \leq 1$.

We will use the convexity of the superlevel sets of $u(x, y)$ in a crucial way to describe their shape near its maximum.

Theorem 1.1.9. *Let Ω and $V(x, y)$ be a domain and potential from Definitions 1.1.1 and 1.1.3. Fix a small absolute constant $c_1 > 0$, and let L_1 and L_2 be as in Definitions 1.1.4 and 1.1.7. In particular, this means that we have fixed the orientation of the set $\Omega_{L_1^{-2}}$. Then, for any fixed absolute constant c, with $c_1 < c < 1 - c_1$, the level set $\{(x, y) \in \Omega : u(x, y) = c\}$ has the following shape: There exists an ellipse E with minor axis in the y-direction of length comparable to L_1 and major axis in the x-direction of length comparable to L_2, such that E is contained inside this level set, and a dilate of E, with a scaling factor bounded by an absolute constant, contains this level set.*

Remark 1.1.10. *The level set $\{(x, y) \in \Omega : u(x, y) = 0\}$ is equal to $\partial\Omega$, the boundary of Ω. We will see that in general the parameters L_1 and L_2 are not comparable to the inner radius and diameter of the original domain Ω. Thus, the result of Theorem 1.1.9 does not remain valid when c becomes close to 0.*

Corollary 1.1.11. *For a convex set W, we define the eccentricity of W, $\text{ecc}(W)$ in the usual way:*

$$\text{ecc}(W) = \frac{\text{diam}(W)}{\text{inradius}(W)}.$$

For $c = 0$, the eccentricity of the superlevel set $\{(x, y) \in \Omega : u(x, y) \geq c\}$ is equal to the eccentricity of Ω, but as c increases (while bounded above by $1 - c_1$), the eccentricity of the superlevel set becomes comparable to L_2/L_1.

The log concavity of the eigenfunction, and resulting convexity of its superlevel sets has been used previously in various situations. For example, in [AC] moduli of convexity and concavity are introduced. Under certain conditions on the potential V, it is then possible to strengthen the log concavity of the first eigenfunction by finding an appropriate modulus of concavity. This allows the

spectral gap for a class of Schrödinger operators to be compared to the case where the potential is identically zero, and allows them to prove the Fundamental Gap Conjecture. In [FJ] the convexity of the superlevel sets of the Green's function are used in a crucial way to prove third derivative estimates on the eigenfunction which are valid up to the boundary of the convex domain.

As well as the convexity of the superlevel sets of $u(x, y)$, a very important part of the proof of Theorem 1.1.9 will be to obtain sufficiently precise eigenvalues bounds for the first eigenvalue λ. For $\mu(x)$ equal to the first eigenvalue of the operator $\mathcal{L}(x)$, we consider the ordinary differential operator

$$\mathcal{A} = -\frac{d^2}{dx^2} + \mu(x), \tag{1.4}$$

and let μ be the first eigenvalue of this operator. Our eigenvalue bounds relate the value of λ to this eigenvalue μ.

Theorem 1.1.12. *Let Ω and $V(x, y)$ be a domain and potential from Definitions 1.1.1 and 1.1.3. If L_2 is defined as in Definition 1.1.7 and μ is the first eigenvalue of the operator $\mathcal{A}$ in* (1.4)*, then the first eigenvalue λ of the operator $\mathcal{L}$ satisfies*

$$\mu \leq \lambda \leq \mu + CL_2^{-2},$$

for an absolute constant C.

Remark 1.1.13. *Theorems 1.1.9 and 1.1.12 are valid for all domains and potentials satisfying the assumptions of Definitions 1.1.1 and 1.1.3, and the bounds are uniform for domains Ω and potentials V leading to the same values for L_1 and L_2.*

While it is much more straightforward to locate the eigenvalue λ to an interval of length comparable to L_1^{-2}, we will see that the more precise bound obtained in Theorem 1.1.12 is necessary to obtain sharp information about the length scale on which the eigenfunction $u(x, y)$ decays in the x-direction, and hence prove Theorem 1.1.9.

Theorem 1.1.12 locates the first eigenvalue λ to an interval of length comparable to L_2^{-2}, provided we know the value of μ. However, μ is also an eigenvalue of a differential operator, and so it may seem like we have only been able to locate the unknown λ in terms of another unknown μ. Another reason why this theorem still has value is that whereas λ is the first eigenvalue of a two dimensional partial differential operator (with a potential), μ is the first eigenvalue of an ordinary differential operator $\mathcal{A}$. Thus, from a computational standpoint, it is much easier to accurately approximate the value of μ compared to λ. Also, we notice that the parameter L_2 depends on the geometric properties

of the domain Ω and potential $V(x,y)$, together with the eigenvalues of the differential operator $\mathcal{L}(x)$ given in (1.3). In other words, L_2 also only depends on knowledge of ordinary differential operators. Thus, the bound given in Theorem 1.1.12 gives information about the eigenvalue of a two dimensional partial differential operator purely in terms of ordinary differential operators.

The idea of relating the eigenfunctions and eigenvalues of a two dimensional problem to an associated ordinary differential operator has also been used extensively by Friedlander and Solomyak in [FS1], [FS2], [FS3]. In these papers, they use this approximate separation of variables to obtain asymptotics for the eigenvalues, and the resolvent of the Dirichlet Laplacian. They use a semiclassical method by sending a small parameter ϵ to 0 in order to give a one-parameter of 'narrow' domains, and then write asymptotics in terms of this small parameter.

Let us now describe how we will proceed in the sections below.

In Section 1.2 we study the length scales L_1 and L_2 from Definitions 1.1.4 and 1.1.7 in more detail. In particular, we will obtain bounds on L_1 and L_2 in terms of the diameter and inner radius of the domain and the potential, and construct domains Ω and potentials $V(x,y)$ to show to what extent these estimates are sharp. We will also give a straightforward bound on λ in terms of L_1 by using the variational formulation for the first eigenvalue.

In Section 1.3 we will prove the eigenvalue bounds in Theorem 1.1.12. For each fixed x, $u(x,y)$ is an admissible test function for the operator $\mathcal{L}(x)$ from (1.3), and the lower bound on λ will follow straightforwardly from this. The proof of the upper bound on λ in Theorem 1.1.12 is more involved. The starting point of the proof is to use the first eigenfunction, $\psi^{(x)}(y)$, of the operator $\mathcal{L}(x)$ to construct a suitable test function in the variational formulation for the first eigenvalue. To obtain the required upper bound on λ it will be necessary to study the first variation of $\psi^{(x)}(y)$ in the cross-sectional variable x. To do this, we will derive the ordinary differential equation that this first variation satisfies for each fixed x. The bounds then follow from using the method of variation of parameters. It will be particularly important to have estimates on the relative size of the first derivative of the potential $V(x,y)$ and the size of $\psi^{(x)}(y)$.

Once we have established the bounds on λ in Theorem 1.1.12, in Section 1.4 we use them to study the first eigenfunction $u(x,y)$ itself. Our first aim is to prove a $L^2(\Omega)$-bound on $u(x,y)$ which is consistent with the shape of the level sets required in Theorem 1.1.9. We begin by using Theorem 1.1.12 to prove a Carleman-type estimate to show how the $L^2(\Omega(x))$-norm of the cross-sections of

$u(x,y)$,

$$H(x) = \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y,$$

decays from its maximum exponentially on a length scale comparable to L_2. To find the required bound on the $L^2(\Omega)$-norm of $u(x,y)$, we then need to estimate the size of the maximum of $H(x)$. We will do this by proving $L^2(\Omega)$-bounds on the first derivatives of $u(x,y)$, which are again consistent with Theorem 1.1.9. We finish Section 1.4 by proving an Agmon-type estimate to give an indication of the behaviour of $u(x,y)$ at points at a large distance from its maximum.

In Section 1.5 we study the shape of the level sets of $u(x,y)$ and complete the proof of Theorem 1.1.9. To do this we will use the results of Section 1.4 on the $L^2(\Omega)$-norms of $u(x,y)$ itself, and also its first derivatives. We will also use the log-concavity of the eigenfunction $u(x,y)$ in a crucial way, since it is this that ensures that the superlevel sets are convex.

Theorem 1.1.9 gives information about the level sets, $\{(x,y) \in \Omega : u(x,y) = c\}$ whenever c is bounded away from 0 and 1. In Section 1.6, we want to study the behaviour of the eigenfunction $u(x,y)$ near its maximum. In particular, we will relate the location of the maximum to the region where $V(x,y) - \lambda$ is bounded above by $-c^* L_1^{-2}$, for an absolute constant $c^* > 0$. We will do this by first using a maximum principle to restrict attention to the part of Ω where $V(x,y) - \lambda$ is at most comparable to L_1^{-2}. This will then be used to convert the $L^2(\Omega)$-bounds on $\nabla_{x,y} u(x,y)$ from Section 1.4 into pointwise bounds near the maximum of $u(x,y)$. These bounds are then in turn used to prove the sharper estimate on the location of the maximum. We finish by giving two consequences of this estimate of the location of the maximum. The first is that we obtain sharper bounds on the derivative $\partial_y u(x,y)$ as we approach the maximum, and we also obtain an improved pointwise bound on $\partial_x u(x,y)$ in a region around the maximum of height comparable to L_1 in the y-direction, and length comparable to L_2 in the x-direction.

In Chapter 2, we will apply the results in this chapter to study the first eigenvalue and eigenfunctions of a class of three dimensional convex domains D. The domains D that we will consider are of the form

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D^*, f_1(x,y) \leq z \leq f_2(x,y)\},$$

where D^* is a two dimensional convex domain satisfying the properties in Definition 1.1.1, and we let ν be the first Dirichlet eigenvalue of D, with corresponding eigenfunction $\psi(x,y,z)$. The functions

$f_1(x,y)$ and $f_2(x,y)$ are convex and concave respectively (ensuring that D is indeed convex), such that defining

$$h(x,y) \coloneqq f_2(x,y) - f_1(x,y),$$

the potential $V(x,y) = h(x,y)^{-2}$ satisfies the assumptions of Definition 1.1.3. In particular, this allows us to find values for the length scales L_1 and L_2 from Definitions 1.1.4 and 1.1.7. It also means that we have an eigenvalue λ with corresponding eigenfunction $u(x,y)$ on D^* with the properties established in the previous sections. If we place an additional assumption on the size of L_2 relative to L_1, then we will be able to use these properties of λ and $u(x,y)$ to obtain an eigenvalue bound for ν that is analogous to the bound on λ in Theorem 1.1.12, and an estimate on the shape of the level sets of the eigenfunction $\psi(x,y,z)$ that is analogous to the estimate for the level sets of $u(x,y)$ in Theorem 1.1.9.

1.2 The length scales L_1 and L_2

Before proving Theorems 1.1.12 and 1.1.9, we first give some more properties of the parameters L_1 and L_2 defined in Definitions 1.1.4 and 1.1.7.

We first want to give upper and lower bounds for L_1, where we recall that L_1 is the largest value for which the sublevel set $\{(x,y) \in \Omega : V(x,y) \leq 1 + L_1^{-2}\}$ has inner radius at least L_1. We can think of this as being analogous to the parameter L from [J1], which we described earlier in (1.1). In [J1], it was shown that this parameter L satisfies

$$N^{1/3} \leq L \leq N,$$

where N is the diameter of the two dimensional domain. The upper bound on L is attained by an exactly rectangular domain, $[0,N] \times [0,1]$, and the lower bound is attained by a right triangle of height 1 and length N. Moreover, any intermediate value for L can be attained by interpolating between these two extreme cases and forming the appropriate trapezoidal shape.

We now give an analogous description for the possible values of L_1. Rather than the potential $V(x,y)$, it will be more convenient to work with the *height function*

$$h(x,y) = V(x,y)^{-1/2}, \tag{1.5}$$

which, by the assumptions on the potential, is a concave function, satisfying

$$0 \leq h(x,y) \leq 1,$$

and attaining its maximum of 1 at the minimum of $V(x,y)$.

Proposition 1.2.1. *Recalling that N_1 is the inner radius of the domain Ω, we have the bounds*

$$cN_1^{1/5} \leq L_1 \leq N_1,$$

for some absolute constant $c > 0$.

Remark 1.2.2. *We will see in the proof of the proposition, that we are using the stronger assumption that $h(x,y) = V(x,y)^{-1/2}$ is concave, instead of just the convexity of $V(x,y)$.*

Proof of Proposition 1.2.1: The proposition follows easily when the inner radius N_1 is comparable to a constant, and so throughout we will assume that $N_1 \gg 1$.

The upper bound follows trivially from the definition of L_1, and is attained, for example, when $V(x,y)$ (and hence $h(x,y)$) is identically equal to 1.

Before proving the lower bound, we recall the following theorem of John, [Jo]:

Theorem 1.2.3. *Let $K \subset \mathbb{R}^m$ be a convex domain. Then, there exists an ellipsoid E such that if $c^* \in \mathbb{R}^m$ is the centre of E, then we have*

$$E \subset K \subset c^* + m(E - c^*).$$

That is, the ellipsoid E is contained within the convex set K, but if it is dilated by a constant depending only on the dimension, then it contains K.

We will also need the following simple property of concave functions:

Lemma 1.2.4. *Suppose $g(x)$ is a concave function on an interval of length M, with $0 \leq g(x) \leq 1$, and $g(0) = 1$. Let $0 < \beta < 1$ and suppose that $g(z) = 1 - \beta$ at some point $z \in (0, M)$. Then, we have the bound*

$$M \leq \beta^{-1} z.$$

Proof of Lemma 1.2.4: By the assumptions on the function $g(x)$, it decreases by at most 1 over an interval of length M. Thus, since it is a concave function, it must satisfy

$$g(x) \geq 1 - \frac{x}{M}.$$

Since $g(z) = 1 - \beta$, this gives

$$1 - \beta \geq 1 - \frac{z}{M}, \qquad \text{or equivalently} \qquad M \leq \beta^{-1} z,$$

as required. □

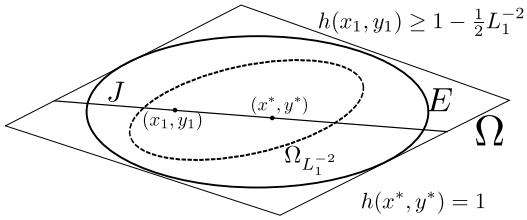

Figure 1.1: The Domain Ω, and other sets appearing in the proof of Proposition 1.2.1

We can now prove Proposition 1.2.1. Let E be the ellipse coming from Theorem 1.2.3 for our two dimensional domain Ω, and let (x^*, y^*) be a point where $h(x, y)$ attains its maximum of 1. Consider the ray J which is the intersection of our domain Ω, and the line containing the point (x^*, y^*) and the centre of the ellipse E (see Figure 1.1).

Since Ω has inner radius equal to N_1, by the properties of the ellipse E, we know that the ray J has length M with

$$M \geq c_1 N_1, \tag{1.6}$$

for some small absolute constant $c_1 > 0$. Now consider the intersection of J with the interior of the sublevel set

$$\Omega_{L_1^{-2}} = \{(x, y) \in \Omega : V(x, y) \leq 1 + L_1^{-2}\}.$$

Let J_1 be this interval. If $V(x, y) = 1 + L_1^{-2}$, then $1 - h(x, y) = 1 - V(x, y)^{-2}$ will be comparable

to L_1^{-2}, and so applying Lemma 1.2.4 with $\beta = L_1^{-2}$, we see that J_1 will be of length A, where

$$M \leq C_1 L_1^2 A, \tag{1.7}$$

for a large absolute constant C_1.

Combining (1.6) and (1.7) gives us

$$c_1 N_1 \leq M \leq C_1 L_1^2 A. \tag{1.8}$$

Thus, the lower bound of the proposition is established unless

$$A \geq C_2 L_1^3, \tag{1.9}$$

for a large constant $C_2 > 0$.

Therefore, we will assume that (1.9) holds, and so in particular, A is large compared to L_1. Let $E_{L_1^{-2}}$ be the ellipse from Theorem 1.2.3 for the set $\Omega_{L_1^{-2}}$, and rotate so that the minor axis of $E_{L_1^{-2}}$ lies in the y-direction. Then, by the definition of L_1, the minor axis of $E_{L_1^{-2}}$ has length comparable to L_1.

This means that the ray of length A must approximately lie in the x-direction. Ω is a convex set with inner radius N_1, and the original ray, J, through Ω is of length M. Therefore, if we pick a point (x_1, y_1) in the interval J_1, which is at a distance of at least $A/4$ from the ends of J_1, then the height of Ω in the y-direction at $x = x_1$ must be at least

$$c_2 A N_1 / M, \tag{1.10}$$

for a constant $c_2 > 0$. In contrast, the height of $\Omega_{L_1^{-2}}$ at $x = x_1$ must be bounded above by $C_3 L_1$, since the minor axis of $E_{L_1^{-2}}$ lies in the y-direction and has length comparable to L_1.

Moreover, the concave function $h(x, y)$ varies from 1 to $1 - L_1^{-2}$ in the interval J_1 of length A. Thus, using Lemma 1.2.4 again, we have

$$h(x_1, y_1) \geq 1 - \frac{3}{4L_1^2}, \tag{1.11}$$

at this point on the ray.

Thus, combining (1.10) and (1.11), we see that, for $x = x_1$ fixed, $h(x_1, y)$ is a concave function

of y, which decreases by at most 1 on an interval of length comparable to AN_1/M, and decreases by $\frac{1}{4}L_1^{-2}$ on an interval of length comparable to L_1. Thus, using Lemma 1.2.4 one more time, we see that

$$\frac{AN_1}{M} \leq C_4 L_1^2 L_1 = C_4 L_1^3, \tag{1.12}$$

for a constant C_4. Combining (1.8) and (1.12) we see that

$$M \leq C_1 L_1^2 A \leq C_1 L_1^2 C_4 L_1^3 \frac{M}{N_1} = C_5 L_1^5 \frac{M}{N_1},$$

for a constant $C_5 > 0$. Rearranging this inequality gives the desired lower bound on L_1.

□

We noted in the proof of Proposition 1.2.1 that it is straightforward to give an example showing that the upper bound on L_1 is sharp. We now want to construct an example showing that the lower bound on L_1 is also optimal.

Lemma 1.2.5. *We can find a domain Ω and potential $V(x,y)$ satisfying the assumptions of Definitions 1.1.1 and 1.1.3 such that*

$$L_1 \geq c N_1^{1/5},$$

for some absolute constant $c > 0$.

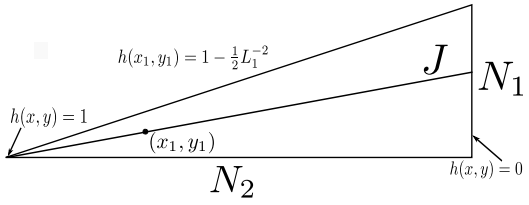

Figure 1.2: The Domain in Lemma 1.2.5

Proof of Lemma 1.2.5: We first construct the domain Ω. We have remarked earlier, that for the two dimensional domain case in [J1], a right triangle gives the smallest possible value for L. Motivated by this, we let Ω be a right triangle of side lengths N_1 in the y-direction, and side length N_2 in the x-direction (see Figure 1.2). We note that while the inner radius of this domain is not identically to N_1, it is comparable to N_1 (independently of the size of N_2), and this is all we need.

We now define the potential $V(x,y)$, via the function $h(x,y) = V(x,y)^{-2}$. We let $h(x,y) = 1$ at the point where the hypotenuse joins the side of length N_2, and set $h(x,y) = 0$ at the midpoint of the side of length N_1. We then require $h(x,y)$ to decay linearly on the interval connecting these two points. Finally, $h(x,y)$ decays linearly to 0 in the y-direction as we move away from this interval. This defines $h(x,y)$ everywhere on Ω, and also ensures that $h(x,y)$ is a concave function. Thus the potential $V(x,y)$ satisfies the required properties.

We define L_1 as usual from Definition 1.1.4 for this domain Ω and potential $V(x,y)$. Consider the line segment J joining the vertex where $h(x,y) = 1$ to the midpoint of the opposite side, and let M be the length of the line segment $J_1 \subset J$ on which $h(x,y) \geq 1 - L_1^{-2}$. Then, since $h(x,y)$ decays linearly, and the whole of J has length comparable to N_2, it is easy to see that

$$M = c_1 L_1^{-2} N_2, \tag{1.13}$$

for a constant $c_1 > 0$.

By the definition of L_1, the set $\{(x,y) \in \Omega : h(x,y) = 1 - L_1^{-2}\}$ has inner radius comparable to L_1. Thus, at the point (x_1, y_1) on the line segment J with

$$h(x_1, y_1) = 1 - \tfrac{1}{2} L_1^{-2}, \tag{1.14}$$

this set has height comparable to L_1 in the y-direction for $x = x_1$ fixed. Moreover, the point (x_1, y_1) is at a distance comparable to M from the vertex where $h(x,y) = 1$, and so the height of Ω at this point is equal to

$$c_2 M \frac{N_1}{N_2}, \tag{1.15}$$

for $c_2 > 0$. Thus, for $x = x_1$ fixed, $h(x_1, y)$ decays linearly to 0 on an interval of length comparable to $L_1 N_2 / N_1$, and by (1.14) decreases linearly by $\frac{1}{2} L_1^{-2}$ on an interval of length comparable to L_1. This tells us that

$$L_1^3 = c_3 M \frac{N_1}{N_2}. \tag{1.16}$$

Combining (1.13) and (1.16) gives

$$L_1^3 = c_3 c_1 L_1^{-2} N_2 \frac{N_1}{N_2} = c_3 c_1 L_1^{-2} N_1,$$

and rearranging gives the desired estimate for L_1. □

Remark 1.2.6. *By combining the two examples which show that the upper and lower bounds on L_1 from Proposition 1.2.1 are sharp, it is easy to construct examples where L_1 attains any intermediate length scale.*

We now want to consider the parameter L_2 introduced in Definition 1.1.7. Before describing the bounds that L_2 must satisfy, we first give a simple bound on the eigenvalue λ.

Proposition 1.2.7. *The first eigenvalue λ satisfies*

$$1 \leq \lambda \leq 1 + C_1 L_1^{-2},$$

for an absolute constant $C_1 > 0$.

Proof of Proposition 1.2.7: We will establish these bounds by using the variational formulation of the first eigenvalue, λ. That is,

$$\lambda = \inf \left\{ \frac{\int_\Omega |\nabla \psi(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega V(x,y)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_\Omega \psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y} \middle| \psi \in W^{1,2}(\Omega), \psi|_{\partial\Omega} = 0, \psi \not\equiv 0 \right\} \quad (1.17)$$

Since $V(x,y) \geq 1$ for all $(x,y) \in \Omega$, the lower bound, $\lambda \geq 1$ follows immediately.

To prove the upper bound, we need to construct a suitable test function $\psi(x,y)$ to use in (1.17). By the definition of L_1, we know that the sublevel set

$$\Omega_{L_1^{-2}} = \{(x,y) : V(x,y) \leq 1 + L_1^{-2}\}$$

has inner radius equal to L_1. Thus, we can choose a point (x_0, y_0) and a constant $c > 0$, such that the set

$$R = \{(x,y) : |x - x_0| \leq cL_1, |y - y_0| \leq cL_1\}$$

is contained in the interior of $\Omega_{L_1^{-2}}$. We then define $\psi(x,y)$ as

$$\psi(x,y) = \cos\left(\frac{\pi(x - x_0)}{2cL_1}\right) \cos\left(\frac{\pi(y - y_0)}{2cL_1}\right)$$

inside the square R, and set $\psi(x,y) = 0$ for all other $(x,y) \in \Omega$. It is then clear that

$$\frac{\int_\Omega |\nabla \psi(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_\Omega \psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y} \leq C_2 L_1^{-2},$$

and since $V(x,y) \leq 1 + L_1^{-2}$ on the support of the test function $\psi(x,y)$, we also have

$$\frac{\int_\Omega V(x,y)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_\Omega \psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y} \leq 1 + C_3 L_1^{-2}.$$

Using these inequalities in (1.17) gives the desired upper bound on λ.

□

We now consider the parameter L_2 from Definition 1.1.7. We recall that the sublevel set $\Omega_{L_1^{-2}}$ has inner radius L_1 and diameter $\tilde{L}_1$, and that we set L_2 to be equal to L_1 unless $\tilde{L}_1 \gg L_1$. The upper and lower bound for L_2 from Definition 1.1.7 that we want to establish is the following:

Proposition 1.2.8. *The parameter L_2 satisfies*

$$c_1 \tilde{L}_1^{1/3} L_1^{2/3} \leq L_2 \leq \frac{1}{c_1} \tilde{L}_1,$$

for some absolute constant $c_1 > 0$.

Remark 1.2.9. *In particular, the lower bound shows us that if we have $\tilde{L}_1 \gg L_1$, then also $L_2 \gg L_1$.*

Proof of Proposition 1.2.8: The value of L_2 depends on the function $\mu(x)$, where $\mu(x)$ is the first eigenvalue of the operator

$$\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x,y). \tag{1.18}$$

L_2 is the largest value such that $\mu(x)$ increases by L_2^{-2} from its minimum value, μ^*, on an interval of length at least L_2. Therefore, before proving the bounds on L_2, we first want to study the properties of the function $\mu(x)$.

We have rotated Ω so that the projection of the set $\Omega_{L_1^{-2}}$ onto the y-axis is of the smallest length amongst the projections onto any line. One immediate consequence of this is that if we set J to be the interval which is the projection of $\Omega_{L_1^{-2}}$ onto the x-axis, then the length of J is comparable to $\tilde{L}_1$, the diameter of $\Omega_{L_1^{-2}}$.

We now give a bound on the eigenvalues $\mu(x)$ for $x \in J$.

Lemma 1.2.10. *For x in the middle half of the interval J, there exists an absolute constant $C_1 > 0$ such that*

$$1 + \frac{1}{C_1 L_1^2} \leq \mu(x) \leq 1 + \frac{C_1}{L_1^2}.$$

Proof of Lemma 1.2.10: Since $\mu(x)$ is the first eigenvalue in the ordinary differential operator in (1.18), we want to apply Lemma 2.4 (a) in [J1]. This lemma implies that

$$1 + \frac{1}{C_1 L(x)^2} \leq \mu(x) \leq 1 + \frac{C_1}{L(x)^2}, \tag{1.19}$$

where $L(x)$ in the length scale associated to $V(x, y)$. In other words, for each fixed x, $L(x)$ is the largest value such that $V(x, y)$ varies from its minimum by $L(x)^{-2}$ on an interval of length at least $L(x)$. Thus, to prove the lemma it is enough to show that $L(x)$ is comparable to L_1 whenever x is in the middle half of the interval J.

The projections of $\Omega_{L_1^{-2}}$ onto the x and y-axes have lengths comparable to $\tilde{L}_1$ and L_1 respectively. It follows from Theorem 1.2.3 that, for those x in the middle half of J, the height of $\Omega_{L_1^{-2}}$ in the y-direction is comparable to L_1. Since the potential $V(x, y)$ is convex, attains its minimum of 1, and is equal to $1 + L_1^{-2}$ on the boundary of $\Omega_{L_1^{-2}}$, we know that for all x in the middle half of J, we must have $V(x, y) \leq 1 + \frac{1}{2} L_1^{-2}$ for some y.

As a result, for all x fixed in the middle half of J, the potential $V(x, y)$ varies by an amount comparable to L_1^{-2}, for y in an interval of length comparable to L_1. Therefore, for each x fixed the length scale $L(x)$ is comparable to L_1, and hence using (1.19) we have the required bound. □

Remark 1.2.11. *Since Lemma 2.4 (a) in [J1] played a key role in the above, let us say a few words about its proof. The upper bound in* (1.19) *follows easily by choosing the appropriate test function, just as in the proof of Proposition 1.2.7. The proof of the lower bound is slightly more complicated and makes use of the convexity of the potential to ensure that it grows at a sufficiently fast rate once we move away from its minimum.*

Before completing the proof of Proposition 1.2.8, we need one more property of the function $\mu(x)$.

Lemma 1.2.12. *The first eigenvalue $\mu(x)$ is a convex function of x.*

Proof of Lemma 1.2.12: This convexity property follows from Corollary 1.15 in [BL1]. The convexity of the eigenvalue is deduced from the log concavity of the fundamental solution of the associated

diffusion operator. □

Remark 1.2.13. *Although in the assumptions of Corollary 1.15 in [BL1], the potential does not depend on the x-variable, the proof of the log concavity of the fundamental solution (and hence the convexity of the first eigenvalue) follows in the same way if $V(x, y)$ is allowed to depend on x, provided it remains a convex function.*

We can now combine Lemmas 1.2.10 and 1.2.12 to complete the proof of Proposition 1.2.8: Since the interval J is of length comparable to $\tilde{L}_1$, Lemma 1.2.10 tells us that $\mu(x)$ varies by an amount at most comparable to L_1^{-2} for x in an interval of length comparable to $\tilde{L}_1$. Thus, since $\mu(x)$ is a convex function, applying the same logic as in Lemma 1.2.4, we immediately obtain the lower bound

$$L_2 \geq c_1 \tilde{L}_1^{1/3} L_1^{2/3}. \tag{1.20}$$

By the convexity of $V(x, y)$, given $C_2 > 0$, we can find $C_3 > 0$ to ensure that

$$V(x, y) \geq 1 + C_2 L_1^{-2},$$

whenever the point (x, y) is at least $C_3 \tilde{L}_1$ from $\Omega_{L_1^{-2}}$. This means that $\mu(x)$ certainly must increase by an amount comparable to L_1^{-2} when x is a distance comparable to $\tilde{L}_1$ from J, and this gives us the upper bound

$$L_2 \leq \frac{1}{c_1} \tilde{L}_1. \tag{1.21}$$

Combining the inequalities in (1.20) and (1.21) completes the proof of the proposition. □

1.3 A bound on the first eigenvalue λ

We recall from Proposition 1.2.7 that the first eigenvalue λ satisfies

$$1 \leq \lambda \leq 1 + C_1 L_1^{-2}.$$

In this section we will assume that we have $\tilde{L}_1 \gg L_1$ (and hence $L_2 \gg L_1$ also), and then prove the improved upper and lower bound on the eigenvalue λ from Theorem 1.1.12. That is, we will

show that λ satisfies

$$\mu \leq \lambda \leq \mu + CL_2^{-2}, \tag{1.22}$$

where μ is the first eigenvalue of the ordinary differential operator

$$\mathcal{A} = -\frac{d^2}{dx^2} + \mu(x). \tag{1.23}$$

The lower bound in (1.22) is more straightforward, and so we establish this bound first.

Proposition 1.3.1 (Lower bound on λ)**.** *The first eigenvalue λ satisfies*

$$\lambda \geq \mu.$$

Proof of Proposition 1.3.1: As before, for each x fixed, let $\Omega(x)$ be the cross-section of Ω at x. Then, the first Dirichlet eigenfunction $u(x, y)$ satisfies $u(x, y) = 0$ whenever y is at the endpoints of the interval $\Omega(x)$. In particular, for each fixed x, the function $u(x, \cdot)$ is an admissible test function for the variational formulation of the first eigenvalue of the operator $\mathcal{L}(x)$. Thus,

$$\int_{\Omega(x)} (\partial_y u(x, y))^2 + V(x, y)u(x, y)^2 \, \mathrm{d}y \geq \mu(x) \int_{\Omega(x)} u(x, y)^2 \, \mathrm{d}y.$$

Integrating this over x, and using

$$\begin{cases} (-\Delta_{x,y} + V(x, y))u(x, y) &= \lambda u(x, y) \qquad \text{in } \Omega \\ u(x, y) &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

we see that

$$\begin{aligned} \lambda \int_\Omega u(x, y)^2 \, \mathrm{d}x \, \mathrm{d}y &= \int_\Omega (\partial_x u(x, y))^2 + (\partial_y u(x, y))^2 + V(x, y)u(x, y)^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\geq \int_\Omega (\partial_x u(x, y))^2 + \mu(x)u(x, y)^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\geq \mu \int_\Omega u(x, y)^2 \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

To get the final inequality, we have defined $u(x, y) = 0$ outside Ω, used Fubini to calculate the interval in x first, and then used the variational formulation for the first eigenvalue μ of the operator $\mathcal{A}$ in (1.23). This gives us the bound $\lambda \geq \mu$, as required. □

We now turn to the upper bound and prove:

Proposition 1.3.2 (Upper bound on λ)**.** *We have an upper bound on the first eigenvalue λ of the form,*

$$\lambda \leq \mu + CL_2^{-2},$$

for an absolute constant $C > 0$.

Remark 1.3.3. *From Lemma 4.2 (e) in [J1], the operator $\mathcal{A}$ defined in* (1.23) *has spectral gap bounded from below by a multiple of L_2^{-2}. Therefore, obtaining bounds on λ up to a precision of CL_2^{-2} is important if we want this separation of variables in the x and y variables to be of use to us.*

Proof of Proposition 1.3.2:

As in the proof of the simple bound on λ in Proposition 1.2.7, we will again make use of the variational formulation for λ given in (1.17). To do this we need to construct an appropriate test function, and our motivation will come from performing an approximate change of variables in the x and y-directions. Before stating our test function, we need some definitions.

Definition 1.3.4. *For each fixed x, we define $\psi_1^{(x)}(y)$ to be the L^2-normalised first eigenfunction of the ordinary differential operator $\mathcal{L}(x)$. That is, $\psi_1^{(x)}(y)$ is L^2-normalised on the cross-section $\Omega(x)$, and satisfies*

$$\begin{cases} \left(-\frac{d^2}{dy^2} + V(x,y)\right)\psi_1^{(x)}(y) &= \mu(x)\psi_1^{(x)}(y) \qquad \text{in } \Omega(x) \\ \psi_1^{(x)}(y) &= 0 \qquad \text{on } \partial\Omega(x). \end{cases}$$

Definition 1.3.5. *Let I be the interval of length L_2 from Definition 1.1.7. We define the cut-off function $\chi(x)$ to be a positive function which is comparable to its maximum in the middle half of the interval I, and supported in the middle three quarters of I, such that it decays smoothly to zero from its maximum. We also require that $\chi(x)$ is L^2-normalised on the interval I. In particular, this allows us to ensure that*

$$|\chi'(x)| \leq C_1 L_2^{-3/2},$$

for some absolute constant C_1.

We can now define the test function $f(x,y)$ which we will use in (1.17).

Definition 1.3.6. *We define the test function $f(x,y)$ by*

$$f(x,y) \coloneqq \chi(x)\psi_1^{(x)}(y).$$

As a first step towards proving Proposition 1.3.2, we prove the following intermediate step.

Proposition 1.3.7. *We have an upper bound for λ of the form*

$$\lambda \leq \mu + \int_\Omega \chi(x)^2(\partial_x\psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_2^{-2},$$

for a constant C_1.

Proof of Proposition 1.3.7: To obtain an upper bound on the first eigenvalue λ, we will calculate the quotient from (1.17)

$$\frac{\int_\Omega |\nabla f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V(x,y)|f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega |f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y}, \tag{1.24}$$

with $f(x,y)$ as in Definition 1.3.6. Since $\psi^{(x)}(y)$ is $L^2(\Omega(x))$-normalised in y for any fixed x, and $\chi(x)$ is $L^2(I)$-normalised in x, first computing the integral in y, and then the integral in x, we see that the denominator in (1.24) is equal to 1. Thus, we have the bound

$$\lambda \leq \int_\Omega |\nabla_{x,y}\left(\chi(x)\psi_1^{(x)}(y)\right)|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V(x,y)\chi(x)^2\psi_1^{(x)}(y)^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{1.25}$$

For each x, the function $\psi^{(x)}(y)$ satisfies

$$\int_{\Omega(x)} \psi_1^{(x)}(y)^2 \,\mathrm{d}y = 1, \tag{1.26}$$

and it is equal to 0 at the endpoints of the interval $\Omega(x)$. Therefore, differentiating (1.26) with respect to x, we obtain the orthogonality relation

$$\int_{\Omega(x)} \partial_x\psi_1^{(x)}(y)\psi_1^{(x)}(y) \,\mathrm{d}y = 0.$$

Thus, calculating the derivatives in the first integral in (1.25), and using this orthogonality

relation, we see that (1.25) becomes

$$\lambda \leq \int_\Omega \chi'(x)^2 \psi_1^{(x)}(y)^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi(x)^2 (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi(x)^2 (\partial_y \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V(x,y)\chi(x)^2 \psi_1^{(x)}(y)^2 \,\mathrm{d}x\,\mathrm{d}y.$$

The eigenfunction $\psi_1^{(x)}(y)$ of $\mathcal{L}(x)$ has eigenvalue $\mu(x)$, and so we have the inequality

$$\lambda \leq \int_I \chi'(x)^2 \,\mathrm{d}x + \int_\Omega \chi(x)^2 (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + \int_I \chi(x)^2 \mu(x) \,\mathrm{d}x.$$

From Definition 1.1.7 we know that

$$|\mu(x) - \mu| \leq L_2^{-2}.$$

Therefore, combining this with the bound on $\chi'(x)$ given in Definition 1.3.5, we obtain the desired upper bound on λ of

$$\lambda \leq \mu + \int_\Omega \chi(x)^2 (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_2^{-2}.$$

□

As a result of Proposition 1.3.7, to obtain an upper bound on λ, we need to consider the derivative with respect to x of the eigenfunction $\psi_1^{(x)}(y)$. In particular, we want to bound

$$\int_{\Omega(x)} (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}y.$$

We will prove the following proposition:

Proposition 1.3.8. *Let x be fixed in the support of the cut-off function $\chi(x)$. Then,*

$$\int_{\Omega(x)} (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}y \leq C_1 L_2^{-2},$$

with the constant C_1 independent of x.

Remark 1.3.9. *Combining Proposition 1.3.7 with Proposition 1.3.8 establishes*

$$\lambda \leq \mu + C_1 L_2^{-2},$$

and finishes the proof of Proposition 1.3.2.

Proof of Proposition 1.3.8:

Throughout the proof of this proposition, $x \in I$ will be fixed in the support of the cut-off function x, and all bounds that appear will be uniform in x. We will also suppress the dependence of certain functions on x where this simplifies the notation.

Since,

$$\left(-\frac{d^2}{dy^2} + V(x,y)\right)\psi_1^{(x)}(y) = \mu(x)\psi_1^{(x)}(y),$$

differentiating with respect to x we find that for $y \in \Omega(x)$, we have

$$\left(-\frac{d^2}{dy^2} + V(x,y) - \mu(x)\right)\partial_x\psi_1^{(x)}(y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y), \tag{1.27}$$

where the notation $'$ denotes differentiation with respect to x. Although, for each fixed x, $\psi_1^{(x)}(y)$ is equal to zero at the endpoints on $\Omega(x)$, the function $\partial_x\psi_1^{(x)}(y)$ will not in general be zero here.

Therefore, we will also need to take into account its boundary values. For those x in the support of the cut-off function $\chi(x)$, we can write the two parts of $\partial\Omega$ below and above in the y-direction as $\{y = g_1(x)\}$ and $\{y = g_2(x)\}$, where $g_1(x)$ and $g_2(x)$ are convex and concave functions respectively. We set $\alpha = \partial_x\psi_1^{(x)}(g_2(x))$, and define

$$g(y) \coloneqq \partial_x\psi_1^{(x)}(y) - \alpha. \tag{1.28}$$

Our aim is to find an expression for the function $g(y)$ using (1.27). To do this we need to make the following definitions (again suppressing the dependence on x throughout).

Definition 1.3.10. *We define the function $F(y)$ by,*

$$F(y) \coloneqq V(x,y) - \mu(x).$$

We know that $\mu(x) \leq 1 + C_1 L_1^{-2}$, and that $\min_y V(x,y) \leq \mu(x)$ for all x in the support of $\chi(x)$. This allows us to define the three points y_1, y_2 and y_3.

Definition 1.3.11. *We fix an absolute constant C. We define y_1 to be the middle point of the 'centre', where the centre is the interval on which $V(x,y) \leq \min_y V(x,y) + CL_1^{-2}$. We then choose $y_2 \geq y_1$ to be the largest value such that $[y_1, y_2]$ is contained in the middle half of the centre. Finally,*

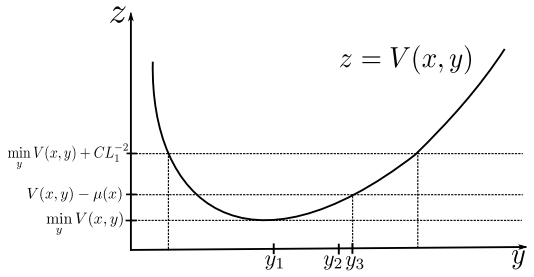

Figure 1.3: The Points y_1, y_2 and y_3 from Definition 1.3.11

we define $y_3 \geq y_2$ to be the value of y for which $F(y_3) = V(x, y_3) - \mu(x) = 0$. (See Figure 1.3)

Definition 1.3.12. *We set $\phi(y)$ to be the first eigenfunction of $\mathcal{L}(x)$, but this time normalised to be positive with a maximum of 1. Note that this function is equal to a multiple of $\psi_1^{(x)}(y)$ (where the multiple depends on the fixed value of x).*

For $y \geq y_1$, we define the function $\tilde{\phi}(y)$ by

$$\tilde{\phi}(y) \coloneqq \phi(y) \int_{y_1}^{y} \phi(t)^{-2} \, \mathrm{d}t.$$

We can now write down an expression for the function $g(y)$.

Lemma 1.3.13. *Let $c_0(x)$ be the value such that*

$$g(y) - c_0(x)\psi_1^{(x)}(y) = 0$$

at $y = y_1$. Then, for $y \geq y_1$, the function $g(y)$ satisfies

$$g(y) - c_0(x)\psi_1^{(x)}(y) = \phi(y) \int_{y_1}^{y} \tilde{\phi}(t) G(x,t) \, \mathrm{d}t + \tilde{\phi}(y) \int_{y}^{g_2(x)} \phi(t) G(x,t) \, \mathrm{d}t, \tag{1.29}$$

where $G(x, y)$ is equal to

$$G(x,y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y) + (V(x,y) - \mu(x))\alpha.$$

Proof of Lemma 1.3.13: We see from the definition of $g(y)$ from (1.28) and the equation that $\partial_x \psi_1^{(x)}(y)$ satisfies in (1.27), that we have

$$\left(-\frac{d^2}{dy^2} + V(x,y) - \mu(x) \right) g(y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y) + (V(x,y) - \mu(x))\alpha.$$

The right hand side of the above equation is equal to $G(x,y)$, so that

$$(\mathcal{L}(x)-\mu(x))(g(y)-c_0(x)\phi(y))=G(x,y). \tag{1.30}$$

Since $\mathcal{L}(x)$ is a second order ordinary differential operator, to find an expression for $g(y)$ we will apply the method of variation of parameters to (1.30). From Definition 1.3.12, we know that

$$(\mathcal{L}(x)-\mu(x))\phi(y)=0,$$

with $\phi(g_2(x))=0$. It is straightforward to check that the function $\tilde{\phi}(y)$ from Definition 1.3.12 also satisfies

$$(\mathcal{L}(x)-\mu(x))\tilde{\phi}(y)=0,$$

for $y \geq y_1$, and is equal to 0 at $y=y_1$. Thus, since the function $g(y)-c_0(x)\phi(y)$ is equal to 0 at $y=y_1$ and $y=g_2(x)$, using (1.30) and variation of parameters, we can write

$$g(y)-c_0(x)\psi_1^{(x)}(y)=\phi(y)\int_{y_1}^{y}\tilde{\phi}(t)G(x,t)\,\mathrm{d}t+\tilde{\phi}(y)\int_{y}^{g_2(x)}\phi(t)G(x,t)\,\mathrm{d}t.$$

□

Looking at this expression for $g(y)$, we see that we will need to study how the magnitude of the functions $\phi(y)$ and $\tilde{\phi}(y)$ depends on the size of the potential $V(x,y)$, and its derivative with respect to x, $\partial_x V(x,y)$. Also, since $g(y)=\partial_x\psi_1^{(x)}(y)-\alpha$, where $\alpha=\partial_x\psi_1^{(x)}(g_2(x))$, we will also need to estimate the size of $\partial_x\psi_1^{(x)}(y)$ at the endpoints of the interval $\Omega(x)$.

1.3.1 Properties of $\phi(y)$ and $\tilde{\phi}(y)$

We first study the function $\phi(y)$, where we recall that it satisfies

$$\left(-\frac{d^2}{dy^2}+V(x,y)-\mu(x)\right)\phi(y)=0.$$

For x fixed in the support of I, let us set $L(x)$ to be the largest value such that $V(x,y)$ varies from its minimum value by $L(x)^{-2}$ on an interval in y of length at least $L(x)$. Then, as we remarked in the proof of Lemma 1.2.10, $L(x)$ is comparable to L_1. Thus, from Lemma 2.4 (b), (d) in [J1], we immediately get the following estimates on $\phi(y)$ (uniformly in x).

Lemma 1.3.14. *There exists an absolute constant C_1 such that the eigenfunction $\phi(y)$ (which we recall will depend on x) satisfies*

$$|\phi'(y)| \leq C_1/L_1 \text{ for all } y \in \Omega(x),$$

and

$$\phi(y) \leq C_1 e^{-c|y-y_1|/L_1},$$

where y_1 is the point in the 'centre' given in Definition 1.3.11.

This second inequality gives an L^∞ exponential decay estimate for $\phi(y)$ as we move away from the minimum of $V(x,y)$ on a length scale comparable to L_1. In particular, this means that the $L^2(\Omega(x))$ norm of $\phi(y)$ is bounded above by a multiple of $L_1^{1/2}$. (In fact, it follows from Lemma 2.4 in [J1] that the $L^2(\Omega(x))$-norm also has a lower bound that is comparable to $L_1^{1/2}$.)

We now want to sharpen this L^∞ exponential decay estimate for $\phi(y)$ as $V(x,y)$ increases from its minimum.

Proposition 1.3.15. *Define the interval J_k by,*

$$J_k = [t_k, t_{k+1}] \coloneqq \{t \geq y_3 : \partial_t V(x,t) \in [2^{-k}, 2^{-k+1}]\}. \tag{1.31}$$

Then, for all $t_k \leq t \leq g_2(x)$,

$$\phi(t) \leq \phi(t_k) \exp(-(t-t_k)2^{-k/3}/10),$$

for all $y_3 \leq t \leq t_{k+1}$,

$$\phi(t_{k+1}) \leq \phi(t) \exp(-(t_{k+1}-t)2^{-k/3}/10)$$

and for all $t \in J_k$,

$$\phi(t) \leq |\phi'(t)| 2^{k/3}.$$

For the interval $\tilde{J}_k$ defined by,

$$\tilde{J}_k = [\tilde{t}_k, \tilde{t}_{k+1}] \coloneqq \{t \geq y_3 : V(x,t) - \min_t V(x,t) \in [2^{-2k/3}, 2^{-2(k-1)/3}]\}, \tag{1.32}$$

we have the analogous bounds on $\phi(t)$.

Remark 1.3.16. *We have the analogous decay estimates for $\phi(y)$ as we move away from the region where $V(x,y) \leq \min_y V(x,y) + L_1^{-2}$ in the other direction.*

Remark 1.3.17. *We recall that $y = y_3$ is the point where $V(x,y) - \mu(x) = 0$. Since $\min_y V(x,y) - \mu(x) \leq -cL_1^{-2}$, by convexity, J_k and $\tilde{J}_k$ are only non-empty for those k satisfying $2^k \leq CL_1^3$, for some absolute constant $C > 0$.*

Proof of Proposition 1.3.15: The proposition follows from the key inequality given in the proof of Theorem A in [J1],

$$\left|(\log \phi(t))'\right| = |\phi'(t)|/\phi(t) \geq 2^{-k/3}/10 \qquad \text{for all } t \in J_k.$$

Integrating this inequality from both $t = t_k$ and $t = t_{k+1}$ gives all of the desired estimates involving the intervals J_k.

By the definition of the intervals $\tilde{J}_k$, we have $V(x,t) - \mu(x) \geq 2^{-2k/3}$ for $t \in \tilde{J}_k$. Therefore, it is straightforward to obtain the same bounds for $(\log \phi(t))'$, and hence $\phi(t)$ itself on $\tilde{J}_k$ as for the intervals J_k. □

We now show to what extent $\phi'(y)$ inherits this exponential decay as we move away from the centre.

Proposition 1.3.18. *Let the intervals J_k be defined as in Proposition 1.3.15. Then, for all $t \geq t_k$,*

$$|\phi'(t)| \leq C|\phi'(t_k)| \exp(-c|t - t_k|2^{-k/3}),$$

for some absolute constants c and $C > 0$.

Proof of Proposition 1.3.18: The function $\phi(t)$ satisfies the equation

$$\phi''(t) = F(t)\phi(t),$$

with the function $F(t) = V(x,t) - \mu(x)$ as before. On the intervals J_k, we know that $t \geq y_3$, and so certainly $F(t) \geq 0$. Also, $\phi'(t) \leq 0$, and so this mean that $|\phi'(t)|$ is decreasing. Thus, for $t \geq t_k$, we have

$$|\phi'(t)| \leq |\phi'(t_k)|.$$

If $|t - t_k| \leq 2^{k/3}$, then this is enough to establish the required bound.

Now suppose that $|t - t_k| \in [N2^{k/3}, (N+1)2^{k/3}]$ for some $N \geq 1$. Then, by Proposition 1.3.15, we know that $\phi(t)$ satisfies

$$\phi(t) \leq C2^{k/3}|\phi'(t_k)| \exp(-cN2^{-k/3}).$$

In particular, $\phi(t)$ changes by at most $C2^{k/3}|\phi'(t_k)| \exp(-cN2^{-k/3})$, as t ranges over this interval of length $2^{k/3}$. Since $\phi'(t)$ is negative here, this gives us a bound on the integral of $|\phi'(t)|$ over this interval.

Moreover, as we noted above, by convexity, $|\phi'(t)|$ decreases as t increases. In particular, since the interval $[N2^{k/3}, (N+1)2^{k/3}]$ has length $2^{k/3}$, this means that

$$|\phi'(t)| \leq C2^{k/3}|\phi'(t_k)| \exp(-cN2^{-k/3}).2^{-k/3} = C|\phi'(t_k)| \exp(-cN2^{-k/3}),$$

for t at the right endpoint of the interval. This concludes the proof of the proposition. □

It will often be important to measure the distance of a point (x,y) from the level sets $\{(x,y) \in \Omega : V(x,y) = 1 + L_1^{-2}\}$.

Definition 1.3.19. *Fix a large absolute constant C^*. Then, suppressing the dependence on x, let $y^* \geq y_1$ be the first point where $V(x,y) \geq 1 + C^* L_1^{-2}$.*

We can now write down an immediate corollary of Proposition 1.3.18.

Corollary 1.3.20. *For any $t \geq t_k$, we have the first derivative estimate*

$$|\phi'(t)| \leq CL_1^{-1} \exp(-c|t - t_k|2^{-k/3}) \exp(-c|t_k - y^*|/L_1).$$

Proof of Corollary 1.3.18: We can apply Proposition 1.3.18 with t replaced by t_k and t_k replaced by

y^* to obtain a bound on $|\phi'(t_k)|$ of the form

$$|\phi'(t_k)| \leq CL_1^{-1} \exp(-c|t_k - y^*|/L_1).$$

We then use this bound in the right hand side of the estimate for $|\phi'(t)|$ in Proposition 1.3.18 to get the desired result. □

From Lemma 1.3.13, we see that as well as $\phi(y)$, it will also be important to study the properties of $\tilde{\phi}(y)$, where we recall that for $y \geq y_1$, we have

$$\tilde{\phi}(y) = \phi(y) \int_{y_1}^{y} \phi(t)^{-2} \, \mathrm{d}t.$$

We recall from Definition 1.3.11 that $y_2 \geq y_1$ is the largest value of y_2 such that $[y_1, y_2]$ is contained in the middle half of the 'centre', where $V(x, y) \leq \min_t V(x, t) + CL_1^{-2}$ and that $y_3 \geq y_2$ is the value of y for which $F(y_3) = V(x, y_3) - \mu(x) = 0$. We now prove:

Lemma 1.3.21. *The function $\tilde{\phi}(y)$ satisfies*

$$\tilde{\phi}(y) \leq C_1 L_1,$$

for $y_1 \leq y \leq y_3$ and

$$\tilde{\phi}(y) \leq C_1 L_1 + C_1 |\phi'(y)|^{-1},$$

for $y_3 \leq y \leq g_2(x)$.

Proof of Lemma 1.3.21: We first consider the interval $[y_1, y_2]$. By the definition of the point y_2, Lemma 2.4 in [J1] implies that we have an absolute lower bound on $\phi(t)$ for $t \in [y_1, y_2]$, and we know that this interval is of length comparable to L_1. Thus, for $y \in [y_1, y_2]$, we have

$$\tilde{\phi}(y) \leq C_1 L_1 \phi(y).$$

Before considering $y \in [y_2, y_3]$, we first assume that $y \geq y_3$. Here $F(y) \geq 0$, and so $|\phi'(y)|$ is decreasing ($\phi'(y)$ is becoming less negative). Therefore, for $t \in [y_3, y]$, we have the lower bound

$$\phi(t) \geq \phi(y) + |\phi'(y)|(y - t).$$

This gives us the bound

$$\int_{y_3}^{y} \phi(t)^{-2}\,\mathrm{d}t \leq C_1\phi(y)^{-1}|\phi'(y)|^{-1}. \tag{1.33}$$

We now want to bound

$$\int_{y_2}^{y_3} \phi(t)^{-2}\,\mathrm{d}t.$$

Since $\phi''(y) = F(y)\phi(y)$, we have

$$\phi'(y) = \int_{\tilde{y}}^{y} F(t)\phi(t)\,\mathrm{d}t,$$

where $\phi(y)$ attains its maximum of 1 at $y = \tilde{y}$. For $t \in [y_2, y_3]$, $F(t) \leq 0$, and so $|\phi'(t)|$ is increasing from 0, $\phi(t)$ is decreasing from 1, and $|y_3 - y_2| \leq C_1L_1$. Therefore, either $\phi(t)$ is bounded below by an absolute constant or else $|\phi'(y)| \geq C_1L_1^{-1}$. This gives us the bound

$$\int_{y_2}^{y_3} \phi(t)^{-2}\,\mathrm{d}t \leq C_1L_1. \tag{1.34}$$

Combining the bounds in (1.33) and (1.34) shows that

$$\tilde{\phi}(y) \leq C_1L_1,$$

for $y \in [y_2, y_3]$, and

$$\tilde{\phi}(y) \leq C_1L_1 + C|\phi'(y)|^{-1}$$

for $y \geq y_3$, as required.

□

1.3.2 An estimate for $\partial_x\psi_1^{(x)}(y)$ at the boundary

We can now bound $\partial_x\psi_1^{(x)}(y)$ at the endpoints of the interval $\Omega(x)$. For each fixed x, $\psi_1^{(x)}(y)$ has zero boundary conditions on $\Omega(x)$. However, since the interval $\Omega(x)$ will in general depend on x, $\partial_x\psi_1^{(x)}(y)$ will not necessarily be zero when y is at the end-points of $\Omega(x)$.

We recall from Definition 1.3.19, that $y^* \geq y_1$ is the first point where $V(x, y) \geq 1 + C^*L_1^{-2}$, for

a fixed large constant C^*. The upper endpoint of the interval $\Omega(x)$ is equal to $g_2(x)$, and we set

$$M \coloneqq g_2(x) - y^*, \tag{1.35}$$

which is the distance between the endpoint of $\Omega(x)$ and the region where the potential $V(x,y)$ is less than $1 + C^* L_1^{-2}$. We can prove a bound on $\partial_x \psi_1^{(x)}(g_2(x))$ in terms of M.

Proposition 1.3.22. *For $y = g_2(x)$ equal to the upper endpoint of the interval $\Omega(x)$, we have the bound*

$$|\alpha| = \left|\partial_x \psi_1^{(x)}(g_2(x))\right| \leq C L_2^{-1} L_1^{-3/2}(L_1 + M)\exp(-cML_1^{-1}).$$

We also have an analogous bound for y equal to the lower endpoint of $\Omega(x)$.

Proof of Proposition 1.3.22: We can view $\psi_1^{(x)}(y)$ as a function of two variables on the domain Ω, with $\psi_1^{(x)}(y)$ identically equal to 0 on $\partial\Omega$. In particular, for those x in the support of the cut-off function $\chi(x)$, we have written the upper boundary of Ω as the graph of the function $y = g_2(x)$, and so $\psi_1^{(x)}(g_2(x))$ is identically zero as a function of x. Differentiating this with respect to x gives

$$\partial_x \psi_1^{(x)}(g_2(x)) = -g_2'(x)\partial_y \psi_1^{(x)}(g_2(x)). \tag{1.36}$$

Thus, to obtain a bound on $\partial_x \psi_1^{(x)}(g_2(x))$, it is enough to consider $\partial_y \psi_1^{(x)}(y)$, and the slope of $\partial\Omega$ at $(x, g_2(x))$.

We remarked in the definition of $\phi(y)$ in Definition 1.3.12 that the eigenfunction $\psi_1^{(x)}(y)$ is equal to a multiple of $\phi(y)$. Since $\phi(y)$ has $L^2(\Omega(x))$-norm comparable to $L_1^{1/2}$, whereas $\psi_1^{(x)}(y)$ is $L^2(\Omega(x))$-normalised, this multiple is comparable to $L_1^{-1/2}$. Thus, by the bound on $\phi'(y)$ from Proposition 1.3.18, with 2^k comparable to L_1^3, we have the bound

$$\left|\partial_y \psi_1^{(x)}(g_2(x))\right| \leq C L_1^{-3/2} \exp\left(-cML_1^{-1}\right).$$

Therefore, by (1.36), to conclude the proof of the proposition it is enough to show that

$$|g_2'(x)| \leq C(L_1 + M)L_2^{-1}, \tag{1.37}$$

for an absolute constant $C > 0$. Recall the set $\Omega_{L_1^{-2}} = \{(x,y) \in \Omega : V(x,y) \leq 1 + L_1^{-2}\}$. This is a convex subset of Ω with height comparable to L_1 in the y-direction, and length comparable to

$\tilde{L}_1$ in the x-direction. Moreover, for x fixed in the support of $\chi(x)$, we are at a distance at least comparable to L_2 from the left and right ends of $\Omega_{L_1^{-2}}$. Therefore, if we write the upper boundary of $\Omega_{L_1^{-2}}$ of this set as the graph of a function $y = v(x)$, then certainly we have the derivative bound

$$|v'(x)| \leq CL_1L_2^{-1}.$$

In particular, if the distance M is bounded above by a multiple of L_1, then by convexity, the part of $\partial\Omega$ for x contained in the support of $\chi(x)$ has slope bounded by a multiple of $L_1L_2^{-1}$. This gives the desired bound for $g_2'(x)$ in (1.37),

$$|g_2'(x)| \leq CL_1L_2^{-1}$$

If the distance M is large compared to L_1, then the domain Ω is convex, and contains an ellipse of height comparable to M in the y-direction, and length comparable to L_2 in the x-direction. Thus, the part of $\partial\Omega$ with x in the support of $\chi(x)$ has slope bounded by a multiple of ML_2^{-1}. Again we get a bound for $g_2'(x)$,

$$|g_2'(x)| \leq CML_2^{-1}$$

which implies the bound in (1.37).

This establishes the estimate in (1.37) in all cases, and completes the proof of the proposition. □

We have now established the properties of the functions $\phi(y)$ and $\tilde{\phi}(y)$ together with the bound required on $\alpha = \partial_x\psi_1^{(x)}(g_2(x))$. Thus, we return to the expression for

$$g(y) = \partial_x\psi_1^{(x)}(y) - \alpha$$

that we derived in Lemma 1.3.13:

$$g(y) - c_0(x)\psi_1^{(x)}(y) = \phi(y)\int_{y_1}^{y}\tilde{\phi}(t)G(x,t)\,\mathrm{d}t + \tilde{\phi}(y)\int_{y}^{g_2(x)}\phi(t)G(x,t)\,\mathrm{d}t, \tag{1.38}$$

where $G(x,y)$ equals

$$G(x,y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y) + (V(x,y) - \mu(x))\alpha. \tag{1.39}$$

We will use (1.38) to obtain the desired bound on $\partial_x \psi_1^{(x)}(y)$:

Proposition 1.3.23. *As usual, for $2^k \leq CL_1^3$, let the intervals J_k be given by*

$$J_k = [t_k, t_{k+1}] = \{t \geq y_3 : \partial_t V(x,t) \in [2^{-k}, 2^{-k+1}]\}.$$

We have the pointwise bound

$$\left|\partial_x \psi_1^{(x)}(y) - c_0(x)\psi_1^{(x)}(y)\right| \leq F_1(y) + F_2(y)$$

for all $y \in \Omega(x)$ with $y \geq y_1$. Here $F_1(y)$ is a positive function on $\Omega(x)$, with a maximum comparable to $L_2^{-1}L_1^{-1/2}$ and decaying exponentially from this maximum on a length scale comparable to L_1 as y moves away from the interval where $V(x,y) \leq 1 + L_1^{-2}$. The function $F_2(y)$ is also a positive function on $\Omega(x)$, with a maximum comparable to $L_2^{-1}L_1^{-1/2}$ but it decays exponentially from this maximum within each interval J_k on a length scale comparable to $2^{k/3}$. We also have the analogous exponential decay estimate on the corresponding intervals as we move away from the 'centre' region where $V(x,y) \leq 1 + L_1^{-2}$ in the opposite direction with $y \leq y_1$.

Before we prove this proposition, let us show how it implies the $L^2(\Omega(x))$-bound on $\partial_x \psi_1^{(x)}(y)$ given in Proposition 1.3.8: We saw in the proof of Proposition 1.3.7 that $\partial_x \psi_1^{(x)}(y)$ and $\psi_1^{(x)}(y)$ satisfy the orthogonality relation

$$\int_{\Omega(x)} \partial_x \psi_1^{(x)}(y)\psi_1^{(x)}(y)\,\mathrm{d}y = 0.$$

Thus, since $\psi_1^{(x)}(y)$ is $L^2(\Omega(x))$-normalised, we have the expression

$$c_0(x) = \int_{\Omega(x)} \left(c_0(x)\psi_1^{(x)}(y) - \partial_x \psi_1^{(x)}(y)\right)\psi_1^{(x)}(y)\,\mathrm{d}y.$$

Using the bound on $c_0(x)\psi_1^{(x)}(y) - \partial_x \psi_1^{(x)}(y)$ in Proposition 1.3.23 we obtain

$$|c_0(x)| \leq C_1 L_2^{-1} L_1^{-1/2} \int_{\Omega(x)} \psi_1^{(x)}(y)\,\mathrm{d}y \leq C_1 L_2^{-1}, \tag{1.40}$$

where the final inequality holds since $\psi_1^{(x)}(y)$ has $L^2(\Omega(x))$-norm equal to 1, and decays exponentially away from its maximum on a length scale comparable to L_1.

Combining this bound on $c_0(x)$ in (1.40) with Proposition 1.3.23, we see that $\partial_x \psi_1^{(x)}(y)$ can be

bounded by functions $F_1(y) + F_2(y)$ with the same properties as in the statement of Proposition 1.3.23. This gives us an $L^2(\Omega(x))$-bound on $\partial_x \psi_1^{(x)}(y)$ of the form

$$\int_{\Omega(x)} \left(\partial_x \psi_1^{(x)}(y)\right)^2 \, \mathrm{d}y \leq C_1 L_2^{-2} + \sum_{2^k \leq C L_1^3} 2^{k/3} L_2^{-2} L_1^{-1} \leq C_1 L_2^{-2}.$$

This completes the proof of Proposition 1.3.8.

□

Since Proposition 1.3.8 implies the desired upper bound on the eigenvalue λ in Proposition 1.3.2, we just need to prove Proposition 1.3.23.

Proof of Proposition 1.3.23:

From Proposition 1.3.22 we know that

$$|\partial_x \psi_1^{(x)}(y) - g(y)| = |\partial_x \psi_1^{(x)}(g_2(x))| = |\alpha| \leq C L_2^{-1} L_1^{-3/2} (L_1 + M) \exp\left(-cML_1^{-1}\right),$$

and this bound has the same properties as the function $F_1(y)$ in the statement of the proposition. Therefore, to prove Proposition 1.3.23, it is enough to show that

$$g(y) - c_0(x) \psi_1^{(x)}(y)$$

has the desired bounds.

To do this, we want to bound the right hand side of (1.38), which contains the functions $\phi(y)$, $\tilde{\phi}(y)$ together with $G(x, y)$. The two remaining functions which we have not discussed above are the functions $\mu'(x)$ and $\partial_x V(x, y)$ appearing in $G(x, y)$. Therefore, let us prove two simple lemmas concerning these functions, and then we will be in a position to bound (1.38).

Lemma 1.3.24. *Let x be in the support of the cut-off function $\chi(x)$. Then, we have the bound*

$$|\mu'(x)| \leq C_1 L_2^{-3},$$

for an absolute constant $C_1 > 0$.

Proof of Lemma 1.3.24: We recall from Lemma 1.2.12 that the function $\mu(x)$ is a convex function of x. Moreover, by the definition of the parameter L_2, we know that $\mu(x)$ varies by L_2^{-2} for x in an interval of length at least L_2. Since the support of $\chi(x)$ is contained within the middle half of this

interval, we immediately obtain the bound

$$|\mu'(x)| \leq C_1 L_2^{-3},$$

by convexity. □

Lemma 1.3.25. *Let x be in the support of $\chi(x)$, and as in Definition 1.3.19 let $y = y^*$ be the first point where $V(x,y) \geq 1 + C^* L_1^{-2}$. Then, for $y \geq y^*$,*

$$|\partial_x V(x,y)| \leq C_1(|y - y^*| + L_1)L_2^{-1}|\partial_y V(x,y)|,$$

and for $y_1 \leq y \leq y^*$,

$$|\partial_x V(x,y)| \leq C_1 L_1^{-2} L_2^{-1} + C_1 L_1 L_2^{-1}|\partial_y V(x,y)|.$$

Proof of Lemma 1.3.25: Given, c, let $y = f(x)$ be a parameterisation of the upper part of the level set $\{(x,y) \in \Omega : V(x,y) = c\}$, so that

$$V(x, f(x)) = c = \text{constant}.$$

Differentiating this with respect to x, we see that

$$\partial_x V(x, f(x)) = -f'(x)\partial_y V(x, f(x)). \tag{1.41}$$

Assume first that $y = f(x) \geq y^*$. The sublevel set $\{(x,y) \in \Omega : V(x,y) \leq 1 + C^* L_1^{-2}\}$ is convex with height comparable to L_1 in the y-direction and length comparable to $\tilde{L}_1$ in the x-direction, and x is at distance comparable to L_2 from the ends of this set. Thus, by the convexity of the sublevel sets, we certainly have a bound on the slope of

$$|f'(x)| \leq C_1(|y - y^*| + L_1)L_2^{-1}.$$

Using this bound in the right hand side of (1.41) gives the desired bound for $y \geq y^*$.

We now suppose that $y_1 \leq y = f(x) \leq y^*$. If y is in the middle half of the interval $\{t : V(x,t) \leq$

$1 + L_1^{-2}\}$, then we certainly have the bound

$$|\partial_x V(x, f(x))| \leq C_1 L_1^{-2} L_2^{-1},$$

by the convexity of the potential $V(x, y)$. For the remaining points $(x, f(x))$ of interest, we can again use the shape of the level set to obtain the desired bound

$$|\partial_x V(x, f(x))| \leq C_1 L_1^{-2} L_2^{-1} + C_1 L_1 L_2^{-1} |\partial_y V(x, f(x))|.$$

This is because for these points we can find a direction $\mathbf{e}$, such that the directional derivative of V at $(x, f(x))$ is bounded by $L_1^{-2} L_2^{-1}$, and this direction makes an angle comparable to $L_1 L_2^{-1}$ with the x-axis.

□

Combining Lemmas 1.3.24 and 1.3.25, we see from (1.39) that

$$\begin{aligned} |G(x,t)| &\leq C_1 L_1^{-2} L_2^{-1} \psi_1^{(x)}(t) + C_1(|t - y^*| + L_1) L_2^{-1} |\partial_t V(x,t)| \psi_1^{(x)}(t) + |V(x,t) - \mu(x)|\alpha \\ &\leq C_1 L_1^{-5/2} L_2^{-1} \phi(t) + C_1(|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \phi(t) + |V(x,t) - \mu(x)|\alpha. \end{aligned} \tag{1.42}$$

The final inequality comes from

$$\psi_1^{(x)}(t) \leq C_1 L_1^{-1/2} \phi(t),$$

which holds since $\psi_1^{(x)}(t)$ is $L^2(\Omega(x))$-normalised, whereas $\phi(t)$ has $L^2(\Omega(x))$-norm comparable to $L_1^{1/2}$.

Everything is now set up to show that the two integrals in (1.38) have the bounds required in the statement of Proposition 1.3.23.

1.3.3 Conclusion of the proof of Theorem 1.1.12

We start by considering the first integral in (1.38),

$$\left| \phi(y) \int_{y_1}^{y} \tilde{\phi}(t) G(x,t) \, \mathrm{d}t \right| \leq \phi(y) \int_{y_1}^{y} \tilde{\phi}(t) |G(x,t)| \, \mathrm{d}t. \tag{1.43}$$

Using (1.42), it is enough to bound

$$\phi(y)\int_{y_1}^{y}\tilde{\phi}(t)\left(C_1L_1^{-5/2}L_2^{-1}\phi(t)+C_1(|t-y^*|+L_1)L_1^{-1/2}L_2^{-1}|\partial_tV(x,t)|\phi(t)+|V(x,t)-\mu(x)|\alpha\right)\,\mathrm{d}t. \tag{1.44}$$

We now bound the three terms in equation (1.44).

Lemma 1.3.26. *We have a bound on the first term in* (1.44)*,*

$$\phi(y)\int_{y_1}^{y}\tilde{\phi}(t)L_1^{-5/2}L_2^{-1}\phi(t)\,\mathrm{d}t\leq C_1L_1^{-1/2}L_2^{-1}.$$

Remark 1.3.27. *We will see in the proof of the lemma that the function decays exponentially from its maximum away from the region where $V(x,y)\leq 1+L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 1.3.23.*

Proof of Lemma 1.3.26: By Lemma 1.3.21, we can bound the left hand side by

$$\phi(y)\int_{y_1}^{y_3}C_1L_1L_1^{-5/2}L_2^{-1}\phi(t)\,\mathrm{d}t+\phi(y)\int_{y_3}^{y}(C_1L_1+C_1|\phi'(t)|^{-1})L_1^{-5/2}L_2^{-1}\phi(t)\,\mathrm{d}t.$$

Using Proposition 1.3.15 we have the bound, $\phi(t)\leq 2^{k/3}|\phi'(t)|\leq C_1L_1|\phi'(t)|$ for $t\in J_k$, and so these integrals can be bounded by

$$C_1\phi(y)\int_{y_1}^{y}L_2^{-1}L_1^{-3/2}\,\mathrm{d}t. \tag{1.45}$$

The eigenfunction $\phi(y)$ has a maximum of 1, and decays exponentially away from this maximum on a length scale comparable to L_1. Thus, we can bound (1.45) by $C_1L_1^{-1/2}L_2^{-1}$ as required, and it also has the decay properties of the function $F_1(y)$. □

Lemma 1.3.28. *We have a bound on the second term in* (1.44)*,*

$$\phi(y)\int_{y_1}^{y}\tilde{\phi}(t)(|t-y^*|+L_1)L_1^{-1/2}L_2^{-1}|\partial_tV(x,t)|\phi(t)\,\mathrm{d}t\leq C_1L_1^{-1/2}L_2^{-1}.$$

Remark 1.3.29. *We will again see in the proof of the lemma that the function decays exponentially from its maximum on a length scale comparable to $2^{k/3}$ within each interval J_k. Therefore we can include this term in the function $F_2(y)$ in the statement of Proposition 1.3.23.*

Proof of Lemma 1.3.28: We first consider the part of this integral over $[y_1, y_3]$. Here, $\tilde{\phi}(t) \leq C_1 L_1$, and by the convexity of the potential

$$\int_{y_1}^{y_3} |\partial_t V(x,t)| \, \mathrm{d}t \leq 2C_1 L_1^{-2}.$$

Therefore, we immediately obtain a bound of $C_1 L_1^{-1/2} L_2^{-1} \phi(y)$. This is certainly at most $C_1 L_1^{-1/2} L_2^{-1}$, and by the properties of $\phi(y)$ it also has the decay properties of the function $F_2(y)$.

We now consider the part of the integral over $[y_3, y]$. Let y be in the interval J_{k^*} for some k^*, where as usual the intervals J_k are as in (1.31). We decompose the integral between y_3 and y as an integral over the relevant intervals J_k where $k \geq k^*$.

By Proposition 1.3.15,

$$\phi(t) \leq |\phi'(t)| 2^{k/3},$$

and so using the bound on $\tilde{\phi}(t)$ from Lemma 1.3.21, to estimate the contribution to the integral from J_k, we have to bound

$$\begin{aligned}
\phi(y) \int_{J_k} \phi(t)(L_1 + |\phi'(t)|^{-1})(|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \, \mathrm{d}t \\
&\leq C_1 \phi(y) \int_{J_k} 2^{k/3} |\phi'(t)| (L_1 + |\phi'(t)|^{-1})(|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} 2^{-k} \, \mathrm{d}t \\
&\leq C_1 2^{-2k/3} \phi(y) \int_{J_k} (|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} \, \mathrm{d}t. \quad (1.46)
\end{aligned}$$

Using Proposition 1.3.15 again, we find that for any $k \geq k^*$,

$$\phi(y) \leq \phi(t_{k^*}) \exp(-(y - t_{k^*}) 2^{-k^*/3}/10) \leq 2^{k^*/3} |\phi'(t_{k^*})| \exp(-(y - t_{k^*}) 2^{-k^*/3}/10).$$

By Corollary 1.3.20, we can bound the factor of $|\phi'(t_{k^*})|$ as

$$|\phi'(t_{k^*})| \leq C L_1^{-1} \exp(-c|t - t_k|/2^{k/3}) \exp(-c|t_k - y^*|/L_1).$$

Inserting these estimates into the integral in (1.46) and integrating over the interval J_k, we have the bound

$$C \exp(-(y - t_{k^*}) 2^{-k^*/3}/10) 2^{k^*/3} 2^{-k/3} L_1^{-1/2} L_2^{-1}.$$

Summing over $k \geq k^*$ gives a bound for the integral over $y_3 \leq t \leq y$ of the form

$$C_1 L_1^{-1/2} L_2^{-1} \exp(-(y - t_{k^*}) 2^{-k^*/3}/10).$$

Note that this quantity is bounded by a multiple of $L_1^{-1/2} L_2^{-1}$, and has the required decay properties that we can include it in the function $F_2(y)$.

□

Lemma 1.3.30. *We have a bound on the final term in* (1.44)*,*

$$\phi(y) \int_{y_1}^{y} \tilde{\phi}(t) |V(x,t) - \mu(x)||\alpha| \, \mathrm{d}t \leq C_1 L_1^{-1/2} L_2^{-1}. \tag{1.47}$$

Remark 1.3.31. *We will see that the function decays exponentially from its maximum away from the region where $V(x,y) \leq 1 + L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 1.3.23.*

Proof of Lemma 1.3.30: We recall from Proposition 1.3.22 that we have an estimate on the boundary value of $\partial_x \psi_1^{(x)}(y)$ of the form

$$|\alpha| = |\partial_x \psi_1^{(x)}(g_2(x))| \leq C L_2^{-1} L_1^{-3/2} (L_1 + M) \exp(-cML_1^{-1}). \tag{1.48}$$

Here M is the distance from $g_2(x)$ to the point y^* where $V(x, y^*) = 1 + C^* L_1^{-2}$.

For the part of the integral in (1.47) over $[y_1, y_3]$, we know that $|V(x,t) - \mu(x)| \leq C_1 L_1^{-2}$, $|y_3 - y_1| \leq C_1 L_1$ and $\tilde{\phi}(t) \leq C_1 L_1$. Combining this with the bound on α from (1.48), immediately gives us the desired bound of $L_2^{-1} L_1^{-1/2} \exp(-cML_1^{-1})$ for this part of the integral in (1.47).

For $t \geq y_3$, we decompose $[y_3, y]$ into the intervals $\tilde{J}_k$ as in (1.32):

$$\tilde{J}_k = [\tilde{t}_k, \tilde{t}_{k+1}] = \{t \geq y_3 : V(x,t) - \min_t V(x,t) \in [2^{-2k/3}, 2^{-2(k-1)/3}]\}.$$

Since $\mu(x) \geq \min_t V(x,t)$, on $\tilde{J}_k$ we know that

$$|V(x,t) - \mu(x)| \leq 2^{-2k/3}.$$

So, for the part of the integral in (1.47) over $\tilde{J}_k$, combining this with the bound on α in (1.48) and

the usual bound on $\tilde{\phi}(t)$ from Lemma 1.3.21, we have

$$\begin{aligned}\phi(y) &\int_{\tilde{J}_k} \tilde{\phi}(t)|V(x,t)-\mu(x)||\alpha|\,\mathrm{d}t \\ &\leq C_1 L_2^{-1} L_1^{-3/2}\phi(y)\int_{\tilde{J}_k}(L_1+|\phi'(t)|^{-1})2^{-2k/3}(L_1+M)\exp(-cML_1^{-1})\,\mathrm{d}t. \end{aligned} \tag{1.49}$$

Similarly to the proof of Lemma 1.3.28, let us assume that $y \in \tilde{J}_{k^*}$ for some k^*. Then, using Proposition 1.3.15 and then Proposition 1.3.18 twice, we obtain

$$\begin{aligned}\phi(y) \leq C_1 2^{k^*/3}|\phi'(y)| &\leq C_1 2^{k^*/3}|\phi'(t_{k^*})|\exp\left(-c|y-t_{k^*}|2^{-k^*/3}\right) \\ &\leq C_1 2^{k^*/3}|\phi'(t)|\exp\left(-c|y-t_{k^*}|2^{-k^*/3}\right)\exp\left(-c|t_{k^*}-t|2^{-k/3}\right)\end{aligned}$$

Inserting this bound for $\phi(y)$ into the right hand side of (1.49) and integrating over $\tilde{J}_k$ gives us the bound for the part of the integral over $\tilde{J}_k$ of

$$C_1 2^{k^*/3}2^{-k/3}L_2^{-1}L_1^{-1/2}\exp(-cML_1^{-1}/2).$$

We finally sum over those k with $k \geq k^*$ to get the desired bound on the part of the integral (1.47) with $y_3 \leq t \leq y$. □

Combining Lemmas 1.3.26, 1.3.28 and 1.3.30, we see that the part of $g(y) - c_0(x)\psi_1^{(x)}(y)$ coming from

$$\phi(y)\int_{y_1}^{y}\tilde{\phi}(t)G(x,t)\,\mathrm{d}t$$

has the bounds required in Proposition 1.3.23.

Therefore to finish the proof of Proposition 1.3.23 we need to establish the analogous estimates for the other part of $g(y) - c_0(x)\psi_1^{(x)}(y)$ in (1.38),

$$\tilde{\phi}(y)\int_{y}^{g_2(x)}\phi(t)G(x,t)\,\mathrm{d}t.$$

The estimates for the various parts of this integral will be similar to the estimates we used above. However, there will be places where we have to use different methods to obtain the desired bounds.

We want to bound

$$\left|\tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)G(x,t)\,\mathrm{d}t\right| \leq \tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)|G(x,t)|\,\mathrm{d}t. \tag{1.50}$$

We again use (1.42) to bound this by

$$\tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)\left(C_1L_1^{-5/2}L_2^{-1}\phi(t)+C_1(|t-y^*|+L_1)L_1^{-1/2}L_2^{-1}|\partial_tV(x,t)|\phi(t)+|V(x,t)-\mu(x)|\alpha\right)\,\mathrm{d}t, \tag{1.51}$$

and we split this into three terms that we need to estimate.

Lemma 1.3.32. *We have a bound on the first term in* (1.51)

$$\tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)L_1^{-5/2}L_2^{-1}\phi(t)\,\mathrm{d}t \leq C_1L_1^{-1/2}L_2^{-1}$$

Remark 1.3.33. *The function also decays exponentially from its maximum away from the region where $V(x,y)\leq 1+L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 1.3.23.*

Proof of Lemma 1.3.32: We know that $\tilde{\phi}(y)\leq\tilde{\phi}(t)$, and $\phi(t)$ decays exponentially on a length scale comparable to L_1 as we move away from y^*. Therefore, this bound follows in a very straightforward manner. □

Before bounding the second term in (1.51), we first want to establish the following lemma.

Lemma 1.3.34. *For any $\tilde{y}\geq y_3$, we have the bound*

$$\int_{\tilde{y}}^{g_2(x)}\phi(t)^2\partial_tV(x,t)\,\mathrm{d}t \leq (\phi'(\tilde{y}))^{1/2}.$$

Proof of Lemma 1.3.34: To prove this lemma, we will consider the 'energy'

$$\mathcal{E}(t) \coloneqq (\phi'(t))^2 - F(x,t)\phi(t)^2. \tag{1.52}$$

Differentiating $\mathcal{E}(t)$ we find that

$$\mathcal{E}'(t) = 2\phi'(t)(\phi''(t)-F(x,t)\phi(t)) - \partial_tF(x,t)\phi(t)^2 = -\partial_tF(x,t)\phi(t)^2,$$

where the final equality holds because $\phi''(t) = F(x,t)\phi(t)$. Since $F(x,t) = V(x,t) - \mu(x)$, we have

$$\partial_t F(x,t) = \partial_t V(x,t),$$

and so

$$\int_{\tilde{y}}^{g_2(x)} \partial_t V(x,t)\phi(t)^2 \,\mathrm{d}t = -\int_{\tilde{y}}^{g_2(x)} \mathcal{E}'(t)\,\mathrm{d}t = \mathcal{E}(\tilde{y}) - \mathcal{E}(g_2(x)).$$

Since $F(x,t) \geq 0$ for $t \geq y_3$, we know that

$$\mathcal{E}(\tilde{y}) = (\phi'(\tilde{y}))^2 - F(x,\tilde{y})\phi(\tilde{y})^2 \leq (\phi'(\tilde{y}))^2.$$

Thus, to finish the proof of the lemma we need to show that $\mathcal{E}(g_2(x)) \geq 0$. We know that

$$\mathcal{E}(g_2(x)) \geq -F(x,g_2(x))\phi(g_2(x))^2,$$

and that $\phi(g_2(x)) = 0$. However, we are not assuming that the potential $V(x,y)$ remains bounded as y approaches $g_2(x)$, and so we cannot immediately deduce that $F(x,g_2(x))\phi(g_2(x))^2 = 0$. Instead we argue as follows. The function $\phi'(t)$ is in $L^\infty(\Omega(x))$, and this has two consequences. First, the eigenfunction $\phi(y)$ decays at least linearly to 0 at $y = g_2(x)$. It also means that $\phi''(y)$ is in $L^1(\Omega(x))$, and hence $F(x,y)\phi(y)$ is in $L^1(\Omega(x))$. This means that $F(x,y)\phi(y)$ cannot grow as fast as $(y - g_2(x))^{-1}$ as we approach the boundary and so

$$\liminf_{y \to g_2(x)} \phi(y)F(x,y)\phi(y) = 0.$$

This implies that $\mathcal{E}(g_2(x)) \geq 0$ and concludes the proof of the lemma. □

Remark 1.3.35. *This energy $\mathcal{E}(t)$ has also been used in [GJ2] in their proof of Theorem 2.1 (B). There they obtain a pointwise estimate comparing the first eigenfunction of the two dimensional domain with the first eigenfunction of the associated ordinary differential operator.*

We can now bound the contribution from the second term in $G(x,t)$.

Lemma 1.3.36. *We have a bound on the second term in* (1.51)*,*

$$\tilde{\phi}(y) \int_y^{g_2(x)} \phi(t)^2(|t - y^*| + L_1)L_1^{-1/2}L_2^{-1}|\partial_t V(x,t)|\,\mathrm{d}t \leq C_1 L_1^{-1/2}L_2^{-1}.$$

Remark 1.3.37. *We will see in the proof that the function also decays exponentially from its maximum away from the region where $V(x,y) \leq 1 + L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 1.3.23.*

Proof of Lemma 1.3.36: If $y \leq y_3$, we first consider the part of the integral where t lies in the interval $[y, y_3]$ of length at most $C_1 L_1$. In this case, we know that $\tilde{\phi}(y) \leq C_1 L_1$ and the estimates follow easily.

For $t \geq y_3$, we first consider the integral between $\tilde{y}$ and $\tilde{y} + L_1$, where $\tilde{y}$ is some point with $\tilde{y} \geq y_3$ and $\tilde{y} \geq y$. Since $\partial_t V(x,t) \geq 0$ here, we have

$$\tilde{\phi}(y) \int_{\tilde{y}}^{\tilde{y}+L_1} \phi(t)^2 (|t - y^*| + L_1) L_2^{-1} |\partial_t V(x,t)| L_1^{-1/2} \, \mathrm{d}t$$
$$\leq C_1 \tilde{\phi}(y) (|\tilde{y} - y^*| + L_1) L_2^{-1} L_1^{-1/2} \int_{\tilde{y}}^{g_2(x)} \phi(t)^2 \partial_t V(x,t) \, \mathrm{d}t. \tag{1.53}$$

Applying Lemma 1.3.34, we can bound the right hand side of (1.53) by

$$C_1 \tilde{\phi}(y) (|\tilde{y} - y^*| + L_1) L_2^{-1} L_1^{-1/2} (\phi'(\tilde{y}))^2.$$

Lemma 1.3.21 shows that

$$\tilde{\phi}(\tilde{y}) |\phi'(\tilde{y})| \leq C_1,$$

and using Proposition 1.3.18 with 2^k comparable to L_1^3, we have the derivative bound

$$|\phi'(\tilde{y})| \leq C_1 L_1^{-1} \exp\left(-c|\tilde{y} - y^*|/L_1\right).$$

Thus the right hand side of (1.53) has the bound

$$C_1 L_2^{-1} L_1^{-1/2} \exp\left(-c|\tilde{y} - y^*|/L_1\right).$$

Summing over $\tilde{y}$ between y and $g_2(x)$ at intervals of length comparable to L_1 then gives the desired bound.

□

We finally have to bound the contribution from the third term in $G(x,t)$.

Lemma 1.3.38. *We have a bound on the third term in* (1.51)

$$\tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)|V(x,t)-\mu(x)||\alpha|\,\mathrm{d}t \le C_1 L_1^{-1/2}L_2^{-1}. \tag{1.54}$$

Remark 1.3.39. *As for the previous two lemmas, we will see in the proof that the function also decays exponentially from its maximum away from the region where $V(x,y) \le 1 + L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 1.3.23.*

Proof of Lemma 1.3.38: From Proposition 1.3.22 we have the estimate on α of the form

$$|\alpha| = |\partial_x \psi_1^{(x)}(g_2(x))| \le CL_2^{-1}L_1^{-3/2}(L_1+M)\exp(-cML_1^{-1}). \tag{1.55}$$

For the part of the integral in (1.44) for t between y and y_3, we know that $|V(x,t)-\mu(x)|$ is at most $C_1L_1^{-2}$, and $\tilde{\phi}(y) \le C_1L_1$. Thus, we immediately get a bound of

$$C_1 L_1^{-1/2}L_2^{-1}\exp(-cML_1^{-1})$$

for this part.

For $t \ge y_3$, we know that $F(x,t) = V(x,t)-\mu(x) \ge 0$. Thus, we can bound this part of the integral in (1.44) by

$$C_1\tilde{\phi}(y)L_2^{-1}L_1^{-3/2}(L_1+M)\exp(-cML_1^{-1})\int_{\max\{y_3,y\}}^{g_2(x)}\phi(t)(V(x,t)-\mu(x))\,\mathrm{d}t. \tag{1.56}$$

Since

$$\phi''(t) = (V(x,t)-\mu(x))\phi(t),$$

and $|\phi'(t)|$ is decreasing for $t \ge y_3$, we find that (1.56) can be bounded by

$$C_1\tilde{\phi}(y)L_2^{-1}L_1^{-3/2}(L_1+M)\exp(-cML_1^{-1})|\phi'(y)| \le C_1L_1^{-1/2}L_2^{-1}\exp(-cML_1^{-1}/2),$$

where the last inequality comes from Lemma 1.3.21 as usual. This concludes the proof of the lemma.

□

We recall that

$$g(y) - c_0(x)\psi_1^{(x)}(y) = \phi(y)\int_{y_1}^{y} \tilde{\phi}(t)G(x,t)\,\mathrm{d}t + \tilde{\phi}(y)\int_{y}^{g_2(x)} \phi(t)G(x,t)\,\mathrm{d}t.$$

Then by the bounds on the right hand side in Lemmas 1.3.26, 1.3.28, 1.3.30 and Lemmas 1.3.32, 1.3.36, 1.3.38, we have shown that

$$\left|g(y) - c_0(x)\psi_1^{(x)}(y)\right| \leq F_1(y) + F_2(y). \tag{1.57}$$

Here the functions $F_1(y)$ and $F_2(y)$ have the desired properties from the statement of Proposition 1.3.23. As we remarked at the beginning of the proof, by the bound on α that we obtained in Proposition 1.3.22, the estimate in (1.57) is sufficient to conclude the proof of Proposition 1.3.23.

□

After the statement of Proposition 1.3.23, we showed that this implied Proposition 1.3.8 and the bound

$$\int_{\Omega(x)} \left(\partial_x \psi_1^{(x)}(y)\right)^2 \,\mathrm{d}y \leq C_1 L_2^{-2}.$$

Combining this with the estimate on the first eigenvalue λ from Proposition 1.3.7

$$\lambda \leq \mu + \int_{\Omega} \chi(x)^2 \left(\partial_x \psi_1^{(x)}(y)\right)^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_2^{-2}$$

gives

$$\lambda \leq \mu + CL_2^{-2}.$$

This completes the proof of the upper bound on λ in Proposition 1.3.2. □

By Propositions 1.3.1 and 1.3.2, we see that the first eigenvalue λ satisfies

$$\mu \leq \lambda \leq \mu + CL_2^{-2},$$

and so we have established Theorem 1.1.12.

1.4 $L^2(\Omega)$ bounds for the first eigenfunction $u(x, y)$

Now that we have established the improved eigenvalue bound on λ in Theorem 1.1.12, we want to use it to study the corresponding eigenfunction $u(x, y)$. We recall that $u(x, y)$ satisfies

$$\begin{cases} (-\Delta_{x,y} + V(x,y))u(x,y) &= \lambda u(x,y) \qquad \text{in } \Omega \\ u(x,y) &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

and is normalised to be positive inside Ω with a maximum of 1. Our main aim is to prove Theorem 1.1.9 and show that the level sets $\{(x, y) \in \Omega : u(x, y) = c\}$ have lengths comparable to L_2 and L_1 in the x and y-directions respectively, whenever c is bounded away from 0 and 1.

1.4.1 A global $L^2(\Omega)$-bound for $u(x, y)$

Before we prove this theorem, in this section we will first establish an $L^2(\Omega)$-bound for $u(x, y)$. More precisely, we will prove the following proposition.

Proposition 1.4.1. *There exists an absolute constant $C > 0$ such that*

$$\int_\Omega u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C L_1 L_2.$$

Remark 1.4.2. *Note that this $L^2(\Omega)$ bound is consistent with the shape of the level sets described in Theorem 1.1.9. We will use the eigenvalue bound on λ from Theorem 1.1.12 in a critical way in the proof.*

Proof of Proposition 1.4.1: Before beginning the proof of this proposition, we make the following definition.

Definition 1.4.3. *We define the function $H(x)$ by*

$$H(x) \coloneqq \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y.$$

That is, $H(x)$ is equal to the square of the $L^2(\Omega(x))$-norm of the cross-section of the eigenfunction $u(x, \cdot)$.

To prove Proposition 1.4.1 we will first study the rate at which the function $H(x)$ decays from its maximum, and we will then prove an estimate for the maximum of $H(x)$.

To study the decay of $H(x)$, we prove a Carleman-type inequality. For the convex function $\mu(x)$ let x^* be a point where it achieves its minimum of μ^*. We now prove:

Proposition 1.4.4. *For any x we have the differential inequality,*

$$H''(x) \geq 2(\mu(x) - \lambda)H(x).$$

In particular, for $|x - x^*| \geq CL_2$, with C a sufficiently large absolute constant, we have

$$H''(x) \geq \frac{1}{L_2^2} H(x).$$

Remark 1.4.5. *This type of Carleman inequality has been used frequently in the study of the ground state Dirichlet eigenfunction of Schrödinger operators. For example, in Lemma 3.9 [GJ2] it has been used to establish the exponential decay of the first Fourier mode of the ground state eigenfunction of the two dimensional convex domain. This first Fourier mode comes from a Fourier decomposition of the cross-section of the domain at each fixed x. A similar argument has also been used in Section 3 of [FS1] to study the decay of the L^2-norm of the cross-section at x of the eigenfunction for a two dimensional domain which is periodic in the x-direction and with height in the y-direction depending on a small parameter $\epsilon > 0$.*

Proof of Proposition 1.4.4: The eigenfunction $u(x, y)$ is equal to 0 when y is at the endpoints of the interval $\Omega(x)$. This allows us to differentiate $H(x)$ twice and pass the derivative inside the integral to obtain

$$\begin{aligned} H''(x) &= 2 \int_{\Omega(x)} u(x,y)\partial_x^2 u(x,y) + (\partial_x u(x,y))^2 \, \mathrm{d}y \\ &= 2 \int_{\Omega(x)} (V(x,y) - \lambda)u(x,y)^2 - u(x,y)\partial_y^2 u(x,y) + (\partial_x u(x,y))^2 \, \mathrm{d}y. \end{aligned}$$

Integrating by parts one time in y in the term containing a factor of $\partial_y^2 u(x, y)$, we can rewrite this as

$$\begin{aligned} H''(x) &= 2 \int_{\Omega(x)} (V(x,y) - \lambda)u(x,y)^2 + (\partial_y u(x,y))^2 + (\partial_x u(x,y))^2 \, \mathrm{d}y \\ &\geq 2 \int_{\Omega(x)} (V(x,y) - \lambda)u(x,y)^2 + (\partial_y u(x,y))^2 \, \mathrm{d}y \end{aligned} \tag{1.58}$$

Since $\mu(x)$ is the first eigenvalue of the operator

$$\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x, y),$$

and $u(x, \cdot)$ vanishes at the endpoints of $\Omega(x)$, (1.58) gives us the lower bound

$$H''(x) \geq 2(\mu(x) - \lambda) \int_{\Omega(x)} u(x, y)^2 \, \mathrm{d}y = 2(\mu(x) - \lambda)H(x). \tag{1.59}$$

Since $\mu(x^*) = \mu^*$ is the minimum value of the function $\mu(x)$, by the definition of the length scale L_2, we know that

$$|\mu(x^*) - \mu| \leq C_1 L_2^{-2}.$$

Thus, applying Theorem 1.1.12, we have the bound

$$|\lambda - \mu(x^*)| \leq C_1 L_2^{-2}. \tag{1.60}$$

The function $\mu(x)$ increases from its minimum by L_2^{-2} as x varies in an interval of length comparable to L_2 from x^*. Moreover, $\mu(x)$ is a convex function. Therefore, provided we choose $C > 0$ sufficiently large, we have

$$\mu(x) - \mu(x^*) \geq (C_1 + 1)L_2^{-2} \tag{1.61}$$

whenever x satisfies $|x - x^*| \geq CL_2$. Combining the inequalities in (1.60) and (1.61) shows that

$$\mu(x) - \lambda \geq L_2^{-2},$$

and using this bound in (1.59) gives

$$H''(x) \geq 2L_2^{-2}H(x)$$

as required. □

Before giving a corollary of this proposition, we recall the generalised maximum principle.

Proposition 1.4.6. *Suppose that the functions v_1 and v_2 satisfy*

$$\Delta v_1 + c(x)v_1 = 0, \qquad \Delta v_2 + c(x)v_2 \leq 0,$$

in a bounded domain D, where $c(x)$ is a continuous function. If in addition v_1 and v_2 are continuous in $\bar{D}$, $v_1 > 0$ in D and $v_2 > 0$ in $\bar{D}$, then

$$\max_{\bar{D}} v_1/v_2 \leq \max_{\partial D} v_1/v_2.$$

This is proven in [PW], Theorem 10, page 73, and follows from applying the usual maximum principle to the function v_1/v_2. We now prove a corollary of Proposition 1.4.4.

Corollary 1.4.7. *Let $A \coloneqq \max_x H(x)$. Then, the function $H(x)$ satisfies the upper bound*

$$H(x) \leq C_1 A \exp\left(-c|x - x^*|/L_2\right).$$

Proof of Corollary 1.4.7: With $C > 0$ as in the statement of Proposition 1.4.4, let $x_1 = x^* + CL_2$. We also define the function $R(x)$ for $x > x_1$ by

$$R(x) \coloneqq Ae^{-(x-x_1)/L_2}.$$

Then, $R(x)$ satisfies $R''(x) = L_2^{-2}R(x)$, and $H(x_1) \leq A = R(x_1)$. By Proposition 1.4.4 we know that

$$H''(x) \geq L_2^{-2}H(x)$$

for all $x \geq x_1$. Therefore, setting D to be the interval $\{x \geq x_1\}$, the conditions of the generalised maximum principle are satisfied and hence

$$H(x) \leq R(x)$$

for all $x \geq x_1$. There is also an analogous bound for $x \leq x^* - CL_2$, and this completes the proof. □

Remark 1.4.8. *In fact, we see from the proof that we can replace A by $H(x_1)$ and conclude that*

for any $x_1 \geq x^* + CL_2$ *we have the bound*

$$H(x) \leq H(x_1)e^{-(x-x_1)/L_2} \tag{1.62}$$

for all $x > x_1$.

In particular, as a result of this corollary, we see that $H(x)$ decays exponentially from its value at $x = x^*$ at least at a length scale comparable to L_2.

Our next aim is to obtain an upper bound for

$$A = \max_x H(x) = \max_x \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y. \tag{1.63}$$

Suppose that we can show that A satisfies

$$A \leq C_1 L_1$$

for an absolute constant C_1. Then, by Proposition 1.4.4 we have

$$H(x) \leq C_1 L_1 e^{-c|x-x^*|/L_2},$$

and so integrating over x gives

$$\int_\Omega u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y = \int H(x) \, \mathrm{d}x \leq CL_1L_2.$$

Therefore to complete the proof of Proposition 1.4.1, it is sufficient to prove this upper bound on A. To do this we first define a cut-off function $\chi_1(x)$ as follows.

Definition 1.4.9. *We define* $\chi_1(x)$ *to be a smooth cut-off function, which satisfies*

$$0 \leq \chi_1(x) \leq 1,$$

and is equal to 1 *on the interval* $[x^* - 2CL_2, x^* + 2CL_2]$ *of length* $4CL_2$*, with* C *as in the statement of Proposition 1.4.4. Moreover, the function* $\chi_1(x)$ *is supported on the interval* $[x^* - 3CL_2, x^* + 3CL_2]$ *and has the derivative estimate*

$$\left|\partial^k \chi_1(x)\right| \leq (CL_2)^{-k},$$

for $k = 1, 2$.

We now prove the following.

Proposition 1.4.10. *Let* $\chi_1(x)$ *be the cut-off function above in Definition 1.4.9. Then,*

$$\int_\Omega \chi_1(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 L_1 L_2,$$

for an absolute constant $C_1 > 0$. *Note that this is consistent with* $u(x,y)$ *decaying on a length scale comparable to* L_1 *in the* y*-direction.*

Proof of Proposition 1.4.10: The first eigenfunction $u(x,y)$ satisfies

$$-\Delta_{x,y} u(x,y) + (V(x,y) - \lambda) u(x,y) = 0 \text{ in } \Omega$$

with zero boundary conditions on $\partial\Omega$. We integrate this against the function $\chi_1(x)u(x,y)$ to obtain

$$\int_\Omega -\chi_1(x) u(x,y) \Delta_{x,y} u(x,y) + \chi_1(x)(V(x,y) - \lambda) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y = 0,$$

and integrating by parts one time in x and y gives

$$\begin{aligned} \int_\Omega \chi_1(x) |\nabla_{x,y} u(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \chi_1'(x) \partial_x u(x,y) u(x,y) \, \mathrm{d}x \, \mathrm{d}y \\ + \int_\Omega \chi_1(x)(V(x,y) - \lambda) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y = 0. \end{aligned} \tag{1.64}$$

In the second integral in (1.64) we can write

$$\chi_1'(x) \partial_x u(x,y) u(x,y) = \tfrac{1}{2} \chi_1'(x) \partial_x (u(x,y)^2)$$

and integrate by parts in x again to rewrite this integral as

$$-\frac{1}{2} \int_\Omega \chi_1''(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y.$$

Thus, from (1.64) we have

$$\begin{aligned} &\int_\Omega \chi_1(x) |\nabla_{x,y} u(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y) - \lambda)_+ u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{2} \int_\Omega \chi_1''(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y) - \lambda)_- u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y, \end{aligned} \tag{1.65}$$

where we have decomposed $V(x,y)-\lambda$ into its positive and negative parts via

$$V(x,y)-\lambda = (V(x,y)-\lambda)_+ - (V(x,y)-\lambda)_-.$$

By the simple eigenvalue bound for λ from Proposition 1.2.7, we know that

$$(V(x,y)-\lambda)_- \leq C_1 L_1^{-2}.$$

This also means that for any fixed x, we can only have $V(x,y)-\lambda \leq 0$ for y in an interval of length at most comparable to L_1. Since the eigenfunction is normalised to have a maximum of 1, and $\chi_1(x)$ is only non-zero in an interval of length comparable to L_2, this gives us a bound on the final term in the right hand side of (1.65) of

$$\int_\Omega \chi_1(x)(V(x,y)-\lambda)_- u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq C_1 L_1^{-2} L_1 L_2 = C_1 L_1^{-1} L_2. \tag{1.66}$$

We now turn to the second integral on the left hand side of (1.65)

$$\int_\Omega \chi_1(x)(V(x,y)-\lambda)_+ u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Fix a large constant $C_2 > 0$. For each fixed x, $V(x,y)-\lambda$ is only bounded above by $C_2 L_1^{-2}$ on an interval in y of length comparable to L_1. Therefore, again combining this with the bound $u(x,y) \leq 1$, we can write

$$C_2 L_1^{-2} \int_\Omega \chi_1(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y - C_1 L_1^{-1} L_2 \leq \int_\Omega \chi_1(x)(V(x,y)-\lambda)_+ u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{1.67}$$

Inserting the estimates in (1.66) and (1.67) back into (1.65) we see that

$$\begin{aligned} &\int_\Omega \chi_1(x)|\nabla_{x,y} u(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y + C_2 L_1^{-2} \int_\Omega \chi_1(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{2}\int_\Omega \chi_1''(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_1^{-1} L_2. \end{aligned} \tag{1.68}$$

The first integral in (1.68) is positive, and so we can drop it from the estimate. Therefore, dividing by $C_2 L_1^{-2}$ gives us

$$\int_\Omega \chi_1(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq \frac{1}{2} C_2^{-1} L_1^2 \int_\Omega \chi_1''(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_1 L_2. \tag{1.69}$$

To conclude the proof of the proposition, we will use Corollary 1.4.7 and the remark following it. By (1.62), for any $x_1 \geq x^* + CL_2$ and any $x \geq x_1$, we have

$$\int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y \leq e^{-(x-x_1)/L_2} \int_{\Omega(x_1)} u(x_1,y)^2 \, \mathrm{d}y.$$

Therefore, we certainly have the estimate

$$\int_{x^*+2CL_2}^{x^*+3CL_2} \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq \int_{x^*+CL_2}^{x^*+2CL_2} \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y, \tag{1.70}$$

and an analogous estimate for $x_1 \leq x^* - CL_2$ and $x \leq x_1$. By the definition of the cut-off function $\chi_1(x)$, the second derivative $\chi_1''(x)$ is supported on the intervals $[x^* - 3CL_2, x^* - 2CL_2]$ and $[x^* + 2CL_2, x^* + 3CL_2]$, and is of order L_2^{-2} here. Also, $\chi_1(x)$ is equal to 1 on the intervals $[x^* - 2CL_2, x^* - CL_2]$ and $[x^* + CL_2, x^* + 2CL_2]$. Therefore, using the estimate in (1.70) the integral on the right hand side of (1.69) is certainly at most $\frac{1}{2}$ the size of the integral on the left hand side. This means that in (1.69) we can bring over the integral to the left hand side and get the bound

$$\int_{\Omega} \chi_1(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 L_1 L_2,$$

as required. □

Corollary 1.4.11. *We have the derivative bound*

$$\int_{\Omega} \chi_1(x) |\nabla_{x,y} u(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 L_1^{-1} L_2.$$

Proof of Corollary 1.4.11: In the proof of Proposition 1.4.10 in (1.68) we established the estimate

$$\begin{aligned} &\int_{\Omega} \chi_1(x) |\nabla_{x,y} u(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y + C_2 L_1^{-2} \int_{\Omega} \chi_1(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\leq \frac{1}{2} \int_{\Omega} \chi_1''(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y + C_1 L_1^{-1} L_2. \end{aligned} \tag{1.71}$$

We also showed that

$$L_1^2 \int_{\Omega} \chi_1''(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y$$

is bounded by $\int_{\Omega} \chi_1(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y$, and hence by Proposition 1.4.10 is bounded by $C_1 L_1^{-1} L_2$. Using this estimate in (1.71) gives the desired result. □

The derivative bound

$$\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \leq C_1 L_1^{-1}L_2$$

is of order L_1^{-2} smaller than the bound we obtained for the eigenfunction $u(x,y)$ itself in Proposition 1.4.10. For the y-derivative $\partial_y u(x,y)$, this bound is consistent with our eventual aim to show that $u(x,y)$ decays away from its maximum on a length scale comparable to L_1. However, in the x-direction, our aim is to show that $u(x,y)$ decays away from its maximum on a length scale comparable to L_2. Therefore, we want to improve the bound on $\partial_x u(x,y)$ given in Corollary 1.4.11.

Proposition 1.4.12. *Let $\chi_1(x)$ be as in Definition 1.4.9. Then, there exists an absolute constant $C_1 > 0$ such that*

$$\int_\Omega \chi_1(x)(\partial_x u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y \leq C L_1 L_2^{-1}.$$

Note that for $L_2 \gg L_1$ this is an improvement on the bound in Corollary 1.4.11.

Proof of Proposition 1.4.12: We begin by proceeding as in the proof of Proposition 1.4.10 to obtain the equality in (1.65):

$$\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y)-\lambda)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{1.72}$$

We know that the integral of $\chi_1(x)u(x,y)^2$ is at most $C_1 L_1 L_2$. Since $|\chi_1''(x)| \leq C_1 L_2^{-2}$ this means that

$$\frac{1}{2}\int_\Omega |\chi_1''(x)|\,u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \leq C_1 L_1 L_2^{-1},$$

and so from (1.72) we have

$$\int_\Omega \chi_1(x)(\partial_x u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(\partial_y u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y)-\lambda)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \leq C_1 L_1 L_2^{-1}. \tag{1.73}$$

For each fixed x, the eigenfunction $u(x,y)$ is an admissible test function for our usual ordinary

differential operator

$$\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x, y).$$

Since this operator has first eigenvalue equal to $\mu(x)$, we obtain the lower bound

$$\int_{\Omega(x)} (\partial_y u(x, y))^2 + (V(x, y) - \lambda)u(x, y)^2 \, \mathrm{d}y \geq (\mu(x) - \lambda) \int_{\Omega(x)} u(x, y)^2 \, \mathrm{d}y. \tag{1.74}$$

Multiplying the inequality in (1.74) by $\chi_1(x)$ and integrating over x, (1.73) becomes

$$\int_{\Omega} \chi_1(x)(\partial_x u(x, y))^2 \, \mathrm{d}x \, \mathrm{d}y + \int_{\Omega} \chi_1(x)(\mu(x) - \lambda)u(x, y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 L_1 L_2^{-1}. \tag{1.75}$$

By the definition of L_2, we have

$$\mu(x) - \mu \geq -C_1 L_2^{-2},$$

and by the eigenvalue bounds in Theorem 1.1.12, we know that

$$\mu - \lambda \geq -C_1 L_2^{-2}.$$

Therefore, (1.75) tells us that

$$\int_{\Omega} \chi_1(x)(\partial_x u(x, y))^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 L_2^{-2} \int_{\Omega} \chi_1(x) u(x, y)^2 \, \mathrm{d}x \, \mathrm{d}y + C_1 L_1 L_2^{-1}.$$

Applying Proposition 1.4.10 then gives the desired bound. □

Now that we have established L^2-bounds for the first derivative, $\nabla_{x,y} u(x, y)$, in Propositions 1.4.10 and 1.4.12, we can return to establishing the required upper bound for

$$A = \max_x H(x) = \max_x \int_{\Omega(x)} u(x, y)^2 \, \mathrm{d}y.$$

Proposition 1.4.13. *A is bounded by L_1 multiplied by an absolute constant.*

Proof of Proposition 1.4.13: Suppose that we have

$$\max_x H(x) = H(x^*) = \int_{\Omega(x^*)} u(x^*, y)^2 \, \mathrm{d}y \geq C^* L_1, \tag{1.76}$$

where $C^* > 0$ is a large absolute constant that we will specify later. Then, for any (x, y), extending $u(x, y)$ to be 0 outside of Ω, we can write

$$u(x, y) = u(x^*, y) + \int_{x^*}^{x} \partial_t u(t, y)\, \mathrm{d}t,$$

and so

$$u(x, y)^2 \geq \tfrac{1}{2} u(x^*, y)^2 - C_1 |x - x^*| \int_{x^*}^{x} (\partial_t u(t, y))^2\, \mathrm{d}t, \tag{1.77}$$

for a fixed constant C_1. Integrating the inequality in (1.77) over y we find that

$$H(x) \geq \tfrac{1}{2} H(x^*) - C_1 |x - x^*| \int_{x^*}^{x} \int_{\Omega(t)} (\partial_t u(t, y))^2\, \mathrm{d}y\, \mathrm{d}t,$$

and so by the assumption on $H(x^*)$ in (1.76), this gives

$$H(x) \geq \tfrac{1}{2} C^* L_1 - C_1 |x - x^*| \int_{x^*}^{x} \int_{\Omega(t)} (\partial_t u(t, y))^2\, \mathrm{d}y\, \mathrm{d}t. \tag{1.78}$$

Let us restrict to those values of x with $|x - x^*| \leq c_1 L_2$ for a small constant $c_1 > 0$. Then by the derivative bound on $\partial_t u(t, y)$ in Proposition 1.4.12, we can ensure that the second term in (1.78) is small compared to $\frac{1}{4} C^* L_1$. Moreover, this constant c_1 can be chosen to be independent of C^*. Therefore, this tells us that for all x in an interval of length $2c_1 L_2$, we have the lower bound

$$H(x) \geq \tfrac{1}{4} C^* L_1.$$

In particular, this shows that

$$\int_{\Omega} \chi_1(x) u(x, y)^2\, \mathrm{d}x\, \mathrm{d}y \geq \int_{x^* - c_1 L_2}^{x^* + c_1 L_2} H(x)\, \mathrm{d}x \geq \tfrac{1}{2} c_1 C^* L_1 L_2.$$

Since c_1 is independent of C^*, we can contradict the $L^2(\Omega)$-bound from Proposition 1.4.10 by choosing C^* sufficiently large. □

By the discussion after the proof of Corollary 1.4.7, this upper bound on A from Proposition 1.4.13 implies the $L^2(\Omega)$-bound

$$\int_{\Omega} u(x, y)^2\, \mathrm{d}x\, \mathrm{d}y \leq C L_1 L_2.$$

This completes the proof of Proposition 1.4.1. □

1.4.2 An Agmon-type decay estimate for $u(x, y)$

In Proposition 1.4.1 we derived an $L^2(\Omega)$-bound for the first eigenfunction $u(x, y)$. For our purposes of studying the shape of the level sets of $u(x, y)$ near to its maximum this will be sufficient. However, another interesting question is to study the rate at which $u(x, y)$ decays from its maximum. Therefore, before continuing with our study of the level sets, let us give some indication about the decay of $u(x, y)$ as we move away from its maximum.

We will do this by using an Agmon-type estimate, but first we need some definitions.

Definition 1.4.14. *Fix a large absolute constant $C > 0$, and let Ω_1 be the subset of Ω given by*

$$\Omega_1 \coloneqq \{(x, y) \in \Omega : V(x, y) \geq 1 + CL_1^{-2}\}.$$

Note that the boundary of Ω_1 consists of parts of the two convex curves coming from $\partial\Omega$ and the level set $\{(x, y) \in \Omega : V(x, y) = 1 + CL_1^{-2}\}$.

Definition 1.4.15. *With $\Omega_1 \subset \Omega$ as above, we also define the distance function*

$$h^* : \Omega_1 \to [0, \infty)$$

as follows. We first define the function $\nu^*(x, y)$ to be equal to $V(x, y) - \lambda$. For (x, y) in Ω_1 we then define $h^*(x, y)$ by

$$h^*(x, y) = \inf_{\gamma} \frac{1}{2} \int_0^1 \nu^*(\gamma(t))^{1/2} |\gamma'(t)| \, \mathrm{d}t,$$

where the infimum is taken over all paths $\gamma : [0, 1] \to \Omega_1$ between the inner boundary of Ω_1 and (x, y).

We are now in a position to state our Agmon-type estimate.

Proposition 1.4.16. *For Ω_1 and $h^*(x, y)$ defined as above, we have*

$$\int_{\Omega_1} u(x, y)^2 e^{2h^*(x,y)} \, \mathrm{d}x \, \mathrm{d}y \leq C_2 L_1 L_2,$$

for some absolute constant $C_2 > 0$.

Remark 1.4.17. *Since we certainly have the lower bound $V(x,y) - \lambda \geq C_1 L_1^{-2}$ on Ω_1, roughly speaking this proposition shows that, in an $L^2(\Omega)$-sense, the function $u(x,y)$ decays at least on a length scale comparable to L_1 as we move away from the region where $V(x,y) \leq 1 + CL_1^{-2}$. However, as $V(x,y) - \lambda$ grows, this rate of exponential decay also increases.*

Proof of Proposition 1.4.16: This proposition will follow from a classical Agmon estimate in [A]. Let us restate Theorem 1.5 from [A] (using slightly different notation).

Theorem 1.4.18 (Theorem 1.5 in [A])**.** *Let D be a bounded connected open set in $\mathbb{R}^2$. Let $q(x,y)$ be a real valued function on D, and suppose that $\nu(x,y)$ is a positive continuous function on D such that*

$$\int_D |\nabla_{x,y}\psi(x,y)|^2 + q(x,y)\psi(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \geq \int_D \nu(x,y)\psi(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \tag{1.79}$$

for all $\psi \in C_0^\infty(D)$.

Fix a point $(x_0, y_0) \in D$, and define the distance $\rho_\nu(x,y)$ by

$$\rho_\nu(x,y) \coloneqq \inf_\gamma \int_0^1 \nu(\gamma(t))^{1/2} |\gamma'(t)| \,\mathrm{d}t, \tag{1.80}$$

where the infimum is taken over all continuous paths $\gamma : [0,1] \to D$ in D between (x_0, y_0) and (x,y). We also define $\rho_\nu((x,y), \{\infty\})$ to be the distance from the point (x,y) to ∂D under the distance function $\rho_\nu(x,y)$, and define D_s by

$$D_s \coloneqq \{(x,y) \in D : \rho_\nu((x,y), \{\infty\}) > s\}.$$

Finally, suppose that

$$-\Delta_{x,y} W(x,y) + q(x,y)W(x,y) = 0$$

and that the function $g(x,y)$ satisfies

$$|\nabla_{x,y} g(x,y)|^2 < \nu(x,y) \tag{1.81}$$

in D. Then, we have the estimate

$$\int_{D_s} W(x,y)^2(\nu(x,y) - |\nabla_{x,y} g(x,y)|^2)e^{2g(x,y)}\,\mathrm{d}x\,\mathrm{d}y \\ \leq \frac{2(1+2s)}{s^2}\int_{D\backslash D_s} W(x,y)^2\nu(x,y)e^{2g(x,y)}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.82}$$

We will now apply this theorem with $W(x,y) = u(x,y)$ and $q(x,y) = V(x,y) - \lambda$. We will choose the set D as follows: We recall that Ω_1 consists of those points (x,y) with $V(x,y) - \lambda \geq 1 + CL_1^{-2}$. We then define D to be all points in $\mathbb{R}^2$ outside of the inner boundary of Ω_1.

Since $u(x,y) = 0$ on $\partial\Omega$, we can extend $u(x,y)$ to D by setting it to be 0 for $D\backslash\Omega_1$, and we extend the potential $V(x,y)$ to D arbitrarily.

We clearly have the estimate

$$\int_D |\nabla_{x,y}\psi(x,y)|^2 + (V(x,y) - \lambda)\psi(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \geq \int_D (V(x,y) - \lambda)\psi(x,y)^2\,\mathrm{d}x\,\mathrm{d}y$$

for all $\psi \in C_0^\infty(D)$. Also, $V(x,y) - \lambda \geq C_1 L_1^{-2}$ for $(x,y) \in D$. As a result of this, from (1.79) we see that we we can set $\nu^*(x,y)$ to be equal to the function described in the definition of $h^*(x,y)$ in Definition 1.4.15.

In Theorem 1.4.18 we are free to choose the value for s, and we will choose $s = 1$. Then, we see that

$$D\backslash D_1 = \{(x,y) \in D : \rho_\nu((x,y), \{\infty\}) \leq 1\}$$

consists of the region near the inner boundary of D with width comparable to at most L_1. This is because we have ensured that $\nu(x,y) \geq cL_1^{-2}$ when the point (x,y) is within a distance L_1 of the boundary of D.

We finally need to choose $g(x,y)$ to ensure that (1.81) holds, and so we need

$$|\nabla_{x,y} g(x,y)|^2 < \nu^*(x,y) = V(x,y) - \lambda.$$

We can achieve this by setting $g(x,y)$ to be equal to the function $h^*(x,y)$ as in Definition 1.4.15. This certainly satisfies the required derivative bound.

Thus, we can apply Theorem 1.4.18 to get

$$\int_{D_1} u(x,y)^2(\nu(x,y) - |\nabla_{x,y}h^*(x,y)|^2)e^{2h^*(x,y)}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq 6\int_{D\setminus D_1} u(x,y)^2\nu(x,y)e^{2h^*(x,y)}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.83}$$

On $D\setminus D_1$, we know that $\nu(x,y) \leq L_1^{-2}$, and $h^*(x,y) \leq 1$. Therefore, by the L^2 bound on $u(x,y)$ from Proposition 1.4.1, the right hand side of (1.83) is bounded by

$$C_1L_1^{-2}L_1L_2 = C_1L_1^{-1}L_2.$$

Since for $(x,y) \in D$, we have $\nu(x,y) - |\nabla_{x,y}h^*(x,y)|^2 \geq c_1L_1^{-2}$, we can therefore conclude from (1.83) that

$$\int_{\Omega_2} u(x,y)^2e^{2h^*(x,y)}\,\mathrm{d}x\,\mathrm{d}y \leq C_1L_1L_2$$

as required. □

1.5 The level sets of $u(x,y)$ and the proof of Theorem 1.1.9

We now return to the problem of studying the shape of the level sets of the first eigenfunction $u(x,y)$. As we have mentioned earlier, since the potential $V(x,y)$ is convex, a theorem of Brascamp and Lieb, [BL2] tells us that $u(x,y)$ is log concave. In particular, this means that the superlevel sets of $u(x,y)$ are convex subsets of Ω.

We will use the results of the previous section to estimate the lengths of the projections of these level sets onto the x and y-axis. In particular, in this section we will establish Theorem 1.1.9 about the shape of the level sets. Throughout this section we let $c_1 > 0$ be a small absolute constant as in the statement of Theorem 1.1.9. The constant $c > 0$ which appears in the propositions below is bounded away from 0 and 1 by satisfying

$$c_1 < c < 1 - c_1,$$

and all other constants will depend on the choice of c_1.

We first use the bound on A from Proposition 1.4.13 to find an upper bound on the behaviour of the level sets of $u(x,y)$ in the y-direction.

Proposition 1.5.1. *Let $0 < c < 1$ be a fixed absolute constant. Then, for any fixed x, the cross-section of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ at x consists of an interval of length at most L_1 multiplied by an absolute constant.*

Proof of Proposition 1.5.1: By Proposition 1.4.13 we know that

$$A = \max_x H(x) = \max_x \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y \leq C_1 L_1.$$

If $u(x,y) \geq c$ for y in an interval of length CL_1 for C sufficiently large, this immediately gives a contradiction. □

We can also prove an upper bound on the length of the projection of the level sets of $u(x,y)$ in the y-direction.

Proposition 1.5.2. *For sufficiently small $\delta > 0$ fixed, there exists an $\eta > 0$ such that if the point (x,y) is within a distance ηL_1 of the level set $\{(x,y) \in \Omega : V(x,y) = 1 + \eta^{-1} L_1^{-2}\}$, then*

$$u(x,y) \leq \delta.$$

In particular, the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ are at a distance comparable to L_1 away from the level set $\{(x,y) \in \Omega : V(x,y) = 1 + CL_1^{-2}\}$, for some absolute constant $C > 0$.

Remark 1.5.3. *The proof of this proposition follows closely the proof of Lemma 3.17 in [GJ2], where an analogous property has been established for the first eigenfunction of a two dimensional convex domain.*

Before proving this proposition, let us show the following corollary:

Corollary 1.5.4. *Let $0 < c < 1$ be a fixed absolute constant. Then, the projection of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y-axis has length bounded from above by an absolute constant multiplied by L_1.*

Proof of Corollary 1.5.4: By the definition of the length scale L_1 and the orientation of the level set $\Omega_{L_1^{-2}} = \{(x,y) \in \Omega : V(x,y) = 1 + L_1^{-2}\}$ we used when defining L_2, we know that the projection of the level set $\Omega_{L_1^{-2}}$ onto the y-axis has length comparable to L_1. Moreover, by the convexity of the potential $V(x,y)$, this is true for any level set $\{(x,y) \in \Omega : V(x,y) = 1 + CL_1^{-2}\}$, for any absolute constant $C > 0$. Therefore, the upper bound on the length of the projection of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y-axis follows from Proposition 1.5.2. □

Proof of Proposition 1.5.2: Let (x', y') be a point which is within a distance ηL_1 of the level set $\{(x,y) \in \Omega : V(x,y) = 1 + \eta^{-1} L_1^{-2}\}$. After a rotation, we may assume that the nearest point of $\{(x,y) \in \Omega : V(x,y) = 1 + \eta^{-1} L_1^{-2}\}$ to (x', y') is equal to (x', y_1), with $y_1 < y'$ and $y' - y_1 < \eta L_1$.

We will need to use two properties of the potential $V(x,y)$. Firstly, by the simple eigenvalue bounds on λ in Proposition 1.2.7 we have seen before that

$$\Delta_{x,y} u(x,y) = (V(x,y) - \lambda) u(x,y) \geq -\frac{C_1^2}{L_1^2} u(x,y) \tag{1.84}$$

for all values of (x,y), for some absolute constant C_1. Moreover, $V(x,y)$ has convex sublevel sets and by the rotation we made above we have $V(x, y_1) = 1 + \eta^{-1} L_1^{-2}$. Therefore,

$$\Delta_{x,y} u(x,y) = (V(x,y) - \lambda) u(x,y) \geq \frac{1}{2\eta L_1^2} u(x,y), \tag{1.85}$$

whenever $y \leq y' - \eta L_1 < y_1$.

We define the comparison function $v_1(x,y)$ by

$$v_1(x,y) = \sin\left(\frac{C_1 (y - y')}{2L_1} + \frac{C_1 \eta}{2} + C_1 \delta \right) \tag{1.86}$$

for $y \geq y' - \eta L_1$, and by

$$v_1(x,y) = (\sin(C_1 \delta)) \exp\left(\frac{\delta}{2} + \frac{(y - y')}{2\delta L_1} \right) \tag{1.87}$$

for $y < y' - \eta L_1$. We make the choice $\eta = \delta^2$, and this ensures that $v_1(x,y)$ is continuous at $y = y' - \eta L_1$ for all values of x.

For $\delta > 0$ sufficiently small, using $\sin(C_1 \delta) > C_1 \delta \cos(C_1 \delta)$, we find that $\partial_y^2 v_1(x,y)$ has a negative delta function along $y = y' - \eta L_1$. Everywhere else, calculating $\Delta_{x,y} v_1(x,y)$ from its definition in (1.86) and (1.87), and using the inequalities for $\Delta_{x,y} u(x,y)$ in (1.84) and (1.85), we see that

$$\frac{\Delta_{x,y} v_1(x,y)}{v_1(x,y)} \leq \frac{\Delta_{x,y} u(x,y)}{u(x,y)}.$$

Moreover, for those $(x,y) \in \partial\Omega$, with $y \leq y' + \left(\frac{\pi}{C_1} - \eta - 2\delta \right) L_1$ we have

$$v_1(x,y) > 0 = u(x,y)|_{\partial\Omega}$$

and for $(x,y) \in \Omega$ with $y = y' + \left(\frac{\pi}{C_1} - \eta - 2\delta\right) L_1$, we have

$$v_1\left(x, y' + (\pi/C_1 - \eta - 2\delta)L_1\right) = 1 \geq u\left(x, y' + (\pi/C_1 - \eta - 2\delta)L_1\right).$$

Thus, applying the generalised maximum principle in Proposition 1.4.6 to those (x,y) in Ω with $y \leq y' + \left(\frac{\pi}{C_1} - \eta - 2\delta\right) L_1$, $v_1(x,y)$ is a positive supersolution, and in particular

$$u(x',y') \leq v_1(x',y') = \sin\left(\frac{C_1\eta}{2} + C_1\delta\right) \leq C_2\delta.$$

Thus, repeating the argument with a suitable multiple of δ gives the desired result.

□

We now want to obtain a lower bound on the height of the level sets in the y-direction.

Proposition 1.5.5. *Let $0 < c < 1$ be a fixed absolute constant. Then, the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ has inner radius bounded below by an absolute constant multiplied by L_1. In particular, the projection of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y-axis has length bounded from below by an absolute constant multiplied by L_1.*

Remark 1.5.6. *The proof of this proposition only considers the parameter L_1, and does not use any properties of the eigenvalue or eigenfunction that depend on L_2. In particular, this means that we do not need to fix the orientation of the level set $\Omega_{L_1^{-2}} = \{(x,y) \in \Omega : V(x,y) = 1 + L_1^{-2}\}$, and we are free to rotate Ω in the course of the proof.*

Proof of Proposition 1.5.5: Let us consider the case $c = 1/4$, and study the level set $\{(x,y) \in \Omega : u(x,y) = \frac{1}{4}\}$. Suppose that the shortest projection of the set onto any direction is of length α. By the convexity of the superlevel sets of $u(x,y)$, after a rotation and a translation, we may then assume that this level set lies between the two lines $y = 0$ and $y = \alpha$.

We will use the comparison function

$$W(x,y) \coloneqq \frac{1}{2}\sin\left(\frac{\pi}{6} + \frac{2\pi}{3\alpha}y\right).$$

This function is equal to $1/4$ when $y = 0$ or $y = \alpha$, and satisfies

$$(\Delta_{x,y} - V(x,y) + \lambda)W(x,y) = -\left(\frac{2\pi}{3\alpha}\right)^2 W(x,y) + (\lambda - V(x,y))W(x,y). \tag{1.88}$$

Since $V(x,y) \geq 1$, by the straightforward eigenvalue bound on λ from Proposition 1.2.7 we have

$$\lambda - V(x,y) \leq C^2 L_1^{-2},$$

for an absolute constant $C > 0$. Therefore, from (1.88) we obtain

$$(\Delta_{x,y} - V(x,y) + \lambda)W(x,y) \leq \left(-\left(\frac{2\pi}{3\alpha}\right)^2 + C^2 L_1^{-2} \right) W(x,y). \tag{1.89}$$

Let us assume that

$$\alpha < \frac{2\pi L_1}{3C}. \tag{1.90}$$

Then, from (1.89) we see that

$$(\Delta_{x,y} - V(x,y) + \lambda)W(x,y) < 0,$$

while $(\Delta_{x,y} - V(x,y) + \lambda)u(x,y) = 0$ in Ω. Also, for all points (x,y) with $y = 0$, α we have

$$u(x,y) \leq W(x,y) = \frac{1}{4},$$

and $u(x,y) = 0 < W(x,y)$ for $(x,y) \in \partial\Omega$, with $0 \leq y \leq \alpha$. Therefore, by the generalised maximum principle in Proposition 1.4.6 we find that

$$u(x,y) \leq W(x,y) \qquad \text{for } (x,y) \in D \text{ with } 0 \leq y \leq \alpha.$$

However, $W(x,y) \leq \frac{1}{2}$, while $u(x,y)$ attains its maximum of 1 at some point (x,y) with $0 \leq y \leq \alpha$. This gives a contradiction, and so from (1.90) we must have

$$\alpha > \frac{2\pi L_1}{3C}.$$

Therefore the projection of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq \frac{1}{4}\}$ onto any direction has length at least comparable to L_1, and this gives us the required lower bound on the inner radius of this superlevel set. We can also repeat the argument above for the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ for any fixed absolute constant c with $c_1 < c < 1 - c_1$ to obtain the same result. □

Corollary 1.5.7. *As an immediate consequence of Proposition 1.5.5, we see that*

$$A = \max_x \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y \geq \tilde{c} L_1,$$

for an absolute constant $\tilde{c} > 0$.

Combining Propositions 1.5.1 and 1.5.5, the height of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ in the y-direction is comparable to L_1. We now turn to the studying the length of the level sets of $u(x,y)$ in the x-direction. We first use Corollary 1.4.7 to obtain an upper bound on the length of the level sets.

Proposition 1.5.8. *Let* $0 < c < 1$ *be a fixed absolute constant. Then, the projection of the level set* $\{(x,y) \in \Omega : u(x,y) = c\}$ *onto the* x*-axis has length bounded by an absolute constant multiplied by* L_2.

Proof of Proposition 1.5.8: Suppose that the length of the projection of $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the x-axis is bounded below by $2CL_2$, where $C > 0$ is a large absolute constant that we will specify later in the proof. For each fixed x, the cross-section of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ at x consists of an interval. Since the superlevel set is convex, the length of this interval is greater than half of its maximum length for x lying in an interval of length CL_2.

By Proposition 1.5.5, this maximum length is bounded below by $2C_1L_1$ for an absolute constant $C_1 > 0$. In other words, $u(x,y) \geq c$ for all (x,y) in a rectangle of height C_1L_1 and width CL_2.

As a result of this, we have

$$H(x) = \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y \geq c^2 C_1 L_1, \tag{1.91}$$

for all x in an interval of length CL_2. By Proposition 1.4.13, A is bounded by an absolute constant multiplied by L_1, and by Corollary 1.4.7 we have the bound

$$H(x) \leq A e^{-c|x-x^*|/L_2}. \tag{1.92}$$

Therefore, combining (1.91) and (1.92), we obtain a contradiction if we choose C to be sufficiently large. This completes the proof of the proposition. □

To complete the proof of Theorem 1.1.9 we finally want to obtain a comparable lower bound on the length of the level set of $u(x,y)$ in the x-direction. To do this we will use the L^2-bound on the first derivative $\partial_x u(x,y)$ from Proposition 1.4.12.

Proposition 1.5.9. *Let $0 < c < 1$ be a fixed absolute constant. Then, the projection of the level set $\{(x, y) \in \Omega : u(x, y) = c\}$ onto the x-axis has length bounded from below by an absolute constant multiplied by L_2.*

Proof of Proposition 1.5.9: We first prove the proposition for $c = 1/4$. By applying Proposition 1.5.5 with $c = \frac{1}{2}$, there exists a point $x = x_0$, and an interval J of length equal to $2c^* L_1$ for a constant $c^* > 0$, such that $u(x_0, y) \geq \frac{1}{2}$ for all y in J. Therefore,

$$\int_J u(x_0, y)^2 \, \mathrm{d}y \geq \tfrac{1}{4} c^* L_1, \tag{1.93}$$

Extending $u(x, y)$ to be zero outside of Ω, for any other x, we can write

$$u(x, y) = u(x_0, y) + \int_{x_0}^{x} \partial_t u(t, y) \, \mathrm{d}t,$$

and so,

$$u(x, y)^2 \geq \tfrac{3}{4} u(x_0, y)^2 - C_1 |x - x_0| \int_{I(x)} (\partial_t u(t, y))^2 \, \mathrm{d}t,$$

where $I(x)$ consists of those points between x_0 and x. Integrating this over $y \in J$, we find that

$$\int_J u(x, y)^2 \, \mathrm{d}y \geq \tfrac{3}{4} \int_J u(x_0, y)^2 \, \mathrm{d}y - C_1 |x - x_0| \int_J \int_{I(x)} (\partial_t u(t, y))^2 \, \mathrm{d}t \, \mathrm{d}y \tag{1.94}$$

By (1.93), the first term on the right hand side of (1.94) is bounded from below by $\frac{3}{8} c^* L_1$. Provided $|x - x_0| \leq c_2 L_2$ for $c_2 > 0$ sufficiently small, we can use Proposition 1.4.12 to show that the second term on the right hand side of (1.94) is bounded above by

$$C_1 |x - x_0| L_1 L_2^{-1},$$

for an absolute constant $C_1 > 0$. Thus, if $|x - x_0| \leq c_3 L_2$ for $c_3 > 0$ sufficiently small, we can ensure that

$$\int_J u(x, y)^2 \, \mathrm{d}y \geq \tfrac{3}{16} c^* L_1 - \tfrac{1}{16} c^* L_1 = \tfrac{1}{8} c^* L_1.$$

Since the interval J has length equal to $c^* L_1$, this means that for each x with $|x - x_0| \leq c_3 L_2$, $u(x, y)$ must be at least $\frac{1}{4}$ at some point $y \in J$. In particular, the level set $\{(x, y) \in \Omega : u(x, y) = 1/4\}$

must have length in the x-direction of at least $c_3 L_2$ as required. A lower bound on the length in the x-direction of the other level sets of $u(x,y)$ follows in an analogous way.

□

Combining Corollary 1.5.4 and Proposition 1.5.5 concerning the height of the level sets in the y-direction with Propositions 1.5.8 and 1.5.9 concerning the length of the level sets in the x-direction we have established the following: For any c with $c_1 < c < 1 - c_1$, the projections of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y and x-axes are of lengths comparable to L_1 and L_2 respectively, and moreover, the inner radius of the corresponding superlevel set is comparable to L_1 while the diameter is comparable to L_2. This implies that the level sets have the desired shape and completes the proof of Theorem 1.1.9.

1.6 The location of the maximum of $u(x,y)$

In Theorem 1.1.9 we have described the shape of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ where c is bounded away from 0 and 1 by $c_1 < c < 1 - c_1$. In Proposition 1.4.16, we gave an indication of the behaviour of $u(x,y)$ as c becomes small. In this section, we instead want to focus on the behaviour of the eigenfunction near its maximum.

The main aim in this section is to prove the following:

Proposition 1.6.1. *Suppose that the eigenfunction $u(x,y)$ attains its maximum at the point (x^*, y^*). Then, there exists an absolute constant $c^* > 0$ such that*

$$V(x^*, y^*) - \lambda \leq -c^* L_1^{-2}.$$

Proof of Proposition 1.6.1: To prove this proposition, we first notice that from Proposition 1.5.2, we can restrict our attention to the region where $V(x,y) \leq 1 + CL_1^{-2}$, for a sufficiently large absolute constant $C > 0$. Before we can prove the sharper estimate on the location of the maximum in Proposition 1.6.1 we need more information about the first derivatives of $u(x,y)$. We will use Proposition 1.5.2 to obtain a pointwise bound on the first derivatives of $u(x,y)$ near its maximum.

To do this, we need to introduce the following function:

Definition 1.6.2. *Let $K_0(r)$ be the 0th modified Bessel function of the second kind. Then, for $r > 0$,*

we define the function $J(r)$ as follows: Let $0 < c_1 < c_2$ be small absolute constants. We first set

$$J(r) \coloneqq K_0(r/L_1),$$

for $0 < r \leq c_1 L_1$, and then we require that $J(r)$ decays smoothly to 0 on a length scale comparable to L_1 for $c_1 L_1 \leq r \leq c_2 L_1$, and is identically 0 for $r > c_2 L_1$.

Lemma 1.6.3 (Properties of $J(r)$)**.** *For $0 < r \leq c_1 L_1$, $J(r)$ satisfies the equation,*

$$\frac{1}{r^2}\left(r^2 \frac{d^2}{dr^2} + r\frac{d}{dr}\right) J(r) = L_1^{-2} J(r),$$

and for $c_1 L_1 \leq r \leq c_2 L_1$,

$$\frac{d^m}{dr^m} J(r) \leq C L_1^{-m}$$

for $m \leq 3$. Moreover, $J(r)$ has a singularity equal to a multiple of $\log r$ as we approach $r = 0$.

Proof of Lemma 1.6.3: These properties follow immediately from the definition of $J(r)$ in Definition 1.6.2 and the corresponding properties of the modified Bessel function $K_0(r)$. □

We will use the function $J(r)$ with r defined by

$$r^2 \coloneqq (x - x')^2 + (y - y')^2 \tag{1.95}$$

to obtain a pointwise bound on the first derivatives of $u(x, y)$.

Proposition 1.6.4. *Fix an absolute constant c with $0 < c < 1$. There exists an absolute constant $C > 0$ such that for any point (x', y') with $u(x', y') \geq c$, we have the bound*

$$|\nabla_{x',y'} u(x', y')| \leq C L_1^{-1}.$$

Proof of Proposition 1.6.4: The strategy of the proof to obtain this pointwise estimate for $\nabla_{x',y'} u(x', y')$ is to use the function $J(r)$ together with the eigenfunction equation

$$-\Delta_{x,y} u(x, y) + (V(x, y) - \lambda) u(x, y) = 0,$$

to obtain an expression for the first derivatives of $u(x', y')$.

We fix (x', y') as in the statement of the proposition, and by Proposition 1.5.2, we see that by choosing $c_1 < c_2$ sufficiently small in the definition of $J(r)$, the support of $J(r)$ is contained in the region where $V(x, y) \leq 1 + C_1 L_1^{-2}$.

With r as in (1.95), we begin by considering the integral

$$\lim_{\epsilon \to 0} \int_{r > \epsilon} \nabla_{x,y} J(r) . \nabla_{x,y} \partial_x u(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.96}$$

Since $J(r)$ has a singularity of the form $\log r$ at $r = 0$, $\nabla_{x,y} J(r)$ is integrable and the above limit is well-defined.

Integrating by parts one time to move the derivative away from $\partial_x u(x, y)$, we obtain a boundary term at $r = 0$, and the integral in (1.96) becomes

$$\partial_x u(x', y') - \lim_{\epsilon \to 0} \int_{r > \epsilon} \Delta_{x,y} J(r) \partial_x u(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.97}$$

Note that by the support properties of $J(r)$, there are no other boundary terms appearing from this integration by parts.

By Lemma 1.6.3, $\Delta_{x,y} J(r) = L_1^{-2} J(r)$ for $r \leq c_1 L_1$ and is $\leq C_1 L_1^{-2}$ elsewhere. Thus, we can integrate by parts again in the integral in (1.97) to get

$$- \lim_{\epsilon \to 0} \int_{r > \epsilon} \partial_x \Delta_{x,y} J(r) u(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.98}$$

The function $\Delta_{x,y} J(r) = L_1^{-2} J(r)$ only has a logarithmic singularity at $r = 0$, and so for this integral we do not get a boundary term at $r = 0$ in the integration by parts. Instead, since $J(r)$ is supported in a region of area L_1^2 and $u(x, y) \leq 1$ everywhere, this integral is bounded by $C L_1^{-1}$ for an absolute constant $C > 0$.

Using this bound we see that the integral in (1.96) is equal to $\partial_x u(x', y')$ plus a contribution which is bounded by $C L_1^{-1}$.

We can also write the integral in (1.96) as

$$\lim_{\epsilon \to 0} \int_{r > \epsilon} \partial_x J(r) \partial_x^2 u(x, y) + \partial_y J(r) \partial_y \partial_x u(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Using the eigenfunction equation, we can rewrite this as

$$\lim_{\epsilon \to 0} \int_{r > \epsilon} -\partial_x J(r) \partial_y^2 u(x, y) + \partial_x J(r) (V(x, y) - \lambda) u(x, y) + \partial_y J(r) \partial_y \partial_x u(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.99}$$

We consider the contribution to this integral from

$$\lim_{\epsilon \to 0} \int_{r>\epsilon} -\partial_x J(r) \partial_y^2 u(x,y) + \partial_y J(r) \partial_y \partial_x u(x,y) \, \mathrm{d}x \, \mathrm{d}y. \tag{1.100}$$

If we integrate by parts in y in the first term and in x in the second term, we find that that the boundary terms at $r = 0$ vanish, and there are no other boundary terms. Therefore, the integral in (1.100) is equal to

$$\lim_{\epsilon \to 0} \int_{r>\epsilon} \partial_y \partial_x J(r) \partial_y u(x,y) - \partial_x \partial_y J(r) \partial_y u(x,y) \, \mathrm{d}x \, \mathrm{d}y = 0.$$

Thus, from (1.99), the original integral in (1.96) becomes

$$\lim_{\epsilon \to 0} \int_{r>\epsilon} \partial_x J(r)(V(x,y) - \lambda) u(x,y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\Omega} \partial_x J(r)(V(x,y) - \lambda) u(x,y) \, \mathrm{d}x \, \mathrm{d}y.$$

This means that we have shown that

$$\partial_x u(x',y') = \int_{\Omega} \partial_x J(r)(V(x,y) - \lambda) u(x,y) \, \mathrm{d}x \, \mathrm{d}y, \tag{1.101}$$

plus a contribution which is bounded by CL_1^{-1}.

Since $u(x',y') \geq c$, we know from Proposition 1.5.2 that $|V(x,y) - \lambda| \leq C_1 L_1^{-2}$ on the support of $J(r)$. Therefore, the integral on the right hand side of (1.101) is bounded by

$$C_1 L_1^{-2} \int_{\Omega} |\partial_x J(r)| \, u(x,y) \, \mathrm{d}x \, \mathrm{d}y \leq C L_1^{-1}.$$

This gives the required bound for $\partial_x u(x',y')$, and the bound for $\partial_y u(x',y')$ follows in exactly the same way. □

We recall from Theorem 1.1.9 that the level sets of $u(x,y)$ are of height comparable to L_1 in the y-direction and of length comparable to L_2 in the x-direction. Therefore, this is consistent with the derivative bound for $\partial_y u(x,y)$ from Proposition 1.6.4 above. However, since in general we have $L_2 \gg L_1$, we want to improve the bound given for $\partial_x u(x,y)$.

To do this we first prove a corollary of Propositions 1.5.2 and 1.6.4 about the location of the level sets of $u(x,y)$ in the x-direction.

Corollary 1.6.5. *Fix an absolute constant c, with $0 < c < 1$. Then, there exists an absolute constant $C > 0$ such that for any point (x,y) in the level set $\{(x,y) \in \Omega : u(x,y) = c\}$, there*

exists points (x_1, y_1) *with* $x_1 - x$ *both positive or negative that* $|x_1 - x|$ *is comparable to* L_2 *and* $V(x_1, y_1) \leq 1 + CL_1^{-2}$.

Proof of Corollary 1.6.5: Let (x', y') be the left most point of the level set $\{(x, y) \in \Omega : u(x, y) = c\}$. That is, $u(x', y') = c$, and $u(x, y) < c$ for any point (x, y) with $x < x'$. Then, by the bound on $\partial_y u(x, y)$ from Proposition 1.6.4, we know that

$$u(x', y) \geq \tfrac{1}{2}c,$$

for all y in an interval J of length comparable to L_1. In particular,

$$\int_J u(x', y)^2 \, \mathrm{d}y \geq \tilde{c} L_1,$$

for some absolute constant $\tilde{c} > 0$. Therefore, using the L^2-bound on $\partial_x u(x, y)$ from Proposition 1.4.12, exactly as in the proof of Proposition 1.5.9, we find that

$$\int_J u(x, y)^2 \, \mathrm{d}y \geq \tilde{c} L_1/2,$$

for all $x < x'$ with $x' - x > c_2 L_2$, for an absolute constant $c_2 > 0$. Thus, for any such $x = x_1$, there exists a $y_1 \in J$ such that $u(x_1, y_1)$ is bounded below by an absolute constant. In particular, by Proposition 1.5.2, this means that $V(x_1, y_1) \leq 1 + CL_1^{-2}$, and this completes the proof of the corollary.

□

This corollary allows us to partially improve the estimate on $\partial_x u(x, y)$ from Proposition 1.6.4. We recall that the maximum of $u(x, y)$ of 1 is achieved at $(x, y) = (x^*, y^*)$.

Proposition 1.6.6. *Consider the level set* $\{(x, y) \in \Omega : u(x, y) = c\}$ *for a fixed constant* c*, with* $c_1 < c < 1 - c_1$*. Then, on part of the upper and lower boundaries of this level set, with* x *in an interval around* $x = x^*$ *of length comparable to* L_2*, we have the pointwise derivative bound*

$$|\partial_x u(x, y)| \leq CL_2^{-1},$$

for some absolute constant $C > 0$.

Proof of Proposition 1.6.6: For fixed c, with $c_1 < c < 1 - c_1$, the level set $\{(x, y) \in \Omega : u(x, y) = c\}$ extends a distance comparable to L_2 in the x-direction on either side of the point x^*. Let $y = g(x)$

be a parametrisation of the upper boundary of the level set. Since this level set is the boundary of a convex set of height comparable to L_1 in the y-direction, this means that

$$|g'(x)| \leq C_1 L_1 L_2^{-1} \tag{1.102}$$

for all x in an interval around x^* of length comparable to L_2. On this part of the level set we have

$$u(x, g(x)) = c,$$

and so differentiating this with respect to x gives

$$\partial_x u(x, g(x)) = -g'(x)\partial_y u(x, g(x)).$$

Combining the estimate for $g'(x)$ in (1.102) with the pointwise bound on $\partial_y u(x, y)$ from Proposition 1.6.4 gives the required bound for $\partial_x u(x, y)$.

□

To prove that the maximum of $u(x, y)$ has the required properties of Proposition 1.6.1, we will also need to improve the derivative estimate of Proposition 1.6.4 for those points (x, y) near the maximum. To do this we will prove the following two propositions.

Proposition 1.6.7. *Let $\epsilon > 0$ be sufficiently small. Then, the convex superlevel set $\{(x, y) \in \Omega : u(x, y) \geq 1 - \epsilon\}$ has inner radius at least $c\epsilon^{1/2} L_1$, for a small absolute constant $c > 0$, which is independent of ϵ.*

Proposition 1.6.8. *Let $u(x', y') = 1 - \epsilon$, where $\epsilon > 0$ is sufficiently small. Then, there exists an absolute constant $C > 0$, which is independent of ϵ, such that*

$$|\nabla_{x,y} u(x', y')| \leq C\epsilon^{1/2} L_1^{-1}.$$

In particular, this is an improvement on Proposition 1.6.4 for small ϵ.

Remark 1.6.9. *Consider a function $f(x)$ defined on the interval $[-L_1, L_1]$ by*

$$f(y) \coloneqq 1 - L_1^{-2} y^2.$$

Then, $f(y) = 1 - \epsilon$ for $y = \pm\epsilon^{1/2} L_1$. In other words, the interval on which $f(y) \geq 1 - \epsilon$ has length

$2\epsilon^{1/2}L_1$. *Thus, the lower bound on the inner radius of the superlevel set of* $u(x,y)$ *in Proposition 1.6.7 is consistent with the eigenfunction* $u(x,y)$ *being bounded from below by such a parabola as we move away from the maximum in the* y*-direction.*

Also, at the two points where $f(y)$ *is equal to* $1-\epsilon$ *we have the derivative bound*

$$f'(y) = -2L_1^{-2}y = \mp 2\epsilon^{1/2}L_1^{-1}.$$

Therefore, the derivative bound in Proposition 1.6.8 is again consistent with the eigenfunction $u(x,y)$ *being bounded from below by such a parabola as we move away from the maximum in the* y*-direction. It is also consistent with having a bound comparable to* L_1^{-2} *on the second derivatives of the eigenfunction.*

Proof of Proposition 1.6.7: Suppose that the proposition does not hold. Then, after a translation and rotation, we may assume that the level set $\{(x,y) \in \Omega : u(x,y) = 1-\epsilon\}$ lies between the lines $y = \pm\alpha$, where $\alpha < c_1\epsilon^{1/2}L_1$ for a small absolute constant c_1 to be chosen later. Note that we are considering the length scale L_1, and we will not use any of the properties of λ and $u(x,y)$ that depend on L_2. This means that we do not have to fix the orientation of $\Omega_{L_1^{-2}} = \{(x,y) \in \Omega : V(x,y) = 1 + L_1^{-2}\}$ and so there is no issue in applying the rotation above.

We will use the comparison function

$$v_2(x,y) \coloneqq \left(1 - \tfrac{1}{2}\epsilon\right) \sin\left(\frac{\pi}{2} + \frac{\epsilon^{1/2}y}{C_1\alpha}\right),$$

where $C_1 > 0$ is chosen so that

$$v_2(x,y) \geq 1 - \epsilon$$

for all (x,y) with $y = \pm\alpha$. This means that

$$u(x,y) \leq v_2(x,y) \tag{1.103}$$

for any $(x,y) \in \Omega$ with $y = \pm\alpha$. Also, for all $(x,y) \in \partial\Omega$ with $-\alpha \leq y \leq \alpha$, we see that

$$u(x,y) = 0 \leq v_2(x,y). \tag{1.104}$$

Moreover, the function $v_2(x,y)$ satisfies

$$(\Delta_{x,y} - V(x,y) + \lambda)\, v_2(x,y) = -\left(\frac{\epsilon}{C_1^2\alpha^2}\right) v_2(x,y) + (\lambda - V(x,y))v_2(x,y).$$

We know that $\lambda - V(x,y) \leq C_2^2 L_1^{-2}$, for some absolute constant $C_2 > 0$, and so

$$(\Delta_{x,y} - V(x,y) + \lambda)\, v_2(x,y) < \left(-\left(\frac{\epsilon}{C_1^2\alpha^2}\right) + C_2^2 L_1^{-2}\right) v_2(x,y). \tag{1.105}$$

Provided $\alpha < c_1 \epsilon^{1/2} L_1$, for c_1 sufficiently small (depending only on C_1 and C_2), we can ensure from (1.105) that

$$(\Delta_{x,y} - V(x,y) + \lambda)\, v_2(x,y) < 0. \tag{1.106}$$

Combining (1.103), (1.104) and (1.106), we see that by the generalised maximum principle in Proposition 1.4.6 that

$$u(x,y) \leq v_2(x,y) \qquad \text{for } (x,y) \in \Omega \text{ with } -\alpha \leq y \leq \alpha.$$

However, $v_2(x,y) \leq 1 - \frac{1}{2}\epsilon$ everywhere, whereas we know that $u(x,y)$ attains its maximum of 1 for some $(x,y) \in \Omega$ with $-\alpha \leq y \leq \alpha$. This contradiction completes the proof of the proposition.

□

Proof of Proposition 1.6.8:

In the proof of Proposition 1.6.8, as well as Proposition 1.6.7, we will also make use of the following proposition:

Proposition 1.6.10. *Suppose that the function $v(x,y)$ satisfies*

$$\Delta_{x,y} v(x,y) + W_1(x,y) v(x,y) = W_2(x,y),$$

in a convex domain D, with $|W_1(x,y)|$, $|W_2(x,y)| \leq C_1$. Let $(x_0,y_0) \in \partial D$, and let B_s be the disc of radius s around (x_0,y_0). If we have $v \geq 0$ in D, then

$$|\nabla_{x,y} v(x_0,y_0)| \leq C \sup_{B_{1/4} \cap D} v.$$

Proof of Proposition 1.6.10: This proposition is a variation of Proposition A.4 in [GJ2], and is proved

in exactly the same way. □

We can now begin the proof of Proposition 1.6.8. After a translation, we may assume that $u(x,y) = 1$ at the point $(x,y) = (0,0)$, and we define the function $u_1(x,y)$ by

$$u_1(x,y) \coloneqq u\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right).$$

This satisfies the equation

$$-\Delta_{x,y}u_1(x,y) + \epsilon L_1^2\left(V\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right) - \lambda\right)u_1(x,y) = 0.$$

We next define $u_2(x,y)$ by

$$u_2(x,y) \coloneqq \epsilon^{-1}(u_1(x,y) - (1-\epsilon)).$$

Consider the convex superlevel set $\{(x,y) \in \Omega : u_1(x,y) \geq 1-\epsilon\}$. By Proposition 1.6.7 and the definition of the function $u_1(x,y)$, this set has inner radius bounded from below by an absolute constant. Moreover, on the boundary of this set the function $u_2(x,y)$ is equal to 0 and inside the superlevel set it takes all values between 0 and 1. It satisfies the equation

$$\begin{aligned}-\Delta_{x,y}u_2(x,y) + \epsilon L_1^2\left(V\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right) - \lambda\right)u_2(x,y)& \\ + (1-\epsilon)L_1^2\left(V\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right) - \lambda\right) = 0.& \qquad (1.107)\end{aligned}$$

From Proposition 1.5.2, we certainly know that inside the convex set $\{(x,y) \in \mathbb{R}^2 : u_1(x,y) = 1-\epsilon\}$ we have the bounds

$$\left|L_1^2\left(V\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right) - \lambda\right)\right| \leq C_1,$$

for an absolute constant $C_1 > 0$. Therefore, in (1.107) we have the bounds

$$\left|\epsilon L_1^2\left(V\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right) - \lambda\right)\right| \leq C_1, \qquad \left|(1-\epsilon)L_1^2\left(V\left(\epsilon^{1/2}L_1 x, \epsilon^{1/2}L_1 y\right) - \lambda\right)\right| \leq C_1.$$

We can thus use Proposition 1.6.10 with $D = \{(x,y) \in \mathbb{R}^2 : u_1(x,y) \geq 1-\epsilon\}$ to conclude that

$$|\nabla_{x,y}u_2(x,y)| \leq C,$$

for (x, y) on the boundary of D. Recalling the definitions of $u_1(x, y)$ and $u_2(x, y)$, this shows that

$$|\nabla_{x,y} u(x', y')| \leq C\epsilon(\sqrt{\epsilon}L_1)^{-1} = C\sqrt{\epsilon}L_1^{-1}$$

as required. □

We need to write down one more consequence of the log concavity of $u(x, y)$, and then we can start to complete the proof of Proposition 1.6.1.

Lemma 1.6.11. *We have an upper bound on $\partial_x^2 u(x, y)$ of the form*

$$\partial_x^2 u(x, y) \leq \frac{(\partial_x u(x, y))^2}{u(x, y)},$$

and an analogous upper bound for $\partial_y^2 u(x, y)$.

Proof of Lemma 1.6.11: Differentiating the function $\log u(x, y)$ twice with respect to x, we find that

$$\partial_x^2(\log u(x, y)) = \frac{\partial_x^2 u(x, y)}{u(x, y)} - \frac{(\partial_x u(x, y))^2}{u(x, y)^2}. \tag{1.108}$$

However, the eigenfunction $u(x, y)$ is log concave, and so

$$\partial_x^2(\log u(x, y)) \leq 0. \tag{1.109}$$

Combining (1.108) and (1.109) gives the desired bound. □

To complete the proof of Proposition 1.6.1 we split into two cases.

1.6.1 Case 1: $L_2 \gg L_1$

We will first assume that $L_2 \gg L_1$, or in other words, $L_2 \geq \tilde{C}L_1$ for a large absolute constant $\tilde{C} > 0$, which we will specify below.

After a translation, we may assume that $u(x, y)$ attains its maximum at the point $(0, 0)$. Let $u(0, -\alpha L_1)$, $u(0, \beta L_1) = 1/2$, where we know from Theorem 1.1.9 that α and β are both comparable to 1. Moreover, without loss of generality, we may assume that $\partial_y V(0, 0) \geq 0$. We want to study the integral

$$\int_0^{\beta L_1} (\beta L_1 - y)(V(0, y) - \lambda)u(0, y)\, \mathrm{d}y. \tag{1.110}$$

We will prove the following two lemmas:

Lemma 1.6.12. *The integral in* (1.110) *is bounded from below by*

$$\tfrac{1}{2}\beta^2 L_1^2 (V(0,0) - \lambda).$$

Lemma 1.6.13. *The integral in* (1.110) *is bounded from above by* $-\frac{1}{4}$.

Combining Lemmas 1.6.12 and 1.6.13, we see that

$$V(0,0) - \lambda \leq -\tfrac{1}{2}\beta^{-2} L_1^{-2},$$

and so we have established Proposition 1.6.1 in the case where $L_2 \gg L_1$. Thus, we are left to prove these two lemmas.

Proof of Lemma 1.6.12: Since $\partial_y V(0,0) \geq 0$, and $V(x,y)$ is convex, we must have $\partial_y V(0,y) \geq 0$ for all $y \geq 0$. Thus, for $0 \leq y \leq \beta L_1$ we have

$$V(0,y) - \lambda \geq V(0,0) - \lambda$$

and $u(0,y) \geq \frac{1}{2}$. The lower bound then follows immediately. □

Proof of Lemma 1.6.13: Since $u(x,y)$ satisfies the eigenfunction equation, we can rewrite (1.110) as

$$\int_0^{\beta L_1} (\beta L_1 - y) \Delta_{x,y} u(0,y) \, \mathrm{d}y. \tag{1.111}$$

Let us first consider the term containing a factor of $\partial_y^2 u(0,y)$. Since $\partial_y u(0,0) = 0$, integrating by parts, this becomes

$$\int_0^{\beta L_1} \partial_y u(0,y) \, \mathrm{d}y = -u(0,0) + u(0,\beta L_1) = -\tfrac{1}{2}.$$

We are left to bound the contribution to (1.111) from

$$\int_0^{\beta L_1} (\beta L_1 - y) \partial_x^2 u(0,y) \, \mathrm{d}y, \tag{1.112}$$

and to do this we will use Proposition 1.6.8 and Lemma 1.6.11.

We first fix $0 < c_1 < \beta$ such that $u(0, c_1 L_1) = 1 - c_2$ for a small constant $c_2 > 0$. Then, by

Proposition 1.6.8, we have the bound

$$|\partial_x u(0,y)| \leq C c_2^{1/2} L_1^{-1}$$

for all y with $0 \leq y \leq c_1 L_1$, with C independent of c_2. In particular, by Lemma 1.6.11, we have

$$\partial_x^2 u(x,y) \leq C c_2 L_1^{-2},$$

and so

$$\int_0^{c_1 L_1} (\beta L_1 - y) \partial_x^2 u(0,y)\, \mathrm{d}y \leq \frac{1}{8}, \tag{1.113}$$

provided $c_1 > 0$ is sufficiently small. We now need to consider the part of the integral in (1.112) with y between $c_1 L_1$ and βL_1. For y in this range, we can use the derivative bound on $\partial_x u(x,y)$ from Proposition 1.6.6, which after applying Lemma 1.6.11 gives

$$\partial_x^2 u(0,y) \leq C L_2^{-2}.$$

Therefore, provided L_2/L_1 is sufficiently large, we also have the bound

$$\int_{c_1 L_1}^{\beta L_1} (\beta L_1 - y) \partial_x^2 u(0,y)\, \mathrm{d}y \leq \frac{1}{8}. \tag{1.114}$$

Combining (1.113) and (1.114), we see that the integral in (1.112) is bounded above by $\frac{1}{4}$, and hence

$$\int_0^{\beta L_1} (\beta L_1 - y) \Delta_{x,y} u(0,y)\, \mathrm{d}y < -\frac{1}{4}.$$

This completes the proof of the lemma.

□

As we discussed after the statement of Lemmas 1.6.12 and 1.6.13, this completes the proof of Proposition 1.6.1 in the case where $L_2 \gg L_1$.

1.6.2 Case 2: L_1 and L_2 are comparable

In this case, we assume that $V(x,y)$ attains its minimum of 1 at $(0,0)$, and we rescale the eigenfunction $u(x,y)$ by L_1 in the x and y-directions. Then, $\tilde{u}(x,y) = u(L_1 x, L_1 y)$ satisfies the equation

$$\Delta_{x,y}\tilde{u}(x,y) = \tilde{F}(x,y)\tilde{u}(x,y),$$

where $\tilde{F}(x,y) = L_1^2\left(V(L_1x, L_1y) - \lambda\right)$. We know that $\tilde{u}(x,y)$ must attain its maximum at some point inside the region where $\tilde{F}(x,y) \leq 0$. We now want to improve this estimate on the location of the maximum of $\tilde{u}(x,y)$.

Lemma 1.6.14. *In the case where L_1 and L_2 are comparable there exists a small absolute constant $\epsilon > 0$ such that $\tilde{u}(x,y)$ attains its maximum at a distance at least $\epsilon^{2/3}$ from the boundary of the region where $\tilde{F}(x,y) \leq 0$.*

The function $\tilde{F}(x,y)$ is convex, has a minimum of $-c$ for some $c > 0$, and the inner radius and diameter of the region where $\tilde{F}(x,y) \leq 0$ are comparable to 1. Therefore, this lemma implies that

$$\tilde{F}(x,y) \leq -c_1$$

for some constant $c_1 > 0$ at the point where $\tilde{u}(x,y)$ attains its maximum. Returning to $u(x,y)$ and $F(x,y) = V(x,y) - \lambda$, this proves Proposition 1.6.1 in the case where L_1 and L_2 are comparable. Thus, we are left to prove Lemma 1.6.14.

Proof of Lemma 1.6.14: Suppose that $\tilde{u}(x,y) = 1$ at a point (x,y) within $\epsilon^{2/3}$ of the boundary of the region $\{(x,y) : \tilde{F}(x,y) \leq 0\}$, where $\epsilon > 0$ is a small constant that we will specify later.

The function $\tilde{F}(x,y)$ attains a negative minimum, is convex, and is negative on a region with diameter comparable to 1. Therefore, $|\nabla_{x,y}\tilde{F}(x,y)| \geq c_1$ on the set where $\tilde{F}(x,y) = 0$. In particular, we have the lower bound

$$\tilde{F}(x,y) \geq \epsilon^{2/3} \tag{1.115}$$

when we are at a distance comparable to $\epsilon^{2/3}$ outside the region where $\tilde{F}(x,y) \leq 0$. Also, by the pointwise derivative bounds on $u(x,y)$ from Proposition 1.6.4, $\tilde{u}(x,y)$ is comparable to 1 at a distance of $\epsilon^{2/3}$ from its maximum.

By the log concavity of $u(x,y)$, we know that

$$\Delta_{x,y} \log u(x,y) = \frac{\Delta_{x,y} u(x,y)}{u(x,y)} - \frac{|\nabla_{x,y} u(x,y)|^2}{u(x,y)^2} \leq 0.$$

Rearranging, and using the eigenfunction equation, this tells us that

$$|\nabla_{x,y} u(x,y)|^2 \geq (V(x,y) - \lambda) u(x,y)^2 = F(x,y) u(x,y)^2. \tag{1.116}$$

Thus, from (1.115) and (1.116), we have the lower bound

$$|\nabla_{x,y} \tilde{u}(x,y)| \geq \tilde{c} \epsilon^{1/3} \tag{1.117}$$

for some point (x_1, y_1) which is at a distance comparable to $\epsilon^{2/3}$ from the point where $\tilde{u}(x,y)$ attains its maximum.

However, by Proposition 1.6.8, we know that when $\tilde{u}(x',y') = 1-\epsilon$, we have the derivative bound

$$|\nabla_{x,y} \tilde{u}(x',y')| \leq C\epsilon^{1/2}. \tag{1.118}$$

For $\epsilon > 0$ sufficiently small, we have $\tilde{c}\epsilon^{1/3} > C\epsilon^{1/2}$, and so from (1.117) and (1.118), we see that $\tilde{u}(x_1, y_1) < 1 - \epsilon$.

In other words, the function $\tilde{u}(x,y)$ changes from 1 to $1-\epsilon$ on a line segment of length comparable to $\epsilon^{2/3}$. However, using Proposition 1.6.8 again, we know that

$$|\nabla_{x,y} \tilde{u}(x,y)| \leq C\epsilon^{1/2}$$

whenever $\tilde{u}(x,y) \geq 1 - \epsilon$, and so $\tilde{u}(x,y)$ can only change by an amount comparable to

$$\epsilon^{1/2} \epsilon^{2/3} = \epsilon^{7/6},$$

on this line segment of length $\epsilon^{2/3}$. For $\epsilon > 0$ sufficiently small, we see that $\epsilon^{7/6} \ll \epsilon$, and so this gives us a contradiction. □

As we discussed after the statement of Lemma 1.6.14 this also completes the proof of Proposition 1.6.1 in the case where L_1 and L_2 are comparable. □

Let us finish by giving two consequences of the location of the maximum of $u(x,y)$ derived in

Proposition 1.6.1. The first is to show that the lower bound on the inner radius of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq 1-\epsilon\}$ given in Proposition 1.6.7 is sharp.

Corollary 1.6.15. *For $\epsilon > 0$ sufficiently small, the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq 1-\epsilon\}$ has inner radius at most $C_1\epsilon^{1/2}L_1$, where $C_1 > 0$ is an absolute constant.*

Proof of Corollary 1.6.15: By Proposition 1.6.1, we know that $V(x,y)-\lambda \leq -c^*L_1^{-2}$ at the maximum of $u(x,y)$. Moreover, by Proposition 1.5.2, inside the level set $\{(x,y) \in \Omega : u(x,y) = 1/2\}$, we have the bound

$$V(x,y) - \lambda \leq CL_1^{-2}.$$

Since this level set has height comparable to L_1 in the y-direction and length comparable to L_2 in the x-direction, by the convexity of the potential, we have

$$V(x,y) - \lambda \leq -\frac{1}{2}c^*L_1^{-2} \tag{1.119}$$

on a region of height c_1L_1 and length c_2L_2 in the y and x-directions around the maximum.

Suppose that the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq 1-\epsilon\}$ has inner radius at least α, where $\alpha = C_1\epsilon^{1/2}L_1$ for some large absolute constant $C_1 > 0$. Then, this superlevel set contains a circle of radius α, and after a translation, centre at $(0,0)$.

Let $J_0(r)$ be the 0th Bessel function of the first kind for $r > 0$. This satisfies $J_0(0) = 1$, $J_0'(0) = 0$ and $J_0''(0) = -1/2$, as well as the equation

$$r^2J_0''(r) + rJ_0'(r) = -r^2J_0(r). \tag{1.120}$$

Setting $r^2 = x^2 + y^2$, we will use the comparison function

$$v(x,y) = (1+\epsilon)J_0(C\epsilon^{1/2}\alpha^{-1}r)$$

for $x^2 + y^2 \leq \alpha$. Here $C > 0$ is chosen so that $(1+\epsilon)J_0\left(C\epsilon^{1/2}\right) \leq 1-\epsilon$. This is possible for $\epsilon > 0$ sufficiently small, since for small r, $J_0(r)$ satisfies

$$J_0(r) = 1 - \tfrac{1}{2}r^2 + O(r^4).$$

In particular, this ensures that

$$v(x,y) \leq u(x,y), \tag{1.121}$$

for $x^2 + y^2 = \alpha^2$. By (1.120), the function $v(x,y)$ also satisfies the equation

$$\Delta_{x,y} v(x,y) = -\frac{C^2\epsilon}{\alpha^2} v(x,y).$$

Thus,

$$\Delta_{x,y} v(x,y) - (V(x,y) - \lambda)v(x,y) = -\frac{C^2\epsilon}{\alpha^2} v(x,y) - (V(x,y) - \lambda)v(x,y). \tag{1.122}$$

Provided that we take $\epsilon > 0$ sufficiently small, we can ensure from (1.119) that

$$V(x,y) - \lambda \leq -\frac{1}{2} c^* L_1^{-2}$$

for $x^2 + y^2 \leq \alpha^2$. Therefore, provided $\alpha = C_1 \epsilon^{1/2} L_1$ for C_1 sufficiently large, and $x^2 + y^2 \leq \alpha^2$, we have

$$-\frac{C^2\epsilon}{\alpha^2} - (V(x,y) - \lambda) \geq \frac{1}{4} c^* L_1^{-2} \geq 0,$$

and so from (1.122) we see that

$$\Delta_{x,y} v(x,y) - (V(x,y) - \lambda)v(x,y) \geq 0 \tag{1.123}$$

for $x^2 + y^2 \leq \alpha^2$. Combining (1.121) and (1.123), we can apply the generalised maximum principle from Proposition 1.4.6 to conclude that

$$v(x,y) \leq u(x,y)$$

whenever $x^2 + y^2 \leq \alpha^2$. However, $v(0,0) = 1 + \epsilon$, while $u(x,y) \leq 1$ everywhere, and so this gives us a contradiction. □

The second consequence of Proposition 1.6.1 is to improve the pointwise bound on $\partial_x u(x,y)$ from Proposition 1.6.6 in the case where $L_2 \gg L_1$.

Corollary 1.6.16. *There exists a constant $c > 0$ such that we have the derivative bound*

$$|\partial_x u(x,y)| \leq C L_2^{-1},$$

for an absolute constant C, for all (x,y) in a rectangle of side lengths cL_2 and cL_1 around the maximum of $u(x,y)$.

Proof of Corollary 1.6.16: From Corollary 1.6.15 above, the superlevel sets $\{(x,y) \in \Omega : u(x,y) \geq 1-\epsilon\}$ have inner radius bounded by $C\epsilon^{1/2}L_1$. Let the maximum of $u(x,y)$ be attained at $(0,0)$. Then, we saw in the proof of Corollary 1.6.15 that the sublevel set

$$\{(x,y) \in \Omega : V(x,y) - \lambda \leq -\tfrac{1}{2}c^* L_1^{-2}\},$$

contains a rectangle R, with centre at $(0,0)$, and of side lengths comparable to L_2 and L_1 in the x and y-directions. We then construct a set $U \subset \Omega$ as follows: It consists of the part of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq 1-\tilde{c}\}$ with x restricted to an interval of length L_2 around 0, and $\tilde{c} > 0$ sufficiently small so that U is contained within the middle half of the rectangle R.

The boundary of this set U then consists of parts of the upper and lower boundaries of the level set $\{(x,y) \in \Omega : u(x,y) = 1-\tilde{c}\}$, and two vertical lines with x fixed. Moreover, by choosing $\tilde{c}$ to be sufficiently small, U is contained between the two lines $y = \pm\frac{1}{2}c_1 L_1$. We then define a comparison function $W(x,y)$ by

$$W(x,y) = \frac{1}{c_2 L_2} \cosh\left(\frac{x \log(L_2/L_1)}{c_3 L_2}\right) \cos\left(\frac{\pi y}{2c_1 L_1}\right).$$

Here c_2 and c_3 are small absolute constants depending on c_1 that we will specify below. Firstly, we choose $c_2 > 0$ sufficiently small so that for all $|y| \leq c_1 L_1/2$, we have

$$W(x,y) \geq C_1 L_2^{-1}.$$

This absolute constant C_1 is chosen so that

$$|\partial_x u(x,y)| \leq W(x,y) \tag{1.124}$$

for all points (x,y) on the curved portion of ∂U consisting of part of the upper and lower boundaries of $\{(x,y) \in \Omega : u(x,y) = 1-\tilde{c}\}$. This is possible due to Proposition 1.6.6.

We now let $x = cL_2$, where $c > 0$ is chosen so that $x = \pm 2cL_2$ is contained in the projection of the set U onto the x-axis. Then, for all $|y| \leq c_1 L_1/2$, we have the lower bound

$$W(cL_2, y) \geq \frac{1}{2c_2 L_2} \cosh\left(\frac{c}{c_3}\log(L_2/L_1)\right) \geq \frac{1}{4c_2 L_2} \exp\left(\frac{c}{c_3}\log(L_2/L_1)\right) = \frac{1}{4c_2 L_2}\left(\frac{L_2}{L_1}\right)^{c/c_3}.$$

We can thus choose $c_3 > 0$ sufficiently small, depending on c only, so that

$$W(cL_2, y) \geq C_2 L_1^{-1}.$$

Here C_2 is chosen so that for $|x| \geq cL_2$, $(x, y) \in U$, we have

$$|\partial_x u(x, y)| \leq W(x, y). \tag{1.125}$$

This is possible due to Proposition 1.6.4.

The function $W(x, y)$ satisfies the equation

$$\Delta_{x,y} W(x, y) = \left(\left(\frac{\log(L_2/L_1)}{c_3 L_2}\right)^2 - \left(\frac{\pi^2}{4c_1^2 L_1^2}\right)\right) W(x, y) \leq -\frac{\pi^2}{8c_1^2 L_1^2} W(x, y), \tag{1.126}$$

provided L_2/L_1 is sufficiently large.

The first derivative $\partial_x u(x, y)$ satisfies

$$(-\Delta_{x,y} + V(x, y) - \lambda)\partial_x u(x, y) = -\partial_x V(x, y) u(x, y). \tag{1.127}$$

By the convexity of $V(x, y)$, we have the bound $|\partial_x V(x, y) u(x, y)| \leq C_3 L_2^{-1} L_1^{-2}$ for all $(x, y) \in U$. Also, $|V(x, y) - \lambda| \leq C_4 L_1^{-2}$.

We will apply the maximum principle to the functions

$$\Psi_{\pm}(x, y) \coloneqq ((\partial_x u)_{\pm}(x, y) + L_2^{-1})/W(x, y),$$

where $\pm$ signifies taking the positive or negative part of the function.

Let $U_{\pm}$ be any connected component of the support of $(\partial_x u)_{\pm}$ in U. Then, inside $U_{\pm}$, the

functions $\Psi_\pm(x,y)$ satisfy

$$\Delta_{x,y}\Psi_\pm(x,y) + 2\nabla_{x,y}\log W(x,y).\nabla_{x,y}\Psi_\pm(x,y) =$$
$$W(x,y)^{-1}\left(\Delta_{x,y}\left((\partial_x u)_\pm(x,y) + L_2^{-1}\right) - \left((\partial_x u)_\pm(x,y) + L_2^{-1}\right)W(x,y)^{-1}\Delta_{x,y}W(x,y)\right),$$

which by (1.126) and (1.127) implies

$$\Delta_{x,y}\Psi_\pm(x,y) + 2\nabla_{x,y}\log W(x,y).\nabla_{x,y}\Psi_\pm(x,y) \tag{1.128}$$
$$\geq W(x,y)^{-1}\left(\partial_x V(x,y)u(x,y) + (V(x,y)-\lambda)\left((\partial_x u)_\pm(x,y) + L_2^{-1}\right) + \tfrac{1}{8}\pi^2 c_1^{-2}L_1^{-2}((\partial_x u)_\pm(x,y) + L_2^{-1})\right).$$

By the bound above on $|\partial_x V(x,y)u(x,y)|$, provided $c_1 > 0$ is sufficiently small, the right hand side of (1.128) is ≥ 0. Combining this with the bounds from (1.124) and (1.125) on the boundary of U, we can apply the maximum principle to conclude that

$$|\partial_x u(x,y)| \leq W(x,y).$$

The function $W(x,y)$ satisfies $W(0,y) \leq CL_2^{-1}$, and we can repeat the argument above with $W(x,y)$ shifted by an amount comparable to L_2 in the x-direction. This gives us the required bound on $\partial_x u(x,y)$ and concludes the proof of the corollary. □

Chapter 2

The Dirichlet Laplacian on a three dimensional convex domain

2.1 Introduction and statement of results

In Section 1.3 when proving the bound on the first eigenvalue λ in Theorem 1.1.12, we made strong use of the properties of the eigenfunctions, $\psi_1^{(x)}(y)$, of the ordinary differential operator

$$\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x, y).$$

Also, in the work of Jerison and Grieser in [GJ1], [GJ2], [J1] on the Dirichlet eigenfunctions of two dimensional convex domains, they use an approximate separation of variables to relate the study of the two-dimensional eigenfunction to the study of related ordinary differential operators. As we remarked in the Introduction, one motivation for studying the operator

$$\mathcal{L} = -\Delta_{x,y} + V(x, y)$$

on the domain Ω with assumptions as in Definitions 1.1.1 and 1.1.3 is that one would expect it to be important in the study of the Dirichlet eigenvalues and eigenfunctions of three dimensional convex domains. In this section, we will give an example of how the work in the previous sections can be used when considering three dimensional domains.

That is, we will consider a class of bounded, three dimensional domains $D \subset \mathbb{R}^3$, and study the Dirichlet eigenfunctions of the Laplacian on this domain, making use of the results of the previous

sections.

As for the operator $\mathcal{L}$ from before, we have an increasing sequence of Dirichlet eigenvalues

$$\nu_1 < \nu_2 \leq \cdots \leq \nu_j \nearrow \infty,$$

with corresponding eigenfunctions $\psi_j(x, y)$ satisfying

$$\begin{cases} (\Delta_{x,y} + \nu_j)\psi_j(x, y, z) &= 0 \qquad \text{in } D \\ \psi_j(x, y, z) &= 0 \qquad \text{on } \partial D. \end{cases}$$

We will in particular study the first eigenvalue $\nu = \nu_1$ and corresponding eigenfunction $\psi(x, y, z) = \psi_1(x, y, z)$. Let us first describe the domains D that we will be considering.

Assumption 2.1.1 (The Domain D)**.** *The domain D is a closed, bounded, convex three dimensional domain in $\mathbb{R}^3$. Letting $E \subset \mathbb{R}^3$ be the John ellipsoid coming from Theorem 1.2.3, the minor axis of E has length comparable to 1, and the other two axes have lengths that are large compared to an absolute constant.*

We now rotate the domain D so that its projection onto the z-axis has the smallest length compared to the projection onto any other line. Then, we can write the domain as

$$D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D^*, f_1(x, y) \leq z \leq f_2(x, y)\} \tag{2.1}$$

where D^* is the two dimensional projection of D onto the (x, y)-plane. By the properties of the John ellipsoid E, we see that $D^* \subset \mathbb{R}^2$ has both inner radius and diameter large compared to an absolute constant. The functions $f_1(x, y)$ and $f_2(x, y)$ are convex and concave functions on D^* respectively. Moreover, after a translation and scaling, we can assume that

$$0 = \min_{D^*} f_1 \leq f_1(x, y) \leq f_2(x, y) \leq \max_{D^*} f_2 = \pi. \tag{2.2}$$

We can deduce some simple properties of the domain D.

Lemma 2.1.2. *For the functions $f_1(x, y)$ and $f_2(x, y)$ as above, we define the* height function $h(x, y)$ *by*

$$h(x, y) \coloneqq f_2(x, y) - f_1(x, y). \tag{2.3}$$

This is a concave function on D^*, which satisfies

$$0 \leq h(x,y) \leq \max_{D^*} h = \pi.$$

Proof of Lemma 2.1.2: The concavity of $h(x,y)$ is immediate from the convexity and concavity of the functions $f_1(x,y)$ and $f_2(x,y)$ respectively. By the properties of $f_1(x,y)$ and $f_2(x,y)$ in (2.2), to show that $\max_{D^*} h = \pi$, we need to show that there exists a point $(x^*, y^*) \in D^*$ with $f_1(x^*, y^*) = 0$ and $f_2(x^*, y^*) = \pi$.

Suppose that no such point (x^*, y^*) exists. Then, we can find the two closest points (x_1, y_1) and (x_2, y_2) in D^* such that $(x_1, y_1) \neq (x_2, y_2)$ and $f_1(x_1, y_1) = 0$, $f_2(x_2, y_2) = \pi$. Consider the line segment in D^* joining these two points and then let l be the perpendicular bisector of this line. Since (x_1, y_1) and (x_2, y_2) are the closest points with the above property, and D is convex, we see that $f_1(x,y)$ is strictly positive on one side of this line, and $f_2(x,y)$ is strictly smaller than π on the other side of this line. Thus, we can rotate D about the axis $l \times \{z = \pi/2\}$ to get a strictly smaller projection onto the z-axis. This gives us a contradiction since we are assuming that the projection of D onto the z-axis is of the smallest length. □

In particular, as a result of Lemma 2.1.2, setting

$$V(x,y) = \frac{\pi^2}{h(x,y)^2},$$

we see that $V(x,y)$ is a convex function on D^*, with $V(x,y)^{-1/2}$ concave, and it attains its minimum of 1. Therefore, the domain $\Omega = D^*$ and potential $V(x,y)$ satisfy the assumptions of Definitions 1.1.1 and 1.1.3. Thus, for the first eigenvalue λ and eigenfunction $u(x,y)$ satisfying

$$\begin{cases} (-\Delta_{x,y} + V(x,y))u(x,y) &= \lambda u(x,y) \qquad \text{in } D^* \\ u(x,y) &= 0 \qquad \text{on } \partial D^*, \end{cases}$$

all the results of the previous sections hold. For example, we can use Definitions 1.1.4 and 1.1.7 to find values for the length scales L_1 and L_2 associated to the domain D^* and potential $V(x,y)$.

We are interested here in studying the cases where we can use this two dimensional eigenvalue problem to study the Dirichlet problem on the three dimensional domain D. To do this we make some further assumptions on the domain D, and the associated height function $h(x,y)$.

Assumption 2.1.3. *With the function $h(x,y)$ defined on D^* as in* (2.3)*, we assume that for $(x,y) \in$*

D^* we have

$$|\partial_x h(x,y)| = |\partial_x f_1(x,y)| + |\partial_x f_2(x,y)|, \qquad |\partial_y h(x,y)| = |\partial_y f_1(x,y)| + |\partial_y f_2(x,y)|.$$

Let $\delta > 0$ be a fixed small constant. Then, we also assume that for the parameters L_1 and L_2 coming from Definitions 1.1.4 and 1.1.7 we have the upper bound on L_2 of the form

$$L_2 \leq C L_1^{3/2-\delta},$$

for an absolute constant $C > 0$.

Before continuing, let us motivate the assumptions made above. The only purpose of the assumption on the first derivatives of $h(x,y)$, $f_1(x,y)$ and $f_2(x,y)$ is to ensure that the first derivatives of $f_1(x,y)$ and $f_2(x,y)$ are controlled by the corresponding first derivatives of $h(x,y)$. The assumption on the size of L_2 is motivated by the following:

We recall that $\mu(x)$ is the first eigenvalue of the ordinary differential operator

$$\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x,y),$$

and L_2 is the largest value such that $\mu(x)$ increases by L_2^{-2} from its minimum on an interval I of length at least L_2. For each x, let

$$D(x) \coloneqq \{(y,z) \in \mathbb{R}^2 : (x,y,z) \in D\} \tag{2.4}$$

be the two dimensional cross-section of the domain D at the point x. We can then define the function $\tilde{\mu}(x)$ to be the first Dirichlet eigenvalue of this two dimensional convex domain $D(x)$. We have the following lemma relating the functions $\mu(x)$ and $\tilde{\mu}(x)$.

Lemma 2.1.4. *For each x, let $L(x)$ be the largest value such that $V(x,y)$ increases from its minimum by $L(x)^{-2}$ on an interval of length at least $L(x)$. Then, we have the bound*

$$\mu(x) \leq \tilde{\mu}(x) \leq \mu(x) + C_1 L(x)^{-3},$$

for an absolute constant C_1.

Proof of Lemma 2.1.4: The function $\tilde{\mu}(x)$ is the first eigenvalue of the two dimensional domain $D(x)$. For each fixed x, this convex domain has height function $h(x,y)$. Thus, since $\mu(x)$ is the first

eigenvalue of the associated ordinary differential operator $\mathcal{L}(x)$, the lemma is exactly the statement of the first part of Theorem A in [J1]. □

For those x in the interval I around the point where $\mu(x)$ attains its minimum, we know that the length scale $L(x)$ is comparable to L_1. Therefore, for these values of x, Lemma 2.1.4 implies that

$$\mu(x) \leq \tilde{\mu}(x) \leq \mu(x) + C_1 L_1^{-3}. \tag{2.5}$$

Analogously to L_2, let $\tilde{L}_2$ be the largest value such that $\tilde{\mu}(x)$ increases by $\tilde{L}_2^{-2}$ on an interval of length at least $\tilde{L}_2$. Under the restriction on the size of L_2 in Assumption 2.1.3, we see that

$$L_2^{-2} \geq C^2 L_1^{-3+2\delta},$$

and by (2.5), the functions $\mu(x)$ and $\tilde{\mu}(x)$ are within $C_1 L_1^{-3}$ of each other. This implies that the length scales L_2 and $\tilde{L}_2$ are comparable, and so it is natural in this case to expect that the two dimensional operator $\mathcal{L}$ will be useful in studying the three dimensional eigenfunction.

We can now state two results for this class of three dimensional convex domains:

Theorem 2.1.5. *Let D be a three dimensional convex domain as above, let L_1 and L_2 be the values from Definitions 1.1.4 and 1.1.7, and suppose that the additional assumptions from Assumption 2.1.3 hold. Fix a small constant $c_1 > 0$, and consider a constant c with $c_1 < c < 1 - c_1$. Then, the level set $\{(x, y, z) \in D : \psi(x, y, z) = c\}$ has the following shape: After the rotation of D^* given in the definition of L_2, there exists an ellipsoid E with axes lying along the coordinate axes of lengths comparable to L_2, L_1 and 1 in the x, y and z-directions respectively, such that E is contained inside this level set, and a dilate of E, with scaling factor bounded by an absolute constant, contains this level set.*

As for the two dimensional case, the key step in establishing such a theorem concerning the level sets of $\psi(x, y, z)$ is to obtain a sufficiently sharp eigenvalue bound for ν:

Theorem 2.1.6. *Let D be a convex three dimensional domain as above, with associated height function $h(x, y)$, and for $V(x, y) = \pi^2 h(x, y)^{-2}$, let*

$$\mathcal{L} = -\Delta_{x,y} + V(x, y)$$

be the associated two dimensional operator on D^* with first eigenvalue λ. For L_1 and L_2 as in Definitions 1.1.4 and 1.1.7, suppose that the additional assumptions in Assumption 2.1.3 hold. Then,

we have a bound on the first Dirichlet eigenvalue of D, ν, of

$$\lambda \leq \nu \leq \lambda + C_1 L_1^{-3+\delta/2},$$

for an absolute constant C_1, depending only on $\delta > 0$.

Remark 2.1.7. *In particular, we see that under the assumption on the relative sizes of L_1 and L_2 from Assumption 2.1.3, Theorem 2.1.6 locates the eigenvalue ν up to a precision that is within L_2^{-2}.*

Before proving these theorems, let us first use it to provide more motivation for the upper bound placed on L_2 in Assumption 2.1.3: As before, we let μ be the first eigenvalue of the ordinary differential operator

$$\mathcal{A} = -\frac{d^2}{dx^2} + \mu(x),$$

and we recall from Theorem 1.1.12 that we have a bound on λ of the form

$$\mu \leq \lambda \leq \mu + C L_2^{-2}.$$

By Lemma 4.2 (e) in [J1], the operator $\mathcal{A}$ has spectral gap bounded from below by cL_2^{-2}, for an absolute constant $c > 0$. That is, letting μ_2 be the second eigenvalue of $\mathcal{A}$, we have

$$\mu_2 - \mu > cL_2^{-2}.$$

Therefore, the bound on λ from Theorem 1.1.12 locates λ to a precision which is comparable to the spectral gap of $\mathcal{A}$. Since the first eigenfunction of $\mathcal{L}$, $u(x, y)$, decays in the x-directions on a length scale comparable to L_2, we also expect that the spectral gap of $\mathcal{L}$ should be comparable to L_2^{-2}. We can prove a simple lower bound on this spectral gap:

Proposition 2.1.8. *Let the domain Ω and potential $V(x, y)$ be as in Definitions 1.1.1 and 1.1.3, and let $\lambda_1 = \lambda$ and λ_2 be the first two eigenvalues of the operator $\mathcal{L}$. Then, given $\delta > 0$, there exists an constant $c > 0$ such that*

$$\lambda_2 - \lambda_1 \geq cL_2^{-2-\delta/2}.$$

Proof of Proposition 2.1.8: We recall that L_2 is the largest value such that the eigenvalue $\mu(x)$ increases by L_2^{-2} from its minimum on an interval I of length at least L_2. Let us consider a

subdomain, $\tilde{\Omega}$, of Ω which contains those points in Ω which are within a distance of $L_2^{1+\delta/4}$ of I in the x-direction, and within a distance of $L_2^{1+\delta/4}$ of the projection of the set $\{(x,y) \in \Omega : V(x,y) \leq 1 + L_1^{-2}\}$ onto the y-axis. We also define the potential $\tilde{V}(x,y)$ on $\tilde{\Omega}$ to be the restriction of $V(x,y)$ to $\tilde{\Omega}$, and this allows us to define an operator $\tilde{\mathcal{L}}$ in the natural way

By domain comparison, if we let $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ be the first two eigenvalues of $\tilde{\mathcal{L}}$, then

$$\lambda_1 \leq \tilde{\lambda}_1 \qquad \text{and} \qquad \lambda_2 \leq \tilde{\lambda}_2. \tag{2.6}$$

The domain $\tilde{\Omega}$ has diameter at most comparable to $L_2^{1+\delta/4}$. Therefore, by the fundamental gap conjecture, established in Corollary 1.4 in [AC], we have the lower bound

$$\tilde{\lambda}_2 - \tilde{\lambda}_1 \geq 2cL_2^{-2-\delta/2}. \tag{2.7}$$

To conclude the proof of the proposition, we need to show that $\lambda_2 - \lambda_1$ has a comparable lower bound. Using (2.6), we see from (2.7) that

$$\tilde{\lambda}_2 - \lambda_1 \geq \tilde{\lambda}_2 - \tilde{\lambda}_1 \geq 2cL_2^{-2-\delta/2}, \tag{2.8}$$

and so we want to compare λ_2 with $\tilde{\lambda}_2$.

Using the same method of proof as in the eigenvalue bound in Theorem 1.1.12, it is straightforward to show that the second eigenvalue λ_2 satisfies

$$\lambda_2 \leq \mu + CL_2^{-2}.$$

This is because we can use the test functions $\chi_1(x)\psi_1^{(x)}(y)$ and $\chi_2(x)\psi_1^{(x)}(y)$ in the variational formulation of the second eigenvalue. Here $\chi_1(x)$ and $\chi_2(x)$ are cutoff functions, with disjoint supports on intervals of length comparable to L_2, where these intervals are contained within the interval I on which $\mu(x)$ is within L_2^{-2} of its minimum.

Let $u_2(x,y)$ be the eigenfunction on Ω with eigenvalue λ_2. As a result of the upper bound on λ_2, we can replicate the proof of Proposition 1.4.4 to show that the L^2-norm on the cross-section,

$$\int_{\Omega(x)} u_2(x,y)^2 \, \mathrm{d}y$$

decays from its maximum on a length scale comparable to L_2. We can also apply the Agmon estimate

from Proposition 1.4.16 to show that the L^2-norm of $u_2(x,y)$ decays on a length scale comparable to L_1 as we move away from the region where $V(x,y) \leq 1+L_1^{-2}$ in the y-direction. In particular, by the definition of the subdomain $\tilde{\Omega}$, this means that the L^2-norm of $u_2(x,y)$ in $\Omega \backslash \tilde{\Omega}$ has an exponentially small bound of $e^{-cL_2^{\delta/4}}$. Therefore, it is easy to see that this gives an upper bound on $\tilde{\lambda}_2$ of

$$\tilde{\lambda}_2 \leq \lambda_2 + Ce^{-cL_2^{\delta/4}}.$$

Using this bound in (2.8) shows that

$$\lambda_2 - \lambda_1 \geq cL_2^{-2-\delta/2}$$

as required. □

By the upper bound on L_2 from Assumption 2.1.3, Theorem 2.1.6 implies that ν is restricted to an interval around λ that is small compared to $L_2^{-2-\delta/2}$. In other words, by Proposition 2.1.8, ν lies in an interval around λ which is small compared to the spectral gap of the operator $\mathcal{L}$. Therefore, again Assumption 2.1.3 seems natural if we want to use the eigenvalues and eigenfunctions of $\mathcal{L}$ to study the three dimensional domain case.

In the next section we will establish the eigenvalue bound on ν from Theorem 2.1.6, and then we will use it to establish the shape of the level sets of $\psi(x,y,z)$ from Theorem 2.1.5 in a similar way to how we established the shape of the level sets of the two dimensional eigenfunction $u(x,y)$.

2.2 A bound on the first eigenvalue ν

We now begin the proof of Theorem 2.1.6.

Proof of Theorem 2.1.6:

The idea of the proof of this eigenvalue bound is similar to the proof of Theorem 1.1.12. We will use the variational formulation of the first eigenvalue,

$$\nu = \inf \left\{ \frac{\int_D |\nabla \Psi(x,y,z)|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{\int_D \Psi(x,y,z)^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z} \middle| \Psi \in W^{1,2}(D), \Psi|_{\partial D} = 0, \Psi \not\equiv 0 \right\} \tag{2.9}$$

and so it is necessary to consider appropriate test functions $\Psi(x,y,z)$.

As usual, the lower bound in Theorem 2.1.6 is more straightforward so we prove it first.

Proposition 2.2.1. *We have a lower bound on ν of the form*

$$\lambda \leq \nu.$$

Proof of Proposition 2.2.1: Any test function $\Psi(x,y,z) \in W^{1,2}(D)$ which vanishes on ∂D satisfies

$$\Psi(x,y,f_1(x,y)) = \Psi(x,y,f_2(x,y)) = 0$$

for all $(x,y) \in D^*$. Therefore, for each fixed (x,y), $\Psi(x,y,\cdot)$ is an admissible test function in the variational formulation of the first eigenvalue $V(x,y) = \pi^2 h(x,y)^{-2}$ for the interval $[f_1(x,y), f_2(x,y)]$. Thus,

$$\int_{f_1(x,y)}^{f_2(x,y)} \left(\partial_z \Psi(x,y,z)\right)^2 \, \mathrm{d}z \geq V(x,y) \int_{f_1(x,y)}^{f_2(x,y)} \Psi(x,y,z)^2 \, \mathrm{d}z.$$

Integrating this over $(x,y) \in D^*$, and using the variational formulation for λ from (1.17) gives the required lower bound, in an analogous way to the proof of Proposition 1.3.1.

□

We now turn to proving the upper bound on ν, which will complete the proof of Theorem 2.1.6.

Proposition 2.2.2. *For $\delta > 0$ as in Assumption 2.1.3, we have an upper bound on ν of the form*

$$\nu \leq \lambda + C_1 L_1^{-3+\delta/2},$$

where C_1 is an absolute constant, depending only on δ.

Proof of Proposition 2.2.2: We will again use the variational formulation for ν from (2.9), and to do this we need to choose the appropriate test function $\Psi(x,y,z)$. We recall that $u(x,y)$ is the first eigenfunction of the two dimensional operator $\mathcal{L}$, and we make the following definition.

Definition 2.2.3. *For each $(x,y) \in D^*$, we define $e_1(x,y,\cdot)$ to be the L^∞-normalised first Dirichlet eigenfunction for the interval $[f_1(x,y), f_2(x,y)]$. In other words,*

$$e_1(x,y,z) \coloneqq \sin\left(\beta(x,y,z)\right),$$

where

$$\beta(x,y,z) = \frac{\pi(z - f_1(x,y))}{h(x,y)}.$$

We want to use an approximate separation of variables and define the function $\Psi(x,y,z)$ to be the product of the two dimensional eigenfunction $u(x,y)$ and this one dimensional eigenfunction $e_1(x,y,z)$ in the z-direction. However, we lose control over the size of the potential $V(x,y)$ and its first derivatives when we are at a large distance from the region where $V(x,y) \leq 1 + L_1^{-2}$. Therefore, to construct $\Psi(x,y,z)$ we will also multiply this product by a cutoff function, and to do this we need more information about the behaviour of $u(x,y)$ near the boundary of D^*.

2.2.1 Estimates on $u(x,y)$ near the boundary of the domain

Lemma 2.2.4. *Let $(x_1,y_1) \in \partial D^*$. After a translation and rotation, we may assume that this point is the origin, the tangent to ∂D^* lies along the x-axis, and that D^* lies above the tangent line. Then, defining the function $R(y)$ by*

$$R(y) \coloneqq \sin\left(\frac{\pi y}{cL_1}\right)$$

for $c > 0$ sufficiently small, we have the bound

$$u(x,y) \leq R(y)$$

for all $y \leq cL_1/2$.

In particular, denoting ∂_n to be the normal derivative on ∂D^*, we have

$$|\partial_n u(x,y)| \leq C_1 L_1^{-1},$$

for all $(x,y) \in \partial D^*$.

Proof of Lemma 2.2.4: Viewing the function $R(y)$ as a function of the two variables (x,y), we see that

$$(-\Delta_{x,y} + V(x,y) - \lambda)\, R(y) = \frac{\pi^2}{c^2 L_1^2} R(y) + (V(x,y) - \lambda) R(y).$$

As we have used many times before, we know that $\lambda - V(x,y) \leq C L_1^{-2}$, and so for $c > 0$

sufficiently small we have

$$(-\Delta_{x,y} + V(x,y) - \lambda)\, R(y) > 0.$$

Moreover, $R(0) = 0 = u|_{\partial D^*}$ and $R(cL_1/2) = 1 = \max_{D^*} u$. Therefore, we can apply the generalised maximum principle from Proposition 1.4.6 to conclude that for $(x,y) \in D^*$ with $0 \leq y \leq cL_1/2$, we have

$$u(x,y) \leq R(y)$$

as required. Since $u(x,y) = 0$ on ∂D^* and u is positive inside D^*, the bound on the normal derivative follows immediately. □

As a result of Lemma 2.2.4, we see that the function $u(x,y)$ decays to 0 at the boundary of D^* in at least a linear fashion with slope comparable to L_1^{-1}. This means that for those points $(x,y) \in D^*$ that are at a distance comparable to 1 from the boundary of D^* we have the bound

$$u(x,y) \leq C_1 L_1^{-1}. \tag{2.10}$$

Remark 2.2.5. *When we use the estimate in* (2.10) *below, we will combine it with the bound*

$$|\nabla_{x,y} h(x,y)| \leq \tfrac{1}{\pi},$$

for those (x,y) *at a distance of at least* 1 *from* ∂D^*. *This bound holds since* $h(x,y)$ *is a concave function, and is restricted to taking values between* 0 *and* π.

We now want to prove a similar estimate for those points (x,y) which satisfy

$$\max_{D^*} h - h(x,y) = \pi - h(x,y) \geq c.$$

Lemma 2.2.6. *At the point* $(x^*, y^*) \in D^*$, *suppose that the height function is strictly smaller than its maximum in such a way that we have the lower bound*

$$\pi - h(x^*, y^*) \geq c.$$

Then, we have the pointwise bound

$$u(x^*, y^*) \leq CL_1^{-1},$$

for an absolute constant $C > 0$.

Proof of Lemma 2.2.6: After a translation and rotation, we may assume that the line joining the point (x^*, y^*) to the closest point on the level set $\{(x, y) \in D^* : h(x, y) = \pi - \frac{1}{2}c\}$ lies in the vertical y-direction. Letting (x^*, b^*) be this point, we can also assume that $b^* < y^*$. We also set (x^*, b) to be the point on ∂D^* with $y^* < b$.

If we have the upper bound $b - y^* \leq C$, then we are done by Lemma 2.2.4 above. Therefore, we will assume that $b - y^* \geq C$, for a large absolute constant $C > 0$. Since $h(x, y)$ is a concave function between 0 and π, we have the lower bound

$$y^* - b^* \geq \tilde{c}M, \tag{2.11}$$

where $M \coloneqq b - b^*$ and $\tilde{c} > 0$ is an absolute constant depending on c. To establish the required bound on $u(x^*, y^*)$ we will again use the generalised maximum principle on the cone-shaped subset of D^* given by

$$D^{(R)} \coloneqq \{(r, s) \in D^* : b^* < s < b, |r - x^*| \leq s - b^*\}.$$

We will use the comparison function

$$R(r, s) \coloneqq C_1 e^{-c_1(s-b^*)} \cosh\left(c_1(r - x^*)\right) \tag{2.12}$$

in $D^{(R)}$. Here the constant $c_1 > 0$ will be chosen to be small relative to the constant c in the statement of the lemma. By choosing C_1 to be large, the smooth function $R(r, s) \geq 1$ is chosen with the following properties: It satisfies

$$R(r, s) \geq C_2, \tag{2.13}$$

for some constant $C_2 > 0$ whenever $|r - x^*|$ is comparable to $s - b^*$, while $R(r, s)$ is equal to

$C_1 e^{-c_1(s-b^*)}$ when $r = x^*$. We also see that

$$|\Delta_{r,s} R(r,s)| \leq C c_1^2 R(r,s)$$

everywhere. In particular, provided we choose c_1 to be sufficiently small compared to c, we have the inequality

$$(\Delta - V(r,s) + \lambda) R(r,s) \leq \left(C c_1^2 - V(r,s) + \lambda\right) R(r,s) < 0, \tag{2.14}$$

whenever $\pi - h(r,s) \geq \frac{c}{2}$. In particular, since $\pi - h(r,s) > \frac{c}{2}$ for all $(r,s) \in D^{(R)}$, the inequality in (2.14) holds for all $(r,s) \in D^{(R)}$. By the property given in (2.13), on the boundary of $D^{(R)}$ we have

$$u(r,s) \leq C R(r,s) \max_{\partial D^{(R)}} u.$$

Combining this with (2.14), we see that we can apply the generalised maximum principle from Proposition 1.4.6 to conclude that

$$u(r,s) \leq C R(r,s) \max_{\partial D^{(R)}} u$$

inside $D^{(R)}$. Setting $(r,s) = (x^*, y^*)$ and using (2.11), this shows that

$$u(x^*, y^*) \leq C e^{-c_1(y^* - b^*)} \max_{\partial D^{(R)}} u \leq C e^{-\tilde{c} c_1 M} \max_{\partial D^{(R)}} u.$$

Finally, using Lemma 2.2.4, we see that

$$\max_{\partial D^{(R)}} u \leq C M L_1^{-1},$$

and so

$$u(x^*, y^*) \leq C e^{-\tilde{c} c_1 M} M L_1^{-1} \leq C L_1^{-1}$$

independent of the value of M. □

Before constructing the test function $\Psi(x,y,z)$ for the variational formulation of ν, we also need to consider the decay of the two dimensional eigenfunction $u(x,y)$ as we move a large distance from its maximum in the x-direction.

Lemma 2.2.7. *Fix a small absolute constant $\alpha > 0$, and let $u(x, y)$ attain its maximum at the point (x^*, y^*). Then, for any t with $|t - x^*| \geq 2L_2 L_1^{\alpha}$ we have*

$$\int_{t-1}^{t+1} \int_{D(x)} u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C \exp\left(-cL_1^{\alpha}\right)$$

and

$$\int_{t-1}^{t+1} \int_{D(x)} \left|\nabla_{x,y} u(x,y)\right|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C \exp\left(-cL_1^{\alpha}\right),$$

for absolute constants c and C. Here $D(x)$ is the cross-section of the two dimensional domain D^* at the point x. In particular, these exponentially small bounds are small compared to any negative power of L_1.

Proof of Lemma 2.2.7: Combining Corollary 1.4.7 and Proposition 1.4.13, we see that

$$\int_{D(x)} u(x,y)^2 \, \mathrm{d}y \leq C_1 L_1 \exp\left(-c_1 |x - x^*| / L_2\right),$$

and so since $|x - x^*| \geq L_2 L_1^{\alpha}$ for all $x \in [t-1, t+1]$, this gives us the desired bound on the eigenfunction itself. To obtain the analogous bound on the first derivatives, we replicate the proof of Corollary 1.4.11 with a cutoff function supported around the interval $[t-1, t+1]$, which shows that the L^2-norm of the first derivative is controlled by the L^2-norm of the eigenfunction itself. $\square$

We now define the cutoff function which will appear in the test function $\Psi(x, y, z)$.

Definition 2.2.8. *Let $\xi(x, y)$ be a smooth cutoff function on D^* with the following properties: It is supported at a distance of at least 1 from the boundary of D^*, and it is also supported in the region $\{(x, y) \in D^* : h(x, y) \geq \frac{1}{4}\}$. Also, with x^* as in Lemma 2.2.7 above, we define $\xi(x, y)$ to be equal to 0 whenever $|x - x^*| \geq L_2 L_1^{\alpha}$, for a given small constant $\alpha > 0$. We then define $\xi(x, y)$ to be equal to its maximum of 1 at a distance of 1 from the boundary of its support, and that its first two derivatives are bounded by absolute constants. In particular, since $h(x, y)$ is a concave function, taking values between 0 and π, we can ensure that if $|x - x^*| \leq L_2 L_1^{\alpha}$, (x, y) is a distance of at least 2 from the boundary and $h(x, y) \geq \frac{1}{2}$, then $\xi(x, y)$ is equal to 1.*

Combining Lemmas 2.2.4, 2.2.6 and 2.2.7, we in particular see that the cutoff function $\xi(x, y)$ is chosen to ensure that $u(x, y)$ is bounded by CL_1^{-1} on the support of the derivatives of $\xi(x, y)$. This bound also holds for all points within a distance comparable to 1 of the support of the derivatives

of $\xi(x,y)$. Since $h(x,y)$ is bounded from below for all such points, then by elliptic regularity, we also obtain a bound of CL_1^{-1} on the first derivatives $|\nabla_{x,y}u(x,y)|$ on the support of the derivatives of $\xi(x,y)$.

2.2.2 The test function $\Psi(x,y,z)$ and its properties

We can now define $\Psi(x,y,z)$.

Definition 2.2.9. *For $e_1(x,y,z)$ as in Definition 2.2.3, and $\xi(x,y)$ as above, we define the test function $\Psi(x,y,z)$ by*

$$\Psi(x,y,z) \coloneqq \xi(x,y)u(x,y)e_1(x,y,z) = \xi(x,y)u(x,y)\sin\left(\beta(x,y,z)\right).$$

Using the variational formulation for ν from (2.9), we get the upper bound

$$\nu \leq \frac{\int_D |\nabla\Psi(x,y,z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\int_D \Psi(x,y,z)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}. \tag{2.15}$$

Therefore, to obtain a bound on ν we need to obtain an upper bound on the numerator and a lower bound on the denominator in (2.15). The lower bound is more straightforward, so we first consider the denominator

$$\int_D \Psi(x,y,z)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_D \xi(x,y)^2u(x,y)^2\sin^2\left(\beta(x,y,z)\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

By Theorem 1.1.9, the eigenfunction $u(x,y)$ is bounded below by an absolute constant on a region of height comparable to L_1 in the y-direction, and length comparable to L_2 in the x-direction. On this region the potential $V(x,y) = \pi^2 h(x,y)^{-2}$ is bounded above by $1 + CL_1^{-2}$, and so in particular the height function $h(x,y)$ is bounded below by an absolute constant. Moreover, the cutoff function $\xi(x,y)$ is equal to 1 on this region. Thus, we have the lower bound

$$\int_D \Psi(x,y,z)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \geq c_1L_1L_2, \tag{2.16}$$

for an absolute constant $c_1 > 0$.

Let us now consider the numerator of (2.15). Integrating by parts we see that

$$\int_D |\nabla\Psi(x,y,z)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\int_D \left(\Delta_{x,y,z}\Psi(x,y,z)\right)\Psi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{2.17}$$

We calculate the second derivatives of $\Psi(x,y,z)$ in the above integral, and integrate by parts in any term which contains a second derivative of the function $\beta(x,y,z)$ in x or y, in an analogous way to the proof of the upper bound in Theorem A in [J1]. Doing this, we find that the right hand side of (2.17) equals

$$\begin{aligned}
&\int_D \xi(x,y)^2 u(x,y) \left\{ -\Delta_{x,y} u(x,y) + \frac{\pi^2}{h(x,y)^2} u(x,y) \right\} \sin^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&+ \int_D \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y}\beta(x,y,z)\right|^2 \cos^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&- 2\int_D \xi(x,y) u(x,y) \nabla_{x,y}\xi(x,y).\nabla_{x,y} u(x,y) \sin^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&- \int_D \xi(x,y) u(x,y)^2 \Delta_{x,y}\xi(x,y) \sin^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (2.18)
\end{aligned}$$

We consider each of the four integrals in (2.18) in turn: By the eigenfunction equation for $u(x,y)$, we can write the first integral as

$$\lambda \int_D \xi(x,y)^2 u(x,y)^2 \sin^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \lambda \int_D \Psi(x,y,z)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (2.19)$$

Before bounding the remaining three integrals let us note that

$$\int_{f_1(x,y)}^{f_2(x,y)} \sin^2\left(\beta(x,y,z)\right) \,\mathrm{d}z = \int_{f_1(x,y)}^{f_2(x,y)} \cos^2\left(\beta(x,y,z)\right) \,\mathrm{d}z = \tfrac{1}{2} h(x,y) \leq \tfrac{1}{2}\pi. \qquad (2.20)$$

The integrands in the third and fourth integrals in (2.18) both contain at least one derivative of the cutoff function $\xi(x,y)$. By Definition 2.2.8, the derivatives of $\xi(x,y)$ are supported on the following two regions: The first is the region of height comparable 1 in the y-direction for each fixed x, and length comparable to $L_1^\alpha L_2$ in the x-direction, which is either within a distance of at least 2 from the boundary, or else $h(x,y) \leq \frac{1}{2}$. Here, using Lemmas 2.2.4 and 2.2.6 together with elliptic estimates, we see that we have the bounds

$$u(x,y) \leq C_1 L_1^{-1}, \qquad \left|\nabla_{x,y} u(x,y)\right| \leq C_1 L_1^{-1}.$$

Therefore, this region gives a contribution of $C L_1^{-2+\alpha} L_2$ to the two finals integrals in (2.18).

The other region is where $|x - x^*| > L_1^\alpha L_2$, and by Lemma 2.2.7 this region gives an exponentially small contribution of $\exp(-cL_1^\alpha)$ to the third and fourth integrals. Thus, we obtain a bound on the

third and fourth integrals in (2.18) of the form

$$CL_1^{-2+\alpha}L_2. \tag{2.21}$$

We finally need to study the second integral in (2.18):

$$\int_D \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y}\beta(x,y,z)\right|^2 \cos^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Recalling that

$$\beta(x,y,z) = \frac{\pi(z - f_1(x,y))}{h(x,y)},$$

we see that

$$\partial_x\beta(x,y,z) = \frac{-\pi\partial_x f_1(x,y)}{h(x,y)} - \frac{\partial_x h(x,y)}{h(x,y)}\beta(x,y,z),$$

and we have a similar equation for $\partial_y\beta(x,y,z)$. For all $z \in [f_1(x,y), f_2(x,y)]$, we know that $|\beta(x,y,z)| \leq \pi$, and by Assumption 2.1.3, the first derivative $\partial_x f_1(x,y)$ is controlled by $\partial_x h(x,y)$. Thus, we can bound the second integral in (2.18) by

$$C_1\int_D \xi(x,y)^2 u(x,y)^2 \left(\frac{|\partial_x h(x,y)| + |\partial_y h(x,y)|}{h(x,y)}\right)^2 \cos^2\left(\beta(x,y,z)\right) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

which by (2.20) is bounded by

$$C_1\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y}h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{2.22}$$

Here, we have also used the lower bound $h(x,y) \geq \frac{1}{4}$ on the support of $\xi(x,y)$.

Using the bounds in (2.19), (2.21) and (2.22), we see that

$$\begin{aligned}\int_D \left|\nabla\Psi(x,y,z)\right|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq \lambda\int_D \Psi(x,y,z)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ + C_1\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y}h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y + CL_1^{-2+\alpha}L_2.\end{aligned}$$

Therefore, combining this with the lower bound

$$\int_D \Psi(x,y,z)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \geq c_1 L_1 L_2,$$

from (2.16), we can use (2.15) to obtain an upper bound on ν of the form

$$\nu \leq \lambda + CL_1^{-1}L_2^{-1}\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y} h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y + CL_1^{-3+\alpha}. \tag{2.23}$$

In particular, provided we choose $\alpha > 0$ so that $\alpha < \delta/2$, the final term in the right hand side of (2.23) has the sufficient smallness required by Proposition 2.2.2. Thus, to complete the proof of Proposition 2.2.2 it is enough to show that

$$L_1^{-1}L_2^{-1}\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y} h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \leq CL_1^{-3+\delta/2}.$$

We will prove the following proposition:

Proposition 2.2.10. *For the cutoff function $\xi(x,y)$ as in Definition 2.2.8, we have the bound*

$$\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y} h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \leq CL_2 L_1^{-2+2\alpha}. \tag{2.24}$$

Let us assume that Proposition 2.2.10 holds. Then, provided that we choose $\alpha > 0$ such that $2\alpha < \delta/2$, (2.23) implies that

$$\nu \leq \lambda + CL_1^{-3+\delta/2},$$

and this completes the proof of Proposition 2.2.2. □

Proof of Proposition 2.2.10: Therefore, to finish the proof of the upper bound on the eigenvalue ν in Proposition 2.2.2 we need to establish the bound in Proposition 2.2.10. Let us begin by discussing the estimate in (2.24). We recall that the potential $V(x,y) = \pi^2 h(x,y)^{-2}$ is a convex function with a minimum of 1, and increases by L_1^{-2} in a region of inner radius at least L_1. In particular, as we have seen before, this means that

$$\left|\nabla_{x,y} h(x,y)\right| \leq CL_1^{-3}, \tag{2.25}$$

whenever (x,y) is at a distance of at least L_1 inside this region. Thus, since $u(x,y)$ has a maximum

of 1, and decays on length scales comparable to L_2 and L_1 in the x and y-directions, this region gives a contribution to the integral in (2.24) of

$$CL_1L_2L_1^{-3} = CL_1^{-2}L_2,$$

which is sufficiently small. However, as we move away from this region, we no longer have the same bound on $|\nabla_{x,y}h(x,y)|$ as in (2.25), and so we cannot immediately conclude the proof of Proposition 2.2.15. Instead we need to show that as we move away from the region where we have this bound on the derivatives, $|\nabla_{x,y}h(x,y)|$, the eigenfunction $u(x,y)$ also decays from its maximum of 1.

Therefore, our strategy for proving the estimate in (2.24) is to decompose the two dimensional domain D^* into a series of subdomains and estimate the contribution to (2.24) from each of these subdomains individually. We will choose these domains based on the size of the function $|\nabla_{x,y}h(x,y)|$.

The motivation for considering this function is twofold. First, we see that the square of this function appears in the integrand in (2.24). The second reason is due to the following lemma, which is an immediate consequence of the concavity of $h(x,y)$.

Lemma 2.2.11. *Let (x,y) be in the support of the cutoff function $\xi(x,y)$. In particular, (x,y) is at a distance of at least 1 from the boundary of D^*. Then, we have the bound*

$$|\nabla_{x,y}h(x,y)| \leq \tfrac{1}{\pi}.$$

We now use this function to construct subdomains $D_k \subset D^*$.

Definition 2.2.12. *For each $k \in \mathbb{Z}$, we define a subset of D^* by*

$$D_k \coloneqq \{(x,y) \in D^* : \xi(x,y) \neq 0, |\nabla_{x,y}h(x,y)| \in [2^{-k}, 2^{-k+1}]\}. \tag{2.26}$$

We note that by Lemma 2.2.11, the sets D_k are only non-empty for $k \geq -C$, for an absolute constant $C > 0$. We will use the sets D_k to study the decay of the eigenfunction $u(x,y)$ away from its maximum, in a similar way to how we used the intervals J_k in the proof of Theorem 1.1.12 to study the decay of the eigenfunctions $\psi_1^{(x)}(y)$ of the ordinary differential operator $\mathcal{L}(x)$.

Also, since $u(x,y)$ decays from its maximum on a length scale comparable to L_1 in the y-direction, and $h(x,y)$ changes by L_1^{-2} on this region, in general we only expect $u(x,y)$ to decay once $\partial_y h(x,y)$ is large compared to L_1^{-3}. Therefore, we also restrict out attention to those values of k such that

$2^{-k} \geq L_1^{-3}$, and so we only consider those k with

$$k \geq -C \text{ and } 2^k \leq L_1^3.$$

In particular, this means that we are only considering a number of distinct values for k which is comparable to $\log(L_1)$.

Remark 2.2.13. *Since $V(x,y) = \pi^2 h(x,y)^{-2}$, and $h(x,y) \leq \pi$, we see that*

$$|\nabla_{x,y} V(x,y)| = 2\pi^2 \frac{1}{h(x,y)^3} |\nabla_{x,y} h(x,y)| \geq 2\pi^{-1} |\nabla_{x,y} h(x,y)|,$$

and so we also have a lower bound on $|\nabla_{x,y} V(x,y)|$ on the domain D_k. We will also see that for the same values of k as above, it will be useful to consider domains $\tilde{D}_k$ given by

$$\tilde{D}_k \coloneqq \{(x,y) \in D^* : \xi(x,y) \neq 0, V(x,y) - \lambda \in [2^{-2k/3}, 2^{-2(k-1)/3}]\}. \tag{2.27}$$

Again, by the properties of the cutoff function $\xi(x,y)$, the sets $\tilde{D}_k$ are only non-zero for $k \geq -C$, and we will also consider these sets for $2^k \leq L_1^3$.

The relationship between the sets D_k and $\tilde{D}_k$ is made clear by the lemma below.

Lemma 2.2.14. *The inner boundary of $\tilde{D}_k$ consists of the points where $V(x,y) - \lambda = 2^{-2k/3}$. Let $\tilde{A}_k$ be the subset of $\{(x,y) \in D^* : V(x,y) - \lambda \geq 2^{-2k/3}\}$ which lies at a distance of at least $2^{k/3} L_1^\alpha$ from this inner boundary. Then the L^2-norm of $u(x,y)$ over $\tilde{A}_k$ is exponentially small in L_1. That is, we have a bound of $Ce^{-cL_1^\alpha}$, for this part of the L^2-norm, where C and c are absolute constants.*

We also have the same bound for the set A_k, where A_k is the analogous set to $\tilde{A}_k$ with $\tilde{D}_k$ replaced by D_k.

Proof of Lemma 2.2.14: To prove this lemma we will use the Agmon estimate established in Proposition 1.4.16. We recall that the function $h^*(x,y)$ in the statement of this proposition is given by

$$h^*(x,y) = \inf_\gamma \frac{1}{2} \int_0^1 \nu^*(\gamma(t))^{1/2} |\gamma'(t)| \, \mathrm{d}t,$$

where the infimum is over all paths $\gamma : [0,1] \to D_1^*$ between (x,y) and the boundary of the set $D_1^* = \{(x,y) : V(x,y) = 1 + CL_1^{-2}\}$, and $\nu^*(x,y) = V(x,y) - \lambda$. For those $(x,y) \in \tilde{A}_k$, we see that these points are separated from this boundary by a distance of $2^{k/3} L_1^\alpha$, on which the function ν^* is

bounded below by $2^{-2k/3}$. Therefore, we have the lower bound

$$h^*(x,y) \geq \frac{1}{2} 2^{k/3} L_1^{\alpha} 2^{-k/3} = \frac{1}{2} L_1^{\alpha},$$

for all $(x,y) \in \tilde{A}_k$. Using this lower bound on $h^*(x,y)$ in the estimate in Proposition 1.4.16 gives the desired bound on the L^2-norm of $u(x,y)$ over $\tilde{A}_k$.

To prove the same estimate for the sets A_k, we argue in the following way: As we noted in Remark 2.2.13, we have the lower bound

$$|\nabla_{x,y} V(x,y)| \geq 2\pi^{-1} 2^{-k}$$

on the sets D_k. Thus, when we are at a distance comparable to $2^{k/3}$ from the inner boundary of D_k, the function $V(x,y) - \lambda$ has increased by an amount comparable to $2^{k/3} 2^{-k} = 2^{-2k/3}$. In other words, the inner boundary of D_k is within a distance comparable to $2^{k/3}$ from the set $\tilde{D}_k$. Therefore, the bounds on $u(x,y)$ in A_k follow from the bounds in $\tilde{A}_k$ established above. □

2.2.3 The decay of $u(x,y)$ and a pointwise derivative bound

We are now in a position to state a proposition about the decay of the eigenfunction $u(x,y)$ away from the set where $V(x,y) \leq 1 + L_1^{-2}$.

Proposition 2.2.15. *Let $\chi_2(x)$ be a smooth cutoff function which is equal to 1 on an interval of length L_1, and decays to zero on a length scale comparable to L_1 such that*

$$|\chi_2'(x)| \leq C L_1^{-1}, \qquad |\chi_2''(x)| \leq C L_1^{-2}.$$

Then, there exists a constant $C > 0$, depending only on α, such that

$$\int_{D_k} \chi_2(x) \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 2^k L_1^{-1+\alpha}. \tag{2.28}$$

The same statement holds for D_k replaced by the set $\tilde{D}_k$ from (2.27).

Before beginning the proof of this proposition, let us discuss the estimate given in (2.28): The eigenfunction has a maximum of 1, and decays on a length scale of at most L_1 in the y-direction. Also, the support of $\chi_2(x)$ is comparable to L_1, and from Lemma 2.2.14, $u(x,y)$ is exponentially small (in an L^2-sense) when we are at a distance large compared to $2^{k/3}$ from the inner boundary

of D_k. Thus, if the eigenfunction $u(x,y)$ has no further decay, we would expect

$$\int_{D_k} \chi_2(x)\xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y$$

to be comparable to $2^{k/3}L_1$. However, the proposition shows that this integral is bounded by $CL_1^{-1+\alpha}2^k$. Therefore, when 2^k is small compared to L_1^3, the proposition does indeed give an improved bound and imply L^2-decay for $u(x,y)$ as we move away from the region where $V(x,y) \leq 1 + L_1^{-2}$.

In fact, suppose that $u(x,y)$ is comparable to its maximum on a subset of D_k of height $2^{k/3}$ in the y-direction, and length L_1 in the x-direction. Then, (2.28) implies that this maximum of $u(x,y)$ on D_k is bounded above by $C2^{k/3}L_1^{-1+\alpha/2}$. We can compare this to the pointwise bound on the eigenfunctions $\phi(t)$ given in Proposition 1.3.15. We saw there that $\phi(t)$ is bounded by $C2^{k/3}L_1^{-1}$ on the intervals J_k, which are analogous to the domains D_k here. Thus, up to the factor of $L_1^{\alpha/2}$, Proposition 2.2.15 indicates a similar decay for the two dimensional eigenfunction $u(x,y)$ as the eigenfunctions $\phi(t)$ as we move away from the region where $V(x,y) \leq 1 + L_1^{-2}$.

We can also use Proposition 2.2.15 to bound the left hand side of (2.24),

$$\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y} h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \tag{2.29}$$

and complete the proof of Proposition 2.2.10: We first note that on the part of D^* where we have

$$|\nabla_{x,y} h(x,y)| \leq CL_1^{-3},$$

we immediately have a bound of

$$CL_1^{-3} \int_{D^*} \xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq CL_2L_1^{-2}, \tag{2.30}$$

where we have used the usual $L^2(D^*)$-bound on $u(x,y)$.

Let us decompose the integral over the remaining part of D^* in (2.29) as a sum of integrals over the regions D_k. By the definition of these sets, we immediately have the bound

$$\int_{D_k} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y} h(x,y)\right|^2 \,\mathrm{d}x\,\mathrm{d}y \leq 2^{-2k+1} \int_{D_k} \xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{2.31}$$

We recall from the definition of the cutoff function $\xi(x,y)$ in Definition 2.2.8 that it is only non-zero

for x supported in an interval of length comparable to $L_2 L_1^{\alpha}$. Therefore, we can decompose the integral on the right hand side of (2.31) into $L_2 L_1^{-1+\alpha}$ integrals, with each of the integrals of the form

$$2^{-2k+1} \int_{D_k} \chi_2(x) \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y, \tag{2.32}$$

with the cutoff function $\chi_2(x)$ supported on an interval of length L_1, and with the properties given in the statement of Proposition 2.2.15. Therefore, we can use Proposition 2.2.15 to obtain the bound

$$2^{-2k+1} \int_{D_k} \chi_2(x) \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C 2^{-2k} 2^k L_1^{-1+\alpha}. \tag{2.33}$$

Summing this bound in (2.33) over the $L_2 L_1^{-1+\alpha}$ cutoff functions gives

$$\int_{D_k} \xi(x,y)^2 u(x,y)^2 \left| \nabla_{x,y} h(x,y) \right|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C 2^{-k} L_2 L_1^{-2+2\alpha}.$$

We then sum this estimate over those k with $k \geq -C$, and combine it with the bound in (2.30) to deduce that

$$\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left| \nabla_{x,y} h(x,y) \right|^2 \, \mathrm{d}x \, \mathrm{d}y \leq C L_2 L_1^{-2+2\alpha}.$$

This completes the proof of Proposition 2.2.10. □

It is left to prove Proposition 2.2.15. To do this we will integrate the eigenfunction equation

$$-\Delta_{x,y} u(x,y) + (V(x,y) - \lambda) u(x,y) = 0$$

against a series of test functions in order to obtain appropriate integral bounds for functions of $u(x,y)$. However, it will be more convenient to work with the function $\xi(x,y)u(x,y)$ rather than $u(x,y)$ itself, and so we first find the equation that $\xi(x,y)u(x,y)$ satisfies.

Lemma 2.2.16. *For $(x,y) \in D^*$, we have*

$$\left(-\Delta_{x,y} + V(x,y) - \lambda\right) \left(\xi(x,y) u(x,y)\right) = F_\xi(x,y),$$

where

$$F_\xi(x,y) = -u(x,y)\Delta_{x,y}\xi(x,y) + 2\nabla_{x,y}u(x,y).\nabla_{x,y}\xi(x,y).$$

In particular, we see that we can use the bounds on $u(x,y)$ and $\nabla_{x,y}u(x,y)$ from Lemmas 2.2.4, 2.2.6 and 2.2.7 for (x,y) in the support of $F_\xi(x,y)$.

Lemma 2.2.17. *For $F_\xi(x,y)$ as above, we have $|F_\xi(x,y)| \leq CL_1^{-1}$, together with the bounds*

$$\int_{D^*} \chi_2(x)F_\xi(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq C$$

and

$$\int_{D^*} \chi_2(x)F_\xi(x,y)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq CL_1^{-1}.$$

Proof of Lemma 2.2.17: The function $F_\xi(x,y)$ is supported on the derivatives of $\xi(x,y)$. Therefore, from Lemmas 2.2.4, 2.2.6 and 2.2.7, together with elliptic estimates, we see that

$$u(x,y) \leq CL_1^{-1}, |\nabla_{x,y}u(x,y)| \leq CL_1^{-1}$$

on the support of $F_\xi(x,y)$, and hence $F_\xi(x,y)$ has this same bound. Moreover, by the definition of $\xi(x,y)$, $\chi_2(x)F_\xi(x,y)$ is supported on two regions, one of which has area bounded by L_1, and the other is where $u(x,y)$ and its first derivatives have the exponentially small bound of $e^{-cL_1^\alpha}$. Therefore, the two required bounds follow immediately. □

To get the required bounds we will see that it will be important to control the first derivative of $u(x,y)$. We recall from Proposition 1.6.4 that, whenever we have the bound $u(x,y) \geq c$ for a fixed absolute constant $c > 0$, the first derivatives satisfy

$$|\nabla_{x,y}u(x,y)| \leq CL_1^{-1}.$$

We also recall from the discussion after Definition 2.2.8 that we have the same bound on the support of the derivatives of the cutoff function $\xi(x,y)$. To prove Proposition 2.2.15 we will need to obtain a pointwise bound on $\nabla_{x,y}u(x,y)$ for all $(x,y) \in D^*$.

Proposition 2.2.18. *There exists an absolute constant C (depending on α) such that*

$$|\partial_x \left(\xi(x,y)u(x,y)\right)|, |\partial_y \left(\xi(x,y)u(x,y)\right)| \leq CL_1^{-1+\alpha/2}.$$

Remark 2.2.19. *As we saw in the proof of Lemma 2.2.14, the inner boundary of D_k is within a distance comparable to $2^{k/3}$ from the set $\tilde{D}_k$. Thus, using these first derivative bounds it is straightforward to see that it is enough to establish the bounds in Proposition 2.2.15 for the integrals over the sets $\tilde{D}_k$.*

Proof of Propositions 2.2.15 and 2.2.18:

We will prove Propositions 2.2.15 and 2.2.18 concerning the decay of $\xi(x,y)u(x,y)$ and pointwise bounds on its first derivatives at the same time. We will obtain a first estimate on the left hand side of (2.28) and the first derivatives of $\xi(x,y)u(x,y)$, and then see that we can iteratively improve them to obtain the desired bounds.

As we mentioned above, we will prove the propositions by integrating the eigenfunction equation against three appropriately chosen test functions. The first test function is $\chi_2(x)\xi(x,y)u(x,y)$:

Lemma 2.2.20. *We have the bound*

$$\int_{D^*} \chi_2(x) \left|\nabla_{x,y} \left(\xi(x,y)u(x,y)\right)\right|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_{D^*} \chi_2(x)(V(x,y)-\lambda)_+ \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1,$$

for an absolute constant $C_1 > 0$. In particular, for $\tilde{D}_k$ as in (2.27) we have

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 2^{2k/3}.$$

Remark 2.2.21. *On the region where $(V(x,y)-\lambda)_+$ is bounded above by L_1^{-2}, the estimate on the second integral follows from the trivial bound $u(x,y) \leq 1$. However, as $(V(x,y)-\lambda)_+$ increases from L_1^{-2} this bound becomes non-trivial.*

Proof of Lemma 2.2.20: This estimate follows by integrating the equation

$$\left(-\Delta_{x,y} + (V(x,y)-\lambda)\right)\left(\xi(x,y)u(x,y)\right) = F_\xi(x,y)$$

from Lemma 2.2.16 against the test function $\chi_2(x)\xi(x,y)u(x,y)$ and then proceeding in an identical

way to the proof of Proposition 1.4.10. We just need to check that

$$\int_{D^*} \chi_2(x)\xi(x,y)u(x,y)F_\xi(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq C_1,$$

but this has been established in Lemma 2.2.17. □

The second test function is $\chi_2(x)\tilde{\chi}(y)$, where $\tilde{\chi}(y)$ is any smooth cutoff function, which is equal to 1 on an interval of length comparable to L_1, supported on a larger interval of length comparable to L_1, and with $|\tilde{\chi}'(y)| \leq CL_1^{-1}$, $|\tilde{\chi}''(y)| \leq CL_1^{-2}$:

Lemma 2.2.22. *We have the bound*

$$\int_{D^*} \chi_2(x)\tilde{\chi}(y)(V(x,y)-\lambda)_+\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq C_1,$$

for an absolute constant $C_1 > 0$. In particular, this shows that

$$\int_{\tilde{D}_k} \chi_2(x)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq C_1 2^{2k/3}.$$

Remark 2.2.23. *Since $u(x,y) \geq 0$, this lemma does give some information about the decay of the eigenfunction: If the set $\tilde{D}_k$ has height comparable to $2^{k/3}$ in the y-direction, then the L^1-bound over the region $\tilde{D}_k$ given by the lemma is consistent with a pointwise bound on $\xi(x,y)u(x,y)$ of $2^{k/3}L_1^{-1}$. In particular, such a pointwise bound would prove Proposition 2.2.15.*

Proof of Lemma 2.2.22: Integrating the equation in Lemma 2.2.16 against the test function $\chi_2(x)\tilde{\chi}(y)$, we find that

$$\begin{aligned}\int_{D^*} \chi_2(x)\tilde{\chi}(y)(V(x,y)-\lambda)_+\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{D^*} \chi_2(x)\tilde{\chi}(y)(\lambda - V(x,y))_+\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{D^*} \chi_2(x)\tilde{\chi}(y)\Delta_{x,y}\left(\xi(x,y)u(x,y)\right)\,\mathrm{d}x\,\mathrm{d}y + \int_{D^*} \chi_2(x)\tilde{\chi}(y)F_\xi(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{2.34}$$

Since $(\lambda - V(x,y))_+ \leq CL_1^{-2}$, the first integral on the right hand side immediately has the desired bound. To deal with the second integral in (2.34) we integrate by parts twice in x and y. This gives us the integral

$$\int_{D^*} \chi_2''(x)\tilde{\chi}(y)\xi(x,y)u(x,y) + \chi_2(x)\tilde{\chi}''(y)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which has the required bound by the properties of $\chi_2(x)$ and $\tilde{\chi}(y)$. Finally, we have shown in Lemma 2.2.17 that the last integral can again be bounded by an absolute constant, and this completes the proof. □

Before using a third test function to obtain another integral bound on the eigenfunction, we obtain a preliminary estimate on the first derivatives of $\xi(x,y)u(x,y)$.

Lemma 2.2.24. *Let $J(r)$ be the cutoff function, decaying on a length scale comparable to L_1, as in Definition 1.6.2, and as before define r by*

$$r^2 \coloneqq (x-x')^2 + (y-y')^2$$

Then, for any (x',y') at a distance at least comparable to 1 from the boundary of D^*, we have the expression

$$\partial_x \left(\xi(x',y')u(x',y')\right) = \int_{D^*} \partial_x J(r) \left(V(x,y)-\lambda\right) \xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y + F(x',y').$$

Here $F(x',y')$ satisfies

$$|F(x',y')| \leq CL_1^{-1}$$

for all such (x',y'). We have an analogous expression for $\partial_y \left(\xi(x',y')u(x',y')\right)$.

Proof of Lemma 2.2.24: Using the equation in Lemma 2.2.16, the expression follows in an identical way to how we established (1.101) in the proof of Proposition 1.6.4, provided we can bound the extra term

$$\int_{D^*} |\nabla_{x,y} J(r)|\, |F_\xi(x,y)|\,\mathrm{d}x\,\mathrm{d}y$$

by CL_1^{-1}. By Lemma 2.2.17, $|F_\xi(x,y)|$ is bounded by CL_1^{-1}, and since it is only non-zero on the support of the derivatives of $\xi(x,y)$, we obtain the desired estimate. □

Thus, by Lemma 2.2.24, to obtain a pointwise bound on $\partial_x \left(\xi(x',y')u(x',y')\right)$ it is enough to bound the integral

$$\int_{D^*} \partial_x J(r) \left(V(x,y)-\lambda\right) \left(\xi(x,y)u(x,y)\right)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.35}$$

As we saw in the proof the Proposition 1.6.4, for the part of the integral in (2.35) over the region where $V(x,y) - \lambda \leq CL_1^{-2}$, we obtain a bound comparable to L_1^{-1}. We now give a first bound for the contribution to the integral from the regions $\tilde{D}_k$.

Lemma 2.2.25. *We have a first bound of*

$$\left| \int_{\tilde{D}_k} \nabla_{x,y} J(r)(V(x,y) - \lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \leq C2^{-k/6} L_1^{-1/2+\alpha/2}.$$

Proof of Lemma 2.2.25: By the definition of $\tilde{D}_k$, we know that $V(x,y) - \lambda$ is comparable to $2^{-2k/3}$ for all $(x,y) \in \tilde{D}_k$. Therefore, by Lemma 2.2.20 we have the bound

$$\left| \int_{\tilde{D}_k} \chi_2(x)(V(x,y) - \lambda)^2 \xi(x,y)^2 u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \right| \leq C2^{-2k/3}. \tag{2.36}$$

Moreover, we know from Lemma 2.2.14 that $u(x,y)$ has an exponentially small L^2 bound of $Ce^{-cL_1^{\alpha/4}}$ over the region that is at a distance of $2^{k/3}L_1^{\alpha/4}$ from the inner boundary of $\tilde{D}_k$. Therefore, up to an exponentially small error in L_1, we may replace the set $\tilde{D}_k$ with a set with height in the y-direction of at most $2^{k/3}L_1^{\alpha/4}$ for any fixed x.

The function $\nabla_{x,y} J(r)$ has a singularity of r^{-1} near $r = 0$, and then decays to 0 on a length scale comparable to L_1. Thus,

$$\int_{\tilde{D}_k} |\nabla_{x,y} J(r)|^{2-\alpha/2}\,\mathrm{d}x\,\mathrm{d}y \leq C2^{k/3} L_1^{-1+\alpha/2}, \tag{2.37}$$

where the constant C only depends on the value of $\alpha > 0$. Combining the bounds in (2.36) and (2.37), and noting that $(V(x,y) - \lambda)\xi(x,y)u(x,y)$ is certainly bounded, we see that

$$\left| \int_{\tilde{D}_k} \nabla_{x,y} J(r)(V(x,y) - \lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \leq C2^{-k/6} L_1^{-1/2+\alpha/2},$$

as required. □

This lemma allows us to obtain a first pointwise bound on $\nabla_{x,y}(\xi(x,y)u(x,y))$:

Lemma 2.2.26. *For any $(x,y) \in D^*$, we have the bound*

$$|\nabla_{x,y}(\xi(x,y)u(x,y))| \leq CL_1^{-1/2+\alpha/2}.$$

Proof of Lemma 2.2.26: As we saw after the definition of the sets $\tilde{D}_k$, we are only interested in those

$k \in \mathbb{Z}$ with

$$k \geq -C, \text{ and } 2^k \leq L_1^3.$$

In particular, this means that for this range of k, we can sum the estimates from Lemma 2.2.25 to obtain

$$\sum_k \left| \int_{\tilde{D}_k} \nabla_{x,y} J(r)(V(x,y) - \lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \leq C \sum_k 2^{-k/6} L_1^{-1/2+\alpha/2} \leq 2CL_1^{-1/2+\alpha/2}.$$

The required bounds therefore follow from the expression for the first derivatives of $\xi(x,y)u(x,y)$ given in Lemma 2.2.24. □

We can also use Lemma 2.2.24 to obtain an L^1-bound on the first derivatives of $\xi(x,y)u(x,y)$.

Lemma 2.2.27. *We have the bound*

$$\int_{\tilde{E}_k} \chi_2(x) \left| \nabla_{x,y} \left(\xi(x,y)u(x,y) \right) \right| \,\mathrm{d}x\,\mathrm{d}y \leq C2^{k/3} \log(L_1)$$

where the set $\tilde{E}_k$ consists of those points of $\tilde{D}_k$ which are within a distance of at most $2^{k/3}$ from the inner boundary of $\tilde{D}_k$.

Proof of Lemma 2.2.27: By Lemma 2.2.24, to obtain the desired estimate it is enough to integrate the expression

$$\left| \int_{D^*} \nabla_{x,y} J(r)(V(x,y) - \lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \right|$$

over the values of (x', y') described in the statement of the lemma. By Lemma 2.2.22 we know that

$$\left| \int_{\tilde{D}_k} \chi_2(x)(V(x,y) - \lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \leq C,$$

and so, since for the range of integration under consideration we have

$$\int \left| \nabla_{x,y} J(r) \right| \,\mathrm{d}x'\,\mathrm{d}y' \leq C2^{k/3} \log(L_1),$$

this gives us the desired bound. □

Before continuing, let us discuss the bounds on the first derivatives of $\xi(x,y)u(x,y)$ that we have

obtained in Lemmas 2.2.25 and 2.2.27:

In Lemma 2.2.25, we have not been able to prove the desired bound on $\nabla_{x,y}(\xi(x,y)u(x,y))$ that we need to finish the proof of Proposition 2.2.18. Looking at the proof of Lemma 2.2.25, at the moment the limiting factor is that we only have the bound

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq C2^{2k/3} \tag{2.38}$$

from Lemma 2.2.20. This is in contrast to the bound of $C2^{k/3}L_1^{-1+\alpha}$ that we require the integral in (2.38) to have in order to prove Proposition 2.2.15. In fact, we see from the proof of Lemma 2.2.25 that if we have a bound on (2.38) of $C2^{k/3}L_1^{-1+\alpha}$, then we also have the required derivative bound. In other words, Proposition 2.2.15 implies Proposition 2.2.18.

In contrast, the estimate from Lemma 2.2.22 of

$$\int_{\tilde{D}_k} \chi_2(x)\tilde{\chi}(y)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq 2^{2k/3},$$

and the L^1-derivative bound from Lemma 2.2.27 are consistent with the bounds we need to prove Propositions 2.2.15 and 2.2.18.

Thus, we want to improve the estimate in (2.38). Our strategy to do this is to return to the eigenfunction equation in Lemma 2.2.16, and use a third test function

$$\chi_2(x)\left(V(x,y)-\lambda+CL_1^{-2}\right)^{1/2}\xi(x,y)u(x,y), \tag{2.39}$$

where the constant $C>0$ is chosen so that

$$V(x,y)-\lambda+CL_1^{-2} \geq \tfrac{1}{2}CL_1^{-2}$$

for all $(x,y)\in D^*$.

Lemma 2.2.28. *We have the bound*

$$\int_{D^*} \chi_2(x)\left(V(x,y)-\lambda+CL_1^{-2}\right)^{1/2}(V(x,y)-\lambda)_+\xi(x,y)^2u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \leq CL_1^{-1/2+\alpha},$$

and so in particular,

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \leq C2^kL_1^{-1/2+\alpha}.$$

Remark 2.2.29. *Comparing this bound to the bound from Lemma 2.2.20, we see that this is an improvement when 2^k is small compared to $L_1^{3/2-3\alpha}$.*

Proof of Lemma 2.2.28: We integrate the eigenfunction equation against the test function in (2.39). Integrating by parts one time, we find that

$$\begin{aligned}
&\int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{1/2} \left|\nabla_{x,y} \left(\xi(x,y)u(x,y)\right)\right|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&+ \int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{1/2} (V(x,y) - \lambda)_+ \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{1/2} (\lambda - V(x,y))_+ \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \qquad (2.40) \\
&+ \int_{D^*} \chi_2'(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{1/2} \partial_x \left(\xi(x,y)u(x,y)\right) \xi(x,y)u(x,y) \, \mathrm{d}x \, \mathrm{d}y \\
&+ \frac{1}{2} \int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{-1/2} \nabla_{x,y} V(x,y) . \nabla_{x,y} \left(\xi(x,y)u(x,y)\right) \xi(x,y)u(x,y) \, \mathrm{d}x \, \mathrm{d}y \\
&+ \int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{1/2} \xi(x,y)u(x,y) F_\xi(x,y) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Since $(\lambda - V(x,y))_+ \leq C_1 L_1^{-2}$, the $L^2(D^*)$-bounds on $\xi(x,y)u(x,y)$ immediately show that the first integral on the right hand side of (2.40) is bounded by CL_1^{-1}, which is sufficiently small.

Also, $V(x,y)$ is bounded by an absolute constant on the support of $\xi(x,y)$, and so the final integral in (2.40) is also bounded by CL_1^{-1} by Lemma 2.2.17.

Integrating by parts in the second integral on the right hand side of (2.40), we see that this integral becomes

$$\begin{aligned}
&- \frac{1}{2} \int_{D^*} \chi_2''(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{1/2} \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&+ \frac{1}{4} \int_{D^*} \chi_2'(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{-1/2} \partial_x V(x,y) \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y. \qquad (2.41)
\end{aligned}$$

For the first integral in (2.41), we can bound the contribution to the integral from $\tilde{D}_k$ by

$$CL_1^{-2} 2^{-k/3} \int_{\tilde{D}_k} \tilde{\chi}_2(x) \xi(x,y)^2 u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq CL_1^{-2} 2^{k/3},$$

for a cutoff function $\tilde{\chi}_2(x)$ with the same properties as $\chi_2(x)$, where in the last inequality we have used Lemma 2.2.20. Therefore, summing over those k with $2^k \leq CL_1^3$, this integral has a bound comparable to L_1^{-1}, which is again sufficiently small.

The remaining terms coming from (2.40) and (2.41) that we need to bound are

$$\int_{D^*} \chi_2'(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{-1/2} \partial_x V(x,y) \xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x \,\mathrm{d}y \tag{2.42}$$

and

$$\int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{-1/2} \nabla_{x,y} V(x,y) . \nabla_{x,y} \left(\xi(x,y) u(x,y)\right) \xi(x,y) u(x,y) \,\mathrm{d}x \,\mathrm{d}y. \tag{2.43}$$

Using Lemma 2.2.26, we can remove a factor of $\chi_2'(x)\xi(x,y)u(x,y)$ in (2.42) and a factor of $\nabla_{x,y}\left(\xi(x,y)u(x,y)\right)$ in (2.43) with a bound of $CL_1^{-1/2+\alpha/2}$. Therefore, to conclude the proof of the lemma, we need to show that

$$CL_1^{-1/2+\alpha/2} \int_{D^*} \chi_2(x) \left(V(x,y) - \lambda + CL_1^{-2}\right)^{-1/2} \left|\nabla_{x,y} V(x,y)\right| \xi(x,y) u(x,y) \,\mathrm{d}x \,\mathrm{d}y \leq CL_1^{-1/2+\alpha}. \tag{2.44}$$

We obtain this bound in the following way: We first decompose into integrals over $\tilde{D}_k$, and we can bound this part of (2.44) by

$$CL_1^{-1/2+\alpha/2} 2^{k/3} \int_{\tilde{D}_k} \chi_2(x) \left|\nabla_{x,y} V(x,y)\right| \xi(x,y) u(x,y) \,\mathrm{d}x \,\mathrm{d}y. \tag{2.45}$$

Fix a point $(x_0, y_0) \in \tilde{D}_k$ such that x_0 is in the support of $\chi_2(x)$. Let us assume that the tangent to the level set $\{(x,y) \in D^* : V(x,y) = V(x_0,y_0)\}$ is in the x-direction, and $\partial_y V(x,y) > 0$ at (x_0,y_0). In particular, we will assume that $\partial_x V(x,y)$ can be controlled by $\partial_y V(x,y)$ in this region. (We will see that we can deal with any other orientation of this level set in the same way.) Given x, let $y_0(x) < y_1(x)$ be such that the points $(x, y_0(x))$ and $(x, y_1(x))$ are on the inner and outer boundary of $\tilde{D}_k$. Then, by the convexity of $V(x,y)$, we have

$$\int_{y_0(x)}^{y_1(x)} \left|\partial_y V(x,y)\right| \,\mathrm{d}y \leq 2^{-2k/3}. \tag{2.46}$$

For any $y \geq y_0$, we can also write

$$\int_{\mathbb{R}} \chi_2(x) \xi(x,y) u(x,y) \,\mathrm{d}x = \int_{\mathbb{R}} \chi_2(x) \xi(x,y_0) u(x,y_0) \,\mathrm{d}x + \int_{\mathbb{R}} \chi_2(x) \left(\int_{y_0}^{y} \partial_t \left(\xi(x,t) u(x,t)\right) \,\mathrm{d}t \right) \mathrm{d}x. \tag{2.47}$$

By the integral bounds on $\partial_y \left(\xi(x,y)u(x,y)\right)$ from Lemma 2.2.27, we can bound the second integral

in (2.47) by $C2^{k/3}\log(L_1)$, provided $y \geq y_0$ and $|y - y_0| \leq 2^{k/3}$. Thus, suppose that

$$\int_{\mathbb{R}} \chi_2(x)\xi(x,y_0)u(x,y_0)\,\mathrm{d}x \geq C^* 2^{k/3}\log(L_1)$$

for a large constant C^* to be chosen below. Then, we can use (2.47) to see that

$$\int_{\mathbb{R}} \chi_2(x)\xi(x,y)u(x,y)\,\mathrm{d}x \geq \tfrac{1}{2}C^* 2^{k/3}\log(L_1)$$

for all $y \geq y_0$ with $|y - y_0| \leq 2^{k/3}$. Thus

$$\int_{\mathbb{R}}\int_{y_0}^{y_0+2^{k/3}} \chi_2(x)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq \tfrac{1}{2}C^* 2^{2k/3}\log(L_1),$$

and this contradicts the bound in Lemma 2.2.22 for C^* sufficiently large. This means that

$$\int_{\mathbb{R}} \chi_2(x)\xi(x,y_0)u(x,y_0)\,\mathrm{d}x \leq C2^{k/3}\log(L_1), \tag{2.48}$$

for some absolute constant $C > 0$. We can also repeat the same argument to replace the factor of $\xi(x,y_0)u(x,y_0)$ in (2.48) by $\sup_y \xi(x,y)u(x,y)$, where for x fixed, the supremum is over those values of y with $(x,y) \in \tilde{D}_k$. Therefore, returning to (2.45), we have the bound

$$\int_{\tilde{D}_k} \chi_2(x)\,|\partial_y V(x,y)|\,\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \left(\int_{\mathbb{R}} \chi_2(x)\sup_y\{\xi(x,y)u(x,y)\}\,\mathrm{d}x\right).\sup_x\left(\int_{y_0(x)}^{y_1(x)} |\partial_y V(x,y)|\,\mathrm{d}y\right). \tag{2.49}$$

Using (2.46) and (2.48), we can bound the right hand side of (2.45) by

$$CL_1^{-1/2+\alpha/2}2^{k/3}\int_{\tilde{D}_k} \chi_2(x)\,|\nabla_{x,y}V(x,y)|\,\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \leq CL_1^{-1/2+3\alpha/4}2^{k/3}2^{-2k/3}2^{k/3} = CL_1^{-1/2+3\alpha/4}.$$

Thus, summing over the $\log(L_1)$ values of k, gives the bound required in (2.44).

This completes the proof of the lemma. □

Remark 2.2.30. *Inspecting the proof of Lemma 2.2.28 above, and looking in particular at the integrals in* (2.42) *and* (2.43)*, we observe the following: If we can improve the pointwise bound of* $CL_1^{-1/2+\alpha/2}$ *on* $\nabla_{x,y}(\xi(x,y)u(x,y))$*, then we can immediately improve on the bound*

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \leq C2^kL_1^{-1/2+\alpha}$$

given in the statement of the lemma. In particular, if we can improve the derivative bound and establish the estimate in Proposition 2.2.18 that $|\nabla_{x,y}(\xi(x,y)u(x,y))| \leq CL_1^{-1+\alpha}$ *everywhere, then we have the estimate*

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq C2^k L_1^{-1+\alpha}.$$

This is what is required to prove Proposition 2.2.15. Therefore, the estimate on the first derivatives in Proposition 2.2.18 implies the decay bound on $\xi(x,y)u(x,y)$ *in Proposition 2.2.15.*

As we stated in Remark 2.2.29, when 2^k is small compared to $L_1^{3/2-3\alpha}$, the bound

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq C2^k L_1^{-1/2+\alpha} \tag{2.50}$$

from Lemma 2.2.28 is an improvement on the estimate in Lemma 2.2.20. For this range of k, we can insert this bound in (2.50) in the proof of Lemma 2.2.25. This improves the estimate of

$$\left|\int_{\tilde{D}_k} \nabla_{x,y} J(r)(V(x,y)-\lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y\right| \leq C2^{-k/6}L_1^{-1/2+\alpha/2}$$

given in Lemma 2.2.25 to

$$\left|\int_{\tilde{D}_k} \nabla_{x,y} J(r)(V(x,y)-\lambda)\xi(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y\right| \leq CL_1^{-3/4+3\alpha/4}. \tag{2.51}$$

Replacing the estimate in Lemma 2.2.25 by (2.51) in the proof of Lemma 2.2.26, we improve the pointwise derivative bound

$$\text{from} \quad |\nabla_{x,y}(\xi(x,y)u(x,y))| \leq CL_1^{-1/2+\alpha} \quad \text{to} \quad |\nabla_{x,y}(\xi(x,y)u(x,y))| \leq CL_1^{-3/4+\alpha}. \tag{2.52}$$

Finally, as we showed in Remark 2.2.30, we can use the improved derivative bound in (2.52) to improve the decay estimate in (2.50) from Lemma 2.2.28. Therefore, we can iteratively improve the estimates in (2.50), (2.51) and (2.52) until we obtain the required bounds of

$$\int_{\tilde{D}_k} \chi_2(x)\xi(x,y)^2 u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq C2^k L_1^{-1+\alpha}$$

and

$$\left|\nabla_{x,y}\left(\xi(x,y)u(x,y)\right)^2\right| \leq CL_1^{-1+\alpha/2}.$$

This completes the proof of Propositions 2.2.15 and 2.2.18. □

As we showed after the statement of Proposition 2.2.15, this proposition implies the bound

$$\int_{D^*} \xi(x,y)^2 u(x,y)^2 \left|\nabla_{x,y} h(x,y)\right|^2 \, \mathrm{d}x \, \mathrm{d}y \leq CL_2 L_1^{-2+2\alpha},$$

and hence establishes Proposition 2.2.10. We also saw after the statement of Proposition 2.2.10, provided we choose $\alpha > 0$ such that $2\alpha < \delta/2$, that this implies the eigenvalue bound

$$\lambda \leq \nu \leq \lambda + C_1 L_1^{-3+\delta/2}.$$

This completes the proof of Theorem 2.1.6.

□

2.3 The level sets of $\psi(x, y, z)$ and the proof of Theorem 2.1.5

In Sections 1.4 and 1.5, we used the eigenvalue bound on λ from Theorem 1.1.12 to deduce information about the shape of the level sets of the corresponding eigenfunction $u(x, y)$ in Theorem 1.1.9. We will now show that, provided the upper bound on L_2 from Assumption 2.1.3 holds, we can use the bound on ν in Theorem 2.1.6 to prove Theorem 2.1.5 concerning the shape of the level sets of the three dimensional eigenfunction $\psi(x, y, z)$. For convenience, let us now restate the theorem here:

Theorem 2.3.1. *Let D be a three dimensional convex domain as above, let L_1 and L_2 be the values from Definitions 1.1.4 and 1.1.7, and suppose that the additional assumptions from Assumption 2.1.3 hold. Fix a small constant $c_1 > 0$, and consider a constant c with $c_1 < c < 1 - c_1$. Then, the level set $\{(x, y, z) \in D : \psi(x, y, z) = c\}$ has the following shape: There exists an ellipsoid E with axes lying along the coordinate axes of lengths comparable to L_2, L_1 and 1 in the x, y and z-directions respectively, such that E is contained inside this level set, and a dilate of E, with scaling factor bounded by an absolute constant, contains this level set.*

Proof of Theorem 2.3.1: We first note that using Theorem 6.1 in [BL2], we know that the superlevel sets of $\psi(x, y, z)$ are convex subsets of D. The proof of this theorem follows in an analogous way

to the proof of Theorem 1.1.9 in Sections 1.4 and 1.5. Therefore, we will just explain the small modifications that we need to make to the work in Sections 1.4 and 1.5 in order to deduce the required shape of the level sets of $\psi(x, y, z)$.

We define the function $\tilde{H}(x)$ by

$$\tilde{H}(x) \coloneqq \int_{D(x)} \psi(x, y, z)^2 \, \mathrm{d}y \, \mathrm{d}z, \tag{2.53}$$

where we recall that $D(x)$ is the two dimensional cross-section of the domain D at the point x. Letting $\tilde{A}$ be the maximum of $\tilde{H}(x)$, we have the estimate

$$\tilde{H}(x) \leq \tilde{A} \exp(-c|x - x^*|/L_2), \tag{2.54}$$

where, as before, x^* is the point where the function $\mu(x)$ attains its minimum. The proof of the bound in (2.54) follows exactly as in the proofs of Proposition 1.4.4 and Corollary 1.4.7 for the following two reasons:

First, since for any x fixed, the eigenfunction $\psi(x, y, z)$ is equal to 0 on the boundary of $D(x)$, we have the lower bound

$$\int_{D(x)} \left(\partial_y \psi(x, y, z)\right)^2 + \left(\partial_z \psi(x, y, z)\right)^2 \, \mathrm{d}y \, \mathrm{d}z \geq \tilde{\mu}(x) \int_{D(x)} \psi(x, y, z)^2 \, \mathrm{d}y \, \mathrm{d}z, \tag{2.55}$$

where $\tilde{\mu}(x)$ is the first Dirichlet eigenvalue of the cross-section $D(x)$. Therefore, if we define $\tilde{L}_2$ to be the largest value such that this convex function $\tilde{\mu}(x)$ increases by $\tilde{L}_2^{-2}$ on an interval of length at least $\tilde{L}_2$, then the proofs of Proposition 1.4.4 and Corollary 1.4.7 show that $\tilde{H}(x)$ decays away from its maximum on a length scale comparable to $\tilde{L}_2$. However, as we noted after Lemma 2.1.4, under the upper bound on L_2 from Assumption 2.1.3, $\tilde{L}_2$ is comparable to L_2. Therefore, $\tilde{H}(x)$ decays on a length scale comparable to $\tilde{L}_2$, and we have the bound in (2.54).

For any fixed $(x, y) \in D^*$, the eigenfunction $\psi(x, y, z)$ is equal to 0 for $z = f_1(x, y)$ and $z = f_2(x, y)$. Therefore, since $V(x, y) = \pi^2 h(x, y)^{-2}$ is the first Dirichlet eigenvalue of the interval $[f_1(x, y), f_2(x, y)]$ of length $h(x, y)$, we have the lower bound

$$\int_{f_1(x,y)}^{f_2(x,y)} \left(\partial_z \psi(x, y, z)\right)^2 \, \mathrm{d}z \geq \frac{\pi^2}{h(x, y)^2} \int_{f_1(x,y)}^{f_2(x,y)} \psi(x, y, z)^2 \, \mathrm{d}z. \tag{2.56}$$

Using the lower bounds in (2.55) and (2.56), we can again replicate the proofs of Propositions 1.4.10, 1.4.12 and Corollary 1.4.11 to show that for a smooth cutoff function $\chi_1(x)$ supported on an interval

of length comparable to L_2 around x^*, we have the estimates

$$\int_D \chi_1(x)\psi(x,y,z)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C_1 L_1 L_2, \tag{2.57}$$

$$\int_D \chi_1(x)\left(\partial_y \psi(x,y,z)\right)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C_1 L_1^{-1} L_2, \tag{2.58}$$

$$\int_D \chi_1(x)\left(\partial_x \psi(x,y,z)\right)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C_1 L_1 L_2^{-1}. \tag{2.59}$$

By an elliptic estimate, we also immediately get the bound on the z-derivative

$$\int_D \chi_1(x)\left(\partial_z \psi(x,y,z)\right)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C_1 L_1 L_2. \tag{2.60}$$

The bound on the first x derivative, $\partial_x \psi(x,y,z)$, in (2.59) can be used to replicate the proof of Proposition 1.4.13 to show that

$$\tilde{A} = \max_x \tilde{H}(x) = \max_x \int_{D(x)} \psi(x,y,z)^2 \,\mathrm{d}y\,\mathrm{d}z \leq C L_1. \tag{2.61}$$

In particular, combining (2.54) and (2.61) gives us the $L^2(D)$ bound

$$\int_D \psi(x,y,z)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C L_1 L_2. \tag{2.62}$$

To establish the shape of the level sets of $\psi(x,y,z)$ and conclude the proof of Theorem 2.1.5, we now also need to show that we can replicate the work of Section 5 for the two dimensional eigenfunction $u(x,y)$. We first note that it is clear that the projection of the level set $\{(x,y,z) \in D : \psi(x,y,z) = c\}$ onto the z-axis has length comparable to 1.

Therefore, the bound on $\tilde{A}$ in (2.61) implies that the for any fixed x the cross-section of the superlevel set has length at most comparable to L_1. To show that the projection of the level set onto the y-axis has length at most comparable to L_1 we want to use the proof of Proposition 1.5.2. In this case, rather than considering $u(x,y)$, we will use $\rho(x,y)$, where

$$\rho(x,y) = \int_{f_1(x,y)}^{f_2(x,y)} \psi(x,y,z)^2 \,\mathrm{d}z.$$

By the Carleson lemma and Harnack inequality, we know that

$$\max_{z \in [f_1(x,y), f_2(x,y)]} \psi(x,y,z)^2 \leq C\rho(x,y), \tag{2.63}$$

Moreover, since $V(x,y) = \pi^2 h(x,y)$ is the first eigenvalue of the interval $[f_1(x,y), f_2(x,y)]$, it is straightforward to calculate that

$$\Delta_{x,y}\rho(x,y) - (V(x,y) - \lambda)\rho(x,y) \geq 0.$$

This means that we can replicate the proof of Proposition 1.5.2 for the function $\rho(x,y)$, and hence by (2.63), we have established the analogue of Proposition 1.5.2 for $\psi(x,y,z)$. This gives us the desired upper bound on the length of the projection of the level set onto the y-axis.

To get the lower bound on the length of this projection, we will use a similar proof to that of the corresponding Proposition 1.5.5 for $u(x,y)$.

Proposition 2.3.2. *For c fixed with $c_1 < c < 1 - c_1$, the projection of the level set $\{(x,y,z) \in D : \psi(x,y,z) = c\}$ onto the y-axis has length bounded from below by an absolute constant multiplied by L_1.*

Proof of Proposition 2.3.2: We will prove the proposition by contradiction in a similar way to the proof of Proposition 1.5.5. However, first we make the following observation: The domain D is contained between the planes $z = 0$ and $z = \pi$, and the first eigenvalue ν certainly satisfies $\nu \leq 1 + CL_1^{-2}$. Therefore, by a simple application of the generalised maximum principle, given any small $c_1 > 0$, provided we choose $c_2 > 0$ such that

$$(1 + c_1)\sin\left(\frac{\pi}{2(1 - c_2)}\right) > 1,$$

the function

$$(1 + c_1)\sin\left(\frac{z}{1 - c_2}\right)$$

is an upper bound for the eigenfunction $\psi(x,y,z)$ for $(x,y,z) \in D$ with $0 \leq z \leq \frac{\pi}{2}$. We have an analogous upper bound for $\frac{\pi}{2} \leq z \leq 1$, and this indicates the decay of $\psi(x,y,z)$ as we approach the boundary.

We now continue with the proof of the proposition. We will establish the result for $c = 1/4$, and see that the same proof works for any other value of c. Suppose that the shortest projection of this level set onto any line in the (x,y)-plane has length less than α, where $\alpha = c^* L_1$, for a small constant c^* that we will specify below. Then, using the convexity of the superlevel sets, we may assume that this level set lies between the planes $y = \alpha$, $y = 2\alpha$, $z = 0$ and $z = \pi$.

Let U be those points in D^* which are at a distance of at most $\alpha = c^* L_1$ from the projection of the level set $\{(x,y,z) \in D : \psi(x,y,z) = \frac{1}{4}\}$ onto the (x,y)-plane. We can then consider a comparison function

$$\tilde{Q}(x,y,z) \coloneqq \frac{1}{2} \sin\left(\frac{z}{0.9}\right) Q(x,y),$$

where $Q(x,y)$ is equal to 1 outside U, and increases on a length scale comparable to α such that $Q(x,y) \geq 3$ whenever (x,y) is inside the projection of the level set $\{(x,y,z) \in D : \psi(x,y,z) = \frac{1}{4}\}$ onto the (x,y)-plane. In particular, we see that $\tilde{Q}(x,y,z) \geq \frac{3}{2} \sin\left(\frac{\frac{\pi}{2}}{0.9}\right) > 1$, when $z = \pi/2$ and (x,y) is inside the projection of this level set. Also, we have $\tilde{Q}\left(x,y,\frac{\pi}{2}\right) > \frac{1}{4}$ everywhere. Therefore, we can use $\tilde{Q}(x,y,z)$ as a comparison function on those points (x,y,z) in D^* with $0 \leq z \leq \pi/2$, and in particular this gives us the upper bound

$$\psi(x,y,z) \leq \frac{1}{2} \sin\left(\frac{z}{0.9}\right) \leq \frac{5}{8} \sin(z), \tag{2.64}$$

for all points (x,y,z) with $(x,y) \notin U$ and with $0 \leq z \leq \pi/2$. We have an analogous bound for $\pi/2 \leq z \leq \pi$.

We will use the comparison function

$$\tilde{W}(x,y,z) = 0.9 \sin\left(\frac{2\pi}{5} + \frac{\pi}{15\alpha} y\right) \sin(z)$$

By (2.64), this function $\tilde{W}(x,y,z)$ is an upper bound for $\psi(x,y,z)$ for all $(x,y,z) \in D^*$ with $y = 0$ or $y = 3\alpha$ and for all $(x,y,z) \in \partial D^*$ with $0 \leq y \leq 3\alpha$. It also satisfies

$$(\Delta + \nu)\, \tilde{W}(x,y,z) = -\left(\frac{\pi}{15\alpha}\right)^2 \tilde{W}(x,y,z) + (\nu - 1)\tilde{W}(x,y,z). \tag{2.65}$$

We have the bound

$$\nu \leq 1 + C L_1^{-2},$$

and since $\alpha = c^* L_1$, provided we choose c^* sufficiently small, we see that

$$(\Delta + \nu)\, \tilde{W}(x,y,z) < 0.$$

Therefore, we can use the generalised maximum principle in Proposition 1.4.6 to deduce that

$$\psi(x,y,z) \leq \tilde{W}(x,y,z)$$

for all $(x,y,z) \in D^*$ with $0 \leq y \leq 3\alpha$. However, $\psi(x,y,z)$ attains its maximum of 1 at some point, and $\tilde{W}(x,y,z) \leq 0.9$ everywhere, and so this gives us a contradiction. Therefore, the level set $\{(x,y,z) \in D : \psi(x,y,z) = \frac{1}{4}\}$ has length at least comparable to L_1 in the y-direction. We can repeat this argument to deduce the same result for any other level set $\{(x,y,z) \in D : \psi(x,y,z) = c\}$, with c satisfying $c_1 < c < 1 - c_1$. □

As a result of this proposition and the discussion beforehand, the projection of the level set $\{(x,y,z) \in D : \psi(x,y,z) = c\}$ onto the y-axis has length comparable to L_1, and

$$\tilde{A} = \max_x \tilde{H}(x) = \max_x \int_{D(x)} \psi(x,y,z)^2 \,\mathrm{d}y \,\mathrm{d}z$$

is comparable to L_1.

We can now finish the proof of Theorem 2.1.5 by establishing that the lengths of the level sets $\{(x,y,z) \in D : \psi(x,y,z) = c\}$ in the x-direction are comparable to L_2. We can do this by replicating the proofs of Propositions 1.5.8 and 1.5.9 for the two dimensional eigenfunction $u(x,y)$.

We first can show that the projections of these level sets onto the x-axis are of length at most comparable to L_2 by using the same argument as in the proof of Proposition 1.5.8. In particular, if the length of the level set is large compared to L_2, then by the convexity of the superlevel sets, we can contradict the upper bound on $\tilde{H}(x)$ from (2.54) and (2.61).

We obtain the desired lower bound on the length of the projection onto the x-axis by replicating the argument from Proposition 1.5.9. We do this by using the lower bound on the lengths of the projection of the level sets in the y-direction from Proposition 2.3.2 and the derivative bound

$$\int_D \chi_1(x) \left(\partial_x \psi(x,y,z)\right)^2 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \leq C_1 L_1 L_2^{-1}$$

from (2.59).

This allows us to conclude that the level sets $\{(x,y,z) \in D : \psi(x,y,z) = c\}$ have lengths comparable to L_2 in the x-direction, and finishes the proof of Theorem 2.3.1. □

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