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# the Iwasawa Theory for Unitary groups

WAN, XIN

Princeton, NJ : Princeton University

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# the Iwasawa Theory for Unitary Groups

XIN WAN

A Dissertation

Presented to the Faculty

of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance

by the Department of

Mathematics

Adviser: Professor Christopher Skinner

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# Abstract

In this thesis we generalize earlier work of Skinner and Urban to construct ($p$-adic families of) nearly ordinary Klingen Eisensten series for the unitary groups $U(r,s) \hookrightarrow U(r+1,s+1)$ and do some preliminary computations of their Fourier Jacobi coefficients. As an application, using the case of the embedding $U(1,1) \hookrightarrow U(2,2)$ over totally real fields in which the odd prime $p$ splits completely, we prove that for a Hilbert modular form $f$ of parallel weight 2, trivial character, and good ordinary reduction at all places dividing $p$, if the central critical $L$-value of $f$ is 0 then the associated Bloch Kato Selmer group has infinite order. We also state a consequence for the Tate module of elliptic curves over totally real fields that are known to be modular.

# Acknowledgements

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To my wife.

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# Chapter 1

# Introduction

## 1.1 Conjectures for Motives

### 1.1.1 Characteristic 0 Conjectures

Let $M$ be a motive over a number field $F$. Suppose $p$ is an odd prime that splits completely in $F$. (We are mainly interested in the $p$-adic realization $H_p(M)$ of $M$, i.e. a Galois representation of $F$ with coefficients a finite extension $L$ of $\mathbb{Q}_p$ and which is unramified outside a finite set of primes and potentially semi-stable at all places dividing $p$.) Let $V$ be $H_p(M)$. Suppose that for each $v|p$ we have defined a subspace $V_v^+ \subset V$ which is invariant under the local Galois group $G_{F,v}$. Then the Selmer group $H^1_f(\mathcal{K}, V)$ of $V$ relative to the $V_v^+$'s is defined to be the kernel of the restriction map

$$H^1(F,V) \to \prod_{v\nmid p} H^1(I_v, V) \times \prod_{v|p} H^1(I_v, V/V_v^+),$$

where $I_v \subset G_{F,v}$ is the inertial group.

Greenberg gave a recipe for choosing such $V_v^+$'s under certain standard conditions. For each $v$ a prime of $F$ dividing $p$, suppose $H_p(M)$ is Hodge-Tate at $v$ with $H_p(M)\otimes_{\mathbb{Q}_p}\mathbb{C}_p \simeq \otimes_i \mathbb{C}_p(i)^{h_i}$ where the $h_i$ are integers and $\mathbb{C}_p(i)$ is the $i$th Tate twist of $\mathbb{C}_p$, i.e. $\mathbb{C}_p := \varprojlim \mu_{p^\infty}^{\otimes i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $d = \dim_{\mathbb{Q}_p}(H_p(M))$ and $d^\pm$ is the dimension of the subspace of $H_p(M)$ on which complex conjugation acts by $\pm 1$, then $d^+ + d^- = d$. We assume that $\sum_{i\geq 1} h_i = d^+$ and that:

Panchishkin Condition:

$H_p(M)$ contains a subspace $F^+H_p(M)$ invariant under $G_{F,v}$ with the property that

$$F^+H_p(M) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \oplus_{i \leq 1} \mathbb{C}_p(i)^{h_i}.$$

Then $V_v^+ := F^+H_p(M)$. Examples of motives for which these conditions hold include:

- all Dirichlet characters and their Tate twists;
- elliptic curves with multiplicative or good ordinary reductions at all places dividing $p$;
- nearly ordinary modular forms.

One can also define the $L$-function $L(M, s)$ for $M$ which, conjecturally is absolutely convergent for $\mathrm{Re}(s)$ is sufficiently large and has a meromorphically continuation to the whole complex plane. A general philosophy is that the size of the Selmer group for $M$ is controlled by the special value $L(M^*(1), 0)$ (up to certain periods and normalization factors), where $*$ means dual and (1) is the Tate twist. More precisely, the characteristic 0 Bloch-Kato conjecture is:

**Conjecture 1.1.1.** *Suppose $V$ is an irreducible Galois representation of $F$, then*

$$ord_{s=0} L(M^*(1), s) = rank_L H^1_f(F, M)$$

### 1.1.2 Iwasawa Main Conjectures

We can choose the coefficient to be the integer ring $\mathcal{O}_L$ instead of $L$ and defined the corresponding 'integral version' Selmer groups as well. We can also deform everything in $p$-adic families. More precisely, on the arithmetic side consider the integral Selmer group $\mathrm{Sel}_M$ for $M$ but over some $\mathbb{Z}_p^d$ extension $F_\infty$ of $F$. This Selmer group has an action of the Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ and can be viewed as interpolating Selmer groups of $H_p(M)$ twisted by characters of $\mathrm{Gal}(F_\infty/F)$. On the analytic side there is a conjectural $p$-adic $L$-function $\mathcal{L}_M \in A[[\mathrm{Gal}(F_\infty/F)]]$ which interpolates special values of $L$-functions for $M$ twisted by Hecke characters. (here $A$ is some finite extension of $\mathcal{O}_L$.) The Iwasawa main conjecture essentially states that:

**Conjecture 1.1.2.** $\mathrm{Sel}_M$ *is a torsion module over* $\mathbb{Z}_p[[\mathrm{Gal}_{F_\infty/F}]]$ *and*

$$\mathrm{Char}(\mathrm{Sel}_M) = (\mathcal{L}_M)$$

*as ideals of the Iwasawa algebra. Here* Char *means the characteristic ideal to be defined later (see section 2.8).*

Note that in the special case when $F$ is totally real and $M$ is 1-dimensional, this is the classical Iwasawa main conjecture which was proved by Mazur-Wiles [28] and Wiles [43]. The strategy is to congruences between $GL_2$ Eisenstein Series, whose associated Galois representations are reducible, and cusp forms, whose Galois representations are irreducible. Recently, this has been generalized successfully by C.Skinner and E.Urban ([35], [36],[37],[38]), proving many cases of the rank 1 and 2 characteristic 0 Bloch-Kato conjectures and the Iwasawa main conjectures for $GL_2$ modular forms as well as some other groups. The method of Skinner and Urban is to study the congruences between cusp forms and Eisenstein series on an even larger group ($GU(2,2)$) to construct the Selmer classes.

## 1.2 Main Results

This thesis is devoted to generalizing some of the work in [35] to other unitary groups. More precisely, starting from a cusp form on $U(r,s)$ we hope to: (1) construct a ($p$-adic family) of nearly ordinary Klingen Eisenstein series on $U(r+1,s+1)$ with the constant terms divisible by the $p$-adic $L$-functions we hope to study; (2) study the $p$-adic properties of the Fourier-Jacobi coefficients of the Klingen Eisenstein families and deduce some congruences between this family and cuspidal families; (3) pass to the Galois side to deduce one divisibility of the Iwasawa main conjecture. The first step is done in the first part of the paper. The second step is the most difficult one and we are only able to achieve this for $U(1,1) \hookrightarrow U(2,2)$ and $U(2,0) \to U(3,1)$. In general we lack general results about non-vanishing modulo $p$ of special $L$-values. The last step is essentially an argument appearing in [35]. As a result we are able to prove one divisibility of the Iwasawa main conjecture for two kinds of Rankin-Selberg L-functions. In the thesis we have only explained the proof of the following theorem due to limited time and leaving the write up of the other results to the future:

**Theorem 1.2.1.** *Let $F$ be a totally real number field. Let $p$ be an odd rational prime that splits completely in $F$. Let $f$ be a Hilbert modular form over $F$ of parallel weight 2 and trivial character. Let $\rho_f$ be the $p$-adic Galois representation associated to $f$ such that $L(\rho_f, s) = L(f, s)$. Suppose:*
*(i) $f$ is good ordinary at all primes dividing $p$;*
*(ii)* ***(irred)*** *and* ***(dist)*** *hold for $\rho_f$.*
*If the central critical value $L(f,1) = 0$, then the Selmer group $H^1_f(F, \rho_f^*)$ is infinite.*

Here ***(irred)*** means the residual Galois representation $\bar{\rho}_f$ of $F$ is irreducible and ***(dist)*** means that for $V = \rho_f$ and each prime $v|p$, the $\mathcal{O}_L^\times$-valued characters giving the actions of $G_{F,v}$ on $V_v^+$ and $V/V_v^+$ are distinct modulo the maximal ideal of $\mathcal{O}_L$.

**Corollary 1.2.1.** *Let $E$ be an elliptic curve over $F$ with the $p$-adic Tate module $\rho_E$. Suppose $E$ has good ordinary reduction at all primes dividing $p$. Suppose also that the residual Galois representation $\bar{\rho}_E$ is modular and satisfies* ***(dist)*** *above. If the central critical value $L(E,1) = 0$, then the Selmer group $H^1_f(F, \rho_E)$ is infinite.*

The corollary follows from the theorem immediately by the modularity lifting results of [40]. We assume that $\bar{\rho}_E$ is modular since we do not know the Serre conjecture in the totally real case. Note that if $f$ is the Hilbert modular form associated to $E$, then according to our convention that the Hecke polynomial is the characteristic polynomial of the image of geometric Frobenius, $\rho_f^* = \rho_E$.

In the special case that $F = \mathbb{Q}$ theorem 1.2.1 is essentially proved in [35], though our result is slightly more general (in particular we do not need to assume that $f$ is special or even square integrable at any finite place).

In the case when the root number is $-1$ the theorem 1.2.1 is a result of Zhang and Nekovar. We prove it when the root number is $+1$. In fact, our theorem, combined with the parity result of Nekovar, implies that when the order of vanishing is even and at least 2, then the rank of the Selmer group is also at least 2. Also note that the method of [37] does not seem to generalize to the totally real field case.

In order to prove theorem 1.2.1 we need to choose a CM extension $\mathcal{K}$ of $F$ and make use of the unitary group $U(1,1)_{/F}$ which is closely related to $GL_2$. We embed $f$ into a Hida family $\mathbf{f}$ and use some $CM$ character $\psi$ to construct a family of forms on $U(1,1)$. Then our proof consists of four steps: (1) from this family on $U(1,1)$ we construct a $p$-adic family of Klingen Eisenstein series on $U(2,2)$ such that the constant term is the divisible by the $p$-adic $L$-function of $\mathbf{f}$ over $\mathcal{K}$; (2) prove (the Fourier expansion of) the Klingen Eisenstein family is co-prime to the $p$-adic $L$-function by a computation using the doubling methods; (3) use the results about the constant terms in step 1 to construct a cuspidal family which is congruent to the Klingen Eisenstein family modulo the $p$-adic $L$-function; (4) pass to the Galois side, using the congruence between the Galois representations for the Klingen Eisenstein family and the cuspidal family to prove the theorem.

We first prove the above theorem assuming that $d$ is even and use a base change trick to remove that condition. A large part of the arguments are straightforward generalizations of [35]. However we do all the computations in the adelic language instead of the mixture of classical and adelic

language of [35]. This simplifies the computations somewhat since we no longer need to compare the classical and adelic pictures. The required non-vanishing modulo $p$ results of some special $L$-values are known thanks to the recent work of of Ming-lun Hsieh [18] and Jeanine Van-Order [45]. Also we use Hida's work on the anticyclotomic main conjecture to compare the CM periods and canonical periods. To construct the cuspidal family in step (3) we explicitly write it down instead of using the geometric argument in [35] Chapter 6. This is a much easier way since we only need to do Hida theory for cuspidal forms (which is already available) if we are only interested in proving the characteristic 0 result. In the future we will generalize the geometric argument in [35] 6.3 to prove the one divisibility of the Iwasawa-Greenberg main conjecture. (In the case when $F \neq \mathbb{Q}$ we need to restrict to a certain subfamily of the whole weight space to have freeness of the nearly ordinary forms over the (sub) weight space and surjectivity to the boundary).

## 1.3 Plan of the Thesis

This thesis consists of two parts: part one is the first 5 chapters, which are computations for general unitary groups, and part two consists of chapters 6-14, which specializes to $U(1,1) \hookrightarrow U(2,2)$ and proves the main theorem.

Part one is devoted to constructing the nearly ordinary Klingen Eisenstein series for unitary groups. The motivation for computations in this generality is for possible future generalization of part two to general unitary groups, by studying the congruences between such Eisenstein Series and cusp forms. In chapter 2 we recall various backgrounds and formulate our main conjectures for unitary groups and Hilbert modular forms. In chapter 3 we recall the notion of Klingen and Siegel Eisenstein series, the pull-back formulas relating them and their Fourier-Jacobi coefficients. In chapter 4 and 5 we construct the nearly ordinary Klingen Eisenstein series by the pullbacks of a Siegel Eisenstein series from a larger group. We manage to take the Siegel sections so that when we are moving our Eisentein datum $p$-adically, these Siegel Eisenstein series also move $p$-adic analytically. The hard part is to choose the sections at $p$-adic places. For the $\ell$-adic cases we just pick one section and might change this choice whenever doing arithmetic applications. At the Archimedean places we restrict ourselves to the parallel scalar weight case which is enough for doing Hida theory. We plan to generalize this to more general weights in the future, which might enable us to do some finite slope arithmetic applications. Also in the first part of this thesis we content ourselves with only computing a single form (instead of a family) and leave the $p$-adic interpolation for future work. We

also do some preliminary computations for Fourier-Jacobi coefficients for the Siegel Eisenstein series on the big unitary group. The Fourier Jacobi coefficients for Klingen Eisenstein series are realized as the Petersson inner-product of that for Siegel Eisenstein series with the cusp form we start with.

The main use for this computation is to prove that the Klingen Eisenstein series is co-prime to the $p$-adic $L$-function and thus giving the congruence relations needed for arithmetic applications.

In part two we apply our computations in part one to the case of $U(1,1) \hookrightarrow U(2,2)$ over totally real fields and deduce our main theorem. For convenience we keep the argument parallel to the [35] paper. In chapter 6 we recall the notion of Hilbert modular forms and record some results on the Iwasawa theory for their Selmer groups. In chapter 7 we recall some results about $p$-adic automorphic forms and Hida theory for the group $U(2,2)$. We prove our main theorem in chapter 14 (corresponding to step (4)). Chapters 8-12 (corresponding to step (1) and (2)) are parallel to chapters 9-13 of [35] and we do the local and global calculations and deduce the required $p$-adic properties needed in chapter 14. Chapter 13 is to construct a cuspidal family from the nearly ordinary Klingen Eisenstein family (step (3)). This is also needed in chapter 14.

We remark that the materials in part one (chapters 2-5) for general unitary groups, especially the $p$-adic computations in section 4.4 are new. Part two (chapters 6-14) differs from the paper [35] only in certain technicalities (the adelic computation and a slightly different choice of the Fourier-Jacobi coefficient in later half of chapter 10, the construction in chapter 13 and the use of different results on non-vanishing modulo $p$ of special $L$-values and comparing periods in proposition 12.3.2).

# Chapter 2

# Background

In this section we recall notations for holomorphic automorphic forms on unitary groups, Eisenstein series and Fourier Jacobi expansions.

## 2.1 Notations

Suppose $F$ is a totally real field such that $[F : \mathbb{Q}] = d$ and $\mathcal{K}$ is a totally imaginary quadratic extension of $F$. For a finite place $v$ of $F$ or $\mathcal{K}$ we usually write $\varpi_v$ for a uniformizer and $q_v$ for $|\varpi_v|^{-1}$. Let $c$ be the non trivial element of $\mathrm{Gal}(\mathcal{K}/F)$. Let $r, s$ be two integers with $r \geq s \geq 0$. We fix an odd prime $p$ that splits completely in $\mathcal{K}/\mathbb{Q}$. We fix $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota : \mathbb{C} \simeq \mathbb{C}_p$ and write $i_p$ for $\iota \circ i_\infty$. Denote $\Sigma_\infty$ to be the set of Archimedean places of $F$. We take a CM type $\Sigma$ of $\mathcal{K}$ (thus $\Sigma \sqcup \Sigma^c$ are all embeddings $\mathcal{K} \to \mathbb{C}$ where $\Sigma^c = \{\tau \circ c, \tau \in \Sigma\}$). There is an associated CM period $\Omega_\infty = (\Omega_{\infty,\sigma})_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma$ (we refer to [Hida07] for the definition). Define: $\Omega_\infty^\Sigma = \prod_{\sigma\in\Sigma} \Omega_{\infty,\sigma}$. We often write $S_m$ to denote the $m$ by $m$ Hermitian matrices either over $F$ or some localization $F_v$.

We use $\epsilon$ to denote the cyclotomic character and $\omega$ the Techimuller character. We will often adopt the following notation: for an idele class character $\chi = \otimes_v \chi_v$ we write $\chi_p(x) = \prod_{v|p} \chi_v(x_v)$. For a character $\psi$ of $\mathcal{K}_v$ or $\mathbb{A}_\mathcal{K}^\times$ we often write $\psi'$ for the restriction to $F_v^\times$ or $\mathbb{A}_F^\times$. For a character $\tau$ of $\mathcal{K}^\times$ or $\mathbb{A}_\mathcal{K}^\times$ we define $\tau^c$ by $\tau^c(x) = \tau(x^c)$.

(Gauss sums) If $v$ is a prime of $F$ with characteristic $\ell$ and $\mathfrak{d}_v\mathcal{O}_{F,v} = (d_v)$ is the different of $F/\mathbb{Q}$ at $v$ and if $\psi_v$ is a character of $F_v^\times$ and $(c_{\psi,v}) \subset \mathcal{O}_{F,v}$ is the conductor then we define the local Gauss

sums:

$$\mathfrak{g}(\psi_v, c_{\psi,v}d_v) := \sum_{a\in(\mathcal{O}_{F,v}/c_{\psi,v})^\times} \psi_v(a)e(Tr_{F_v/\mathbb{Q}_\ell}(\frac{a}{c_{\psi,v}d_v})).$$

If $\otimes\psi_v$ is an idele class character of $\mathbb{A}_F^\times$ then we set the global Gauss sum:

$$\mathfrak{g}(\otimes\psi_v) := \prod_v \psi_v^{-1}(c_{\psi,v}d_v)\mathfrak{g}(\psi, c_{\psi,v}d_v).$$

This is independent of all the choices. Also if $F_v \simeq \mathbb{Q}_p$ and $(p^t)$ is the conductor for $\psi_v$, then we write $\mathfrak{g}(\psi_v) := \mathfrak{g}(\psi_v, p^t)$. We define the Gauss sums for $\mathcal{K}$ similarly.

Let $\mathcal{K}_\infty$ be the maximal abelian $\mathbb{Z}_p$ extension of $\mathcal{K}$. Write $\Gamma_\mathcal{K} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. We define: $\Lambda_\mathcal{K} := \mathbb{Z}_p[[\Gamma_\mathcal{K}]]$. For any $A$ a finite extension of $\mathbb{Z}_p$ define $\Lambda_{\mathcal{K},A} := A[[\Gamma_\mathcal{K}]]$. Let $\boldsymbol{\varepsilon}_\mathcal{K} : G_\mathcal{K} \to \Gamma_\mathcal{K} \hookrightarrow \Lambda_\mathcal{K}^\times$ be the canonical character. We define $\Psi_\mathcal{K}$ to be the composition of $\boldsymbol{\varepsilon}_\mathcal{K}$ with the reciprocity map of global class field theory, which we denote as $rec_\mathcal{K}$. Here we used the geometric normalization of class field theory. We make the corresponding definitions for $F$ as well.

## 2.2 Unitary Groups

We define:

$$\theta_{r,s} = \begin{pmatrix} & & 1_s \\ & \theta & \\ -1_s & & \end{pmatrix}$$

where $\theta = \zeta 1_{r-s}$ with some totally imaginary element $\zeta \in \mathcal{K}$. Let $V = V(r,s)$ be the skew Hermitian space over $\mathcal{K}$ with respect to this metric, i.e. $\mathcal{K}^{r+s}$ equipped with the metric given by $< u, v >:= u\theta_{r,s}{}^t\overline{v}$. We define algebraic groups $G := GU(r,s)$ and $U(r,s)$ as follows: for any $F$-algebra $R$, the $R$ points are:

$$G(R) = GU(r,s)(R) := \{g \in GL_{r+s}(\mathcal{K}\otimes_F R) | g\theta_{r,s}g^* = \mu(g)\theta_{r,s}, \mu(g) \in R^\times\}$$

($\mu : GU(r,s) \to \mathbb{G}_m$ is called the similitude character) and

$$U(r,s)(R) := \{g \in GU(r,s)(R) | \mu(g) = 1\}.$$

Sometimes we write $GU_n$ and $U_n$ for $GU(n,n)$ and $U(n,n)$.

We have the following embedding:

$$GU(r,s) \times \mathrm{Res}_{\mathcal{O}_\mathcal{K}/\mathcal{O}_F}\mathbb{G}_m \to GU(r+1,s+1)$$

$$g \times x = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & k \end{pmatrix} \times x \mapsto \begin{pmatrix} a & & b & c & \\ & \mu(g)\bar{x}^{-1} & & & \\ d & & e & f & \\ h & & l & k & \\ & & & & x \end{pmatrix}$$

We write $m(g,x)$ for the right hand side. The image of the above map is the Levi subgroup of the Klingen parabolic subgroup $P$ of $GU(r+1,s+1)$, which consists of matrices in $GU(r+1,s+1)$ such that the off diagonal entries of the $s+1$th column and the last row are 0, of $GU(r,s)$. We denote this levi by $M_P$ . We also write $N_P$ for the unipotent radical of $P$. We also define $B = B(r,s)$ to be the standard Borel consisting of matrices $\begin{pmatrix} A_g & B_g \\ & D_g \end{pmatrix}$ where the blocks are with respect to the partition $r+s$ and we require that $A_g$ is lower triangular and $D_g$ is upper triangular.

We write $-V = -V(r,s)$ for the hermitian space whose metric is $-\theta_{r,s}$. We define some embeddings of $GU(r+1,s+1) \times GU(-V(r,s))$ into some larger groups. This will be used in the doubling method. First we define $G(r+s+1,r+s+1)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} & & & 1_b & & & \\ & & & & 1 & & \\ & & \theta & & & & \\ & & & & & & -1_b \\ -1_b & & & & & & \\ & -1 & & & & & \\ & & & & & -\theta & \\ & & & 1_b & & & \end{pmatrix}$$

and $G(r+s,r+s)'$ to be associated to

$$\begin{pmatrix} & & 1_b & & \\ & \theta & & & \\ & & & & -1_b \\ -1_b & & & & \\ & & & -\theta & \\ & & 1_b & & \end{pmatrix}.$$

We define an embedding

$$\alpha : \{g_1 \times g_2 \in GU(r+1,s+1) \times GU(-V(r,s)), \mu(g_1) = \mu(g_2)\} \to GU(r+s+1,r+s+1)'$$

as follows: we consider $g_1$ as a block matrix with respect to the partition $s+1+(r-s)+s+1$ (this means we use this partition to divide both all the rows and all the columns into blocks) and $g_2$ as a block matrix with respect to $s+(r-s)+s$, then we define $\alpha$ by requiring the $1,2,3,4,5$th (block wise) rows and columns of $GU(r+1,s+1)$ embeds to the $1,2,3,5,6$th (block wise) rows and columns of $GU(r+s+1,r+s+1)'$ and the $1,2,3$th (block wise) rows and columns of $GU(-V(r,s)$ embeds to the $8,7,4$th rows and columns (block-wise) of $GU(r+s+1,r+s+1)'$.

We also define an embedding:

$$\alpha' : \{g_1 \times g_2 \in GU(r,s) \times GU(-V(r,s)), \mu(g_1) = \mu(g_2)\} \to GU(r+s,r+s)'$$

in a similar way as above: consider $GU(r,s)$ and $GU(-V(r,s))$ as block matrices with respect to the partition $s+(r-s)+s$. Putting the 1,2,3th (block wise) rows and columns of the first $GU(r,s)$ into the 1,2,4th (block wise) rows and columns of $GU(r+s,r+s)'$ and putting the 1,2,3th (block wise) rows and columns of the second $GU(r,s)$ into the 6,5,4th rows and columns of $GU(r+s,r+s)'$.

We also define an isomorphism:

$$\beta : GU(r+s+1,r+s+1)' \xrightarrow{\sim} GU(r+s+1,r+s+1)$$

and

$$\beta' : GU(r+s,r+s)' \xrightarrow{\sim} GU(r+s,r+s)$$

by:

$$g \mapsto S^{-1} g S$$

or

$$g \mapsto S'^{-1} g S'$$

where

$$S = \begin{pmatrix} 1 & & & & & & -\frac{1}{2} \\ & 1 & & & & & \\ & & 1 & & & -\frac{\zeta}{2} & \\ & & & -1 & \frac{1}{2} & & \\ & & & 1 & \frac{1}{2} & & \\ & & & & & 1 & \\ & & -1 & & & -\frac{\zeta}{2} & \\ -1 & & & & & & -\frac{1}{2} \end{pmatrix}$$

and

$$S' = \begin{pmatrix} 1 & & & & & -\frac{1}{2} \\ & 1 & & & -\frac{\zeta}{2} & \\ & & -1 & \frac{1}{2} & & \\ & & 1 & \frac{1}{2} & & \\ & -1 & & & -\frac{\zeta}{2} & \\ -1 & & & & & -\frac{1}{2} \end{pmatrix}$$

**Remark 2.2.1.** *(About Unitary Groups) In order to have Shimura varieties for doing p-adic modular forms and Galois representations, we need to use a unitary group defined over $\mathbb{Q}$. More precisely consider $V$ as a Hermitian space over $\mathbb{Q}$ and still denote $\theta_{r,s}$ to be the metric on it. Then the correct unitary similitude group should be:*

$$GU(A) := \{g \in GL_{\mathcal{K}\otimes A}(V \otimes_{\mathbb{Q}} A) | g \text{ is } \mathcal{K}\text{-linear}, g\theta_{r,s}g^* = \mu(g)\theta_{r,s}, \mu(g) \in A\}$$

*This group is smaller than the one we defined before. However this group is not convenient for local computations since we can not treat the primes of $F$ each independently. So what we do (implicitly) is to construct forms on the larger unitary similitude group defined above and then restrict to the smaller one.*

## 2.3 Hermitian Symmetric Domain

Suppose $r \geq s > 0$. Then the Hermitian symmetric domain for $GU(r,s)$ is

$$X^+ = X_{r,s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}^\Sigma), y \in M_{(r-s)\times s}(\mathbb{C}^\Sigma), i(x^* - x) > -iy^*\theta^{-1}y\}.$$

For $\alpha \in GU(r,s)(F_\infty))$ (here $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R}$) we write

$$\alpha = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & d \end{pmatrix}$$

according to the standard basis of $V$ together with the block decomposition with respect to $s + (r-s) + s$. There is an action of $\alpha \in G(F_\infty)^+$ (here the superscript $+$ means the component with positive determinant at all Archimedean places) on $X_{r,s}$ defined by:

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by + c \\ gx + ey + f \end{pmatrix} (hx + ly + d)^{-1}$$

If $rs = 0$, $X_{r,s}$ consists of a single point written $\boldsymbol{x}_0$ with the trivial action of $G$. For an open compact subgroup $U$ of $G(\mathbb{A}_{F,f})$ put

$$M_G(X^+, U) := G(F)^+ \backslash X^+ \times G(\mathbb{A}_{F,f})/U$$

where $U$ is an open compact subgroup of $G(\mathbb{A}_{F,f})$.

### 2.3.1 Automorphic forms

We will mainly follow [17] to define the space of automorphic forms, with slight modifications. We define a cocycle: $J : R_{F/\mathbb{Q}}G(\mathbb{R})^+ \times X^+ \to GL_r(\mathbb{C}^\Sigma) \times GL_s(\mathbb{C}^\Sigma) := H(\mathbb{C})$ by $J(\alpha, \tau) = (\kappa(\alpha, \tau), \mu(\alpha, \tau))$

where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & d \end{pmatrix}$,

$$\kappa(\alpha,\tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\theta} \\ -\bar{\theta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\theta}^{-1}\bar{g}^t y + \bar{\theta}^{-1}\bar{e}\bar{\theta} \end{pmatrix},\ \mu(\alpha,\tau) = hx + ly + d.$$

Let $\boldsymbol{i}$ be the point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ on the Hermitian symmetric domain for $GU(r,s)$ (here 0 means the $(r-s)\times s$ matrix 0). Let $GU(r,s)(\mathbb{R})^+$ be the subgroup of $GU(r,s)(\mathbb{R})$ whose similitude factor is positive. Let $K_\infty^+$ be the compact subgroups of $U(r,s)(\mathbb{R})$ stabilizing $\boldsymbol{i}$ and let $K_\infty$ be the groups generated by $K_\infty^+$ and $diag(1_{r+s}, -1_s)$. Then $J : K_\infty^+ \to H(\mathbb{C}), k_\infty \mapsto J(k_\infty, \boldsymbol{i})$ defines an algebraic representation of $K_\infty^+$.

**Definition 2.3.1.** *A weight $\underline{k}$ is defined by a set $\{\underline{k}_\sigma\}_{\sigma\in\Sigma_\infty}$ where each*

$$\underline{k}_\sigma = (c_{r+s,\sigma}, ..., c_{s+1,\sigma}; c_{1,\sigma}, ..., c_{s,\sigma})$$

*with $c_{1,\sigma} > ... > c_{r+s,\sigma}$ for the $c_{i,\sigma}$'s in $\mathbb{Z}$*

**Remark 2.3.1.** *Our convention is different from others in the literature. For example in [17] the $a_{r+1-i}$ there for $\leq i \leq r$ is our $c_{s+i}$ and $b_{s+1-j}$ there for $1 \leq j \leq s$ is our $c_j$. Also our $c_i$ is the $-c_{r+s+1-i}$ in [37]. We also note that if each $\underline{k}_\sigma = (0, ..., 0; \kappa, ..., \kappa)$ then $L^{\underline{k}}(\mathbb{C})$ is one dimensional of $\rho^{\underline{k}}(h) = \det \mu(h,i)^\kappa$.*

We refer to [17] for the definition of the definition of the algebraic representation $L_{\underline{k}}(\mathbb{C})$ with the action denoted by $\rho_{\underline{k}}$ (note the different index for weight) and define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h) = \rho_{\underline{k}}({}^t h^{-1}), h \in H(\mathbb{C}).$$

For a weight $\underline{k}$, define $\|\underline{k}\| = \{\|\underline{k}\|_\sigma\}_{\sigma\in\Sigma} \in \mathbb{Z}[\Sigma]$ by:

$$\|\underline{k}\| := -c_{s+1,\sigma} - ... - c_{s+r,\sigma} + c_{1\sigma} + ... + c_{s,\sigma}$$

and $|\underline{k}| \in \mathbb{Z}^{\Sigma \sqcup \Sigma^c}$ by:

$$|\underline{k}| = \sum_{\sigma \in \Sigma} (c_{1,\sigma} + ... + c_{s,\sigma}).\sigma - (c_{s+1,\sigma} + ... + c_{s+r,\sigma}).\sigma c.$$

Let $\chi$ be a Hecke character of $\mathcal{K}$ with infinite type $|\underline{k}|$, i.e. the Archimedean part of $\chi$ is given by:

$$\chi(z_\infty) = (\prod_\sigma z_\sigma^{(c_{1,\sigma}+...+c_{s,\sigma})}.z_{\sigma^c}^{-(c_{s+1,\sigma}+...+c_{s+r,\sigma})}).$$

**Definition 2.3.2.** *Let $U$ be an open compact subgroup in $G(\mathbb{A}_{F,f})$. We denote by $M_{\underline{k}}(U, \mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions $f$ on $X^+ \times G(\mathbb{A}_{F,f})$ such that for $\tau \in X^+$, $\alpha \in G(F)^+$ and $u \in U$ we have:*

$$f(\alpha\tau, \alpha g u) = \mu(\alpha)^{-\|\underline{k}\|} \rho^{\underline{k}}(J(\alpha, \tau)) f(\tau, g).$$

Now we consider automorphic forms on unitary groups in the adelic language. Let $\boldsymbol{i} \in X^+$ and $K_\infty^+ \subset U(r,s)(F_\infty)$ be the stabilizer of $\boldsymbol{i}$. The space of automorphic forms of weight $\underline{k}$ and level $U$ with central character $\chi$ consists of smooth and slowly increasing functions $F : G(\mathbb{A}_F) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(F) \times K_\infty^+ \times U \times Z(\mathbb{A}_F)$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, \boldsymbol{i})^{-1}) F(g) \chi^{-1}(z).$$

## 2.4 Galois representations Associated to Cuspidal Representations

In this section we follow [34] to state the result of associating Galois representations to cuspidal automorphic representations on $GU(r,s)(\mathbb{A}_F)$. First of all let us fix the notations. Let $\bar{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ and let $G_{\mathcal{K}} := \mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$. For each finite place $v$ of $\mathcal{K}$ let $\bar{\mathcal{K}}_v$ be an algebraic closure of $\mathcal{K}_v$ and fix an embedding $\bar{\mathcal{K}} \hookrightarrow \bar{\mathcal{K}}_v$. The latter identifies $G_{\mathcal{K}_v} := \mathrm{Gal}(\bar{\mathcal{K}}_v/\mathcal{K}_v)$ with a decomposition group for $v$ in $G_{\mathcal{K}}$ and hence the Weil group $W_{\mathcal{K}_v} \subset G_{\mathcal{K}_v}$ with a subgroup of $G_{\mathcal{K}}$. Let $\pi$ be a holomorphic cuspidal irreducible representation of $U(r,s)(\mathbb{A}_F)$ with weight $\underline{k} = (c_{r+s,\sigma}, ..., c_{s+1,\sigma}; c_{1,\sigma}, ..., c_{s,\sigma})_{\sigma \in \Sigma}$ and central character $\chi_\pi$. Let $\sum(\pi)$ be a finite set of primes of $F$ containing all the primes at which $\pi$ is unramified and all the primes dividing $p$. Then for some $L$ finite over $\mathbb{Q}_p$, there is a Galois representation (by [33], [26] and [34]):

$$R_p(\pi) : G_{\mathcal{K}} \to GL_n(L)$$

such that:

(a)$R_p(\pi)^c \simeq R_p(\pi)^\vee \otimes \rho_{p,\chi_\pi^{1+c}}\epsilon^{1-n}$ where $\chi_\pi$ is the central character of $\pi$, $\rho_{p,\chi_\pi^{1+c}}$ denotes the associated Galois character by class field theory and $\epsilon$ is the cyclotomic character.

(b)$R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi) \cup \{$ primes dividing $p)$, and for such a place $w$:

$$\det(1 - R_p(\pi)(\mathrm{frob}_w q_w^{-s}) = L(BC(\pi)_w \otimes \chi_{\pi,w}^c, s + \frac{1-n}{2})^{-1}$$

Here the $\mathrm{frob}_w$ is the geometric Frobenius and $BC$ means the base change from $U_{r,s}$ to $GL_{r+s}$. We write $V$ for the representation space and it is possible to take a Galois stable $\mathcal{O}_L$ lattice which we denote as $T$. Suppose $\pi_v$ is nearly ordinary at all primes $v$ dividing $p$ with respect to $\underline{k}$ (to be defined later, see section 4.4). Suppose $v|p$ correspond to $\sigma \in \Sigma$ under $\iota : \mathbb{C} \simeq \mathbb{C}_p$, then if we write $\kappa_{i,\sigma} = s - i + c_{i,\sigma}$ for $\leq i \leq s$ and $\kappa_{i\sigma} = c_{i,\sigma} + s + r + s - i$ for $s + 1 \leq i \leq r + s$, then:

$$R_p(\pi)|G_{\mathcal{K},v} \simeq \begin{pmatrix} \xi_{r+s,v}\epsilon^{-\kappa_{r+s,\sigma}} & * & & * & * \\ & \xi_{r+s-1,v}\epsilon^{\kappa_{r+s-1,\sigma}} & & & * \\ & & & \ldots & * \\ & 0 & & & \\ & & & & \xi_{1,v}\epsilon^{-\kappa_{1,\sigma}} \end{pmatrix}$$

where $\xi_{i,v}$ are unramified characters. Using the fact (a) above we know that $R_p(\pi)_{\bar{v}}$ is equivalent to an upper triangular representation as well.

## 2.5 Selmer Groups

We recall the notion of $\Sigma$-primitive Selmer groups, following [35, 3.1] with some modifications. In this section $F$ is a subfield of $\bar{\mathbb{Q}}$, not necessarily finite over $\mathbb{Q}$. For $T$ a free module over a profinite $\mathbb{Z}_p$-algebra $A$ and assume that $T$ is equipped with a continuous action of the absolute Galois group $G_F$ of $F$. Assume that for each place $v|p$ of $F$ we are given a $G_v$-stable free $A$-direct summand $T_v \subset T$. Let $A^*$ be the Pontryagin dual of $A$. For any finite set of primes $\Sigma$ we denote by $\mathrm{Sel}_F^\Sigma(T, (T_v)_{v|p})$ the kernel of the restriction map:

$$H^1(F, T \otimes_A A^*) \to \prod_{v \notin \Sigma, v \nmid p} H^1(I_v, T \otimes_A A^*) \times \prod_{v|p} H^1(I_v, T/T_v \otimes_A A^*),$$

We also define:

$$\mathbf{X}_F^{\Sigma}(T,(T_v)_{v|p}) := Hom_A(Sel_F^{\Sigma}(T,(T_v)_{v|p}),A^*).$$

Now let us take $T$ to be the Galois representation defined in section 2.4. Then for each $v \in \Sigma_p$ suppose $R_p(\pi)_v$ is of the above form with respect to the basis $v_{r+s,v},...,v_{1,v}$ then we define $T_v$ to be the $\mathcal{O}_L$ span if $v_{r+s},...,v_{s+1,v}$. Also, if $R_p(\pi)_{\bar{v}}$ is upper triangular with respect to the basis $v_{1,\bar{v}},...,v_{r+s,\bar{v}}$ then we define $T_{\bar{v}}$ to be the $\mathcal{O}_L$ span of $v_{1,\bar{v}},...v_{s,\bar{v}}$.

**Remark 2.5.1.** *The Selmer group defined here is not quite correct. In fact $T$ does not always satisfy Greenberg's Panchishkin condition. But it is correct in the "Iwasawa theoretic sense". We will explain this in a moment (remark 2.6.1).*

## 2.6 Iwasawa Theory

Let $\mathcal{K}_\infty$ be the maximal abelian $\mathbb{Z}_p$ extension of $\mathcal{K}$. Write $\Gamma_{\mathcal{K}} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. We define: $\Lambda_{\mathcal{K}} := \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]]$. For any $A$ a finite extension of $\mathbb{Z}_p$ define $\Lambda_{\mathcal{K},A} := A[[\Gamma_{\mathcal{K}}]]$. Let $\boldsymbol{\varepsilon}_{\mathcal{K}} : G_{\mathcal{K}} \to \Gamma_{\mathcal{K}} \hookrightarrow \Lambda_{\mathcal{K}}^{\times}$ be the canonical character. Then by Shapiro's lemma we have:

$$\mathrm{Sel}_{\mathcal{K}_\infty}^{\Sigma}(T) \simeq \mathrm{Sel}_{\mathcal{K}}^{\Sigma}(T \otimes_A \Lambda_{\mathcal{K},A}(\boldsymbol{\varepsilon}_{\mathcal{K}}^{-\mathbf{1}}))$$

So we have a $\Lambda_{\mathcal{K},A}$ module structure for $\mathbf{X}_{\mathcal{K}_\infty}^{\Sigma}(T)$. One can define the Selmer groups for intermediate fields between $\mathcal{K}$ and $\mathcal{K}_\infty$ as well.

**Remark 2.6.1.** *Later on we will see some control theorems for Selmer groups relating the big Selmer groups for $\mathcal{K}_\infty$ to those of its subfields. However if we take $T$ to be the Galois representation coming from automorphic forms then $\mathrm{Sel}_{\mathcal{K}}^{\Sigma}(T)$ itself is not a Selmer group since $T$ does not satisfy Greenberg's Panchinshkin conditions (by checking the Hodge-Tate weights). (According to our convention the dual of such $T$ satisfies this condition.) But by twisting $T$ by certain Galois character we can make $T$ satisfy this condition. Also the $T_v$'s we put at $v|p$ are indeed Selmer conditions for such twists in the sense of Greenberg. Therefore our Iwasawa module is indeed interpolating Selmer groups for $T$ twisted by certain characters.*

## 2.7 $p$-adic $L$-Functions

In a recent work of [5] Eischen, Harris, Li and Skinner construct the $p$-adic $L$-function $\mathcal{L}_{\pi,\mathcal{K},\psi}^{\Sigma} \in A[[\Gamma_{\mathcal{K}}]]$ (where $\psi$ is some fixed Hecke character for $\mathcal{K}$) interpolating special values of $L^{\Sigma}(\pi,\psi\otimes\chi_\phi,s)$

up to some periods and normalization factors. Here $\phi \in \mathrm{Spec}\Lambda_{\mathcal{K},A}$ is an arithmetic point (See definition 7.1.1 for a special case) and $\chi_\phi$ corresponds to $\phi \circ \boldsymbol{\varepsilon}_{\mathcal{K}}$ under the reciprocity map.

## 2.8 Characteristic Ideals and Fitting Ideals

In this subsection we let $A$ be a Noetherian ring. We write $\mathrm{Fitt}_A(X)$ for the Fitting ideal in $A$ of a finitely generated $A$-module $X$. This is the ideal generated by the determinant of the $r \times r$ minors of the matrix giving the first arrow in a given presentation of $X$:

$$A^s \to A^r \to X \to 0.$$

If $X$ is not a torsion $A$-module then $\mathrm{Fitt}(X) = 0$.

Fitting ideals behave well with respect to base change. For $I \subset A$ an ideal, then:

$$\mathrm{Fitt}_{A/I}(X/IX) = \mathrm{Fitt}_A(X) \bmod I$$

Now suppose $A$ is a Krull domain (a domain which is Noetherian and normal), then the characteristic ideal is defined by:

$$\mathrm{Char}_A(X) := \{x \in A : \mathrm{ord}_Q(x) \geq \ell_Q(X) \text{ for any } Q \text{ a height one prime }\},$$

where $\ell_Q(X)$ is the length of $X$ at $Q$. Again if $X$ is not torsion then we define $\mathrm{Char}_A(X) = 0$.

## 2.9 Main Conjectures

Now we are in a position for formulate the Iwasawa-Greenberg main conjecture, we write $\mathrm{Char}^{\Sigma}_{\pi,\mathcal{K},\psi}$ for the characteristic ideal for $\mathbf{X}^{\Sigma}_{\pi,\mathcal{K},\psi}$, then:

**Conjecture 2.9.1.** $\mathrm{Char}^{\Sigma}_{\pi,\mathcal{K},\psi}$ *is principal and generated by* $\mathcal{L}^{\Sigma}_{\pi,\mathcal{K},\psi}$.

In our attempt to prove cases of this main conjecture we need to embed some nearly ordinary $f \in \pi$ into a Hida family $\mathbf{f}$ of nearly ordinary forms with some coefficient ring $\mathbb{I}$ (taken to be a normal domain). We have a Galois representation $R_p(\mathbf{f})$ on some $\mathbf{T}$ a free module over $\mathbb{I}$ of finite rank. It satisfies local conditions at $v|p$ similar to that for $f$ and we define the corresponding Selmer

conditions and thus $\mathrm{Sel}^{\Sigma}_{\mathbf{f},\mathcal{K},\psi}$ and $X^{\Sigma}_{\mathbf{f},\mathcal{K},\psi}$ which is a module over $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$. On the automorphic side there is still a $p$-adic $L$-function $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\psi}$. Then we have the main conjecture for Hida families as well:

**Conjecture 2.9.2.**

$$\mathrm{Char}^{\Sigma}_{\mathbf{f},\mathcal{K},\psi} = (\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\psi})$$

*as ideals of* $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$.

## 2.10 Hilbert Modular Forms

As mentioned in the introduction we can use unitary groups to study the Iwasawa theory for Hilbert modular forms. Let $f$ ($\mathbf{f}$) be a nearly ordinary Hilbert modular form (or Hida family). Then the associated galois representations satisfy similar local conditions at $v|p$, namely isomorphic to upper triangular representations and one can define Selmer groups $Sel^{\Sigma}_{f,\mathcal{K},\chi}$, $\mathbf{X}^{\Sigma}_{f,\mathcal{K},\chi}$. (see chapter 6 for details). Also the $p$-adic $L$-functions $\mathcal{L}^{\Sigma}_{f,\mathcal{K},\chi}$, $\tilde{\mathcal{L}}^{\Sigma}_{f,\mathcal{K},\chi}$ (see chapter 11) are essentially those for $U(1,1)$ with some modifications for interpolation formulas (since now we are using $GL_2$ $L$-functions of $f$ instead of that for base change of unitary group automorphic forms). We can formulate the following main conjecture as well.

**Conjecture 2.10.1.** *As ideals of* $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$,

$$(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\chi}) = \mathrm{Char}^{\Sigma}_{f,\mathcal{K},\chi}.$$

We can construct a non-integral $p$-adic $L$-function $\tilde{\mathcal{L}}^{\Sigma}_{\mathbf{f},\mathcal{K},\chi}$ in great generality. (This is an element in $F_{\mathbb{I}}[[\Gamma_{\mathcal{K}}]]$ instead of $\mathbb{I}[\Gamma_{\mathcal{K}}]$ which is also interpolating special values of the $L$-functions with a slightly different interpolation formula, see chapter 11). This is enough for proving the characteristic 0 results (theorem 1.2.1) . However we need certain Gorenstein properties of some Hecke algebras to construct the integral $p$-adic $L$-function that appears in the conjecture above. Let us briefly discuss this issue. Let $\mathbf{f}$ be a Hida family of nearly ordinary Hilbert modular eigenforms with tame level $M$. Let $\mathbb{I}$ be some finite extension of $\Lambda_W$, let $\mathfrak{m}_{\mathbf{f}}$ be the maximal ideal of the Hecke algebra $\mathbb{T}(M,\mathbb{I})$ with $\mathbb{I}$ coefficients correspond to $\mathbf{f}$. Let $\mathbb{T}_{\mathfrak{m}_f} := \mathbb{T}(M,A)_{\mathfrak{m}_{\mathbf{f}}}$ be the localization. Then we say that it is Gorenstein if $\mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathfrak{m}_{\mathbf{f}}},\mathbb{I})$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}_{\mathbf{f}}}$. This is used to guarantee the existence of a generator of the congruence module. In the case when $F=\mathbb{Q}$ Wiles [Wiles95] proved that this is true whenever the (irred) and (dist) in [35] (see theorem 1.2.1.) are satisfied. In general the situation is complicated. We record here a theorem of Fujiwara which gives sufficient conditions for $\mathbb{T}_{\mathfrak{m}_{\mathbf{f}}}$ to

be Gorenstein:

**Theorem 2.10.1.** *(Fujiwara) Let $\bar{\rho}$ be the modulo $p$ Galois representation associated to $\mathbf{f}$. Suppose*

- *$p \geq 3$ and $\bar{\rho}|F(\zeta_p)$ is absolutely irreducible. When $p = 5$ the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}$ is isomorphic to $PGL_2(\mathbb{F}_p)$ and the* mod$p$ *cyclotomic character $\bar{\chi}_{cycle}$ factors through $G_F \to \bar{G}^{ab} \simeq \mathbb{Z}/2$;*
- *There is a minimal modular lifting of $\bar{\rho}$.*
- *The case $0_E$ defined in [8] section 3.1 does not occur for any finite place $v$.*
- *In the case when $d := [F : \mathbb{Q}]$ is odd the Ihara's lemma is true for Shimura curves.*

*Then the ring $\mathbb{T}_{\mathfrak{m}_f}$ is Gorenstein.*

This is [8, theorem 11.2]. The third condition is put to ensure that the quaternion algebra considered by Fujiwara is not ramified at any finite places so that the quaternionic Hecke algebra is the same as the $GL_2$ Hecke algebra. Recall that $0_E$ in called "exceptional" by Fujiwara and means that $\bar{\rho}_{I_{F,v}}$ is absolutely irreducible and $q_v \equiv -1 \mod p$.

# Chapter 3

# Eisenstein Series and Fourier-Jacobi Coefficients

The materials of this chapter are straightforward generalizations of parts of [35, chapter 9 and 11] and I use the same notations as *loc.cit*; so everything in this chapter should eventually be the same as [35] when specializing to the group $GU(2,2)_{/\mathbb{Q}}$.

## 3.1 Klingen Eisenstein Series

Recall that in chapter 2 we let $GU$ be $GU(r,s)$ defined there. Let $\mathfrak{gu}(\mathbb{R})$ be the Lie algebra of $GU(r,s)(\mathbb{R})$. We will often use the notation $a, b$ for $a = r - s$ and $b = s$.

### 3.1.1 Archimedean Picture

Let $v$ be an infinite place of $F$ so that $F_v \simeq \mathbb{R}$. Let $\boldsymbol{i}'$ and $\boldsymbol{i}$ be the points on the Hermitian symmetric domain for $GU(r,s)$ and $GU(r+1,s+1)$ which are $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} i1_{s+1} \\ 0 \end{pmatrix}$ respectively (here 0 means the $(r-s)\times s$ or $(r-s)\times(s+1)$ matrix 0). Let $GU(r,s)(\mathbb{R})^+$ be the subgroup of $GU(r,s)(\mathbb{R})$ whose similitude factor is positive. Let $K_\infty^+$ and $K_\infty^{+,'}$ be the compact subgroups of $U(r+1,s+1)(\mathbb{R})$ and $U(r,s)(\mathbb{R})$ stabilizing $\boldsymbol{i}$ or $\boldsymbol{i}'$ and let $K_\infty$ ($K'_\infty$) be the groups generated by $K_\infty^+$ ($K_\infty^{+,'}$) and $diag(1_{r+s+1}, -1_{s+1})$ ($diag(1_{r+s}, -1_s)$).

Now let $(\pi, H)$ be a unitary Hilbert representation of $GU(\mathbb{R})$ with $H_\infty$ the space of smooth vectors. We define a representation of $P(\mathbb{R})$ on $H_\infty$ as follows: for $p = mn, n \in N_P(\mathbb{R}), m =$

$m(g,a) \in M_P(\mathbb{R})$ with $a \in \mathbb{C}^\times, g \in GU(\mathbb{R})$, put

$$\rho(p)v := \tau(a)\pi(g)v, v \in H_\infty.$$

We define a representation by smooth induction $I(H_\infty) := \mathrm{Ind}_{P(\mathbb{R})}^{GU(r+1,s+1)(\mathbb{R})}\rho$ and denote $I(\rho)$ as the space of $K_\infty$-finite vectors in $I(H_\infty)$. We also define for each $z \in \mathbb{C}$ a function

$$f_z(g) := \delta(m)^{(a+2b+1)/2+z}\rho(m)f(k), g = mk \in P(\mathbb{R})K_\infty,$$

where $\delta$ is such that $\delta^{a+2b+1} = \delta_P$ for $\delta_P$ the modulus character of $P$. and an action of $GU(r+1, s+1)(\mathbb{R})$ on it by

$$(\sigma(\rho,z)(g)f)(k) := f_z(kg).$$

Let $(\pi^\vee, V)$ be the irreducible $(\mathfrak{gu}(\mathbb{R}), K'_\infty)$-module given by $\pi^\vee(x) = \pi(\eta^{-1}x\eta)$ for $\eta = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}$ and $x$ in $\mathfrak{gu}(\mathbb{R})$ or $K'_\infty$, and denote $\rho^\vee, I(\rho^\vee), I^\vee(H_\infty)$ and $\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with $\pi, \tau$ replaced by $\pi^\vee \otimes (\tau \circ \det), \bar{\tau}^c$. We are going to define an intertwining operator. Let $w = \begin{pmatrix} & & 1_{b+1} \\ & 1_a & \\ -1_{b+1} & & \end{pmatrix}$. For any $z \in \mathbb{C}$, $f \in I(H_\infty)$ and $k \in K_\infty$ consider the integral:

$$A(\rho, z, f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn. \tag{3.1}$$

This is absolutely convergent when $Re(z) > \frac{a+2b+1}{2}$ and $A(\rho, z, -) \in Hom_{\mathbf{C}}(I(H_\infty), I^\vee(H_\infty))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

Suppose $\pi$ is the holomorphic discrete series representation associated to the (scalar) weight $(0, ..., 0; \kappa, ..., \kappa)$, then it is well known that there is a unique (up to scalar) vector $v \in \pi$ such that $k.v = \det \mu(k,i)^{-\kappa}$ (here $\mu$ means the second component of the automorphic factor $J$ instead of the similitude character) for any $k \in K_{\infty,v}^{+,'}$ (notation as in section 3.1). Then by Frobenius reciprocity there is a unique (up to scalar) vector $\tilde{v} \in I(\rho)$ such that $k.\tilde{v} = \det \mu(k,i)^{-\kappa}\tilde{v}$ for any $k \in K_\infty^+$. We fix $v$ and scale $\tilde{v}$ such that $\tilde{v}(1) = v$. In $\pi^\vee$, $\pi(w)v$ ($w$ is defined in section 3.1) has the action of $K_\infty^+$ given by multiplying by $\det \mu(k,i)^{-\kappa}$. We define $w' \in U(a+b+1, b+1)$ by

$w' = \begin{pmatrix} 1_b & & & \\ & & & 1 \\ & 1_a & & \\ & & 1_b & \\ -1 & & & \end{pmatrix}$. There is a unique vector $\tilde{v}^\vee \in I(\rho^\vee)$ such that the action of $K_\infty^+$ is given by $\det \mu(k,i)^{-\kappa}$ and $\tilde{v}^\vee(w') = \pi(w)v$. Then by uniqueness there is a constant $c(\rho, z)$ such that $A(\rho, z, \tilde{v}) = c(\rho, z)\tilde{v}^\vee$.

**Definition 3.1.1.** *We define $F_\kappa \in I(\rho)$ to be the $\tilde{v}$ as above.*

### 3.1.2 Prime to $p$ Picture

Our discussion here follows [35, 9.1.2]. Let $(\pi, V)$ be an irreducible, admissible representation of $GU(F_v)$ which is unitary and tempered. Let $\psi$ and $\tau$ be unitary characters of $\mathcal{K}_v^\times$ such that $\psi$ is the central character for $\pi$. We define a representation $\rho$ of $P(F_v)$ as follows. For $p = mn, n \in N_P(F_v)$, $m = m(g,a) \in M_P(F_v), a \in K_v^\times, g \in GU(F_v)$ let

$$\rho(p)v := \tau(a)\pi(g)v, v \in V.$$

Let $I(\rho)$ be the representation defined by admissible induction: $I(\rho) = \mathrm{Ind}_{P(F_v)}^{GU(r+1,s+1)(F_v)}\rho$. As in the Archimedean case, for each $f \in I(\rho)$ and each $z \in \mathbb{C}$ we define a function $f_z$ on $GU(F_v)$ by

$$f_z(g) := \delta(m)^{(a+2b+1)/2+z}\rho(m)f(k), g = mk \in P(F_v)K_v$$

and a representation $\sigma(\rho, z)$ of $GU(r+1, s+1)(F_v)$ on $I(\rho)$ by

$$(\sigma(\rho, z)(g)f)(k) := f_z(kg).$$

Let $(\pi^\vee, V)$ be given by $\pi^\vee(g) = \pi(\eta^{-1}g\eta)$. This representation is also tempered and unitary. We denote by $\rho^\vee, I(\rho^\vee)$, and $(\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with $\pi$ and $\tau$ replaced by $\pi^\vee \otimes (\tau \circ \det)$, and $\bar{\tau}^c$, respectively.

For $f \in I(\rho), k \in K_v$, and $z \in \mathbb{C}$ consider the integral

$$A(\rho, z, v)(k) := \int_{N_P(F_v)} f_z(wnk)dn. \tag{3.2}$$

As a consequence of our hypotheses on $\pi$ this integral converges absolutely and uniformly for $z$ and $k$ in compact subsets of $\{z : Re(z) > (a+2b+1)/2\} \times K_v$. Moreover, for such $z$, $A(\rho, z, f) \in I(\rho^\vee)$ and the operator $A(\rho, z, -) \in Hom_{\mathbb{C}}(I(\rho), I(\rho^\vee))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

For any open subgroup $U \subseteq K_v$ Let $I(\rho)^U \subseteq I(\rho)$ be the finite-dimensional subspace consisting of functions satisfying $f(ku) = f(k)$ for all $u \in U$. Then the function $z \in \mathbb{C} : Re(z) > (a+2b+1)/2 \to Hom_{\mathbb{C}}(I(\rho)^U, I(\rho^\vee)^U)$, $z \mapsto A(\rho, z, -)$, is holomorphic. This map has a meromorphic continuation to all of $\mathbb{C}$.

We finally remark that when $\pi$ and $\tau$ are unramified then there is a unique up to scalar unramified vector $F_{\rho_v} \in I(\rho)$.

### 3.1.3 Global Picture

We follow [35, 9.1.4] for this part. Let $(\pi, V)$ be an irreducible cuspidal tempered automorphic representation of $GU(\mathbb{A}_F)$. It is an admissible $(\mathfrak{gu}(\mathbb{R}), K'_\infty)_{v|\infty} \times GU(\mathbb{A}_f)$-module which is a restricted tensor product of local irreducible admissible representations. Let $\psi, \tau : \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ be Hecke characters such that $\psi$ is the central character of $\pi$. Let $\tau = \otimes \tau_w$ and $\psi = \otimes \psi_w$ be their local decompositions, $w$ running over places of $F$. Define a representation of $(P(F_\infty) \cap K_\infty) \times P(\mathbb{A}_{F,f})$ by putting:

$$\rho(p)v := \otimes(\rho_w(p_w)v_w),$$

Let $I(\rho)$ be the restricted product $\otimes I(\rho_w)$'s with respect to the $F_{\rho_w}$'s at those $w$ at which $\tau_w, \psi_w, \pi_w$ are unramified. As before, for each $z \in \mathbb{C}$ and $f \in I(\rho)$ we define a function $f_z$ on $GU(r+1, s+1)(\mathbb{A})$ as

$$f_z(g) := \otimes f_{w,z}(g_w)$$

where $f_{w,z}$ are defined as before and an action $\sigma(\rho, z)$ of $(\mathfrak{gu}, K_\infty) \otimes GU(r+1, s+1)(\mathbb{A}_f)$ on $I(\rho)$ by $\sigma(\rho, z) := \otimes \sigma(\rho_w, z)$. Similarly we define $\rho^\vee, I(\rho^\vee)$, and $\sigma(\rho^\vee, z)$ but with the corresponding things replaced by their $\vee$'s.

**Definition 3.1.2.** *Let $\Sigma$ be a finite set of primes of $F$ containing all the infinite places, primes dividing $p$ and places where $\pi$ or $\tau$ is ramified. Then we call the triple $\mathcal{D} = (\pi, \tau, \Sigma)$ an Eisenstein Datum.*

I am sorry to use the same notation as the CM type in section 2.1. The meaning should be clear in the context.

### 3.1.4 Klingen-Type Eisenstein Series on G

We follow [35, 9.1.5] as in this subsection. Let $\pi, \psi$, and $\tau$ be in the above subsection. For $f \in I(\rho), z \in \mathbb{C}$, there are maps from $I(\rho)$ and $I(\rho^\vee)$ to spaces of automorphic forms on $P(\mathbb{A}_F)$ given by

$$f \mapsto (g \mapsto f_z(g)(1)).$$

In the following we often write $f_z$ for the automorphic form given by this recipe.

If $g \in GU(r+1, s+1)(\mathbb{A}_F)$ it is well known that

$$E(f, z, g) := \sum_{\gamma \in P(F)\backslash G(F)} f_z(\gamma g) \tag{3.3}$$

converges absolutely and uniformly for $(z, g)$ in compact subsets of $\{z \in \mathbb{C} : Re(z) > \frac{a+2b+1}{2}\} \times GU(r+1, s+1)(\mathbb{A})$. Therefore we get some automorphic forms which are called Klingen Eisenstein series.

**Definition 3.1.3.** *For any parabolic subgroup $R$ of $GU(r+1, s+1)$ and an automorphic form $\varphi$ we define $\varphi_R$ to be the constant term of $\varphi$ along $R$.*

The following lemma is well-known (see [35, lemma 9.1.6]).

**Lemma 3.1.1.** *Let $R$ be a standard $F$-parabolic of $GU(r+1, s+1)$ (i.e, $R \supseteq B$ where $B$ is the standard Borel). Suppose $Re(z) > \frac{a+2b+1}{2}$.*
*(i) If $R \neq P$ then $E(f, z, g)_R = 0$;*
*(ii) $E(f, z, -)_P = f_z + A(\rho, f, z)_{-z}$.*

## 3.2 Siegel Eisenstein Series on $G_n$

### 3.2.1 Local Picture

Our discussion in this section follows [35, 11.1-11.3] closely. Let $Q = Q_n$ be the Siegel parabolic subgroup of $GU_n$ consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n \times n$ block is zero. For a place $v$ of $F$ and a character $\chi$ of $\mathcal{K}_v^\times$ we let $I_n(\chi)$ be the space of smooth

$K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(\mathcal{O}_{F,v})$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi(detD_q)f(k)$ for all $q \in Q_n(F_v) \cap K_{n,v}$ (we write $q$ as block matrix $q = \begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z \in \mathbb{C}$ and $f \in I(\chi)$ we also define a function $f(z,-) : G_n(F_v) \to \mathbb{C}$ by $f(z, qk) := \chi(detD_q))|detA_qD_q^{-1}|_v^{z+n/2}f(k), q \in Q_n(F_v)$ and $k \in K_{n,v}$.

Let $\chi$ be a unitary character of $\mathcal{K}_v^\times$, $v$ a place of $F$. For $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in K_{n,v}$, the intertwining integral is defined by:

$$M(z,f)(k) := \bar{\chi}^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n r k)dr.$$

For $z$ in compact subsets of $\{Re(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in $k$. In this case it is easy to see that $M(z,f) \in I_n(\bar{\chi}^c)$. A standard fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.

Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\chi)$ taking values in a finite dimensional subspace $V \subset I_n(\chi)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

### 3.2.2 Gloabl Picture

For an idele class character $\chi = \otimes\chi_v$ of $\mathbb{A}_{\mathcal{K}}^\times$ we define a space $I_n(\chi)$ to be the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\chi_v), f_v^{sph}(K_{n,v}) = 1$, at the finite places $v$ where $\chi_v$ is unramified.

For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(F)\backslash G_n(F)} f(z, \gamma g).$$

This series converges absolutely and uniformly for $(z, g)$ in compact subsets of $\{Re(z) > n/2\} \times G_n(\mathbb{A}_F)$. The defined automorphic form is called Siegel Eisenstein series.

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in $z$ to all of $\mathbb{C}$ in the following sense. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well known that it can be meromorphically

continued to all $z \in \mathbb{C}$.

Now for $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in \prod_{v\nmid\infty} K_{n,v} \prod_{v|\infty} K_\infty$ there is a similar intertwining integral $M(z,f)(k)$ as above but with the integration being over $N_{Q_n}(\mathbb{A}_F)$. This again converges absolutely and uniformly for $z$ in compact subsets of $\{Re(z) > n/2\} \times K_n$. Thus $z \mapsto M(z,f)$ defines a holomorphic section $\{Re(z) > n/2\} \to I_n(\bar{\chi}^c)$. This has a continuation to a meromorphic section on $\mathbb{C}$. For $Re(z) > n/2$, we have

$$M(z,f) = \otimes_v M(z,f_v), f = \otimes f_v.$$

The functional equation for Siegel Eisenstein series is:

$$E(f,z,g) = \chi^n(\mu(g))E(M(z,f);-z,g)$$

in the sense that both sides can be meromorphically continued to all $z \in \mathbb{C}$ and the equality is understood as of meromorphic functions of $z \in \mathbb{C}$.

### 3.2.3 the Pull-Back Formulas

Let $\chi$ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$. Given a cuspform $\phi$ on $GU(r,s)$ we consider

$$F_\phi(f;z,g) := \int_{U(r,s)(\mathbb{A}_F)} f(z, S^{-1}\alpha(g,g_1h)S)\bar{\chi}(\det g_1 g)\phi(g_1h)dg_1,$$

$$f \in I_{r+s+1}(\chi), g \in GU(r+1,s+1)(\mathbb{A}_F), h \in GU(r,s)(\mathbb{A}_F), \mu(g) = \mu(h)$$

This is independent of $h$. The pull-back formulas are the identities in the following proposition.

**Proposition 3.2.1.** *Let $\chi$ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$.*
*(i) If $f \in I_{r+s}(\chi)$, then $F_\phi(f;z,g)$ converges absolutely and uniformly for $(z,g)$ in compact sets of $\{Re(z) > r+s\} \times GU(r,s)(\mathbb{A}_F)$, and for any $h \in GU(r,s)(\mathbb{A}_F)$ such that $\mu(h) = \mu(g)$*

$$\int_{U(r,s)(F)\backslash U(r,s)(\mathbb{A}_F)} E(f;z,S'^{-1}\alpha(g,g_1h)S')\bar{\chi}(\det g_1 h)\phi(g_1h)dg_1 = F_\phi(f;z,g). \tag{3.4}$$

*(ii) If $f \in I_{r+s+1}(\chi)$, then $F_\phi(f;z,g)$ converges absolutely and uniformly for $(z,g)$ in compact sets*

*of* $\{Re(z) > r+s+1/2\} \times GU(r+1,s+1)(\mathbb{A}_F)$ *such that* $\mu(h) = \mu(g)$

$$\int_{U(r,s)(F)\backslash U(r,s)(\mathbb{A}_F)} E(f; z, S^{-1}\alpha(g, g_1 h)S)\bar{\chi}(\det g_1 h)\phi(g_1 h)dg_1 = \sum_{\gamma \in P(F)\backslash G(r+1,s+1)(F)} F_\phi(f; z, \gamma g), \tag{3.5}$$

*with the series converging absolutely and uniformly for* $(z, g)$ *in compact subsets of* $\{Re(z) > r+s+1/2\} \times G(r+1, s+1)(\mathbb{A}_F)$.

*Proof.* (i) is proved by Piatetski-Shapiro and Rallis and (ii) is a straight-forward generalization by Shimura [31]. See also [35, proposition 11.2.3]. □

## 3.3 Fourier-Jacobi Expansion

We will usually use the notation $e_{\mathbb{A}}(x) = e_{\mathbb{A}_\mathbb{Q}}(\mathrm{Tr}_{\mathbb{A}_F/\mathbb{A}_\mathbb{Q}} x)$ for $x \in \mathbb{A}_F$. For any automorphic form $\varphi$ on $GU(r,s)(\mathbb{A}_F)$, $\beta \in S_m(F)$ for $m \leq s$. We define the Fourier-Jacobi coefficient at $g \in GU(r,s)(\mathbb{A}_F)$:

$$\varphi_\beta(g) = \int_{S_m(F)\backslash S_m(\mathbb{A}_F)} \varphi\left(\begin{pmatrix} & & S & 0 \\ 1_s & 0 & & \\ & & 0 & 0 \\ 0 & 1_{r-s} & 0 & \\ 0 & 0 & & 1_s \end{pmatrix} g\right) e_{\mathbb{A}}(-\mathrm{Tr}(\beta S))dS.$$

In fact we are mainly interested in two cases: $m = s$ or $r = s$ and arbitary $m \leq s$. In particular, suppose $G = G_n = GU(n,n)$, $0 \leq m \leq n$ are integers, $\beta \in S_m(F)$. Let $\varphi$ be a function on $G(F)\backslash G(\mathbb{A})$. The $\beta$-th Fourier-Jacobi coefficient $\varphi_\beta$ of $\varphi$ at $g$ is defined by

$$\varphi_\beta(g) := \int \varphi\left(\begin{pmatrix} & S & 0 \\ 1_n & 0 & 0 \\ & & 1_n \end{pmatrix} g\right) e_{\mathbb{A}}(-Tr\beta S)dS.$$

Now we prove a useful formula on the Fourier Jacobi coefficients for Siegel Eisenstein series.

**Definition 3.3.1.** *let:*

$$Z := \{\begin{pmatrix} & z & 0 \\ 1_n & 0 & 0 \\ 0_n & & 1_n \end{pmatrix} | z \in \mathrm{Her}_m(\mathcal{K})\}$$

$$V := \{ \begin{pmatrix} 1_m & x & z & y \\ & 1_{n-m} & y^* & 0 \\ & & 1_m & \\ & 0_n & & \\ & & -x^* & 1_{n-m} \end{pmatrix} | x, y \in M_{m(n-m)}(\mathcal{K}), z - xy^* \in \mathrm{Her}_m(\mathcal{K}) \}$$

$$X := \{ \begin{pmatrix} 1_m & x & & 0_n \\ & 1_{n-m} & & \\ & & 1_m & \\ & 0_n & & \\ & & -x^* & 1_{n-m} \end{pmatrix} | x \in M_{m(n-m)}(\mathcal{K}) \}$$

$$Y := \{ \begin{pmatrix} & z & y \\ 1_n & y^* & 0 \\ 0_n & & 1_n \end{pmatrix} | y \in M_{m(n-m)}(\mathcal{K}) \}$$

**Proposition 3.3.1.** *Suppose $f \in I_n(\tau)$ and $\beta \in S_m(F)$, $\beta$ is totally positive. If $E(f; z, g)$ is the Siegel Eisenstein Series on GU defined by $f$ for some $Re(z)$ sufficiently large then the $\beta$-th Fourier-Jacobi coefficient $E_\beta(f; z, g)$ satisfies:*

$$E_\beta(f; z, g) = \sum_{\gamma \in Q_{n-m}(F) \backslash GU_{n-m}(F)} \sum_{y \in Y} \int_{S_m(\mathbb{A})} f(w_n \begin{pmatrix} & S & y \\ 1_n & {}^t\bar{y} & 0 \\ & & 1_n \end{pmatrix} \alpha_{n-m}(1, \gamma) g) e_{\mathbb{A}}(-Tr\beta S) dS$$

.

*Proof.* We follow [21, section 3]. Let $H$ be the normalizer of $V$ in $G$. Then

$$G_n(F) = \sqcup_{i=1}^m Q_n(F) \xi_i H(F)$$

for $\xi_i := \begin{pmatrix} 0_{m-i} & 0 & -1_{m-i} & 0 \\ 0 & 1_{n-m+i} & 0 & 0 \\ 1_{m-i} & 0 & 0_{m-i} & 0 \\ 0 & 0_{n-m+i} & 0 & 1_{n-m+i} \end{pmatrix}$. then unfold the Eisenstein series we get:

$$E_\beta(f;z,g) = \sum_{i>0} \sum_{\gamma \in Q_n(F)\backslash Q_n(F)\xi_i H(F)} \int f(\gamma \begin{pmatrix} & S & 0 \\ 1_n & 0 & 0 \\ & & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S))dS$$
$$+ \sum_{\gamma \in Q_n(F)\backslash Q_n(F)\xi_0 H(F)} \int f(\gamma \begin{pmatrix} & S & 0 \\ 1_n & 0 & 0 \\ & & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S))dS$$

by in [21, lemma(3.1)] (see *loc.cit* P628), the first term vanishes. Also, we have (*loc.cit*)

$$\begin{aligned} & Q_n(F)\backslash Q_n(F)\xi_0 H(F) \\ = \quad & \xi_0 Z(F)X(F)Q_{n-m}(F)\backslash G_{n-m}(F) \\ = \quad & \xi_0 X(F).Q_{n-m}(F)\backslash G_{n-m}(F).Z(F) \\ = \quad & w_n Y(F) S_m(F) w_{n-m} Q_{n-m}(F)\backslash G_{n-m}(F) \end{aligned}$$

(note that $S_m$ commutes with $X$ and $G_{n-m}$). So

$$E_\beta(f;z,g) = \sum_{\gamma \in Q_{n-m}(F)\backslash G_{n-m}(F)} y \sum_{y \in Y(F)} \int_{S_m(\mathbb{A})} f(w_n \begin{pmatrix} & S & y \\ 1_n & {}^t\bar{y} & 0 \\ & & 1_n \end{pmatrix} \alpha_{n-m}(1,\gamma)g) e_{\mathbb{A}}(-Tr(\beta S))$$

Note that the final integral is essentially a product of local ones. □

Now we record some useful formulas:

**Definition 3.3.2.** *If $g_v \in U_{n-m}(F_v), x \in GL_m(\mathcal{K}_v)$, then define:*

$$FJ_\beta(f_v;z,y,g,x) = \int_{S_m(F_v)} f(w_n \begin{pmatrix} & S & y \\ 1_n & {}^t\bar{y} & 0 \\ & & 1_n \end{pmatrix} \alpha(\mathrm{diag}(x, {}^t\bar{x}^{-1}), g)) e_{F_v}(-Tr\beta S)dS$$

*where if* $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, g_2 = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ *then:*

$$\alpha(g_1, g_2) = \begin{pmatrix} A & & B & \\ & D' & & C' \\ C & & D & \\ & B' & & A' \end{pmatrix}$$

Since

$$\begin{pmatrix} & S & X \\ 1_n & & \\ & {}^t\bar{X} & \\ & & 1_n \end{pmatrix} \begin{pmatrix} 1_m & & & \\ & \bar{A}^{-1} & & \\ & & & 1_m \\ & B\bar{A}^{-1} & & A \end{pmatrix} = \begin{pmatrix} 1_m & XB\bar{A}^{-1} & & \\ & \bar{A}^{-1} & & \\ & & & 1_m \\ & B\bar{A}^{-1} & & A \end{pmatrix} \begin{pmatrix} & S - XB^t\bar{X} & XA \\ 1_n & \bar{A}^t\bar{X} & \\ & & 1_n \\ & & \end{pmatrix}.$$

it follows that:

$$FJ_\beta(f; z, X, \begin{pmatrix} A & B\bar{A}^{-1} \\ & \bar{A}^{-1} \end{pmatrix} g, Y) =$$
$$\tau_v^c (\det A)^{-1} |det A\bar{A}|_v^{z+n/2} e_v(-tr({}^t\bar{X}\beta X B)) FJ_\beta(f; z, XA, g, Y)$$

Also we have:

$$FJ_\beta(f; z, y, g, x) = \tau_v(\det x)|\det x\bar{x}|_{\mathbb{A}}^{-(z+\frac{n}{2}-m)} FJ_{t_{\bar{x}\beta x}}(f; z, x^{-1}y, g, 1)$$

### 3.3.1 Weil Representations

The local set-up.

Let $v$ be a place of $F$. Let $h \in S_m(F_v), \det h \neq 0$. Let $U_h$ be the unitary group of this matric and denote $V_v$ to be the corresponding Hermitian space. Let $V_{n-m} := \mathcal{K}_v^{(n-m)} \oplus \mathcal{K}_v^{(n-m)} := X_v \oplus Y_v$ be the Hermitian space associated to $U(n-m, n-m)$. Let $W := V_v \otimes_{\mathcal{K}_v} V_{n-m,v}$. Then $(-,-) := Tr_{\mathcal{K}_v/F_v}(< -,- >_h \otimes_{\mathcal{K}_v} < -,- >_{n-m})$ is a $F_v$ linear pairing on $W$ that makes $W$ into an $4m(n-m)$-dimensional symplectic space over $F_v$. The canonical embeding of $U_h \times U_{n-m}$ into $Sp(W)$ realizes the pair $(U_h, U_{n-m})$ as a dual pair in $Sp(W)$. Let $\lambda_v$ be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{F_v^\times} = \chi_{\mathcal{K}/F,v}^m$. It is well known (see [23]) that there is a splitting $U_h(F_v) \times U_{n-m}(F_v) \hookrightarrow Mp(W, F_v)$

of the metaplectic cover $Mp(W,F_v) \to Sp(W,F_v)$ determined by the character $\lambda_v$. This gives the Weil representation $\omega_{h,v}(u,g)$ of $U_h(F_v) \times U_{n-m}(F_v)$ where $u \in U_h(F_v)$ and $g \in U_{n-m}(F_v)$, via the Weil representation of $Mp(W,F_v)$ on the space of Schwartz functions $\mathcal{S}(V_v \otimes_{\mathcal{K}_v} X_v)$. Moreover we write $\omega_{h,v}(g)$ to mean $\omega_{h,v}(1,g)$. For $X \in M_{m\times(n-m)}(\mathcal{K}_v)$, we define $< X, X >_h := {}^t\bar{X}\beta X$ (note that this is a $(n-m)\times(n-m)$ matrix). We record here some useful formulas for $\omega_{h,v}$ which are generalizations of the formulas in [35, chapter 10].

- $\omega_{h,v}(u,g)\Phi(X) = \omega_{h,v}(1,g)\Phi(u^{-1}X)$
- $\omega_{h,v}(diag(A, {}^t\bar{A}^{-1}))\Phi(X) = \lambda(\det A)|\det A|_{\mathcal{K}}\Phi(XA)$,
- $\omega_{h,v}(r(S))\Phi(x) = \Phi(x)e_v(\mathrm{tr} < X, X >_h S)$,
- $\omega_{h,v}(\eta)\Phi(x) = |\det h|_v \int \Phi(Y)e_v(Tr_{\mathcal{K}_v/F_v}(\mathrm{tr} < Y, X >_h))dY$.

global setup:

Let $h \in S_m(F), h > 0$. We can define global versions of $U_h, GU_h, W$, and $(-,-)$, analogously to the above. Fixing an idele class character $\lambda = \otimes\lambda_v$ of $\mathbb{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times}$ such that $\lambda|_{F^\times} = \chi_{\mathcal{K}/F}^m$, the associated local splitting described above then determine a global splitting $U_h(\mathbb{A}_F) \times U_1(\mathbb{A}_F) \hookrightarrow Mp(W,\mathbb{A}_F)$ and hence an action $\omega_h := \otimes\omega_{h,v}$ of $U_h(\mathbb{A}_F) \times U_1(\mathbb{A}_F)$ on the Schwartz space $\mathcal{S}(V_{\mathbb{A}_{\mathcal{K}}} \otimes X)$.

# Chapter 4

# Local Computations

In this chapter we do the local computations for Klingen Eisenstein sections realized as the pullbacks of Siegel Eisenstein sections. We will mainly compute the Fourier and Fourier-Jacobi coefficients for the Siegel sections and the pullback Klingen Eisenstein section. From now on we will usually write $w_n$ for $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.

## 4.1 Archimedean Computations

Let $v$ be an Archimedean place of $F$.

### 4.1.1 Fourier Coefficients

Now we recall a lemma from [35, 11.4.2]. Let $J_n(g, i1_n) := \det(C_g i + D_g)$ for $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$.

**Lemma 4.1.1.** *If we define $f_{\kappa,n}(z,g) = J_n(g,i1_n)^{-\kappa}|J_n(g,i1_n)|^{\kappa-2z-n}$,suppose $\beta \in S_n(\mathbb{R})$. Then the function $z \to f_{\kappa,\beta}(z,g)$ has a meromorphic continuation to all of $\mathbb{C}$. Furthermor, if $\kappa \geq n$ then $f_{\kappa,n,\beta}(z,g)$ is holomorphic at $z_\kappa := (\kappa-n)/2$ and for $y \in \mathrm{GL}_n(\mathbb{C})$, $f_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = 0$ if $\det\beta \leq 0$ and if $\det\beta > 0$ then*

$$f_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa-j-1)!} e(i\mathrm{Tr}(\beta y {}^t\bar{y}))\det(\beta)^{\kappa-n}\det\bar{y}^{\kappa}.$$

Later on our $f_{\kappa,n}$ will be defined differently, but it is just the one defined above translated by matrices of the form $\mathrm{diag}(y, {}^t\bar{y}^{-1})$. So the Fourier coefficient can be deduced from the above lemma.

### 4.1.2 Pullback Sections

Now we assume that our $\pi$ is the holomorphic discrete series representation associated to the (scalar) weight $(0,...,0;\kappa,...,\kappa)$ and let $\phi$ be the unique (up to scalar) vector such that the action of $K_\infty^{+,\prime}$ (see section 3.1) is given by $\det \mu(k,i)^{-\kappa}$. Recall also that in section 3.1 we have defined the Klingen section $F_\kappa(z,g)$ there. Recall that:

$$S = \begin{pmatrix} 1 & & & & & & & -\frac{1}{2} \\ & 1 & & & & & & \\ & & 1 & & & & -\frac{\zeta}{2} & \\ & & & -1 & \frac{1}{2} & & & \\ & & & 1 & \frac{1}{2} & & & \\ & & & & & 1 & & \\ & & -1 & & & & -\frac{\zeta}{2} & \\ -1 & & & & & & & -\frac{1}{2} \end{pmatrix}$$

and

$$S' = \begin{pmatrix} 1 & & & & & -\frac{1}{2} \\ & 1 & & & -\frac{\zeta}{2} & \\ & & -1 & \frac{1}{2} & & \\ & & 1 & \frac{1}{2} & & \\ & -1 & & & -\frac{\zeta}{2} & \\ -1 & & & & & -\frac{1}{2} \end{pmatrix}.$$

Let $\mathbf{i} := \begin{pmatrix} \frac{i}{2}1_b & & \\ & i & \\ & & \frac{\zeta}{2}1_a \end{pmatrix}$ be a point in the symmetric domain for $GU(n,n)$ or $GU(n+1,n+1)$ for $n = a+2b$, where the block matrices $i$ are of size $b \times b$ or $(b+1)\times(b+1)$. We define archimedean sections to be:

$$f_\kappa(g) = J_{n+1}(g,\mathbf{i})^{-\kappa}|J_{n+1}(g,\mathbf{i})|^{\kappa-2z-n-1}$$

and

$$f'_\kappa(g) = J_n(g,\mathbf{i})^{-\kappa}|J_n(g,\mathbf{i})|^{\kappa-2z-n}$$

and the pull back sections on $GU(a+b+1,b+1)$ and $GU(a+b,a)$ to be

$$F_\kappa(z,g) := \int_{U_{a+b,b}(\mathbf{R})} f_\kappa(z, S^{-1}\alpha(g,g_1)S)\bar{\tau}(\det g_1)\pi(g_1)\phi dg_1$$

and

$$F'_\kappa(z,g) := \int_{U_{a+b,b}(\mathbf{R})} f'_\kappa(z, S'^{-1}\alpha(g,g_1)S')\bar{\tau}(\det g_1)\pi(g_1)\phi dg_1$$

**Lemma 4.1.2.** *The integrals are absolutely convergent for $Re(z)$ sufficiently large and for such $z$, we have:*

*(i)*

$$F_\kappa(z,g) = c_\kappa(z)F_{\kappa,z}(g);$$

*(ii)*

$$F'_\kappa(z,g) = c'_\kappa(z)\pi(g)\phi;$$

*where*

$$c'_\kappa(z,g) = 2^\nu |\det\theta|_v^b \begin{cases} \pi^{(a_v+b_v)b_v}\Gamma_{b_v}(z+\frac{n+\kappa}{2}-a_v-b_v)\Gamma_{b_v}(z+\frac{n+k}{2})^{-1}, & b>0 \\ 1, & \textit{otherwise.} \end{cases}$$

*and $c_\kappa(z,g) = c'_\kappa(z+\frac{1}{2},g)$. Here $\Gamma_m := \pi^{\frac{m(m+1)}{2}}\prod_{k=0}^{m-1}\Gamma(s-k)$ and $\nu := (a+2b)db$ (recall that $d=[F:\mathbb{Q}]$).*

*Proof.* See [31, 22.2, A2.9]. Note that the action of $(\beta,\gamma)\in U(r,s)\times U(r,s)$ is given by $(\beta',\gamma')$ defined there. Taking this into consideration, our conjugation matrix $S$ are Shimura's $S$ times $\Sigma^{-1}$, which is defined in (22.1.2) in [31]. Also our result differs from [35, 11.4.4] by some powers of 2 since we are using a different $S$ here. □

### 4.1.3 Fourier-Jacobi Coefficients

**Lemma 4.1.3.** *Let $z_\kappa = \frac{\kappa-n}{2}$, $\beta\in S_m(\mathbb{R})$, $m<n$, $\det\beta>0$. then:*

*(i) $FJ_{\beta,\kappa}(z_\kappa,x,\eta,1) = f_{\kappa,m,\beta}(z_\kappa+\frac{n-m}{2},1)e(iTr({}^t\bar{X}\beta X))$;*

*(ii) If $g\in U_{n-m}(\mathbb{R})$, then*

$$FJ_{\beta,\kappa}(z_\kappa,X,g,1) = e(iTr\beta)c_m(\beta,\kappa)f_{\kappa-m,n-m}(z_\kappa,g')w_\beta(g')\Phi_{\beta,\infty}(x).$$

*where* $g' = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}$, $c_t(\beta,\kappa) = \frac{(-2)^{-t}(2\pi i)^{t\kappa}(2/\pi)^{t(t-1)/2}}{\prod_{j=0}^{t-1}(\kappa-j-1)} \det \beta^{\kappa-t}$ *and* $\Phi_{\beta,\infty} = e^{-2\pi \mathrm{Tr}(<x,x>_\beta)}$.

*Proof.* For (i) we first assume that $m \leq n/2$, then there is a matrix $U \in U_{n-m}$ such that $XU = (0, A)$ for $A$ a $(m \times m)$ positive semi-definite Hermitian matrix . It then follows that $FJ_{\beta,\kappa}(z, X, \eta, 1) = FJ_\beta(z, (0, A), \eta, 1)$ and $e(iTr({}^t\bar{X}\beta X)) = e(iTr(U^{-1}{}^t\bar{X}\beta XU))$, so we are reduced to the case when $X = (0, A)$.

Let $C$ be a $(m \times m)$ positive definite Hermitian matrix defined by $C = \sqrt{A^2 + 1}$. (Since $A$ is positive semi-definite Hermitian, this $C$ exists by linear algebra.)

$$\begin{pmatrix} & & A \\ 1_n & & \\ & A & \\ & & 1_n \end{pmatrix} = \begin{pmatrix} C & & & & & \\ & 1 & & & & \\ & & C & & & \\ & & AC^{-1} & C^{-1} & & \\ & & & & 1 & \\ AC^{-1} & & & & & C^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & & & & & C^{-1}A \\ & 1 & & & & \\ & & C^{-1} & C^{-1}A & & \\ & & -C^{-1}A & C^{-1} & & \\ & & & & 1 & \\ -C^{-1}A & & & & & C^{-1} \end{pmatrix}$$

write $k(a)$ for the second matrix in the right of above which belongs to $K^+_{n,\infty}$, then as in [35, lemma 11.4.3],

$$w_n \begin{pmatrix} & S & X \\ 1_n & {}^t\bar{X} & \\ & & 1_n \end{pmatrix} = \begin{pmatrix} C^{-1} & & & \times & \times & \times \\ & 1 & & \times & \times & \times \\ \times & \times & C^{-1} & \times & \times & \times \\ & & & C & & \\ & & & & 1 & \\ & & & \times & \times & C \end{pmatrix} w_n \begin{pmatrix} & U^{-1}SU^{-1} & \\ 1_n & & \\ & & 1_n \end{pmatrix} k(a).$$

Thus

$$\begin{aligned} FJ_{\beta,\kappa}(z_\kappa, (0, A), \eta, 1) &= (\det C)^{2m-2\kappa} FJ_{\beta',\kappa}(z_\kappa, 0, \eta, 1), \beta' = C\beta C \\ &= (\det C)^{2m-2\kappa} f_{\kappa,m,\beta'}(z_\kappa + \frac{n-m}{2}, 1) \\ &= f_{\kappa,m,\beta}(z + \frac{n-m}{2}, 1) e(i\mathrm{Tr}(C\beta C - \beta)). \end{aligned}$$

But

$$e(iTr(C\beta C-\beta))=e(iTr(C^2\beta-\beta))=e(iTr((C^2-1)\beta))=e(iTr(A^2\beta))=e(iTr(A\beta A)).$$

This proves part (i).

Part (ii) is proved completely the same as in [35, lemma 11.4.3].

In the case when $m>\frac{n}{2}$ we proceed similarly as in [35, lemma 11.4.3], replacing $a$ and $u$ there by corresponding block matrices just as above. We omit the details.

□

## 4.2 Finite Primes, Unramified Case

### 4.2.1 Fourier-Jacobi Coefficients

Let $v$ be a prime of $F$ not dividing $p$ and $\tau$ a character of $\mathcal{K}_v^\times$. For $f\in I_n(\tau)$ and $\beta\in S_m(F_v), 0\leq m\leq n$, we have defined the local Fourier-Jacobi coefficient to be

$$f_\beta(z;g):=\int_{S_m(F_v)} f(z,w_n\begin{pmatrix} & S & 0\\ 1_n & 0 & 0\\ & & 1_n\end{pmatrix}g)e_v(-Tr\beta S)dS.$$

We first record a generalization of [35, lemma 11.4.6] to any fields (Proposition 18.14 and 19.2 of [31]).

**Lemma 4.2.1.** *Let $\beta\in S_n(F_v)$ and let $r:=rank(\beta)$. Then for $y\in GL_n(\mathcal{K}_v)$,*

$$\begin{aligned} f_{v,\beta}^{sph}(z,diag(y,{}^t\bar{y}^{-1}))= &\quad \tau(dety)|dety\bar{y}|_v^{-z+n/2}D_v^{-n(n-1)/4}\\ &\times\frac{\prod_{i=r}^{n-1}L(2z+i-n+1,\bar{\tau}'\chi_{\mathcal{K}}^i)}{\prod_{i=0}^{n-1}L(2z+n-i,\bar{\tau}\chi_{\mathcal{K}}^i)}h_{v,{}^t\bar{y}\beta y}(\bar{\tau}'(\varpi)q_v^{-2z-n}).\end{aligned}$$

*where $h_{v,{}^t\bar{y}\beta y}\in\mathbb{Z}[X]$ is a monic polynomial depending on $v$ and ${}^t\bar{y}\beta y$ but not on $\tau$. If $\beta\in S_n(\mathcal{O}_{F,v})$ and $\det\beta\in\mathcal{O}_{F,v}^\times$, then we say that $\beta$ is $v$-primitive and in this case $h_{v,\beta}=1$.*

**Lemma 4.2.2.** *Suppose $v$ is unramified in $\mathcal{K}$. Let $\beta\in S_m(F_v)$ such that $\det\beta\neq 0$. Let $y\in GL_{n-m}(\mathcal{K}_v)$ such that ${}^t\bar{y}\beta y\in S_m(\mathcal{O}_{F_v})$, let $\lambda$ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v{}^\times}=1$.*

*(i) If $\beta, y \in GL_m(\mathcal{O}_v)$, then for $u \in U_\beta(F_v)$:*

$$FJ_\beta(f_n^{sph}; z, x, g, uy) = \tau(\det u)|\det u\bar{u}|_v^{-z+1/2} \frac{f_{n-m}^{sph}(z, g)\omega_\beta(u, g)\Phi_{0,y}(x)}{\prod_{i=0}^{m-1} L(2z + n - i, \bar{\tau}'\chi_{\mathcal{K}}^i)}.$$

*(ii) If ${}^t\bar{y}\beta y \in GL_m(\mathcal{O}_v)$, then for $u \in U_\beta(F_v)$;*

$$FJ_\beta(f_n^{sph}; z, x, g, uy) = \tau(\det uy)|\det uy\bar{u}|_{\mathcal{K}}^{-z+1/2} \frac{f_{n-m}^{sph}(z, g)\omega_\beta(u, g)\Phi_{0,y}(x)}{\prod_{i=0}^{m-1} L(2z + n - i, \bar{\tau}'\chi_{\mathcal{K}}^i)}.$$

### 4.2.2 Pull-Back Integrals

**Lemma 4.2.3.** *Suppose $\pi, \psi$ and $\tau$ are unramified and $\phi \in \pi$ is a newvector. If $Re(z) > (a + b)/2$ then the pull back integral converges and*

$$F_\phi(f_v^{sph}; z, g) = \frac{L(\tilde{\pi}, \bar{\tau}^c, z + 1)}{\prod_{i=0}^{a+2b-1} L(2z + a + 2b + 1 - i, \bar{\tau}'\chi_{\mathcal{K}}^i)} F_{\rho,z}(g)$$

*where $F_\rho$ is the spherical section.*

This is computed in [25, proposition 3.3].

## 4.3 Prime to $p$ Ramified Case

### 4.3.1 Pull Back integrals

Again let $v$ be a prime of $F$ not dividing $p$. The choices in this section are not quite important. In fact in applications we are going to change it according to our needs. The purpose for this section is only to convince the reader that such kinds of sections do exist. We define $f^\dagger$ to be the Siegel section supported on the cell $Q(F_v)w_{a+2b+1}N_Q(\mathcal{O}_{F,v})$ where $w_{a+2b+1} = \begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & \end{pmatrix}$ and the value at $N_Q(\mathcal{O}_{F,v})$ equals 1. We fix some $x$ and $y$ in $\mathcal{K}$ which are divisible by some high power of $\varpi_v$. (When we are moving things $p$-adically the $x$ and $y$ are not going to change).

**Definition 4.3.1.**

$$f_{v,sieg}(g) := f(\begin{pmatrix} 1_{a+2b+1} & \\ & \frac{1}{2}1_{a+2b+1} \end{pmatrix} g \begin{pmatrix} 1_{a+2b+1} & \\ & 2.1_{a+2b+1} \end{pmatrix} \tilde{\gamma}_v)$$

*where $\tilde{\gamma}_v$ is defined to be:*

$$\begin{pmatrix} 1 & & & & & & & \frac{1}{x} \\ & 1 & & & & & & \\ & & 1 & & & & \frac{1}{y\bar{y}} & \\ & & & 1 & \frac{1}{\bar{x}} & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

**Lemma 4.3.1.** *Let $K_v^{(2)}$ be the subgroup of $G(F_v)$ of the form* $\begin{pmatrix} 1 & & & & d \\ a & 1 & f & b & c \\ & & 1 & & g \\ & & & 1 & e \\ & & & & 1 \end{pmatrix}$ *where $e = -{}^t\bar{a}$, $b = {}^t\bar{d}$, $g = -\theta {}^t\bar{f}$, $c = \bar{c}$, $a \in (x)$, $e \in (\bar{x})$, $f \in (y\bar{y})$, $g \in (2\zeta y\bar{y})$. Then $F_\phi(z; g, f)$ is supported in $PwK_v^{(2)}$ and is invariant under the action of $K_v^{(2)}$.*

*Proof.* Let $S_{x,y}$ consists of matrices: $S := \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$ in the space of Hermitian $(a+2b+1) \times (a+2b+1)$ matrices (the blocks are with respect to the partition $b+1+a+b$) such that the entries of $S_{13}, S_{23}$ are divisible by $y$, the entries of $S_{14}, S_{24}$ are divisible by $x$, the entries of $S_{31}, S_{32}$ are divisible by $\bar{y}$, the entries of $S_{41}, S_{42}$ are divisible by $\bar{x}$, the entries of $S_{33}$ are divisible by $y\bar{y}$, the entries of $S_{34}$ are divisible by $x\bar{y}$, the entries of $S_{43}$ are divisible by $\bar{x}y$, and the entries of $S_{44}$ are divisible by $x\bar{x}$. Let $Q_{x,y} := Q(F_v)$. $\begin{pmatrix} 1 & \\ S_{x,y} & 1 \end{pmatrix}$.

For $g = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$, we have:

$$\begin{aligned} & \gamma(g,1) \in \mathrm{supp} f_{v,sieg} \\ \Leftrightarrow \quad & \gamma(g,1) w_{a+2b+1} d_{x,y} \tilde{\gamma}^{-1} \in Q_{x,y} \\ \Leftrightarrow \quad & \gamma(gw, \eta \mathrm{diag}(\bar{x}^{-1}, 1, x)) w' d_y \tilde{\delta}^{-1} \in Q_{x,y}. \end{aligned}$$

Here $d_{x,y} = \mathrm{diag}(1,1,y,x,1,1,\bar{y}^{-1},\bar{x}^{-1})$ and $d_y = \mathrm{diag}(1,1,y,1,1,1,\bar{y}^{-1},1)$. where $x$ and $y$ here stand for the corresponding block matrices of the corresponding size. Recall that $\gamma((m(g_1,1),g_1) \in Q$, by multiplying this to the left for $g_1 = \mathrm{diag}\bar{x},1,x^{-1})\eta^{-1}$, we are reduced to proving that if $\gamma(g,1)w'd_y\tilde{\gamma}^{-1} \in Q_{x,y}$, then $g \in PwK_v^{(2)}w^{-1}$. A computation tells us that: $\gamma(g,1)w'd_y\tilde{\gamma}^{-1}$ equals:

$$\begin{pmatrix} a_1 & a_2 & \zeta a_3 y & -b_1 & b_1 & b_2 & a_3\bar{y}^{-1} & \\ a_4 & a_5 & \zeta a_6 y - a_6\bar{y}^{-1} & -b_3 & b_3 & b_4 & a_6\bar{y}^{-1} & \\ a_7/2 & a_8/2 & \frac{\zeta y(a_9-1)}{2} - \frac{(a_9+1)\bar{y}^{-1}}{2} & -\frac{b_5}{2} & \frac{b_5}{2} & \frac{b_6}{2} & \frac{(a_9+1)\bar{y}^{-1}}{2} & \\ & & & 1 & & & & \\ c_1 & c_2 & \zeta c_3 y - c_3\bar{y}^{-1} & 1-d_1 & d_1 & d_2 & c_3\bar{y}^{-1} & \\ c_4 & c_5 & \zeta c_6 y - c_6\bar{y}^{-1} & -d_3 & d_3 & d_4 & c_6\bar{y}^{-1} & \\ -\frac{\zeta^{-1}}{2}a_7 & -\frac{\zeta^{-1}}{2}a_8 & -\frac{(a_9+1)}{2}y + \frac{\zeta^{-1}}{2}(a_9-1)\bar{y}^{-1} & \frac{\zeta^{-1}}{2}b_5 & -\frac{\zeta^{-1}}{2}b_5 & -\frac{\zeta^{-1}}{2}b_6 & \frac{\zeta^{-1}}{2}(1-a_9)\bar{y}^{-1} & \\ a_1-1 & a_2 & \zeta a_3 y - a_3\bar{y}^{-1} & -b_1 & b_1 & b_2 & a_3\bar{y}^{-1} & 1 \end{pmatrix}$$

One first proves that $d_4 \neq 0$ by looking at the second row of the lower left of the above matrix, so by left multiplying $g$ by some matrix in $N_P$, we may assume that $d_2 = b_2 = b_4 = b_6 = 0$, then the result follows by an argument similarly as lemma 4.4.11 later on. □

Now recall that $g = \begin{pmatrix} a_5 & a_6 & a_4 \\ a_8 & a_9 & a_7 \\ a_2 & a_3 & a_1 \end{pmatrix}$. Let $\mathfrak{Y}$ be the set of $g$'s so that the entries of $a_2$ are integers, the entries of $a_3$ are divisible by $y\bar{y}$, the entries of $a_1-1$ are divisible by $\bar{x}$, the entries of $1-a_5$ are divisible by $x$, the entries of $a_6$ are divisible by $\bar{x}y$, the entries of $a_4$ are divisible by $x\bar{x}$, $\frac{1-a_9}{2} = y\bar{y}\zeta(1+y\bar{y}N)$ for some $N$ with integral entries, the entries of $a_8$ are divisible by $\bar{y}y\zeta$, and the entries of $a_7$ are

divisible by $\bar{y}yx\zeta$.

**Lemma 4.3.2.** *Let $\phi_x = \pi(\mathrm{diag}(\bar{x}, 1, x^{-1})\eta^{-1})\phi$ where $\phi$ is invariant under the action of $\mathfrak{Y}$ defined above, then $F_{\phi_x}(z,w) = \tau(y\bar{y}x)|(y\bar{y})^2 x\bar{x}|_v^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\mathfrak{Y}).\phi$.*

*Proof.*

$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \frac{1}{2} & & & & \frac{1}{2} & \\ & & & 1 & & & & \\ & & & 1 & 1 & & & \\ & & & & & 1 & & \\ & & -\frac{\zeta^{-1}}{2} & & & & -\frac{\zeta^{-1}}{2} & \\ 1 & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & a_1 & & & a_3 & a_2 \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & a_7 & & & a_9 & a_6 \\ & & & a_4 & & & a_6 & a_5 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & -\zeta \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & -1 & & & & -\zeta & \\ & & & & & & & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & -1 & & & & & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & y & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \bar{y}^{-1} & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & -1 & 1 & & \\ & & & & & & 1 & \\ & & -1 & & & & & 1 \\ -1 & & & & & & & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ \frac{a_8}{2} & \frac{\zeta y}{2}(1-a_9) - \frac{\bar{y}^{-1}}{2}(1+a_9) & -\frac{a_7}{2} & & & \frac{\bar{y}^{-1}}{2}(1+a_9) & -\frac{a_8}{2} \\ -a_2 & a_3\zeta y + a_3\bar{y}^{-1} & a_1 & & & -a_3\bar{y}^{-1} & a_2 \\ -a_2 & a_3\zeta y + a_3\bar{y}^{-1} & a_1 - 1 & 1 & & -a_3\bar{y}^{-1} & a_2 \\ & & & & 1 & & \\ \frac{\zeta^{-1}}{2}a_8 & -\frac{y}{2}(1+a_9) + \frac{\zeta^{-1}\bar{y}^{-1}}{2}(1-a_9) & -\frac{\zeta^{-1}}{2}a_7 & & & -\frac{\zeta^{-1}\bar{y}^{-1}}{2}(1-a_9) & -\frac{\zeta^{-1}}{2}a_8 \\ 1-a_5 & a_6\zeta y + a_6\bar{y}^{-1} & a_4 & & & -a_6\bar{y}^{-1} & a_5 \end{pmatrix}$$

One checks the above matrix belongs to $Q_{x,y}$ if and only if the $a_i$'s satisfy the conditions required by the definition of $\mathfrak{Y}$. The lemma follows by a similar argument as in lemma 4.4.12 below. $\square$

**Definition 4.3.2.** *We fix a constant $C_v$ such that $C_v\mathrm{Vol}(\mathfrak{Y})$ is a p-adic integer.*

When we are moving things in $p$-adic families, this constant is not going to change.

### 4.3.2 Fourier-Jacobi Coefficient

Here we record a lemma on the Fourier-Jacobi coefficient for $f_v^\dagger \in I_n(\tau_v)$ and $\beta \in S_m(F_v)$.

**Lemma 4.3.3.** *If $\beta \notin S_m(\mathcal{O}_{F_v})^*$ then $FJ_\beta(f^\dagger; z, u, g, hy) = 0$. If $\beta \in S_n(\mathcal{O}_{F_v})^*$ then*

$$FJ_\beta(f^\dagger; z, u, g, 1) = f^\dagger(z, g'\eta)\omega_\beta(h, g'\eta^{-1})\Phi_{0,y}(u).\mathrm{Vol}(S_m(\mathcal{O}_{F_v})),$$

*where* $g' = \begin{pmatrix} 1_{n-m} & \\ & -1_{n-m} \end{pmatrix} g \begin{pmatrix} 1_{n-m} & \\ & -1_{n-m} \end{pmatrix}$.

The proof is similar to [35, lemma 11.4.16].

## 4.4 $p$-adic Computations

In this section we first prove that under some 'generic conditions' the unique up to scalar nearly ordinary vector in $I(\rho)$ is just the unique up to scalar vector with certain prescribed level action. Then we construct a section $F^\dagger$ in $I(\rho^\vee)$ which is the pull back of a Siegel section $f^\dagger$ supported in the big cell. We can understand the level action of this section. Then we define $F^0$ to be the image of $F^\dagger$ under the intertwining operator. By checking the level action of $F^0$ we can prove that it is just the nearly ordinary vector.

### 4.4.1 Nearly Ordinary Sections

Let $\lambda_1, ..., \lambda_n$ be $n$ characters of $\mathbb{Q}_p^\times$, $\pi = Ind_B^{GL_n}(\lambda_1, ..., \lambda_n)$.

**Definition 4.4.1.** *Let $n = r + s$ and $\underline{k} = (c_{r+s}, ..., c_{s+1}; c_1, ..., c_s)$ be a weight. We say $(\lambda_1, ...\lambda_n)$ is nearly ordinary with respect to $\underline{k}$ if the set:*

$$\{\mathrm{val}_p\lambda_1(p), ..., \mathrm{val}_p\lambda_n(p)\}$$

$$= \{c_1 + s - 1 - \tfrac{n}{2} + \tfrac{1}{2}, c_2 + s - 2 - \tfrac{n}{2} + \tfrac{1}{2}, ..., c_s - \tfrac{n}{2} + \tfrac{1}{2}, c_{s+1} + r + s - 1 - \tfrac{n}{2} + \tfrac{1}{2}, ..., c_{r+s} + s - \tfrac{n}{2} + \tfrac{1}{2}\}$$

We denote the set as $\{\kappa_1, ..., \kappa_{r+s}\}$. Thus $\kappa_1 > ... > \kappa_{r+s}$.

Let $\mathcal{A}_p := \mathbb{Z}_p[t_1, t_2, ..., t_n, t_n^{-1}]$ be the Atkin-Lehner ring of $G(\mathbb{Q}_p)$, where $t_i$ is defined by $t_i = N(\mathbb{Z}_p)\alpha_i N(\mathbb{Z}_p)$, $\alpha_i = \begin{pmatrix} 1_{n-i} & \\ & p1_i \end{pmatrix}$. Then $t_i$ acts on $\pi^{N(\mathbb{Z}_p)}$ by

$$v|t_i = \sum_{x \in N|\alpha_i^{-1} N\alpha_i} x_i \alpha_i^{-1} v.$$

We also define a normalized action with respect to the weight $\underline{k}$ following ([12]):

$$v\|t_i := \delta(\alpha_i)^{-1/2} p^{\kappa_1 + ... + \kappa_i} v|t_i$$

**Definition 4.4.2.** *A vector $v \in \pi$ is called nearly ordinary if it is an eigenvector for all $\|t_i$'s with eigenvalues that are p-adic units.*

We identify $\pi$ as a set of smooth functions on $GL_n(\mathbb{Q}_p)$:

$$\pi = \{f : GL_n(\mathbb{Q}_p) \to \mathbb{C}, f(bx) = \lambda(b)\delta(b)^{1/2} f(x)\}.$$

Let $w_\ell$ be the longest Weyl element $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & ... & & \\ 1 & & & \end{pmatrix}$, and let $f^\ell$ be the element in $\pi$ such that $f^\ell$ is supported in $Bw_\ell N(\mathbb{Z}_p)$ and invariant under $N(\mathbb{Z}_p)$; $f^\ell$ is unique up to scalar. We have:

**Lemma 4.4.1.** *$f^\ell$ is an eigenvector for all $t_i$'s.*

*Proof.* Note that for any $i$, $f^\ell|t_i$ is invariant under $N(\mathbb{Z}_p)$. By looking at the definition of $v|t_i$ for the above model for $\pi$ it is not hard to see that $f^\ell|t_i$ is supported in $B(\mathbb{Q}_p)w_\ell B(\mathbb{Z}_p)$. So $f^\ell|t_i$ must be a multiple of $f^\ell$. □

**Lemma 4.4.2.** *Suppose that $(\lambda_1, ..., \lambda_n)$ is nearly ordinary with respect to $\underline{k}$ and suppose*

$$\nu_p(\lambda_1(p)) > \nu_p(\lambda_2(p)) > ... > \nu_p(\lambda_n(p))$$

*then the eigenvalues of $\|t_i$ acting on $f^\ell$ are p-adic units. In other words $f^\ell$ is an ordinary vector.*

*Proof.* A straightforward computation gives that

$$f^\ell\|t_i = \lambda_1 ... \lambda_i(p^{-1}) p^{\kappa_1 + ... + \kappa_i} f^\ell$$

which is clearly a $p$-adic unit by the definition of $(\lambda_1, ..., \lambda_n)$ to be nearly ordinary with respect to $\underline{k}$. □

**Lemma 4.4.3.** *Let $\lambda_1, ..., \lambda_{a+2b}$ be a set characters of $\mathbb{Q}_p^\times$ such that $cond(\lambda_{a+2b}) > ..., > cond(\lambda_{b+1}) > cond(\lambda_1) > ... > cond(\lambda_b)$. In this case we say $\lambda := (\lambda_1, ..., \lambda_{a+2b})$ is generic. We define a subgroup: $K_\lambda$ of $GL_{a+2b}(\mathbb{Z}_p)$ to be those matrices whose below diagonal entries of the ith column are divisible by $cond(\lambda_{a+2b+1-i})$ for $1 \leq i \leq a+b$, and the left to diagonal entries of the jth row are divisible by $cond(\lambda_{a+2b+1-j})$ for $a+b+2 \leq j \leq a+2b$. Let $\lambda^{op}$ be the character of $K_\lambda$ defined by:*

$$\lambda_{a+2b}(g_{11})\lambda_{a+2b-1}(g_{22})...\lambda_1(g_{a+2b\ a+2b}).$$

*Then $f^\ell$ is the unique (up to scalar) vector in $\pi$ such that the action of $K_\lambda$ is given by multiplying $\lambda^{op}$.*

*Proof.* This can be proven in the same way as [35, 9.2.6]. □

We let $w_1 := \begin{pmatrix} 1 & & & & & \\ & ... & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & ... & \\ & & & 1 & & \end{pmatrix}$. Now let $\tilde{B} = B^{w_1}$ and $\tilde{K}_\lambda = K_\lambda^{w_1}$.

**Corollary 4.4.1.** *Denote $a_i := \nu_p(\lambda_i(p))$. Suppose $\lambda_1, ..., \lambda_{a+2b}$ are such that $cond(\lambda_1) > ... > cond(\lambda_{a+2b})$ and $a_1 < ... < a_{a+b} < a_{a+2b} < ... < a_{a+b+1}$. Then the unique (up to scalar) ordinary section with respect to $\tilde{B}$ is*

$$f^{ord}(x) = \begin{cases} \lambda_1(g_{11})...\lambda_{a+2b}(g_{a+2b,a+2b}), & g \in \tilde{K}_\lambda, \\ 0 & otherwise\ . \end{cases}$$

*Proof.* We only need to prove that $\pi(w_1)f^{ord}(x)$ is ordinary with respect to $\tilde{B}^{w_1} = B$ . Let $\lambda_1' = \lambda_{a+b+1}, ..., \lambda_b' = \lambda_{a+2b}, \lambda_{b+1}' = \lambda_{a+b}, ..., \lambda_{a+2b}' = \lambda_1$. Then $\lambda'$ satisfies lemma 4.4.2 and thus the ordinary section for $B$ (up to scalar) is $f_{\lambda'}^\ell$. Since $\lambda'$ also satisfies the assumptions of lemma 4.4.3, $f_{\lambda'}^\ell$ is the unique section such that the action of $K_\lambda$ is given by $\lambda_{a+2b}'(g_{11})...\lambda_1'(g_{a+2b,a+2b})$. But $\lambda$ is clearly regular, so $Ind_B^{GL_{a+2b}}(\lambda) \simeq Ind_B^{GL_{a+2b}}(\lambda')$. So the ordinary section of $Ind_B^{GL_{a+2b}}(\lambda)$ for $B$ also has the action of $K_\lambda$ given by this character. It is easy to check that $\pi(w_1)f^{ord}$ has this property and the uniqueness (up to scalar) gives the result. □

### 4.4.2 Pull Back Sections

In this section we construct a Siegel section on $U(a+2b+1, a+2b+1)$ which pulls back to the nearly ordinary Klingen sections on $U(a+b+1, b+1)$. We need to re-index the rows and columns since we are going to study large block matrices and the new index will greatly simplify the explanation. One can check that the Klingen Eisenstein series we construct in this section, when going back to our previous index, is nearly ordinary with respect to the Borel : $\begin{pmatrix} * & * & & * & * \\ & * & & * & * \\ & * & * & * & * \\ & & & * & \\ & & & * & * \end{pmatrix}$, where the diagonal blocks are upper, upper, upper, lower, lower triangular, while the one we need is nearly ordinary with respect to the Borel $\begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & & * \\ & & & * & * \end{pmatrix}$ (it is for this one we can use the $\Lambda$-adic Fourier-Jacobi expansions). (here the blocks are with respect to the partition: $b+1+a+b+1$.) However we will see that the nearly ordinary sections with respect to different Borels only differ by right translation by some Weyl element depending on $a$ and $b$. We will specify this Weyl element when doing arithmetic applications.

Now we explain the new index. Let $V_{s,b}$ be the hermitian space with metric $\begin{pmatrix} \zeta 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix}$ and $V_{a,b+1}$ be the hermitian space with metric $\begin{pmatrix} \zeta 1_a & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}$. The matrix $S$ for the embedding:

$U(V_{a,b}) \times U(V_{a,b+1}) \hookrightarrow U(V_{a+2b+1})$ becomes:

$$\begin{pmatrix} 1 & & & & -\frac{\zeta}{2} & & & \\ & 1 & & & & & & \frac{1}{2} \\ & & 1 & & & & & \\ & & & -1 & & \frac{1}{2} & & \\ -1 & & & & -\frac{\zeta}{2} & & & \\ & & & 1 & & \frac{1}{2} & & \\ & & & & & & 1 & \\ & -1 & & & & & & -\frac{1}{2} \end{pmatrix}$$

Siegel-Weil section at $p$

Let $\tau$ be character of $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Suppose $\tau = (\tau_1, \tau_2^{-1})$ and let $p^{s_i}$ be the conductor of $\tau_i, i = 1, 2$. Let $\chi_1, ...\chi_a, \chi_{a+1}, ...\chi_{a+2b}$ be characters of $\mathbb{Q}_p^\times$ whose conductors are $p^{t_1}, ..., p^{t_{a+2b}}$. Suppose we are in the:

Generic case:

$$t_1 > t_2 > ... > t_{a+b} > s_1 > t_{a+b+1} > ... > t_{a+2b} > s_2$$

Also, let $\xi_i = \chi_i \tau_1^{-1}$ for $1 \leq i \leq a+b$, $\xi_j = \chi_j^{-1} \tau_2$ for $a+b+2 \leq j \leq a+2b+1$. Let $\xi_{a+2b+1} = 1$.

Let $\Phi_1$ be the following Schwartz function: let $\Gamma$ be the subgroup of $GL_{a+2b+1}(\mathbb{Z}_p)$ consists of matrices $\gamma = (\gamma_{ij})$ such that $p^{t_k}$ divides the below diagonal entries of the $k^{th}$ column for $1 \leq k \leq a+b$ and $p^{s_1}$ divides $\gamma_{ij}$ when $a+b+2 \leq j \leq a+2b+1$, $i \leq a+b+1$ or $i > j$. Let $\xi_i' = \chi_i \tau_2^{-1}$, $1 \leq i \leq a+b$, $\xi_j = \chi_j^{-1} \tau_1$, $a+b+2 \leq j \leq a+2b+1$, and $\xi_{a+b+1}' = \tau_1 \tau_2^{-1}$. (Thus $\xi_k' = \xi_k \tau_1 \tau_2^{-1}$ for any $k$).

**Definition 4.4.3.**

$$\Phi_1(x) = \begin{cases} 0 & x \notin \Gamma \\ \prod_{k=1}^{a+b+1} \xi_k'(x_{kk}) & x \in \Gamma \end{cases}$$

Now we define another Schwartz function $\Phi_2$.

Let $\mathfrak{X}$ be the following set: if $\mathfrak{X} \ni x = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$ is in the block form with respect to the partition: $a+2b+1 = a+b+1+b$, then:

- $x$ has entries in $\mathbb{Z}_p$;
- $\begin{pmatrix} A_{11} & A_{14} \\ A_{21} & A_{24} \end{pmatrix}$ has $\ell$-upper-left minors $A_\ell$ so that $(\det A_\ell) \in \mathbb{Z}_p^\times$ for $\ell = 1, 2, ..., a+b$;
- and $A_{42}$ has $\ell$-upper-left minors $B_\ell$ so that $(\det B_\ell) \in \mathbb{Z}_p^\times$ for $\ell = 1, 2, ..., b$.

We define:

$$\Phi_\xi(x) = \begin{cases} 0 & x \notin \mathfrak{X} \\ \xi_1/\xi_2(\det A_1)...\xi_{a+b-1}/\xi_{a+b}(\det A_{a+b}) & \\ \times\xi_{a+b+2}/\xi_{a+b+3}(\det B_1)...\xi_{a+2b}/\xi_{a+2b+1}(\det B_{b-1})\xi_{a+2b+1}(\det B_b). & x \in \mathfrak{X} \end{cases}$$

Let

$$\Phi_2(x) := \hat{\Phi}_\xi(x) = \int_{M_{a+2b+1}} (\mathbb{Q}_p)\Phi_\xi(y)e_p(-try x^t)dy.$$

Let $\Phi$ be the Schwartz function on $M_{a+2b+1,2(a+2b+1)}(Q_p)$ defined by:

$$\Phi(X, Y) := \Phi_1(X)\Phi_2(Y),$$

and define a Siegel-(Weil) section by:

$$f^\Phi(g) = \tau_2(\det g)|\det g|_p^{-s+\frac{a+2b+1}{2}} \times \int_{GL_{a+2b+1}(Q_p} \Phi((0, X)g)\tau_1^{-1}\tau_2(\det X)|\det X|_p^{-2s+a+2b+1}d^\times X.$$

**Lemma 4.4.4.** *If $\gamma \in \Gamma$, then:*

$$\Phi_\xi({}^t\gamma X) = \prod_{k+1}^{a+2b+1} (\xi_k(\gamma_{kk}))\Phi_\xi(X)$$

*Proof.* Straightforward. □

Fourier Coefficients (at $p$)

If $\beta \in \mathrm{Herm}_{a+2b+1}(K)$ the Fourier coefficient is defined by:

$$\begin{aligned} f_\beta^\Phi(1,s) &= \int_{M_{a+2b+1}} (\mathbb{Q}_p) f^\Phi( \begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & \end{pmatrix} \begin{pmatrix} 1 & N \\ & 1 \end{pmatrix}) e_p(-tr\beta N) dN \\ &= \int_{M_{a+2b+1}(\mathbb{Q}_p)} \int_{GL_{a+2b+1}\mathbb{Q}_p)} \Phi((0,X) \begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & -N \end{pmatrix}) \tau_1^{-1}\tau_2(\det X) \\ &\quad \times |\det X|_p^{-2s+a+2b+1} e_p(-tr\beta N) dN d^\times X \\ &= \int_{GL_{a+2b+1}(\mathbb{Q}_p)} \Phi_1(-X)\Phi_\xi(-{}^tX^{-1t}\beta)\tau_1^{-1}\tau_2(\det X)|\det X|_p^{-2s} d^\times X \\ &= \tau_1^{-1}\tau_2(-1) vol(\Gamma)\Phi_\xi({}^t\beta). \end{aligned}$$

**Definition 4.4.4.** *Let $\tilde{f}^\dagger = \tilde{f}^\dagger_{a+2b+1}$ be the Siegel section supported on $Q(\mathbb{Q}_p)w_{a+2b+1} \begin{pmatrix} 1 & M_{a+2b+1}(\mathbb{Z}_p) \\ & 1 \end{pmatrix}$ and $\tilde{f}^\dagger(w \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}) = 1$ for $X \in M_{a+2b+1}(\mathbb{Z}_p)$.*

**Lemma 4.4.5.**

$$\tilde{f}^\dagger_\beta(1) = \begin{cases} 1 & \beta \in M_{a+2b+1}(\mathbb{Z}_p) \\ 0 & \beta \notin M_{a+2b+1}(\mathbb{Z}_p) \end{cases}$$

*(here we used the projection of $\beta$ into its first component in $\mathcal{K}_v = F_v \times F_v$) where the first component correspond to the element inside our CM-type $\Sigma$ under $\iota := \mathbb{C} \simeq \mathbb{C}_p$ (see section 2.1).*

**Definition 4.4.5.**

$$f^\dagger := \frac{f^\Phi}{\tau_1^{-1}\tau_2(-1)\mathrm{Vol}(\Gamma)}$$

*Thus $f^\dagger_\beta = \Phi_\xi({}^t\beta)$.*

**Remark 4.4.1.** *This ensures that when we are moving our Eisenstein datum p-adically, the Siegel Eisenstein series also move p-adic analytically.*

Now we recall a lemma from [35, 11.4.12] which will be useful later.

**Lemma 4.4.6.** *Suppose $v|p$ and $\beta \in S_n(\mathbb{Q}_v), \det\beta \neq 0$.*

*(i) If $\beta \notin S_n(\mathbb{Z}_v)$ then $M(z, \tilde{f}^\dagger_n)_\beta(-z,1) = 0$;*

*(ii) Suppose $\beta \in S_n(\mathbb{Z}_v)$. Let $c := \mathrm{ord}_v(\mathrm{cond}(\tau'))$. Then:*

$$M(z, \tilde{f}^\dagger_n)_\beta(-z,1) = \tau'(\det\beta)|\det\beta|_v^{-2z}\mathfrak{g}(\bar{\tau}')^n c_n(\tau', z).$$

*where*

$$c_n(\tau', z) := \begin{cases} \tau'(p^{nc})p^{2ncz - cn(n+1)/2} & c > 0 \\ p^{2nz - n(n+1)/2} & c = 0. \end{cases}$$

Note that our $\tilde{f}^\dagger$ is the $f^\dagger$ in [35] and our $\tau$ is their $\chi$.

Now we want to write down our Siegel-Weil section $f^\Phi$ in terms of $\tilde{f}^\dagger$. First we prove the following:

**Lemma 4.4.7.** *Suppose $\Phi_\xi$ is the function on $M_n(\mathbb{Q}_p)$ defined as follows: if cond$(\xi_i) = (p^{t_i})$ for $i = 1, 2, ..., n$, let*

$$\tilde{\mathfrak{X}}_\xi := N(\mathbb{Z}_p) \begin{pmatrix} p^{-t_1}\mathbb{Z}_p^\times & & \\ & ... & \\ & & p^{-t_n}\mathbb{Z}_p^\times \end{pmatrix} N^{opp}(\mathbb{Z}_p).$$

*then the Fourier transform $\hat{\Phi}_\xi$ of $\Phi_\xi$ is the following function:*

$$\frac{\hat{\Phi}_\xi(x)}{\prod_{i=1}^n G(\xi_i)} = \begin{cases} 0 & x \notin \tilde{\mathfrak{X}}_\xi \\ \prod_{i=1}^n \bar{\xi}_i(x_i p^{t_i}) & \tilde{\mathfrak{X}}_\xi \ni x = \begin{pmatrix} 1 & & \\ ... & ... & \\ ... & ... & 1 \end{pmatrix} \begin{pmatrix} x_1 & & \\ & ... & \\ & & x_n \end{pmatrix} \begin{pmatrix} 1 & ... & ... \\ & ... & ... \\ & & 1 \end{pmatrix} \end{cases}$$

*Proof.* First suppose $x$ is supported in the "big cell": $N(\mathbb{Q}_p)T(\mathbb{Q}_p)N^{opp}(\mathbb{Q}_p)$ where the superscript 'opp' means the opposite parabolic. It is easily seen that we can write $x$ in terms of block matrices:

$$x = \begin{pmatrix} 1_{n-1} & u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v & 1 \end{pmatrix}$$

where $z \in M_{n-1}(\mathbb{Q}_p)$ and $w \in \mathbb{Q}_p$. A first observation is that $\tilde{\Phi}_\xi$ is invariant under right multiplication by $N^{opp}(\mathbb{Z}_p)$ and left multiplication by $N(\mathbb{Z}_p)$. We show that $v \in M_{1\times(n-1)}(\mathbb{Z}_p)$ if $\tilde{\Phi}_\xi(x) \neq 0$.

By definition:

$$\begin{aligned}
\tilde{\Phi}_\xi(x) &= \int_{M_n(Q_p)} \Phi_\xi(y) e_p(try^t x)dy \\
&\quad (y = \begin{pmatrix} 1_{n-1} & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & \\ & b \end{pmatrix} \begin{pmatrix} a_{n-1} & m \\ & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & a_{n-1}m \\ \ell a_{n-1} & \ell a_{n-1}m + b \end{pmatrix}) \\
&= \int_{a\in\mathfrak{X}_{\xi,n-1}, m\in M(\mathbb{Z}_p), \ell\in M(\mathbb{Z}_p), b\in\mathbb{Z}_p^\times} \Phi_\xi(\begin{pmatrix} 1 & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}) \\
&\times \quad e_p(tr \begin{pmatrix} 1 & \\ {}^t m & 1 \end{pmatrix} \begin{pmatrix} {}^t a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & {}^t\ell \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix} \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix})dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(tr \begin{pmatrix} 1 & \\ {}^t m + v & 1 \end{pmatrix} \begin{pmatrix} {}^t a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & {}^t\ell + u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix})dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(tr \begin{pmatrix} a & a(\ell+u) \\ (m+v)a & (m+v)a(\ell+u)+b \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix})dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(az + ((m+v)a(\ell+u)+b)w)dy
\end{aligned}$$

(Note that $\Phi_\xi$ is invariant under transpose.)

If $\tilde{\Phi}_\xi(x) \neq 0$, then it follows from the last expression that: $w \in p^{-t_n}\mathbb{Z}_p^\times$ and suppose $v \notin M_{1\times(n-1)}(\mathbb{Z}_p)$, then $m + v \notin M_{1\times(n-1)}(\mathbb{Z}_p)$. We let $a, m, b$ to be fixed and let $\ell$ to vary in $M_{1\times(n-1)}(\mathbb{Z}_p)$, we find that this integral must be 0. (Notice that $a \in \mathfrak{X}_{\xi,n-1}$ and $w \in p^{-t_n}\mathbb{Z}_p^\times$, thus $(m + v)aw \notin M_{1\times n-1}(\mathbb{Z}_p)$) Thus a contradiction. Therefore, $v \in M_{1\times n-1}(\mathbb{Z}_p)$, similarly $u \in M_{n-1,1}(\mathbb{Z}_p)$. Thus by the observation at the beginning of the proof we may assume $u = 0$ and $v = 0$ without lose of generality.

Thus if we write $\phi_{\xi,n-1}$ as the restriction of $\Phi_\xi$ to the up-left $(n-1)\times(n-1)$ minor,

$$\begin{aligned}
\tilde{\Phi}_\xi(x) &= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(az + (ma\ell + b)w)dy \\
&= p^{-nt_n}\mathfrak{g}(\xi_n)\bar{\xi}_n(wp^{t_n}) \int_{a\in\mathfrak{X}_{\xi,n-1}} \Phi_{\xi,n-1}(a) e_p(az)dy
\end{aligned}$$

by an induction procedure one gets:

$$\tilde{\Phi}_{\xi}(x) = \begin{cases} 0 & x \notin \tilde{\mathfrak{X}}_{\xi,n} \\ p^{-\sum_{i=1}^{n} ic_i} \prod_{i=1}^{n} \mathfrak{g}(\xi_i) \prod_{i=1}^{n} \bar{\xi}_i(x_i p^{t_i}) & x \in \tilde{\mathfrak{X}}_{\xi}. \end{cases}$$

Since $\tilde{\mathfrak{X}}_{\xi,n}$ is compact, now that we have proved that $\tilde{\Phi}_{\xi,n}$ when restricting to the "big cell" has support in $\tilde{\mathfrak{X}}_{\xi,n}$, therefore $\tilde{\Phi}_{\xi,n}$ itself must be supported in $\tilde{\mathfrak{X}}_{\xi,n}$. □

**Lemma 4.4.8.** *Let $\tilde{\mathfrak{X}}_{\xi}$ be the support of $\Phi_2 = \hat{\Phi}_{\xi}$, then a complete representative of $\tilde{\mathfrak{X}}_{\xi}$ mod $M_{a+2b+1}(\mathbb{Z}_p)$ is given by:*

$$\begin{pmatrix} A & & B \\ C & & D \\ & & \\ & E & \end{pmatrix}$$

*where the blocks are with respect to the partition $a+b+1+b$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over the following set:*

$$\begin{pmatrix} 1 & m_{12} & \dots & m_{1,a+b} \\ & \dots & \dots & \dots \\ & & \dots & m_{a+b-1,a+b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & x_{a+b} \end{pmatrix} \begin{pmatrix} 1 & & & \\ n_{21} & \dots & & \\ \dots & \dots & \dots & \\ n_{a+b,1} & \dots & n_{a+b,a+b-1} & 1 \end{pmatrix}$$

*where $x_i$ runs over $p^{-t_i}\mathbb{Z}_p^{\times}$ mod $\mathbb{Z}_p$, $m_{ij}$ runs over $\mathbb{Z}_p$ mod $p^{t_j}$ and $n_{ij}$ runs over $\mathbb{Z}_p$ mod $p^{t_i}$, and $E$ runs over the following set:*

$$\begin{pmatrix} 1 & k_{12} & \dots & k_{1,b} \\ & \dots & \dots & \dots \\ & & \dots & k_{b-1,b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & y_b \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_{21} & \dots & & \\ \dots & \dots & \dots & \\ \ell_{b,1} & \dots & \ell_{b,b-1} & 1 \end{pmatrix}$$

*where $y_i$ runs over $p^{-t_{i+a+b}}\mathbb{Z}_p$ mod $\mathbb{Z}_p$; $k_{ij}$ runs over $\mathbb{Z}_p$ mod $p^{t_{a+b+j}}$; $\ell_{ij}$ runs over $\mathbb{Z}_p$ mod $p^{t_{a+b+i}}$.*

*Proof.* This is elementary and we omit it here. □

Now we define several sets: Let $\mathfrak{B}'$ be the set of $(a+b) \times (a+b)$ upper triangular matrices of

the form

$$\begin{pmatrix} 1 & m_{12} & \dots & m_{1,a+b} \\ & \dots & \dots & \dots \\ & & \dots & m_{a+b-1,a+b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & x_{a+b} \end{pmatrix}$$

where $x_i$ runs over $p^{-t_i}\mathbb{Z}_p^{\times}$ *mod* $\mathbb{Z}_p$, $m_{ij}$ runs over $\mathbb{Z}_p$ *mod* $p^{t_j}$.

Let $\mathfrak{C}'$ be the set of $b \times b$ lower triangular matrices of the form

$$\begin{pmatrix} 1 & & & \\ n_{21} & \dots & & \\ \dots & \dots & \dots & \\ n_{a+b,1} & \dots & n_{a+b,a+b-1} & 1 \end{pmatrix}$$

where $n_{ij}$ runs over $\mathbb{Z}_p$ *mod* $p^{t_i}$

Let $\mathfrak{E}'$ be the set of $b \times b$ upper triangular matrices of the form

$$\begin{pmatrix} 1 & k_{12} & \dots & k_{1,b} \\ & \dots & \dots & \dots \\ & & \dots & k_{b-1,b} \\ & & & 1 \end{pmatrix}$$

where $k_{ij}$ runs over $\mathbb{Z}_p$ *mod* $p^{t_{a+b+j}}$.

Let $\mathfrak{D}'$ be the set of $(a+b) \times (a+b)$ lower triangular matrices of the form

$$\begin{pmatrix} y_1 & & & \\ & \dots & & \\ & & \dots & \\ & & & y_b \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_{21} & \dots & & \\ \dots & \dots & \dots & \\ \ell_{b,1} & \dots & \ell_{b,b-1} & 1 \end{pmatrix}$$

where $y_i$ runs over $p^{-t_{i+a+b}}\mathbb{Z}_p$ *mod*$\mathbb{Z}_p$; $\ell_{ij}$ runs over $\mathbb{Z}_p$ *mod* $p^{t_{a+b+i}}$. Also we define for $g \in GL_{a+2b}(\mathbb{Q}_p)$, $g^{\iota} = \begin{pmatrix} & 1_{a\times a} & \\ & & 1_{b\times b} \\ 1_{b\times b} & & \end{pmatrix} g \begin{pmatrix} & & 1_{b\times b} \\ 1_{a\times a} & & \\ & 1_{b\times b} & \end{pmatrix}$.

**Corollary 4.4.2.**

$$f^\dagger(z,g) = p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)$$
$$\times \sum_{A,B,C,D,E} \prod_{i=1}^{a} \bar{\xi}_i(A_{ii}) \prod_{i=1}^{b} \bar{\xi}_{a+i,a+i}(D_{ii}) \times \prod_{i=1}^{b} \bar{\xi}_{a+b+1+i}(E_{ii}) \tilde{f}^\dagger(z, g \begin{pmatrix} & A & & B \\ & C & & D \\ 1_{a+2b+1} & & & \\ & & E & \\ & & 1_{a+2b+1} & \end{pmatrix})$$

*Proof.* using the lemma above, we see that both hand sides have the same $\beta$'th fourier coefficients for all $\beta \in S_{a+2b+1}(\mathbb{Q}_p)$ thus they must be the same. □

thus if $B', C', D', E'$ runs over the set $\mathfrak{B}', \mathfrak{C}', \mathfrak{D}', \mathfrak{E}'$, then

$$\begin{aligned} f^\dagger(z,g) &= p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\ &\quad \times \sum_{B',C',D',E'} \prod_{i=1}^{a+b} \bar{\xi}_i(B'_{ii}) \prod_{i=1}^{b} \xi_{a+b+i}(D'_{ii}) \\ &\quad \times \tilde{f}^\dagger(z, g\alpha(\begin{pmatrix} B' & & \\ & 1 & \\ & & C' \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^\iota) \begin{pmatrix} 1 & A' \\ & 1 \end{pmatrix} \alpha(\begin{pmatrix} B' & & \\ & 1 & \\ & & C' \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^\iota)^{-1}) \\ &= p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\ &\quad \times \sum_{B',C',D',E'} \prod_{i=1}^{a+b} \bar{\xi}_i(B'_{ii}) \prod_{i=1}^{b} \xi_{a+b+i}(D'_{ii}) \prod_{i=1}^{B} \bar{\tau}_1(B'_{ii}) \prod_{i=1}^{b} \bar{\tau}_2(D'_{ii}) \\ &\quad \times \tilde{f}^\dagger(z, g\alpha(\begin{pmatrix} B' & & \\ & 1 & \\ & & C' \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^\iota) \begin{pmatrix} 1 & A' \\ & 1 \end{pmatrix}) \end{aligned}$$

where $A' =$
$$\begin{pmatrix} p^{-t_1} & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & p^{-t_2} & & & & & & & \\ & & & & & p^{-t_{a+1}} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & p^{-t_{a+b}} & & \\ & & & p^{-t_{a+b+1}} & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & p^{-t_{a+2b}} & & & & \end{pmatrix}$$

We let
$$\gamma = \begin{pmatrix} \zeta^{-1} & & & & & -\zeta^{-1} & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ \frac{1}{2} & & & & & \frac{1}{2} & & & & \\ & & & 1 & & & 1 & & & \\ & & & & & & & 1 & & \\ & 1 & & & & & & & 1 & \end{pmatrix}.$$

**Definition 4.4.6.** *(pull back section) If $f$ is a Siegel section and $\phi \in \pi_p$, then*

$$F_\phi(z, f, g) := \int_{GL_{a+2b}(\mathbb{Q}_p)} f(z, \gamma\alpha(g, g_1)\gamma^{-1})\bar{\tau}(\det g_1)\rho(g_1)\phi dg_1$$

Now we define a subset $K$ of $GL_{a+2b+2}(\mathbb{Z}_p)$ to be so that $k \in K$ if and only if $p^{t_i}$ divides the below diagonal entries of the $i$th column for $1 \leq i \leq a+b$, $p^{s_1}$ divides the below diagonal entries of the $(a+b+1)$th column, and $p^{t_{a+b+j}}$ divides the right to diagonal entries of the $(a+b+1+j)$th row for $1 \leq j \leq b-1$. We also define $\nu$, a character of $K$ by:

$$\nu(k) = \tau_1(k_{a+b+1,a+b+1})\tau_2(k_{a+2b+2,a+2b+2}) \prod_{i=1}^{a+b} \chi_i(k_{ii}) \prod_{i=1}^{b} \chi_{a+b+i}(k_{a+b+i+1,a+b+i+1})$$

for any $k \in K$, we also define $\tilde{\nu}$ a character of $\tilde{K}$ by:

$$\tilde{\nu}(k) = \prod_{i=1}^{b} \chi_{a+i}(k_{i,i}) \prod_{i=1}^{a} \chi_i(k_{b+i,b+i}) \prod_{i=1}^{b} \chi_{a+b+i}(k_{a+b+i,a+b+i}).$$

We also define $\Upsilon$ to be the element in $U(n,n)(F_v)(= U(n,n)(\mathbb{Q}_p))$ such that the projection to the first component of $\mathcal{K}_v = F_v \times F_v$ equals that of $\gamma$ (note that $\gamma \notin U(n,n)(F_v)$).

**Lemma 4.4.9.** *Let $K' \subset K$ be the compact subset defined by:*

$K' \ni k = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$ *(here the blocks are with respect to $a+b+1+b+1$) if and only if: $p^{t_{a+b+i}+t_j}$ divides the $(i,j)$th entry of $c_1$ for $1 \le i \le b$, $1 \le j \le a$ and $p^{t_{a+b+i}+t_{a+j}}$ divides the $(i,j)$th entry of $c_2$ for $1 \le i \le b$, $1 \le j \le b$.(it is not hard to check that this is a group).*
*Then: $F_\phi(z, \rho(\Upsilon)f^\dagger, gk) = \nu(k)F_\phi(z, \rho(\Upsilon)f^\dagger, g)$ for any $\phi \in \pi$ and $k \in K'$*

*Proof.* this follows directly from the action of $K'$ on the Siegel Weil section $f^\dagger$. □

We define $K''$ to be the subset of $K$ consists of matrices

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ c_1 & c_2 & & 1 & \\ & & & & 1 \end{pmatrix}$$

such that $p^{t_j}$ divides the $(i,j)$th entry of $c_1$ for $1 \le i \le b$, $1 \le j \le a$ and $p^{t_{a+j}}$ divides the $(i,j)$th entry of $c_2$ for $1 \le i \le b$, $1 \le j \le b$.

**Definition 4.4.7.** *Let $\tilde{K} \subset GL_{a+2b}(\mathbb{Z}_p)$ be the set of matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}$ (blocks are with respect to $(b+a+b)$) such that the column's of $a_3, a_6$ are divisible by $p^{t_1}, ..., p^{t_a}$; the column's of $a_4$ are divisible by $p^{t_{a+1}}, ..., p^{t_{a+b}}$; $p^{t_{a+j}}$ divides the below diagonal entries of the $i$'th column of $a_1$ $(1 \le i \le b)$; $p^{t_j}$ divides the below diagonal entries of the $j$'s column of $a_9$ $(1 \le j \le a)$; $p^{t_{a+b+k}}$ divides the above diagonal entries of the $k$'th row of $a_5$.*

*Let* $\tilde{K}' \subset \tilde{K}$ *be the set such that* $p^{t_{a+b+i}+t_{a+j}}$ *divides the* $(i,j)$*th entry of* $a_4$ *for* $1 \leq i \leq b$, $1 \leq j \leq b$ *and* $p^{t_{a+b+i}+t_j}$ *divides the* $(i,j)$*th entry of* $a_6$ *for* $1 \leq i \leq b$, $1 \leq j \leq a$. *We also define* $\tilde{K}''$ *to be the subset of* $\tilde{K}$ *consisting of matrices:*

$$\begin{pmatrix} 1 & & \\ & 1 & \\ a_4 & a_6 & 1 \end{pmatrix}$$

*such that* $p^{t_{a+j}}$ *divides the* $(i,j)$*th entry of* $a_4$ *for* $1 \leq i \leq b$, $1 \leq j \leq b$ *and* $p^{t_j}$ *divides the* $(i,j)$*th entry of* $a_6$ *for* $1 \leq i \leq b$, $1 \leq j \leq a$.

The following lemma would be useful in identifying our pull back section:

**Lemma 4.4.10.** *Suppose* $F_\phi(z, \rho(\Upsilon)f^\dagger, g)$ *as a function of* $g$ *is supported in* $PwK$ *and*

$$F_\phi(z, \rho(\Upsilon)f^\dagger, gk) = \nu(k)F_\phi(z, \rho(\Upsilon)f^\dagger, g)$$

*for* $k \in K'$, *and* $F_\phi(z, \rho(\Upsilon)f^\dagger, w)$ *is invariant under the action of* $(\tilde{K}'')^\iota$. *then* $F_\phi(a, \rho(\Upsilon)f^\dagger, g)$ *is the unique section (up to scalar) whose action by* $k \in K$ *is given by multiplying by* $\nu(k)$.

*Proof.* This is easy from the fact that $K = K'K'' = K''K'$. The uniqueness follows from lemma 4.4.3. □

We define a matrix $w$ to be $\begin{pmatrix} 1_a & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}$.

**Lemma 4.4.11.** *If* $\gamma\alpha(g,1)\gamma^{-1} \in supp(\rho(\Upsilon)f^\dagger)$ *then* $g \in PwK$. *(Here* $\rho$ *denotes the action of* $GU_{a+2b+1}(F_v)$ *on the Siegel sections given by right translation.)*

*Proof.* since $f^\dagger$ is of the form $\sum_{A \in \mathfrak{X}} \tilde{f}^\dagger(-\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix})$, where $\mathfrak{X}$ is some set, we only have to check the lemma for each term.

First recall we defined: $A' = \begin{pmatrix} -p^{-t_1} & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & p^{-t_a} & & & & & & & \\ & & & & & -p^{t_{a+1}} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & -p^{t_{a+b}} & & \\ & & & -p^{-t_{a+b+1}} & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & -p^{-t_{a+2b}} & & & & \end{pmatrix}$

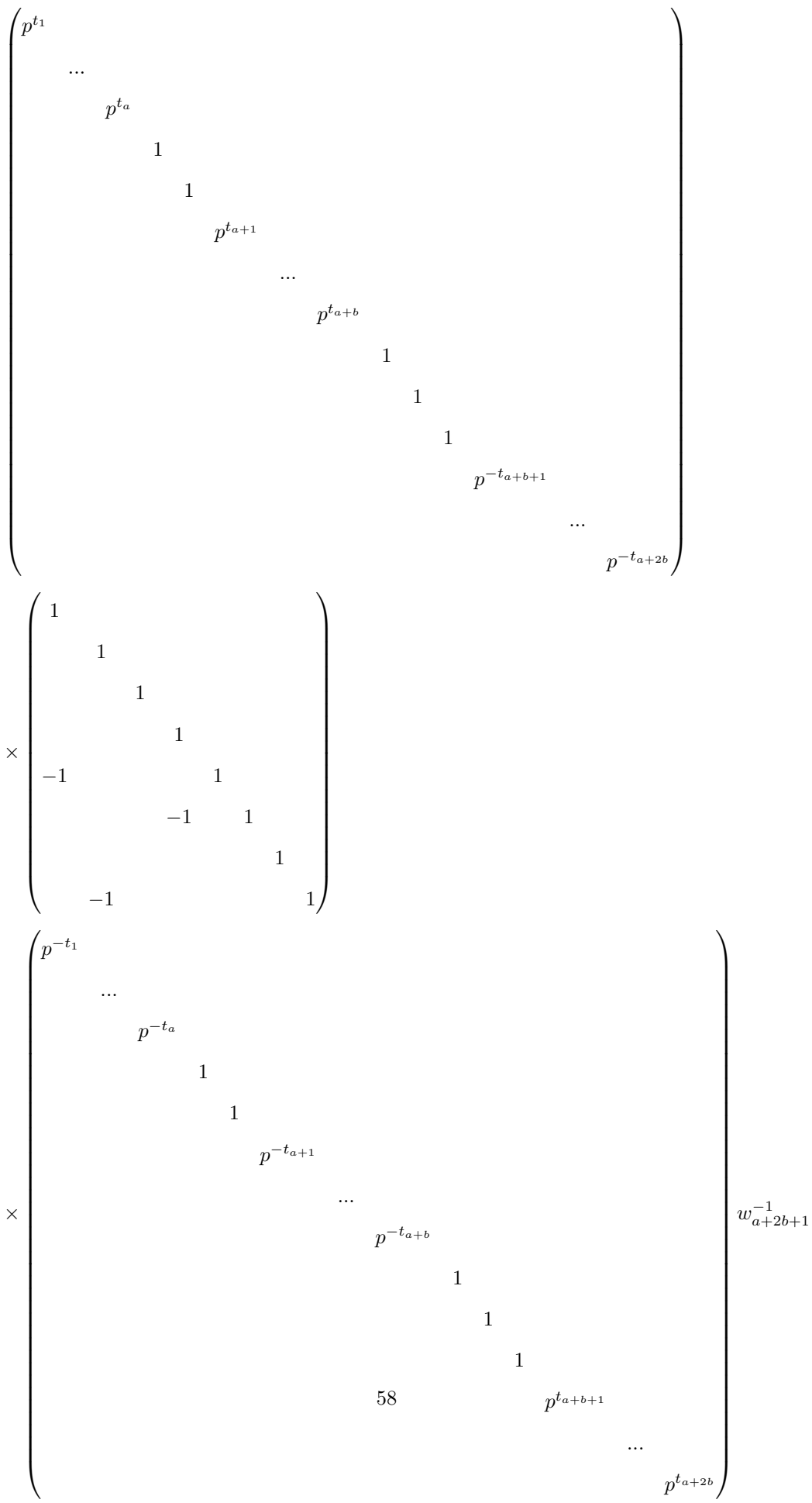
where the blocks are with respect to $a+b+1+b$. Let $\zeta_v$ and $\gamma_v$ be the projection of $\zeta$ and $\gamma_v$ to the first component of $\mathcal{K}_v \simeq F_v \times F_v$, then:

$$\gamma_v = \begin{pmatrix} \zeta_v^{-1} & & & & & & & -\zeta_v^{-1} \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \frac{1}{2} & & & & \frac{1}{2} & & & \\ & & 1 & & & 1 & & \\ & & & & & & 1 & \\ 1 & & & & & & & 1 \end{pmatrix} = \begin{pmatrix} 2\zeta_v^{-1} & & & & & & & -\zeta_v^{-1} \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \frac{1}{2} & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & 1 & & & \\ & & 1 & & & 1 & & \\ & & & & & & 1 & \\ 1 & & & & & & & 1 \end{pmatrix}$$

we denote the last term $\tilde{\gamma}_v$. Some times we omit the subscript $v$ if no confusions arise.

Using the expression for $f^\dagger$ involving the $B', C', D', E'$'s as above and the fact that $\gamma(m(g,1),g) \in Q$ and that $K$ is invariant under the right multiplication of $B$'s and $C$'s, we only need to check that if $\tilde{\gamma}_v \alpha(g,1) \tilde{\gamma}_v^{-1} \in supp\rho(\Upsilon)\rho(\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix} \tilde{f}^\dagger$, then $g \in PwK$. If $gw = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$ then this is equivalent to

$$\begin{pmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
1 & & & & 1 & & & \\
 & & & 1 & & 1 & & \\
 & & & & & & 1 & \\
 & 1 & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & a_3 & & & b_1 & b_2 & \\
a_4 & a_5 & a_6 & & & b_3 & b_4 & \\
a_7 & a_8 & a_9 & & & b_5 & b_6 & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
c_1 & c_2 & c_3 & & & d_1 & d_2 & \\
c_4 & c_5 & c_6 & & & d_3 & d_4 & \\
 & & & & & & & 1
\end{pmatrix} w_{a+2b+1} \times$$



being in $supp\tilde{f}^{\dagger}$, which is equivalent to

$$\tilde{\gamma}\alpha(g,\begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}\begin{pmatrix} p^{t_{a+b+1}} & & & & & & \\ & \ddots & & & & & \\ & & p^{-t_1} & & & & \\ & & & \ddots & & & \\ & & & & p^{-t_{a+1}} & & \\ & & & & & \ddots & \end{pmatrix})w_{a+2b+1}\tilde{\gamma}^{-1},$$

and thus also, $\tilde{\gamma}\alpha(g,1)w_{a+2b+1}\tilde{\gamma}^{-1}$, belonging to:

$$supp\rho(\begin{pmatrix} p^{-t_1} & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & p^{-t_a} & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & 1 & & & & & & & \\ & & & & & p^{-t_{a+1}} & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & p^{-t_{a+b}} & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & & & 1 & & \\ & & & & & & & & & & 1 & \\ & & & & & & & & & & & p^{t_{a+b+1}} \\ & & & & & & & & & & \ddots & \\ & & & & & & & & & & & p^{t_{a+2b}} \end{pmatrix} w_{a+2b+1})\tilde{f}^{\dagger}.$$

The right hand side is contained in: $Q_t := Q.\{\begin{pmatrix} 1 & \\ S & 1 \end{pmatrix} : S \in S_t = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}\}$ where the blocks for $S_t$ are with respect to $a+b+1+b$ and consist of matrices such that $S_{ij} \in M(\mathbb{Z}_p)$, $p^{t_i}$ divides the $i$th column for $1 \leq i \leq a$, $p^{t_{a+i}}$ divides the $(a+b+1+i)$th column for $1 \leq i \leq b$, $p^{t_{a+b+i}}$ divides the $(a+b+1+i)$th row for $1 \leq i \leq b$, and the $(i,j)$-th entry of $S_{41}$ and $S_{44}$ are divisible by $p^{t_{a+b+i}+t_j}$ and $p^{t_{a+b+i}+t_{a+j}}$ respectively. Observe that we have only to show that if

$\tilde{\gamma}\alpha(gw,1)\begin{pmatrix} & & & 1 & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & & 1 & & \\ 1 & & & & & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & \end{pmatrix}\tilde{\gamma}^{-1} \in Q_t$ then $g \in PwK$, i.e. $gw \in PK^w$ for $K^w := wKw$.

Let

$$\tilde{\gamma}\alpha(g,1)w\tilde{\gamma}^{-1} = \begin{pmatrix} -a_1 & a_2 & a_3 & -b_1 & a_1 & b_1 & b_2 & \\ -a_4 & a_5 & a_6 & -b_3 & a_4 & b_3 & b_4 & \\ -a_7 & a_8 & a_9 & -b_5 & a_7 & b_5 & b_6 & \\ & & & 1 & & & & \\ 1-a_1 & a_2 & a_3 & -b_1 & a_1 & b_1 & b_2 & \\ -c_1 & c_2 & c_3 & -d_1 & c_1 & d_1 & d_2 & \\ -c_4 & c_5 & c_6 & -d_3 & c_4 & d_3 & d_4 & \\ -a_4 & a_5-1 & a_6 & -b_3 & a_4 & b_3 & b_4 & 1 \end{pmatrix} := H.$$

Thus if $H \in Q_t$, then there exists $S \in S_t$ such that:

$$\begin{pmatrix} 1-a_1 & a_2 & a_3 & -b_1 \\ -c_1 & c_2 & c_3 & 1-d_1 \\ -c_4 & c_5 & c_6 & -d_3 \\ -a_4 & a_5-1 & a_6 & -b_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & b_2 & \\ c_! & d_1 & d_2 & \\ c_4 & d_3 & d_4 & \\ a_4 & b_3 & b_4 & 1 \end{pmatrix} S$$

By looking at the 3rd row (block-wise), one finds $d_4 \neq 0$, so by multiplying $g$ by a matrix $\begin{pmatrix} 1 & & & & \times \\ & 1 & & & \times \\ & & 1 & \times & \times \\ & & & 1 & \times \\ & & & & d_4^{-1} \end{pmatrix}$ (which does not change the assumption and conclusion) we may assume that $d_4 = 1$ and $d_2 = 0, b_2 =$

$0, b_4 = 0, b_6 = 0, b_5 = 0$. So we assume that $gw$ is of the form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & b_1 & \\ a_4 & a_5 & a_6 & b_3 & \\ a_7 & a_8 & a_9 & & \\ c_1 & c_2 & c_3 & d_1 & \\ c_4 & c_5 & c_6 & d_3 & 1 \end{pmatrix}.$$

Next by looking at the 2nd row (block-wise) and noting that $d_2 = 0$ we find that $d_1$ is of the form

$$\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & ... & ... & \mathbb{Z}_p \\ p^{t_{a+1}}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... & \mathbb{Z}_p \\ ... & p^{t_{a+2}}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... \\ ... & ... & ... & ... & ... \\ p^{t_{a+1}}\mathbb{Z}_p & ... & ... & ... & \mathbb{Z}_p^\times \end{pmatrix}.$$

And by looking at the 3rd row again we see $c_4 = (p^{t_1}\mathbb{Z}_p, ..., p^{t_a}\mathbb{Z}_p)$, $d_3 \in (p^{t_{a+1}}, ..., p^{t_{a+b}})$., $c_1 \in M_{b\times 1}(p^{t_1}\mathbb{Z}_p), M_{b\times 1}(p^{t_2}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_a}\mathbb{Z}_p))$, $c_2 \in M_{b\times b}(\mathbb{Z}_p)$, $c_3 \in M_{b\times 1}(\mathbb{Z}_p)$.

By looking at the 1st row and note that $b_2 = 0$ we know $a_1 \in \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & ... & ... & \mathbb{Z}_p \\ p^{t_1}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... & \mathbb{Z}_p \\ ... & p^{t_2}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... \\ ... & ... & ... & ... & ... \\ p^{t_1}\mathbb{Z}_p & ... & ... & ... & \mathbb{Z}_p^\times \end{pmatrix}$,
$b_1 \in (M_{a\times 1}(p^{t_{a+1}}\mathbb{Z}_p), M_{a\times 1}(p^{t_{a+2}}\mathbb{Z}_p), ..., M_{a\times 1}(p^{t_{a+b}}\mathbb{Z}_p))$. Finally looking at the 4th row (block-wise), we note that $b_4 = 0$. Similarly, $a_4 \in (M_{b\times 1}(p^{t_1}\mathbb{Z}_p), M_{b\times 1}(p^{t_2}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_a}\mathbb{Z}_p))$,
$b_3 \in (M_{b\times 1}(p^{t_{a+1}}\mathbb{Z}_p), M_{b\times 1}(p^{t_{a+2}}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_{a+b}}\mathbb{Z}_p))$,
and $a_5 - 1 \in \begin{pmatrix} M_{1\times b}(p^{t_{a+b+1}}\mathbb{Z}_p) \\ M_{1\times b}(p^{t_{a+b+2}}\mathbb{Z}_p) \\ ... \\ M_{1\times b}(p^{t_{a+2b}}\mathbb{Z}_p) \end{pmatrix}$,$a_6 \in \begin{pmatrix} p^{t_{a+b+1}}\mathbb{Z}_p \\ p^{t_{a+b+2}}\mathbb{Z}_p \\ ... \\ p^{t_{a+2b}}\mathbb{Z}_p \end{pmatrix}$, $a_2 \in M_{a\times b}(\mathbb{Z}_p)$, $a_3 \in M_{a\times 1}(\mathbb{Z}_p)$.
Now we prove that $gw \in PK^w$ using the properties proven above. First we multiply $gw$ by

$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ -d_1^{-1}c_1 & -d_1^{-1}c_2 & -d_1^{-1}c_3 & d_1^{-1} & \\ -c_4 & -c_5 & -c_6 & -d_3 & 1 \end{pmatrix} \in K^w$, which does not change the above properties or what needs to be proven, so without loss of generality we assume that $c_4 = 0, c_5 = 0, c_6 = 0, d_3 = 0, c_1 = 0, c_2 = 0, c_3 = 0, d_1 = 0$. Moreover we set $\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix}^{-1} \begin{pmatrix} a_3 \\ a_6 \end{pmatrix} := T$. Then $\begin{pmatrix} 1 & & T_1 & & \\ & 1 & T_2 & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \in K^w$, and it is now clear that $gw \in PK^w$. □

Now suppose that $\pi$ is nearly ordinary with respect to $\underline{k}$. We denote $\phi$ to be the unique (up to scalar) nearly ordinary vector in $\pi$. Let $\phi_w = \pi(w)\phi$, $\phi_{aux} = \sum_{x \in J} \pi \begin{pmatrix} 1 & & \\ & 1 & \\ & & x \end{pmatrix} \phi_w$ where $x = \begin{pmatrix} 1 & x_{12} & ... & x_{1b} \\ & ... & ... & ... \\ & & ... & ... \\ & & & 1 \end{pmatrix}$ with $x_{ij}$ running through representatives of $[\mathbb{Z}_p : p^{t_{a+b+i}-t_{a+b+j}}\mathbb{Z}_p]$. Note that $\phi_{aux}$ apparently depends on the choices of the representatives.

Now write

$$\phi' = \rho( \begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{t_1} & & & \\ & & & ... & & \\ & & & & p_{t_{a+1}} & \\ & & & & & ... \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota} ) \phi_{aux}.$$

We want to compute the value $F_{\phi'}(z, \rho(\Upsilon)f^\dagger, w)$. In fact it is equal to:

$$\sum_{B,C,D,E} \int_{GL_{a+2b}(Q_p)} \tilde{f}^\dagger(\tilde{\gamma}\alpha(w \begin{pmatrix} B & & & \\ & 1 & & \\ & & C & \\ & & & 1 \end{pmatrix} w, (g_1 \begin{pmatrix} E & \\ & D \end{pmatrix} \begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & \ddots & & & & \\ & & p^{t_1} & & & \\ & & & \ddots & & \\ & & & & p^{t_{a+1}} & \\ & & & & & \ddots \end{pmatrix}$$

$$\times \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix})^\iota) w' \tilde{\gamma}^{-1} \begin{pmatrix} p^{-t_1} & & & & & & & & & & & & \\ & \ddots & & & & & & & & & & & \\ & & p^{-t_a} & & & & & & & & & & \\ & & & 1 & & & & & & & & & \\ & & & & 1 & & & & & & & & \\ & & & & & p^{-t_{a+1}} & & & & & & & \\ & & & & & & \ddots & & & & & & \\ & & & & & & & p^{-t_{a+b}} & & & & & \\ & & & & & & & & 1 & & & & \\ & & & & & & & & & 1 & & & \\ & & & & & & & & & & 1 & & \\ & & & & & & & & & & & p^{t_{a+b+1}} & \\ & & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & & p^{t_{a+2b}} \end{pmatrix}$$

$$\times w_{a+2b+1}^{-1}) \bar{\tau}(\det g_1) \rho(g_1^\iota) \phi' dg_1$$

where $w' = \begin{pmatrix} & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ -1 & & & & & & & \\ & & & & & & & 1 \\ & & & & & & 1 & \end{pmatrix}$ and the sum is over $B \in \mathfrak{B}, C \in \mathfrak{C}, D \in \mathfrak{D}, E \in \mathfrak{E}$. A

direct computation gives: $\tilde{\gamma}\alpha(1, \begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota})w'\tilde{\gamma}^{-1}$ equals

$$\begin{pmatrix} -1 & & & & 1 & & \\ & 1 & & & & & \\ & & 1 & & & & \\ -a_3 & -a_2 & a_1 & & & a_2 & \\ -a_9-1 & -a_8 & a_7 & 1 & & a_8 & \\ -a_3 & -a_2 & a_1-1 & & 1 & a_2 & \\ & & & & & 1 & \\ -a_6 & 1-a_5 & a_4 & & & a_5 & \end{pmatrix}$$

Now we define $\mathfrak{Y}$ to be the subset of $GL_{a+2b}(\mathbb{Z}_p)$ consisting of block matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}$ such that $\tilde{\gamma}\alpha(1, \begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota})w'\tilde{\gamma}^{-1}$ is in the $Q_t$ defined in the proof of lemma 4.4.11. It is not hard to prove that it can be described as follows: the $i$-th column of $a_9 - 1$ and $a_3$ are divisible by $p^{t_i}$ for $1 \leq i \leq a$, the $i$-th column of $a_7, a_1 - 1$ are divisible by $p^{t_{a+i}}$, the $(i,j)$-th entry of $a_6$ is divisible by $p^{t_{a+b+i}+t_j}$, the $(i,j)$-th entry of $a_4$ is divisible by $p^{t_{a+b+i}+t_{a+j}}$, the $i$-th row of $1 - a_5$ is

divisible by $p^{t_{a+b+i}}$. The entries in $a_2$ and $a_8$ are in $\mathbb{Z}_p$. Then the pull back section is equal to

$$\sum_{B,C,D,E}\int \tilde{f}^{\dagger}(\tilde{\gamma}\alpha(1,g_1)w'\tilde{\gamma}^{-1}\begin{pmatrix} p^{-t_1} & & & & & & & & & & & & & & & \\ & \ddots & & & & & & & & & & & & & & \\ & & p^{-t_a} & & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & & \\ & & & & & p^{-t_{a+1}} & & & & & & & & & & \\ & & & & & & \ddots & & & & & & & & & \\ & & & & & & & p^{-t_{a+b}} & & & & & & & & \\ & & & & & & & & 1 & & & & & & & \\ & & & & & & & & & 1 & & & & & & \\ & & & & & & & & & & 1 & & & & & \\ & & & & & & & & & & & p^{t_{a+b+1}} & & & & \\ & & & & & & & & & & & & \ddots & & & \\ & & & & & & & & & & & & & p^{t_{a+2b}} & & \end{pmatrix}$$
$$\times w_{a+2b+1}^{-1})\bar{\tau}(\det g_1)\rho(g_1^t)\phi dg_1$$

where the integration is over the set:

$$g_1 \in \begin{pmatrix} B & \\ & C \end{pmatrix}\mathfrak{Y}\begin{pmatrix} E & \\ & D \end{pmatrix}_{conj}\begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}\begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \ddots & & & & \\ & & p^{-t_1} & & & \\ & & & \ddots & & \\ & & & & p^{-t_{a+b}} & \\ & & & & & \ddots \end{pmatrix}$$

for:

$$\begin{pmatrix} E & \\ & D \end{pmatrix}_{conj} := \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \ddots & & & & \\ & & p^{-t_1} & & & \\ & & & \ddots & & \\ & & & & p^{-t_{a+1}} & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} E & \\ & D \end{pmatrix}$$

$$\times \begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & \ddots & & & & \\ & & p^{t_1} & & & \\ & & & \ddots & & \\ & & & & p^{t_{a+1}} & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}$$

The value of $\tilde{f}^{\dagger}$ when $g_1 = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \ddots & & & & \\ & & p^{-t_1} & & & \\ & & & \ddots & & \\ & & & & p^{-t_{a+b}} & \\ & & & & & \ddots \end{pmatrix}$ is

$$\tau((p^{t_{a+b+1}+\ldots+t_{a+2b}}, p^{t_1+\ldots+t_{a+b}}))|p^{t_1+\ldots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}},$$

thus straightforward computation tells us the following:

**Lemma 4.4.12.** *If $\phi$ and $\phi'$ are defined as after the proof of lemma 4.4.11 then:*

$$\begin{aligned} F_{\phi'}(z, \rho(\Upsilon)f^{\dagger}, w) = \quad & \tau((p^{t_1+\ldots+t_{a+b}}, p^{t_{a+b+1}+\ldots+t_{a+2b}}))|p^{t_1+\ldots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}') \\ & \times p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\phi_w. \end{aligned}$$

Combining the 3 lemmas above, we get the following:

**Proposition 4.4.1.** *Assumptions are as in the above lemma. $F_{\phi'}(z, \rho(\Upsilon)f^{\dagger}, g)$ is the unique section supported in $PwK$ such that the right action of $K$ is given by multiplying the character $\nu$ and its*

*value at $w$ is:*

$$F_{\phi'}(z,\rho(\Upsilon)f^\dagger,w) = \tau((p^{t_1+\ldots+t_{a+b}},p^{t_{a+b+1}+\ldots+t_{a+2b}}))|p^{t_1+\ldots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}') \times p^{-\sum_{i=1}^{a+b}it_i-\sum_{i=1}^{b}it_{a+b+1+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\phi_w$$

*Proof.* $\phi_w$ is clearly invariant under $(\tilde{K}'')^\iota$. □

This $F_{\phi'}(z,\rho(\Upsilon)f^\dagger,g)$ we constructed is not going to be the nearly ordinary vector unless we apply the intertwining operator to it. So now we start with a $\rho=(\pi,\tau)$. We require that $\rho^\vee=(\pi^\vee,\bar{\tau}^c)$ satisfies the conditions at the beginning of this section about the conductors. We define our Siegel section $f^0\in I_{a+2b+1}(\tau)$ to be:

$$f^0(z;g) := M(-z,f^\dagger)_z(g)$$

where $f^\dagger\in I_{a+2b+1}(\bar{\tau}^c)$. We recall the following proposition from [35] (in a generalized form)

**Proposition 4.4.2.** *Suppose our data $(\pi,\tau)$ comes from the local component at $v$ of a global data. Then there is a meromorphic function $\gamma^{(2)}(\rho,z)$ such that*

$$F_{\phi^\vee}(-z,M(z,f),g)=\gamma^{(2)}(\rho,z)A(\rho,z,F_\phi(f;z,-))_{-z}(g)$$

*moreover if $\pi_v\simeq\pi(\chi_1,\ldots,\chi_{a+2b})$ then if we write $\gamma^{(1)}(\rho,z)=\gamma^{(2)}(\rho,z-\frac{1}{2})$ then*

$$\gamma^{(1)}(\rho,z)=\psi(-1)c\epsilon(\tilde{\pi},\bar{\tau}^c,z+\frac{1}{2})\frac{L(\pi,\tau^c,1/2-z)}{L(\tilde{\pi},\bar{\tau}^c,z+1/2)}$$

*where $c$ is the constant appearing in lemma 4.4.6*

*Proof.* The same as [35, 11.4.13]. □

**Remark 4.4.2.** *Note that here we are using the L-factors for the base change from the unitary groups while [35] uses the $GL_2$ L-factor for $\pi$ so our formula appears slightly different.*

Now we are going to show that:

$$F_v^0(z;g) := F_{\phi'}(z,\rho(\Upsilon)f^0,g)$$

is a constant multiple of the nearly ordinary vector if our $\rho$ comes from the local component of the global Eisenstein data (see section 3.1). Return to the situation of our Eisenstein Data. Suppose

that at the archimedean places our representation is a holomorphic discrete series associated to the (scalar) weight: $\underline{k} = (0,...0;\kappa,...\kappa)$ with $r$ 0's and $s$ $\kappa$'s. Here $r = a+b, s = b$. Suppose $\pi \simeq \mathrm{Ind}(\chi_1,...,\chi_{a+2b})$ is nearly ordinary with respect to the weight $\underline{k}$. We suppose $\nu_p(\chi_1(p)) = s-\frac{n}{2}+\frac{1}{2},...,\nu_p(\chi_r(p)) = r+s-1-\frac{n}{2}+\frac{1}{2},\nu_p(\chi_{r+1}(p)) = \kappa-\frac{n}{2}+\frac{1}{2},...,\nu_p(\chi_{r+s}(p)) = \kappa+s-1-\frac{n}{2}+\frac{1}{2}$, and $\nu_p(\tau_1(p)) = \frac{\kappa}{2},\nu_p(\tau_2(p)) = \frac{\kappa}{2}$, so

$$\nu_p(\chi_1(p)) < ... < \nu_p(\chi_{a+b}(p)) < \nu_p(\tau_1(p)p^{-z_\kappa}) < \nu_p(\tau_2(p)p^{z_\kappa}) < \nu_p(\chi_{a+v+1}(p)) < ... < \nu_p(\chi_{a+2b}(p))$$

where $z_\kappa = \frac{\kappa-r-s-1}{2}$. It is easy to see that $I(\rho_v,z_\kappa) \simeq \mathrm{Ind}(\chi_1,...\chi_{r+s},\tau_1|.|^{z_\kappa},\tau_2|.|^{-z_\kappa})$. By definition $I(\rho_v,z_\kappa)$ is nearly ordinary with respect to the weight $(0,...,0;\kappa,...,\kappa)$ with $(r+1)$ 0's and $s$ $\kappa$'s.

First of all from the form of $F_{\phi'}(z,f^\dagger;g)$ and the above proposition we have a description for $F_v^0(z,g)$: it is supported in $P(\mathbb{Q}_p)K_v$,

$$\begin{aligned} F_v^0(z,1) = \quad & \gamma^{(2)}(\rho,-z)\bar{\tau}^c((p^{t_1+...+t_{a+b}},p^{t_{a+b+1}+...+t_{a+2b}}))|p^{t_1+...+t_{a+2b}}|^{z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}') \\ & \times p^{-\sum_{i=1}^{a+b}(i+1)t_i-\sum_{i=1}^{b}(i+1)t_{a+b+1+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\phi \end{aligned}$$

and the right action of $K_v$ is given by the character

$$\begin{aligned} & \chi_1(g_{11})...\chi_{a+b}(g_{a+b\ a+b})\tau_1(g_{a+b+1\ a+b+1})\chi_{a+b+1}(g_{a+b+2\ a+b+2})... \\ & \times \chi_{a+2b}(g_{a+2b+1\ a+2b+1})\tau_2(g_{a+2b+2\ a+2b+2}). \end{aligned}$$

(It is easy to compute $A(\rho,z,F_{\phi'}(\rho(\Upsilon)f^\dagger;z,-))_{-z}(1)$ and we use the uniqueness of the vector with the required $K_v$ action. Here on the second row of the above formula for $F_v^0(z,1)$ the power for $p$ is slightly different from that for the section $F(z,f^\dagger,w)$. This comes from the computations for the intertwining operators for Klingen Eisenstein sections.)

Thus Corollary 4.4.1 tells us that $F_v^0(z,g)$ is a nearly ordinary vector in $I(\rho)$.

Now we describe $f^0$:

**Definition 4.4.8.** *Suppose $(p^t) = \mathrm{cond}(\tau')$ for $t \geq 1$ then define $f_t$ to be the section supported in $Q(\mathbb{Q}_p)K_Q(p^t)$ and $f_t(k) = \tau(d_k)$ on $K_Q(p^t)$.*

**Lemma 4.4.13.**

$$f_x^0 := M(-z, \tilde{f}^\dagger)_z = f_{t,z}.$$

*Proof.* This is just [35, 11.4.10]. □

### 4.4.3 Fourier Coefficients for $f^0$

We record a formula here for the Fourier Coefficients for $f^0$ which will be used in $p$-adic interpolation.

**Lemma 4.4.14.** *Suppose* $|\det \beta| \neq 0$ *then:*
*(i) If* $\beta \notin S_{a+2b+1}(\mathbb{Z}_p)$ *then* $f_\beta^0(z,1) = 0$*;*
*(ii) Let* $t := \mathrm{ord}_p(\mathrm{cond}(\tau')$*. If* $\beta \in S_{a+2b+1}(\mathbb{Z}_p)$*, then:*

$$f_\beta^0(z,1) = \bar{\tau}'(\det \beta)|\det \beta|_p^{2z}\mathfrak{g}(\tau')^{a+2b+1}c_{a+2b+1}(\bar{\tau}', z)\Phi_\xi({}^t\beta).$$

*where* $c_{a+2b+1}$ *is defined in lemma 4.4.6 and* $\Phi_\xi$ *is defined at the beginning of this section.*

*Proof.* This follows from [35, 11.4.12]. and the argument of corollary 4.4.2 where we deduce the form of $f^\dagger$ from the section $\tilde{f}^\dagger$. □

### 4.4.4 Fourier-Jacobi Coefficients

Now let $m = b + 1$. For $\beta \in S_m(F_v) \cap GL_m(\mathcal{O}_v)$ we are going to compute the Fourier Jacobi coefficient for $f_t$ at $\beta$

**Lemma 4.4.15.** *Let* $x := \begin{pmatrix} 1 & \\ D & 1 \end{pmatrix}$ *(this is a block matrix with respect to* $(a+b)+(a+b)$*).*
*(a)* $FJ_\beta(f_t; -z, v, x\eta^{-1}, 1) = 0$ *if* $D \in p^t M_{a+b}(\mathbb{Z}_p)$*;*
*(b) if* $D \in p^t M_n(\mathbb{Z}_p)$ *then* $FJ_\beta(f_t; -z, v, x\eta^{-1}, 1) = c(\beta, \tau, z)\Phi_0(v)$*, where*

$$c(\beta, \tau, z) := \bar{\tau}(-\det \beta)|\det \beta|_v^{2z+n-m}\mathfrak{g}(\tau')^m c_m(\tau', -z - \frac{n-m}{2})$$

*and* $c_m$ *is defined in lemma 4.4.6*

*Proof.* We only give the detailed proof for the case when $a = 0$. The case when $a > 0$ is even easier to treat.

Assuming $a = 0$, we temporarily write $n$ for $b$ and save the letter $b$ for other use. We have:

$$w_{2n+1}\begin{pmatrix} & S & v \\ 1_{2n+1} & {}^t\bar{v} & D \\ & & 1_{2n+1} \end{pmatrix}\alpha(1,\eta^{-1}) = \begin{pmatrix} & & 1_{n+1} & \\ & -1_n & & \\ -1_{n+1} & v & -S & \\ & D & -{}^t\bar{v} & -1_n \end{pmatrix}.$$

This belongs to $Q_{2n+1}(\mathbb{Q}_p)K_{Q_{2n+1}}(p^t)$ if and only if $S$ is invertible, $S^{-1} \in p^t M_{n+1}(\mathcal{O}_v)$, $S^{-1}v \in p^t M_{(n+1)\times n}(\mathcal{O}_v)$ and ${}^t\bar{v}S^{-1}v - D \in p^t M_n(\mathbb{Z}_p)$. Since $v = \gamma\,{}^t(b,0)$ for some $\gamma \in SL_{n+1}(\mathcal{O}_v)$ and $b \in M_n(\mathcal{K}_v)$ we are reduced to the case $v = {}^t(b,0)$. Writing $b = (b_1, b_2)$ with $b_i \in M_n(\mathbb{Q}_p)$ and $S = (T, {}^tT)$ with $T \in M_{n+1}(\mathbb{Q}_p)$ and $T^{-1} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ where $a_1 \in M_n(\mathbb{Q}_p)$, $a_2 \in M_{n\times 1}(\mathbb{Q}_p)$,$a_3 \in M_{1\times n}(\mathbb{Q}_p)$, $a_4 \in M_1(\mathbb{Q}_p)$, the conditions on $S$ and $v$ can be rewritten as:

(*) $\det T \neq 0$, $a_i \in p^t M_n(\mathbb{Z}_p)$, $a_1 b_1 \in p^t M_n(\mathbb{Z}_p)$, $a_3 b_1 \in p^t M_{1\times n}(\mathbb{Z}_p)$, ${}^t a_1 b_2 \in p^t M_n(\mathbb{Z}_p)$, ${}^t a_2 b_2 \in p^t\mathbb{Z}_p$, ${}^t b_2 a_1 b_1 - D \in p^t M_n(\mathbb{Z}_p)$

Now we prove that: if the integral for $FJ_\beta$ is non-zero then $b_1, b_2 \in M_n(\mathbb{Z}_p)$. Suppose otherwise, then without lose of generality we assume $b_1$ has an entry which has the maximal $p$-adic absolute value among all entries of $b_1$ and $b_2$, Suppose it is $p^w$ for $w > 0$ (throughout the paper $w$ means this only inside this lemma). Also, for any matrix $A$ of given size we say $A \in {}^t b_2^\vee$ if and only ${}^t b_2 A$ has all entries in $\mathbb{Z}_p$ (of course we assume the sizes of the matrices are correct so that the product makes sense).

Now,let

$$\Gamma := \left\{ \begin{array}{ll} \gamma\begin{pmatrix} h & j \\ k & l \end{pmatrix} \in GL_{n+1}(\mathbb{Z}_p): & h \in GL_{n+1}(\mathbb{Z}_p), l \in \mathbb{Z}_P^\times, \\ h - 1 \in {}^t b_2^\vee \cap p^t M_n(\mathbb{Z}_p), & j \in \mathbb{Z}_p^n \cap {}^t b_2^\vee, k \in p^t M_{1\times n}(\mathbb{Z}_p) \end{array} \right\}$$

Suppose that our $b_1, b_2, D$ are such that there exist $a_i$'s satisfying (*), then one can check that $\Gamma$ is a subgroup, and if $T$ satisfies (*), so does $T\gamma$ for any $\gamma \in \Gamma$. Let $\mathcal{T}$ denote the set of $T \in M_{n+1}(\mathbb{Q}_p)$ satisfying (*). Then $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ D & 1 \end{pmatrix}\eta^{-1}, 1)$ equals

$$\sum_{T\in\mathcal{T}/\Gamma} |\det T|_p^{3n+2-2z} \int_\Gamma \tau'(-\det T\gamma) e_p(-tr\beta T\gamma) d\gamma.$$

Let $T' := \beta T = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ (blocks with respect to $n+1$), then the above integral is zero unless $c_1 \in p^{-t}M_n(\mathbb{Z}_p) + [{}^t b_2]_{n\times n}, c_4 \in p^{-t}\mathbb{Z}, c_2 \in p^{-t}M_{n+1}(\mathbb{Z}_p), c_3 \in [{}^t b_2]_{1\times n} + M_{1\times n}(\mathbb{Z}_p)$. Here $[{}^t b_2]_{i\times n}$ means the set of $i \times n$ matrices such that each row is a $\mathbb{Z}_p$-linear combination of the rows of ${}^t b_2$.

But then

$$\beta \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = T'T^{-1} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 a_1 b_1 + c_2 a_2 b_1 \\ c_3 a_1 b_1 + c_4 a_3 b_1 \end{pmatrix}$$

since $\beta \in GL_{n+1}(\mathbb{Z}_p)$, the left must contain some entry with $p$-adic absolute value $p^w$. But it is not hard to see that all entries on the right hand side have $p$-adic values strictly less than $p^w$, a contradiction, thus we conclude that $b_1 \in M_n(\mathbb{Z}_p)$ and $b_2 \in M_n(\mathbb{Z}_p)$. By (*): $b_2 {}^t a_1 b_1 - D \in p^t M_n(\mathbb{Z}_p), a_1 \in p^t M_n(\mathbb{Z}_p)$ so $D \in p^t M_n(\mathbb{Z}_p)$.

The value claimed in part (ii) can be deduced similarly as in [35, 11.4.22] □

# Chapter 5

# Global Computations

## 5.1 Klingen Eisenstein Series

Now we are going to construct the nearly ordinary Klingen Eisenstein series (and will $p$-adically interpolate in families). First of all, recall that for a Hecke character $\tau$ which is of infinite type $(\frac{\kappa}{2}, -\frac{\kappa}{2})$ at all infinite places (here the convention is that the first infinite place of $\mathcal{K}$ is inside our CM type) we construct a Siegel Eisenstein series $E$ associated to the Siegel section:

$$f = \prod_{v|\infty} f_\kappa \prod_{v|p} \rho(\Upsilon_v) f_v^0 \prod_{v\in\Sigma, v\nmid p} f_{v,sieg} \prod_v f_v^{sph} \in I_{a+2b+1}(\tau, z).$$

Here $\Upsilon_v$ is the $\Upsilon$ defined in the section of $p$-adic computations. We warn the reader that the matrix form of it is not quite the one given there, due to our re-index of the rows and columns mentioned at the beginning of subsection 4.4.2. Recall that we write $\mathcal{D} := \{\pi, \tau, \Sigma\}$ for the Eisenstein datum where $\Sigma$ (see definition 3.1.2) is a finite set containing all the infinite places, primes dividing $p$ and the places where $\pi$ or $\tau$ is ramified.then define the normalization factor:

$$\begin{aligned} B_{\mathcal{D}} : \quad &= \frac{1}{\Omega_\infty^{2\kappa\Sigma}} \left(\frac{(-2)^{-d(a+2b+1)}(2\pi i)^{d(a+2b+1)\kappa}(2/\pi)^{d(a+2b+1)(a+2b)/2}}{\prod_{j=0}^{a+2b}(\kappa-j-1)^d}\right)^{-1} \prod_{i=0}^{(a+2b)} L^\Sigma(2z_\kappa + a + 2b + 1 - i, \bar{\tau}'\chi_{\mathcal{K}}^i) \\ &\prod_{v|p} (\mathfrak{g}(\bar{\tau}_v')^{a+2b+1} c_{a+2b+1}(\tau_v', z_\kappa))^{-1} \prod_{v\nmid p, v\in\Sigma} \tau^{-1}(y_v\bar{y}_v x_v) |(y_v\bar{y}_v)^2 x_v \bar{x}_v|_v^{z_\kappa + \frac{a+2b+1}{2}} C_v \end{aligned}$$

Here $\Omega_\infty$ is the CM period in section 2.1. First note that since $\pi$ is nearly ordinary with respect to the scalar weight $\kappa$ and $\phi = \phi^{ord}$ is a holomorphic nearly ordinary vector. Then its contragradient is also nearly ordinary. (But the nearly ordinary vector is not the one whose neben-type is the inverse of $\phi^{ord}$). We denote this representation as $\tilde{\pi}$. We choose a nearly ordinary vector of this

representation which we choose to be "$p$-adically primitive", i.e. integral but not divisible by $p$ in terms of Fourier Jacobi expansion. In general we will need some Gorenstein properties of certain Hecke algebras to make primitive forms in Hida families. But we do not touch this at the moment. We consider $E(\gamma(g,-))$ as an automorphic form on $U(a+b,b)$. For each $v \nmid p$ there is a level group $\tilde{K}_{v,s} \subset U(a+b,b)_v$ such that

$$\prod_{v\nmid p} \rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-)) \otimes \bar{\tau}(\det -))$$

is invariant under its action. Now let $\tilde{\pi}$ be the contragradient automorphic representation of $\pi$. Suppose $\tilde{\phi}^{ord}$ is a "new form" with level group $\tilde{K}_v$ so that $\tilde{\pi}_v^{\tilde{K}_v}$ is 1-dimensional for each $v$ and that its infinity components are in the contragradient $K_\infty$-type of $\phi^{ord}$. (We will make everything explicit in a future work. Right now let us content ourselves with these vague definitions for simplicity.) Assume also that there is a Hecke action $1_{\tilde{\pi}}$ with respect to the level group $\prod_v \tilde{K}_v$ which takes any nearly ordinary automorphic form for this level to its $\tilde{\pi}$ component (which is a multiple of $\tilde{\phi}^{ord}$).

**Remark 5.1.1.** *In a future work we will see that when deforming everything in families, the (Fourier coefficients of the) Siegel Eisenstein series $B_{\mathcal{D}}E$ moves p-adic analytically. This enables us to construct the p-adic analytic family $E_{Kling}$. This is the reason for introducing $B_{\mathcal{D}}$.*

We define $E_{Kling}$ by:

$$B_{\mathcal{D}} 1_{\pi^c}^{low} e^{low} \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_{v,s}} \rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) = E_{Kling}(g) \boxtimes \tilde{\phi}^{ord}$$

Here we used the superscript *low* to mean that under $U(a+b+1,b+1)\times U(a+b,b) \hookrightarrow U(a+2b+1,a+2b+1)$ the action is for the group $U(a+b,b)$. We remark that when $a \neq 0$, this $E_{Kling}$ is nearly ordinary with some Borel which is not quite the one we want. Therefore we need to translate both the Siegel Eisenstein series and the Klingen Eisenstein series by some Weyl element at primes dividing $p$. We leave the precise computations for this to the future when doing arithmetic applications in the $a \neq 0$ cases.

Recall that for $v|p$ we have defined $\phi_{aux,v} = \sum_{x\in J} \pi \begin{pmatrix} 1 & & \\ & 1 & \\ & & x \end{pmatrix} \phi_{w,v}$ where $x$ runs through

$\begin{pmatrix} 1 & x_{12} & ... & x_{1b} \\ & ... & ... & ... \\ & & ... & ... \\ & & & 1 \end{pmatrix}$ with $x_{ij}$ running through representatives of $[\mathbb{Z}_p : p^{t_{a+b+i}-t_{a+b+j}}\mathbb{Z}_p]$. $\phi_{aux,v}$ apparently depends on the choices of the representatives. We write $\phi_w$ and $\phi_{aux}$ as the vector obtained by applying the corresponding operators at all primes dividing $p$ to $\phi = \phi^{ord}$. We write $< -, - >$ as the integration the product of the two forms over $U(F)\backslash U(\mathbb{A}_F)$. We use the Haar measure here, normalized so that it is the product of Haar measures of local groups in which the measure of each $U(r,s)(\mathcal{O}_{F,v})$ is 1 and the Archimedean measures are the ones chosen in [31]. This induces natural pairing between $\pi$ and $\tilde{\pi}$, then

$$< \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),$$

$$\rho(\prod_{v|p}(\begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{t_1} & & & \\ & & & ... & & \\ & & & & p^{t_{a+1}} & \\ & & & & & ... \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota}))\phi_{aux} >$$

$$= \quad < \rho^{low}(\prod_{v|p} \begin{pmatrix} p^{t_{a+b+1}} & & & & \\ & ... & & & \\ & & p^{t_{a+2b}} & & \\ & & & 1_a & \\ & & & & 1_b \end{pmatrix}^{\iota}) \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -),$$

$$\rho(\prod_{v|p}(\begin{pmatrix} 1_b & & & & \\ & p^{t_1} & & & \\ & & ... & & \\ & & & p^{t_{a+1}} & \\ & & & & ... \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota}))\phi_{aux} >$$

Since $E(\gamma(g,-))$ satisfies the property that if $\tilde{K}'''$ is the subgroup of $\tilde{K}$ (defined in the last chapter)

consisting of matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^\iota$ such that the $(i,j)$-th entry of $a_7$ is divisible by $p_{t_i+t_{a+b+j}}$ and the $(i,j)$-th entry of $a_4$ is divisible by $p^{t_{a+i}+t_{a+b+j}}$, the $i$-th row of $a_8$ and the right to diagonal entries of $a_9$ are divisible by $p^{t_i}$ for $i = 1, ..., a$, the $i$-th column of the below diagonal entries of $a_1$ are divisible by $p^{t_{a+b+i}}$, the $i$-th row of the up to diagonal entries of $a_5$ are divisible by $p^{t_{a+i}}$. Then the right action of $h^\iota$ for $h \in \tilde{K}'''$ on $E(\gamma(g,-))$ is given by the character $\lambda(h^\iota) = \bar{\chi}_{a+b+1}(h_{11})...\bar{\chi}_{a+2b}(h_{bb})\bar{\chi}_1(h_{b+1,b+1})..\bar{\chi}(h_{a+b,a+b})\bar{\chi}_{a+1}(h_{a+b+1,a+b+1})...\bar{\chi}_{a+b}(h_{a+2b,a+2b})$. It is elementary to check that the above expression equals:

$$(\prod_{v|p} \frac{1}{\prod_{i=1}^{t} p^{t_{a+b+i}(a+b)}}) \qquad < \prod_{v|p} (\sum_y \rho^{low}(y)\rho^{low}(\begin{pmatrix} p^{t^{a+b+1}} & & & \\ & ... & & \\ & & 1_a & \\ & & & 1_b \end{pmatrix}^\iota)$$

$$\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta \mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),$$

$$\rho(\prod_{v|p}(\begin{pmatrix} 1_b & & & \\ & p^{t_1} & & \\ & & ... & \\ & & & p^{t_{a+1}} \\ & & & & ... \end{pmatrix}^\iota \begin{pmatrix} & & & -1_b \\ & & 1_a & \\ 1_b & & & \end{pmatrix}^\iota))\phi_w >$$

where $y$ runs over $N(\mathbb{Z}_p)/\beta N(\mathbb{Z}_p)\beta^{-1}$ and $\beta = \begin{pmatrix} p^{t_{a+b+1}} & & & \\ & ... & & \\ & & 1_a & \\ & & & 1_b \end{pmatrix}$. For $v|p$ define $T^{low}_{\beta,v}$ to be the Hecke action corresponding to $\beta$ just in terms of double cosets. (no normalization factors involved). By checking the level actions we can see that the $\tilde{\pi}$ component of the left part when viewed as an automorphic form on $U(a+b,a)$ is a multiple of $\tilde{\phi}^{ord}$ defined right before remark 5.1.1.

Suppose its eigenvalue for the Hecke operator $T^{low}_{\beta,v}$ is $\tilde{\lambda}_{\beta,v}$. Let

$$\phi' = \prod_{v|\infty}\phi_v \prod_{v\notin\Sigma}\phi^{sph} \prod_{v\in\Sigma, v\nmid p}\phi_v \prod_{v|p}\rho(\begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{t_1} & & & \\ & & & \cdots & & \\ & & & & p_{t_{a+1}} & \\ & & & & & \cdots \end{pmatrix}^{\iota} \begin{pmatrix} & & & -1_b \\ & 1_a & & \\ 1_b & & & \end{pmatrix}^{\iota})\phi_{aux,v}$$

and

$$\phi'' = \prod_{v|\infty}\phi_v \prod_{v\notin\Sigma}\phi^{sph} \prod_{v\in\Sigma, v\nmid p}\phi_v \prod_{v|p}\rho(\begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{t_1} & & & \\ & & & \cdots & & \\ & & & & p_{t_{a+1}} & \\ & & & & & \cdots \end{pmatrix}^{\iota} \begin{pmatrix} & & & -1_b \\ & 1_a & & \\ 1_b & & & \end{pmatrix}^{\iota})\phi_w.$$

Here for $v|\infty$ the $\phi_v$ is the unique up to scalar vector mentioned before definition 3.1.1. Then we have

**Proposition 5.1.1.** *With these notations we have:*

$$E_{Kling}(g) = B_{\mathcal{D}} \prod_{v|p} \frac{\prod_{i=0}^{t} p^{t_{a+b+i}(a+b)}}{\tilde{\lambda}_{\beta,v}} . \frac{< \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))E(\gamma(g,-)), \phi' >}{< \tilde{\phi}^{ord}, \phi'' >}$$

*Proof.* The $\tilde{\pi}$-component of the left part of the inner product above is:

$$\begin{aligned} & \prod_{v|p} T^{low}_{\beta,v}.\tilde{\pi} - \text{component of } \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\ = \quad & e^{low}.\prod_{v|p} T^{low}_{\beta,v}.\tilde{\pi} - \text{component of } \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\ = \quad & \prod_{v|p} T^{low}_{\beta,v}.e^{low}.\tilde{\pi} - \text{component of } \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\ = \quad & \prod_{v|p} T^{low}_{\beta,v}.\tilde{\pi} - \text{component of } e^{low}\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\ = \quad & \prod_{v|p} \tilde{\lambda}_{\beta,v}.\tilde{\pi} - \text{component of } e^{low}\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \end{aligned}$$

Thus

$$\begin{aligned}
&\tilde{\pi}-\text{component of } e^{low}\textstyle\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -))\\
=\quad & \frac{<\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),\phi''>}{<\tilde{\phi}^{ord},\phi''>}\phi^{c,ord}\\
=\quad & \prod_{v|p}\frac{\prod_{i=0}^{t} p^{t_{a+b+i}(a+b)}}{\tilde{\lambda}_{\beta,v}}\cdot\frac{<\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),\phi'>}{<\tilde{\phi}^{ord},\phi''>}\phi^{c,ord}
\end{aligned}$$

Thus we get the proposition. □

## 5.2 Constant Terms

### 5.2.1 Archimedean Computation

Suppose $\pi$ is associated to the weight $(0,...,0;\kappa,...,\kappa)$, then it is well known that there is a unique (up to scalar) vector $v\in\pi$ such that $k.v=\det\mu(k,i)^{-\kappa}$ for any $k\in K_\infty^{+,\prime}v$ (notation as in section 3.1). Then by Frobenius reciprocity law there is a unique (up to scalar) vector $\tilde{v}\in I(\rho)$ such that $k.\tilde{v}=\det\mu(k,i)^{-\kappa}\tilde{v}$ for any $k\in K_\infty^+$. We fix $v$ and scale $\tilde{v}$ such that $\tilde{v}(1)=v$. In $\pi^\vee$, $\pi(w)v$ ($w$ is defined in section 3.1) has the action of $K_\infty^+$ given by multiplying by $\det\mu(k,i)^{-\kappa}$. We define $w'\in U(a+b+1,b+1)$ by $w'=\begin{pmatrix} 1_b & & & \\ & & & 1\\ & & 1_a & \\ & & & 1_b\\ & -1 & & \end{pmatrix}$. Then there is a unique vector $\tilde{v}^\vee\in I(\rho^\vee)$ such that the action of $K_\infty^+$ is given by $\det\mu(k,i)^{-\kappa}$ and $\tilde{v}^\vee(w')=\pi(w)v$. Then by uniqueness there is a constant $c(\rho,z)$ such that $A(\rho,z,\tilde{v})=c(\rho,z)\tilde{v}^\vee$.

**Lemma 5.2.1.** *Assumptions are as above, then:*

$$\begin{aligned}
c(\rho,z)\quad =\pi^{a+2b+1}&\textstyle\prod_{i=0}^{b-1}(\frac{1}{z+\frac{\kappa}{2}-\frac{1}{2}-i-a})(\frac{1}{z-\frac{\kappa}{2}+\frac{1}{2}-i})\prod_{i=0}^{a-1}(\frac{1}{-1+i-2z+2b})\\
&\times\frac{\Gamma(2z+a)2^{-1-2z+2b}}{\Gamma(\frac{a+1}{2}+z+\frac{\kappa}{2})\Gamma(\frac{a+1}{2}+z-\frac{\kappa}{2})}\det(i\theta/2)^{-2}.
\end{aligned}$$

*Proof.* It follows the same way as [35, 9.2.2]. □

**Corollary 5.2.1.** *In case when $\kappa>\frac{3}{2}a+2b$ or $\kappa\geq 2b$ and $a=0$, we have $c(\rho,z)=0$ at the point $z=\frac{\kappa-a-2b-1}{2}$.*

Let $F$ be the Klingen section which is the tensor product of the local Klingen sections defined

in the last Chapter by pulling back of the corresponding Siegel sections. In the case when $\kappa$ is sufficiently large the intertwining operator:

$$A(\rho, z_\kappa, F) = A(\rho_\infty, z_\kappa, F_\kappa) \otimes A(\rho_f, z_\kappa, F_f)$$

and all terms are absolutely convergent. Thus as a consequence of the above corollary we have $A(\rho, z_\kappa, F) = 0$. Therefore the constant term of $E_{Kling}$ is just $B_{\mathcal{D}} F_{z_\kappa}$. It is essentially

$$\frac{L^\Sigma(\tilde{\pi}, \bar{\tau}^c, z_\kappa + 1)}{\Omega_\infty^{2\kappa\Sigma} < \tilde{\phi}^{ord}, \phi'' >} . L^\Sigma(2z_\kappa + 1, \bar{\tau}' \chi_{\mathcal{K}}^{a+2b}) \phi.$$

up to a product of normalization factors at local places.

# Chapter 6

# Hilbert Modular Forms and Selmer Groups

From now on we are in part two where we specialize to $U(1,1) \hookrightarrow U(2,2)_{/F}$ and prove our main theorem.

## 6.1 More Notations

Now we are going to define more notations that we are going to use from now on. We define $\mu_{p^\infty}$ as the set of roots of unity with order powers of $p$. Let $\delta_{\mathcal{K}}, \mathfrak{d} = \mathfrak{d}_F, D_{\mathcal{K}}, D_F$ be the different and discriminant of $\mathcal{K}$ and $F$. Let $\tilde{\delta}_{\mathcal{K}}$ be the different from $\mathcal{K}$ to $F$ and $\tilde{D}_{\mathcal{K}} = \mathrm{Nm}_{\mathcal{K}/F}(\tilde{\delta}_{\mathcal{K}})$. We denote $N$ to be the level of $f$ and $M$ the prime to $p$ part of it. Here $N, M, \delta_{\mathcal{K}}, \mathfrak{d}, D_{\mathcal{K}}, D_F$ are all elements in the ideles of $F, \mathcal{K}$ or $\mathbb{Q}$ supported at the finite primes (also the $M_{\mathcal{D}}$ defined later)! This is much more convenient when working in the adelic language. For each $v|p$ we suppose $p^{r_v} \| N_v$ (we save the notation $r_v$ for other use). We assume that $\mathcal{K}$ is split over all primes dividing the $\mathfrak{d}_F$. This assumption makes the computation of Fourier-Jacobi coefficients easier. Let $h = h_F$ be the narrow ideal class number of $F$, we divide the ideal classes $Cl(\mathcal{K})$ into $I_1 \sqcup ... \sqcup I_h$ corresponding to the image of the norm map to $Cl_n(F)$ and suppose $I_1$ are those mapping to the trivial class. (Here $n$ stands for narrow). We assume that $\mathcal{K}$ is disjoint from the narrow Hilbert class field of $F$ and thus it is easy to see that the norm map above is surjective. Also we write $< f, g >$ (integration over $U(1,1)(F)\backslash U(1,1)(\mathbb{A}_F)$) to be the integration of $f.\bar{g}$ along $U(1,1)(F)\backslash U(1,1)(\mathbb{A}_F)$. Note that this convention is different from part one since we are taking the complex conjugation of $g$ here. For $f$

and $g$ Hilbert modular forms such that the product of the central characters of $f$ and $\bar{g}$ are trivial then we denote $<,>_{GL_2}$ to be the inner product on $GL_2$ (integration over $GL_2(F)\mathbb{A}_F^\times\backslash GL_2(\mathbb{A}_F)$, note that we need to mod out by the center here). We also write, for example $<,>_{U_\mathcal{D}}$, $<,>_{GL_2,\Gamma_0(N)}$ the inner product with respect to the indicated level group, i.e. $[U(1,1)(\mathcal{O}_F) : U_\mathcal{D}]$. $<,>$, etc.

For $v$ a finite prime of $F$, as in [35, chapter 8] we define the level group $K_{r,t} \subset GU(2,2)(F_v)$ for $r,t > 0$ as follows: for $Q$ and $P$ being the Siegel and Klingen parabolic, $K_{r,t} = K_{Q,v}(\varpi_v^r) \cap w_2' K_P(\varpi_v^t) w_2'$ where $\varpi_v$ is a uniformizer for $v$, $K_{Q,v}(\varpi_v^r)$ means the matrices which are in $Q(\mathcal{O}_{F,v})$ modulo $\varpi_v^r$ and $K_{P,v}(\varpi_v^t)$ means matrices which are in $P(\mathcal{O}_{F,v})$ modulo $\varpi_v^t$, and $w_2' = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$.

Finally for any domain $A$ we usually write $F_A$ for the fraction field of $A$.

## 6.2 Hilbert Modular Forms

### 6.2.1 Hilbert Modular Forms

We set up the basic notions of Hilbert modular forms, following [11] with minor modifications. Let $I$ be the set of all field embedding of $F$ into $\bar{\mathbb{Q}}$. We may regard $I$ as the set of infinite places of $F$ via $\iota_\infty$. The weight of a Hilbert modular form over $F$ is a pair of elements $(\kappa, w)$ in the free module $\mathbb{Z}[I]$ generated by $I$. We identify $F_\infty = F \otimes_\mathbb{Q} \mathbb{R}$ with $\mathbb{R}^I$ and embed $F$ into $\mathbb{R}^I$ via the diagonal map $a \mapsto (a^\sigma)_{\sigma\in I}$. Then the identity component $G_\infty^+$ of $GL_2(F_\infty)$ naturally acts on $\mathscr{L} = \mathfrak{h}^I$ with $\mathfrak{h}$ the Poincare half plane. We write $C_\infty^+$ for the stabilizer in $G_\infty^+$ of the center point $z_0 = (\sqrt{-1}, \sqrt{-1}, \cdots, \sqrt{-1}) \in \mathscr{L}$. Then for each open compact subgroup $U$ of $GL_2(F_{\mathbb{A}_f})$, we denote by $M_{\kappa,w}(U;\mathbb{C})$ the space of holomorphic modular forms of weight $(\kappa, w)$ with respect to $S$. Namely $M_{\kappa,w}(U;\mathbb{C})$ is the space of smooth functions $f : GL_2(\mathbb{A}_F) \to \mathbb{C}$ satisfying the automorphic condition:

$$f(\alpha x u) = f(x) j_{\kappa,w}(u_\infty, z_0)^{-1} \text{ for } \alpha \in GL_2(F) \text{ and } u \in UC_\infty^+,$$

where $j_{\kappa,w}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = (ad-bc)^{-w}(cz+d)^\kappa$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\infty)$ and $z \in \mathscr{L}$ and such that for any $g_f \in GL_2(\mathbb{A}_f)$ the associated classical form defined by $f_{cl}(z, g_f) := f(g).j_{\kappa,w}(g_\infty, z_0)$ $g$ such that $g_\infty.z_0 = z$ is holomorphic on the symmetric domain and at all cusps. We write $S_{\kappa,w}(U;\mathbb{C})$

for the subspace of $M_{\kappa,w}(U;\mathbb{C})$ consisting of cusp forms. Here we have used the convention that $c^s = \prod_{\sigma\in I} c_\sigma^{s_\sigma}$ for $c = (c_\sigma)_{\sigma\in I} \in \mathbb{C}^I$ and $s = \sum s_\sigma \sigma \in \mathbb{C}[I]$. Setting $t = \sum_\sigma \sigma$, we sometimes use another pair $(n, v)$ to denote the weight, for $n = \kappa - 2t$ and $v = t - w$. Each automorphic representation $\pi$ spanned by forms in $S_{\kappa,w}(U;\mathbb{C})$ has central character $|\cdot|_{\mathbb{A}}^{-m}$ up to a finite order character. The twist $\pi^u = \pi \otimes |\cdot|_{\mathbb{A}}^{m/2}$ is called the unitarization of $\pi$.

Let $h$ be the narrow class number of $F$ and decompose

$$\mathbb{A}_F^\times = \sqcup_{i=1}^h F^\times a_i (\hat{\mathcal{O}_F})^\times F_{\infty+}^\times \text{ with } a_i \in \mathbb{A}_{F,f}^\times.$$

Then by strong approximation

$$G(\mathbb{A}_F) = \cup_{i=1}^h GL_2(F) t_i U_0(N) G_{\infty+} \text{ for } t_i = \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

For any ideal $N$ of $\mathcal{O}_F$ let $U_0(N)$ be the open compact subgroup of $GL_2(\hat{\mathcal{O}_F})$ whose image modulo $N$ is inside $B(\hat{\mathcal{O}_F})$. Let $\varepsilon$ be a neben character of $T(\hat{\mathcal{O}_F^\times})$ whose conductor contains $N$. Any automorphic form in the space $M_{\kappa,w}(U_0(N), \varepsilon; \mathbb{C})$ is determined by its restriction to the connected component of $t_i$ in $GL_2(F) \setminus GL_2(\mathbb{A}_F)/U_0(N)G_{\infty+}$. So we identify the above space with the space of $h$-tuples: $\{f_i\}$ where $f_i$ are forms in $M_{\kappa,w}(\Gamma_i, \mathbb{C})$ for $\Gamma_i := t_i U_0(N) t_i^{-1}$ with $f_i(g_\infty) := f(g_\infty t_i)$. Each $f_i$ has a $q$-expansion:

$$f_i(z) = a(0, f_i) + \sum_{0<<\xi\in F^\times} a(\xi, f_i) e_F(\xi z).$$

Let $\mathbb{A}_{F^+}^\times$ be the set of ideles whose Archimedean parts are all positive. We have the following theorem in [11] about the $q$-expansion for Hilbert modular forms:

**Theorem 6.2.1.** *Each $f \in M_{\kappa,w}(U;\mathbb{C})$ has a Fourier expansion of the following type:*

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}} \{a_0(y\mathfrak{d}, f)|y|_{\mathbb{A}}^{-[v]} + \sum_{0<<\xi\in F^\times} a(\xi y\mathfrak{d}, f)\{(\xi y d)^v\}(\xi y_\infty)^{-v} e_F(i\xi y_\infty) e_F(\xi x)\},$$

*where $\mathbb{A}_{F^+}^\times \ni y \mapsto a_0(y, f)$ is a function invariant under $F_+^\times U_F(N) F_{\infty+}^\times$ (here $+$ and $0 <<$ means totally positive) and vanishes identically unless $w \in \mathbb{Z}\cdot t$, and $\mathbb{A}_{F+}^\times \ni y \mapsto a(y, f)$ is a function vanishing outside $\hat{\mathcal{O}}_F F_{\infty+}^\times$ and depending only on the coset of $y_f U_F(N)$.*

This adelic $q$-expansion is deduced from the usual $q$-expansions. We omit the details and refer to

[11].

### 6.2.2 Hida Families

First of all let us define the weight space for Hilbert modular Hida families. For $A$ some finite extension of $\mathbb{Q}_p$ let $\Lambda'_W := A[[\{W_{1,v}, W_{2,v}\}_{v|p}]]$. A point $\phi \in \mathrm{Spec}(\Lambda'_W)$ is called arithmetic if $(1+W_{1,v}) \mapsto \zeta_{1,,v,\phi} \in \mu_{p^\infty})$ and $(1+W_{2,v}) \mapsto (1+p)^{\kappa_{\phi,v}-2}\zeta_{2,v,\phi}$ where $\zeta_{2,v,\phi} \in \mu_{p^\infty}$ and $\kappa_{\phi,v} \geq 2$ is some integer. We also require that $\kappa_{\phi,v}$ to be the same for all $v$. (this means we only consider Hilbert modular forms of parallel weight, which is already enough for constructing the whole Hida family.)

Define $\Lambda_W$ such that $\mathrm{Spec}\Lambda_W$ is the closed subspace of $\mathrm{Spec}\Lambda'_W$ defined as the Zariski closure of the arithmetic points satisfying: $\phi((1+W_{1,v})(1+W_{2,v}))$ is equal for all $v|p$. It is naturally a power series ring with $d+1$ variables. We only consider this weight space for simplicity. In fact if the Leopoldt Conjecture is true, then this is the whole weight space for Hida families of Hilbert modular forms.

Now we define the neben-typus associated to $\phi$: let

$$\varepsilon'_{1,\phi,v}(1+p) = \zeta_{1,\phi_v}, \varepsilon'_{2,\phi,v}(1+p) = \zeta_{2,\phi,v}.$$

We extend these to be characters on $\mathcal{O}_v^\times$ as follows: for $a$ such that $a \equiv 1 \bmod p$ it is obvious how to extend and then we require them to be trivial on the torsion part of $\mathcal{O}_v^\times$. Define:

$$\varepsilon_{\phi,v}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \varepsilon'_{1,v,\phi}(a)\varepsilon'_{2,v,\phi}(b)\omega^{\kappa_\phi - 2}(b)$$

for $a, b \in \mathcal{O}_v^\times$ (recall that $\omega$ is the Techimuller character). It is well known that in the Hilbert modular form case there is a nearly ordinary idempotent $e$ defined by Hida.

**Definition 6.2.1.** *A Hilbert modular form $f$ is called nearly ordinary if $ef = f$. We define $M^{ord}_{\kappa,w}(U;\mathbb{C})$ and $S^{ord}_{\kappa,w}(U;\mathbb{C})$ to be the space of nearly ordinary modular forms and cusp forms with level group $U$.*

**Remark 6.2.1.** *Suppose $f$ is a nearly ordinary unitary eigenform of weight $(\kappa_\phi, \frac{\kappa}{2})$ and nebentypus $\varepsilon_\phi$. Then we can assume that for each $v|p$ the $v$-component is $\pi(\mu_{1,v}, \mu_{2,v})$ where $\mathrm{val}\mu_{1,v}(p) = p^{-\frac{\kappa_\phi - 1}{2}}, \mathrm{val}_p\mu_{2,v}(p) = p^{\frac{\kappa_\phi - 1}{2}}$. In this case $\mu_{1,v}$, $\mu_{2,v}$ have the same restriction to $\mathcal{O}^\times_{F,v}$ as $\varepsilon'_{1,\phi,v}$ and $\varepsilon'_{2,\phi,v}\omega^{\kappa_\phi - 2}$ respectively.*

Let $\mathbb{I}$ be a finite integral extension of $\Lambda_W$.

**Definition 6.2.2.** *Let $M$ be an ideal of $F$ prime to $p$. An $\mathbb{I}$-adic ordinary cusp form $\mathbf{f}$ of level $V_1(M)$ is a set of elements of $\mathbb{I}$:*

$$\{c_i(\xi,\mathbb{I}) \text{ for } \xi \in F^\times, c_i(0,\mathbb{I}) \in \mathbb{I} \text{ for } i = 1,\cdots,h\}$$

*with the property that for a Zariski dense set of primes $\phi$ of $\mathbb{I}$ which map to an arithmetic point in* $\mathrm{Spec}(\Lambda_W)$*, the specialization $\mathbf{f}_\phi := \{\sum_{\xi\in F^\times} \phi(c_i(\xi,\mathbb{I}))e_F(\xi,z), i = 1,...,h\}$ is the $q$-expansion of a nearly ordinary cusp form of weight $(\kappa_\phi, \frac{\kappa_\phi}{2})$, prime to $p$ level $M$ and nebentypus $\varepsilon_\phi$ at primes dividing $p$.*

### 6.2.3 Galois Representations of Hilbert Modular Forms

Let $A$ be a finite extension of $\mathbb{Q}_p$. One can also define the space of Hilbert modular forms $M_{\kappa,w}(U_0(N),\varepsilon,A)$ and the corresponding cuspidal spaces $S_{\kappa,w}(U_0(N),\varepsilon,A)$. For $f \in S_{\kappa,w}(U_0(N),\varepsilon,A)$, $\kappa \geq 2$, a normalized eigenform, we fix $L \subset \bar{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$ containing all the Fourier coefficients of $f$. Let $\mathcal{O}_L$ be the integer ring of $L$ and $\mathbb{F}$ its residue field. Then we have a continuous semi-simple 2-dimensional Galois representation $(\rho_f, V_f)$: $\rho_f : G_Q \to GL_L(V_f)$, characterized by being unramified at primes $v \nmid p$ such that $\pi_v$ is unramified and satisfying:

$$\mathrm{tr}\rho_f(\mathrm{frob}_v) = a(v,f)$$

where $a(v,f)$ is the Hecke eigenvalue of $f$ under the Hecke operator $T_v$ (recall that this is associated to $\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}$ where $\varpi_v$ is a uniformizer at $v$). Furthermore, if $f$ is nearly ordinary at all primes dividing $p$, then we have the following description of $\rho_f$ restricted to the decomposition groups for all primes $v$ dividing $p$:

$$\rho|_{G_{F_v}} \simeq \begin{pmatrix} \sigma_{\mu_{1,v}} & * \\ & \sigma_{\mu_{2,v}} \end{pmatrix}.$$

Here $\sigma$ is the local reciprocity map (we use the geometric normalization) and $\pi_v \simeq \pi(\mu_{1,v},\mu_{2,v})$ where $\mu_{1,v}(p)$ has smaller $p$-valuation than $\mu_{2,v}(p)$.

Therefore for each $v|p$ we have a one-dimensional subspace $V^+_{f,v} \subset V_f$ such that the action of $G_v$ on $V^+_{f,v}$ is given by the character $\sigma_{\mu_{1,v}}$ and $G_v$ acts on the quotient $V^-_{f,v} := V_{f,v}/V^+_{f,v}$ by $\sigma_{\mu_{2,v}}$.

We choose a Galois stable $\mathcal{O}_L$ lattice $T_f$ of $V_f$.

Similarly, for a Hida family $\mathbf{f}$ of eigenforms with coefficient ring $\mathbb{I}$. Suppose the residue Galois representation $\bar{\rho}_{f_\phi}$ for some member $f_\phi$ of this family is irreducible (***irred***) defined in the introduction). Then it is well known that we have a Galois representation

$$\rho_{\mathbf{f}} : G_F \to GL_2(\mathbb{I})$$

and we denote the representation space (a free rank 2 module over $\mathbb{I}$) by $T_{\mathbf{f}}$.

## 6.3 Selmer Groups

We recall the notion of $\Sigma$-primitive Selmer groups, emphasizing the case of Hilbert modular case, following [35, 3.1] with some modifications. Let $F$ be a totally real number field as before. Let $T$ be a free module of finite rank over a profinite $\mathbb{Z}_p$-algebra $A$ and assume that $T$ is equipped with a continuous action of $G_F$. Denote by $A^*$ as the Pontryagin dual of $A$. Assume furthermore that for each place $v|p$ of $F$ we are given a $G_v$-stable free $A$-direct summand $T_v \subset T$. For any finite set of primes $\Sigma$ we denote by $Sel_F^\Sigma(T, (T_v)_{v|p})$ the kernel of the restriction map:

$$H^1(F, T \otimes_A A^*) \to \prod_{v \notin \Sigma, v \nmid p} H^1(I_v, T \otimes_A A^*) \times \prod_{v|p} H^1(I_v, T/T_v \otimes_A A^*).$$

We always assume that $\Sigma$ contains all primes at which $T$ is ramified. We put

$$\mathbf{X}_F^\Sigma(T, (T_v)_{v|p}) := Hom_A(Sel_F^\Sigma(T, (T_v)_{v|p}), A^*).$$

If $E/F$ is a finite extension, we put $Sel_E^\Sigma(T) := Sel_E^{\Sigma_E}(T, (T_w)_{w|p})$ and $\mathbf{X}_E^\Sigma(T) := \mathbf{X}_E^{\Sigma_E}(T, (T_w)_{w|p})$, where $\Sigma_E$ is the set of places of $E$ over those in $\Sigma$ and if $w|v|p$ then $T_w = g_w T_v$ for $g_w \in G_F$ such that $g_w^{-1} G_{E,w} g_w \subseteq G_{F.v}$. If $E/F$ is infinite we set: $Sel_E^\Sigma(T) = \varinjlim_{E \subseteq F' \subseteq E} Sel_{F'}^\Sigma(T)$ and $\mathbf{X}_E^\Sigma(T) = \varprojlim_{F \subseteq F' \subseteq E} \mathbf{X}_{F'}^\Sigma(T)$, where $F'$ suns over the finite extensions of $F$ contained in $E$.

Suppose $\mathcal{K}/F$ is a CM number field over its maximal totally real subfield, $c$ being the nontrivial element of $G_F/G_{\mathcal{K}}$. Then we have an action of $c$ on the Selmer groups of $F$. We have the following lemma as in [35, 3.1.5]. (Recall that we have assumed $p \neq 2$.)

**Lemma 6.3.1.** *There is a decomposition*

$$Sel_F^\Sigma(T) = Sel_F^\Sigma(T)^+ \oplus Sel_F^\Sigma(T)^-,$$

*according to the $\pm 1$ eigenspaces of the action by c. Also, restriction induces isomorphisms*

$$Sel_F^{\Sigma}(T) \to Sel_{\mathcal{K}}^{\Sigma_{\mathcal{K}}}(T)^+ \qquad Sel_F^{\Sigma}(T \otimes \chi_{\mathcal{K}}) \to Sel_{\mathcal{K}}^{\Sigma_{\mathcal{K}}}(T)^-.$$

## 6.4 Iwasawa Theory of Selmer Groups

We let $F_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $F$. The Galois group, which we denote as $\Gamma_F$, is isomorphic to $\mathbb{Z}_p$. Let $\mathcal{K}_\infty^-$ be the maximal anticyclotomic (the complex conjugation acting by $-1$ on the Galois group) unramified outside $p$ abelian $\mathbb{Z}_p$-extension of $\mathcal{K}$ with Galois group denoted as $\Gamma_{\mathcal{K}}^-$. This is isomorphic to $\mathbb{Z}_p^d$. Write $\mathcal{K}_\infty^+$ for $F_\infty\mathcal{K}$ with $\Gamma_{\mathcal{K}}^+$ the Galois group over $\mathcal{K}$ (identified with $\Gamma_F$). Let $\mathcal{K}_\infty := \mathcal{K}_\infty^- F_\infty$. This is a galois extension with Galois group $\mathbb{Z}_p^{d+1}$. Conjecturally (Leopoldt) this is the maximal unramified outside $p$ abelian $\mathbb{Z}_p$-extension of $\mathcal{K}$. Recall that in chapter 2 we have defined the Iwasawa algebras $\Lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K},A}$. We define more Iwasawa algebras $\Lambda_F$, $\Lambda_{\mathcal{K}}^-$, $\Lambda_{\mathcal{K}}^+$, $\Lambda_{\mathcal{K},A}^-$, $\Lambda_{\mathcal{K},A}^+$ in an analogous way. We let $\boldsymbol{\varepsilon}_F$ be the canonical character $G_F \to \Gamma_F \hookrightarrow \Lambda_F^\times$ of $G_F$.

We fix topological generators for each group above: $\gamma := \mathrm{rec}_F(\prod_{v|p}(1+p)_v)$, $\gamma^+ := \mathrm{rec}_{\mathcal{K}}(\prod_{v|p}(1+p, 1+p)_v^{\frac{1}{2}})$ and $\gamma_v^- := \mathrm{rec}_{\mathcal{K}}((1+p, (1+p)^{-1})_v^{\frac{1}{2}})$. Here rec means the reciprocity map of class field theory normalized by the geometric Frobenius.

### 6.4.1 Control of Selmer Groups

We recall some results in [35, 3.2] with minor modifications for the totally real situation. These will be used in proving the main theorem in the last chapter. In this subsection (only in this subsection) we let $A$ be any profinite $\mathbb{Z}_p$ algebra and $\mathfrak{a}$ an ideal of $A$. Let $T$ be a free $A$-module equipped with a continuous $G_F$-action and let $T^* := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^*$. It is noted in [35, 3.2.7] that there is a canonical map:

$$Sel_F^{\Sigma}(T/\mathfrak{a}T) \to Sel_F^{\Sigma}(T)[\mathfrak{a}].$$

Here $[\mathfrak{a}]$ on the right hand side means the $\mathfrak{a}$-torsion part.

**Proposition 6.4.1.** *Suppose there is no nontrivial $A$-subquotient of $T^*$ on which $G_{\mathcal{K}_\infty^+}$ acts trivially. Suppose also that for any prime $\mathfrak{p}|p$ of $F$ the action of $I_\mathfrak{p}$ on $T/T_\mathfrak{p}$ factors through the image of $I_\mathfrak{p}$ in $\Gamma_F$ and that $\Sigma \cup \{p\}$ contains all primes at which $T$ is ramified. Let $\tilde{F} = F_\infty, \mathcal{K}_\infty^+$. Then the above map induces isomorphisms:*

$$Sel_{\tilde{F}}^{\Sigma}(T/\mathfrak{a}T) \simeq Sel_{\tilde{F}}^{\Sigma}(T)[\mathfrak{a}]$$

*and*

$$X_{\tilde{F}}^{\Sigma}(T) \simeq X_{\tilde{F}}^{\Sigma}(T)/\mathfrak{a}X_{\tilde{F}}^{\Sigma}(T).$$

Descent from $\mathcal{K}_\infty$ to $\mathcal{K}_\infty^+$.

We have the following corollary of the above proposition:

**Corollary 6.4.1.** *Under the hypotheses of the above proposition. If $\tilde{F}$ is $F_\infty$ or $\mathcal{K}_\infty^+$ then:* $\mathrm{Fitt}_{\tilde{F},A/\mathfrak{a}}^{\Sigma}(T/\mathfrak{a}T) = \mathrm{Fitt}_{\tilde{F},A}^{\Sigma}(T)$ *mod* $\mathfrak{a}$*;*

**Corollary 6.4.2.** *Let $I^-$ be the kernel of the natural map $\Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{K}}^+$. Then under the hypotheses of the above proposition, we have an isomorphism:*

$$X_{\mathcal{K}_\infty}^{\Sigma}(T)/I^- X_{\mathcal{K}_\infty}^{\Sigma}(T) \xrightarrow{\sim} X_{\mathcal{K}_\infty^+}^{\Sigma}(T)$$

*of $\Lambda_{\mathcal{K},A}^+$-modules.*

From $\mathcal{K}_\infty^+$ and $F_\infty$ to $\mathcal{K}$ and $F$.

Let $(T, T_v^+(v|p))$ be as above. Let $\phi$ be a algebra homomorphism $\Lambda_F \to \mathbb{C}_p$ and $I_\phi$ be its kernel.

**Proposition 6.4.2.** *Let $(T', T_v'^+|_{v|p})$ be $(T, T_v^+|_{v|p})$ twisted by $\phi \circ \boldsymbol{\varepsilon}_F$. Suppose there is no nontrivial $A$-subquotient of $T'^*$ on which $G_F$ acts trivially. Assume:*
*(i) $\Sigma \cup \{$ primes above $p\}$ contains all primes at which $T$ is ramified;*
*(ii) For any $v|p$, $(H^0(I_v, T/T_v^+ \otimes_A \Lambda_{F,A}^*(\boldsymbol{\varepsilon}_F^{-1})) \otimes_{\Lambda_F} \Lambda_F/I_\phi)^{G_v} = 0$.*
*Then restriction yields isomorphisms:*

$$Sel_F^{\Sigma}(T') \to Sel_{F_\infty}^{\Sigma}(T)[I_\phi] \quad \text{and} \quad Sel_{\mathcal{K}}^{\Sigma}(T') \to Sel_{\mathcal{K}_\infty^+}^{\Sigma}(T)[I_\phi]$$

*Here we have identified $\Lambda_{\mathcal{K}}^+$ with $\Lambda_F$.*

This is only a slight generalization of [35] Proposition 3.2.13 and the proofs are identical.

# Chapter 7

# Hida Theory for Unitary Hilbert Modular Forms

In this chapter we recall basic results about ordinary Hida families for Unitary groups over totally real fields. We also recall generalizations of certain results in [35, chapter 6] which are mostly due to Hida. Some results are only stated for cuspidal forms since this is enough for our use. However as a trade off we make the ad hoc construction in chapter 13 in which we explicitly write down a cuspidal family given the Klingen Eisenstein family.

## 7.1 Iwasawa Algebras

We let $\mathbb{I}_{\mathcal{K}} := \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ and $\Lambda_{\mathcal{D}} := \mathbb{I}[[\Gamma_{\mathcal{K}}^- \times \Gamma_{\mathcal{K}}]] = \mathbb{I}_{\mathcal{K}}[[\Gamma_{\mathcal{K}}^-]]$. Here we used the notation $\mathcal{D}$ which stands for the Eisenstein datum to be defined in the beginning of chapter 11. Let

$$\alpha : A[[\Gamma_{\mathcal{K}}]] \to \mathbb{I}_{\mathcal{K}}^-, \alpha(\gamma^+) = (1+W_{1,v})^{\frac{1}{2}}(1+W_{2,v})^{\frac{1}{2}}(1+p)^1, \gamma_v^- \to \gamma_v^-(1+p)^{\frac{\kappa}{2}}$$

$$\beta : \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]] \to \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]], \beta(\gamma^+) = \gamma^+, \beta(\gamma_v^-) = \gamma_v^-$$

for each $v$. We also let $\Lambda := \Lambda_W[[\Gamma_{\mathcal{K}}^-, \Gamma_{\mathcal{K}}]]$. Thus $\Lambda_{\mathcal{D}}$ is finite over $\Lambda$.

**Definition 7.1.1.** *A $\bar{\mathbb{Q}}_p$ point $\phi \in \mathrm{Spec}\mathbb{I}[[\Gamma_{\mathcal{K}}]]$ is called arithmetic if $\phi|_{\mathbb{I}}$ is arithmetic and $\phi(\gamma^+) = (1+p)^{\frac{\kappa}{2}}\zeta^+$ for $\zeta^+ \in \mu_{p^\infty}$ and $\phi(\gamma_v^-) = \zeta_v^-$ for $\zeta_v^- \in \mu_{p^\infty}$. Here $\kappa = \kappa_{\phi|\mathbb{I}}$.*

We write $\mathcal{X}^a_{\mathbb{I}_{\mathcal{K}}}$ for the set of arithmetic points. Next let $W_2 := \prod_{v|p}(1+p\mathbb{Z}_p^\times)_v^4$ and $\Lambda_2$ be the

completed group algebra of it. We give a $\Lambda_2$-algebra structure for $\Lambda_{\mathcal{D}}$ by: for each $v|p$,

$$(t_1,t_2,t_3,t_4)\to(\alpha\otimes\beta)(\mathrm{rec}_{\mathcal{K}_v}(t_3t_4,t_1^{-1}t_2^{-1})\times\mathrm{rec}_{\mathcal{K}_v}(t_4^{-1},t_2)(1+W_{1,v})^{\log_{1+p}(t_1t_3^{-1})}.$$

Here rec means the reciprocity map in class field theory. This way $\Lambda$ becomes a quotient of $\Lambda_2$.

**Remark 7.1.1.** *When $F=\mathbb{Q}$ then $\Lambda_2=\Lambda$. In general $\Lambda$ is of lower dimension. In other words we are only considering a subfamily of the whole weight space.*

## 7.2 Igusa Tower and $p$-adic Automorphic Forms

We refer the definition of Shimura varieties $S(K)$ for the unitary similitude group and open compact $K$ and the automorphic sheaves $\omega_{\underline{k}}$ and the universal differential sheaf $\omega$ to [24], [12] and [17] respectively. Recall that a weight $\underline{k}=\{\underline{k}_\sigma\}_{\sigma\in\Sigma}$ where $\underline{k}_\sigma=(c_{s+1,\sigma},...,c_{r+s,\sigma};c_{1,\sigma},...,c_{s,\sigma})$. We write $M_{\underline{k}}(K,R)$ for the space of automorphic forms with weight $\underline{k}$, level $K$ and coefficient $R$. We write $M^0_{\underline{k}}(K,R)$ for the cuspidal part.

For any $v|p$, $U(2,2)\simeq\mathrm{GL}_4(\mathbb{Z}_p)$ under projection to the first factor of $\mathcal{K}_v=F_v\times F_v$. (Recall that our convention is the first factor correspond to the Archimedean place inside the CM type under $\iota:\mathbb{C}\simeq\mathbb{C}_p$.) Define $B$ to be the standard Borel $\begin{pmatrix}\times&\times&\times&\times\\&\times&\times&\times\\&&\times&\\&&\times&\times\end{pmatrix}$ and $B^u$ the unipotent radical. Let $I_{0,s}$ ($I_{1,s}$) consists of elements in $U(n,n)(\mathbb{Z}_p)$ which are in $B(\mathbb{Z}_p/p^s\mathbb{Z}_p)$ ($B^u(\mathbb{Z}_p/p^s)$) modulo $p^s$. (see [35, 5.3.6].)

Let $L$ be a finite extension of $\mathbb{Q}_p$. Recall that as in [35, 6.1], if $K$ is neat and maximal at all primes dividing $p$, we have $\mathcal{S}=\mathcal{S}_K$ a fixed toroidal compactification of $S_G(K)$ over $\mathcal{O}_L$. Let $\mathcal{I}_{\mathcal{S}}$ be the ideal of the boundary of $\mathcal{S}$. There is a section $H$ of $det(\omega)$, called the Hasse invariant. Since $\det(\omega)$ is ample on the minimal compactification $\mathcal{S}^*$, one finds $E$, a lifting of $H^m$ over $\mathcal{O}_L$ for sufficiently large $m$. Then $\mathcal{S}^*[1/E]$ is affine. For any positive integer $m$, set $S_m:=\mathcal{S}[1/E]\times\mathcal{O}_L/p^m$. Let $H=\mathrm{GL}_2\times\mathrm{GL}_2$. For any integers $s\geq m$, we have the Igusa variety $T_{s,m}$ which is an etale Galois covering of $S_m$ with Galois group canonically isomorphic to $\prod_{v|p}GL_2(\mathcal{O}_{F,v}/p^s)^+\times GL_2(\mathcal{O}_{F,v}/p^s)^-=H(\prod_{v|p}\mathcal{O}_{F,v}/p^s)$. We put $V_{s,m}:=\Gamma(T_{s,m},\mathcal{O}_{T_{s,m}}\otimes_{\mathcal{O}_{\mathcal{S}}}\mathcal{I}_{\mathcal{S}})$. For $j=0,1$ let $I^H_{j,s}:=I_{j,s}\cap H(\prod_{v|p}\mathcal{O}_v/p^s)$, define

$$W_{s,m}:=H^0(I^H_{1,s},V_{s,m})$$

and

$$\mathcal{W} := \varinjlim_{m}(\varprojlim_{s} W_{s,m}).$$

We also write $V^0_{s,m}$, $W^0_{s,m}$, $\mathcal{W}^0$ to be the cuspidal part of the corresponding spaces.

For $q = 0$ or $\phi$ we also defines the space of $p$-adic automorphic forms on $G$ of weight $\underline{k}$ and level $K = K_p^0 K^p$ with $p$ divisible coefficients:

$$V^q_{\underline{k}}(K, L/\mathcal{O}_L) := \varinjlim_{m} \Gamma(S_m, \omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}_S).$$

and similarly, if $A$ is an $\mathcal{O}_L$-algebra the the space of $p$-adic automorphic forms with coefficients in $A$ are defined as the inverse limits:

$$V_{\underline{k}}(K, A) := \varprojlim_{m} \Gamma(S_m, (\omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}_S) \otimes_{\mathcal{O}_L} A).$$

Finally for any $\underline{a} = \{a_v\}_{v|p}$ where each $a_v \in (\mathbb{F}_p^\times)^4$ we define the modules: $V^q_{\underline{a},\underline{k}}(K, L/\mathcal{O}_L)$, etc, in the same way as [35, 6.2].

## 7.3 Nearly Ordinary Automorphic Forms

Hida defined an idempotent $e_{ord}$ on the space of $p$-adic automorphic forms (see [35] chapter 6) and we define $W_{ord}$, $\mathcal{W}_{ord}$ $V_{\underline{k},ord}(K, A)$ to be the image of $e_{ord}$ acting on the corresponding spaces. Now we recall the following important theorem of Hida (see [35, 6.2.10]):

**Theorem 7.3.1.** *For any sufficiently regular weight $\underline{k}$ there is a constant $C(\underline{k}) > 0$ depending on $\underline{k}$ such that for any integer $l > C(\underline{k})$, the canonical map:*

$$e_{ord} M^0_{\underline{k}+l(p-1)\underline{t}}(K, L/\mathcal{O}_L) \hookrightarrow V^0_{\underline{k}+l(p-1)\underline{t},ord}(K, L/\mathcal{O}_L)$$

*with $\underline{t} = (0, 0; 1, 1)$ at all infinite places $\sigma$.*

From this theorem we know that there are enough classical forms in our family and thus can construct families of (pseudo)-Galois representations from the classical ones. This is also used in the proof of theorem 14.2.1 where we used Harris' result that there are no (CAP) form with sufficiently regular weight.

**Lemma 7.3.1.** *For any weight $\underline{k}$, we have canonical isomorphisms;*

$$V^q_{\underline{k},ord}(K,\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathcal{W}^q_{ord}[\underline{k}] := \{w \in \mathcal{W}^q :: t.w = t^{\underline{k}}w \forall t \in T_H(\mathbb{Z}_p)\}$$

*and*

$$V^q_{\underline{k},ord}(K^pI_s,\psi,\mathbb{Q}_p/\mathbb{Z}_p) \simeq (\mathcal{W}^q \otimes_{\mathbb{Z}} A)[\psi_{\underline{k}}] := \{w \in \mathcal{W}^q \otimes_{\mathbb{Z}} A : t.w = \psi_{\underline{k}}(t)w \forall t \in T_H(\mathbb{Z}_p)\}$$

*for any $\mathbb{Z}_p(\psi)$-algebra $A$.*

*Proof.* the same as [35, 6.2.3]. □

**Proposition 7.3.1.** *For $q = 0$ we have for any sufficiently regular weight $\underline{k} \geq 0$, the canonical base-change morphism*

$$e_{ord}.\Gamma(\mathcal{S}^*[1/E],\pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q) \otimes \mathbb{Z}/p^m\mathbb{Z}) \to e_{ord}.\Gamma(\mathcal{S}^*[1/E],\pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q \otimes \mathbb{Z}/p^m\mathbb{Z}))$$

*is an isomorphism.*

This proposition fails for $q \neq 0$, thus we can't get a good control theorem for non-cuspidal Hida families.

The following corollary is immediate from the above proposition.

**Corollary 7.3.1.** *For $q = 0$ and any sufficiently regular weight $\underline{k}$ the module $V^q_{\underline{k},ord}(K,\mathbb{Q}_p/\mathbb{Z}_p)$ is divisible.*

## 7.4 Λ-adic Ordinary Automorphic Forms

Recall that we have defined the Iwasawa algebra $\Lambda_2$. There is an action of it on the space of $p$-adic automorphic forms given by neben characters and the weights. (see [35, chapter 6]) We define $\mathbf{V}_{ord}$ ($\mathbf{V}^0_{ord}$) to be the Pontrjagin dual of $\mathcal{W}_{ord}$ ($\mathcal{W}^0_{ord}$). As in [35] we have the following theorem by the above corollary:

**Theorem 7.4.1.** *$\boldsymbol{V}_{ord}$ is finite over $\Lambda_n$ and $\boldsymbol{V}^0_{ord}$ is free of finite rank over $\Lambda_2$.*

*Proof.* This is proved by Hida. See [35, theorem 6.3.3]. Note that the freeness is no longer true if the base field is not $\mathbb{Q}$. □

Now we define the space of ordinary cuspidal $\Lambda_n$-adic forms to be

$$\mathcal{M}^0_{ord}(K^p, \Lambda_2) = \mathrm{Hom}_{\Lambda_n}(\mathbf{V}^0_{ord}, \Lambda_2).$$

Recall that in 7.1 we have defined a quotient $\Lambda$ of $\Lambda_2$. Then $\boldsymbol{V}^0_{ord} \otimes_{\Lambda_2} \Lambda$ is also free over $\Lambda$. So we define the space of $\Lambda$-adic forms to be:

$$\mathcal{M}^0_{ord}(K^p, \Lambda) = \mathrm{Hom}_{\Lambda}(\mathbf{V}^0_{ord} \otimes_{\Lambda_2} \Lambda, \Lambda).$$

This is a closed subfamily of $\Lambda_2$-adic forms. For any finite $\Lambda$ algebra $A$ we also define the space of $A$-adic forms to be:

$$\mathcal{M}^0_{ord}(K^p, A) = \mathcal{M}^0_{ord}(K^p, \Lambda) \otimes_{\Lambda} A.$$

## 7.5 $q$-Expansions

The $q$-expansion principle will be crucial for our later argument. Similar as in [35], for $x$ running through a (finite) set of representatives of $G(F)\backslash G(\mathbb{A}_{F,f})/K$ with $x_p \in Q(\mathcal{O}_{F,p})$, we have that the $\Lambda_n$-adic $q$-expansion map

$$\mathcal{M}^0_{ord}(K^p, \Lambda) \hookrightarrow \oplus_x \Lambda[[q^{S^+_x}]]$$

is injective. Here $S^+_x$ is the set of Hermitian matrices $h$ in $M_2(\mathcal{K})$ such that $\mathrm{Tr}_{F/\mathbb{Q}}\mathrm{Tr}hh' \in \mathbb{Z}$ for all Hermitian matrices $h'$ such that $\begin{pmatrix} 1 & h' \\ & 1 \end{pmatrix} \in N_Q(F) \cap xKx^{-1}$ and $K$ is the open compact of $G(\hat{\mathcal{O}}_F)$ maximal at primes dividing $p$ which we fix from the very beginning. This follows from the irreducibility of the Igusa Tower. Let $A$ be a finite torsion-free $\Lambda$-free algebra finite over $\Lambda$ and let $\Sigma$ be a Zariski dense subset of primes of $A$ such that $Q \cap \Lambda = P_{\psi_{\underline{k}}}$ for some pair $(\underline{k}, \psi)$ (we refer the definitions to [35, chapter 6]). Let $\mathcal{N}^0_{\Sigma,ord}(A)$ be the set of elements $(F_x)_x \in \oplus_x A[[q^{S^+_x}]]$ such that for each $Q \in \Sigma$ above $P_{\psi_{\underline{k}}}$ the reduction of $(F_x)_x$ is the $q$-expansion of some element $f \in V^0_{\underline{k},ord}(K^p I_s, \psi, A/Q)$. Then we have:

**Lemma 7.5.1.** *the inclusion:*

$$\mathcal{M}^0_{\underline{a},ord}(K^p, A) \hookrightarrow \mathcal{N}^0_{\underline{a},\Sigma,ord}(A)$$

*is an equality.*

*Proof.* See [35, 6.3.7]. □

We will use this lemma to see that the family constructed in the last chapter by formal $q$-expansions comes from some $\Lambda$-adic form.

# Chapter 8

# Klingen Eisenstein Series

Now we recall some notions for Klingen Eisenstein Series in this setting. The situation is slightly different from Chapter 3 since we need to pass from the $GL_2$ picture to the unitary group $U(1,1)$ picture (see to [35] chapter 9). The $p$-adic constructions are just as in part one (this is more general than [35] since we allow nearly ordinary forms instead of only ordinary forms). For the $\ell$-adic construction the one used by [35] is much better than the one used in part one so we just follow [35]. We also remark that as in section 3.1, this chapter follows closely [35] chapter 9.

## 8.1 Induced Representations

### 8.1.1 Archimedean Picture

Let $(\pi, H)$ be a unitary Hilbert representation of $GL_2(\mathbb{R})$ and $H_\infty$ be the smooth vectors. Let $\chi$ be the central character of $\pi$ and let $\psi$ and $\tau$ be unitary characters of $\mathbb{C}^\times$ such that $\psi|_{\mathbb{R}^\times} = \chi$. It is well known that we can use $\pi$ and $\psi$ to define a representation of $GU(1,1)$ which we denote as $\pi_\psi$. Now we define a representation $\rho$ of $P(\mathbb{R})$ in $H$: for $p = mn, n \in N_P(\mathbb{R}), m = m(bx, a) \in M_P(\mathbb{R})$ with $a, b \in \mathbb{C}^\times, x \in GL_2(\mathbb{R})$, we define

$$\rho(p)v := \tau(a)\psi(b)\pi(x)v, v \in H.$$

Recall that in section 3.1 we have defined $I(H_\infty)$ and $I(\rho)$, $f_z$ for $f \in I(\rho)$ and $\sigma(\rho, z)$ which we omit here.

Denote $\eta = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Let $(\pi^\vee, V)$ be $H$ but with the action given by $\pi^\vee(x) = \pi(\eta^{-1}x\eta)$ for $x$ in $GL_2(\mathbb{R})$. Denote $\rho^\vee, I(\rho^\vee), I^\vee(H_\infty)$ and $\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with $\pi, \psi, \tau$ replaced by $\pi^\vee \otimes (\tau \circ \det), \psi\tau\tau^c, \bar{\tau}^c$. Also, recall that for any $z \in \mathbb{C}$, $f \in I(H_\infty)$ and $k \in K_\infty$ we have defined the intertwining operator

$$A(\rho, z, f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn. \tag{8.1}$$

Then $A(\rho, z, -) \in Hom_{\mathbb{C}}(I(H_\infty), I^\vee(H_\infty))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

In section 3.1 we defined the Klingen section $F_\kappa$. This is the same as the section defined in [35, 9.2.1].

### 8.1.2 Prime to $p$ Picture

Again we modify section 3.1 a little to pass from $GL_2$ to $U(1,1)$ picture. Let $v$ be a prime of $F$ and $(\pi, V)$ an irreducible, admissible representation of $GL_2(F_v)$ which is unitary and tempered, with central character $\chi$. Let $\psi$ and $\tau$ be unitary characters of $\mathcal{K}_v^\times$ such that $\psi|_{F_v^\times} = \chi$. We extend $\pi$ to a representation $\rho$ of $P(F_v)$ on $V$ as follows. For $p = mn, n \in N_P(F_v)$, $m = m(bx, a) \in M_P(F_v), a, b \in K_v^\times, x \in GL_2(F_v)$, put

$$\rho(g)v := \tau(a)\psi(b)\pi(s)v, v \in V.$$

We have defined in section 3.1 $I(\rho)$, $f_z$ for $f \in I(\rho)$ and $\sigma(\rho, z)$.

Let $(\pi^\vee, V)$ be $V$ but the action given by $\pi^\vee(g) = \pi(\eta^{-1}g\eta)$. This representation is also tempered and unitary. We denote by $\rho^\vee, I(\rho^\vee)$, and $(\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined by replacing $\pi, \psi$ and $\tau$ by $\pi^\vee \otimes (\tau \circ det), \psi\tau\tau^c$, and $\bar{\tau}^c$, respectively.

Also in section 3.1 we have defined for $f \in I(\rho), k \in K_v$, and $z \in \mathbb{C}$ the intertwining operator

$$A(\rho, z, v)(k) := \int_{N_P(F_v)} f_z(wnk)dn. \tag{8.2}$$

As a consequence of our hypotheses on $\pi$ this integral converges absolutely and uniformly for $z$ and $k$ in compact subsets of $\{z : Re(z) > 3/2\} \times K_v$. As in [35, 9.1.3], this has a meromorphic continuation (in the sense defined there) to $\mathbb{C}$ and the poles can only occur when $Re(z) = 0, \pm\frac{1}{2}$.

### 8.1.3 $p$-adic Picture

Now assume $v|p$. We need to study the relations between the $GL_2$ picture and the computations in part one for $U(1,1)$. Suppose $\pi_v \simeq \pi(\mu_1, \mu_2)$ where $\mathrm{val}_p(\mu_1(p)) = -\frac{\kappa-1}{2}$ and $\mathrm{val}(\mu_2(p)) = \frac{\kappa-1}{2}$. Later we might write $\mu_{1,v}$ and $\mu_{2,v}$ to indicate the dependence on $v$. From now on we write $\xi = \frac{\psi}{\tau}$ and $\xi = (\xi_1, \xi_2)$ with respect to $\mathcal{K}_v \equiv F_v \times F_v$. Similarly for $\tau_1, \tau_2, \psi_1, \psi_2$. Note that our $\xi$ here is different from part one. In fact the $\xi_1, \xi_2, \chi_1, \chi_2$ there are $\mu_1\bar{\xi}_2, \mu_1\bar{\xi}_1, \mu_1\psi_2^{-1}, \mu_2\psi_2^{-1}$. Note that $\psi_1\psi_2 = \mu_1\mu_2$.

Generic Case:

The generic case mentioned in part one correspond to: $\mathrm{cond}(\chi_1) > \mathrm{cond}(\tau_2) > \mathrm{cond}(\tau_1) > \mathrm{cond}(\chi_2)$. (note that the $\tau$ in $\rho^\vee$ is $\bar{\tau}^c$. We assume

$$\mathrm{cond}(\psi_2) > \mathrm{cond}(\tau_2) > \mathrm{cond}(\tau_1) > \mathrm{cond}(\psi_1) > \mathrm{cond}(\mu_1).$$

Then the datum is generic in the sense of part one.

### 8.1.4 Global Picture

Let $(\pi, V)$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ and let $\tau, \psi : \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ be Hecke characters such that $\psi|_{\mathbb{A}_F^\times} = \chi_\pi$. We let $\tau = \otimes\tau_w$ and $\psi = \otimes\psi_w$ be their local decompositions, $w$ over places of $F$. We define $I(\rho)$ to be the restricted product $\otimes I(\rho_w)$ with respect to the $F_{\rho_w}$'s at those $w$ at which $\tau_w, \psi_w, \pi_w$ are unramified.

As in section 3.1 for each $z \in \mathbb{C}$ and $f \in I(\rho)$ we define a function $f_z$ on $G(\mathbb{A}_F)$ as

$$f_z(g) := \otimes f_{w,z}(g_w)$$

where $f_{w,z}$ are defined as before. Also we define an action $\sigma(\rho, z)$ of $(\mathfrak{gu}, K_\infty) \otimes G(\mathbb{A}_f)$ on $I(\rho)$ by $\sigma(\rho, z) := \otimes\sigma(\rho_w, z)$. Similarly we define $\rho^\vee, I(\rho^\vee)$, and $\sigma(\rho^\vee, z)$ but with the corresponding things replaced by their $\vee$'s.

### 8.1.5 Klingen-Type Eisenstein Series on G

Let $\pi, \psi$, and $\tau$ be as above. As in section 3.1, for $f \in I(\rho), z \in \mathbb{C}$, and $g \in GU(2,2)(\mathbb{A}_F)$ the series

$$E(f,z,g) := \sum_{\gamma \in P(F)\backslash G(F)} f_z(\gamma g) \tag{8.3}$$

is known to converge absolutely and uniformly for $(z,g)$ in compact subsets of $\{z \in \mathbb{C} : Re(z) > 3/2\} \times GU(2,2)(\mathbb{A}_F)$. We call this series the Klingen Eisenstein series.

## 8.2 Explicit Local Sections

### 8.2.1 Archimedean Sections

The Klingen section at each place dividing $\infty$ is the $F_\kappa$ defined in chapter 3.

### 8.2.2 Prime to $p$ Sections

Let $v$ be a prime of $F$ not dividing $p$. Let $(\varpi_v^{r_\psi})$ and $(\varpi_v^s)$ be the conductors of $\psi$ and $\xi$. The sections chosen here are the same as in [35, chapter 9] which we briefly recall. For $K \subseteq K_{r,s}$ with $r \geq \max(r_\psi, s)$ we define a character $\nu$ of $K_{r,s}$ by

$$\nu\left(\begin{pmatrix} a & b & & \\ c & d & & * \\ & & * & \\ & * & & * \end{pmatrix}\right) := \psi(ad-bc)\bar{\xi}(d).$$

Let $\phi \in V$ be any vector having a conductor with respect to $\pi^\vee$ and let $(\lambda^{r_\phi}) := cond_{\pi^\vee}(\phi)$. For any $K_{r,t}$ with $r \geq max(r_\psi, r_\phi, s)$ and $t \geq s$ we define $F_{\phi,r,t} \in I(\rho)$ by

$$F_{\phi,r,t}(g) := \begin{cases} \nu(k)\rho(p)\phi & g = pmk \in P(\mathcal{O}_{F,v})wK_{r,t} \\ 0 & otherwise. \end{cases}.$$

### 8.2.3 $p$-adic Sections

We define our $p$-adic section to be the $F_v^0$ defined right after remark 4.4.2. This is nearly ordinary as proved there.

## 8.3 Good Eisenstein Series

### 8.3.1 Eisenstein Data

Let $(\pi, V)$ be an irreducible cuspidal unitary automorphic representation of $GL_2(\mathbb{A}_F)$ with central character $\chi$ and let $V = \otimes V_\pi$ and $\pi = \otimes \pi_w$. By an Eisenstein datum for $\pi$ we will mean a 4-tuple $\mathcal{D} = \{\Sigma, \varphi, \psi, \tau\}$ consisting of a finite set of primes $\Sigma$, a cuspform $\varphi \in V$ that is completely reducible $\varphi = \otimes \phi_v$, and unitary Hecke characters $\psi = \otimes \psi_w$ and $\tau = \otimes \tau_w$ of $\mathbb{A}_{\mathcal{K}}^\times / \mathcal{K}^\times$ satisfying:

- $\Sigma$ contains all primes dividing $p$, primes ramified in $\mathcal{K}/\mathbb{Q}$, and all primes $v$ such that $\pi_v, \psi_v$ or $\tau_v$ is ramified
- for all $k \in K_\infty^{+'}, \pi_\infty(k)\phi_\infty = j(k,i)^{-\kappa}\phi_\infty$;
- if $v \notin \Sigma$ then $\phi_v$ is the newvector;
- if $v \in \Sigma$,$v \nmid p$, then $\phi_v$ has a conductor with respect to $\pi_v^\vee$;
- if $v|p$, then $\phi_v$ is the nearly ordinary vector.
- $\psi|_{\mathbb{A}_F^\times} = \chi$;
- $\tau_v(x) = (x/|x|)^{-\kappa} = \psi_v(x)$ for any $x \in F_v$ and $v|\infty$.

We remark that all the above are similar to [35, 9.3.1] except that for $v|p$ we are allowing nearly ordinary (not just ordinary) vectors.

Let $\xi = \otimes \xi_w = \psi/\tau$ and define $F := \otimes_{v|\infty} F_\kappa \prod_{v \notin \Sigma} F^{sph} \otimes_{v \in \Sigma, v \nmid p} F_{\phi,r,t} \otimes_{v|p} F_v^0$. We define $E_{\mathcal{D}}(z,g) = E(F,z,g)$ to be the Klingen Eisenstein series associated to the section $F$. Then we have the following straightforward generalization of [35, lemma 9.3.2]. The proofs are completely the same.

**Lemma 8.3.1.** *Suppose $\kappa > 6$ and let $z_\kappa := (\kappa - 3)/2$. Let $F = F_\kappa \otimes F_f \in I(\rho) = I(\rho_\infty) \otimes I(\rho_f)$.*
*(1) $A(\rho, z_\kappa, F) = 0$.*
*(2) $E(F, z_\kappa, g)_P = F_{z_\kappa}(g)$.*

Let $\kappa > 6$. Then for any $F = F_\kappa \otimes F_f \in I(\rho)$ we define a function of $(Z, x) \in X_{2,2}^+ \otimes G(\mathbb{A}_{F,f})$:

$$E(Z, x; F) := J(g, \boldsymbol{i})^\kappa \mu(g)^\kappa E(F, z_\kappa, gx), g \in GU(2,2)^+(F_\infty), g(\boldsymbol{i}) = Z.$$

Here $J(g, \boldsymbol{i}) = \det(C_g \boldsymbol{i} + D_g)$ for $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$. The following proposition is essentially [35, 9.3.3].

**Proposition 8.3.1.** *Suppose $\kappa > 6$ and $F = F_\kappa \otimes F_f$. Then $E(Z, x; F)$ is a holomorphic modular form of weight $\kappa$.*

## 8.4 Hecke Operators

We will recall the definitions for the Hecke operators at the unramified primes at the beginning of Chapter 14, which are essentially those given in [35, 9.5]. We let $\mathcal{H}^\Sigma$ be the abstract Hecke algebra generated by Hecke operators introduced there at primes outside $\Sigma$. Define $\lambda_{\mathcal{D}}$: $\mathcal{H}^\Sigma \to \mathbb{C}$ by $h.E_{\mathcal{D}} = \lambda_{\mathcal{D}}(h)E_{\mathcal{D}}$. Also for $v \notin \Sigma$ we are going to introduce a Hecke polynomial $Q_v$ there. We record the following generalization of proposition ([35, 9.6.1]):

**Proposition 8.4.1.** *Suppose $\kappa > 6$ and $v \notin \Sigma$. Then $\lambda_{\mathcal{D}}(Q_v)(q_v^{-s})$ is given by the Euler factor at $v$:*

$$L_{\mathcal{K}}^{\Sigma}(f, \bar{\xi}^c \psi^c, s-2) L^{\Sigma}(\psi^c, s-3) L^{\Sigma}(\chi \bar{\xi}' \psi^c, s-\kappa)$$

*where $L_{\mathcal{K}}^{\Sigma}(f, \bar{\xi}^c \psi^c, s-2)$ is the corresponding $L$-function for $f$ twisted by the character $\bar{\xi}^c \psi^c$ over $\mathcal{K}$, with the Euler factors at primes dividing $\Sigma$ removed.*

The proof is completely the same (there is no difference between the local situations for $F = \mathbb{Q}$ and general $F$). This explains the reason why the Galois representation associated to the Klingen Eisenstein series is the one given in the last chapter.

# Chapter 9

# Hermitian Theta Functions

Recall that we have discussed the Weil representations in subsection 3.3.1. Now we specialize the situation there to $n = 3$ and $m = 2$ and continue to use the notations there.

Theta Functions

Given $\Phi \in \mathcal{S}(V_{\mathbb{A}_F})$ we let

$$\Theta_h(u, g; \Phi) := \sum_{x \in V} \omega_h(u, g)\Phi(x).$$

This is an automorphic form on $U_h(\mathbb{A}_F) \times U_1(\mathbb{A}_F)$.

## 9.1 Some Useful Schwartz Functions.

Suppose $v|\infty$. We now record some Schwartz functions that show up in later formulas and their properties. These are straightforward generalizations of [35, chapter 10]. We have been keeping the presentation parallel to *loc.cit* for convenience.

### 9.1.1 Archimedean Schwartz Functions

Let $\Phi_{h,v} \in \mathcal{S}(V \otimes \mathbb{R})$ be

$$\Phi_{h,v}(x) = e^{-2\pi < x, x >_h}.$$

and that

$$\lambda_v(z) = (z/|z|)^{-2}.$$

Recall that $\mathfrak{h}$ is the Poincare half plane.

**Lemma 9.1.1.** *Given $z \in \mathfrak{h}$, let $\Phi_{h,z}(x) := e(< x, x >_h z)$ (so $\Phi_{h,i} = \Phi_{h,\infty}$). For any $g \in U_1(\mathbb{R})$,*

$$\omega_h(g)\Phi_{h,z} = J_1(g,z)^{-2}\Phi_{h,g(z)}.$$

*In particular, if $k \in K_{\infty,1}^+$ then $\omega_h(k)\Phi_{h,\infty} = J_1(k,i)^{-2}\Phi_{h,\infty}$.*

*Proof.* This is just [35, 10.2.2]. □

### 9.1.2 Schwartz Functions at Finite Places.

For a finite place $v$ of $F$ dividing a rational prime $\ell$, let $\Phi_0 \in \mathcal{S}(V_v)$ be the characteristic function of the set of column vectors with entries in $\mathcal{O}_{\mathcal{K},v}$. For $y \in GL_2(\mathcal{K}_v)$ we let $\Phi_{0,y}(x) := \Phi_0(y^{-1}x)$.

**Lemma 9.1.2.** *Let $h \in S_2(F_v)$, $\det h \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$. Suppose $\bar{y}^t h y \in S_2(\mathcal{O}_{F,v})^\times$.*
*(i) If $\lambda$ is unramified, $v$ is unramified in $\mathcal{K}$, and $h, y \in GL_2(\mathcal{O}_{F,v})$, then*

$$\omega_h(U_1(\mathcal{O}_{F,v}))\Phi_{0,y} = \Phi_{0,y}.$$

*(ii) If $D_v \det \bar{y}^t h y | \varpi_v^r$, $r > 0$, then*

$$\omega_h(k)\Phi_{0,y} = \lambda(a_k)\Phi_{0,y},\ k \in \{k \in U_1(\mathcal{O}_{F,v}) : \varpi_v^r | c_k\}.$$

.

*Proof.* See [35, 10.2.4]. □

Let $\theta$ be a character of $\mathcal{K}_v^\times$ and let $0 \neq x \in cond(\theta)$. Let

$$\Phi_{\theta,x}(u) := \sum_{a \in (\mathcal{O}_{\mathcal{K},v}/x)^\times} \theta(a)\Phi_0((u_1 + a/x, u_2)^t), u = (u_1, u_2)^t.$$

For $y \in GL_2(\mathcal{K}_v)$ we let $\Phi_{\theta,x,y}(u) := \Phi_{\theta,x}(y^{-1}u)$. We let $\Phi_{h,\theta,x} := \omega_h(\eta^{-1})\Phi_{\theta,x}$ and $\Phi_{h,\theta,x,y} := \omega_h(\eta^{-1})\Phi_{\theta,x,y}$.

**Lemma 9.1.3.** *Let $h \in S_2(F_v), \det h \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$. Suppose $\bar{y}^t h y \in S_2(\mathcal{O}_{F,v})^\times$. Let $\theta$ be a character of $\mathcal{K}_v^\times$ and Let $0 \neq x \in cond(\theta)$ be such that $\varpi_v | x$. Let $(c) := cond(\theta) \cap (\tilde{\varpi}_v)$ where $\tilde{\varpi}_v = \varpi_v$ if $v$ splits in $\mathcal{K}$ (i.e. $\tilde{\varpi}_v = (\varpi_v, \varpi_v)$ for $\varpi_v$ the uniformizer of $F_v$) and otherwise a uniformizer of $\mathcal{K}_v$ at $v$.*

*(i) If* $cD_v \det {}^t\bar{y}hy \| x$ *and* $y^{-1}hy \in GL_2(\mathcal{O}_{F,v})$ *and* $D_v = 1$ *or* $y^{-1}h^{-1}y^{-1^t} = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$ *with* $d \in \mathcal{O}_{F,v}$, *then*

$$\omega_h(k)\Phi_{\theta,x,y} = \lambda\theta(a_k)\Phi_{\theta,x,y},\ k \in U_1(\mathcal{O}_{F,v}), \mathfrak{d}^{-1}D_v|c_k, x\bar{x}|b_k$$

*(ii) If* $h = diag(\alpha, \beta)$, *then* $\Phi_{h,\theta,x,y}$ *is supported on the lattice* $h^{-1t}y^{-1}L^*_{\theta,x}$ *where if* $v$ *is non split in* $\mathcal{K}$ *then*

$$L^*_{\theta,x} = \{(u_1,u_2)^t : u_2 \in {\delta_{\mathcal{K}}}^{-1}\mathcal{O}_{\mathcal{K},v}, \bar{u}_1 \in \frac{x}{c\delta_{\mathcal{K}}} \begin{cases} \mathcal{O}_{\mathcal{K},v}, & cond(\theta) = \mathcal{O}_{\mathcal{K},v}, \\ \mathcal{O}^\times_{\mathcal{K},v}, & cond(\theta) \neq \mathcal{O}_{\mathcal{K},v}. \end{cases}\} \tag{9.1}$$

*and if* $v$ *splits in* $\mathcal{K}$, *then*

$$L^*_{\theta,x} := \{(u_1,u_2)^t : u_2 \in \delta_{\mathcal{K}}^{-1}\mathcal{O}_{\mathcal{K},v}, \bar{u}_{1,i} \in \frac{x_i}{c_i\delta_{\mathcal{K}}} \begin{cases} \mathcal{O}_{F,v}, & cond(\theta) = \mathcal{O}_{F,v}, \\ \mathcal{O}^\times_{F,v}, & cond(\theta) \neq \mathcal{O}_{F,v}. \end{cases}\} \tag{9.2}$$

*with* $\bar{u}_1 = (\bar{u}_{1,1}, \bar{u}_{1,2}), x = (x_1, x_2), c = (c_1, c_2) \in \mathcal{K}_v = F_v \times F_v$ *and* $\theta = (\theta_1, \theta_2)$. *Furthermore for* $v = h^{-1t}y^{-1}u$ *with* $u \in L^*_{\theta,x}$,

$$\Phi_{h,\theta,x,y}(v) = |det\ hy\bar{y}|_v D_v^{-1}\lambda(-1) \sum_{a \in (\mathcal{O}_{\mathcal{K},v}/x)^\times} \theta(s)e_\ell(\mathrm{Tr}_{\mathcal{K}/\mathbb{Q}}a\bar{u}_1/x)$$

.

*Proof.* See [35, 10.2.5]. □

**Lemma 9.1.4.** *Suppose* $v|p$ *splits completely in* $\mathcal{K}$. *Let* $(c) := cond(\theta)$ *and suppose* $c = (p^r, p^s)$ *with* $r, s > 0$. *Let* $\gamma = (\eta, 1) \in SL_2(\mathcal{O}_{\mathcal{K},v}) = SL_2(\mathcal{O}_{F,v}) \times SL_2(\mathcal{O}_{F,v})$. *Suppose* $h = diag(\alpha, \beta)$ *with* $\alpha, \beta \in F_v^\times$. *Then*

*(i)* $\Phi_{h,\theta,c,\gamma}$ *is supported on*

$$L' := \{u = (a,b)^t : a \in \mathcal{O}^\times_{F,v} \times \mathcal{O}_{F,v}, b \in \mathcal{O}_{F,v} \times \mathcal{O}^\times_{F,v}\}$$

*and for* $u \in L'$

$$\Phi_{h,\theta,c,\gamma}(u) = \theta_1^{-1}(\alpha a_2)\mathfrak{g}(\theta_1)\theta_2^{-1}(\beta b_1)\mathfrak{g}(\theta_2)$$

*where* $a = (a_1, a_2), b = (b_1, b_2) \in \mathcal{O}_{F,v} \times \mathcal{O}_{F,v}$, *and* $\theta = (\theta_1, \theta_2)$.

*(ii)* $\omega_h(u,k)\Phi_{h,\theta,c} = \theta_1^{-1}(a_g)\theta_2(d_g)\lambda\theta(d_k)\Phi_{h,\theta,c}$ *for* $u = (g, g') \in U_h(\mathbb{Z}_p)$ *with* $p^{max(r,s)}|c_g$ *and for* $k \in U_1(\mathbb{Z}_p)$ *such that* $p^{max(r,s)}|c_k$.

*Proof.* See [35, 10.2.6]. □

# Chapter 10

# Siegel Eisenstein Series and Their Pull-Backs

Now we continue our discussion from section 3.2 for Siegel Eisenstein series and make all the computations explicit in the special case of $GU(2,2)_{/F}$. For the reader's convenience we try to present in a parallel way to [35] chapter 11. Also along the way we modify our conventions (for example definition for the matrix $S$ and $S'$) from our section 3.1 a little so as to keep things the same as [35] chapter 11.

## 10.1 Some Isomorphisms and Embeddings.

We recall the notations of [35, 11.1]. Let $V_n := \mathcal{K}^{2n}$. Then $w_n$ defines a skew-hermitian pairing $< -,- >_n$ on $V_n :< x,y >_n := xw_n\bar{y}^t$. The group $G_n/F$ is the unitary similitude group $GU(V_n)$ of the hermitian space $(V_n, < -,- >_n)$. Let $W_n := V_{n+1} \oplus V_n$ and $W_n' : V_n \oplus V_n$. The matrices $w_{n+1} \oplus -w_n$ and $w_n \oplus w_n$ define define hermitian pairings on $W_n$ and $W_n'$, respectively.

One can define isomorphisms: $\alpha_n : GU(W_n) \simeq G_{2n+1}$, $\alpha_n' : GU(W_n') \simeq G_{2n}$, $\gamma_n : GU(W_n) \simeq G_{2n+1}$ and $\gamma_n' : GU(W_n') \simeq G_{2n}$. We omit the details and refer to [35, 11.2.1]. Also as in [35] we use $S$ and

$S'$ to denote the matrices

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -1 & 1 & & \\ & & & & 1 & \\ -1 & & & & & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & -1 & 1 & & \\ -1 & & & 1 & \end{pmatrix}.$$

This is different from the convention of part one of this thesis.

## 10.2 Pull-Backs of Siegel Eisenstein Series.

Now we follow [35, 11.2] closely to recall the pull-back formulas. It has certain overlap with the materials in section 3.1. However we repeat it here since the conventions are slightly different.

### 10.2.1 The pull-back formulas

Let $\chi$ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$. Given a cuspform $\phi$ on $G_n$ we consider

$$F_{\phi}(f;z,g) := \int_{U_n(\mathbb{A}_F)} f(z,\gamma(g,g_1h))\bar{\chi}(\det g_1 g)\phi(g_1h)dg_1,$$

$$f \in I_{m+n}(\chi), g \in G_m(\mathbb{A}_F), h \in G_n(\mathbb{A}_F), \mu_m(g) = \mu_n(h), m = n+1 \text{ or } n,$$

with $\gamma = \gamma_n$ or $\gamma'$ depending on whether $m = n+1$ or $m = n$. This is independent of $h$. The pull-back formulas are the identities in the following proposition.

**Proposition 10.2.1.** *Let $\chi$ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$.*
*(i) If $f \in I_{2n}(\chi)$, then $F_{\phi}(f;z,g)$ converges absolutely and uniformly for $(z,g)$ in compact sets of $\{Re(z) > n\} \times G_n(\mathbb{A}_F)$, and for any $h \in G_n(\mathbb{A}_F)$ such that $\mu_n(h) = \mu(g)$*

$$\int_{U_n(F)\backslash U_n(\mathbb{A}_F)} E(f;z,\gamma_n'(g,g_1h))\bar{\chi}(detg_1h)\phi(g_1h)dg_1 = F_{\phi}(f;z,g).$$

*(ii) If $f \in I_{2n+1}(\chi)$, then $F_\phi(f;z,g)$ converges absolutely and uniformly for $(z,g)$ in compact sets of $\{Re(z) > n+1/2\} \times G_{n+1}(\mathbb{A}_F)$ such that $\mu_n(h) = \mu_{n+1}(g)$*

$$\int_{U_n(F)\backslash U_n(\mathbb{A}_F)} E(f;z,\gamma_n(g,g'h))\bar{\chi}(\det g_1 h)\phi(g_1 h)dg_1 = \sum_{\gamma \in P_{n+1}(F)\backslash G_{n+1}(F)} F_\phi(f;z,\gamma g),$$

*with the series converging absolutely and uniformly for $(z,g)$ in compact subsets of $\{Re(z) > n+1/2\} \times G_{n+1}(\mathbb{A}_F)$.*

*Proof.* See [35, 11.2.3]. □

## 10.3 Fourier-Jacobi Expansions: Generalities.

Let $0 < r < n$ be an integer. Recall that we have defined the Fourier-Jacobi expansion

$$E(f;z,g) = \sum_{\beta \in S_{n-r}(F)} E_\beta(f;z,g). \tag{10.1}$$

where

$$E_\beta(f;z,g) := \int_{S_{n-r}(F)\backslash S_{n-r}(\mathbb{A}_F)} E(f;z,\begin{pmatrix} & S & 0 \\ 1_n & 0 & 0 \\ & & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S))dS. \tag{10.2}$$

**Lemma 10.3.1.** *Let $f = \otimes_v f_v \in I_n(\chi)$ be such that for some prime $v$ the support of $f_v$ is in $Q_n(F_v)w_n Q_n(F_v)$. Let $\beta \in S_n(F)$ and $q \in Q_n(\mathbb{A}_F)$. If $Re(z) > n/2$ then*

$$E_\beta(f;z,g) = \prod_v \int_{S_n(F_v)} f_v(z, w_n r(S_v) q_v) e_v(-Tr\beta S_v)dS_v. \tag{10.3}$$

*In particular, the integrals on the right-hand side converge absolutely for $Re(z) > n/2$.*

*Proof.* see [35, 11.3.1]. □

**Lemma 10.3.2.** *Suppose $f \in I_3(\chi)$ and $\beta \in S_2(F), \beta > 0$. Let $V$ be the two-dimensional $\mathcal{K}$-vector space of column vectors. If $Re(z) > 3/2$ then*

$$E_\beta(f;z,g)=\sum_{\gamma\in Q_1(F)\backslash G_1(F),\gamma\in U_1(F)}\sum_{x\in V}\int_{S_2(\mathbb{A}_F)} f(w_3\begin{pmatrix}1_3 & \begin{matrix}S & x\\ \bar{x}^t & 0\end{matrix}\\ & 1_3\end{pmatrix}\alpha_1(1,\gamma)g)$$
$$\times e_{\mathbb{A}}(-Tr_{\mathcal{K}/\mathbb{Q}}\beta S)dS.$$

Recall that $e_{\mathbb{A}}(x)=e_{\mathbb{A}_{\mathbb{Q}}}(\mathrm{Tr}_{F/\mathbb{Q}}x)$ for $x\in\mathbb{A}_F$.

*Proof.* See [35, 11.3.2]. □

We also recall a few identities which are special cases or consequences of the formulas right before subsection 3.3.1. These are also straightforward generalizations of [35, 11.3.2(a)-(d)]. Letting:

$$FJ_\beta(f;z,x,g,y):=\int_{S_2(F_v)} f(z,w_3\begin{pmatrix}1_n & \begin{pmatrix}S & x\\ \bar{x}^t & 0\end{pmatrix}\\ & 1_n\end{pmatrix}\alpha_1(diag(y,y^{t^{-1}}),g))e_v(-Tr\beta S)dS,$$

then:

$$FJ_\beta(f;z,x,\begin{pmatrix}a & \bar{a}^{-1}b\\ & \bar{a}^{-1}\end{pmatrix}g,y)=\chi_v^c(a)^{-1}|a\bar{a}|_v^{z+3/2}e_v(\bar{x}^t\beta xb)FJ_\beta(f;z,xa,g,y).$$

For $u\in U_\beta(\mathbb{A}_F)$, $U_\beta$ being the unitary group associated to $\beta$,

$$FJ_\beta(f;z,x,g,uy)=\chi(\det u)|\det u\bar{u}|_{\mathbb{A}_F}^{-z+1/2}FJ-\beta(f;z,u^{-1}x,g,y).$$

If as a function of $x$, $FJ_\beta(f;z,x,g,y)\in\mathcal{S}(V\otimes F_v)$, then:

$$FJ_\beta(f;z,x,\begin{pmatrix}a & \bar{a}^{-1}b\\ & \bar{a}^{-1}\end{pmatrix}g,y)$$
$$=(\lambda_v/\chi_v^c)(a)|a\bar{a}|_v^{z+1/2}\omega_\beta(\begin{pmatrix}a & \bar{a}^{-1}b\\ & \bar{a}^{-1}\end{pmatrix})FJ_\beta(f:z,x,g,y).$$

## 10.4 Some Good Siegel Sections

### 10.4.1 Archimedean Siegel Sections

We summarize the results of [35, 11.4.1]. Let $k \geq 0$ be an integer. Then $\chi(x) = (x/|x|)^{-k}$ is a character of $\mathbb{C}^\times$.

The sections. We let $f_{\kappa,n} \in I_n(\chi)$ be defined by $f_{\kappa,n}(k) := J_n(k,i)^{-\kappa}$. Then

$$f_{\kappa,n}(z, qk) = J_n(k,i)^{-\kappa}\chi(\det D_q)|\det A_q D_q^{-1}|^{z+1/2}, q \in Q_n(\mathbb{R}), k \in K_{n,\infty}.$$

If $g \in U_n(\mathbb{R})$ then $f_{\kappa,n}(z,g) = J_n(g,i)^{-\kappa}|J_n(g,i)|^{\kappa-2z-n}$.

Fourier-jacobi coefficients. Given a matrix $\beta \in S_n(\mathbb{R})$ we consider the local fourier coefficient:

$$f_{\kappa,n,\beta}(z,g) := \int_{S_n(\mathbb{R})} f_\kappa(z, w_n \begin{pmatrix} 1_n & S \\ & 1_n \end{pmatrix} g) e_\infty(-Tr\beta S) dS.$$

This converges absolutely and uniformly for $z$ in compact sets of $\{Re(z) > n/2\}$.

**Lemma 10.4.1.** *Suppose $\beta \in S_n(\mathbb{R})$. The function $z \mapsto f_{\kappa,\beta}(z,g)$ has a meromorphic continuation to all of $\mathbb{C}$. Furthermore, if $\kappa \geq n$, then $f_{\kappa,n,\beta}(z,g)$ is holomorphic at $z_\kappa := (\kappa - n)/2$ and for $y \in GL_n(\mathbb{C})$, $f_{\kappa,n,\beta}(z_\kappa, diag(y, \bar{y^t}^{-1})) = 0$ if $det\beta \leq 0$, and if $det\beta > 0$ then*

$$f_{\kappa,n,\beta}(z_\kappa, diag(y, {}^t\bar{y}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa - j - 1)!} e(iTr(\beta y \bar{y}^t))(det\beta)^{\kappa-n} det\bar{y}^\kappa.$$

*Proof.* See [35, 11.4.2]. □

Suppose now that $n = 3$. For $\beta \in S_2(\mathbb{R})$ let $FJ_{\beta,\kappa}(z,x,g,y) := FJ_\beta(f_\kappa; z, x, g, y)$.

**Lemma 10.4.2.** *Let $z_\kappa := (\kappa - 3)/2$. Let $\beta \in S_2(\mathbb{R})$, $det\beta > 0$.*
*(i)$FJ_{\beta,\kappa}(z_\kappa, x, \eta, 1) = f_{\kappa,2,\beta}(z_\kappa + 1/2, 1)e(i < x, x >_\beta)$.*
*(ii) For $g \in U_1(\mathbb{R})$*

$$FJ_{\beta,\kappa}(z_\kappa, x, g, y) = e(i\mathrm{Tr}\beta y\bar{y}^t)\det\bar{y}^\kappa c(\beta,\kappa) f_{\kappa-2,1}(z_\kappa, g')\omega_\beta(g')\Phi_{\beta,\infty}(x),$$

*where* $g' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ *and*

$$c(\beta,\kappa) = \frac{(2\pi i)^{2\kappa}(2/\pi)}{4(\kappa-1)!(\kappa-2)!}\det\beta^{\kappa-2}$$

*and the Weil representation* $\omega_\beta$ *is defined using the character* $\lambda_\infty(z) = (z/|z|)^{-2}$.

This is just [35, lemma 11.4.3].

Pull-back integrals. The Archimedean situation is completely the same as the situation in [35]. Let $f_\kappa \in I_3(\tau)$ be as before and let

$$F_\kappa(z,g) := \int_{U_1(\mathbb{R})} f_\kappa(z, S^{-1}\alpha_1(g,g_1h))\bar\tau(\det g_1h)\pi_1(g_1h)\phi dg_1,$$
$$g \in G_2(\mathbb{R}), h \in G_1(\mathbb{R}), \mu_1(h) = \mu_2(g).$$

Similarly, for $f_\kappa \in I_2(\tau)$ and $g \in G_1(\mathbb{R})$ we let

$$F_\kappa(z,g) := \int_{U_1(\mathbb{R})} f_\kappa(z, S'^{-1}\alpha'_1(g,g_1h))\bar\tau(\det g_1h)\pi_1(g_1h)\phi dg_1,$$
$$g,h \in G_1(\mathbb{R}), \mu_1(h) = \mu_1(g).$$

**Lemma 10.4.3.** *([35, 11.4.4].) The integrals converge if* $Re(z) \geq (\kappa - m - 1)/2$ *and* $Re(z) > (m-1-\kappa)/2$, $m = 2$ *and* $1$, *respectively (according to the convention of subsection 10.2.1 and for such* $z$ *we have:*
*(i)* $F_\kappa(z,g) = \pi 2^{-2z-1}\frac{\Gamma(z+(1+\kappa)/2)}{\Gamma(z+(3+\kappa)/2)}F_{\kappa,z}(g)$;
*(ii)* $F'_\kappa(z,g) = \pi 2^{-2z}\frac{\Gamma(z+\kappa/2)}{\Gamma(z+1+\kappa/2)}\pi_\psi(g)\phi$.

### 10.4.2 Prime to $p$ Siegel Sections: the Unramified Case

**Lemma 10.4.4.** *Let* $\beta \in S_n(F_v)$ *and let* $r := rank(\beta)$. *Then for* $y \in GL_n(\mathcal{K}_v)$.

$$f^{sph}_{v,\beta}(z, diag(y, {}^{t}y^{-1})) = \chi(det\ y)|det\ y\bar y|_v^{-z+n/2}Vol(S_n(\mathcal{O}_{F,v}))$$
$$\times\frac{\prod_{i=r}^{n-1}L(2z+i-n+1,\bar\chi'\chi^i_{\mathcal{K}})}{\prod_{i=0}^{n-1}L(2z+n-i,\bar\chi'\chi^i_{\mathcal{K}})}h_{v,\bar y^t\beta y}(\bar\chi'(\varpi_v)q_v^{-2z-n})$$

*where* $h_{v,\bar y^t\beta y}$ *is a monic polynomial depending on* $v$ *and* $\bar y^t\beta y$ *but not on* $\chi$.

*Proof.* This is proven in [31]. See [35, 11.4.6]. □

**Lemma 10.4.5.** *Suppose $v$ is unramified in $\mathcal{K}$, let $\beta \in S_2(F_v)$ such that $\det \beta \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$ such that $\bar{y}^t \beta y \in S_2(\mathcal{O}_{F,v})$. Let $\lambda$ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v^\times} = 1$.*
*(i) If $\beta, y \in GL_2(\mathcal{O}_{\mathcal{K},v})$ then for $u \in U_\beta(F_v)$.*

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(det\ u)|det\ u\bar{u}|_v^{-z+1/2} \frac{f_1^{sph}(z, g)\omega_\beta(u, g)\Phi_{0,y}(x)}{\prod_{i=o}^{1} L(2z + 3 - i, \bar{\chi}'\chi_{\mathcal{K}}^i)}$$

*(ii) If $\bar{y}^t \beta y \in GL_2(\mathcal{O}_{\mathcal{K},v})$, then for $u \in U_\beta(F_v)$.*

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(det\ uy)|det\ uy|_{\mathcal{K}}^{-z+1/2} \frac{f_1^{sph}(z, g)\omega_\beta(u, g)\Phi_{0,y}(s)}{\prod_{i=0}^{1} L(2z + 3 - i, \bar{\chi}'\chi_{\mathcal{K}}^i)}$$

.

*Proof.* (i) is the same as [35, 11.4.7]. Note that in (ii) we have removed the assumption in *loc.cit* that $g$ is of the form $\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}$. In fact since

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z-\frac{1}{2}} FJ_{{}^t\bar{y}\beta y}(f_3^{sph}; z, y^{-1}u^{-1}x, g, 1)$$

by (i) we have only to prove that

$$\omega_{{}^t\bar{y}\beta y}(1, g)\Phi_0(y^{-1}u^{-1}x) = \omega_\beta(u, g)\Phi_{0,y}(x) = (\omega_\beta(1, g)\Phi_{0,y})(u^{-1}x),$$

i.e.

$$(\omega_{{}^t\bar{y}\beta y}(1, g)\Phi_0)_{,y}(x) = (\omega_\beta(1, g)\Phi_{0,y})(x).$$

Here for any Schwartz function $\Phi$ we write $\Phi_{,y}$ to be the function defined by: $\Phi_{,y}(x) = \Phi(y^{-1}x)$. By definition one checks that for any $\Phi$

$$\omega_\beta(g)\Phi_{,y} = (\omega_{{}^t\bar{y}\beta y}(1, g)\Phi)_{,y}(x)$$

for $g$ of the forms $\begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$, $\eta$, thus for all $g \in U_1(F_v)$. In particular, for $\Phi = \Phi_0$ □

Pull-back integrals

Recall that we have $(\pi, \psi, \tau)$ as in chapter 8. Let $\phi \in V$. Let $m = 1$ or 2 according to the convention

of subsection 10.2.1. Given $f \in I_{m+1}(\tau)$ we consider the integral:

$$F_\phi(f;z,g) := \int_{U_1(F_v)} f(z,\gamma(g,g_1h))\bar{\tau}(detg_1h)\pi_\psi(g_1h)\phi dg_1, \tag{10.4}$$

where $\gamma = \gamma_1$ or $\gamma_1'$ depending on whether $m=2$ or $m=1$. (similar to [35, 11.4].)

**Lemma 10.4.6.** *Suppose $\pi, \psi$ and $\tau$ are unramified and $\phi$ is a newvector. If $Re(z) > (m+1)/2$ then the above integral converges and*

$$F_\phi(f_v^{sph};z,g) = \begin{cases} \frac{L(\bar{\pi},\xi,z+1/2)}{\prod_{i=0}^1 L(2z+2-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}\pi_\psi(g)\phi & m=1 \\ \frac{L(\bar{\pi},\xi,z+1)}{\prod_{i=0}^1 L(2z+3-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}F_{\rho,z}(g) & m=2. \end{cases}$$

*Here, $F_\rho$ is the spherical section.*

*Proof.* This is proved in [25, proposition 3.3]. See [35, 11.4.8]. □

### 10.4.3 Siegel Sections at Ramified Primes

The sections.

Let $v$ be a finite prime of $F$. We are going to define two important Siegel sections.

(1) Let $f_n^\dagger \in I_n(\chi)$ be the function supported on $Q_n(\mathcal{O}_{F,v})w_nN_{Q_n}(\mathcal{O}_{F,v})$ such that $f_n^\dagger(w_nr) = 1, r \in N_{Q_n}(\mathcal{O}_{F,v})$.

(2) Given $(\lambda^u) \subseteq \mathcal{O}_{\mathcal{K},v}$ contained in the conductor of $\chi$, we let $f_{u,n} \in I_n(\chi)$ be the function such that $f_{u,n}(k) = \chi(\det D_k)$ if $k \in K_{Q_n}(\lambda^u)$ and $f_{u,n}(k) = 0$ otherwise.

**Lemma 10.4.7.** *Suppose $v$ is not ramified in $\mathcal{K}$ and suppose $\chi$ is such that $\mathcal{O}_{\mathcal{K},v} \neq cond(\chi) \supseteq cond(\chi\chi^c)$. Let $(\lambda^u) := cond(\chi)$. Then*

$$M(z, f_n^\dagger) = f_{u,n} \cdot \mathrm{Vol}(S_n(\mathcal{O}_{F,v})) \in I_n(\bar{\chi}^c)$$

*for all $z \in \mathbb{C}$.*

*Proof.* See [35, 11.4.10]. □

**Lemma 10.4.8.** *Let $A \in GL_n(\mathcal{K}_v)$. If $\det\beta \neq 0$, then*

$$f_{n,\beta}^\dagger(z, diag(A, {}^t\bar{A}^{-1})) = \begin{cases} \chi(\det\ A)|\det\ A|_v^{-z+n/2}\mathrm{Vol}(S_n(\mathcal{O}_{F,v})) & {}^t\bar{A}\beta A \in S_n(\mathcal{O}_{F,v})^* \\ 0, & otherwise. \end{cases} \tag{10.5}$$

*Proof.* See [35, 11.4.11]. □

**Lemma 10.4.9.** *Suppose $\beta \in S_n(F_v)$, $det\ \beta \neq 0$, $char(v) = \ell$ and $\ell$ splits completely in $\mathcal{K}$.*

*(i) if $\beta \notin S_n(\mathcal{O}_{F,v})$, then $M(z, f_n^\dagger)_\beta(-z, 1) = 0$.*

*(ii) Suppose $\beta \in S_n(\mathcal{O}_{F,v})$. Let $c := ord_v(cond(\chi'))$. If $c > 0$, then*

$$M(z, f_n^\dagger)_\beta(-z, 1) = \chi'(det\ \beta)|det\ \beta|_v^{-2z} g(\bar{\chi}')^n c_n(\chi', z).$$

*where*

$$c_n(\chi', z) = \begin{cases} \chi'(\ell^{nc})\ell^{2ncz - cn(n+1)/2} & c > 0 \\ \ell^{2nz - n(n+1)/2} & c = 0 \end{cases} \tag{10.6}$$

*Proof.* See [35, 11.4.12]. □

Now we use the convention for $m = 1$ or $2$ in subsection 10.2.1.

**Proposition 10.4.1.** *Let $m = 1$ or $2$. There exists a meromorphic function $\gamma^{(m)}(\rho, z)$ on $\mathbb{C}$ such that:*

*(i) If $m = 1$, then $F_{\phi^\vee}(M(z, f); -z.g) = \gamma^{(1)}(\rho, z)\tau(\mu_1(g))F_\phi(f; z, \eta g)$*

*Moreover, if $\pi \simeq \pi(\chi_1, \chi_2)$ and $v$ splits in $\mathcal{K}$, then*

$$\gamma^{(1)}(\rho, z) = \Psi(-1)g(\bar{\tau}', \varpi_v^e)^2 \cdot \tau'(\varpi_v^{nc})|\varpi_v|_v^{-2ncz + n(n+1)c/2} \cdot \epsilon(\tilde{\pi} \otimes \xi^c, z + 1/2)\frac{L(\pi \otimes \bar{\xi}^c, 1/2 - z)}{L(\tilde{\pi} \otimes \xi^c, z + 1/2)}.$$

*(ii) If $m = 2$ and $\pi, \Psi, \tau$ are the $v$ constituents of a global triple, then*

$$F_{\phi^\vee}(M(z, f); -z, g) = \gamma^{(2)}(\rho, z)A(\rho, z, F_\phi(f; z, -))_{-z}(g).$$

*Each of these equalities is an identity of meromorphic functions of $z$.*

*(iii) Suppose moreover that $\mathcal{O}_v \neq \mathrm{cond}(\tau) \supset \mathrm{cond}(\tau\tau^c)$. Then:*

$$\gamma^{(2)}(\rho, z) = \gamma^{(1)}(\rho, z - \frac{1}{2}).$$

*Proof.* See [35, 11.4.13]. □

### 10.4.4 Sections at Ramified Primes Again

The sections.

Again let $v$ be a finite prime of $F$. As in [35, 11.4.14], we define modified version of the sections $f^\dagger$.

Let $m = 1$ or 2. For $x \in \mathcal{O}_{\mathcal{K}.v} \cap \mathcal{K}_v^\times$ let

$$f_x^{\dagger,(m)}(z,g) = f_{m+1}^\dagger(z, g \begin{pmatrix} 1 & & & & & & 1/x \\ & 1_{m-1} & & & & 0_{m-1} & \\ & & 1 & 1/\bar{x} & & & \\ & & & 1_{m-1} & & & \\ & & & & 1 & & \\ & & & & & 1 & \end{pmatrix}).$$

**Lemma 10.4.10.** *Let $\beta = (b_{i,j}) \in S_{m+1}(F_v)$. Then for all $z \in \mathbb{C}, f_{x,\beta}^{\dagger,(m)}(z,1) = 0$ if $\beta \notin S_{m+1}(\mathcal{O}_{F,v})^*$. If $\beta \in S_{m+1}(\mathcal{O}_{F,v})^*$, then*

$$f_{x,\beta}^{\dagger,(m)}(z,1) = \mathrm{Vol}(S_{m+1}(\mathcal{O}_{F,v}))e_\ell(Tr_{\mathcal{K}_v/\mathbb{Q}_v}(b_{m+1,1}/x))$$

.

*Proof.* See [35, 11.4.15]. □

**Lemma 10.4.11.** *Let $\beta \in S_2(F_v)$, $\det \beta \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$ and suppose $\bar{y}^t \beta y \in S_2(\mathcal{O}_{F,v})^*$. Let $\lambda, \theta$ be characters of $\mathcal{K}_v^\times$ and suppose $\lambda|_{F_v^\times} = 1$. Let $(c) := cond(\lambda) \bigcap cond(\theta) \bigcap (\varpi_v)$. Let $x \in \mathcal{K}_v^\times$ be such that $D_v|x$, $cond(\chi^c)|x$, and $cD_v \det\ \bar{y}^t \beta y|x$, where $D_v := \mathrm{Nm}_{\mathcal{K}/F}(\delta_{\mathcal{K}/\mathbb{Q}})$. Suppose $y^{-1}\beta^{-1}\bar{y}^{t-1} = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$ with $d \in F_v$. For $\tilde{D}_v := \mathrm{Nm}_{\mathcal{K}/F}(\delta_{\mathcal{K}/F})$ then for $h \in U_\beta(F_v)$,*

$$\sum_{a \in (\mathcal{O}_v/x)^\times} \theta\bar{\chi}^c(a) FJ_\beta(f_x^{\dagger,(2)}; z, u, g \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix}, hy)$$

$$= \chi(\det hy)|\det\ hy|_{\mathcal{K}}^{-z+1/2} \mathrm{Vol} S_2(\mathcal{O}_{F,v}) \cdot \sum_{b \in (\mathcal{O}_v/\tilde{D}_v \mathfrak{d}\mathcal{O}_v)} f_{-b}(z, g'\eta)\omega_\beta(h, g' \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}) \Phi_{\theta,x,y}(u)$$

*where* $g' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ *and*

$$f_b(g) = \begin{cases} \chi\lambda^{-1}(d_p), & g = p\eta \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}, p \in B_1(\mathcal{O}_{F,v}), m - b \in \tilde{D}_v \mathfrak{d}\mathcal{O}_{F,v} \\ 0 & \text{otherwise .} \end{cases}$$

*Proof.* See [35, 11.4.16]. □

Pull-back integrals

Let $\mathcal{T}$ denote a triple $(\phi, \psi, \tau)$ with $\phi \in V$ having a conductor with respect to $\tilde{\pi}$. Let $\phi_x := \pi_\psi(\eta diag(\bar{x}^{-1}, x))\psi$ and let

$$F^{(m)}_{\mathcal{T},x}(z, g) := \int_{U_1(F_v)} f^{\dagger,(m)}_x(z, S^{-1}\alpha(g, g'h))\bar{\tau}(\det g'h)\pi_\psi(g'h)\phi_x dg',$$

where $\alpha = \alpha_1$ or $\alpha'_1$ depending on whether $m = 2$ or 1. If $f(z, g) = f^{\dagger,(m)}_x(z, gS^{-1})$ then $F_{\mathcal{T},x}(z, g) = F_{\phi_x}(f; z, g)$.

**Proposition 10.4.2.** *Suppose $x = \lambda^t, t > 0$ is contained in the conductors of $\tau$ and $\psi$ and $x\bar{x} \in (\lambda^{r_\phi}) = cond_{\tilde{\pi}}(\phi)$. Then $F^{(m)}_{\mathcal{T},x}(z, g)$ converges for all $z$ and $g$ and*

$$F^{(1)}_{\mathcal{T},x}(z, \eta) = [U_1(\mathcal{O}_{F,v}) : K_x]^{-1}\tau(x)|x\bar{x}|^{-z-1}_v \phi$$

*and*

$$F^{(2)}_{\mathcal{T},x} = [U_1(\mathcal{O}_{F,v}) : K_x]^{-1}\tau(x)|x\bar{x}|^{-z-3/2}_v F_{\phi,r,t}.$$

*for any $r \geq max\{r_\phi, t\}$. Here $K_x$ is the subgroup defined as:*

$$K_x := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(\mathcal{O}_{F,v}) : a - 1 \in (\bar{x}), b \in (x\bar{x}), c \in \mathcal{O}_v, d - 1 \in (x)\}.$$

*Proof.* See [35, 11.4.17]. □

**Proposition 10.4.3.** *For $m = 1$ or 2, Let $\gamma^{(m)}(\rho, z)$ be as in proposition 10.4.1. Assuming $char(v) = \ell$ which is unramified in $\mathcal{K}$. If $\mathcal{O}_v \neq cond(\tau) \supseteq cond(\tau\tau^c)$ then $\gamma^{(2)}(\rho, z) = \gamma^{(1)}(\rho, z - 1/2)$.*

*Proof.* See [35, 11.4.18]. □

### 10.4.5 $p$-adic Sections

Now let $v|p$ be a prime of $F$. Since we have done a large part of this in part one we only record the formulas for Fourier-Jacobi coefficients and pull-back sections below. These are only slightly different from [35, 11.4] (the $\xi^c$'s appeared there are replaced by $\xi^c\mu_{1,v}^{-1}$ in our cases).

**Lemma 10.4.12.** *Suppose our data (subsection 8.1.3) is in the Generic case and let $(p^m) := cond(\tau')$. Let $\beta \in S_2(F_v)$, $\det\beta \neq 0$, and suppose $\beta \in GL_2(\mathcal{O}_v)$. Let $y \in GL_2(\mathcal{O}_v)$. Let $\lambda$ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v^\times} = 1$. Then for $h \in U_\beta(F_v)$*

$$\sum_{a\in(\mathcal{O}_v/x)^\times} \mu_{1,v}^{-1}\xi^c\tau(a)FJ_\beta(f_z^{0,(2)}; -z, u, gdiag(a_{-1}, \bar{a}), hy)$$
$$\xi(-1)c(\beta,\tau,z)\tau(dethy)|deth\bar{h}|_v^{-z+1/2} f_{m,1}(z, g\eta)\omega_\beta(h,g)\Phi_{\mu_{1,v}^{-1}\xi^c,x,y}(u), \tag{10.7}$$

*where $\omega$ is defined using $\lambda$,and*

$$c(\beta,\tau,z) := \bar{\tau}'(-det\beta)|det\beta|_v^{2z+1}\mathfrak{g}(\tau')^2\bar{\tau}'(p^2m)p^{-4mz-5m}.$$

*Proof.* See [35, 11.4.22]. □

Now we use the convention for $m = 1$ or 2 as in subsection 10.4.2.

**Proposition 10.4.4.** *Let $\phi \in V$ be an eigenvector for $\pi$ such that $v|cond_\pi(\phi)$. Let $(x) := cond(\xi_1) = (\varpi_v^t) = (\varpi_v^{t_1}, \varpi_v^{t_2})$. Suppose again that we are in the Generic case. Let $\phi_x^\vee := \psi(-1)\pi(diag(x, \bar{x}^{-1}))\phi$. Then*

$$F_{\phi_x^\vee}(\tilde{f}_{-z}^{0,(m)}; z, g) = \gamma^{(m)}(\rho^\vee, -z)[U_1(\mathcal{O}_{F,v}) : K_x]^{-1}\bar{\tau}^c(x)|x\bar{x}|_v^{z-\frac{m+1}{2}} \begin{cases} F_{\phi,z}^0(g), & m = 2 \\ \pi_\psi(g)\phi, & m = 1. \end{cases} \tag{10.8}$$

*where $\tilde{f}_{-z}^{0,(m)}(z,g) = f_{-z}^{0,m}(z, gS^{-1})$.*

*Proof.* The same as [35, proposition 11.4.23]. □

## 10.5 Good Siegel Eisenstein Series

From now on we assume that the characters $\psi$ and $\tau$ are unramified outside $p$. Let $(\pi, V) = (\otimes\pi_v, \otimes V_v)$ be as before and let $\mathcal{D} = (\Sigma, \varphi, \psi, \tau)$ be an Eisenstein datam as defined in section 8.3. We augment this datum with a choice of an idele $M_{\mathcal{D}}$ satisfying

- $M_{\mathcal{D}}$ is divisible only by primes in $\Sigma \setminus \{v|p\}$;
- for $v \in \Sigma \setminus \{v|p\}$, $M_{\mathcal{D}}$ is contained in $\delta_{\mathcal{K}}, \mathrm{cond}(\xi_v), \mathrm{cond}(\psi_v), \mathrm{cond}(\tau_v)$, and $\mathrm{cond}_{\pi_v^\vee}(\phi_v)$.

We remark that we have freedom to choose such $M_{\mathcal{D}}$. This is crucial for proving the $p$-adic properties of the Eisenstein series.

the Siegel Eisenstein Series

For $m = 1$ or 2 we define a meromorphic section $f_{\mathcal{D}}^{(m)} : \mathbb{C} \to I_{m+1}(\tau)$ as follows: $f_{\mathcal{D}}^{(m)}(z) = \otimes f_{\mathcal{D},w}^{(m)}(z)$ where

- $f_\infty^{(m)}(z) := f_\kappa \in I_{m+1}(\tau_\infty)$ for any infinite place;
- $v \nmid \Sigma$ then $f_{\mathcal{D},v}^{(m)}(z) := f_v^{sph} \in I_{m+1}(\tau_v)$.
- if $v|\Sigma, v \nmid p$, then $f_{\mathcal{D},v}^{(m)}(z) := f_{M_{\mathcal{D}},v}^{(m)} \in I_{m+1}(\tau_v)$;
- for $v|p, f_{\mathcal{D},v}^{(m)}(z) := f_{-z}^{0,(m)} \in I_{m+1}(\tau_v)$, where $x_v$ is used to define $f_{-z}^{0,(m)}$.

**Definition 10.5.1.** $H_{\mathcal{D}}^{(m)}(z, g) := E(f_{\mathcal{D}}^{(m)}; z, g)$.

Now we define a level group for $U(1,1)$

$$U_{\mathcal{D}} := \prod_{v|p} K_{x_v,v} \prod_{v \notin \Sigma \setminus \{v|p\}} K_{M_{\mathcal{D}},v} \prod_{v \notin \Sigma} U_1(\mathcal{O}_{F,v}),$$

with $K_{x_v}$ defined in proposition 10.4.2.

**Remark 10.5.1.** *Later we will use $U_{\mathcal{D}}$ to denote the corresponding level groups in $GL_2$ as well.*

We also let

$$K_{\mathcal{D}}^{(m)} := \{k \in G_{m+1}(\hat{\mathcal{O}}_F) : 1 - k \in M_{\mathcal{D}}^2 \prod_{v|p} (x_v \bar{x}_v).M_{2(m+1)}(\hat{\mathcal{O}}_F)\}.$$

Then it easily follows from the definition of the $f^{(m)}_{\mathcal{D},v}(z)$'s that

$$H^{(m)}_{\mathcal{D}}(z,gk) = H^{(m)}_{\mathcal{D}}(z,g), k \in K^{(m)}_{\mathcal{D}}, \tag{10.9}$$

and that

$$H^{(m)}_{\mathcal{D}}(z,g\alpha(1,k)) = \tau(a_{k_p})H^{(m)}_{\mathcal{D}}(z,g), k \in U_{\mathcal{D}}.$$

For $u \in GL_{m+1}(\mathbb{A}_{\mathcal{K},f})$ let

$$L^{(m)}_v := \{\beta \in S_{m+1}(F) : \beta \geq 0, Tr\beta\gamma \in \hat{\mathcal{O}}_F, \gamma \in uS_{m+1}(\hat{\mathcal{O}}_F)^t\bar{u}\}.$$

We record the following formulas which are slight generalizations of the results in [35, 11.5]. These will be used to construct the $p$-adic families of $L$-functions and Eisenstein series.

**Lemma 10.5.1.** *(i) If $\kappa \geq m+1$, then $H^{(m)}_{\mathcal{D}}$ is holomorphic at $z_\kappa := (\kappa - m - 1)/2$;*
*(ii) If $\kappa \geq m+1$ and if $g \in Q_{m+1}(\mathbb{A}_F)$ then*

$$H^m_{\mathcal{D}}(z_\kappa, g) = \sum_{\beta \in S_{m+1}(F), \beta > 0} H^{(m)}_{\mathcal{D},\beta}(z_\kappa, g)$$

*Furthermore, if $\beta > 0$, $g_{\infty,i} = r(X_i)diag(Y_i, {}^t\bar{Y}_i^{-1})$ and $g_f = r(a)diag(u, {}^t\bar{u}^{-1}) \in G_{m+1}(\mathbb{A}_{F,f})$, then $H^{(m)}_{\mathcal{D},\beta}(z_\kappa, g) = 0$ if $\beta \notin L^{(m)}_u$ and otherwise*

$$H^{(m)}_{\mathcal{D},\beta}(z_k, g) = e(tr\beta a)\frac{(-2)^{-(m+1)d}(2\pi i)^{(m+1)d\kappa}(2/\pi)^{m(m+1)d/2}\prod_j(\det\beta_j^{\kappa-(m+1)}, \det\bar{Y}_j^{\kappa})}{(\prod_{j=0}^m(\kappa-j-1)!)^d\prod_{j=0}^m L^S(\kappa-j, \bar{\tau}'\chi_{\mathcal{K}}^j)}$$

$$\times\prod_{j\in I} e(Tr\beta_j(X_j + iY_j\bar{Y}_j^t))\prod_{v\notin S} f_{\mathcal{D},\beta u,v}(z_\kappa, 1)$$

$$\times\tau(\det u)|\det u\bar{u}|_F^{m+1-\kappa/2}\prod_{v\notin S} H_{v,\beta}(\bar{\tau}'_v(\varpi_v)q_v^{-2z-n})$$

*where $\beta_u = {}^t\bar{u}\beta u$, $\beta_j = j(\beta)$, $\iota_j$ is the embedding $F \hookrightarrow \mathbb{R}$ and $S \supseteq \Sigma$ is a finite set of primes such that $g_v \in K_{m+1,v}$ if $v \notin S$.*

*Proof.* Same as [35, 11.5.1]. □

If $\kappa \geq m+1$, define a function $H^m_{\mathcal{D}}(Z,x)$ on $X^+_{m+1,m+1} \times G_{m+1}(\mathbb{A}_{F,f})$ by

$$H^{(m)}_{\mathcal{D}}(Z,x) := \prod_{j=1}^{d} \mu_{m+1}(g_{\infty,j})^{(m+1)\kappa/2} \prod_{j=1}^{d} J_{m+1}(g_{\infty,j},i)^{-\kappa} H^{(m)}_{\mathcal{D}}((\kappa-m-1)/2, g_\infty x)$$

where $g_\infty \in G^+_{m+1}(\mathbb{R}), g_\infty(i) = Z$ and define $A^{(m)}_{\mathcal{D},\beta}(x)$ as the $\beta$-th Fourier coefficient of $H^{(m)}_{\mathcal{D}}(Z,x)$.

**Lemma 10.5.2.** *Suppose $\kappa \geq m+1$. Then $H^{(m)}_{\mathcal{D}}(\mathcal{Z},x) \in M_\kappa(K^{(m)}_{\mathcal{D}})$ (notation as in section 2.3, where $\kappa$ stands for the scalar weight $\underline{k} := (0,...,0;\kappa,...,\kappa)$).*

*Proof.* Same as [35, 11.5.1]. □

**Lemma 10.5.3.** *Suppose $\kappa \geq m+1$ and that $x = diag(u, \bar{u}^{-1})$, $u \in GL_{m+1}(\mathbb{A}_{F,f})$ with $u_v = diag(1_m, \bar{a}_v)$, $a_v \in \mathcal{O}_v^\times$, if $v \in \Sigma$. If $\beta \notin L^{(m)}_u$ or if $\det\beta = 0$ then $A_{\mathcal{D},\beta}(x) = 0$, and for $\beta = (\beta_{i,j}) \in L^{(m)}_u$ with $\det\beta > 0$*

$$\begin{aligned} A^{(m)}_{\mathcal{D},\beta}(x) = |\delta_{\mathcal{K}}|^{m(m+1)/4}_{\mathcal{K}} |\delta_F|^{(m+1)/2}_F \frac{(-2)^{-(m+1)d}(2\pi i)^{(m+1)\kappa d}(2/\pi)^{m(m+1)d/2} \prod_{v|p}(\det\beta|\beta|_v)^{\kappa-m-1}}{\prod_{j=0}^m (\kappa-j-1)!^d \prod_{j=0}^m L^\Sigma(\kappa-j, \bar{\tau}'\chi^j_{\mathcal{K}})} \\ \times \prod_{v|p} \bar{\tau}_v(a_v \det(\beta)) \mathfrak{g}(\tau'_v)^{m+1} c(\bar{\tau}'_v, -(\kappa-m-1)/2) e_v(Tr_{\mathcal{K}_v/\mathbf{Q}_v}(a_v b_{m+1,1}/x_v) \\ \times \prod_{v\in\Sigma, v\nmid p} \tau^c_v(a_v) e_v(Tr_{\mathcal{K}_v/\mathbf{Q}_v}(a_v b_{m+1,1}/M_{\mathcal{D}})) \\ \times \prod_{v\notin\Sigma} \tau_v(\det u_v) |u_v \bar{u}_v|^{m+1-\kappa/2}_v h_{v,\bar{u}^t_v \beta u_v}(\bar{\tau}_v(\varpi_v) q_v^{-\kappa}). \end{aligned} \tag{10.10}$$

*Proof.* See [35, 11.5.3]. □

## 10.6 $E_{\mathcal{D}}$ via Pull-Back

As in [35, 11.6] we let $\varphi_0$ be defined by: $\varphi_0(g) = \varphi_\psi(gy)$ for

$$y_v = \begin{cases} 1, & v = \infty, v \notin \Sigma \\ \eta^{-1}\mathrm{diag}(M_{\mathcal{D}}^{-1}, M_{\mathcal{D}})\eta & v \in \Sigma, v \nmid p \\ \mathrm{diag}(x_v, \bar{x}_v^{-1}), & v|p \end{cases}$$

Here $\varphi \in V$ and $\varphi_\psi$ is the form on $GU(1,1)(\mathbb{A}_F)$ given by $\varphi$ and $\psi$.

**Proposition 10.6.1.** *Let $m = 1$ or $2$. Suppose that for any $v|p$ $(x_v) = (p^{t_v})$ with $t_v > 0$ and that $x_v \in cond(\psi)$ and $x_v\bar{x}_v \in cond_{\pi_v}(\phi_v)$ where $\phi_v$ is defined by $\varphi = \otimes\varphi_v$. Let $g \in G_m(\mathbb{A}_F)$ and*

*$h \in G_1(\mathbb{A}_F)$ be such that $\mu_1(h) = \mu_m(g)$. If $k \geq m+1$ then*

$$\begin{aligned} &\int_{U_1(F)/U_1(\mathbb{A}_F)} H_{\mathcal{D}}^{(m)}(z, \alpha(g, g'h))\bar{\tau}(\det g'h)\varphi_0(g'h)dg' \\ &= [U_1(\hat{\mathcal{O}}_F) : U_{\mathcal{D}}]^{-1} \begin{cases} c_{\mathcal{D}}^{(1)}(z)\phi(g) & m = 1 \\ c_{\mathcal{D}}^{(2)}(z)E_{\mathcal{D}}(z, g) & m = 2 \end{cases} \end{aligned} \tag{10.11}$$

*where*

$$\begin{aligned} c_{\mathcal{D}}(x) := \pi^d 2^{(-2z-m+1)d} |M_{\mathcal{D}}|_F^{(2z+m+1)} \prod_{v|p} |x_v \bar{x}_v|_v^{z-(m+1)/2} \bar{\tau}_v^c(x_v) \prod_{v\nmid p} \tau_v(M_{\mathcal{D}}) \\ \times \frac{\Gamma(z + (m-1+\kappa)/2)^d L^{\Sigma}(\tilde{\pi}, \xi, z + m/2)}{\Gamma(z + (m+1+\kappa)/2)^d \prod_{i=0}^{1} L^{\Sigma}(\bar{\tau}' \epsilon_{\mathcal{K}}^i, 2z + m + 1 - i)} \prod_{v|p} \gamma^{(m)}(\rho_{1,v}, -z) \end{aligned} \tag{10.12}$$

This is just a summary of the previous computations, similar to [35, proposition 11.6.1]. We also have the following results for the Fourier coefficients which is an immediate consequence of the above proposition.

**Proposition 10.6.2.** *Let $m = 1$ or $2$. Suppose that for each $v|p$, $(x_v) = (p^{t_p})$ with $t_p > 0$ and that $x_v \in cond(\psi)$ and $x_v \bar{x}_v \in cond_{\pi_v}(\phi_v)$. Let $g \in G_m(\mathbb{A}_F)$ and $h \in G_1(\mathbb{A}_F)$ be such that $\mu_1(h) = \mu_m(g)$. Let $\beta \in S_m(F)$. If $\kappa \geq m+1$ then*

$$\begin{aligned} &\int_{U_1(F)/U_1(\mathbb{A}_F)} H_{\mathcal{D},\beta}^{(m)}(z, \alpha(g, g'h))\bar{\tau}(\det g'h)\varphi_0(g'h)dg' \\ &= [U_1(\hat{\mathcal{O}}_F) : U_{\mathcal{D}}]^{-1} \begin{cases} c_{\mathcal{D}}^{(1)}(z)\varphi_\beta(g) & m = 1 \\ c_{\mathcal{D}}^{(2)}(z)\mu_{\mathcal{D}}(\beta, z, g) & m = 2 \end{cases} \end{aligned} \tag{10.13}$$

*where $c_{\mathcal{D}}^{(m)}(z)$ is as defined above.*

Recall that $a_1, ..., a_{h_{\mathcal{K}}} \in \hat{\mathcal{O}}_{\mathcal{K}}$ are representatives for the class group of $\mathcal{K}$. We assume that each $a_i = (\varpi_v, 1) \in \mathcal{O}_{\mathcal{K},v}$ for some prime $v \notin \Sigma$ that splits in $\mathcal{K}$. Let

$$\Gamma_{\mathcal{D}} := U_1(F) \cap U_{\mathcal{D}}, \Gamma_{\mathcal{D},i} := U_1(F) \cap \begin{pmatrix} a_i^{-1} & \\ & \bar{a}_i \end{pmatrix} U_{\mathcal{D}} \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix}.$$

We often write $\Gamma_{\mathcal{D}}$ for the $GL_2(\mathbb{A}_{F,f})$ open compact group with the same congruence requirement as $U_{\mathcal{D}}$. Also, we write $\Gamma_{\mathcal{D},0} \supseteq \Gamma_{\mathcal{D}}$ by removing the congruence conditions required for diagonal entries.

(Similar to $\Gamma_0(N) \supset \Gamma_1(N)$ in the classical case). For any $v|p$ let

$$(p_{u_v}) := (x_v) \cap \mathcal{O}_{F,v}, (p^{r_v})_v := (x_v \bar{x}_v).$$

It follows easily from the strong approximation that if we let $\mathcal{Y} \subset \hat{\mathcal{O}}$ be any set of representatives for $(\hat{\mathcal{O}}_{\mathcal{K}}/\prod_{v|p} \bar{x}_v M_{\mathcal{D}})^\times/(\hat{\mathcal{O}}_F/p^{u_p} M_{\mathcal{D}})^\times$, then

$$U_1(\mathbb{A}_F) = \sqcup_{i=1}^{h_{\mathcal{K}}} \sqcup_{a \in \mathcal{Y}} U_1(F) U_1(F_\infty) \begin{pmatrix} a_i^{-1} a^{-1} & \\ & \bar{a}_i \bar{a} \end{pmatrix} U_{\mathcal{D}},$$

with each element appearing $2^{v_{\mathcal{K}}} h_F$ times where $v_{\mathcal{K}}$ is a number depending only on $\mathcal{K}$. Define:

$$\tilde{H}^{(m)}_{\mathcal{D},\beta}(z,g) := \sum_{a \in (\hat{\mathcal{O}}_{\mathcal{K}}/(\prod_{v|p} x_v M_{\mathcal{D}}))^\times} (\prod_{v|p} \mu_{1,v}^{-1}) \xi^c \tau(a) H^{(m)}_{\mathcal{D},\beta}(z, g\alpha(1, diag(a^{-1}, \bar{a}))).$$

## 10.7 Neben-Typus

In this section we discuss the relations between $U(1,1)$ automorphic forms and $GL_2$ automorphic forms. This will be useful later. In the [35] case the situation is easier since they assumed the forms are newforms, i.e. invariant under the action of matrices: $\begin{pmatrix} * & * \\ & 1 \end{pmatrix}$. Since we are going to work with the full dimensional Hida family so we do not assume this anymore. A principle for this issue is: we assume the neben characters at places not dividing $p$ and the torsion part at $p$-adic places to be similar to the new form and let the free part of the $p$-adic neben characters to vary arbitrarily. Let $\varepsilon' = \otimes_v \varepsilon'_v$ be a character of $T_{U(1,1)}(\hat{\mathcal{O}}_F)$. First look at the $p$-adic places. Note that $\mathbb{Z}_p = \Delta \times \Gamma$ for $\Delta \simeq \mathbb{F}_p^\times$ and $\Gamma = 1 + p\mathbb{Z}_p$. $T_{U(1,1)}(\mathbb{Z}_p) = \{ \begin{pmatrix} \bar{a}^{-1} & \\ & a \end{pmatrix} | a \in \mathcal{O}_{\mathcal{K},v}^\times \}$, $T_{GL_2}(\mathbb{Z}_p) \simeq \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. For $v|p$ $\varepsilon'_v$ is a character of $T_{U(1,1)}(\mathcal{O}_{F,v})$ can be written as $\varepsilon'_{v,tor}.\varepsilon'_{v,fr}$ with respect to $\Delta \times \Gamma$. Let $\psi$ be a Hecke character we can define $\psi_{v,tor}$ and $\psi_{v,fr}$ be characters of $\mathcal{O}_{\mathcal{K},v}^\times$ in the same way. Since $\varepsilon'_{v,fr}$ and $\psi_{v,fr}$ have order powers of $p$ so there are unique square roots $\varepsilon'^{\frac{1}{2}}_{v,fr}$ and $\psi^{\frac{1}{2}}_{v,fr}$ of them. Now suppose for each $v|p$ we have:

$$\varepsilon'_{v,tor}(\begin{pmatrix} \bar{a}^{-1} & \\ & a \end{pmatrix}) = \psi_{v,tor}(a)$$

for all $a \in \mathcal{O}_{\mathcal{K},v}^{\times}$ and that for all $v \nmid p$,

$$\varepsilon'_v(\begin{pmatrix} \bar{a}^{-1} & \\ & a \end{pmatrix}) = \psi_v(a)$$

for all $a \in \mathcal{O}_{\mathcal{K},v}^{\times}$. Then we define a neben character of $T_{GL_2}(\hat{\mathcal{O}}_F)$ by: for $v \nmid p$,

$$\varepsilon_v(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \psi_v(b),$$

for $v|p$

$$\varepsilon_{v,tor}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \psi_{v,tor}(b)$$

and

$$\varepsilon_{v,fr}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \varepsilon'^{\frac{1}{2}}_{v,fr}(\begin{pmatrix} \frac{a}{b} & \\ & \frac{b}{a} \end{pmatrix})\psi^{\frac{1}{2}}_{v,fr}(ab)$$

and

$$\varepsilon = \otimes_v \varepsilon_v$$

. Now let $\psi$ and $\varepsilon'$ be as above and $I$ be an ideal contained in the conductor of $\varepsilon'$. Let $\varphi$ be an automorphic form on $U(1,1)(\mathbb{A}_F)$ such that the action of $k \in U_0(I)$ (recall that $U_0(I)$ consists of matrices $\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ such that $c_k \in I$) is given by $\varepsilon'(\begin{pmatrix} a_k & \\ & b_k \end{pmatrix})$. Surippose moreover that it satisfies the condition that:

(*) for any totally positive global unit $b \in \mathcal{O}_F^{\times}$ we have:

$$\varphi(\begin{pmatrix} b & \\ & 1 \end{pmatrix} g \begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix}) = \varphi(g)\varepsilon'(\begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix}).$$

This condition is necessary for a $SL_2$ Hilbert modular form to be able to extend to $GL_2$ with given nebentypus)

We define a map $\alpha_\psi$ from $\varphi$'s on $U(1,1)(\mathbb{A}_F)$ as above to automorphic forms on $GL_2(\mathbb{A}_F)$.

**Definition 10.7.1.**

$$\alpha_\psi = \alpha_{\psi,\varepsilon,\varepsilon'}(\varphi)(g) = \sum_{j: a_j \bar{a}_j \sim g} \varphi(h_\infty \begin{pmatrix} \bar{a}_j & \\ & a_j^{-1} \end{pmatrix}) \varepsilon(k) \psi(z_\infty a_j)$$

*for* $g = \gamma z_\infty h_\infty \begin{pmatrix} \bar{a}_j \bar{a}_j & \\ & 1 \end{pmatrix} k \in GL_2(\mathbf{A}_F)$ *where* $\gamma \in GL_2(F), h_\infty \in SL_2(F_\infty), z_\infty \in Z(F_\infty), k \in \Gamma_0(I)_{GL_2}$.

**Lemma 10.7.1.** *Assumptions are as above. Suppose* $\varphi_1, \varphi_3$ *are automorphic forms on* $GU(1,1)(\mathbb{A}_F)$, $\varphi_2$ *is an automorphic form on* $U(1,1)$. *Let* $\psi_1, \psi_2, \psi_3$ *be Hecke characters for* $\mathcal{K}$. *Suppose* $\psi_1 \psi_2 \bar{\psi}_3 = 1$ *and the central characters of* $\varphi_1, \varphi_3$ *are* $\psi_1, \psi_3$. *Suppose also that* $\varepsilon_1', \varepsilon_2', \varepsilon_3'$ *are neben typus of* $\alpha_1|_{U(1,1)}, \alpha_2, \alpha_3|_{U(1,1)}$. *Assume that* $\varepsilon_1' \varepsilon_2' \bar{\varepsilon}_3' = 1$ *and the* $\varepsilon_i'$*'s and* $\psi_i$*'s satisfy the assumptions above. Then*

$$2^{u_\mathcal{K}} [\mathcal{O}_\mathcal{K}^\times : \mathcal{O}_F^\times] < \varphi_1 \varphi_2, \varphi_3 >_{U(1,1)} = < \varphi_1 \alpha_{\psi_2}(\varphi_2), \varphi_3 >_{GL_2}$$

*where* $u_\mathcal{K}$ *is some number depending only on* $\mathcal{K}$. *This factor comes out when considering* $GL_2/F$ *modulo the center and involves also* $v_\mathcal{K}$.

Here we implicitly identified $GL_2$ as a subgroup of $GU(1,1)$ in the obvious way. The proof is straightforward.

## 10.8 Formulas

**Definition 10.8.1.**

$$f^c(g) := \overline{f(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}$$

$$\tilde{f}^c := f^c \otimes \psi(det -)$$

**Definition 10.8.2.**

$$\tilde{g}_{\mathcal{D},\beta}^{(m)}(-, x) := (\tilde{H}_{\mathcal{D},\beta}^{(m)}(\alpha(x, -)) \otimes \xi(\det -))$$

*and*

$$g_{\mathcal{D},\beta}^{(m)} := tr_{\Gamma_0(M)/\Gamma_0(M_\mathcal{D}^2)} \pi(\begin{pmatrix} 1 & \\ & \frac{M_\mathcal{D}^2}{M} \end{pmatrix}) (\tilde{g}_{\mathcal{D},\beta}^{(m)})$$

**Proposition 10.8.1.** *Notations as above. Let $\beta \in S_m(F)$.*

*(i) There exists a constant $C_{\mathfrak{D}}^{(m)}$ depending only on $\mathcal{D}$ and $m$ such that*

$$< \tilde{g}^{(m)}_{\mathcal{D},\beta}(-,x), \rho(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f \begin{pmatrix} M_{\mathcal{D}}^2 \prod_{v|p} x_v \bar{x}_v & \\ & 1 \end{pmatrix}_f) \tilde{f}^c >_{\Gamma_{\mathcal{D},0}} = C_{\mathcal{D}}^{(m)} \begin{cases} a_{\mathfrak{D}}(\beta,x) & m=1 \\ c_{\mathfrak{D}}(\beta,x) & m=2. \end{cases}$$

*(the $\xi$ showing up here is the difference of the $\psi$ which we twisted $f^c$ and the $\tau$ in the pull back formula.)*

*(ii) If $\mu_{v,1}(p) \neq 0$ for any $v|p$ and if $p|\mathfrak{f}_\chi$ and $p|f_{\chi^{-1}\xi}$ then*

$$\begin{aligned} C_{\mathcal{D}}^{(1)} = \quad & (-\pi 2^{2-\kappa} i^{-\kappa})^d \prod_{v\nmid p, v\in\Sigma} \psi_v^c \tau_v(M_{\mathcal{D}}) |M_{\mathcal{D}}|_F^{\kappa} \gamma^{(1)}(\rho_p, -z_\kappa) \\ & \times \prod_{v|p} \xi_v^c(x_v) p^{r_v + n_v(\kappa-2) - \frac{\kappa}{2} r_v} \cdot \frac{\Gamma(\kappa-1)^d L_K^{\Sigma}(f, \chi^{-1}\xi, \kappa-1)}{\Gamma(\kappa)^d \prod_{j=0}^{1} L^{\Sigma}(\chi^{-1}\xi' \chi_K^j, \kappa-j)} \end{aligned}$$

*where*

$$\begin{aligned} \gamma^{(1)}(\rho_p, z_\kappa) = \quad & \bar{\psi}_p(-1) \prod_{v|p} c_2(\bar{\tau}'_{0,v}, 1-\kappa/2) \bar{\xi}_v^c(x_v) \mathfrak{g}(\tau'_{0,v})^2 \\ & \times \prod_{v|p} \mu_{1,v}(p)^{r_v - n_v} \mathfrak{g}(\mu_{1,v}^{-1} \xi_v^c, x_v) . \chi_v \bar{\xi}_v(y_v) \mathfrak{g}(\mu_{1,v} \bar{\chi}_v \xi_v^c, y_v) \end{aligned}$$

*and $(y_v) := \mathrm{cond}(\bar{\chi}_v \xi_v^c)$ and $(p^{n_v}) := (y_v \bar{y}_v)$.*

*(iii) If $\mathcal{O}_v \neq cond(\xi_v \psi_v^{-2} \psi_v^c) \supseteq cond(\bar{\chi}_v \xi_v \xi_v^c)$ for any $v|p$ then $C_{\mathcal{D}}^{(2)} = C_{\mathcal{D}}^{(1)} \prod_{v|p} p^{r_v}$.*

*Proof.* One argues similarly to [35, 11.7.1] and the end of [35, 11.6]. □

Now we define a normalization constant:

$$\begin{aligned} B_{\mathcal{D}}^{(m)} := & \frac{|M_{\mathcal{D}}|_K^{\frac{\kappa}{2}} \prod_{j=0}^{m} (\kappa-j-1)!^d \prod_{j=0}^{m} L^{\Sigma}(\kappa-j, \chi\xi' \epsilon_K^j) \prod_{v\in\Sigma, v\nmid p} \chi_v \bar{\xi}_v^c(y_v \delta_K) \mathfrak{g}(\bar{\chi}_v \xi_v^c, y_v \delta_K) |y_v \delta_K|_{K,v}^{2-\kappa}}{\prod_{v\nmid p, v\in\Sigma} \psi_v^c \tau_v(M_{\mathcal{D}}) \bar{\psi}_p(-1) \prod_{v|p} c_{m+1}(\bar{\tau}'_{0,v}, -(\kappa-m-1)/2) \mathfrak{g}(\tau'_{0,v})^{m+1} \mathfrak{g}(\xi_v^c, x_v)} \\ & \times i^d (-1)^m d 2^{m(m+2)d} (2\pi i)^{-(m+1)d\kappa} (\pi/2)^{m(m+2)d/2} \\ \times & \begin{cases} \prod_{v\in\Sigma, v\nmid p} \chi_v \bar{\xi}'_v(\varpi_v^{e_v}) q_v^{e_v(\kappa-2)} \mathfrak{g}(\chi_v \bar{\xi}'_v)^{-1} & m=2 \\ 1 & m=1 \end{cases} \end{aligned} \tag{10.14}$$

Now for $m = 1$ or 2 we define:

$$L_{\mathcal{D}}^{(m)} = \frac{2^{-3d}(2i)^{d(\kappa+1)}}{\prod_{v|p} p^{r_v(1-\kappa/2)}} B_{\mathcal{D}}^{(m)} C_{\mathcal{D}}^{(m)}.$$

and

$$S(f) := \prod_{v|p} \mu_{1,v}(p)^{-r^v} p^{r^v(\kappa/2-1)} W'(f)$$

where $W'(f)$ is the prime to $p$ part of the root number of $f$ with $|W'(f)|_p = 1$ (see [35, 11.7.3].) Recall that in the section for notations we defined $r^v$ such that $p^{r^v} \| N_v$ for $v|p$.

**Proposition 10.8.2.** *Assumptions are as before. Suppose $\kappa \geq 2$ if $m = 1$ and $\kappa > 6$ if $m = 2$. Suppose $x = diag(u, {}^t\bar{u}^{-1})$ with $GL_m(\mathbb{A}_{\mathcal{K},f})$. Suppose $p|\mathfrak{f}_{\bar{\chi}\xi}$ and $p^r|Nm(\mathfrak{f}_\xi)$. Suppose also $cond(\psi_p)|\mathfrak{f}_\xi^c \mathcal{O}_{\mathcal{K},p}$.*

*(i)*

$$\frac{< (f_{\mathcal{D},\beta,x}^{(m)} \otimes \xi), \rho(M \prod_{v|p} \begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix}_f) \tilde{f}^c >_{\Gamma_{\mathcal{D},0}}}{< f, \rho(M \prod_{v|p} \begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix}) \tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D},0}}} = \frac{L_{\mathcal{D}}^{(m)}}{2^{-3d}(2i)^{d(\kappa+1)} S(f) < f, \rho(\begin{pmatrix} & -1 \\ N & \end{pmatrix}) \tilde{f}^c >_{GL_2,\Gamma_0(N)}} \times W'(f)^{-1} \begin{cases} a_{\mathcal{D}}(T,x) & m = 1 \\ c_{\mathcal{D}}(T,x) & m = 2. \end{cases}$$

*(ii)*

$$L_{\mathcal{D}}^{(1)} = \prod_{v|p} a(v,f)^{-ord_v(Nm(\mathfrak{f}_{\bar{\chi}\xi}))} (\frac{(\kappa-2)!}{(-2\pi i)^{\kappa-1}})^{2d} \mathfrak{g}(\bar{\chi}\xi) Nm(\mathfrak{f}_{\bar{\chi}\xi}\delta_{\mathcal{K}})^{\kappa-2} L_{\mathcal{K}}^{\Sigma}(f, \bar{\chi}\xi, \kappa-1).$$

*where $L_{\mathcal{K}}^{\Sigma}(f, \bar{\chi}\xi, \kappa-1)$ is the $\Sigma$-primitive $L$-function for $f$ twisted by $\bar{\chi}\xi$ over $\mathcal{K}$. (iii) Under the hypotheses of Proposition 10.4.1 (iii)*

$$L_{\mathcal{D}}^{(2)} = \prod_v p^{r_v} \times L(3-\kappa, \chi\bar{\xi}') \prod_{v\in\Sigma} (1 - \bar{\chi}\xi'(\varpi_v) q_v^{2-\kappa}) L_{\mathcal{D}}^{(1)}.$$

See [35, proposition 11.7.4].

**Corollary 10.8.1.** *Under the hypotheses above*

$$\frac{< (f^{(1)}_{\mathcal{D},1,x_M} \otimes \xi), \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D},0}}}{< f, \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D},0}}} = \frac{L^{(1)}_{\mathcal{D}}}{(2^{-3}(2i)^{\kappa+1})^d S(f) < f, \rho(\begin{pmatrix} & -1 \\ N & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,U_0(N)}}.$$

See [35, corollary 11.7.5]. For any $x \in G(\mathbb{A}_{F,f})$ let

$$G_{\mathcal{D}}(Z,x) := W'(f)^{-1} L^{(2)}_{\mathcal{D}} |\mu(x)|_F^{-\kappa} E_{\mathcal{D}}(Z,x). \tag{10.15}$$

and let $C_{\mathcal{D}}(\beta, x)$ be its $\beta$th Fourier coefficient.

**Corollary 10.8.2.** *Under the hypotheses as above,*

$$\frac{< (f^{(2)}_{\mathcal{D},\beta,x} \otimes \xi), \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D},0}}}{< f, \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D},0}}} = \frac{C_{\mathcal{D}}(\beta,x)}{(2^{-3}(2i)^{\kappa+1})^d S(f) < f, \rho(\begin{pmatrix} & -1 \\ N & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_0(N)}}.$$

See [35, corollary 11.7.6].

## 10.9 a Formula for Fourier Coefficients

Now we express certain Fourier coefficients of $G_{\mathcal{D}}(Z,x)$ as essentially Rankin-Selberg convolutions of $f$ and sums of theta functions. This is used later to prove various $p$-adic properties of these coefficients.

### 10.9.1 the Formula

Let $\mathcal{D} = (f, \psi, \xi, \Sigma)$ be an Eisenstein datum. We assume:

$$\text{for any } v|p, \pi_v, , \phi_v, \psi_v, \tau_v \text{ are in the Generic Case .} \tag{10.16}$$

Let $\lambda$ be an idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$ such that

- $\lambda|_{\mathbb{A}_F^{\times}} = 1$;
- $\lambda_v(x) = (x_v/|x_v|)^{-2}$ for all $v|\infty$;
- $\lambda_v$ is unramified if $v \nmid \Sigma \setminus \{v|p\}$.

Let $a_1, .., a_{h_{\mathcal{K}}} \in \mathbb{A}_{\mathcal{K}}^{\times}$ be representatives of the class group of $\mathcal{K}$ as in the previous sections; so $a_i = (\varpi_{v_i}, 1)$ for some place $v_i$ of $F$ splitting in $\mathcal{K}$. Also for $i \in I_1$, $a_i\bar{a}_i$ is trivial in the narrow class group. For such $i$ we assume $\varpi_{v_i} = q_i$ for some totally positive $q_i \in F$. Let $\mathcal{Q} = \{v_i\}_{i \in I_1}$.

Let $\beta \in S_2(F), \beta > 0$, and $u \in GL_2(\mathbb{A}_{\mathcal{K},f})$ be such that

- $u_v \in GL_2(\mathcal{O}_{\mathcal{K},v})$for $v \notin \mathcal{Q}$;
- ${}^t\bar{u}\beta u \in S_2(\mathcal{O}_{F,v})^*$for all primes $v$;
- ${}^t\bar{u}\beta u$ is $v$-primitive for all $v \notin \Sigma \setminus \{v|p\}$;
- if $u^{-1}\beta^{-1}{}^t\bar{u}^{-1} = \begin{pmatrix} * & * \\ & d \end{pmatrix}$ then $d_v \in \mathcal{O}_F$for all $v \in \Sigma \setminus \{v|p\}$.

Let $M_{\mathcal{D}}$ be as before and also satisfying:

$$cond(\lambda)|M_{\mathcal{D}} \text{ and } D_{\mathcal{K}} \det {}^t\bar{u}\beta u|M_{\mathcal{D}}. \tag{10.17}$$

All Weil representations that show up in the following are defined using the splitting determined by the character $\lambda$. By our choice of $\mathcal{K}$, there is an idele $\mathfrak{d}_1$ of $\mathbb{A}_{\mathcal{K}}$ so that $\mathfrak{d}_1\bar{\mathfrak{d}}_1 = \mathfrak{d}$. Later we are going to choose $u$ and $\beta$ such that they do not belong to $GL_2(\mathcal{O}_v)$ only when $v = v_i$ for some $v_i$ above. Recall that we have proved:
for $v = v_i$

$$FJ_{\beta,v}(f; z_\kappa, x, g_v, y) = \frac{\tau(\det r_v y_v)|\det r_v y_v|_{\mathcal{K}}^{-z+\frac{1}{2}}}{\prod_{j=0}^{1} L(2z+3-j, \bar{\tau}'_{0,v}\chi_{\mathcal{K},v}^{j})} f_1^{sph}(g_v)\omega_\beta(r_v, g_v)\Phi_{0,y_v}(x),$$

for $v \in \Sigma \setminus \{v|p\}$, (notice that we have restricted ourselves to the case when the local characters $\psi_v, \tau_v$ are trivial):

$$\sum_{a\in(\mathcal{O}_{\mathcal{K},v}/M_{\mathcal{D}})^\times} F_{\beta,v}(z;x,g_v\begin{pmatrix}a^{-1}&\\&\bar{a}\end{pmatrix},r_vu_v)=\sum_{b\in\mathcal{O}_{F,v}/\tilde{D}_v\mathfrak{d}} f_{-b}(z,g_v'\eta)\omega_\beta(r_v,g_v'\begin{pmatrix}1&\\-b&1\end{pmatrix})\Phi_{1,M_{\mathcal{D}},u_v}(x),$$

for $v|p$, then

$$\begin{aligned}&\sum_{a\in(\mathcal{O}_{\mathcal{K},v}/x_v)^\times}\xi_v^c\tau(a)\mu_{1,v}^{-1}F_{\beta,v}(-z;x,g_v\begin{pmatrix}a^{-1}&\\&\bar{a}\end{pmatrix},r_vu_v)\\&=\psi_v(-1)\bar{\tau}_v'(deth)\mathfrak{g}(\tau_v')^2\bar{\tau}_v'(p^{2u_v})p^{-4u_vz-5u_v}\tau(\det r_vy_v)|\det r_v|_{\mathcal{K}}^{-z+1/2}|\det h|_v^{2z+1}\\&\times f_{u_v,1}(z,g_v\eta)\omega_\beta(r_v,g_v)\Phi_{\xi^c\mu_{1,v}^{-1},x_v,u_v(x)}(x),\end{aligned}$$

(Sorry for the bad notation $u_v$, the last one is of different meaning as at the beginning of this section while the others are the the conductors of $\xi_v'$.)

Now for $v\in\Sigma\setminus\{v|p\}$, we have

$$\begin{aligned}f_{-b,v}(\begin{pmatrix}1&\\n&1\end{pmatrix}g'\eta)&=(\tau_v\lambda_v^{-1})(-1)f_{-b,v}(\begin{pmatrix}1&\\-n&1\end{pmatrix}g_v\eta)\\&=(\tau_v\lambda_v^{-1})(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1}|\delta_{\mathcal{K}}\bar{\mathfrak{d}}_1|^{\frac{\kappa}{2}}f_v^\dagger(\begin{pmatrix}1&\\-n&1\end{pmatrix}g_v\eta\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1&\\&\bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\bar{\mathfrak{d}}_1^{-1}\end{pmatrix})\end{aligned}$$

For $h\in U_\beta(\mathbb{A}_F),u\in GL_2(\mathbb{A}_{\mathcal{K}})$, we define

$$\tilde{\Phi}:=\tilde{\Phi}_{\mathcal{D},\beta,u}=\otimes_{v|\infty}\Phi_{\beta,v}\prod_{v|p}\Phi_{\beta,\xi_v^c\mu_{1,v}^{-1},x_v,u_v}\otimes_{v\nmid p}\Phi_{\beta,1,M_{\mathcal{D}},u_v}\otimes_{v\notin\Sigma}\Phi_{0,u_v}$$

and $\Phi:=\Phi_{\mathcal{D},\beta,u}=\lambda(\mathfrak{d}_1\delta_{\mathcal{K}}^{-1})^{-1}|\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}|_{\mathcal{K}}^{-1}\omega((\begin{pmatrix}\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}&\\&\bar{\mathfrak{d}}_1^{-1}\bar{\delta}_{\mathcal{K}}\end{pmatrix}\eta^{-1})\tilde{\Phi}_{\mathcal{D},\beta,u}$, and define $\Theta_{\mathcal{D},\beta}(h,g;u):=\Theta_\beta(h,g;\Phi_{\mathcal{D},\beta,u})$.

The following formula will be useful

$$\begin{aligned}
& \omega_{\beta\mathfrak{d}^{-1},v}(-\eta \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g_v^\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \eta)\tilde{\Phi}(v\mathfrak{d}_1) \\
= \quad & \omega_{\beta,v}(-\begin{pmatrix} \delta_1 & \\ & \bar{\delta}_1^{-1} \end{pmatrix} \begin{pmatrix} \mathfrak{d}^{-1} & \\ & 1 \end{pmatrix} \eta \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g_v^\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \eta^{-1} \begin{pmatrix} \mathfrak{d} & \\ & 1 \end{pmatrix})\tilde{\Phi}(v) \\
= \quad & \lambda(\mathfrak{d}_1\delta_{\mathcal{K}}^{-1})|\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}|_{\mathcal{K}}\omega_{\beta,v}(-\begin{pmatrix} \bar{\delta}_1^{-1} & \\ & \bar{\delta}_1^{-1} \end{pmatrix} \eta \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g_v^\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \mathfrak{d} \end{pmatrix} \begin{pmatrix} \delta_{\mathcal{K}}\mathfrak{d}_1^{-1} & \\ & \bar{\delta}_{\mathcal{K}}^{-1}\bar{\mathfrak{d}}_1 \end{pmatrix})\Phi(v) \\
= \quad & \lambda(\mathfrak{d}_1\delta_{\mathcal{K}}^{-1})|\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}|_{\mathcal{K}}\omega_{\beta,v}(-\eta \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g_v^\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta_{\mathcal{K}}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix})\Phi(v)
\end{aligned}$$

To see this, observe that

$$\tilde{\Phi} = \omega_\beta(\eta \begin{pmatrix} \delta_{\mathcal{K}}\mathfrak{d}_1^{-1} & \\ & \bar{\delta}_{\mathcal{K}}^{-1}\bar{\mathfrak{d}}_1 \end{pmatrix})\Phi$$

and

$$\omega_{\beta\mathfrak{d}^{-1}}(g) = \omega_\beta(\begin{pmatrix} \mathfrak{d}^{-1} & \\ & 1 \end{pmatrix} g \begin{pmatrix} \mathfrak{d} & \\ & 1 \end{pmatrix})$$

**Definition 10.9.1.** *Let*

$$\otimes f_v = f_{\mathcal{D}} := \otimes_{v|\infty} f_\kappa \otimes_{v|p} f_{u_v} \otimes_{v\in\Sigma, v\nmid p} f_v^\dagger \otimes_{v\notin\Sigma} f_v^{sph}$$

*and define $\mathcal{E}_{\mathcal{D}}$ to be the corresponding Eisenstein series on $U(1,1)(\mathbb{A}_F)$.*

Let $g^\eta = \eta^{-1}g\eta$ and $g' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. If $x = \begin{pmatrix} hu & \\ & {}^t(\bar{hu})^{-1} \end{pmatrix}$ for $u \in GL_2(\mathbb{A}_{\mathcal{K}}), h \in U_\beta(\mathbb{A}_F)$ satisfying the assumptions at the beginning of this section then using the useful formula

$$\tilde{H}_{\mathcal{D},\beta}(z_k,\alpha(x,g')\mathrm{diag}(\mathfrak{d}_1^{-1}\mathfrak{d}_1^{-1},1,\bar{\mathfrak{d}}_1,\bar{\mathfrak{d}}_1,1))$$

$$=\sum_{a\in(\hat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times}\xi^c\tau(a)H_{\mathcal{D},\beta}(\alpha(x,g'\begin{pmatrix}a^{-1}&\\&\bar{a}\end{pmatrix})\mathrm{diag}(\mathfrak{d}_1^{-1}\mathfrak{d}_1^{-1},1,\bar{\mathfrak{d}}_1,\bar{\mathfrak{d}}_1,1))$$

$$=C_{\mathcal{D}}(\beta,r,u)\sum_n\sum_v\sum_b\prod_v f_{-b,v}(\begin{pmatrix}1&\\-n&1\end{pmatrix}g_v\eta)\omega_{\beta\mathfrak{d}^{-1},v}(h,\begin{pmatrix}1&\\n&1\end{pmatrix}'g_v\begin{pmatrix}1&\\-b&1\end{pmatrix})\tilde{\Phi}_{\mathcal{D},\beta,u}(v\mathfrak{d}_1)$$

$$=|\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1}C_{\mathcal{D}}(\beta,r,u)\sum_n\sum_v\sum_b\prod_v f_v(\begin{pmatrix}1&\\-n&1\end{pmatrix}g_v\eta\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}\tilde{\delta}_{\mathcal{K}}&\\&\bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\end{pmatrix}\begin{pmatrix}\bar{\mathfrak{d}}_1&\\&\mathfrak{d}_1^{-1}\end{pmatrix})$$

$$\times\omega_{\beta,v}(\eta\begin{pmatrix}1&n\\&1\end{pmatrix}g_v^\eta\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}\tilde{\delta_{\mathcal{K}}}&\\&\bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\end{pmatrix})\Phi_{\mathcal{D},\beta,u}(v)$$

$$=|\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1}C_{\mathcal{D}}(\beta,r,u)\sum_b\rho(\eta\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}\tilde{\delta_{\mathcal{K}}}&\\&\bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\end{pmatrix})\Theta_{\mathcal{D},\beta}(h,g;u)$$

$$\times\rho(\eta\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}\tilde{\delta}_{\mathcal{K}}&\\&\bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\end{pmatrix}\begin{pmatrix}\bar{\mathfrak{d}_1}&\\&\mathfrak{d}_1^{-1}\end{pmatrix})\mathcal{E}_{\mathcal{D}}(g).$$

The last step is because $\Theta_\beta$ is an automorphic form.

Now let $x=\begin{pmatrix}hu\mathfrak{d}_1^{-1}&\\&{}^t\bar{h}^{-1}{}^t\bar{u}^{-1}{}^t\bar{\mathfrak{d}}_1\end{pmatrix}$. Then:

$$< \tilde{g}^{(m)}_{\mathcal{D},\beta}(-,x), \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} 1 & \\ & M^2_{\mathcal{D}} \prod_{v|p} (p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f ) \tilde{f}^c >_{\Gamma_{\mathcal{D}}}$$

$$= |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1} \tau(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1)^{-1} < \sum_b \rho(\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix}) (\Theta_{\mathcal{D},\beta} \otimes \xi)(h,-,u) \cdot$$

$$\rho(\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} {\mathfrak{d}_1}^{-1} \end{pmatrix}) \mathcal{E}_{\mathcal{D}}, \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} 1 & \\ & M^2_{\mathcal{D}} \prod_{v|p} (p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f ) \tilde{f}^c >$$

$$= |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1} \tau(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1)^{-1} < \sum_b \rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix}) (\Theta_{\mathcal{D},\beta} \otimes \xi)(h,-;u) \cdot$$

$$\rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} {\mathfrak{d}_1}^{-1} \end{pmatrix}) \mathcal{E}_{\mathcal{D}}, \rho(\begin{pmatrix} 1 & \\ & M^2_{\mathcal{D}} \prod_{v|p} (p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f ) \tilde{f}^c >$$

$$= |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1} \tau(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1)^{-1} < \sum_b \rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix}) (\Theta_{\mathcal{D},\beta} \otimes \xi)(h,-;u) \cdot$$

$$\rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} {\mathfrak{d}_1}^{-1} \end{pmatrix}) \mathcal{E}_{\mathcal{D}}, \rho(\begin{pmatrix} 1 & \\ & M^2_{\mathcal{D}} \prod_{v|p} (p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f ) \tilde{f}^c >$$

$$= |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1} \tau(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1)^{-1} < \rho(\begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix}) A'_{\beta}(h,-;u) \cdot \mathcal{E}_{\mathcal{D}},$$

$$\rho(\begin{pmatrix} \tilde{\delta}_{\mathcal{K}}^{-1} \bar{\mathfrak{d}}_1^{-1} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}} \mathfrak{d}_1 \end{pmatrix} \begin{pmatrix} 1 & \\ & M^2_{\mathcal{D}} \prod_{v|p} (p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f ) \tilde{f}^c >_{\Gamma_0(M^2_{\mathcal{D}} \tilde{D}_{\mathcal{K}} \mathfrak{d} \prod_{v|p} p^{r_v})}$$

$$= |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1} \xi(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1) < \begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix}) A'_{\beta}(h,-;u) \cdot \mathcal{E}_{\mathcal{D}},$$

$$\rho(\begin{pmatrix} 1 & \\ & \tilde{D}_{\mathcal{K}} \mathfrak{d} \end{pmatrix} \begin{pmatrix} 1 & \\ & M^2_{\mathcal{D}} \prod_{v|p} (p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f ) \tilde{f}^c >_{\Gamma_0(M^2_{\mathcal{D}} \tilde{D}_{\mathcal{K}} \mathfrak{d} \prod_{v|p} p^{r_v})}$$

where $A'_{\beta} = (\Theta_{\mathcal{D},\beta} \otimes \xi)(h,-;u))$. Now for $v_i$ with $i \in I_1$, by definition we have $v_i \bar{v}_i$ is an ideal of $F$ generated by a totally positive global element. Let us take such a generator $q_i$. Also we take representatives $\{b_j\}_j$ of the coset:

$$\{b : \text{ totally positive units in } \mathcal{O}_F^{\times}\} / \{c\bar{c} \text{ for } c \text{ a unit in } \mathcal{O}_{\mathcal{K}}^{\times}\}$$

Then we define

$$A' := \sum_{i,j,k} \Theta_{\mathcal{D},\beta_{ijk}} \otimes \xi(h,-,u).$$

where $\beta_{ijk} = \begin{pmatrix} b_j & \\ & q_i b_k \end{pmatrix}$.

**Remark 10.9.1.** *The reason for introducing such $b_j$ is to make sure that the $A'$ satisfy $(*)$ in the section for neben typus (see also [Hida91] on top of page 324 for the q-expansion) and can be identified later with some theta functions on $GL_2$.*

**Definition 10.9.2.** *Let $\alpha_{\xi\lambda}$ be the operator defined in the section for nebentypus. We define*

$$A := \alpha_{\xi\lambda} A'.$$

We are in a position to state our formula for the Fourier coefficients for Klingen Eisenstein series. Before this let us do some normalizations:

$$\begin{aligned} C_{\mathcal{D}}(\beta,r,u) = {} & \frac{(2\pi i)^{2\kappa d}(2/\pi)^d |\delta_{\mathcal{K}}|_{\mathcal{K}}^{-1/2} |\delta_F|_F^{-1} \chi\bar{\xi}(\det ru)|\det ru|_{\mathcal{K}}^{-\frac{\kappa}{2}+2} \prod_{v_i|\infty}(\det \beta_{v_i}^{\kappa-2})^d}{(\prod_{j=0}^{1}(\kappa-1-j)!)^d L^{\Sigma}(\kappa-j,\bar{\chi}\xi'\chi_{\mathcal{K}}^j)} \\ & \times \Psi_p(-1)\bar{\chi}_p\xi'_p(\det\beta)|\det\beta|_p^{\kappa-2} \prod_{v|p} \mathfrak{g}^2(\chi_v\bar{\xi}'_v,p)\bar{\chi}_p\xi_p(p^{2u_p})p^{u_p(1-2\kappa)} \end{aligned} \tag{10.18}$$

$$B_{\mathfrak{D},1} = \frac{(\kappa-3)!^d L^{\Sigma}(\kappa-2,\bar{\chi}\xi')}{(-2)^d(2\pi i)^{(\kappa-2)d}\prod_{v|p}(\mathfrak{g}(\chi_v\bar{\xi}'_v,p))\bar{\chi}_v\xi'_v(p^{u_p})p^{(2-\kappa)u_p}}$$

$$B_{\mathfrak{D},2} := \frac{|M_{\mathcal{D}}^2|_{\mathcal{K}}^{\frac{\kappa}{2}} 2^{3d} i^{-2d} |\delta_{\mathcal{K}}|_{\mathcal{K}}^{-1/2} |\delta_F|_F^{-1} \prod_{v|\Sigma/\{v|p\}} \chi_v\bar{\xi}_v^c(y_v\delta_{\mathcal{K}})\mathfrak{g}(\bar{\chi}_v\xi_v^c, y_v\delta_{\mathcal{K}})|y_v\delta_{\mathcal{K}}|_{\mathcal{K},v}^{2-\kappa}}{\bar{\chi}_p\xi_p^c(M_{\mathcal{D}})\mathfrak{g}(\xi_p^c,x_p)}$$

$$B_{\mathfrak{D}}(\beta,r,u) := \frac{\psi\xi(\det ru)|\det ru|_{\mathcal{K}}^{\frac{\kappa}{2}+2}\bar{\chi}_p\xi'_p(\det\beta)|\det\beta|_p^{\kappa}\det\beta^{\kappa-2}}{\prod_{v|\Sigma/\{v|p\}}\bar{\chi}_v\xi'_v(\varpi_v^{e_v})q_v^{e_v(2-\kappa)}\mathfrak{g}(\chi_v\bar{\xi}'_v)}.$$

Then

$$B_{\mathfrak{D}}^{(2)} C_{\mathcal{D}}(\beta,r,u) = B_{\mathfrak{D}}(\beta,r,u) B_{\mathfrak{D},1} B_{\mathfrak{D},2}.$$

The following proposition follows immediately from proposition 10.8.1 and the previous calculation

**Proposition 10.9.1.** *With the assumptions at the beginning of this section. Let $\beta \in S_2(F), \beta > 0$, $u, h, x$ as before, Then:*

$$\frac{C_{\mathcal{D}}(\beta, x)}{2^{-3d}(2i)^{(\kappa-1)d} S(f) < f, \rho(\begin{pmatrix} & & -1 \\ & & \\ N & & \end{pmatrix}) \tilde{f}^c >_{GL_2, \Gamma_0(N)}}$$

$$= |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1} \xi(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1) B_{\mathcal{D}}(\beta, h, u) \times$$

$$\frac{< B_{\mathcal{D},1} \mathcal{E}_{\mathcal{D}}(-) B_{\mathcal{D},2} \begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix}) A'_\beta(h, -; u), \rho(\begin{pmatrix} & & -1 \\ & & \\ M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}} \mathfrak{d} & & \end{pmatrix} \begin{pmatrix} & & -1 \\ & & \\ \prod_{v|p} p^{r_v} & & \end{pmatrix}) \tilde{f}^c >_{\Gamma_0(\prod_{v|p} p^{r_v} M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}} \mathfrak{d})}}{< f, \rho(\begin{pmatrix} & & -1 \\ & & \\ \prod_{v|p} p^{r_v} & & \end{pmatrix} \begin{pmatrix} & & -1 \\ & & \\ M & & \end{pmatrix}) \tilde{f}^c >_{GL_2, \Gamma_{\mathcal{D}}}}$$

Now let us make some choices for the $u$ and $\beta$ and record some formulas for the Theta kernel functions. We remark that our convention for $\Phi$ is slightly different from [35] in the $F = \mathbb{Q}$ case (ours is given by applying $\omega(\begin{pmatrix} \mathfrak{d}_1 \delta_{\mathcal{K}}^{-1} & \\ & \bar{\mathfrak{d}_1}^{-1} \bar{\delta}_{\mathcal{K}} \end{pmatrix})$ to the one chosen by [35]).

Let $\gamma_0 \in GL_2(\mathbf{A}_{\mathcal{K},f})$ be such that $\gamma_{0,v} = (\eta, 1)$ for $v|p$ and $\gamma_{0,v} = 1$ otherwise. We let $u_i = \gamma_0 \begin{pmatrix} 1 & \\ & a_i^{-1} \end{pmatrix}$. Then $\beta_{ijk}, u_i$ satisfy the assumptions at the beginning of this section.

for $v|p$, then

$$\Phi_{\beta_{ijk}, \xi_v^c \mu_{1,v}^{-1}, x_v, \gamma_{0,p}}(x) = \begin{cases} \bar{\xi}_{v,2} \mu_{1,v}^{-1}(b_k q_i x_2'') g(\xi_{v,2}) \bar{\xi}_{v,1} \mu_{1,v}^{-1}(x_1' b_j) g(\xi_{v,1}) & x_1 = (x_1', x_2'') \in \mathbb{Z}_p^\times \times \mathbb{Z}_p \\ & x_2 = (x_2', x_2'') \in \mathbb{Z}_p^\times \times \mathbb{Z}_p \\ 0 & otherwise \end{cases} \tag{10.19}$$

for $j \in I$, then

$$\omega_{\beta_{ijk}}(g_{\infty_j}) \Phi_{\beta_{ijk}, \infty_j}(x) = e(Nm(x_1) b_j w) e(Nm(x_2) b_k q_i w) j(g_{\infty_j}, i)^{-2}$$

also if $v \nmid p$,

$$\Phi_{\beta_{ijk},1,M_{\mathcal{D}},1}(x)=|D_v|^{-1}\lambda_v(-1)|M_{\mathcal{D}}^2|_v^{-1}\begin{cases}1-\frac{1}{q_v'}, & x_1\in M_{\mathcal{D}}\mathcal{O}_v, x_2\in\mathcal{O}_v\\ -\frac{1}{q_v'}, & x_2\in\mathcal{O}_v, x_1\in\frac{M_{\mathcal{D}}}{\varpi_v}\mathcal{O}_v^\times\\ 0, & otherwise.\end{cases}\tag{10.20}$$

for $v$ non split, then

$$\Phi_{\beta_{ijk},1,M_{\mathcal{D}},1}(x)=|D_v|^{-1}\lambda_v(-1)|M_{\mathcal{D}}|_v^{-1}\begin{cases}(1-\frac{1}{q_v})^2, & x_1\in\frac{M_{\mathcal{D}}\mathcal{O}_v}{\mathfrak{d}_1}, x_2\in\frac{\mathcal{O}_v}{\mathfrak{d}_1}.\\ -\frac{1}{q_v}(1-\frac{1}{q_v}), & x_2\in\frac{\mathcal{O}_v}{\mathfrak{d}_1}, x_1\in(\frac{M_{\mathcal{D}}}{\varpi_{v'}\mathfrak{d}_1}\mathcal{O}_{F_v}^\times\times\frac{\mathcal{O}_{F_v}^\times}{\mathfrak{d}_1})\\ & or\ (\frac{\mathcal{O}_{F_v}^\times}{\mathfrak{d}_1}\times\frac{M_{\mathcal{D}}}{\varpi_{v'}\mathfrak{d}_1}\mathcal{O}_{F_v}^\times)\\ \frac{1}{q_v^2}, & x_2\in\frac{\mathcal{O}_v}{\mathfrak{d}_1}, x_1\in\frac{M_{\mathcal{D}}}{\varpi_v\mathfrak{d}_1}\mathcal{O}_v^\times\times\frac{M_{\mathcal{D}}}{\varpi_v\mathfrak{d}_1}\mathcal{O}_v^\times\\ 0 & otherwise.\end{cases}\tag{10.21}$$

for $v$ split.

If $v=v_i$:

$$\Phi_{0,u_i,v}(x)=\begin{cases}1 & x_1\in\mathcal{O}_{\mathcal{K},v}, x_2\in a_i^{-1}\mathcal{O}_{\mathcal{K},v}\\ 0 & otherwise.\end{cases}\tag{10.22}$$

## 10.10 Identify with Rankin-Selberg Convolutions

From now on we assume that all characters are unramified outside $p$.

Let $\alpha=(a_v)_v\in GL_2(\mathbb{A}_{F,f})$ be defined by $a_v=\begin{pmatrix}&-1\\M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}&\end{pmatrix}$ of $v\in\sum\setminus\{p\}$ and $a_v=1$ otherwise. For $m\geq 0$ let $b_m\in GL_2(\mathbb{A}_{F,f})$ be defined by $b_{m,v}=\begin{pmatrix}&-1\\p^m&\end{pmatrix}$ and $b_{m,v}=1$ if $v\nmid p$ then

$$\rho(\alpha)\mathcal{E}_{\mathcal{D}}=E(f'_{\mathcal{D}},z_\kappa;\gamma_\infty)$$

where $f'_{\mathcal{D}}(z,g):=f_{\mathcal{D}}(z,g\alpha_f^{-1})\in I_1(\tau/\lambda)$. It follows that $f'_{\mathcal{D}}(z,g)$ is supported on

$$B_1(\mathbb{A}_F)\eta K_{1,\infty}^+N_{B_1}(\hat{\mathcal{O}}_F)\alpha=B_1(\mathbb{A}_F)K_{1,\infty}^+K_1(p^{u_p}M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}})$$

and that for $g = bk_\infty k_f$ in the support we have :

$$f'_{\mathcal{D}}(z,g) = (M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}})^{d(\kappa/2-1)} \tau\bar{\lambda}(d_b d_{k_f}) |a_b/d_b|_{\mathbb{A}_F}^{z+1/2} J_1(k_\infty, i)^{2-\kappa}.$$

Now we recall the notion of Rankin-Selberg convolution for Hilbert modular forms, following [11]. Let $f$ and $g$ be two Hilbert automorphic forms (as functions on $GL_2(\mathbb{A}_F)$) with level $\mathfrak{m}$. For simplicity, we assume that both $f$ and $g$ have unitary central characters $\chi$ and $\xi$ and have parallel weight $k$ and $\kappa$ such that $k > \kappa$. Let $\tau = \chi/\xi$ and define

$$\mathscr{E}(x;s) = \sum_\gamma \tau(\gamma x)\eta(\gamma x)^s |j(\gamma, x_\infty(z_0))^{k-\kappa}| j(\gamma, x_\infty(z_0))|^{\kappa-k}$$

as in [11, p341 (4.5)] where $\eta$ is defined at the bottom of [11, P341]. Suppose that the neben typus of $g$ and $f$ differ by the nebentypus of $\mathscr{E}$. (this satisfies [11, 4.5]). Consider the following integral:

$$Z(s, f^c, g) = \int_{F_{\mathbb{A}_+}^\times / F_+} \int_{F_\mathbb{A}/F} \Phi(f^c, g)(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) |y|_\mathbb{A}^s dx d^\times y$$

where $\Phi(f^c, g)(x) = \overline{f^u(x)} g^u(x) |j(x_\infty, z_0)^{-k-\kappa}|$, $f^u(x) = D^{-1} f(x) j(x_\infty, z_0)^k$ and $g^u(x) = D^{-1} g(x) j(x_\infty, z_0)^\kappa$, with $D$ the discriminant of $F/\mathbb{Q}$. Note that there are minor differences between the notations here and [11], and the $m$ and $\mu$ there are 0 in our case. Then:

$$Z(s, f^c, g) = D^{(1+2s)/2} \tau(\mathfrak{d})^{-1} (4\pi)^{-d(s+(k+\kappa)/2)} \Gamma(s + k/2 + \kappa/2)^d D(s, f^c, g)$$

where

$$D(x, f^c, g) = \sum_{\mathcal{O}_F \supset \mathfrak{a} \neq 0} a(\mathfrak{a}, f^c) a(\mathfrak{a}, g) \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{a})^{-s}.$$

By (4.7) in *loc.cit*, up to a non-zero constant

$$\begin{aligned} Z(s, f, g, \tau) =& D^{-2} \int_{X_0} \bar{f} g(x) \\ & \times \mathscr{E}(x; s+1) j(x_\infty, z_0)^{\kappa-k} |j(x_\infty, z_0)^{k-\kappa}| dx, \end{aligned}$$

where $X_0 := GL_2(F) \backslash GL_2(\mathbb{A}_F) / U_0(\mathfrak{m}) \mathbb{A}_F^\times C_\infty^+$. Note that our formula is a special case of *loc.cit* (4.7) and is easier due to our assumptions on the nebentypus.

Suppose $h \in S_2(p^{r_p} M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}})$ such that the neben typus of $\mathcal{E}_{\mathcal{D}}.h$ is the same as $f$. Then:

$$B_{\mathcal{D},1} < \mathcal{E}_{\mathcal{D}} \cdot \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid,\infty} \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)h, \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v|\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f \tilde{f}^c >$$

$$= B_{\mathcal{D},1} < \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty}^{-1} \mathcal{E}_{\mathcal{D}} \cdot h, \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty}^{-1}$$

$$\times \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)\tilde{f}^c >$$

$$= B_{\mathcal{D},1} < \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty})\mathcal{E}_{\mathcal{D}} \cdot h, \rho(\begin{pmatrix} & -1 \\ p^{r_p} & \end{pmatrix}_p)\tilde{f}^c >$$

$$= B_{\mathcal{D},1} < \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)\mathcal{E}_{\mathcal{D}} \cdot h, (\chi_p(p)a_p(\tilde{f}_p^c))^{r_p-u_p}\rho(\begin{pmatrix} & -1 \\ p^{u_p} & \end{pmatrix}_p)\tilde{f}^c >$$

$$= |M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}|_F^{\frac{\kappa}{2}-1}(\bar{\chi}_p(p)a(f_p))^{r_p-u_p}B_{\mathcal{D},1}(4\pi)^{(1-\kappa)d}\Gamma(\kappa-1)^d D(\rho(\begin{pmatrix} & -1 \\ p^{u_p} & \end{pmatrix}_p)\tilde{f}^c, h; \kappa-1)$$

$$= |M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}|_F^{\frac{\kappa}{2}-1}(\bar{\chi}_p(p)a_p(f_p))^{r_p-u_p}B_{\mathcal{D},1}c(\bar{f})(\xi_p(p)a_p(h_p))^{u_p-r}(4\pi)^{(1-k)d}\Gamma(\kappa-1)^d$$

$$\times L^{\Sigma}(\kappa-2, \bar{\chi}\xi')^{-1}L(f_1^c \times h, \kappa-1).$$

**Lemma 10.10.1.** *Assumptions are as above. Suppose $h \in S_2(p^{r_p}M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}})$ is a normalized eigen form on $GL_2(\mathbb{A}_F)$ then*

$$< B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}} \cdot \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty} \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)h, \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d} \prod_{v\nmid\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f \tilde{f}^c >$$

$$= B_{\mathcal{D},3}L(f_1^c \times h, \kappa-1)$$

*where:*

$$B_{\mathcal{D},3} := |M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}|_F^{\frac{\kappa}{2}-1}(\bar{\chi}_p(p)a_p(f_p))^{r_p-u_p}B_{\mathfrak{D},1}c(\bar{f})(\xi_p(p)a_p(h_p))^{u_p-r}(4\pi)^{(1-\kappa)d}\Gamma(\kappa-1)^d \times L^{\Sigma}(\kappa-2, \bar{\chi}\xi')^{-1}.$$

# Chapter 11

# $p$-adic Interpolations

## 11.1 $p$-adic Eisenstein Datum

As in [35, chapter 12] we define the $p$-adic Eisenstein datum to be $\mathcal{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \Sigma)$ consists of:

- The integer ring $A$ of a finite extension of $\mathbb{Q}_p$;
- $\mathbb{I}$ a finite integral domain over $\Lambda_{W,A}$;
- a nearly ordinary $\mathbb{I}$-adic form $\mathbf{f}$ which is new at all $v \nmid p$ and has the tame part of the character equal to 1;
- a finite order Hecke character $\psi$ of $\mathbb{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times}$ and cond$\psi|p$ and $\psi|_{\mathbb{A}_F^{\times}} \equiv 1$;
- a finite set $\Sigma$ of primes containing all primes dividing $N\delta_{\mathcal{K}}$.

**Remark 11.1.1.** *For simplicity we have assumed $\psi$ is unramified outside $p$ and that the $\boldsymbol{\chi}_{\mathbf{f}}$ and $\xi$ in [35, 12.1] are trivial.*

Recall also that we have defined in section 7.1 the maps $\alpha$ and $\beta$. Let $\boldsymbol{\psi} := \alpha \circ \omega\psi\Psi_{\mathcal{K}}^{-1}$ and $\boldsymbol{\xi} := \beta \circ \Psi_{\mathcal{K}}$ .

**Definition 11.1.1.** *We define the set of arithmetic weights $\phi$ such that $\kappa_\phi > 6$ to be $\mathcal{X}^a$ and $\mathcal{X}_{\mathcal{D}}^{gen} \subset \mathcal{X}^a$ to be the subset such that the local Eisenstein datum is Generic as defined in subsection 8.1.3.*

For $\phi \in \mathcal{X}^a$ we define:

$$\psi_\phi(x) := (\prod_{\sigma \in \Sigma} x_\sigma^{-\kappa_\phi} x_{v_\sigma}^{\kappa_\phi})(\phi \circ \boldsymbol{\psi}(x))|x|_{\mathcal{K}}^{\kappa_\phi}.$$

Here $v_\sigma$ is the $p$-adic place corresponding to $\sigma$ under $\iota : \mathbb{C} \simeq \mathbb{C}_p$. We also define:

$$\xi_\phi := \phi \circ \boldsymbol{\xi}.$$

We always assume **(*irred*)** and **(*dist*)** defined in the introduction hold for our Hilbert modular form $f$ or Hida family $\mathbf{f}$.

## 11.2 Interpolation

### 11.2.1 Congruence Module and the Canonical Period

Suppose $R$ is a finite extension of $\mathbb{Z}_p$ we let $\mathbb{T}_\kappa^{ord}(N, \varepsilon; R)$ ($\mathbb{T}_\kappa^{0,ord}(Mp^r, \varepsilon; R)$) be the $R$-sub-algebra of $End_R(M_\kappa^{ord}(Mp^r, \varepsilon; R))$ (respectively, $End_R(S_\kappa^{ord}(Mp^r, \varepsilon; R))$) generated by the Hecke operators $T_v$ (these are Hecke operators defined using the double coset $U_1(N)_v \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} U_1(N)_v$ for the $v$'s). For any $f \in S_\kappa^{ord}(N, \varepsilon; R)$ a nearly ordinary eigenform. Then we have $1_f \in \mathbb{T}_\kappa^{0,ord}(N, \varepsilon; R) \otimes_R F_R = \mathbb{T}'_\kappa \times F_R$ the projection onto the second factor.

Suppose that the localization of the Hecke algebra at $\mathfrak{m}_f$ satisfies the Gorenstein property. Then $\mathbb{T}^{ord,0}(M, \varepsilon; R)_{\mathfrak{m}_f}$ is a Gorenstein $R$-algebra, so $\mathbb{T}^{ord,0}(M, \varepsilon; R) \cap (0 \otimes F_R)$ is a rank one $R$-module. We let $\ell_f$ be a generator; so $\ell_f = \eta_f 1_f$ for some $\eta_f \in R$.

Suppose $\mathbf{f} \in \mathcal{M}^{ord}(M, \mathbb{I})$ is a nearly ordinary $\mathbb{I}$-adic cuspidal newform. Then as above $\mathbb{T}^{ord,0}(M, \mathbb{I}) \otimes F_\mathbb{I} \simeq \mathbb{T}' \times F_\mathbb{I}$, $F_\mathbb{I}$ being the fraction field of $\mathbb{I}$ where projection onto the second factor gives the eigenvalues for the actions on $\mathbf{f}$. Again let $1_\mathbf{f}$ be the idempotent corresponding to projection onto the second factor. Then for an $\mathbf{g} \in S^{ord}(M, \mathbb{I}) \otimes_\mathbb{I} F_\mathbb{I}$, $1_\mathbf{f}\mathbf{g} = c\mathbf{f}$ for some $c \in F_\mathbb{I}$. As above, under the Gorenstein property for $\mathbf{f}$, we can define $\ell_\mathbf{f}$ and $\eta_\mathbf{f}$.

**Definition 11.2.1.** *For a classical point $f_\phi$ of $\mathbf{f}$ the canonical period of $f_\phi$ is defined by*

$$\Omega_{can} := \frac{< f_\phi, \tilde{f}_\phi^c >_{\Gamma_0(N)}}{\eta_{f_\phi}}.$$

**Remark 11.2.1.** *This "canonical" period is not quite canonical since it depends on the generator $\ell_\mathbf{f}$.*

Now we define $\mathcal{M}_\mathcal{X}(M, \Lambda_\mathcal{D})$ to be the space of (finite set of) formal $q$ expansions which when

specializing to $\phi \in \mathcal{X}^a$ is a classical modular form with the neben typus $\varepsilon_\phi$. Lemma 12.2.4 in [35] is true as well for the Hilbert modular forms: (the character $\theta$ there is assumed to be trivial in our situation.)

**Lemma 11.2.1.** *There exists an idempotent $e \in End_{\Lambda_{\boldsymbol{D}}}(\mathcal{M}_{\mathcal{X}}(M;\Lambda_{\boldsymbol{D}}))$ such that for any $\mathbf{g} \in \mathcal{M}_{\mathcal{X}}(M;\Lambda_{\boldsymbol{D}}), (e\mathbf{g})_\phi = e\mathbf{g}_\phi \in M^{ord}_{\kappa_\phi}(Mp^t_\phi, \varepsilon_\phi; \phi(\Lambda_{\boldsymbol{D}}))$ for all $\phi \in \mathcal{X}^a$*

We also have an analogue of [35, lemma12.2.7] (the kay interpolation lemma) in the Hilbert modular case and the proofs are completely the same. This is used in constructing the $p$-adic $L$-functions and $p$-adic Eisenstein series in the next two sections.

## 11.3 $p$-adic $L$-Functions

Now we state the main theorems for the existence of the non-integral and integral $p$-adic $L$-functions following [35, 12.3].

**Theorem 11.3.1.** *Let $A, \mathbb{I}, \mathbf{f}, \xi$, and $\Sigma$ as above. Suppose that there exists a finite $A$-valued idele class character $\psi$ of $\mathbb{A}_{\mathcal{K}}^\times$ such that $\psi|_{\mathbb{A}_F^\times} = \chi_{\mathbf{f}}$ and $\psi$ is unramified outside $\Sigma$.*
*(i)There exists $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}^{gen}_{\mathcal{D}}$, $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ is finite at $\phi$ and*

$$\begin{aligned}&\phi(\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi}) \\ &\quad = \prod_{v|p} \mu_{1,v,\phi}(p)^{-ord_v(Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}))} \frac{((\kappa_\phi-2)!)^{2d}\mathfrak{g}(\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi)\mathrm{Nm}(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}\delta_{\mathcal{K}})^{\kappa_\phi-2} L^\Sigma_{\mathcal{K}}(\mathbf{f}_\phi, \bar{\chi}_{\mathbf{f}_\phi}\xi_\phi, \kappa_\phi-1)}{(-2\pi i)^{2d(\kappa_\phi-1)}2^{-3d}(2i)^{d(\kappa_\phi+1)}S(\mathbf{f}_\phi)<\mathbf{f}_\phi, \tilde{\mathbf{f}}^c_\phi>}.\end{aligned}$$

*(ii) Suppose that the localization of the Hecke algebra at $\mathfrak{m}_f$ is Gorenstein. Then There exists $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in \mathbb{I}_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}^{gen}_{\mathcal{D}}$, $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}$ is finite at $\phi$ and*

$$\begin{aligned}&\phi(\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}) \\ &\quad = \prod_{v|p} \mu_{1,v,\phi}(p)^{-ord_v(Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}))} \frac{((\kappa_\phi-2)!)^{2d}\mathfrak{g}(\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi)Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}\delta_{\mathcal{K}})^{\kappa_\phi-2} L^\Sigma_{\mathcal{K}}(\mathbf{f}_\phi, \bar{\chi}_{\mathbf{f}_\phi}\xi_\phi, \kappa_\phi-1)}{(-2\pi i)^{2d(\kappa_\phi-1)}\Omega_{can}}.\end{aligned}$$

Recall that the $\mu_{1,v}$ are defined by $\pi_v \simeq \pi(\mu_{1,v}, \mu_{2,v})$ and $\mu_{1,v}(p)$ has lower $p$-adic valuation than $\mu_{2,v}(p)$.

*Proof.* See [35, 12.3.1]. The point is the Fourier coefficients of the normalized Siegel Eisenstein series constructed in the last chapter are elements in $\Lambda_{\mathcal{D}}$ (using the formulas in section 10.5) and thus the fourier jacobi coefficients (the $B^{(2)}_{\mathcal{D}} g^{(2)}_{\mathcal{D},\beta}(-,x)$)'s there) are $\Lambda_{\mathcal{D}}$-adic forms. A difference is that: the Fourier Jacobi coefficients are only forms on $U(1,1)$, which we do not know how to compare the

unitary group inner product with the $GL_2$ unless it satisfies $(*)$ as defined in the section for neben typus. So we use

$$\sum_j B_{\mathcal{D}}^{(2)} g_{\mathcal{D},\beta}^{(2)}(\begin{pmatrix} b_j^{-1} & \\ & 1 \end{pmatrix} g \begin{pmatrix} b_j & \\ & 1 \end{pmatrix})\varepsilon'(\begin{pmatrix} b_j & \\ & 1 \end{pmatrix})$$

instead, where $b_j$'s are defined before remark 10.9.1 and $\varepsilon'$ is some neben character. This satisfies (*) and one can apply the constructions in the last section to the image of the above expression under $\alpha_\psi$. □

We will often write $\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma}$ and $\mathcal{L}_{\mathcal{D}}^{\Sigma}$ for $\tilde{\mathcal{L}}_{\mathbf{f},\mathcal{K},\xi}^{\Sigma}$ and $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}^{\Sigma}$. Here $\mathcal{D}$ stands for the Eisenstein datum.

**Remark 11.3.1.** *[11] also constructed a full dimensional p-adic L-functions for Hilbert modular Hida families. In fact his p-adic L-function corresponds to our $\tilde{\mathcal{L}}$ except for local Euler factors at $\Sigma$. Our interpolation points are not quite the same as his. In fact he used the differential operators to get the whole family while we instead allowed more general neben typus at p. (Recall that he used the Rankin-Selberg method and required the difference of the p-parts of the neben typus of $\boldsymbol{f}$ and $\boldsymbol{g}$ comes from a global character.) Hida is able to interpolate more general critical values. In particular, the points $\phi_0$ corresponding to the special value $L(f_2, 1)$ where $f_2$ is the element in $\mathbf{f}$ with parallel weight 2 and trivial neben typus is an interpolation point. Our $\tilde{\mathcal{L}}_{\mathbf{f},\mathcal{K},1}^{\Sigma}$ coincides with his along a subfamily containing the cyclotomic 1-dimensional family containing $\phi_0$. This is very useful in proving some characteristic 0 results for Selmer groups.*

We also have the $\Sigma$ primitive $p$-adic $L$-functions $\tilde{L}_{f,\mathcal{K},\xi}^{\Sigma}$ and $L_{f,\mathcal{K},\xi}^{\Sigma}$ for a single $f$ by specializing the one for $\mathbf{f}$ to $f$. (See [35, 12.3.2])

### 11.3.1 Connections with Anticyclotomic $p$-adic $L$-Functions

Let $\beta : \Lambda_{\mathcal{K},A} \to \Lambda_{\mathcal{K},A}^{-}$ be the homomorphism induced by the canonical projection $\Gamma_{\mathcal{K}} \to \Gamma_{\mathcal{K}}^{-}$. For $A$ reduced, $\beta$ extends to $F_A \otimes_A \Lambda_{\mathcal{K},A} \to F_A \otimes_A \Lambda_{\mathcal{K},A}^{-}$, $F_A$ the ring of fractions of $A$.
Now we define the anticyclotomic $p$-adic L-function:

$$\mathcal{L}_{f,\mathcal{K},\xi}^{\Sigma,-} := \beta(\mathcal{L}_{f,\mathcal{K},\xi}^{\Sigma}) \in \Lambda_{\mathcal{K},A}^{-}$$

and

$$\tilde{\mathcal{L}}_{f,\mathcal{K},\xi}^{\Sigma,-} := \beta(\tilde{\mathcal{L}}_{f,\mathcal{K},\xi}^{\Sigma}) \in \Lambda_{\mathcal{K},A}^{-} \otimes_A F_A$$

For $v|p$ we can further specialize $\gamma_{v'} = 1$ for all $v' \neq v$ to get $\mathcal{L}_{f,\mathcal{K},\xi,v}^{\Sigma,-}$ and $\tilde{\mathcal{L}}_{f,\mathcal{K},\xi,v}^{\Sigma,-}$.

We define two notions concerning the anticyclotomic $p$-adic L-function which would be useful.

**Definition 11.3.1.** *For some $v|p$, writing $\tilde{\mathcal{L}}^{\Sigma,-}_{f,\mathcal{K},\xi,v} = \tilde{a}_0 + \tilde{a}_1(\gamma_{-,v} - 1) + \cdots, \tilde{a}_i \in F_A$ (the fraction field), and $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} = a_0 + a_1(\gamma_{-,v} - 1) + \cdots, a_i \in A$, when we have the Gorenstein property required to construct it, then we say $f$ satisfies*

*(NV1) if at least one of the $\tilde{a}_i$ is non-zero.*

*(NV2) if at least one of the $a_i$ is a p-adic unit.*

We denote $f_2$ to be an ordinary form in the family $\boldsymbol{f}$ of parallel weight 2 and trivial neben typus and characters. Also let $\phi_0$ be the 1-dimensional prime which is the map given by $f_2$ when restricting to $\mathbb{I}$ and maps all the $W_{i,v}$'s and $\Gamma_{\mathcal{K}}$ to 1. The corresponding special $L$-value interpolated at $\phi_0$ is $L(f_2, 1)$.

Now we state two theorems giving sufficient condition for that (NV1) and (NV2) to be satisfied.

**Theorem 11.3.2.** *([4]) $f$ is a Hilbert modular form of parallel weight 2 and trivial Neben typus and character. If the conductor of $\chi_{\mathcal{K}/F}$ and $f$ are disjoint and the $S(1)$ defined in [4, p123] has even number of primes, then picking any $v|p$ we have (NV1) is satisfied for $f$.*

Also Jeanine Van-Order constructed an anti-cyclotomic $p$-adic $L$-function $\mathcal{L}_{f,\mathcal{K},1}$. We state the following theorem of [45]:

**Theorem 11.3.3.** *(Jeanine Van-Order) Suppose the level of $f_2$ is $M = M^+M^-$ where $M^+$ and $M^-$ are products of split and inert primes respectively. Suppose:*

*(1) $M^-$ is square free with the number of prime factors $\equiv d(\mathrm{mod}2)$;*

*(2) $\bar{\rho}_f$ is ramified at all $v|M^-$.*

*then for any $v|p$ the anti-cyclotomic $\mu$ invariant at $v$ defined by her is 0.*

In fact in her paper [45] the result is not stated this way. First of all her formula is stated in an implicit say since she is using [46]. However she informed the author that in our situation it is not hard to get the above theorem using the special value formula in [49] instead. Note also that her period is not our canonical period. However the difference of the periods is a $p$-adic unit under the second hypothesis above. So we can relate our $\Sigma$-primitive anticylotomic $L$-function to hers similar to [35, 12.3.5]. Thus by the argument in *loc.cit* the $\mu$ invariant of our $p$-adic $L$-function is also 0. Thus (NV2) is OK for $f_2$.

## 11.4 $p$-adic Eisenstein Series

We state some theorems which are straight generalizations of the section 12.4 of [35].

**Theorem 11.4.1.** *Assumptions as in theorem 11.3.1 (ii). Let $\mathcal{D} = (A, \mathbb{I}, \boldsymbol{f}, \psi, \xi, \Sigma)$ be a p-adic Eisenstein datum. Suppose that (irred) and (dist) hold. Then for each $x = diag(u, {}^t\bar{u}^{-1}) \in G(\mathbb{A}_{F,f}^{\Sigma})$ there exists a formal q-expansion*

$$\boldsymbol{E}_{\mathcal{D}}(x) := \sum_{\beta \in S(F), \beta \geq 0} c_{\mathcal{D}}(\beta, x) q^{\beta}$$

*$c_{\mathcal{D}}(\beta, z) \in \Lambda_{\mathcal{D}}$, with the property that for each $\phi \in \mathcal{X}_{\mathcal{D}}^{gen}$:*

$$\boldsymbol{E}_{\mathcal{D},\phi}(x) := \sum_{\beta \in S(F), \beta \geq 0} \phi(c_{\mathcal{D}}(\beta, x)) e(Tr \beta Z)$$

*is the q expansion at x for $\frac{G_{\mathcal{D}_\phi}}{\Omega_{can}}$.*

**Remark 11.4.1.** *There is also a $\tilde{\boldsymbol{E}}_{\mathcal{D}}$ version of the above theorem under the hypothesis of theorem 11.3.1 (i) using $1_{\mathbf{f}}$ instead of $\ell_{\mathbf{f}}$. We omit it here. Also from the definition of $1_{\mathbf{f}}$ and the fact that the congruence number of $f_2$ is finite, we can find an element $\mathbf{b} \in \Lambda_{\mathcal{D}}$ such that $\mathbf{b}1_{\mathbf{f}}$ in a Hecke operator with integra ($\Lambda_{\mathcal{D}}$) coefficients and that $\mathbf{b}(\phi_0) \neq 0$. This remark will be useful later.*

# Chapter 12

# $p$-adic Properties of Fourier coefficients of $E_{\mathcal{D}}$

In this chapter, following [SU]chapter 13, using the theta correspondence between different unitary groups, we prove that certain Fourier coefficient of $E_{\mathcal{D}}$ is not divisible by certain hight one prime $P$.

## 12.1 Automorphic Forms on Some Definite Unitary Groups

### 12.1.1 generalities

Let $\beta \in S_2(F)$, $\beta > 0$. Let $H_\beta$ be the unitary group of the pairing determined by $\beta$. We write $H$ for $H_\beta$ sometimes for simplicity.

For an open compact subgroup $U \subseteq H(\mathbb{A}_{F,f})$ and any $Z$-algebra $R$ we let:

$$\mathcal{A}(U,R) := \{f : H(\mathbb{A}_F) \to R : f(\gamma hku) = f(h), \gamma \in H(F), k \in H(F_\infty), u \in U\}.$$

and for any subgroup $K \subseteq (\mathbb{A}_{F,f})$ let

$$\mathcal{A}_H(K;R) := \varinjlim_{U \supseteq K} \mathcal{A}_H(U;R),$$

### 12.1.2 Hecke operators.

For a prime $v$ let $U, U' \subset H(F_v)$ be open compact subgroups and let $h \in H(F_v)$. We can define Hecke operators $[UhU'] : \mathcal{A}(U,R) \to \mathcal{A}_H(U';R)$ in the usual way.

We will be mainly interested in two cases:

unramified split case.

Suppose $v$ splits in $\mathcal{K}$. The identification $GL_2(\mathcal{K}_v) = GL_2(F_v) \times GL_2(F_v)$ yields an identification of $H(F_v)$ with $GL_2(F_v)$ via projection ont the first factor: $H(F_v) = \{(A, \beta^{-1t}A^{-1}\beta) \in GL_2(\mathcal{K}_v)\}$. We let $H_v \subset H(F_v)$ be the subgroup identified with $GL_2(\hat{\mathcal{O}}_{F,v})$. For $U = H_v$ we Define $T_v^H$ for the Hecke operator $[Uh_vU], h_v := \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} \in GL_2(F_v) = H(F_v)$, where $\varpi_v$ is a uniformizer at $v$.

primes dividing $p$.

If $v|p$, for a positive integer $n$ we let $I_{n,v} \subset H_v$ be the subgroup identified with the set of $g \in GL_2(\mathbb{Z}_p)$ such that $g \ mod \ p^n$ belongs to $N_B(\mathbb{Z}/p^n\mathbb{Z}_p)$. For $U = I_{n,v}$, we write $U_v^H$ for the Hecke operator $[Uh_vU]$ where $h_v := \begin{pmatrix} p & \\ & 1 \end{pmatrix}$. This operator respects variation in $n$ and $U'$ and commutes with the $T_v^H$'s for $v \nmid p$. Let $U_p := \prod_{v|p} U_v$.

Now we define the nearly ordinary projector. First recall that we have fixed an $\iota : \mathbb{C}_p \simeq \mathbb{C}$. Let $R$ be either a $p$-adic ring or of the form $R = R_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $R_0$ a $p$-adic ring. Then we define

$$e_H := \varinjlim_m U_p^{H,m!} \in End_R(\mathcal{A}_H(U;R)).$$

It is well known that this exists and is an idempotent.

## 12.2 Applications to Fourier Coefficients

### 12.2.1 Forms on $H \times U_1$

If $v$ splits in $\mathcal{K}$ then we view representations of $H(F_v)$ via the respective identifications of these groups with $GL_2(F_v)$ (projection onto the first factor of $GL_2(\mathcal{K}_v) = GL_2(F_v) \times GL_2(F_v)$). Let $\lambda$ be a character of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda_\infty(z) = (z/|z|)^{-2}$ and $\lambda|_{\mathbb{A}_F^\times} = 1$. Let $(\pi, \mathcal{V}), \mathcal{V} \subseteq \mathcal{A}_H$, be an irreducible representation of $H(\mathbb{A}_{F,f})$ and let $(\sigma, \mathcal{W}), \mathcal{W} \subseteq \mathcal{A}(U_1)$, be an irreducible representation

of $U_1(\mathbb{A}_{F,f})$. Let $\chi_\pi$ and $\chi_\sigma$ be their respective central characters. We assume that:

- $\chi_\sigma = \lambda\chi_\pi^{-1}$;
- if $v$ splits in $\mathcal{K}$ then $\sigma_v \simeq \tilde{\pi}_v \otimes \lambda_{v,1}$ as representations of $GL_2(F_v)$.
- we fix a finite set $S$ of primes outside of which $\lambda$ is unramified.

Let $\varphi \in \mathcal{V} \otimes \mathcal{W}$. We assume that

- if $v \notin S$ then $\varphi(hu, g) = \varphi(h, g)$ for $u \in H_v$.
- there is a character $\varepsilon$ of $T_{U(1,1)}(\hat{\mathcal{O}}_F)$ and an ideal $N$ divisible only by primes in $S$ such that $\varphi(h, gk) = \varepsilon(\begin{pmatrix} a_k & \\ & d_k \end{pmatrix})\varphi(h, g)$ for all $k \in U_1(\hat{\mathcal{O}}_F)$ satisfying $N|c_k$. ($k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$).

Now for the group $U_1$ we can similarly define Hecke operators $T_v^U$ for unramified split $v$'s using the double coset action for $\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}$ and $U_v$ operators for $v|p$ and the nearly ordinary projector $e_U$. The following lemma follows immediately from our assumptions for $\pi$ and $\sigma$.

**Lemma 12.2.1.** *Suppose above assumptions are valid, then for any $v \notin S$ that splits in $\mathcal{K}$.*

$$(\chi_{\pi,v,1}^{-1}\lambda_{v,1}(\varpi_v)T_v^H\varphi)(h, u) = T_v^U(\varphi)(h, u),$$

*where we have wrote $\chi_{\pi,v}$ as $(\chi_{\pi,v,1}, \chi_{\pi,v,1}^{-1})$ and $\lambda_v = (\lambda_{v,1}, \lambda_{v,2})$ with respect to $\mathcal{K}_v = F_v \times F_v$.*

Now we consider the $p$-adic ordinary idempotents $e_H$ and $e$. For any $v|p$ suppose $\varepsilon_v(k) = \varepsilon_{1,v}(a_k)\varepsilon_{2,v}(d_k)$. Suppose additionally that for such $v$

$$\begin{aligned} &\bullet\ \lambda_v \text{ is unramifed at } v. \\ &\bullet\ \mathrm{cond}(\varepsilon_{2,v}) = (p^r),\ \mathrm{cond}(\varepsilon_{1,v}) = (p^s), r > s \text{ for any } v|p. \\ &\bullet\ p^r||N. \\ &\bullet\ \phi(hk, g) = \varepsilon_{2,v}^{-1}(a_k)\varepsilon_{1,v}^{-1}(d_k)\varphi(h, g) \text{ for } k = (k_1, k_2) \in H_p,\ p^r|c_{k_1}.. \end{aligned} \tag{12.1}$$

**Lemma 12.2.2.** *Assuming all the above assumptions. Then*

$$(e_H\varphi)(h, -) = e_U(\varphi)(h, -).$$

*Proof.* Completely the same as [35, 13.2.3]. □

### 12.2.2 Consequences for Fourier Coefficients

We return to the notation and setup of chapter 10. In particular $\mathcal{D} = (\varphi, \psi, \tau, \Sigma)$is a Eisenstein datum. Letting $\Theta_{ijk}(h,g) := \Theta_{\beta_{ijk}}(h, g; \Phi_{\mathcal{D},\beta_{ijk},u_i})$. Now we decompose each $\Theta_{ijk}(h,g)$ with respect to irreducible automorphic representations $\pi_H$ of $H_{ijk}(\mathbb{A}_{F,f})$:

$$\Theta_{ijk}(h,g) = \sum_{\pi_H} \varphi^{(ijk)}_{\pi_H}(h,g).$$

Then, as in [35, p 202], using general consequences of theta correspondences in the split case we may decompose:

$$\Theta_{ijk}(h,g) = \sum_{(\pi_H,\sigma)} \varphi^{(ijk)}_{(\pi_H,\sigma)}(h,g), \varphi^{(ijk)}_{(\pi_H,\sigma)} \in \pi_H \otimes \sigma,$$

$$\sigma_v \simeq \tilde{\pi}_{H,v} \otimes \lambda_{v,1} \text{ as representations of } GL_2(F_v) \text{ for all } v \text{ splits in } \mathcal{K},$$

and such $\varphi^{(ijk)}_{(\pi_H,\sigma)}(h,g)$ satisfies the assumptions about the nebentypus in the last subsection.

For $i \in I_1$, let

$$C_{\mathcal{D},ijk}(h) := \bar{\tau}(\det h) C_{\mathcal{D}}(\beta_{ijk}, diag(u_i, {}^t\bar{u_i}^{-1}); h) \in \mathcal{A}_{H_{ijk}},$$

Recall that we have defined $A' := \sum_{ijk} A'_{\beta_{ijk}}$.

**Proposition 12.2.1.** *Let $\mathcal{L} = \{v_1, v_2, \cdots, v_m\}$ be a set of primes that split in $\mathcal{K}$ and do not belong to $\Sigma \cup \mathcal{Q}$. Let $P \in \mathbb{C}[X_1, ..., X_m]$. Let $P_{H_{ijk}} := P(\xi_{v_1,1}(\varpi_{v_1})T_{v_1}^{H_{ijk}}, \cdots, \xi_{v_m,1}(\varpi_{v_m})T_{v_m}^{H_{ijk}})$ and $P_{U_1} := P(\xi_{v_1,2}\lambda_{v_1,1}^{-1}(\varpi_{v_1})T_{v_1}^{U}, ..., \xi_{v_m,2}\lambda_{v_m,1}^{-1}(\varpi_{v_m}))$ Then*

$$\frac{\sum_{ijk} e_{H_{ijk}} P_{H_{ijk}} C_{\mathcal{D},ijk}(h) B_{\mathcal{D}}(\beta_{ijk}, h, u_i)^{-1}}{2^{-3d}(2i)^{d(k+1)} S(f) < f, \rho(\begin{pmatrix} & N \\ -1 & \end{pmatrix}) \tilde{f}^c >}$$

$$= \bar{\tau}(\det h) |\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1|^{\frac{\kappa}{2}-1} \xi(\tilde{\delta}_{\mathcal{K}} \bar{\mathfrak{d}}_1)$$

$$\times \frac{< B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}} \cdot B_{\mathcal{D},2} \rho(\begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix}) e_U P_{U_1} A', \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}} \mathfrak{d} & \end{pmatrix} \prod_{v|p} \begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix}) \tilde{f}^c >}{< f, \rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix} \prod_{v|p} \begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix}) \tilde{f}^c >_f}$$

*Proof.* It follows from lemma 12.2.1 and lemma 12.2.2 in the same way as [35, 13.2.4] and [35, 13.2.5].

Observe that $\rho(\begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix})$ commutes with $eP_1$. $\square$

Now $A'$ satisfies the condition $(*)$ defined in the section of Nebentypus. So we can apply the operator $\alpha_{\lambda\xi}$ on it to get $A := \alpha_{\lambda\xi}(A')$. The following corollary follows easily from the above proposition by comparing the $GL_2$ and $U(1,1)$ Hecke eigenvalues for unramified split primes on automorphic forms with central character $\lambda\xi$, and apply lemma 10.7.1.

**Corollary 12.2.1.** *Let $P_1 = P(T_{v_1}, ..., T_{v_m})$ be a $GL_2$ Hecke operator for the polynomial $P$ in the above proposition and let $e$ be the $GL_2$ nearly ordinary projector. Then the last expression of the above proposition equals:*

$$\bar{\tau}(\det h)|\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|^{\frac{\kappa}{2}-1}\xi(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)2^{-u_{\mathcal{K}}}[\mathcal{O}_{\mathcal{K}}^{\times}:\mathcal{O}_F^{\times}]^{-1}$$
$$\times \frac{< B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}} \cdot B_{\mathcal{D},2}\rho(\begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix})eP_1A, \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}\mathfrak{d} & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix})\tilde{f}^c >_{GL_2}}{< f, \rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix})\tilde{f}^c >_{GL_2}}$$

## 12.3 $p$-adic Properties of Fourier Coefficients

In this section we put the operations above in $p$-adic families. Let $\mathcal{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \sum)$ be a $p$-adic Eisenstein datum as in the last chapter. and $\mathbf{E}_{\mathcal{D}} \in \mathcal{M}_{\underline{a},ord}(K'_{\mathcal{D}}, \Lambda_{\mathcal{D}})$ or $\tilde{\mathbf{E}}_{\mathcal{D}}$ be as there. For $x \in G(\mathbb{A}_{F,f})$ with $x \in Q(\mathcal{O}_{F,v})$ for all primes $v|p$ we let $\boldsymbol{c}_{\mathcal{D}}(\beta, x) \in \Lambda_{\mathcal{D}}$ or $\tilde{\boldsymbol{c}}_{\mathcal{D}}(\beta, x) \in \Lambda_{\mathcal{D}}$ be the $\beta$-fourier coefficient of $\mathbf{E}_{\mathcal{D}}$ or $\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}}$ at $x$ (here $\mathbf{b}$ is defined in remark 11.4.1). So for $\phi \in \mathcal{X}_{\mathcal{D}}^a$, $c_{\mathcal{D},\phi}(\beta, x) := \phi(\boldsymbol{c}_{\mathcal{D}}(\beta, x))$ is the $\beta$-fourier expansion at $x$ of a holomorphic hermitian modular form $E_{\mathcal{D},\phi}(Z, x)$ and define the $\tilde{c}_{\mathcal{D},\phi}(\beta, x)$ correspondingly. Define

$$\varphi_{\mathcal{D},\beta,x,\phi}(h) := \chi_{\mathbf{f}}\psi_{\phi}^{-1}\xi_{\phi}(\det h)c_{\mathcal{D},\phi}(\beta, \begin{pmatrix} h & \\ & {}^t\bar{h}^{-1} \end{pmatrix}x)$$

and

$$\tilde{\varphi}_{\mathcal{D},\beta,x,\phi}(h) := \chi_{\mathbf{f}}\psi_{\phi}^{-1}\xi_{\phi}(\det h)\tilde{c}_{\mathcal{D},\phi}(\beta, \begin{pmatrix} h & \\ & {}^t\bar{h}^{-1} \end{pmatrix}x).$$

As in [35, 13.3.1], recall that $\beta_{ijk} = \begin{pmatrix} b_j & \\ & q_i b_k \end{pmatrix}$ and $u_i = \gamma_0 \begin{pmatrix} 1 & \\ & a_i^{-1} \end{pmatrix}$. For $h \in GL_2(\mathbb{A}_{\mathcal{K},f})$ with $h_v \in GL_2(\mathcal{O}_{\mathcal{K},v})$ for all $v|p$ let

$$\varphi_{\mathcal{D},ijk} := \chi_f \boldsymbol{\psi}^{-1}\boldsymbol{\xi}^{-1}(\det h) c_{\mathcal{D}}(\beta_{ijk}, \begin{pmatrix} hu_i\mathfrak{d}_1^{-1} & \\ & {}^t\bar{h}^{-1}{}^t\bar{u}_i\bar{\mathfrak{d}}_1 \end{pmatrix}) B_{\mathcal{D}}(\beta_{ijk}, h, u_i)^{-1} \in \Lambda_{\mathcal{D}}.$$

(Note that by our choices $B_{\mathcal{D}}(\beta_{ijk}, h, u_i)^{-1}$ moves as a unit in $\Lambda_{\mathcal{D}}$.)

and for $\phi \in \mathcal{X}_{\mathcal{D}}^a$ and $h \in GL_2(\mathbb{A}_{\mathcal{K},f})$ let

$$\varphi_{\mathcal{D},ijk,\phi}(h) := \varphi_{\mathcal{D},\beta_{ijk},diag(u_i\mathfrak{d}_1^{-1}, {}^t\bar{u}_i{}^{-1}\bar{\mathfrak{d}}_1),\phi}(h).$$

We define the $\tilde{\varphi}$ versions of the above objects correspondingly when the local Hecke algebra for $\mathbf{f}$ is not Gorenstein. Now we have the following lemma interpolating the Hecke operators, completely as in [35, 13.3.2].

**Lemma 12.3.1.** *Let $\mathcal{L} := \{v_1, \cdots, v_m\}$ be a finite set of primes that split in $\mathcal{K}$ and do not belong to $\Sigma \cup \mathcal{Q}$. Let $P \in \Lambda_{\mathcal{D}}[X_1, \cdots, X_m]$. For $h \in H_i(\mathbf{A}_{F,f})$ with $h_p \in H_{i,p}$, there exists $\varphi_{\mathcal{D},i}(\mathcal{L}, P; h) \in \Lambda_{\mathcal{D}}$ such that:*

*(a) For all $\phi \in \mathcal{X}_{\mathcal{D}}^a$,*

$$\phi(\varphi_{\mathcal{D},ijk}(\mathcal{L}, P; h)) = P_\phi(\xi_{\phi,v_1,1}(\varphi_{v_1})T_{v_1}^{H_{ijk}}, \cdots, \xi_{\phi,v_m,1}(\varphi_{v_m})T_{v_m}^{H_{ijk}}) e_{H_{ijk}} \varphi_{\mathcal{D},ijk,\phi}(h),$$

*where $P_\phi$ is the polynomial obtained by applying $\phi$ to the coefficients of $P$.*

*(b) If $M \subseteq \Lambda_{\mathcal{D}}$ is a closed $\Lambda_{\mathcal{D}}$-submodule and $\varphi_{\mathcal{D},ijk}(h) \in M$ for all $h$ with $h_p \in H_{i,p}$, then $\varphi_{\mathcal{D},ijk}(\mathcal{L}, P; h) \in M$.*

Observe that the neben typus of $\alpha_{\xi\lambda}(A)$ at $v|p$ are given by:

$$\varepsilon'(\begin{pmatrix} a_v & \\ & d_v \end{pmatrix}) \to \mu_{1,v}(a_v)\mu_{2,v}(d_v)\tau_{1,v}^{-1}\tau_{2,v}^{-1}(d_v).$$

for any $a_v, d_v \in \mathcal{O}_{F_v}^\times$. From the definition of the theta functions ($q$-expansion) we know that $\alpha_{\xi\lambda}(A)$ is a $\Lambda_{\mathcal{D}}$ adic form. Also for each arithmetic weight $\phi$ we consider the resulting form at $\phi$ is a form of parallel weight 2 and neben-typus at $v|p$ only depend on the restriction of $\phi$ to $R^+ := \mathbb{I}[[\Gamma_{\mathcal{K}}^+]]$.

Now let $\boldsymbol{g} \in M^{ord}(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}, 1; \Lambda_{W,A})$ be a Hida family of forms which are new at primes not dividing $p$ and such that $\boldsymbol{g} \otimes \chi_{\mathcal{K}} = \boldsymbol{g}$. Suppose also that the localization of the Hecke algebra at the maximal ideal corresponding the $\boldsymbol{g}$ is Gorenstein so that $\ell_{\boldsymbol{g}}$ makes sense. Now following the remark of [35] before 13.3.4, one can change the weight homomorphism and view $\boldsymbol{g}$ as an element of $M^{ord}(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}, 1; R^+)$ such that at any $\phi$ we consider it is a normalized nearly ordinary form of parallel weight 2 and neben-typus at $v|p$ the same as $\alpha_{\xi\lambda}(A)$. Also as in *loc.cit* one can find a polynomial of the Hecke actions $P_{\boldsymbol{g}} := P(T_{v_1}, ..., T_{v_m}) \in \mathbb{T}^{ord}(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}, 1; R^+)$ such that $P_{\boldsymbol{g}} = a_{\boldsymbol{g}}\ell_{\boldsymbol{g}}$ with $0 \neq a_{\boldsymbol{g}} \in R^+$.

With these preparations we can prove the following proposition in the same way as [35, 13.3.4].

**Proposition 12.3.1.** *Under the above hypotheses, (1) If the local Hecke algebra for* $\mathbf{f}$ *is Gorenstein, then we have:*

$$\sum_{i,j,k} \varphi_{\mathcal{D},ijk}(\mathcal{L}, P_{\boldsymbol{g}}; 1) = \mathcal{A}_{\mathcal{D},\boldsymbol{g}}\mathcal{B}_{\mathcal{D},\boldsymbol{g}}.$$

*with* $\mathcal{A}_{\mathcal{D},\boldsymbol{g}} \in \mathbb{I}[[\Gamma_{\mathcal{K}}^+]]$ *and* $\mathcal{B}_{\mathcal{D},\boldsymbol{g}} \in \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ *such that for all* $\phi \in \mathcal{X}'_{\mathcal{D}}$*:*

$$\phi(\mathcal{A}_{\mathcal{D},\boldsymbol{g}}) = \frac{2^{-u_{\mathcal{K}}}[\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_F^\times]^{-1}|\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\xi(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)\phi(a_{\boldsymbol{g}})\eta_{\mathbf{f}_\phi} <B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}_\phi}\rho(\begin{pmatrix}1 & \\ & \mathfrak{d}\end{pmatrix})\boldsymbol{g}_\phi, \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}\mathfrak{d} & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{v,\phi}} & \end{pmatrix})\boldsymbol{f}_\phi^c>_{GL_2}}{<f_\phi, \rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{v,\phi}} & \end{pmatrix})f_\phi^c>_{GL_2}}$$

*and for* $\phi \in \mathcal{X}^a$*,*

$$\phi(\mathcal{B}_{\mathcal{D},\boldsymbol{g}}) = \eta_{\boldsymbol{g}_\phi}\frac{< B_{\mathcal{D},2}\alpha_{\xi\lambda}(\sum_{ijk}\Theta_{\mathcal{D},\beta_{ijk}}\otimes\xi)_\phi, \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}} & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{v,\phi}} & \end{pmatrix})\boldsymbol{g}_\phi^c >_{GL_2}}{< \boldsymbol{g}_\phi, \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}} & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{v,\phi}} & \end{pmatrix})\boldsymbol{g}_\phi^c >_{GL_2}}.$$

*Furthermore,* $\mathcal{A}_{\mathcal{D},\boldsymbol{g}} \neq 0$*.*

*(2) In general we have*

$$\sum_{i,j,k} \tilde{\varphi}_{\mathcal{D},ijk}(\mathcal{L}, P_{\boldsymbol{g}}; 1) = \tilde{\mathcal{A}}_{\mathcal{D},\boldsymbol{g}}\mathcal{B}_{\mathcal{D},\boldsymbol{g}}$$

*where* $\phi(\tilde{\mathcal{A}}_{\mathcal{D},\mathbf{g}})$ *is the expression in (1) with* $\eta_{\mathbf{f}_\phi}$ *replaced by* $\mathbf{b}_\phi$ *and* $\mathcal{B}_{\mathcal{D},\mathbf{g}}$ *the same as in (1).*

**Definition 12.3.1.** *Suppose we have a Hida family* $\mathbf{f}$ *of ordinary Hilbert modular forms and* $\mathcal{K}$ *is*

*a CM extension of $F$ as before. Let $f_2$ be an element in $\mathbf{f}$ of parallel weight 2 and trivial character. We also denote $\phi_0$ to be the point on the weight space corresponding to the special $L$-value $L(f_2, 1)$.*

Now we prove the following key proposition:

**Proposition 12.3.2.** *Let $A$ be the integer ring of a finite extension of $\mathbb{Q}_p$, $\mathbb{I}$ a domain and a finite $\Lambda_{W,A}$-algebra, and $\mathbf{f} \in \mathcal{M}^{ord}(M, \mathbf{1}; \mathbb{I})$ an $\mathbb{I}$-adic newform such that (irred) and **(dist)** hold.*

*(1) Suppose $\mathbb{T}_{\mathfrak{m}_\mathbf{f}}$ is Gorenstein. Then by possibly enlarging the $\Sigma$ in our Eisenstein datum, there exists an integer $M_\mathcal{D}$ as before and divisible by all primes dividing $\Sigma$ such that the following hold for the associated $\Lambda_\mathcal{D}$-adic Eisenstein series $\mathbf{E}_\mathcal{D}$ and the set $\mathcal{C}_\mathcal{D} = \{c_\mathcal{D}(\beta_{ijk}, x); x \in G(\mathbb{A}_{F,f}) \cap Q(F_p)\}$ of fourier coefficients of $\mathbf{E}_\mathcal{D}$. If $R \subseteq \Lambda_\mathcal{D}$ is any height-one prime containing $\mathcal{C}_\mathcal{D}$, then $R = P\Lambda_\mathcal{D}$ for some height-one prime $P \subset \mathbb{I}[[\Gamma_\mathcal{K}^+]]$.*
*(ii) In general the conclusion in (1) is still true with the $\mathcal{C}_\mathcal{D}$ and $c_\mathcal{D}(\beta_{ijk}, x)$ by $\tilde{\mathcal{C}}_\mathcal{D}$ and $\tilde{c}_\mathcal{D}(\beta_{ijk}, x)$.*

*Proof.* We follow [35, 13.4.1] closely.
As in loc.cit, we only need to find an $M_\mathcal{D}$ so that there is an $\boldsymbol{g}$ with $\mathcal{B}_{\mathcal{D},\boldsymbol{g}}$ is a p-adic unit.
First we find an idele class character $\theta$ of $\mathbb{A}_\mathcal{K}^\times$ such that:

- $\theta_\infty(z) = \prod_{v \in \Sigma} z_v^{-1}$;
- $\theta|_{\mathbb{A}_F^\times} = |\cdot|_F \chi_{\mathcal{K}/F}$;
- $Nm(\mathfrak{f}_\theta) = M_\theta^2$for some $M_\theta \in F^\times$ prime to $p$ and such that $D_\mathcal{K} M | M_\theta$ and $v | M_\theta$ for all $v \in \Sigma \setminus \{p\}$;
- for some $v | \tilde{D}_\mathcal{K}$, the anticyclotomic part of $\theta|_{\mathcal{O}_{\mathcal{K},q}}^\times$ has order divisible by $q_v$.
- $\Omega_\infty^{-\Sigma} L(1, \theta)$ is a $p$-adic unit, where $\Omega_\infty$ is the CM period defined in [12];
- $\theta_{v,2}(p) - 1$ is a $p$-adic unit for any $v|p$.
- $\psi$ has order prime to $p$.
- the local character $\psi$ is nontrivial over $\mathcal{K}_\mathfrak{P}^\times$ for all $\mathfrak{P} \in \Sigma_p$
- the restriction of $\psi$ to $Gal(\bar{F}/\mathcal{K}[\sqrt{p^*}])$ is nontrivial.

Here $\psi$ is the "torsion part" (as defined in [13]) of the anticyclotomic part of $\theta^a := \theta^c/\theta$, $p^*$ is $(-1)^{(p-1)/2}p$.

The existence is proven in a similar way as in [35, 13.4.1], using the main theorem of [18] instead of [7]. (The result in [Hsieh11] is not stated in the generality we need since he put a condition (C)

there requiring that the non split part of the CM character is square free. But M-L Hsieh informed the author that, as mentioned in that paper, this condition is removed later.) Now using the main result of [13] and [15], (we thank Hida for informing us his results in *loc.cit*), under the last three conditions above (which are put to apply Hida's result), we have

$$\eta_{g_\theta} | \frac{(g_\theta, g_\theta)}{\Omega_\infty^{2\Sigma}}$$

Thus

$$L(1,\theta)^2/\Omega_{can} | L(1,\theta)^2/\Omega_\infty^{2\Sigma} \tag{12.2}$$

where $\Omega_{can}$ is the canonical period associated to $g_\theta$.

If $g_\theta$ is the CM newform associated with $\theta$. It has parallel weight 2, level $M_\theta^2 \tilde{D}_{\mathcal{K}}$, and trivial neben character. Similarly as in [35] p210, we see that it satisfies $\boldsymbol{(irred)}$ and $\boldsymbol{(dist)}$. Let $\boldsymbol{g} \in \mathcal{M}^{ord}(M_\theta^2 \tilde{D}_{\mathcal{K}}, \mathbf{1}; R^+)$ be the ordinary CM newform associated with $\theta$. (this is constructed in [20, p133-134]. one need to first construct the automorphic representation generated by some theta series and then pick up the nearly ordinary vector inside that representation space.) The Gorenstein properties are also true as remarked by [13]. Recall that we have defined $A := \alpha_{\xi\lambda}(\sum_{i,j,k} \Theta_{\beta_{ijk}} \otimes \xi)$. Now we evaluate $B_{\mathcal{D},g}$ at the $\phi$ which restricts trivially to $W_{i,v}$'s and $\Gamma_{\mathcal{K}}$. In this case the argument in [35, 11.9.3] gives that:

$$A_\phi = (B_{\mathcal{D},4})_\phi E'(\chi_{\mathcal{K}}) \rho( \begin{pmatrix} & -1 \\ M_{\mathcal{D}} & \end{pmatrix} ) E'(\chi_{\mathcal{K}})$$

where $(B_{\mathcal{D},4})_\phi = |M_{\mathcal{D}}^2|_F^{-1} |\delta_{\mathcal{K}}|_{\mathcal{K}} 2^{3d} i^{-2d} |\delta_{\mathcal{K}}|_{\mathcal{K}}^{\frac{1}{2}}$ which is a $p$-adic unit. Here

$$E' = \prod_{v|p} (1 - p^{\frac{1}{2}} (\rho( \begin{pmatrix} 1 & \\ & p \end{pmatrix}_v )) E(\chi_{\mathcal{K}})$$

for $E(\chi_{\mathcal{K}})$ being the weight 1 Eisenstein series whose $L$-function is $L(F,s).L(F,\chi_{\mathcal{K}},s)$. We write

$$h = E'(\chi_{\mathcal{K}}) \rho( \begin{pmatrix} & -1 \\ M_{\mathcal{D}} & \end{pmatrix} ) E'(\chi_{\mathcal{K}})$$

Then the argument in [35, 13.4.1] tells us that:

$$< h, \rho(\prod_{v|p} \begin{pmatrix} & -1 \\ p & \end{pmatrix}_v \begin{pmatrix} & -1 \\ M_{\mathcal{D}}\tilde{D}_{\mathcal{K}} & \end{pmatrix}) g^c >= \frac{\pm|\tilde{D}_{\mathcal{K}}|_F \prod_{v|p} \theta_{v,2}(p)^{-2}}{i^d(-2\pi i)^{2d}\mathfrak{g}(\chi_{\mathcal{K}})} L(1,\theta)^2 \prod_{v|p}(1-\theta_{v,2}(p))^3$$

Thus

$$\phi(\mathcal{B}_{\boldsymbol{D},\boldsymbol{g}}) = \frac{\pm|\tilde{D}_{\mathcal{K}}|_F \prod_{v|p} \theta_{v,2}(p)^{-2}}{i^d(-2\pi i)^{2d}\mathfrak{g}(\chi_{\mathcal{K}})\Omega_{\text{can}}} L(1,\theta)^2 \prod_{v|p}(1-\theta_{v,2}(p))^3.$$

By definition $\phi(\mathcal{B}_{\boldsymbol{D},\boldsymbol{g}})$ is $p$-integral. But as noted before, $\frac{L(1,\theta)^2}{\Omega_{can}}$ divides a $p$-adic unit, thus itself must also be a $p$-adic unit. Therefore, $\mathcal{B}_{\boldsymbol{D},\boldsymbol{g}}$ is a unit. This proves (i). (ii) is just an easy consequence of (i). □

# Chapter 13

# Construction of the Cuspidal Family

In this chapter we construct a $\Lambda_D$-adic cusp form which is prime to the $p$-adic $L$-function by explicitly writing down some $\Lambda_D$-adic forms with the same boundary restriction as the Klingen Eisenstein family constructed before.

## 13.1 Certain Eisenstein series on GU(2,2)

### 13.1.1 Siegel Eisenstein Series

In this chapter we use $\mathbf{P}$ instead of $P$ to denote the Klingen parabolic and save the letter $P$ for the height one prime. Consider the $p$-adic family of CM characters of $\mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times$ which are unramified outside $p$. In the component containing the trivial character, there is one element $\tau_{\kappa_0}$ which is unramified everywhere and has infinite types $(\frac{\kappa_0}{2}, -\frac{\kappa_0}{2})$ at all infinite places for some $\kappa_0 > 6$ divisible by $(p-1)$. Define a Siegel Eisenstein series $E_{\kappa_0}$ on $GU(2,2)$ by choosing the local sections as follows: Let $f_v^\dagger$ be the section supported on $Qw_2K_Q(\varpi_v)$ and equals 1 on $K_Q(\varpi_v)$. (Here $K_Q(\varpi_v)$ means matrices with $v$-integral entries that belong to $Q(\mathcal{O}_{F,v})$ modulo $\varpi_v$.) If $v|p$ let

$$f_v'(g) = \begin{cases} \tau_0(\det D_1)|A_q D_q^{-1}|^s & \text{if } g = qw_{13}k \in Qw_{13}K_Q(\varpi_v) \\ 0 & \text{otherwise} \end{cases}$$

where $w_{13} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix}$. We define $f_v = f_{\kappa_0}$ for $v|\infty$, $f_v = f_v^\dagger$ for $v \in \Sigma, v \nmid p$, and $f_v = f'_v$ for $v|p$. Now we want to compute the constant terms of $E_{\kappa_0}$ along $\mathbf{P}$ at $g$ as an automorphic form on $M_{\mathbf{P}} = \{m(a,x)\}$, i.e. compute $E_{\kappa_0,\mathbf{P}}(m(a,x)g)$ as an automorphic form on $m(a,x) \in M_{\mathbf{P}}(\mathbb{A}_F)$ where $a \in GU(1,1)(\mathbb{A}_F)$ and $x \in \mathbb{A}_{\mathcal{K}}^\times$.

First note that

$$G(F) = Q(F)\mathbf{P}(F) \sqcup Q(F)w_2\mathbf{P}(F).$$

Thus

$$\begin{aligned} E_{\kappa_0}(g) &= \sum_{\gamma \in Q(F)\backslash G(F)} f(\gamma g) \\ &= \sum_{\gamma \in Q(F)\backslash Q(F)\mathbf{P}(F)} f(\gamma g) + \sum_{\gamma \in Q(F)\backslash (F)w_2\mathbf{P}(F)} f(\gamma g) \end{aligned}$$

Suppose the above summation is in the absolute convergent region, then

$$\begin{aligned} E_{\kappa_0,\mathbf{P}}(z,g) &= \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_f)} E_{\kappa_0}(ng)dn \\ &= \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_F)} \sum_{\gamma \in Q(F)\backslash Q(F)\mathbf{P}(F)} f_z(\gamma ng)dn \\ &+ \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_F)} \sum_{\gamma \in Q(F)\backslash Q(F)w_2\mathbf{P}(F)} f_z(\gamma ng)dn \\ &= \mathbb{I}_1 + \mathbb{I}_2. \end{aligned}$$

We claim that $\mathbb{I}_2(g) = 0$. By [27] we have

$$\begin{aligned} \mathbb{I}_2(g) &= \sum_{m' \in M_{\mathbf{P}}(F)\cap w^{-1}Q(F)w^{-1}\backslash M_{\mathbf{P}}(F)} \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_F)} \sum_{n' \in N_{\mathbf{P}}(F)\cap m'^{-1}w^{-1}Q(F)wm'} f_z(wm'n'ng)dn \\ &= \sum_{m'} \int_{N_{\mathbf{P}}(F)\cap w^{-1}Q(F)w\backslash N_{\mathbf{P}}(\mathbf{A}_F)} f_z(wnm'g)dn \\ &= \sum_{m'} \int_{N_1(F)\backslash(\mathbf{A}_F)} \int_{N_1(\mathbf{A}_F)\backslash N_{\mathbf{P}}(\mathbf{A})} f_z(wnm'g)dn \\ &= \sum_{m'} \int_{N_2(\mathbf{A}_F)} f_z(wnm'g)dn. \end{aligned}$$

For any $g_0$ such that $g_{0,\infty}$ is of the form $\begin{pmatrix} y & \\ & \bar{y}^t \end{pmatrix}$, $\mathbb{I}_2(gg_0)$ as a function of $g$ is an automorphic form on $M_{\mathbf{P}}$. In order to prove this is zero, we show that all the Fourier coefficients of $\mathbb{I}_2$ along the unipotent group $\begin{pmatrix} 1 & & \times & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ are zero. But this is nothing but

$$\int_{N_Q(\mathbf{A}_F)} f_z(w \begin{pmatrix} 1 & S \\ & 1 \end{pmatrix} g_0) e(-tr\beta S) dS$$

for $\beta = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$, which we have proven to be 0 (See lemma 4.1.1) at $z = z_{\kappa_0} = \frac{\kappa_0 - 2}{2}$ for all $g_0$ with the required $\infty$ part.

Next we consider $\mathbb{I}_1$. We define a Siegel Eisenstein series $E^1_{\kappa_0}$ on $GU(1,1)$ by choosing the local sections by $f_v = f_v^\dagger$ for $v$ finite (recall that this is the Siegel section supported in $Q(F_v)wN(\mathcal{O}_{F,v})$ and is 1 on $wN(\mathcal{O}_{F,v})$) and $f_v = f_{\kappa_0}$ for $v|\infty$. For each finite prime $v$ of $F$ let $K_v$ to be $K_{r,t}$ defined in chapter 8 if $v \nmid p$ and $K_v := K$ defined in chapter 4 for $v|p$. Then it is easy to see that if $g$ is such that:

$$g_v \in \begin{cases} K_v & \text{if } v|p \\ w_2 K_v & \text{if } v \nmid p, \infty \\ 1 & \text{if } v|\infty \end{cases}$$

Then $\mathbb{I}_1(m(g_1,1)g)$ is given by

$$E^1_{\kappa_0}(g_1 \prod_{v \nmid p,\infty} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_v)$$

for $g_1 \in U(1,1)(\mathbb{A}_F)$.

## 13.2 Hecke Operators

In this section we study the relations between the $GU(2,2)$ and $GU(1,1)$ Hecke operators via the restriction to the boundary. For an arithmetic point $\phi$ we write $E_\phi$ for the specialization of the $\tilde{\mathbf{E}}_{\mathcal{D}}$ to $\phi$. For any automorphic form $F$ on $G(\mathbf{A}_F)$ and some $g_0 \in G(\mathbf{A}_F)$ we consider $F_{\mathbf{P}}$ as an automorphic form on $GU(1,1)$: the value at $g' \in GU(1,1)(\mathbf{A}_F)$ is given by $F_{\mathbf{P}}(m(g',1)g_0)$.

### 13.2.1 Unramified Cases

Suppose $v$ is a place unramified in $\mathcal{K}/F$.

split case

If $v$ splits in $\mathcal{K}/F$, then $U(\mathcal{O}_{F_v}) \simeq GL_4(\mathcal{O}_{F_v})$. We write $\tau_v = (\tau_1, \tau_2)$ and $\tau_{0,v} = (\tau_1^\circ, \tau_2^\circ)$ with respect to $\mathcal{K}_v = F_v \times F_v$. Recall in this case we have defined

$$d_v := diag((\varpi_v, 1), 1, (1, \varpi_v^{-1}), 1).$$

Via projection onto the first component, $t_2^{(1)} = \begin{pmatrix} \varpi_v & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $B(F_v)$ ($B$ is the Borel) is identified with the matrices $\begin{pmatrix} \times & & \times & \times \\ \times & \times & \times & \times \\ & & \times & \times \\ & & & \times \end{pmatrix}$. Let $K$ be identified with $GU(2,2)(\mathcal{O}_{F_v})$. Then

$$Kt_2^{(1)}K = \sqcup n_{1i} d_1 K \sqcup n_{2j} d_2 K \sqcup n_{3k} d_3 K \sqcup d_4 K$$

where $n_{1i}$ goes through $\begin{pmatrix} 1 & & \times & \times \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $n_{2j}$ goes through $\begin{pmatrix} 1 & & & \\ \times & 1 & \times & \times \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $n_{3k}$ goes through $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \times \\ & & & 1 \end{pmatrix}$, and $d_1, d_2, d_3, d_4$ are $\begin{pmatrix} \varpi_v & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & \varpi_v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi_v & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi_v \end{pmatrix}$

respectively. If $g = m(g', 1)$ for some $g' \in U(1,1)$, then

$$E_{\phi,\mathbf{P}}(gn_{2j}d_2) = E_{\phi,\mathbf{P}}(gd_2) = E_{\phi,\mathbf{P}}(g)\tau_2^{-1}(\varpi_v),$$
$$E_{\phi,\mathbf{P}}(gd_4) = E_{\phi,\mathbf{P}}(g)\tau_1(\varpi_v),$$
$$E_{\kappa_0,\mathbf{P}}(gn_{2j}d_2) = E_{\kappa_0,\mathbf{P}}(g, d_2) = (\tau_2^\circ)^{-1}(\varpi_v),$$
$$E_{\kappa_0,\mathbf{P}}(gd_4) = \tau_1^\circ(\varpi_v)E_{\kappa_0,\mathbf{P}}(g).$$

Thus one sees:

**Lemma 13.2.1.** *Let $g_{0,v} = 1$. For $g$ such that $g_v = m(g_{1,v}, 1)$ for some $g_{1,v} \in U(1,1)(F_v)$ we have:*

$$\begin{aligned}
&(T_v(t_2^{(1)})(E_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g) \\
= \quad &(q_v^3(\tau_2^{-1}(\varpi_v) \cdot (\tau_2^\circ)^{-1}(\varpi_v)) + \tau_1\tau_1^\circ(\varpi_v))(E_\phi \cdot E_{\kappa_0})_{\mathbf{P}}(g) \\
&+q_v T_v(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix})((E_\phi \cdot E_{\kappa_0})_{\mathbf{P}})(g)
\end{aligned}$$

*where $T_v(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix})$ is as a Hecke action on $GU(1,1)$ and we consider $(E_\phi \cdot E_{\kappa_0})_{\mathbf{P}}$ as an automorphic form on $U(1,1)$ using $g_0$ by the remark at the beginning of this section.*

unramified inert case

Suppose $v$ is inert in $\mathcal{K}/F$ and take $K$ to be $G(\mathcal{O}_{F_v})$. Define: $d_v = \begin{pmatrix} \varpi_v & & & \\ & \varpi_v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. Then:

$$Kd_vK = \sqcup_i n_{1i}d_1K \sqcup n_{2j}d_2K \sqcup_k n_{3k}d_3K \sqcup \{d_4\}K$$

where

$$d_1 = \begin{pmatrix} \varpi_v & & & \\ & \varpi_v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, d_2 = \begin{pmatrix} 1 & & & \\ & \varpi_v & & \\ & & \varpi_v & \\ & & & 1 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} \varpi_v & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi_v \end{pmatrix}, d_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi_v & \\ & & & \varpi_v \end{pmatrix}$$

where $n_{1i}$ runs over matrices of the form $\begin{pmatrix} 1 & & \times & \times \\ & 1 & \times & \times \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $n_{2j}$ over $\begin{pmatrix} 1 & & & \\ \times & 1 & & \times \\ & & 1 & \times \\ & & & 1 \end{pmatrix}$, $n_{3k}$ over $\begin{pmatrix} 1 & & \times & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. As in the split case (actually even simpler), we see that

**Lemma 13.2.2.** *Again let $g_{0,v} = 1$. For $g$ such that $g_v = m(g_{1,v}, 1)$ for some $g_{1,v} \in U(1,1)(F_v)$:*

$$(T_v(d_v) \cdot (E_\phi \cdot E_{k_0}))_{\mathbf{P}}(g) = (q_v^3 + 1) T_v(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix})((E_\phi \cdot E_{k_0})_{\mathbf{P}})(g)$$

*where $T_v(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix})$ is the Hecke action on $GU(1,1)$.*

### 13.2.2 $v|p$ Case

Suppose $d = d_v = (\begin{pmatrix} p^3 & & & \\ & p^2 & & \\ & & 1 & \\ & & & p \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & p^{-1} & & \\ & & p^{-3} & \\ & & & p^{-2} \end{pmatrix})$, we study $(T_{d_v}(E_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g)$. Using the decomposition $KdK = \sqcup n_i dK$ where $n_i$ runs over

$$\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \alpha & \beta \\ & 1 & \bar{\beta} & \gamma \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and $x, \alpha, \beta, \gamma$ runs over congruence classes modulo: $\mathcal{O}_v/(p,p)$, $\mathbb{Z}_v/p^3, \mathcal{O}_v/(p^2,p^2), \mathbb{Z}_v/p$ respectively. First notice that if $x \neq 0$, $E_{\phi,\mathbf{P}}(gn_id) = 0$, so we may ignore such terms while summing up. So

$$(T_{d_v}(E_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g) = \sum_{\alpha,\beta,\gamma} E_{\phi,\mathbf{P}}(gn_id)E_{\kappa_0,\mathbf{P}}(gn_id).$$

Observe that for all choices of $\beta, \gamma$, the above expression does not change. Therefore the summation is essentially only over $\alpha$'s.

If $g = m(g_1, 1)$ for some $g_1 \in U(1,1)$ then an easy computation taking into account Hida's normalization factors given by [35, 6.2.2.a] for $U(2,2)$ and $U(1,1)$ gives:

$$(U_{d_v}(E_\phi.E_{\kappa_0}))_{\mathbf{P}}(g) = p^{-\frac{3}{2}(\kappa_\phi+\kappa_0)}\tau_{\phi,v}((p,p^{-2}))\tau_{\kappa_0,v}((p,p^{-2}))U_{p^3}((E_\phi.E_{\kappa_0})_{\mathbf{P}})(g).$$

Here $U_{p^3}$ is the $U(1,1)$ normalized Hecke operator associated to $(\begin{pmatrix} p^3 & \\ & 1 \end{pmatrix}, 1)$.

Recall if we define $U_p^{(2,2)} := \prod_{v|p} U_{d_v}$, then $e^{ord}_{U(2,2)} = \lim_n U_p^{n!}$.
The above calculation told us that

**Lemma 13.2.3.** *Let $g_{0,v} = 1$ For $g$ such that $g_v = m(g_{1,v}, 1)$ for some $g_{1,v} \in U(1,1)(F_v)$, then:*

$$(e^{ord}_{(2,2)}(\tilde{E}_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g) = e^{ord}_{(1,1)}((\tilde{E}_\phi \cdot E_{\kappa_0})_{\mathbf{P}})(g).$$

### 13.2.3 Construction of the Family

Now we define an automorphism $\gamma : \Lambda_W \to \Lambda_W$ such that for any arithmetic weight $\phi$ , $\gamma \circ \phi$ is an arithmetic weight with the same nebentypus at $p$ but $\kappa_{\gamma\circ\phi} = \kappa_\phi + \kappa_0$. The formula is given by: $(\gamma(1 + W_{1,v}) = (1 + W_{1,v}), \gamma(1 + W_{2,v}) = (1 + W_{2,v})(1 + p)^{\kappa_0}$. Then we consider $\mathbb{I} \otimes_{\Lambda_W,\gamma} \mathbb{I}$ (the $\Lambda_W$-algebra structure on the right $\mathbb{I}$ is given by the natural one composed with $\gamma$). We choose a reduced irreducible component $\mathbb{J}'$ whose spectrum maps surjectively onto Spec$\Lambda_W$. Then it is easy to see that both $\mathbb{I}$'s inject to $\mathbb{J}'$. We define $\mathbb{J}$ to be the normalization of $\mathbb{J}'$. (Intuitively $\mathbb{J}$ is parameterizing pairs of forms with weight $\kappa_\phi$ and $\kappa_{\gamma\circ\phi}$). We write $j_1 : \mathbb{I} \to \mathbb{J}$ and $j_2 : \mathbb{I} \to \mathbb{J}$ for the two embeddings. We have $j_2 \circ \gamma = j_1$. We also define an automorphism $\Lambda_{\mathcal{D}} \to \Lambda_{\mathcal{D}}$ which we again denote as $\gamma$ such that the Eisenstein data we get by $\phi$ and $\phi \circ \gamma$ have the same "finite order part" and $\kappa_{\phi\circ\gamma} = \kappa_\phi - \kappa_0$. Thus $\xi_\phi = \xi_{\phi\circ\gamma}$ and $\psi_\phi = \psi_{\phi\circ\gamma} \times \tau_{\kappa_0}$.

Now recall that we have defined a point $\phi_0$ (subsection 11.3.1) on Spec$\Lambda_{\mathcal{D}}$ which induces the point on the weight space mapping all the $W_{i,v}$'s and $\Gamma_{\mathcal{K}}$ to 1. We denote $\Lambda_{\mathcal{D},\mathbb{J}} = \Lambda_{\mathcal{D}} \otimes_{\mathbb{I},j_1} \mathbb{J}$. We also

denote some point (choose any) on $\mathrm{Spec}\Lambda_{\mathcal{D},\mathbb{J}}$ inducing $\phi_0$ on $\mathrm{Spec}\Lambda_{\mathcal{D}}$ via $j_1$ still by $\phi_0$. Moreover, recall that by the definition of $1_{\mathbf{f}}$ in chapter 11 and the fact that the congruence number for $f_2$ is finite, we have defined a $\mathbf{b}$ in remark 11.4.1 such that $\mathbf{b}(\phi_0) \neq 0$ and $\mathbf{b}1_{\mathbf{f}}$ is an integral Hecke action. Therefore $\mathbf{b}\tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}} \in \Lambda_{\mathcal{D}}$ and $\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}}$ is an $\Lambda_{\mathcal{D}}$-adic (in other words integral) form. Thus both $j_1(\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}})$ and $j_2(\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}}.E_{\kappa_0})$ are $\Lambda_{\mathcal{D},\mathbb{J}}$-adic forms. We consider $\mathbf{f}$ as a $\mathbb{J}$-adic Hida family.

Similarly as above we can let $P_{\mathbf{f}} := P_{\mathbf{f}}(T_{v_1}, \cdots, T_{v_m})$ be a polynomial with coefficients in $\mathbb{J}$ such that $P_{\mathbf{f}} = \mathbf{a}_{\mathbf{f}}1_{\mathbf{f}}$ for some $a_{\mathbf{f}} \in \mathbb{J}$ such that $a_{\mathbf{f}}(\phi_0) \neq 0$. We define a $\Lambda_D$-coefficient formal $q$-expansion:

$$\mathbf{E}^{\circ} := (P_{\mathbf{f}}(\tilde{T}_{v_1,d_{v_1}}, \cdots, \tilde{T}_{v_m,d_{v_m}})e^{ord}_{U(2,2)}(\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}} \cdot j_2(E_{\kappa_0})))$$

where

$$\tilde{T}_v(d_v) := \begin{cases} (\frac{1}{q_v}(T_v(d_v) - q_v^3(\bar{\tilde{\tau}}_2)(\varpi_v) \cdot \bar{\tau}_2^{\circ}(\varpi_v)) - \tilde{\tau}_1\tau_1^{\circ}(\varpi_v))\Psi_{2,v}(\varpi_v) & \text{if } v \text{ splits in } \mathcal{K}/F \\ \frac{1}{q_v^3+1}T_v(d_v) & \text{if } v \text{ is inertial in } \mathcal{K}/F \end{cases}$$

Here $d_v$'s are defined as before and $\Psi_{2,v}$ is to take care of the difference between Hecke eigenvalues between $U(1,1)$ and $GL_2$. We should justify the operator $e^{\mathrm{ord}}_{U(2,2)}$ acting on $\Lambda_{\mathcal{D},\mathbb{J}}$-adic forms. This can be done in the same way as [35, 12.2.4] (i).

We have already computed that if $g$ is such that $g_v = 1$ for $v|p$ and $g_v = w$ for $v \nmid p, v \in \Sigma$ and $g_v = 1$ for $v|\infty$ then the constant term $j_1(\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}})_{\mathbf{P}}(m(g_1,1)g)$ is given by

$$j_1(\mathbf{b}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\xi}'}\tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}})\mathbf{f}(g_1 \prod_{v\nmid p,\infty} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_v)$$

for each $g_1 \in U(1,1)(\mathbb{A}_F)$. Therefore the constant terms

$$j_2(\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}}.E_{\kappa_0})_{\mathbf{P}}(m(g_1,1)g) = j_2(\mathbf{b}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\xi}'}\tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}})\mathbf{f}(g_1(\prod_{v\nmid p,\infty} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_v)E^1_{\kappa_0}(g_1 \prod_{v\nmid p,\infty} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_v).$$

Now we consider a 1-dimensional subspace of $\Lambda_{\mathcal{D}}$ defined by the Zariski closure of the arithmetic $\phi$'s such that $\xi_\phi$ is trivial and $f_\phi$ has trivial neben typus at $p$ (but allow the weight of $f_\phi$ to vary). Observe that $j_1(\boldsymbol{b}\tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}})$ is not identically 0 along this family by the interpolation properties and the temperedness of $f_\phi$'s. (Recall also that we do not know that these points are interpolations points.

But by comparing with Hida we know that his and our constructions coincide along a subfamily containing the 1-dimensional family above and we can use Hida's interpolation formula.) So we can choose $\kappa_0$ so that $j_2(\mathcal{L}_{\mathcal{D}}^{\Sigma})$ does not belong to $\phi_0$. (only need to avoid a finite number of points). (Note that $j_2(\mathcal{L}_{\mathcal{D}}^{\Sigma})$ does not interpolate any classical $L$-values at $\phi_0$ since the weight is $2-\kappa_0$.) Let $P$ be any height one prime of $\Lambda_{\mathcal{D},\mathbb{J}}$ contained in $\phi_0$. Then $P$ is prime to $j_2(\mathcal{L}_{\mathcal{D}}^{\Sigma})$.

To sum up, for $\prod_v g_v = g$, if:

$$g_v \in \begin{cases} \mathbf{P}(F_v)K_v. & v|p \\ \mathbf{P}(F_v)wK_v, & v \in \Sigma \setminus \{v|p\}. \\ Q(F_v), & v|\infty \\ 1, & \text{otherwise} \end{cases}$$

then there is an $\mathbf{a} \in \mathbb{J}$ satisfying:

$$(j_1(\mathcal{L}_{\mathcal{D}}^{\Sigma}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'}\mathbf{E}^{\circ}))_{\phi,\mathbf{P}}(g) = (\mathbf{a}\cdot\mathbf{a}_{\mathbf{f}})_{\phi}(j_2(\mathcal{L}_{\mathcal{D}}^{\Sigma}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'}\mathbf{E}))_{\phi,\mathbf{P}}(g)$$

and $\mathbf{a}_{\phi} = \mathbf{b}_{\phi}\dfrac{<\mathbf{f}_{\phi\circ\gamma^{-1}}\cdot E^1_{\kappa_0},\rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_\phi} & \end{pmatrix})\mathbf{f}^c_{\phi}>}{<\mathbf{f}_{\phi\circ\gamma^{-1}},\rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_\phi} & \end{pmatrix})\mathbf{f}^c_{\phi}>}$. (This can be seen easily from our computations for the constant terms for $E_{\kappa_0}$ along $\mathbf{P}$ in the first section.) In the case when $\kappa_{\phi} >> \kappa_0$ our previous computations on Rankin-Selberg convolutions told us that $\mathbf{a}_{\phi} \neq 0$ by the temperedness of $\mathbf{f}_{\phi\circ\gamma^{-1}}$ and $\rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_\phi} & \end{pmatrix})\mathbf{f}_{\phi}$.

**Theorem 13.2.1.** *There is a $\Lambda_{\mathcal{D},\mathbb{J}}$-coefficients formal $q$ expansion $\boldsymbol{F}$, such that:*

*(i) For a Zariski dense set of arithmetic points $\phi$, $\boldsymbol{F}_{\phi}$ is an ordinary cusp form on $GU(2,2)(\mathbb{A}_F)$.*

*(ii) $\mathbf{F} \equiv \mathbf{a}\mathbf{a}_{\mathbf{f}} j_2(\boldsymbol{b}\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'})\mathbf{b}\tilde{\mathbf{E}}_{\mathcal{D}}(\bmod j_1(\mathbf{b}\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'}))$ for some $0 \neq \mathbf{a}\mathbf{a}_{\mathbf{f}} \in \mathbb{J}(\Gamma_{\mathcal{K}}^{+})$.*

*(iii) For any height 1 prime $P$ of $\Lambda_{\mathcal{D}}$ containing $j_1(\mathbf{b}\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma})$ passing through $\phi_0$ which is not a pull back of a height 1 prime of $\mathbb{J}[[\Gamma_{\mathcal{K}}^{+}]]$, there is a coefficient of $\mathbf{F}$ not contained in $P$.*

*Proof.* Take $\mathbf{F} := j_1(\boldsymbol{b}\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'})\mathbf{E}^{\circ} - \mathbf{a}\mathbf{a}_{\mathbf{f}} j_2(\boldsymbol{b}\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'})\boldsymbol{b}\tilde{\mathbf{E}}_{\mathcal{D}}$. From our computations we know that the constant terms for $\mathbf{F}$ along $\mathbf{P}$ is 0. The constant terms along other standard parabolic subgroups are 0 by the corresponding properties of Klingen Eisenstein series. Thus $\mathbf{F}$ is cuspidal. For (iii) we need to use proposition 12.3.2.

□

# Chapter 14

# Proof of the Main Results

We prove the main results in this chapter. Recall that we use $f$ and $\mathbf{f}$ to denote a nearly ordinary Hilbert modular form or Hida family with some coefficient ring $\mathbb{I}$. Let $\psi, \tau$ be Hecke characters of $\mathbb{A}_{\mathcal{K}}^{\times}$ and $\boldsymbol{\psi}$, $\boldsymbol{\tau}$ be $p$-adic families of Hecke characters of $\mathbb{A}_{\mathcal{K}}^{\times}$. We require that the restrictions of $\psi$ and $\boldsymbol{\psi}$ to $\mathbb{A}_F^{\times}$ to be the same as the central character of the $f$ or $\mathbf{f}$. Let $\xi$ and $\boldsymbol{\xi}$ be $\frac{\psi}{\tau}$ and $\frac{\boldsymbol{\psi}}{\boldsymbol{\tau}}$. These are part of the Eisenstein datum $\mathcal{D}$ defined at the beginning of chapter 11.

## 14.1 the Eisenstein Ideal

### 14.1.1 Hecke Operators

Let $K' = K'_{\Sigma} K^{\Sigma} \subset G(\mathbb{A}_f^p)$ be an open compact subgroup with $K^{\Sigma} = G(\hat{\mathcal{O}}_F^{\Sigma})$ and such that $K := K' K_p^0$ is neat. The Hecke operators we are going to consider are at the unramified places and at primes dividing $p$. We follow closely to [35, 9.5,9.6].

Unramified Inert Case

Let $v$ be a prime of $F$ inert in $\mathcal{K}$. Recall that as in [35, 9.5.2] that $Z_{v,0}$ is the Hecke operator associated to the matrix $z_0 := \mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v$ by the double coset $K z_0 K$ where $K$ is the maximal compact subgroup of $G(\mathcal{O}_{F,v})$. Let $t_0 := \mathrm{diag}(\varpi_v, \varpi_v, 1, 1)$, $t_1 := \mathrm{diag}(1, \varpi_v, 1, \varpi_v^{-1})$ and $t_2 := \mathrm{diag}(\varpi_v, 1, \varpi_v^{-1}, 1)$. We define the Hecke operators $T_i$ for $i = 1, 2, 3, 4$ by requiring that

$$1 + \sum_{i=1}^{4} T_i X^i = \prod_{i=1}^{2} (1 - q_v^{\frac{3}{2}}[t_i]X)(1 - q_v^{\frac{3}{2}}[t_i]^{-1}X)$$

is an equality of polynomials of the variable $X$. Here $[t_i]$ means the Hecke operator defined by the double coset $Kt_iK$. We also define:

$$Q_v(X) := 1 + \sum_{i=1}^{4} T_i (Z_0 X)^i.$$

Unramified Split Case

Suppose $v$ is a prime of $F$ split in $\mathcal{K}$. In this case we define $z_0^{(1)}$ and $z_0^{(2)}$ to be $(\mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v), 1)$ and $(1, \mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v)$ and define the Hecke operators $Z_0^{(1)}$ and $Z_0^{(2)}$ as above but replacing $z_0$ by $z_0^{(1)}$ and $z_0^{(2)}$. Let $t_1^{(1)} := \mathrm{diag}(1, (\varpi_v, 1), 1, (1, \varpi_v^{-1}))$, $t_2^{(1)} := \mathrm{diag}((\varpi_v, 1), 1, (1, \varpi_v^{-1}), 1)$. Define $t_i^{(2)} := \bar{t}_i^{(1)}$ and $t_i = t_i^{(1)} t_i^{(2)}$ for $i = 1, 2$. Then we define Hecke operators $T_i^{(j)}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ by requiring that the following

$$1 + \sum_{i=1}^{4} T_i^{(j)}) X^i = \prod_{i=1}^{2} (1 - q_v^{\frac{3}{2}} [t_i^{(j)}] X)(1 - q_v^{\frac{3}{2}} [t_i^{(j')}]^{-1} X)$$

to be equalities of polynomials of the variable $X$. Here $j' = 3 - j$ and $[t_i^{(j)}]$'s are defined similarly to the inert case. Now let $v = w\bar{w}$ for $w$ a place of $\mathcal{K}$. Define $i_w = 1$ or $w$ depending on whether the valuation associated to 2 comes from the projection onto the first or second factor of $\mathcal{K}_v = F_v \times F_v$. Then we define:

$$Q_w(X) := 1 + \sum_{i=1}^{4} T_i^{(i_w)} (Z_0^{(3-i_w)} X)^i.$$

$p$-adic Case

Let $t = \mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_4}, p^{a_3})$, let $u_t$ be Hida's normalized operator defined in [35, 6.2.2.a].

Let $\mathbf{h}_{\mathcal{D}} = \mathbf{h}_{\mathcal{D}}(K')$ be the reduced quotient of the universal ordinary cuspidal Hecke algebra which is defined by the ring of elements in $\mathrm{End}_{\Lambda_{\mathcal{D}}}(S^{ord}(K', \Lambda_{\mathcal{D}}))$ generated by the Hecke operators $Z_{v,0}$, $Z_{v,0}^{(i)}$, $T_{i,v}$, $T_{i,v}^{(j)}$,$u_{t,v}$ defined above. This is a finite reduced $\Lambda_{\mathcal{D}}$-algebra. Now we define for each prime $w$ of $\mathcal{K}$ a polynomial $Q_{w,\mathcal{D}}(X) = \det(1 - \rho_{\mathcal{D}}(\mathrm{frob}_w)X)$ where $\mathcal{D}$ is the Eisenstein datum mentioned at the beginning of this chapter and $\boldsymbol{\rho}_{\mathcal{D}}$ is the Galois representation defined in subsection 14.2.2.

We define the Eisenstein ideal $I_{\mathcal{D}}$ (which is actually the kernel of homomorphism from the abstract Hecke algebra to $\Lambda_{\mathcal{D}}$ determined by the Eisenstein family) generated by:

- the coefficients of $Q_w(X) - Q_{w,\mathcal{D}}(X)$ for all finite places $v$ of $\mathcal{K}$ and not dividing a prime in $\Sigma$.
- $Z_{v,0} - \boldsymbol{\sigma}_\psi \boldsymbol{\sigma}_\xi^{-1}(\mathrm{frob}_v)$ for $v$ a finite place outside $\Sigma$.

- $Z_{v,0}^{(i)} - \boldsymbol{\sigma}_\psi \boldsymbol{\sigma}_\xi^{-1}(\mathrm{frob}_{w_i})$ for all $v$ outside $\Sigma$ such that $v = w_1 w_2$ being the factorization of $\mathcal{K}_v = F_v \times F_v$.

- For all $v|p$, $u_{t,v} - \lambda_{\mathbf{E}_\mathcal{D}}(u_{t,v})$ with $a_1 \leq \cdots \leq a_4$

Here $\boldsymbol{\sigma}$ is the reciprocity map of class field theory, $\lambda_{\mathbf{E}_\mathcal{D}}$ is the Hecke eigenvalue for $u_{v,t}$ acting on $\mathbf{E}_\mathcal{D}$. It follows from the computations in part one (lemma 4.4.2) that these are elements in $\mathbb{I}[[\Gamma_\mathcal{K}]]$ (the $\lambda_{i,v}(p)$'s can be expressed in terms of the Hecke eigenvalues of $U_v$ on $\mathbf{f}$ and $\xi_{v,1}(p)$ and $\xi_{v,2}$'s in our situation). We omit the precise formulas. We remark that the elements in $I_\mathcal{D}$ all annihilate $E_D$ and that, in fact, $I_D$ is the image of the annihilator of $E_D$ in the abstract Hecke algebra generated by the Hecke operators at unramified primes and primes dividing $p$ as above.
The structure map $\Lambda_\mathcal{D} \to \mathbf{h}_\mathcal{D}/I_\mathcal{D}$ is surjective and we denote $\mathcal{E}_\mathcal{D} \subset \Lambda_\mathcal{D}$ to be kernel of this map so that:

$$\Lambda_\mathcal{D}/\mathcal{E}_\mathcal{D} \xrightarrow{\sim} \mathbf{h}_\mathcal{D}/I_\mathcal{D}.$$

Recall that we have defined $\phi_0$ to be the point on the weight space in subsection 11.3.1 such that the special $L$-value interpolated is $L(f_2, 1)$ where $f_2$ is an ordinary form in our Hida family of parallel weight 2 and trivial neben typus at primes dividing $p$. (In fact this notion is a little bit ambiguous since we might have several $f_2$'s inside the Hida family and what we are going to prove is true for any of such point $\phi_0$).

Recall that in chapter 13 we have chosen a $\boldsymbol{b} \in \mathbb{I}$ such that $\boldsymbol{b}(\phi_0) \neq 0$ and $\boldsymbol{b}1_\mathbf{f}$ is an integral element of Hecke action. Then $\boldsymbol{b}\tilde{\mathcal{L}}_\mathcal{D}^\Sigma$ and $\boldsymbol{b}\tilde{\mathbf{E}}_\mathcal{D}$ are all integral. Recall also that we have defined a finite extension $\mathbb{J}$ of $\mathbb{I}$ and map $j_1$ in 13. We can replace $\mathbb{I}$ by $\mathbb{J}$ and make all the constructions above for $\mathcal{E}_{\mathcal{D},\mathbb{J}}$, $\mathbf{h}_{\mathcal{D},\mathbb{J}}$, $I_{\mathcal{D},\mathbb{J}}$, etc. We will write $\tilde{\mathcal{L}}_{\mathcal{D}^\Sigma}$ for $j_1(\tilde{\mathcal{L}}_\mathcal{D}^\Sigma)$ as well. We have the following theorem which is the analogue of [35, 6.5.4] in our situation:

**Theorem 14.1.1.** *There is a finite normal extension $\mathbb{J}$ of $\mathbb{I}$ such that if $P \subset \Lambda_{\mathcal{D},\mathbb{J}}$ is a height one prime of $\Lambda_{\mathcal{D},\mathbb{J}}$ contained in $\phi_0$ (Recall that in chapter 13 we choose any point on $\mathrm{Spec}\Lambda_{\mathcal{D},\mathbb{J}}$ inducing $\phi_0$ in $\mathrm{Spec}\Lambda_\mathcal{D}$ via $j_1$ defined there) such that $\boldsymbol{b}\tilde{\boldsymbol{E}}_{\mathcal{D},\mathbb{J}}$ is non-zero modulo $P$ (i.e. if the ideal generated by the Fourier coefficient of $\boldsymbol{b}\boldsymbol{E}_\mathcal{D}$ is not contained in $P$), and that $P$ is not a pull back of a height one prime from $\mathbb{J}[[\Gamma_\mathcal{K}^+]]$ then:*

$$ord_P(\mathcal{E}_{\mathcal{D},\mathbb{J}}) \geq ord_P(j_1(\tilde{\mathcal{L}}_\mathcal{D}^\Sigma))$$

*where $j_1$ is the map defined in chapter 13.*

*Proof.* The proof is completely the same as [35, 6.5.4]. Let $\mathbf{F}$ be the $\Lambda_{\mathcal{D},\mathbb{J}}$-adic cuspidal family constructed in chapter 13. Then for some $\beta \in S_2(F)$ and $x \in U(2,2)(\mathbb{A}_F)$ unramified at primes

dividing $p$ we have the Fourier coefficient $c(\beta, x; F)$ is not in $P$. Suppose $P^r \| j_1(\tilde{\mathcal{L}}_{\mathcal{D}}^{\Sigma})$ then the map $\mu : \boldsymbol{h}_{\mathcal{D}} \mapsto \Lambda_{\mathcal{D},\mathbb{J}}$ defined by $\mu(h) = c(\beta, x; h\mathbf{F})/c(\beta, x; \mathbf{F})$ is $\Lambda_{\mathcal{D},\mathbb{J}}$-linear and by theorem 14, for any $h \in \boldsymbol{h}_{\mathcal{D}}$ we have

$$c(\beta, x; h\mathbf{F}) \equiv c(\beta, x; h\mathbf{E}_{\mathcal{D}}) \equiv \lambda_{\mathcal{D}}(h) c(\beta, x; \mathbf{E}_{\mathcal{D}}) \equiv \lambda_{\mathcal{D}}(h) c(\beta, x; \mathbf{F}) (\ mod\ P^r).$$

Here $\lambda_{\mathcal{D}}(h)$ is the Hecke eigenvalue of $h$ for $\mathbf{E}_{\mathcal{D}}$ and note that $\boldsymbol{b} \notin P$. Thus $\mu$ gives

$$\Lambda_{\mathcal{D},\mathbb{J}}/\mathcal{E}_{\mathcal{D},\mathbb{J}}\Lambda_{\mathcal{D},\mathbb{J},P} \twoheadrightarrow \Lambda_{\mathcal{D},\mathbb{J},P}/P^r\Lambda_{\mathcal{D},\mathbb{J},P}.$$

This gives the theorem. □

## 14.2 Galois Representations

### 14.2.1 Galois Theoretic Argument

In this section, for ease of reference we repeat the set-up and certain results from [35, chapter 4], which are used to construct elements in the Selmer group.

Let $G$ be a group and $C$ a ring. $r :\to Aut_C(V)$ a representation of $G$ with $V \simeq C^n$. This can be extended to $r : C[G] \to End_C(V)$. For any $x \in C[G]$, define: $Ch(r, x, T) := det(id - Tr(x)) \in C[T]$.

Let $(V_1, \sigma_1)$ and $(V_2, \sigma_2)$ be two $C$ representations of $G$. Assume both are defined over a local henselian subring $B \subseteq C$, we say $\sigma_1$ and $\sigma_2$ are residually disjoint modulo the maximal ideal $\mathfrak{m}_B$ if there exists $x \in B[G]$ such that $Ch(\sigma_1, x, T)$ mod $\mathfrak{M}_B$ and $Ch(\sigma_2, x, , T)$ mod $\mathfrak{m}_B$ are relatively prime in $\kappa_B[T]$, where $\kappa_B := B/\mathfrak{m}_B$.

Let $H$ be a group with a decomposition $H = G \rtimes \{1, c\}$ with $c \in H$ an element of order two normalizing $G$. For any $C$ representations $(V, r)$ of $G$ we write $r^c$ for the representation defined by $r^c(g) = r(cgc)$ for all $g \in G$.

Polarizations:

Let $\theta : G \to GL_L(V)$ be a representation of $G$ on a vector space $V$ over field $L$ and let $\psi : H \to L^{\times}$

be a character. We assume that $\theta$ satisfies the $\psi$-polarization condition:

$$\theta^c \simeq \psi \otimes \theta^\vee.$$

By a $\psi$-polarization of $\theta$ we mean an $L$-bilinear pairing $\Phi_\theta : V \times V \to L$ such that

$$\Phi_\theta(\theta(g)v, v') = \psi(g)\Phi_\theta(v, \theta^c(g)^{-1}v').$$

Let $\Phi_\theta^t(v, v') := \Phi_\theta(v', v)$, which is another $\psi$-polarization. We say that $\psi$ is compatible with the polarization $\Phi_\theta$ if

$$\Phi_\theta^t = -\psi(c)\Phi_\theta.$$

Suppose that:

(1)$A_0$ is a pro-finite $\mathbf{Z}_p$ algebra and a Krull domain;

(2)$P \subset A_0$ is a height one prime and $A = \hat{A}_{0,P}$ is the completion of the localization of $A_0$ at $P$. This is a DVR.

(3) $R_0$ is local reduced finite $A_0$-algebra;

(4) $Q \subset R_0$ is prime such that $Q \cap A_0 = P$ and $R = \hat{R_{0,Q}}$;

(5) there exist ideals $J_0 \subset A_0$ and $I_0 \subset R_0$ such that $I_0 \cap A_0 = J_0, A_0/J_0 = R_0/I_0, J = J_0A, I = I_0R, J_0 = J \cap A_0$ and $I_0 = I \cap R_0$;

(6) $G$ and $H$ are pro-finite groups; we have subgroups $D_i \subset G$ for $i = 1, \cdots, d$.

the set up: suppose we have the following data:

(1) a continuous character $\nu : H \to A_0^\times$;

(2) a continuous character $\xi : G \to A_0^\times$ such that $\bar{\chi} \neq \bar{\nu}\bar{\chi}^{-c}$; Let $\chi' := \nu\chi^{-c}$;

(3) a representation $\rho : G \to Aut_A(V), V \simeq A^n$, which is a base change from a representation over $A_0$, such that:

$$a.\rho^c \simeq \rho^\vee \otimes \nu,$$

$$\bar{\rho} \text{ is absolutely irreducible ,}$$

$$\rho \text{ is residually disjoint from } \chi \text{ and } \chi';$$

(4) a representation $\sigma : G \to Aut_{R \otimes_A F}(M), M \simeq (R \otimes_A F)^m$ with $m = n + 2$, which is defined over

the image of $R_0$ in $R$, such that:

*a.* $\sigma^c \simeq \sigma^\vee \otimes \nu$,

*b.* $tr\sigma(g) \in R$ for all $g \in G$ ,

*c.* for any $v \in M$,$\sigma(R[G])v$ is a finitely-generated $R$-module

(5) a proper ideal $I \subset R$ such that $J := A \cap I \neq 0$, the natural map $A/J \to R/I$ is an isomorphism, and

$$tr\sigma(g) \equiv \chi'(g) + tr\rho(g) + \chi(g) \ mod\ I$$

for all $g \in G$.

(6) $\rho$ is irreducible and $\nu$ is compatible with $\rho$.

(7) (local conditions for $\rho$) For each $v|p$ there is a $G_v$ stable sub $A_0$ module $V_{0,v}^+ \subset V_0$ such that $V_{0,v}^+$ and $V_{0,v}^- := V_{0,v}/V_{0,v}^+$ are free $A_0$ modules.

(8) (local conditions for $\sigma$). For each $v|p$ there is a $G_v$-stable sub-$R \otimes_A F$-module $M_v^+ \subseteq M$ such that $M_v^+$ and $M_v^- := M/M_v^+$ are free $R \otimes_A F$ modules.

(9) (compatibility with the congruence condition) Assume that for all $x \in R[G_v]$, we have congruence relation:

$$Ch(M_v^+, x, T) \equiv Ch(V_v^+, x, T)(1 - T\chi(x)) \ mod\ I$$

(then we automatically have:

$$Ch(M_v^-, x, T) \equiv Ch(V_v^-, x, T)(1 - T\chi'(x)) \ mod\ I$$

(10) For each $F$-algebra homomorphism $\lambda : R \otimes_A F \to K$, $K$ a finite field extension of $F$, the representation $\sigma_\lambda : G \to GL_m(M \otimes_{R\otimes F} K)$ obtained from $\sigma$ via $\lambda$ is either absolutely irreducible or contains an absolutely irreducible two-dimensional sub $K$-representation $\sigma'_\lambda$ such that $tr\sigma'_\lambda(g) \equiv \chi(g) + \chi'(g) \ mod\ I$.

One defines the Selmer groups $\mathbf{X}_H(\chi'/\chi) := ker\{H^1(G, A_0^*(\chi'/\chi)) \to H^1(D, A_0^*(\chi'\chi))\}^*$. and $\mathbf{X}_G(\rho_0 \otimes \chi^{-1}) := ker\{H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})) \to H^1(D, V_0^- \otimes_{A_0} A_0^*(\chi^{-1}))\}^*$

**Proposition 14.2.1.** *Under the above assumptions, if* $ord_P(Ch_H(\chi'/\chi)) = 0$ *then*

$$ord_P(Ch_G(\rho_0 \otimes \chi^{-1}) \geq ord_P(J).$$

This is just [35, Corollary 4.5.7].

We record here an easy lemma about Fitting ideals and characteristic ideals which will be useful in proving main conjectures.

**Lemma 14.2.1.** *Let* $A$ *be a Krull domain and* $T$ *an* $A$*-module. Suppose* $f \in A$ *is such that for any height one primes* $P$ *of* $A$, $\mathrm{ord}_P(\mathrm{Fitt}_A T) \geq \mathrm{ord}_P(f)$, *then* $\mathrm{Char}_A(T) \subset (f)$.

*Proof.* for any $g \in \mathrm{char}_A(T)$ the assumption and the definition for characteristic ideals ensures that for any height 1 prime $P$, $\mathrm{ord}_P(\frac{g}{f}) \geq 0$. Since $A$ is normal this implies $\frac{g}{f} \in A$. Thus $g \in (f)$. □

### 14.2.2 Galois Representations

Now we are going to apply the result in the last subsection to our situation. First we define a semi-simple representation

$$\boldsymbol{\rho}_{\mathcal{D}} := \boldsymbol{\sigma}_{\boldsymbol{\psi}}^c \epsilon^{-3} \oplus (\rho_{\boldsymbol{f}} \otimes \boldsymbol{\xi}^{-c} \boldsymbol{\psi}^c \epsilon^{-2}) \oplus \epsilon^{-1} det \rho_{\boldsymbol{f}} \boldsymbol{\sigma}_{\boldsymbol{\xi}'}^{-1} \boldsymbol{\sigma}_{\boldsymbol{\psi}}^c.$$

Recall that here $\boldsymbol{\sigma}$ means the reciprocity map. We will see that this is the Galois representation associated to the Eisenstein family constructed in the next chapter.

On the other hand, recall that we have fixed some prime to $p$ level $K^p$ and let $K = \prod_{v|p} K_v^0 K^p$ by an argument completely the same as [35] Proposition 7.2.1 there exists a pseudo-representation $T_{K^p}^{\Sigma} : G_{\mathcal{K}} \to \mathbf{h}_{\mathcal{D}}$ such that for each for each automorphic representation on $GU(2,2)_{/F}$ of weight $\underline{k}$ and $\pi_f^{K^p} \neq 0$ then:

$$\mathrm{tr}(R_p(\pi)) = \lambda_{\pi}^{ord} \circ T_{K^p}^{\Sigma}.$$

Here $\lambda_{\pi}^{ord}$ is the character of $\mathbf{h}_{\mathcal{D}}$ determined by the Hecke eigenvalues for $\pi$.

As in [35, 7.3] we let $T_{\mathcal{D}}$ be the pseudo-character $T_{K^p}^{\Sigma}$. We have defined $\boldsymbol{h}_{\mathcal{D},\mathbb{J}}$ and let $B_{\mathcal{D}} := \mathbf{h}_{\mathcal{D},\mathbb{J}} \otimes_{\Lambda_{\mathcal{D},\mathbb{J}}} F_{\Lambda_{\mathcal{D},\mathbb{J}}}$. Let $\mathbf{T}_{\mathcal{D}}$ and $\mathbf{T}_{\mathcal{D},\mathbb{J}}$ be the localization of $\mathbf{h}_{\mathcal{D}}$ and $\mathbf{h}_{\mathcal{D},\mathbb{J}}$ at the maximal ideal containing the Eisenstein ideal.

For any prime $v|p$ of $F$ we let $T^+_{\mathbf{f},v} \subseteq T_{\mathbf{f}}$ be the rank one $\mathbb{J}$-summand of $T_{\mathbf{f}}$ that is $G_v$-stable. Given a height one prime $P$ of $\Lambda_{\mathcal{D},\mathbb{J}}$ containing $\mathcal{E}_{\mathcal{D},\mathbb{J}}$ let:

- $H := G_{F,\Sigma}, G := G_{\mathcal{K},\Sigma}, c =$ the usual complex conjugation;
- $A_0 := \Lambda_{\mathcal{D},\mathbb{J}}, A := \hat{\Lambda}_{\mathcal{D},\mathbb{J},P}$;
- $J_0 := \mathcal{E}_{\mathcal{D},\mathbb{J}}, J := \mathcal{E}_{\mathcal{D},\mathbb{J}}A$;
- $R_0 := \boldsymbol{T}_{\mathcal{D},\mathbb{J}}, I_0 := I_{\mathcal{D},\mathbb{J}}$;
- $Q \subset R_0$ is the inverse image of $P$ *mod* $\mathcal{E}_{\mathcal{D},\mathbb{J}}$ under $\boldsymbol{T}_{\mathcal{D},\mathbb{J}} \to \boldsymbol{T}_{\mathcal{D},\mathbb{J}}/I_{\mathcal{D},\mathbb{J}} = \Lambda_{\mathcal{D},\mathbb{J}}/\mathcal{E}_{\mathcal{D},\mathbb{J}}$ ;
- $R := \hat{\boldsymbol{T}}_{\mathcal{D},\mathbb{J},Q}, I := I_{\mathcal{D},\mathbb{J}}R$;
- $V_0; = T_{\mathcal{O}} \otimes_{\mathbb{J}} \Lambda_{\mathcal{D},\mathbb{J}}, \rho := \rho_{\mathbf{f}} \otimes \boldsymbol{\sigma}_{\xi}^{-c}\boldsymbol{\sigma}_{\psi}^{c}\epsilon^{-2}$;
- for any $v|p$, $V^+_{0,v} := T^+_{\mathbf{f},v} \otimes_{\mathbb{I}} A_0, V^-_{0,v} := (T_{\mathbf{f}}/T^+_{\mathbf{f},v}) \otimes_{A_0} A$;
- $V = V_0 \otimes_{A_0} A, \rho = \rho_0 \otimes_{A_0} A, V^{\pm}_v := V^{\pm}_{0,v} \otimes_{A_0} A$;
- $\chi := \epsilon^{-1} \det \rho_{\boldsymbol{f}}\boldsymbol{\sigma}_{\xi'}^{-1}\boldsymbol{\sigma}_{\psi}^{c}, \nu := \boldsymbol{\sigma}_{\psi'}\boldsymbol{\sigma}_{\xi'}^{-1}\epsilon^{-4}$;
- $\chi' := \boldsymbol{\sigma}_{\psi}^{c}\epsilon^{-3}$ so $\chi' = \nu\chi^{-c}$;
- $M := (R \otimes_A F_A)^4$, $F_A$ is the field of fractions of $A$;
- $\sigma$ is the representation on $M$ obtained from $T_{\mathcal{D}}$ in the same way as [35, 7.3].

Let $\mathcal{T} := (T_{\mathbf{f}} \otimes_{\mathbb{J}} \mathbb{J}[[\Gamma_{\mathcal{K}}]])(\boldsymbol{\varepsilon}_{\mathcal{K}})$ and $\mathcal{T}^+_v := (T^+_{\mathbf{f},v} \otimes_{\mathbb{J}} \mathbb{J}[[\Gamma_{\mathcal{K}}]])(\boldsymbol{\varepsilon}_{\mathcal{K}})$ for each $v|p$. Let $Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}}) \subset \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ be the characteristic ideal of the dual Selmer group $X^{\Sigma}_{\mathcal{K}}(\mathcal{T}, \mathcal{T}^+_v|_{v|p})$.

**Theorem 14.2.1.** *Suppose $\mathbb{I}$ is an integrally closed domain. Let $P_0 \subset \mathbb{J}[[\Gamma_{\mathcal{K}}]]$ be a height one prime that is not a pullback of one of $\mathbb{J}[[\Gamma^+_{\mathcal{K}}]]$ and let $P = P_0\Lambda_{\mathcal{D}}$ be the height one prime of $\Lambda_{\mathcal{D}}$ it generates. Suppose also that:*

$$V^+ \oplus A(\chi) \text{ and } V^- \oplus A(\chi') \text{ modulo } P \text{ do not have common irreducible pieces .} \tag{14.1}$$

*Then*

$$ord_{P_0}(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\mathbf{f}} \otimes \boldsymbol{\varepsilon}_{\mathcal{K}})) \geq ord_P(\mathcal{E}_{\mathcal{D},\mathbb{J}}).$$

*Proof.* One just applies proposition 14.2.1. The condition (10) there is guaranteed by an argument similar to [35] theorem 7.3.1: we use the modularity lifting results in [40] for ordinary Galois representations satisfying **(*irred*)** and **(*dist*)** and Harris's result that there is no (CAP) forms when the weight $\underline{k}$ is sufficiently regular. We also use the main conjecture for totally real field $F$ proven in [43] to conclude that $\mathrm{ord}_P(Ch_H(\chi'/\chi)) = 0$. (Since the $p$-adic $L$-function for a Dirichlet character involves only the cyclotomic direction and is non-zero, so it is not in $P$. By [43] we know that the characteristic ideal is controlled by this $p$-adic $L$-functions and thus is not contained in $P$. □

## 14.3 Proof of the Main Result

Now for $f$ a Hilbert modular form with trivial nebentypus. Recall that we have define **(*irred*)** and **(*dist*)** in the introduction. Then:

**Theorem 14.3.1.** *Let $p$ be a rational odd prime that splits completely in $F$. Let $f$ be a Hilbert modular form over $F$ of parallel weight 2 and trivial character. Suppose:*
*(i) $f$ is ordinary at all primes of $F$ dividing $p$;*
*(ii)* **(*irred*)** *and* **(*dist*)** *hold for $\rho_f$.*
*If the central critical value $L(f,1) = 0$, then the Selmer group $H^1_f(F, \rho_f^*)$ is infinite.*

**Remark 14.3.1.** *The reason for assuming that $p$ splits completely is that we only have the various non-vanishing modulo $p$-results in this case. The computation part of the argument could be done more generally.*

*Proof.* We only need to prove the theorem in the case when the root number for $f$ is $+1$ since otherwise it is a well known result of Nekovar [29], (which crucially uses the work of S. Zhang).

First suppose that $d = [F : \mathbb{Q}]$ is even. Then we choose an imaginary quadratic extension $\mathcal{K}$ of $F$ so that $\mathcal{K}/F$ is split at all primes at which $f$ is ramified and such that $L(f, \chi_{\mathcal{K}/F}, 1) \neq 0$ where $\chi_{\mathcal{K}/F}$ is the quadratic character of $\mathbb{A}_F^\times$ associated to $\mathcal{K}/F$. This is possible by a well known result of Waldspurger [41], [42]. Then the $S(1)$ defined in [4, p123] consists of exactly all the infinite places, and since $d$ is even we are in the definite case there.

The (normalized ordinary) form $f$ belongs to a Hida family. (This is well known to experts. One first considers $f$ as an ordinary $p$-adic modular form and then from [10] the specialization map from the space of Hida families defined in *loc.cit* to ordinary $p$-adic modular forms of given weight and

neben-typus is an isomorphism.) Now that we have chosen $\mathcal{K}$, recall we have defined the point $\phi_0$ in $\mathrm{Spec}\Lambda_{\mathcal{D}}$ (subsection 11.3.1). We remark that later we will use $\phi_0$ to denote the point (or prime) not only of $\mathrm{Spec}\Lambda_{\mathcal{D}}$ but also subspaces of it.

Now we do not know the Gorenstein properties for the Hecke rings associated with $\mathbf{f}$ in general, so we have to use $1_{\mathbf{f}}$ instead of $\ell_{\mathbf{f}}$ everywhere and the non-integral $p$-adic $L$-functions $\tilde{\mathcal{L}}^{\Sigma}_{\mathbf{f},\mathcal{K},1}$ (in $F_{\mathbb{J}} \otimes_{\mathbb{J}} \mathbb{J}[[\Gamma_{\mathcal{K}}]]$ actually) and non-integral Klingen Eisenstein serie $\tilde{\mathbf{E}}_{\mathcal{D}}$. Suppose $\tilde{\mathcal{L}}^{\Sigma}_{\mathbf{f},\mathcal{K},1} = \frac{h}{g}$, it follows from the definition for $1_{\mathbf{f}}$ in chapter 11 and the fact that the congruence number for $f_2$ is finite that we may choose $g \in \mathbb{J}$ so that $g(\phi_0) \neq 0$. Start with the 1-dimensional family of cylotomic twists of $f$, i.e. the subspace $\mathrm{Spec}A[[\Gamma^+_{\mathcal{K}}]]$ with $\Lambda_{\mathcal{D},\mathbb{J}} \to A[[\Gamma^+_{\mathcal{K}}]]$ where the map $\mathbb{J} \to A$ is the specialization map at $f$. Since $L(f,1) = 0$,$h(\phi_0) = 0$. Then there is a height 1 prime $P_0$ of the 1-dimensional space contained in $\phi_0$ and containing the image of $h$. Here we again note that our construction did not include $\phi_0$ as an interpolation point of $\tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}}$, but by the interpolation property we know that our $\tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}}$ is the same as Hida's ([11]) up to Euler factors at $\Sigma$ at a sub-family containing the above 1-dimensional family.

Now we consider the specialization step by step. At each step the Iwasawa algebra is a Krull domain. Suppose we have 2 Iwasawa algebras $\Lambda_1$ and $\Lambda_2$ such that both are quotients of $\Lambda_{\mathcal{D},\mathbb{J}}$ and the corresponding closed spaces contain the point $\phi_0$. Suppose $\mathrm{Spec}\Lambda_1 \hookrightarrow \mathrm{Spec}\Lambda_2$ where $\Lambda_2$ has one more variable than $\Lambda_1$, i.e. $\Lambda_1 = \Lambda_2/x\Lambda_2$ for some variable $x$. If $P_1$ is a height 1 prime of $\Lambda_1$ passing through $\phi_0$ and containing the image of $h$ in $\Lambda_1$ then we can find $P_2$ a height one prime of $\Lambda_2$ also contained in $\phi_0$ and contains the image of $h$ in $\Lambda_2$ such that $\mathrm{Supp}P_1 \subset \mathrm{Supp}P_2$ under $\mathrm{Spec}\Lambda_1 \hookrightarrow \mathrm{Spec}\Lambda_2$. Finally we found some $P$ a height one prime of $\Lambda_{\mathcal{D}}$ contained in $\phi_0$ and containing $h$. Note also that $P$ does not contain $g$ since $g(\phi_0) \neq 0$. In chapter 11 we have seen that (NV1) is satisfied in our situation and thus $h$ is not contained in any height one prime of $\mathbb{J}[[\Gamma^+_{\mathcal{K}}]]$ contained in $\phi_0$. So $P$ is not the pullback of a height one prime of $\mathbb{J}[[\Gamma^+_{\mathcal{K}}]]$. Then proposition 12.3.2 (i) and the argument of theorem 14.1.1 involving the construction of chapter 13 gives:

$$1 \leq \mathrm{ord}_P \tilde{\mathcal{L}}^{\Sigma}_{\mathcal{D}} \leq \mathrm{ord}_P(\mathcal{E}_{\mathcal{D},\mathbb{J}}).$$

By theorem 14.2.1:

$$\mathrm{ord}_P \mathrm{Fitt}^{\Sigma}_{\mathbf{f},\mathcal{K},1} \geq 1.$$

Then we need to specialize the variables back step by step to prove the theorem. Using the control theorem for the Selmer groups (results in subsection 6.4.1) we have at each step: we get $\mathrm{ord}_{P_i}\mathrm{Fitt}_{\mathcal{K}}^{\Sigma} \geq 1$ here the $P_i$ and the Selmer modules are interpreted in the context of each step. Finally we specialize to the point $\phi_0$ to get that the $\Sigma$-primitive Selmer group over $\mathcal{K}$ is infinity. But this implies that $H_f^1(F, \rho_f^*)$ is itself infinity since the local Euler factors $L_\Sigma(f_2, 1)$ is non zero. However by lemma 6.3.1 this Selmer group is the product of Selmer groups for $f$ and $f \otimes \chi_{\mathcal{K}}$. By [46] and our choice of $\mathcal{K}$, we know that the Selmer group for $f \otimes \chi_{\mathcal{K}}$ is finite. So our theorem is true.

Finally we assume $d$ is odd. Then again by Waldspurger we can find a real quadratic character $\chi_{F,/F}$ such that $F'$ is split at all primes at which $f$ is ramified and $L(f, \chi_{F'/F}, 1) \neq 0$. We consider $f_{F'}$ the base change of $f$ to $F'$. Then $[F' : \mathbb{Q}]$ is even and we deduce that at least one of $H_f^1(F, \rho_f^*)$ and $H_f^1(F, \rho_f^* \otimes \chi_{F'/F})$ is infinite. But by [46] we know that $H_f^1(F, \rho_f^* \otimes \chi_{F'/F})$ is finite. So $H_f^1(F, \rho_f^*)$ must be infinite. $\square$

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