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Space of Filtrations and Singularities

Qi, Lu

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SPACE OF FILTRATIONS AND SINGULARITIES

LU QI

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Abstract

In this thesis, we systematically study the structure of the space of saturated filtrations on a Noetherian local domain, and consider its application in commutative algebra, K -stability and birational geometry. Introduced in the joint work with Harold Blum and Yuchen Liu, saturation can be used to characterize when two filtrations have equal multiplicities, generalizing Rees' theorem concerning Hilbert-Samuel multiplicity and integral closure of ideals. Next, we present the joint work with Blum and Liu on the convexity of multiplicities along geodesics. This result allows us to prove the convexity of volumes on a single simplex in the valuation space, generalizing the result of Boucksom, Favre and Jonsson. Moreover, it gives a new proof to the uniqueness of normalized volume minimizers originally proven by Xu and Zhuang, which is part of the Stable Degeneration Conjecture. In order to justify the term geodesic, we introduce a metric d_1 , mimicking the Darvas metric in complex geometry and global non-archimedean geometry. The constructions endow the space of saturated filtrations with the structure of a geodesic metric space. We also define several other topologies on the space. As an application, we show some continuity properties of the log canonical threshold function, in the spirit of Kollár-Demailly.

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Chapter 1

Introduction

1.1 Background

1.1.1 Kähler-Einstein metric and K -stability

A fundamental problem in complex geometry is the existence of some canonical metrics on Kähler manifolds. As an important class of such canonical metrics, Kähler-Einstein metrics are those whose Ricci curvature is proportional to the Kähler class.

The Yau-Tian-Donaldson Conjecture for Fano manifolds predicts that a Fano manifold X admits a Kähler-Einstein metric if and only if X is K -polystable, which was proved by Chen-Donaldson-Sun, and Tian [CDS15, Tia15].

More effort was later put into the understanding of the singular and algebraic case. After the work of [Fuj19, Li17, BJ20, LXZ22], we now have a clear picture of the theory of algebraic K -stability for Fano varieties. Combined with the work [BBJ21, LTW22], we now know that the Yau-Tian-Donaldson Conjecture holds for projective Fano varieties.

By the recent work of Blum, C. Li, Y. Liu, Xu, Zhuang and many others [Blu18b, Li18, LL19, LX20, XZ21], besides its own interest, the local stability theory for klt singularities also plays an important role in the global theory.

1.1.2 Stable degeneration of singularities and local stability

The local stability theory was initiated in [Li18]. The theory centers around minimizing the normalized volume function

$$\widehat{\text{vol}} : \text{Val}_{X,x} \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$$

defined on the space of valuations on X centered at x . See Section 2.3 for the definition. The complete conjectural picture, the *Stable Degeneration Conjecture* proposed by in [Li18, LX18], has now been proven after the work of Blum, Li, Liu, Wang, Xu and Zhuang. Here we state the following version of the theorem from [Xu18]. (See [LXZ22] for a more comprehensive version.)

Theorem 1.1.1 (Stable degeneration theorem). *[Li18, Blu18a, LX18, Xu20, XZ21, LWX21, XZ22] Let $x \in (X, D)$ be a klt singularity over an algebraically closed field of characteristic 0. Then there exists a unique minimizer v of the function $\widehat{\text{vol}}$ up to scaling. Moreover, v is quasi-monomial, with a finitely generated associated graded ring $R_0 := \text{gr}_v(R)$, which induces a degeneration to a K -semistable log Fano cone singularity.*

We will not use definitions related to log Fano cone singularities, so we refer to [LWX21]. See Section 2 for the other definitions.

By the following result of Liu [Liu18], graded sequences of ideals, or filtrations, appear naturally in the study of the local volumes.

Theorem 1.1.2. *[Liu18, Theorem 27] If $x \in (X, D)$ is a klt singularity of dimension n , then*

$$\inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) = \inf_{\mathfrak{a} \text{ m-primary}} \text{lct}(\mathfrak{a})^n \cdot e(\mathfrak{a}) = \inf_{\mathfrak{a}_\bullet \in \text{Fil}_{X,x}} \text{lct}(\mathfrak{a}_\bullet)^n \cdot e(\mathfrak{a}_\bullet).$$

1.2 Main results

This dissertation is centered around $\mathfrak{m}$ -filtrations with positive multiplicities on a fairly general class of local domains, and concerns the applications in commutative algebra and the theory of singularities. The aim is to set up a general framework to study the structure, in particular, the metric geometry the space of such filtrations.

We will work with a Noetherian local domain $(R, \mathfrak{m})$ which is analytically irreducible, that is, the $\mathfrak{m}$ -adic completion $\hat{R}$ is again a domain. Some examples are the formal power series ring $\mathbb{k}[[x_1, \dots, x_n]]$ over an arbitrary field $\mathbb{k}$, the ring of convergent power series $\mathbb{C}\{\{x_1, \dots, x_n\}\}$ over the complex numbers, and the local ring $\mathcal{O}_{X,x}$ of a normal variety X over an arbitrary field $\mathbb{k}$ at a point $x \in X$.

An $\mathfrak{m}$ -filtration on R is a sequence of $\mathfrak{m}$ -primary ideals graded by $\mathbb{R}_{\geq 0}$ in the term of [ELS03] with certain continuity condition. To be more precise, an $\mathfrak{m}$ -filtration on R is a collection $\mathfrak{a}_\bullet = (\mathfrak{a}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ of $\mathfrak{m}$ -primary ideals of R that is decreasing, graded, and left continuous. The latter three conditions mean $\mathfrak{a}_\lambda \subset \mathfrak{a}_\mu$ when $\lambda > \mu$, $\mathfrak{a}_\lambda \cdot \mathfrak{a}_\mu \subset \mathfrak{a}_{\lambda+\mu}$, and $\mathfrak{a}_\lambda = \mathfrak{a}_{\lambda-\epsilon}$ when $0 < \epsilon \ll 1$, respectively.

An important invariant of an $\mathfrak{m}$ -filtration is its *multiplicity*, introduced by Ein, Lazarsfeld, and Smith [ELS03], which is a generalization of the Hilbert-Samuel multiplicity of an $\mathfrak{m}$ -primary ideal. The multiplicity of $\mathfrak{a}_\bullet$ defined by

$$e(\mathfrak{a}_\bullet) := \lim_{\mathbb{Z}_{>0} \ni m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!} = \lim_{\mathbb{Z}_{>0} \ni m \rightarrow \infty} \frac{e(\mathfrak{a}_m)}{m^n},$$

where the existence of the above limit and the equality were proven in increasing generality by [ELS03, Mus02, LM09, Cut13, Cut14]. We will denote the set of all $\mathfrak{m}$ -filtrations on R with positive multiplicity by $\text{Fil}_{R,\mathfrak{m}}$.

1.2.1 Saturation of a filtration and Rees' Theorem

Theorem A and Corollary B below are the results of [BLQ24] joint with Harold Blum and Yuchen Liu. We thank them for allowing the results to be reproduced here.

A theorem of Rees [Ree61] states that for two $\mathfrak{m}$ -primary ideals $\mathfrak{a} \subset \mathfrak{b}$, the following are equivalent:

1. $e(\mathfrak{a}) = e(\mathfrak{b})$.
2. $\overline{\bigoplus_{m \in \mathbb{N}} \mathfrak{a}^m} = \overline{\bigoplus_{m \in \mathbb{N}} \mathfrak{b}^m}$.
3. $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$.

The symbol $\bar{\cdot}$ in 2. denotes the algebraic closure in $R[t]$, while in 3. it denotes the integral closure of an ideal. The equivalence between 2. and 3. follows from definitions.

It is natural to ask for a generalization of the above result for $\mathfrak{m}$ -filtrations $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$. In [Cut21], Cutkosky studies the problem and proved some partial results. But the statement in general does not give an equivalence between the conditions.

To remedy this issue, we introduce the *saturation* of an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$, which is the $\tilde{\mathfrak{a}}_\bullet$ defined by

$$\tilde{\mathfrak{a}}_\lambda := \{f \in \mathfrak{m} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{DivVal}_{R, \mathfrak{m}}\}$$

for each $\lambda \in \mathbb{R}_{>0}$. We say $\mathfrak{a}_\bullet$ is *saturated* if $\mathfrak{a}_\bullet = \tilde{\mathfrak{a}}_\bullet$. The definition is analogous to the valuative definition of the integral closure of an ideal, see Section 2.4.1.

Using this notion, we prove a version of Rees' theorem for filtrations.

Theorem A (Rees' Theorem). *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian local domain. For $\mathfrak{m}$ -filtrations $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$, $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$ if and only if $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_\bullet$.*

By work of Teissier [Tei78], Rees and Sharp [RS78], and D. Katz [Kat88] in increasing generality, for two $\mathfrak{m}$ -primary ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R ,

$$e(\mathfrak{a}\mathfrak{b})^{1/n} \leq e(\mathfrak{a})^{1/n} + e(\mathfrak{b})^{1/n}$$

and the equality holds if and only if there exists $c, d \in \mathbb{Z}_{>0}$ such that $\overline{\mathfrak{a}^c} = \overline{\mathfrak{b}^d}$.

For two $\mathfrak{m}$ -filtrations $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$, let $\mathfrak{a}_\bullet \mathfrak{b}_\bullet$ denote the $\mathfrak{m}$ -filtration $(\mathfrak{a}_\lambda \mathfrak{b}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$. Using the saturation of a filtration, we characterize when the Minkowski inequality for filtrations is an equality.

Corollary B (Minkowski Inequality). *For $\mathfrak{m}$ -filtrations $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ with positive multiplicity,*

$$e(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)^{1/n} \leq e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n} \tag{1.1}$$

and the equality holds if and only if $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_{c\bullet}$ for some $c \in \mathbb{R}_{>0}$.

The inequality statement of the above corollary is not new and is due to Mustața [Mus02], Kaveh and Khovanskii [KK14], and Cutkosky [Cut15] in increasing levels of generality. In the equality statement, the forward implication follows easily from [Cut21, Theorem 10.3] and the definition of the saturation, while the reverse implication relies on Theorem A.

Lattice structure

A classical result in ring theory is that the semiring of all ideals of a ring R with the partial order by inclusion forms a modular lattice, where meet is given by intersection and join is given by sum. We denote the lattice of all ideals (all $\mathfrak{m}$ -primary ideals when $(R, \mathfrak{m})$ is a local ring, resp.) by $\mathcal{I}_R$ ($\mathcal{I}_{R, \mathfrak{m}}$, resp.).

The spaces of filtrations are equipped with a natural partial order by inclusion, that is, $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ if and only if $\mathfrak{a}_\lambda \subset \mathfrak{b}_\lambda$ for any $\lambda \in \mathbb{R}_{>0}$. One can similarly define the

intersection of two filtration termwisely, and in Definition-Lemma 3.1.3 (Definition-Lemma 3.5.5, resp.), we define the *join* $\vee$ (*saturated join* $\vee_s$, resp.), which is the analogue of the sum of two ideals. The next theorem asserts that these operations define natural lattice structures on the spaces of $\mathfrak{m}$ -filtrations. Moreover, the lattice of saturated filtrations is distributive.

Theorem C (Lattice structure). *Let $(R, \mathfrak{m})$ be a Noetherian local domain that is analytically irreducible. Then*

1. *the set of all $\mathfrak{m}$ -filtrations with the partial order by inclusion is a lattice, where meet is given by $\cap$ and join is given by $\vee$, and there is an injective join morphism from $\mathcal{I}_{R, \mathfrak{m}}$ to its sub-lattice $(\text{Fil}_{R, \mathfrak{m}}, \subset, \cap, \vee)$,*
2. *$(\text{Fil}_{R, \mathfrak{m}}^s, \subset, \cap, \vee_s)$ is a distributive lattice. Moreover, $\mathfrak{a} \mapsto \widetilde{\mathfrak{a}}^\bullet$ is an injective join morphism $\mathcal{I}_{R, \mathfrak{m}}^{\text{ic}} \rightarrow \text{Fil}_{R, \mathfrak{m}}^s$, and the saturation $\text{Fil}_{R, \mathfrak{m}} \rightarrow \text{Fil}_{R, \mathfrak{m}}^s$ is a surjective join morphism.*

Here $\mathcal{I}_{R, \mathfrak{m}}^{\text{ic}}$ is the set of integrally closed $\mathfrak{m}$ -primary ideals. For definitions related to lattices, see Section 2.4.3.

Theorem C together with Theorem G can be viewed as a higher-dimensional analogue of the results of Favre and Jonsson [FJ04].

1.2.2 Convexity of multiplicities along geodesics

The main results in this section are the results of [BLQ24] joint with Harold Blum and Yuchen Liu. We thank them for allowing the results to be reproduced here.

Given two $\mathfrak{m}$ -filtrations $\mathfrak{a}_{\bullet, 0}$ and $\mathfrak{a}_{\bullet, 1}$, we define a segment of $\mathfrak{m}$ -filtrations $(\mathfrak{a}_{\bullet, t})_{t \in [0, 1]}$ interpolating between $\mathfrak{a}_{\bullet, 0}$ and $\mathfrak{a}_{\bullet, 1}$ by setting

$$\mathfrak{a}_{\lambda, t} := \sum_{\lambda = (1-t)\mu + t\nu} \mathfrak{a}_{\mu, 0} \cap \mathfrak{a}_{\nu, 1}. \quad (1.2)$$

We call $(\mathbf{a}_{\bullet,t})_{t \in [0,1]}$ the *geodesic* (the term will be justified below in Theorem G) between $\mathbf{a}_{\bullet,0}$ and $\mathbf{a}_{\bullet,1}$, which is the local analogue of the geodesic between two filtrations of the section ring of a polarized variety studied by Blum, Liu, Xu and Zhuang and Reboulet [BLXZ23, Reb23]. This definition is also related to a construction in [XZ21].

In [BLXZ23], it was shown that several non-Archimedean functionals from the theory of K-stability are strictly convex along geodesics in the global setting. In a similar spirit, we prove a convexity result in the local setting for the multiplicity along geodesics.

Theorem D. *Let $(R, \mathfrak{m})$ be an analytically irreducible Noetherian local domain. Assume that R contains a field. If $\mathbf{a}_{\bullet,0}$ and $\mathbf{a}_{\bullet,1}$ are $\mathfrak{m}$ -filtrations with positive multiplicity, then the function $E(t) : [0, 1] \rightarrow \mathbb{R}$ defined by $E(t) := e(\mathbf{a}_{\bullet,t})$ satisfies the following properties:*

1. $E(t)$ is smooth;
2. $E(t)^{-1/n}$ is concave, meaning

$$E(t)^{-1/n} \geq (1-t)E(0)^{-1/n} + tE(1)^{-1/n} \quad \text{for all } t \in [0, 1];$$

3. $E(t)^{-1/n}$ is linear if and only if $\tilde{\mathbf{a}}_{\bullet,0} = \tilde{\mathbf{a}}_{c\bullet,1}$ for some $c \in \mathbb{R}_{>0}$.

The term smooth here means $E(t)$ extends to a C^∞ function on $(-\epsilon, 1 + \epsilon)$ for some $\epsilon > 0$.

The proof of Theorem D is inspired by a related argument in the global setting [BLXZ23] and relies on constructing a measure on $\mathbb{R}^2$ that encodes the multiplicities of the filtrations along the geodesic. In the special case when $\mathbf{a}_{\bullet,0}$ is the $\mathfrak{m}$ -filtration of a valuation minimizing the normalized volume function over a klt singularity, the proof of [XZ21] can be used to show Theorem D(2). Theorem D removes these strong

restrictions and is proven without the theory of K-stability for valuations introduced in [XZ21, Section 3.1].

Applications

Theorem D can be applied to show that the volume of a valuation is strictly log convex on simplices of quasi-monomial valuations in the valuation space of $(R, \mathfrak{m})$. This gives an affirmative answer to a question of Li, Liu and Xu [LLX20, Question 6.23]. We note that the volume was previously shown to be Lipschitz continuous on such a simplex by work of Boucksom, Favre and Jonsson [BFJ14, Corollary D].

Corollary E (Convexity of volume). *Assume R contains a field. Let $\eta \in (Y, D_1 + \cdots + D_r)$ be a log smooth birational model of $(R, \mathfrak{m})$. For any $\alpha, \beta \in \mathbb{R}_{>0}^r$ and $t \in (0, 1)$,*

$$\text{vol}(v_{(1-t)\alpha+t\beta})^{-1/n} \geq (1-t)\text{vol}(v_\alpha)^{-1/n} + t\text{vol}(v_\beta)^{-1/n}$$

and equality holds if and only if $\alpha = c\beta$ for some $c \in \mathbb{R}_{>0}$.

Using Theorem D, we revisit one part of Theorem 1.1.1 that the minimizer of the normalized volume function is unique up to scaling, which is due to Xu and Zhuang [XZ21]. (The uniqueness was also previously proven in [LX18] under the assumption that the minimizer has finitely generated associated ring. The latter was recently shown in [XZ22].) In particular, we give a proof of the uniqueness result independent of the theory of K-semistability for valuations developed in [XZ21].

Corollary F (Uniqueness of minimizer). *If $x \in (X, D)$ is a klt singularity defined over an algebraically closed field of characteristic 0, then any minimizer of $\widehat{\text{vol}}_{X,D,x}$ is unique up to scaling.*

The new proof of Corollary F takes the following direct approach. Fix two valuations v_0 and v_1 that minimize $\widehat{\text{vol}}_{X,D,x}$ and consider the geodesic $(\mathbf{a}_{\bullet,t})_{t \in [0,1]}$ between

$\mathbf{a}_\bullet(v_0)$ and $\mathbf{a}_\bullet(v_1)$. Using Theorem D(2), a characterization of the infimum of the normalized volume function in terms of normalized multiplicities [Liu18, Theorem 27], and an inequality of log canonical thresholds [XZ21, Theorem 3.11], we show that $e(\mathbf{a}_{\bullet,t})$ is linear. Theorem D(3) then implies $cv_0 = v_1$ for some $c > 0$.

1.2.3 Metric geometry of spaces of filtrations

We now turn to the geometry of the space Fil .

Darvas metric

Given a compact Kähler manifold (X, ω) , Darvas [Dar15] introduced various Finsler metrics on the space $\mathcal{H}$ of smooth Kähler potentials, generalizing the metric studied in [Mab87]. One of the metrics, the *Darvas metric* d_1 , has found numerous applications in complex geometry. In particular, the geodesic metric space $(\mathcal{H}, d_1)$ and its completion, $(\mathcal{E}^1, d_1)$, the space of Kähler potentials of finite energy, are closely related to Kähler-Einstein metrics by work of Darvas, Rubinstein and Lu [BBE⁺19, DR17, DL20]. See [Dar19] for a comprehensive survey about this metric space and its applications in the proof of Tian’s properness conjecture.

More recently, a similar construction in non-archimedean geometry was developed by Boucksom and Jonsson [BJ21], where a pseudo-metric d_1 , also called the Darvas metric, was defined on the space $\mathcal{N}$ of norms on the section ring $R(X, -mrK_X)$ of a projective Fano variety X . Geodesics on this space have been studied in [Reb23, BLXZ23], and have found successful applications in the algebraic theory of K -stability and algebraic Kähler-Ricci soliton degenerations.

Our major motivation is to introduce an analogue of the above constructions in the local setting.

Define a function $d_1 : \text{Fil}_{R,\mathfrak{m}} \times \text{Fil}_{R,\mathfrak{m}} \rightarrow \mathbb{R}_{\geq 0}$ by

$$d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) := 2e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) - e(\mathfrak{a}_\bullet) - e(\mathfrak{b}_\bullet), \quad (1.3)$$

which is a similar expression to the Darvas metrics in the global case.

Our next result is that the Hausdorff quotient of $(\text{Fil}_{R,\mathfrak{m}}, d_1)$ has the structure of a geodesic metric space, and thus justify the term for (1.2).

Theorem G. *Let $(R, \mathfrak{m})$ be a Noetherian local domain that is analytically irreducible. Denote the set of all linearly bounded $\mathfrak{m}$ -filtrations on R by $\text{Fil}_{R,\mathfrak{m}}$. Then*

1. *the function d_1 given by (1.3) is a pseudometric on $\text{Fil}_{R,\mathfrak{m}}$,*
2. *the Hausdorff quotient of $(\text{Fil}_{R,\mathfrak{m}}, d_1)$ is $(\text{Fil}_{R,\mathfrak{m}}^s, d_1)$, and*
3. *$(\text{Fil}_{R,\mathfrak{m}}^s, d_1)$ is a convex metric space. Moreover, if R contains a field, then it is a geodesic metric space, and a geodesic between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ can be given by $\mathfrak{a}_{\bullet,t}$ for $t \in [0, 1]$, where $\mathfrak{a}_{\bullet,t}$ is defined by (1.2).*

In the above theorem, $\text{Fil}_{R,\mathfrak{m}}^s$ denotes the subset of $\text{Fil}_{R,\mathfrak{m}}$ consisting of saturated $\mathfrak{m}$ -filtrations.

Supnorm metric

As in the global case [BJ21], there is also a correspondence between filtrations and norms in the local setting, see Definition 2.2.1 for the definition of norms and Definition-Lemma 3.1.7, Lemma 3.2.5 and Lemma 5.3.2 for the dictionary. Despite the equivalence, it is sometimes more convenient to use one of the languages.

As an example, we will consider another pseudometric d_∞ on the space $\mathcal{N}_{R,\mathfrak{m}}$ of $\mathfrak{m}$ -norms on R , which behaves like the uniform metric on the space of continuous functions on a compact space. This can also be viewed as the local analogue of the d_∞ metric introduced by Boucksom and Jonsson [BJ21].

Theorem H. *Let $(R, \mathfrak{m})$ be a Noetherian local domain that is analytically irreducible. Then*

1. *there is a function $d_\infty : \mathcal{N}_{R, \mathfrak{m}} \times \mathcal{N}_{R, \mathfrak{m}} \rightarrow \mathbb{R}_{\geq 0}$ which is a pseudometric on $\mathcal{N}_{R, \mathfrak{m}}$, and*
2. *the restriction of d_∞ to $\mathcal{N}_{R, \mathfrak{m}}^h$ is a metric, and the metric space $(\mathcal{N}_{R, \mathfrak{m}}^h, d_\infty)$ is complete.*

In the above theorem, $\mathcal{N}_{R, \mathfrak{m}}^h$ denotes the subset of $\mathcal{N}_{R, \mathfrak{m}}$ of homogeneous norms. For the definition of d_∞ and homogeneous norms, see Section 5.3.

Besides the metrics introduced, we also consider the filtrations, or equivalently, norms, as functions on various spaces, and thus define some weak topologies on the spaces of filtrations. For more details, see Section 5.4.2.

1.2.4 Continuity of log canonical thresholds

We now view filtrations as functions on the spaces of valuations, and prove that the log canonical threshold function (see Section 2.3.2 for the definition) satisfies certain continuity properties, in the spirit of [DK01].

Theorem I (Continuity of log canonical threshold). *Let $(X, x) = (\text{Spec} R, \mathfrak{m})$ be a klt singularity over a field $\mathbb{k}$ of characteristic 0. Then*

1. *If $\mathfrak{a}_{\bullet, k} \in \text{Fil}$ converges weakly to $\mathfrak{a}_\bullet \in \text{Fil}$, then*

$$\text{lct}(\mathfrak{a}_\bullet) \leq \liminf_{k \rightarrow \infty} \text{lct}(\mathfrak{a}_{\bullet, k}),$$

2. *if $\mathfrak{a}_{\bullet, k} \in \text{Fil}^s$ converges to $\mathfrak{a}_\bullet \in \text{Fil}^s$ in the $+$ -topology, then*

$$\text{lct}(\mathfrak{a}_\bullet) \geq \limsup_{k \rightarrow \infty} \text{lct}(\mathfrak{a}_{\bullet, k}),$$

and

3. if $\mathbf{a}_{\bullet,k} \in \text{Fil}^s$ converges to $\mathbf{a}_{\bullet} \in \text{Fil}^s$ in the d_{∞} -topology, then

$$\text{lct}(\mathbf{a}_{\bullet}) = \lim_{k \rightarrow \infty} \text{lct}(\mathbf{a}_{\bullet,k}).$$

Moreover, given $\mathbf{a}_{\bullet} \in \text{Fil}^s$, for any $\epsilon > 0$, there exists $\delta := \delta(\mathbf{a}_{\bullet}, \epsilon) > 0$ such that for any $\mathbf{b}_{\bullet}$ with $d_{\infty}(\mathbf{a}_{\bullet}, \mathbf{b}_{\bullet}) < \delta$, we have

$$|\text{lct}(\mathbf{b}_{\bullet}) - \text{lct}(\mathbf{a}_{\bullet})| \leq \epsilon.$$

Chapter 2

Preliminaries

2.1 Notation and conventions

Throughout we work over a Noetherian local domain $(R, \mathfrak{m})$ which is analytically irreducible, that is, the $\mathfrak{m}$ -adic completion $\hat{R}$ is again a domain. Denote by $\kappa := R/\mathfrak{m}$ the residue field of R and $K := \text{Frac}(R)$ its fraction field. We will always denote the Krull dimension of R by $n := \dim R$.

We write $c := c(a, \alpha, \dots)$ to mean that $c \in \mathbb{R}_{>0}$ is a constant depending only on $a, \alpha, \dots$.

A *variety* over a field $\mathbb{k}$ is a separated, reduced, irreducible scheme of finite type over $\mathbb{k}$. A *log pair* (X, D) is a normal Variety X with a $\mathbb{R}$ -Weil divisor D such that $K_X + D$ is $\mathbb{R}$ -Cartier.

2.2 Norms and valuations

Definition 2.2.1. A *seminorm* on R is a function $\chi : R \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that

1. $\chi(0) = +\infty$ and $\chi(-f) = \chi(f)$,
2. $\chi(f + g) \geq \min\{\chi(f), \chi(g)\}$ for any $f, g \in R$, and

3. $\chi(fg) \geq \chi(f) + \chi(g)$ for any $f, g \in R$.

A seminorm χ is a *norm* if $\chi(f) = +\infty$ if and only if $f = 0$. A (semi)norm χ is called an $\mathfrak{m}$ -(semi)norm if it further satisfies

4. $\chi(f) \geq 0$ for any $f \in R$ and $\chi(f) > 0$ if and only if $f \in \mathfrak{m}$.

If χ is a $(\mathfrak{m})$ -norm on R , and we replace condition (3) above by

3'. $\chi(fg) = \chi(f) + \chi(g)$ for any $f, g \in R$,

then χ is a real valuation on R (centered at $\mathfrak{m}$). Denote the space of real valuations on R centered at $\mathfrak{m}$ by $\text{Val}_{R,\mathfrak{m}}$.

As in the global case, we adopt the additive terminology here. For a norm χ on R , $\|\cdot\|_\chi := e^{-\chi(\cdot)}$ defines a non-Archimedean norm on R in the sense of [BGR84].

A valuation v of R induces a *valuation ring* $R_v := \{f \in K \mid v(f) \geq 0\}$. We write $\mathfrak{m}_v$ for the maximal ideal of R_v and $\kappa_v := R_v/\mathfrak{m}_v$.

In geometric settings, we will instead denote the set by $\text{Val}_{X,x}$.

For $v \in \text{Val}_{R,\mathfrak{m}}$ and $\lambda \in \mathbb{R}_{>0}$, we define the *valuation ideal*

$$\mathfrak{a}_\lambda(v) := \{f \in R \mid v(f) \geq \lambda\}$$

for each $\lambda \in \mathbb{R}$. It is not hard to show $\mathfrak{a}_\bullet(v)$ is an $\mathfrak{m}$ -filtration.

For $v \in \text{Val}_{R,\mathfrak{m}}$ and an ideal $\mathfrak{a} \subset R$, set $v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a}\}$.

The set of divisorial valuations over $(R, \mathfrak{m})$ is denoted by $\text{DivVal}_{R,\mathfrak{m}}$ (or $\text{DivVal}_{X,x}$ in the geometric setting).

2.2.1 Divisorial valuations

A valuation $v \in \text{Val}_{R,\mathfrak{m}}$ is *divisorial* if

$$\text{tr.deg}_\kappa(\kappa_v) = n - 1.$$

We write $\text{DivVal}_{R,\mathfrak{m}} \subset \text{Val}_{R,\mathfrak{m}}$ for the set of such valuations.

Divisorial valuations appear geometrically. If $\mu : Y \rightarrow X$ is a proper birational morphism with Y normal and $E \subset Y$ a prime divisor, then there is an induced valuation $\text{ord}_E : \text{Frac}(R)^\times \rightarrow \mathbb{Z}$. If $\mu(E) = x$ and $c \in \mathbb{R}_{>0}$, then $c \cdot \text{ord}_E \in \text{DivVal}_{R,\mathfrak{m}}$. When R is excellent, all divisorial valuations are of this form; see e.g. [CS22, Lemma 6.5].

2.2.2 Quasi-monomial valuations

In the following construction, we always assume R contains a field. Let $\mu : Y := \text{Spec}(S) \rightarrow X = \text{Spec}(R)$ be a birational morphism with $R \rightarrow S$ finite type and $\eta \in Y$ a not necessarily closed point such that $\mathcal{O}_{Y,\eta}$ is regular and $\mu(\eta) = x$. Given a regular system of parameters $y_1, \dots, y_r$ of $\mathcal{O}_{Y,\eta}$ and $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \setminus \mathbf{0}$, we define a valuation v_α as follows. For $0 \neq f \in \mathcal{O}_{Y,\eta}$, we can write f in $\widehat{\mathcal{O}}_{Y,\eta} \simeq k(\eta)[[y_1, \dots, y_r]]$ as $\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_\beta y^\beta$ and set

$$v_\alpha(f) := \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}. \quad (2.1)$$

A valuation of the above form is called *quasi-monomial*.

Let $D = D_1 + \dots + D_r$ be a reduced divisor on Y such that $y_i = 0$ locally defines D_i and $\mu(D_i) = x$ for each i . We call $\eta \in (Y, D)$ a *log smooth birational model* of X . We write $\text{QM}_\eta(Y, E) \subset \text{Val}_{X,x}$ for the set of quasi-monomial valuations that can be described at η with respect to $y_1, \dots, y_r$ and note that $\text{QM}_\eta(Y, D) \simeq \mathbb{R}_{\geq 0}^r \setminus \mathbf{0}$. α is called the *weight* of the valuation defined in (2.1) with respect to the log smooth pair (Y, D) .

The set of quasi-monomial valuations over $(R, \mathfrak{m})$ is denoted by $\text{QMVal}_{R,\mathfrak{m}}$ (or $\text{QMVal}_{X,x}$ in the geometric setting).

2.2.3 Izumi's inequality

The order function $R \setminus 0 \rightarrow \mathbb{N}$ is defined by

$$\text{ord}_{\mathfrak{m}}(f) := \max\{k \in \mathbb{Z}_{\geq 0} \mid f \in \mathfrak{m}^k\}.$$

The following version of Izumi's inequality compares $\text{ord}_{\mathfrak{m}}$ to a fixed quasi-monomial valuation.

Lemma 2.2.2. *Let $v \in \text{Val}_{R,\mathfrak{m}}$. If (i) v is divisorial or (ii) R contains a field and v is quasi-monomial, then there exists a constant $c > 0$ such that*

$$v(\mathfrak{m}) \cdot \text{ord}_{\mathfrak{m}}(f) \leq v(f) \leq c \cdot \text{ord}_{\mathfrak{m}}(f).$$

for all $f \in R$.

Proof. Since $f \in \mathfrak{m}^{\text{ord}_{\mathfrak{m}}(f)}$ by definition, $v(\mathfrak{m}) \cdot \text{ord}_{\mathfrak{m}}(f) = v(\mathfrak{m}^{\text{ord}_{\mathfrak{m}}(f)}) \leq v(f)$. It remains to prove the existence of $c > 0$ such that $v(f) \leq c \cdot \text{ord}_{\mathfrak{m}}(f)$ for all $f \in R$.

If (i) holds, the the existence of c follows from Izumi's Theorem for divisorial valuations as phrased in [RS14, Remark 1.6]. If (ii) holds, then there exists a log smooth birational model $\eta \in (Y, D)$ of $x \in X$ and $\alpha \in \mathbb{R}^r$ such that $v = v_{\alpha}$. Choose $\gamma \in \mathbb{Z}_{>0}^r$ such that $\alpha_i \leq \gamma_i$ for each $i = 1, \dots, r$ and consider the valuation $w := v_{\gamma}$. We claim that $w \in \text{DivVal}_{R,\mathfrak{m}}$. Assuming the claim, then (i) implies that there exists $c > 0$ such that $w(f) \leq c \cdot \text{ord}_{\mathfrak{m}}(f)$ for all $f \in R$. Since $v(f) \leq w(f)$ for all $f \in R$, (ii) then follows.

To verify that $w \in \text{DivVal}_{R,\mathfrak{m}}$, note that $w : \text{Frac}(R)^{\times} \rightarrow \mathbb{R}$ is the composition

$$\text{Frac}(R)^{\times} \hookrightarrow \text{Frac}(\widehat{\mathcal{O}}_{Y,\eta})^{\times} \xrightarrow{\widehat{w}} \mathbb{R},$$

where $\widehat{w}$ is the valuation that sends $\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_{\beta} y^{\beta}$ to $\min\{\langle \alpha, \beta \rangle \mid c_{\beta} \neq 0\}$. Using that

w is $\mathbb{Z}$ -valued, a computation (see e.g. [JM12, Proposition 3.7]) shows $\widehat{w}$ is a divisorial valuation of $\widehat{\mathcal{O}}_{Y,\eta}$. Hence w is a divisorial valuation of $\mathcal{O}_{Y,\eta}$ by [HS06, Proposition 9.3.5]. Now, we compute

$$\begin{aligned} \mathrm{tr.deg}_{\kappa}(\kappa_w) &= \mathrm{tr.deg}_{\kappa}(k(\eta)) + \mathrm{tr.deg}_{k(\eta)}(\kappa_v) \\ &= \mathrm{tr.deg}_{\kappa}(k(\eta)) + (\dim \mathcal{O}_{Y,\eta} - 1) = \dim R. \end{aligned}$$

To see the last equality holds, note that the complete local ring $\widehat{R}$ is a domain by assumption and thus equidimensional. Therefore R is universally catenary by [Sta18, Tag 0AW6] and so the dimension formula [Sta18, Tag 02II] gives the last equality. Therefore, w is divisorial as desired. $\square$

2.3 Invariants in birational geometry

In this section, we assume $(R, \mathfrak{m})$ is the local ring $\mathcal{O}_{x,X}$ of a closed point x on an algebraic variety X defined over an algebraically closed field of characteristic 0.

For an $\mathfrak{m}$ -filtration $\mathfrak{a}_{\bullet}$, set

$$v(\mathfrak{a}_{\bullet}) := \lim_{\mathbb{N} \ni m \rightarrow \infty} \frac{v(\mathfrak{a}_m)}{m} = \inf_{m \in \mathbb{N}} \frac{v(\mathfrak{a}_m)}{m},$$

where the existence of the limit and second equality is [JM12, Proposition 2.3].

2.3.1 Log discrepancy

For a divisorial valuation $v = \mathrm{ord}_E \in \mathrm{Val}_{X,x}$, choose a log smooth model $Y \rightarrow X$ such that $E \subset Y$ is a Weil divisor. The *log discrepancy* of v is defined by

$$A_X(\mathrm{ord}_E) = A_X(E) := \mathrm{ord}_E(K_{Y/X}) + 1,$$

here $\text{ord}_E(\bullet)$ is the coefficient of E in the divisor.

If $v \in \text{Val}_{X,x}$ is the quasi-monomial valuation with weight $\alpha = (\alpha_1, \dots, \alpha_n)$ with respect to a log smooth model $(Y, D = \sum D_i)$, then the *log discrepancy* of v is defined to be

$$A_X(v) := \sum_{i=1}^n \alpha_i A_X(\text{ord}_{D_i}).$$

For any log smooth model (Y, D) over X , there is a retraction map $r_{Y,D} : \text{Val}_{X,x} \rightarrow \text{QM}(Y, D)$, and the *log discrepancy* of a general valuation $v \in \text{Val}_{X,x}$ is defined to be

$$A_X(v) := \sup_{(Y,D)} A_X(r_{Y,D}(v)) \in \mathbb{R} \cup \{+\infty\}.$$

We refer to [JM12] for the definition of $r_{Y,D}$ and more details.

Denote $\text{Val}_{X,x}^{<+\infty} := \{v \in \text{Val}_{X,x} \mid A_X(v) < +\infty\}$.

We say that $x \in X$ is a *klt singularity* if X is normal, K_X is $\mathbb{Q}$ -Cartier and $A_X(v) > 0$ for any $v \in \text{Val}_{X,x}$.

2.3.2 Log canonical threshold

Let $x \in X$ be a klt singularity. The *log canonical threshold (lct)* of an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ is defined to be

$$\text{lct}(X; \mathfrak{a}_\bullet) := \inf_{v \in \text{Val}_{X,x}^{<+\infty}} \frac{A_X(v)}{v(\mathfrak{a}_\bullet)}.$$

We will write $\text{lct}(\mathfrak{a}_\bullet)$ when there is no ambiguation. If $v \in \text{Val}_{X,x}^{<+\infty}$ satisfies $\text{lct}(\mathfrak{a}_\bullet) = \frac{A_X(v)}{v(\mathfrak{a}_\bullet)}$, then we say that v *computes* the log canonical threshold of $\mathfrak{a}_\bullet$. By [Blu18a], the infimum in the definition is indeed a minimum.

Theorem 2.3.1. [Blu18a, Theorem B.1] *If $x \in X$ is a klt singularity and $\mathfrak{a}_\bullet$ is an $\mathfrak{m}$ -filtration such that $\text{lct}(\mathfrak{a}_\bullet) < +\infty$, then there exists $v_0 \in \text{Val}_{X,x}^{<+\infty}$ that computes $\text{lct}(\mathfrak{a}_\bullet)$.*

2.3.3 Local volume

The following invariant was first introduced in [Li18] and plays an important role in the study of K-stability of Fano varieties and Fano cone singularities.

Definition 2.3.2. For a klt singularity $x \in (X, \Delta)$, the *normalized volume function* $\widehat{\text{vol}}_{(X, \Delta), x} : \text{Val}_{X, x} \rightarrow (0, +\infty]$ is defined by

$$\widehat{\text{vol}}_{(X, \Delta), x}(v) := \begin{cases} A_{X, \Delta}(v)^n \cdot \text{vol}(v), & \text{if } A_{X, \Delta}(v) < +\infty \\ +\infty, & \text{if } A_{X, \Delta}(v) = +\infty, \end{cases}$$

where $A_{X, \Delta}(v)$ is the log discrepancy of v as defined in [JM12, BdFFU15]. The *local volume* of a klt singularity $x \in (X, \Delta)$ is defined as

$$\widehat{\text{vol}}(x, X, \Delta) := \inf_{v \in \text{Val}_{X, x}} \widehat{\text{vol}}_{(X, \Delta), x}(v).$$

The above infimum is indeed a minimum by [Blu18a, Xu20].

Lemma 2.3.3. (c.f. [Zhu22, Lemma 3.4]) *There exists $c_0 = c_0(n) > 0$ depending only on n such that for any n -dimensional klt singularity $x \in X$ and $v \in \text{Val}_{X, x}$,*

$$\frac{A_X(v)}{v(\mathbf{a}_\bullet)} \leq c_0 \cdot \frac{\widehat{\text{vol}}_{x, X}(v)}{\widehat{\text{vol}}(x, X)} \cdot \text{lct}(X; \mathbf{a}_\bullet) \quad (2.2)$$

for any $\mathbf{a}_\bullet \in \text{Fil}_x(X)$.

Proof. Note that $v(\mathbf{a}_\bullet) = \lim_m v(\mathbf{a}_m)/m$ and $\text{lct}(X; \mathbf{a}_\bullet) = \lim_m m \cdot \text{lct}(X; \mathbf{a}_m)$, it is enough to prove (2.2) for $\mathfrak{m}$ -primary ideals,

$$\frac{A_X(v)}{v(\mathbf{a})} \leq c_0 \cdot \frac{\widehat{\text{vol}}_{x, X}(v)}{\widehat{\text{vol}}(x, X)} \cdot \text{lct}(X; \mathbf{a}).$$

Assume $\mathbf{a} = (f_1, \dots, f_m)$. By lifting $\mathbf{a}$ to a power, we may assume that $\text{lct}(\mathbf{a}) < 1$. Then by Bertini's theorem, for general $c_1, \dots, c_m \in \mathbb{k}$, $\text{lct}(\sum_{i=1}^m c_i f_i) = \text{lct}(\mathbf{a})$.

Moreover, by definition, for $f \neq g \in \mathfrak{a}$, $v(f + cg) > \min\{v(f), v(g)\}$ for at most one $c \in \mathbb{k}$. Hence one can choose the $c_i \in \mathbb{k}$ such that $v(\sum c_i f_i) = \min_{1 \leq i \leq m} \{v(f_i)\} = v(\mathfrak{a})$. So we are further reduced to the case of a principal ideal $\mathfrak{a}$. Now the lemma follows from [Zhu22, Lemma 3.4]. $\square$

2.4 Miscellaneous materials

2.4.1 Integral closure

Let $\mathfrak{a} \subset R$ be an $\mathfrak{m}$ -primary ideal. Recall, the *integral closure* $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ can be characterized valuably by

$$\bar{\mathfrak{a}} = \{f \in R \mid v(f) \geq v(\mathfrak{a}) \text{ for all } v \in \text{DivVal}_{R, \mathfrak{m}}\}.$$

See for example [HS06, Theorem 6.8.3] and [Laz04, Example 9.6.8].

2.4.2 Intersection number

The theory of intersection numbers of line bundles on a proper scheme over an algebraically closed fields was developed in [Kle66]. We will use a more general framework developed in [Kle05, Appendix B].

Let Z be a proper scheme over an Artinian ring Λ . For line bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$ on Z , the function

$$\mathbb{Z}^r \ni (m_1, \dots, m_r) \rightarrow \sum_{i \geq 0} (-1)^i \ell_{\Lambda} \left(H^i(Z, \mathcal{L}_1^{m_1} \otimes \dots \otimes \mathcal{L}_r^{m_r}) \right) \in \mathbb{Z}$$

is a polynomial of degree $\leq \dim Z$ [Kle05, Theorem B.7]. The *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_r)$ is defined to be the coefficient of $m_1 \cdots m_r$ in the above polynomial. When the choice of Λ is unclear, we will write $(\mathcal{L}_1 \cdots \mathcal{L}_r)_{\Lambda}$.

Now let $\mu : Y \rightarrow X = \operatorname{Spec}(R)$ be a proper birational morphism with Y normal. For Cartier divisors $F_1, \dots, F_{n-1}$ on Y and a Weil divisor $D := \sum_{i=1}^r a_i D_i$ on Y with support contained in Y_κ , we set

$$F_1 \cdot \dots \cdot F_{n-1} \cdot D := \sum_{i=1}^r a_i (\mathcal{O}_Y(F_1)|_{D_i} \cdot \dots \cdot \mathcal{O}_Y(F_{n-1})|_{D_i})_\kappa.$$

This is well defined, since each prime divisor D_i is proper over $\operatorname{Spec}(\kappa)$.

Proposition 2.4.1. *Assume R is complete. If $F_1, \dots, F_n$ are Cartier divisors on Y with support contained in Y_κ , then $F_1 \cdot \dots \cdot F_n$ is independent of the ordering of the F_i .*

Since the intersection product is symmetric, $F_1 \cdot \dots \cdot F_n$ is independent of the order of $F_1, \dots, F_{n-1}$. To deduce the full result, we rely on intersection theory [Ful98].

A subtle issue is that the results in [Ful98, §1-18] are stated for schemes of finite type over a field and, hence, do not immediately apply to Y . Fortunately, the Chow group of a scheme of finite type over a regular base scheme can be defined and the results of [Ful98, §2] extend to this setting by [Ful98, §20.1] (see also [Sta18, Chapter 02P3] for the results in an even more general setting).

Proof. By Cohen's Structure Theorem, there exists a surjective map $A \twoheadrightarrow R$, where A is a regular local ring. Since the composition $Y \rightarrow X := \operatorname{Spec}(R) \rightarrow \operatorname{Spec}(A)$ is finite type, the framework of [Ful98, §20.1] applies. Using intersection theory on Y and its subschemes, we compute

$$\begin{aligned} F_1 \cdot \dots \cdot F_n &= \sum_{i=1}^r a_i \int_{D_i} c_1(\mathcal{M}_1) \cap \dots \cap c_1(\mathcal{M}_{n-1}) \cap [D_i] \\ &= \int_{Y_\kappa} c_1(\mathcal{M}_1) \cap \dots \cap c_1(\mathcal{M}_{n-1}) \cap [F_n] = \int_{Y_\kappa} F_1 \cdot \dots \cdot F_{n-1} \cdot [F_n], \end{aligned}$$

where $\int_{D_i}$ and $\int_{Y_\kappa}$ denote the degree maps induced by the proper morphisms $D_i \rightarrow \operatorname{Spec}(\kappa)$ and $Y_\kappa \rightarrow \operatorname{Spec}(\kappa)$ [Ful98, Definition 1.4] and $\mathcal{M}_i := \mathcal{O}_Y(F_i)$. The first

equality holds by [Ful98, Example 18.3.6] with the fact that D_i is a proper scheme over κ , the second by the equality $[F_n] = \sum_{i=1}^r a_i [D_i]$, and the last by the definition of the first Chern class. Since

$$F_1 \cdot \dots \cdot F_{n-1} \cdot [F_n] = F_1 \cdot \dots \cdot F_n \cdot [Y] \in A_*(Y_\kappa)$$

and the latter is independent of the order of the F_i by [Ful98, Corollary 2.4.2], the proposition holds. $\square$

2.4.3 Partially ordered set and lattice

We recall some basic definitions regarding lattices in order theory.

Definition 2.4.2. A partially ordered set $(\mathcal{S}, \leq)$ is called a *lattice* if for any $a, b \in \mathcal{S}$,

1. there exists $a \vee b \in \mathcal{S}$, called the *supremum*, or *join*, such that for any $c \in \mathcal{S}$, $c \geq a$ and $c \geq b$ imply $c \geq a \vee b$, and
2. there exists $a \wedge b \in \mathcal{S}$, called the *infimum*, or *meet*, such that for any $c \in \mathcal{S}$, $c \leq a$ and $c \leq b$ imply $c \leq a \wedge b$.

A lattice $\mathcal{S}$ is *modular* if for any $a, b, c \in \mathcal{S}$ with $a \leq b$, we have $a \vee (c \wedge b) = (a \wedge c) \vee b$.

A lattice $\mathcal{S}$ is *distributive* if for any $a, b, c \in \mathcal{S}$, we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

A lattice $\mathcal{S}$ is *complete* if any subset $A \subset \mathcal{S}$ has an infimum $a = \inf A$ and a supremum $b = \sup A$.

Definition 2.4.3. A map $f : L \rightarrow M$ between two lattices is a *join morphism* if for any $a, b \in L$, $f(a \vee_L b) = f(a) \vee_M f(b)$, where $\vee_L$ and $\vee_M$ denote the join in L and M , respectively.

2.4.4 Metric space and geodesic

Recall that a *pseudometric* on a set S is a function $d : S \times S \rightarrow \mathbb{R}_{\geq 0}$ such that

- (non-negative) $d(x, y) \geq 0$ for any $x, y \in S$,
- (symmetric) $d(x, y) = d(y, x)$ for any $x, y \in S$, and
- (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in S$.

A pseudometric is a *metric* if and only if it is non-degenerate, that is, $d(x, y) = 0$ if and only if $x = y$.

Definition 2.4.4. A point z in a metric space (S, d) is said to be *between* two points x and y , if all three points are distinct, and $d(x, y) = d(x, z) + d(z, y)$.

A metric space (S, d) is *convex* if for any two distinct points $x, y \in S$, there exists a point z between x and y .

Given a pseudometric space (S, d) , its *Hausdorff quotient* is the quotient space $(\bar{S} := S / \sim, \bar{d})$, where $x \sim y$ if and only if $d(x, y) = 0$, and $\bar{d}$ is the natural induced metric on $\bar{S}$.

Recall that the *length* of a continuous curve $\gamma : [a, b] \rightarrow S$ in a metric space (S, d) is defined by

$$\ell(\gamma) := \sup_{\mathcal{P}} \sum_{i=1}^n d(x_{i-1}, x_i),$$

where the supremum is taken over all finite partitions $\mathcal{P} = \{a = x_0 \leq x_1 \leq \dots \leq x_n = b\}$ of the interval $[a, b]$.

Definition 2.4.5. A metric space (S, d) is a *length space* if d coincides with the *intrinsic metric*, that is, $d(x, y) = \inf_{\gamma} \ell(\gamma) =: d_I(x, y)$ for any $x, y \in S$, where the infimum is taken over all continuous curves $\gamma : [0, 1] \rightarrow S$ with $\gamma(0) = x$ and $\gamma(1) = y$.

(S, d) is a *geodesic space* if the above infimum is a minimum, that is, the distance can always be calculated by a continuous curve.

Chapter 3

Filtrations, multiplicities and saturations

In this section we introduce the *saturation* of an $\mathfrak{m}$ -filtration, and prove its first applications, Theorem A and Theorem C. The main results of Section 3.2 through Section 3.4 are joint with Harold Blum and Yuchen Liu. We thank them for allowing the results to be reproduced here.

3.1 Filtrations on a local ring

A filtration on R is a continuously indexed version of a graded sequence of ideals.

Definition 3.1.1. A *filtration* on R is a collection $\mathfrak{a}_\bullet = (\mathfrak{a}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ of ideals of R such that

1. (decreasing) $\mathfrak{a}_\lambda \subset \mathfrak{a}_\mu$ for any $\lambda > \mu$,
2. (left-continuity) $\mathfrak{a}_\lambda = \mathfrak{a}_{\lambda-\epsilon}$ for any $\lambda \in \mathbb{R}_{>0}$ and $0 < \epsilon \ll 1$, and
3. (multiplicativity) $\mathfrak{a}_\lambda \cdot \mathfrak{a}_\mu \subset \mathfrak{a}_{\lambda+\mu}$ for any $\lambda, \mu \in \mathbb{R}_{>0}$.

A filtration $\mathfrak{a}_\bullet$ is called an $\mathfrak{m}$ -filtration if $\mathfrak{a}_\lambda$ is $\mathfrak{m}$ -primary for any $\lambda \in \mathbb{R}_{>0}$. By convention, we always set $\mathfrak{a}_0 := R$.

The definition is a local analogue of a filtration on the section ring of a polarized variety in [BHJ17]. We will exclusively work with $\mathfrak{m}$ -filtrations in the sequel.

For $\lambda \in \mathbb{R}_{\geq 0}$, set $\mathfrak{a}_{>\lambda} := \cup_{\mu > \lambda} \mathfrak{a}_\mu$. If $\lambda \in \mathbb{R}_{>0}$ satisfies $\mathfrak{a}_{>\lambda} \subsetneq \mathfrak{a}_\lambda$, then we call λ a *jumping number* of $\mathfrak{a}_\bullet$. The *scaling* of an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ by $c \in \mathbb{R}_{>0}$ is $\mathfrak{a}_{c\bullet} := (\mathfrak{a}_{c\lambda})_{\lambda \in \mathbb{R}_{>0}}$, which is an $\mathfrak{m}$ -filtration. For two $\mathfrak{m}$ -filtrations $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$, we say that $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ if $\mathfrak{a}_\lambda \subset \mathfrak{b}_\lambda$ for any $\lambda \in \mathbb{R}_{>0}$. This defines a partial order on the set of all $\mathfrak{m}$ -filtrations. There is a maximal element among all $\mathfrak{m}$ -filtrations, $\{\mathfrak{m}\}_{\lambda \in \mathbb{R}_{>0}}$. We will sometimes abuse the notation and denote this $\mathfrak{m}$ -filtration by $\mathfrak{m}$.

Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be two $\mathfrak{m}$ -filtrations. Their *intersection* is $\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet := (\mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$. Their *product* can be defined in the same way. It is not hard to check that these are again an $\mathfrak{m}$ -filtrations.

Definition-Lemma 3.1.2. Let $\{\mathfrak{a}_{\bullet,i}\}_{i \in I}$ be an arbitrary non-empty set of $\mathfrak{m}$ -filtrations. Assume that there exists an $\mathfrak{m}$ -filtration $\mathfrak{b}_\bullet$ such that $\mathfrak{b}_\bullet \subset \mathfrak{a}_{\bullet,i}$ for any $i \in I$. Then the collection of $\mathfrak{m}$ -primary ideals, $(\cap_{i \in I} \mathfrak{a}_{\lambda,i})_{\lambda \in \mathbb{R}_{>0}}$, is an $\mathfrak{m}$ -filtration, called the *intersection* of the $\mathfrak{a}_{\bullet,i}$, and is denoted by $\cap_{i \in I} \mathfrak{a}_{\bullet,i}$. Moreover, if an $\mathfrak{m}$ -filtration $\mathfrak{c}_\bullet$ satisfies $\mathfrak{c}_\bullet \subset \mathfrak{a}_{\bullet,i}$ for any $i \in I$, then we have $\mathfrak{c}_\bullet \subset \cap_{i \in I} \mathfrak{a}_{\bullet,i}$.

Proof. By assumption, for any $\lambda > 0$ we have $\mathfrak{b}_\lambda \subset \mathfrak{a}_{\lambda,i}$ and hence $\cap_{i \in I} \mathfrak{a}_{\lambda,i}$ is $\mathfrak{m}$ -primary. It is straightforward to verify conditions (1) and (3) in the definition.

To prove (2), note that $\mathfrak{b}_\lambda \subset \cap_i \mathfrak{a}_{\lambda,i} \subset \cap_i \mathfrak{a}_{\mu,i}$ for any $\mu \leq \lambda$. In $R/\mathfrak{b}_\lambda$ we have

$$\begin{aligned} \cap_{i \in I} \mathfrak{a}_{\lambda,i} &= \cap_{i \in I} \cap_{\epsilon > 0} \mathfrak{a}_{\lambda-\epsilon,i} = \cap_{\epsilon > 0} \cap_{i \in I} \mathfrak{a}_{\lambda-\epsilon,i} \\ &= \cap_{l \in \mathbb{Z}_{>1/\lambda}} \cap_{i \in I} \mathfrak{a}_{\lambda-\frac{1}{l},i} = \cap_{i \in I} \mathfrak{a}_{\lambda-\frac{1}{l},i} \end{aligned}$$

for some $l \in \mathbb{Z}_{>1/\lambda}$, where the first equality follows from the left-continuity of $\mathfrak{a}_{\bullet,i}$, and the last equality uses the fact that $R/\mathfrak{b}_\lambda$ is Artinian. This proves that condition

(2) holds for $\{\cap_{i \in I} \mathfrak{a}_{\lambda, i}\}$, and thus it is an $\mathfrak{m}$ -filtration.

The last assertion follows by definition, and the proof is finished. $\square$

Definition-Lemma 3.1.3. Let $\{\mathfrak{a}_{\bullet, i}\}_{i \in I}$ be an arbitrary set of $\mathfrak{m}$ -filtrations. Then there is a filtration, called the *join* of the $\mathfrak{a}_{\bullet, i}$, denoted by $\vee_{i \in I} \mathfrak{a}_{\bullet, i}$, such that $\mathfrak{a}_{\bullet, i} \subset \vee_{i \in I} \mathfrak{a}_{\bullet, i}$ for any $i \in I$. Moreover, if an $\mathfrak{m}$ -filtration $\mathfrak{c}_{\bullet}$ satisfies $\mathfrak{a}_{\bullet, i} \subset \mathfrak{c}_{\bullet}$ for any $i \in I$, then we have $\vee_{i \in I} \mathfrak{a}_{\bullet, i} \subset \mathfrak{c}_{\bullet}$.

Proof. If $I = \emptyset$, then $\vee_{i \in I} \mathfrak{a}_{\bullet, i}$ is the maximal $\mathfrak{m}$ -filtration $\mathfrak{m}$.

Assume now $I \neq \emptyset$ and fix $0 \in I$. The set

$$J := \{\mathfrak{b}_{\bullet} \mid \mathfrak{a}_{\bullet, i} \subset \mathfrak{b}_{\bullet} \text{ for any } i \in I\}$$

is non-empty, since the maximal $\mathfrak{m}$ -filtration $\mathfrak{m} \in J$. Moreover, by definition $\mathfrak{a}_{\bullet, 0} \subset \mathfrak{b}_{\bullet}$ for any $\mathfrak{b}_{\bullet} \in J$. Hence we can apply Definition-Lemma 3.1.2 to get an $\mathfrak{m}$ -filtration

$$\vee_{i \in I} \mathfrak{a}_{\bullet, i} := \cap_{\mathfrak{b}_{\bullet} \in J} \mathfrak{b}_{\bullet}.$$

The last assertion follows by definition, and the proof is finished. $\square$

Example 3.1.4. Given an $\mathfrak{m}$ -primary ideal $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$, we have an $\mathfrak{m}$ -filtration $\mathfrak{a}^{\bullet}$, the *$\mathfrak{a}$ -adic filtration*, defined by $\{\mathfrak{a}^{[\lambda]}\}_{\lambda \in \mathbb{R}_{>0}}$. Thus we get an injection from the set $\mathcal{I}_{\mathfrak{m}}$ to the set of all $\mathfrak{m}$ -filtrations by $\mathfrak{a} \mapsto \mathfrak{a}^{\bullet}$. In particular, there is a canonical $\mathfrak{m}$ -filtration $\mathfrak{m}^{\bullet}$.

For $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_{\mathfrak{m}}$, we claim that $\mathfrak{a}^{\bullet} \vee \mathfrak{b}^{\bullet} = (\mathfrak{a} + \mathfrak{b})^{\bullet}$. Indeed, since $\mathfrak{a}^{\bullet} \subset (\mathfrak{a} + \mathfrak{b})^{\bullet}$ and $\mathfrak{b}^{\bullet} \subset (\mathfrak{a} + \mathfrak{b})^{\bullet}$, by Definition-Lemma 3.1.3 we have $\mathfrak{c}_{\bullet} := \mathfrak{a}^{\bullet} \vee \mathfrak{b}^{\bullet} \subset (\mathfrak{a} + \mathfrak{b})^{\bullet}$. On the other hand, we have $\mathfrak{a} \subset \mathfrak{c}_1$ and $\mathfrak{b} \subset \mathfrak{c}_1$ by definition. Hence for any $\lambda \in \mathbb{R}_{>0}$, we have

$$(\mathfrak{a} + \mathfrak{b})^{[\lambda]} \subset \mathfrak{c}_1^{[\lambda]} \subset \mathfrak{c}_{[\lambda]} \subset \mathfrak{c}_{\lambda}.$$

This proves $(\mathfrak{a} + \mathfrak{b})^{\bullet} \subset \mathfrak{c}_{\bullet}$ and thus the equality holds.

However, in general we only have $(\mathfrak{a} \cap \mathfrak{b})^\bullet \subset \mathfrak{a}^\bullet \cap \mathfrak{b}^\bullet$ and the inclusion can be strict. Indeed, let $R = \mathbb{k}[[x, y]]$ for some field $\mathbb{k}$, $\mathfrak{a} = (x, y^2)$ and $\mathfrak{b} = (x^2, y)$. Then it is not hard to see that $\mathfrak{a} \cap \mathfrak{b} = (x^2, xy, y^2) = \mathfrak{m}^2$ and $(\mathfrak{a} \cap \mathfrak{b})^2 = \mathfrak{m}^4$. Hence $x^2y \in \mathfrak{a}^2 \cap \mathfrak{b}^2 \setminus (\mathfrak{a} \cap \mathfrak{b})^2$.

Unlike the intersection, it can be hard to determine the join of two general filtrations. Later in Definition 3.5.5, we will define a similar notion, the *saturated join*, and give a formula for it. The following proposition, which is a combination of Definition-Lemma 3.1.2 and Definition-Lemma 3.1.3, justifies the term *join*.

Proposition 3.1.5. *The set of all $\mathfrak{m}$ -filtrations with the partial order by inclusion form a lattice, where the meet of two elements is given by their intersection, and the join is defined as in Definition-Lemma 3.1.3.* $\square$

Remark 3.1.6. Note that by Example 3.1.4, the lattice $\mathcal{I}_{\mathfrak{m}}$ of $\mathfrak{m}$ -primary ideals of R does not embed in the lattice of all $\mathfrak{m}$ -filtrations, since the meet is not preserved.

$\mathfrak{m}$ -seminorms and $\mathfrak{m}$ -filtrations can be identified as follows.

Definition-Lemma 3.1.7. Given an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$, its *order function* $\text{ord}_{\mathfrak{a}_\bullet}$ is defined by

$$\text{ord}_{\mathfrak{a}_\bullet}(f) := \sup\{\lambda \in \mathbb{R}_{>0} \mid f \in \mathfrak{a}_\lambda\}, \quad (3.1)$$

which is a seminorm on R . If $\mathfrak{a}_\lambda = \mathfrak{m}$ for $0 < \lambda \ll 1$, then it is an $\mathfrak{m}$ -seminorm.

Conversely, given an $\mathfrak{m}$ -seminorm χ , define $\mathfrak{a}_\bullet(\chi)$ by

$$\mathfrak{a}_\lambda(\chi) := \{f \in R \mid \chi(f) \geq \lambda\}. \quad (3.2)$$

Then $\mathfrak{a}_\bullet(\chi)$ is an $\mathfrak{m}$ -filtration, called the *associated filtration* of χ .

The above maps give a 1-1 correspondence between $\mathfrak{m}$ -seminorms and $\mathfrak{m}$ -filtrations satisfying $\mathfrak{a}_\lambda = \mathfrak{m}$ for $0 < \lambda \ll 1$. Moreover, $\mathfrak{m}$ -norms correspond to $\mathfrak{m}$ -filtrations with $\bigcap_{\lambda>0} \mathfrak{a}_\lambda = \{0\}$.

Note that for $v \in \text{Val}_{R, \mathfrak{m}}$, its associated filtration is the *valuation ideals* of v .

Proof. Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$ -filtration. Clearly $\text{ord}_{\mathfrak{a}_\bullet}$ satisfies condition (1) of Definition 2.2.1. Condition (2) follows from the fact that each $\mathfrak{a}_\lambda$ is an ideal. Condition (3) follows from the multiplicativity of a filtration. If $\mathfrak{a}_\lambda = \mathfrak{m}$ for $0 < \lambda \ll 1$, then condition (4) follows from the convention $\mathfrak{a}_0 = R$.

Conversely, let χ be an $\mathfrak{m}$ -seminorm. We first show that for any $\lambda > 0$, $\mathfrak{a}_\lambda(\chi)$ defined in (3.2) is an $\mathfrak{m}$ -primary ideal. It is an ideal since $\chi(f-g) \geq \min\{\chi(f), \chi(-g)\} = \min\{\chi(f), \chi(g)\}$ and $\chi(fg) \geq \chi(f) + \chi(g) \geq \min\{\chi(f), \chi(g)\}$. Choose a set of generators $f_1, \dots, f_r$ of $\mathfrak{m}$, then for any $f \in \mathfrak{m}$ we have $\chi(f) \geq \chi(\mathfrak{m}) := \min_{1 \leq i \leq r} \{\chi(f_i)\} > 0$. Let $C := \lceil \lambda / \chi(\mathfrak{m}) \rceil$, then $\chi(f^C) \geq C\chi(f) \geq \lambda$, that is, $\mathfrak{m}^C \subset \mathfrak{a}_\lambda(\chi)$. Note that the above argument also shows that $\mathfrak{a}_\lambda(\chi) = \mathfrak{m}$ for $0 < \lambda < \chi(\mathfrak{m})$. Clearly the collection of $\mathfrak{m}$ -primary ideals $\{\mathfrak{a}_\lambda(\chi)\}$ is decreasing, and it is multiplicative by condition (3) of Definition 2.2.1. To show left continuity, note that we can write $\mathfrak{a}_\lambda(\chi)$ as an decreasing intersection

$$\mathfrak{a}_\lambda(\chi) = \bigcap_{\epsilon > 0} \mathfrak{a}_{\lambda - \epsilon}(\chi),$$

which terminates since the equality holds in the Artinian ring $R/\mathfrak{a}_\lambda(\chi)$. $\square$

3.1.1 Asymptotic invariants

Recall that given a filtration $\mathfrak{a}_\bullet$ on R and $v \in \text{Val}_{R, \mathfrak{m}}$, we have $v(\mathfrak{a}_\bullet) := \lim_{\lambda \rightarrow \infty} \frac{v(\mathfrak{a}_\lambda)}{\lambda}$.

Lemma 3.1.8. (cf. [JM12, Lemma 2.4]) *Let $\chi \in \mathcal{N}_{R, \mathfrak{m}}$ and $w \in \text{Val}_{R, \mathfrak{m}}$. Then*

$$w(\mathfrak{a}_\bullet(\chi)) = \inf_{f \in \mathfrak{m}} \frac{w(f)}{\chi(f)}.$$

Proof. Let $c := \inf_{f \in \mathfrak{m}} \frac{w(f)}{\chi(f)}$. Then for any $f \in \mathfrak{m}$ we have $w(f) \geq c \cdot \chi(f)$, hence by definition, $w(\mathfrak{a}_\lambda(\chi)) \geq c \cdot \chi(\mathfrak{a}_\lambda(\chi)) \geq c\lambda$ for any $\lambda \in \mathbb{R}_{>0}$. Letting $\lambda \rightarrow \infty$ and dividing by λ , we get $w(\mathfrak{a}_\bullet(\chi)) \geq c$.

Conversely, for any $\epsilon > 0$, by definition there exists $f \in \mathfrak{m}$ such that $0 < w(f) < (c + \epsilon)\chi(f)$. Note that for any $k \in \mathbb{Z}_{>0}$, $f^d \in \mathfrak{a}_{d \cdot \chi(f)}(\chi)$ as $\chi(f^d) \geq d \cdot \chi(f)$. Thus for $d \in \mathbb{Z}_{>0}$ we have

$$\frac{w(\mathfrak{a}_{d \cdot \chi(f)}(\chi))}{d \cdot \chi(f)} \leq \frac{w(f^d)}{d \cdot \chi(f)} = \frac{d \cdot w(f)}{d \cdot \chi(f)} < c + \epsilon.$$

Letting $d \rightarrow \infty$ we get $w(\mathfrak{a}_\bullet(\chi)) \leq c + \epsilon$. This finishes the proof. $\square$

3.2 Multiplicities

3.2.1 Multiplicities of an ideal

The *multiplicity* of an $\mathfrak{m}$ -primary ideal $\mathfrak{a}$ is

$$e(\mathfrak{a}) := \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}^m)}{m^n/n!}.$$

The following intersection formula for multiplicities commonly appears in the literature when $x \in X$ is a closed point on a quasi-projective variety [Laz04, pp. 92]. In the generality stated below, it follows from [Ram74].

Proposition 3.2.1. *Let $\mathfrak{a} \subset R$ be an $\mathfrak{m}$ -primary ideal and $Y \rightarrow \operatorname{Spec}(R)$ a proper birational morphism with Y normal. If $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ for a Cartier divisor $E = \sum_i a_i D_i$, then*

$$e(\mathfrak{a}) = (-E)^{n-1} \cdot E.$$

Proof. By [Ram74, Theorem and Remark 1], $e(\mathfrak{a}) = (\mathcal{L}^{n-1})_{R/\mathfrak{a}}$, where $\mathcal{L} := \mathcal{O}_Y(-E)|_E$. Observe

$$(\mathcal{L}^{n-1})_{R/\mathfrak{a}} = \sum \ell_{\mathcal{O}_{E, \xi_i}}(\mathcal{O}_{E, \xi_i})(\mathcal{L}_i^{n-1})_{R/\mathfrak{a}} = \sum a_i (\mathcal{L}_i^{n-1})_{R/\mathfrak{a}},$$

where $E = \sum a_i D_i$ and $\mathcal{L}_i := \mathcal{L}|_{D_i}$. Indeed, the first equality is [Kle05, Lemma B.12]

and the second equality holds, since $\mathcal{O}_{Y,\xi_i}$ is a DVR and $\mathcal{O}_{E,\xi_i} = \mathcal{O}_{Y,\xi}/(\pi^{a_i})$, where π is a uniformizer of the DVR.

To finish the proof, notice that D_i is defined over κ , since there is a natural inclusion $D_i \hookrightarrow Y_{\kappa,\text{red}} \rightarrow Y_\kappa$. Since the residue field of the local ring $R/\mathfrak{a}$ is κ , any finite length κ -module M satisfies $\ell_{R/\mathfrak{a}}(M) = \ell_\kappa(M)$ by [Sta18, Lemma 02M0]

$$\sum_{i=1}^r a_i (\mathcal{L}_i^{n-1})_{R/\mathfrak{a}} = \sum_{i=1}^r a_i (\mathcal{L}_i^{n-1})_\kappa = (-E)^{n-1} \cdot E,$$

which completes the proof. $\square$

3.2.2 Multiplicities of a filtration

Following [ELS03], the *multiplicity* of a graded sequence of $\mathfrak{m}$ -primary ideals $\mathfrak{a}_\bullet$ is

$$e(\mathfrak{a}_\bullet) := \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!} \in [0, \infty).$$

By [ELS03, Mus02, LM09, Cut13, Cut14] in increasing generality, the above limit exists and

$$e(\mathfrak{a}_\bullet) = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m)}{m^n} = \inf_m \frac{e(\mathfrak{a}_m)}{m^n}; \quad (3.3)$$

see in particular [Cut14, Theorem 6.5]. Also defined in [ELS03], the *volume* of a valuation $v \in \text{Val}_{R,\mathfrak{m}}$ is

$$\text{vol}(v) := e(\mathfrak{a}_\bullet(v)).$$

We set

$$\text{Val}_{R,\mathfrak{m}}^+ := \{v \in \text{Val}_{R,\mathfrak{m}} \mid \text{vol}(v) > 0\} \subset \text{Val}_{R,\mathfrak{m}}.$$

Proposition 3.2.2. *Let $v \in \text{Val}_{R,\mathfrak{m}}$. If (i) v is divisorial or (ii) R contains a field and v is quasi-monomial, then $\text{vol}(v) > 0$.*

Proof. By Lemma 2.2.2, there exists $c > 0$ such that $\mathfrak{a}_\bullet(v) \subset \mathfrak{m}^{c\bullet}$. Thus, $\text{vol}(v) \geq$

$$e(\mathfrak{m}^{c\bullet}) = c^{-n}e(\mathfrak{m}) > 0.$$

□

3.2.3 Linear boundedness

Definition 3.2.3. An $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ is *linearly bounded* if there exists a constant $c \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_\bullet \subset \mathfrak{m}^{c\bullet}$.

An $\mathfrak{m}$ -norm χ is *linearly bounded* if there exists $c \in \mathbb{R}_{>0}$ such that $\chi(f) \leq c^{-1}\text{ord}_{\mathfrak{m}}(f)$ for any $f \in R$.

Lemma 3.2.4. Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$ -filtrations. If $\mathfrak{a}_\bullet$ is linearly bounded, then there exists $c \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_{c\lambda} \subset \mathfrak{b}_\lambda$ for all real numbers $\lambda \geq 1$

Proof. Since $\mathfrak{a}_\bullet$ is linearly bounded, there exists $c_0 \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_\lambda \subset \mathfrak{m}^{\lceil c_0\lambda \rceil}$ for all $\lambda > 0$. Since $\mathfrak{b}_1$ is an $\mathfrak{m}$ -primary ideal, there exists $d \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^d \subset \mathfrak{b}_1$. If we set $c := 2d/c_0$, then

$$\mathfrak{a}_{c\lambda} \subset \mathfrak{m}^{\lceil 2d\lambda \rceil} \subset \mathfrak{m}^{d\lceil \lambda \rceil} \subset (\mathfrak{b}_1)^{\lceil \lambda \rceil} \subset \mathfrak{b}_{\lceil \lambda \rceil} \subset \mathfrak{b}_\lambda$$

for all $\lambda \geq 1$. (The second inclusion uses that $d\lceil \lambda \rceil \leq d(\lambda + 1) \leq 2d\lambda \leq \lceil 2d\lambda \rceil$). □

By Krull intersection theorem, a linearly bounded $\mathfrak{m}$ -filtration satisfies $\bigcap_{\lambda \in \mathbb{R}_{\geq 0}} \mathfrak{a}_\lambda = \{0\}$. In particular, by Definitin-Lemma 3.1.7, we have the following

Lemma 3.2.5. An $\mathfrak{m}$ -filtration is linearly bounded if and only if $\text{ord}_{\mathfrak{a}_\bullet}$ is a linearly bounded $\mathfrak{m}$ -norm.

Later in Corollary 3.4.3, we will show that an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ is linearly bounded if and only if $e(\mathfrak{a}_\bullet) > 0$.

3.3 Saturation

3.3.1 Saturation

We define the saturation of an $\mathfrak{m}$ -filtration in a similar manner to the integral closure of an $\mathfrak{m}$ -primary ideal.

Definition 3.3.1. The *saturation* $\tilde{\mathfrak{a}}_\bullet$ of an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ is defined by

$$\tilde{\mathfrak{a}}_\lambda := \{f \in \mathfrak{m} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{DivVal}_{R,\mathfrak{m}}\}$$

for each $\lambda \in \mathbb{R}_{>0}$. We say $\mathfrak{a}_\bullet$ is *saturated* if $\mathfrak{a}_\bullet = \tilde{\mathfrak{a}}_\bullet$.

Remark 3.3.2. Corollary 3.3.16 shows that it is equivalent to define the saturation using all positive volume valuations, rather than only divisorial valuations.

Remark 3.3.3. The saturation is a local analogue of the maximal norm of a multiplicative norm of the section ring of a polarized variety, which was defined and studied in [BJ21].

Remark 3.3.4. Definition 3.3.1 differs from the definition of *saturation* used in [Mus02, Section 2] for monomial ideals, which coincides with the ideals in Lemma 3.3.6 defined using the integral closure of the Rees algebra.

Remark 3.3.5. After the first version of this paper was posted on the arXiv, Cutkosky and Praharaj introduced an operation on $\mathbb{R}$ -filtrations that is defined using certain asymptotic Hilbert–Samuel functions (see the definition in [CP24, Theorem 1.3]). As shown by [CP24, Example 7.2], their operation does not always coincide with the saturation.

Lemma 3.3.6 ([Cut21, Lemma 5.6]). *If $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$ -filtration, then the integral closure of $\text{Rees}(\mathfrak{a}_\bullet) := \bigoplus_{m \in \mathbb{N}} \mathfrak{a}_m t^m$ in $R[t]$ is given by $\bigoplus_{m \in \mathbb{N}} \mathfrak{a}'_m t^m$, where*

$$\mathfrak{a}'_m = \{f \in R \mid \text{there exists } r > 0 \text{ such that } f^r \in \bar{\mathfrak{a}}_{rm}\}.$$

Example 3.3.7. In general, the ideals $\tilde{\mathfrak{a}}_m$ and $\mathfrak{a}'_m$ appearing above do not coincide. For example, let $R := k[x]_{(x)}$ and $\mathfrak{m} = (x)$. Consider the $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ defined by $\mathfrak{a}_\lambda := (x^{\lceil \lambda + 1 \rceil})$. Using that $\text{ord}_{\mathfrak{m}}(\mathfrak{a}_\lambda) = \lceil \lambda + 1 \rceil$ and $\text{ord}_{\mathfrak{m}}(\mathfrak{a}_\bullet) = 1$, we compute

$$\mathfrak{a}'_m = (x^{m+1}) \subsetneq (x^m) = \tilde{\mathfrak{a}}_m.$$

for each $m \in \mathbb{Z}_{>0}$. In particular, $\text{Rees}(\mathfrak{a}_\bullet)$ is integrally closed, but $\mathfrak{a}_\bullet$ is not saturated.

The following lemma states basic properties of the saturation.

Lemma 3.3.8. *For any $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$, the following hold:*

1. $\mathfrak{a}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$,
2. $v(\mathfrak{a}_\bullet) = v(\tilde{\mathfrak{a}}_\bullet)$ for all $v \in \text{DivVal}_{R, \mathfrak{m}}$, and
3. $\tilde{\mathfrak{a}}_\bullet$ is saturated.

Proof. For any $\lambda \in \mathbb{R}_{>0}$,

$$v(\mathfrak{a}_\lambda) = \lim_{m \rightarrow \infty} \frac{v(\mathfrak{a}_\lambda^m)}{m} \geq \lim_{m \rightarrow \infty} \left(\frac{v(\mathfrak{a}_{\lfloor \lambda m \rfloor})}{\lfloor \lambda m \rfloor} \cdot \frac{\lfloor \lambda m \rfloor}{m} \right) = \lambda v(\mathfrak{a}_\bullet),$$

where the inequality uses that $(\mathfrak{a}_\lambda)^m \subset \mathfrak{a}_{\lambda m} \subset \mathfrak{a}_{\lfloor \lambda m \rfloor}$. Therefore, $\mathfrak{a}_\lambda \subset \tilde{\mathfrak{a}}_\lambda$, which is (1).

For (2), note that $v(\mathfrak{a}_\bullet) \leq \frac{v(\tilde{\mathfrak{a}}_m)}{m} \leq \frac{v(\mathfrak{a}_m)}{m}$ for each $m \in \mathbb{Z}_{>0}$, where the first inequality follows from the definition of $\tilde{\mathfrak{a}}_m$ and the second from (1). Sending $m \rightarrow \infty$ gives $v(\mathfrak{a}_\bullet) \leq v(\tilde{\mathfrak{a}}_\bullet) \leq v(\mathfrak{a}_\bullet)$, which implies (2). Statement (3) follows immediately from (2). $\square$

We say two $\mathfrak{m}$ -filtrations $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ are *equivalent* if $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_\bullet$. The following proposition gives a characterization of when two filtrations are equivalent after possible scaling.

Proposition 3.3.9. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$ -filtrations and $c \in \mathbb{R}_{>0}$. The following are equivalent:*

1. $\widetilde{\mathfrak{a}}_\bullet = \widetilde{\mathfrak{b}}_{c\bullet}$.
2. $v(\mathfrak{a}_\bullet) = cv(\mathfrak{b}_\bullet)$ for all $v \in \text{DivVal}_{R,\mathfrak{m}}$.

Proof. First, note that $v(\mathfrak{b}_{c\bullet}) = cv(\mathfrak{b}_\bullet)$ for all $v \in \text{Val}_{R,\mathfrak{m}}$. Therefore, (1) implies (2) follows from Lemma 3.3.8(2), while (2) implies (1) follows from the definition of the saturation. $\square$

3.3.2 Saturation and completion

Let $(\widehat{R}, \widehat{\mathfrak{m}})$ denote the $\mathfrak{m}$ -adic completion of $(R, \mathfrak{m})$ and write $\varphi : R \rightarrow \widehat{R}$ for the natural morphism. (Note that $\widehat{R}$ is a domain by the assumption throughout the paper that R is analytically irreducible.) For an $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$, we set $\mathfrak{a}_\bullet \widehat{R} := (\mathfrak{a}_\lambda \widehat{R})_{\lambda > 0}$, which is an $\widehat{\mathfrak{m}}$ -filtration.

Proposition 3.3.10. *If $\mathfrak{a}_\bullet$ is an $\mathfrak{m}$ -filtration and $\mathfrak{b}_\bullet := \mathfrak{a}_\bullet \widehat{R}$, then $\widetilde{\mathfrak{a}}_\bullet \widehat{R} = \widetilde{\mathfrak{b}}_\bullet$.*

The proposition shows that completion and saturation commute, as is the case with the integral closure of ideal [HS06, Proposition 1.6.2]. As a consequence of the proposition, many results regarding saturations reduce to the case when R is a complete local domain.

Proof. By [HS06, Theorem 9.3.5], there is a bijective map $\varphi_* : \text{DivVal}_{\widehat{R}, \widehat{\mathfrak{m}}} \rightarrow \text{DivVal}_{R, \mathfrak{m}}$ that sends $\hat{v}$ to the valuation v defined by composition

$$\text{Frac}(R)^\times \hookrightarrow \text{Frac}(\widehat{R})^\times \xrightarrow{\hat{v}} \mathbb{R}.$$

Note that $\hat{v}(\mathfrak{a}\widehat{R}) = v(\mathfrak{a})$ for any ideal $\mathfrak{a} \subset R$.

Fix $\lambda \in \mathbb{R}_{>0}$. Using the previous observation twice and the definition of the saturation, we compute

$$\hat{v}(\tilde{\mathfrak{a}}_\lambda \hat{R}) = v(\tilde{\mathfrak{a}}_\lambda) \geq \lambda v(\mathfrak{a}_\bullet) = \lambda \hat{v}(\mathfrak{b}_\bullet).$$

Thus, $\tilde{\mathfrak{a}}_\lambda \hat{R} \subset \tilde{\mathfrak{b}}_\lambda$. To prove the reverse inclusion, note that $\mathfrak{c}R = \tilde{\mathfrak{b}}_\lambda$ for some ideal $\mathfrak{c} \subset R$, since $\tilde{\mathfrak{b}}_\lambda$ is $\hat{\mathfrak{m}}$ -primary. As before, we compute

$$v(\mathfrak{c}) = \hat{v}(\tilde{\mathfrak{b}}_\lambda) \geq \lambda \hat{v}(\mathfrak{b}_\bullet) = \lambda v(\mathfrak{a}_\bullet).$$

Therefore, $\tilde{\mathfrak{b}}_\lambda = \mathfrak{c}\hat{R} \subset \tilde{\mathfrak{a}}_\lambda \hat{R}$. □

Proposition 3.3.11. *If $\mathfrak{a}_\bullet$ is an $\mathfrak{m}$ -filtration and $\mathfrak{b}_\bullet = \mathfrak{a}_\bullet \hat{R}$, then $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$.*

Proof. Since $R/\mathfrak{a}_\lambda$ and $\hat{R}/\mathfrak{b}_\lambda$ are isomorphic as Artinian rings, $\ell(R/\mathfrak{a}_\lambda) = \ell(\hat{R}/\mathfrak{b}_\lambda)$ for each $\lambda > 0$. Therefore, the equality of multiplicities holds. □

3.3.3 Saturation and multiplicity

In this section we prove Theorem A and a number of corollaries. The theorem is a consequence of the following two propositions.

Proposition 3.3.12. *Let $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ be $\mathfrak{m}$ -filtrations. If $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$, then $v(\mathfrak{a}_\bullet) = v(\mathfrak{b}_\bullet)$ for all $v \in \text{Val}_{R,\mathfrak{m}}^+$.*

The proposition was shown when v is divisorial in [Cut21, Theorem 7.3] using Okounkov bodies. The proof below instead follows the strategy of [LX20, Proposition 2.7] and [XZ21, Lemma 3.9].

Proof. Suppose the statement is false. Then there exists $v \in \text{Val}_{R,\mathfrak{m}}^+$ such that $v(\mathfrak{b}_\bullet) < v(\mathfrak{a}_\bullet)$. After scaling v , we may assume $v(\mathfrak{a}_\bullet) = 1$. Therefore, there exists $l \in \mathbb{Z}_{>0}$ so that $v(\mathfrak{b}_l)/l < v(\mathfrak{a}_\bullet) = 1$. Hence, we may choose $f \in \mathfrak{b}_l \setminus \mathfrak{a}_l$ such that $k := v(f) < l$.

Now, consider the map

$$\phi : R \rightarrow \mathfrak{b}_{lm}/\mathfrak{a}_{lm} \quad \text{sending} \quad g \mapsto f^m g.$$

We claim $\ker(\phi) \subset \mathfrak{a}_{(l-k)m}(v)$. Indeed, if $g \in \ker \phi$, then $f^m g \in \mathfrak{a}_{lm}$. Hence,

$$v(g) = v(f^m g) - v(f^m) \geq v(\mathfrak{a}_{lm}) - km \geq lm - km = (l - k)m$$

as desired. Using the claim, we deduce

$$\ell(\mathfrak{b}_{lm}/\mathfrak{a}_{lm}) \geq \ell(R/\ker \phi) \geq \ell(R/\mathfrak{a}_{(l-k)m}(v)). \quad (3.4)$$

Finally, we compute

$$e(\mathfrak{a}_\bullet) - e(\mathfrak{b}_\bullet) = \lim_{m \rightarrow \infty} \frac{\ell(\mathfrak{b}_{lm}/\mathfrak{a}_{lm})}{(lm)^n/n!} \geq \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_{(l-k)m}(v))}{(lm)^n/n!} = \frac{(l-k)^n \text{vol}(v)}{l^n} > 0,$$

where the first inequality is by (3.4) and the second uses that v has positive volume.

This contradicts our assumption that $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$. $\square$

Proposition 3.3.13. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$ -filtrations. Assume $(R, \mathfrak{m})$ is complete. If $v(\mathfrak{b}_\bullet) \leq v(\mathfrak{a}_\bullet)$ for all $v \in \text{DivVal}_{R, \mathfrak{m}}$, then $e(\mathfrak{b}_\bullet) \leq e(\mathfrak{a}_\bullet)$.*

The proposition and its proof are local analogues of [Szé15, Lemma 22], which concerns the volumes of graded linear series of projective varieties.

Remark 3.3.14. The assumption that R is complete in Proposition 3.3.13 implies that R is Nagata [Sta18, Lemma 032W](see [Sta18, Definition 033S]). The latter property will be used in the proof of Proposition 3.3.13 to ensure certain normalization morphisms are proper.

Proof. We claim that there exists a sequence of proper birational morphisms

$$\cdots \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 = X$$

such that the each X_i is normal and the sheafs $\mathfrak{a}_l \cdot \mathcal{O}_{X_m}$ and $\mathfrak{b}_l \cdot \mathcal{O}_{X_m}$ are line bundles when $l \leq m$. Such a sequence can be constructed inductively as follows. Let $X_i \rightarrow X_{i-1}$ be defined by the composition

$$X_i \rightarrow X_{i,2} \rightarrow X_{i,1} \rightarrow X_{i-1},$$

where $X_{i,1}$ is the blowup of $\mathfrak{a}_i \cdot \mathcal{O}_{X_{i-1}}$, $X_{i,2}$ is the blowup of $\mathfrak{b}_i \cdot \mathcal{O}_{X_{i,1}}$, and X_i the normalization of $X_{i,2}$. Note that $X_{i,2} \rightarrow X$ is proper, since $X_{i,2} \rightarrow X_{i-1}$ is a blowup and $X_{i-1} \rightarrow X$ is proper by our inductive assumption. Since X is Nagata and $X_{i,2} \rightarrow X$ is finite type, $X_{i,2}$ is Nagata by [Sta18, Lemma 035A]. Therefore, $X_i \rightarrow X_{i,2}$ is finite by the definition of Nagata, and we conclude the composition $X_i \rightarrow X_{i-1}$ is proper, which completes the proof of the claim.

Next, consider a proper birational morphism $Y \rightarrow X$ with Y normal and factoring as

$$Y \xrightarrow{\pi_m} X_m \longrightarrow X,$$

For $l \leq m$, there exist Cartier divisors G_l and G'_l on Y such that

$$\mathfrak{a}_l \cdot \mathcal{O}_Y = \mathcal{O}_Y(G_l) \quad \text{and} \quad \mathfrak{b}_l \cdot \mathcal{O}_Y = \mathcal{O}_Y(G'_l).$$

Set $F_l := \frac{G_l}{l}$ and $F'_l := \frac{G'_l}{l}$, which are relatively nef over X and satisfy

$$F_l = - \sum_{E \subset Y} \frac{\text{ord}_E(\mathfrak{a}_l)}{l} E \quad \text{and} \quad F'_l = - \sum_{E \subset Y} \frac{\text{ord}_E(\mathfrak{b}_l)}{l} E,$$

where the sums run through prime divisors $E \subset Y$. Throughout the proof, we will

without mention replace Y with higher birational models so it factors through certain $X_m \rightarrow X$.

Given $\epsilon > 0$, by Lemma 3.2.1 and (3.3), there exists $m_0 > 0$ such that

$$-F_{m_0}^{n-1} \cdot F_{m_0} = e(\mathbf{a}_{m_0})/m_0^n < e(\mathbf{a}_\bullet) + \epsilon. \quad (3.5)$$

For a multiple m_1 of m_0 ,

$$-(F_{m_0})^{n-1} \cdot F_{m_1} < e(\mathbf{a}_\bullet) + \epsilon,$$

since $F_{m_0} \leq F_{m_1}$ and F_{m_0} is nef over X . Now, we compute

$$\begin{aligned} -(F_{m_0})^{n-1} \cdot F_{m_1} + (F_{m_0})^{n-1} \cdot F'_{m_1} &= \sum_{E \subset Y} (-\text{ord}_E(F_{m_1}) + \text{ord}_E(F'_{m_1})) (F_{m_0})^{n-1} \cdot E \\ &= \sum_{E \subset X_{m_0}} (-\text{ord}_E(F_{m_1}) + \text{ord}_E(F'_{m_1})) (\pi_{m_0*} F_{m_0})^{n-1} \cdot E, \end{aligned}$$

where the second equality uses that F_{m_0} is the pullback of $\pi_{m_0*} F_{m_0}$ and the projection formula [Kle05, Proposition B.16]. Since $\pi_{m_0*} F_{m_0}$ is nef over X and

$$\lim_{m \rightarrow \infty} (-\text{ord}_E(F_m) + \text{ord}_E(F'_m)) = \text{ord}_E(\mathbf{a}_\bullet) - \text{ord}_E(\mathbf{b}_\bullet) \geq 0,$$

we may choose m_1 sufficiently large and divisible by m_0 such that

$$-(F_{m_0})^{n-1} \cdot F'_{m_1} \leq -F_{m_0}^{n-1} \cdot F_{m_1} + \epsilon.$$

Therefore,

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_0} = -(F_{m_0})^{n-1} \cdot F'_{m_1} \leq e(\mathbf{a}_\bullet) + 2\epsilon,$$

where the first equality is Proposition 2.4.1.

For any multiple m_2 of m_1 ,

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_2} \leq e(\mathbf{a}_\bullet) + 2\epsilon,$$

since $F_{m_0} \leq F_{m_2}$ and the terms F'_{m_1} and F_{m_0} are nef over X . Similar to the previous paragraph, we compute

$$\begin{aligned} & -(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_2} - (F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F'_{m_2} \\ &= \sum_{E \subset X_{m_1}} (-\text{ord}_E(F_{m_2}) + \text{ord}_E(F'_{m_2})) ((\pi_{m_1*} F_{m_0})^{n-1} \cdot \pi_{m_1*} F'_{m_1}) \cdot E. \end{aligned}$$

and, hence, we may choose m_2 sufficiently large and divisible by m_1 so that

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F'_{m_2} < -(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_2} + \epsilon.$$

Thus,

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F'_{m_2} < e(\mathbf{a}_\bullet) + 3\epsilon.$$

Repeating in this way gives

$$-F'_{m_1} \cdot F'_{m_2} \cdot \dots \cdot F'_{m_n} < e(\mathbf{a}_\bullet) + (n+1)\epsilon,$$

where each m_i divides m_{i+1} . Since each F'_{m_i} is nef over X and $F'_{m_i} \leq F'_{m_{i+1}}$, we see

$$-(F'_{m_n})^n < e(\mathbf{a}_\bullet) + n\epsilon.$$

Using that $e(\mathbf{b}_\bullet) = \inf_m e(\mathbf{b}_m)/m^n = \inf_m -(F'_m)^n$ by Lemma 3.2.1 and (3.3), we see $e(\mathbf{b}_\bullet) < e(\mathbf{a}_\bullet) + (n+1)\epsilon$. Therefore, $e(\mathbf{b}_\bullet) \leq e(\mathbf{a}_\bullet)$. $\square$

Remark 3.3.15. When $x \in X$ is an isolated singularity on a normal variety, Proposition 3.3.13 follows easily from the intersection theory for nef b -divisors developed

in [BdFF12]. Indeed, the assumption $v(\mathfrak{a}_\bullet) \leq v(\mathfrak{b}_\bullet)$ implies $Z(\mathfrak{a}_\bullet) \geq Z(\mathfrak{b}_\bullet)$, where $Z(\mathfrak{a}_\bullet)$ is the nef b -Weil $\mathbb{R}$ -divisor associated to $\mathfrak{a}_\bullet$. The proposition then follows from [BdFF12, Remark 4.17]. However, when x is not an isolated singularity, a satisfactory intersection theory for nef b -divisors seems missing from the literature.

3.3.4 Saturation and finite volume valuations

Using results from the previous section, we show that the saturation can be defined using positive volume valuations, rather than only divisorial valuations.

Proposition 3.3.16. *If $\mathfrak{a}_\bullet$ is an $\mathfrak{m}$ -filtration and $\lambda \in \mathbb{R}_{>0}$, then*

$$\tilde{\mathfrak{a}}_\lambda = \{f \in \mathfrak{m} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{Val}_{R,\mathfrak{m}}^+\}$$

The proposition will be deduced from the following lemma.

Lemma 3.3.17. *If $\{v_i\}_{i \in I}$ is a collection of valuations in $\text{Val}_{R,\mathfrak{m}}^+$ and $\{c_i\}_{i \in I}$ of non-negative real numbers, then the $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ defined by*

$$\mathfrak{a}_\lambda := \{f \in \mathfrak{m} \mid v_i(f) \geq \lambda c_i \text{ for all } i \in I\}$$

is saturated. Hence, $\mathfrak{a}_\bullet(v)$ is saturated for any $v \in \text{Val}_{R,\mathfrak{m}}^+$.

Proof. Suppose the statement is false. Then there exists some $\lambda \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_\lambda \subsetneq \tilde{\mathfrak{a}}_\lambda$. Thus, there exists $f \in \tilde{\mathfrak{a}}_\lambda$ and $i \in I$ such that $v_i(f) < \lambda c_i$. We claim $v_i(\tilde{\mathfrak{a}}_\bullet) < v_i(\mathfrak{a}_\bullet)$. Indeed,

$$v_i(\tilde{\mathfrak{a}}_\bullet) = \lim_{m \rightarrow \infty} \frac{v_i(\tilde{\mathfrak{a}}_{\lfloor \lambda m \rfloor})}{\lfloor \lambda m \rfloor} \leq \lim_{m \rightarrow \infty} \frac{v_i(f^m)}{\lfloor \lambda m \rfloor} = \frac{v_i(f)}{\lambda} < c_i,$$

where the second equality uses that $f^m \in (\tilde{\mathfrak{a}}_\lambda)^m \subset \tilde{\mathfrak{a}}_{\lambda m} \subset \tilde{\mathfrak{a}}_{\lfloor \lambda m \rfloor}$. Since $v_i(\mathfrak{a}_\bullet) \geq c_i$, the claim holds.

By Proposition 3.3.12 and the claim, $e(\tilde{\mathfrak{a}}_\bullet) < e(\mathfrak{a}_\bullet)$. The latter contradicts Theorem A. $\square$

Proof of Proposition 3.3.16. Let $\mathfrak{b}_\bullet$ denote the $\mathfrak{m}$ -filtration defined by

$$\mathfrak{b}_\lambda := \{f \in \mathbb{R} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{Val}_{R,\mathfrak{m}}^+\}.$$

Notice that $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$, where the second inclusion uses that all divisorial valuations have positive volume. Taking saturations gives $\tilde{\mathfrak{a}}_\bullet \subset \tilde{\mathfrak{b}}_\bullet \subset \tilde{\tilde{\mathfrak{a}}}_\bullet$. Since $\mathfrak{b}_\bullet$ and $\tilde{\mathfrak{a}}_\bullet$ are saturated by Lemma 3.3.17, we conclude $\tilde{\mathfrak{a}}_\bullet \subset \mathfrak{b}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$. $\square$

3.4 Rees' Theorem and Minkowski's inequality

In this section we prove Theorem A and a number of corollaries, including Corollary B.

Proof of Theorem A. By Propositions 3.3.10 and 3.3.11, we may assume $(R, \mathfrak{m})$ is complete. This condition will be needed below to apply Proposition 3.3.13.

By Propositions 3.3.12 and 3.3.13, $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$ if and only if $v(\mathfrak{a}_\bullet) = v(\mathfrak{b}_\bullet)$ for all $v \in \text{DivVal}_{R,\mathfrak{m}}$. By Proposition 3.3.9, the latter condition holds if and only if $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_\bullet$. $\square$

Corollary 3.4.1. *If $\mathfrak{a}_\bullet$ is an $\mathfrak{m}$ -filtration, then $e(\mathfrak{a}_\bullet) = e(\tilde{\mathfrak{a}}_\bullet)$.*

Proof. By Lemma 3.3.8, $\mathfrak{a}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$ and $\tilde{\mathfrak{a}}$ is saturated. Thus, Theorem A implies $e(\mathfrak{a}_\bullet) = e(\tilde{\mathfrak{a}}_\bullet)$. $\square$

Corollary 3.4.2. *Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$ -filtrations. The following are equivalent:*

1. $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_\bullet$.
2. $e(\mathfrak{a}_\bullet) = e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) = e(\mathfrak{b}_\bullet)$.

Proof of Corollary 3.4.2. Assume (1) holds. Observe that

$$e(\mathfrak{a}_\bullet) = e(\tilde{\mathfrak{a}}_\bullet) = e(\tilde{\mathfrak{b}}_\bullet) = e(\mathfrak{b}_\bullet),$$

where the first and third equality follow from Corollary 3.4.1. Thus, it remains to show that $e(\mathfrak{b}_\bullet) = e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet)$. First, note that $e(\mathfrak{b}_\bullet) \leq e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet)$ holds trivially, since $\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet \subset \mathfrak{b}_\bullet$. For the reverse inequality, we compute

$$\begin{aligned} e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) &= \lim_{m \rightarrow \infty} \frac{\ell(R/(\mathfrak{a}_m \cap \mathfrak{b}_m))}{m^n/n!} = \lim_{m \rightarrow \infty} \left(\frac{\ell(R/(\mathfrak{a}_m))}{m^n/n!} + \frac{\ell(R/\mathfrak{b}_m)}{m^n/n!} - \frac{\ell(R/(\mathfrak{a}_m + \mathfrak{b}_m))}{m^n/n!} \right) \\ &= e(\mathfrak{a}_\bullet) + e(\mathfrak{b}_\bullet) - \lim_{m \rightarrow \infty} \frac{\ell(R/(\mathfrak{a}_m + \mathfrak{b}_m))}{m^n/n!} \\ &\leq e(\mathfrak{a}_\bullet) + e(\mathfrak{b}_\bullet) - \lim_{m \rightarrow \infty} \frac{\ell(R/\tilde{\mathfrak{a}}_m)}{m^n/n!} \\ &= e(\mathfrak{a}_\bullet) + e(\mathfrak{b}_\bullet) - e(\tilde{\mathfrak{a}}_\bullet) \\ &= e(\mathfrak{b}_\bullet), \end{aligned}$$

where the inequality uses that $\mathfrak{a}_m + \mathfrak{b}_m \subset \tilde{\mathfrak{a}}_m + \tilde{\mathfrak{b}}_m = \tilde{\mathfrak{a}}_m + \tilde{\mathfrak{a}}_m = \tilde{\mathfrak{a}}_m$ by the assumption that (1) holds. Therefore, (2) holds.

Conversely, assume (2) holds. Applying Theorem A to both $(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) \subset \mathfrak{a}_\bullet$ and $(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) \subset \mathfrak{b}_\bullet$, we see $\widetilde{\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet} = \tilde{\mathfrak{a}}_\bullet$ and $\widetilde{\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet} = \tilde{\mathfrak{b}}_\bullet$, which implies (1) holds. $\square$

The following result was proven when R is regular in [Mus02, Theorem 1.7.2].

Corollary 3.4.3. *An $\mathfrak{m}$ -filtration $\mathfrak{a}_\bullet$ is linearly bounded if and only if $e(\mathfrak{a}_\bullet) > 0$.*

Proof. If $\mathfrak{a}_\bullet$ is linearly bounded, then there exists $c > 0$ such that $\mathfrak{a}_\lambda \subset \mathfrak{m}^{\lceil c\lambda \rceil}$ for all $\lambda > 0$. Thus, $e(\mathfrak{a}_\bullet) \geq c^n e(\mathfrak{m}) > 0$ as desired.

Next, assume $e(\mathfrak{a}_\bullet) > 0$. We claim that there exists $v \in \text{DivVal}_{R,\mathfrak{m}}$ such that $v(\mathfrak{a}_\bullet) > 0$. If not, then $\tilde{\mathfrak{a}}_\lambda = \mathfrak{m}$ for all $\lambda > 0$. Using Corollary 3.4.1, we then see $e(\mathfrak{a}_\bullet) = e(\tilde{\mathfrak{a}}_\bullet) = 0$, which is a contradiction. Now, fix $v \in \text{DivVal}_{R,\mathfrak{m}}$ with $v(\mathfrak{a}_\bullet) > 0$. Using that $\mathfrak{a}_\bullet \subset \mathfrak{a}_\bullet(v)$ and Proposition 3.2.2, we conclude $e(\mathfrak{a}_\bullet) \geq e(\mathfrak{a}_\bullet(v)) > 0$. $\square$

Proposition 3.4.4. *[Cut21, Theorem 10.3] Let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be $\mathfrak{m}$ -filtrations with positive multiplicity. If $e(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)^{1/n} = e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n}$, then there exists $c > 0$ such that $v(\mathfrak{a}_\bullet) = cv(\mathfrak{b}_\bullet)$ for all $v \in \text{DivVal}_{R, \mathfrak{m}}$.*

Proof of Corollary B. By [Cut15, Theorem 3.2],

$$e(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)^{1/n} \leq e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n}. \quad (3.6)$$

Thus, it remains to analyze when the inequality is an equality.

If (3.6) is an equality, then Proposition 3.4.4 and Proposition 3.3.9 imply there exists $c > 0$ such that $\widetilde{\mathfrak{a}}_\bullet = \widetilde{\mathfrak{b}}_{c\bullet}$. Conversely, assume there exists $c > 0$ such that $\widetilde{\mathfrak{a}}_\bullet = \widetilde{\mathfrak{b}}_{c\bullet}$. Note that

$$v(\mathfrak{a}_\bullet \mathfrak{b}_\bullet) = v(\mathfrak{a}_\bullet) + v(\mathfrak{b}_\bullet) = (c + 1)v(\mathfrak{b}_\bullet)$$

where the first equality uses that $\frac{v(\mathfrak{a}_m \mathfrak{b}_m)}{m} = \frac{v(\mathfrak{a}_m)}{m} + \frac{v(\mathfrak{b}_m)}{m}$ for all $m \in \mathbb{N}$ and the second is by Proposition 3.3.9. Therefore, Proposition 3.3.9 implies $\widetilde{\mathfrak{a}_\bullet \mathfrak{b}_\bullet} = \widetilde{\mathfrak{b}}_{(c+1)\bullet}$. We now compute

$$e(\mathfrak{a}_\bullet \mathfrak{b}_\bullet)^{1/n} = e(\mathfrak{b}_{(c+1)\bullet})^{1/n} = e(\mathfrak{b}_{c\bullet})^{1/n} + e(\mathfrak{b}_\bullet)^{1/n} = e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n},$$

where the first and third equality hold by Corollary 3.4.2. □

3.5 Lattice structure

Denote the subset of Fil of saturated filtrations by $\text{Fil}^s := \{\mathfrak{a}_\bullet \in \text{Fil} \mid \mathfrak{a}_\bullet \text{ is saturated}\}$.

There is a canonical element in Fil^s , that is, the saturation $\widetilde{\mathfrak{m}}^\bullet$ of the $\mathfrak{m}$ -adic filtration.

Denote its order function by $\chi_0 := \text{ord}_{\widetilde{\mathfrak{m}}^\bullet} = \widetilde{\text{ord}}_\mathfrak{m}$.

The following lemma asserts that for saturated filtrations, the partial order by

inclusion can be detected by evaluating on all divisorial valuations, which is not the case for general filtrations.

Lemma 3.5.1. *Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$ -filtration and $\mathfrak{b}_\bullet \in \text{Fil}^s$. Then $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ if and only if $v(\mathfrak{a}_\bullet) \geq v(\mathfrak{b}_\bullet)$ for any $v \in \text{DivVal}_{X,x}$.*

Proof. If $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ then by definition $v(\mathfrak{a}_\bullet) \geq v(\mathfrak{b}_\bullet)$ for any $v \in \text{Val}_{X,x}$.

Conversely, assume that $\mathfrak{a}_\bullet \not\subset \mathfrak{b}_\bullet$. Then there exists $\lambda \in \mathbb{R}_{>0}$ and $f \in \mathfrak{a}_\lambda$ such that $f \notin \mathfrak{b}_\lambda$. This implies that there exists $\epsilon > 0$ and $v \in \text{DivVal}_{X,x}$ such that $v(f) < \lambda \cdot v(\mathfrak{b}_\bullet) - \epsilon$. Hence for any $m \in \mathbb{Z}_{>0}$, we have

$$v(\mathfrak{a}_{m\lambda}) \leq v(\mathfrak{a}_\lambda^m) = mv(\mathfrak{a}_\lambda) \leq mv(f) < m(\lambda \cdot v(\mathfrak{b}_\bullet) - \epsilon),$$

where the first inequality follows from $\mathfrak{a}_\lambda^m \subset \mathfrak{a}_{m\lambda}$. Dividing by $m\lambda$ and letting $m \rightarrow \infty$, we get $v(\mathfrak{a}_\bullet) \leq v(\mathfrak{b}_\bullet) - \epsilon/\lambda < v(\mathfrak{b}_\bullet)$. $\square$

In other words, $\tilde{\mathfrak{a}}_\bullet$ is the unique maximal $\mathfrak{m}$ -filtration $\mathfrak{b}_\bullet \supset \mathfrak{a}_\bullet$ such that $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$. We now prove a formula for saturations.

Lemma 3.5.2. *Let $\mathfrak{a}_\bullet \in \text{Fil}$. Then we have*

$$\tilde{\mathfrak{a}}_\bullet = \bigcap_{v(\mathfrak{a}_\bullet)=1} \mathfrak{a}_\bullet(v) = \bigcap_{v \geq \text{ord}_{\mathfrak{a}_\bullet}} \mathfrak{a}_\bullet(v),$$

where the first intersection is taken over all $v \in \text{DivVal}_{R,\mathfrak{m}}$ with $v(\mathfrak{a}_\bullet) = 1$ and the second over all $v \in \text{DivVal}_{R,\mathfrak{m}}$ such that $v(f) \geq \text{ord}_{\mathfrak{a}_\bullet}(f)$ for any $f \in \mathfrak{m}$.

Proof. Note that $v(\mathfrak{a}_\bullet) > 0$ for any $v \in \text{DivVal}_{R,\mathfrak{m}}$, by Lemma 3.4.3. For $v \in \text{DivVal}_{R,\mathfrak{m}}$ with $v(\mathfrak{a}_\bullet) = 1$, by Proposition 3.3.9 we know that $v(\tilde{\mathfrak{a}}_\bullet) = 1$, so $\tilde{\mathfrak{a}}_\bullet \subset \mathfrak{a}_\bullet(v)$ by definition. Hence $\tilde{\mathfrak{a}}_\bullet \subset \bigcap_{v(\mathfrak{a}_\bullet)=1} \mathfrak{a}_\bullet(v)$. Conversely, if $f \in \bigcap_{v(\mathfrak{a}_\bullet)=1} \mathfrak{a}_\lambda(v)$, then for any $v \in \text{DivVal}_{R,\mathfrak{m}}$, since $v'(\mathfrak{a}_\bullet) = 1$, where $v' := v/v(\mathfrak{a}_\bullet)$, we have

$$v(f) = \frac{v}{v(\mathfrak{a}_\bullet)}(f) \cdot v(\mathfrak{a}_\bullet) \geq \lambda \cdot v(\mathfrak{a}_\bullet).$$

Hence $\cap_{v(\mathfrak{a}_\bullet)=1} \mathfrak{a}_\lambda(v) \subset \tilde{\mathfrak{a}}_\bullet$ and this proves the first equality.

If $v(\mathfrak{a}_\bullet) = 1$ then for any $f \in \mathfrak{a}_m$, we have $v(f) \geq v(\mathfrak{a}_m) \geq mv(\mathfrak{a}_\bullet)$. Hence $v(f) \geq \text{ord}_{\mathfrak{a}_\bullet}(f)$, that is, $v \geq \text{ord}_{\mathfrak{a}_\bullet}$, and we get $\cap_{v(\mathfrak{a}_\bullet)=1} \mathfrak{a}_m(v) \supset \cap_{v \geq \text{ord}_{\mathfrak{a}_\bullet}} \mathfrak{a}_m(v)$. Conversely, if $v \geq \text{ord}_{\mathfrak{a}_\bullet}$, then $v(\mathfrak{a}_m) \geq \text{ord}_{\mathfrak{a}_\bullet}(\mathfrak{a}_m) \geq m$, which implies $v(\mathfrak{a}_\bullet) \geq 1$, that is, $v \geq v' := v/v(\mathfrak{a}_\bullet)$, where $v'(\mathfrak{a}_\bullet) = 1$. Thus we get

$$\cap_{v(\mathfrak{a}_\bullet)=1} \mathfrak{a}_m(v) \subset \cap_{v'} \mathfrak{a}_m(v') \subset \cap_{v \geq \text{ord}_{\mathfrak{a}_\bullet}} \mathfrak{a}_m(v).$$

This proves the second equality. $\square$

Based on the observation above, we give a characterization for saturated filtrations.

Proposition 3.5.3. *A filtration $\mathfrak{a} \in \text{Fil}$ is saturated if and only if it is of the form*

$$\mathfrak{a}_\bullet = \cap_{v \in \Sigma} \mathfrak{a}_\bullet(v),$$

where Σ is a nonempty subset of either $\text{DivVal}_{R,\mathfrak{m}}$ or $\text{Val}_{R,\mathfrak{m}}^+$.

Proof. If $\mathfrak{a}_\bullet = \cap_{v \in \Sigma} \mathfrak{a}_\bullet(v)$ for some $\Sigma \subset \text{Val}_{R,\mathfrak{m}}^+$, then by Lemma 3.3.17, $\mathfrak{a}_\bullet$ is saturated.

Conversely, if $\mathfrak{a}_\bullet$ is saturated, then by Lemma 3.5.2, $\mathfrak{a}_\bullet = \tilde{\mathfrak{a}}_\bullet = \cap_{v \in \Sigma_v} \mathfrak{a}_\bullet(v)$, where

$$\Sigma_v := \{v \in \text{DivVal}_{R,\mathfrak{m}} \mid v(\mathfrak{a}_\bullet) = 1\} \subset \text{DivVal}_{R,\mathfrak{m}}.$$

Since $\text{DivVal}_{R,\mathfrak{m}} \subset \text{Val}_{R,\mathfrak{m}}^+$, the proof is finished. $\square$

Corollary 3.5.4. *Let $\mathfrak{a}_{\bullet,i} \in \text{Fil}^s$ for $i \in I$. If $\cap_{i \in I} \mathfrak{a}_{\bullet,i} \in \text{Fil}$, then $\cap_{i \in I} \mathfrak{a}_{\bullet,i} \in \text{Fil}^s$.*

Proof. By Proposition 3.5.3, for any $i \in I$ we may write $\mathfrak{a}_{\bullet,i} = \cap_{v \in \Sigma_i} \mathfrak{a}_\bullet(v)$ for $\Sigma_i \subset \text{DivVal}_{X,x}$. Thus

$$\cap_{i \in I} \mathfrak{a}_{\bullet,i} = \cap_{v \in \cup_{i \in I} \Sigma_i} \mathfrak{a}_\bullet(v) \in \text{Fil}^s$$

by Proposition 3.5.3 again. $\square$

Definition-Lemma 3.5.5. Let $\mathfrak{a}_{\bullet,i} \in \text{Fil}$ for $i \in I \neq \emptyset$. If there exists $\mathfrak{b}_{\bullet} \in \text{Fil}$ such that $\mathfrak{a}_{\bullet,i} \subset \mathfrak{b}_{\bullet}$ for any $i \in I$, then there exists an $\mathfrak{m}$ -filtration, called the *saturated join* of the $\mathfrak{a}_{\bullet,i}$, denoted by $\bigvee_{s,i \in I} \mathfrak{a}_{\bullet,i}$, such that the following holds. If $\mathfrak{c}_{\bullet} \in \text{Fil}^s$ satisfies $\mathfrak{a}_{\bullet,i} \subset \mathfrak{c}_{\bullet}$ for any $i \in I$, then we have $\bigvee_{s,i \in I} \mathfrak{a}_{\bullet,i} \subset \mathfrak{c}_{\bullet}$. Moreover, $\bigvee_{s,i \in I} \mathfrak{a}_{\bullet,i} \in \text{Fil}^s$.

Proof. Fix $0 \in I$. By assumption, the set

$$J := \{\mathfrak{c}_{\bullet} \in \text{Fil}^s \mid \mathfrak{a}_{\bullet,i} \subset \mathfrak{c}_{\bullet} \text{ for any } i \in I\} \subset \text{Fil}^s$$

is non-empty, and $\mathfrak{a}_{\bullet,0} \subset \mathfrak{c}_{\bullet}$ for any $\mathfrak{c}_{\bullet} \in J$. Thus by Definition-Lemma 3.1.2, the intersection

$$\bigvee_{s,i \in I} \mathfrak{a}_{\bullet,i} := \bigcap_{\mathfrak{c}_{\bullet} \in J} \mathfrak{c}_{\bullet} \in \text{Fil},$$

where the linearly boundedness is because $\bigvee_{s,i \in I} \mathfrak{a}_{\bullet,i} \subset \mathfrak{b}_{\bullet}$ by construction. Now $\bigvee_{s,i \in I} \mathfrak{a}_{\bullet,i} \in \text{Fil}^s$ by Corollary 3.5.4. $\square$

We now relate the join, the saturated join and the saturation, and give a formula for the saturated join of two saturated filtrations.

Proposition 3.5.6. *If $\mathfrak{a}_{\bullet}, \mathfrak{b}_{\bullet} \in \text{Fil}$, then $\mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet} = \widetilde{\mathfrak{a}_{\bullet} \vee \mathfrak{b}_{\bullet}} = \widetilde{\mathfrak{a}_{\bullet}} \vee_s \widetilde{\mathfrak{b}_{\bullet}}$, and*

$$\mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet} = \bigcap_{v \in \text{DivVal}_{R,\mathfrak{m}}} \mathfrak{a}_{\bullet} \left(\frac{v}{\min\{v(\mathfrak{a}_{\bullet}), v(\mathfrak{b}_{\bullet})\}} \right). \quad (3.7)$$

Proof. By definition, $\widetilde{\mathfrak{a}_{\bullet} \vee \mathfrak{b}_{\bullet}}$ is a saturated filtration containing both $\mathfrak{a}_{\bullet}$ and $\mathfrak{b}_{\bullet}$, so by Definition-Lemma 3.5.5 we have $\mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet} \subset \widetilde{\mathfrak{a}_{\bullet} \vee \mathfrak{b}_{\bullet}}$. Conversely, by Definition-Lemma 3.5.5 again $\mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet}$ is saturated and $\mathfrak{a}_{\bullet} \vee \mathfrak{b}_{\bullet} \subset \mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet}$ by definition, hence $\widetilde{\mathfrak{a}_{\bullet} \vee \mathfrak{b}_{\bullet}} \subset \mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet}$. The inclusion $\mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet} \subset \widetilde{\mathfrak{a}_{\bullet}} \vee_s \widetilde{\mathfrak{b}_{\bullet}}$ is obvious. Conversely, if $\mathfrak{a}_{\bullet} \subset \mathfrak{c}_{\bullet}$ for $\mathfrak{c}_{\bullet} \in \text{Fil}^s$ then $\widetilde{\mathfrak{a}_{\bullet}} \subset \mathfrak{c}_{\bullet}$ and similarly for $\mathfrak{b}_{\bullet}$. Thus $\widetilde{\mathfrak{a}_{\bullet}} \vee_s \widetilde{\mathfrak{b}_{\bullet}} \subset \mathfrak{a}_{\bullet} \vee_s \mathfrak{b}_{\bullet}$. This proves the first equality.

To prove the formula (3.7), we first rewrite Lemma 3.5.2 as

$$\mathbf{a}_\bullet = \bigcap_{v \in \text{DivVal}_{R, \mathbf{m}}} \mathbf{a}_\bullet \left(\frac{v}{v(\mathbf{a}_\bullet)} \right) \quad (3.8)$$

for $\mathbf{a}_\bullet \in \text{Fil}^s$. Let $\mathbf{c}_\bullet$ be the $\mathbf{m}$ -filtration defined by the right-handed side of (3.7).

Since

$$\frac{v}{\min\{v(\mathbf{a}_\bullet), v(\mathbf{b}_\bullet)\}} = \max\left\{\frac{v}{v(\mathbf{a}_\bullet)}, \frac{v}{v(\mathbf{b}_\bullet)}\right\} = \max\left\{\frac{v}{v(\tilde{\mathbf{a}}_\bullet)}, \frac{v}{v(\tilde{\mathbf{b}}_\bullet)}\right\},$$

applying the formula (3.8) to $\tilde{\mathbf{a}}_\bullet$ and $\tilde{\mathbf{b}}_\bullet$ respectively, we know that $\mathbf{a}_\bullet \subset \tilde{\mathbf{a}}_\bullet \subset \mathbf{c}_\bullet$ and similarly $\mathbf{b}_\bullet \subset \mathbf{c}_\bullet$. Hence $v(\mathbf{c}_\bullet) \leq \min\{v(\mathbf{a}_\bullet), v(\mathbf{b}_\bullet)\}$ for any $v \in \text{Val}_{R, \mathbf{m}}$. By (3.7), for any $v \in \text{DivVal}_{X, x}$ we have $v(\mathbf{c}_\bullet) \geq v(\mathbf{a}_\bullet(v / \min\{v(\mathbf{a}_\bullet), v(\mathbf{b}_\bullet)\})) = \min\{v(\mathbf{a}_\bullet), v(\mathbf{b}_\bullet)\}$.

Thus

$$v(\mathbf{c}_\bullet) = \min\{v(\mathbf{a}_\bullet), v(\mathbf{b}_\bullet)\}. \quad (3.9)$$

We now verify that $\mathbf{c}_\bullet \in \text{Fil}$. Take $C \in \mathbb{R}_{>0}$ such that $\mathbf{a}_{C\bullet} \subset \mathbf{m}^\bullet$ and $\mathbf{b}_{C\bullet} \subset \mathbf{m}^\bullet$. Since $\mathbf{m}^\bullet \subset \tilde{\mathbf{m}}^\bullet$, for any $v \in \text{DivVal}_{R, \mathbf{m}}$, we have $Cv(\mathbf{a}_\bullet) \geq v(\mathbf{m}^\bullet) = v(\tilde{\mathbf{m}}^\bullet)$ and similarly $Cv(\mathbf{b}_\bullet) \geq v(\tilde{\mathbf{m}}^\bullet)$. By (3.9), $Cv(\mathbf{c}_\bullet) \geq v(\tilde{\mathbf{m}}^\bullet)$. So by Lemma 3.5.1 $\mathbf{c}_\bullet \subset \tilde{\mathbf{m}}^\bullet$ is linearly bounded.

By Corollary 3.5.4, $\mathbf{c}_\bullet \in \text{Fil}^s$, which implies $\mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet \subset \mathbf{c}_\bullet$. By definition and (3.9) we have $v(\mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet) \leq \min\{v(\mathbf{a}_\bullet), v(\mathbf{b}_\bullet)\} = v(\mathbf{c}_\bullet)$ for any $v \in \text{DivVal}_{R, \mathbf{m}}$. Since $\mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet \in \text{Fil}^s$ by Definition-Lemma 3.5.5, we may apply Lemma 3.5.1 to conclude that $\mathbf{c}_\bullet \subset \mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet$. This proves the formula (3.7) and finishes the proof of the Proposition. $\square$

We now show that the lattice operations on Fil^s satisfy the distributive law.

Proposition 3.5.7. *If $\mathbf{a}_\bullet, \mathbf{b}_\bullet, \mathbf{c}_\bullet \in \text{Fil}^s$, then we have $\mathbf{a}_\bullet \vee_s (\mathbf{b}_\bullet \cap \mathbf{c}_\bullet) = (\mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet) \cap (\mathbf{a}_\bullet \vee_s \mathbf{c}_\bullet)$.*

Proof. We have $\mathbf{a}_\bullet \vee_s (\mathbf{b}_\bullet \cap \mathbf{c}_\bullet) \subset (\mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet) \cap (\mathbf{a}_\bullet \vee_s \mathbf{c}_\bullet)$, since $\mathbf{a}_\bullet \vee_s (\mathbf{b}_\bullet \cap \mathbf{c}_\bullet) \subset \mathbf{a}_\bullet \vee_s \mathbf{b}_\bullet$ and $\mathbf{a}_\bullet \vee_s (\mathbf{b}_\bullet \cap \mathbf{c}_\bullet) \subset \mathbf{a}_\bullet \vee_s \mathbf{c}_\bullet$.

Conversely, assume that $f \in (\mathfrak{a}_\bullet \vee_s \mathfrak{b}_\bullet)_\lambda \cap (\mathfrak{a}_\bullet \vee_s \mathfrak{c}_\bullet)_\lambda$. Then for any $v \in \text{DivVal}_{R,\mathfrak{m}}$ we know that

$$\begin{aligned} v(f)/\lambda &\geq \max\{v(\mathfrak{a}_\bullet \vee_s \mathfrak{b}_\bullet), v(\mathfrak{a}_\bullet \vee_s \mathfrak{c}_\bullet)\} \\ &\geq \max\{\min\{v(\mathfrak{a}_\bullet), v(\mathfrak{b}_\bullet)\}, \min\{v(\mathfrak{a}_\bullet), v(\mathfrak{c}_\bullet)\}\} \\ &\geq \min\{v(\mathfrak{a}_\bullet), \max\{v(\mathfrak{b}_\bullet), v(\mathfrak{c}_\bullet)\}\}, \end{aligned}$$

where the second inequality follows from Proposition 3.5.6, and the last inequality is easy to check. By Proposition 3.5.6 again, we get $f \in (\mathfrak{a}_\bullet \vee_s (\mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet))_\lambda$. This shows that $\mathfrak{a}_\bullet \vee_s (\mathfrak{c}_\bullet \cap \mathfrak{b}_\bullet) \supset (\mathfrak{a}_\bullet \vee_s \mathfrak{c}_\bullet) \cap \mathfrak{b}_\bullet$ and the equality must hold. $\square$

Example 3.5.8. We continue to consider the adic filtration $\mathfrak{a}^\bullet$ of $\mathfrak{a} \in \mathcal{I}_\mathfrak{m}$. By definition,

$$(\widetilde{\mathfrak{a}^\bullet})_\lambda = \{f \in \mathfrak{m} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}^\bullet) \text{ for any } v \in \text{DivVal}_{R,\mathfrak{m}}\}.$$

Since $v(\mathfrak{a}^\bullet) = \lim_{\lambda \rightarrow \infty} v(\mathfrak{a}^{[\lambda]})/\lambda = \lim_{\lambda \rightarrow \infty} \lceil \lambda \rceil v(\mathfrak{a})/\lambda = v(\mathfrak{a})$, for $m \in \mathbb{Z}_{>0}$ we have

$$(\widetilde{\mathfrak{a}^\bullet})_m = \{f \in \mathfrak{m} \mid v(f) \geq mv(\mathfrak{a}) = v(\mathfrak{a}^m) \text{ for any } v \in \text{DivVal}_{R,\mathfrak{m}}\}.$$

Hence $(\widetilde{\mathfrak{a}^\bullet})_m = \overline{\mathfrak{a}^m}$ is the integral closure of $\mathfrak{a}^m$. See for example [HS06, Theorem 6.8.3]. Thus we have a map $\mathcal{I}_\mathfrak{m} \rightarrow \text{Fil}^s$, $\mathfrak{a} \mapsto \widetilde{\mathfrak{a}^\bullet}$, which is an injection when restricted to the set $\mathcal{I}_\mathfrak{m}^{\text{ic}}$ of integrally closed $\mathfrak{m}$ -primary ideals. Note that in general, $\widetilde{\mathfrak{a}^\bullet}$ is not an adic filtration. For example, $(\widetilde{\mathfrak{m}^2})^\bullet$ has $1/2$ as a jumping number.

Even if $\mathfrak{a}$ is integrally closed, the powers of $\mathfrak{a}$ are in general not integrally closed, hence an $\mathfrak{a}$ -adic filtration is in general *not* saturated. By Example 3.1.4, the join of two adic filtrations is still an adic filtration, which means that the inclusion $\mathfrak{a}_\bullet \vee \mathfrak{b}_\bullet \subset \mathfrak{a}_\bullet \vee_s \mathfrak{b}_\bullet$ can be strict for general $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet \in \text{Fil}$. It is unclear to the author whether $\mathfrak{a}_\bullet \vee \mathfrak{b}_\bullet = \mathfrak{a}_\bullet \vee_s \mathfrak{b}_\bullet$ if $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet \in \text{Fil}^s$.

It is easy to see that $\widetilde{\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet} \subset \widetilde{\mathfrak{a}_\bullet} \cap \widetilde{\mathfrak{b}_\bullet}$, and the inclusion can be strict. Indeed, let

$R = \mathbb{k}[[x, y]]$, $\mathfrak{a} = (x^2, y^{10})$ and $\mathfrak{b} = (x^{10}, y^2)$, then it is not hard to get $\overline{\mathfrak{a}} = (x^2, xy^5, y^{10})$ and $\overline{\mathfrak{b}} = (x^{10}, x^5y, y^2)$. By looking at the Newton polytope of $\mathfrak{a} \cap \mathfrak{b} = (x^2y^2, x^{10}, y^{10})$, one can compute

$$\overline{\mathfrak{a} \cap \mathfrak{b}} = (x^2y^2, xy^6, x^6y, x^{10}, y^{10}).$$

In particular, by the calculation for adic filtrations above, $f := x^5y \in \overline{\mathfrak{a}} \cap \overline{\mathfrak{b}} = (\widetilde{\mathfrak{a}^\bullet} \cap \widetilde{\mathfrak{b}^\bullet})_1$, but $f \notin \widetilde{\overline{\mathfrak{a} \cap \mathfrak{b}}} = (\widetilde{\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet})_1$.

Next we compute the saturation of an ideal-adic filtration. First note that to prove two saturated filtrations are equal, it suffices to check the equality along an arbitrary unbounded sequence.

Lemma 3.5.9. *Let $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet \in \text{Fil}^s$. If there exists $\lambda_k \in \mathbb{R}_{>0}$ with $\lambda_k \rightarrow \infty$ such that $\mathfrak{a}_{\lambda_k} = \mathfrak{b}_{\lambda_k}$, then $\mathfrak{a}_\bullet = \mathfrak{b}_\bullet$.*

Proof. By assumption, for any $v \in \text{DivVal}_{R, \mathfrak{m}}$ we have

$$v(\mathfrak{a}_\bullet) = \lim_{k \rightarrow \infty} v(\mathfrak{a}_{\lambda_k})/\lambda_k = \lim_{k \rightarrow \infty} v(\mathfrak{b}_{\lambda_k})/\lambda_k = v(\mathfrak{b}_\bullet).$$

Hence by Lemma 3.5.1, $\mathfrak{a}_\bullet = \mathfrak{b}_\bullet$. □

Recall that for an ideal $\mathfrak{a} \in \mathcal{I}_R$, the set $\text{RV}(\mathfrak{a})$ of *Rees valuations* of $\mathfrak{a}$ is the minimal set of divisorial valuations such that for any $m \in \mathbb{Z}_{>0}$,

$$\overline{\mathfrak{a}^m} = \{f \in R \mid v(f) \geq mv(\mathfrak{a}) \text{ for any } v \in \text{RV}(\mathfrak{a})\}.$$

This is a finite set for any $\mathfrak{a}$. We refer to [HS06, Chapter 10] for more about Rees valuations.

Proposition 3.5.10. *Let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$ be an $\mathfrak{m}$ -primary ideal. Then*

$$\widetilde{\mathfrak{a}^\bullet} = \bigcap_{v \in \text{RV}(\mathfrak{a})} \mathfrak{a}_\bullet \left(\frac{v}{v(\mathfrak{a})} \right), \quad (3.10)$$

where $RV(\mathfrak{a})$ is the set of Rees valuations of $\mathfrak{a}$.

Proof. Denote $\mathfrak{b}_\bullet := \cap_{v \in RV(\mathfrak{a})} \mathfrak{a}_\bullet(\frac{v}{v(\mathfrak{a})})$, which is saturated by Proposition 3.5.3. By the definition of Rees valuations and the calculation in Example 3.5.8, we know that $(\tilde{\mathfrak{a}}^\bullet)_m = \mathfrak{b}_m$ for $m \in \mathbb{Z}_{>0}$. Thus (3.10) follows from Lemma 3.5.9. $\square$

We now prove Theorem C.

Proof of Theorem C. (1) By Definition-Lemma 3.1.2 and Definition-Lemma 3.1.3, the set of $\mathfrak{m}$ -filtrations is a lattice. It is easy to see that the intersection and join of two linearly bounded $\mathfrak{m}$ -filtrations are again linearly bounded, hence $(\text{Fil}, \subset, \cap, \vee)$ is a sublattice of all $\mathfrak{m}$ -filtrations. Since $\mathfrak{a} \subset \mathfrak{m}$, we know that $\mathfrak{a}^{[\lambda]} \subset \mathfrak{m}^{[\lambda]}$ and the filtration $\mathfrak{a}^\bullet$ is linearly bounded. Now the map $\mathcal{I}_\mathfrak{m} \rightarrow \text{Fil}, \mathfrak{a} \mapsto \mathfrak{a}^\bullet$ is clearly injective, and by Example 3.1.4, it is a join morphism.

(2) It follows from Definition-Lemma 3.1.2, Definition-Lemma 3.5.5 and Proposition 3.5.7 that $(\text{Fil}^s, \subset, \cap, \vee_s)$ is a distributive lattice. By Proposition 3.5.6, the saturation from Fil to Fil^s is a joint morphism, which is clearly surjective. By Example 3.5.8, the map $\mathcal{I}_\mathfrak{m}^{\text{ic}} \rightarrow \text{Fil}^s, \mathfrak{a} \mapsto \tilde{\mathfrak{a}}^\bullet$ is injective, and is a join morphism as the composition of two join morphisms. $\square$

Chapter 4

Convexity of multiplicities along geodesics

In this section we prove Theorem D and its applications Corollary E and Corollary F. The results in this section are taken from the joint work with Harold Blum and Yuchen Liu. We thank them for allowing these results to be reproduced in this thesis. Throughout this section, $(R, \mathfrak{m}, \kappa)$ denotes an n -dimensional, analytically irreducible, Noetherian local domain containing a field of arbitrary characteristic.

4.1 Duistermaat-Heckman measures in the local setting

4.1.1 Geodesics

Fix two $\mathfrak{m}$ -filtrations $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ with $e(\mathfrak{a}_{\bullet,i}) > 0$ for $i = 0, 1$.

Definition 4.1.1. For each $t \in (0, 1)$, we define an $\mathfrak{m}$ -filtration $\mathfrak{a}_{\bullet,t}$ by setting

$$\mathfrak{a}_{\lambda,t} = \sum_{\lambda=(1-t)\mu+t\nu} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1},$$

where the sum runs through all $\mu, \nu \in \mathbb{R}$ satisfying $\lambda = (1-t)\mu + t\nu$. We call $(\mathfrak{a}_{\bullet,t})_{t \in [0,1]}$ the *geodesic* between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$.

This definition is a local analog of the geodesic between two filtrations of the section ring of a polarized variety [BLXZ23, Reb23].

Example 4.1.2. Let $R := k[x, y]_{(x,y)}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_{>0}^2$, let $v_\alpha: \text{Frac}(R)^\times \rightarrow \mathbb{R}$ be the monomial valuation with weight α_1 and α_2 with respect to x and y , i.e.

$$v\left(\sum_{m,n \geq 0} c_{m,n} x^m y^n\right) = \min\{\alpha_1 m + \alpha_2 n \mid c_{m,n} \neq 0\}.$$

Note that $\mathfrak{a}_\lambda(v_\alpha) = \{x^m y^n \mid m\alpha_1 + n\alpha_2 \geq \lambda\}$. Now, fix $\alpha, \beta \in \mathbb{R}_{\geq 0}^2$ and consider the geodesic $\mathfrak{a}_{\bullet,t}$ between $\mathfrak{a}_{\bullet,0} := \mathfrak{a}_\bullet(v_\alpha)$ and $\mathfrak{a}_{\bullet,1} := \mathfrak{a}_\bullet(v_\beta)$. A short computation shows

$$\mathfrak{a}_{\bullet,t} = \mathfrak{a}_\bullet(v_{(1-t)\alpha + t\beta}).$$

This computation unfortunately does not generalize to the case of quasi-monomial valuations. See [LXZ22, Section 4] for a study of this failure in the global setting.

To prove Theorem D, we define a measure on $\mathbb{R}^2$ that encodes the multiplicities along the geodesic. The argument may be viewed as a local analogue of the construction in [BLXZ23, Section 3.1] for the geodesic between two filtrations of the section ring of a polarized variety. The latter global construction is motivated by [BC11] and [BHJ17, Theorem 4.3], which constructs a measure on $\mathbb{R}$ associated to a single filtration of the section ring of polarized variety.

Before proceeding with the proof, fix integers $C > 0$ and $D > 0$ such that

$$\begin{aligned} \mathfrak{m}^{[C\lambda]} &\subset \mathfrak{a}_{\lambda,0} \cap \mathfrak{a}_{\lambda,1}, \\ \mathfrak{a}_{D\lambda,0} &\subset \mathfrak{a}_{\lambda,1} \quad \text{and} \quad \mathfrak{a}_{D\lambda,1} \subset \mathfrak{a}_{\lambda,0}. \end{aligned}$$

for all $\lambda \geq 1$. The existence of C follows from Lemma 3.2.4, while the existence

of D follows from Lemma 3.2.4 and that $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ are both linearly bounded by Corollary 3.4.3.

4.1.2 Sequences of measures

For each $m \in \mathbb{Z}_{>0}$, consider the function $H_m: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$H_m(x, y) := \frac{\ell(R/(\mathfrak{a}_{mx,0} \cap \mathfrak{a}_{my,1}))}{m^n/n!}.$$

Notice that H_m is non-decreasing and left continuous in each variable. Using the sequence $(H_m)_{m \geq 1}$, we define a sequence of measures on $\mathbb{R}^2$.

Proposition 4.1.3. *The distributional derivative $\mu_m := -\frac{\partial^2 H_m}{\partial x \partial y}$ is a discrete measure on $\mathbb{R}^2$ and has support contained in $\{\frac{1}{D}x \leq y \leq Dx\} \cup \{0 \leq x \leq \frac{1}{m}\} \cup \{0 \leq y \leq \frac{1}{m}\}$.*

Before proving the proposition, we prove the following lemma allowing us to reduce to the case of complete local rings.

Lemma 4.1.4. *Let $(\hat{R}, \hat{\mathfrak{m}})$ denote the completion of $(R, \mathfrak{m})$, $\mathfrak{b}_{\bullet,i} = \mathfrak{a}_{\bullet,i} \cdot \hat{R}$ for $i = 0, 1$, and $(\mathfrak{b}_{t,\bullet})_{t \in [0,1]}$ the geodesic between $\mathfrak{b}_{\bullet,0}$ and $\mathfrak{b}_{\bullet,1}$. Then the following hold:*

1. $\ell(R/\mathfrak{a}_{a,0} \cap \mathfrak{a}_{b,1}) = \ell(\hat{R}/\mathfrak{b}_{a,0} \cap \mathfrak{b}_{b,1})$ for each $a, b \geq 0$.
2. $\ell(R/\mathfrak{a}_{a,t}) = \ell(\hat{R}/\mathfrak{b}_{a,t})$ for each $a \geq 0$ and $t \in [0, 1]$.

Proof. For $\mathfrak{m}$ -primary ideals $\mathfrak{c} \subset R$ and $\mathfrak{d} \subset R$,

$$(\mathfrak{c} + \mathfrak{d})\hat{R} = \mathfrak{c}\hat{R} + \mathfrak{d}\hat{R} \quad \text{and} \quad (\mathfrak{c} \cap \mathfrak{d})\hat{R} = \mathfrak{c}\hat{R} \cap \mathfrak{d}\hat{R}.$$

Therefore,

$$(\mathfrak{a}_{a,0} \cap \mathfrak{a}_{b,1})\hat{R} = \mathfrak{b}_{a,0} \cap \mathfrak{b}_{b,1} \quad \text{and} \quad \mathfrak{a}_{a,t}\hat{R} = \mathfrak{b}_{a,t}.$$

Statements (1) and (2) follow from these equalities. □

Proof of Proposition 4.1.3. By Lemma 4.1.4, it suffices to prove the statement when $(R, \mathfrak{m})$ is complete. Since R contains a field by assumption and is complete, there is an inclusion $\kappa \hookrightarrow R$ such that the composition $\kappa \hookrightarrow R \rightarrow \kappa$ is the identity. This will be helpful, since any finite length R -module is naturally a finite dimensional κ -vector space via restriction of scalars.

Fix an integer $N > 0$ and consider the finite dimensional κ -vector space $V := R/\mathfrak{m}^{NCm}$. The $\mathfrak{m}$ -filtrations $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ induce decreasing filtrations $\mathcal{F}_0^\bullet$ and $\mathcal{F}_1^\bullet$ of V defined by

$$\mathcal{F}_0^\lambda V := \text{im}(\mathfrak{a}_{\lambda,0} \rightarrow V) \quad \text{and} \quad \mathcal{F}_1^\lambda V := \text{im}(\mathfrak{a}_{\lambda,1} \rightarrow V).$$

Observe that when $x < N$ and $y < N$,

$$\ell(R/(\mathfrak{a}_{mx,0} \cap \mathfrak{a}_{my,1})) = \dim_\kappa(V/(\mathcal{F}_0^{mx,0}V \cap \mathcal{F}_1^{my,1}V)) = \dim_\kappa V - \dim_\kappa(\mathcal{F}_0^{mx,0}V \cap \mathcal{F}_1^{my,1}V),$$

where the first equality uses that $\mathfrak{m}^{NCm} \subset \mathfrak{a}_{mx,0} \cap \mathfrak{a}_{my,1}$.

To analyze the dimension of $\mathcal{F}_0^{mx}V \cap \mathcal{F}_1^{my}V$, we use that any finite dimensional vector space with two filtrations admits a basis simultaneously diagonalizing both filtrations [BE21, Proposition 1.14] [AZ22, Lemma 3.1]. This means there exists a basis $(s_1, \dots, s_\ell)$ for V such that each $\mathcal{F}_j^\lambda V$ is the span of some subset of the basis elements. Hence, if we set $\lambda_{i,j} := \sup\{\lambda \in \mathbb{R} \mid s_i \in \mathcal{F}_j^\lambda V\}$, then

$$\mathcal{F}_0^\lambda V = \text{span}\langle s_i \mid \lambda_{i,0} \geq \lambda \rangle \quad \text{and} \quad \mathcal{F}_1^\lambda V = \text{span}\langle s_i \mid \lambda_{i,1} \geq \lambda \rangle.$$

Using that $\dim_\kappa \mathcal{F}_0^{mx}V \cap \mathcal{F}_1^{my}V = \#\{i \mid \lambda_{i,0} \geq mx \text{ and } \lambda_{i,1} \geq my\}$, we compute

$$\frac{\partial^2}{\partial x \partial y} \dim_\kappa(\mathcal{F}_0^{mx}V \cap \mathcal{F}_1^{my}V) = \sum_{i=1}^{\ell} \delta_{(m^{-1}\lambda_{i,0}, m^{-1}\lambda_{i,1})}.$$

Therefore, the restriction of μ_m to $(-\infty, N) \times (-\infty, N)$ equals

$$\mu_m|_{(-\infty, N) \times (-\infty, N)} = \frac{1}{m^n/n!} \sum_{i \mid \lambda_{i,0}, \lambda_{i,1} < mN} \delta_{(m^{-1}\lambda_{i,0}, m^{-1}\lambda_{i,1})}. \quad (4.1)$$

Since $N > 0$ was arbitrary, μ_m is a discrete measure on $\mathbb{R}^2$.

It remains to analyze the support of μ_m . Note that $\mathcal{F}_0^{\lambda D} V \subset \mathcal{F}_1^\lambda V$ and $\mathcal{F}_1^{\lambda D} V \subset \mathcal{F}_0^\lambda V$ for all $\lambda \geq 1$. Therefore,

(i) If $\lambda_{i,1} > 1$, then $\lambda_{i,0} \leq D\lambda_{i,1}$.

(ii) If $\lambda_{i,0} > 1$, then $\lambda_{i,1} \leq D\lambda_{i,0}$.

Using (i), (ii), and (4.1), we see $\text{supp}(\mu_m) \subset \{\frac{1}{D}x \leq y \leq Dx\} \cup \{0 \leq x \leq \frac{1}{m}\} \cup \{0 \leq y \leq \frac{1}{m}\}$. $\square$

Using (4.1), we show that μ_m encodes the colengths of $\mathbf{a}_{\bullet, t}$.

Proposition 4.1.5. *For each $t \in [0, 1]$ and $a, b \in \mathbb{R}_{\geq 0}$,*

$$\begin{aligned} \mu_m(\{(1-t)x + ty < a\}) &= \frac{\ell(R/\mathbf{a}_{ma, t})}{m^n/n!} \\ \mu_m(\{x < a\} \cup \{y < b\}) &= \frac{\ell(R/(\mathbf{a}_{ma, 0} \cap \mathbf{a}_{mb, 1}))}{m^n/n!} \end{aligned}$$

Proof. By Lemma 4.1.4, it suffices to prove the result when $(R, \mathfrak{m})$ is complete. In this case we continue with the notation from the previous proof, but additionally fix $N > \max\{\frac{a}{1-t}, \frac{b}{t}\}$. Now, consider the filtration $\mathcal{F}_t^\bullet$ of V defined by $\mathcal{F}_t^\lambda V := \text{im}(\mathbf{a}_{\lambda, t} \rightarrow V)$ for each $\lambda \in \mathbb{R}$. Observe that

$$\mathcal{F}_t^{ma} V = \sum_{ma = (1-t)\mu + t\nu} \mathcal{F}_0^\mu V \cap \mathcal{F}_1^\nu V = \text{span}\langle s_i \mid (1-t)\lambda_{i,0} + t\lambda_{i,1} \geq ma \rangle, \quad (4.2)$$

where the sum runs through all $\mu, \nu \in \mathbb{R}$ satisfying $ma = (1-t)\mu + t\nu$. Now, we

compute

$$\begin{aligned}\ell(R/\mathfrak{a}_{ma,t}) &= \dim_{\kappa} V/\mathcal{F}_t^{ma}V = \dim_{\kappa} V - \dim_{\kappa} \mathcal{F}_t^{ma}V \\ &= \#\{i \mid (1-t)\lambda_{i,0} + t\lambda_{i,1} < ma\} = \frac{m^n}{n!} \mu_m(\{(1-t)x + ty < a\})\end{aligned}\quad (4.3)$$

where the first equality uses that $\mathfrak{m}^{NCm} \subseteq \mathfrak{a}_{ma/(1-t),0} + \mathfrak{a}_{mb/t,1} \subset \mathfrak{a}_{ma,t}$, the third (4.2), and the fourth (4.1). Therefore, the first formula in the proposition holds.

To verify the second formula, we fix $N > \max\{a, b\}$ and similarly compute

$$\begin{aligned}\dim_{\kappa} R/(\mathfrak{a}_{ma,0} \cap \mathfrak{a}_{mb,1}) &= \dim_{\kappa} V/(\mathcal{F}_0^{ma}V \cap \mathcal{F}_1^{mb}V) = \dim_{\kappa} V - \dim_{\kappa} \mathcal{F}_0^{ma}V \cap \mathcal{F}_1^{mb}V \\ &= \#\{j \mid \lambda_{j,0} < ma \text{ or } \lambda_{j,1} < mb\} = \frac{m^n}{n!} \mu_m(\{x < a\} \cup \{y < b\}),\end{aligned}$$

where the first equality uses that $\mathfrak{m}^{NCm} \subset \mathfrak{a}_{ma,0} \cap \mathfrak{a}_{mb,1}$. □

4.1.3 Limit Measure

We now construct a limit of the sequence of measures $(\mu_m)_{m \geq 1}$ that encodes the multiplicity along the geodesic.

Consider the function $H: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$H(x, y) = e(\mathfrak{a}_{x\bullet,0} \cap \mathfrak{a}_{y\bullet,1}).$$

Above, we use the convention that $e(R) = 0$, which occurs when $x \leq 0$ and $y \leq 0$.

Proposition 4.1.6. *The distributional derivative $\mu := -\frac{\partial^2 H}{\partial x \partial y}$ is a measure on $\mathbb{R}^2$ and the sequence of measures $(\mu_m)_{m \geq 1}$ converges weakly to μ as $m \rightarrow \infty$.*

Proof. We claim that H_m converges to H in $L_{\text{loc}}^1(\mathbb{R}^2)$. Assuming the claim, then H_m converges to H as distributions, and, hence, μ_m converges to $\mu := -\frac{\partial^2 H}{\partial x \partial y}$ as distributions as well. Since each μ_m is a measure, [Hör03, Theorem 2.1.9] implies μ

is a measure and μ_m converges to μ weakly as measures.

It remains to prove the above claim. First, observe that H_m converges to H pointwise by the definition of the multiplicity as a limit. Next, fix an integer $N > 0$. For $-N \leq x, y < N$, observe

$$0 \leq H_m(x, y) \leq \frac{\ell(R/\mathfrak{m}^{NCm})}{m^n/n!} \leq \sup_{m \geq 1} \frac{\ell(R/\mathfrak{m}^{NCm})}{m^n/n!} < +\infty.$$

Therefore, the dominated convergence theorem implies

$$\lim_{m \rightarrow \infty} \int_{(-N, N) \times (-N, N)} |H_m - H| = 0.$$

Since $N > 0$ was arbitrary, H_m converges to H in $L^1_{\text{loc}}(\mathbb{R}^2)$ as desired. $\square$

Proposition 4.1.7. *For each $t \in [0, 1]$ and $a, b \in \mathbb{R}_{\geq 0}$,*

$$(1) \ a^n e(\mathfrak{a}_{\bullet, t}) = e(\mathfrak{a}_{a\bullet, t}) = \mu(\{x(1-t) + yt < a\})$$

$$(2) \ e(\mathfrak{a}_{a\bullet, 0} \cap \mathfrak{a}_{b\bullet, 1}) = \mu(\{x < a\} \cup \{y < b\}).$$

Proof. Fix $\epsilon > 0$. Since $(\mu_m)_{m \geq 1}$ converges to μ weakly as measures by Proposition 4.1.6,

$$\begin{aligned} \limsup \mu_m(\{x(1-t) + yt < a - \epsilon\}) &\leq \mu(\{x(1-t) + yt < a\}) \\ &\leq \liminf \mu_m(\{x(1-t) + yt < a\}) \end{aligned}$$

Using Proposition 4.1.5(1) to compute the limsup and liminf, we deduce

$$(a - \epsilon)^n e(\mathfrak{a}_{\bullet, t}) = e(\mathfrak{a}_{(a-\epsilon)\bullet, t}) \leq \mu(\{x(1-t) + yt < a\}) \leq e(\mathfrak{a}_{a\bullet, t}) = a^n e(\mathfrak{a}_{\bullet, t}).$$

Sending $\epsilon \rightarrow 0$ completes the proof of (1). The proof of the second is similar, but uses Proposition 4.1.5.2. $\square$

Lemma 4.1.8. *For any real number $c > 0$, the following are equivalent:*

1. $\text{supp}(\mu) \subset \{cx = y\}$.
2. $e(\mathfrak{a}_{\bullet,0}) = e(\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{c\bullet,1}) = e(\mathfrak{a}_{c\bullet,1})$.

Proof. If (1) holds, then

$$\mu(\{x < 1\}) = \mu(\{x < 1\} \cup \{y < c\}) = \mu(\{y < c\}).$$

Proposition 4.1.7 then implies (2) holds.

For the reverse implication, assume (2) holds. We claim

$$H(x, y) = \begin{cases} e(\mathfrak{a}_{x\bullet,0}) & \text{if } cx \geq y \\ e(\mathfrak{a}_{y\bullet,1}) & \text{if } y \geq cx \end{cases}.$$

Indeed, if $cx \geq y$, then

$$e(\mathfrak{a}_{x\bullet,0}) \leq H(x, y) \leq e(\mathfrak{a}_{x\bullet,0} \cap \mathfrak{a}_{cx\bullet,1}) = e(\mathfrak{a}_{x\bullet,0}).$$

Thus, the claim holds when $cx \geq y$ and similar reasoning treats the case when $cx < y$.

Using the claim, we conclude (1) holds. $\square$

Proposition 4.1.9. *The following hold:*

1. *The support of μ is contained in $\{\frac{1}{D}x \leq y \leq Dx\}$.*
2. *μ is homogeneous of degree n (i.e. $\mu(c \cdot A) = c^n \mu(A)$ for any Borel set $A \subset \mathbb{R}^2$ and real number $c > 0$).*

Proof. By Propositions 4.1.3 and 4.1.6, $\text{supp}(\mu) \subset \{\frac{1}{D}x \leq y \leq Dx\} \cup \{x = 0\} \cup \{y = 0\}$. Note that

$$\mu(\{x = 0\}) = \lim_{a \rightarrow 0^+} \mu(\{0 \leq x < a\}) = \lim_{a \rightarrow 0^+} e(\mathfrak{a}_{a\bullet,0}) = \lim_{a \rightarrow 0^+} a^n e(\mathfrak{a}_{\bullet,0}) = 0,$$

where the second equality is by Proposition 4.1.7. Similar reasoning shows $\mu(\{y = 0\})$. Thus, (1) holds. Statement (2), follows from the fact that H is homogeneous of degree n , meaning $H(cx, cy) = c^n H(x, y)$ for each $c > 0$ and $(x, y) \in \mathbb{R}^2$. $\square$

4.1.4 Measure on the segment

To prove Theorem D, it will be convenient to work with a measure induced by μ on the interval $[0, 1]$.

Consider the embedding

$$j: [0, 1] \hookrightarrow \mathbb{R}^2 \quad \text{defined by} \quad j(z) := (z, 1 - z).$$

Note that $j([0, 1])$ is the line segment between $(0, 1)$ and $(1, 0)$ in $\mathbb{R}^2$. For $A \subset [0, 1]$, set

$$\text{Cone}(A) := \mathbb{R}_{\geq 0} \cdot j(A).$$

We define a measure $\tilde{\mu}$ on $[0, 1]$ by setting

$$\tilde{\mu}(A) := \mu(\text{Cone}(A) \cap \{x + y < 1\})$$

for any Borel set $A \subset [0, 1]$. Note that $\tilde{\mu}$ is indeed a measure, since μ is a measure by Proposition 4.1.6 and $\mu(\{\mathbf{0}\}) = 0$, which follows from Proposition 4.1.7. Additionally, $\text{supp}(\tilde{\mu}) \subset [\frac{1}{D+1}, \frac{D}{D+1}]$, since $\text{supp}(\mu) \subset \{\frac{1}{D}x \leq y \leq Dx\}$.

Lemma 4.1.10. *For any real number $c > 0$, the following are equivalent:*

1. $\text{supp}(\tilde{\mu}) = \frac{1}{c+1}$.
2. $e(\mathfrak{a}_{\bullet,0}) = e(\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{c\bullet,1}) = e(\mathfrak{a}_{c\bullet,1})$.

Proof. Using that $\mathbb{R} \cdot j(\frac{1}{c+1}) = \{cx = y\}$ and the definition of $\tilde{\mu}$, we see (1) holds if and only if $\text{supp}(\mu|_{\{0 \leq x+y < 1\}}) \subset \{cx = y\}$. Since μ is homogeneous by Proposition

4.1.9(2), $\text{supp}(\mu|_{\{0 \leq x+y < 1\}}) \subset \{xc = y\}$ if and only if $\text{supp}(\mu) \subset \{cx = y\}$. By Proposition 4.1.8, the latter condition holds if and only if (2) holds. $\square$

Lemma 4.1.11. *For $t \in [0, 1]$,*

$$e(\mathbf{a}_{\bullet, t}) = \int ((1-t)z + t(1-z))^{-n} d\tilde{\mu}.$$

Proof. For simplicity, we assume $t \leq \frac{1}{2}$, which insures $g_t(z) := (1-t)z + t(1-z)$ is non-decreasing. (The case when $t > \frac{1}{2}$ can be proved similarly, but using left hand approximations below.) For each $m \geq 1$, consider the simple function

$$g_{t,m}(z) = \sum_{k=1}^{2^m} g_t(k/2^m) \cdot 1_{I_{k,m}}.$$

where $I_{1,m} := [0, \frac{1}{2^m}]$ and $I_{k,m} := (\frac{k-1}{2^m}, \frac{k}{2^m}]$ when $1 < k \leq 2^m$. Since $g_{t,m} \geq g_{t,m+1}$ and $g_{t,m} \rightarrow g$ pointwise as $m \rightarrow \infty$, the Monotone Convergence Theorem implies

$$\int g_t(z)^{-n} d\tilde{\mu} = \lim_{m \rightarrow \infty} \int g_{t,m}(z)^{-n} d\tilde{\mu}.$$

(Above we are using that $g_t(z)^{-n}$ is defined and continuous on $(z, t) \in \text{supp}(\tilde{\mu}) \times [0, 1]$.) Now, set $A_{k,m} := g_t(k/2^m)^{-1} \cdot (\text{Cone}(I_{k,m}) \cap \{x+y < 1\}) \subset \mathbb{R}^2$ and $A_m = \cup_{k=1}^{2^m} A_{k,m}$. We compute

$$\int g_{t,m}(z)^{-n} d\tilde{\mu} = \sum_{k=1}^{2^m} g_t(k/2^m)^{-n} \tilde{\mu}(I_{k,m}) = \sum_{k=1}^{2^m} \mu(A_{k,m}) = \mu(A_m),$$

where the second equality uses Proposition 4.1.9.2. Next, notice that $A_m \subset A_{m+1}$

and

$$\begin{aligned} \bigcup_{m \geq 1} A_m &= \bigcup_{m \geq 1} \bigcup_{z \in [0,1]} g_{t,m}(z)^{-1} \cdot (\text{Cone}(\{z\}) \cap \{x + y < 1\}) \\ &= \bigcup_{z \in [0,1]} g_t(z)^{-1} \cdot (\text{Cone}(\{z\}) \cap \{x + y < 1\}) = \mathbb{R}_{>0}^2 \cap \{(1-t)x + ty < 1\}. \end{aligned}$$

Combining the previous two computations, we conclude

$$\int g_t(z)^{-n} d\tilde{\mu} = \lim_{m \rightarrow \infty} \mu(A_m) = \mu(\mathbb{R}_{>0}^2 \cap \{(1-t)x + ty < 1\}) = e(\mathbf{a}_\bullet, t),$$

where the final equality is Proposition 4.1.7(1). □

4.2 Convexity along geodesics

Now we are ready to prove Theorem D.

Proof of Theorem D. Set $g(z, t) := (1-t)z + t(1-z)$. By Lemma 4.1.11,

$$E(t) = \int g(z, t)^{-n} d\tilde{\mu}.$$

Since $g(z, t) = 0$ if and only if $z = \frac{t}{2t-1}$, there exists $\epsilon > 0$ such that

$$g(z, t) \neq 0 \text{ for all } (z, t) \in \left[\frac{1}{D+1} - \epsilon, \frac{D}{D+1} + \epsilon\right] \times [-\epsilon, 1 + \epsilon].$$

Since $g(z, t)$ is also smooth on $\mathbb{R}^2$, $g(z, t)^{-n}$ is smooth and has bounded derivatives on $(\frac{1}{D+1} - \epsilon, \frac{D}{D+1} + \epsilon) \times (-\epsilon, 1 + \epsilon)$. Using that $\text{supp} \tilde{\mu} \subset [\frac{1}{D+1}, \frac{D}{D+1}]$, the Leibniz integral formula implies $\int g(z, t)^{-n} d\tilde{\mu}$ is smooth on $(-\epsilon, 1 + \epsilon)$, which proves (1). The formula additionally implies

$$E'(t) = n \int (2z-1)g(z, t)^{-n-1} d\tilde{\mu} \quad \text{and} \quad E''(t) = n(n+1) \int (2z-1)^2 g(z, t)^{-n-2} d\tilde{\mu}.$$

To prove (2), it suffices to show $\frac{d^2}{dt^2} (E(t)^{-1/n}) \leq 0$ for all $t \in (0, 1)$. Note that

$$\frac{d^2}{dt^2} (E(t)^{-1/n}) = \frac{(n+1)E'(t)^2 - nE(t)E''(t)}{n^2E(t)^{1/n+2}}.$$

To show the latter quantity is non-positive, we compute

$$\begin{aligned} nE(t)E''(t) &= n^2(n+1) \left(\int |g(z, t)^{-n}| \, d\tilde{\mu} \right) \left(\int |(2z-1)^2 g(z, t)^{-n-2}| \, d\tilde{\mu} \right) \\ &\geq n^2(n+1) \left(\int |(2z-1)g(z, t)^{-n-1}| \, d\tilde{\mu} \right)^2 \geq (n+1)E'(t)^2, \end{aligned} \quad (4.4)$$

where the first inequality is by the Cauchy-Schwarz inequality. Therefore, $\frac{d^2}{dt^2} (E(t)^{-1/n}) \leq 0$ for all $t \in (0, 1)$, which proves (2).

To prove (3), note that $E(t)$ is linear iff $\frac{d^2}{dt^2} (E(t)^{-1/n}) = 0$ for all $t \in (0, 1)$. The latter holds iff the inequalities in (4.4) are all equalities. The inequalities are equalities iff $(2z-1)g(z, t)^{-(n+2)/2}$ and $g(z, t)^{-n/2}$ are linearly dependent in $L^1(\tilde{\mu})$. Equivalently, $2z-1$ and $g(z, t)$ are linearly dependent in $L^1(\tilde{\mu})$. The linear dependence holds exactly when $\tilde{\mu}$ is supported at a single point. By Lemma 4.1.10 and Corollary 3.4.2, the latter condition is equivalent to the existence of $c \in \mathbb{R}$ such that $\tilde{\mathbf{a}}_{\bullet,0} = \tilde{\mathbf{a}}_{c\bullet,1}$. $\square$

4.3 Applications

4.3.1 Convexity of volumes

Proof of Corollary E. Let $(\mathbf{a}_{\bullet,t})_{t \in [0,1]}$ denote the geodesic between $\mathbf{a}_{\bullet,0} := \mathbf{a}_{\bullet}(v_{\alpha})$ and $\mathbf{a}_{\bullet,1} := \mathbf{a}_{\bullet}(v_{\beta})$. Since $v_{(1-t)\alpha+t\beta}(f) \geq (1-t)v_{\alpha}(f) + tv_{\beta}(f)$ for any $f \in \mathcal{O}_{X,x}$, $\mathbf{a}_{\bullet,t} \subset \mathbf{a}_{\bullet}(v_{(1-t)\alpha+t\beta})$. We compute

$$\begin{aligned} \text{vol}(v_{(1-t)\alpha+t\beta})^{-1/n} &:= e(\mathbf{a}_{\bullet}((v_{(1-t)\alpha+t\beta}))^{-1/n} \geq e(\mathbf{a}_{\bullet,t})^{-1/n} \\ &\geq (1-t)e(\mathbf{a}_{\bullet,0})^{-1/n} + te(\mathbf{a}_{\bullet,1})^{-1/n} = (1-t)\text{vol}(v_0)^{-1/n} + t\text{vol}(v_1)^{-1/n}, \end{aligned}$$

where the second inequality is Theorem D(2). Additionally, by Theorem D(3) and Lemma 3.3.17, the second inequality is strict unless there exists $c \in \mathbb{R}_{>0}$ such that $\mathbf{a}_{\bullet,0} = \mathbf{a}_{c\bullet,1}$. Since $\mathbf{a}_{\bullet,0} = \mathbf{a}_{c\bullet,1}$ if and only if $c\alpha_0 = \alpha_1$, the result follows. $\square$

4.3.2 Uniqueness of minimizers

The proof of Corollary F relies on Theorem D and the following proposition, which will be deduced from [XZ21, Theorem 3.11]

Proposition 4.3.1. *Let $x \in (X, D)$ be a klt singularity. If $\mathbf{a}_{\bullet,0}$ and $\mathbf{a}_{\bullet,1}$ are $\mathbf{m}$ -filtrations, then*

$$\mathrm{lct}(X, D; \mathbf{a}_{\bullet,t}) \leq (1-t) \cdot \mathrm{lct}(X, D; \mathbf{a}_{\bullet,0}) + t \cdot \mathrm{lct}(X, D; \mathbf{a}_{\bullet,1}),$$

for all $t \in (0, 1)$.

Proof. We claim $v(\mathbf{c}_{\bullet}) = v(\mathbf{a}_{\bullet,t})$ for all $v \in \mathrm{Val}_{R,\mathbf{m}}$, where $\mathbf{c}_{\bullet}$ is the $\mathbf{m}$ -filtration defined by $\mathbf{c}_{\lambda} := \sum_{i=0}^{\lceil \lambda \rceil} (\mathbf{a}_{(\lceil \lambda \rceil - i)/(1-t),0} \cap \mathbf{a}_{i/t,0})$. Assuming the claim holds,

$$\begin{aligned} \mathrm{lct}(X, D; \mathbf{a}_{\bullet,t}) &= \mathrm{lct}(X, D; \mathbf{c}_{\bullet}) \\ &\leq \mathrm{lct}(X, D; \mathbf{a}_{\bullet/(1-t),0}) + \mathrm{lct}(X, D; \mathbf{a}_{\bullet/t,1}) = (1-t)\mathrm{lct}(X, D; \mathbf{a}_{\bullet,0}) + t\mathrm{lct}(X, D; \mathbf{a}_{\bullet,1}), \end{aligned}$$

where the inequality is [XZ21, Theorem 3.11]. It remains to verify the claim. Since $\mathbf{c}_m \subset \mathbf{a}_{t,m}$ for each integer $m > 0$, $v(\mathbf{c}_{\bullet}) \geq v(\mathbf{a}_{t,\bullet})$ for all $v \in \mathrm{Val}_{R,\mathbf{m}}$. Next fix $\mu, \nu \in \mathbb{R}$ such that $\mu(1-t) + \nu t = m$. Set $r := \mu(1-t) = m - \nu t$. For each $\ell \in \mathbb{Z}_{>0}$,

$$(\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1})^{\ell} \subset (\mathbf{a}_{\ell\mu,0} \cap \mathbf{a}_{\ell\nu,1}) = (\mathbf{a}_{\ell r/(1-t),1} \cap \mathbf{a}_{\ell(m-r)/t,1}) \subset (\mathbf{a}_{\lfloor \ell r \rfloor/(1-t),0} \cap \mathbf{a}_{(\ell m - 1 - \lfloor \ell r \rfloor)/t,1}) \subset \mathbf{c}_{\ell m - 1},$$

Using the previous inclusions, we see

$$v(\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1}) = \lim_{\ell \rightarrow \infty} \frac{v((\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1})^\ell)}{\ell} \geq \lim_{\ell \rightarrow \infty} \frac{v(\mathbf{c}_{\ell m-1})}{\ell} = v(\mathbf{c}_\bullet).$$

We can now compute

$$v(\mathbf{a}_{\bullet,t}) = \lim_{m \rightarrow \infty} \frac{v(\mathbf{a}_{m,t})}{m} = \lim_{m \rightarrow \infty} \min_{\mu(1-t) + \nu t = m} \left\{ \frac{v(\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1})}{m} \right\} \geq v(\mathbf{c}_\bullet).$$

Therefore, $v(\mathbf{a}_{\bullet,t}) = v(\mathbf{c}_\bullet)$ as desired. $\square$

Proof of Corollary F. Let $v_0, v_1 \in \text{Val}_{R,\mathbf{m}}$ be minimizers of $\widehat{\text{vol}}_{(X,D),x}$. By rescaling, we may assume $A_{X,\Delta}(v_i) = 1$ and, hence, $\text{vol}(v_i) = \widehat{\text{vol}}(x, X, D)$ for $i = 0, 1$. We seek to show $v_0 = v_1$.

Consider the geodesic $(\mathbf{a}_{t,\bullet})_{t \in [0,1]}$ between $\mathbf{a}_{\bullet,0} := \mathbf{a}_\bullet(v_0)$ and $\mathbf{a}_{\bullet,1} := \mathbf{a}_\bullet(v_1)$. For $t \in (0, 1)$,

$$\begin{aligned} \text{lct}(X, D; \mathbf{a}_{\bullet,t}) &\leq (1-t)\text{lct}(X, D; \mathbf{a}_{\bullet,0}) + t\text{lct}(X, D; \mathbf{a}_{\bullet,1}) \\ &\leq (1-t)A_{X,D}(v_0) + tA_{X,D}(v_1) = 1, \end{aligned} \quad (4.5)$$

where the first inequality is Proposition 4.3.1 and the second uses that $v_i(\mathbf{a}_{\bullet,i}) = 1$ by [Blu18a, Lemma 3.5]. Observe that

$$\begin{aligned} \widehat{\text{vol}}(x, X, D)^{1/n} &\leq \text{lct}(\mathbf{a}_{\bullet,t})e(\mathbf{a}_{\bullet,t})^{1/n} \leq e(\mathbf{a}_{\bullet,t})^{1/n} \\ &\leq ((1-t)e(\mathbf{a}_{\bullet,0})^{-1/n} + te(\mathbf{a}_{\bullet,1})^{-1/n})^{-1} = \widehat{\text{vol}}(X, D, x)^{1/n} \end{aligned}$$

where the first inequality holds by [Liu18, Theorem 27], the second by (4.5), and the third by Theorem D(2). Therefore, every inequality above, in particular the third one, is an equality. By Theorem D(3) and Lemma 3.3.17, there exists $c > 0$ such that $\mathbf{a}_{\bullet,0} = \mathbf{a}_{c\bullet,1}$, which translates to $\mathbf{a}_\bullet(cv_0) = \mathbf{a}_\bullet(v_1)$. Therefore, $cv_0 = v_1$. Since

$c = A_{X,D}(cv_0) = A_{X,D}(v_1) = 1$, we see $v_0 = v_1$. □

Chapter 5

Metric geometry of spaces of filtrations

5.1 Darvas metrics

Throughout this section we will use the language of filtrations. Recall that $\text{Fil} = \text{Fil}_{R, \mathfrak{m}}$ denotes the space of all linearly bounded $\mathfrak{m}$ -filtrations on R .

Definition 5.1.1. We define a function $d_1 : \text{Fil} \times \text{Fil} \rightarrow \mathbb{R}_{\geq 0}$ as follows.

Given $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet \in \text{Fil}$. If $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$, then let

$$d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = d_1(\mathfrak{b}_\bullet, \mathfrak{a}_\bullet) := e(\mathfrak{a}_\bullet) - e(\mathfrak{b}_\bullet). \quad (5.1)$$

In general, define

$$d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) := d_1(\mathfrak{a}_\bullet, \mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) + d_1(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet, \mathfrak{b}_\bullet). \quad (5.2)$$

Proposition 5.1.2. d_1 satisfies the triangle inequality, that is, for $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet, \mathfrak{c}_\bullet \in \text{Fil}$, we have

$$d_1(\mathfrak{a}_\bullet, \mathfrak{c}_\bullet) \leq d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) + d_1(\mathfrak{b}_\bullet, \mathfrak{c}_\bullet). \quad (5.3)$$

Hence d_1 is a pseudometric on Fil .

Proof. By definition, we have

$$d_1(\mathfrak{a}_\bullet, \mathfrak{c}_\bullet) = 2e(\mathfrak{a}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{a}_\bullet) - e(\mathfrak{c}_\bullet)$$

and

$$d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) + d_1(\mathfrak{b}_\bullet, \mathfrak{c}_\bullet) = 2e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) - e(\mathfrak{a}_\bullet) - e(\mathfrak{b}_\bullet) + 2e(\mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{b}_\bullet) - e(\mathfrak{c}_\bullet)$$

Hence to prove (5.3), it suffices to show that

$$e(\mathfrak{a}_\bullet \cap \mathfrak{c}_\bullet) \leq e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) + e(\mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{b}_\bullet),$$

which follows immediately from the following comparison on colengths

$$\ell(R/\mathfrak{a}_\lambda \cap \mathfrak{c}_\lambda) \leq \ell(R/\mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda) + \ell(R/\mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda) - \ell(R/\mathfrak{b}_\lambda) \quad (5.4)$$

for any $\lambda \in \mathbb{R}_{>0}$. But we have

$$\begin{aligned} \ell(R/\mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda) + \ell(R/\mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda) - \ell(R/\mathfrak{b}_\lambda) - \ell(R/\mathfrak{a}_\lambda) &= \ell(\mathfrak{a}_\lambda / \mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda) + \ell(\mathfrak{b}_\lambda / \mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda) \\ &= \ell((\mathfrak{a}_\lambda + \mathfrak{b}_\lambda) / \mathfrak{b}_\lambda) + \ell(\mathfrak{b}_\lambda / \mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda) \\ &= \ell((\mathfrak{a}_\lambda + \mathfrak{b}_\lambda) / \mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda) \\ &\geq \ell((\mathfrak{a}_\lambda + \mathfrak{b}_\lambda) / (\mathfrak{a}_\lambda + \mathfrak{b}_\lambda) \cap \mathfrak{c}_\lambda) \\ &= \ell((\mathfrak{a}_\lambda + \mathfrak{b}_\lambda + \mathfrak{c}_\lambda) / \mathfrak{c}_\lambda) \\ &\geq \ell((\mathfrak{a}_\lambda + \mathfrak{c}_\lambda) / \mathfrak{c}_\lambda) \\ &= \ell(\mathfrak{a}_\lambda / \mathfrak{a}_\lambda \cap \mathfrak{c}_\lambda) = \ell(R/\mathfrak{a}_\lambda \cap \mathfrak{c}_\lambda) - \ell(R/\mathfrak{a}_\lambda). \end{aligned}$$

This proves (5.4), and hence the triangle inequality (5.3) holds.

Clearly $d_1(\mathfrak{a}_\bullet, \mathfrak{a}_\bullet) = 0$, and $d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = d_1(\mathfrak{b}_\bullet, \mathfrak{a}_\bullet)$ by definition, hence we know that (Fil, d_1) is a pseudometric space. $\square$

Using the identification between Fil and $\mathcal{N}$, we get a pseudometric d_1 on $\mathcal{N}$, which is a metric on $\mathcal{N}^s$. Compare the following Lemma with [BJ21, Lemma 3.2] and [Dar19, Proposition 3.35].

Lemma 5.1.3. *For $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet, \mathfrak{c}_\bullet \in \text{Fil}$, we have*

$$d_1(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet, \mathfrak{a}_\bullet \cap \mathfrak{c}_\bullet) \leq d_1(\mathfrak{b}_\bullet, \mathfrak{c}_\bullet). \quad (5.5)$$

Proof. By definition, we have

$$d_1(\mathfrak{b}_\bullet, \mathfrak{c}_\bullet) = e(\mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{b}_\bullet) + e(\mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{c}_\bullet)$$

and

$$d_1(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet, \mathfrak{a}_\bullet \cap \mathfrak{c}_\bullet) = e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) + e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{a}_\bullet \cap \mathfrak{c}_\bullet).$$

So by symmetry, it suffices to prove the inequality

$$e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) \leq e(\mathfrak{b}_\bullet \cap \mathfrak{c}_\bullet) - e(\mathfrak{b}_\bullet),$$

which follows from the inclusion

$$\begin{aligned} (\mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda) / (\mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda) &\cong (\mathfrak{a}_\lambda \cap \mathfrak{b}_\lambda + \mathfrak{c}_\lambda) / \mathfrak{c}_\lambda \\ &\subset (\mathfrak{b}_\lambda + \mathfrak{c}_\lambda) / \mathfrak{c}_\lambda \cong \mathfrak{b}_\lambda / (\mathfrak{b}_\lambda \cap \mathfrak{c}_\lambda). \end{aligned}$$

This finishes the proof. $\square$

We give a naive example of the metric space (Fil^s, d_1) .

Example 5.1.4. If R is a DVR, then $\text{Fil}_{R,\mathfrak{m}}^s \cong \mathbb{R}_{>0}$. Let $\pi \in \mathfrak{m}$ a uniformizer. For $\mathfrak{a}_\bullet \in \text{Fil}_{R,\mathfrak{m}}^s$, let $c := \text{ord}_\pi(\mathfrak{a}_\bullet) \in \mathbb{R}_{>0}$. Since ord_π is the only divisorial valuation up to scaling, for $\lambda \in \mathbb{R}_{>0}$ we have

$$\mathfrak{a}_\lambda = \{f \in R \mid \text{ord}_\pi(f) \geq \lambda \cdot c\} = (\pi^{\lceil c\lambda \rceil}) = \mathfrak{m}^{\lceil c\lambda \rceil},$$

that is, $\mathfrak{a}_\bullet = \mathfrak{m}^{c\bullet}$. Thus we have a bijection of sets $\phi : \text{Fil}^s \rightarrow \mathbb{R}_{>0}$, $\mathfrak{m}^{c\bullet} \mapsto c$. It is easy to see that $e(\mathfrak{m}^{c\bullet}) = c \cdot e(\mathfrak{m}^\bullet) = c$, hence the bijection ϕ is an (order-reversing) isometry.

5.2 Measures and geodesics

Recall that given $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}$, the *geodesic* between them is the segment $\{\mathfrak{a}_{\bullet,t}\}_{t \in [0,1]}$, defined by

$$\mathfrak{a}_{\lambda,t} = \sum_{(1-t)\mu + t\nu = \lambda} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}.$$

The goal of this section is to justify the term *geodesic*, in the sense that it is a distance-minimizing continuous curve with respect to the metric d_1 .

First we give a formula for the geodesic between two saturated filtrations, and use it to show that $\mathfrak{a}_{\bullet,t} \in \text{Fil}^s$ when $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}^s$.

Proposition 5.2.1. *Let $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}^s$. Then for any $\lambda \in \mathbb{R}_{>0}$ and $t \in (0,1)$, we have*

$$\mathfrak{a}_{\bullet,t} = \bigcap_{v \in \text{DivVal}_{R,\mathfrak{m}}} \mathfrak{a}_\bullet\left(\frac{v}{c_{t,v}}\right), \tag{5.6}$$

where

$$c_{t,v} := \frac{v(\mathfrak{a}_{\bullet,0})v(\mathfrak{a}_{\bullet,1})}{tv(\mathfrak{a}_{\bullet,0}) + (1-t)v(\mathfrak{a}_{\bullet,1})}.$$

In particular, $\mathfrak{a}_{\bullet,t} \in \text{Fil}^s$ for any $t \in (0,1)$.

Proof. Denote the right-handed side of (5.6) by $\mathfrak{c}_{\bullet,t}$. First assume $f \in \mathfrak{c}_{\lambda,t}$. Then for

any $v \in \text{DivVal}_{R,\mathfrak{m}}$, $v(f) \geq \lambda c_{t,v}$. Since $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}^s$, this implies that $f \in \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}$, where $\mu := \frac{\lambda v(\mathfrak{a}_{\bullet,1})}{tv(\mathfrak{a}_{\bullet,0}) + (1-t)v(\mathfrak{a}_{\bullet,1})}$ and $\nu := \frac{\lambda v(\mathfrak{a}_{\bullet,0})}{tv(\mathfrak{a}_{\bullet,0}) + (1-t)v(\mathfrak{a}_{\bullet,1})}$. Since $(1-t)\mu + t\nu = \lambda$, we get $f \in \mathfrak{a}_{\lambda,t}$ and hence $\mathfrak{c}_{\bullet,t} \subset \mathfrak{a}_{\bullet,t}$.

Conversely, if $f \in \mathfrak{a}_{\lambda,t}$, then for any $v \in \text{DivVal}_{R,\mathfrak{m}}$, we have

$$\begin{aligned} v(f) &\geq v \left(\sum_{(1-t)\mu + t\nu = \lambda} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1} \right) = \min_{(1-t)\mu + t\nu = \lambda} \{v(\mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1})\} \\ &\geq \min_{(1-t)\mu + t\nu = \lambda} \max\{v(\mathfrak{a}_{\mu,0}), v(\mathfrak{a}_{\nu,1})\} \\ &= \min_{0 \leq \mu \leq \lambda/(1-t)} \max\{\mu v(\mathfrak{a}_{\bullet,0}), \frac{\lambda - (1-t)\mu}{t} v(\mathfrak{a}_{\bullet,1})\}, \end{aligned}$$

where in the last line we used the assumption that $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}^s$. Now it is not hard to check that the last minimum of the piecewise linear function (of μ) is achieved at $\mu_0 := \frac{\lambda v(\mathfrak{a}_{\bullet,1})}{tv(\mathfrak{a}_{\bullet,0}) + (1-t)v(\mathfrak{a}_{\bullet,1})} \in (0, \lambda/(1-t))$, with value $\mu_0 v(\mathfrak{a}_{\bullet,0}) = \lambda c_{t,v}$. This shows that $f \in \mathfrak{a}_{\lambda}(v/c_{t,v})$. Since v is arbitrary, we get $f \in \mathfrak{c}_{\lambda,t}$. Thus $\mathfrak{a}_{\bullet,t} \subset \mathfrak{c}_{\bullet,t}$ and equality holds.

Since $\mathfrak{a}_{\bullet,t} \in \text{Fil}$ for any $t \in (0,1)$, by Proposition 3.5.3 we know that $\mathfrak{a}_{\bullet,t} \in \text{Fil}^s$. The proof is finished. $\square$

5.2.1 Distance-minimizing property

We first make the following easy but useful observation.

Lemma 5.2.2. *Let $\mathfrak{a}_{\bullet,i} \in \text{Fil}$ for $i = 0, 1$, and let $\{\mathfrak{a}_{\bullet,t}\}_{t \in [0,1]}$ be the geodesic between them. For $0 \leq t < t' \leq 1$, we have $\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{\bullet,t'} \subset \mathfrak{a}_{\bullet,t}$.*

Proof. For any $\lambda \in \mathbb{R}_{>0}$, we compute

$$\begin{aligned}
\mathfrak{a}_{\lambda,t'} \cap \mathfrak{a}_{\lambda,0} &= \left(\sum_{(1-t')\mu+t'\nu=\lambda} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1} \right) \cap \mathfrak{a}_{\lambda,0} \\
&= \sum_{\mu \leq \lambda} \mathfrak{a}_{\lambda,0} \cap \mathfrak{a}_{\nu,1} + \sum_{\mu > \lambda} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1} \\
&= \mathfrak{a}_{\lambda,0} \cap \mathfrak{a}_{\lambda,1} + \sum_{\mu > \lambda} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}.
\end{aligned}$$

Since $\mathfrak{a}_{\lambda,0} \cap \mathfrak{a}_{\lambda,1} \subset \mathfrak{a}_{\lambda,s}$ for any $s \in [0, 1]$, it suffices to show that for any $(\mu, \nu) \in \mathbb{R}_{\geq 0}^2$ satisfying $(1-t')\mu + t'\nu = \lambda$ and $\mu > \lambda$, we have

$$\mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1} \subset \mathfrak{a}_{\lambda,t}. \quad (5.7)$$

To see this, let $\mu' := \frac{(1-t')\mu + (t'-t)\nu}{1-t}$, then $(1-t)\mu + t\nu = (1-t')\mu + t'\nu = \lambda$, and we have

$$\mu - \mu' = \frac{(1-t)\mu - (1-t')\mu - (t'-t)\nu}{1-t} = \frac{(t'-t)(\mu - \nu)}{1-t} > 0,$$

where the last inequality follows from the condition $\mu > \lambda > \nu$. This implies

$$\mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1} \subset \mathfrak{a}_{\mu',0} \cap \mathfrak{a}_{\nu,1} \subset \mathfrak{a}_{\lambda,t},$$

hence (5.7) holds and the proof is finished. $\square$

Next we prove that the geodesic is always distance-minimizing.

Proposition 5.2.3. *Let $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}$ and let $\{\mathfrak{a}_{\bullet,t}\}_{t \in [0,1]}$ be the geodesic between them. Then for any $0 \leq t < t' \leq 1$, we have*

$$d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,t'}) = d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,t}) + d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{a}_{\bullet,t'}). \quad (5.8)$$

Proof. For simplicity, denote $\mathfrak{b}_{\bullet} := \mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{\bullet,t}$, $\mathfrak{b}'_{\bullet} := \mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{\bullet,t'}$ and $\mathfrak{c}_{\bullet} := \mathfrak{a}_{\bullet,t} \cap \mathfrak{a}_{\bullet,t'}$.

By Lemma 5.2.2 we know that $\mathfrak{b}'_\bullet \subset \mathfrak{b}_\bullet \subset \mathfrak{a}_{\bullet,0}$ and $\mathfrak{b}'_\bullet \subset \mathfrak{c}_\bullet \subset \mathfrak{a}_{\bullet,t'}$. We claim that for any $\lambda \in \mathbb{R}_{>0}$,

$$\mathfrak{a}_{\lambda,t}/\mathfrak{b}_\lambda \cong \mathfrak{c}_\lambda/\mathfrak{b}'_\lambda. \quad (5.9)$$

Granting the claim for now, we know that $d_1(\mathfrak{b}_\bullet, \mathfrak{a}_{\bullet,t}) = d_1(\mathfrak{b}'_\bullet, \mathfrak{c}_\bullet)$. So we get

$$d_1(\mathfrak{b}_\bullet, \mathfrak{a}_{\bullet,t}) + d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{c}_\bullet) = d_1(\mathfrak{b}'_\bullet, \mathfrak{c}_\bullet) + d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{c}_\bullet) = d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{b}'_\bullet) = d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{b}_\bullet) + d_1(\mathfrak{b}_\bullet, \mathfrak{b}'_\bullet),$$

which implies $d_1(\mathfrak{b}_\bullet, \mathfrak{b}'_\bullet) = d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{c}_\bullet)$. Now we compute

$$\begin{aligned} d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,t'}) &= d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{b}'_\bullet) + d_1(\mathfrak{b}'_\bullet, \mathfrak{a}_{\bullet,t'}) \\ &= d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{b}_\bullet) + d_1(\mathfrak{b}_\bullet, \mathfrak{b}'_\bullet) + d_1(\mathfrak{b}'_\bullet, \mathfrak{c}_\bullet) + d_1(\mathfrak{c}_\bullet, \mathfrak{a}_{\bullet,t'}) \\ &= d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{b}_\bullet) + d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{c}_\bullet) + d_1(\mathfrak{b}_\bullet, \mathfrak{a}_{\bullet,t}) + d_1(\mathfrak{c}_\bullet, \mathfrak{a}_{\bullet,t'}) \\ &= d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,t}) + d_1(\mathfrak{a}_{\bullet,t}, \mathfrak{a}_{\bullet,t'}) \end{aligned}$$

It remains to prove the claim (5.9). Indeed, by definition, any $f \in \mathfrak{a}_{\lambda,t}$ can be written as

$$f = \sum_{(1-t)\mu + t\nu = \lambda} f_{\mu,\nu},$$

where $f_{\mu,\nu} \in \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}$. If $\mu \geq \lambda$, then $f_{\mu,\nu} \in \mathfrak{a}_{\mu,0} \subset \mathfrak{a}_{\lambda,0}$. If $\mu \leq \lambda$, then $\nu \geq \lambda$, and hence $f_{\mu,\nu} \in \mathfrak{a}_{\nu,1} \subset \mathfrak{a}_{\lambda,1}$. Thus we have

$$f = \sum_{\mu \leq \lambda} f_{\mu,\nu} + \sum_{\mu \geq \lambda} f_{\mu,\nu} \in \mathfrak{a}_{\lambda,t} \cap \mathfrak{a}_{\lambda,1} + \mathfrak{a}_{\lambda,0},$$

and hence $\mathfrak{c}_\lambda \subset \mathfrak{a}_{\lambda,t} \subset \mathfrak{a}_{\lambda,t} \cap \mathfrak{a}_{\lambda,1} + \mathfrak{a}_{\lambda,0} \subset \mathfrak{c}_\lambda + \mathfrak{a}_{\lambda,0}$, where the last inclusion follows from Lemma 5.2.2. So we get

$$\mathfrak{a}_{\lambda,t}/\mathfrak{b}_\lambda \cong (\mathfrak{a}_{\lambda,t} + \mathfrak{a}_{\lambda,0})/\mathfrak{a}_{\lambda,0} = (\mathfrak{c}_\lambda + \mathfrak{a}_{\lambda,0})/\mathfrak{a}_{\lambda,0} \cong \mathfrak{c}_\lambda/(\mathfrak{c}_\lambda \cap \mathfrak{a}_{\lambda,0}) = \mathfrak{c}_\lambda/\mathfrak{b}'_\lambda,$$

where the last equality follows again from Lemma 5.2.2. This proves (5.9), and finishes the proof of the proposition. $\square$

Remark 5.2.4. When R contains a field, the proposition can be proved (easily) using the measure as in the next section. We present the above more tedious but elementary argument, which does not rely on this assumption.

5.2.2 Duistermaat-Heckman measures and continuity

In this section R is assumed to contain a field $\mathbb{k}$ of arbitrary characteristic. We use the measure introduced in Section 4.1 encoding multiplicities and its properties.

Given $\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,1} \in \text{Fil}$. Define the function $H : \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$ by

$$H(x, y) := e(\mathfrak{a}_{x\bullet,0} \cap \mathfrak{a}_{y\bullet,1}),$$

where we adopt the convention $e(R) = 0$. Fix $D \in \mathbb{Z}_{>1}$ such that $\mathfrak{a}_{D\bullet,0} \subset \mathfrak{a}_{\bullet,1}$ and $\mathfrak{a}_{D\bullet,1} \subset \mathfrak{a}_{\bullet,0}$.

Lemma 5.2.5. *Fix $(a, b) \in \mathbb{R}^2$. Let $A := [a + \alpha] \times [b + \beta] \subset \mathbb{R}^2$ be a rectangle, where $\alpha, \beta \in \mathbb{R}_{>0}$. Then there exists $M := M(a, b) \in \mathbb{R}_{>0}$ such that $\mu(A) \leq M\alpha\beta$ when $\alpha, \beta \leq 1$.*

Proof. We may assume that $(a, b) \in \mathbb{R}_{\geq 0}^2$. Using the inequality

$$(x + \delta)^n - x^n \leq n(x + \delta)^{n-1}\delta$$

for $x \geq 0$ and $\delta > 0$, we compute

$$\begin{aligned}
\mu(A) &= \mu([0, a + \alpha] \times [b, b + \beta]) - \mu([0, a] \times [b, b + \beta]) \\
&= ((a + \alpha)^n - a^n) \mu([0, 1] \times [b, b + \beta]) \\
&= ((a + \alpha)^n - a^n) ((b + \beta)^n - b^n) \mu([0, 1] \times [0, 1]) \\
&\leq ((a + \alpha)^n - a^n) ((b + \beta)^n - b^n) e(\mathfrak{a}_{\bullet, 0} \cap \mathfrak{a}_{\bullet, 1}) \\
&\leq n^2 (a + 1)^{n-1} (b + 1)^{n-1} e(\mathfrak{a}_{\bullet, 0} \cap \mathfrak{a}_{\bullet, 1}) \cdot \alpha \beta,
\end{aligned}$$

where the second and the third equality follow from Proposition 4.1.9, and the first inequality follows from Proposition 4.1.7. Hence proof is finished if we take $M := n^2 (a + 1)^{n-1} (b + 1)^{n-1} e(\mathfrak{a}_{\bullet, 0} \cap \mathfrak{a}_{\bullet, 1})$. $\square$

Now we can prove that the function $t \mapsto \mathfrak{a}_{\bullet, t}$ is Lipschitz continuous with respect to d_1 .

Lemma 5.2.6. *Suppose R contains a field. Given $\mathfrak{a}_{\bullet, 0}, \mathfrak{a}_{\bullet, 1} \in \text{Fil}$, the map $[0, 1] \rightarrow \text{Fil}$, $t \mapsto \mathfrak{a}_{\bullet, t}$ is Lipschitz continuous, where $[0, 1]$ is equipped with the Euclidean topology and Fil is equipped with the topology induced by the pseudometric d_1 .*

Proof. Given $0 \leq t_1 < t_2 \leq 1$, denote $\Delta_j := \{(x, y) \in \mathbb{R}_{\geq 0}^2 \mid (1 - t_j)x + t_j y \leq 1\}$ for $j = 1, 2$. An easy calculation using Proposition 4.1.7 gives

$$d_1(\mathfrak{a}_{\bullet, t_1}, \mathfrak{a}_{\bullet, t_2}) = \mu(\Delta_1 \setminus \Delta_2) + \mu(\Delta_2 \setminus \Delta_1). \quad (5.10)$$

First assume that $0 < t_1 < t_2 < 1$. By Proposition 4.1.9 we have

$$\mu(\Delta_1 \setminus \Delta_2) = \mu((\Delta_1 \setminus \Delta_2) \cap \{y \leq Dx\}) = \mu(\Delta)$$

where $\Delta := \{(x, y) \in \mathbb{R}_{\geq 0}^2 \mid 0 \leq x \leq 1, y \leq Dx, (1 - t_2)x + t_2 y \leq 1 \leq (1 - t_1)x + t_1 y\}$.

By an elementary calculation, the area of Δ is at most

$$\text{Area}(\Delta) \leq S := \frac{D(D-1)(t_2 - t_1)}{(1 - t_1 + t_1 D)(1 - t_2 + t_2 D)} \leq D(D-1)(t_2 - t_1).$$

Note that $(x, y) \in \Delta$ satisfies $x \leq 1$ and $y \leq D$. So Δ can be covered by finitely many rectangles R_j of the form $[a_j, a'_j] \times [b_j, b'_j]$, satisfying

$$a_j \leq 1, \ b_j \leq D, \text{ and } \sum_j \text{Area}(R_j) \leq 2S.$$

Hence we may apply Lemma 5.2.5 to get

$$\begin{aligned} \mu(\Delta_1 \setminus \Delta_2) &\leq n^2 2^{n-1} (D+1)^{n-1} e(\mathbf{a}_{\bullet,0} \cap \mathbf{a}_{\bullet,1}) \sum_j \text{Area}(R_j) \\ &= n^2 2^n (D+1)^{n-1} e(\mathbf{a}_{\bullet,0} \cap \mathbf{a}_{\bullet,1}) S \\ &\leq n^2 2^n (D+1)^{n-1} D(D-1) e(\mathbf{a}_{\bullet,0} \cap \mathbf{a}_{\bullet,1}) (t_2 - t_1) \end{aligned}$$

Similarly $\mu(\Delta_1 \setminus \Delta_1)$ satisfies the same estimate. Hence by (5.10), we have

$$d_1(\mathbf{a}_{\bullet,t_1}, \mathbf{a}_{\bullet,t_2}) \leq n^2 2^{n+1} (D+1)^{n-1} D(D-1) e(\mathbf{a}_{\bullet,0} \cap \mathbf{a}_{\bullet,1}) (t_2 - t_1). \quad (5.11)$$

Now assume that $t_1 = 0$. We may assume $t_2 < \frac{1}{2}$. We have

$$\begin{aligned} \mu(\Delta_1 \setminus \Delta_2) &= \mu((\Delta_1 \setminus \Delta_2) \cap \{y \leq Dx\}) \\ &\leq \mu\left(\left[\frac{1 - t_2 D}{1 - t_2}, 1\right] \times [1, D]\right) \\ &\leq n^2 \left(\frac{1 - t_2 D}{1 - t_2}\right)^{n-1} 2^{n-1} (D-1) \frac{(D-1)t_2}{1 - t_2} \\ &\leq n^2 2^{2n-1} (D-1)^2 e(\mathbf{a}_{\bullet,0} \cap \mathbf{a}_{\bullet,1}) t_2, \end{aligned}$$

where the first equality follows from Proposition 4.1.9 and the second inequality

follows from Lemma 5.2.5, and

$$\begin{aligned}
\mu(\Delta_2 \setminus \Delta_1) &\leq \mu\left([1, \frac{1}{1-t_2}] \times [0, 1]\right) \\
&\leq n^2 2^{n-1} \frac{t_2}{1-t_2} \\
&\leq n^2 2^n e(\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{\bullet,1}) t_2
\end{aligned}$$

where the second inequality follows from Lemma 5.2.5. Hence

$$d_1(\mathfrak{a}_{\bullet,0}, \mathfrak{a}_{\bullet,t_2}) \leq n^2 2^n (2^{n-1} (D-1)^2 + 1) e(\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{\bullet,1}) t_2. \quad (5.12)$$

The case when $t_2 = 1$ can be treated similarly. Combining the cases together, we get

$$d_1(\mathfrak{a}_{\bullet,t_1}, \mathfrak{a}_{\bullet,t_2}) \leq L(t_2 - t_1).$$

where the Lipschitz constant can be taken to be

$$L := n^2 2^{n+1} (D+1)^{n-1} D(D-1) e(\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{\bullet,1}),$$

where we used the assumption that $D \in \mathbb{Z}_{>1}$. □

We now prove Theorem G.

Proof of Theorem G. (1) By Proposition 5.1.2, d_1 is a pseudometric on Fil .

By [BLQ24, Corollary 3.17], $\tilde{\mathfrak{a}}_{\bullet} = \tilde{\mathfrak{b}}_{\bullet}$ if and only if $e(\mathfrak{a}_{\bullet}) = e(\mathfrak{b}_{\bullet}) = e(\mathfrak{a}_{\bullet} \cap \mathfrak{b}_{\bullet})$, which holds if and only if $d_1(\mathfrak{a}_{\bullet}, \mathfrak{b}_{\bullet}) = 0$ by definition. This proves $\text{Fil}^s = \text{Fil} / \sim_1$.

(2) Assume that $\mathfrak{a}_{\bullet,0} \neq \mathfrak{a}_{\bullet,1}$. We first assume that there exists $f \in R$ and $\lambda \in \mathbb{R}_{>0}$ such that $f \in \mathfrak{a}_{\lambda,0} \setminus \mathfrak{a}_{\lambda,1}$ and $f \notin \mathfrak{a}_{\lambda',0}$ for $\lambda' > \lambda$. Take $t \in (0, 1)$ and μ, ν with $(1-t)\mu + t\nu = \lambda$. We claim that $f \notin \mathfrak{a}_{\lambda,t}$. Indeed, if $\mu \leq \lambda$, then $\nu \geq \lambda$, so $f \notin \mathfrak{a}_{\nu,1}$ since $f \notin \mathfrak{a}_{\lambda,1}$. If $\mu > \lambda$, then by the choice of f and λ we have $f \notin \mathfrak{a}_{\mu,0}$. So in either

case, $f \notin \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}$. This shows that $\mathfrak{a}_{\lambda,t} \neq \mathfrak{a}_{\lambda,1}$. If $\mathfrak{a}_{\bullet,1} \not\subseteq \mathfrak{a}_{\bullet,0}$, then we can repeat the above construction to show that $\mathfrak{a}_{\bullet,t} \neq \mathfrak{a}_{\bullet,0}$ and hence $\mathfrak{a}_{\bullet,t}$ is between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$.

Otherwise, $\mathfrak{a}_{\bullet,1} \subsetneq \mathfrak{a}_{\bullet,0}$. We claim that there exists $\lambda \in \mathbb{R}_{>0}$ and $\epsilon > 0$ such that $\mathfrak{a}_{\lambda+\epsilon,1} \supsetneq \mathfrak{a}_{\lambda,1}$ and $\mathfrak{a}_{\lambda-\epsilon,1} \supsetneq \mathfrak{a}_{\lambda,1}$. Indeed if all the jumping numbers of $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ coincide then the claim is clear. Otherwise, choose a jumping number a of $\mathfrak{a}_{\bullet,1}$ with $\mathfrak{a}_{a,1} \subsetneq \mathfrak{a}_{a,0}$ which is not a jumping number of $\mathfrak{a}_{\bullet,0}$. Then there exists $a' > a$ such that $\mathfrak{a}_{a',0} = \mathfrak{a}_{a,0}$. Let $\lambda := (a + a')/2$ and $\epsilon := \lambda - a$, then $\mathfrak{a}_{\lambda-\epsilon,1} = \mathfrak{a}_{a,1} \supsetneq \mathfrak{a}_{\lambda,1}$ since a is a jumping number and $\mathfrak{a}_{\lambda+\epsilon,1} = \mathfrak{a}_{a',1} = \mathfrak{a}_{a,1} \supsetneq \mathfrak{a}_{a,0} \supset \mathfrak{a}_{\lambda,0}$. Thus we showed $\mathfrak{a}_{\lambda,t} \neq \mathfrak{a}_{\lambda,1}$ and hence $\mathfrak{a}_{\bullet,t}$ is between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$. Now it follows from Proposition 5.2.3 that (Fil^s, d_1) is convex.

When R contains a field, by Lemma 5.2.6, the curve $\gamma : [0, 1] \rightarrow \text{Fil}^s$, $t \mapsto \tilde{\mathfrak{a}}_{\bullet,t}$ is continuous. It computes the distance between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ by Proposition 5.2.3. Hence Fil^s is a geodesic metric space. $\square$

5.3 Homogeneous filtrations and supnorm metrics

In this section we consider homogeneous filtrations and prove Theorem H. Throughout this section we will use the language of norms.

5.3.1 Homogeneous norms

Definition 5.3.1. A norm $\chi \in \mathcal{N}$ is said to be *power-multiplicative*, or *homogeneous*, if it satisfies $\chi(f^d) = d \cdot \chi(f)$ for any $f \in R$ and $d \in \mathbb{Z}_{>0}$.

A filtration $\mathfrak{a}_{\bullet}$ is said to be *homogeneous* if $f^d \in \mathfrak{a}_{\lambda}$ implies $f \in \mathfrak{a}_{\lambda/d}$ for any $\lambda > 0$ and $d \in \mathbb{Z}_{>0}$.

It is easy to see that a saturated filtration is homogeneous, and that a homogeneous filtration satisfies the additional condition in Definition-Lemma 3.1.7.

Lemma 5.3.2. *Let $\mathfrak{a}_{\bullet}$ be an $\mathfrak{m}$ -filtration. Then*

1. if $\mathfrak{a}_\bullet$ is saturated, then it is homogeneous, and
2. if $\mathfrak{a}_\bullet$ is homogeneous, then $\mathfrak{a}_\lambda = \mathfrak{m}$ for $0 < \lambda \ll 1$. In particular, $\mathfrak{a}_\bullet$ is homogeneous if and only if $\text{ord}_{\mathfrak{a}_\bullet}$ is homogeneous.

Proof. (1) If $\mathfrak{a}_\bullet$ is saturated and $f^d \in \mathfrak{a}_\lambda$, then $v(f^d) = dv(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet)$, which implies $v(f) \geq \lambda/d \cdot v(\mathfrak{a}_\bullet)$. Hence $f \in \mathfrak{a}_{\lambda/d}$ by definition.

(2) Choose a set $\{f_1, \dots, f_r\}$ of generators of $\mathfrak{m}$ and $D \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^D \subset \mathfrak{a}_1$. Then $f_i^D \in \mathfrak{a}_1$ and hence $f_i \in \mathfrak{a}_{1/D}$ by assumption. So $\mathfrak{m} \subset \mathfrak{a}_{1/D}$ and hence $\mathfrak{a}_\lambda = \mathfrak{m}$ for $0 < \lambda < 1/D$. The last assertion is straightforward by Definition-Lemma 3.1.7. $\square$

Definition 5.3.3. [CP24, Definition 3.5] Given a filtration $\mathfrak{a}_\bullet$ on R , its *asymptotic Samuel function* $\widehat{\text{ord}}_{\mathfrak{a}_\bullet} : R \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is defined by $\widehat{\text{ord}}_{\mathfrak{a}_\bullet}(f) := \lim_{k \rightarrow \infty} \frac{\text{ord}_{\mathfrak{a}_\bullet}(f^k)}{k}$ for any $f \in R$.

Denote the corresponding filtration by $\widehat{\mathfrak{a}}_\bullet$, called the *homogeneization* of $\mathfrak{a}_\bullet$.

The limit exists by Feteke's lemma, see [CP24, Theorem 3.4]. Similar to Theorem A for the saturation, the homogeneization has the following property

Theorem 5.3.4. [CP24, Theorem 5.5] Let $\mathfrak{a}_\bullet$ be an $\mathfrak{m}$ -filtration. Then $\widehat{\mathfrak{a}}_\bullet$ is the unique largest $\mathfrak{m}$ -filtration $\mathfrak{b}_\bullet$ such that $\widehat{\text{ord}}_{\mathfrak{a}_\bullet} = \widehat{\text{ord}}_{\mathfrak{b}_\bullet}$.

We provide an alternative characterization for homogeneous norms, similar to Proposition 3.5.3.

Lemma 5.3.5. A norm $\chi \in \mathcal{N}$ is homogeneous if and only if it is of the form

$$\chi = \inf_{v \in I} v,$$

where $I \subset X^{\text{an}}$.

Proof. If $\chi(f) = \inf_{v \in I} v(f)$, then χ is homogeneous, since the same is true for each v .

The reverse implication is a restatement of the Berkovich maximum modulus principle, applied to the seminormed ring (R, χ) . See, for example, [Jon20, Theorem 2.2.1] $\square$

We will denote the subspace of Fil consisting of homogeneous filtrations by $\text{Fil}^h := \{\mathbf{a}_\bullet \in \text{Fil} \mid \mathbf{a}_\bullet \text{ is homogeneous}\}$.

5.3.2 The d_∞ metric

Definition 5.3.6. Fix a norm $\rho \in \mathcal{N}^h$. We define a function $d_{\infty, \rho} : \mathcal{N} \times \mathcal{N} \rightarrow \mathbb{R}_{\geq 0}$ by

$$d_{\infty, \rho}(\chi, \chi') := \limsup_{\lambda \rightarrow \infty} \sup_{\rho(f) \geq \lambda} \left\{ \frac{|\chi(f) - \chi'(f)|}{\rho(f)} \right\}.$$

We will denote $d_{\infty, \chi_0} =: d_\infty$.

Lemma 5.3.7. For any $\rho \in \mathcal{N}$, $d_{\infty, \rho}$ defines a pseudometric on $\mathcal{N}$. Moreover, for any $\rho, \rho' \in \mathcal{N}$, there exists $M \in \mathbb{R}_{>0}$ such that $M^{-1}d_{\infty, \rho} \leq d_{\infty, \rho'} \leq Md_{\infty, \rho}$.

Proof. By definition, $d_{\infty, \rho}$ is symmetric and non-negative for any ρ . Given $\chi, \chi', \chi'' \in \mathcal{N}$, since $|\chi(f) - \chi''(f)| \leq |\chi(f) - \chi'(f)| + |\chi'(f) - \chi''(f)|$, the triangle inequality holds. This proves that $d_{\infty, \rho}$ is a pseudometric.

The second claim follows from the fact that given $\rho, \rho' \in \mathcal{N}$, there exists $M \in \mathbb{R}_{>0}$ such that $M^{-1}\rho \leq \rho' \leq M\rho$. $\square$

Lemma 5.3.8. If $\chi, \chi' \in \mathcal{N}^h$, then

$$d_\infty(\chi, \chi') = \sup_{f \in \mathfrak{m}} \frac{|\chi(f) - \chi'(f)|}{\chi_0(f)}.$$

Proof. For any $\epsilon > 0$, take $g \in \mathfrak{m}$ such that $\frac{|\chi(g) - \chi'(g)|}{\chi_0(g)} > \sup_{f \in \mathfrak{m}} \frac{|\chi(f) - \chi'(f)|}{\chi_0(f)} - \epsilon =: d - \epsilon$.

Denote $a := \chi_0(g) > 0$. Since χ, χ' and χ_0 are homogeneous, we have

$$\sup_{\chi_0(f) = ka} \frac{|\chi(f) - \chi'(f)|}{\chi_0(f)} \geq \frac{|\chi(g^k) - \chi'(g^k)|}{\chi_0(g^k)} = \frac{|\chi(g) - \chi'(g)|}{\chi_0(g)} > d - \epsilon.$$

So

$$d_\infty(\chi, \chi') = \limsup_{\lambda \rightarrow \infty} \sup_{\chi_0(f) \geq \lambda} \frac{|\chi(f) - \chi'(f)|}{\chi_0(f)} \geq d - \epsilon.$$

Thus $d_\infty(\chi, \chi') \geq d$. Since the reverse inequality is obvious, this finishes the proof. $\square$

Lemma 5.3.9. *Let $\chi, \chi' \in \mathcal{N}$, then $d_\infty(\widehat{\chi}, \widehat{\chi}') \leq d_\infty(\chi, \chi')$.*

Proof. Denote $d := d_\infty(\widehat{\chi}, \widehat{\chi}')$. By Lemma 5.3.8, we may assume that $d = \sup \frac{\widehat{\chi}(f) - \widehat{\chi}'(f)}{\chi_0(f)}$.

For any $\epsilon > 0$, there exists $f \in \mathfrak{m}$ such that

$$(d - \epsilon)\chi_0(f) < \widehat{\chi}(f) - \widehat{\chi}'(f) = \lim_{k \rightarrow \infty} \frac{\chi(f^k)}{k} - \lim_{k \rightarrow \infty} \frac{\chi'(f^k)}{k}.$$

Thus for $k \gg 0$ we have

$$\chi(f^k) - \chi'(f^k) \geq (d - 2\epsilon)k\chi_0(f) = (d - 2\epsilon)\chi_0(f^k)$$

since χ_0 is homogeneous. Similar to the proof of Lemma 5.3.8 we get $d_\infty(\chi, \chi') \geq d$. $\square$

Lemma 5.3.10. *Let $\{\chi_k\}_{k \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in $(\mathcal{N}^h, d_\infty)$. Then there exists $\chi \in \mathcal{N}^h$ such that*

$$\lim_{k \rightarrow \infty} d_\infty(\chi_k, \chi) = 0.$$

Proof. Given any $\epsilon > 0$, by assumption there exists $k \in \mathbb{Z}_{>0}$ such that for any $k, k' > K$, $d_\infty(\chi_k, \chi_{k'}) < \epsilon$. For any $f \in \mathfrak{m}$ and $k, k' > K$, by Lemma 5.3.8 we know that

$$|\chi_k(f) - \chi_{k'}(f)| \leq \chi_0(f)d_\infty(\chi_k, \chi_{k'}) < \chi_0(f) \cdot \epsilon.$$

Hence $\{\chi_k(f)\}$ is a Cauchy sequence and we may define $\chi(f) := \lim_{k \rightarrow \infty} \chi_k(f)$.

It is elementary to verify that $\chi \in \mathcal{N}^h$. Then we can apply Lemma 5.3.8 to get $\lim_{k \rightarrow \infty} d_\infty(\chi_k, \chi) = 0$. $\square$

Now we can prove Theorem H

Proof of Theorem H. Assertion (1) follows from Lemma 5.3.7.

By Lemma 5.3.8, d_∞ is non-degenerate when restricted to $\mathcal{N}^h$. Moreover, by Lemma 5.3.10, the metric space $(\mathcal{N}^h, d_\infty)$ is complete. Thus assertion (2) is proved. $\square$

5.4 Comparison of different topologies

5.4.1 Space of valuations and filtrations

From the previous sections we have the following subspaces of $\text{Val}_{X,x}$:

- the space of divisorial valuations, denoted by $\text{DivVal}_{R,\mathfrak{m}}$,
- the space of quasi-monomial valuations when R contains a field, denoted by $\text{QMVal}_{R,\mathfrak{m}}$,
- the space of valuations with finite log discrepancy, when $x \in X = \text{Spec}R$ is a klt singularity over a field of characteristic 0, denoted by $\text{Val}_{X,x}^{<+\infty}$, and
- the space of valuations with positive volume, denoted by $\text{Val}_{X,x}^+$, which is equal to the space of linearly bounded valuations by Corollary 3.4.3.

Moreover, we have the following spaces of linearly bounded filtrations on R :

- the space of linearly bounded $\mathfrak{m}$ -filtrations, denoted by $\text{Fil}_{R,\mathfrak{m}}$,
- the space of power-multiplicative $\mathfrak{m}$ -filtrations which are linearly bounded, denoted by $\text{Fil}_{R,\mathfrak{m}}^h$, and
- the space of saturated $\mathfrak{m}$ -filtrations which are linearly bounded, denoted by $\text{Fil}_{R,\mathfrak{m}}^s$.

We have the following inclusions

$$\text{DivVal}_{X,x} \subset \text{QMVal}_{X,x} \subset \text{Val}_{X,x}^{<+\infty} \subset \text{Val}_{X,x}^+ \subset \text{Fil}_{R,\mathfrak{m}}^s \subset \text{Fil}_{R,\mathfrak{m}}^h \subset \text{Fil}_{R,\mathfrak{m}} \quad (5.13)$$

where the third inclusion follows from Li's properness estimate [Li18, Theorem 1.4]. By Proposition 3.3.16, to compute the saturation of a filtration, one can use any of the above spaces of valuations.

5.4.2 Weak topologies

There are more ways to define a topology on the space Fil (and Fil^s and Fil^h , respectively), depending on the point of view.

For example, a filtration $\mathfrak{a}_\bullet$ acts naturally on the ring R as a function $\chi_{\mathfrak{a}_\bullet}$ (or dually). The product topology, or the “weak-* topology” induced by this action will be called the *weak topology*, which is the coarsest topology such that for any $f \in R \setminus \{0\}$, the function $\phi_f : \text{Fil} \rightarrow \mathbb{R}_{\geq 0}$, $\mathfrak{a}_\bullet \mapsto \text{ord}_{\mathfrak{a}_\bullet}(f)$ is continuous. In other words, a *subbase* of the weak topology is given by $\{\phi_f^{-1}(U) \mid U \subset \mathbb{R}_{\geq 0} \text{ is open, } f \in R \setminus \{0\}\}$. Under this topology, all valuation spaces in (5.13) equipped with the weak topology, defined in [JM12], are subspaces of Fil . It is not hard to check that a sequence $\{\mathfrak{a}_{\bullet,k}\} \subset \text{Fil}$ converges weakly to $\mathfrak{a}_\bullet \in \text{Fil}$ if and only if $\chi_k(f) \rightarrow \chi(f)$ for any $f \in R \setminus \{0\}$.

Similarly, a valuation has a natural action on the spaces of filtrations (or dually), that is, any $v \in \text{Val}_{X,x}$ gives a function $\text{ev}_v : \text{Fil} \rightarrow \mathbb{R}_{\geq 0}$, $\mathfrak{a}_\bullet \mapsto v(\mathfrak{a}_\bullet)$. Hence one can consider the “weak topology” induced by this action, that is, the coarsest topology on Fil^s such that ev_v is continuous for any $v \in X^{\text{an}}$, $\text{Val}_{X,x}$, $\text{Val}_{X,x}^+$ or $\text{DivVal}_{X,x}$, respectively. Similarly one can check that a sequence $\{\mathfrak{a}_{\bullet,k}\}$ converges to $\mathfrak{a}_\bullet$ under these topologies if and only if $v(\mathfrak{a}_{\bullet,k}) \rightarrow v(\mathfrak{a}_\bullet)$ for any v in the corresponding valuation space. In this paper, we will mainly consider the topology defined using $\text{Val}_{X,x}^+$, and call it the *+topology*. The product topology defined using $\text{DivVal}_{X,x}$ as above

was considered in [BdFFU15] for b -divisors. We will thus call it the *coefficientwise topology*.

Remark 5.4.1. Since $v(\mathbf{a}_\bullet) = v(\widetilde{\mathbf{a}}_\bullet)$ for $v \in \text{DivVal}_{X,x}$ by Proposition 3.3.9, the functions ev_v do not distinguish $\mathbf{a}_\bullet$ and its saturation. Hence the $+$ -topology (and the coefficientwise topology) only makes sense on Fil^s . Similar issues appear if we define the topology on Fil using other valuation spaces. But all the topologies are well-defined on Fil^s .

5.4.3 Comparison of the topologies

Now we would like to compare the topologies on the space Fil^s .¹ First we show that the a d_1 -converging sequence converges in the $+$ -topology.

Lemma 5.4.2. *Let $\mathbf{a}_\bullet, \mathbf{a}_{\bullet,k} \in \text{Fil}^s$ for $k \in \mathbb{Z}_{>0}$. If $\lim_k d_1(\mathbf{a}_{\bullet,k}, \mathbf{a}_\bullet) = 0$, then $\mathbf{a}_{\bullet,k} \rightarrow \mathbf{a}_\bullet$ in the $+$ -topology.*

Proof. We claim that if $\mathbf{a}_{\bullet,k} \subset \mathbf{b}_{\bullet,k}$ satisfy $\lim_{k \rightarrow \infty} d_1(\mathbf{a}_{\bullet,k}, \mathbf{b}_{\bullet,k}) = 0$, then for any $v \in \text{Val}_{X,x}^+$, we have $\lim_{k \rightarrow \infty} (v(\mathbf{a}_{\bullet,k}) - v(\mathbf{b}_{\bullet,k})) = 0$.

For any $v \in \text{Val}_{X,x}^+$, applying the claim to $\mathbf{a}_{\bullet,k}, \mathbf{a}_{\bullet,k} \cap \mathbf{a}_\bullet$ and $\mathbf{a}_{\bullet,k} \cap \mathbf{a}_\bullet, \mathbf{a}_\bullet$ separately yields

$$\lim_{k \rightarrow \infty} |v(\mathbf{a}_{\bullet,k}) - v(\mathbf{a}_\bullet)| \leq \lim_{k \rightarrow \infty} |v(\mathbf{a}_{\bullet,k}) - v(\mathbf{a}_{\bullet,k} \cap \mathbf{a}_\bullet)| + \lim_{k \rightarrow \infty} |v(\mathbf{a}_{\bullet,k} \cap \mathbf{a}_\bullet) - v(\mathbf{a}_\bullet)| = 0.$$

Now it remains to prove the claim, which follows essentially from the same argument as Proposition 3.3.12. We include a sketch for the reader's convenience. Assume to the contrary that there exists $\epsilon > 0$ and $v \in \text{Val}_{X,x}^+$ such that $v(\mathbf{a}_{\bullet,k}) > v(\mathbf{b}_{\bullet,k}) + \epsilon$

¹Since the product topology is in general not first countable and hence not obviously sequential, we avoid using the term *finer* for them, but only compare convergence for sequences.

for any k . Then for $k \in \mathbb{Z}_{>0}$, we may choose $\ell_k \in \mathbb{Z}_{>0}$ and $f_k \in \mathfrak{b}_{\ell_k, k}$ such that

$$\frac{v(f_k)}{\ell_k} = \frac{v(\mathfrak{b}_{\ell_k, k})}{\ell_k} < v(\mathfrak{a}_{\bullet, k}) - \epsilon.$$

For $m \in \mathbb{Z}_{>0}$, consider the morphism $\phi_{k, m} : R \rightarrow R/\mathfrak{a}_{m\ell_k, k}$, $g \mapsto f_k^m g + \mathfrak{a}_{m\ell_k, k}$. Since $f_k^m \in \mathfrak{b}_{\ell_k, k}^m \subset \mathfrak{b}_{m\ell_k, k}$, the image is contained in $\mathfrak{b}_{m\ell_k, k}/\mathfrak{a}_{m\ell_k, k}$. Moreover, one can check that $\ker(\phi_{k, m}) \subset \mathfrak{a}_{m\ell_k \epsilon}(v)$. Hence

$$\ell(\mathfrak{b}_{m\ell_k, k}/\mathfrak{a}_{m\ell_k, k}) \geq \ell(R/\ker(\phi_{k, m})) \geq \ell(R/\mathfrak{a}_{m\ell_k \epsilon}(v)).$$

Letting $m \rightarrow \infty$ we get $e(\mathfrak{a}_{\bullet, k}) - e(\mathfrak{b}_{\bullet, k}) \geq \epsilon^n \text{vol}(v) > 0$, which is a contradiction. $\square$

Next we compare the two metrics d_1 and d_∞ .

Lemma 5.4.3. *Let $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet \in \text{Fil}$ be filtrations satisfying $\mathfrak{m}^{C_\bullet} \subset \mathfrak{a}_\bullet$ and $\mathfrak{m}^{C_\bullet} \subset \mathfrak{b}_\bullet$ for some $C \in \mathbb{R}_{>1}$. Then there exists a constant $M := M(C) > 0$ such that*

$$d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) \leq M \cdot d_\infty(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet). \quad (5.14)$$

As a consequence, the d_∞ -topology is finer than the d_1 -topology.

Proof. By Lemma 5.3.9 we may assume $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet \in \text{Fil}^h$. Since $d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) = d_1(\mathfrak{a}_\bullet, \mathfrak{c}_\bullet) + d_1(\mathfrak{b}_\bullet, \mathfrak{c}_\bullet)$ and $d_\infty(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) \geq \max\{d_\infty(\mathfrak{a}_\bullet, \mathfrak{c}_\bullet), d_\infty(\mathfrak{b}_\bullet, \mathfrak{c}_\bullet)\}$, where $\mathfrak{c}_\bullet := \mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet$, we may further assume that $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ (but replace M by $2M$).

Denote $d := d_\infty(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)$. Then we know that

$$0 \leq \frac{\chi_{\mathfrak{b}}(f)}{\chi_{\mathfrak{a}}(f)} - 1 = \frac{\chi_0(f)}{\chi_{\mathfrak{a}}(f)} \cdot \frac{\chi_{\mathfrak{b}}(f) - \chi_{\mathfrak{a}}(f)}{\chi_0(f)} \leq Cd,$$

which implies

$$\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet \subset \mathfrak{a}_\bullet/(1+Cd).$$

Hence we get

$$e(\mathfrak{a}_\bullet) \geq e(\mathfrak{b}_\bullet) \geq (1 + Cd)^{-n} e(\mathfrak{a}_\bullet),$$

that is

$$\begin{aligned} d_1(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) &= e(\mathfrak{a}_\bullet) - e(\mathfrak{b}_\bullet) \\ &\leq (1 - (1 + Cd)^{-n}) e(\mathfrak{a}_\bullet) \\ &\leq nCd \cdot e(\mathfrak{a}_\bullet) \\ &\leq nC^{n+1}e(\mathfrak{m}) \cdot d, \end{aligned}$$

where in the last inequality we used the fact that $(1 + x)^{-n} \geq 1 - nx$ for $x \geq 0$. This proves (5.14) with $M := 2nC^{n+1}e(\mathfrak{m})$.

Given $\mathfrak{a}_\bullet \in \text{Fil}$, we may take $C \in \mathbb{Z}_{>1}$ such that $\mathfrak{m}^C \subset \mathfrak{a}_1$ which implies $\mathfrak{m}^{C\bullet} \subset \mathfrak{a}_\bullet$. Now for $0 < \epsilon < 1/2C$ and $\mathfrak{b}_\bullet \in \text{Fil}$ with $d_\infty(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet) < \epsilon$, by definition we have $\mathfrak{m}^{2C\bullet} \subset \mathfrak{b}_\bullet$. Hence the second assertion follows from (5.14). $\square$

To summarize, we have the following diagram for convergence of sequences

$$d_\infty\text{-convergence} \left\{ \begin{array}{l} \implies \text{weak convergence} \\ \implies d_1\text{-convergence} \implies +\text{-convergence} \implies \text{coefficientwise} \end{array} \right.$$

Here the upper line is obvious; the first two arrows of the lower line follow from Lemma 5.4.3 and Lemma 5.4.2 respectively; and the last arrow follows from the fact that $\text{DivVal}_{X,x} \subset \text{Val}_{X,x}^+$. It is not hard to see that the upper line is not an equivalence. We do not have any counterexamples for the other reverse arrows.

Chapter 6

Continuity of log canonical thresholds

In this section we consider the continuity properties of the log canonical threshold function in various topologies.

We first prove that the log canonical thresholds is locally Lipschitz with respect to the d_∞ -topology.

Proposition 6.0.1. *Let $\chi \in \mathcal{N}$. For any $\epsilon > 0$, there exists $\delta = \delta(\chi, \epsilon) > 0 \in \mathbb{R}_{>0}$ such that for any $\chi' \in \mathcal{N}$ with $d_\infty(\chi, \chi') < \delta$, we have $|\text{lct}(\chi') - \text{lct}(\chi)| \leq \epsilon$.*

Proof. Since $\text{lct}(\chi) = \text{lct}(\widehat{\chi})$ for any $\chi \in \mathcal{N}$, by Lemma 5.3.9, we may assume that $\chi, \chi' \in \mathcal{N}^h$. Fix $c = c(\chi) := \inf_{f \in \mathfrak{m}} \frac{\chi(f)}{\chi_0(f)} > 0$ and denote $a := \text{lct}(\chi) > 0$. $\square$

Proof. Assume that $d_\infty(\chi, \chi') < \delta$ for some $\delta \in (0, c)$. By Lemma 5.3.8, we know that for any $f \in R$,

$$\frac{\chi'(f)}{\chi_0(f)} \geq \frac{\chi(f)}{\chi_0(f)} - \delta \geq c - \delta.$$

Now we compute

$$\begin{aligned}
\frac{v(f)}{\chi'(f)} &= \frac{v(f)}{\chi(f)} \left(1 + \frac{\chi(f) - \chi'(f)}{\chi'(f)} \right) \\
&= \frac{v(f)}{\chi(f)} \left(1 + \frac{\chi_0(f)}{\chi'(f)} \frac{\chi(f) - \chi'(f)}{\chi_0(f)} \right) \\
&\leq \frac{v(f)}{\chi(f)} \left(1 + \frac{\delta}{c - \delta} \right)
\end{aligned} \tag{6.1}$$

For $\delta \in (0, c/2)$, by Lemma 3.1.8 we get that $v(\mathfrak{a}_\bullet(\chi')) \leq (1 + 2\delta/c)v(\mathfrak{a}_\bullet(\chi))$. Now let

$$\delta := \min\left\{\frac{c}{2}, \frac{c\epsilon}{a}\right\} > 0,$$

which depends only on ϵ and χ . Then for any $\chi' \in \text{Fil}^s$ with $d_\infty(\chi, \chi') < \delta$ and any $v \in \text{Val}_{X,x}^{\leq +\infty}$, we have

$$\frac{A(v)}{v(\mathfrak{a}_\bullet(\chi'))} > \left(1 - \frac{\delta}{2c}\right) \frac{A(v)}{v(\mathfrak{a}_\bullet(\chi))} \geq \left(1 - \frac{\delta}{2c}\right) \text{lct}(\chi) = a - \frac{a}{2c}\delta > a - \epsilon.$$

Taking infimum among all v gives $\text{lct}(\chi') \geq a - \epsilon$.

Interchanging χ and χ' in (6.1), by Lemma 3.1.8 we get $v(\mathfrak{a}_\bullet(\chi)) \leq (1 + \delta/c)v(\mathfrak{a}_\bullet(\chi'))$.

By Theorem 2.3.1, there exists $v_0 \in \text{Val}_{X,x}^{\leq +\infty}$ computing the lct of $\mathfrak{a}_\bullet(chi)$, thus

$$\text{lct}(\chi') \leq \frac{A(v_0)}{v_0(\mathfrak{a}_\bullet(\chi'))} \leq \left(1 + \frac{\delta}{c}\right) \frac{A(v_0)}{v_0(\mathfrak{a}_\bullet(\chi))} = \left(1 + \frac{\delta}{c}\right)a \leq a + \epsilon.$$

The proof is finished. □

Proposition 6.0.2. *Let $\mathfrak{b}_{\bullet,k} \in \text{Fil}^s$ be a sequence of filtrations that converges weakly to $\mathfrak{a}_\bullet \in \text{Fil}^s$. Then*

$$\text{lct}(\mathfrak{a}_\bullet) \leq \liminf_{k \rightarrow \infty} \text{lct}(\mathfrak{b}_{\bullet,k}). \tag{6.2}$$

Proof. Denote $\rho_k = \text{ord}_{\mathfrak{b}_{\bullet,k}}$ and $\chi = \text{ord}_{\mathfrak{a}_\bullet}$. For $k, l \in \mathbb{Z}_{>0}$, define

$$\mathfrak{a}_{\bullet,k}^{(l)} := \bigcap_{j=k}^{k+l} \mathfrak{b}_{\bullet,j}.$$

Then for any $k \in \mathbb{Z}_{>0}$, $\{\mathfrak{a}_{\bullet,k}^{(l)}\}_{l \in \mathbb{Z}_{>0}}$ is a decreasing sequence of filtrations. Let

$$\mathfrak{a}_{\bullet,k} := \cap_l \mathfrak{a}_{\bullet,k}^{(l)} \subset \mathfrak{b}_{\bullet,k}.$$

Clearly the sequence of filtrations $\{\mathfrak{a}_{\bullet,k}\}_{k \in \mathbb{Z}_{>0}}$ is increasing. Let $\chi_k = \text{ord}_{\mathfrak{a}_{\bullet,k}}$ be the corresponding norm. By definition, $\chi_k = \inf_{j \geq k} \rho_j \leq \chi$. Since $\rho_j \rightarrow \chi$ weakly, given any $f \in \mathfrak{m}$ and any $\epsilon > 0$, there exists $J \in \mathbb{Z}_{>0}$ such that $\rho_j(f) > \chi(f) - \epsilon$ for any $j \geq J$. Thus

$$\chi(f) \geq \rho_k(f) = \inf_{j \geq k} \rho_j(f) \geq \chi(f) - \epsilon$$

as long as $k \geq J$. This proves $\chi_k \rightarrow \chi$ weakly as an increasing sequence. Note that for any $k \in \mathbb{Z}_{>0}$, $\text{lct}(\mathfrak{a}_{\bullet,k}) \leq \text{lct}(\mathfrak{b}_{\bullet,k})$, so $\lim_{k \rightarrow \infty} \text{lct}(\mathfrak{a}_{\bullet,k}) \leq \liminf_{k \rightarrow \infty} \text{lct}(\mathfrak{b}_{\bullet,k})$. Thus to prove (6.2), it suffices to show that

$$\text{lct}(\mathfrak{a}_{\bullet}) \leq \lim_{k \rightarrow \infty} \text{lct}(\mathfrak{a}_{\bullet,k}). \quad (6.3)$$

Assume to the contrary that $\text{lct}(\mathfrak{a}_{\bullet}) > \lim_k \text{lct}(\mathfrak{a}_{\bullet,k}) =: c$. Fix $\epsilon > 0$ such that $\text{lct}(\mathfrak{a}_{\bullet}) > c + \epsilon$. Since $\text{lct}(\mathfrak{a}_{\bullet}) = \lim_{\lambda} \lambda \cdot \text{lct}(\mathfrak{a}_{\lambda})$, there exists $\Lambda \in \mathbb{R}_{>0}$ such that $\lambda \cdot \text{lct}(\mathfrak{a}_{\lambda}) > c + \epsilon$ for any $\lambda \geq \Lambda$. As $\mathfrak{a}_{\bullet,k}$ converges to $\mathfrak{a}_{\bullet}$ weakly, that is, $\mathfrak{a}_{\Lambda} = \cup_k \mathfrak{a}_{\Lambda,k}$, there exists $k_1 \in \mathbb{Z}_{>0}$ such that $\mathfrak{a}_{\Lambda} = \mathfrak{a}_{\Lambda,k_1}$. In particular,

$$\Lambda \cdot \text{lct}(\mathfrak{a}_{\Lambda,k_1}) > c + \epsilon. \quad (6.4)$$

Now take $k_2 \in \mathbb{Z}_{>0}$ such that $\text{lct}(\mathfrak{a}_{\bullet,k}) = \sup_{\lambda} \lambda \cdot \text{lct}(\mathfrak{a}_{\lambda,k}) < c + \epsilon$ for any $k \geq k_2$. For $k \geq \max\{k_1, k_2\}$, combining this with (6.4) we get

$$c + \epsilon < \Lambda \cdot \text{lct}(\mathfrak{a}_{\Lambda,k_1}) = \Lambda \cdot \text{lct}(\mathfrak{a}_{\Lambda,k}) \leq \sup_{\lambda} \lambda \cdot \text{lct}(\mathfrak{a}_{\lambda,k}) < c + \epsilon,$$

where the equality follows from $\mathfrak{a}_{\Lambda,k_1} = \mathfrak{a}_{\Lambda} = \mathfrak{a}_{\Lambda,k}$. This is a contradiction, so (6.3)

holds and the proof is finished. $\square$

We are now ready to prove Theorem I.

Proof of Theorem I. (1) follows immediately from Proposition 6.0.2.

To see (2), assume that $\mathfrak{a}_{\bullet,k} \rightarrow \mathfrak{a}_{\bullet} \in \text{Fil}^s$ in the $+$ -topology. By Theorem 2.3.1, there exists $v \in \text{Val}_{X,x}^{<+\infty}$ such that $\text{lct}(\mathfrak{a}_{\bullet}) = \frac{A_X(v)}{v(\mathfrak{a}_{\bullet})}$. By definition, we have $\lim_k v(\mathfrak{a}_{\bullet,k}) = v(\mathfrak{a}_{\bullet})$, hence

$$\limsup_{k \rightarrow \infty} \text{lct}(\mathfrak{a}_{\bullet,k}) = \limsup_{k \rightarrow \infty} \inf_{w \in \text{Val}_{X,x}^{<+\infty}} \frac{A_X(w)}{w(\mathfrak{a}_{\bullet,k})} \leq \lim_{k \rightarrow \infty} \frac{A_X(v)}{v(\mathfrak{a}_{\bullet,k})} = \frac{A_X(v)}{v(\mathfrak{a}_{\bullet})} = \text{lct}(\mathfrak{a}_{\bullet}).$$

(3) follows from Proposition 6.0.1. The proof is finished. $\square$

Bibliography

- [AZ22] Hamid Abban and Ziquan Zhuang. K-stability of Fano varieties via admissible flags. *Forum Math. Pi*, 10:Paper No. e15, 43, 2022.
- [BBE⁺19] Robert J. Berman, Sebastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties. *J. Reine Angew. Math.*, 751:27–89, 2019.
- [BBJ21] Robert J. Berman, Sébastien Boucksom, and Mattias Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture. *J. Amer. Math. Soc.*, 34(3):605–652, 2021.
- [BC11] Sébastien Boucksom and Huayi Chen. Okounkov bodies of filtered linear series. *Compos. Math.*, 147(4):1205–1229, 2011.
- [BdFF12] Sebastien Boucksom, Tommaso de Fernex, and Charles Favre. The volume of an isolated singularity. *Duke Math. J.*, 161(8):1455–1520, 2012.
- [BdFFU15] S. Boucksom, T. de Fernex, C. Favre, and S. Urbinati. Valuation spaces and multiplier ideals on singular varieties. In *Recent advances in algebraic geometry*, volume 417 of *London Math. Soc. Lecture Note Ser.*, pages 29–51. Cambridge Univ. Press, Cambridge, 2015.

- [BE21] Sébastien Boucksom and Dennis Eriksson. Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry. *Adv. Math.*, 378:Paper No. 107501, 124, 2021.
- [BFJ14] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. A refinement of Izumi’s theorem. In *Valuation theory in interaction*, EMS Ser. Congr. Rep., pages 55–81. Eur. Math. Soc., Zürich, 2014.
- [BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.
- [BHJ17] Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson. Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. *Ann. Inst. Fourier (Grenoble)*, 67(2):743–841, 2017.
- [BJ20] Harold Blum and Mattias Jonsson. Thresholds, valuations, and K-stability. *Adv. Math.*, 365:107062, 57, 2020.
- [BJ21] Sébastien Boucksom and Mattias Jonsson. A non-archimedean approach to k-stability, i: Metric geometry of spaces of test configurations and valuations. *arXiv preprint arXiv:2107.11221*, 2021.
- [BLQ24] Harold Blum, Yuchen Liu, and Lu Qi. Convexity of multiplicities of filtrations on local rings. *Compos. Math.*, 160(4):878–914, 2024.
- [Blu18a] Harold Blum. Existence of valuations with smallest normalized volume. *Compos. Math.*, 154(4):820–849, 2018.
- [Blu18b] Harold Blum. *Singularities and K-stability*. PhD thesis, University of Michigan, 2018.

- [BLXZ23] Harold Blum, Yuchen Liu, Chenyang Xu, and Ziquan Zhuang. The existence of the Kähler-Ricci soliton degeneration. *Forum Math. Pi*, 11:Paper No. e9, 28, 2023.
- [CDS15] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds. I, II, III: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, 28(1):183–197, 199–234, 235–278, 2015.
- [CP24] Steven Dale Cutkosky and Smita Praharaj. The asymptotic samuel function of a filtration. *Acta Math. Viet.*, pages 1–21, 2024.
- [CS22] Steven Dale Cutkosky and Parangama Sarkar. Multiplicities and mixed multiplicities of arbitrary filtrations. *Res. Math. Sci.*, 9(1):Paper No. 14, 33, 2022.
- [Cut13] Steven Dale Cutkosky. Multiplicities associated to graded families of ideals. *Algebra Number Theory*, 7(9):2059–2083, 2013.
- [Cut14] Steven Dale Cutkosky. Asymptotic multiplicities of graded families of ideals and linear series. *Adv. Math.*, 264:55–113, 2014.
- [Cut15] Steven Dale Cutkosky. Asymptotic multiplicities. *J. Algebra*, 442:260–298, 2015.
- [Cut21] Steven Dale Cutkosky. The Minkowski equality of filtrations. *Adv. Math.*, 388:Paper No. 107869, 63, 2021.
- [Dar15] Tamás Darvas. The Mabuchi geometry of finite energy classes. *Adv. Math.*, 285:182–219, 2015.

- [Dar19] Tamás Darvas. Geometric pluripotential theory on Kähler manifolds. In *Advances in complex geometry*, volume 735 of *Contemp. Math.*, pages 1–104. Amer. Math. Soc., [Providence], RI, [2019] ©2019.
- [DK01] Jean-Pierre Demailly and János Kollár. Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. École Norm. Sup. (4)*, 34(4):525–556, 2001.
- [DL20] Tamás Darvas and Chinh H. Lu. Geodesic stability, the space of rays and uniform convexity in Mabuchi geometry. *Geom. Topol.*, 24(4):1907–1967, 2020.
- [DR17] Tamás Darvas and Yanir A. Rubinstein. Tian’s properness conjectures and Finsler geometry of the space of Kähler metrics. *J. Amer. Math. Soc.*, 30(2):347–387, 2017.
- [ELS03] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. Uniform approximation of Abhyankar valuation ideals in smooth function fields. *Amer. J. Math.*, 125(2):409–440, 2003.
- [FJ04] Charles Favre and Mattias Jonsson. *The valuative tree*, volume 1853 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.
- [Fuj19] Kento Fujita. A valuative criterion for uniform K-stability of $\mathbb{Q}$ -Fano varieties. *J. Reine Angew. Math.*, 751:309–338, 2019.
- [Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

- [Hör03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].
- [HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.
- [JM12] Mattias Jonsson and Mircea Mustața. Valuations and asymptotic invariants for sequences of ideals. *Ann. Inst. Fourier (Grenoble)*, 62(6):2145–2209, 2012.
- [Jon20] Mattias Jonsson. *Lecture Notes on Berkovich Spaces*. <https://dept.math.lsa.umich.edu/~mattiasj/715/>, 2020.
- [Kat88] Daniel Katz. Note on multiplicity. *Proc. Amer. Math. Soc.*, 104(4):1021–1026, 1988.
- [KK14] Kiumars Kaveh and Askold Khovanskii. Convex bodies and multiplicities of ideals. *Proc. Steklov Inst. Math.*, 286(1):268–284, 2014. Reprint of Tr. Mat. Inst. Steklova **286** (2014), 291–307.
- [Kle66] Steven L. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math. (2)*, 84:293–344, 1966.
- [Kle05] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series*

- of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.
- [Li17] Chi Li. K-semistability is equivariant volume minimization. *Duke Math. J.*, 166(16):3147–3218, 2017.
 - [Li18] Chi Li. Minimizing normalized volumes of valuations. *Math. Z.*, 289(1-2):491–513, 2018.
 - [Liu18] Yuchen Liu. The volume of singular Kähler-Einstein Fano varieties. *Compos. Math.*, 154(6):1131–1158, 2018.
 - [LL19] Chi Li and Yuchen Liu. Kähler-Einstein metrics and volume minimization. *Adv. Math.*, 341:440–492, 2019.
 - [LLX20] Chi Li, Yuchen Liu, and Chenyang Xu. A guided tour to normalized volume. In *Geometric analysis—in honor of Gang Tian’s 60th birthday*, volume 333 of *Progr. Math.*, pages 167–219. Birkhäuser/Springer, Cham, [2020] ©2020.
 - [LM09] Robert Lazarsfeld and Mircea Mustață. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.
 - [LTW22] Chi Li, Gang Tian, and Feng Wang. The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties. *Peking Math. J.*, 5(2):383–426, 2022.
 - [LWX21] Chi Li, Xiaowei Wang, and Chenyang Xu. Algebraicity of the metric tangent cones and equivariant K-stability. *J. Amer. Math. Soc.*, 34(4):1175–1214, 2021.

- [LX18] Chi Li and Chenyang Xu. Stability of valuations: higher rational rank. *Peking Math. J.*, 1(1):1–79, 2018.
- [LX20] Chi Li and Chenyang Xu. Stability of valuations and Kollár components. *J. Eur. Math. Soc. (JEMS)*, 22(8):2573–2627, 2020.
- [LXZ22] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang. Finite generation for valuations computing stability thresholds and applications to K-stability. *Ann. of Math. (2)*, 196(2):507–566, 2022.
- [Mab87] Toshiki Mabuchi. Some symplectic geometry on compact Kähler manifolds. I. *Osaka J. Math.*, 24(2):227–252, 1987.
- [Mus02] Mircea Mustață. On multiplicities of graded sequences of ideals. *J. Algebra*, 256(1):229–249, 2002.
- [Ram74] C. P. Ramanujam. On a geometric interpretation of multiplicity. *Invent. Math.*, 22:63–67, 1973/74.
- [Reb23] Rémi Reboulet. The space of finite-energy metrics over a degeneration of complex manifolds. *J. Éc. polytech. Math.*, 10:659–701, 2023.
- [Ree61] D. Rees. $\mathfrak{a}$ -transforms of local rings and a theorem on multiplicities of ideals. *Proc. Cambridge Philos. Soc.*, 57:8–17, 1961.
- [RS78] D. Rees and R. Y. Sharp. On a theorem of B. Teissier on multiplicities of ideals in local rings. *J. London Math. Soc. (2)*, 18(3):449–463, 1978.
- [RS14] G. Rond and M. Spivakovsky. The analogue of Izumi’s theorem for Abhyankar valuations. *J. Lond. Math. Soc. (2)*, 90(3):725–740, 2014.
- [Sta18] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>, 2018.

- [Szé15] Gábor Székelyhidi. Filtrations and test-configurations. *Math. Ann.*, 362(1-2):451–484, 2015. With an appendix by Sebastien Boucksom.
- [Tei78] B. Teissier. On a Minkowski-type inequality for multiplicities. II. In *C. P. Ramanujam—a tribute*, volume 8 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 347–361. Springer, Berlin-New York, 1978.
- [Tia15] Gang Tian. K-stability and Kähler-Einstein metrics. *Comm. Pure Appl. Math.*, 68(7):1085–1156, 2015.
- [Xu18] Chenyang Xu. Interaction between singularity theory and the minimal model program. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pages 807–830. World Sci. Publ., Hackensack, NJ, 2018.
- [Xu20] Chenyang Xu. A minimizing valuation is quasi-monomial. *Ann. of Math. (2)*, 191(3):1003–1030, 2020.
- [XZ21] Chenyang Xu and Ziquan Zhuang. Uniqueness of the minimizer of the normalized volume function. *Camb. J. Math.*, 9(1):149–176, 2021.
- [XZ22] Chenyang Xu and Ziquan Zhuang. Stable degenerations of singularities. *arXiv preprint arXiv:2205.10915*, 2022.
- [Zhu22] Ziquan Zhuang. On boundedness of singularities and minimal log discrepancies of kollár components. *arXiv preprint arXiv:2202.06455*, 2022.