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Doctoral Dissertations, 2011-2026 Mathematics

2020

# On (1,1)-knots and L-space conjecture

Nie, Zipei

Princeton, NJ : Princeton University

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# On $(1,1)$-knots and L-space conjecture

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A Dissertation

Presented to the Faculty

of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance

by the Department of

Mathematics

Adviser: Professor Zoltán Szabó

June 2020

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# Abstract

In this thesis, we present some results about $(1,1)$-knots and L-space conjecture. In particular, we prove that (1) L-space twisted torus knots of form $T^{l,m}_{p,kp\pm1}$ are closures of 1-bridge braids; (2) the L-space conjecture holds for the L-spaces obtained from Dehn surgery on closures of iterated 1-bridge braids, and for 3-manifolds obtained from Dehn fillings on the hyperbolic $\mathbf{Q}$-homology solid torus $v2503$; (3) there are infinitely many $(1,1)$-knots which are topologically slice but not smoothly slice.

# Acknowledgements

I would like to extend my genuine gratitude to my advisor Zoltán Szabó who offered me tremendous freedom to explore problems in the field of low dimensional topology. His kindness and encouragement have made me highly motivated during Ph.D.. I would like to thank David Gabai and Peter Ozsváth for serving my thesis committee, and Steven Boyer for being my thesis reader.

I would like to thank my peers Weibo Fu, Zhaorong Jin, Maggie Miller, Congling Qiu, Antoine Song, Jun Su, Konstantinos Varvarezos, Qingcan Wang, Lingfu Zhang, Ruixiang Zhang and Fan Zheng. I have many mathematical discussions with them in the Fine Hall common room. In addition, I would like to thank Jonathan Hanselman, Kazuhiro Ichihara, Yi Ni, Carola Wenk, Fan Ye and Boyu Zhang, with whom I spent much time discussing my research papers during my Ph.D. studies. I would like to thank Jill LeClair for administrative supports.

I would like to offer special thanks to all the faculty and staff of the mathematics department of Princeton University for building such an intellectual, pleasant and productive place to learn and work together, especially John Conway, who is no longer with us.

I would like to dedicate this thesis to my wife, Fei Xia, for making my days happier during my Ph.D. in Princeton, and being a good listener of all my random thoughts. Finally, I thank my father and mother, for their unconditional and continual care and support.

To my lovely wife Fei.

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# Chapter 1

# Introduction

The purpose of the thesis is to present some works [38, 39] of the author on $(1,1)$-knots and the L-space conjecture.

The L-space conjecture is a conjectural relation of having the simplest Heegaard Floer homology (that is, being L-spaces), the left-orderability of their fundamental groups and the existence of taut foliation for irreducible rational homology 3-spheres. The existence of taut foliation is a central open problem in the foliation theory of 3-manifolds, and if this conjecture holds true, we will have a group theoretic criteria and a Heegaard Floer homological criteria for the taut foliation condition. While the L-space conjecture is verified for geometric non-hyperbolic 3-manifolds, 2-fold branched covers of non-split alternating links and graph manifolds, there are still many 3-manifolds for which the problem is open. A general way to construct 3-manifolds is via Dehn surgery.

$(1,1)$-knots are knots which admit $(1,1)$-decompositions in $S^3$. They are defined with a compatible Heegaard splitting of $S^3$ of genus 1. As a result, the knot Floer homology of $(1,1)$-knots is arguably the simplest to compute.

On one hand, nontrivial knots which are closures of 1-bridge braids are $(1,1)$ L-space knots, which means the Dehn surgeries yield many L-spaces for which we can test the L-space conjecture.

In Chapter 2, we will prove that the L-space twisted torus knots of form $T^{l,m}_{p,kp\pm1}$, which form another widely studied family of $(1,1)$ L-space knots, are closures of 1-bridge braids.

In Chapter 3, we will prove that the L-space conjecture holds for the L-spaces obtained from Dehn surgery on closures of iterated 1-bridge braids.

In Chapter 4, we will prove that the L-space conjecture holds for 3-manifolds obtained from Dehn fillings on the hyperbolic $\mathbf{Q}$-homology solid torus $v2503$ using the same technique as in Chapter 3.

On the other hand, in Chapter 5, we will use the Alexander polynomial and the $\tau$ invariant from knot Floer homology to prove that infinitely many $(1,1)$-knots are topologically slice but not smoothly slice.

## 1.1 L-space conjecture

### 1.1.1 Heegaard Floer homology and knot Floer homology

The Heegaard Floer homology is a collection of 3-manifold invariants constructed by combining the ideas of Heegaard splitting and pseudo-holomorphic disk counting. The knot Floer homology is a closely related collection of knot invariants. Among all different versions, the hat version $\widehat{HF}$ and $\widehat{HKF}$ are the simplest. A basic construction of $\widehat{HF}$ and $\widehat{HKF}$ by multi-pointed Heegaard diagrams can be found in Section 2 in Rácz's thesis [45]. As defined in [42], the $\tau$ invariant is the least integer $m$ with non-trivial induced map

$$H_*(\mathcal{F}(K,m)) \to H_*(\widehat{CF}(S^3))$$

where the chain complex $\widehat{CF}(S^3)$ and the subcomplex $\mathcal{F}(K,m) \subset \widehat{CF}(S^3)$ generated by intersection points with Alexander grading less than or equal to $m$ are defined in the construction of Heegaard Floer homology and knot Floer homology.

For the Heegaard Floer homology $\widehat{HF}$ of a rational homology 3-sphere, the following relation was proved.

**Theorem 1.1.1.** *For a rational homology 3-sphere $Y$, we have $\chi(\widehat{HF}(Y)) = |H_1(Y;\mathbf{Z})|$.*

In particular, for a rational homology 3-sphere $Y$, the rank of $\widehat{HF}(Y)$ is at least $|H_1(Y;\mathbf{Z})|$. We define the L-spaces as the rational homology 3-spheres with minimal Heegaard Floer homology in the following sense.

**Definition 1.1.1.** *An L-space is a rational homology 3-sphere $Y$ with the rank of $\widehat{HF}(Y)$ equal to $|H_1(Y;\mathbf{Z})|$.*

Many L-spaces are constructed via Dehn surgery, so we define the L-space knots as follows.

**Definition 1.1.2.** *A (positive) L-space knot is a knot $K$ in $S^3$ such that there exists a positive Dehn surgery along $K$ which yields an L-space.*

For any knot in $S^3$, the set of L-space surgery slopes is always an interval. More specifically, the following theorem was proved in [24, 43].

**Theorem 1.1.2.** *For any nontrivial L-space knot $K$ in $S^3$, the Dehn surgery along $K$ with slope $\frac{p}{q}$ yields an L-space if and only if $\frac{p}{q} \geq 2g(K) - 1$.*

### 1.1.2 Taut foliations

In foliation theory, a codimension $k$ foliation $\mathcal{F}$ of an $n$-manifold $M^n$ is defined as follows.

**Definition 1.1.3.** *A codimension $k$ foliation $\mathcal{F}$ of an $n$-manifold $M^n$ is a partition of $M^n$ into leaves $\{L_\alpha\}_{\alpha\in A}$, such that, there exists a cover of $M^n$ by open sets $U_i$ with diffeomorphisms $\phi : U_i \to \mathbf{R}^n$ sending each $L_\alpha \cap U$ into a union of parallel copies of the standard $\mathbf{R}^k$ in $\mathbf{R}^n$.*

Taut foliation is a foliation with the following property.

**Definition 1.1.4.** *A taut foliation is a codimension one foliation $\mathcal{F}$ of a 3-manifold $Y$ such that there exists a transverse circle intersecting every leaf $L_\alpha$ ($\alpha \in A$).*

There are also other equivalent topological and geometric formulations of tautness. Having a taut foliation restricts the topology of the 3-manifold. For a long time, mathematicians wonder what exactly the restriction is. Later it was proved that [3, 31, 40] an L-space does not admit co-orientable taut foliations.

**Theorem 1.1.3.** *L-spaces do not admit co-orientable taut foliations.*

However, the converse of Theorem 1.1.3 remains open.

### 1.1.3 Left orderable groups

The left-orderability is a combinatorial property of groups defined as follows.

**Definition 1.1.5.** *A left order $\leq$ on $G$ is a total order which is invariant under left multiplication, that is, $g \leq h$ implies $fg \leq fh$ for any $f, g, h$ in $G$.*

**Definition 1.1.6.** *A left orderable group $G$ is a nontrivial group which admits a left order.*

Note that the trivial group is not left orderable, which is intentional.

The studies of left-orderable 3-manifold groups brings interests among topologist after the works [5] of Boyer, Rolfsen and Wiest. Later Boyer, Gordon and Watson conjectured [47] that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

### 1.1.4 Formulation of the L-space conjecture

Combining the conjectures in Section 1.1.2 and in Section 1.1.3, we have the following L-space conjecture.

**Conjecture 1.1.1.** *For an irreducible rational homology 3-sphere $Y$, the following statements are equivalent,*

*(a) $Y$ is an L-space;*

*(b) $\pi_1(Y)$ is not left orderable;*

*(c) $Y$ does not admit a co-orientable taut foliation.*

Boyer, Gordon and Watson also proved that [47] Conjecture 1.1.1 holds for geometric, non-hyperbolic 3-manifolds and 2-fold branched covers of non-split alternating links.

## 1.2 Definition and basic properties of $(1,1)$-knots

The following definition of $(g,b)$-knot is from [14].

**Definition 1.2.1.** *A knot $K$ in $S^3$ is called a $(g,b)$-knot, if there is a Heegaard splitting $S^3 = U_0 \cup U_1$ of genus $g$, such that each of $K \cap U_0$ and $K \cap U_1$ consists of $b$ trivial arcs.*

By definition, the $(0,b)$-knots are the $b$-bridge knots. So the $(1,1)$-knots can be seen as 1-bridge knots on the standard torus. All 2-bridge knots and torus knots are $(1,1)$-knots.

For a $(1,1)$-knot $K$, the intersection of $K$ and the standard torus consists two basepoints $w$ and $z$. The information we need to determines the knot is the trivial arcs connecting $w$ and $z$ inside and outside the standard torus. However, knowing how to embed the two trivial arcs on the standard torus is more than enough. The

meridian disks of the 1-handles inside and outside the standard torus which does not intersect $K$ determines the knot $K$. The boundary of the meridian disks are called the $\alpha$ curve and the $\beta$ curve. A torus with a pair of basepoints $w$ and $z$ and a pair of curves $\alpha$ and $\beta$ on it, is called a $(1,1)$-diagram, if $\alpha$ and $\beta$ are embedded closed curves on the complement of $\{w,z\}$ and have the algebraic intersection number $\pm 1$. We can use four parameters to specify a $(1,1)$-diagram or a $(1,1)$-knot, which is called the Rasmussen's notation $K(p,q,r,s)$, as in [44].

The general methods of computing the knot groups and knot Floer homology of $(1,1)$-knots are discussed in Subsection 3.4.2 and Section 3.5 in Rácz's thesis [45] respectively.

Given a $(1,1)$-knot $K = K(p,q,r,s)$, we can find [19] the boundary operator of the chain complex $\mathrm{CFK}^\infty(S^3,K)$. Moreover, the $\tau$ invariant can be found as it only relies on $\mathrm{CFK}^\infty(S^3,K)$.

## 1.3 Main results

### 1.3.1 Families of $(1,1)$ L-space knots

The 1-bridge braids are defined as follows. They are also called Berge-Gabai knots.

**Definition 1.3.1.** *A (positive) 1-bridge braid in $S^1 \times D^2$ is a positive braid isotopic to a union of two arcs $\gamma \cup \delta$, where $\gamma \subset \partial(S^1 \times D^2)$ transverses to each meridian, and the bridge $\delta$ is properly embedded in some meridional disk $D$.*

Alternatively, we can equivalently [17] define the 1-bridge braid as the braid $B(\omega,t,b)$ represented by the word

$$(\sigma_b\sigma_{b-1}\cdots\sigma_2\sigma_1)(\sigma_{\omega-1}\sigma_{\omega-2}\cdots\sigma_2\sigma_1)^t$$

on $\omega$ strands, where $0 \leq b \leq \omega - 2$ and $t \geq 1$, given that the braid is a knot, rather than a link.

By definition, closures of 1-bridge braids are $(1,1)$-knots. It was shown [22] that they are also L-space knots.

In the literature, another family of $(1,1)$ L-space knots was studied extensively. To define this family, we first define the twisted torus knots as follows.

**Definition 1.3.2.** *A twisted torus knot $T^{l,m}_{p,q}$ is the knot obtained from the $(p,q)$-torus knot by twisting $l$ strands $m$ full times.*

Alternatively, we can equivalently define the twisted torus knot $T^{l,m}_{p,q}$ as the closure of the braid

$$(\sigma_{l-1}\sigma_{l-2}\cdots\sigma_2\sigma_1)^{lm}(\sigma_{p-1}\sigma_{p-2}\cdots\sigma_2\sigma_1)^q$$

on $q$ strands. For positive integers $p, k, l, m$ with $0 < l < p$, Vafaee proved that [50] the twisted torus knot $T^{l,m}_{p,kp\pm1}$ is an L-space knot if and only if

(a) $l = p - 1$, or

(b) $m = 1$ and $l = 2$, or

(c) $m = 1$ and $l = p - 2$.

In addition, he proved that $T^{l,m}_{p,kp\pm1}$ is a $(1,1)$-knot in the above cases.

The $(-2,3,2n+1)$-pretzel knots with $n \geq 3$ and their mirror images are the only hyperbolic L-space knots among all Montesinos knots by [2]. These knots lie in the family of L-space twisted torus knots $T^{l,m}_{p,kp\pm1}$. Explicitly, the $(-2,3,2n+1)$-pretzel knot and the twisted torus knot $T^{2,n-2}_{3,5}$ are the same.

In Chapter 2, we will prove the following theorem.

**Main Theorem 1.** *If one of the following conditions holds for the positive integers $p, q, l, n$,*

*(a)* $q \geq l+1 = p$,

*(b)* $q = l$ *divides* $n$, *and* $\gcd(p,q) = 1$,

*(c)* $l = n = 2$ *and* $\gcd(p,q) = 1$,

*(d)* $l = n = p-2$ *and* $q = kp \pm 1$ *for some positive integer* $k$,

*then there exist integers* $\omega, t, b$ *with* $0 \leq b \leq \omega - 2$ *and* $t \geq 1$, *such that the closure of the braid*

$$(\sigma_{l-1}\sigma_{l-2}\cdots\sigma_2\sigma_1)^n(\sigma_{p-1}\sigma_{p-2}\cdots\sigma_2\sigma_1)^q$$

*and the closure of the braid*

$$(\sigma_b\sigma_{b-1}\cdots\sigma_2\sigma_1)(\sigma_{\omega-1}\sigma_{\omega-2}\cdots\sigma_2\sigma_1)^t$$

*represent the same oriented link in* $S^3$.

In particular, all the $(1,1)$ L-space knots constructed in this section are closures of 1-bridge braids.

### 1.3.2 Non-left-orderability for 1-bridge braids

A number of research papers [6, 8, 27, 28, 30, 36, 37, 48] have been devoted to study non-left-orderable surgeries on subfamilies of the L-space twisted torus knots $T^{l,m}_{p,kp\pm1}$. Recently, Tran proved [49] that the fundamental group of the manifold obtained by Dehn surgery on an L-space twisted torus knot of form $T^{l,m}_{p,kp\pm1}$ with slope at least $2g(T^{l,m}_{p,kp\pm1}) - 1$ is not left orderable. Combined with Theorem 1.1.2 and Theorem 1.1.3, Tran's result verifies the L-space conjecture for the L-spaces obtained from Dehn surgery on twisted torus knots of form $T^{l,m}_{p,kp\pm1}$.

As for 1-bridge braids, Liang proved [33] that if

(a) $t = 1 + m\omega$ and $b = 2k$, or

(b) $\omega = 2n+1$, $t = 2n-1+m\omega$ and $b = 2k$, or

(c) $\omega = 2n$, $t = 2n-2+m\omega$ and $b = 2k-1$,

then fundamental group of the manifold obtained by Dehn surgery on the closure of $B(\omega, t, b)$ with slope at least $\omega t + b - t$ is not left orderable.

In Section 3.4, we will prove the following theorem.

**Main Theorem 2.** *The fundamental group of the manifold obtained by Dehn surgery on the nontrivial knot $K$ which is the closure of a 1-bridge braid $B(\omega, t, b)$ with slope $\frac{p}{q} \geq \omega t + b - t - \omega$ is not left orderable.*

As the surface obtained by applying Seifert's algorithm to a positive diagram is a minimal genus Seifert surface [10], the genus of $K$ equals $\frac{1}{2}(\#\text{crossings} - \#\text{strands} + 1) = \frac{1}{2}((\omega-1)t + b - \omega + 1)$. So we have $\omega t + b - t - \omega = 2g(K) - 1$. Therefore, Main Theorem 2 generalizes Tran's result on L-space twisted torus knots of form $T^{l,m}_{p,kp\pm1}$ and Liang's result on closures of certain 1-bridge braids. Combined with Theorem 1.1.2 and Theorem 1.1.3, the L-space conjecture holds for the L-spaces obtained from Dehn surgery on closures of 1-bridge braids.

Let us list some approaches of proving the non-left-orderability in the literature.

1. Brute force search in possible left orders — Whenever we find a selected element for which we cannot decide its positivity, we consider the problem in two scenarios, in one we assume the element is positive, in the other we assume it is negative. See [1] as an example of extensive use of brute force search. For any finitely generated non left orderable group, the brute force search always terminates and produce a proof of non-left-orderability [9].

2. The fixed point method — It is known that [25] a countable group is left orderable if and only if it acts on the real line by orientation preserving homeomorphisms without global fixed points. For a certain element $k$ in the group, if

there exists a real number $s$ fixed by $k$, then we try to prove that $s$ is a global fixed point; otherwise, all conjugates of $k$ have the same sign. See [27, 49] for example.

3. [8, Theorem 2.1] gives a criteria to deduce the non left orderability of $\pi_1(M(\frac{p}{q}))$ from the signs of $\mu^{p_0}\lambda^{q_0}$ and $\mu^{p_1}\lambda^{q_1}$ where $\frac{p_0}{q_0} < \frac{p}{q} < \frac{p_1}{q_1}$ under certain conditions. This theorem is used to derive the non left orderability of a two-parameter family of groups.

4. For a Seifert fibred space $M$, one can construct [5] a horizontal foliation on $M$ from the representation $\pi_1(M) \to \mathrm{Homeo}^+(\mathbf{R})$ under certain conditions. Since it is known [15, 29, 35] which Seifert bundles admit horizontal foliations, one can derive the non-left-orderability of $\pi_1(M)$.

To prove Main Theorem 2, we make full use of the second technique as follows. Assume the theorem does not hold. First, we find an element $k$ in the group which represents a self-homeomorphism on $\mathbf{R}$ without fixed points. By continuity, the self-homeomorphism is pointwise greater than or pointwise less than the identity map. Naturally, we define the pointwise partial order $\leq_C$ on the set of self-homeomorphisms, or equivalently, on the group with a left order, then the partial order $\leq_C$ is invariant not only under left multiplication but also under right multiplication.

Suppose $k$ represents a self-homeomorphism which is pointwise greater than (resp., pointwise less than) the identity map, by $k$ we generate a subset of the left orderable group, which is closed under multiplication, conjugation and taking roots, and whose elements have the same property. Then, this subset does not contain the identity element. Equivalently, we take this subset as a positive cone to define a preorder $\leq_k$ on the group. As $\leq_C$ is an extension of $\leq_k$ (resp., the inverse of $\leq_k$), the preorder $\leq_k$ has to be a partial order. However, we prove the negation.

Notice that the preorder $\leq_k$ does not depend on the choice of left order, but only on the group element $k$. So our method is computationally lighter and more human-readable than the first approach.

### 1.3.3 Non-left-orderability for iterated 1-bridge braids

Knot cabling is one way to produce L-space knots. By definition, the $(p, q)$-cable of a knot $K$ is the satellite knot $P(K)$ with pattern $P$ being the $(p, q)$-torus knot. The $(p, q)$-cable of a nontrivial knot $K$ is an L-space knot if and only if [24, 26] $K$ is an L-space knot and $\frac{q}{p} \geq 2g(K) - 1$.

Hom, Lidman and Vafaee extended [34] the above result to the satellite knot $P(K)$ with pattern $P$ being 1-bridge braid. They proved that, if the pattern $P$ is the 1-bridge braid $B(\omega, t, b)$ and the companion $K$ is nontrivial, then the satellite knot $P(K)$ is an L-space knot if and only if $K$ is an L-space knot and $\frac{t}{\omega} \geq 2g(K) - 1$. (The formula in the original paper is $\frac{b+t\omega}{\omega^2} \geq 2g(K) - 1$. As $0 \leq b \leq \omega - 2$ and $\omega, t, b$ and $g(K)$ are integers, it is equivalent to the formula here.) They also proved that if the pattern $P$ is a negative braid, then the satellite knot $P(K)$ is not a (positive) L-space knot.

Clay and Watson introduced [7] the concept of $r$-decayed knots. Then they proved that the $(p, q)$-cable of an $r$-decayed knot is $pq$-decayed if $\frac{q}{p} > r$, and that sufficiently positive surgeries on decayed knots yield manifolds with non-left-orderable fundamental groups. The decayed knots were defined as follows to relate the left orders on the knot group and the left orders on the fundamental group of the manifold obtained by Dehn surgery.

**Definition 1.3.3.** *For a nontrivial knot $K$ with $\mu$ and $\lambda$ representing a meridian and a longitude in the knot group, we say $K$ is $r$-decayed, if for any positive cone $P$ in the knot group, either $P \cap S_r = S_r$ or $P \cap S_r = \emptyset$, where $S_r$ is the set $\left\{\mu^p \lambda^q | \frac{p}{q} \geq r\right\}$.*

As an analog of the concept of decayed knots, we define the property (D) for nontrivial knots as follows.

**Definition 1.3.4.** *For a nontrivial knot $K$ with $\mu$ and $\lambda$ representing a meridian and a longitude in the knot group, we say $K$ has property (D) if*

*(a) for any homomorphism $\rho$ from $\pi_1(S^3 \setminus K)$ to $Homeo_+(\mathbf{R})$, if $s \in \mathbf{R}$ is a common fixed point of $\rho(\mu)$ and $\rho(\lambda)$, then $s$ is a fixed point of every element in $\pi_1(S^3 \setminus K)$, and,*

*(b) $\mu$ is in the root-closed, conjugacy-closed submonoid generated by $\mu^{2g(K)-1}\lambda$ and $\mu^{-1}$.*

Two major differences between our definition of property (D) and the definition of decayed knots are the follows.

1. In the viewpoint of fixed points, there is a common condition on left orders on the fundamental groups of the manifolds obtained by all nonzero surgeries, but not on left orders on the knot group: $\rho(\mu)$ and $\rho(\lambda)$ have the same set of fixed points. Without referring to the homomorphism $\rho$, the condition can be rewritten as, for any element $g$, we have $\mu g \leq g$ if and only if $g \leq \lambda g$ if the surgery slope is positive, and $\mu g \leq g$ if and only if $\lambda g \leq g$ if the surgery slope is negative.

2. We impose our condition on a preorder instead of the set of left orders. One technical difficulty of using the left orders is that they are total orders, so the relation between left orders on the knot group and left orders on the fundamental group of the manifold obtained by Dehn surgery is complicated.

If a knot $K$ has property (D), we will prove that the fundamental group of the manifold obtained by Dehn surgery on $K$ with slope $\frac{p}{q} \geq 2g(K) - 1$ is not left

orderable. Combined with Theorem 1.1.2 and Theorem 1.1.3, the L-space conjecture holds for the L-spaces obtained from Dehn surgery on knots with property (D).

The following theorem is slightly stronger than Main Theorem 2. We only need to work in the knot group, instead of the fundamental group of the manifold obtained by Dehn surgery.

**Main Theorem 3.** *Nontrivial knots which are closures of* 1*-bridge braids have property (D).*

The purpose of defining decayed knots was to prove the non-left-orderability for L-space cable knots. Likewise, the purpose of defining property (D) is to prove the non-left-orderability for L-space satellite knots with 1-bridge braid patterns.

In Section 3.5, we will prove the following theorem by utilizing an element without fixed points.

**Main Theorem 4.** *Let the pattern $P$ be a* 1*-bridge braid $B(\omega, t, b)$, and the companion $K$ be a nontrivial knot with property (D). Suppose that $\frac{t}{\omega} \geq 2g(K) - 1$, then the satellite knot $P(K)$ has property (D).*

Combining Main Theorem 3 and Main Theorem 4, we can generate many knots with property (D), which we may call them L-space iterated 1-bridge braid knots. In particular, the L-space iterated torus knots have property (D). It is known that [21] surgeries on iterated torus knots always yield graph manifolds, and that [4, 23] a graph manifold has a non-left-orderable fundamental group if and only if it is an L-space. Historically, the proof of the if part used the fourth method listed in Section 1.3.2, and the third approach was also exercised [32] on certain pairs of $(p, q)$-cable knots of torus knots.

### 1.3.4 Non-left-orderability for $v2503$

Culler and Dunfield [12] found that every non-longitudinal Dehn filling on the hyperbolic $\mathbf{Q}$-homology solid torus $v2503$ is an L-space, and suggested the readers to prove that every Dehn filling on $v2503$ yields a manifold with a non-left-orderable fundamental group. Varvarezos [51] proved that, for a certain homological framing, the fundamental group of the manifold obtained by Dehn filling on $v2503$ with slope $\frac{p}{q} < -1$ is not left orderable.

In Section 4.1, we will extend the result to every Dehn filling on $v2503$ also by utilizing an element without fixed points.

**Main Theorem 5.** *The fundamental group of an manifold obtained by Dehn filling on $v2503$ is not left orderable.*

Combined with Theorem 1.1.3, the L-space conjecture holds for the 3-manifolds obtained from Dehn filling on $v2503$.

Motivated by these results, we speculate that the existence of a specific element without fixed points could be a key step to prove the non-left-orderability for fundamental groups of L-spaces obtained by Dehn surgeries. The author wonders whether the following statement holds.

**Conjecture 1.3.1.** *Let $K$ be an L-space knot. Let $\mu$ and $\lambda$ represent a meridian and a longitude in the knot group. Then there exists an element $k$ in the knot group, such that for any homomorphism $\rho$ from $\pi_1(S^3 \setminus K)$ to $Homeo_+(\mathbf{R})$ without global fixed points, if $\rho(\mu)$ and $\rho(\lambda)$ have the same set of fixed points, then $\rho(k)$ does not have fixed points.*

### 1.3.5 $(1,1)$-knots which are topologically slice but not smoothly slice

The smooth (resp., topological) slice genus of a knot $K$ is the minimal genus of a connected, orientable 2-manifold smoothly (resp., locally flatly) embedded in the 4-ball $D^4$ whose boundary is the knot $K$. A knot $K$ is called smoothly (resp.,topologically) slice, if its smooth (resp., topological) slice genus is zero. While it is known that [20] there exist infinitely many topologically slice knots which are not smoothly slice, we prove that it is still true if we restrict the knots to be $(1,1)$-knots, which was a conjecture proposed in Rácz's thesis [45]. In other words, we are going to prove the following theorem.

**Main Theorem 6.** *There are infinitely many $(1,1)$-knots which are topologically slice, but not smoothly slice.*

To prove Main Theorem 6, we will construct a one-parameter family of $(1,1)$-knots $K_n$ $(n = 0, 1, \ldots)$ as a valid example.

In Section 5.1, we construct the family of knots and prove they are indeed $(1,1)$-knots by deriving the $(1,1)$-diagrams for them.

In Section 5.2, we compute the knot Floer homology of each $K_n$. Then we prove these knots have trivial Conway polynomials, and therefore are topologically slice.

In Section 5.3, we first compute the $\tau$ invariant of $K_0$. By certain inequalities for $\tau$ invariants in [42], we prove the $\tau$ invariant and the smooth slice genus of each $K_n$ are 1, and therefore these knots are not smoothly slice.

The pictures of knots are generated by the software KnotPlot [46].

# Chapter 2

# L-space twisted torus knots of form $T^{l,m}_{p,kp\pm1}$ are closures of 1-bridge braids

## 2.1 Main Theorem 1

In the subsequent sections, we will prove the following theorem algebraically using the Markov braid theorem.

**Theorem 2.1.1.** *If one of the following conditions holds for the positive integers* $p, q, l, n$,

*(a)* $q \geq l + 1 = p$,

*(b)* $q = l$ *divides* $n$, *and* $\gcd(p, q) = 1$,

*(c)* $l = n = 2$ *and* $\gcd(p, q) = 1$,

*(d)* $l = n = p - 2$ *and* $q = kp \pm 1$ *for some positive integer* $k$,

*then there exist integers* $\omega, t, b$ *with* $0 \leq b \leq \omega - 2$ *and* $t \geq 1$*, such that the closure of the braid*

$$(\sigma_{l-1}\sigma_{l-2}\cdots\sigma_2\sigma_1)^n(\sigma_{p-1}\sigma_{p-2}\cdots\sigma_2\sigma_1)^q$$

*and the closure of the braid*

$$(\sigma_b\sigma_{b-1}\cdots\sigma_2\sigma_1)(\sigma_{\omega-1}\sigma_{\omega-2}\cdots\sigma_2\sigma_1)^t$$

*represent the same oriented link in* $S^3$*.*

In particular, the following corollary follows directly from Theorem 2.1.1 and Vafaee's result [50].

**Corollary 2.1.1.** *L-space twisted torus knots of form* $T^{l,m}_{p,kp\pm1}$ *for positive integers* $p, k, l, m$ *with* $0 < l < p$ *are closures of* 1*-bridge braids.*

## 2.2 Braid group identities

For any positive integer $m$, let $\pi_m$ denote the word $\sigma_m\sigma_{m-1}\cdots\sigma_1$, and let $\Pi_m$ denote the word $\pi_1\pi_2\cdots\pi_m$. As $\Pi_m$ denotes the half twist on the first $m-1$ strands and $\pi_m^{m+1}$ denotes the full twist on the first $m-1$ strands, we have

$$\pi_m^{m+1} = \Pi_m^2.$$

We also have

$$\sigma_i\pi_m^{m+1} = \pi_m^{m+1}\sigma_i$$

and

$$\sigma_i\Pi_m = \Pi_m\sigma_{m+1-i}$$

for each integer $i$ with $1 \leq i \leq m$.

For each integer $i$ with $1 \leq i \leq m-1$, we have

$$\begin{aligned}
\sigma_i \pi_m &= \sigma_i(\sigma_m \sigma_{m-1} \cdots \sigma_1) \\
&= (\sigma_m \sigma_{m-1} \cdots \sigma_{i+2})(\sigma_i \sigma_{i+1} \sigma_i)(\sigma_{i-1} \sigma_{i-2} \cdots \sigma_1) \\
&= (\sigma_m \sigma_{m-1} \cdots \sigma_{i+2})(\sigma_{i+1} \sigma_i \sigma_{i+1})(\sigma_{i-1} \sigma_{i-2} \cdots \sigma_1) \\
&= (\sigma_m \sigma_{m-1} \cdots \sigma_1)\sigma_{i+1} \\
&= \pi_m \sigma_{i+1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\pi_m^2 &= \pi_m(\sigma_m \sigma_{m-1} \cdots \sigma_1) \\
&= \sigma_{m-1}\pi_m(\sigma_{m-1} \sigma_{m-2} \cdots \sigma_1) \\
&= (\sigma_{m-1}\sigma_{m-2})\pi_m(\sigma_{m-2} \sigma_{m-3} \cdots \sigma_1) \\
&= \cdots \\
&= (\sigma_{m-1} \sigma_{m-2} \cdots \sigma_1)\pi_m \sigma_1 \\
&= \pi_{m-1}\pi_m \sigma_1,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_m \pi_m^2 &= \sigma_m \pi_{m-1} \pi_m \sigma_1 \\
&= \pi_m^2 \sigma_1.
\end{aligned}$$

For a positive integer $s$ with $2 \leq s \leq m$, we have

$$\begin{aligned}
\pi_m^s &= \pi_{m-1}\pi_m \sigma_1 \pi_m^{s-2} \\
&= \pi_{m-1}\pi_m^2 \sigma_2 \pi_m^{s-3} \\
&= \cdots
\end{aligned}$$

$$= \pi_{m-1}\pi_m^{s-1}\sigma_{s-1}.$$

Then we have

$$\begin{aligned}
\pi_m^s &= \pi_{m-1}\pi_m^{s-1}\sigma_{s-1} \\
&= \pi_{m-1}^2\pi_m^{s-2}\sigma_{s-2}\sigma_{s-1} \\
&= \cdots \\
&= \pi_{m-1}^{s-1}\pi_m(\sigma_1\sigma_2\cdots\sigma_{s-1}).
\end{aligned}$$

## 2.3 Condition (a) and condition (b)

By Markov braid theorem, the closure of two braids $f$ and $g$ represent the same oriented link if and only if one can be transform into another by a sequence of Markov moves. In this case, we say $f \sim g$.

Suppose that $t \leq b \leq \omega - 1$, then we have

$$\begin{aligned}
\pi_b\pi_{\omega-1}^t &= \pi_b\pi_{\omega-2}^{t-1}\pi_{\omega-1}(\sigma_1\sigma_2\cdots\sigma_{t-1}) \\
&\sim \pi_b\pi_{\omega-2}^t(\sigma_1\sigma_2\cdots\sigma_{t-1}) \\
&= \pi_b\pi_{\omega-3}^{t-1}\pi_{\omega-2}(\sigma_1\sigma_2\cdots\sigma_{t-1})^2 \\
&\sim \pi_b\pi_{\omega-3}^t(\sigma_1\sigma_2\cdots\sigma_{t-1})^2 \\
&\sim \cdots \\
&\sim \pi_b^{t+1}(\sigma_1\sigma_2\cdots\sigma_{t-1})^{\omega-b-1} \\
&\sim \cdots \\
&\sim \pi_t^{t+1}(\sigma_1\sigma_2\cdots\sigma_t)^{b-t}(\sigma_1\sigma_2\cdots\sigma_{t-1})^{\omega-b-1} \\
&\sim \Pi_t\pi_t^{t+1}(\sigma_1\sigma_2\cdots\sigma_t)^{b-t}(\sigma_1\sigma_2\cdots\sigma_{t-1})^{\omega-b-1}\Pi_t^{-1} \\
&= \pi_t^{b+1}(\sigma_t\sigma_{t-1}\cdots\sigma_2)^{\omega-b-1}
\end{aligned}$$

$$= \pi_t^b \pi_{t-1}^{\omega-b-1} \pi_t$$
$$\sim \pi_{t-1}^{\omega-b-1} \pi_t^{b+1}.$$

When $q \geq l+1 = p$, we have $\pi_{l-1}^n \pi_{p-1}^q = \pi_{p-2}^n \pi_{p-1}^q \sim \pi_{q-1} \pi_{n+q-1}^{p-1}$. Therefore Theorem 2.1.1 holds with condition (a).

When $q = l$ divides $n$ and $\gcd(p,q) = 1$, by definition, the twisted torus knot $T_{p,q}^{q,\frac{n}{q}}$ is the torus knot $T_{p+n,q}$. So Theorem 2.1.1 holds with condition (b).

## 2.4 Condition (c)

If $p = 2$, then the twisted torus knot $T_{2,q}^{2,1}$ is the torus knot $T_{2,q+2}$. If $q = 1$, then the twisted torus knot $T_{p,1}^{2,1}$ is the torus knot $T_{2,3}$. If $q = 2$, then the twisted torus knot $T_{p,2}^{2,1}$ is the torus knot $T_{p+2,2}$. So we assume $p, q \geq 3$.

We prove the following elementary number theoretic lemma.

**Lemma 2.4.1.** *Suppose $p, q \geq 3$ are coprime positive integers. Let $x$ be the multiplicative inverse of $p$ modulo $q$ with $0 \leq x < q$, and $y$ be the multiplicative inverse of $q$ modulo $p$ with $0 \leq y < p$. Then $x < \frac{q}{2}$ or $y < \frac{p}{2}$.*

*Proof.* The integer $2pq - px - qy + 1$ is a multiple of $p$ and a multiple of $q$, so $2pq - px - qy + 1$ is a multiple of $pq$. If $\frac{q}{2} \leq x \leq q-1$ and $\frac{p}{2} \leq y \leq p-1$, as either $x \geq \frac{q+1}{2}$ or $y \geq \frac{p+1}{2}$, we have $0 < p+q+1 \leq 2pq - xq - yp + 1 \leq pq - \frac{1}{2}\min(p,q) + 1 < pq$, which is a contradiction. □

For each integer $i, j$ with $1 \leq i, j \leq p-1$ and $j \equiv i+q \pmod{p}$, we have

$$\sigma_i \pi_{p-1}^q = \pi_{p-1} \sigma_{i+1} \pi_{p-1}^{q-1}$$
$$= \pi_{p-1}^2 \sigma_{i+2} \pi_{p-1}^{q-2}$$
$$= \cdots$$

$$
\begin{aligned}
&= \pi_{p-1}^{kp-1-i}\sigma_{p-1}\pi_{p-1}^{q-(kp-1-i)} \\
&= \pi_{p-1}^{kp+1-i}\sigma_1\pi_{p-1}^{q-(kp+1-i)} \\
&= \cdots \\
&= \pi_{p-1}^{q}\sigma_j.
\end{aligned}
$$

If there exists a multiplicative inverse $y$ of $q$ modulo $p$ with $0 \leq y < \frac{p}{2}$, then $1 + sq \not\equiv 0 \pmod{p}$ for $s = 1, 2, \ldots, y-1$. Therefore there exists a sequence $1 = i_0, i_1, i_2, \ldots, i_y = 2$ such that $i_s \equiv i_{s-1}+q \pmod{p}$ for $s = 1, 2, \ldots, y$ and $3 \leq i_s \leq p-1$ for $s = 1, 2, \ldots, y-1$.

$$
\begin{aligned}
\sigma_1^2\pi_{p-1}^q &= \sigma_1\sigma_{i_0}\pi_{p-1}^q \\
&= \sigma_1\pi_{p-1}^q\sigma_{i_1} \\
&\sim \sigma_{i_1}\sigma_1\pi_{p-1}^q \\
&= \sigma_1\sigma_{i_1}\pi_{p-1}^q \\
&\sim \cdots \\
&\sim \sigma_{i_s}\sigma_1\pi_{p-1}^q \\
&= \sigma_2\sigma_1\pi_{p-1}^q \\
&= \pi_2\pi_{p-1}^q
\end{aligned}
$$

Suppose that a multiplicative inverse $x$ of $p$ modulo $q$ satisfies $0 \leq x < \frac{q}{2}$. By definition, the twisted torus knot $T_{p,q}^{2,1}$ is the twisted torus knot $T_{q,p}^{2,1}$. So we have

$$
\begin{aligned}
\sigma_1^2\pi_{p-1}^q &\sim \sigma_1^2\pi_{q-1}^p \\
&\sim \pi_2\pi_{q-1}^p.
\end{aligned}
$$

Therefore Theorem 2.1.1 holds with condition (c).

## 2.5 Condition (d)

Since we have

$$\begin{aligned}
\pi_{p-3}^{p-2}\pi_{p-1}^{kp+1} &= \pi_{p-3}^{p-3}\pi_{p-1}^{kp+1}(\sigma_{p-2}\sigma_{p-3}\cdots\sigma_2)\\
&\sim (\sigma_{p-2}\sigma_{p-3}\cdots\sigma_2)\pi_{p-3}^{p-3}\pi_{p-1}^{kp+1}\\
&= \pi_{p-2}\pi_{p-3}^{p-4}\pi_{p-4}\pi_{p-1}^{kp+1}\\
&= \pi_{p-2}\pi_{p-3}^{p-4}\pi_{p-1}^{kp+1}(\sigma_{p-3}\sigma_{p-4}\cdots\sigma_2)\\
&\sim (\sigma_{p-3}\sigma_{p-4}\cdots\sigma_2)\pi_{p-2}\pi_{p-3}^{p-4}\pi_{p-1}^{kp+1}\\
&= \pi_{p-2}(\sigma_{p-2}\sigma_{p-3}\cdots\sigma_3)\pi_{p-3}^{p-4}\pi_{p-1}^{kp+1}\\
&= \pi_{p-2}^{2}\pi_{p-3}^{p-5}\pi_{p-5}\pi_{p-1}^{kp+1}\\
&\sim \cdots\\
&\sim \pi_{p-2}^{p-3}\pi_{p-1}^{kp+1},
\end{aligned}$$

Theorem 2.1.1 with condition (a) implies Theorem 2.1.1 with condition (d) and $q = kp + 1$.

By definition, the twisted torus knot $T_{p,kp-1}^{p-3,1}$ is the twisted torus knot $T_{kp-1,p}^{p-3,1}$, so we have

$$\begin{aligned}
\pi_{p-3}^{p-2}\pi_{p-1}^{kp-1} &\sim \pi_{p-3}^{p-2}\pi_{kp-2}^{p}\\
&= \pi_{p-3}^{p-3}\pi_{kp-2}^{p}(\sigma_{2p-3}\sigma_{2p-4}\cdots\sigma_{p+1})\\
&\sim (\sigma_{2p-3}\sigma_{2p-4}\cdots\sigma_{p+1})\pi_{p-3}^{p-3}\pi_{kp-2}^{p}\\
&= \pi_{p-3}^{p-3}(\sigma_{2p-3}\sigma_{2p-4}\cdots\sigma_{p+1})\pi_{kp-2}^{p}\\
&\sim \cdots\\
&\sim \pi_{p-3}^{p-3}(\sigma_{kp-3}\sigma_{kp-4}\cdots\sigma_{(k-1)p+1})\pi_{kp-2}^{p}\\
&= \pi_{p-3}^{p-3}\pi_{kp-2}^{p}(\sigma_{p-2}\sigma_{p-3}\cdots\sigma_2).
\end{aligned}$$

As in the case with condition (d) and $q = kq + 1$, we have

$$\begin{aligned}
\pi_{p-3}^{p-2}\pi_{p-1}^{kp-1} &\sim \pi_{p-3}^{p-2}\pi_{kp-2}^{p} \\
&\sim \pi_{p-3}^{p-3}\pi_{kp-2}^{p}(\sigma_{p-2}\sigma_{p-3}\cdots\sigma_2) \\
&\sim \cdots \\
&\sim \pi_{p-2}^{p-3}\pi_{kp-2}^{p}.
\end{aligned}$$

If $k = 1$, then the closure of the braid is a torus knot.

Suppose that $k \geq 2$. As in the case with condition (a), we have

$$\begin{aligned}
\pi_{p-3}^{p-2} &\sim \pi_{p-2}^{p-3}\pi_{kp-2}^{p} \\
&\sim \pi_{p-2}^{p-3}\pi_{kp-3}^{p}(\sigma_1\sigma_2\cdots\sigma_{p-1}) \\
&\sim \pi_{p-2}^{p-3}\pi_{kp-4}^{p}(\sigma_1\sigma_2\cdots\sigma_{p-1})^2 \\
&\sim \cdots \\
&\sim \pi_{p-2}^{p-3}\pi_{p-1}^{p}(\sigma_1\sigma_2\cdots\sigma_{p-1})^{(k-1)p-1} \\
&= \pi_{p-2}^{2p-4}\pi_{p-1}(\sigma_1\sigma_2\cdots\sigma_{p-1})^{(k-1)p} \\
&= \pi_{p-2}^{2p-4}\pi_{p-1}(\Pi_{p-1}\pi_{p-1}\Pi_{p-1}^{-1})^{(k-1)p} \\
&= \pi_{p-2}^{2p-4}\pi_{p-1}\Pi_{p-1}\pi_{p-1}^{(k-1)p}\Pi_{p-1}^{-1} \\
&= \pi_{p-2}^{2p-4}\pi_{p-1}\Pi_{p-1}\Pi_{p-1}^{2(k-1)}\Pi_{p-1}^{-1} \\
&= \pi_{p-2}^{2p-4}\pi_{p-1}\Pi_{p-1}^{2(k-1)} \\
&= \pi_{p-2}^{2p-4}\pi_{p-1}^{(k-1)p+1}.
\end{aligned}$$

Theorem 2.1.1 with condition (a) implies Theorem 2.1.1 with condition (d) and $q = kp - 1$.

Therefore, Theorem 2.1.1 is proved.

# Chapter 3

# Non-left-orderability for iterated 1-bridge braids

## 3.1 Knot group of 1-bridge braids

In this section, we compute the knot group of a 1-bridge braid.

Let $B(\omega, t, b)$ be a 1-bridge braid in $S^1 \times D^2$. The meridional disk $D$ of the solid torus $S^1 \times D^2$, the arc $\gamma$ and the bridge $\delta$ are explained in the definition of 1-bridge braid in Section 1.3.1. Let the solid torus $S^1 \times D^2$ lie in the interior of a slightly larger solid torus $N(S^1 \times D^2)$. Then the knot group of $B(\omega, t, b)$ is $\pi_1(N(S^1 \times D^2) \setminus (\gamma \cup \delta))$.

The bridge $\delta$ divides $D$ into two oriented embedded disks $D_x$ and $D_y$, where $\gamma$ intersects $\partial D_x$ positively in $b$ points, and $\gamma$ intersects $\partial D_y$ positively in $\omega - b - 1$ points. The fundamental group of $(S^1 \times D^2) \setminus \delta$ is the free group generated by $x$ and $y$, where $x$ represent a loop which intersects $D_x$ positively once and does not intersect $D_y$, and $y$ represent a loop which does not intersect $D_x$ and intersects $D_y$ positively once.

The space $N(S^1 \times D^2) \setminus (S^1 \times D^2)$ is homotopy equivalent to a torus, so its fundamental group is $\mathbf{Z} \times \mathbf{Z}$, generated by a meridian $r_\mu$ and a longitude $r_\lambda$ of $N(S^1 \times D^2)$ in the knot group.

The space $(S^1 \times D^2) \setminus \delta$ is a genus two handlebody, so its fundamental group is freely generated by two elements $x$ and $y$, where $x$ represents a curve in $(S^1 \times D^2) \setminus \delta$ which intersects $D_x$ positively once and does not intersect $D_y$, and $y$ represents a curve in $(S^1 \times D^2) \setminus \delta$ which intersects $D_y$ positively once and does not intersect $D_x$.

The space $\partial(S^1 \times D^2) \setminus \gamma$ is homotopy equivalent to a punctured torus, so its fundamental group is freely generated by two elements, which represent a homotopical meridian and a homotopical longitude of $S^1 \times D^2$ which does not intersect the arc $\gamma$. In particular, the inclusion map $\partial(S^1 \times D^2) \setminus \gamma \hookrightarrow N(S^1 \times D^2) \setminus (S^1 \times D^2)$ induces an epimorphism in fundamental groups.

The knot complement $N(S^1 \times D^2) \setminus (\gamma \cup \delta)$ can be obtained by gluing $(S^1 \times D^2) \setminus \delta$ with $N(S^1 \times D^2) \setminus (S^1 \times D^2)$ along $\partial(S^1 \times D^2) \setminus \gamma$. By Seifert-van Kampen theorem, the knot group of $B(\omega, t, b)$ is generated by $x$ and $y$, with a relation $r_\mu r_\lambda = r_\lambda r_\mu$, where $r_\mu$ and $r_\lambda$ are elements generated by $x$ and $y$, which represent a homotopical meridian and a homotopical longitude of $S^1 \times D^2$ which does not intersect the arc $\gamma$, i.e., we have

$$\pi_1(N(S^1 \times D^2) \setminus (\gamma \cup \delta)) = \langle x, y | r_\mu r_\lambda = r_\lambda r_\mu, r_\mu = r_\mu(x, y), r_\lambda = r_\lambda(x, y) \rangle.$$

By definition, the element

$$\mu = xy^{-1}$$

represent a meridian of $B(\omega, t, b)$.

To find the functions $r_\mu(x, y)$ and $r_\lambda(x, y)$, we first consider a meridian and a longitude of $S^1 \times D^2$. Assume that the meridian intersects the arc $\gamma$ in $\omega$ points, namely $Q_1, Q_2, \ldots, Q_\omega$, and it does not intersect $\partial D$. Assume that the longitude intersects

$\partial D_y$ positively once and then intersects the arc $\gamma$ in $t$ points, namely $P_t, P_{t-1}, \ldots, P_1$ in order. Then, we isotope the curve so that the intersection points $Q_1, Q_2, \ldots, Q_\omega$ and $P_t, P_{t-1}, \ldots, P_1$ move in the positive direction of the arc $\gamma$ past its endpoint. The resulting curves are represented by $r_\mu$ and $r_\lambda$.

Let $P_0$ be the starting point of the arc $\gamma$. For each integer $i$ with $0 \leq i \leq t$ (resp., each integer $j$ with $1 \leq j \leq \omega$), in the process when the crossing point $P_i$ (resp., $Q_j$) moves in the positive direction of the arc $\gamma$ towards its endpoint, starting from identity, each time the point goes past $\partial D_x$ we right multiply the element by $x$, and each time the point goes past $\partial D_y$ we right multiply the element by $y$. Let the resulting element be $g_i$ (resp., $h_j$). Then we have

$$r_\mu = (h_1 \mu h_1^{-1})(h_2 \mu h_2^{-1}) \cdots (h_\omega \mu h_\omega^{-1}),$$

and

$$r_\lambda = y(g_t \mu^{-1} g_t^{-1})(g_{t-1} \mu^{-1} g_{t-1}^{-1}) \cdots (g_1 \mu^{-1} g_1^{-1}).$$

Furthermore, we have the following properties.

1. $g_i$ $(1 \leq i \leq t)$ and $h_j$ $(1 \leq j \leq \omega)$ are suffixes of $g_0$;

2. If $i \equiv j \pmod{\omega}$, then $g_i = h_j$.

The longitude of $B(\omega, t, b)$ is represented by $\lambda = y g_0 \mu^k$ for some integer $k$ to cancel out the linking number. In $H_1(N(S^1 \times D^2) \setminus (\gamma \cup \delta))/[r_\lambda]$, we have $0 = [r_\lambda] = [y] - t[\mu]$, so $[y] = t[\mu]$. We also have $[\lambda] = [y] + [g_0] + k[\mu] = b[x] + (\omega - b)[y] + k[\mu] = \omega[y] + (k + b)[\mu] = (\omega t + k + b)[\mu]$. So we have $k = -\omega t - b$. Therefore we have

$$\lambda = y g_0 \mu^{-\omega t - b}.$$

## 3.2 Partial order $\leq_C$, root-closed, conjugacy-closed submonoid and preorder $\leq_k$

In this section, we will define a few terms and prove relevant theorems to describe our method of proving non-left-orderability.

Let $G$ be a group with a left order $\leq$. We define a binary relation $\leq_C$ as follows.

**Definition 3.2.1.** *For $f, g \in G$, we say $f \leq_C g$ when $fh \leq gh$ for every $h \in G$.*

Then $\leq_C$ has the following properties.

**Theorem 3.2.1.** *The binary relation $\leq_C$ is a partial order which is invariant under left multiplication and right multiplication.*

*Proof.* The reflexivity, transitivity and antisymmetry of $\leq$ implies the reflexivity, transitivity and antisymmetry of $\leq_C$ respectively. Therefore $\leq_C$ is a partial order.

Suppose $f \leq_C g$. By definition, $fh \leq gh$ for every $h \in G$. Because $\leq$ is invariant under left multiplication, we have $h'fh \leq h'gh$ for every $h', h \in G$. Because $h$ is arbitrary, we replace $h$ with $h''h$ to deduce that $h'fh''h \leq h'fh''h$ for every $h', h'', h \in G$. By definition, $h'fh'' \leq_C h'fh''$ for every $h', h'' \in G$. □

**Theorem 3.2.2.** *For $n \in \mathbf{N}^+$ and $f \in G$, if $f^n \leq_C 1$, then $f \leq_C 1$.*

*Proof.* Suppose $f^n \leq_C 1$. Let $g$ be an arbitrary element in $G$. By definition, we have $f^n g \leq g$.

If $g \leq fg$, then $f^k g \leq f^{k+1} g$ for any integer $k$. Hence we have $fg \leq f^2 g \leq \cdots \leq f^n g \leq g$. Since we have $fg \leq g$ if $g \leq fg$, and $\leq$ is a total order, we always have $fg \leq g$. By definition, we have $f \leq_C 1$. □

**Theorem 3.2.3.** *The total order $\leq$ is a linear extension of $\leq_C$.*

*Proof.* By taking $h = 1$ in the definition of $\leq_C$, we have that $f \leq_C g$ implies $f \leq g$. □

We summarize the properties of its positive cone in the following definition.

**Definition 3.2.2.** *The root-closed, conjugacy-closed submonoid $M$ of $G$ generated by a set $S$ is defined as the minimal subset of $G$ containing $S$ and identity, with the following properties:*

*(a) If $f, g \in M$, then $fg \in M$;*

*(b) If $f \in M$ and $g \in G$, then $gfg^{-1} \in M$;*

*(c) If $n \in \mathbf{N}^+$ and $f^n \in M$, then $f \in M$.*

It was shown [25] that there exists a monomorphism $\rho : G \to \mathrm{Homeo}_+(\mathbf{R})$ without global fixed points, such that $f \leq g$ if and only of $\rho(f)(0) \leq \rho(g)(0)$ for every $f, g \in G$. Suppose that, for some specific set $S_k$ of elements $g$ in $G$, we have proved the homeomorphisms $\rho(g)$ $(g \in S_k)$ are either all pointwise not less than the identity map, or all pointwise not greater than the identity map. Then we define another useful binary relation as follows.

**Definition 3.2.3.** *For $f, g \in G$, we say $f \leq_k g$ when $f^{-1}g$ is in the root-closed, conjugacy-closed submonoid generated by the subset $S_k$ of $G$.*

Notice that $\leq_k$ does not depend on the choice of the left order $\leq$, but only depend on the generating set $S_k$. The following two properties of $\leq_k$ are similar to Theorem 3.2.1 and Theorem 3.2.2, and they even do not rely on the existence of a left order.

**Theorem 3.2.4.** *The binary relation $\leq_k$ is a preorder which is invariant under left multiplication and right multiplication.*

*Proof.* Let the root-closed, conjugacy-closed submonoid generated by $S_k$ be $M_k$. $M_k$ has an identity element, so $\leq_k$ is reflexive. $M_k$ is closed under multiplication, so $\leq_k$ is transitive. Therefore $\leq_k$ is a preorder.

Suppose $f \leq_k g$. When $f$ and $g$ are both left multiplied by an element, $f^{-1}g$ does not change. So $\leq_k$ is invariant under left multiplication. When $f$ and $g$ are both right multiplied by an element, $f^{-1}g$ changes to its conjugation. $M_k$ is closed under conjugation, so $\leq_k$ is invariant under right multiplication. □

**Theorem 3.2.5.** *For $n \in \mathbf{N}^+$ and $f \in G$, if $f^n \leq_k 1$, then $f \leq_k 1$.*

*Proof.* Since a root-closed, conjugacy-closed submonoid is closed under taking roots, the theorem holds. □

For a left orderable group $G$, the preorder $\leq_k$ has the following property, which is similar to Theorem 3.2.3.

**Theorem 3.2.6.** *For the group $G$ with a left order $\leq$ and a compatible monomorphism $\rho$, suppose that the homeomorphisms $\rho(g)$ $(g \in S_k)$ are either all pointwise not less than the identity map, or all pointwise not greater than the identity map. Then the partial order $\leq_C$ is either an extension of $\leq_k$, or an extension of the inverse of $\leq_k$.*

*Proof.* If the homeomorphisms $\rho(g)$ $(g \in S_k)$ are all pointwise not less than the identity map, then for an arbitrary choice of $h \in G$, we have $\rho(gh)(0) = \rho(g)\rho(h)(0) > \rho(g)(0)$. By definition of $\rho$, we have $gh > h$ for every $g \in S_k$ and $h \in G$, that is, $g >_C 1$ for any $g \in S_k$. By Theorem 3.2.1 and Theorem 3.2.2, $h \geq_C 1$ for every $h$ in the root-closed, conjugacy-closed submonoid generated by $S_k$. By taking $h = f^{-1}g$, we have that $f \leq_k g$ implies $f \leq_C g$.

If the homeomorphisms $\rho(g)$ $(g \in S_k)$ are all pointwise not greater than the identity map, similarly we have that $f \leq_k g$ implies $g \leq_C f$. □

**Corollary 3.2.1.** *With the conditions of Theorem 3.2.6, the preorder $\leq_k$ is a partial order.*

*Proof.* The antisymmetry of $\leq$ implies the antisymmetry of $\leq_k$. □

## 3.3 Definition of property (D) and a relevant theorem

As in Section 1.3.2, we define property (D) for nontrivial knots as follows.

**Definition 3.3.1.** *For a nontrivial knot $K$ with $\mu$ and $\lambda$ representing a meridian and a longitude in the knot group, we say $K$ has property (D) if*

*(a) for any homomorphism $\rho$ from $\pi_1(S^3 \setminus K)$ to $Homeo_+(\mathbf{R})$, if $s \in \mathbf{R}$ is a common fixed point of $\rho(\mu)$ and $\rho(\lambda)$, then $s$ is a fixed point of every element in $\pi_1(S^3 \setminus K)$, and,*

*(b) $\mu$ is in the root-closed, conjugacy-closed submonoid generated by $\mu^{2g(K)-1}\lambda$ and $\mu^{-1}$.*

Now we prove that property (D) implies the non-left-orderability.

**Theorem 3.3.1.** *Suppose a nontrivial knot $K$ has property (D), then the fundamental group of the manifold obtained by Dehn surgery on $K$ with slope $\frac{p}{q} \geq 2g(K) - 1$ is not left orderable.*

*Proof.* Let $G$ denote the fundamental group of the manifold obtained by Dehn surgery on $K$ with slope $\frac{p}{q} \geq 2(K)-1$. Let $\mu$ and $\lambda$ represent a meridian and a longitude. Let $\rho : G \to \text{Homeo}_+(\mathbf{R})$ be a monomorphism without global fixed points. As a pullback by inclusion, $\rho$ induces a homomorphism from $\pi_1(S^3 \setminus K)$ to $\text{Homeo}_+(\mathbf{R})$ without global fixed points. Since $\mu^p = \lambda^{-q}$, we have $\rho(\mu)^p = \lambda(\lambda)^{-q}$, so $\rho(\mu)$ and $\rho(\lambda)$ have the same set of fixed points. Because $K$ has property (D), $\rho(\mu)$ does not have fixed points.

By continuity, either $\rho(\mu)$ is pointwise greater than identity map or $\rho(\mu)$ is pointwise less than identity map. We define the preorder $\leq_k$ on $G$ generated by $\mu$. Since $1 = \mu^p\lambda^q = \mu^{p-(2g(K)-1)q}(\mu^{2g(K)-1}\lambda)^q \geq_k (\mu^{2g(K)-1}\lambda)^q$, by Theorem 3.2.5, we have

$\mu^{2g(K)-1}\lambda \leq_k 1$. Since $\mu^{-1} \leq_k 1$ ,$\mu^{2g(K)-1}\lambda \leq_k 1$ and $\mu$ is in the root-closed, conjugacy-closed submonoid generated by $\mu^{2g(K)-1}\lambda$ and $\mu^{-1}$, we have $\mu \leq_k 1$. As $\rho(\mu)$ is different from the identity map, we know $\mu$ and 1 are different, but $\mu \leq_k 1$ and $1 \leq_k \mu$. Therefore $\leq_k$ is not a partial order. By Corollary 3.2.1, $G$ is not left orderable. □

## 3.4 Main Theorem 2 and Main Theorem 3

In this section, we will prove the following two theorems.

**Theorem 3.4.1.** *The fundamental group of the manifold obtained by Dehn surgery on the nontrivial knot $K$ which is the closure of a 1-bridge braid $B(\omega, t, b)$ with slope $\frac{p}{q} \geq \omega t + b - t - \omega$ is not left orderable.*

**Theorem 3.4.2.** *Nontrivial knots which are closures of 1-bridge braids have property (D).*

Let the nontrivial knot $K$ be the closure of a 1-bridge braid $B(\omega, t, b)$. The fundamental group $G$, meridian $\mu$, longitude $\lambda$ are defined in the proof of Theorem 3.3.1. As in Section 3.1, the group $G$ has a presentation

$$G = \langle x, y | r_\lambda = 1, \mu^p \lambda^q = 1 \rangle.$$

Before we prove the part (a) of the property (D), we establish a theorem which serves as a step towards a global fixed point.

**Theorem 3.4.3.** *Suppose $\rho$ is a homomorphism from a group $G$ to $Homeo_+(\mathbf{R})$. Let $g_1, g_2$ be elements in $G$, $g_3$ be represented by a word with alphabet $\{g_1, g_2\}$ and at least one letter $g_1$, and $g_4$ be represented by a word with alphabet $\{g_1^{-1}, g_2\}$ and at least one letter $g_1^{-1}$. If $s \in \mathbf{R}$ is a common fixed point of $\rho(g_3)$ and $\rho(g_4)$, then $s$ is a fixed point of $\rho(g_1)$.*

*Proof.* The element $g_3$ is represented by a word with alphabet $\{g_1, g_2\}$ and at least one letter $g_1$, hence the first condition is that $g_1$ is in the monoid generated by $\{g_1^{-1}, g_2^{-1}, g_3\}$. The element $g_4$ is represented by a word with alphabet $\{g_1^{-1}, g_2\}$ and at least one letter $g_1^{-1}$, hence the second condition is that $g_1$ is in the monoid generated by $\{g_1^{-1}, g_2, g_4^{-1}\}$.

Suppose that $s$ is a real number with $\rho(g_3)s = \rho(g_4)s = s$.

If $\rho(g_2)s \geq s$ and $\rho(g_1)s \geq s$, then by the first condition we have $\rho(g_1)s \leq s$.

If $\rho(g_2)s \geq s$ and $\rho(g_1)s \leq s$, then by the second condition we have $\rho(g_1)s \geq s$.

If $\rho(g_2)s \leq s$ and $\rho(g_1)s \geq s$, then by the second condition we have $\rho(g_1)s \leq s$.

If $\rho(g_2)s \leq s$ and $\rho(g_1)s \leq s$, then by the first condition we have $\rho(g_1)s \geq s$.

In all cases, we have $\rho(g_1)s = s$. □

Now, we prove the part (a) of the property (D).

**Lemma 3.4.1.** *For any homomorphism $\rho$ from $\pi_1(S^3 \setminus K)$ to $Homeo_+(\mathbf{R})$, if $s \in \mathbf{R}$ is a common fixed point of $\rho(\mu)$ and $\rho(\lambda)$, then $s$ is a fixed point of every element in $\pi_1(S^3 \setminus K)$.*

*Proof.* Since $s$ is a fixed point of $\rho(xy^{-1}) = \rho(\mu)$ and $\rho(yg_0) = \rho(\mu^{\omega t+b}\lambda) = \rho(\mu)^{\omega t+b}\rho(\lambda)$, and $yg_0$ is represented by a word with $b$ letter $x$'s and $\omega - b$ letter $y$'s, by Theorem 3.4.3, we have $\rho(y)s = s$. Since $x = \mu y$, we have $\rho(x)s = s$. Since $G$ is generated by $x$ and $y$, $s$ is a fixed point of every element in $\pi_1(S^3 \setminus K)$. □

Then, we prove the part (b) of the property (D).

Let $\tilde{g}_0 = 1, \tilde{g}_1, \ldots, \tilde{g}_{\omega-1} = g_0, \tilde{g}_\omega = yg_0$ be all suffixes of the word $yg_0$, ordered by length. For each integer $i$ with $0 \leq i \leq \omega - 1$, suppose that $\tilde{g}_i$ appears $b_i$ times in $g_1, g_2, \ldots, g_t$. Let $l$ be the index with $\tilde{g}_l = g_1$.

We define the preorder $\leq_k$ generated by $\mu$ and $(\mu^{2g(K)-1}\lambda)^{-1}$ on the knot group

$$\pi_1(S^3 \setminus K) = \langle x, y | r_\lambda = 1 \rangle.$$

Since $xy^{-1} = \mu$, we have $x \geq_k y$. For each $0 \leq i \leq \omega - 1$, since $\tilde{g}_{i+1} = x\tilde{g}_i$ or $\tilde{g}_{i+1} = y\tilde{g}_i$, we have $\tilde{g}_{i+1} \geq_k y\tilde{g}_i$. Because $r_\lambda = 1$, we have

$$\begin{aligned} y &= (g_1\mu g_1^{-1})(g_2\mu g_2^{-1})\cdots(g_t\mu g_t^{-1}) \\ &\geq_k (g_1\mu g_1^{-1})(\tilde{g}_i\mu\tilde{g}_i^{-1})^{b_i} \end{aligned}$$

for each integer $i$ with $0 \leq i \leq \omega - 1$ and $i \neq l$. Therefore

$$\begin{aligned} \tilde{g}_{i+1} &\geq_k y\tilde{g}_i \\ &\geq_k g_1\mu g_1^{-1}\tilde{g}_i\mu^{b_i}. \end{aligned} \tag{3.1}$$

By applying the inequality (3.1) for $0 \leq i \leq l - 1$, we have

$$\begin{aligned} g_1 &= \tilde{g}_l \\ &\geq_k (g_1\mu g_1^{-1})^l\tilde{g}_0\mu^{\sum_{i=0}^{l-1} b_i} \\ &= g_1\mu^l g_1^{-1}\mu^{\sum_{i=0}^{l-1} b_i}, \end{aligned}$$

$$g_1 \geq_k \mu^{l+\sum_{i=0}^{l-1} b_i}.$$

By applying the inequality (3.1) for $l + 1 \leq i \leq \omega - 1$, we have

$$\begin{aligned} yg_0 &= \tilde{g}_{\omega+1} \\ &\geq_k (g_1\mu g_1^{-1})^{\omega-l-1}\tilde{g}_{l+1}\mu^{\sum_{i=l+1}^{\omega-1} b_i} \\ &= g_1\mu^{\omega-l-1}g_1^{-1}\tilde{g}_{l+1}\mu^{\sum_{i=l+1}^{\omega-1} b_i} \\ &\geq_k \mu^{\omega-1+\sum_{i=0}^{l-1} b_i}g_1^{-1}y\tilde{g}_l\mu^{\sum_{i=l+1}^{\omega-1} b_i} \\ &= \mu^{\omega-1+\sum_{i=0}^{l-1} b_i}g_1^{-1}yg_1\mu^{\sum_{i=l+1}^{\omega-1} b_i}. \end{aligned}$$

Since $2g(K)-1 = \omega t + b - t - \omega$, have

$$\begin{aligned} yg_0 &= \mu^{\omega t+b}\lambda \\ &= \mu^{t+\omega}(\mu^{2g(K)-1}\lambda) \\ &\leq_k \mu^{t+\omega}. \end{aligned}$$

Since $\sum_{i=0}^{\omega-1} b_i = t$, we have

$$yg_1 \leq_k g_1\mu^{b_l+1}.$$

We first consider the case where $t > b_l$. In this case, let $l'$ be an index with $0 \leq l' \leq \omega - 1$ and $l' \neq l$ such that $b_{l'} \geq \frac{t-b_l}{\omega-1} > 0$. There exists an integer $j$ with $0 \leq j \leq b_l$, such that

$$\begin{aligned} g_1\mu^{b_l+1} &\geq_k yg_1 \\ &= (g_1\mu g_1^{-1})(g_2\mu g_2^{-1})\cdots(g_t\mu g_t^{-1})g_1 \\ &\geq_k (g_1\mu g_1^{-1})^j(\tilde{g}_{l'}\mu\tilde{g}_{l'}^{-1})(g_1\mu g_1^{-1})^{b_l-j}g_1 \\ &= g_1\mu^j g_1^{-1}\tilde{g}_{l'}\mu\tilde{g}_{l'}^{-1}g_1\mu^{b_l-j}, \end{aligned}$$

which is

$$g_1\mu g_1^{-1} \geq_k \tilde{g}_{l'}\mu\tilde{g}_{l'}^{-1}.$$

Therefore

$$\begin{aligned} y\tilde{g}_{l'} &= (g_1\mu g_1^{-1})(g_2\mu g_2^{-1})\cdots(g_t\mu g_t^{-1})\tilde{g}_{l'} \\ &\geq_k (\tilde{g}_{l'}\mu\tilde{g}_{l'}^{-1})^{b_l+b_{l'}}\tilde{g}_{l'} \\ &= \tilde{g}_{l'}\mu^{b_l+b_{l'}}. \end{aligned}$$

Since $\tilde{g}_{l'}$ is a suffix of $\tilde{g}_{l'} y g_0$, we have

$$\begin{aligned}
\tilde{g}_{l'} \mu^{t+\omega} &\geq_k \tilde{g}_{l'} y g_0 \\
&\geq_k y^{\omega} \tilde{g}_{l'} \\
&\geq_k \tilde{g}_{l'} \mu^{\omega(b_l + b_{l'})},
\end{aligned}$$

which is

$$\mu^{t+\omega-\omega(b_l+b_{l'})} \geq_k 1. \tag{3.2}$$

Furthermore, we have

$$\begin{aligned}
t + \omega - \omega(b_l + b_{l'}) &\leq t + \omega - \omega b_l + (b_l - t - b_{l'}) \\
&\leq -(\omega - 1)(b_l - 1) \\
&\leq 0.
\end{aligned}$$

Then we consider the edge case where $t = b_l$, that is, $y = (g_1 \mu g_1^{-1})^t = g_1 \mu^t g_1^{-1}$. Because $g_1$ is a suffix of $g_1 y g_0$, we have

$$\begin{aligned}
g_1 \mu^{t+\omega} &\geq_k g_1 y g_0 \\
&\geq_k y^{\omega} g_1 \\
&= (g_1 \mu^t g_1^{-1})^{\omega} g_1 \\
&= g_1 \mu^{\omega t},
\end{aligned}$$

which is

$$\mu^{t+\omega-\omega t} \geq_k 1. \tag{3.3}$$

The exponent is not positive if and only if $t > 1$. Now assume $t = 1$. Since $K$ is not trivial, we have $b \geq 1$ Because $g_1$ is a suffix of $g_1 y g_0$, there exists an integer $j$ with

$0 \leq j \leq \omega - 1$, such that

$$\begin{aligned} g_1\mu^{\omega+1} &\geq_k g_1 y g_0 \\ &\geq_k y^j x y^{\omega-j-1} g_1 \\ &= (g_1\mu g_1^{-1})^j \mu (g_1\mu g_1^{-1})^{\omega-j} g_1 \\ &= g_1\mu^j g_1^{-1}\mu g_1 \mu^{\omega-j}, \end{aligned}$$

which is

$$y \geq_k \mu.$$

Since $x = \mu y \geq_k \mu^2$, we have

$$\mu^{\omega+1} \geq_k y g_0 \geq_k \mu^{\omega+b} \geq_k \mu^{\omega+1}. \tag{3.4}$$

In every case, we have reached a loop (3.2) (3.3) (3.4) in the preorder $\leq_k$. By the definition of $\leq_k$, we can construct a computation tree where each leaf is 1, $\mu$ or $(\mu^{2g(K)-1}\lambda)^{-1}$, each non-leaf node represents multiplication, conjugation or taking roots, and the root node is 1. Furthermore, although there are many edge cases, we leave it to readers to check that in all cases, $\mu$ appears in the leaves at least once. Then every node in the computation tree is in the root-closed, conjugacy-closed submonoid generated by $\mu$ and $(\mu^{2g(K)-1}\lambda)^{-1}$. We prove the following statement.

**Lemma 3.4.2.** *Every ancestor of a leaf $\mu$ is in the root-closed, conjugacy-closed submonoid generated by $\mu^{-1}$ and $\mu^{2g(K)-1}\lambda$.*

*Proof.* Use induction on the depth of the node. When the depth is 0, the node is the root node, which is identity. Suppose that the lemma is proved for nodes of depth $d-1$, consider a node $P$ of depth $d$. Then consider the parent of $P$, we have three cases. First, suppose it is a multiplication node. Then $P$ is the multiplication of its parent and the inverse of its sibling. Since its parent is in the root-closed, conjugacy-

closed submonoid generated by $\mu^{-1}$ and $\mu^{2g(K)-1}\lambda$, and its sibling is in the root-closed, conjugacy-closed submonoid generated by $\mu$ and $(\mu^{2g(K)-1}\lambda)^{-1}$, the statement is true for $P$. Second, suppose it is a conjugation node. Then $P$ is a conjugate element of its parent. Therefore the statement is true for $P$. Last, suppose it is a taking root node. Then $P$ is a positive power of its parent. Therefore the statement is true for $P$. □

**Corollary 3.4.1.** *$\mu$ is in the root-closed, conjugacy-closed submonoid generated by $\mu^{-1}$ and $\mu^{2g(K)-1}\lambda$.*

Combining Lemma 3.4.1 and Corollary 3.4.1, we proved Theorem 3.4.2.

Combining Theorem 3.4.2 and Theorem 3.3.1, we proved Theorem 3.4.1.

## 3.5 Main Theorem 4

In this section, we will prove the following theorem.

**Theorem 3.5.1.** *Let the pattern $P$ be a 1-bridge braid $B(\omega, t, b)$, and the companion $K$ be a nontrivial knot with property (D). Suppose that $\frac{t}{\omega} \geq 2g(K) - 1$, then the satellite knot $P(K)$ has property (D).*

Let $P$ denote the 1-bridge braid pattern $B(\omega, t, b)$, and $K$ denote a nontrivial knot with property (D). Suppose that $2g(K)-1 \leq \frac{t}{\omega}$. Let $\mu_K$ and $\lambda_K$ represent a meridian and a longitude of $K$ in the knot group of $K$. Let $\mu$ and $\lambda$ represent a meridian and a longitude of $P(K)$ in the knot group of $P(K)$. As in Section 3.1, by Seifert–van Kampen theorem, the knot group of $P(K)$ is

$$\pi_1(S^3 \setminus P(K)) = (\pi_1(S^3 \setminus K) * \langle x, y \rangle)/(\mu_K = r_\mu, \lambda_K = r_\lambda).$$

First, we prove the part (a) of the property (D).

**Lemma 3.5.1.** *For any homomorphism $\rho$ from $\pi_1(S^3 \setminus P(K))$ to $Homeo_+(\mathbf{R})$, if $s \in \mathbf{R}$ is a common fixed point of $\rho(\mu)$ and $\rho(\lambda)$, then $s$ is a fixed point of every element in $\pi_1(S^3 \setminus P(K))$.*

*Proof.* By reasoning similar to the proof of Lemma 3.4.1, we have $\rho(x)s = \rho(y)s = s$. Since $\mu_K = r_\mu$ and $\lambda_K = r_\lambda$ are in the subgroup generated by $x$ and $y$, we have $\rho(\mu_K)s = \rho(\lambda_K)s = s$. Since $K$ has property (D), $s$ is a fixed point of every element in $\pi_1(S^3 \setminus K)$. Because $\pi_1(S^3 \setminus P(K))$ is generated by $x$, $y$ and elements of $\pi_1(S^3 \setminus K)$, $s$ is a fixed point of every element in $\pi_1(S^3 \setminus P(K))$. □

Then, we prove the part (b) of the property (D).

We define the preorder $\leq_k$ generated by $\mu$ and $(\mu^{2g(P(K))-1}\lambda)^{-1}$ on the knot group $\pi_1(S^3 \setminus P(K))$. Since $xy^{-1} = \mu$, we have $x \geq_k y$. For each $0 \leq i \leq \omega - 1$, since $\tilde{g}_{i+1} = x\tilde{g}_i$ or $\tilde{g}_{i+1} = y\tilde{g}_i$, we have $\tilde{g}_{i+1} \geq_k y\tilde{g}_i$.

As in Section 3.1, we have

$$r_\mu = (h_1 \mu h_1^{-1})(h_2 \mu h_2^{-1}) \cdots (h_\omega \mu h_\omega^{-1}),$$

$$r_\lambda = y(g_t \mu^{-1} g_t^{-1})(g_{t-1} \mu^{-1} g_{t-1}^{-1}) \cdots (g_1 \mu^{-1} g_1^{-1}).$$

We also have that $g_i = h_j$ if $i \equiv j \pmod{\omega}$. Therefore, we have

$$r_\mu^{2g(K)-1} r_\lambda = y(g_t \mu^{-1} g_t^{-1})(g_{t-1} \mu^{-1} g_{t-1}^{-1}) \cdots (g_{\omega(2g(K)-1)+1} \mu^{-1} g_{\omega(2g(K)-1)+1}^{-1})$$

for each integer $i$ with $\omega(2g(K)-1) < i \leq t$. Here we use the condition $2g(K)-1 \leq \frac{t}{\omega}$.

As in Section 3.4, let $\tilde{g}_0 = 1, \tilde{g}_1, \ldots, \tilde{g}_{\omega-1} = g_0, \tilde{g}_\omega = yg_0$ be all suffixes of the word $yg_0$, ordered by length. For each integer $i$ with $0 \leq i \leq \omega - 1$, suppose that $\tilde{g}_i$ appears $b_i$ times in $g_{\omega(2g(K)-1)+1}, g_{\omega(2g(K)-1)+2}, \ldots, g_t$. Then we have

$$r_\mu^{2g(K)-1} r_\lambda \leq_k y(\tilde{g}_i \mu^{-1} \tilde{g}_i^{-1})^{b_i}$$

$$= y\tilde{g}_i \mu^{-b_i} \tilde{g}_i^{-1},$$

which implies

$$\begin{aligned} \tilde{g}_{i+1} &\geq_k y\tilde{g}_i \\ &\geq_k (r_\mu^{2g(K)-1} r_\lambda)\tilde{g}_i \mu^{b_i} \end{aligned} \tag{3.5}$$

for each integer $i$ with $0 \leq i \leq \omega - 1$.

By applying the inequality (3.5) for $0 \leq i \leq \omega - 1$, we have

$$\begin{aligned} \mu^{\omega t+b}\lambda &= yg_0 \\ &= \tilde{g}_\omega \\ &\geq_k (r_\mu^{2g(K)-1} r_\lambda)^\omega \tilde{g}_0 \mu^{\sum_{i=0}^{\omega-1} b_i} \\ &= (r_\mu^{2g(K)-1} r_\lambda)^\omega \mu^{\sum_{i=0}^{\omega-1} b_i} \end{aligned}$$

By [11, Theorem 5.8.1], we have

$$g(P(K)) \geq g(P) + \omega g(K).$$

Because

$$\sum_{i=0}^{\omega-1} b_i = t - \omega(2g(K) - 1)$$

and

$$g(P) = \frac{1}{2}((\omega - 1)(t - 1) + b),$$

we have

$$\begin{aligned} \omega t + b - \sum_{i=0}^{\omega-1} b_i &= \omega t - t - \omega + b + 2\omega g(K) \\ &= 2g(P) + 2\omega g(K) - 1 \end{aligned}$$

$$\leq 2g(P(K)) - 1.$$

Therefore, we have

$$\begin{aligned}(r_\mu^{2g(K)-1} r_\lambda)^\omega &\leq_k \mu^{\omega t + b - \sum_{t=0}^{\omega-1} b_i} \lambda \\ &\leq_k \mu^{2g(P(K))-1} \lambda \\ &\leq_k 1.\end{aligned}$$

By Theorem 3.2.5, we have $r_\mu^{2g(K)-1} r_\lambda \leq_k 1$.

Because $K$ has property (D), $r_\mu$ is in the root-closed, normal submonoid generated by $r_\mu^{2g(K)-1} r_\lambda$ and $r_\mu^{-1}$, as this relation is invariant under free multiplication and quotient maps. Therefore we have $r_\mu \leq_k 1$, as $r_\mu^{2g(K)-1} r_\lambda \leq_k 1$ and $r_\mu^{-1} \leq_k 1$.

Since $r_\mu$ is the product of $\omega$ conjugates of $\mu$, $\mu$ is in the root-closed, normal submonoid generated by $r_\mu$ and $\mu^{-1}$. Therefore we have $\mu \leq_k 1$, as $r_\mu \leq_k 1$ and $\mu^{-1} \leq_k 1$.

By the definition of $\leq_k$, we completed the proof of the part (a) of the property (D). Combined with Lemma 3.5.1, we proved Theorem 3.5.1.

# Chapter 4

# Non-left-orderability for Dehn fillings on the manifold $v2503$

## 4.1 Main Theorem 5

In this section, we will prove the following theorem by using similar techniques as in Chapter 3.

**Theorem 4.1.1.** *The fundamental group of an manifold obtained by Dehn filling on* $v2503$ *is not left orderable.*

According to the software SnapPy [13] and Varvarezos's computation [51], the fundamental group of the manifold $v2503$ has the presentation

$$\pi_1(v2503) = \langle a, b | a^2b^{-2}ab^{-2}a^2ba^2baba^2b = 1 \rangle,$$

and a meridian $\mu$ and a longitude $\lambda$ can be written as

$$\mu = b^{-2}ab^{-2}ab^{-1},$$

and

$$\lambda = a^{-2}b^{-1}a^{-2}b.$$

The following lemma is similar to Lemma 3.4.1 and Lemma 3.5.1.

**Lemma 4.1.1.** *For any homomorphism $\rho$ from $\pi_1(v2503)$ to $Homeo_+(\mathbf{R})$, if $s \in \mathbf{R}$ is a fixed point of $\rho(\mu)$, then $s$ is a fixed point of every element in $\pi_1(v2503)$.*

*Proof.* Since

$$\begin{aligned} 1 &= a^2b^{-2}ab^{-2}a^2ba^2baba^2b \\ &= a^2\mu baba^2baba^2b, \end{aligned}$$

we have

$$\mu^{-1} = baba^2baba^2ba^2.$$

By taking $g_1 = b, g_2 = a, g_3 = \mu^{-1}, g_4 = \mu$ in Theorem 3.4.3, we get $\rho(b)s = s$.

By taking $g_1 = a, g_2 = b, g_3 = \mu^{-1}, g_4 = \mu^{-1}$ in Theorem 3.4.3, we get $\rho(a)s = s$.

Since the group $\pi(v2503)$ is generated by $a$ and $b$, $s$ is a fixed point of every element in $\pi_1(v2503)$. □

The following theorem is an analog of the part (b) in the definition of property (D).

**Theorem 4.1.2.** *$\lambda$ is in the root-closed, conjugacy-closed submonoid generated by $\lambda^{-1}$.*

*Proof.* We define the preorder $\leq_k$ generated by $\lambda$ on $\pi_1(v2503)$. Then $\lambda \geq_k 1$ implies that $a^2ba^2 \leq_k b$. Therefore, we have

$$\begin{aligned} 1 &= a^2b^{-2}ab^{-2}a^2ba^2baba^2b \\ &= a^2b^{-2}ab^{-2}(a^2ba^2)bab(a^2ba^2)a^{-2} \end{aligned}$$

$$\leq_k a^2b^{-2}ab^{-2}bbabba^{-2}$$
$$= a^2b^{-2}a^2b^2a^{-2},$$

which is $a^2 \geq_k 1$. Therefore, we have

$$\lambda = a^{-2}b^{-1}a^{-2}b$$
$$\leq_k b^{-1}b$$
$$= 1.$$

By the definition of preorder $\leq_k$, $\lambda$ is in the root-closed, conjugacy-closed submonoid generated by $\lambda^{-1}$. □

Now we prove Theorem 4.1.1.

By Seifert-van Kampen theorem, the fundamental group of an manifold obtained by Dehn filling on $v2503$ with slope $\frac{p}{q}$, denoted by $G$, is the quotient group of $\pi_1(v2503)$ by the relation $\mu^p\lambda^q = 1$. Let $\rho : G \to \mathrm{Homeo}_+(\mathbf{R})$ be a monomorphism without global fixed points. As a pullback by inclusion, $\rho$ induces a homomorphism from $\pi_1(v2503)$ to $\mathrm{Homeo}_+(\mathbf{R})$ without global fixed points. Since $\mu^p = \lambda^{-q}$, we have $\rho(\mu)^p = \rho(\lambda)^{-q}$. By Lemma 4.1.1, $\rho(\mu)$ does not have fixed points.

If $p \neq 0$, then $\rho(\lambda)$ does not have fixed points. By Corollary 3.2.1, $\leq_k$ is a partial order. By Theorem 4.1.2, we have $1 \leq_k \lambda \leq_k 1$, so $\lambda = 1$ which contradicts to that $\rho(\lambda)$ does not have fixed points.

If $p = 0$, then we get the equalities $\lambda = 1$, $a^2ba^2 = b$ and $a^2 = 1$. Then $G$ can be simplified to $\mathbf{Z} * (\mathbf{Z}/2\mathbf{Z})$. The existence of a nontrivial torsion element implies the non-left-orderability.

Therefore, Theorem 4.1.1 holds.

# Chapter 5

# Topologically slice $(1,1)$-knots which are not smoothly slice

## 5.1 The $(1,1)$-knots $K_n$

For each non-negative integer $n$, let $K_n$ be the knot with the following planar projection.

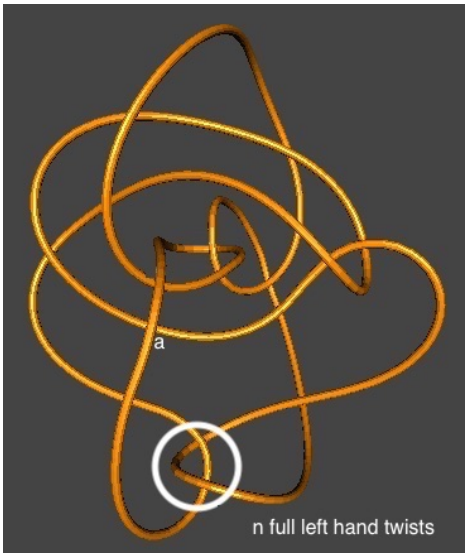



Figure 5.1: Knot $K_1$ as shown. The knot $K_n$ has $n$ full left hand twists inside the white circle.

**Theorem 5.1.1.** *$K_n$ is the $(1,1)$-knot $K(64n+31, 24n+12, 16n+6, 32n+18)$ in Rasmussen's notation.*

*Proof.* A $(1,1)$-diagram of $K_n$ is essentially a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$, where $(\Sigma, \alpha, \beta)$ is a genus one Heegaard splitting of $S^3$ and $w, z$ are base points on the torus $\Sigma$. The following figure gives the torus $\Sigma$ as the blue torus and the base points $w, z$ as labeled. The $\alpha$ circle is immediately known. The only thing left to compute is the $\beta$ circle. This can be found via a sequence of isotopies of the space.

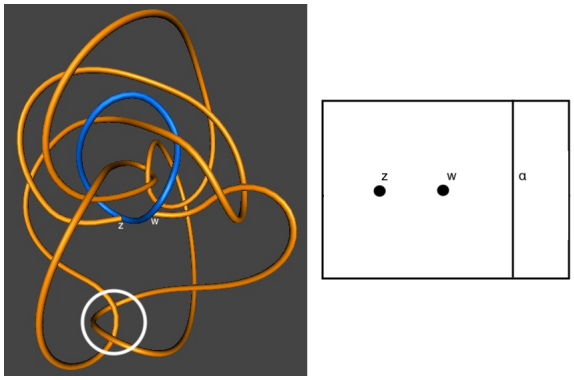



Figure 5.2: First Step.

Then, we move $z$ along a longitude of the blue torus clockwise and move $z$ around $w$ clockwise. Now we get the following figure.

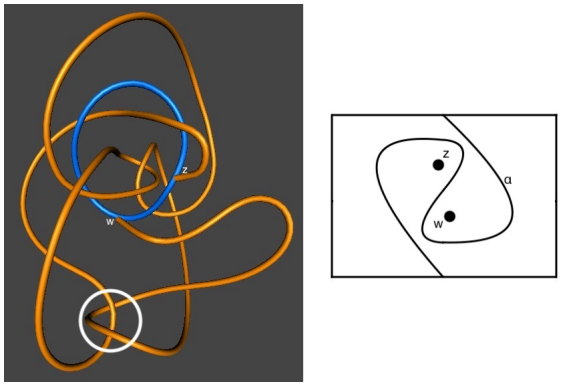



Figure 5.3: Second Step.

Then, we move $z$ along a longitude of the blue torus counterclockwise, and then move $z$ along a meridian of the blue torus into the paper. Now we get the following figure.

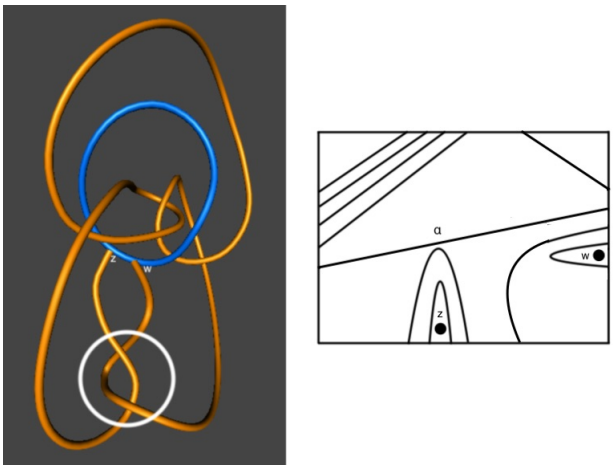



Figure 5.4: Third Step.

Now we move $w$ around $z$ clockwise $n$ times, and we get the following figure.

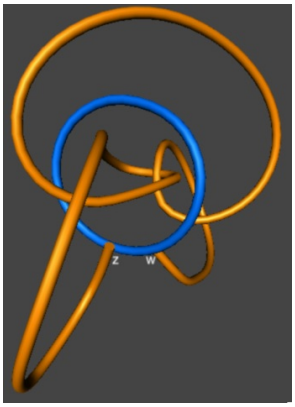



Figure 5.5: Fourth Step.

Then, we move $w$ along a longitude of the blue torus clockwise and and then move $w$ along a meridian of the blue torus out of the paper. Then we get the following figure. Here an arc with a red number $s$ on it represents a family of $s$ parallel curves.

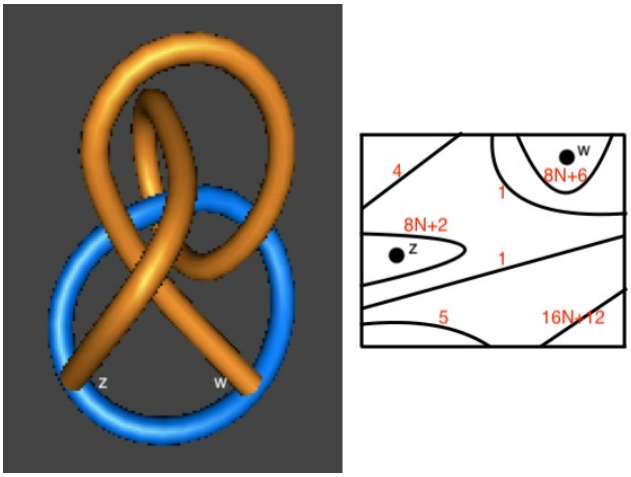



Figure 5.6: Fifth Step.

Before we proceed to find the $\beta$ curve, we simplify the $\alpha$ curve without moving the basepoints $w$ and $z$ and get the following diagram.

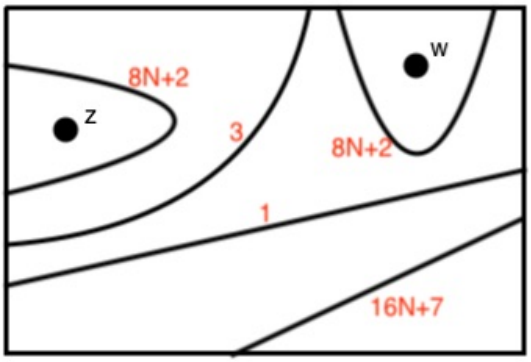



Figure 5.7: Sixth Step.

Finally, it is straightforward to get the triviality of the outside arc, and a $\beta$ curve can be constructed as follows (the blue curve).

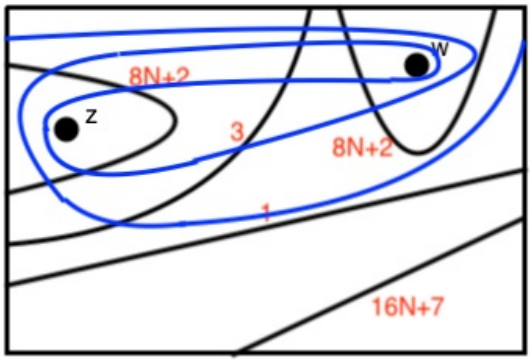



Figure 5.8: Seventh Step.

By straightening the $\beta$ curve, we obtain the following $(1,1)$-diagram.

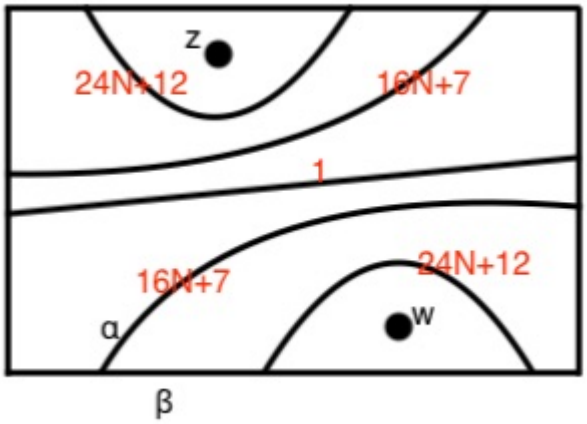



Figure 5.9: Eighth Step.

Equivalently, we have the following $(1,1)$-diagram. In Rasmussen's notation, $K_n$ is the $(1,1)$-knot $K(64n+31, 24n+12, 16n+6, 32n+18)$.

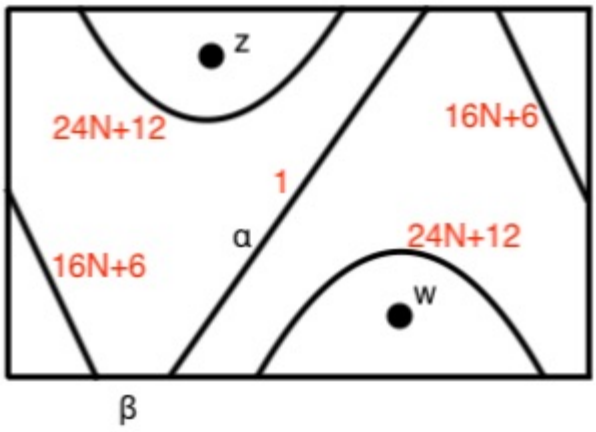



Figure 5.10: Ninth Step.

□

## 5.2 The knot Floer homology of $K_n$

**Theorem 5.2.1.** *The Poincare polynomial of the knot Floer homology of $K_n$ is*

$$\begin{aligned} HF_{K_n}(q,t) &= \sum_{m,a\in\mathbf{Z}} q^m t^a \, rank_{\mathbf{Z}}(\widehat{HFK}_m(S^3,K_n,a)) \\ &= -q^{-1} + (2n+1)q^{-3}t^{-2}(1+q)(1+qt)^4. \end{aligned}$$

*Proof.* It is clear that the total rank of the knot Floer homology of $K_n$ is $64n+31$, where the crossing points $x_i$ $(i = 1, \ldots, 64n+31)$ between the $\alpha$ curve and the $\beta$ curve represent a basis of the knot Floer homology. To compute the knot Floer homology, we need to find the Alexander grading $a_i$ and the Maslov grading $m_i$ of each crossing point $x_i$ $(i = 1, \ldots, 64n+31)$.

From each Whitney disk of unit Maslov index, we derive a equation for the Alexander and Maslov grading of the vertices of the disk. With sufficient number of such disks, we can obtain the relative Alexander and Maslov gradings.

First, there is a disk from $x_i$ to $x_{48n+25-i}$ for each $i = 1, \ldots, 24n + 12$, with a single basepoint $z$ inside. And there is a disk from $x_i$ to $x_{80n+39-i}$ for each $i = 16n + 8, \ldots, 40n + 19$, with a single basepoint $w$ inside.

Then, we concentrate on the disk from $x_{24n+12}$ to $x_{24n+13}$, and try to extend it into larger disks while keeping the basepoint on it unchanged. Sequentially, we obtain disks from $x_{8(n-i)+2}$ to $x_{8(n-i)+1}$ and disks from $x_{8(n+i)+9}$ to $x_{8(n+i)+10}$ for $i = 0, \ldots, n-1$. Then we obtain a disk from $x_2$ to $x_1$. After that, we sequentially obtain disks from $x_{8(2n-i)+5}$ to $x_{8(2n-i)+4}$ and disks from $x_{8i+6}$ to $x_{8i+7}$ for $i = 0, \ldots, 2n$. All the disks obtained here are Whitney disk of unit Maslov index with a single basepoint $z$ inside.

Finally, we keep extending to obtain the disk from $x_{64n+31}$ to $x_{48n+25}$. To argue that the last one we got is an embedded disk in the universal covering space, we notice that, in every previous step, the segment where the endpoints lie is moving diagonally in the universal covering space, but in the last step, it is moving vertically, so the disk has no self intersections. Therefore the last disk we obtained is a Whitney disk of unit Maslov index with two $z$'s inside.

Similarly, we extend the disk from $x_{40n+20}$ to $x_{40n+19}$ in the same way. Then we sequentially obtain disks from $x_{8(n+i)+5}$ to $x_{8(n+i)+6}$ and disks from $x_{8(n-i)-2}$ to $x_{8(n-i)-3}$ for $i = 0, \ldots, n-1$. And then we obtain a disk from $x_{16n+5}$ to $x_{16n+6}$. Then we sequentially obtain disks from $x_{8i+2}$ to $x_{8i+3}$ and disks from $x_{8(2n-i)+1}$ to $x_{8(2n-i)}$ for $i = 0, \ldots, 2n-1$. And then we obtain a disk from $x_{16n+2}$ to $x_{16n+3}$. All the disks obtained here are Whitney disk of unit Maslov index with a single basepoint $w$ inside. And similarly, we obtain a Whitney disk of unit Maslov index with two $w$'s inside from $x_1$ to $x_{16n+7}$ in the final step.

Now we extend the disk from $x_{24n+10}$ to $x_{24n+15}$ in the same way. Then we sequentially obtain disks from $x_{8(n-i)+4}$ to $x_{8(n-i)-1}$ and disks from $x_{8(n+i)+7}$ to $x_{8(n+i)+12}$ for $i = 0, \ldots, n-1$. All the disks obtained here are Whitney disk of unit Maslov index with a single basepoint $z$ inside.

Similarly, we extend the disk from $x_{40n+22}$ to $x_{40n+17}$ in the same way. Then we sequentially obtain disks from $x_{8(n+i)+3}$ to $x_{8(n+i)+8}$ and disks from $x_{8(n-i)}$ to $x_{8(n-i)-5}$ for $i = 0, \ldots, n-1$. Then we obtain a disk from $x_{16n+3}$ to $x_{16n+8}$. All the disks obtained here are Whitney disk of unit Maslov index with a single basepoint $w$ inside.

Now we have found sufficient Whitney disks to get the relative Alexander and Maslov gradings for the first $16n+8$ crossing points. By symmetry, we also get the relative Alexander and Maslov gradings for the last $16n+8$ crossings. Via the Whitney disks from $x_i$ to $x_{80n+39-i}$ ($i = 16n+8, \ldots 40n+19$), because we have a common element $x_{16n+8}$ in the sets of crossing points, we get the relative Alexander and Maslov gradings for the first $32n+15$ crossing points. By using the symmetry and the Whitney disks for a second time, we get the relative Alexander and Maslov gradings for all the crossing points.

The following is a solution to the relative Alexander gradings.

$$
\begin{aligned}
(a_{8i+1}, \ldots, a_{8i+8}) &= (-1,0,1,0,1,0,-1,0) \text{ for } i = 0, \ldots, n-1, \\
a_{8n+1} &= -1, \\
(a_{8n+8i+2}, \ldots, a_{8n+8i+9}) &= (0,1,0,1,2,1,2,1) \text{ for } i = 0, \ldots, 2n, \\
(a_{24n+8i+10}, \ldots, a_{24n+8i+17}) &= (0,1,0,-1,0,-1,0,1) \text{ for } i = 0, \ldots, n-1, \\
(a_{32n+10}, \ldots, a_{32n+22}) &= (0,1,0,-1,0,-1,0,1,0,1,0,-1,0), \\
(a_{32n+8i+23}, \ldots, a_{32n+8i+30}) &= (-1,0,1,0,1,0,-1,0) \text{ for } i = 0, \ldots, n-1, \\
(a_{40n+8i+23}, \ldots, a_{40n+8i+30}) &= (-1,-2,-1,-2,-1,0,-1,0) \text{ for } i = 0, \ldots, 2n, \\
a_{56n+31} &= 1, \\
(a_{56n+8i+32}, \ldots, a_{56n+8i+39}) &= (0,1,0,-1,0,-1,0,1) \text{ for } i = 0, \ldots, n-1.
\end{aligned}
$$

By symmetry, the above is also a solution to the absolute Alexander gradings.

The following is a solution to the relative Maslov gradings.

$$
\begin{aligned}
(m_{8i+1},\dots,m_{8i+8}) &= (-2,-1,0,0,1,0,-1,-1) \text{ for } i=0,\dots,n-1\\
m_{8n+1} &= -2,\\
(m_{8n+8i+2},\dots,m_{8n+8i+9}) &= (-1,0,0,1,2,1,1,0) \text{ for } i=0,\dots,n,\\
(m_{16n+8i+10},\dots,m_{16n+8i+17}) &= (-1,1,0,1,2,0,1,0) \text{ for } i=0,\dots,n-1,\\
(m_{24n+8i+10},\dots,m_{24n+8i+17}) &= (-1,1,0,-1,0,-2,-1,0) \text{ for } i=0,\dots,n-1,\\
(m_{32n+10},\dots,m_{32n+22}) &= (-1,1,0,-1,0,-2,-1,0,0,1,0,-1,-1),\\
(m_{32n+8i+23},\dots,m_{32n+8i+30}) &= (-2,-1,0,0,1,0,-1,-1) \text{ for } i=0,\dots,n-1,\\
(m_{40n+8i+23},\dots,m_{40n+8i+30}) &= (-2,-3,-2,-2,-1,0,-1,-1) \text{ for } i=0,\dots,n-1,\\
(m_{48n+8i+23},\dots,m_{48n+8i+30}) &= (-2,-3,-1,-2,-1,0,-2,-1) \text{ for } i=0,\dots,n,\\
m_{56n+31} &= 0,\\
(m_{56n+8i+32},\dots,m_{56n+8i+39}) &= (-1,1,0,-1,0,-2,-1,0) \text{ for } i=0,\dots,n-1.
\end{aligned}
$$

To find the absolute Maslov gradings, we take a step back to look at Figure 5.9. If we remove basepoint $z$, most of the crossing points can be easily reduced. In fact, only the last three crossing points might survive. By analyzing the twisting part of the $\alpha$ curve, we find that the crossing points $x_{64n+29}$ and $x_{64n+30}$ can be reduced from above. Therefore, the only survived crossing point $x_{64n+31}$ has absolute Maslov grading zero. So the relative Maslov gradings we got is also the absolute Maslov gradings.

The Poincare polynomial is derived from counting the number of crossing points with given Alexander and Maslov grading. □

**Corollary 5.2.1.** *$K_n$ and $K_m$ are non-isomorphic if $m \neq n$.*

*Proof.* This is because they have different knot Floer homology. □

**Corollary 5.2.2.** *The Conway polynomial of $K_n$ is* 1.

*Proof.* The Conway polynomial is [41] the graded Euler characteristic of the knot Floer homology, so we have

$$\Delta_{K_n}(t) = \mathrm{HF}_{K_n}(-1, t) = 1.$$

□

**Corollary 5.2.3.** *$K_n$ is topologically slice.*

*Proof.* Any knot with trivial Conway polynomial is topologically slice, as proved in [16, 18]. □

## 5.3 $\tau$ invariant and smooth slice genus

**Lemma 5.3.1.** *The invariant $\tau(K_0)$ is* 1.

*Proof.* To calculate the invariant $\tau(K_0)$, we first compute the chain complex $\mathrm{CFK}^\infty(S^3, K_0)$. By definition, the generators of $\mathrm{CFK}^\infty(S^3, K_0)$ is given by $[x_i, j, j+a_i]$ $(i = 1, \ldots, 64n + 31, j \in \mathbf{Z})$, and the boundary operator on $\mathrm{CFK}^\infty(S^3, K_0)$ is given by

$$\begin{aligned}
\partial[x_1, i+1, i] &= [x_7, i-1, i] - [x_{10}, i, i] + [x_{21}, i, i-1] - [x_{24}, i+1, i-1],\\
\partial[x_2, i+1, i+1] &= [x_1, i+1, i] - [x_3, i, i+1] - [x_9, i, i+1] - [x_{23}, i+1, i],\\
\partial[x_3, i, i+1] &= [x_7, i-1, i] - [x_8, i-1, i+1] + [x_{21}, i, i-1] - [x_{22}, i, i],\\
\partial[x_4, i, i] &= -[x_7, i-1, i] - [x_{21}, i, i-1],\\
\partial[x_5, i, i+1] &= [x_4, i, i] - [x_6, i-1, i+1] - [x_{20}, i, i],\\
\partial[x_6, i-1, i+1] &= -[x_7, i-1, i] - [x_{19}, i-1, i],\\
\partial[x_7, i-1, i] &= -[x_{18}, i-1, i-1],
\end{aligned}$$

$$\partial[x_8, i-1, i+1] = -[x_{17}, i-1, i],$$
$$\partial[x_9, i, i+1] = [x_8, i-1, i+1] - [x_{10}, i, i] - [x_{16}, i, i],$$
$$\partial[x_{10}, i, i] = -[x_{15}, i, i-1],$$
$$\partial[x_{11}, i-1, i] = -[x_{14}, i-1, i-1],$$
$$\partial[x_{12}, i, i] = [x_{11}, i-1, i] - [x_{13}, i, i-1],$$
$$\partial[x_{13}, i, i-1] = -[x_{14}, i-1, i-1],$$
$$\partial[x_{14}, i-1, i-1] = 0,$$
$$\partial[x_{15}, i, i-1] = 0,$$
$$\partial[x_{16}, i, i] = [x_{15}, i, i-1] - [x_{17}, i-1, i],$$
$$\partial[x_{17}, i-1, i] = 0,$$
$$\partial[x_{18}, i-1, i-1] = 0,$$
$$\partial[x_{19}, i-1, i] = [x_{18}, i-1, i-1],$$
$$\partial[x_{20}, i, i] = [x_{19}, i-1, i] - [x_{21}, i, i-1],$$
$$\partial[x_{21}, i, i-1] = [x_{18}, i-1, i-1],$$
$$\partial[x_{22}, i, i] = [x_{17}, i-1, i],$$
$$\partial[x_{23}, i+1, i] = [x_{16}, i, i] + [x_{22}, i, i] - [x_{24}, i+1, i-1],$$
$$\partial[x_{24}, i+1, i-1] = [x_{15}, i, i-1],$$
$$\partial[x_{25}, i, i-1] = [x_{14}, i-1, i-1],$$
$$\partial[x_{26}, i+1, i-1] = [x_{13}, i, i-1] + [x_{25}, i, i-1],$$
$$\partial[x_{27}, i+1, i] = [x_{12}, i, i] + [x_{26}, i+1, i-1] - [x_{28}, i, i],$$
$$\partial[x_{28}, i, i] = [x_{11}, i-1, i] + [x_{25}, i, i-1],$$
$$\partial[x_{29}, i+1, i] = [x_{10}, i, i] - [x_{11}, i-1, i] + [x_{24}, i+1, i-1] - [x_{25}, i, i-1],$$
$$\partial[x_{30}, i+1, i+1] = [x_9, i, i+1] + [x_{23}, i+1, i] + [x_{29}, i+1, i] - [x_{31}, i, i+1],$$
$$\partial[x_{31}, i, i+1] = [x_8, i-1, i+1] - [x_{11}, i-1, i] + [x_{22}, i, i] - [x_{25}, i, i-1],$$

by counting all Whitney disks of unit Maslov index, where the signs are by an orientation of the $\beta$ curve.

By definition, the chain complex $\widehat{\mathrm{CF}}(S^3)$ is generated by $[x_i, 0, a_i]$ ($i = 1, \ldots, 64n+31$), and the boundary operator on $\widehat{\mathrm{CF}}(S^3)$ is given by

$$
\begin{aligned}
\partial[x_1, 0, -1] &= -[x_{24}, 0, -2], \\
\partial[x_2, 0, 0] &= [x_1, 0, -1] - [x_{23}, 0, -1], \\
\partial[x_3, 0, 1] &= [x_{21}, 0, -1] - [x_{22}, 0, 0], \\
\partial[x_4, 0, 0] &= -[x_{21}, 0, -1], \\
\partial[x_5, 0, 1] &= [x_4, 0, 0] - [x_{20}, 0, 0], \\
\partial[x_6, 0, 2] &= -[x_7, 0, 1] - [x_{19}, 0, 1], \\
\partial[x_7, 0, 1] &= -[x_{18}, 0, 0], \\
\partial[x_8, 0, 2] &= -[x_{17}, 0, 1], \\
\partial[x_9, 0, 1] &= -[x_{10}, 0, 0] - [x_{16}, 0, 0], \\
\partial[x_{10}, 0, 0] &= -[x_{15}, 0, -1], \\
\partial[x_{11}, 0, 1] &= -[x_{14}, 0, 0], \\
\partial[x_{12}, 0, 0] &= -[x_{13}, 0, -1], \\
\partial[x_{13}, 0, -1] &= 0, \\
\partial[x_{14}, 0, 0] &= 0, \\
\partial[x_{15}, 0, -1] &= 0, \\
\partial[x_{16}, 0, 0] &= [x_{15}, 0, -1], \\
\partial[x_{17}, 0, 1] &= 0, \\
\partial[x_{18}, 0, 0] &= 0, \\
\partial[x_{19}, 0, 1] &= [x_{18}, 0, 0],
\end{aligned}
$$

$$\begin{aligned}
\partial[x_{20},0,0] &= -[x_{21},0,-1],\\
\partial[x_{21},0,-1] &= 0,\\
\partial[x_{22},0,0] &= 0,\\
\partial[x_{23},0,-1] &= -[x_{24},0,-2],\\
\partial[x_{24},0,-2] &= 0,\\
\partial[x_{25},0,-1] &= 0,\\
\partial[x_{26},0,-2] &= 0,\\
\partial[x_{27},0,-1] &= [x_{26},0,-2],\\
\partial[x_{28},0,0] &= [x_{25},0,-1],\\
\partial[x_{29},0,-1] &= [x_{24},0,-2],\\
\partial[x_{30},0,0] &= [x_{23},0,-1]+[x_{29},0,-1],\\
\partial[x_{31},0,1] &= [x_{22},0,0]-[x_{25},0,-1].
\end{aligned}$$

The homology of is generated by the cycle $[x_3,0,1]+[x_4,0,0]+[x_{28},0,0]+[x_{31},0,1]$. Since $[x_3,0,1]+[x_4,0,0]+[x_{28},0,0]+[x_{31},0,1]\in\mathcal{F}(K_0,1)$, we have $\tau(K_0)\leq 1$.

Consider the projection $p:\widehat{\mathrm{CF}}(S^3)\to\mathbf{Z}$ that sends all generators to zero except for $p([x_3,0,1])=1$. Then $p$ is a chain map which induces isomorphism on homology. Since $p$ maps all elements in $\mathcal{F}(K_0,0)$ to zero, we have $\tau(K_0)\geq 1$.

Therefore the invariant of $\tau(K_0)$ is 1. □

**Theorem 5.3.1.** *The $\tau$ invariant and the smooth slice genus of $K_n$ are* 1.

*Proof.* After changing $n$ of the $2n$ positive crossings in white circle in Figure 5.1 to negative crossings, the new knot becomes $K_0$, therefore by [42] and Lemma 5.3.1, we have $g_4(K_n)\geq\tau(K_n)\geq\tau(K_0)=1$.

By resolving one of the $2n$ positive crossings in white circle and the crossing $a$ in Figure 5.1, we get a new knot as follows.

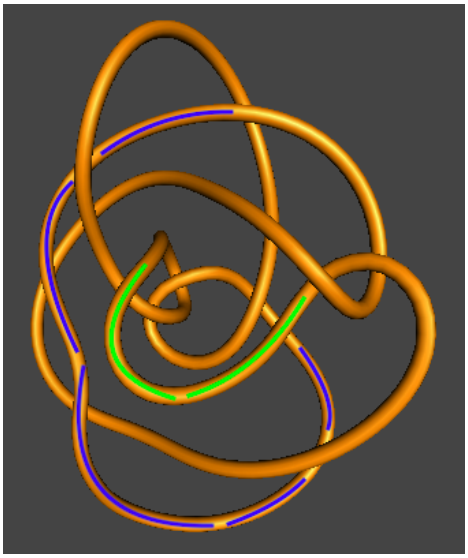


Figure 5.11: The knot obtained by resolving two crossings in $K_n$.

By straightening the blue part and the green part in Figure 5.11, we get the following isotopic knot.

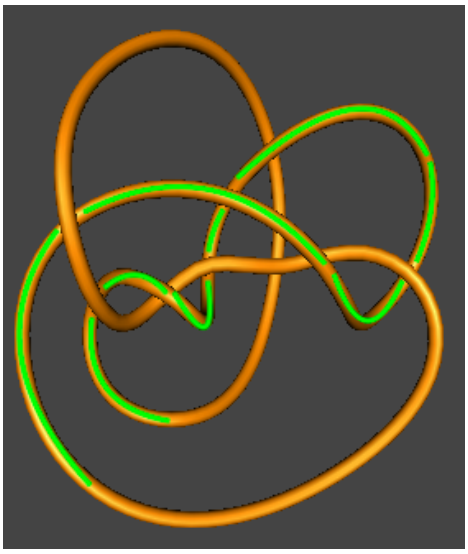


Figure 5.12: The knot obtained by resolving two crossings in $K_n$.

By straightening the green part in Figure 5.12, it is now clear that our new knot is the unknot.

Hence, by resolving two crossings, we can construct a torus cobordism (a split cobordism followed by a merge cobordism) between $K_n$ and the unknot. By definition, we have $g_4(K_n) \leq 1$.

Therefore, we have $\tau(K_n) = g_4(K_n) = 1$. □

## 5.4 Main Theorem 6

Now we can finally prove Main Theorem 6.

**Theorem 5.4.1.** *There are infinitely many* $(1,1)$*-knots which are topologically slice, but not smoothly slice.*

*Proof.* By Theorem 5.1.1, Corollary 5.2.1, Corollary 5.2.3 and Theorem 5.3.1, the family of knots $K_n$ serves as an example. □

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