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Essays on Maturity Structure of Sovereign Debt

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ESSAYS ON MATURITY STRUCTURE OF SOVEREIGN DEBT

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Abstract:

In the first chapter, I develop a tractable model to study the optimal debt maturity structure and fiscal policy in an environment with incomplete markets, lack of commitment, and opportunity to default by the government. The default on public debt is endogenous and the real interest rate reflects the default risk and the marginal rate of substitution between present and future consumption. I show that the maturity is used to resolve the time-consistency problem: The present government can incentivize future governments to stick to an ex ante optimal sequence of fiscal policies and interest rates.

In the second chapter, I show that if both risk-free interest rates and risk premiums can be manipulated, the optimal maturity structure tends to have a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. Debt maturity data across countries are consistent with model predictions.

In the last chapter, I study the sovereign debt maturity structure of a small-open economy in a model with stochastic interest rates and opportunity to default by the government. If default premiums are perfectly foreseen, the optimal debt policy is to issue only one-period debt. Short-term debt disciplines the future governments not to over borrow compared to ex ante optimal allocations because, otherwise, the sovereign has an incentive to dilute the value of long-term debt ex post. If default premiums are stochastic but locally independent of level of debt, sovereign issues consol bonds or maturity is flat. Flat maturity hedges the government against unpredictable swings in interest rates and smooths consumption over states of the world. If default premiums are stochastic (so that maturity can be used as a hedging against changes in interest rates) and continuously increasing in outstanding debt (so that sovereign has an incentive to use short-term debt to minimize dilution of long-term debt in the future), the optimal maturity is mostly short-term debt as minimizing costs associated with

lack of commitment is quantitatively more important compared to minimizing costs associated with lack of insurance.

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To my family

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Chapter 1

Government Debt Maturity Structure, Fiscal Policy, and Default

1.1 Introduction

Debt maturity structure is an important element of optimal fiscal policy, especially in light of recent sovereign debt crises. The consensus is that debt maturity is used to minimize the costs of lack of commitment. In their seminal paper, Lucas and Stokey (1983) derive the classic result that in an environment with endogenous risk-free interest rates and no default the government should issue consol bonds, i.e., the optimal maturity is spread out or flat. In contrast, Aguiar et al. (2016) study an open economy with default but exogenous risk-free interest rates. The authors demonstrate that the time-consistency problem can be resolved if the government issues only short-term debt and abstains from any active issuance or repurchase of long-term liabilities.

In this chapter, I combine both sources of time inconsistency - manipulation of risk-free interest rates and debt dilution due to option to default - within a unified framework. I develop a tractable model to study the optimal fiscal policy and optimal debt maturity structure in an environment with incomplete markets, lack of commitment to fiscal policies, and endogenous default on public debt, and show that, if a government can alter both risk-free rates and risk premiums, the optimal maturity structure exhibits a decaying profile, i.e., total payments due at a later maturity date are lower. This prediction is in line with empirical data as observed term structures of most countries are neither flat nor short but skewed toward the short end.

The model features a benevolent government and a continuum of atomistic households with strictly concave utility functions over private consumption. Households are the only lenders to the government. The government cannot commit to either future fiscal policies or to repay its debt, and sets fiscal policies, restructures its debt portfolio, and decides whether or not to default sequentially. The markets are incomplete, and the set of financial instruments is limited to bonds with various maturities. Interest rates reflect both the probability of default and marginal rate of substitution between present and future consumption.

Default is modeled as a stochastic outside option that can be exercised at the beginning of every period. Whenever the value of the outside option exceeds the value of repaying debt, the default option is triggered. Default is costly to sustain a positive amount of debt in equilibrium. The value of the outside option is the only shock in the model. In addition, the value of default is continuously distributed to allow smoothness in default probability.

I analyze the Markov perfect competitive equilibrium in which all decisions are made sequentially and are functions of payoff-relevant state variables: the outstanding debt at various maturities and the value of the outside option. I characterize the op-

timal allocation by considering the modified commitment problem as the benchmark: A contract that allows the government to commit to predetermined fiscal policies but not to abstain from default. In other words, the planner simultaneously makes the fiscal decisions for all future periods and can promise to pursue the plan: however, it cannot promise to repay debt if the outside option is preferred. The optimal allocation of the modified commitment problem defines the fiscal plan: the sequence of budget surpluses needed to repay debt contingent on no prior default.

In this model, the Markov perfect competitive equilibrium is efficient in the sense that the sequential policy maker follows the ex ante optimal fiscal plan and sticks to ex ante optimal risk-free interest rates and default probabilities. Even though the government cannot commit to future policies, it can set the term structure of its liabilities so that it has no incentive to deviate from the plan in the future. Why might the government be willing to distort the ex ante optimal allocation in the future? At every date, the value of outstanding debt must be financed by future budget surpluses. Therefore, a deviation from the plan can be ex post beneficial if the market value of outstanding debt is decreased. For example, if debt is mostly short term, then reallocation of budget surpluses leading to a decrease in the value of short-term debt at the expense of an increase in the value of long-term debt might be optimal ex post, and vice versa. However, the government can structure its debt maturity such that any such distortion strictly reduces the budget set of the government by increasing the value of outstanding debt and, hence, deviations are not optimal.

In the second chapter I demonstrate that in the presence of default risk, the government issues more short-term debt than long-term debt. Moreover, the optimal maturity structure has a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. The average maturity depends on the relative sensitivity of risk-free interest rates and risk premiums. The term structure is shorter if risk-free interest rates are

less responsive to changes in government policies. On the other hand, if a deviation from a fiscal plan has a negligible effect on default risk, then the optimal maturity structure is approximately flat.

To gain intuition, suppose that the government can distort only risk-free rates and the default risk is absent. Then the optimal maturity is flat, meaning that the total payments due at different maturity dates are constant. Any deviation from the fiscal plan that increases the budget surplus in one period and decreases it in another period does not lead to a decrease in the value of outstanding debt; this is because changes in risk-free rates are proportional and offset each other. However, if debt is skewed toward the short- or long-term end, decreasing the price of a larger stock of debt at the expense of increasing the price of a lower stock of debt allows the government to reduce the value of total debt.

Now consider an environment in which risk-free rates are exogenous, but the default risk is positive and increasing in total debt issued. The next period, the government can affect default probabilities in future periods by increasing or decreasing the budget surplus. Thus, it can affect the value of debt that matures in subsequent periods. However, the government cannot manipulate the price of a one-period debt issued in the preceding period, because it cannot alter the default probability in the current period. The reason is that all government fiscal policies are conducted conditional on no prior default in that period. Therefore, the optimal debt policy prescribes issuance of one-period bonds only.

Finally, suppose the government can manipulate risk-free interest rates and default probabilities. In such an environment, the price of debt with longer maturity is more sensitive to potential future distortions compared to the price of debt with shorter maturity. Consider a deviation from ex ante optimal fiscal plan that implies reallocation of budget surpluses between two subsequent periods, keeping the market value of budget surpluses constant. This perturbation causes proportional changes

in risk-free interest rates. The probability of default in the later period changes as a higher or lower budget surplus in that period corresponds to a higher or lower value of pursuing the fiscal plan and, hence, to a lower or higher default risk. However, this deviation does not affect the default probability in the earlier period, as the deviation described does not change government welfare in the earlier period. Therefore, change in the price of debt with longer maturity reflects distortions in both the risk-free interest rate and risk premium, while the price of debt with shorter maturity varies due only to changes in the risk-free interest rate. A deviation from the ex ante fiscal plan has an offsetting effect on that value of shorter- and longer-term debt if the stock of debt with shorter maturity is larger. Extending this result to a finite-period model leads to the conclusion that the optimal term structure must be decreasing in maturity date.

The benefit of using the modified commitment problem is that it allows me to characterize the optimal maturity structure in a multi-period model with various maturities available to the government. First, I solve for the optimal path of fiscal policies. Then I recursively solve for the maturity structure in every period that renders the ex ante plan incentive-compatible for future governments.

In a quantitative exercise, I consider a six-period model. In the initial period the government is hit with a high taste parameter for government spending and optimally chooses the budget deficit. This budget deficit must be financed by bonds with different maturity dates. I show that the optimal maturity structure is not only tilted toward the short end, but also has a decaying profile: Total payments owed by the government decrease as the maturity date goes up. Moreover, this downward-sloping structure persists in subsequent periods, and future governments issue positive debt with all remaining maturities.

My analysis implies that the data on the maturity structure of developed economies is broadly consistent with normative analysis of the optimal debt policy under lack

of commitment and opportunity to default. According to this model, lengthening government debt maturity would cause an increase in long-term risk-free rates and default probabilities, as such term restructuring would incentivize future governments to over-borrow compared to the ex ante optimal policy.

Related Literature

As already mentioned, the paper bridges the gap between two literatures that study lack of commitment due to risk-free rates manipulation and risk premiums manipulation in isolation. I build on the work of Aguiar, Amador, Hopenhayn and Werning (2016) by introducing endogenous risk-free interest rates as in Lucas and Stokey (1983).

This paper also relates to the literature that investigates time consistency of fiscal *and* monetary policy. Alvarez et al. (2004) show that Ramsey policy can be made time consistent under the Friedman rule, i.e., zero nominal interest rate is optimal. Persson et al. (2006) argue that time consistency can be achieved by structuring government nominal and indexed debt in an environment where positive nominal interest rates are optimal. In my paper, the focus is on the option of outright government default which is missing from the discussed studies, however, nominal debt and, hence, government's ability to inflate away debt is absent in my paper. I find that the fiscal policy is time-consistent in a weaker sense, as discussed in Aguiar et al. (2016): A government follows an optimal sequence of fiscal policy decisions conditional on no prior default.

Maturity structure can be also used to hedge a government against fiscal shocks. Angeletos (2002) shows that in an environment with perfect commitment but incomplete markets, state-contingent debt can be replicated by maturity structure of non-contingent debt providing complete insurance to the government. According to quantitative exercises discussed in Buera and Nicolini (2004), such an insurance

requires very large debt positions relative to GDP. However, Debortoli et al. (2017) show that such large debt positions are not sustainable in an environment with lack of commitment as a government has an incentive to distort risk-free interest rates to alter the value of outstanding debt. Moreover, the authors find that the optimal maturity structure is approximately flat because minimizing the costs associated with the lack of commitment is quantitatively much more important than minimizing the costs associated with the lack of insurance. The latter conclusion rationalizes the focus of the paper on the commitment problem and abstraction from hedging motive by setting deterministic fiscal shocks.

Maturity has been studied in international quantitative sovereign debt models. Aguiar and Gopinath (2006) were among the first to present a quantitative model with endogenous default decision in an environment with incomplete markets, as in the seminal paper by Eaton and Gersovitz (1981). Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) find that exogenously lengthening debt maturity by introducing a consol bond with a decaying coupon rate improves quantitative fit of such models. Arellano and Ramanarayanan (2012) extend their framework by allowing a sovereign to choose between consol bonds with different decaying rates, and the authors show that average maturity shortens in an event of a crisis. Short-term debt in these models minimizes an incentive to dilute the value of longer-term debt, while long-term debt serves as a hedging against income shocks. In these models, maturity structure of debt has a decaying profile by construction, while in my model I show that such debt structure is optimal. However, in contrast to the aforementioned studies, the role of long-term debt in this paper is to minimize risk-free interest rate distortions, while hedging motive is absent due to deterministic fiscal shocks and constant endowment.

Open economy and corporate finance literature often emphasizes the disciplining role of short-term debt. Jeanne (2009) demonstrates that short-term debt can

incentivize a government to pursue a creditor-friendly policy as debt is rolled-over conditional on policy implementation. In Calomiris and Kahn (1991) and Diamond and Rajan (1991) short-term debt provides a creditor an option to liquidate project. In this model, lenders are atomistic and cannot directly affect government's decisions, instead, a time-inconsistent government uses debt maturity to discipline itself in the future.

1.2 A Finite-period Model without Fiscal Shocks

In this section, I describe a model with the two sources of time inconsistency: manipulation of the risk-free rate and debt dilution due to the probability of default. The economy is closed and consists of a government and a unit mass of atomistic households. The time is discrete and finite $t = 0, 1, \dots, T$.

Preferences and Endowment. A representative household values private consumption and government spending:

$$\mathbb{E} \sum_{t=0}^T \beta^t (u(c_t) + \theta_t \omega(g_t)) \quad (1.1)$$

where u and ω are continuously differentiable, strictly increasing, concave functions and $\beta \in (0, 1]$ is the discount factor. $\theta_t > 0$ represents taste parameter for public spending. Larger θ implies higher marginal utility of government expenditures and, hence, households would prefer more resources to be spent on public goods. In the basic model all θ_t are deterministic, known at date 0 and $\theta_t \in [\theta_{min}, \theta_{max}] \forall t$. The government is benevolent and shares the same preferences.

There is no capital in the economy. Each period a representative household is endowed with $1 - \tau$ units of consumption and the government is endowed with τ units of consumption. Every period the resource constraint has to be satisfied

$$c_t + g_t = 1 \forall t \tag{1.2}$$

I make the following assumptions about u and ω :

Assumption 1. $\theta_{min}\omega'(\tau) \geq u'(1 - \tau)$.

Assumption 1 ensures that the government has an incentive to raise funds from households. Moreover, it implies that government utility is strictly decreasing in the budget surplus if the latter is positive.

Bond Markets and Default. The government can borrow from households. I assume that state-contingent bonds are not available, and the set of financial instruments is limited to bonds with different maturities. Define by b_t^{t+k} the government debt held by a household that is issued at date t and promises to pay one unit of consumption at $t + k$ and let q_t^{t+k} be the price of the bond. Without loss of generality the government rebalances its portfolio each period, i.e., it buys back all the outstanding debt and issues new debt at all maturities.

The government can default on its debt and the value of default is stochastic. Partial default is not allowed. Following Aguiar et al. (2016), I assume that in every period the government has an outside option V_t^{def} that can be achieved by default. V_t^{def} is defined as $V_t^{def} = \frac{1-\beta^{T-t+1}}{1-\beta} \cdot v_t^d$, where v_t^d is drawn from continuous distribution F that has bounded support $[v_{min}; v_{max}]$. I make the following assumptions about the outside option.

Assumption 2. *Outside option:*

- (i) $v_{max} < u(1 - \tau) + \theta_{min}\omega(\tau)$;
- (ii) $\exists g_{min} > 0$: $v_{min} > u(1 - g_{min}) + \theta_{min}\omega(g_{min})$;

- (iii) F is strictly increasing on (v_{min}, v_{max}) and $f(v_{max}) = 0$;
- (iv) v_t^d is independent across time and independent of debt portfolio.

Restriction (i) ensures that the government will never choose an outside option if debt positions are zero. In addition, it guarantees that some positive level of debt can be sustained in equilibrium. Restriction (ii) implies that the government always defaults if the debt position is high enough and government spending is sufficiently low. Assumption (iii) allows to avoid kinks in the pricing functions which ensures that the equilibrium can be characterized by first-order necessary conditions. The assumption of independence in (iv) is made to abstract from hedging motives.

One way to justify such a structure of outside options is to assume that default excludes governments from bond markets and incurs endowment cost χ in every period. The value of default is defined as $V_t^{def} = \frac{1-\beta^{T-t+1}}{1-\beta} \cdot (u(1-\tau) + \omega(\tau - \chi))$. Then it is possible to find a stochastic endowment cost $\chi \sim F_\chi$, $\chi \in [\chi_{min}; \chi_{max}]$ that defines the outside value objects $(v_{min}, v_{max}$ and F): $v_{min} = u(1-\tau) + \omega(\tau - \chi_{max})$, $v_{max} = u(1-\tau) + \omega(\tau - \chi_{min})$ and $F(v) = 1 - F_\chi(\chi(v))$ where $\chi(v)$ satisfies $v = u(1-\tau) + \omega(\tau - \chi(v))$.¹

Timing and Government Problem. At the beginning of every period, the government whether or not to default on its debt. If it defaults, it receives the outside option value V_t^{def} . Otherwise, the government sets government expenditures, buys back existing debt and issues new debt. The default decision precedes any fiscal decisions, and the government is not allowed to default until the beginning of the next period once new debt has been issued. This timing rules out the possibility of self-fulfilling debt crises, as discussed by Cole and Kehoe (2000).

¹Alternatively, one can assume that default implies utility loss ξ so that $V_t^{def} = \frac{1-\beta^{T-t+1}}{1-\beta} \cdot (u(1-\tau) + \omega(\tau) - \xi)$. Then analogically to χ shock, it is straightforward to define $\xi \in [\xi_{min}, \xi_{max}]$, $\xi \sim F_\xi$ that leads to the outside value objects.

I focus on a Markov perfect competitive equilibrium in which the government makes decisions sequentially as functions of payoff-relevant variables: the outstanding bond holdings and period t . Denote by $\mathbf{b}_t = (b_t^{t+1}, \dots, b_t^T)$ the vector of bond holdings issued at period t where T stands for the index of the final period. The size of the vector decreases by one every period and reduces to zero in the very last period. Let $\mathbf{q}_t = (q_t^{t+1}, \dots, q_t^T)$ be the vector of corresponding bond prices.

To simplify notation, it is useful to define the contingent budget surplus as the difference between endowment of and spending by the government if it does not default:

$$s_t = \tau - g_t$$

Consumption is then defined as $c_t = 1 - \tau + s_t$ and government spending is $g_t = \tau - s_t$. Therefore, setting contingent budget surpluses is equivalent to choosing government expenditures. Conditional on no default the budget constraint in every period satisfies:

$$s_t + \mathbf{q}_t(s_t, \mathbf{b}_t) \cdot \mathbf{b}_t \geq (1, \mathbf{q}_t(s_t, \mathbf{b}_t)) \cdot \mathbf{b}_{t-1} \quad (1.3)$$

Let $V_t(\mathbf{b}_{t-1})$ be the value of the government if it does not prefer the outside option V_t^{def} . Then at any $t < T$ it can be defined recursively as

$$V_t(\mathbf{b}_{t-1}) = \max_{s_t, \mathbf{b}_t} \left\{ u(1 - \tau + s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \left\{ V_{t+1}(\mathbf{b}_t); V_{t+1}^{def} \right\} \right\} \quad (1.4)$$

subject to $s_t \in (-(1 - \tau), \tau)$ and the budget constraint (1.3)

At $t = T$, if government does not default, it just pays the outstanding debt b_{T-1}^T and $V_T(b_{T-1}^T) = u(1 - \tau + b_{T-1}^T) + \theta_T \omega(\tau - b_{T-1}^T)$. Denote by $\rho_t(\mathbf{b}_{t-1}) = \{s_t^*(\mathbf{b}_{t-1}), \mathbf{b}_t^*(\mathbf{b}_{t-1})\}$ the optimal government fiscal and debt policies, conditional on no default at t .

Household Optimization and Bond Prices. In any competitive equilibrium, household optimality conditions must be satisfied. A representative household takes into account the future government policies that are reflected in risk-free interest rates and risk premiums. The price of a bond that matures in $k \geq 1$ periods can be defined recursively as

$$q_t^{t+k}(s_t, \mathbf{b}_t) = \beta \frac{u'(1 - \tau + s_{t+1}^*(\mathbf{b}_t))}{u'(1 - \tau + s_t)} \cdot (1 - \pi_{t+1}(\mathbf{b}_t)) \cdot q_{t+1}^{t+k}(\rho_{t+1}(\mathbf{b}_t)) \quad (1.5)$$

where $\pi_{t+1}(\mathbf{b}_t)$ defines the probability of default in the next period:

$$\pi_{t+1}(\mathbf{b}_t) = \text{Prob}(V_{t+1}^{def} > V_{t+1}(\mathbf{b}_t)) = 1 - F(V_{t+1}(\mathbf{b}_t) \cdot \frac{1 - \beta}{1 - \beta^{T-t}}) \quad (1.6)$$

Definition of Markov Perfect Competitive Equilibrium. The Markov Perfect Competitive Equilibrium of the economy consists of the value function $V_t(\mathbf{b}_{t-1})$, the fiscal policy function $\rho_t(\mathbf{b}_{t-1})$ and the pricing function $\mathbf{q}_t(s_t, \mathbf{b}_t)$ such that:

- (i) the value function $V_t(\mathbf{b}_{t-1})$ solves the Bellman equation (1.4) given the fiscal policy function $\rho_t(\mathbf{b}_{t-1})$ and the pricing function $\mathbf{q}_t(s_t, \mathbf{b}_t)$;
- (ii) the fiscal policy function $\rho_t(\mathbf{b}_{t-1})$ maximizes the right-hand side of (1.4) subject to the budget constraint (1.3), taking into account the pricing function $\mathbf{q}_t(s_t, \mathbf{b}_t)$;
- (iii) the pricing function $\mathbf{q}_t(s_t, \mathbf{b}_t)$ satisfies the first-order condition of household utility maximization (1.5) given the fiscal policy function $\rho_t(\mathbf{b}_{t-1})$.

1.3 Time Consistency of a Markov Perfect Competitive Equilibrium

In this section, I show that the time consistency of fiscal policy carries over in environments in which interest rates reflect both the default probability, as in Aguiar et al. (2016), and the endogenous marginal rate of substitution between present and future consumption, as in Lucas and Stokey (1983).

Lucas and Stokey (1983) argue that if debt commitments are binding, then in an environment with no capital the discretionary fiscal policy is time consistent. However, they show that time consistency does not carry over in a monetary economy, in which future governments can inflate away the value of outstanding debt. Analogous conclusion applies to the model studied in this paper: If default risk is positive, it is reflected in bond prices and, hence, the first-best allocation is generally not attainable for a Markov perfect competitive equilibrium. Nevertheless, the optimal fiscal policy can still be characterized by considering the modified commitment problem in which fiscal commitments are binding but debt commitments are not. Then the fiscal policy is “time consistent,” in a sense that the sequential government sticks to the ex ante optimal fiscal plan and follows it as long as the government does not default.

1.3.1 The Modified Commitment Problem

To characterize the optimal maturity debt structure of a government that cannot commit (a Markov government), it is useful to consider the following planning problem. The government (a planner) can commit to fiscal policies conditional on sequential default decisions not being preferred. In other words, at date 0 the planner simultaneously makes fiscal decisions for all periods, and it can promise to follow the plan; however, it defaults whenever the value of the outside option is higher than the value

of pursuing the fiscal plan. I call this the modified commitment problem due to the planner's inability to commit to pay its debt.

Suppose that the planner does not opt to default at the beginning of $t = 0$. Then the planner sets fiscal policies for current and all future periods. Define by a fiscal plan $\mathbf{s}_t = (s_t, s_{t+1}, \dots, s_T)$ a sequence of contingent budget surpluses from period t to T . Let $W_t(\mathbf{s}_t)$ be the value of fiscal plan $\mathbf{s}_t$, that is defined recursively as

$$W_t(\mathbf{s}_t) = u(1 - \tau + s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \{W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def}\} \quad (1.7)$$

Any fiscal plan $\mathbf{s}_0$ uniquely determines bond prices. Iterating equation (1.5) forward we can derive the bond prices at the initial period:

$$q_0^t(\mathbf{s}_0) = \beta^t \frac{u'(1 - \tau + s_t)}{u'(1 - \tau + s_0)} \cdot Pr_0^t(\mathbf{s}_0) \quad (1.8)$$

where $Pr_0^t(\mathbf{s}_0)$ defines the probability of repaying debt issued at date 0, which matures at t , i.e., the probability that the planner does not default at dates 1, 2, ..., t with $Pr_0^0(\mathbf{s}_0) = 1$:

$$Pr_0^t(\mathbf{s}_0) = \prod_{k=1}^t \text{Prob}(W_k(\mathbf{s}_k) \geq V_k^{def}) = \prod_{k=1}^t F(W_k(\mathbf{s}_k) \cdot \frac{1 - \beta}{1 - \beta^{T-k+1}}) \quad (1.9)$$

Any fiscal plan $\mathbf{s}_0$ must satisfy the dynamic budget constraint:

$$s_0 + \sum_{k=1}^T \beta^k \frac{u'(1 - \tau + s_k)}{u'(1 - \tau + s_0)} \cdot Pr_0^k(\mathbf{s}_0) \cdot s_k \geq b_{-1}^0 + \sum_{k=1}^T \beta^k \frac{u'(1 - \tau + s_k)}{u'(1 - \tau + s_0)} \cdot Pr_0^k(\mathbf{s}_0) \cdot b_{-1}^k$$

or equivalently

$$\sum_{k=0}^T \beta^k u'(1 - \tau + s_k) \cdot Pr_0^k(\mathbf{s}_0) \cdot s_k \geq \sum_{k=0}^T \beta^k u'(1 - \tau + s_k) \cdot Pr_0^k(\mathbf{s}_0) \cdot b_{-1}^k \quad (1.10)$$

The left-hand side represents the present value of contingent budget surpluses (adjusted by $u'(1 - \tau + s_0)$), while the right-hand side is the market value of outstanding debt, i.e., any outstanding debt must be financed by future budget surpluses.

Let $\hat{V}_0(\mathbf{b}_{-1})$ be the value of the planner at the initial period if the planner prefers not to default:

$$\hat{V}_0(\mathbf{b}_{-1}) = \max_{\{\mathbf{s}_0\}} W_0(\mathbf{s}_0) \quad (1.11)$$

subject to $s_t \in (-(1 - \tau), \tau)$, $\forall t$ and the dynamic budget constraint (1.10)

Therefore, the allocation associated with the modified commitment problem consists of the fiscal plan $\hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ and the value function of the planner $\hat{V}_0(\mathbf{b}_{-1})$ such that:

- (i) the fiscal plan $\hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ maximizes (1.11) subject to the dynamic budget constraint (1.10);
- (ii) the value function $\hat{V}_0(\mathbf{b}_{-1})$ satisfies (1.11).

Importantly, the maturity structure is completely irrelevant for the planner. As long as the government can commit to the sequence of budget surpluses, default probabilities and risk-free interest rates remain constant. Therefore, bond prices do not change and there are infinitely many ways to implement the allocation, with multiple maturities available every period. This does not apply to the Markov

government, as the inherited maturity structure affects government decisions in future periods.

Characterization of the Modified Commitment Problem

Throughout the paper, I will assume that the solution to the modified commitment problem (i) is interior, (ii) can be described by first-order necessary conditions, and (iii) the budget constraint holds with equality. Assumption 2 (iii) guarantees that the value and pricing functions are continuously differentiable.

Define by $B_0(\mathbf{s}_0, \mathbf{b}_{-1})$ the right-hand side of (1.10), i.e. the market value of outstanding debt $\mathbf{b}_{-1}$ at date 0 if the planner pursues fiscal plan $\mathbf{s}_0$. Similarly, let $S_0(\mathbf{s}_0)$ define the market value of budget surpluses (the left-hand side of (1.10)):

$$B_0(\mathbf{s}_0, \mathbf{b}_{-1}) = \sum_{k=0}^T \beta^k u'(1 - \tau + s_k) \cdot Pr_0^k(\mathbf{s}_0) \cdot b_{-1}^k,$$

$$S_0(\mathbf{s}_0) = \sum_{k=0}^T \beta^k u'(1 - \tau + s_k) \cdot Pr_0^k(\mathbf{s}_0) \cdot s_k$$

Let $MRS_{t,t+k}^0(\mathbf{s}_0)$ be the marginal rate of substitution between contingent budget surpluses at period t and $t+k$ from the perspective of the planner at period 0, $0 \leq t < t+k$. The marginal rate of substitution shows by how much the budget surplus at period t can be decreased if the planner increases the budget surplus at $t+k$ by one unit, keeping value in (1.7) constant. Note that the marginal rate of substitution depends only on fiscal plan $\mathbf{s}_0$ and does not depend on the initial debt composition $\mathbf{b}_{-1}$:

$$MRS_{t,t+k}^0(\mathbf{s}_0) = - \frac{\Delta s_t}{\Delta s_{t+k}} \Big|_{\text{keeping } W_0(s_0^T) \text{ constant}} =$$

$$= \frac{\frac{\partial}{\partial s_{t+k}} W_0(\mathbf{s}_0)}{\frac{\partial}{\partial s_t} W_0(\mathbf{s}_0)} \quad (1.12)$$

Define by $MRT_{t,t+k}^0(\mathbf{s}_0, \mathbf{b}_{-1})$ the marginal rate of transformation between contingent budget surpluses at period t and $t+k$, i.e. the rate at which the planner at period $0 \leq t < t+k$ can transfer budget surpluses from $t+k$ to t , keeping its budget constraint satisfied with equality:

$$\begin{aligned} MRT_{t,t+k}^0(\mathbf{s}_0, \mathbf{b}_{-1}) &= -\frac{\Delta s_t}{\Delta s_{t+k}} \Big|_{\text{budget constraint (10) at } t=0 \text{ is satisfied with equality}} = \\ &= \frac{\frac{\partial}{\partial s_{t+k}} (S_0(\mathbf{s}_0) - B_0(\mathbf{s}_0, \mathbf{b}_{-1}))}{\frac{\partial}{\partial s_t} (S_0(\mathbf{s}_0) - B_0(\mathbf{s}_0, \mathbf{b}_{-1}))} \end{aligned} \quad (1.13)$$

In equilibrium, marginal rates of substitution must be equal to the corresponding marginal rates of transformation, i.e. $MRS_{t,t+1}^0(\mathbf{s}_0) = MRT_{t,t+1}^0(\mathbf{s}_0, \mathbf{b}_{-1}) \quad \forall t = 0, 1, \dots, T-1$. Therefore, the optimal fiscal plan $\hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ solves

$$\begin{cases} MRS_{t,t+1}^0(\mathbf{s}_0) = MRT_{t,t+1}^0(\mathbf{s}_0, \mathbf{b}_{-1}) & \text{for } t = 0, \dots, T-1 \\ S_0(\mathbf{s}_0) = B_0(\mathbf{s}_0, \mathbf{b}_{-1}) \end{cases} \quad (1.14)$$

Intuitively, any marginal deviation from the optimal fiscal plan keeping the budget constraint (1.10) satisfied with equality should not lead to an increase in the planner's value.

1.3.2 Equivalence of the Markov and Modified Commitment Problems

In this subsection, I demonstrate that the Markov government can replicate the planner's allocation. First of all, note that the value of the modified commitment problem is an upper bound for the value of the Markov problem.

Lemma 1. $\hat{V}_0(\mathbf{b}_{-1}) \geq V_0(\mathbf{b}_{-1}) \forall T \geq 1.$

The proof of Lemma 1 is simple. Any Markov allocation has to satisfy budget constraint (1.3) at every period $t = 0, 1, \dots, T$. Combining all sequential budget constraints (1.3) yields dynamic budget constraint (1.10). Therefore, any Markov allocation is feasible to the planner and can be replicated by the modified commitment contract. The allocation associated with the modified commitment problem is, thus, at least as good as the Markov perfect competitive equilibrium allocation. Lemma 1 also implies that if the Markov government can replicate the planner's optimal allocation, then this allocation must be optimal for the Markov government.

At date 0 the Markov government can choose budget deficit as under the modified commitment contract. However, it cannot commit to the sequence of ex ante optimal budget surpluses, as future decisions will be made by the future governments and their actions depend on the maturity of outstanding debt.

Let $\mathbf{s}_0^*(\mathbf{b}_{-1}) = (s_0^*, s_1^*, \dots, s_T^*)$ be the sequence of contingent budget surpluses associated with the Markov perfect competitive equilibrium and let $\hat{\mathbf{s}}_0(\mathbf{b}_{-1}) = (\hat{s}_0, \hat{s}_1, \dots, \hat{s}_T)$ be the optimal fiscal plan associated with the modified commitment problem. Proposition 1 states that the Markov government achieves exactly the same allocation as the planner.

Proposition 1. *Suppose the solution to (1.11) is interior and given by (1.14). Then $\mathbf{s}_0^*(\mathbf{b}_{-1}) = \hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ and $V_0(\mathbf{b}_{-1}) = \hat{V}_0(\mathbf{b}_{-1}) \forall T \geq 1$.*

The proof of Proposition 1 is inductive. Notice that Proposition 1 holds trivially for $T = 1$. If default is not optimal at the beginning of the last period, a government only repays the outstanding debt. Moreover, Proposition 1 is then satisfied for $T = 2$, as no fiscal decision is made in the last period. Therefore, the Markov government and the planner's problem are identical for $T = 2$. To complete the proof, it is sufficient to show that if Proposition 1 holds for a T period model then it is also true for a $T + 1$ period model.

Consider a T period economy with time indexed as $t = 1, 2, \dots, T$. Suppose that Proposition 1 holds for T . Define by $\mathbf{s}_1^*(\mathbf{b}_0) = (s_1^*(\mathbf{b}_0), \dots, s_T^*(\mathbf{b}_0))$ the allocation under Markov Perfect Competitive Equilibrium and by $\hat{\mathbf{s}}_1(1, \mathbf{b}_0) = (\hat{s}_1(1, \mathbf{b}_0), \dots, \hat{s}_T(1, \mathbf{b}_0))$ the optimal fiscal plan made by a planner at period 1 given outstanding debt $\mathbf{b}_0$. The Markov allocation is, therefore, equivalent to the fiscal plan made at $t = 1$, i.e. $\mathbf{s}_1^*(\mathbf{b}_0) = \hat{\mathbf{s}}_1(1, \mathbf{b}_0)$, moreover, $V_1(\mathbf{b}_0) = \hat{V}_1(\mathbf{b}_0) \equiv W_1(\hat{\mathbf{s}}_1(1, \mathbf{b}_0))$.

Now consider a $T + 1$ period model with $t = 0, 1, \dots, T$. The Markov problem at period 0, conditional on default not being optimal at $t = 0$, can be rewritten as follows:

$$V_0(\mathbf{b}_{-1}) = \max_{\{s_0, \mathbf{b}_0\}} \left\{ u(1 - \tau + s_0) + \theta_0 \omega(\tau - s_0) + \beta \cdot \mathbb{E} \max \left\{ W_1(\hat{\mathbf{s}}_1(1, \mathbf{b}_0)); V_1^{def} \right\} \right\} \quad (1.15)$$

subject to $s_0 \in (-(1 - \tau), \tau)$ and

$$u'(1-\tau+s_0) \cdot s_0 = u'(1-\tau+s_0) \cdot b_{-1}^0 + \sum_{k=1}^T \beta^k u'(1-\tau+\hat{s}_k(1, \mathbf{b}_0)) \cdot Pr_0^k(\hat{\mathbf{s}}_1(1, \mathbf{b}_0)) \cdot (b_{-1}^k - b_0^k) \quad (1.16)$$

The government optimally chooses the budget deficit at date 0 that equals to the change in the market value of debt. Bond prices depend on the fiscal plan set by the next period planner and the fiscal plan in turn is affected by the maturity choice of the current period government. Lemma 2 states that the Markov government can use maturity to discipline the future policy maker.

Lemma 2. *Suppose the solution to (1.11) is interior and given by (1.14). Then there exists a (generally unique) maturity structure $\mathbf{b}_0^*$ such that if $s_0 = \hat{s}_0(\mathbf{b}_{-1})$ budget constraint (13) is satisfied and $\hat{\mathbf{s}}_1(1, \mathbf{b}_0^*) = \hat{\mathbf{s}}_1(\mathbf{b}_{-1})$.*

Therefore, any optimal fiscal plan $\hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ can be replicated by the Markov government. The government can structure its debt portfolio such that the next period planner chooses exactly the same allocation as the planner at period 0. Every period, the number of maturities is sufficient so that all first-order necessary conditions and the budget constraint of the next-period planner are satisfied by the ex ante optimal fiscal plan. This completes the proof of Proposition 1.

1.3.3 The Optimal Maturity Structure

To better understand the intuition behind Lemma 2 and the role of maturity, first, note that for a given budget surplus at period 0 $\hat{s}_0$, the planners at period 0 and period 1 maximize exactly the same objective function $W_1(\mathbf{s}_1)$. Therefore, the marginal rates of substitution given by (1.12) are identical for the planners. Lemma A.2 in Appendix A formally proves this statement.

However, the marginal rates of transformation as defined by (1.13) are generally different. The reason is that by adjusting the fiscal plan, the government at period 1 not only alters the market value of contingent budget surpluses but also affects the market value of outstanding debt, which is generally different from the outstanding debt in preceding period. To see this, rewrite the budget constraint at $t = 0$ (1.10) as

$$\sum_{k=1}^T \beta^{k-1} \cdot u'(1 - \tau + s_k) \cdot Pr_1^k(\mathbf{s}_1) \cdot (s_k - b_{-1}^k) = -\frac{\beta}{Pr_0^1(\mathbf{s}_1)} \cdot u'(1 - \tau + s_0) \cdot (s_0 - b_{-1}^0) \quad (1.17)$$

where the left hand-side represents the difference between market values² of budget surpluses and outstanding debt from period 1 to T , and the right-hand side is analogical difference for period 0. As defined in (1.9), $Pr_1^k(\mathbf{s}_1)$ is the probability of repaying debt issued at date 1 which matures at period k , $Pr_0^1(\mathbf{s}_0)$ defines the probability of no default at $t = 1$ given fiscal plan $\mathbf{s}_0$.

Analogously, the budget constraint (1.10) at $t = 1$ can be rewritten as

$$\sum_{k=1}^T \beta^{k-1} \cdot u'(1 - \tau + s_k) \cdot Pr_1^k(\mathbf{s}_1) \cdot (s_k - b_{-1}^k) = \sum_{k=1}^T \beta^{k-1} \cdot u'(1 - \tau + s_k) \cdot Pr_1^k(\mathbf{s}_1) \cdot (b_0^k - b_{-1}^k) \quad (1.18)$$

Note that the left-hand sides of (1.17) and (1.18) are identical. Combining the right-hand sides of the equations yields the budget constraint of Markov government at $t = 0$ (1.16), i.e., the value of the budget deficit at $t = 0$ equals to the difference in the market values of outstanding debt at $t = 0$ and $t = 1$.

Consider a marginal deviation from the optimal plan $\hat{\mathbf{s}}_1 = (\hat{s}_1, \dots, \hat{s}_T)$ keeping $\hat{s}_0$ and $W_1(\hat{\mathbf{s}}_1)$ constant. The planner's optimality conditions imply that budget constraint (1.17) remains satisfied with equality. Importantly, the right-hand side of

²adjusted by $\frac{\beta}{Pr_0^1(\mathbf{s}_1)} \cdot u'(1 - \tau + s_0)$

equation (1.17) is unaffected, because the no default probability $Pr_0^1(\hat{\mathbf{s}}_1) = F(W_1(\hat{\mathbf{s}}_1) \cdot \frac{1-\beta}{1-\beta^{T-1}})$ depends exclusively on $W_1(\hat{\mathbf{s}}_1)$, which remains constant. Consequently, the left-hand side of (1.17) must also be unaffected.

Let us now discuss how the same deviation can affect the budget constraint at $t = 1$ (1.18). Note that the left-hand side of (1.18) stays constant because it is identical to the left-hand side of (1.17). However, the right-hand side of (1.18)—the value of net debt issued at period 0—can be manipulated. For example, suppose that the debt is mostly short term. Then a reallocation in budget surpluses that decreases the price of short-term debt at the expense of an increase in the price of long-term debt could decrease the market value of total debt.

The optimal conditions of modified commitment problem at period 1 require that any marginal deviations from $\hat{\mathbf{s}}_1$ that keep $W_1(\hat{\mathbf{s}}_1)$ constant do not allow the government to relax budget constraint (1.18). Consequently, the optimal maturity structure $\mathbf{b}_0^*$ has to be such that any changes in $\hat{\mathbf{s}}_1$ that keep $W_1(\hat{\mathbf{s}}_1)$ constant do not cause a reduction in the value of debt issued at period 1.

Proposition 2. The Optimal Maturity Structure. *Suppose the solution to (1.11) is interior and given by (1.14). Then the optimal maturity structure $\mathbf{b}_0^*(\mathbf{b}_{-1}) = (b_0^1(\mathbf{b}_{-1}), \dots, b_0^T(\mathbf{b}_{-1}))$ satisfies:*

$$\frac{\frac{\partial \beta \cdot u'(1-\tau+\hat{s}_{t+1}) \cdot Pr_t^{t+1}(\hat{\mathbf{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{\frac{\partial u'(1-\tau+\hat{s}_t)}{\partial \hat{s}_t}} \cdot \frac{b_0^{t+1}(\mathbf{b}_{-1}) - b_{-1}^{t+1}}{b_0^t(\mathbf{b}_{-1}) - b_{-1}^t} = MRS_{t,t+1}^0(\hat{\mathbf{s}}_t(\mathbf{b}_{-1})) \quad (1.19)$$

for $t = 0, 1, \dots, T-1$, where $\hat{\mathbf{s}}_t(\mathbf{b}_{-1})$ is the optimal fiscal plan and $MRS_{t,t+1}^0(\hat{\mathbf{s}}_t(\mathbf{b}_{-1}))$ is given by (1.12).

The complete proof of Proposition 2 is provided in Appendix A. Intuitively, debt maturity is structured such that any marginal alterations in bond prices cannot reduce the value of debt issued in the initial period. For example, consider an increase in the

budget surplus at $t + 1$, Δs_{t+1} , and a decrease in the budget surplus at t , $-\Delta s_t$, such that government value $W_1(\mathbf{s}_1)$ stays constant. Note that such perturbation does not affect the value of debt maturing before t , because marginal utilities $u'(1 - \tau + s_k)$ for $k < t$ are unchanged. In addition, since $W_k(\mathbf{s}_k)$ for $k < t$ stays constant the default probabilities in periods $k = 0, 1, \dots, t - 1$ are unaffected. Similarly, this deviation does not alter the prices of bonds that mature after $t + 1$.

Thus, a variation in budget surpluses s_t and s_{t+1} keeping $W_1(\mathbf{s}_1)$ constant affects only the market value of debt maturing at t and $t + 1$. It is worth highlighting that the default probability at t is constant because $W_t(\mathbf{s}_t)$ is unchanged. However, as $W_{t+1}(\mathbf{s}_{t+1})$ depends on s_{t+1} but not on s_t , the probability of default at date $t + 1$ changes. Therefore, the described deviation from a fiscal plan manipulates the risk-free rate of debts maturing at t and $t + 1$ and the risk premium of debt maturing at $t + 1$ but not at t . The latter implies that if default probability is positive the price of debt maturing at a later period is more elastic compared to an obligation with shorter maturity. In other words, a marginal deviation from a path of fiscal policies has a higher impact on longer-term debt.

Along with the elasticities of bond prices, the optimal maturity structure depends on the ex ante optimal fiscal plan that is designed by taking into account the initial period inherited debt and future public spending taste parameters. In Sections 4 and 5, I consider an example with zero initial debt and stationary future taste parameters that allows me to characterize the shape of optimal maturity analytically.

Appendix

Appendix A. Proofs.

Proof of Lemma 1.

Let $\mathbf{s}_0^*(\mathbf{b}_{-1}) = (s_0^*, s_1^*, \dots, s_T^*)$ be the sequence of contingent budget surpluses associated with the Markov perfect competitive equilibrium and $\mathbf{b}_t^*(\mathbf{b}_{t-1})$ be the optimal debt issuances at $t = 0, \dots, T$. $\mathbf{s}_0^*(\mathbf{b}_{-1})$ must satisfy budget constraint (1.3) at every period $t = 0, \dots, T$. Combining budget constraints (1.3) at every period yields the dynamic budget constraint (1.10). It implies that $\mathbf{s}_0^*(\mathbf{b}_{-1})$ is always feasible for the planner. Therefore, the value of a planner is at least as good as the value of a Markov government: $\hat{V}(\mathbf{b}_{-1}) \geq V(\mathbf{b}_{-1})$. ■

Lemma A1. For $1 \leq k \leq T - t$

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = -\beta^k \cdot Pr_t^{t+k}(\mathbf{s}_{t+1}) \cdot (\theta_{t+k} \omega'(\tau - s_{t+k}) - u'(1 - \tau + s_{t+k})).$$

Proof.

Recall that $W_t(\mathbf{s}_t)$ is defined recursively as

$$W_t(\mathbf{s}_t) = u(1 - \tau + s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \left\{ W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def} \right\}.$$

Note that

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_t} = -(\theta_t \omega'(\tau - s_t) - u'(1 - \tau + s_t))$$

because s_t does not affect budget surpluses or default probabilities in future periods. I first show that

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1}(\mathbf{s}_{t+1}) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}$$

for $1 \leq k \leq T - t$ where

$$Pr_t^{t+1}(\mathbf{s}_{t+1}) = F(W_{t+1}(\mathbf{s}_{t+1}) \cdot \frac{1 - \beta}{1 - \beta^{T-t}}).$$

Decompose the last component of $W_t(\mathbf{s}_t)$ as

$$\mathbb{E} \max \{W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def}\} = (1 - Pr_t^{t+1}(\mathbf{s}_{t+1})) \cdot \mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})] + Pr_t^{t+1}(\mathbf{s}_{t+1}) \cdot W_{t+1}(\mathbf{s}_{t+1})$$

where

$$\mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})] = \frac{1}{1 - Pr_t^{t+1}(\mathbf{s}_{t+1})} \cdot \frac{1 - \beta^{T-t}}{1 - \beta} \cdot \int_{v'(\mathbf{s}_{t+1})}^{v_{max}} v dF(v)$$

is the conditional expected value of outside value option, and $v'(\mathbf{s}_{t+1}) = W_{t+1}(\mathbf{s}_{t+1}) \cdot \frac{1 - \beta}{1 - \beta^{T-t}}$ is the minimum value of per-period payoff of the outside option, at which the economy enters the crisis region. Then

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta \cdot \frac{\partial(1 - Pr_t^{t+1}(\mathbf{s}_{t+1})) \cdot \mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})]}{\partial s_{t+k}} + \beta \cdot \frac{\partial Pr_t^{t+1}(\mathbf{s}_{t+1}) \cdot W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}$$

$$\frac{\partial(1 - Pr_t^{t+1}(\mathbf{s}_{t+1})) \cdot \mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})]}{\partial s_{t+1}} = -v'(\mathbf{s}_{t+1}) \cdot f(v'(\mathbf{s}_{t+1})) \cdot \frac{\partial v'(\mathbf{s}_{t+1})}{\partial s_{t+k}} =$$

$$= -\frac{1 - \beta}{1 - \beta^{T-t}} \cdot W_{t+1}(\mathbf{s}_{t+1}) \cdot f\left(W_{t+1}(\mathbf{s}_{t+1}) \cdot \frac{1 - \beta}{1 - \beta^{T-t}}\right) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}},$$

$$\begin{aligned} \frac{\partial Pr_t^{t+1}(\mathbf{s}_{t+1}) \cdot W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}} &= Pr_t^{t+1}(\mathbf{s}_{t+1}) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}} + \\ &+ \frac{1 - \beta}{1 - \beta^{T-t}} \cdot W_{t+1}(\mathbf{s}_{t+1}) \cdot f\left(W_{t+1}(\mathbf{s}_{t+1}) \cdot \frac{1 - \beta}{1 - \beta^{T-t}}\right) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}. \end{aligned}$$

Thus yielding

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1}(\mathbf{s}_{t+1}) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}.$$

Iterating forward leads to

$$\begin{aligned} \frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} &= \beta^k \cdot \prod_{j=1}^k Pr_{t+j-1}^{t+j}(\mathbf{s}_{t+j}) \cdot \frac{\partial W_{t+k}(\mathbf{s}_{t+k})}{\partial s_{t+k}} = \\ &= -\beta^k \cdot Pr_t^{t+k}(\mathbf{s}_t) \cdot (\theta_{t+k} \omega'(\tau - s_{t+k}) - u'(1 - \tau + s_{t+k})). \end{aligned}$$

■

Lemma A.2. *Let $\mathbf{s}_1 = (s_1, \dots, s_T)$. Then $MRS_{1,1+t}^0(\mathbf{s}_1) = MRS_{1,1+t}^1(\mathbf{s}_1)$.*

Proof.

Recall from (1.12) and (1.13) that

$$MRS_{1,1+t}^0(\mathbf{s}_1) = \frac{\frac{\partial W_0((s_0, \mathbf{s}_1))}{\partial s_{1+t}}}{\frac{\partial W_0((s_0, \mathbf{s}_1))}{\partial s_1}} \text{ and } MRS_{1,1+t}^1(\mathbf{s}_1) = \frac{\frac{\partial W_1(\mathbf{s}_1)}{\partial s_{1+t}}}{\frac{\partial W_1(\mathbf{s}_1)}{\partial s_1}}$$

The proof follows from Lemma A.1 which implies $\frac{\partial W_0(s_0, \mathbf{s}_1)}{\partial s_{1+t}} = \beta Pr_0^1(\mathbf{s}_1) \cdot \frac{\partial W_1(\mathbf{s}_1)}{\partial s_{1+t}}$ for $0 \leq t \leq T-1$, hence,

$$\begin{aligned} \frac{\frac{\partial W_0((s_0, \mathbf{s}_1))}{\partial s_{1+t}}}{\frac{\partial W_0((s_0, \mathbf{s}_1))}{\partial s_1}} &= \frac{\beta Pr_0^1(\mathbf{s}_1) \cdot \frac{\partial W_1(\mathbf{s}_1)}{\partial s_{1+t}}}{\beta Pr_0^1(\mathbf{s}_1) \cdot \frac{\partial W_1(\mathbf{s}_1)}{\partial s_1}} = \\ &= \frac{\frac{\partial W_1(\mathbf{s}_1)}{\partial s_{1+t}}}{\frac{\partial W_1(\mathbf{s}_1)}{\partial s_1}}. \end{aligned}$$

■

Proof of Lemma 2.

Let $\hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ be the optimal fiscal plan designed by a planner at period 0 given the outstanding debt $\mathbf{b}_{-1}$. Let $\hat{\mathbf{s}}_1(1, \mathbf{b}_0)$ be the optimal fiscal plan chosen by a planner at period 1 given the outstanding debt $\mathbf{b}_0$.

$\hat{\mathbf{s}}_0(\mathbf{b}_{-1})$ has to satisfy the first-order necessary conditions:

$$\left\{ \begin{array}{l} MRS_{0,1}^0(\mathbf{s}_0) = MRT_{0,1}^0(\mathbf{s}_0, \mathbf{b}_{-1}) \\ MRS_{t,t+1}^0(\mathbf{s}_0) = MRT_{t,t+1}^0(\mathbf{s}_0, \mathbf{b}_{-1}) \quad \text{for } t = 1, \dots, T-1 \\ S_0(\mathbf{s}_0) = B_0(\mathbf{s}_0, \mathbf{b}_{-1}) \end{array} \right.$$

where $MRS_{t,t+1}^0(\mathbf{s}_0)$ and $MRT_{t,t+1}^0(\mathbf{s}_0)$ are as defined by (1.12) and (1.13).

In turn, $\hat{\mathbf{s}}_1(1, \mathbf{b}_0)$ satisfies the first-order necessary conditions at period 1:

$$\begin{cases} MRS_{t,t+1}^1(\mathbf{s}_1) = MRT_{t,t+1}^1(\mathbf{s}_1, \mathbf{b}_0) & \text{for } t = 1, \dots, T-1 \\ S_1(\mathbf{s}_1) = B_1(\mathbf{s}_1, \mathbf{b}_0) \end{cases}$$

where $MRS_{t,t+1}^1(\mathbf{s}_0)$ and $MRT_{t,t+1}^1(\mathbf{s}_1)$ are defined analogously to (1.12) and (1.13).

Recall from Lemma A.2 that if $\mathbf{s}_1 \in \mathbf{s}_0$ then $MRS_{t,t+1}^0(\mathbf{s}_0) = MRS_{t,t+1}^1(\mathbf{s}_1)$, $t = 1, \dots, T-1$.

Therefore, if $\mathbf{b}_0$ is such that the following system of equations is satisfied where $\hat{\mathbf{s}}_1(\mathbf{b}_{-1}) \in \hat{\mathbf{s}}_0(\mathbf{b}_{-1})$, i.e., $\hat{\mathbf{s}}_0(\mathbf{b}_{-1}) = (\hat{s}_0(\mathbf{b}_{-1}), \hat{\mathbf{s}}_1(\mathbf{b}_{-1}))$:

$$\begin{cases} MRT_{t,t+1}^0(\hat{\mathbf{s}}_1(\mathbf{b}_{-1}), \mathbf{b}_0) = MRT_{t,t+1}^1(\hat{\mathbf{s}}_1(\mathbf{b}_{-1}), \mathbf{b}_{-1}) & \text{for } t = 1, \dots, T-1 \\ S_1(\hat{\mathbf{s}}_1(\mathbf{b}_{-1})) = B_1(\hat{\mathbf{s}}_1(\mathbf{b}_{-1}), \mathbf{b}_0) \end{cases} \quad (1.20)$$

then the planner chooses $\hat{\mathbf{s}}_1(\mathbf{b}_{-1})$ at $t = 1$ because the plan solves the planner's optimality conditions. Note that this system is linear. To see it, Note that (I set $\hat{\mathbf{s}}_t(\mathbf{b}_{-1}) = \hat{\mathbf{s}}_t$ to simplify notation)

$$MRT_{t,t+1}^0(\hat{\mathbf{s}}_t(\mathbf{b}_{-1}), \mathbf{b}_{-1}) = \frac{\frac{\partial \sum_{k=0}^T \beta^k u'(1-\tau+\hat{s}_k) \cdot Pr_0^k(\hat{s}_0) \cdot (\hat{s}_k - b_{-1}^k)}{\partial \hat{s}_{t+1}}}{\frac{\partial \sum_{k=0}^T \beta^k u'(1-\tau+\hat{s}_k) \cdot Pr_0^k(\hat{s}_0) \cdot (\hat{s}_k - b_{-1}^k)}{\partial \hat{s}_t}} = \beta \cdot Pr_t^{t+1}(\hat{\mathbf{s}}_{t+1}) \times$$

$$\times \frac{u'(1-\tau+\hat{s}_{t+1}) + \left[u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1}) \frac{\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{s}_{t+1})} \right] \cdot (\hat{s}_{t+1} - b_{-1}^{t+1})}{u'(1-\tau+\hat{s}_t) + u''(1-\tau+\hat{s}_t) \cdot (\hat{s}_t - b_{-1}^t)} \quad (1.21)$$

The reason is that changes in budget surpluses at dates t and $t + 1$ keeping the value $W_1(\hat{\mathbf{s}}_1(\mathbf{b}_{-1}))$ constant does not causes changes in default probabilities in all periods except $t + 1$.

Analogously, the MRT for the planner at period 1 is

$$MRT_{t,t+1}^1(\hat{\mathbf{s}}_t(\mathbf{b}_{-1}), \mathbf{b}_0) = \frac{\frac{\partial \sum_{k=1}^T \beta^{k-1} u'(1-\tau+\hat{s}_k) \cdot Pr_1^k(\hat{\mathbf{s}}_1) \cdot (\hat{s}_k - b_0^k)}{\partial \hat{s}_{t+1}}}{\frac{\partial \sum_{k=1}^T \beta^{k-1} u'(1-\tau+\hat{s}_k) \cdot Pr_1^k(\hat{\mathbf{s}}_1) \cdot (\hat{s}_k - b_0^k)}{\partial \hat{s}_t}} = \beta \cdot Pr_t^{t+1}(\hat{\mathbf{s}}_{t+1}) \times$$

$$\times \frac{u'(1-\tau+\hat{s}_{t+1}) + \left[u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1}) \frac{\frac{\partial Pr_t^{t+1}(\hat{\mathbf{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{\mathbf{s}}_{t+1})} \right] \cdot (\hat{s}_{t+1} - b_0^{t+1})}{u'(1-\tau+\hat{s}_t) + u''(1-\tau+\hat{s}_t) \cdot (\hat{s}_t - b_0^t)} \quad (1.22)$$

Equating (1.21) and (1.22), and given that $MRS_{t,t+1}^0(\hat{\mathbf{s}}_t(\mathbf{b}_{-1})) = MRT_{t,t+1}^9(\hat{\mathbf{s}}_t(\mathbf{b}_{-1}), \mathbf{b}_{-1})$ yields (1.19):

$$\frac{\frac{\partial \beta \cdot u'(1-\tau+\hat{s}_{t+1}) \cdot Pr_t^{t+1}(\hat{\mathbf{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{\frac{\partial u'(1-\tau+\hat{s}_t)}{\partial \hat{s}_t}} \cdot \frac{b_0^{t+1}(\mathbf{b}_{-1}) - b_{-1}^{t+1}}{b_0^t(\mathbf{b}_{-1}) - b_{-1}^t} = MRS_{t,t+1}^0(\hat{\mathbf{s}}_t(\mathbf{b}_{-1}))$$

Therefore, the system (1.20) consists of T equations with T unknowns.

Let $\mathbf{b}_0^*(\mathbf{b}_{-1})$ be the solution. Finally, I show that $(\hat{s}_0(\mathbf{b}_{-1}), \mathbf{b}_0^*(\mathbf{b}_{-1}))$ satisfies the budget constraint (1.16). Note that $S_0(\mathbf{s}_0) = u'(1-\tau+s_0) \cdot s_0 + \beta \cdot (1-\pi_1(\mathbf{s}_1)) \cdot S_1(\mathbf{s}_1)$. Plugging in $S_1(\hat{\mathbf{s}}_1(\mathbf{b}_{-1})) = B_1(\hat{\mathbf{s}}_1(\mathbf{b}_{-1}), \mathbf{b}_0^*(\mathbf{b}_{-1}))$ and $S_0(\hat{\mathbf{s}}_0(\mathbf{b}_{-1})) = B_0(\hat{\mathbf{s}}_0(\mathbf{b}_{-1}), \mathbf{b}_{-1})$ yields (1.16). ■

Proof of Proposition 1.

The proof is inductive. Proposition 1 holds trivially for $T = 1$ and $T = 2$.

Suppose the Proposition holds for a T period model with timing $t = 1, \dots, T$, i.e.

$$V_1(\mathbf{b}_0) = \hat{V}_1(\mathbf{b}_0) \text{ and } \mathbf{s}_1^*(\mathbf{b}_0) = \hat{\mathbf{s}}_1(\mathbf{b}_0).$$

Consider a $T + 1$ model. According to Lemma 2 there exists feasible $\mathbf{b}_0$ such that $\hat{\mathbf{s}}_1(\mathbf{b}_0) = \hat{\mathbf{s}}_1(\mathbf{b}_{-1})$, hence, $V_0(\mathbf{b}_{-1}) = \hat{V}_0(\mathbf{b}_{-1})$ and $\mathbf{s}_0^*(\mathbf{b}_{-1}) = \hat{\mathbf{s}}_0(\mathbf{b}_{-1})$. ■

Proof of Proposition 2.

See proof of Lemma 2. Follows from (1.21) and (1.22), and given that $MRS_{t,t+1}^0(\hat{\mathbf{s}}_t(\mathbf{b}_{-1})) = MRT_{t,t+1}^9(\hat{\mathbf{s}}_t(\mathbf{b}_{-1}), \mathbf{b}_{-1})$ in equilibrium.

■

Chapter 2

Decaying Maturity Profile

2.1 A Three-period Analytical Model: Issue More Short-Term Debt

In the previous chapter I demonstrated that maturity is used to discipline future governments. The question that arises is how maturity should be structured if both risk-free interest rates and risk premiums can be altered. In this section I consider a tractable three-period model and show that if default risk is increasing in debt, then the optimal maturity is skewed toward the short-term end.

2.1.1 The Model

Consider a three-period model, as in Section 2. In contrast to the benchmark model, I make one simplifying assumption: the only uncertainty in this model concerns the outside value at period 2; default never occurs in periods 0 or 1. More formally, V_0^{def} and V_1^{def} are deterministic, known at $t = 0$ and $V_0^{def} = V_1^{def} \rightarrow -\infty$.

Notice that this assumption does not affect the implications for the optimal maturity structure because any perturbations in budget surpluses at $t = 1$ and $t = 2$ do not change the default probability at $t = 1$. The Markov government at period 1

makes its fiscal decisions conditional on no prior default. Once the government has decided not to take the outside option at $t = 1$, it cannot manipulate the risk of default in the intermediate period.

Let $V_2^{def} = v^d$ where v^d is as defined in Section 2: it is drawn from continuous distribution F , $v^d \in [v_{min}, v_{max}]$ and Assumption 2 is satisfied. To abstract away from effect of exogenously given inherited debt, I set the initial debt to be equal to zero. Taste parameter for government spending $\theta_1 = \theta_2 = 1$, but $\theta_0 > 1$ to ensure that the government has an incentive to issue some positive debt, which must be repaid at the future dates.

At the beginning of period 2, the government decides whether to default or not. Notice that the budget constraint at $t = 2$ always holds with equality for any $b_1^2 \geq 0$ due to Assumption 1. Hence, conditional on no default the government sets the budget surplus equal to the outstanding debt $s_2 = b_1^2$. The government defaults whenever $V_2^{def} > u(1 - \tau + b_1^2) + \omega(\tau - b_1^2)$.

Given that $V_2^{def} \in [v_{min}, v_{max}]$ and taking into account Assumption 2 (i) and (ii), the debt space can be divided into three regions. Let $\underline{b}$ denote the largest value of debt for which the government never defaults and $\bar{b}$ be the smallest amount of debt for which the government defaults with probability 1, i.e. $\underline{b}$ and $\bar{b}$ satisfy:

$$u(1 - \tau + \underline{b}) + \omega(\tau - \underline{b}) = v_{max}$$

$$u(1 - \tau + \bar{b}) + \omega(\tau - \bar{b}) = v_{min}$$

Then for any $b_1^2 \leq \underline{b}$ default never occurs because $u(1 - \tau + b_1^2) + \omega(\tau - b_1^2) \geq v_{max}$. Analogously, for any $b_1^2 > \bar{b}$ the government always defaults because the value of outside option is higher. For $b_1^2 \in (\underline{b}, \bar{b})$ the default decision depends on the realization of the outside option value shock, and the ex-ante default probability is

$\pi(b_1^2) = 1 - F(u(1 - \tau + b_1^2) + \omega(\tau - b_1^2))$. Following the literature, I refer to these regions as to safe, default and crisis regions respectively.

The intermediate-period government inherits legacy debt¹ (b_0^1, b_0^2) and chooses what fraction of outstanding debt is repaid in the current period (the budget surplus at period 1 s_1) and leaves the rest to be paid in the last period (contingent budget surplus at period 2 $s_2 = b_1^2$), taking into account the default decision of the next-period government. The budget constraint implies that the present value of budget surpluses has to be equal to the present value of outstanding debt. The government's problem at $t = 1$ is

$$V_1(b_0^1, b_0^2) = \max_{s_1, s_2} u(1 - \tau + s_1) + \omega(\tau - s_1) + \int \max\{u(1 - \tau + s_2) + \omega(\tau - s_2); v^d\} dF(v^d) \quad (2.1)$$

subject to $s_0 \in (-(1 - \tau), \tau)$ and

$$u'(1 - \tau + s_1) \cdot (s_1 - b_0^1) + u'(1 - \tau + s_2)(1 - \pi(s_2)) \cdot (s_2 - b_0^2) \geq 0 \quad (2.2)$$

Let $s_1^*(b_0^1, b_0^2)$ and $s_2^*(b_0^1, b_0^2)$ be the optimal policy functions of problem (2.1). The period 0 government takes into account the decisions made by the future governments, and chooses the budget deficit at date 0 $-s_0$, short-term debt b_0^1 and long-term debt b_0^2 to solve:

¹I assume that the debt position (b_0^1, b_0^2) is feasible in a sense that there exist s_1 and s_2 , $s_{1,2} \in (-(1 - \tau), \tau)$ that satisfy the budget constraint (2.2). Notice that if the government at period 0 issues large positions of debt such that it can not be repaid by the future government and default occurs then the bond prices are zeros. The initial period government is then strictly better-off by issuing no debt at all.

$$V_0 = \max_{s_0, b_0^1, b_0^2} u(1 - \tau + s_0) + \theta_0 \omega(\tau - s_0) + V_1(b_0^1, b_0^2) \quad (2.3)$$

subject to $s_0 \in (-(1 - \tau), \tau)$ and

$$u'(1 - \tau + s_0) \cdot s_0 + u'(1 - \tau + s_1^*(b_0^1, b_0^2)) \cdot b_0^1 + u'(1 - \tau + s_2^*(b_0^1, b_0^2))(1 - \pi(s_2^*(b_0^1, b_0^2))) \cdot b_0^2 \geq 0 \quad (2.4)$$

According to Proposition 1, the Markov government can structure debt maturity such that the next government is incentivized to follow the ex ante optimal sequence of contingent budget surpluses. Thus, in order to characterize the optimal maturity structure $\mathbf{b}_0^* = (b_0^{1*}, b_0^{2*})$ I first solve for the optimal fiscal plan of the modified commitment contract as explained in Section 3. Then I find the maturity that makes the optimal fiscal plan incentive-compatible for the next-period government.

2.1.2 Optimal Fiscal Plan

Consider the modified commitment problem, as discussed in Section 3. The planner can commit to a fiscal plan; however, it defaults whenever the outside option exceeds the value of pursuing the fiscal plan. At period 0 the planner chooses the sequence of contingent budget surpluses $\mathbf{s}_0 = (s_0, s_1, s_2)$ that satisfies the dynamic budget constraint:

$$u'(1 - \tau + s_0) \cdot s_0 + u'(1 - \tau + s_1) \cdot s_1 + u'(1 - \tau + s_2) Pr_1^2(s_2) \cdot s_2 \geq 0 \quad (2.5)$$

where $Pr_1^2(s_2) \equiv (1 - \pi(s_2)) = F(u(1 - \tau + s_2) + \omega(\tau - s_2))$ is the probability of no default in period 2.

The intuition for the above constraint is that the budget deficit at period 0 is financed by the budget surplus at period 1 and the contingent budget surplus at period 2. Notice that the maturity structure (b_0^1, b_0^2) does not enter the budget constraint, and, therefore, is completely irrelevant for the planner. As long as the government can commit to the sequence of budget surpluses, bond prices do not change and the budget constraint is satisfied. This does not apply to the Markov government, as the inherited maturity structure affects the government's decision at period 1 and, hence, the prices at period 0.

Let $W_0(\mathbf{s}_0)$ be the expected value of pursuing the fiscal plan $\mathbf{s}_0 = (s_0, s_1, s_2)$:

$$\begin{aligned} W_0(\mathbf{s}_0) = & u(1 - \tau + s_0) + \theta_0 \omega(\tau - s_0) + u(1 - \tau + s_1) + \omega(\tau - s_1) \\ & + \int \max\{u(1 - \tau + s_2) + \omega(\tau - s_2); v^d\} dF(v^d) \end{aligned} \quad (2.6)$$

Then the planner's problem in the initial period is

$$\hat{V}_0 = \max_{\mathbf{s}_0} W_0(\mathbf{s}_0) \quad (2.7)$$

subject to $s_0 \in (-(1 - \tau), \tau)$ and (2.5)

Let $\hat{\mathbf{s}}_0 = (\hat{s}_0, \hat{s}_1, \hat{s}_2)$ be the optimal allocation under the modified commitment problem. As discussed in Section 3.2, any optimal plan satisfies the following optimality condition: the marginal rate of substitution between budget surpluses at period 1 and period 2 ($MRS_{1,2}^0(\mathbf{s}_0)$) equals to the marginal rate of transformation ($MRT_{1,2}^0(\mathbf{s}_0)$):

$$MRS_{1,2}^0(\mathbf{s}_0) = -\frac{\Delta s_1}{\Delta s_2} \Big|_{\text{keeping } W_0(\mathbf{s}_0) \text{ constant}} =$$

$$= \frac{\omega'(\tau - s_2) - u'(1 - \tau + s_2)}{\omega'(\tau - s_1) - u'(1 - \tau + s_1)} Pr_1^2(s_2) \quad (2.8)$$

$$MRT_{1,2}^0(\mathbf{s}_0) = -\frac{\Delta s_1}{\Delta s_2} \Big|_{\text{budget constraint (25) is satisfied with equality}} =$$

$$= \frac{u'(1 - \tau + s_2) + \left[u''(1 - \tau + s_2) - u'(1 - \tau + s_2) \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)} \right] \cdot s_2}{u'(1 - \tau + s_1) + u''(1 - \tau + s_1) \cdot s_1} Pr_1^2(s_2) \quad (2.9)$$

Suppose that a representative household has a constant relative risk aversion function:

Assumption 3. $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C \geq 0$.

Proposition 3 summarizes the optimal allocation in the intermediate and last periods.

Proposition 3. The Optimal Fiscal Plan

- (i) if $\theta_0 \in (1, \bar{\theta}]$, where $\bar{\theta}$ is such that $\hat{s}_2(\bar{\theta}) = \underline{b}$, the planner sets $\hat{s}_1 = \hat{s}_2$;
- (ii) if $\theta_0 > \bar{\theta}$, the optimal fiscal plan implies $\hat{s}_1 > \hat{s}_2$.

The first result of Proposition 3 relates to the safe region. Suppose θ_0 is relatively small so that the government never defaults at the beginning of period 2. Then periods 1 and 2 are identical and the optimal policy for the planner is to smooth consumption and government spending over time. As θ_0 goes up and the government sets a higher budget deficit in period 0, the planner eventually enters the crisis region in which a marginal increase in $\hat{s}_2$ continuously raises the default risk. Then setting equal

budget surpluses is no longer optimal. If $\hat{s}_1 = \hat{s}_2$ the marginal rate of substitution (2.8) equals the probability of repaying debt $Pr_1^2(\hat{s}_2)$. However, the marginal rate of transformation (2.9) is less than $Pr_1^2(\hat{s}_2)$, since a marginal decrease in contingent budget surplus s_2 decreases the default probability. Intuitively, the planner prefers to repay a larger fraction of debt in the intermediate period to decrease default probability in the last period.

2.1.3 The Optimal Maturity Structure

Recall that dynamic budget constraint (2.5) is

$$u'(1 - \tau + s_1) \cdot s_1 + u'(1 - \tau + s_2)Pr_1^2(s_2) \cdot s_2 \geq -u'(1 - \tau + s_0) \cdot s_0 \quad (2.10)$$

and the budget constraint of period 1 (2.2) can be rewritten as

$$u'(1 - \tau + s_1) \cdot s_1 + u'(1 - \tau + s_2)Pr_1^2(s_2) \cdot s_2 \geq u'(1 - \tau + s_1) \cdot b_0^1 + u'(1 - \tau + s_2)Pr_1^2(s_2) \cdot b_0^2 \quad (2.11)$$

The left-hand sides of (2.10) and (2.11) are identical and represent the market value of budget surpluses. The right-hand side of (2.11) is fixed: The planner optimally chooses budget surpluses to finance some optimally chosen budget deficit in the initial period. However, the right-hand side of (2.11), the market value of outstanding debt, can be manipulated by the intermediate government.

Following the discussion in Section 3.3, the optimal debt structure (b_0^{1*}, b_0^{2*}) has to be such that any marginal deviations from the ex ante optimal fiscal plan along the indifference curve cannot decrease the market value of outstanding debt (the left-hand side of (2.11)). More formally, (b_0^{1*}, b_0^{2*}) has to satisfy

$$\frac{\partial}{\partial s_1} u'(1 - \tau + \hat{s}_1) \cdot b_0^{1*} \cdot MRS_{1,2}^0(\hat{s}_0) + \frac{\partial}{\partial s_2} u'(1 - \tau + \hat{s}_2) Pr_1^2(\hat{s}_2) \cdot b_0^{2*} = 0 \text{ or}$$

$$\frac{\frac{\partial}{\partial s_2} u'(1 - \tau + s_2) Pr_1^2(\hat{s}_2)}{\frac{\partial}{\partial s_1} u'(1 - \tau + \hat{s}_1)} \cdot \frac{b_0^{2*}}{b_0^{1*}} = MRS_{1,2}^0(\hat{s}_0) \quad (2.12)$$

The focus of my analysis is on the crisis region, where the government can manipulate both risk-free interest rates and the risk premium, i.e. $\theta_0 > \bar{\theta}$. Proposition 4 states that the optimal maturity is tilted toward the short-term end, i.e., the government issues more short-term debt than long-term debt.

Proposition 4. The Optimal Maturity Structure in a Three-period Model. *If the economy is in the crisis region ($\theta_0 > \bar{\theta}$), the government issues positive short-term and long-term debt, but the term structure is skewed toward the short end:*

$$b_0^{1*} > 0, b_0^{2*} > 0 \text{ and } \frac{b_0^{2*}(\theta_0)}{b_0^{1*}(\theta_0)} \in (0, 1). \text{ In addition, } b_0^{1*} > \hat{s}_1 > \hat{s}_2 > b_0^{2*}.$$

To better understand this result, it is useful to consider two limiting cases. The first corresponds to the safe region ($\pi_t = \pi'_t = 0$): The government can manipulate risk-free interest rates, but it cannot affect the default risk. The second limiting case is $\gamma_C \rightarrow 0$: Lenders are risk-neutral, risk-free interest rates equal β , and cannot be manipulated by the government. However, the default risk is positive, and the government can dilute the value of outstanding debt.

The first limiting case mirrors the findings of Stokey and Lucas (1983). According to Proposition 3, if the probability of default is zero then the government prefers to smooth budget surpluses over time, $s_1 = s_2$. Given the fiscal plan and that the default risk is absent, the price of a one-period bond and a two-period bond are equal to each other. Then the only maturity structure that does not allow the future government to

decrease the value of outstanding debt is flat: $b_0^1 = b_0^2$. Under such a term structure, any marginal change in risk-free rates has an equal effect on the value of outstanding short-term debt and long-term debt, i.e. $u''(1 - \tau + \hat{s}_1) \cdot b_0^1 = u''(1 - \tau + \hat{s}_2) \cdot b_0^2$, and therefore the government has no incentive to deviate from ex-ante optimal allocation. Also, notice that the outstanding debt equals to the corresponding budget surplus, $\hat{s}_1 - b_0^1 = \hat{s}_2 - b_0^2 = 0$, and the government does not have to actively manage its debt in the intermediate period.

The second limiting case reflects the result of Aguiar et al. (2016). Assume that risk-free interest rates are exogenous, and only risk premiums can be manipulated. Therefore, the future government can affect the value of outstanding long-term debt but cannot alter the value of short-term debt. As already mentioned, the short-term default premium cannot be manipulated by the intermediate period government. Therefore, the only maturity that prevents the future government from being able to decrease the market value of outstanding debt is issuing one-period debt only. Whenever $b_0^2 > 0$, the government at period 1 has an incentive to dilute the value of long-term debt by deviating from the ex ante optimal allocation and issuing more b_1^2 . If, on the other hand, $b_0^2 < 0$, the government would want to issue less debt to increase the market value of its long-term savings. More formally, $MRT_{1,2}^0(\mathbf{s}_0) = Pr_1^2(s_2) + \frac{\partial Pr_1^2(s_2)}{\partial s_2} \cdot s_2$ but $MRT_{1,2}^1(\mathbf{s}_1) = Pr_1^2(s_2) + \frac{\partial Pr_1^2(s_2)}{\partial s_2} \cdot (s_2 - b_0^2)$. The optimal maturity structure is then $b_0^{2*} = 0$ implying positive net debt issuance in the intermediate period.

Now suppose that the government can affect both risk-free interest rates and default risk. Recall that the government in the intermediate period can manipulate only the long-term risk premium and cannot manipulate the short-term risk premium. Consequently, the price of long-term bonds is more elastic than the price of short-term debt. The flat maturity is not optimal in this case. If the structure were nearly flat, then marginally increasing the contingent budget surplus at period 2 and decreasing

the budget surplus at period 1 would lead to a larger decrease in the market value of long-term debt compared to an increase in the value of short-term debt. Issuing only short-term debt is also not optimal, as the government can distort the short-term risk-free rate. The higher elasticity of the long-term debt price implies that the stock of short-term debt must be larger than the long-term debt position.

Moreover, the net issuance of debt in the intermediate period is positive, i.e. $b_1^2 = s_2 > b_0^2$. Suppose by contradiction that $b_0^1 = s_1 > s_2 = b_0^2$. Recall from Proposition 2 that the government prefers to repay a larger fraction of debt at period 1 to decrease default probability at period 2, i.e. $s_1 > s_2$. In equilibrium, the marginal rate of substitution is below the default probability: $\frac{\omega'(\tau-s_2)-u'(1-\tau+s_2)}{\omega'(\tau-s_1)-u'(1-\tau+s_1)} Pr_1^2(s_2) < Pr_1^2(s_2)$. However, the ex post marginal rate of transformation is larger than default probability: $\frac{u'(1-\tau+s_2)}{u'(1-\tau+s_1)} Pr_1^2(s_2) > Pr_1^2(s_2)$. Intuitively, if net debt issued is zero at the intermediate period then the Markov government does not internalize the adverse effect of increasing default risk on the value of already issued debt. Therefore, optimal (b_0^{1*}, b_0^{2*}) has to satisfy $b_0^1 > s_1 > s_2 > b_0^2$ concluding that the net debt issued is positive in the intermediate period.

2.2 The Decaying Profile of Maturity

In this section I extend the three-period model to a general $T+1$ period model where default can occur in any period and $\beta \leq 1$. I continue to assume that the initial debt is zero and $\theta_0 > 1$, $\theta_1 = \theta_2 = \dots = \theta_T = 1$ to ensure that the government has an incentive to issue some positive debt that must be repaid at the future dates. I show that the optimal maturity structure is decaying: Debt position is lower if maturity date is later.

Obviously, if $\theta_0 = 1$ then the government does not need to borrow any debt from households and $s_0^* = s_1^* = \dots = s_T^* = 0$. Importantly, the default probability is zero because due to Assumption 2 (i), $u(1 - \tau) + \omega(\tau) > v_{max}$. If $\theta_0 > 1$, then the government prefers to run a budget deficit in the initial period that has to be financed by budget surpluses in future periods. As long as default probability is zero, the government prefers to smooth consumption over time and sets equal budget surplus in periods 1 to T . Let $\hat{s}(\theta_0)$ be the budget surplus in consequent periods as a function of θ_0 .

Similarly to Section 4, there is $\bar{\theta}$ that satisfies $u(1 - \tau + \hat{s}(\bar{\theta})) + \omega(\tau - \hat{s}(\bar{\theta})) = v_{max}$ so if $\theta_0 > \bar{\theta}$, the default probability becomes positive. Let $\hat{\mathbf{s}} = (\hat{s}_0, \hat{s}_1, \dots, \hat{s}_T)$ be the optimal fiscal plan as defined in Section 3. Proposition 5 states that if $\theta_0 > \bar{\theta}$ then the government prefers to repay a larger fraction of debt in earlier periods. Thus, the result in Proposition 3 for the three-period model extends to the model.

Proposition 5. The Optimal Fiscal Plan. *Suppose that $\theta_0 > \bar{\theta}$. Then $\hat{s}_1 > \hat{s}_2 > \dots > \hat{s}_T$.*

This result reflects the trade-off between smoothing consumption over time and decreasing default probability. Evenly distributed budget surpluses are not optimal, because a marginal increase in $\hat{s}_t$ and marginal decrease in $\hat{s}_{t+1}$ would decrease the default probability in period $t + 1$ while having no effect on default risk in period t . This increases the expected value of the outside option conditional on default, while having a negligible effect on government's value conditional on no default. Therefore, the planner has an incentive to frontload net payments to creditors.

Now we are ready to characterize the optimal term structure. Let $\mathbf{b}_0^* = (b_0^1, \dots, b_0^T)$ be the optimal debt issuances by the Markov government at date 0. Proposition 6 establishes that if the economy is in the crisis region, with debt probability being

positive, then the optimal maturity has a decaying profile: Total payments are lower if the maturity date is later.

Proposition 6. The Optimal Maturity Structure. *Suppose that $\theta_0 > \bar{\theta}$. Then $b_0^1 > b_0^2 > \dots > b_0^T > 0$.*

The government structures its debt portfolio so that the future policy maker cannot benefit from altering the market value of debt. Recall from (1.19) that the optimal maturity given no initial debt satisfies

$$\frac{\frac{\partial \beta \cdot u'(1-\tau+\hat{s}_{t+1}) \cdot Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{\frac{\partial u'(1-\tau+\hat{s}_t)}{\partial \hat{s}_t}} \cdot \frac{b_0^{t+1}}{b_0^t} = MRS_{t,t+1}^0(\hat{s}_t) \quad (2.13)$$

The proposition 5 implies that $MRS_{t,t+1}^0(\hat{s}_t) < \beta \cdot Pr_t^{t+1}(\hat{s}_{t+1})$ because $\hat{s}_t > \hat{s}_{t+1}$. Therefore, (2.13) can be rewritten as

$$\frac{u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1}) \cdot \frac{\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{s}_{t+1})}}{u''(1-\tau+\hat{s}_t)} \cdot \frac{b_0^{t+1}}{b_0^t} < 1 \quad (2.14)$$

The numerator on the left-hand side of (2.14) consists of two parts. The first part² corresponds to the change in the risk-free interest rate at date $t+1$. The second part³ represents the change in the risk premium at $t+1$. The denominator is the change in the risk-free interest rate at date t . As already discussed, a marginal perturbation is s_t and s_{t+1} that keeps the value $W_1(\hat{s}_1)$ constant has no effect on default probability at t .

The distribution of debt over maturity is therefore skewed toward the short end. The reason is that a marginal perturbation in the fiscal plan has a larger effect on the

$$\begin{aligned} &^2 u''(1-\tau+\hat{s}_{t+1}) \\ &^3 u'(1-\tau+\hat{s}_{t+1}) \cdot \frac{\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{s}_{t+1})} \end{aligned}$$

price of bonds with longer maturity. Any such deviation affects the long-term default risk, while the effect on the short-term risk premium is negligible, as the government's value of pursuing the fiscal plan is not changed. Therefore, to preserve the value of debt from being manipulated by future governments, the optimal term structure must have a decaying profile: Payments due are decreasing in maturity date.

Importantly, the frontloading result of Proposition 5 is not crucial for the decaying maturity structure. Suppose that $\hat{s}_1 = \hat{s}_2 = \dots = \hat{s}_T = \hat{s}$. Then after simplifying (2.13) the optimal maturity structure satisfies

$$\left(1 + \frac{u'(1 - \tau + \hat{s})}{u''(1 - \tau + \hat{s})} \cdot \frac{\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{s}_{t+1})}\right) \cdot \frac{b_0^{t+1}}{b_0^t} = 1.$$

The maturity scheme is still skewed to the short end because $\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}} < 0$ reflecting the fact that the longer-term debt price is more sensitive to deviations from the fiscal plan.

The optimal maturity structure thus depends on the sensitivity of risk-free interest rates and risk premiums. If risk-free interest rates are almost constant, while default risk changes considerably, then the optimal maturity is very short. At the limit, the government issues only short-term debt because $u''(1 - \tau + \hat{s}_t) = 0$. On the other hand, if a marginal deviation from the fiscal plan has significant impact on risk-free interest rates but negligible effect on the default risk, then the optimal term structure is approximately flat.

2.3 Quantitative Exercise

In this section I numerically solve for the optimal maturity and discuss how it is affected by the debt-to-GDP ratio, risk aversion, and default risk. In addition, the time consistency of fiscal policy allows me to analyze the evolution of debt maturity in a multi-period model with multiple maturities available for the government.

2.3.1 Functional Forms

Throughout this section I assume that conditional on no prior default, the government's per-period payoff is

$$\frac{c^{1-\gamma_C} - 1}{1 - \gamma_C} + \kappa \frac{g^{1-\gamma_G} - 1}{1 - \gamma_G}$$

where $\gamma_C = \gamma_G = 1$, $\tau = 0.25$ and κ satisfies $(1 - \tau)^{-\gamma_C} = \kappa \tau^{-\gamma_G}$. I assume that the outside option V_t^{def} is equivalent to being excluded from the bond market and incurring output loss χ in every period. I model output loss as the proportional decrease in consumption and government spending, i.e. $V_t^{def} = \frac{1-\beta(T-t+1)}{1-\beta} \cdot u^d(\chi)$, where $u^d(\chi)$ is defined as

$$u^d(\chi) = \frac{((1 - \tau) \cdot (1 - \chi))^{1-\gamma_C} - 1}{1 - \gamma_C} + \kappa \frac{(\tau \cdot (1 - \chi))^{1-\gamma_G} - 1}{1 - \gamma_G}$$

Estimates of output drops following default vary vastly in the empirical literature. For example, Aguiar and Gopinath (2006) use 2% output loss in their model, while Hebert and Schreger (2016) find that Argentina's cost of default corresponds to a 9.4% permanent decrease in output. For this exercise I set $\chi_{min} = 0.005$ and $\chi_{max} = 0.095$, i.e., the permanent loss of output varies from 0.5% to 9.5%, and thereby $v_{min} = u^d(\chi_{max})$ and $v_{max} = u^d(\chi_{min})$. The per-period payoff of being in default v^d has a probability density distribution f :

$$f(v) = \alpha_F \cdot (v - v_{min})^2 \cdot (v - v_{max})^2 \text{ for } v \in [v_{min}, v_{max}]$$

where α_F is such that $\int_{v_{min}}^{v_{max}} \alpha_F \cdot (v - v_{min})^2 \cdot (v - v_{max})^2 = 1$. The distribution satisfies $f(v_{min}) = f(v_{max}) = 0$; it also implies a smooth increase in default probability as the economy enters the crisis region.

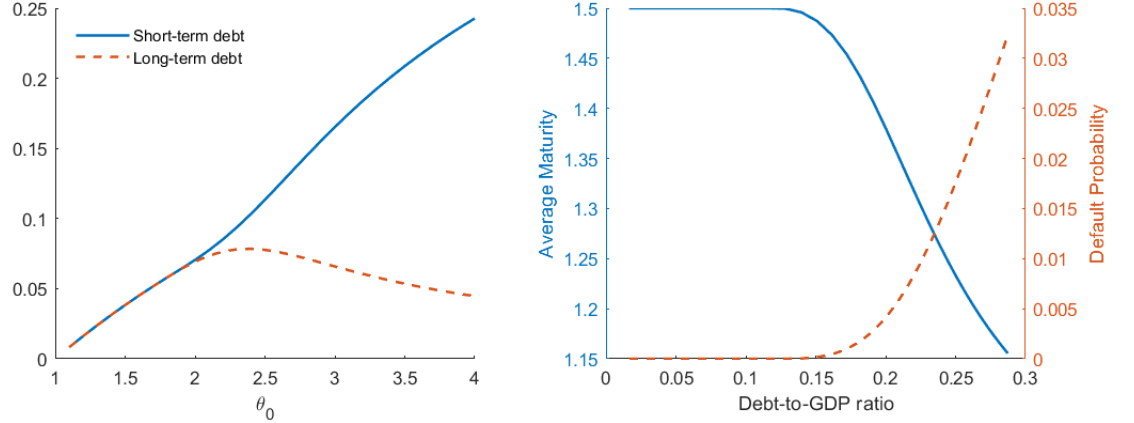
2.3.2 Three-period Model

The Benchmark Model

I first consider the three-period problem as discussed in Section 2.1. The default can occur only at date 2. The discount factor β is 1.

Figure 2.1 displays the optimal maturity structure in the three-period model. The left panel shows the short-term and long-term debt positions as taste parameter θ_0 increases. The right panel shows the average maturity and default probability for different levels of debt-to-GDP ratio. If the debt maturity is flat, then the average maturity is 1.5 years; if debt consists of short-term bonds only, then the average maturity is 1 year.

Figure 2.1: The optimal maturity structure in the three-period model



For a relatively low θ_0 the economy is in the safe region, the default probability is zero. The optimal maturity structure is flat, i.e., the government issues an equal amount of one-period and two-period debt. However, as θ_0 increases and debt-to-GDP ratio goes up the government eventually faces an increasing default risk. For $\theta_0 \approx 2$

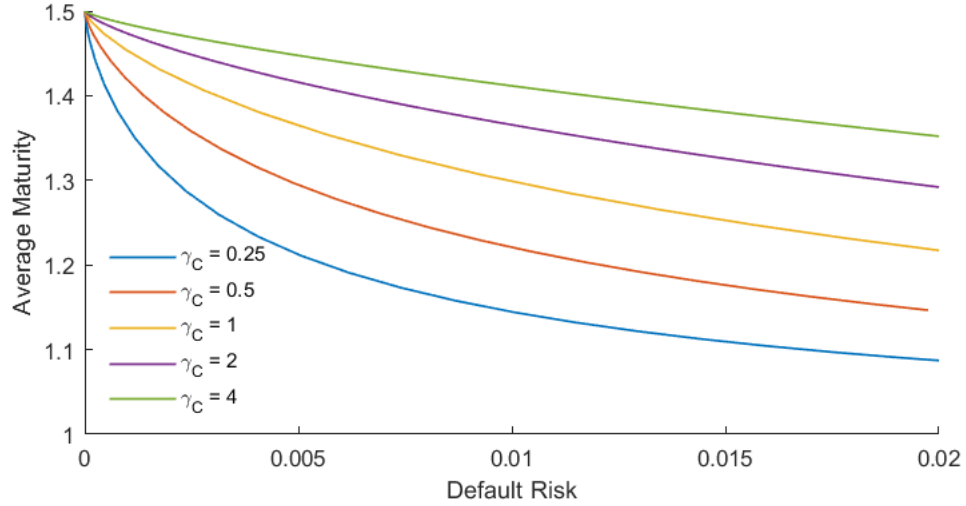
and debt-to-GDP ≈ 0.13 the economy enters the crisis region. The optimal maturity then shortens significantly, and therefore covariates negatively with the default probability. At a 23% debt-to-GDP ratio the average maturity is approximately 1.25 years; this implies that about 75% of debt is in one-period bonds. Importantly, such a skewed maturity structure corresponds to a default risk of only (approximately) 1%.

Relative Risk Aversion and the Hazard Rate

The two limiting cases considered in Section 2.4.2 imply that the optimal maturity structure is flat if marginal reallocation of budget surpluses does not change the default risk, and the optimal debt policy implies the issuance of only one-period bonds if lenders are risk-neutral. Combining strict concavity of household utility functions and continuously increasing default probability leads to an intermediate maturity structure, i.e., the average maturity is in between 1 year and 1.5 years. In the next two numerical exercises I demonstrate the effect of change in risk aversion and the default hazard rate on the optimal term structure.

Figure 2.2 shows how average maturity depends on relative risk aversion γ_C . Since changing γ_C also shifts the crisis region—i.e. the debt-to-GDP ratio at which the default probability becomes positive—I make the comparison for different levels of default risk. The middle line ($\gamma_C = 1$) corresponds to the benchmark model. If default risk is zero, then the optimal term structure is flat. As default risk increases, the average maturity gradually decreases. This feature is common for various values of γ_C ; however, keeping the probability of default fixed the average maturity is different. A larger γ_C is associated with a flatter term structure. For example, if default risk is 1% and $\gamma_C = 1$, then the average maturity is approximately 1.3 years. However, if $\gamma_C = 4$, the average maturity is about 1.45 years, and if $\gamma_C = 0.25$, the average maturity drops below 1.2 years.

Figure 2.2: Average Maturity under Different Relative Risk Aversion



Intuitively, as risk aversion increases, a deviation from the ex ante optimal fiscal plan has a larger impact on risk-free interest rates. Therefore, a flatter maturity minimizes government's ability to decrease the market value of outstanding debt by manipulating risk-free rates. As γ_C goes to zero, the risk-free interest rates become almost constant, and therefore the government can manipulate only the risk premium. The optimal maturity then shortens so that the government could not alter the market value of outstanding debt by manipulating the default risk.

In the second exercise, I manipulate the default hazard rate $\frac{\pi'(s_2)}{1-\pi(s_2)}$ by considering different values of maximum output loss χ_{max} . As χ_{max} decreases and gets close to χ_{min} , a marginal increase in the contingent budget surplus leads to a larger rise in the default risk, which implies a higher default hazard rate and vice versa. Also, notice that $\chi_{max} \rightarrow 1$, so that $v_{min} \rightarrow -\infty$ would correspond to a no-default case, as for any finite $v > -\infty$, $f(v) = 0$.

Figure 2.3: Average Maturity under Different Default Hazard Rates

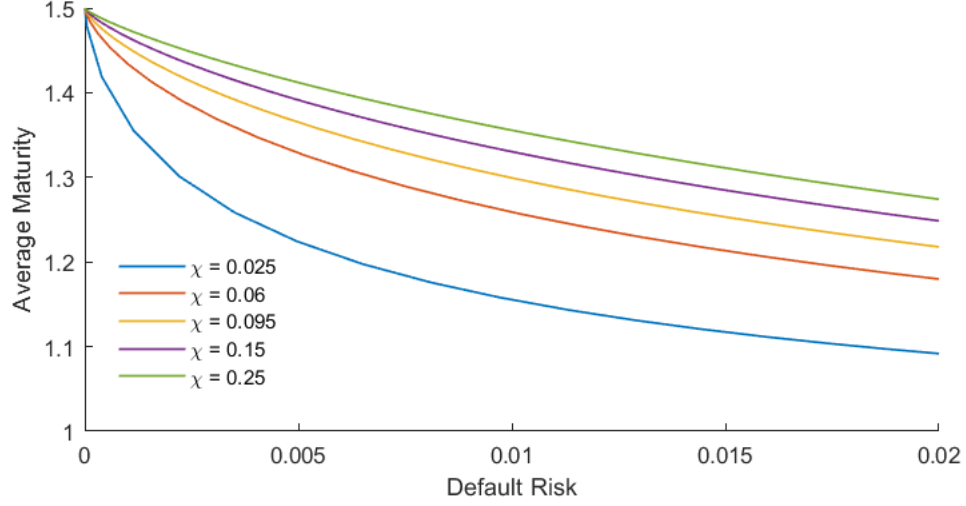


Figure 2.3 shows the average maturity structure at given default risk for different values of χ_{max} . The middle line corresponds to the benchmark model parameters. As χ_{max} increases and, hence, the default hazard rate decreases, the average maturity increases. The reason is that a marginal increase in the budget surplus at period 2 leads to a relatively lower increase in default probability and, therefore, the price of long-term debt becomes relatively less elastic.

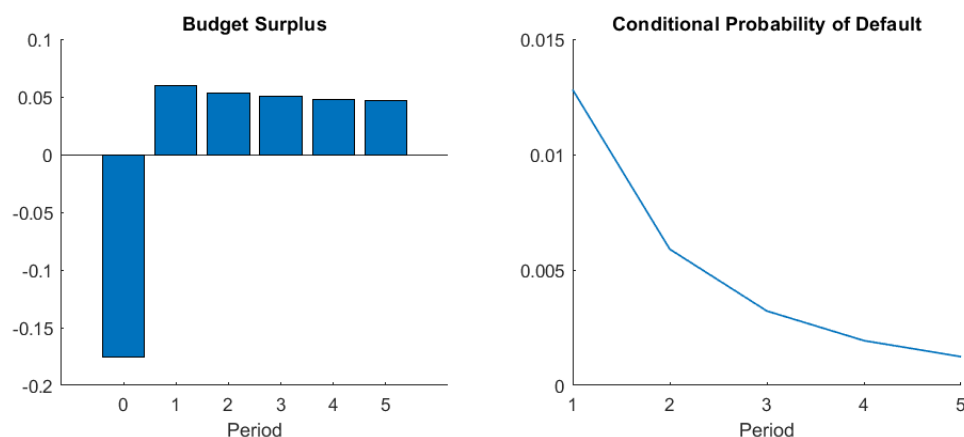
2.3.3 A Six-period Model

The Benchmark Model

In this subsection I study the optimal fiscal and debt policy in a six period model. Consider a general model as discussed in Section 2. In contrast to the previous section, the government can default in every period. I assume that at the initial date the government has a temporary higher taste parameter for public spending, $\theta_0 = 4$, followed by $\theta_t = 1 \forall t = 1, \dots, 5$. The initial debt is zero and $\beta = 0.99$. For this exercise, I set $\chi_{min} = 0.005$ and $\chi_{max} = 0.03$.

The focus of the analysis is on optimal maturity structure if multiple maturities are available and how term structure changes over time. I start by discussing the optimal fiscal plan and optimal endogenous default risk. I then solve for the maturity structure at every period that makes the ex ante optimal fiscal plan incentive-compatible for the Markov government.

Figure 2.4: Optimal Fiscal Plan and Default Probabilities Conditional on No Prior Default



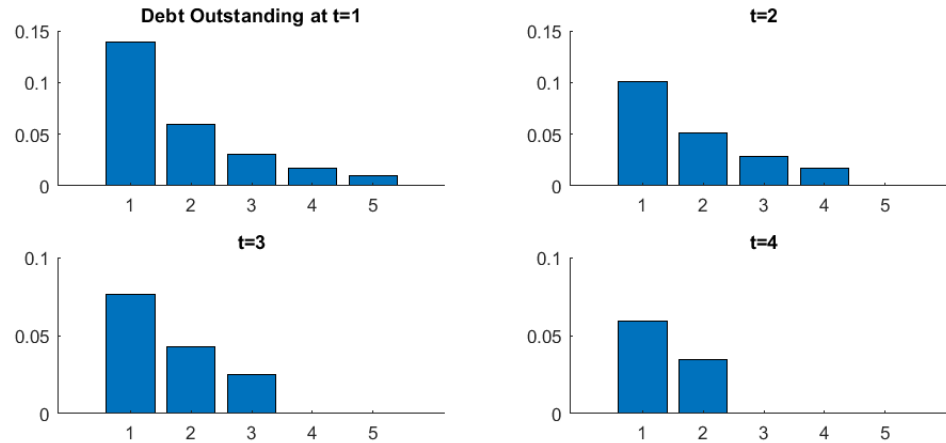
The right panel of Figure 2.4 presents the optimal fiscal plan. The left panel displays the default probability conditional on no default in preceding periods. At period 0 the government runs a budget deficit of 17.5%. In subsequent periods the government chooses positive contingent budget surpluses to finance the deficit. Budget surpluses are slightly skewed toward the left-end as established in Proposition 5. The budget surplus at period 1, 6%, gradually decreases in the following periods and is equal to 4.6% in the last period. Such skewness reflects decaying default risk. The conditional default risk in period 1 is 1.27%, and only 0.1% in the last period.

As discussed in Section 2.3, the government prefers such paths for budget surpluses for the following reason. By reallocating budget surpluses from the last period to the

first period, the government can decrease the default probability in the last period as it increases the government's utility in that period. However, such perturbation has a negligible effect on default risk in the first period. The reason is that an increase in utility in the last period is compensated by a decrease in utility in the first period. Thus, such reallocation does not change the value of pursuing the fiscal plan in period 1. By distributing budget surpluses more evenly the government would not be able to decrease default risk in period 1; however, it would increase default risk in the last period. Therefore, the government faces a trade-off between smoothing consumption over time and decreasing default probabilities in later periods.

Figure 2.5 displays the maturity structure at different dates. The top left panel shows the term structure of outstanding debt at the beginning of period 1. The X axis corresponds to maturity.

Figure 2.5: Optimal Maturity Structure

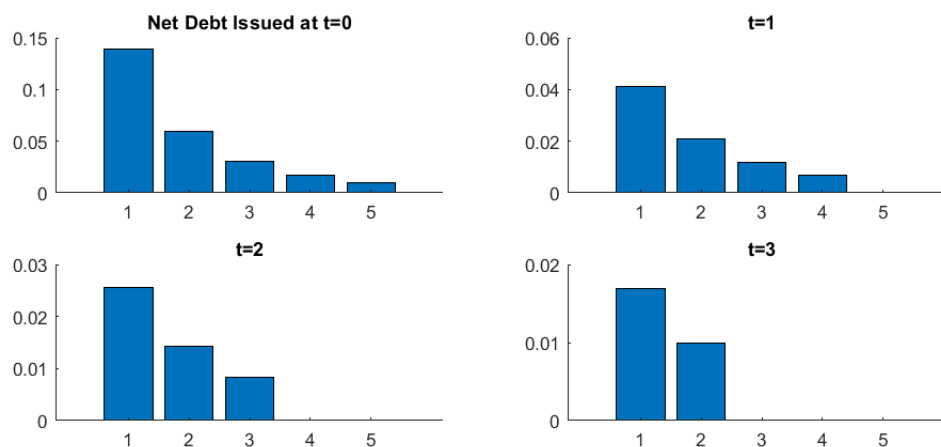


As predicted by the analytical model, maturity is tilted toward the short-term end and has a decaying profile: Payments due at later dates are lower than payments at earlier dates. For example, at period 0 the government issues 0.14 units of one-period

bond, 0.06 units of two-period bond and only 0.01 of five-period bond. In addition, as seen from the other panels, term structure remains decaying in the subsequent periods as well. However, it is worth noting that maturity structure flattens over time. This tendency reflects the decreasing conditional probability of default as highlighted in the right panel of Figure 2.4.

Figure 2.6 demonstrates the net debt issued at dates 0, 1, 2 and 3. The top left panel is identical to the outstanding debt at $t = 1$ due to zero initial debt. Importantly, at every date the government issues positive net debt at all maturities. Moreover, net debt issued is also skewed towards the short end.

Figure 2.6: Net Debt Issued



Persistent Fiscal Shock

In the previous exercise the positive fiscal shock at the initial date is assumed to be temporary. In reality, many shocks are persistent over time. In this exercise, I analyze how optimal maturity changes if the initial shock to taste parameter is persistent. I

set $\theta_0 = 3$ and $\theta_1 = 2$. At date 2 and at all future dates the parameter is one: $\theta_2 = \dots = \theta_5 = 1$. Initial debt is zero.

Figure 2.7. Persistent Fiscal Shock

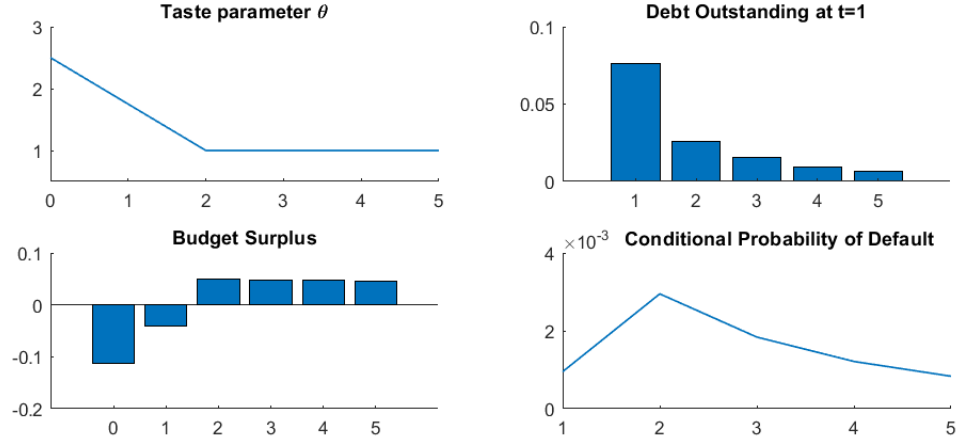


Figure 2.7 displays taste parameter θ , the optimal maturity structure, the fiscal plan, and the conditional default risk at every date. The bottom left panel demonstrates that the government prefers to run a budget deficit in first two periods, which must be repaid in the future. Due to this reason, the conditional risk of default is higher in period 2 compared to the default probability in period 1, as presented on the right bottom panel. Nonetheless, the right top panel shows the debt issued at $t = 0$ still has a decaying structure.

The exercise demonstrates that a persistent fiscal shock does not alter the implications for the optimal maturity structure. In the previous exercise, the optimal fiscal plan consists of budget surpluses that slightly decrease over time causing the conditional probability of default to decrease over time. In this exercise, the next period budget surplus is negative and, thus, the conditional default risk is not monotonically decreasing. Still, the stock of one-period debt is much larger than the stock

of two-period debt. Such maturity is optimal because the government at date 1 cannot manipulate the default risk at date 1.

2.3.4 The Maturity Structure of Developed Countries

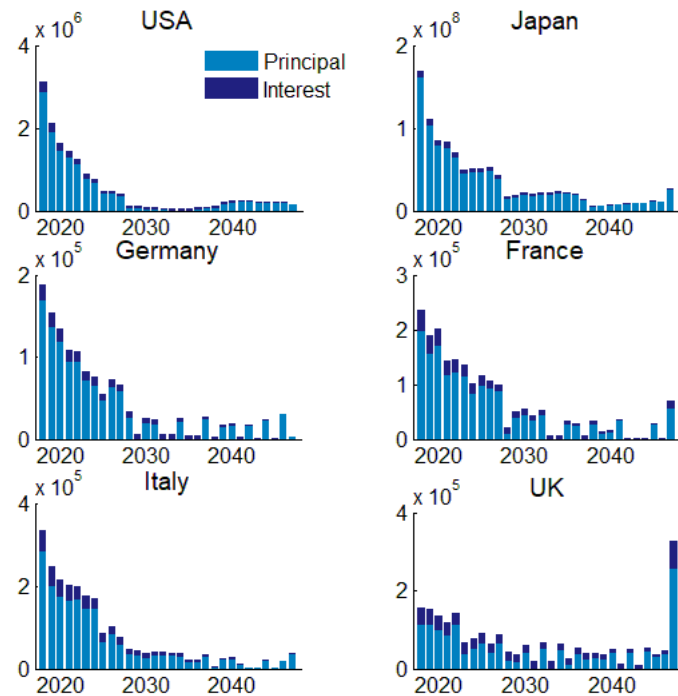
Figure 2.8 displays the maturity structure of marketable bonds for the following countries: the US, Japan, Germany, France, Italy and the UK. The US debt is in millions USD, the debts of the German, Italian and French governments are in millions EUR, the UK debt is in millions GBP and the Japanese debt is in millions JPY. The data was collected on 17th of October, 2017 and each bar shows the principal and interest payments owed by a government that has to be paid by the government in a given year as of October 17, 2017. I, thus, skip the payments due in the end of October, November and December 2017 and start with 2018. In each panel the first bar represents the total payments owed by a government due in 2018, the second bar represents the total payments due in 2019 and so on.

The exception is the very last bar in each panel that includes payments due in 2047 and all future years. The last bar is somewhat higher for Italy, Japan and France, but most importantly it represents a considerable part of debt for the United Kingdom. The main reason is that the British Government actively issued consol bonds during the Industrial Revolution (see Mokyr, 2011). Due to this aggregation of debt I ignore the last bar in the discussion of maturity data.

The maturity statistics is broadly consistent with the predictions of the model. First of all, debt is skewed toward the short-term end. Even though the countries issue bonds maturing in 30 years and later, the average maturity for the US, Japan, Germany, France and Italy is 5.79, 7.74, 6.8, 7.83 and 6.8 years respectively. It is worth noting that the maturity structure of the UK government debt is much flatter, and the average maturity is 14.97 years. For each country the stock of debt maturing

in one year is the largest⁴. In 2018 the US government has to pay (or roll-over) more than 20% of the total debt. The debt to be paid by the US government in the next five years constitutes 62% of the total debt. Similarly, total amount of debt maturing in the next 5 years constitutes approximately 50% of total debt for Japan, Germany, France and Italy⁵.

Figure 2.8: Maturity Structure of the USA, Japan, Germany, France, Italy and the UK (Source: Bloomberg)



Moreover, maturity structure has a decaying profile as predicted by the model. The US maturity structure exhibits decaying profile for 15 years: payments due in 2018-2033 are strictly decreasing. Then the term structure does display some

⁴recall that I ignore the last bar which aggregates total payments due in 2047 and later years.

⁵it is 48%, 53%, 46% and 52% for Japan, Germany, France and Italy respectively.

increasing trend, however, note that the total debt due in 2034 and all later years is lower than the debt due in 2018. Debt term structures of Japan, Germany, France and Italy have similar patterns. Even much flatter UK debt has a tendency to decline over maturity date: The total debt maturing in 1-5 years amounts to approximately 32%, while the total debt to be paid in 6-10 years constitutes less than 18%.

One of the reasons maturity structure is not perfectly decaying is that the number of issuance is limited and most of them are short-term. For example, the number of French debt active issuances is only 95⁶. There is no principal payments due at 2033, 2034, 2037 and some subsequent years. The number of issuances can be limited due to some fixed costs or other frictions.

2.4 Summary

The first two chapters show that in an environment with endogenous risk-free interest rates and endogenous default premiums the optimal maturity structure has a decaying profile. An important assumption is that marginal changes in fiscal policies lead to a marginal change in the risk-free interest rate and the default risk.

In this model, fiscal policy is time consistent in a sense that the government pursues the ex ante optimal fiscal plan. However, the consistency depends on deterministic nature of fiscal shocks assumed in the model. If fiscal shocks are stochastic then the allocation associated with the modified commitment problem is generally not feasible for the Markov planner at least if markets are incomplete. To study the optimal maturity structure in such an environment, ideally, we want to consider a multiple period model with multiple maturities available to the government. However, such analysis is infeasible in infinite horizon model with an infinite choice of maturities. Moreover, the problem is complicated even if there are more than two instruments with different maturities.

⁶Source: Bloomberg

There are several interesting avenues for future research. First, this paper assumes that the government cannot default within the period once new debt has been issued. The optimal debt policy under lack of commitment and positive default risk implies issuance of a large stock of short-term debt. This in turn increases the likelihood of self-fulfilling debt crisis if the latter is possible. Thus, allowing for self-fulfilling debt crises could lead to an interesting trade-off between short-term and long-term debt in such an environment. Second, the government is assumed to be able to default on its debt, but partial default is not allowed in this model. Therefore, it would be interesting to investigate the optimal fiscal and debt policies if the government issues nominal debt that can be inflated away rather than real debt.

Appendix

Lemma A.3. *If $u'(1-\tau+s) \cdot s$ is increasing in s for $s \in (-(1-\tau), \tau)$ then $\frac{\partial u'(1-\tau+s) \cdot s}{\partial s}$ is strictly decreasing in s .*

Proof

Recall from Assumption 3 that $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C > 0$. Then

$$\frac{\partial u'(1-\tau+s) \cdot s}{\partial s} = (1-\tau+s)^{-\gamma_C-1} \cdot (1-\tau+s-\gamma_C \cdot s)$$

$\frac{\partial u'(1-\tau+s) \cdot s}{\partial s} \geq 0$ implies $1-\tau+s-\gamma_C \cdot s \geq 0$, therefore,

$$\frac{\partial^2 u'(1-\tau+s) \cdot s}{\partial s^2} = 2 \cdot u''(1-\tau+s) + (-1-\gamma_C)u''(1-\tau+s) \cdot \frac{s}{1-\tau+s} =$$

$$= u''(1-\tau+s) \cdot (1-\tau+(1-\tau+s-\gamma_C \cdot s)) < 0.$$

■

Lemma A.4. *Suppose the solution to (1.11) is interior and given by (1.14). Besides, suppose that the budget constraint holds with equality, initial debt is zero and $\hat{s}_t > 0$ for $t \geq 1$. Then in equilibrium $u'(1-\tau+\hat{s}_t) \cdot \hat{s}_t$ is increasing in $\hat{s}_t$.*

Proof

The proof is by contradiction. Suppose that $u'(1 - \tau + \hat{s}_t) \cdot \hat{s}_t$ is decreasing in $\hat{s}_t$ for some optimal fiscal plan $\hat{\mathbf{s}}_0$. Then marginal decrease in $\hat{s}_t$ decreases the market value of budget surpluses:

$$\frac{\partial \sum_{k=0}^T \beta^k u'(1 - \tau + \hat{s}_k) \cdot Pr_0^k(\hat{\mathbf{s}}_0) \cdot \hat{s}_k}{\partial \hat{s}_t} = \sum_{k=1}^t \beta^k u'(1 - \tau + \hat{s}_k) \cdot \frac{\partial Pr_0^k(\hat{\mathbf{s}}_0)}{\partial \hat{s}_t} \cdot \hat{s}_k +$$

$$+ \beta^t \cdot Pr_0^t(\hat{\mathbf{s}}_0) \cdot (u'(1 - \tau + \hat{s}_t) + u''(1 - \tau + \hat{s}_t) \cdot \hat{s}_t) \leq 0$$

The first term represents decrease in market value of budget surpluses before t . It is negative because default probabilities (weakly) increase due to decrease in $W_k(\hat{\mathbf{s}}_k)$ for $k = 1, \dots, t$:

$$\frac{\partial Pr_0^k(\hat{\mathbf{s}}_0)}{\partial \hat{s}_t} = \sum_{k=1}^t \frac{f(W_k(\mathbf{s}_k) \cdot \frac{1-\beta}{1-\beta^{T-k+1}})}{F(W_k(\mathbf{s}_k) \cdot \frac{1-\beta}{1-\beta^{T-k+1}})} \cdot Pr_0^k(\hat{\mathbf{s}}_0) \cdot \frac{1-\beta}{1-\beta^{T-k+1}} \cdot \frac{\partial W_k(\hat{\mathbf{s}}_k)}{\partial \hat{s}_t} \leq 0$$

The second term is the change in market value of budget surplus at period t (keeping the default probability constant). The second term is (weakly) negative if $u'(1 - \tau + \hat{s}_t) \cdot \hat{s}_t$ is decreasing in $\hat{s}_t$.

Therefore, a marginal decrease in $\hat{s}_t$ keeping all other surpluses constant is feasible. Such deviation strictly increases government's value $W_0(\hat{\mathbf{s}}_0)$. However, it contradicts the optimality of $\hat{\mathbf{s}}_0$. Therefore, in equilibrium $u'(1 - \tau + \hat{s}_t) \cdot \hat{s}_t$ is increasing in $\hat{s}_t$. ■

Proof of Proposition 3.

(i) Suppose that $\frac{\partial Pr_1^2(s_2)}{\partial s_2} = Pr_1^2(s_2) = 0$. Then the optimality condition ((2.8) and (2.9)) implies :

$$\frac{\omega'(\tau - s_2) - u'(1 - \tau + s_2)}{\omega'(\tau - s_1) - u'(1 - \tau + s_1)} = \frac{u'(1 - \tau + s_2) + u''(1 - \tau + s_2) \cdot s_2}{u'(1 - \tau + s_1) + u''(1 - \tau + s_1) \cdot s_1}$$

As s_2 increases and s_1 decreases the left-hand side of the above equation strictly goes up, however, according to Lemma A.3. the right-hand side strictly goes down. The only solution is $s_1 = s_2$.

(ii) Now suppose that $\frac{\partial Pr_1^2(s_2)}{\partial s_2} > 0$ and $Pr_1^2(s_2) > 0$. The optimality condition requires

$$\begin{aligned} & \frac{\omega'(\tau - s_2) - u'(1 - \tau + s_2)}{\omega'(\tau - s_1) - u'(1 - \tau + s_1)} Pr_1^2(s_2) = \\ & = \frac{u'(1 - \tau + s_2) + \left[u''(1 - \tau + s_2) + \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)} u'(1 - \tau + s_2) \right] \cdot s_2}{u'(1 - \tau + s_1) + u''(1 - \tau + s_1) \cdot s_1} Pr_1^2(s_2) \end{aligned}$$

If $s_1 = s_2$ then the left-hand side (MRS) equals $Pr_1^2(s_2)$, however, the right-hand side (MRT) is

$$\left[1 + \frac{u'(1 - \tau + s_2)}{u'(1 - \tau + s_2) + u''(1 - \tau + s_2) \cdot s_2} \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)} \cdot s_2 \right] \cdot Pr_1^2(s_2) < Pr_1^2(s_2)$$

because $u'(1 - \tau + s_2) + u''(1 - \tau + s_2) \cdot s_2 > 0$ and $\frac{\partial Pr_1^2(s_2)}{\partial s_2} < 0$.

Therefore, in the equilibrium the marginal rate of substitution is lower than $Pr_1^2(s_2) \Rightarrow s_1 > s_2$. ■

Proof of Proposition 4.

The optimal maturity structure has to satisfy (2.12) or

$$\frac{u''(1 - \tau + s_2) + u'(1 - \tau + s_2) \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)}}{u''(1 - \tau + s_1)} \cdot \frac{b_0^2}{b_0^1} = MRT_{s_1, s_2}^0(\hat{s}_0) \cdot \frac{1}{Pr_1^2(s_2)}$$

Recall from Proposition 2 that $s_1 > s_2$. From Assumption 3 it follows that $-u''(1 - \tau + s_1) \leq -u''(1 - \tau + s_2)$ leading to

$$\frac{u''(1 - \tau + s_2) + u'(1 - \tau + s_2) \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)}}{u''(1 - \tau + s_1)} > 1$$

On other hand, the (negative) marginal rate of transformation that equals marginal rate of substitution is below $(1 - \pi(s_2))$:

$$MRT_{s_1, s_2}^0(\hat{s}_0) = \frac{\omega'(\tau - s_2) - u'(1 - \tau + s_2)}{\omega'(\tau - s_1) - u'(1 - \tau + s_1)} Pr_1^2(s_2) < Pr_1^2(s_2)$$

This yields $\frac{b_0^2}{b_0^1} < 1$. Besides, recall that $\omega'(\tau - s_2) - u'(1 - \tau + s_2) > 0$ for any $s_2 > 0 \Rightarrow \frac{b_0^2}{b_0^1} > 0$ (The government never chooses s_2 such that $Pr_1^2(s_2) = 0$ because then the government is strictly better off by setting $s_2 = 0$).

Next notice that if $s_1 \geq b_0^1$ and $s_2 \leq b_0^2$ then $MRT_{1,2}^1(s_1, s_2)$ is

$$\begin{aligned} & \frac{u'(1 - \tau + s_2) + \left[u''(1 - \tau + s_2) + u'(1 - \tau + s_2) \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)} \right] \cdot (s_2 - b_0^2)}{u'(1 - \tau + s_1) + u''(1 - \tau + s_1) \cdot (s_1 - b_0^1)} \cdot Pr_1^2(s_2) \geq \\ & \geq \frac{u'(1 - \tau + b_0^2)}{u'(1 - \tau + b_0^1)} Pr_1^2(s_2) > Pr_1^2(s_2) \end{aligned}$$

that contradicts the optimality condition because $MRS_{1,2}^1(s_1, s_2) < Pr_1^2(s_2)$, therefore, $s_1 < b_0^1$ and $s_2 > b_0^2$. ■

Proof of Proposition 5

It is sufficient to prove that if $\theta_0 > \bar{\theta}$ then $\hat{s}_t > \hat{s}_{t+1}$.

Suppose by contradiction that $\hat{s}_{t+1} \geq \hat{s}_t$. It then implies that

$$MRS_{t,t+1}^0 = \beta P_t^{t+1}(\hat{\mathbf{s}}_{t+1}) \cdot \frac{\omega'(\tau - \hat{s}_{t+1}) - u'(1 - \tau + \hat{s}_{t+1})}{\omega'(\tau - \hat{s}_t) - u'(1 - \tau + \hat{s}_t)} \geq \beta P_t^{t+1}(\hat{\mathbf{s}}_{t+1})$$

However, $\hat{s}_{t+1} \geq \hat{s}_t$ also implies that

$$MRT_{t,t+1}^0 =$$

$$\begin{aligned} & u'(1 - \tau + \hat{s}_{t+1}) + \left[u''(1 - \tau + \hat{s}_{t+1}) + u'(1 - \tau + \hat{s}_{t+1}) \cdot \frac{\frac{\partial P_t^{t+1}(\hat{\mathbf{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{P_t^{t+1}(\hat{\mathbf{s}}_{t+1})} \right] \cdot \hat{s}_{t+1} \\ = & \frac{u'(1 - \tau + \hat{s}_{t+1}) + \left[u''(1 - \tau + \hat{s}_{t+1}) + u'(1 - \tau + \hat{s}_{t+1}) \cdot \frac{\frac{\partial P_t^{t+1}(\hat{\mathbf{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{P_t^{t+1}(\hat{\mathbf{s}}_{t+1})} \right] \cdot \hat{s}_{t+1}}{u'(1 - \tau + \hat{s}_t) + u''(1 - \tau + \hat{s}_t) \cdot \hat{s}_t} \beta P_t^{t+1}(\hat{\mathbf{s}}_{t+1}) < \\ < & \beta P_t^{t+1}(\hat{\mathbf{s}}_{t+1}). \end{aligned}$$

The last inequality follows from Lemma A.3 and A.4 because

$$s_{t+1} \geq s_t \Rightarrow u'(1 - \tau + s_t) + u''(1 - \tau + s_t) \cdot s_t \geq u'(1 - \tau + s_{t+1}) + u''(1 - \tau + s_{t+1}) \cdot s_{t+1} >$$

$$> u'(1 - \tau + s_{t+1}) + \left[u''(1 - \tau + s_{t+1}) + u'(1 - \tau + s_{t+1}) \cdot \frac{\frac{\partial P_t^{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+1}}}{P_t^{t+1}(\mathbf{s}_{t+1})} \right] \cdot s_{t+1}.$$

■

Proof of Proposition 6

It is sufficient to show that $b_0^{t+1} > b_0^t \forall t = 1, 2, \dots, T$. The proposition 5 implies that $MRS_{t,t+1}^0(\hat{s}_t) = \beta P_t^{t+1}(\hat{s}_{t+1}) \cdot \frac{\omega'(\tau - \hat{s}_{t+1}) - u'(1 - \tau + \hat{s}_{t+1})}{\omega'(\tau - \hat{s}_t) - u'(1 - \tau + \hat{s}_t)} < \beta P_t^{t+1}(\hat{s}_{t+1})$ because $\hat{s}_t > \hat{s}_{t+1}$. Therefore, (2.13) can be rewritten as

$$\frac{b_0^{t+1}}{b_0^t} < \frac{u''(1 - \tau + \hat{s}_t)}{u''(1 - \tau + \hat{s}_{t+1}) - u'(1 - \tau + \hat{s}_{t+1}) \cdot \frac{\frac{\partial P_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{P_t^{t+1}(\hat{s}_{t+1})}}$$

Assumption 3 implies that if $\hat{s}_t > \hat{s}_{t+1}$ then $-u''(1 - \tau + \hat{s}_t) < -u''(1 - \tau + \hat{s}_{t+1}) < -u''(1 - \tau + \hat{s}_{t+1}) + u'(1 - \tau + \hat{s}_{t+1}) \cdot \frac{\frac{\partial \pi_{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{1 - \pi_{t+1}(\hat{s}_{t+1})} \Rightarrow \frac{b_0^{t+1}}{b_0^t} < 1$. Additionally, note that $\frac{b_0^{t+1}}{b_0^t} > 0$. Therefore, $b_0^1 > b_0^2 > \dots > b_0^T > 0$. ■

Chapter 3

Sovereign Debt Maturity Structure of a Small-Open Economy: Short or Long?

3.1 Introduction

There are two major motives to use maturity of sovereign debt: the incentive motive and hedging motive. The first motive implies that if there is a lack of commitment then maturity can discipline future governments to pursue ex ante optimal policies. The second motive implies that in absence of state-contingent bonds different maturities of debt can be used to complete the markets. However, when studied in isolation these two motives can lead to very different implications of the optimal maturity structure. According to the literature¹, a small-open economy which lacks commitment to repay its debt should issue short-term debt to minimize the so-called debt dilution problem. On the other hand, long-term debt can be used to hedge economy against exogenous shocks in interest rates and to smooth consumption over

¹see, for example, Aguiar et al. (2018), Arellano and Ramanayanan (2012), Fernandez and Martin (2015) etc.

different states of the world. The question is how a small open economy should structure the maturity of its debt if both problems are present and, particularly, what problem is quantitatively more important: the problem of lack of commitment or the problem of lack of insurance?

Similar question is answered in a recent paper by Debortoli et al. (2017) but for a closed economy with endogenous risk-free interest rates and no default. They find that the maturity of government debt should structure its maturity so that maturity minimizes the costs of lack of commitment, i.e., the future governments do not have an incentive to distort risk-free interest rates ex post, while problem of lack of insurance is quantitatively much less important. Following Lucas and Stokey (1983) we know that to resolve the problem of lack of commitment a government issues approximately flat maturity structure. However, consumption smoothing in absence of state-contingent bonds require very large and tilted positions of short-term and long-term debt, for example, around -10000% of GDP of short-term debt and 5000% of GDP of long-term debt². Not surprisingly that given such large positions of debt a government has a huge incentive to distort risk-free interest rates. In other words, marginal deviations from flat maturity structure (or whatever maturity structure that minimizes the problem of lack of commitment) have extremely small benefit of hedging compared to costs of distortion of interest rates ex post.

In contrast to a model with endogenous risk-free interest rates and no default, a small open-economy model (with exogenous risk-free interest rates but endogenous default risk) does not require such large and tilted positions of debt to achieve full hedging. A three-period example in this paper shows that while the problem of lack of commitment can be resolved with only one-period debt, the hedging problem can be resolved using 50% of one-period debt and 50% of long-term debt. Therefore, it is not clear whether the conclusion of Debortoli et al. (2017) that minimizing the costs

²see Buera and Nicolini (2004) for numerical examples of Angeletos (2002) model

of commitment is quantitatively more important than the costs of insurance applies to a small-open economy model as well.

I consider a classical small-open economy model with exogenous risk-free interest rates and opportunity to default on sovereign debt. Economy is endowed with a deterministic but not constant stream of output. Sovereign can borrow from foreign risk-neutral lenders issuing bonds with different maturities. The default is modeled similar to Aguiar et al. (2018). In the beginning of a period, a sovereign has an option to default on its debt and to receive an outside option. The value of outside option accounts for potential costs of default, it is stochastic and drawn from a continuous distribution. The latter is very important for the incentive motive: a marginal increase in government debt marginally increases default risk, thus, maturity matters for the debt-dilution problem. However, in contrast to Aguiar et al. (2018) I assume that the value of outside option can be driven from two different distributions which correspond to two different states of the world, one of which is associated with higher default risk. I call it the “crisis” state of the world and the other state is the “normal” state of the world. The government learns the state of the world at date 1 but does not know it initially. Thus, I introduce the hedging motive for using maturity.

There are several contributions of this paper. First, I develop an algorithm of numerical solution of the model using bicubic interpolation approach. The benefit of bicubic interpolation compared to discretization approach is that instead of a step function the price function remains continuously differentiable with respect to both state variables (short-term debt and long-term debt). Therefore, to solve for the optimal solution we can use first-order conditions instead of iteration on a grid that increases preciseness and saves time significantly.

Second, I solve the model numerically and demonstrate that if equilibrium default risk is positive, then the sovereign issues mostly short-term debt. The ideal solution for the government would be to use short-term state-contingent bonds. Obviously,

the amounts are not generally the same as the government would prefer to pay less debt in the state of the world with higher default risk. However, since the state-contingent bonds are not allowed, the sovereign is restricted to choose between short and long-term debt. If the government chooses only short-term debt, then in the “crisis” state of the world the default risk is above ex ante optimal level of default risk in this state of the world while in the “normal” state of the world default risk is below the analogous level. If the government decides to lengthen maturity of its debt by issuing some fraction of long-term debt, then it marginally increases default risk in both states of the world, however, the effect is bigger for the “crisis” state of the world making the problem of the default premium distortion in this state of the world even worse. If the difference between default probabilities in the different states of the world increases, the cost of lack insurance does go up so the share of long-term debt rises as well, however, the share still remains insignificant.

Finally, I demonstrate that standard numerical methods that approximate continuous distribution with a discrete distribution can lead to biased results exaggerating the role of long-term debt. To show it I solve a three-period version of Aguiar et al. (2018) model. The analytical solution of this model implies that the government issues only short-term debt. However, if continuous distribution is approximated with a discrete, then numerical solution implies issuance of positive and significant amount of long-term debt contradicting the analytical solution. As the number of possible values of discrete distribution goes up the bias decreases, however, in practice distributions are approximated with a small number of points³ which questions the accuracy of the numerical solutions.

The structure of the chapter is as follows. Section 2 describes the model. In Section 3 I consider three benchmark models which can be solved numerically: a model with state-contingent bonds, a model with one state of the world, and a model

³for example, Arellano and Ramanayanan (2012) discretize continuous distribution with only 6 possible values

with a discrete distribution. Section 4 explains the numerical algorithm and provides numerical solution to the problem. Section 5 discusses limitations of the standard numerical approaches and potential bias in the solution. Last section concludes.

3.2 A Three-Period Model

Consider a small open-economy and a sovereign who makes all decisions on a behalf of a representative household who values private consumption. Every period economy is endowed with y_t units of a single tradable good, $t = 0, 1, 2$. The stream of endowment $\{y_t\}_{t=0}^2$ is deterministic, however, y_t can be different in different periods. The sovereign can reallocate consumption across time periods only through borrowing and lending on international financial markets. Particularly, I assume that $y_0 < y_1 = y_2 = 1$ so that the sovereign has an incentive to issue some stock of debt at date 0 to smooth consumption over time. The key friction of the model is that the sovereign lacks commitment and can default on its debt. For simplicity (but without loss of generality) I assume that government can default only at date 2 and there is no option to default in previous periods.

The sovereign's utility function is

$$u(c_0) + \beta u(c_1) + \beta^2 \mathbb{E} \max \{u(c_2), V^{def}\} \quad (3.1)$$

where $\beta \in (0, 1)$ is a discount factor, c_t is consumption at period t , $u()$ is a strictly concave, continuously differentiable function, and V^{def} is the value of outside option that can be achieved if sovereign defaults at date 2.

The international lenders are risk-neutral and risk-free interest rate is $R = \frac{1}{\beta}$. Sovereign can issue discount bonds with different maturities. Let denote by b_t^{t+k} the amount of discount bonds issued at period t maturing at period $t+k$ and let q_t^{t+k} be the corresponding bond price. Suppose that the initial debt is zero. Thus, at date

0 government issues b_0^1 of a one-period (or short-term) debt and b_0^2 of two-period (or long-term) debt. The total market value of debt issued at $t = 0$ is $q_0^1 b_0^1 + q_0^2 b_0^2$. At the intermediate date the sovereign pays b_0^1 and issues $q_1^2(b_1^2 - b_0^2)$. At the beginning of the last period, conditional on no default, the sovereign pays b_1^2 to the international lenders.

I model default as in Aguiar et al. (2018). The sovereign can choose an outside option V^{def} that can be achieved upon default, where V^{def} is stochastic and drawn from a continuous distribution F . However, in contrast to Aguiar et al. (2018), I assume that F is not deterministic. More specifically, suppose that $F \in \{F^N, F^C\}$, where "N" stands for a "normal" state of the world which corresponds to relatively lower default risk, and "C" stands for a "crisis", riskier state of the world. At date 0 the government does not know the distribution of V^{def} but the sovereign learns the distribution at period 1.

The other important assumptions about outside option are summarized in Assumption 1.

Assumption 1. Outside Option:

- (i) $V^{def} \in [v^{min}(F), v^{max}(F)]$;
- (ii) $v^{max}(F) \leq u(y_2)$;
- (iii) $f(v^{min}(F)) = f(v^{max}(F)) = 0$;
- (iv) $p_0(F = F^C) = \lambda$;
- (v) $F^C(v) \leq F^N(v) \forall v \in R$.

First assumption implies that any F has a bounded support. Assumption (ii) states that an upper bound of outside option is always below the value achieved by consuming endowment only ensuring that it is never optimal to choose the outside option if there is no debt. The third assumption states that the density of the

boundaries equals zero. This assumption is made to avoid kinks in value, pricing and policy functions. The fourth assumption implies that at period 0 the probability of “crisis” distribution of outside option is λ and, hence, the probability of “normal” distribution is $1 - \lambda$. The last assumption implies that default is weakly more likely if $F = F^C$.

Such structure of outside option is isomorphic to assuming that default is costly, i.e, in case of default the sovereign does not repay its debt back, but there is an endowment loss associated with foreign lenders sanctions or other type of punishment. For example, a popular assumption is that upon default economy loses a fraction χ of its output and, hence, the outside option equals to $V_t^{def} = u(y_2(1 - \chi))$. Assuming that χ is continuously distributed and has a bounded support is equivalent to the assumption that V_t^{def} is continuously distributed with a bounded support.

Bellman Equations and Break-Even Conditions

Suppose that the sovereign does not default at the beginning of period 2, then its value is simply $u(y_2 - b_1^2)$. At date 1 the sovereign chooses consumption today c_1 and the stock of debt to be paid in the next period b_1^2 to solve

$$V_1(b_0^1, b_0^2, s) = \max_{c_1, b_1^2} u(c_1) + \beta \mathbb{E} V_2(b_1^2, s) \quad (3.2)$$

$$\text{s.t. } c_1 = y_1 - b_0^1 + q_1^2(b_1^2, s) \cdot (b_1^2 - b_0^2) \quad (3.3)$$

$$\mathbb{E} V_2(b_1^2, s) = \int_{v^{min}(s)}^{v^{max}(s)} \max \{u(y_2 - b_1^2), v\} dF^s(v) \quad (3.4)$$

where $s = \{C, N\}$ is the state of the world at date 2 known at $t = 1$. As the lenders are risk-neutral, bond prices depend on exogenous risk-free interest rate and default risk:

$$q_1^2(b_1^2, s) = \frac{1}{R} \cdot \Pr(V_2(b_1^2) > V^{def} | F = F^s) = \frac{1}{R} \cdot F^s(u(y_2 - b_1^2)) \quad (3.5)$$

At date 0 the sovereign makes consumption and debt decisions to solve:

$$V_0 = \max_{c_0, b_0^1, b_0^2} u(c_0) + \beta\lambda \cdot V_1(b_0^1, b_0^2, C) + \beta(1 - \lambda) \cdot V_1(b_0^1, b_0^2, N) \quad (3.6)$$

$$\text{s.t. } c_0 = y_0 + \frac{1}{R} \cdot b_0^1 + q_0^2(b_0^1, b_0^2) \cdot b_0^2 \quad (3.7)$$

Because the government never defaults at date 1, the price of a short-term debt is just $\frac{1}{R}$ ⁴. Since the state of the world s is unknown at $t = 0$, the long-term debt price reflects expected default risk:

$$\begin{aligned} q_0^2(b_0^1, b_0^2) &= \frac{1}{R} \cdot \mathbb{E} [q_1^2(b_1^2, s)] = \\ &= \frac{1}{R^2} \cdot \lambda \cdot F^C(u(y_2 - b_1^{2*}(b_0^1, b_0^2, C))) + \frac{1}{R^2} \cdot (1 - \lambda) \cdot F^N(u(y_2 - b_1^{2*}(b_0^1, b_0^2, N))) \end{aligned} \quad (3.8)$$

where $b_1^{2*}(b_0^1, b_0^2, s)$ is the optimal debt policy at $t = 1$ as a function of outstanding debt and state of the world.

⁴I implicitly assume that the government is not allowed to issue such large positions of debt so that it cannot pay it back next period

The key object of the analysis is the maturity structure (b_0^1, b_0^2) . I will first proceed with the benchmark models for which we can derive analytical solutions to discuss the so called “incentive” and “hedging” motives of maturity.

3.3 Benchmark Models

In this section I consider three benchmark models. First, I consider the model described in Section 2 if state-contingent bonds were available. I show that the solution is efficient in the sense that the sovereign can smooth consumption over states of the world at date 1 and achieve ex ante optimal level of default risk. However, the same allocation cannot generally be achieved without state-contingent bonds. Then I consider two specific examples for which planner’s allocation can be achieved even without state-contingent bonds but they lead to opposite conclusions regarding the optimal maturity structure.

3.3.1 A Model with State-Contingent Bonds

Suppose that the sovereign at date 0 could issue state-contingent bonds $(b_0^1(s), b_0^2(s))$ $s = \{C, N\}$. Then at period 1 outstanding sovereign debt equals to $(b_0^1(C), b_0^2(C))$ if $s = C$ and $(b_0^1(N), b_0^2(N))$ if $s = N$. Then the sovereign’s problem at $t = 0$ is

$$V_0^{SCB} = \max_{c_0, b_0^1(s), b_0^2(s)} u(c_0) + \beta\lambda \cdot V_1(b_0^1(C), b_0^2(C), C) + \beta(1 - \lambda) \cdot V_1(b_0^1(N), b_0^2(N), N) \quad (3.9)$$

$$\text{s.t. } c_0 = y_0 + \lambda \frac{1}{R} \cdot b_0^1(C) + q_0^2(b_0^1(C), b_0^2(C)) \cdot b_0^2(C) + (1 - \lambda) \frac{1}{R} \cdot b_0^1(N) + q_0^2(b_0^1(N), b_0^2(N)) \cdot b_0^2(N) \quad (3.10)$$

where $\lambda \frac{1}{R}$ and $(1 - \lambda) \frac{1}{R}$ are the prices of state-contingent short-term bonds, and $q_0^2(b_0^1(s), b_0^2(s), s)$ is the price of state-contingent long-term bond:

$$q_0^2(b_0^1(s), b_0^2(s), s) = \frac{1}{R} \cdot p_0(F = F^s) \cdot F^s \left(u(y_2 - b_1^{2*}(b_0^1(s), b_0^2(s), s)) \right)$$

To characterize the problem (3.9) consider the following modified commitment problem (MCP). Suppose there is a planner who can commit to future fiscal policies but cannot commit to repay its debt and defaults strategically. More specifically, the planner at date 0 not only chooses state-contingent short-term and long-term debt $(b_0^1(s), b_0^2(s))$, but also chooses state-contingent debt to be paid in the last period $b_1^2(s)$. However, the planner cannot promise to repay debt at $t = 2$ and does it if $u(y_2 - b_1^2(s)) < V_2^{def}$. Thus, the planner's problem is

$$\begin{aligned} \hat{V}_0^{SCB} = & \max_{c_0, c_1(s), b_1^2(s)} u(c_0) + \beta\lambda \cdot \left(u(c_1(C) + \beta\mathbb{E}V_2(b_1^2(C), C)) \right) + \\ & + \beta(1 - \lambda) \cdot \left(u(c_1(N) + \beta\mathbb{E}V_2(b_1^2(N), N)) \right) \end{aligned} \quad (3.11)$$

$$\text{s.t. } c_0 + \frac{\lambda}{R}c_1(C) + \frac{1 - \lambda}{R}c_1(N) = y_0 + \frac{1}{R}y_1 +$$

$$+ \frac{\lambda}{R^2}F^C(u(y_2 - b_1^2(C)) \cdot b_1^2(C) + \frac{1 - \lambda}{R^2}F^N(u(y_2 - b_1^2(N)) \cdot b_1^2(N)) \quad (3.12)$$

Lemma 1 implies that the planner prefers to smooth consumption at period 0 and each state of the world at period 1. The proof is presented in Appendix A.1.

Lemma 1. Optimal Allocation of MCP. $c_0 = c_1(C) = c_1(N)$.

Proposition 1 states that the MCP allocation can be achieved by the government with lack of commitment. In order to achieve it, the sovereign issues only short-term debt and issues no long-term debt.

Proposition 1. Efficiency with State-Contingent Bonds. If state-contingent bonds are available, the solution is

- (i) efficient, i.e., $V_0^{SCB} = \hat{V}_0^{SCB}$;
- (ii) if $F^s(u(y_2 - b_1^2(s))) > 0$, sovereign issues only short-term bonds, i.e., $b_0^2(s) = 0$.

The complete proof of Proposition 1 is presented in Appendix A.2. Below I highlight the key reason for using only one-period bonds. Notice that the first-order necessary condition of problem (3.11) is

$$-\beta \cdot \frac{\partial \mathbb{E} V_2(b_1^2(s), s)}{\partial b} = u'(c_1(s)) \cdot \frac{1}{R} \left[F^s(u(y_2 - b_1^2(s))) + \frac{\partial}{\partial b} F^s(u(y_2 - b_1^2(s))) \cdot b_1^2(s) \right] \quad (3.13)$$

The left-hand side of (3.13) shows a marginal decrease in expected value of the sovereign at $t = 2$ for state of the world s if it issues an extra marginal unit of debt $b_1^2(s)$. The right-hand side corresponds to a marginal increase in sovereign utility at $t = 1$ if it issues extra unit of debt $b_1^2(s)$. Importantly, the planner takes into account how a marginal increase in stock of debt $b_1^2(s)$ affects the market value of debt $q_1^2(b_1^2, s) \cdot b_1^2(s)$ and, henceforth, $c_1(s)$ keeping consumption at period 0 constant. Below is the analogical first-order necessary condition of the MPCE problem (3.9):

$$-\beta \cdot \frac{\partial \text{EV}_2(b_1^2(s), s)}{\partial b} = u'(c_1(s)) \cdot \frac{1}{R} \left[F^s(u(y_2 - b_1^2(s))) + \frac{\partial}{\partial b} F^s(u(y_2 - b_1^2(s))) \cdot (b_1^2(s) - b_0^2(s)) \right] \quad (3.14)$$

The left-hand sides of (3.13) and (3.14) are identical because the objective functions of the MCP and MPCE at $t = 1$ (keeping c_0 constant) are exactly the same. However, the right-hand sides are not because the budget constraints of these two problems are different. The key difference is in that the Markov government does not internalize how its actions influence the market value of debt and consumption at period 0.

An increase in $b_1^2(s)$ affects the market value of net debt issued due to two effects. The first is simply due to an increase in debt: for any extra bond issued the sovereign receives the price of a bond, i.e., $q_1^2(b_1^2, s)$. However, an increase in debt also (weakly) decreases bond price due to increased default risk which decreases the market value of net debt issued. From the planner's perspective, the market value of already issued debt changes by $\frac{\partial}{\partial b} q_1^2(b_1^2(s), s) \cdot b_1^2(s)$ as it realizes how an extra unit of debt issued affects its lifetime budget constraint. However, from the Markov government point of view, the market value of net debt issued at $t = 1$ changes by only $\frac{\partial}{\partial b} q_1^2(b_1^2(s), s) \cdot (b_1^2(s) - b_0^2(s))$ as it does not internalize how an increase in $b_1^2(s)$ affects the market value of long-term debt issued at $t = 0$: $\frac{1}{R} \cdot q_1^2(b_1^2(s), s) \cdot b_0^2(s)$. The latter decreases consumption at the initial period c_0 which is not taken into account at $t = 1$ because c_0 is already a history.

As a result, the only optimal strategy for the government is to issue no long-term debt at all. Whenever a positive stock of long-term debt is issued, the government at $t = 1$ over borrows relative to the planner's allocation. If a government decides to issue negative stock of long-term debt, the government at $t = 1$ borrows less. In both cases, the Markov government cannot achieve exactly the same allocation

as the planner which means that issuing positive or negative stocks of long-term debt are not optimal. If government issues no long-term debt (and right amounts of state-contingent short-term debt $b_0^1(s)$) then the Markov government faces exactly the same problem as the planner, the optimality conditions (3.13) and (3.14) are identical and, therefore, the government implements the planner's allocation.

Importantly, the difference matters only for so called "crisis" region when default risk is positive and continuously increasing in debt. For a "safe" region and no default risk (or if F is not continuously increasing), a marginal change in the probability of default $\frac{\partial}{\partial b} F^s(u(y_2 - b_1^2(s)))$ as b goes up equals zero. In this case, the optimality conditions are equivalent, so assuming that the solution is given by first-order conditions and keeping bond prices and market value of outstanding debt constant, any marginal perturbations in maturity structure does not matter and maturity structure is (locally) irrelevant. However, the solution is unique for any continuously increasing F .

In addition, notice that $b_0^1(C)$ does not generally equal to $b_0^1(N)$. It implies that if state-contingent bonds are not available then the Markov government cannot generally implement the social planner allocation, i.e., $V_0 < \hat{V}_0^{SCB}$. In the next subsections I consider two examples for which the best allocation can be achieved without state-contingent bonds.

3.3.2 A Model without Hedging

First, let consider a model with $F^C \equiv F^N$. The goal of this exercise is to highlight the importance of short-term debt in disciplining the future governments. The model with no uncertainty is basically a simplified version of Aguiar et al. (2018). In addition, notice that this model is equivalent to the model with state-contingent bonds and only one state of the world, and, therefore, the optimal strategy is to issue only short-term

debt.

Corollary 1. Consider a model with $F^C \equiv F^N$. Then if $F(b_1^2) > 0$, the optimal maturity strategy implies $b_0^2 = 0$.

The proof follows from Proof of Proposition 1. The intuition is also analogous. The government issues only short-term debt because, otherwise, it has an incentive to dilute the value of outstanding long-term debt in the next period. The short-term debt, thus, serves as an incentive for the next period government to stick to ex ante optimal level of default risk.

3.3.3 A Model without Incentive

Now consider again the model with two states of the world, but instead of a continuous distribution consider the following discrete distribution. Suppose in each state of the world there are only two possible values of outside option and they are the same for all states of the world: $v^{max}(C) = v^{max}(N) = v^{max} \equiv u(y_2)$ and $v^{min}(C) = v^{min}(N) = v^{min} \equiv \lim_{c \rightarrow 0} u(c)$. Thus, v^{max} and v^{min} imply that the default is either absolutely cost less and the sovereign will default on any positive debt, or default is infinitely costly and any feasible amount of debt will be repaid. However, the states of the world are different in the probability of the outcomes. Let p^s be the probability of $V^{def} = v^{max}$ if state of the world is s . Then $p^C > p^N$ meaning that there is a higher chance of default at the crisis state of the world .

MPCE of this problem can also be characterized by considering MCP. Lemma A.2 in Appendix proves that for any $y_0 < y_1 = y_2$ the government always opts to be in the crisis region. The reason is that by escaping a crisis region in any state of the world s , i.e., by setting $c_2(s) = y_2$, an increase in present value of income is exactly compensated by an increase in present value of consumption. Therefore, there is no

benefit of decreasing default risk. Instead, the government distorts consumption in different states of the world that strictly reduces its value.

Given discrete distributions F^s we can rewrite the MCP (3.11) as follows:

$$\begin{aligned}\hat{V}_0 = & \max_{c_0, c_1(s), c_2(s)} u(c_0) + \beta\lambda \left(u(c_1(C)) + \beta(1 - p^C) \cdot u(c_2(C)) + \beta p^C v^{max} \right) \\ & + \beta(1 - \lambda) \left(u(c_1(N)) + \beta(1 - p^N) \cdot u(c_2(N)) + \beta p^N v^{max} \right)\end{aligned}\quad (3.15)$$

$$\begin{aligned}\text{s.t. } c_0 + \frac{\lambda}{R} \left(c_1(C) + \frac{1 - p^C}{R} c_2(C) \right) + \frac{1 - \lambda}{R} \left(c_1(N) + \frac{1 - p^N}{R} c_2(N) \right) = \\ = y_0 + \frac{y_1}{R} + \left(\lambda(1 - p^C) + (1 - \lambda)(1 - p^N) \right) \cdot \frac{y_2}{R^2}\end{aligned}$$

The best allocation is to smooth consumption over time and states of the world.

Lemma 2. Optimal Allocation of MCP with discrete distribution. $c_0 = c_1(s) = c_2(s) = c$.

Notice that compared to Lemma 1 the planner smooths consumption not only in period 0 and 1 but in period 2 as well. The reason is that if F is continuous then the planner has an incentive to marginally decrease default risk in the last period by decreasing consumption in that period. If F is a discrete distribution with such simple structure as discussed above, the incentive motive disappears completely.

Proposition 2 states that the planner's allocation can be implemented with flat maturity structure:

Proposition 2. Hedging Motive of Maturity. Suppose that $y_0 < y_1 = y_2 = y$, and F^s is a discrete distribution with two outcomes: $v^{max} = u(y_2)$ and $v^{min} = \lim_{c \rightarrow 0} u(c) \forall s$. Then the optimal maturity structure is unique and flat:

$$b_0^1 = b_0^2$$

As discussed in Section 3.1, if a marginal increase in b does not change the default risk, i.e., $f(v) = 0$ for $v \in (v^{min}; v^{max})$, then the optimality conditions for both the planner and the Markov government are identical and does not depend on maturity structure of debt. In other words, maturity structure is not used as a disciplining device since the intermediate period Markov government has no incentive to dilute the outstanding long-term debt anyway. However, the maturity still enters the intermediate period budget constraints and the perfect consumption smoothing is possible only if it is structured correctly. The budget constraint at $t = 1$ can be rewritten as follows:

$$c_1(s) = y_1 - b_0^1 + \frac{1 - p^s}{R} ((y_2 - c_2(s)) - b_0^2)$$

The consumption smoothing ($c_1(s) = c_2(s) = c$) and $y_1 = y_2 = y$ then imply that $b_0^1 - \frac{1 - p^s}{R} ((y - c) - b_0^2) = y - c$ for any $p^s \in [0, 1]$ leading to conclusion that $b_0^1 = b_0^2 = y - c$.

Thus, the flat maturity structure allows to hedge the sovereign against any changes in interest rates. Notice that this result can be extended to a model with multiple states of the world because under flat maturity structure the sovereign does not roll over any short-term debt in the intermediate period so the interest rate is basically irrelevant⁵.

⁵this is true if being in crisis region is optimal.

Discussion

To conclude, the implications for the optimal maturity structure depend crucially on the assumptions of the model. If we consider a model with no hedging motive (there is no uncertainty regarding the process of outside option in the last period) but with a hedging motive (default risk is continuously increasing in outstanding level of debt), then the government issues only one-period debt. However, in a model with hedging motive (multiple states of the world) but no incentive motive (default risk does not locally depend on level of debt) the optimal policy is to issue consol bonds so that the maturity is flat. In a more general model, both incentive and hedging motives are important but in absence of state-contingent bonds the government lacks instruments to both smooth consumption and minimize debt dilution problem.

3.4 Optimal Maturity Structure: Numerical Exercises

In order to better understand the optimal maturity structure in a general model I consider a number of numerical exercises. In this section I explain the choice of parameters and functional forms and numerical algorithm which is different from what is common in the literature.

3.4.1 Functional Forms and Parameters

I assume CRRA per-period utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

where γ is the relative risk aversion coefficient and I set $\gamma = 2$. Output at period 1 and 2 is normalized to one: $y_1 = y_2 = y = 1$. For simplicity, $\beta = R = 1$. At date

0, the probabilities of a crisis state and a normal state of the world are equal, i.e., $\lambda = 0.5$.

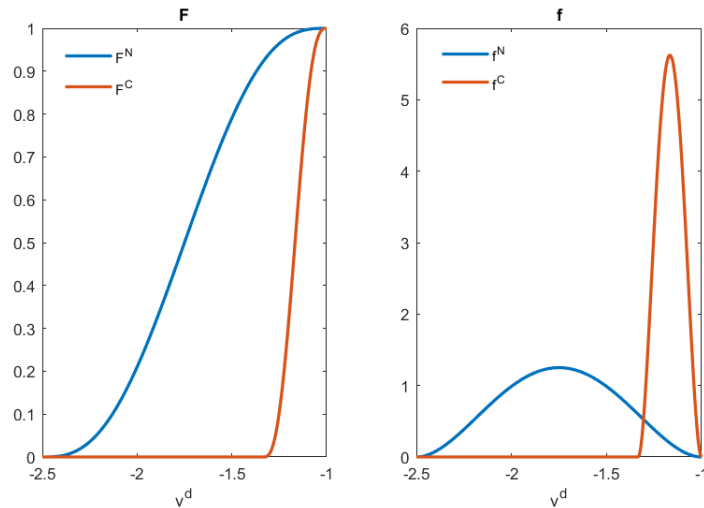
The distribution density function of the value of outside option at period is a polynomial of order 4 which satisfies:

$$f_2^s(v) = \begin{cases} a_2(s) \cdot (v - v_2^{\min}(s))^2 \cdot (v - v_2^{\max}(s))^2 & \text{if } v \in (v_2^{\min}(s), v_2^{\max}(s)) \\ 0 & \text{otherwise} \end{cases} \quad (3.16)$$

where $a(s)$ satisfies $\int_{v_2^{\min}(s)}^{v_2^{\max}(s)} a(s) \cdot (v - v_2^{\min}(s))^2 \cdot (v - v_2^{\max}(s))^2 dv = 1$.

I implicitly assume that default is equivalent to loosing a fraction of output. I set $v_2^{\max}(C) = v_2^{\max}(N) = u(y)$ meaning that for any positive debt at the beginning of date 2 the probability of default is positive as well. In the benchmark case, I set $v_2^{\min}(C) = u(0.75y)$ and $v_2^{\min}(N) = u(0.4y)$. The probability and cumulative density functions are presented on Figure 3.1.

Figure 3.1. Density Functions of Outside Value Option at $t = 2$



In addition to default at date 2, I allow the government to default at period 1 as well. Analogously, $V_1^{def} \sim F_1$ where F_1 has a bounded support $[v_1^{min}; v_1^{max}]$, $v_1^{min} = (1 + \beta)v_2^{min}(N)$, $v_1^{max} = (1 + \beta)v_2^{max}(N)$, and $f_1 = a_1 \cdot (v - v_1^{min})^2 \cdot (v - v_1^{max})^2$ such that $\int_{v_1^{min}}^{v_1^{max}} a_1 \cdot (v - v_1^{min})^2 \cdot (v - v_1^{max})^2 dv = 1$.

3.4.2 Numerical Algorithm

The key distinction between the computational approach of this paper and other approaches used in the literature is that I do not discretize the continuous distributions F_1 and F_2^s . In addition, instead of finding the maximum value on the grid I approximate pricing and value functions with bicubic splines and solve the first-order necessary optimality conditions. One benefit of bicubic interpolation is that an approximated function and its first-order derivatives are continuous functions.

First, I find functions which approximate $V_1(b_0^1, b_0^2, s)$, and $q_1^2(b_0^1, b_0^2, s)$. To approximate any function $g(x, y)$ on $[0, 1] \times [0, 1]$ with a multivariate polynomial function of order three $p(x, y)$ we find a vector of 16 coefficients $\alpha = \{a_{ij}\}_{i=0, \dots, 3, j=0, \dots, 3}$ for which the values of g and its derivatives g_x , g_y , and g_{xy} equal to p , p_x , p_y , and p_{xy} at the four corners $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Then

$$g(x \in [0, 1], y \in [0, 1]) \approx p(x, y) = \sum_{i=0}^3 \sum_{j=0}^3 a_{ij} \cdot x^i \cdot y^j \quad (3.17)$$

In addition, such approach allows to get an approximate for $\frac{\partial g(x, y)}{\partial x}$ and $\frac{\partial g(x, y)}{\partial y}$ as follows

$$\frac{\partial}{\partial x} g(x \in [0, 1], y \in [0, 1]) \approx p_x(x, y) = \sum_{i=0}^3 \sum_{j=0}^3 a_{ij} \cdot i x^{i-1} \cdot y^j \quad (3.18)$$

$$\frac{\partial}{\partial y} g(x \in [0, 1], y \in [0, 1]) \approx p_y(x, y) = \sum_{i=0}^3 \sum_{j=0}^3 a_{ij} \cdot x^i \cdot j y^{j-1} \quad (3.19)$$

Let $[b_0^{1min}, b_0^{1max}]$ and $[b_0^{2min}, b_0^{2max}]$ be the range of possible values of short-term debt and long-term debt. Let N_s and N_l be the number of nodes for short-term debt and long-term debt respectively. Then for each node $(i \in [1, N_s], j \in [1, N_l], s \in \{C, N\})$ such that

$$(b_0^1)_i = b_0^{1min} + (b_0^{1max} - b_0^{1min}) \cdot \frac{i-1}{N_s}$$

$$(b_0^2)_j = b_0^{2min} + (b_0^{2max} - b_0^{2min}) \cdot \frac{j-1}{N_l}$$

we find the values of $V_1((b_0^1)_i, (b_0^2)_j, s)$ and $q_1^2((b_0^1)_i, (b_0^2)_j, s)$ ⁶. In addition, I find the derivatives $\frac{\partial}{\partial b_0^1} V_1((b_0^1)_i, (b_0^2)_j, s)$, $\frac{\partial}{\partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s)$, and $\frac{\partial^2}{\partial b_0^1 \partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s)$ (and analogously the derivatives of $q_1^2((b_0^1)_i, (b_0^2)_j, s)$) by finding the values at the points $((b_0^1)_i + \epsilon, (b_0^2)_j, s)$, $((b_0^1)_i - \epsilon, (b_0^2)_j, s)$, $((b_0^1)_i, (b_0^2)_j + \epsilon, s)$, $((b_0^1)_i, (b_0^2)_j - \epsilon, s)$, $((b_0^1)_i + \epsilon, (b_0^2)_j + \epsilon, s)$, and $((b_0^1)_i - \epsilon, (b_0^2)_j - \epsilon, s)$ where $\epsilon > 0$ is a very small number so that

$$\frac{\partial}{\partial b_0^1} V_1((b_0^1)_i, (b_0^2)_j, s) = \frac{V_1((b_0^1)_i + \epsilon, (b_0^2)_j, s) - V_1((b_0^1)_i - \epsilon, (b_0^2)_j, s)}{2\epsilon}$$

$$\frac{\partial}{\partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s) = \frac{V_1((b_0^1)_i, (b_0^2)_j + \epsilon, s) - V_1((b_0^1)_i, (b_0^2)_j - \epsilon, s)}{2\epsilon}$$

$$\frac{\partial^2}{\partial b_0^1 \partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s) =$$

⁶technically, q_1^2 is a function of b_1^2 and s but b_1^2 is a function of b_0^1 , b_0^2 , and s

$$\begin{aligned}
&= \frac{V_1((b_0^1)_i + \epsilon, (b_0^2)_j + \epsilon, s) - V_1((b_0^1)_i + \epsilon, (b_0^2)_j, s) - V_1((b_0^1)_i, (b_0^2)_j + \epsilon, s)}{2\epsilon^2} + \\
&+ \frac{2V_1((b_0^1)_i, (b_0^2)_j, s) - V_1((b_0^1)_i - \epsilon, (b_0^2)_j, s) - V_1((b_0^1)_i, (b_0^2)_j - \epsilon, s) + V_1((b_0^1)_i - \epsilon, (b_0^2)_j - \epsilon, s)}{2\epsilon^2}
\end{aligned}$$

Then for each square $[(b_0^1)_i, (b_0^1)_{i+1}] \times [(b_0^2)_j, (b_0^2)_{j+1}]$ and $s \in \{C, N\}$ I find the vectors of coefficients $\alpha_{ijs}(V_1)$ and $\alpha_{ijs}(q_1^2)$ to approximate the value and policy functions and their derivatives analogously to (3.17) - (3.19).

To find $V(b_0^1, b_0^2, s)$ and $q_1^2(b_0^1, b_0^2, s)$ I solve the system of equations consisting of the budget constraint at period 1 (3.3) and the first-order optimality condition⁷

$$-\beta \cdot \frac{\partial \mathbb{E}V_2(b_1^2, s)}{\partial b} = u'(c_1) \cdot \frac{1}{R} \left[F^s(u(y_2 - b_1^2)) + \frac{\partial}{\partial b} F^s(u(y_2 - b_1^2)) \cdot (b_1^2 - b_0^2) \right] \quad (3.20)$$

where

$$\frac{\partial \mathbb{E}V_2(b, s)}{\partial b} = -\frac{\partial u(y - b)}{\partial b} \cdot F^s(u(y - b))$$

and find $b_1^2(b_0^1, b_0^2, s)$ and $c_1^2(b_0^1, b_0^2, s)$. Then the value function is just

$$V_1(b_0^1, b_0^2, s) = u(c_1^2(b_0^1, b_0^2, s)) + \beta \mathbb{E}V_2(b_1^2(b_0^1, b_0^2, s), s)$$

where

$$\mathbb{E}V_2(b, s) = \int_{u(y-b)}^{v^{max}(s)} v \cdot dF^s(v) + u(y - b) \cdot F^s(u(y - b))$$

⁷optimality condition (3.20) is different from (3.14) because we use b_1^2 and b_0^2 instead of $b_1^2(s)$ and $b_0^2(s)$.

The first component can be easily computed given that F^s is a polynomial. The pricing function is

$$q_1^2(b_0^1, b_0^2, s) = \frac{1}{R} \cdot F^s(u(y - b_1^2(b_0^1, b_0^2, s)))$$

The final step is to find the optimal maturity structure (b_0^1, b_0^2) by solving the system of equations consisting of the budget constraint at period 0 and first-order optimality conditions with respect to short-term debt and long-term debt:

$$\left\{ \begin{array}{l} c_0 = y_0 + q_0^1(b_0^1, b_0^2) \cdot b_0^1 + q_0^2(b_0^1, b_0^2) \cdot b_0^2 \\ -\frac{\partial}{\partial b_0^1} \beta \mathbb{E}W_1(b_0^1, b_0^2) = u'(c_0) \cdot \left(q_0^1(b_0^1, b_0^2) + \frac{\partial}{\partial b_0^1} q_0^1(b_0^1, b_0^2) \cdot b_0^1 + \frac{\partial}{\partial b_0^1} q_0^2(b_0^1, b_0^2) \cdot b_0^2 \right) \\ -\frac{\partial}{\partial b_0^2} \beta \mathbb{E}W_1(b_0^1, b_0^2) = u'(c_0) \cdot \left(q_0^2(b_0^1, b_0^2) + \frac{\partial}{\partial b_0^2} q_0^1(b_0^1, b_0^2) \cdot b_0^1 + \frac{\partial}{\partial b_0^2} q_0^2(b_0^1, b_0^2) \cdot b_0^2 \right) \end{array} \right.$$

The left-hand sides of the optimality conditions shows a marginal decrease in discounted expected value tomorrow $\beta \mathbb{E}W_1(b_0^1, b_0^2)$ if the sovereign increases issuance of debt by one unit where

$$\mathbb{E}W_1(b_0^1, b_0^2) = \lambda \left(\int_{V_1(b_0^1, b_0^2, C)}^{v_1^{max}} v dF_1(v) + V_1(b_0^1, b_0^2, C) \cdot F_1(V_1(b_0^1, b_0^2, C)) \right) +$$

$$(1 - \lambda) \left(\int_{V_1(b_0^1, b_0^2, N)}^{v_1^{max}} v dF_1(v) + V_1(b_0^1, b_0^2, N) \cdot F_1(V_1(b_0^1, b_0^2, N)) \right)$$

and

$$\frac{\partial}{\partial b_0^1} \mathbb{E}W_1(b_0^1, b_0^2) = \lambda \frac{\partial}{\partial b_0^1} V_1(b_0^1, b_0^2, C) \cdot F_1(V_1(b_0^1, b_0^2, C)) + (1 - \lambda) \frac{\partial}{\partial b_0^1} V_1(b_0^1, b_0^2, N) \cdot F_1(V_1(b_0^1, b_0^2, N))$$

$$\frac{\partial}{\partial b_0^2} \mathbb{E}W_1(b_0^1, b_0^2) = \lambda \frac{\partial}{\partial b_0^2} V_1(b_0^1, b_0^2, C) \cdot F_1(V_1(b_0^1, b_0^2, C)) + (1-\lambda) \frac{\partial}{\partial b_0^2} V_1(b_0^1, b_0^2, N) \cdot F_1(V_1(b_0^1, b_0^2, N))$$

The right-hand side of the optimality conditions shows how an increase in debt by a unit increases the sovereigns utility in the initial period. Notice that it takes into account how such action affects the market value of already issued debt. The bond prices are computed as follows

$$q_0^1(b_0^1, b_0^2) = \frac{1}{R} \lambda F_1(V_1(b_0^1, b_0^2, C)) + \frac{1}{R} (1-\lambda) F_1(V_1(b_0^1, b_0^2, N))$$

$$q_0^2(b_0^1, b_0^2) = \frac{1}{R} \lambda F_1(V_1(b_0^1, b_0^2, C)) \cdot q_1^2(b_0^1, b_0^2, C) + \frac{1}{R} (1-\lambda) F_1(V_1(b_0^1, b_0^2, N)) \cdot q_1^2(b_0^1, b_0^2, N)$$

and their derivatives are

$$\frac{\partial}{\partial b_0^x} q_0^1(b_0^1, b_0^2) = \frac{1}{R} \lambda f_1(V_1(b_0^1, b_0^2, C)) \cdot \frac{\partial}{\partial b_0^x} V_1(b_0^1, b_0^2, C) + \frac{1}{R} (1-\lambda) f_1(V_1(b_0^1, b_0^2, N)) \cdot \frac{\partial}{\partial b_0^x} V_1(b_0^1, b_0^2, N)$$

$$\frac{\partial}{\partial b_0^x} q_0^2(b_0^1, b_0^2) = \frac{1}{R} \lambda f_1(V_1(b_0^1, b_0^2, C)) \cdot \frac{\partial}{\partial b_0^x} V_1(b_0^1, b_0^2, C) \cdot q_1^2(b_0^1, b_0^2, C) +$$

$$+ \frac{1}{R} \lambda F_1(V_1(b_0^1, b_0^2, C)) \cdot \frac{\partial}{\partial b_0^x} q_1^2(b_0^1, b_0^2, C) + \frac{1}{R} (1-\lambda) f_1(V_1(b_0^1, b_0^2, N)) \cdot \frac{\partial}{\partial b_0^x} V_1(b_0^1, b_0^2, N) \cdot q_1^2(b_0^1, b_0^2, N) +$$

$$+\frac{1}{R}(1-\lambda)F_1(V_1(b_0^1, b_0^2, N)) \cdot \frac{\partial}{\partial b_0^x} q_1^2(b_0^1, b_0^2, N)$$

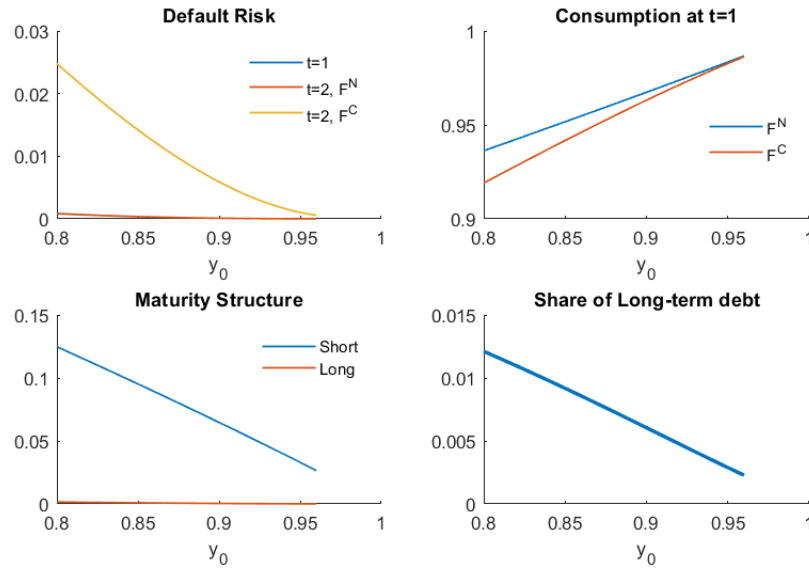
for $x = \{1, 2\}$.

3.4.3 Optimal Maturity Structure

Benchmark Model

In the first numerical exercise, I study the optimal maturity structure as a function of output in the initial period. If output is lower the sovereign runs a larger budget deficit at period 0 resulting in a higher default risk as well as higher interest rate spread between different states of the world. Figure 3.2 displays the results of the benchmark model.

Figure 3.2. Maturity Structure in Benchmark Model



The left top panel shows the equilibrium ex ante default risk at period 1 and for each state of the world at period 2. For the parameter of the model, the default risk

at $t = 1$ and in the normal state of the world at $t = 2$ are negligibly small. The default risk in the crisis state of the world is more substantial. The right top panel shows equilibrium consumption at period 1 in different states of the world. It is clear that the sovereign cannot or does not prefer to smooth consumption between the states of the world.

The left bottom panel shows the optimal maturity structure of sovereign debt. It is clear that the maturity structure is almost short. There is approximately linear relationship between output at period 0 and short-term debt issued. The stock of long-term is negligibly small. The right bottom panel displays the share of long-term debt. Even though the share increases linearly, it is very insignificant with slightly above 1% for $y_0 = 0.8$ which corresponds to approximately 2% default risk in the crisis state of the world next period.

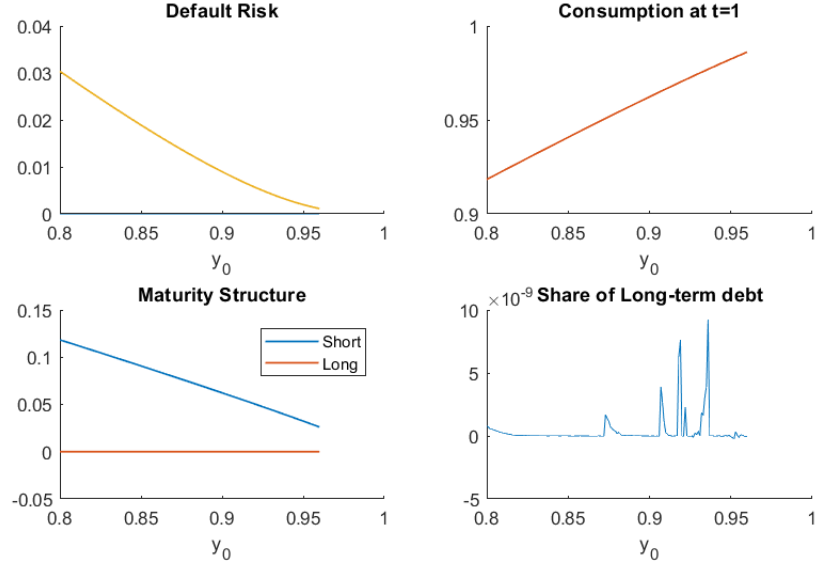
3.5 Limitations of Discretization Approach

The literature studying maturity structure of sovereign debt often discretize the continuous distribution as, for example, Arellano and Ramanarayanan (2012). They approximate continuous distribution with quadrature procedure (Tauchen and Hussey, 1991) using only six-state Markov chain. In this section I demonstrate that the numerical algorithm matters for the solution of the maturity structure and discretization can lead to a biased results.

Consider a model discussed in Section 3.2 but with only one state of the world, i.e., $F^C = F^N = F$. Importantly, I assume again that there is no default in the intermediate period. From Corollary 1 we know that if equilibrium default risk is positive the sovereign should issue only one-period bonds and does not issue two-period bonds at all. Figure 3.3 shows the solution to such an example using the algorithm explained in Section 3.4.2.

We see that the numerical solution coincides with the analytical solution and is very accurate. The share of long-term debt does not exceed 10^{-8} which corresponds to computational error. However, as we will see, discretization approach does not lead to such results.

Figure 3.3. Solution to a single-state model using bicubic approximation approach



Suppose that f has a functional form as in (3.16) and $V_2^{def} \in [v^{min}; v^{max}]$ where $v^{max} = u(y)$ and $v^{min} = u(0.5y)$. Let construct discrete distribution $\tilde{f}$ with N_f points such that

$$x_i = v^{min} + \frac{(v^{max} - v^{min})}{N_f} \cdot \left(\frac{1}{2} + (i - 1) \right), i = 1, \dots, N_f$$

$$\tilde{f}_i = \frac{f(x_i)}{\sum_{i=1}^{N_f} f(x_i)}, i = 1, \dots, N_f$$

and solve the problem finding the maximum on the fine grid. Figure 3.4 shows the approximated distribution for $N_f = 5$.

Figure 3.4. Discrete distribution, $N_f = 5$

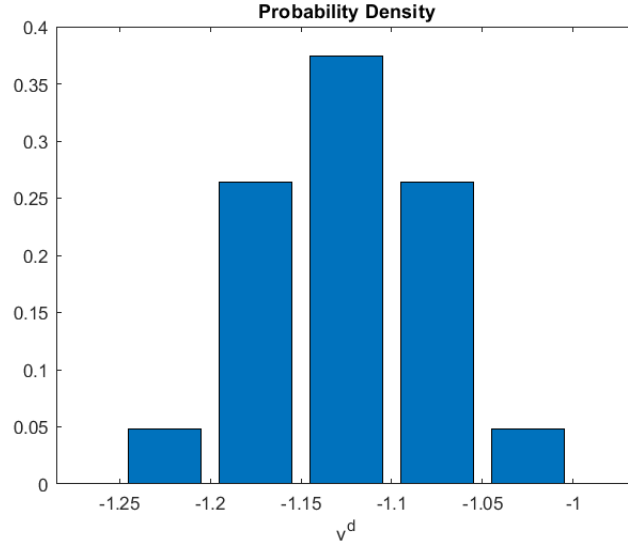
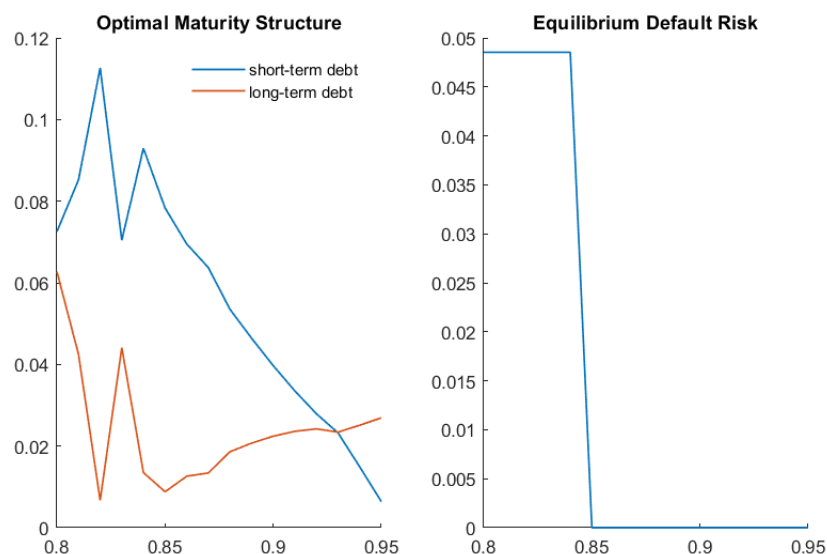


Figure 3.5 displays the optimal maturity structure and equilibrium default risk at period 2. We can see that the government issues more short-term debt. However, it also issues positive and substantial amount of long-term debt. For example, if $y_0 = 0.8$ then the sovereign issues more than 0.06 of long-term debt while short-term debt amounts to less than 0.08. Obviously, it contradicts the analytical solution to the problem that implies issuance of short-term debt only.

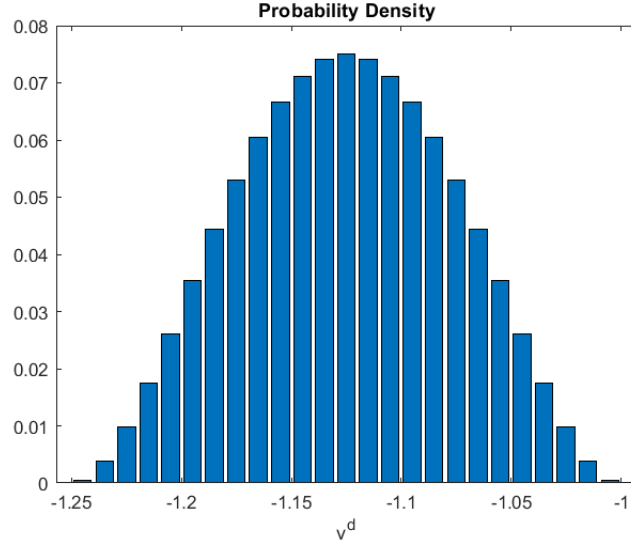
Figure 3.5. Solution to a single-state model using discretization approach for $N_f = 5$



In addition, notice that the total debt issued and default risk presented on Figure 5 does not correspond to the results displayed on Figure 3. On Figure 3 default risk increases smoothly and reaches approximately 3% at $y_0 = 0$. If we consider the discretization solution, then default risk “jumps” to about 5% at $y_0 = 0.85$. As a result, not only the maturity structure is wrong but also the total level of debt.

Obviously, as N_f goes up and thus the approximated discrete distribution because closer to the original continuous distribution, the bias decreases. Let increase N_f to 25. Figure 3.6 shows the discrete distribution.

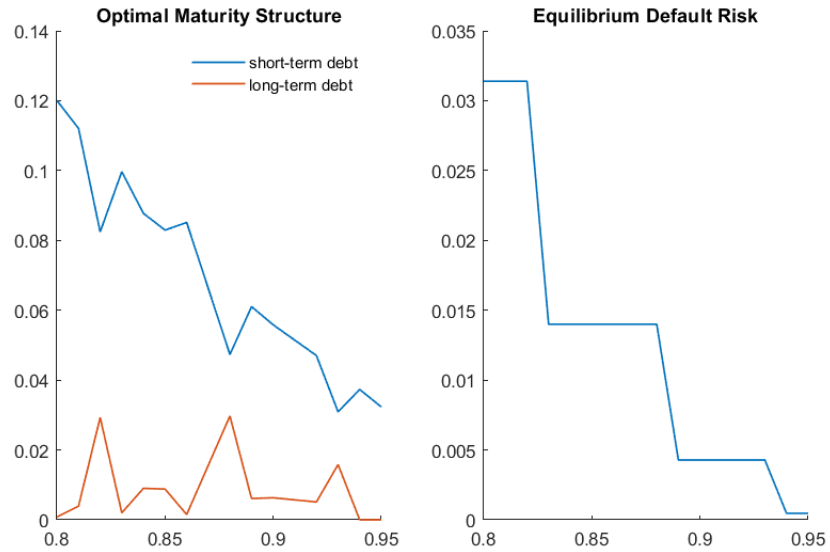
Figure 3.6. Discrete distribution, $N_f = 25$



The optimal maturity structure and equilibrium default risk are presented on Figure 3.7. We can see issuance of short-term debt and default risk schedule are much closer to Figure 7 compared to the results presented on Figure 3.5.

Figure 3.7. Solution to a single-state model using discretization approach for

$$N_f = 25$$



Nevertheless, the solution to the problem does not imply that the issuance of long-term debt is zero or, at least, significantly close to zero. For example, it exceeds 30% for $y_0 = 0.87$. Therefore, precise solution to the optimal maturity structure problem requires that continuous probability distributions are approximated with a significantly high number of nodes. Otherwise, the results can be biased and indicate exaggerated share of long-term debt.

3.6 Summary

In this chapter I show that for a small-open economy it is numerically more important to minimize the costs of lack of commitment than the costs of lack of insurance or, loosely speaking, a sovereign should issue mostly short-term debt. In addition, I demonstrate that the computational algorithm matters for the solution of the maturity structure. Specifically, an approach involving discretization of continuous distribution results in a longer optimal maturity compared to known analytical solution. Instead I use bicubic spline interpolation approach which is much more accurate method.

Appendix

A.1. Proof of Lemma 1.

Let μ be the Lagrange multiplier on the budget constraint (3.12), then the first-order necessary with respect to c_0 and $c_1(s)$ ($s = \{C, N\}$) conditions are:

$$u'(c_0) = \mu,$$

$$\beta p_0(F = F^s)u'(c_1(s)) = \mu \frac{p_0(F = F^s)}{R},$$

implying that $u'(c_0) = \beta R u'(c_1(s)) \Rightarrow c_0 = c_1(C) = c_1(N)$. ■

A.2. Lemma A.1. The value of modified commitment problem with state-contingent bonds is weakly greater than the value of MPCE with state-contingent bonds: $\hat{V}_0^{SCB} \geq V_0^{SCB}$.

Proof.

Let $(c_0^*, c_1^*(s), b^*(s))$ be the solution to MPCE problem (3.9) and $(\hat{c}_0, \hat{c}_1(s), \hat{b}(s))$ be the solution to MCP problem (3.11). Note that any MPCE allocation is feasible to the planner. Therefore, the planner can always achieve the value at least as high as V_0^{SCB} . ■

A.3. Proof of Proposition 1.

Lemma A.1. implies that $\hat{V}_0^{SCB} \geq V_0^{SCB}$. Below I show that if $b_0^2(s) = 0 \forall s$ then the Markov government replicates the planner's allocation.

The first-order optimality conditions of the MCP imply that

$$-\beta \cdot \frac{\partial \text{EV}_2(b_1^2(s), s)}{\partial b} = u'(c_1(s)) \cdot \frac{1}{R} \left[F^s(u(y_2 - b_1^2(s))) + \frac{\partial}{\partial b} F^s(u(y_2 - b_1^2(s))) \cdot b_1^2(s) \right]$$

and the MPCE optimality condition is

$$-\beta \cdot \frac{\partial \text{EV}_2(b_1^2(s), s)}{\partial b} = u'(c_1(s)) \cdot \frac{1}{R} \left[F^s(u(y_2 - b_1^2(s))) + \frac{\partial}{\partial b} F^s(u(y_2 - b_1^2(s))) \cdot (b_1^2(s) - b_0^2(s)) \right]$$

The optimality conditions are identical only if $b_0^2(s) = 0$. In addition, it is trivial to show that there exist $b_0^1(s) \forall s$ such that MPCE allocation is equivalent to MCP allocation. ■

A.4. Lemma A.2. Consider the Modified Commitment Problem discussed in Section 3.3:

$$\begin{aligned} \hat{V}_0 = & \max_{c_0, c_1(s), c_2(s)} u(c_0) + \beta \lambda \left(u(c_1(C)) + \beta(1 - p^C) \cdot u(c_2(C)) + \beta p^C \max \{u(c_2(C)); v^{max}\} \right) \\ & + \beta(1 - \lambda) \left(u(c_1(N)) + \beta(1 - p^N) \cdot u(c_2(N)) + \beta p^N \max \{u(c_2(N)); v^{max}\} \right) \end{aligned}$$

$$\begin{aligned} \text{s.t. } c_0 + \frac{\lambda}{R} \left(c_1(C) + \frac{1 - p^C}{R} c_2(C) \right) + \frac{1 - \lambda}{R} \left(c_1(N) + \frac{1 - p^N}{R} c_2(N) \right) = \\ = y_0 + \frac{y_1}{R} + \left(\lambda(1 - p^C) + (1 - \lambda)(1 - p^N) \right) \cdot \frac{y_2}{R^2} + \end{aligned}$$

$$+\frac{\lambda p^C}{R^2} \mathbb{I}_{\{c_2(C) \geq y_2\}} \cdot (y_2 - c_2(C)) + \frac{(1-\lambda)p^N}{R^2} \mathbb{I}_{\{c_2(N) \geq y_2\}} \cdot (y_2 - c_2(N))$$

where $\mathbb{I}_{\{c_2 \geq y_2\}}$ is the indicator function that equals 1 if $c_2 \geq y_2$ and 0 otherwise.

If $y_0 < y_1 = y_2$ then $c_2(s) < y_2 \forall s$.

Proof.

Basically, the planner has 4 options:

- (i) $c_2(C) \geq y_2, c_2(N) \geq y_2$,
- (ii) $c_2(C) < y_2, c_2(N) \geq y_2$,
- (iii) $c_2(C) \geq y_2, c_2(N) < y_2$,
- (iv) $c_2(C) < y_2, c_2(N) < y_2$,

that correspond to no default risk, default risk in the crisis state of the world and no default risk in the normal state of the world, default risk in the normal state of the world and no default risk in the crisis state of the world, or default risk in both states of the world.

First, notice that if the government decides to avoid default risk in state s it sets $c_2(s) = y_2$. If $c_2(s) > y_2$ the government can be better off by decreasing $c_2(s)$ and by smoothing consumption, i.e., increasing consumption in other periods.

For each case (i)-(iv) the first-order necessary conditions imply that $c_0 = c_1(s) = c \forall s$. In addition, if the government does not restrict $c_2(s)$ to be equal to y_2 then $c_2(s) = c$:

$$u'(c_0) = \mu$$

$$\beta p_0(F = F^s) u'(c_1(s)) = p_0(F = F^s) \frac{\mu}{R} \Rightarrow u'(c_1(s)) = \mu$$

$$\beta^2 p_0(F = F^s) \left(1 - p^s + p^s \cdot \mathbb{I}_{\{c_2(s) \geq y_2\}}\right) u'(c_2(s)) = \mu \frac{p_0(F = F^s) \left(1 - p^s + p^s \cdot \mathbb{I}_{\{c_2(s) \geq y_2\}}\right)}{R^2}$$

$$\Rightarrow u'(c_2(s)) = \mu$$

where μ is the Lagrange multiplier and $p^s = p^C$ if $s = \{C\}$ and $p^s = p^N$ otherwise.

Notice that the cases (i)-(iv) have exactly the same objective functions (because $u(c_2(s)) = v^{max}$) and budget constraints (because if $\mathbb{I}_{\{c_2(s) \geq y_2\}} > 0$ then $c_2(s) = y$). The only difference is that the cases (i)-(iii) impose additional restrictions on $c_2(s)$ as the government chooses to avoid default risk: (i) implies $c_2(C) = c_2(N) = y_2$, (ii) imposes $c_2(N) = y_2$, and (iii) imposes $c_2(C) = c$. Thus, the government is strictly better off by choosing $c_2(C) = c_2(N) = c < y_2$.

■

A.5. Proof of Lemma 2.

Result follows from the first-order necessary conditions (see proof of Lemma A.2 for details).

■

A.6. Proof of Proposition 2.

Analogously to Lemma A.1, one can show that the MCP is an upper bound for the MPCE problem. Consider the problem of the intermediate period government with the state (b_0^1, b_0^2, s)

$$V_1(b_0^1, b_0^2, s) = \max_{c_1, c_2} u(c_1) + \beta(1 - p^s) \cdot u(c_2(s)) + \beta p^s \max \{u(c_2(s)); v^{max}\}$$

$$\text{s.t. } y - c_1 - b_0^1 + \frac{1}{R} \cdot \left(1 - p^s + p^s \mathbb{I}_{\{c_2(s) \geq y\}}\right) (y - c_2(s) - b_0^2) = 0$$

First, notice that for any $b_0^1 > 0$ and $b_0^2 > 0$ if the government decides to avoid default risk by choosing $c_2(s) = y$, it reduces its budget set because then $y - c_2(s) - b_0^2 < 0$. Therefore, if debt is positive the equilibrium default risk is also positive. The optimal consumption is given by the first-order necessary conditions:

$$c_1(s) = c_2(s) = y - \frac{b_0^1 + \frac{1-p^s}{R} b_0^2}{1 + \frac{1-p^s}{R}}$$

Recall that Lemma 2 implies that $c_0 = c_1(s) = c_2(s) \forall s$. Therefore, the Markov government can implement the planner's allocation if $\frac{b_0^1 + \frac{1-p^s}{R} b_0^2}{1 + \frac{1-p^s}{R}} = \text{const}$ for any $p^s \in [0, 1]$. The latter is true if and only if $b_0^1 = b_0^2$. ■

References

Aguiar, Mark, Manuel Amador, Hugo Hopenhayn, and Ivan Werning, “Take the Short Route: Equilibrium Default and Debt Maturity”. *Working Paper*, 2016.

Alvarez, Fernando, Patrick J. Kehoe, and Pablo Andres Neumeyer, “The Time Consistency of Optimal Monetary and Fiscal Policies,” *Econometrica*, 72 (2004), 541–567.

Angeletos, George-Marios, “Fiscal Policy with Non-Contingent Debt and the Optimal Debt Maturity Structure,” *Quarterly Journal of Economics*, 2002, 117.

Arellano, Cristina, and Ananth Ramanarayanan, “Default and the Maturity Structure in Sovereign Bonds,” *Journal of Political Economy*, 120 (2012), 1105-1131.

Bhandari, Anmol, David Evans, Mikhail Golosov, and Thomas J. Sargent, “Taxes, Debts, and Redistributions with Aggregate Shocks,” *Quarterly Journal of Economics*, forthcoming.

Broner, Fernando, Guido Lorenzoni, and Sergio Schmukler, “Why Do Emerging Economies Borrow Short Term?,” *Journal of European Economics Association*, 2013, 11, 67-100.

Buera, Francisco, and Juan Nicolini, “Optimal Maturity of Government Debt without State Contingent Bonds,” *Journal of Monetary Economics*, 51 (2004), 531-554.

Calomiris, Charles and Charles Kahn, “The Role of Demandable Debt in Structuring Optimal Banking Arrangements,” *American Economic Review*, 1991, 81, 497–513.

Chatterjee, Satyajit and Burcu Eyigungor, “Maturity, Indebtedness, and Default Risk,” *American Economic Review*, 2012, 102, 2674–99.

Cochrane, John H., “A New Structure for US Federal Debt,” in *The 13 Trillion Question: Managing the U.S. Government’s Debt*, Wessel David ed. (Washington, DC: Brookings Institution Press, 2015).

Cole, Harold and Timothy Kehoe, “Self-Fulfilling Debt Crises,” *The Review of Economic Studies*, 2000, 67, 91–116.

Diamond, Douglas and Raghuram Rajan, “Liquidity Risk, Liquidity Creation and Financial Fragility: A Theory of Banking,” *Journal of Political Economy*, 2001, 109, 287–327.

Debortoli, Davide, Ricardo Nunes, and Pierre Yared, “Optimal Time-Consistent Government Debt Maturity,” *Quarterly Journal of Economics*, 2017, 132, 55–102.

Dovis, Alessandro, “Efficient Sovereign Default”, *Working paper*, 2012.

Eaton, Jonathan and Mark Gersovitz, “Debt with Potential Repudiation: Theoretical and Empirical Analysis,” *The Review of Economic Studies*, 1981, 48(2), 289–309.

Fernandez, Raquel, and Alberto Martin, “The Long and the Short of It: Sovereign Debt Crises and Debt Maturity,” *Barcelona GSE Working Paper*, No. 818, 2015.

Hatchondo, Juan Carlos and Leonardo Martinez, “Long-Duration Bonds and Sovereign Defaults,” *Journal of International Economics*, 2009, 79, 117–125.

Hebert, Benjamin and Jesse Schreger, “The Costs of Sovereign Default: Evidence from Argentina.” *Tech. rep.*, *National Bureau of Economic Research*, 2016.

Jeanne, Olivier, “Debt Maturity and the International Financial Infrastructure,” *American Economic Review*, 2009, 99, 2135-2148.

Lucas, Robert E., and Nancy L. Stokey, “Optimal Fiscal and Monetary Policy in an Economy without Capital,” *Journal of Monetary Economics*, 12 (1983), 55-93.

Lustig, Hanno, Christopher Sleet, and Sevin Yeltekin, “Fiscal Hedging with Nominal Assets,” *Journal of Monetary Economics*, 55 (2008), 710-727.

Mokyr, Joel, *The Enlightened Economy: Britain and the Industrial Revolution, 1700–1850* (London: Penguin Books, 2011).

Niepelt, Dirk, “Debt Maturity without Commitment”, *Journal of Monetary Economics*, 68 (2014), S37-S54.

Persson, Mats, Torsten Persson, and Lars E.O. Svensson, “Time Consistency of Fiscal and Monetary Policy: A Solution,” *Econometrica*, 74 (2006), 193–212.

Tauchen, George, and Robert Hussey, “Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models,” *Econometrica*, 59 (1991), 371–96.