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Doctoral Dissertations, 2011-2026 Economics

2019

# Essays on Financial Frictions in Macroeconomic Models

Mendo Lopez, Fernando Jerico

Princeton, NJ : Princeton University

http://arks.princeton.edu/ark:/88435/dsp01b5644v42b

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# ESSAYS ON FINANCIAL FRICTIONS IN MACROECONOMIC MODELS

FERNANDO JERICO MENDO LOPEZ

A DISSERTATION

PRESENTED TO THE FACULTY

OF PRINCETON UNIVERSITY

IN CANDIDACY FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE

BY THE DEPARTMENT OF ECONOMICS

ADVISER: MARKUS BRUNNERMEIER

JUNE 2019

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# Abstract

This collection of essays investigates the role of financial frictions in macroeconomic models. Chapter 1 entitled "Risk to Control Risk" explores the interactions between two fragilities that arise due to incomplete markets: amplification of real shocks and exposure to systemic bank runs. In a continuous-time macroeconomic model, I show that low measured volatility may disguise the buildup of hidden systemic run risk. Runs trigger large drops in asset prices and real production and propel the economy into a highly unstable crisis regime. Agents take on too much leverage in the hidden risk regime because they disregard their contribution to the economy's exposure to systemic runs. Surprisingly, a leverage cap can increase hidden run risk by deepening crises that follow runs. Economies exposed to less volatile fluctuations due to real shocks are more prone to systemic runs: stability breeds instability.

Chapter 2 entitled "Financial Frictions: Amplifying or Hedging Fundamental Shocks?" highlights that financial frictions lead to multiplicity in the response of economic outcomes to fundamental shocks. In particular, asset prices can amplify or hedge the effect of real shocks. The hedging equilibrium Pareto dominates the one with amplification and both are observationally equivalent when the productive agents are well capitalized. However, after a sequence of bad shocks, equilibrium selection is key. I propose a policy that ensures the hedging equilibrium prevails: commitment to support asset prices for an intermediate range of capitalization of productive agents.

In Chapter 3 entitled "Safety Traps in a Global Economy" joint with Julius Vutz, we investigate the consequences of global safe asset shortages for aggregate economic activity. The model has two countries, Home and Foreign, and emphasizes two heterogeneities: Home has (i) more developed financial markets and (ii) a smaller share of risk-averse agents compared to Foreign. Safe asset demand by Foreign causes a safety trap, i.e. a liquidity trap in the market for safe assets, which depresses output

in both countries. Safe public debt provision expands economic activity in both countries but the associated tax distortions are borne only by the issuing country. This externality results in a global under-provision of safe public debt.

# Acknowledgments

I would like to thank my advisor, Markus Brunnermeier, for his invaluable guidance and support throughout my years at Princeton. I am also indebted to Nobuhiro Kiyotaki and Mark Aguiar for insightful conversations that shaped and improved my projects. Thanks also to Oleg Itskhoki for his guidance in one of my earliest papers. I would like to thank my fellow doctoral students—those who have moved on, those in the quagmire, and those just beginning—for their support, feedback, and friendship. For chapter 1, financial support of the Alfred P. Sloan Foundation and the Macro Financial Modeling project is gratefully acknowledged.

Last, but certainly not least, I would like to thank my parents and brothers for their love, support, and encouragement, without them my journey would not have been possible.

*To my parents and brothers*

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# Chapter 1

# Risk to Control Risk

## 1.1 Introduction

*Fighting fire with fire*, a principle effectively exploited in fields such as medicine (vaccines) or firefighting (controlled burns), can be the key to controlling risk in economics. The hypothesis that prolonged periods of stability incite the buildup of hidden risks is a first-order concern among policymakers and academics, yet there is no framework that illustrates the internal consistency of this concern or the mechanism underlying it. Such a framework is necessary to incorporate its effects into risk measurement and policy design.

The seemingly paradoxical idea that stability breeds instability is widespread in policy and academic discussions. However, we have limited knowledge about the interactions between the different forms of risk. Several questions regarding these interactions remain unresolved. Do prolonged periods of macroeconomic stability (e.g. the Great Moderation) disguise the risk of sudden collapses such as the last financial crisis? If so, how can we identify this vulnerability in seemingly safe low-volatility environments? Is there a trade-off between business cycle stabilization and financial stability, i.e., do less volatile real fluctuations foster financial fragility? How effective are macroprudential policies to simultaneously control different forms of risk?

To study these questions, I set up a continuous-time macro model with risk-averse banks and households. Banks obtain higher returns from managing capital than households and these returns are subject to small exogenous shocks to capital quality which are the only real shocks in the economy. There are no equity markets, so banks finance capital holdings beyond their net worth by issuing short-term non-contingent

debt (deposits) to households. Households occasionally consider whether or not to roll over deposits (according to a sunspot that follows a Poisson process). If banks are unable to meet their deposit obligations by liquidating capital, households refuse to roll over their deposits, triggering a systemic run. Bank runs wipe out large parts of the banking system and trigger large collapses in asset prices and output.

The economy exhibits two endogenous instabilities: 1) amplification of real shocks on output and asset prices, and 2) exposure to systemic runs. The main contribution of this paper is to include both instabilities in a tractable framework in order to characterize their joint dynamics and study how they respond to changes in the underlying volatility of real shocks.

The presence and intensity of the endogenous instabilities vary according to the key endogenous state variable: the ratio of the banking sector's wealth to the aggregate value of capital (or banks' wealth share). In particular, the joint dynamics unveil three different risk regimes: 1) a highly unstable *crisis regime* characterized by the amplification of real shocks and occasional runs, 2) a *safe regime* with no endogenous risks, and 3) a *hidden risk regime* in which real shocks are not amplified but systemic runs can occur.

The crisis regime corresponds to situations in which the wealth share of the banking sector is low enough to prevent banks from holding all capital in the economy. This misallocation depresses output and asset prices. Managing the entire stock of capital would require such high leverage and risk exposure that banks refrain from doing so despite their productivity advantage. Then, banks' capital demand depends on their risk-bearing capacity, which in turn depends on their wealth. Real shocks are amplified in this regime precisely because of their impact on banks' risk-bearing capacity, which translates into changes in allocative efficiency.

Within the crisis regime, the economy is exposed to systemic runs only if the banking sector's wealth share is sufficiently large. For runs to be possible, the difference between contemporaneous asset prices and their liquidation value (i.e. their price just after a run when households manage almost all capital) needs to be large enough to imply bankruptcy. When banks' wealth share is exceptionally low, this difference is minuscule, and the economy is shielded from systemic runs. This region with amplification risk but no run risk is the exact opposite of the hidden risk regime in terms of the endogenous instabilities present.

The safe regime is associated with situations in which the banking sector's wealth share is sufficiently large to allow banks to absorb all capital in the economy and yet

maintain low leverage. The absence of misallocation implies no amplification. The soundness of banks' balance sheets prevents exposure to systemic runs in spite of the large difference between contemporaneous high asset prices and their liquidation value.

The hidden risk regime corresponds to an intermediate range of the banking sector's wealth share. It is high enough to allow banks to hold the entire capital stock but low enough to require them to choose weak balance sheets, i.e., high leverage, in order to do so. The latter combined with the high asset prices implied by the absence of misallocation exposes the economy to systemic runs, which immediately propel the system into the unstable crisis regime.

Importantly, prior to the realization of a run, the hidden risk regime is observationally equivalent to the safe regime in terms of the volatility of real and financial variables. In this sense, run risk is hidden.[1] This illustrates the potential fragility of low-volatility environments and highlights the necessity of a *risk topography* to identify risks. In particular, the model suggests the following warning signals of run risk when measured volatility is low: the joint presence of high asset prices and a highly (or, at least, moderately) levered banking sector, high excess returns per unit of measured volatility, and a low risk-free rate.

Systemic instabilities are an ergodic phenomenon in this economy. That is, the risk regimes associated with amplification and systemic runs are frequently visited by the system. In fact, the two peaks of the bimodal stationary distribution of banks' wealth share may lie within these risky regimes. The highest peak corresponds to the stochastic steady state, i.e., the balance point to which the system tends to return, and it can be located within the hidden risk regime. The economy spends prolonged periods in this regime because, in contrast to amplification risk, run risk has a limited impact on excess returns on capital over the deposit rate. Then, wealth accumulation by levered banks is not particularly accelerated in the hidden risk regime. The second peak corresponds to the situation that follows a systemic run, which is characterized by a severely undercapitalized banking sector, i.e., it lies within the crisis regime. This peak reflects that the initial phase of the recovery after a run is slow.

A comparison across economies with different levels of exogenous risk (fundamental volatility of real shocks) unveils the key insight of the model: stability breeds instability or its contrapositive, risk controls risk. Economies with the lowest exogenous risk have the most severe endogenous instabilities. Lower fundamental risk prompts

[1] Run risk is not hidden for agents in the economy.

banks to issue more debt in order to expand capital holdings. This translates into a more efficient allocation of resources and therefore into higher asset prices. In other words, lower exogenous risk fuels high leverage and asset prices, which lead to both greater amplification risk and greater run risk.

Crucially, the increase in amplification risk is not enough to overcome the initial reduction of exogenous risk, so total volatility due to real shocks decreases alongside exogenous risk. Then, the main risk for economies with low fundamental volatility of real shocks is an elevated exposure to systemic runs. On the other end of the spectrum, economies exhibiting sufficiently high exogenous risk are completely shielded from run risk, i.e., *risk controls risk*. The weak reaction of amplification risk follows from the presence of an automatic stabilization mechanism, which is absent in the case of run risk. An increase in amplification risk feeds back into more prudent equilibrium portfolios for banks, while larger exposure to systemic runs has no effect on banks' equilibrium leverage. This is ultimately related to the fact that banks disregard their contribution to run risk when making portfolio decisions (run externality).

An important insight is that the welfare ranking with respect to exogenous risk is non-monotonic. The key effect driving the result is that lower exogenous risk induces higher bank leverage (capital holdings). This has two opposing effects on welfare: 1) reduced misallocation, and 2) increased exposure to endogenous instabilities. Banks fail to adequately internalize these forces, so laissez-faire leverage can be excessive or overly restrained with respect to its efficient level. The total effect on welfare depends on the relative strengths of externalities affecting banks' portfolio decisions.

If the externality associated with misallocation is stronger than those related to instabilities, then banks' capital holdings are inefficiently restrained, and a marginal reduction of exogenous risk helps minimize this inefficiency. Otherwise, a marginal reduction of exogenous risk aggravates the economy's inefficiencies. Numerical results suggest that the run externality is the decisive factor in this comparison between externalities. This is the key behind the non-monotonicity since run risk is present only in economies where exogenous risk is sufficiently low.

Regarding policy interventions, the model reveals that a constant leverage constraint may fail to control endogenous instabilities. A leverage cap depresses the liquidation price of capital by deepening the recession that follows a run. This fuels run risk and is particularly detrimental for the economy since leverage is limited when it is needed the most. This force dampens the reduction of run vulnerability due to lower leverage in the crisis regime and, most importantly, it leads to greater exposure to runs in the

hidden risk regime (unless the constraint is tight enough to completely rule out runs). A constant leverage constraint also increases amplification risk in situations where it is not binding and households still hold some capital. Despite these shortcomings, welfare analysis suggests that a constant leverage cap can benefit both agents as long as it reduces the economy's overall exposure to runs.

The optimal state-contingent leverage cap overcomes the mentioned drawbacks by restricting banks' portfolio decisions only when the economy would be exposed to runs, i.e., in an intermediate range of banks' wealth share. Interestingly, a binding optimal leverage cap in the hidden risk regime can limit run risk but introduces amplification risk. This emphasizes the relative importance of the run externality.

I also explore a different approach to control run risk. In particular, I study how a benevolent policymaker would adjust the complete markets prescription of full stabilization when considering exposure to runs. The exercise ignores any cost of stabilization by assuming the policymaker can directly adjust fundamental volatility conditional on the strength of banks' balance sheets. This allows me to focus on the optimal adjustment given the presence of run risk. Results suggest that it is optimal to allow risk associated with real shocks if the economy is exposed to runs and to implement complete stabilization if it is not. The exercise illustrates the direction in which the possibility of runs should affect stabilization policy; however, it is not a direct policy recommendation since the model does not capture known benefits of stabilizing the economy.

***Literature***. The idea that *stability breeds instability* has been discussed at least since Minsky (1986). He argues that instability is a normal result of modern financial capitalism and casts doubts on the "fine-tuning" approach to policy. Minsky conjectures that even if policy does manage to achieve transitory stability, this stability would render financial markets complacent and susceptible to the reemergence of instability. This paper formalizes one such mechanism: a larger exposure to systemic runs.

The literature has extensively explored, separately, the two endogenous risks studied in this paper. In the case of amplification risk, the financial accelerator mechanism behind it was first discussed by Kiyotaki and Moore (1997) (henceforth KM), and Bernanke et al. (1999) (henceforth BGG). Later work by Brunnermeier and Sannikov (2014) (henceforth BruSan) identifies that the intensity of this risk varies in a highly nonlinear fashion along the business cycle. In the case of run risk, there is an ample literature studying bank runs following the seminal contribution of Diamond and Dybvig (1983). This literature works with partial equilibrium or short-horizon

macroeconomic frameworks, which are not suitable for studying endogenous risk dynamics.

Only after both risks arguably played an important role during the last financial crisis did economists start to explore them together within macroeconomic models. Prominent examples are Gertler and Kiyotaki (2015) (henceforth GK) and Gertler et al. (2017). The main interest in these papers is to compare the propagation mechanisms of real shocks with and without bank runs. In contrast, this paper focuses on the interactions between these risks and their implications for macroeconomic stability. Moreover, while the literature concentrates on the economy's response to shocks in some specific situations, this paper provides a full characterization of the endogenous risks necessary to uncover the different risk regimes.

The "volatility paradox" result in BruSan shares with this paper the general idea that low exogenous volatility of real shocks can fuel endogenous risks. However, their results differ from mine along several dimensions. First, their model predicts that every time the economy is subject to risks associated with the financial sector, the risks translate into observable volatility of financial and macroeconomic variables. That is, their model is unable to identify the buildup of hidden risks. Second, this paper raises much stronger concerns about low exogenous volatility. In BruSan, endogenous amplification persists as exogenous volatility vanishes, but this effect is not enough to increase overall volatility, so welfare improves with lower exogenous volatility. This paper shows that the welfare ranking reverses once run risk is considered since this risk is magnified as exogenous fluctuations are reduced.

The modeling approach of this paper builds on BruSan and GK. The continuous time techniques used to characterize the dynamics of risks expand work by BruSan, while the approach to include systemic runs in an infinite-horizon economy follows GK. In an international context, Brunnermeier and Sannikov (2015) study amplification risk together with a different class of discrete collapses: sudden stops of international capital flows. Their work is technically the closest to this paper since they also consider a continuous time framework with self-fulfilling runs; yet they only study unanticipated runs, i.e., their analysis disregards any feedback effects from run risk to agents' decisions or equilibrium dynamics. These feedback effects are key in this paper.

From a technical point of view, this paper expands the literature on continuous-time macro-finance models, e.g., BruSan and Hansen et al. (2018), by considering (anticipated) aggregate jump risk. This is necessary to include systemic runs, and

it is technically challenging because aggregate jump risk breaks the key feature that makes continuous-time models tractable: their focus on local perturbations.

## 1.2 Model

### 1.2.1 Environment

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that satisfies the usual conditions, and assume all stochastic processes are adapted. The economy evolves in continuous time with $t \in [0, \infty)$ and is populated by a continuum of banks and households as well as a government. There are two goods: the final consumption good and physical capital.

*A. Technology*

The production technology is linear and reflects that bankers are more efficient than households at managing capital. In particular, capital held by bankers produces

$$y_t = ak_t$$

while households' production function is $y_t = \underline{a}k_t$ with $a > \underline{a}$. In general, I use the underbar notation for parameters and variables associated with the household sector.

Physical capital evolves according to[2]

$$dk_t = (\Phi(\iota_t) - \delta)\, k_t dt + \sigma k_t dZ_t \ , \tag{1.1}$$

where $\iota_t$ is the investment per unit of capital, $\delta$ is the depreciation rate, $\sigma > 0$ is the exogenous volatility of capital growth, and $Z_t$ is a Brownian motion associated with capital quality shocks. The investment technology allows for transforming $\iota_t k_t$ units of final goods into $\Phi(\iota_t)k_t$ units of capital. Function $\Phi$ represents technological illiquidity or adjustment costs and satisfies $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$. The Brownian shock is the only source of exogenous aggregate uncertainty in the model. It affects capital quantity but not its productivity, which can be motivated by stochastic depreciation or news about capital quality. Further discussion follows in subsection 1.2.4.

[2] As is standard in the literature, this law of motion does not include capital purchases.

### *B. Preferences*

Preferences are symmetric for both types of agent. In particular, the utility that agents obtain from a stochastic consumption path $\{c_t\}_{t\geq 0}$ is

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log(c_t)dt\right],$$

where $\rho$ is the preference discount rate.

### *C. Switching identities*

Agents experience idiosyncratic Poisson shocks that switch their types, i.e., turn bankers into households and viceversa. Let $\lambda$ be the intensity of the shock that turns bankers into households and $\underline{\lambda}$ be the intensity of the inverse process. These dynamics prevent either sector from taking over the entire wealth of the economy indefinitely and therefore ensure the existence of a non-degenerate stationary distribution.

### *D. Markets and financial friction*

There are markets for physical capital and final goods. Deposit contracts are the only financial assets in this economy. A deposit contract ensures the repayment of principal and interest as long as the borrower remains solvent, and it transforms into a claim on borrowers' assets in the case of default. Note that deposits held at different banks are distinct assets because banks' exposures to default and recovery values may differ. The absence of equity markets represents a financial friction that can be motivated by a "skin in the game" constraint.

Also, I assume that banks that underperform with respect to their peers incur in a prohibitively high cost $\phi$ (measured in final goods). This allows me to focus on symmetric equilibriums among banks. Further discussion of these assumptions is presented in subsection 1.2.4.

### *E. Bank runs*

Households are occasionally concerned about the soundness of banks. In particular, they "wake up" to consider whether or not to roll over their deposits according to a Poisson process $\hat{J}_t$ with arrival rate $\hat{p}_t$. The timing of events within the "$dt$ period" is as follows.

1. Households decide whether or not to roll over their deposits. Banks sell their assets to repay deposits not rolled over. The liquidation price of capital is the one at which capital can be sold at the end of the $dt$ period.

2. Banks that are not able to fully repay deposits go into bankruptcy. Failed banks' assets are liquidated and the resources are used to pay depositors. Depositors who decided not to roll over deposits are served first.[3]

3. Failed banks receive a small transfer, which is financed by taxes on the household sector. This transfer allows them to resume operations.

4. Agents optimally rebalance their portfolios.

Whenever a bank run equilibrium is feasible, there is equilibrium multiplicity since the no-run equilibrium is always a possibility. I assume depositors coordinate on a bank run using a sunspot $\omega_t \in \{0, 1\}$. If $\omega_t = 1$, then the run equilibrium is realized. This can be interpreted as a panic among households. Since the no-run equilibrium is equivalent to households not evaluating the soundness of banks, the only relevant wake-up calls are the ones linked to panics, i.e., to a run equilibrium. I define the Poisson process $J_t$ associated with wake-up calls linked to panics as

$$d\check{J}_t = \mathbb{1}_{\{\omega_t=1\}} d\hat{J}_t$$

with arrival rate $\check{p}_t = \Pr(\omega_t = 1)\hat{p}_t$. In general, the probability of coordinating on a run equilibrium $\Pr(\omega_t = 1)$ can be a function of endogenous variables. From now on, I work with Poisson process $\check{J}_t$ and later I discuss the equilibrium selection mechanism directly in terms of $\check{p}_t$.

### *F. Transfer policy*

Let $\mathcal{T}_t K_t$ be the total transfer from the household sector to banks that go bankrupt due to a systemic run, where $K_t$ is the aggregate capital in the economy and $\mathcal{T}_t$ is chosen by the government. The transfer is financed via taxes levied on households, and the government runs a balanced budget at each point in time.

[3]If depositors recover the same value independently of their rollover decision, they are always indifferent as to whether banks fail (no loss on deposits' value) or not (recovery rate depends on liquidation value). There are several (off-equilibrium) assumptions that break the indifference if the bank will fail. I assume sequential services in two stages: depositors who do not roll over deposits are paid before the rest. Alternatively, I could assume that banks can divert a fraction of deposits that are rolled over just before collapse.

Taxes and transfers are proportional to the net worth agents had before the systemic run in order to preserve tractability. Denote $\tau_t$ as the transfer per unit of net worth to banks and $\underline{\tau}_t$ the tax per unit of net worth levied on households. The transfer can be interpreted as a direct bailout of the banking sector or as recognition that banks are able to rescue some value after a collapse.

### 1.2.2 Agents' problems

*A. Debt rollover decision and run vulnerability*

***Liquidation value.*** The liquidation value of capital corresponds to the price of capital after a systemic run, i.e., the price when almost all capital is managed by unproductive households (except for the capital banks can acquire using the small transfer). I denote the liquidation value of capital in terms of the numeraire (final goods) as $\tilde{q}_{t-}$.

I use notation $z_{t-} \equiv \lim_{s \nearrow t} z_s$ to refer to the value of variables just before a jump.[4] The liquidation price is known before the uncertainty regarding the Poisson shock is unveiled. The liquidation price is equal to the capital price $q_t$ if the systemic run is realized, i.e., $\tilde{q}_{t-} = q_t$ if $d\check{J}_t = 1$ and the economy is vulnerable to runs (see condition below).

***Debt rollover decision***. If a bank is not able to meet its deposit obligations at the liquidation price for its assets, i.e.,

$$\tilde{q}_{t-} k_{t-} < b_{t-},$$

where $b_{t-}$ is the value of deposits at the bank and $k_{t-}$ are the banks' capital holdings, then households decide not to roll over their deposits at this bank. In this case, households recover $\tilde{q}_{t-} k_{t-}/b_{t-}$ per deposit. If households had rolled over deposits, they wouldn't have been able to recover any value from their deposits since there are no resources left after serving depositors who decided not to roll over deposits. Importantly, households are only willing to pay the liquidation value $\tilde{q}_{t-}$ to banks selling capital to meet their deposit obligations. If this run condition is not met, then households roll over their deposits in this bank after the wake-up call.

[4]This means $z_{t-}$ is measurable with respect to the filtration associated with $\{Z_s, J_s : s \in [0, t)\}$.

The run condition can also be written in terms of the percentage loss on deposits in case of a run, i.e.,

$$\ell^d_{t-}(k_{t-}, b_{t-}) \equiv \left[1 - \frac{\tilde{q}_{t-}k_{t-}}{b_{t-}}\right]^+ > 0, \tag{1.2}$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$. I use $\ell^d_{t-}$ to denote the loss on deposits evaluated at banks' equilibrium portfolio $(k_t, b_t)$. For convenience, also define the capital price drop following a systemic run as

$$\ell^q_{t-} \equiv 1 - \frac{\tilde{q}_{t-}}{q_{t-}},$$

where $q_t$ is the contemporaneous capital price.

***Run vulnerability.*** The liquidation price $\tilde{q}_t$ at which banks' solvency is evaluated assumes that the entire banking sector fails (and resumes operation only with the net worth provided by the transfer). Therefore, consistency requires that this is indeed the case, i.e., that all banks collapse in case of a run. This condition can be characterized in terms of the aggregate percentage loss on deposits after a systemic run[5]

$$\mathcal{L}^d_{t-} \equiv \left[1 - \frac{\tilde{q}_{t-}K^b_{t-}}{B_{t-}}\right]^+ > 0,$$

where $B_t$ is the aggregate value of deposits at the banking sector and $K^b_t$ represents banks' aggregate capital holdings.[6] Systemic runs occur if and only if there is a wake-up call linked to a panic ($d\check{J}_t = 1$) and the banking sector is sufficiently vulnerable ($\mathcal{L}^d_{t-} > 0$).

***Equilibrium selection mechanism.*** I assume

$$\check{p}_{t-} = \Gamma(\mathcal{L}^d_{t-}), \tag{1.3}$$

where $\Gamma(.)$ is continuous and satisfies $\Gamma(0) = 0$ and $\Gamma'(\cdot) \geq 0$, i.e., households are more likely to coordinate on the bank run equilibrium if potential losses for them are higher. Note that $\mathcal{L}^d_{t-}$ is an aggregate variable, which cannot be influenced by any individual agent alone. The equilibrium selection mechanism chosen has a key role for welfare analysis and numerical exercises, but qualitative features about joint dynamics of instabilities will be independent from it. Further discussion about this assumption is presented in subsection 1.2.4.

[5]Given that in equilibrium all banks will choose the same portfolio, the average loss will be equal to the individual loss of each agent.

[6]The formal definition of aggregate variables is provided in subsection 1.2.3.

### *B. Households*

Households choose their consumption rate $c_t$, investment rate $\iota_t$, capital holdings $\underline{k}_t$, and deposits $b_t$. Let $\underline{n}_t$ denote a household's net worth, which is the only individual state variable for this problem.[7] Also, let $\underline{T}$ be the stochastic arrival time of the idiosyncratic shock that turns a household into a bank. For a given $\underline{n}_s$, a household solves

$$\underline{V}_s\left(\underline{n}_s\right) \equiv \max_{\{\underline{c}_t, \underline{\iota}_t \geq 0, \underline{k}_t \geq 0, \underline{b}_t\}_s^{\underline{T}}} \mathbb{E}_s\left[\int_s^{\underline{T}} e^{-\rho(t-s)} \log(\underline{c}_t) dt + V_T\left(\underline{n}_T\right)\right] \tag{1.4}$$

s.t.

$$\begin{aligned} d\underline{n}_t = & \left[\left(\underline{a} - \underline{\iota}_t\right)\underline{k}_t + r_t \underline{b}_t - \underline{c}_t\right] dt + d(q_t \underline{k}_t)\Big|_{d\check{J}_t = 0} \\ & - \mathbb{1}_{\{\mathcal{L}^d_{t-} > 0\}}\left[\ell^d_{t-}\underline{b}_{t-} + \ell^q_{t-} q_{t-}\underline{k}_{t-} + \underline{\tau}_{t-}\underline{n}_{t-}\right] d\check{J}_t \qquad \forall t \leq \underline{T} \\ \underline{n}_t = & \, q_t \underline{k}_t + \underline{b}_t, \end{aligned}$$

where $V_t\left(.\right)$ is banks' value function, defined below. The last term of the dynamic budget constraint represents the cost households bear in case of a systemic run: losses on deposits $\ell^d_{t-}\underline{b}_{t-}$, on capital value $\ell^q_{t-} q_{t-}\underline{k}_{t-}$, and taxes levied to bail out banks $\underline{\tau}_{t-}\underline{n}_{t-}$. The other terms are standard and correspond to production net of investment $\left(\underline{a} - \underline{\iota}_t\right)\underline{k}_t$, return on deposits $r_t \underline{b}_t$, consumption expenditure $\underline{c}_t$ and capital gains absent runs $d(q_t \underline{k}_t)\Big|_{dJ_t = 0}$.

For simplicity, this formulation assumes that all banks choose the same portfolio and therefore there is a unique deposit contract available for households with return $r_t$ conditional on no default. In general, deposits at banks with different portfolios render different returns, i.e., they are different assets, because the portfolio determines the recovery rate of deposits in case of a run.

### *C. Banks*

Banks choose their consumption rate $c_t$, investment rate $\iota_t$, capital holdings $k_t$, and debt $b_t$ (deposits). Let $n_t$ denote the household's net worth, which is the only

[7] The shock structure (capital quality shocks instead of productivity shocks) and the linear production technology allow for reducing individual states from capital and deposits to only net worth.

individual state variable for this problem. Also, let $T$ be the stochastic arrival time of the shock that turns the bank into a household. Given $n_s$, a bank solves

$$V_s(n_s) \equiv \max_{\{c_t,\iota_t\geq 0,k_t\geq 0,b_t\}_s^T} \mathbb{E}_s\left[\int_s^T e^{-\rho(t-s)}\log(c_t)dt + \underline{V}_T(n_T)\right] \tag{1.5}$$

s.t.

$$\begin{aligned} dn_t = &\left[(a-\iota_t)k_t - r_t(k_t,b_t)b_t - c_t\right]dt + d(q_t k_t)\Big|_{d\check{J}_t=0} \\ &- \mathbb{1}_{\{\mathcal{L}_{t-}^d>0\}}\left[\mathbb{1}_{\{\ell_{t-}^d(k_{t-},b_{t-})>0\}}(1-\tau_{t-})n_{t-}d\check{J}_t + \mathbb{1}_{\{\ell_{t-}^d(k_{t-},b_{t-})\leq 0\}}\phi dt\right] \\ &\forall t \leq T \\ n_t = &\, q_t k_t - b_t \end{aligned}$$

The last term of the dynamic budget constraint is the one associated with runs, and it is relevant only if the economy is vulnerable to runs, i.e., $\mathcal{L}_t^d > 0$. In these situations, the bank can choose a portfolio $(k_t, b_t)$ that implies bankruptcy and losses for depositors after a systemic run, i.e., $\ell_t^d(k_t, b_t) > 0$. In this case, a run wipes out the bank's net worth $n_{t-}$ but allows it to receive a transfer $\tau_{t-}n_{t-}$. Alternatively, the bank can choose a safe portfolio that allows it to avoid bankruptcy in case the run is realized ($\ell_t^d(k_t, b_t) \leq 0$) and pay the underperforming cost $\phi$. A safe portfolio implies underperforming with respect to banks choosing a risky portfolio and exposing the economy to a systemic run.

The other terms in the constraint are standard and analogous to that described for the household. The only difference is that the banks' portfolio decision affects the rate paid on deposits $r_t(k_t, b_t)$ because it influences the depositors' loss rate in case of a run. This function needs to be consistent with households' asset-pricing conditions.

### 1.2.3 Equilibrium

***Notation.*** Denote the set of banks by the interval $\mathbb{I}_t = [0, \nu_t]$ and index individual banks by $i \in \mathbb{I}_t$. Similarly, let the set of households be the interval $\mathbb{J}_t = (\nu_t, 2]$ and index individual households by $j \in \mathbb{J}_t$. Let $\mathbb{I}$ be the interval including the index of all agents who are banks at some point and $\mathbb{J}$ be the corresponding interval for households.[8]

[8]The threshold $\nu_t$ captures the transition between types and satisfies $d\nu_t = (\underline{\lambda}(1-\nu_t) - \lambda\nu_t)\,dt$ with $\nu_0 = 1$. This representation assumes agents don't know their own index so they cannot anticipate their switching time. It also assumes only net flows between agents type take place. Alternative representations are possible, but imply reassigning indexes to keep intervals connected.

The following notation is necessary because agents alternate types. Let $T_n^u$ be the $n^{th}$ point in time in which agent $u \in [0,2]$ experiences an idiosyncratic shock that changes his type. Define $\mathbb{B}^u \equiv \{[T_{n-1}^u, T_n^u] : \ u \text{ is a banker}\}$ as the set of all intervals in which agent $u$ is a banker, and denote an element of this set as $\mathcal{B}_m^u$. Similarly, define set $\mathbb{H}^u$ with elements $\mathcal{H}_m^u$ to be the set of time intervals in which the agent is a household. I abuse notation and also use $\mathbb{B}^u = \cup_m \mathcal{B}_m^u$ and $\mathbb{H}^u = \cup_m \mathcal{H}_m^u$.

***Equilibrium definition.***[9] Fix a transfer policy $\{\mathcal{T}_t\}$. For any initial endowment of capital $\{k_0^i, \underline{k}_0^j : i \in \mathbb{I}_0, j \in \mathbb{J}_0\}$ such that

$$\int_{\mathbb{I}_0} k_t^i di + \int_{\mathbb{J}_0} \underline{k}_0^j dj = K_0$$

an equilibrium is a set of stochastic functions[10] on the filtered probability space defined by aggregate shocks $\{Z_t, J_t : t \geq 0\}$: capital price $\{q_t\}$ and liquidation value $\{\tilde{q}_t\}$, deposit rate function $\{r_t(\cdot)\}$, losses function $\{\ell_t^d\}$, arrival intensity $\{\check{p}_t\}$, transfers and taxes $\{\tau_t, \underline{\tau}_t\}$, households' decisions $\{\underline{c}_t^j, \underline{\iota}_t^j, \underline{k}_t^j, \underline{b}_t^j\}_{t \in \mathbb{H}^j}$ and value function $\{\underline{V}_t^j(\cdot)\}$ for $j \in \mathbb{J}$, and banks' decisions $\{c_t^i, \iota_t^i, k_t^j, b_t^j\}_{t \in \mathbb{B}^i}$ and value function $\{V_t^i(\cdot)\}$ for $i \in \mathbb{I}$ such that

1. Initial net worths satisfy $n_0^i = q_0 k_0^i$ and $n_0^j = q_0 k_0$ for all $i \in \mathbb{I}_0$ and $j \in \mathbb{J}_0$.
2. Agents optimize
   (a) Households: Given stochastic functions $\left\{q_t, \tilde{q}_t, r_t, \ell_t^d, \check{p}_t, V_t^i(\cdot)\right\}$, decisions $\{\underline{c}_t^j, \underline{\iota}_t^j, \underline{k}_t^j, \underline{b}_t^j\}_{t \in \mathcal{H}_m^j}$ solve households' problem for all $j \in \mathbb{J}$, $\mathcal{H}_m^j \in \mathbb{H}^j$.
   (b) Banks: Given stochastic functions $\{q_t, \tilde{q}_t, r_t(\cdot), \check{p}_t, \underline{V}_t^j(\cdot)\}$, decisions $\{c_t^i, \iota_t^i, k_t^j, b_t^j\}_{t \in \mathcal{B}_m^i}$ solve banks' problem for all $i \in \mathbb{I}$, $\mathcal{B}_m^i \in \mathbb{B}^j$.
3. Markets clear
   (a) Goods

$$\int_{\mathbb{I}_t} c_t^i di + \int_{\mathbb{J}_t} \underline{c}_t^j dj + \phi \mathbb{1}_{\{\mathcal{L}_t^d > 0\}} \int_{\mathbb{I}_t} \mathbb{1}_{\ell_t^d < 0} di = \int_{\mathbb{I}_t} (a - \iota_t^i) k_t^i di + \int_{\mathbb{J}_t} (\underline{a} - \underline{\iota}_t^j) \underline{k}_t^j dj$$

[9] The definition includes the symmetry assumptions embedded in agents' problems. In particular, when defining the asset prices available to an agent type, it assumes a symmetric portfolio decision of the other one. This simplifies considerable notation and exposition.

[10] Stochastic functions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are mappings $X : S \times \Omega \to \mathbb{R}$ such that $X(s,.)$ is random variable $\forall s \in S$. Stochastic processes are the special case where $S = \mathbb{R}_+$, which is usually interpreted as time. In this definition, $\left\{q_t, \tilde{q}_t, p_t, \ell_t^d\right\}$ are stochastic processes, but other objects are associated with different definitions of $S$: $\{V_t(\cdot), \underline{V}_t(\cdot)\}$ with $S = \mathbb{R}^2$,$\{r_t(\cdot)\}$ with $S = \mathbb{R}^3$, $\{c_t^i, \iota_t^i, x_{t-}^i\}$ with $S = \mathbb{B}^i$, and $\{\underline{c}_t^j, \underline{\iota}_t^j, \underline{x}_{t-}^j\}$ with $S = \mathbb{H}^j$ .

(b) Capital

$$\underbrace{\int_{\mathbb{I}_t} k_t^i di}_{\equiv K_t^b} + \underbrace{\int_{\mathbb{J}_t} \underline{k}_t^j dj}_{\equiv \underline{K}_t} = K_t$$

(c) Deposits

$$\int_{\mathbb{I}_t} b_t^i di = \int_{\mathbb{J}_t} \underline{b}_t^j dj$$

4. The law of motion of aggregate capital is

$$dK_t = \left( \int_{\mathbb{I}_t} \Phi(\iota_t^i) k_t^i di + \int_{\mathbb{J}_t} \Phi(\underline{\iota}_t^j) \underline{k}_t^j di - \delta K_t \right) dt + \sigma K_t dZ_t$$

5. Consistency conditions:

   - Value functions $\{V_t^i(\cdot), \underline{V}_t^j(\cdot)\}$ are consistent with agents' optimal decisions, i.e., they satisfy (1.4) and (1.5).
   - Deposit rate function $r_t(k_t, b_t)$ is consistent with asset-pricing conditions of households.
   - Arrival intensity of Poisson process $\{\check{p}_t\}$ is consistent with the selection mechanism chosen.
   - Liquidation value $\{\tilde{q}_t\}$ is consistent with the capital price process $\{q_t\}$, i.e. $\tilde{q}_{t-} = q_t$ if $\mathbb{1}_{\{\mathcal{L}_{t-}^a > 0\}} d\check{J}_t = 1$.

6. Total transfers are consistent with the government's policy and its budget is balanced

$$\mathcal{T}_t K_t = \tau_t \int_{\mathbb{I}_t} n_t^i di = \underline{\tau}_t \int_{\mathbb{J}_t} \underline{n}_t^j dj$$

### 1.2.4 Discussion of assumptions

*Productivity difference.* Banks' higher productivity captures the advantage that the banking sector has in allocating resources to productive agents. It summarizes the real contribution of the financial sector to the economy. Ideally, the banking sector would intermediate all resources to achieve the most efficient allocation.

*Capital quality shocks.* The model features shocks to the quantity of capital instead of the more standard productivity shocks. This assumption is made for tractability and is typical in the macro-finance literature. This formulation together with the linearity

of the production function allows the economy to scale with the capital stock; i.e., it reduces the state space by eliminating the aggregate capital stock as a state variable. An interpretation is that capital is measured in efficiency units and the shocks are news about its quality. A positive capital quality shock behaves similarly to a very persistent positive productivity shock that induces a higher utilization rate.

*Financial friction.* The financial friction is that agents cannot issue equity to raise funds. This can be thought of as a limited case of a "skin in the game" constraint, which can be microfounded by an agency problem in which banks are able to divert a fraction of asset returns at the expense of households (see, for example, BruSan). Due to this constraint, markets are incomplete, so agents cannot write contracts conditional on the aggregate capital quality shock. Market incompleteness links the allocation of productive resources to the allocation of risk since agents need to bear the risk associated with capital in order to use it for production.

*Cost of underperforming the market.* If the economy is exposed to systemic runs, banks' portfolio choices will have two local maxima: 1) one that exploits excess returns but exposes them to bankruptcy in case the run is realized, and 2) one that allows them to survive a run but forgo high contemporaneous excess returns. Given that systemic runs are rare events, consistently choosing the second investment strategy implies underperforming the market for a prolonged period, e.g., 40 years with baseline parameters. The cost of underperformance captures the fact that no asset manager can stay in his job long following such a strategy. Recent history is replete with examples of financial institutions that have chosen to underperform the market while waiting for a large collapse but were not able to sustain the strategy long enough, such as Tiger Management funds[11] during the dot-com bubble.

The underperformance cost does not force the economy to be exposed to systemic runs. It simply allows me to focus on symmetric equilibria among banks. Absent this cost, the economy would be exposed to runs in the same situations, but only a fraction of the banking sector would fail after a run.

*Transfer policy.* The transfer policy ensures that, after a bankruptcy, banks are able to recover some value at the expense of their creditors. This policy allows banks to be included in the welfare analysis since the obvious alternative is that, after a systemic run, some banks' wealth drops to zero, leading to a minus infinity value for utility.

[11] In March 2000, Tiger Management LLC closed its six hedge funds after poor return performances that followed from staying out of Internet and technology stocks. Julian H. Robertson, manager and co-founder, correctly anticipated the collapse to come but could not stay in business long enough to profit from his prediction.

A common solution is to assign an arbitrary utility value in these situations, e.g., assuming that bankers die and their continuation payoff is zero. This is not suitable with an endogenous arrival intensity of the death event, since the arbitrary value influences the preference ranking between situations with different exposures to such a event. Alternatively, one can focus on households for welfare analysis and disregard the transfer policy, i.e., set $\mathcal{T} = 0$.[12]

*Equilibrium selection mechanism.* The dependence of the selection mechanism on the aggregate average loss of deposits relates the fundamental that determines the existence of a run equilibrium to the likelihood such an equilibrium would be realized. This follows the spirit of global games which application would deliver the probability of a run as a function of the relevant fundamental without the necessity of a sunspot.[13] Importantly, for a systemic run to be self-fulfilling, it needs to generate a drop in asset prices, which cannot be influenced by individual decisions. This implies that the relevant fundamental needs to be an aggregate variable. The approach follows work by GK. Welfare results depend on this assumption, but qualitative results about joint dynamics of risk do not.

## 1.3 Recursive equilibrium solution

*A. Aggregate state*

As mentioned before, the shock structure and the linear production function eliminate capital $k$ as a state variable (scale invariant) for each individual household and bank. Therefore, the only individual state variable of an agent's problem is his net worth. So, in general, the aggregate state of this economy will be the distribution of net worth. Fortunately, the state space of this economy can be simplified to a single aggregate state variable. First, all agents' decisions will be linear in their net worth, which makes the net worth heterogeneity within sectors irrelevant. In other words, it reduces the set of aggregate states to banks' aggregate net worth $N_t$ and households' aggregate net worth $\underline{N}_t$. Second, I can instead use total capital in the economy $K_t$ and the net worth share of banks

$$\eta_t \equiv \frac{N_t}{\underline{N}_t + N_t}$$

[12] In this case, the off-equilibrium cost of underperforming would need to diverge toward infinity.
[13] A direct application of global games in this model is not pursued due to technical complications.

as states of the economy. The reason is that capital is the only asset in positive net supply, and therefore aggregate net worth is equal to the total value of the capital stock. Third, I can dispense with $K_t$ as an aggregate state because allocations (scaled by capital) do not depend on the level of the capital stock. This result follows from the linear production technology and the linearity of decisions in net worth. To summarize, the only aggregate state is the banking sector's wealth share $\eta_t$.

From now on, I switch to recursive notation, i.e., time subindexes are suppressed. All equilibrium objects are functions of the aggregate state of the economy $\eta$, but this dependence is left implicit. Also, I denote by $\tilde{z}$ the value that variable $z$ would take if the sunspot were realized when the aggregate state is $\eta$, i.e., $\tilde{z} = z(\tilde{\eta}(\eta))$, where $\tilde{\eta}(\eta)$ is banks' wealth share just after a systemic run is realized.

### *B. Risks*

The economy is subject to two different types of aggregate risk. The first one corresponds to the exogenous real shock and its propagation mechanism. In this case, the risk measure associated with a variable of interest is its sensitivity to that shock, which I refer to as ***total diffusion risk*** or ***diffusion volatility.*** This risk can be decomposed into an exogenous component and an endogenous one. The former represents the direct effect of the shock absent any change in the behavior of agents, while the latter is the additional sensitivity generated by agents' endogenous responses, which generates amplification. For concreteness, I refer to these components as ***exogenous risk*** and ***amplification risk***, respectively. Note that these risks are defined with respect to a variable of interest, e.g., returns on capital or output growth.

The second type of aggregate risk corresponds to vulnerabilities that arise due to the system's internal dynamics without any real exogenous impulse, e.g., systemic runs. This type of risk can materialize whenever the economy is sufficiently fragile. In the case of runs, this fragility is captured by the losses depositors would experience in case a run is realized, and it manifests through the arrival rate of runs.[14] I refer to the latter as ***run risk***, and it constitutes a risk measure not associated with a particular variable of interest. Of course, the sensitivity of a variable with respect to the realization of the run is also informative about the risk a run represents, but I focus the discussion on the the arrival rate of runs.

[14]In fact, losses for depositors and the arrival rate of runs are positively related through the equilibrium selection mechanism chosen. As discussed before, this assumption is natural since it links the vulnerability's determinant with the likelihood of the risk being realized.

Importantly, the two types of risks are qualitatively different. While exogenous real shocks are small and frequent, systemic runs are rare events that occur suddenly and have a discrete impact on real and financial variables. Continuous time allows the model to sharply distinguish between these two types of risk by associating real shocks to a diffusion process and systemic runs to a jump process.

### *C. Capital price and returns*

*Poisson process.* To alleviate notation, from now on I work with Poisson process $J_t$ that captures the realization of systemic runs. Runs require a panic wake-up call $d\check{J}_t = 1$ and a vulnerable economy $\mathcal{L}^d > 0$, i.e. $dJ_t = \mathbb{1}_{\{\mathcal{L}^d>0\}} d\check{J}_t$. The associated arrival rate is $p = \mathbb{1}_{\{\mathcal{L}^d>0\}}\check{p}$. Note that the selection mechanism considered implies $p_t = \hat{p}_t = \Gamma(\mathcal{L}^d)$ because $\Gamma(0) = 0$.

*Physical capital.* Return on capital managed by banks is

$$dr^k \equiv \underbrace{\frac{(a-\iota)\,k}{qk}}_{\text{dividend}} dt + \underbrace{\frac{d(qk)}{qk}}_{\text{capital gains}}, \tag{1.6}$$

where the first term represents capital's dividend yield net of investment and the second term is the capital gains rate, which includes price and capital quantity fluctuations. Returns on capital managed by households, $d\underline{r}^k$, are defined analogously using their lower productivity $\underline{a}$ and corresponding investment rate $\underline{\iota}$.

Given the uncertainty driving the economy, I conjecture the following dynamics for the capital price process

$$\frac{dq}{q} = \mu^q dt + \sigma^q dZ_t - \ell^q dJ_t, \tag{1.7}$$

where $\mu_t^q$ is the capital price growth conditional on no aggregate shocks, $\sigma^q$ is the sensitivity of the capital price growth to the aggregate capital quality shock $dZ_t$, and $\ell^q$ is the percentage loss on capital value when a bank run is realized. $\{\mu^q, \sigma^q, \ell^q\}$ are endogenous objects, i.e., functions of the aggregate state. Liquidation value corresponds to $\tilde{q} = q(\tilde{\eta}) = q(\eta)\,(1 - \ell^q(\eta))$.

Then, using capital evolution (1.1), capital return for banks can be written as

$$dr^k = \mu^R dt + \underbrace{(\underbrace{\sigma}_{\text{exogenous}} + \underbrace{\sigma^q}_{\text{amplification}})}_{\text{difussion risk}} \underbrace{dZ_t}_{\text{real shock}} - \underbrace{\ell^q}_{\text{loss}}\,\underbrace{dJ_t}_{\text{runs}} \tag{1.8}$$

where

$$\mu^R \equiv \frac{a-\iota}{q} + \Phi(\iota) - \delta + \mu^q + \sigma\sigma^q,$$

The expected return conditional on no aggregate shocks, $\mu^R$, includes a dividend component (first term) and deterministic capital gains. The latter are associated with deterministic quantity gains $\Phi(\iota)-\delta$, deterministic price gains $\mu^q$, and the interaction between unexpected quantity and price gains (Ito's term, related to the nonlinearity of $value = price \times quantity$).

The dynamics of capital return show the different risks discussed above. In this economy, the exogenous aggregate shocks are the capital quality shocks, $dZ_t$, and the endogenous aggregate shocks are the bank runs coordinated by Poisson shocks $dJ_t$. Total diffusion risk of capital returns, $|\sigma + \sigma^q|$, includes exogenous risk, $\sigma$, and amplification risk, $\sigma^q$. The former represents the sensitivity of capital quantity to the shock, while the latter represents the sensitivity of capital price. Given that $\sigma > 0$, the price response amplifies the effect of the initial shock on returns whenever a price increase follows a positive quality shock, i.e., $\sigma^q > 0$.[15] If the economy experiences a systemic run, i.e. $dJ_t = 1$, total capital value decreases proportionally to the percentage price drop $\ell^q$ because there is no effect on quantity (no direct real effect).

The capital returns (in the absence of shocks) obtained by households are identical to those obtained by banks in terms of capital gains, but the dividend component is different due to a lower productivity and a potentially different investment rate, i.e.,

$$\underline{\mu}^R \equiv \frac{\underline{a}-\underline{\iota}}{q} + \Phi(\underline{\iota}) - \delta + \mu^q + \sigma\sigma^q.$$

### *D. Consumption and real investment*

*Consumption.* Optimal consumption for both agents is proportional to their wealth, i.e.,

$$\frac{c}{n} = \frac{\underline{c}}{\underline{n}} = \rho. \tag{1.9}$$

Logarithmic preferences imply that consumption decisions are independent of asset returns.

[15] The literature focuses on the equilibrium that exhibits amplification, i.e., $\sigma^q > 0$, but there is another one where capital price movements hedge real shocks, i.e., $\sigma^q < 0$. Mendo (2018) discusses hedging equilibrium in financial frictions models.

*Investment.* The return on capital for both agents is maximized by choosing the investment rate that solves

$$\max_{\iota} \; q\Phi(\iota) - \iota.$$

The first-order condition $q\Phi'(\iota) = 1$ (marginal Tobin's Q) equates the marginal benefit of investment (extra capital $\Phi'(\iota)$ times its price $q$) with its marginal cost (a unit of final goods). The concavity of adjustment cost $\Phi(\cdot)$ implies that investment is an increasing function of the price of capital (if $\iota > 0$), which can be written as

$$\iota = \underline{\iota} = \underbrace{\left[\Phi'^{-1}\left(\frac{1}{q}\right)\right]^{+}}_{\equiv \iota(q)}. \tag{1.10}$$

The investment decision is a completely static problem (it only depends on the current capital price) because the investment process has no delays.

### *E. Households' portfolio decision*

*Capital portfolio share.* The portfolio decisions, i.e., capital holdings $\underline{k}_t$ and deposits $\underline{b}_t$, can be summarized by the capital portfolio share $\underline{x} \equiv \frac{q\underline{k}}{\underline{n}}$. Given $\underline{x}$ and individual state $\underline{n}$, capital holdings are $\underline{k} = \underline{x}\underline{n}/q$ and deposits $\underline{b} = (1-\underline{x})\underline{n}$. Also, note that $\ell^d(k, b)$ presented in equation (1.2) can be written in terms of $x$ alone as

$$\ell^d(x) = \left[1 - \frac{q(\tilde{\eta})}{q}\left(\frac{x}{x-1}\right)\right]^{+}. \tag{1.11}$$

The optimal portfolio share of capital for households $\underline{x}$ satisfies

$$\underline{\mu}^R - r \leq \underbrace{\underline{x}\left(\sigma + \sigma_q\right)^2}_{\text{diffusion risk compensation}} + \underbrace{p\frac{\tilde{\underline{\Lambda}}}{\underline{\Lambda}}\left(\ell^q - \ell^d\right)}_{\text{run risk compensation}}, \tag{1.12}$$

where $\underline{\Lambda} \equiv \underline{c}^{-1} = \rho\underline{n}^{-1}$ denotes household's stochastic discount factor. The LHS is the market excess return (conditional on no aggregate shocks) of capital over deposits and the RHS is the compensation for risk required by the households to hold capital. If $\underline{x} > 0$, the condition holds with equality and the excess return needs to compensate for diffusion risk and run risk.

Both compensations can be expressed as *risk price* times *risk quantity.* The prices of all risks are associated with the agent's stochastic discount factor. In the case of

diffusion risk, the price is measured by the sensitivity of the agent's net worth to this risk, i.e. $\underline{x}(\sigma+\sigma^q)$, while its quantity is the sensitivity of the asset return to diffusion risk, i.e., $\sigma+\sigma^q$. In the case of run risk, the price is the arrival rate of the run $p$ times a measure of the drop in net worth when a run is realized, i.e.,

$$\frac{\underline{\Lambda}}{\Lambda} = \frac{\underline{n}}{\tilde{\underline{n}}} = \left(1-\ell^d-\underline{x}(\ell^q-\ell^d)-\underline{\tau}\right)^{-1}$$

while the quantity of risk is the excess loss of capital over deposits, i.e., $\ell^q-\ell^d$. The following lemma provides a formal characterization of households' portfolio decision.

**Lemma 1.1.** *Household's optimal capital portfolio share is*

$$\underline{x} = \frac{1}{2}\left[\left(x^{nr}+x^{th}\right) - \sqrt{(x^{th}-x^{nr})^2+4p(\sigma+\sigma^q)^{-2}}\right]^+, \tag{1.13}$$

*where* $x^{nr} = \frac{\underline{\mu}^R-r}{(\sigma+\sigma^q)^2}$ *is the optimal portfolio decision absent runs, i.e.* $p=0$*, and* $x^{th} = \frac{1-\ell^d-\underline{\tau}}{\ell^q-\ell^d}$ *is the threshold capital share above which the household's net worth collapses to zero in case of a systemic run. Moreover,* $\underline{x} \leq min\{x^{nr}, x^{th}\}$*. Households' capital demand is decreasing in the run intensity* $p$*, diffusion volatility* $|\sigma+\sigma^q|$*, and capital loss* $\ell^q$*.*

*Pricing of deposits.* Recall that the recovery value of a bank's deposits depends on the composition of its portfolio. In equilibrium, households have access to a unique type of deposit because all banks choose the same portfolio; however, banks need to know how households would price deposits if they were to choose a different portfolio. The asset-pricing condition for a bank's portfolio with capital share $x$ can be written as

$$r(x)-r^f = p\frac{\tilde{\underline{\Lambda}}}{\underline{\Lambda}}\ell^d(x), \tag{1.14}$$

where $r^f \equiv -\mathbb{E}\left[\frac{d\underline{\Lambda}}{\underline{\Lambda}}\right]$ is the return households require for a risk-free asset. The spread of deposits over the risk-free rate needs to compensate households for the run risk and is increasing in the likelihood of the run $p$, the net worth drop $\tilde{\underline{\Lambda}}/\underline{\Lambda}$, and the loss rate on deposits $\ell^d(x)$. An intuitive derivation of this condition follows from including a risk-free asset with return $r^f$ and a deposit with return $r(x)$ and loss $\ell^d(x)$ in households' problem.

*F. Banks' portfolio decision*

Capital portfolio share for banks is $x \equiv \frac{qk}{n}$. Given $x$ and the individual state $n$, capital holdings are given by $k = xn/q$ and debt (deposits) by $b = (x-1)n$. The optimal portfolio share for banks satisfies

$$\mu^R - r(x) = \underbrace{x(\sigma + \sigma_q)^2}_{\text{diffusion risk compensation}} + \underbrace{(x-1)r'(x)}_{\text{increase in deposits' cost}} . \tag{1.15}$$

The LHS is the market excess return (conditional on no aggregate shocks) of capital over deposits and the RHS is the excess return required by households in order to hold capital. The logic behind the compensation for diffusion risk is analogous to that in the case of households. Recall that banks' portfolio decisions influence the deposit rate they need to pay households due to their effect on potential losses for depositors. The compensation for the increase in the cost of deposits is the marginal increase in the deposit rate, $r'(x) > 0$, times the portfolio share of deposits, $(x-1)$.

The first-order condition presented assumes that the cost of underperforming $\phi$ is large enough, i.e., it satisfies condition (1.32) in Appendix A. When the economy is exposed to runs, the optimization with respect to capital's portfolio share features two local maxima: a safe one that implies no failure after a systemic run, and a risky one that triggers bankruptcy if the systemic run is realized. In the absence of a cost $\phi > 0$ of underperforming the market, this feature can imply an asymmetric equilibrium among banks, i.e., one in which a fraction of banks choose the optimal risky portfolio and the rest choose the optimal safe one. Choosing the optimal safe portfolio implies underperforming the market and assuming the corresponding cost. The assumption regarding the cost $\phi$ rules out the safe portfolio choice. The following lemma characterizes the optimal portfolio decision.

**Lemma 1.2.** *For a sufficiently large underperforming cost $\phi$, banks' optimal portfolio weight on capital $x$ satisfies*

$$x = \frac{1}{(\sigma + \sigma^q)^2}\left[\mu^R - r^f - p\frac{\tilde{\Lambda}}{\underline{\Lambda}}\ell^q\right]. \tag{1.16}$$

*banks' capital demand is decreasing in the run intensity $p$, capital loss $\ell^q$, and diffusion volatility $|\sigma + \sigma^q|$. A sufficiently large underperforming cost is defined by equation (1.32) in Appendix A.*

### G. Wealth dynamics

***Evolution of aggregate state.*** Agents' decisions, i.e., on capital, investment, and consumption, are linear in their net worth. Therefore, aggregation is immediate. The dynamics of household and banking sector net worths are

$$\frac{d\underline{N}}{\underline{N}} = \frac{d\underline{n}}{\underline{n}} - \underline{\lambda} + \frac{N}{\underline{N}}\lambda$$

and

$$\frac{dN}{N} = \frac{dn}{n} - \lambda + \frac{\underline{N}}{N}\underline{\lambda},$$

respectively. The only difference between the evolution of a sector's net worth and that of an individual agent within that sector comes from the Poisson type-switching processes. The following lemma characterizes the evolution of the aggregate state in the economy.

**Lemma 1.3.** *The banking sector's wealth share dynamics are*

$$\frac{d\eta}{\eta} = \mu_\eta dt + \sigma_\eta dZ_t - \ell_\eta dJ_t,$$

*where*

$$\eta\mu_\eta = \eta(1-\eta)\left[x\left(\mu^R - r\right) - \underline{x}\left(\underline{\mu}^R - r\right)\right] - \eta\sigma_\eta(\sigma + \sigma_q) + \underline{\lambda}(1-\eta) - \lambda\eta \tag{1.17}$$

$$\eta\sigma_\eta = \eta(1-\eta)(x - \underline{x})(\sigma + \sigma_q) \tag{1.18}$$

$$\eta\ell_\eta = 1 - \tilde{\eta} \tag{1.19}$$

*and $\tilde{\eta}(\eta)$ represents the wealth share of banks after a systemic run and satisfies*

$$\tilde{\eta}q(\tilde{\eta}) = \mathcal{T} \tag{1.20}$$

*whenever $p > 0$.*

These dynamics determine the ergodic distribution of the share of wealth held by banks. Importantly, the dynamics are mainly driven by the portfolio decisions $x$ and $\underline{x}$. Absent the realization of any shocks, i.e. considering just the drift of $\eta$,

the wealth share of banks moves according to 1) the difference in portfolio expected returns, i.e., share times market excess return $x\left(\mu^R - r\right)$, 2) an adjustment term due to the nonlinearity of wealth share as a function of the wealth levels of each sector, $\eta\sigma_\eta(\sigma + \sigma_q)$, and 3) the type-switching processes.

The wealth share of banks is exposed to capital quality shocks as long as agents choose different portfolio shares. In particular, if banks choose a larger capital portfolio share than households, a positive capital quality shock increases their wealth share. The impact depends on the sensitivity of returns: it includes the direct quantity effect and the amplification effect on the price of capital. The condition that determines the wealth share after a systemic run (1.20) establishes that banks' new wealth share $\tilde{\eta}$ is equal to the banking sector's new wealth $\mathcal{T}K$ divided by the new value of total wealth $q(\tilde{\eta})K$.

### *H. Markov Equilibrium*

***Market clearing.*** For convenience, define $\psi \equiv x\eta$ as the capital share managed by banks. Also, since capital is the only asset in positive net supply in this economy, total wealth in the economy is $qK = N + \underline{N}$.

Market clearing for goods (scaled by total capital) can be written as[16]

$$\rho q + \iota(q) = \psi a + (1 - \psi)\underline{a}, \tag{1.21}$$

where the LHS represents aggregate demand, and the RHS is aggregate supply. The first term on the left is aggregate consumption, which is a constant fraction $\rho$ times total wealth $qK$, and the second term represents aggregate investment $\iota(q)K$. Total production per unit of capital is equal to agents' productivities weighted by the share of capital that they manage.

Market clearing for capital (scaled by total wealth) is

$$x\eta + \underline{x}(1 - \eta) = 1, \tag{1.22}$$

where $xN = x\eta qK$ is total wealth invested in capital by banks, and $\underline{xN} = \underline{x}(1-\eta)qK$ is the corresponding quantity for households.

[16]The market clearing conditions for goods assumes that no bank pays the underperforming cost in equilibrium.

***Consistency.*** Capital price dynamics need to be consistent with the dynamics of the aggregate state, i.e.

$$q\mu_q = q_\eta \mu_\eta + \frac{1}{2} q_{\eta\eta} \left(\sigma_\eta \eta\right)^2 \tag{1.23}$$

$$q\sigma_q = q_\eta \sigma_\eta \eta \tag{1.24}$$

$$\ell^q = 1 - \frac{q(\tilde{\eta})}{q} \tag{1.25}$$

These conditions follow from applying Ito's lemma to function $q(\eta)$. Also, the equilibrium loss on deposits in case of a run needs to be consistent with banks' portfolio decision $x$ and capital price dynamics, i.e., equation (1.11) has to hold. Finally, run intensity needs to be consistent with depositors' average losses in case of a systemic run, i.e., equation (1.3) evaluated at $\mathcal{L}^d = \ell^d$ or

$$p = \Gamma(\ell^d). \tag{1.26}$$

***Markov equilibrium.*** Given an exogenous transfer policy $\mathcal{T}(\eta)$, an equilibrium is a set of functions of banks' share of net worth $\eta$ for prices $\{q, \mu_q, \sigma_q, \ell^q, r, \ell^d\}$, allocations $\{c/n, (\underline{c}/\underline{n}), \iota, \underline{\iota}\}$, portfolio decisions $\{x, \underline{x}\}$, run intensity $\{p\}$, and the dynamics of banks' wealth share $\{\mu_\eta, \sigma_\eta, \ell_\eta, \tilde{\eta}\}$ such that

- Agents optimize. Given prices, allocations and portfolio decisions solve the agents' problem: (1.9), (1.10), (1.12), and (1.16).
- Markets clear: (1.21) and (1.22).
- The dynamics of the aggregate state are consistent with optimal decisions: (1.17) – (1.20).
- Capital price dynamics are consistent with the dynamics of the aggregate state: (1.23) – (1.25).
- Depositors' losses are consistent with banks' portfolio choices and capital price dynamics: (1.11).
- Run intensity is consistent with depositors' losses: (1.26).

***Value functions***. The Markov equilibrium definition does not include agents' value functions but they are necessary for welfare analysis. The following lemma characterizes them.

**Lemma 1.4.** *The value function for a bank with individual net worth $n$ can be written as*

$$V(n;\eta) = v(\eta) + \frac{1}{\rho}\log(K) + \frac{1}{\rho}\log\left(\frac{n}{N}\right) \tag{1.27}$$

*while the corresponding value for a household with individual net worth $\underline{n}$ is*

$$\underline{V}(\underline{n};\eta) = \underline{v}(\eta) + \frac{1}{\rho}\log(K) + \frac{1}{\rho}\log\left(\frac{\underline{n}}{\underline{N}}\right), \tag{1.28}$$

*where $v(\eta)$ and $\underline{v}(\eta)$ satisfy functional equations (1.35) and (1.34) in Appendix A, respectively.*

The first term in (1.27) is the indirect utility (scaled by total capital) of a banker that owns the entire sector's net worth, i.e., $n = N$, and it is the relevant welfare measure discussed below. The second term is the adjustment for aggregate capital level and captures the non-stationary component of the value function since the economy is constantly growing. The third term represents a simple adjustment to consider the wealth share of the agent within the sector. If there were no switching identity shocks, i.e., $\lambda = \underline{\lambda} = 0$, the latter would be a constant. In general, it is time-varying but its dynamics are taken into account by $v(\eta)$. The description of the value function for households is completely analogous.

*I. Numerical approach*

The Markov equilibrium defines a functional equation for $q(\eta)$. Given the presence of jumps, this functional equation is not an ODE, as is the case in basically all continuous-time macroeconomic models in the literature. Besides $\{q(\eta), q'(\eta), q''(\eta)\}$, the functional equation also includes $q(\tilde{\eta}(\eta))$, the capital price if a systemic run were to arrive at state $\eta$. Importantly, the post-run banks' wealth share $\tilde{\eta}(\eta)$ is endogenous to the system. There is no general approach to solving this class of functional equations.

The key to the two-step numerical approach I implement is that a subset of equilibrium conditions defines a standard ODE for capital price. In particular, it is possible find a map from $\{\eta, q(\eta)\}$ to $q'(\eta)$ that is independent from non-local changes $\{\tilde{\eta}(\eta), q(\tilde{\eta}(\eta))\}$. The first step is to solve this ODE. Given function $q(\eta)$, the second step is to solve for $\tilde{\eta}(\eta)$. Finally, all other equilibrium objects are recovered from these two functions. Due to logarithmic preferences, the equilibrium definition does

not include value functions, so the numerical procedure does not require iterations. Given the solution for the Markov equilibrium, I use an iterative approach to solve for the value functions. I also develop a numerical strategy to solve for the stationary distribution of the state variable without resorting to simulations.

Importantly, the numerical procedures to solve for the Markov equilibrium, the value functions, and the stationary distribution of the state variable are sufficiently fast to be embedded into an optimization procedure. This makes it possible to implement a direct numerical approach to welfare analysis.

### *J. Baseline Parameters*

The exposition of results combines theoretical insights and numerical illustrations, so I briefly discuss the baseline parametrization summarized in Table 1.1. I use a logarithmic adjustment cost function, $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$ , and a linear function for the sunspot arrival rate, $\Gamma(\ell^d) = \gamma^\ell \ell^d$. Time is measured in years. I use a standard value for the discount rate $\rho$ (2 percent) and a slightly low value for the depreciation rate $\delta$ (3 percent) given that capital quality shocks are similar to persistent productivity shocks with endogenous capital utilization, i.e., depreciation increases during booms since the model measures capital in efficiency units. The Poisson arrival rate for the transition from households to banks $\underline{\lambda}$ could be set to zero, but I choose a positive small number (1 percent) to prevent the economy from becoming trapped near $\eta = 0$ due to low volatility. The choice of a nonzero value for $\underline{\lambda}$ can be viewed as conservative in that it reduces the importance of bank runs for welfare. I also set the total transfer $\mathcal{T}$ to a small number (0.1 percent of total capital).

For the set of parameters $\{a, \underline{a}, \sigma, \kappa\}$, I match the set of moments presented in Table 1.1 at the stochastic steady state, i.e., at the point in the state space at which the economy converges absent any shocks. The moments in the data are calculated using U.S. data for the post-World War II period, except for the largest drop in asset prices, which I set to 30 percent. The last two parameters require information about the dynamics in vulnerable regions and are therefore matched using the entire stationary distribution. The rate $\lambda$ at which banks become households targets a conservative leverage estimate of 5. Larger values of leverage translate into a larger fraction of time spent in vulnerable regions. Finally, the sensitivity of the arrival rate of runs to potential losses in deposit value $\gamma^\ell$ is set so that runs happen on average once every 40 years. This paper focuses on mechanisms, so I provide robustness exercises

Table 1.1: Baseline parameters

| Parameter | | Value | Moment | Target | Model |
|---|---|---|---|---|---|
| | | | | | |
| Depreciation rate | $\delta$ | 0.03 | – | – | – |
| Discount rate | $\rho$ | 0.02 | – | – | – |
| Transition hh $\rightarrow$ banks | $\underline{\lambda}$ | 0.01 | – | – | – |
| Transfer after run | $\mathcal{T}$ | 0.001 | – | – | – |
| | | | Stochastic steady state | | |
| Productivity banks | $a$ | 0.125 | Investment/Capital | 8.8% | 8.8% |
| Productivity hh | $\underline{a}$ | 0.058 | Max. price drop | 30% | 29% |
| Exogenous volatility | $\sigma$ | 0.023 | GDP growth volatility | 2.31% | 2.31% |
| Inv. adjustment costs | $\kappa$ | 10.19 | GDP growth | 3.24% | 3.24% |
| | | | Including vulnerable regions | | |
| Transition banks $\rightarrow$ hh | $\lambda$ | 0.047 | Financial sector leverage | 5 | 5 |
| Eq. selection mechanism | $\gamma^{\ell}$ | 0.742 | Run probability | 2.5% | 2.5% |

for numerical illustrations but leave more ambitious quantitative exercises to less stylized models.

## 1.4 Instabilities

The full dynamic solution allows me to address the following economic questions about the different forms of risk: 1) when is the economy more vulnerable to real shocks? 2) is this vulnerability affected by the possibility or realization of rare collapses such as systemic runs? 3) can the economy be exposed to systemic runs when real shocks are not amplified and volatility is limited? (iv) if so, how do we identify this *hidden risk*? and 5) is the economy prone to spending large periods of time exposed to systemic runs and/or amplification of real shocks?

The economy is faced with two endogenous instabilities. First, the response to real shocks is larger than the exogenous fundamental impulse due to the presence of *endogenous amplification risk*. Second, the economy is exposed to *systemic runs,* which trigger collapses in output and asset prices and propel the economy to an unstable regime characterized by large amplification risk and occasional runs.

These two risks vary in distinct ways according to the key endogenous state variable: the wealth share of the banking sector. Amplification risk, which operates through

changes in misallocation, is present when this share is sufficiently low to prevent banks from holding all capital in the economy due to the large risk exposure it would involve. In contrast, run risk manifests in the intermediate region of banks' wealth share and depends on potential losses for depositors. The economy is shielded from runs if banks' wealth share is too high because their balance sheets are sound in these situations. Run risk is also absent for exceptionally low levels of the state variable because of low asset prices, i.e., small potential price drops.

The distinct dynamics of the two instabilities unveil a *hidden risk regime* in which volatility of macroeconomic and financial variables is low due to the absence of amplification, yet the economy is exposed to sudden collapses because of runs. In order to distinguish this regime from a truly safe low-volatility environment, it is necessary to consider a set of indicators that can flag the economy's vulnerabilities, i.e., a *risk topography*. The model suggests the following as indicators of run risk: the joint presence of high asset prices and a highly (or, at least, moderately) levered banking sector, high excess returns per unit of measured volatility, and a low risk-free rate.

There is a sharp contrast between the feedback among the two forms of endogenous risk. The presence of amplification risk $\sigma^q$ induces banks to lower their leverage in equilibrium, which lowers the economy's exposure to systemic runs. In contrast, the arrival rate of systemic runs $p$ or the losses they generate, $\ell^q$, have no influence on banks' equilibrium risk taking.

The system frequently visits the crisis and hidden risk regimes. In fact, absent aggregate shocks, the economy gravitates toward the hidden risk regime for the baseline parameter configuration. Furthermore, the economy passes through the crisis regime while recovering from systemic runs.

### *A. Financial and macroeconomic amplification risk*

The exposition focuses on the effect of amplification risk on capital returns due to fluctuations in asset prices as presented in equation (1.8). In this model, the volatility of asset prices translates into volatility of output due to the financial accelerator mechanism discussed below. Therefore, the amplification $\sigma^q$ that affects asset returns is also a macroeconomic risk, not just a financial one. In order to emphasize this point, I also discuss results in terms of output growth.

Let $Y \equiv yK \equiv (\psi(a-\underline{a})+\underline{a})\,K$ represent total output in the economy. Then its dynamics can be written as

$$\frac{dY}{Y_-} \equiv \mu^Y dt + \underbrace{(\underbrace{\sigma}_{\text{exogenous}} + \underbrace{\sigma^y}_{\text{amplification}})}_{\text{diffusive volatility}} \underbrace{dZ_t}_{\text{real shock}} - \underbrace{\ell^Y}_{\text{loss}}\,\underbrace{dJ_t}_{\text{runs}},$$

where $\sigma^y$ is the amplification associated with changes in allocative efficiency. Changes in the fraction of capital allocated to banks $\psi$ are directly related to the price of capital in equilibrium through (1.21). The following lemma characterizes the equilibrium relation between the amplification risk affecting capital returns and output growth.

**Lemma 1.5.** *Amplification risk of output growth $\sigma^y$ can be written as $\sigma^y = \varepsilon_{y,q}\sigma^q$, where $\varepsilon_{y,q} > 0$ represents the elasticity of aggregate demand (per unit of capital), i.e., $y^d \equiv \rho q + \iota(q)$, with respect to capital price.*

### *B. Amplification risk*

Amplification risk for capital return $dr^k$ corresponds to fluctuations in capital price $\sigma^q$ generated by the real shock driving the economy. Given that shocks in this economy are to the quantity of capital and not to its productivity, a unit of capital always has the same productive capacity. Then, an efficient allocation would imply a constant capital price, i.e., any fluctuation in capital price in response to the real shock operates through the internal dynamics of the system. In particular, these changes follow from portfolio adjustments due to precautionary motives, which vary with the state of the system. This translates into state-dependent amplification risk despite a constant level of exogenous risk $\sigma$.

The financial accelerator mechanism behind amplification was initially identified in KM and BGG, and the dynamics of this endogenous risk are discussed in BruSan using a similar model but without considering systemic runs. My main interest lies in the effect of run risk on the financial accelerator mechanism and amplification dynamics.

*(i) Invariance to run risk*

The following proposition shows that run risk has no effect on amplification risk in this economy.

**Proposition 1.1.** *Amplification risk $\sigma^q$ is independent of the presence of run risk. That is, it is independent of the feasibility of a run equilibrium, the sunspot arrival rate function $\Gamma(\cdot)$, and the potential losses associated with its realization $\ell^q$. Moreover, the equilibrium capital allocation $\psi$ and capital price $q$ are also independent of run risk.*

The amplification mechanism crucially depends on banks' leverage $x$ (assets to net worth). Given that capital is the asset with the largest value drop in a systemic run, it should be expected that agents' capital demand decreases when the economy is exposed to runs. Lemmas 1.1 and 1.2 show that this is indeed the case. However, the key determinant of amplification risk is the change in banks' capital demand *relative* to households', which turns out to be invariant with respect to run risk. Simple manipulation of optimal portfolio conditions (1.12), (1.15), and (1.14) delivers an illustration of the independence of capital allocation with respect to runs, i.e.,

$$x - \underline{x} \leq \left(\frac{a - \underline{a}}{q}\right) \frac{1}{(\sigma + \sigma^q)^2}$$

with equality for $\underline{x} > 0$. The excess capital demand of banks $x - \underline{x}$ depends on their excess return due to the productivity advantage $(a-\underline{a})/q$, and the risk associated with real shocks $\sigma+\sigma^q$, but not on run risk. Recall both agents require excess compensation due to run risk: households because capital losses exceed those on deposits, and banks due to the increase in the cost of deposits. The last equation implies that these excess compensations are the same.

There are two reasons to expect that banks' capital demand decreases more than households due to run risk. First, runs affect banks more than households since they practically wipe out all their net worth and therefore they should be less willing to expose themselves. Second, it is precisely banks' leverage decision that exposes the economy to systemic runs. However, neither line of reasoning is correct. Regarding the former, banks do not bear losses beyond bankruptcy due to limited liability, meaning *marginal* changes in banks' portfolio choices do not affect their net worth after a run. Regarding the latter, systemic runs depend on aggregate leverage of the banking sector, so no agent internalizes this effect. In other words, there is a run externality: individual banks make their portfolio decisions disregarding the fact that, as a sector, they influence run risk. Externalities associated with banks' leverage decision are discussed in section 1.5*B*.

*(ii) Mechanism and dynamics*

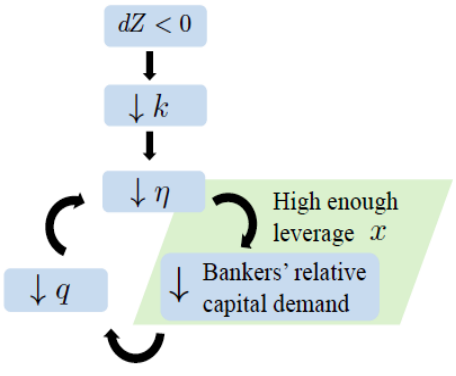
The irrelevance of runs for amplification risk implies that the mechanisms behind amplification risk and its dynamics will be similar to those in previous work.[17] However, it is necessary to understand and characterize both in order to study the joint dynamics of the different forms of risk and the potential influence of amplification risk on run risk.

Endogenous amplification risk depends on 1) bank leverage $x$ and 2) the sensitivity of the capital price to changes in banks' wealth share $q'(\eta)$. The latter captures the sensitivity of banks' capital demand to net worth, which in equilibrium is reflected by $\psi'(\eta)$. Recall banks' capital share $\psi$ and the capital price $q$ move together to preserve equilibrium in the goods market.

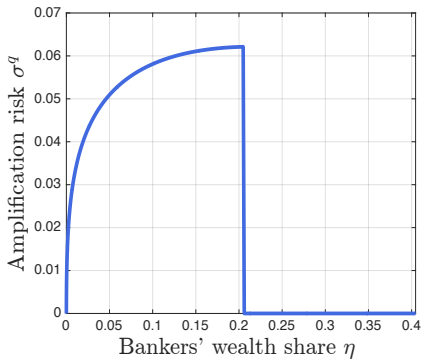
In equilibrium, banks finance capital holdings beyond their net worth by issuing short-term debt to households, i.e., $x > 1$, in order to exploit their higher efficiency in allocating resources. Then, a negative aggregate capital quality shock $dZ_t < 0$ directly reduces wealth share of banks $\eta$. If banks' capital demand is not linked to their net worth, i.e., if they just hold the entire capital stock, then there is no amplification. On the contrary, if banks' capital demand is limited by their net worth, initial losses trigger a downward spiral. The reduction of relative capital demand by banks implies a drop in its price since households are less efficient at managing capital. The drop in asset prices generates a further reduction in $\eta$ due to leverage, which translates again into lower capital demand by banks. Figure 1.1(a) illustrates the spiral.

Figure 1.1: Amplification risk

(a) Amplification spiral




(b) Amplification risk $\sigma^q$




[17] The description of the spiral behind amplification risk closely follows BruSan.

The continuous-time framework allows a simple characterization of the spiral. Consider an exogenous shock that reduces capital quantity by 1 percent; the direct effect on capital price then is a

$$\zeta \equiv \underbrace{\frac{q'(\eta)\eta}{q(\eta)}}_{\equiv \varepsilon_{q,\eta}} \underbrace{(x-1)}_{\text{direct drop in } \eta}$$

percent drop. The shock directly reduces the value of capital holdings by 1 percent, which translates in a reduction of $(x-1)$ percent of banks' wealth share $\eta$. The change in the state implies a price change governed by the elasticity of capital price with respect to the banks' wealth share, $\varepsilon_{q,\eta}$. The direct effect on the price of capital feeds back into $\eta$ and its effect is identical to the effect of the initial quantity shock because price and quantity have the same impact on total value. Then, the second-round effect on capital price is $\zeta^2$, and the total effect can be found by summing the corresponding geometric series. Therefore, the total effect of a unit shock to capital quantity on capital price, i.e., *amplification risk*, can be written as

$$\sigma^q = \underbrace{\frac{\zeta}{1-\zeta}}_{\text{spiral effect}} \underbrace{\sigma}_{\text{exogenous risk}}.$$

The effect of the initial impulse, $dZ_t$, on capital quantity depends on exogenous risk $\sigma$, and the change in capital quantity triggers the spiral described above.

There are two different regimes relevant for amplification risk that are characterized by the strength of the banking sector. In normal times, banks are well-capitalized and are willing to absorb the risk of managing all capital in the economy, so $\psi = 1$ and $q'(\eta) = 0$. There is no amplification spiral, i.e., $\sigma^q = \zeta = 0$, since the equilibrium demand of banks does not depend on their net worth. However, when banks become poorly capitalized, the risk of managing all capital stock is too great and it is optimal to sell some to households despite their lower productivity, so in this regime $\psi < 1$ and $q'(\eta) > 0$. In these situations, banks' capital demand depends on their net worth because, with larger net worth, their risk-bearing capacity increases, and therefore the spiral described is triggered after negative capital shocks. These dynamics are formalized by the following proposition and illustrated in Figure 1.1(b).

**Proposition 1.2.** *There exists a threshold $\eta^\psi \in (0, 1]$ such that amplification risk is present, i.e., $\sigma^q > 0$, if and only if banks' wealth share is below this threshold, i.e.,*

$\eta < \eta^{\psi}$ *(otherwise,* $\sigma^q = 0$*). Moreover, the positive amplification region is precisely the misallocation region, i.e. the region where* $\psi < 1$*.*

The independence result implies that run risk can affect the manifestation of amplification risk only through its influence on the shape of the stationary distribution.[18] There are two competing forces at play: 1) when the economy is exposed to run risk, runs cause banks' wealth share to be almost completely erased, sending the economy into the amplification region, and 2) run risk increases the risk premium on capital, allowing levered banks to increase their wealth share faster. Under the baseline parameter specification, the former effect dominates, and the economy spends more time in the crisis regime when I allow for the possibility of bank runs.

### *C. Run risk*

Systemic runs represent a different class of risk faced by the economy. Runs are not driven by a real exogenous impulse; rather, they are the outcome of agents' behavior and can arrive whenever the financial system is sufficiently fragile. The impetus for a run is a coordinating signal that arrives at a rate determined by the equilibrium selection mechanism. Hence, the possibility of a run is driven purely by the system's internal dynamics and coordination among agents.

There are three relevant measures of the risk that systemic runs pose: 1) the region in the state space in which a run is possible, 2) the losses they impose in terms of variables of interest such as asset values or output, and 3) the frequency of runs. The following proposition shows that, out of these three measures, only the frequency of runs depends on the equilibrium selection device.

**Proposition 1.3.** *Let* $\mathbb{S} = \{\eta \in [0,1] : \ell^d > 0\}$ *be the set of aggregate states in which a run equilibrium is feasible, i.e., the run vulnerability region. Then,* $\mathbb{S}$ *is independent of the equilibrium selection device* $\Gamma(\cdot)$*. Moreover, the capital value losses* $\ell^q$ *and output growth drop* $\ell^y$ *that a systemic run generates are independent of* $\Gamma(\cdot)$ *as well.*

A run may occur whenever the economy is fragile enough that a run on the banking system would imply losses for depositors. Under these circumstances, households'

[18] The independence result shows that amplification risk function $\sigma^q(\eta)$ does not depend on run risk. Nevertheless, run risk will affect the time the economy spends at different $\eta$ values.

decision not to roll over deposits yields a self-fulfilling equilibrium. Potential losses for depositors are determined by two factors: 1) bank leverage $x$ and 2) the drop in asset prices $\ell^q$ conditional on a run, as evidenced by the equation

$$\ell^d = \left[1 - (1 - \ell^q)\left(\frac{x}{x-1}\right)\right]^+,$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$. Importantly, the preceding proposition demonstrates that the region in the state space in which $\ell^d > 0$ is independent of the coordination device chosen. The intuition underlying this result is that, as argued earlier, relative capital demand is independent of the equilibrium selection mechanism because households and banks require equal compensation to bear run risk. Therefore, bank leverage $x$ is independent of the possibility of a run, so the price function $q(\eta)$ (and hence the potential loss in the event of a run) is independent of the equilibrium selection mechanism as well.

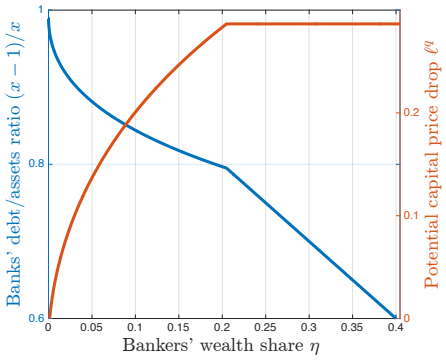
The run vulnerability region corresponds to intermediate levels of banks' wealth share. At one extreme, when banks hold a large share of total wealth, the leverage required to hold the entire capital stock is low. Even though the potential fall in asset prices is large, bank equity is sufficient to absorb any losses in the event of a run. On the other hand, when banks hold a very small share of wealth, misallocation is severe and asset prices are depressed. Despite the fact that bank leverage is high, the economy is shielded from runs because the potential drop in asset prices is not large enough to wipe out bank equity. The next corollary formalizes this intuition.

**Corollary 1.1.** *There exist wealth share levels for banks* $\{\eta^L, \eta^H\} \in (0,1)$ *with* $\eta^L \leq \eta^L$ *such that the economy is not vulnerable to runs if banks' wealth share is lower than* $\eta^L$ *or larger than* $\eta^H$*, i.e.,* $\eta \notin \mathbb{S}$ *if* $\eta < \eta^L$ *or* $\eta > \eta^H$*.*

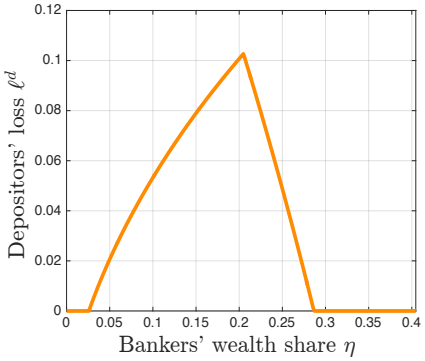
Figure 1.2 illustrates the state-dependence of depositors' losses $\ell^d$ as well as its determinants: the potential drop $\ell^q$ in the price of capital and bank leverage $x$. In the region where the economy is exposed to amplification risk, $\ell^q$ increases in $\eta$ as asset prices increase, but leverage declines in $\eta$ as banks' balance sheets become stronger. The increase in $\ell^q$ dominates in all numerical exercises, so in this region the potential losses faced by depositors are increasing in $\eta$. In the region where the economy is exposed only to run risk, $\ell^d$ is decreasing as a function of $\eta$ as asset prices are constant but leverage declines.

Figure 1.2: Run risk

(a) Determinants of run risk




(b) Depositors' loss $\ell^d$




Within the region where runs are possible, the frequency of runs depends on the sunspot intensity function $\Gamma(\cdot)$. In the baseline specification, $\Gamma(\cdot)$ is a linear function of depositors' losses, so the state-dependence of run frequency coincides with that of depositors' potential losses.

The intuition behind capital price losses increasing in banks' wealth share is the following. When a run occurs, banks' wealth share jumps to $\eta \approx 0$, so households are forced to operate essentially the entire capital stock, and the price of capital falls to its minimum value. The drop in the price of capital is larger when the initial capital price is higher, i.e., when $\eta$ is higher and misallocation is lower. Allocative efficiency also jumps to a minimum after a run, so the potential drop in output is increasing in $\eta$ as well. This implies runs are more costly in the region where the economy is only exposed to runs. This logic is formalized below.

**Corollary 1.2.** *For a non-contingent transfer policy, i.e., $\mathcal{T}(\eta) = \bar{\mathcal{T}}$, potential losses in terms of output $\ell^Y$ and capital value $\ell^q$ are weakly increasing in $\eta$ within the region vulnerable to runs.*

The economy is not necessarily exposed to run risk in any region of the state space. Whether run risk emerges depends on the losses to which depositors are exposed, which in turn are a function of bank leverage and the potential fall in asset prices. When either the exposure of asset prices to runs or leverage is sufficiently low, depositors never rationally decide to run because the levered losses incurred by banks are too small to wipe out their equity. For instance, if banks' productivity approaches

that of households ($a \to \underline{a}$), the potential drop in prices goes to zero, which itself is enough to rule out runs. More importantly, as will be shown below, large values of exogenous risk $\sigma$ will endogenously restrain bank leverage to the point that households never have to fear a loss on their deposits. This result stands in stark contrast to Proposition 1.2, which establishes that amplification risk is *always* present in some region of the state space.

### *D. Joint dynamics and the "hidden risk" regime*

The economy's state space is comprised of three distinct risk regimes: 1) a highly unstable *crisis regime*, 2) an apparently stable *hidden risk regime*, and 3) a truly stable *safe regime*. The regimes are characterized by the economy's exposure to endogenous risks (amplification risk and run risk) and correspond to distinct regions of the state space. As a consequence, the qualitative features of the economy's dynamics and the distribution of variables of interest, e.g. capital returns or output growth, vary dramatically across these three regions.

Figure 1.3(a) presents the distributions of output growth conditional on the state $\eta$, and Figure 1.3(b) presents one example of a distribution per risk regime. These graphs illustrate how the different endogenous risks translate into outcomes for variables of interest. Amplification risk implies large variability, but it cannot generate a distribution of output growth with asymmetries or fat tails given the normality of real shocks. In contrast, run risk generates an output growth distribution with negative skewness and fat tails. Systemic runs are responsible for the second mode at large negative growth rates, so this mass in the left tail appears only when the economy is vulnerable to runs, i.e., at intermediate values of $\eta$. Figure 1.3(c) presents the stationary density for banks' wealth share $\eta$ and distinguishes the three risk regimes.

The *safe* region, amplification risk and systemic runs are absent. In this region, banks' share of wealth is large and they hold the entire capital stock without exposing depositors to losses in the event of a run. There is no misallocation, so the price of capital is constant. Banks' wealth share is exposed only to the direct effect of real shocks, i.e., the change in the quantity of capital. Output growth inherits the distribution of the growth of the capital stock, which is normal with volatility $\sigma$.

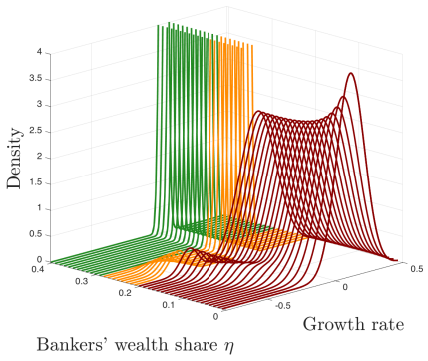
In the *hidden risk* region, the economy is exposed to systemic runs, yet insulated from amplification risk. Banks' wealth share must lie in an intermediate range: it must be small enough for depositors to rationally fear losses in the event of a run, but

large enough that banks do not wish to fire-sell assets to households. Apart from the economy's response to the realization of systemic runs, the dynamics in this region are identical to those in the safe regime. In particular, in the absence of runs, the dynamics of output and asset prices are observationally equivalent in the two regimes. Banks hold the entire capital stock, so capital price is constant and banks' portfolios are exposed only to exogenous risk and systemic runs. The distribution of output growth is similar to that in the safe regime, except for a second mode at large negative growth rates resulting from the economy's exposure to runs.

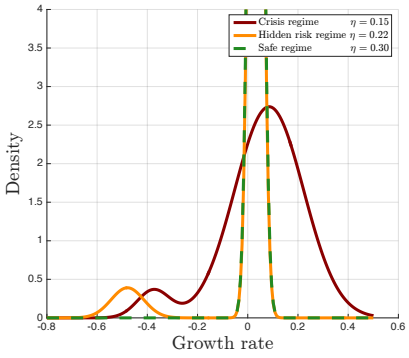
In the *crisis* region, banks' share of aggregate wealth is low enough that the economy is exposed to amplification risk. Negative real shocks force banks to sell assets, which increases misallocation, lowers asset prices, and sets off a loss spiral. As a result, the volatility of macroeconomic and financial variables is high in this region. Systemic

Figure 1.3: Risk regimes

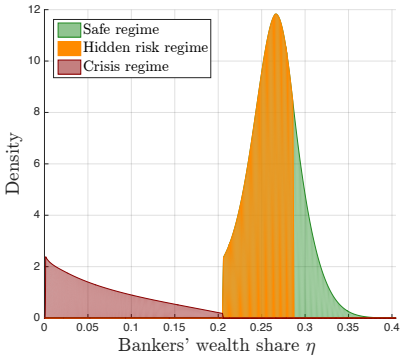
(a) Conditional distributions of output growth



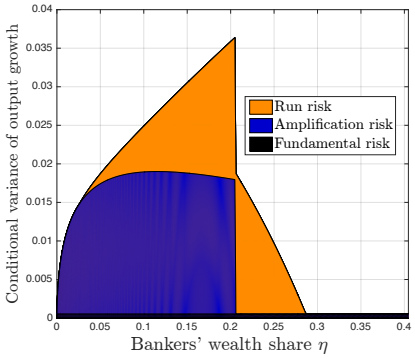

(b) Distributions by risk regime (examples)




(c) Stationary distribution of $\eta$




(d) Decomposition of conditional variance



runs may also be possible in this region due to the weakness of bank balance sheets, but their incidence is not guaranteed because asset prices in this region may be so low that the potential loss during a mass liquidation is insufficient to wipe out bank equity. The distributions of output growth and returns on capital have high variance and are bimodal when runs are possible.

Relative to earlier work such as BruSan, the main qualitative difference brought about by the introduction of systemic runs lies in the dynamics of the hidden risk regime. Unlike the safe region and the crisis region, the hidden risk region may not always exist. Indeed, as argued earlier, there are some parameter combinations for which the economy is never exposed to runs. Therefore, the relevant question is whether a hidden risk regime is always present, conditional on the economy being exposed to systemic runs, i.e., $\mathbb{S} \neq \emptyset$. The following corollary describes a necessary and sufficient condition for the existence of the hidden risk regime: positive losses for depositors (conditional on a run) when the state is at the boundary $\eta^{\psi}$ of the crisis regime.

**Corollary 1.3.** *Given a constant transfer policy $\mathcal{T}$, there exists a hidden risk regime if and only if a run equilibrium is feasible at the threshold banks' wealth share level $\eta^{\psi}$, i.e., $\ell^d\left(\eta^{\psi}\right) > 0$.*

The condition mentioned in corollary 1.3 is satisfied for all parameter combinations for which the economy is exposed to run risk. Moreover, in each such numerical exercise, depositors' losses $\ell^d$ reach a maximum at $\eta^{\psi}$. Absent a formal proof, this illustrates that the *hidden risk* regime is a robust feature of the model.

Continuous-time methods allow me to analytically characterize the stochastic evolution of the variables of interest at any point in the state space and, therefore, to calculate any relevant conditional moment. In order to illustrate the relative contributions of each risk to variability, Figure 1.3(d) presents the decomposition of the variance of output growth into components associated with exogenous risk, amplification risk, and run risk.

For baseline parameters, the contribution of exogenous risk to the conditional variance of output is dwarfed by the contributions of endogenous risks. The variability associated with run risk is at least as important as that due to amplification risk in the crisis regime. Furthermore, it is clearly the dominant source of risk in the hidden risk regime, where the potential drops in output and asset prices are largest.

*E. Ergodic instabilities*

The stationary distribution of the state of the economy has several interesting properties. To begin with, the economy is highly prone to instability under the baseline parametrization, as evidenced by the stationary distribution in Figure 1.3(c). In fact, the economy spends more time in the hidden risk region than it does in the safe region. More subtly, the stationary distribution exhibits a counterintuitive feature: despite the fact that, more often than not, the state of the economy is in a region where endogenous risks are present, it is also true that most of the time there is no observable amplification. This can be seen in Figure 1.3(c) by noting that the economy spends a considerably larger span of time in the hidden risk region than in the crisis region.

Another striking property of the stationary distribution is that there is a significant amount of mass in the extreme left tail near $\eta = 0$. The following lemma shows that this mass is entirely due to runs.

**Lemma 1.6.** *Consider the model without runs, i.e.,* $\Gamma(\cdot) \equiv 0$*. If there is a positive flow of agents from the household to the banking sector* $\underline{\lambda} > 0$*, then the stationary distribution has density zero at* $\eta = 0$*.*

This lemma shows that exogenous risk and amplification risk together do not generate enough volatility to keep banks' wealth near zero when there is a stabilizing force on their wealth. The implication is that systemic runs on the financial system are wholly responsible for the most severe crises. Further, the size of the mass near $\eta = 0$ in the stationary distribution indicates that runs are not only severe but subject to a slow recovery.

There is a sharp contrast between the shape of the ergodic distribution in the crisis regime and outside it. The reason is that, in the crisis regime, excess returns on capital include compensation for amplification risk, which favors wealth accumulation in the banking sector. Therefore, the endogenous dynamics imply a relatively fast though highly volatile transition out of the crisis regime. This explains the relatively low mass in this region. Once amplification disappears, banks' accumulation of wealth is significantly slower, as reflected by the large increase in the density at threshold banks' wealth share $\eta^{\psi}$.

The shape of the $\eta$-density within the crisis regime follows from the dynamics of endogenous risks: when risks are small or absent, excess returns on capital are also limited and banks accumulate wealth slowly relative to households. Therefore, recovery is slowest immediately after a systemic run, i.e., close to $\eta = 0$, since there is no run risk and amplification risk is small (because for low levels of $\eta$, even a large percentage movement implies a small absolute change, which determines the extent of the loss spiral). As $\eta$ increases in the crisis regime, amplification and run risk increase, so the transition out of this highly unstable regime speeds up as long as no run is realized. The downward slope of the density in the crisis regime follows from this logic.

### *F. Risk topography*

The hidden risk regime and the safe regime are observationally equivalent in several respects. For one, the level of asset prices, output per unit of capital, and real investment per unit of capital are all identical across the two regimes. Moreover, in each of the two regimes the volatility of output is equal to the exogenous volatility of the quantity of capital, and the volatility of asset prices is equal to zero.

In order to distinguish between these two regimes, it is necessary to develop a *risk topography* to describe the joint behavior of macroeconomic variables, financial indicators, asset prices, and returns. In this environment, three important quantities suggest presence of hidden run risk. First, the simultaneous incidence of high asset prices and high bank leverage directly implies that depositors will take large losses in case of a run, since these are precisely what drive depositors' potential losses $\ell^d$. Second, high excess returns on capital per unit of observed volatility can be indicative of run risk as well. Even though run risk does not affect relative capital demand, it does affect the risk premium required by both types of agents. Hence, risk premia relative to asset price volatility will be high in the hidden risk regime because observable volatility in that regime is low. Third, a low risk-free rate indicates the possibility of a run. Although there is no risk-free asset in the model, an asset-pricing exercise shows that the risk-free rate tends to be low in the hidden risk regime. This is because the possibility of a collapse due to a run drags down agents' expected consumption growth.

## 1.5 Risk to control risk

The following examines how the model's endogenous instabilities change as the amount of exogenous risk $\sigma$ varies. The interactions between the different types of risks are key to understanding the main insight of the model: *stability breeds instability* and its contrapositive, *risk controls risk.* Economies with the lowest level of exogenous risk will be most exposed to systemic runs in the sense that the region in which they can occur is largest for low $\sigma$. Importantly, this result will hold independently of the equilibrium selection device. On the other hand, total volatility $\sigma + \sigma^q$ will decrease alongside exogenous risk.

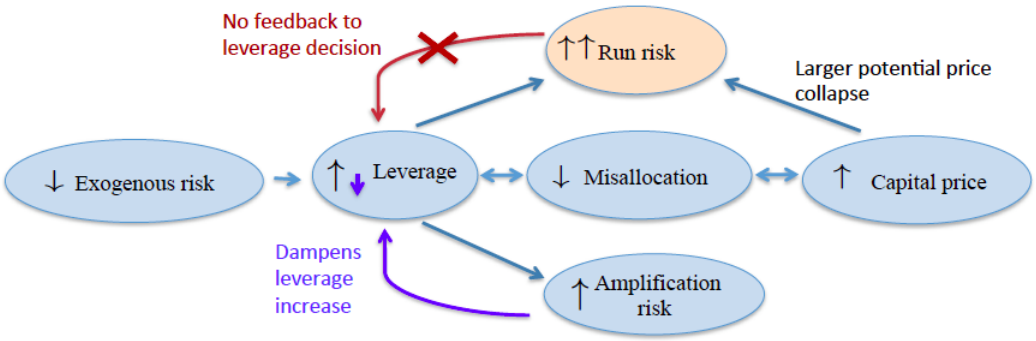
These results lead to a non-monotonic welfare ranking with respect to exogenous risk. For high levels of exogenous risk, the economy is shielded from runs and agents dislike increases in exogenous risk since they translate into a larger probability of entering the crisis regime and worse crises, i.e., higher misallocation and amplification. However, for low levels of exogenous volatility, the economy is exposed to runs and agents favor increases in exogenous volatility because they restrain risk-taking by banks and prevent runs. The welfare analysis assumes the equilibrium selection mechanism is independent of exogenous volatility.

*A. Stability breeds instability*

A brief examination of the mechanisms at play illustrates why low exogenous volatility leads to lower total volatility but greater run risk. When exogenous volatility is low, the capital demand of both types of agents increases, but banks' demand increases relative to households' because of their greater exposure to the exogenous risk. Therefore, banks' equilibrium leverage increases. The increase in leverage is dampened (but not completely reversed) by the corresponding increase in amplification risk. Higher leverage reduces misallocation and increases asset prices. However, higher leverage and asset prices also increase the economy's exposure to a run. Unlike the increase in amplification risk, the increase in run risk does not feed back into banks' equilibrium leverage because of 1) limited liability, which makes the banks' fate after a run – bankruptcy – insensitive to their risk-taking at the margin, and 2) banks' failure to internalize their contribution to aggregate run risk. Figure 1.4 illustrates this logic.

Next I describe in detail how each type of endogenous vulnerability responds to lower exogenous risk and provide formal results.

Figure 1.4: Exogenous risk effect over endogenous instabilities

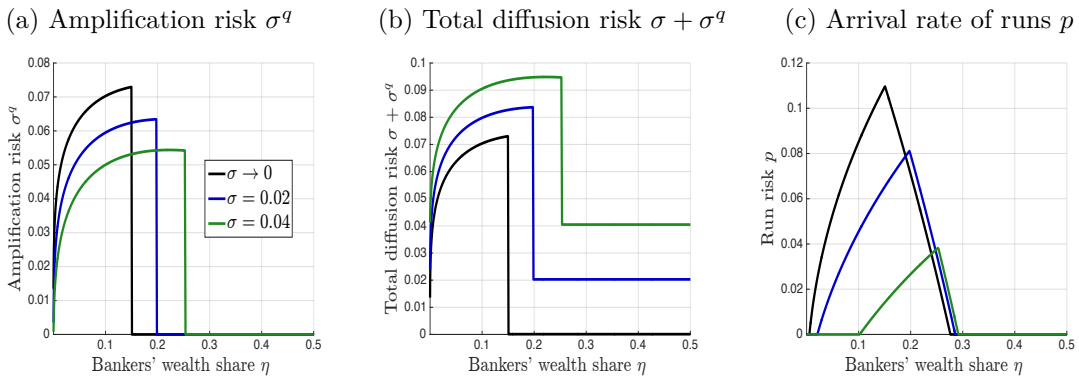



***Amplification risk***. Lower exogenous risk has two opposing effects on the loss spiral that generates amplification risk. First, it decreases the impact of a given exogenous impulse on the quantity of capital. Second, it increases the sensitivity of banks' relative capital demand to changes in net worth, i.e., $\psi'(\eta)$ and $q'(\eta)$ are higher in the crisis regime. As a result, the crisis regime shrinks, but amplification risk is higher in the new crisis regime. Nevertheless, the increase in amplification is never enough to undo the initial reduction in exogenous volatility, so total risk associated with real shocks decreases. Panels (a) and (b) in Figure 1.5 illustrate this effect and the following proposition formalizes it.

**Proposition 1.4.** *Total diffusion risk $\sigma + \sigma^q$ is weakly increasing in exogenous risk $\sigma$ for every $\eta \in [0, 1]$. Also, capital price $q$ and banks' capital share $\psi$ (or leverage $x$) are weakly decreasing in $\sigma$ for every $\eta \in [0, 1]$.*

***Run risk***. When exogenous risk is reduced, total risk associated with real shocks decreases as well. Banks then increase their leverage, resulting in less misallocation and higher capital prices. The greater fragility of bank balance sheets and higher capital prices both contribute to a larger potential loss for depositors and hence a greater exposure to systemic runs (holding the liquidation price fixed). The reduction in exogenous volatility does not have a large effect on the liquidation price because the transfer to reboot the banking sector is small enough that the liquidation price is close to $q(0)$, which is independent of exogenous volatility. Nevertheless, in general the small increase in the liquidation price due to lower exogenous risk dampens the increase in run risk. This effect is illustrated in Figure 1.5(c) and made precise by the following proposition.

Figure 1.5: Endogenous instabilities and exogenous risk




**Proposition 1.5.** *For a constant liquidation price, the size of the run vulnerability region, i.e., the Lebesgue measure of* $\mathbb{S}$*, is weakly decreasing in* $\sigma$*. A constant liquidation corresponds, for example, to the case where there are no transfers to banks after their collapse, i.e.,* $\mathcal{T} \to 0$*.*

In this model, both endogenous risks remain as exogenous volatility vanishes. For amplification risk this effect is simply the "volatility paradox" of BruSan: as the exogenous impulse vanishes ($\sigma \to 0$), the spiral effect grows unboundedly ($\zeta/(1-\zeta) \to \infty$) in a way that endogenous amplification risk persists ($\sigma^q \to \hat{\sigma}^q \in \mathbb{R}$). The persistence of run risk as exogenous risk vanishes in this model is actually more powerful than the volatility paradox. First, given the previous result, when exogenous risk vanishes run risk is actually maximized. Second, if exogenous risk is initially high enough that the economy is not exposed to runs, a reduction of exogenous risk may actually introduce the possibility of runs to the economy. This intuition is partially formalized below.

**Corollary 1.4.** *Amplification risk does not vanish as exogenous risk disappears, i.e., threshold level for banks' wealth share* $\eta^{\psi} \to \eta^{+} > 0$ *as* $\sigma \to 0$*.*

### *B. Externalities and welfare measures*

In order to understand how exogenous risk affects welfare in this economy, it is first necessary to examine the externalities arising from banks' portfolio decisions and discuss the relevant welfare measures.

*Externalities.* Absent runs, there are two pecuniary externalities related to banks' portfolio decisions (or capital demand). First, there is a *static pecuniary externality* associated with the *level of capital price*. Banks do not take into account that a decrease in their leverage has a negative impact on the price of capital, which reduces the risk-bearing capacity (or net worth) of other banks, forcing them to decrease their capital positions and increase misallocation. In this sense, banks' leverage is inefficiently low. Second, there is a *dynamic pecuniary externality* associated with the *volatility of capital price*. Banks disregard that, by increasing their leverage, they increase the amplification of future real shocks in the economy. In this other sense, banks' leverage is inefficiently high. Therefore, these externalities bias leverage in different directions.

Bank runs introduce two additional externalities in banks' portfolio decisions. First, banks do not internalize that their choice of leverage affects the economy's exposure to systemic runs. Larger leverage increases potential losses for depositors directly and through asset prices: higher leverage increases the price of capital and therefore the size of the drop to its liquidation price. Agents fail to internalize this effect because individual decisions have zero impact on aggregate variables and run risk depends on aggregate losses of depositors if the bank run is realized. Second, banks disregard the cost of transfers after their collapse. These *run externalities* lead to excessive leverage. Importantly, even though their presence does not depend on the equilibrium selection mechanism (as long as the economy is exposed to runs), their magnitude does.

*Welfare measures.* The solution to the model provides a detailed characterization of agents' well-being. In particular, it delivers a solution for the value function (scaled by total capital) conditional on each aggregate state, i.e., $v(\eta)$ for banks and $\underline{v}(\eta)$ for households, and also describes the frequency with which the economy visits each of these states, i.e., the stationary distribution. Then, a welfare comparison across economies needs to select a relevant summary statistic. For a given agent, the most intuitive statistic is the expected indirect utility where the expectation is taken with respect to the stationary distribution, i.e., $\mathbb{E}\left[v(\eta)\right]$ and $\mathbb{E}\left[\underline{v}(\eta)\right]$. For the entire economy, a Pareto-weighted average of these statistics is a natural measure. The non-monotonic welfare ranking with respect to exogenous risk (emphasized later in this section) will hold not only for any Pareto weights but will be robust to alternative welfare measures such as the indirect utility conditional on a particular value of $\eta$.

### *C. Exogenous risk and welfare*

The welfare consequences of changing the level of exogenous risk in the economy can be conceptually decomposed into three components: 1) a fundamental effect, 2) a redistributive effect, and 3) the impact on externalities.

*Fundamental effect.* The fundamental effect of exogenous risk on welfare is the impact it would have in a frictionless economy. In this model, the frictionless benchmark corresponds to a version that allows for equity issuance. In this case, the model collapses to a growth model with a representative agent that has access to the most productive technology. All financing happens through equity contracts, so there is no short-term debt and no runs. Larger exogenous risk decreases the utility of the representative agent due to risk aversion. The following lemma formalizes this intuition.

**Lemma 1.7.** *With complete markets, i.e., allowing for equity issuances, the indirect utility function of the representative agent is decreasing in exogenous volatility $\sigma$.*

*Redistributive effect.* The redistributive effect corresponds to shifts in the wealth distribution. Without externalities, any wealth redistribution has a positive welfare impact for the agent whose net worth increases and a negative one for the agent whose wealth decreases. In comparisons across economies, this effect is related to shifts of stationary distribution. For example, a switch toward a stationary distribution that spends more time at states where banks own a small fraction of wealth would represent a redistribution toward households. This is precisely the case when considering a reduction of exogenous risk.

Lower $\sigma$ increases the exposure to systemic runs – see Proposition 1.5 – so the economy is more frequently propel to situations where households own a large fraction of wealth. It also decreases excess capital returns in the safe regime, making the economy less likely to visit this region of low wealth share for households. Therefore, if reducing exogenous risk has a negative welfare effect for households, it will be in spite of the redistributive impact.

*Impact on externalities.* A reduction of exogenous risk increases banks' leverage – see Proposition 1.4 below – so the impact of $\sigma$ on externalities in this model depends on whether banks' leverage is inefficiently high or inefficiently low. If leverage is

inefficiently low, the main effect of a marginal reduction of $\sigma$ is to increase allocative efficiency (reduce the *static pecuniary externality*), and therefore it benefits both agents. If leverage is inefficiently high, the principal effect of a marginal decrease of $\sigma$ is the increase of endogenous instabilities (larger *dynamic pecuniary externality* and *run externalities*), so it is detrimental for both agents.

Whether leverage is inefficiently low or high depends on the relative strengths of externalities (and potentially on the welfare measure adopted). When exogenous risk is high enough to shield the economy against runs, leverage tends to be too low, so reducing exogenous volatility reduces inefficiencies in the economy. Moreover, as discussed before, the larger endogenous amplification risk does not overcome the fundamental effect of lower $\sigma$, so total risk associated with real shocks decreases.

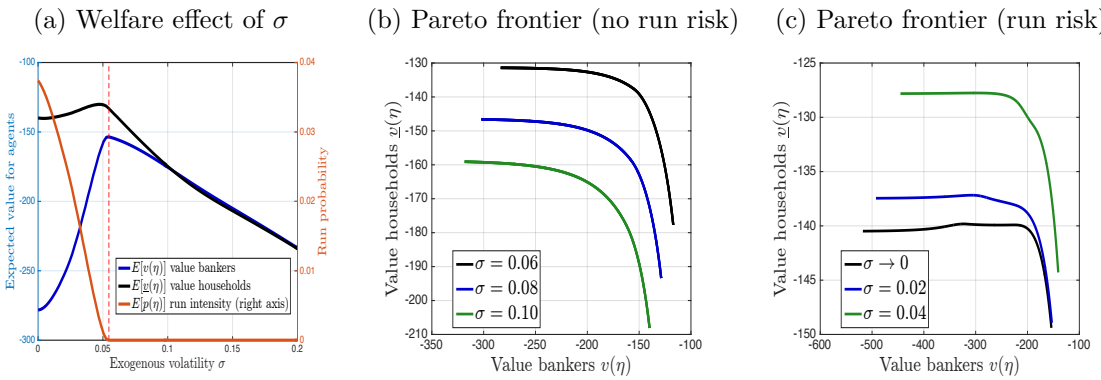
In contrast, when exogenous volatility is low enough to expose the economy to systemic runs, equilibrium leverage tends to be too high and a marginal decrease in $\sigma$ is costly for agents. Intuitively, the inefficiencies caused by systemic runs tend to be stronger because there is no feedback from larger run risk to portfolio decisions (as is the case when considering risk associated with real shocks).

*Non-monotonic welfare ranking and run risk.* The overall effect of exogenous risk $\sigma$ on each agents' welfare is illustrated in Figure 1.6(a). It shows the expected indirect utility of agents, i.e., $\mathbb{E}[v(\eta)]$ and $\mathbb{E}[\underline{v}(\eta)]$, and the annual probability of systemic runs $\mathbb{E}[p(\eta)]$ for economies with different levels of exogenous risk. The non-monotonicity of the welfare measure for both agents is evident. Moreover, the change in the relationship between the welfare measures and exogenous risk coincides with the presence of systemic runs. It is important to mention that results presented in this section assume that the equilibrium selection mechanism $\Gamma(\cdot)$ is invariant to exogenous volatility $\sigma$.

The reversal of the relationship between welfare and exogenous risk in the presence of systemic runs indicates that the main driving force is the effect of $\sigma$ on externalities. The directions of the other two effects discussed do not depend on the presence of run risk: lower $\sigma$ implies a positive fundamental effect and a redistribution toward households. In contrast, leverage switches from being inefficiently low to inefficiently high in the presence of run risk, which implies that lower $\sigma$ exacerbates externalities.

The non-monotonicity result and its dependence on the presence of runs not only hold on average, i.e., considering the expectation operator over the stationary distribution, but also hold conditioning on any particular wealth distribution, i.e., taking $\eta$ as given. Figure 1.6(b) shows the expansion of the Pareto frontier of the economy as

Figure 1.6: Welfare effect of exogenous risk




$\sigma$ decreases when the economy is not exposed to runs (high exogenous risk), while Figure 1.6(c) illustrates the exact opposite effect when the economy is exposed to runs (low exogenous risk). The frontier represents combinations $(v(\eta), \underline{v}(\eta))$ for each $\eta \in [0, 1]$.[19] [20]

## 1.6 Policy

Given the financial friction, i.e., the restriction not to issue equity, short-term debt fulfills the key role of allowing banks to finance larger capital holdings in order to improve productive efficiency in the economy. However, this comes at the cost of systemic instability due to amplification and run risk. As discussed before, banks do not appropriately internalize all the costs or benefits of their portfolio decisions, which opens the possibility for a welfare-improving policy intervention.

First, I consider a simple and intuitive macroprudential policy: a constant leverage cap $\bar{L} \geq 1$. The analysis unveils important shortcomings of this policy as a tool to limit endogenous risks. Surprisingly, a leverage constraint increases amplification risk whenever it is not binding and exacerbates exposure to systemic runs within the hidden risk regime. Despite these limitations, a leverage cap can improve welfare by reducing overall run risk.

[19] The expansion (contraction) of the frontier does not necessarily imply that both agents are better (worse) off at each aggregate state $\eta$ but that is indeed the case for Pareto frontiers shifts discussed.

[20] Absent runs, the welfare ranking might be non-monotonic in exogenous risk for banks because an increase of $\sigma$ redistributes wealth toward them.

Second, I study a leverage cap contingent on the wealth share of the banking sector: $\bar{L}(\eta) \geq 1$. The optimal state-contingent leverage cap restrains banks' risk-taking when the economy would be exposed to runs, i.e., in the intermediate range of banks' wealth share, but allows for large leverage when banks' wealth share is particularly low, i.e., during downturns.

Third, I study economies in which exogenous risk varies along with the wealth share of the banking sector and explore which of these relations, $\sigma(\eta)$, benefits agents the most. Results indicate that agents would prefer to live in an economy in which exogenous risk increases in regions of $\eta$ where the economy would be exposed to systemic runs. Beyond the model, this suggest that exposure to systemic runs can be an unintended effect of policies that stabilize real fluctuations.

### 1.6.1 Constant leverage constraint

The aim of implementing a leverage cap, despite the increase in misallocation it implies, is to limit the endogenous risks that leverage fosters, i.e., amplification and run risk. However, there are important shortcomings in using a constant leverage cap as a policy tool to control risk.

*A. Misallocation and risk control*

*Misallocation.* The main cost of implementing a leverage constraint is to decrease allocative efficiency, i.e., to reduce the capital share of banks. This drop is the largest during downturns when banks are poorly capitalized and prices signal they should use substantial leverage. The drop in equilibrium leverage is not limited to situations in which the constraint is binding (the low $\eta$ region) but extends to other regions of the state space because banks reduce their capital demand in anticipation of a deeper recession. Since capital price is directly related with allocative efficiency, the leverage constraint also reduces it. The following proposition formalizes the increase in misallocation.

**Proposition 1.6.** *The share of capital $\psi$ held by banks weakly decreases with the implementation of a leverage cap, i.e., $\psi(\eta) \geq \psi\left(\eta; \bar{L}\right)$ for $\eta \in [0,1]$. The same applies to the capital price function $q$.*

*Amplification risk.* The leverage cap controls amplification risk whenever it is binding. This is the case during downturns, i.e., the region where banks' wealth share is low.

However, this leverage constraint actually increases amplification risk whenever both: 1) the economy exhibits misallocation and 2) the constraint is not binding. This increase in amplification risk is necessary to ensure both agents still want to hold capital. The intuition is the following. A lower capital price implies a larger gap between market returns for banks and households $(a - \underline{a})/q$ while a lower capital share for banks implies their relative capital demand $(x - \underline{x})$ is lower, so the only way that both agents could still be willing to hold capital is if total diffusion risk $\sigma + \sigma^q$ increases. This logic is illustrated by

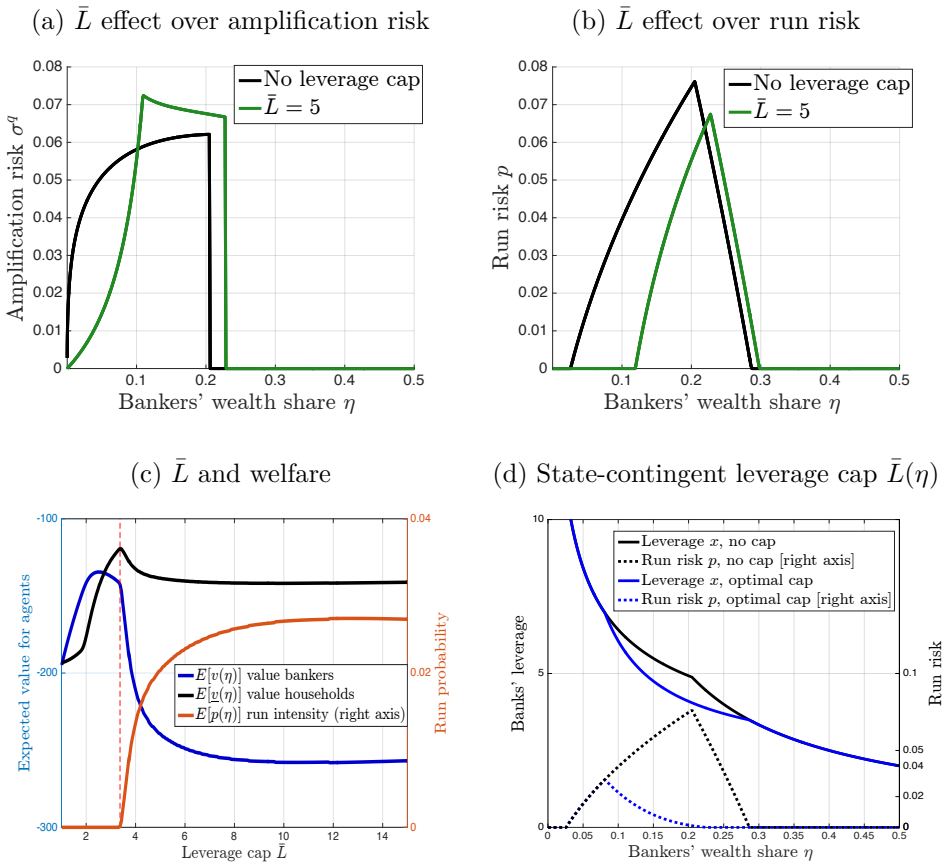
$$\underbrace{(x - \underline{x})}_{\downarrow}\underbrace{(\sigma + \sigma^q)^2}_{\uparrow\uparrow} = \underbrace{\frac{a - \underline{a}}{q}}_{\uparrow}.$$

An alternative interpretation is that the only way banks will reduce their relative capital demand (with respect to households') given the increase in the return gap is if diffusion risk increases. Moreover, the leverage constraint also enlarges the region of positive amplification risk. The following proposition formalizes the limitations of a constant leverage cap to contain amplification risk, and Figure 1.7(a) shows how amplification risk varies with the inclusion of a leverage cap.

**Proposition 1.7.** *Amplification risk weakly increases when including a leverage cap, i.e.,* $\sigma^q(\eta) \leq \sigma^q\left(\eta; \bar{L}\right)$*, for the* $\eta$*-region in which there is misallocation in both environments and the leverage cap is not binding, i.e.,* $\left\{\psi\left(\eta; \bar{L}\right), \psi(\eta)\right\} \in [0, 1)$ *and* $x\left(\eta; \bar{L}\right) < \bar{L}$*. Moreover, the region with positive amplification risk weakly increases, i.e.,* $\eta^\psi \leq \eta^\psi\left(\bar{L}\right)$*.*

*Run risk.* The leverage cap achieves lower equilibrium leverage, which dampens run risk. However, the inclusion of such a constraint decreases the liquidation price of capital, increasing potential drops in asset prices and the economy's exposure to systemic runs. The leverage effect usually dominates when both forces are present, i.e., in the *crisis regime*, but only the second force is present in the *hidden risk* regime (where there is no misallocation, so leverage is the same with and without the constraint). Therefore, a leverage cap increases run risk in this regime. The following proposition formalizes the dimension in which a leverage cap increases run risk in the economy, and Figure 1.7(b) illustrates how this risk changes with the inclusion of a leverage cap.

Figure 1.7: Leverage constraint




**Proposition 1.8.** *(a) The liquidation price weakly decreases with the inclusion of a leverage cap, i.e.,* $q(\tilde{\eta}(\eta)) \geq q(\tilde{\eta}(\eta; \bar{L}); \bar{L})$.

*(b) For the* $\eta$*-region that belongs to the hidden risk regime in both solutions, i.e.* $\psi(\eta) = \psi\left(\eta; \bar{L}\right) = 1$ *and* $\left\{p(\eta), p(\eta; \bar{L})\right\} \in \mathbb{R}_{++}$*, the solution with the leverage constraint is weakly more prone to runs, i.e.,* $p(\eta) \leq p(\eta; \bar{L})$.

*Resilience of endogenous risks to leverage caps.* While the only way to eliminate all amplification risk using a leverage cap is to ban any debt issuance, i.e., $\bar{L} = 1$, it is possible that the economy is completely shielded from systemic runs with $\bar{L} > 1$.

### *B. Leverage constraint and welfare*

Despite the shortcomings of a constant leverage constraint in controlling endogenous risks, it can still improve agents' welfare by reducing the economy's exposure to runs.

Figure 1.7(c) summarizes the comparison between economies with different leverage caps in terms of welfare and exposure to systemic runs. It shows that, as long the leverage cap helps reduce run risk ($\bar{L} > 3.4$ for baseline parameters), both agents prefer a tighter leverage constraint. However, once the leverage cap is tight enough to shield the economy completely from runs, there is a disagreement among agents about reducing $\bar{L}$ any further. Banks would prefer an even tighter leverage constraint while households prefer $\bar{L}$ to be just small enough to avoid runs.

A leverage cap poses a trade-off between misallocation and endogenous instabilities: a tighter leverage cap increases misallocation but potentially reduces endogenous instabilities. Results suggest it is worthwhile to impose the larger misallocation when the reduction of endogenous risk includes shielding the economy with respect to runs. For leverage caps low enough to prevent any chance of systemic runs, a tighter leverage constraint might benefit banks because of the redistributive effect: economies with lower $\bar{L}$ spend more time in regions where banks own a larger fraction of wealth. The reason is that risk-adjusted capital excess return increases for the region where the constraint is binding (lowest $\eta$ region). The leverage cap limits competition among banks, allowing larger risk-adjusted returns that translate into faster recoveries.

The same exercise assuming away run risk (not shown), i.e., $\Gamma(\cdot) \equiv 0$, further illustrates its role. Absent runs, households prefer not to have any leverage constraint (banks still like it due to the redistributive effect). This means that the misallocation cost tends to be stronger than the imperfect reduction in amplification risk, at least for the agent relatively less exposed to capital.

Given that welfare results weight opposing forces, they depend on the magnitude and sensitivity of the arrival rate of runs, i.e., they are sensitive to the equilibrium selection mechanism. Also, an implicit assumption in this exercise is that the equilibrium selection mechanism is invariant to the leverage cap and exogenous risk.

### 1.6.2 State-contingent leverage cap

A state-contingent leverage cap $\bar{L}(\eta)$ can overcome some of the shortcomings highlighted in the case of the constant leverage constraint. In particular, it allows for controlling banks' risk-taking when the economy would be exposed to runs, i.e., the middle region of the banking sector's wealth share, without limiting leverage during downturns, i.e., when the aggregate state variable is particularly low. The latter avoids deepening recessions that follow a run, i.e., it prevents fostering run risk by

depressing the liquidation price of assets. This intuition is illustrated by the following numerical exercise.

I search for the state-contingent leverage constraint that maximizes an equally Pareto-weighted average of expected indirect utilities.[21] Figure 1.7(d) shows banks' leverage, $x$, and the arrival rate of runs, $p$, for the laissez-faire equilibrium and the one with the optimal leverage constraint. The difference between leverage levels corresponds to the region where the optimal leverage constraint is binding. This optimal state-contingent leverage constraint unequivocally reduces run vulnerability, i.e., the $p$ decreases for every value of the aggregate state variable because it reduces leverage and capital price without having an effect over its liquidation value.

Interestingly, the optimal leverage constraint reintroduces amplification risk in the hidden risk regime of the laissez-faire equilibrium in order to limit risk-taking by banks and shield the economy from run risk. This highlights the larger detrimental effects of systemic runs, especially the ones that occur when the economy is seemingly stable since these generate the largest collapses. This point is also illustrated by the fact that the optimal state-contingent leverage constraint allows for run risk only for relatively low levels of banks' wealth share, i.e., it eliminates completely run risk for levels of the aggregate state variable associated with the hidden risk regime in the laissez-faire equilibrium.

A parallel exercise in an environment without runs (not shown), i.e., $\Gamma(\cdot) \equiv 0$, shows that the optimal state-contingent leverage constraint only slightly limits risk-taking around the banks' wealth share level that divides the crisis regime and the safe one, $\eta^{\psi}$, around which amplification risk is the highest. This indicates that only for extreme amplification risk it is worthwhile to increase misallocation by using a leverage constraint to control this instability.

### 1.6.3 Stabilization of real shocks

Previous results showed that stability can breed instability, and they illustrated that, given this trade-off, agents prefer an economy with strictly positive exogenous risk. This suggested that a full stabilization of real shocks has a cost in terms of financial fragility and might not be an appropriate policy objective. In order to further illustrate this point, I assume a benevolent policymakers are able to influence

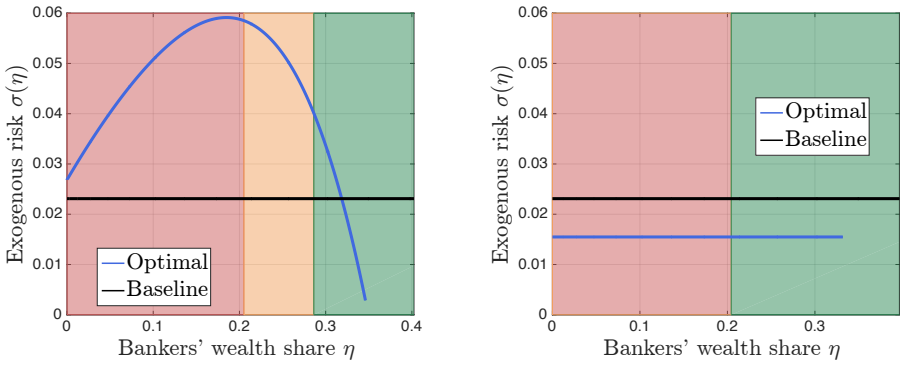
[21] In particular, I consider the family of functions described by $L(\eta) = \sum_{i=-1}^{5} c_i x^i$ and use as a welfare measure $0.5\mathbb{E}(v(\eta)) + 0.5\mathbb{E}(\underline{v}(\eta))$, where the expectation is with respect to the stationary distribution of the state variable.

fluctuations due to real shocks directly and at no cost, and evaluate how they would stabilize real fluctuations depending on the aggregate state variable of the economy: the wealth share of the banking sector.

The exercise searches for the exogenous risk function[22] $\sigma(\eta)$ that maximizes an equally Pareto-weighted average of expected indirect utilities. Results reinforce the message that full stabilization is not optimal in the presence of run risk: it is optimal to allow relatively large fluctuations due to real shocks whenever the economy is exposed to runs, i.e., in the crisis and hidden risk regimes for baseline parameters, to control exposure to runs. Nevertheless, it is optimal to stabilize, i.e., insure against real fluctuations in the economy, when there is no exposure to runs, such as in the safe regime. Figure 1.8(a) illustrates this exercise (the colors indicate the regimes in the economy without the policy).

Figure 1.8: Stabilization policy

(a) Baseline model (run risk) (b) Model without runs, $\Gamma(\cdot) \equiv 0$.




A similar exercise conducted in the version of the economy without runs delivers the result that a low non-contingent level of exogenous risk is optimal, as shown in Figure 1.8(b).[23] This is aligned with the common stabilization prescription and indicates that the misallocation cost and the negative fundamental effect of larger real fluctuations are stronger than the benefit of reducing amplification risk alone. However, the trade-off changes once real fluctuations also reduce the likelihood of large collapses. It is important to mention that, absent financial frictions, full stabilization is always the optimal policy prescription with respect to real fluctuations.

[22]The optimization is performed over fifth-degree polynomial functions.

[23]It is not optimal to set $\sigma \to 0$ because I include banks in the welfare measure and such a policy redistributes wealth toward households. If Pareto-weight on banks is zero, then it is indeed optimal to fully stabilize the economy, i.e., set $\sigma \to 0$.

## 1.7 Conclusions

This paper presents a tractable macroeconomic framework that includes two different endogenous risks related to the financial sector: amplification of real shocks and systemic runs. The model demonstrates that economies with smaller exposure to real shocks are more prone to systemic runs, i.e., *stability breeds instability.* This interaction can be exploited by using *risk to control risk*, i.e., by eschewing the full stabilization prescription and permitting small disruptions in order to prevent excessive risk-taking. Welfare analysis suggests that such an approach can benefit all agents in the economy.

The analysis also illustrates important challenges in the identification of systemic risks. In particular, the threat of large collapses due to systemic runs can be present even when volatility of real and financial variables is low. In other words, systemic risk may be *hidden.* The model provides guidance on how to differentiate such a regime from safe, low-volatility environments.

Finally, the study of a simple and intuitive macro-prudential policy such as a leverage cap exemplifies the difficulties in controlling two different dimensions of risk at the same time, even in an economy in which the financial sector invests in a single asset. This suggests that policies that discourage agents' risk-taking by influencing (the appropriate type of) risk in the economy rather than controlling portfolio decisions may be more successful at promoting systemic stability.

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# Appendix

## 1.A Analytical results

***Proof Lemma 1.1.*** *Households' problem.* The HJB equation for households' problem is

$$\rho \underline{V}(\underline{n},\eta) = \max_{\underline{c},\underline{\iota}\geq 0,\underline{x}\geq 0} \log(c) + \mathbb{E}\left[\frac{d\underline{V}(\underline{n},\eta)}{dt}\right] + \underline{\lambda}\left(V(\underline{n},\eta) - \underline{V}(\underline{n},\eta)\right)$$

s.t.

$$\frac{d\underline{n}}{\underline{n}} = \underline{x}d\underline{r}^k + (1-\underline{x})r - \frac{\underline{c}}{\underline{n}}dt - \left[(x-1)\ell^d + \underline{\tau}\right]dJ_t \ ,$$

where the last term in the HJB equation is associated with idiosyncratic Poisson shock that turns the household into a bank and $V$ represents the value function of banks. The dynamics of aggregate state $\eta$ are taken as given by the household.

*Solution.* Take as given $V(n,\eta) = \rho^{-1}\log(n) + \xi(\eta)$, which is proved below. Conjecture $\underline{V}(n,\eta) = \rho^{-1}\log(\underline{n}) + \underline{\xi}(\eta)$ with $d\underline{\xi} = \underline{\mu}^{\xi}\underline{\xi}dt + \underline{\sigma}^{\xi}\underline{\xi}dZ_t + \left(\underline{\xi}(\tilde{\eta}) - \underline{\xi}\right)dJ_t$. These dynamics will need to satisfy consistency conditions with the aggregate state's law of motion, detailed below. Given the conjecture and using Ito's lemma, the evolution of $\underline{V}$ can be written as

$$d\underline{V} = \underline{V}_n\left(\underline{\mu}^n n dt + \underline{\sigma}^n n dZ_t\right) + \underline{V}_{\xi}\left(\underline{\mu}^{\xi}\underline{\xi}dt + \underline{\sigma}^{\xi}\underline{\xi}dZ_t\right) + \frac{1}{2}\underline{V}_{nn}\left(\underline{\sigma}^n n\right)^2 dt ...$$
$$... + \frac{1}{2}\underline{V}_{\underline{\xi}\underline{\xi}}\left(\underline{\sigma}^{\xi}\underline{\xi}\right)^2 dt + \underline{V}_{\xi n}\underline{\sigma}^n\underline{\sigma}^{\xi}n\underline{\xi}dt + \left[\underline{V}(\tilde{\underline{n}},\tilde{\eta}) - \underline{V}(\underline{n},\eta)\right]dJ_t$$

Then, the HJB reduces to

$$\log(\underline{n}) + \rho\underline{\xi}(\eta) = \max_{\underline{c},\underline{\iota}\geq 0,\underline{x}\geq 0} \underbrace{\log(c) + \frac{1}{\rho}\underline{\mu}_n - \frac{1}{2\rho}\underline{\sigma}_n^2 + p\left[\frac{1}{\rho}\log\left(\frac{\tilde{\underline{n}}}{\underline{n}}\right)\right]}_{\text{only these terms depend on decisions}} ...$$
$$... + p\left(\underline{\xi}(\tilde{\eta}) - \underline{\xi}(\eta)\right) + \underline{\lambda}\left[\xi(\eta) - \underline{\xi}(\eta)\right] + \underline{\mu}_{\xi}\underline{\xi}$$

where the tilde notation refers to the value of a variable just after a systemic run. Note that decisions only affect the first four terms on the RHS so, if the conjecture is correct, the maximization is equivalent to optimize over these terms. First-order conditions deliver $\underline{c} = \rho\underline{n}$, $\iota = [\Phi'(1/q)]^+$, and

$$\underline{\mu}^R - r \leq \underline{x}\left(\sigma + \sigma_q\right)^2 + p\frac{\left(\ell^q - \ell^d\right)}{1 - \ell^d - \underline{x}(\ell^q - \ell^d) - \underline{\tau}} \ , \tag{1.29}$$

which holds with equality for $\underline{x} > 0$. Solving the latter for $\underline{x}$ delivers two roots but only the smaller one, explicitly displayed in equation (1.13) in the main text, is a solution to the maximization problem (the other violates $\underline{n} \geq 0$ after a run). Second-order conditions are satisfied and ensure a unique global maximum.

Replacing these optimal decision on the HJB equation,

$$\begin{aligned}\rho\underline{\xi} = \log(\rho) + \frac{1}{\rho}\left[\underline{x}\left(\underline{\mu}^R - r\right) + r - \rho\right] - \frac{1}{2\rho}\underline{x}^2\left(\sigma + \sigma^q\right)^2 ... \\ ... + p\left[\frac{1}{\rho}\log\left(1 - \ell^d - \underline{x}(\ell^q - \ell^d) - \underline{\tau}\right) + \underline{\xi}(\tilde{\eta}) - \underline{\xi}(\eta)\right] ... \\ ... + \underline{\lambda}\left[\xi(\eta) - \underline{\xi}(\eta)\right] + \underline{\mu}_{\xi}\underline{\xi} \ ,\end{aligned} \tag{1.30}$$

where $\underline{x}$ is given by equation (1.13) and $\underline{\mu}_{\xi}$ satisfies

$$\underline{\mu}_{\xi}\underline{\xi} = \underline{\xi}_{\eta}\left(\mu^{\eta}\eta\right) + \frac{1}{2}\underline{\xi}_{\eta\eta}\left(\sigma^{\eta}\eta\right)^2$$

which follows from Ito's lemma. Since equation (1.30) does not depend on individual state $\underline{n}$, function $\underline{\xi}(\eta)$ can be chosen to ensure it is always satisfied. This verifies the conjecture.

I have characterized the solution of the HJB equation but I still need to verify that process

$$\underline{G}_t \equiv \int_0^t e^{-\rho s}\log(\underline{c}_s)ds + e^{-\rho t}\underline{V}(\underline{n}_t, \eta_t)$$

is martingale under the optimal policy and a supermartingale under any other admissible policy. I omit this technical discussion.

The relation of $\underline{x}$ with respect to arrival rate of a run $p$ and total diffusion volatility $\sigma + \sigma^q$ follow from (1.29). Note that the RHS is strictly increasing in $\underline{x}$ (for solutions that imply positive net worth after the run $\tilde{\underline{n}}$), run intensity $p$, diffusion volatility $|\sigma + \sigma^q|$, and capital loss $\ell^q$. So $\underline{x}$ is decreasing in the mentioned variables.

***Proof Lemma 1.2.*** *Banks' problem.* The HJB equation for the banks' problem is

$$\rho \underline{V}(n,\eta) = \max_{c,\iota \geq 0, x \geq 0} \log(c) + \mathbb{E}\left[\frac{dV(n,\eta)}{dt}\right] + \lambda\left(\underline{V}(n,\eta) - V(n,\eta)\right)$$

s.t.

$$\begin{aligned}\frac{dn}{n} = x dr^k - (x-1)\, r(x)\, dt - \frac{c}{n} dt + \mathbb{1}_{\left\{\ell^d(x)>0\right\}} \left(x\ell^q - 1 + \tau\right) dJ_t ... \\ ... - \mathbb{1}_{\{\mathcal{L}^d > 0 \ \wedge \ \ell^d(x) < 0\}} \phi dt\end{aligned}$$

where the last term in the HJB equation is associated with idiosyncratic Poisson shock that turns the bank into a household and $\underline{V}$ represents the value function of households. The law of motion of aggregate state $\eta$ is taken as given by the banker.

*Solution.* Take as given $\underline{V}(n,\eta) = \rho^{-1}\log(\underline{n}) + \underline{\xi}(\eta)$. Conjecture $V(n,\eta) = \rho^{-1}\log(n) + \xi(\eta)$ with $d\xi = \mu^\xi \xi dt + \sigma^\xi \xi dZ_t + \left(\xi(\tilde{\eta}) - \xi\right) dJ_t$. These dynamics will need to satisfy consistency conditions with the aggregate state's law of motion, detailed below. Given the conjecture and using Ito's lemma, the evolution of $V$ can be written as

$$\begin{aligned}dV = V_n\left(\mu^n n dt + \sigma^n n dZ_t\right) + V_\xi\left(\mu^\xi \xi dt + \sigma^\xi \xi dZ_t\right) + \frac{1}{2} V_{nn}\left(\sigma^n n\right)^2 dt ... \\ ... + \frac{1}{2} V_{\xi\xi}\left(\sigma^\xi \xi\right)^2 dt + V_{\xi n}\sigma^n \sigma^\xi n \xi dt + \left[V(\tilde{n},\tilde{\eta}) - V(n,\eta)\right] dJ_t\end{aligned}$$

Then, the HJB reduces to

$$\begin{aligned}\log(n) + \rho\xi(\eta) = \max_{c,\iota \geq 0, x \geq 0} \underbrace{\log(c) + \frac{1}{\rho}\mu_n - \frac{1}{2\rho}\sigma_n^2 + p\left[\frac{1}{\rho}\log\left(\frac{\tilde{n}}{n}\right)\right]}_{\text{only these terms depend on decisions}} \\ + p\left[\xi(\tilde{\eta}) - \xi(\eta)\right] + \lambda\left[\underline{\xi}(\eta) - \xi(\eta)\right] + \mu_\xi \xi\end{aligned}$$

Note that decisions only affect the first four terms on the RHS. So, if the conjecture is correct, the maximization is equivalent to optimize over these terms. First-order conditions for consumption and investment are symmetric to households', i.e., $c =$

$\rho n$, and $\iota = [\Phi'(1/q)]^+$. There are two local maxima with respect to $x$, one safe that satisfies $x^s \leq (\ell^q)^{-1}$ and another risky that satisfies $x^r > (\ell^q)^{-1}$. The optimal condition for $x^s$ is

$$\mu^R - r(x^s) \leq x^s \left(\sigma + \sigma_q\right)^2 + p\frac{\ell^q}{1 - x^s \ell^q} \ ,$$

which holds with equality for $x^s > 0$. The optimal condition for $x^r$ is

$$\mu^R - r\left(x^r\right) \leq x^r \left(\sigma + \sigma_q\right)^2 + (x^r - 1) r'\left(x^r\right) \ , \tag{1.31}$$

which holds with equality for $x^r > (\ell^q)^{-1}$. The risky solution is optimal if

$$\begin{aligned}(x^r - x^s)(\mu^R - r^f) + \phi \geq (x^r - 1)(r(x^r) - r^f) + \frac{1}{2}\left((x^r)^2 - (x^s)^2\right)(\sigma + \sigma^q)^2 ... \\ ... + p\left(\log\left(1 - x^s \ell^q\right) - \log(\tau)\right)\end{aligned} \tag{1.32}$$

The assumption is that the cost of underperforming $\phi$ is sufficiently large for this condition to be satisfied and therefore ensure that equilibrium is symmetric (avoid that a fraction of banks choose the safe leverage while the others choose the risky one). The optimal risky leverage condition can be rearranged to render equation (1.16) using the pricing condition for deposits (1.14).

Replacing these optimal decisions on the HJB equation,

$$\begin{aligned}\rho\xi = \log(\rho) + \frac{1}{\rho}\left[x\left(\mu^R - r\right) + r - \rho\right] - \frac{1}{2\rho}x^2\left(\sigma + \sigma^q\right)^2 ... \\ ... + p\left[\frac{1}{\rho}\log(\tau) + \xi(\tilde{\eta}) - \xi(\eta)\right] + \lambda\left[\underline{\xi}(\eta) - \xi(\eta)\right] + \mu_\xi \xi \ ,\end{aligned} \tag{1.33}$$

where $x$ is given by equation (1.16) and $\underline{\mu}_\xi$ satisfies

$$\mu_\xi \xi = \xi_\eta\left(\mu^\eta \eta\right) + \frac{1}{2}\xi_{\eta\eta}\left(\sigma^\eta \eta\right)^2 \ ,$$

which follows from Ito's lemma. Since equation (1.33) does not depend on individual state $n$, function $\xi(\eta)$ can be chosen to ensure it is always satisfied. This verifies the conjecture. I omit the technical discussion about the verification argument.

***Proof Lemma 1.3.*** *Aggregate state dynamics.* The proof follows directly from applying Ito's formula for processes with jumps to $\eta\left(N, \underline{N}\right) = N/\left(N + \underline{N}\right)$ and the law of motions of $N$ and $\underline{N}$.

***Proof Lemma*** 1.4**.** *Scaled value functions. Households.* The value function of a households with net worth $\underline{n}$ can be written as

$$\begin{aligned}\underline{V}(n,\eta) &= \rho^{-1}\log(\underline{n}) + \underline{\xi}(\eta) \\ &= \rho^{-1}\log\left(\frac{\underline{n}}{\underline{N}}(1-\eta)qK\right) + \underline{\xi}(\eta) \\ &= \rho^{-1}\left[\log\left(\frac{\underline{n}}{\underline{N}}\right) + \log(K)\right] + \underbrace{\rho^{-1}\log\left((1-\eta)q\right) + \underline{\xi}(\eta)}_{\equiv \underline{v}(\eta)}\end{aligned}$$

where the first line follows from Lemma 1.1, the second from $\underline{N} = (1-\eta)qK$, and the third just illustrates how $\underline{v}(\eta)$ is defined in terms of equilibrium objects already derived. Using this definition to replace $\underline{\xi}(\eta)$ with $\underline{v}(\eta)$ and market clearing conditions, equation (1.30) becomes

$$\begin{aligned}\rho\underline{v}(\eta) = \log(\rho(1-\eta)q) + \frac{1}{\rho}\left[\Phi(\iota) - \delta - \frac{1}{2}\sigma^2 + \underline{\lambda} - \lambda\left(\frac{\eta}{1-\eta}\right)\right] + \underline{v}\underline{\mu}_v ... \\ ... + p(\eta)\left[\underline{v}(\tilde{\eta}) - \underline{v}(\eta)\right] + \underline{\lambda}\left[\frac{1}{\rho}\log\left(\frac{\eta}{1-\eta}\right) + v(\eta) - \underline{v}(\eta)\right] \qquad (1.34)\end{aligned}$$

where

$$\underline{\mu}_v\underline{v} = \underline{v}_\eta\left(\mu^\eta\eta\right) + \frac{1}{2}\underline{v}_{\eta\eta}\left(\sigma^\eta\eta\right)^2$$

Alternatively, this expression can be derived directly by conjecturing directly (1.28) and replacing this together with optimal decisions in the HJB equation. In this approach, it is necessary to take into account that the share of individual wealth to sector's wealth has non-trivial dynamics due to the idiosyncratic shocks that switch agents' type. I verified my results using both procedures.

*Banks.* Derivation is symmetric using value function in Lemma 1.2 and $N = \eta qK$. In this case, $\underline{v}(\eta) \equiv \rho^{-1}\log(\eta q) + \xi(\eta)$. Using this definition to replace $\xi(\eta)$ with $v(\eta)$ and market clearing conditions, equation (1.33) becomes

$$\begin{aligned}\rho v(\eta) = \log(\rho\eta q) + \frac{1}{\rho}\left[\Phi(\iota) - \delta - \frac{1}{2}\sigma^2 + \lambda - \underline{\lambda}\left(\frac{1-\eta}{\eta}\right)\right] + v\mu_v ... \qquad (1.35) \\ ... + p(\eta)\left[v(\tilde{\eta}) - v(\eta)\right] + \lambda\left[\frac{1}{\rho}\log\left(\frac{\eta}{1-\eta}\right) + \underline{v}(\eta) - v(\eta)\right]\end{aligned}$$

where

$$\mu_v v = v_\eta\left(\mu^\eta\eta\right) + \frac{1}{2}v_{\eta\eta}\left(\sigma^\eta\eta\right)^2$$

The alternative procedure described for value function of households also applies for banks.

***Proof Lemma*** 1.5***.*** *Output growth and capital returns.* Aggregate demand per unit of capital is

$$\begin{aligned} y^d(q) &= \frac{C + \underline{C} + \iota K^b + \underline{\iota K}}{K} \\ &= \rho q + \iota(q) \end{aligned}$$

where the second line follows from optimal consumption and investment decisions. Then, the result follows directly from applying Ito's lemma to function $y^d(q)$.

***Proof Proposition 1.1.*** *Invariance to run risk.* I characterize functions $\{q(\eta), \psi(\eta), \sigma^q(\eta)\}$ independently from selection mechanism $\Gamma(\cdot)$.

First, consider the case in which households hold some capital, $\psi < 1$. In this case, expression (1.12) holds with equality. Subtract the latter from (1.15) and use equation (1.14) to find the explicit expression for $r'(x)$. This yields

$$(x - \underline{x}) \left(\sigma + \sigma^q\right)^2 = \frac{a - \underline{a}}{q}. \tag{1.36}$$

Equations (1.24) and (1.18) render

$$\sigma + \sigma^q = \frac{\sigma}{1 - \frac{q_\eta}{q}\eta(1-\eta)(x - \underline{x})} \ . \tag{1.37}$$

Replacing $x = \psi/\eta$, and $\underline{x} = (1-\psi)/(1-\eta)$ into these two conditions and using goods market clearing (1.21), I find an ODE for $q(\eta)$, i.e.,

$$\frac{\psi(q) - \eta}{\eta(1-\eta)} \left( \frac{\sigma}{1 - \frac{q_\eta}{q}\left(\psi(q) - \eta\right)} \right)^2 = \frac{a - \underline{a}}{q} \tag{1.38}$$

where $\psi(q) \equiv \left[q\left(\iota(q) + \rho\right) - \underline{a}\right]/(a - \underline{a})$. The boundary condition for this first-order ODE follows from goods market clearing condition evaluated at $\eta = 0$, i.e.,

$$q(0)\left[\iota(q(0)) + \rho\right] = \underline{a} \ .$$

The latter equation has a unique solution due to the concavity of $\Phi(\cdot)$. This ODE describes the solution for $q(\eta)$ as long as respects the non-negativity constraint for households' capital holdings, i.e., $\underline{x} \geq 0$.

Second, consider the case in which households hold no capital, i.e., $\psi = 1$. In this case, goods market clearing condition implies $q(\eta) = \bar{q}$ which is defined by

$$\bar{q}\left[\iota(\bar{q}) + \rho\right] = a \ .$$

Then, $q_\eta = 0$ and equation (1.37) implies $\sigma^q = 0$. This is a solution as long as households do not desire to hold any capital, i.e.,

$$x\sigma^2 < \frac{a - \underline{a}}{\bar{q}}.$$

Note that in both cases the equations that characterize $q(\eta)$ are independent from selection mechanism $\Gamma(\cdot)$. The same applies for $\{\psi, \sigma^q\}$ which are related to $q$ through static conditions presented above. Also, the following endogenous objects do not appear on the ODE or boundary condition: arrival rate of runs, $p$, losses conditional on runs, $\ell^q$ and $\ell^d$, and the state of the economy after the run $\tilde{\eta}$.

***Proof Proposition 1.2.*** *Amplification region.* The first step of the proof is to show that $q(\eta)$ is a weakly increasing function. The ODE (1.38) can be written as

$$q_\eta = \frac{q}{\psi(q) - \eta}\left(1 \pm \sigma\sqrt{\frac{(\psi(q) - \eta)\, q}{\eta(1-\eta)(a - \underline{a})}}\right) \tag{1.39}$$

Equation (1.36) implies that $\psi > \eta$. If both agents are holding capital, they price it the same. For this to be possible banks who earn a larger return on capital need to have a larger exposure. Then, if we consider the solution with the plus sign, which is associated with $\sigma + \sigma^q < 0$, we have that $q_\eta > 0$ immediately. However, this paper (and the literature) does not focus on this solution. For a discussion about the type of equilibria that includes this solution, see Mendo (2018).

For the solution with the minus sign, which is characterized by $\sigma + \sigma^q > 0$, a sufficient condition for $q_\eta > 0$ is the presence of amplification, i.e. $\sigma^q > 0$. This is evident from writing the ODE as

$$q_\eta = \frac{q}{\psi(q) - \eta}\left(1 - \frac{\sigma}{\sigma + \sigma^q}\right)$$

Since, in this paper, I only consider solutions with $\sigma^q \geq 0$, we have that $q_\eta \geq 0$. In fact, the inequality must be strict, i.e. $\sigma^q > 0$, because the original ODE (1.38) is not satisfied by a constant $q$.

Then, I have proved that $q_\eta > 0$ if $\psi < 1$ (for the equilibria of interest). For the region in which $\psi = 1$, we know that $q = \bar{q}$ and $q_\eta$ =0. See proof of proposition 1.1.

The second step is to show that $\exists \eta^{\psi} \in (0,1]$ s.t. $\psi < 1$ if and only if $\eta < \eta^{\psi}$. In equilibrium, $\psi'(\eta)$ has the same sign as $q'(\eta)$ so $\psi_\eta \geq 0$. This follows from goods market clearing. We also know that $\psi \leq 1$ because no agent can short capital. Therefore, if $\psi(\eta_a) = 1$, then $\psi(\eta) = 1$ for all $\eta > \eta_a$. I define the minimum $\eta$ such that banks manage all capital as $\eta^{\psi}$, i.e., $\eta^{\psi} = \inf_{\eta \in \Psi} \eta$ where $\Psi = \{\eta \in [0,1] \text{ s.t. } \psi(\eta) = 1\}$. If $\eta < \eta^{\psi}$ , then the ODE above holds and $q_\eta > 0$. Note that $\eta^{\psi} > 0$ since without net worth banks cannot hold any capital $\psi(0) = 0$. The proof is completed by noting the following. First, $q_\eta = 0$ implies $\sigma^q = 0$. Second, $q_\eta > 0$ implies $\sigma^q > 0$.

***Proof Proposition 1.3.*** *Run vulnerability region.* Proposition 1.1 establishes that $\{q(\eta), \psi(\eta)\}$ are independent from selection mechanism $\Gamma(\cdot)$. Losses for depositors $\ell^d$ depend on banks' leverage $x = \psi/\eta$, capital price $q(\eta)$, and liquidation value $q(\tilde{\eta})$. Then, it is only left to show that liquidation value does not depend on $\Gamma(\cdot)$. This follows directly from equation (1.20), which determines $\tilde{\eta}$ given $q(\eta)$ and transfer policy $\mathcal{T}$. Note that $\{\ell^q, \ell^y\}$ only depend on $\{q(\eta), \psi(\eta), q(\tilde{\eta})\}$ so they are also independent from selection mechanism.

***Proof Corollary 1.1.*** Propositions 1.1 shows that $\{q, \psi\}$ are continuous functions of state $\eta$. Then, equation (1.20) implies function $\tilde{\eta}$ is also a continuous function if policy $\mathcal{T}(\eta)$ is continuous. Consider the uncapped potential losses $\hat{\ell}^d = 1 - \frac{q(\tilde{\eta})}{q}\left(\frac{x}{x-1}\right)$ which are equivalent to $\ell^d(\eta)$ except they can be negative. It follows that $\hat{\ell}^d(\eta)$ is a continuous function.

Note that $\lim_{\eta \to 0} \hat{\ell}^d(\eta) < 0$ because there are no potential losses since $q(\tilde{\eta}) \geq q(0)$ and $\lim_{\eta \to 1} \hat{\ell}^d(\eta) < 0$ because banks don't use leverage anymore as $\eta \to 1$. Then, the result follows directly from the continuity of $\hat{\ell}^d(\eta)$.

***Proof Corollary 1.2.*** Given equation (1.20), a constant $\mathcal{T}$ implies a constant liquidation value $q(\tilde{\eta})$. Then, results follow from the fact that $q(\eta)$ and $\psi(\eta)$ are weakly increasing in $\eta$ as shown in proposition *1.2.*

***Proof Corollary 1.3.*** Note that for $\eta > \eta^{\psi}$, the expression for depositors' losses is

$$\ell^d = \left[1 - \frac{q(\tilde{\eta})}{\bar{q}}\left(\frac{1}{1-\eta}\right)\right]^+$$

which is strictly decreasing in $\eta$ for a constant $q(\tilde{\eta})$. Therefore, if $\ell^d\left(\eta^{\psi}\right) = 0$, then $\ell^d = 0$ for all $\eta > \eta^{\psi}$ and there is no hidden risk region. Conversely, if $\ell^d\left(\eta^{\psi}\right) > 0$, by the continuity of $\ell^d$, there exists a $d_a > 0$ s.t. $\ell^d(\eta) > 0$ for all $|\eta - \eta^{\psi}| < d_a$.

***Proof Lemma 1.6.*** Consider the following geometric process (with constant coefficients)

$$\frac{dw_t}{w_t} = \mu_w dt + \sigma_w dZ_t$$

If the associated stationary distribution is not degenerate, it satisfies satisfies $g(w) = c_w w^{2\left(\frac{\mu_w}{\sigma_w^2}-1\right)}$ where $c_w$ is a scaling constant. This follows directly from the Kolmogorov Forward Equation (KFE). For details, see for example BruSan.

I show that the dynamics of the aggregate state $\eta$ converge to a geometric process as $\eta \to 0$ and that the associated drift and volatility satisfy $\lim_{\eta\to 0} \mu_\eta > \lim_{\eta\to 0} \sigma_\eta$ . This implies that stationary distribution $g(\eta) = c_\eta \eta^{2\left(\frac{\mu_\eta}{\sigma_\eta^2}-1\right)}$ has mass zero at $\eta = 0$ since exponent is positive. I consider the model l without runs, i.e. $\Gamma(\cdot) \equiv 0$. The existence of a non degenerate stationary distribution follows from the idiosyncratic Poisson shocks that switch agents' type.

Using lemma 1.3, market clearing and first order conditions, the geometric drift and diffusion of $\eta$ can be written as

$$\mu_\eta = \sigma_\eta^2 + (1-\psi)\left(\frac{a-\underline{a}}{q}\right) + \underline{\lambda}(1/\eta - 1) - \lambda$$
$$\sigma_\eta = \left(\frac{\psi}{\eta} - 1\right)(\sigma + \sigma_q)$$

I now work on the limits as $\eta \to 0$ of the necessary expressions. First, note that equation (1.37) and the fact that $\lim_{\eta\to 0} \psi \to 0$ implies that $\lim_{\eta\to 0} \sigma^q \to 0$. From goods market clearing, we have that $\lim_{\eta\to 0} q = \underline{q}$ where

$$\underline{q}\left(\iota(\underline{q}) + \rho\right) = \underline{a}$$

It is left to define the limit of $\psi/\eta$. Equation (1.36) can be written as

$$\frac{a-\underline{a}}{q} = \left(\frac{\psi}{\eta} - 1\right)(\sigma + \sigma^q)^2$$

so taking the limit as $\eta \to 0$, we have that

$$\lim_{\eta\to 0} \frac{\psi}{\eta} = \frac{a-\underline{a}}{\underline{q}\sigma^2} + 1$$

Therefore,

$$\lim_{\eta\to 0}\sigma_\eta = \left(\frac{a-\underline{a}}{\underline{q}\sigma}\right)$$
$$\lim_{\eta\to 0}\mu_\eta = \left(\frac{a-\underline{a}}{\underline{q}\sigma}\right)^2 + \left(\frac{a-\underline{a}}{q}\right) + \underline{\lambda}\left[\lim_{\eta\to 0}\frac{1}{\eta} - 1\right] - \lambda$$

Clearly, if $\underline{\lambda} > 0$, we have that $\lim_{\eta\to 0}\mu_\eta > \lim_{\eta\to 0}\sigma_\eta$ and the stationary distribution has mass zero at $\eta = 0$.

***Proof Proposition 1.4.*** *Exogenous risk and total diffusion risk.* First, I show that $q(\eta;\sigma)$ is weakly decreasing in $\sigma$ for $\eta \in [0,1]$. I only consider the solution with $\sigma + \sigma^q > 0$, i.e. the one associated with the minus sign for ODE (1.39). Consider $\sigma^H > \sigma^L$. Since for $\eta = 0$, the price of capital does not depend on $\sigma$, we have $q(0;\sigma^L) = q(0;\sigma^H)$. I prove that $q(\eta;\sigma^L) \geq q(\eta;\sigma^H)$ by contradiction. Assume $q(\hat{\eta};\sigma^L) < q(\hat{\eta};\sigma^H)$ for some $\hat{\eta}$. Since both functions are continuous in $\eta$, this means that $\exists\check{\eta} \in (0,\hat{\eta})$ such that $q(\check{\eta};\sigma^L) = q(\check{\eta};\sigma^H)$ and $q_\eta(\check{\eta};\sigma^L) < q_\eta(\check{\eta};\sigma^H)$. This is a contradiction since $q_\eta$ is decreasing in $\sigma$ (and all other values are the same). Since $\psi(\eta)$ is directly related to $q(\eta)$ through market clearing condition, it is also weakly decreasing in $\sigma$.

To prove that $\sigma + \sigma^q$ is weakly increasing in $\sigma$ for every $\eta$, note that equation (1.36) can be written as

$$(\sigma + \sigma^q)^2 = \frac{\eta}{\psi - \eta}\left(\frac{a-\underline{a}}{q}\right).$$

Then, the result follows from $q$ and $\psi$ being weakly decreasing in $\sigma$. Note this implies that the RHS of the latter equation weakly increasing in $\sigma$.

***Proof Proposition 1.5.*** *Exogenous risk and run risk.* It is sufficient to prove that depositors losses are weakly decreasing in $\sigma$ for a given liquidation price, which I call $q^{liq}$. Then, losses can be written as

$$\ell^d = \left[1 - \frac{q^{liq}}{q}\left(\frac{1}{1-\eta/\psi}\right)\right]^+.$$

The result follows from $q$ and $\psi$ being weakly decreasing in $\sigma$, which is proved in proposition 1.4. Lower capital price and lower leverage shield the economy from runs.

***Proof Corollary 1.4.*** The follows from noting that the ODE for $q(\eta)$, i.e., equation (1.39), converges to

$$q_\eta = \frac{q}{\psi(q) - \eta}$$

when $\sigma \to 0$. The relevant boundary conditions $q(0)$ is the same as before. The solution to this differential equation reaches $\psi = 1$ at $\eta^+ > 0$ because $q(0) < \bar{q}$ where $\bar{q}$ is the capital prices associated with $\psi = 1$.

***Proof Lemma 1.7.*** The equilibrium with complete markets is standard. I provide a brief overview of the equilibrium and focus on the welfare result. The economy collapses to a representative single agent model with production technology $y_t = ak_t$, i.e. all capital is managed by banks and risk is perfectly shared among agents. Since all agents have the same preferences, perfect risk sharing implies symmetric positions (no deposits). Capital price $\bar{q}$ solves $\bar{q}(\iota(\bar{q}) + \rho) = a$.

The single agent problem can be written as

$$\rho V(k) = \log(c) + \mathbb{E}\left[dV(k)\right]$$

s.t. (1.1). It is straightforward to guess and verify that the value function can be written as

$$V(k) = \rho^{-1} \log(\bar{q}k) + \bar{v}$$

where

$$\bar{v} = \rho \left( \log(\rho) + \Phi(\iota(\bar{q}) - \delta - \frac{1}{2}\sigma^2 \right)$$

Note that value for agents is decreasing in exogenous risk $\sigma$.

***Proof Proposition 1.6.*** When the leverage constraint is not binding the ODE that characterizes the equilibrium is still (1.38) for $\psi < 1$. If the leverage constraint is binding, then $\psi = \bar{L}\eta$, and capital price $q(\eta)$ can be solved from goods market clearing condition,

$$q(\iota(q) + \rho) = \bar{L}\eta(a - \underline{a}) + \underline{a} \ ,$$

which delivers a strictly increasing function of $\eta$. Then, amplification risk $\sigma^q$ solves (1.37).

I proceed to prove the result by contradiction. Consider $\eta_1$ as the minimum $\eta$ such that $\psi(\eta; \bar{L}) > \psi(\eta)$. Given that $\psi(0; \bar{L}) = \psi(0) = 0$, by continuity, $\exists \eta_2 \in [0, \eta_1)$ such that $\psi(\eta_2; \bar{L}) = \psi(\eta_2)$ and $\lim_{\eta \downarrow \eta_2} \psi_\eta(\eta; \bar{L}) > \lim_{\eta \downarrow \eta_2} \psi_\eta(\eta)$. Since the laissez-faire leverage $x = \psi/\eta$ is a decreasing function of $\eta$, we know it is a feasible allocation for $\eta > \eta_2$. Therefore, the characterization of both solutions in this region is given by ODE (1.38) with boundary condition $q(\eta_2)$. This implies $\lim_{\eta \downarrow \eta_2} \psi_\eta(\eta; \bar{L}) = \lim_{\eta \downarrow \eta_2} \psi_\eta(\eta)$ which is a contradiction. Therefore, $\psi(\eta; \bar{L}) \leq \psi(\eta)$ for all $\eta$.

Given the direct relation between capital price $q$ and allocation $\psi$, it follows that $q(\eta; \bar{L}) \leq q(\eta)$.

***Proof Proposition 1.7.*** Consider the $\eta$-region in which there is misallocation in both environments and the leverage cap is not binding, i.e., $\left\{\psi\left(\eta; \bar{L}\right), \psi\left(\eta\right)\right\} \in [0, 1)$ and $x\left(\eta; \bar{L}\right) < \bar{L}$. Then, both solutions need to satisfy

$$(\sigma + \sigma^q)^2 = \left(\frac{a - \underline{a}}{q}\right) \frac{\eta(1-\eta)}{\psi - \eta}$$

Then, the result follows from $\psi$ and $q$ weekly decreasing when a leverage cap in introduced, see proposition 1.6.

***Proof Proposition 1.8.*** (a) Given that $q(\eta)$ weakly decreases with the introduction of a leverage cap, the $\tilde{\eta}$ that solves $\mathcal{T}(\eta) = q(\tilde{\eta})\tilde{\eta}$ increases. Given that $\mathcal{T}(\eta)$ is not affected by the inclusion of the leverage cap, the new liquidation price needs to be weakly lower, i.e. $q(\tilde{\eta}(\eta; \bar{L}); \bar{L}) \leq q(\tilde{\eta}(\eta))$.

(b) The result follows from the fact that $q(\eta)$ weakly decreases with the introduction of a leverage cap.

# Chapter 2

# Financial Frictions: Amplifying or Hedging Fundamental Shocks?

## 2.1 Introduction

Understanding the role of financial frictions in macroeconomic fluctuations has been a topic of first-order interest for the profession since the last financial crisis. A central lesson is that financial frictions amplify the effect of fundamental shocks by triggering a spiral between asset prices and the distribution of wealth. This paper makes a simple yet important theoretical precision about this logic: financial frictions make the economy vulnerable to the mentioned spiral but this does not necessarily imply amplification of real shocks.

The spiral is a self-contained mechanism and the shock is a device that allows agents to coordinate their asset demands (or beliefs about asset prices) to trigger the spiral. I illustrate this point in two ways using a stripped-down macroeconomic model with financial frictions. First, I show the existence of a hedging equilibrium in which agents coordinate asset prices drops on negative fundamental shocks. Since the main role of the real shock is to synchronize beliefs about asset prices, agents can coordinate asset price movements in the counter-intuitive direction too. Second, I show that the mentioned spiral can be triggered by extrinsic uncertainty (sunspots). There is no need for a fundamental impulse.

I also show that in the hedging equilibrium agents achieve greater welfare than in the standard amplification one and suggest a policy intervention (an asset purchase program) that ensures coordination on this hedging equilibrium. The broad intuition

behind the welfare ranking is that total volatility of economic outcomes is lower when fundamental shocks hedge fluctuations in asset prices.

**Model.** I set up a simple continuous-time macro model with risk-averse experts and households. Experts obtain higher returns from managing capital than households and these returns are subject to capital quality shocks, which are the only source of fundamental uncertainty in the economy. There are no equity markets, so experts finance capital holdings beyond their net worth by issuing short-term non-contingent debt to households. The financial friction, i.e., the absence of financial instruments to trade aggregate risk, implies that production and risk exposure decisions are interdependent: the only way to expand production is to hold more capital and bear the associated risk.

The economy has two equilibria: the amplification equilibrium and the hedging one. Both present a safe regime and a crisis regime. The safe regime is associated with situations in which the experts' capitalization is sufficiently large to allow banks to absorb all capital in the economy and maintain low leverage. Experts are levered so negative capital quality shocks hurt their net worth. However, there is no effect over their capital demand. Experts just increase their leverage and keep holding the entire capital stock. In the safe regime, both equilibria are observationally equivalent.

The crisis regime corresponds to situations in which the wealth share of experts is sufficiently low to prevent banks from holding all capital in the economy. Managing the entire stock of capital would require such high leverage and risk exposure that experts refrain from doing so despite their productivity advantage. In this region, experts' capital demand depends on their net worth because it determines their risk-bearing capacity. Larger net worth allows expert to hold the same capital with a lower risk exposure (leverage).

**Spiral.** The spiral between wealth distribution and asset prices follows from experts' capital demand depending on their net worth. A reduction in capital prices generates larger losses for the experts than for households because the former use debt to finance capital beyond their net worth (experts are levered). Then, experts reduce their capital demand (fire-sale capital to households) despite their productivity advantage, which depresses capital price further.

**Amplification vs. hedging.** In the crisis region, equilibria differ in how agents coordinate their shifts in capital demand (and therefore capital prices) when the capital quality shocks are realized. As consequence, economic outcomes such as total

volatility of capital returns, the range of experts' capitalizations that correspond to the crisis region, capital misallocation and price, and welfare differ across equilibria.

In the amplification equilibrium, capital price drops following a negative capital quality shock so the endogenous response of asset prices amplify the effect of the real shock over capital returns leading to highly volatile economic outcomes. In the hedging equilibrium, agents coordinate an increase in their capital demand, and therefore in capital price, upon a negative fundamental shock. The rise in their demand is sufficiently large to overcome the effect of the real shock and generate a wealth gains for levered experts. In this case, the fundamental shock and the capital price movement hedge each other. As a result, total volatility of capital returns is lower than in the amplification equilibrium.

Emboldened by the lower total volatility, experts choose larger leverage in the hedging equilibrium, which translates into higher allocation efficiency. Lower volatility also means that experts are willing to hold the entire capital stock with a lower share of wealth, i.e., the range of experts' capitalizations that correspond to the crisis region is smaller in the hedging equilibrium. The higher leverage also helps the economy to transit faster out of the crisis region. The lower volatility, higher allocation efficiency, and faster transition out of the crisis region make the hedging equilibrium superior in terms of welfare.

**Policy.** Given the welfare ranking, a policy intervention that can ensure coordination on the hedging equilibrium is desirable. The commitment to an asset purchase program that sustains asset prices for an intermediate range of experts' capitalization achieves this. Moreover, the policy only acts as a credible off-equilibrium threat, which implies that the policy measure is costless. It is important not to attempt to sustain asset prices for extremely low levels of experts' capitalization, otherwise the government would be forced to hold capital which is largely inefficient.

Consider the situation in which experts are well capitalized in both equilibria. After a sequence of negative fundamental shocks, the economy is at the threshold in which further deterioration of experts' net worth leads to a price drop only in the amplification equilibrium. Experts are not willing to hold the entire capital stock if total capital return volatility is endogenously amplified by asset price movement but would do so if such additional volatility were not present. If the government credibly commits to sustain capital price, it eliminates asset price fluctuations and experts now hold the entire capital stock at the efficient price. There is no need for the government to effectively purchase capital. The policy implies coordination in

the hedging equilibrium as long as there is no attempt to sustain capital price for experts' capitalization levels that correspond to the crisis region even in the hedging equilibrium.

**Volatility paradox.** The volatility paradox (Brunnermeier and Sannikov (2014)) establishes that as the exogenous volatility of fundamental shocks decreases, endogenous amplification (the sensitivity of capital price to the shock) increases because experts take larger leverage. This result corresponds to the standard amplification equilibrium. In the hedging equilibrium, a decrease in the volatility of the fundamental shock leads to a reduction of experts' leverage since it represents a larger hedging for asset price fluctuations. Lower leverage decreases the sensitivity of asset prices to the fundamental shock.[1]

**Extrinsic uncertainty.** In order to illustrate that the spiral between wealth distribution and asset prices does not need to be triggered by a fundamental shock, I consider the economy when the aggregate shock does not have any direct real effect, i.e. the aggregate shock becomes a sunspot. In this case, one equilibrium is the first-best outcome in which experts own the entire capital stock and capital price is high and constant. The economy is completely deterministic.

However, there is a second equilibrium in which agents coordinate on the sunspot to trigger the discussed spiral. This highlights again that the fundamental shock is not essential to generate economic fluctuations, an exogenous coordination device is enough. Interestingly, the sunspot equilibrium is the one towards both the hedging and the amplification equilibrium converge as fundamental volatility vanishes.

**Literature.** This article fits within the literature that studies financial frictions in infinite-horizon macroeconomic models going back to seminal contribution by Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997), and Bernanke et al. (1999). These papers illustrate how financial frictions allow for a spiral mechanism between wealth distribution and asset prices and how this spiral amplifies the effect of real shocks. Brunnermeier and Sannikov (2014) use continuous-time techniques to provide a full characterization of the amplification mechanism and emphasize that it is highly non-linear and manifest strongly when borrowers are not well capitalized. This paper provides an important insight about this mechanism: financial frictions indeed expose the economy to the mentioned spiral but this does not necessarily imply amplification

[1] In the amplification equilibrium, total volatility of capital return decreases along with fundamental volatility while in the case of the hedging equilibrium it increases.

of real shocks. The real shock main role is not its initial direct effect but to serve as coordination device.

This paper relates to the literature that highlights similar spirals in finance. Brunnermeier and Pedersen (n.d.) illustrate how margin constraints that depend on volatility of asset prices open the door for a spiral between net worth of market makers and asset price deviations from fundamental value. Di Tella (n.d.) shows a spiral between the wealth share of productive agents and their investment opportunities (relative to unproductive agents). Several other papers discuss spirals: Shleifer and Vishny (1997); Xiong (2001); Kyle and Xiong (2001); Gromb and Vayanos (2002), among others. This article suggests that the mechanisms unveiled by this literature can present a similar decomposition between initial impulse and spiral, as well as alternative predictions.

This article also relates to the literature on sunspot equilibria introduced by Shell (1977) and Cass and Shell (1982, 1983). In these equilibria extrinsic uncertainty has a non-trivial influence over allocations and is rationally understood by agents. This article illustrates that such equilibrium exists in infinite-horizon macroeconomic models with financial frictions and uses the example to isolate the role of coordination from the real initial impulse of the aggregate shock.

Cass (1992, 1989) illustrates that in an economy with nominal assets, market incompleteness lead to vast multiplicity of sunspot equilibria. Woodford (1986) shows that a stationary sunspot equilibrium exists in a production economy with infinite lived agents and money, when it is not possible to borrow against future income. These papers rely on the potential multiplicity of the price level for their results. Different price levels translate into different market incompleteness in real terms, which unsurprisingly leads to different allocations. The present paper contributes to this literature by illustrating that sunspot equilibria can originate due to incomplete markets in when considering real assets. This is relevant since Antinolfi and Keister (1998) established that market incompleteness does not necessarily imply the existence of a sunspot equilibria.

## 2.2 Model

Time is continuous $t \in [0, \infty)$. The economy is populated by a unit mass of two type of agents: experts and households.

**Preferences.** Preferences are symmetric

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_t)dt\right] \tag{2.1}$$

**Technology.** Production technology is linear in physical capital and experts are more productive than households. In particular, experts produce $ak_t$ units of final goods with their capital holdings while households only $\underline{a}k_t$ where $\underline{a} < a$. Physical capital (absent purchases) evolves according to

$$dk_t = \sigma k_t dZ_t \tag{2.2}$$

where $\sigma > 0$ is the exogenous volatility of capital growth and $dZ_t$ is the aggregate capital quality shock. This is the only source of fundamental uncertainty in the economy.

**Financial markets.** The only financial assets in the economy are non-contingent bonds. The financial friction is the absence of financial contracts that allow to trade aggregate risk, $dZ_t$, e.g. equity.

Later, in order to prevent any type of agent to overcome the economy, I introduce idiosyncratic Poisson shocks that turn households into experts (with intensity $\lambda$) and vice-versa (intensity $\underline{\lambda}$).[2]

### 2.2.1 Agent's problem

The net worth of an agent with productivity $a^i \in \{a, \underline{a}\}$ evolves according to

$$dn_t = \left[a^i k_t + r_t b_t - c_t\right]dt + d\left(q_t k_t\right) \tag{2.3}$$

where $q_t$ represents the capital price, $r_t$ the risk-free interest rate on bonds, and $n_t = q_t k_t + b_t$ the agent's net worth. Then, the agents problem is to choose $\{c_t, b_t, k_t\}_{t=0}^\infty$, given initial wealth $n_0$, in order to maximize (2.1) subject to (2.3), and $k_t \geq 0$.[3]

[2] These shocks will not alter optimality conditions presented below.

[3] Law of motion (2.2) determines capital gains $d(q_t k_t)$ but does not describe the evolution of capital holdings of individual agents since it does not include capital purchases. I abuse notation by using $k_t$ in both (2.2) and (2.3).

### 2.2.2 Equilibrium

Denote the set of experts by the interval $\mathbb{I} = [0, 1]$ and index individual experts by $i \in \mathbb{I}$. Similarly, let the set of households be the interval $\mathbb{J} = (1, 2]$ and index individual households by $j \in \mathbb{J}$.

For any initial endowment of capital $\{k_0^i, k_0^j : i \in \mathbb{I}, j \in \mathbb{J}\}$ such that

$$\int_{\mathbb{I}} k_0^i di + \int_{\mathbb{J}} k_0^j dj = K_0$$

an equilibrium is a set of stochastic functions[4] on the filtered probability space defined by aggregate shock $\{Z_t : t \geq 0\}$: capital price $\{q_t\}$, risk-free rate $\{r_t\}$, households' decisions $\{c_t^j, k_t^j, b_t^j\}$ for $j \in \mathbb{J}$, and experts' decisions $\{c_t^i, k_t^i, b_t^i\}$ for $i \in \mathbb{I}$ such that:

- Agents' decisions solve their problems.
- Markets for final goods, capital and bonds clear.

### 2.2.3 Characterization and recursive equilibrium

This section provides a succinct characterization of the recursive equilibrium in the model. The reader interested in a detailed derivation is referred to Appendix A or Brunnermeier and Sannikov (2016).

**Capital price.** Conjecture that capital price evolves according to

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t$$

where $\mu_t^q$ is the expected capital price growth, and $\sigma_t^q$ is the sensitivity of the capital price growth to the aggregate capital quality shock $dZ_t$. Processes $\{\mu_t^q, \sigma_t^q\}$ are endogenous equilibrium objects. Literature usually refers to $\sigma_t^q$ as endogenous volatility since $|\sigma + \sigma_t^q|$ is the total volatility of returns, however, in the alternative equilibrium emphasize in this paper, $\sigma_t^q < 0$, i.e. it price fluctuations hedge real shocks and can dampen total volatility.

**Optimal decisions.** Given this capital price process, the following lemma characterizes the optimal consumption and portfolio decisions.

[4]Stochastic functions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are mappings $X : S \times \Omega \to \mathbb{R}$ such that $X(s, .)$ is random variable $\forall s \in S$. Stochastic processes are the special case where $S = \mathbb{R}_+$, which is usually interpreted as time.

**Lemma 2.1.** *Optimal decisions of an agent with productivity $a^i$ are characterized by*

$$c_t = \rho n_t \tag{2.4}$$

$$\frac{a^i}{q_t} + \underbrace{\mu_t^q + \sigma\sigma_t^q}_{\mathbb{E}(d(q_t k_t)/dt)} - r_t \leq x_t \left(\sigma + \sigma_t^q\right)^2 \tag{2.5}$$

*where $x_t \equiv q_t k_t / n_t$ corresponds to the fraction of wealth invested in capital. The inequality holds with equality if $x_t > 0$.*

Agents consume their net worth at a constant rate due to logarithmic preferences. The expected excess return of capital over bonds (LHS of (2.5)) equates the compensation the agent requires for holding this risky asset (the RHS) when the agent is a marginal buyer.

**Aggregate state.** The shock structure and the linear production function allow to reduce individual states of an agent's problem, i.e. their capital $k_t$ and bond positions $b_t$, to a single individual state variable: their net worth. So, in general, the aggregate state of this economy will be the distribution of net worth.

Fortunately, the state space of this economy can be simplified to a single aggregate state variable. First, all agents' decisions will be linear in their net worth, which makes the net worth heterogeneity within sectors irrelevant: only aggregate net worths of each sector,

$$N_t = \int_{\mathbb{I}} n_t^i di \quad \text{and} \quad \underline{N}_t = \int_{\mathbb{J}} n_t^j dj \ ,$$

are aggregate states. Second, since capital is the only asset in positive net supply, total wealth in the economy is equal to the total value of the capital stock. Then, the aggregate states of the economy $(N_t, \underline{N}_t)$ are equivalent to $(K_t, \eta_t)$ where

$$\eta_t \equiv \frac{N_t}{N_t + \underline{N}_t}$$

represents the wealth share of experts. Third, I can dispense with $K_t$ as an aggregate state because allocations (scaled by capital) do not depend on the level of the capital stock. Then, the only aggregate state is the wealth share of experts $\eta_t$.

Denote the law of motion of the aggregate state as

$$\frac{d\eta_t}{\eta_t} = \mu_t^\eta dt + \sigma_t^\eta dZ_t$$

where $\{\mu^\eta, \sigma^\eta\}$ are endogenous processes.

**Recursive equilibrium.** A Markov equilibrium is a set of of functions $f(\eta) : [0,1] \to \mathbb{R}$, i.e., prices $\{q, \mu^q, \sigma^q, r\}$, allocations $\{x, \underline{x}, c/n, \underline{c}/\underline{n}\}$, and law of motion for aggregate state $\{\mu^\eta, \sigma^\eta\}$, such that

- Given prices and the law of motion (LoM) of the aggregate state, allocations solve agents' problems.
- Markets for goods, capital and bonds clear.
- Dynamics of prices are consistent with LoM of the aggregate state.
- LoM of the aggregate state is consistent with optimal decisions.

**Characterization.** The following Lemma characterizes the equilibrium capital price function $q(\eta)$ and capital share owned by experts $\psi(\eta)$. The rest of the equilibrium objects are (uniquely) derived from these functions. See Appendix A for details.

**Lemma 2.2.** *An equilibrium capital price function $q(\eta)$ satisfies one of the following ODE for $\eta \in [0, \eta^\psi]$*

$$q' = \frac{q}{\psi - \eta}\left[1 \pm \sigma\sqrt{\frac{(\psi-\eta)}{\eta(1-\eta)}\frac{q}{(a-\underline{a})}}\right]$$

*where $\psi(\eta) = (\rho q(\eta) - \underline{a}) / (a - \underline{a})$ and boundary conditions $q(0) = \underline{a}/\rho$ and $q(\eta^\psi) = a/\rho$ for some threshold $\eta^\psi \in (0,1]$. Moreover, $q(\eta) = a/\rho$ for $\eta \in [\eta^\psi, 1]$.*

## 2.3 The *hedging* equilibrium

The model presented is representative of a large class of models (e.g. Kiyotaki and Moore (1997); Bernanke et al. (1999); Brunnermeier and Sannikov (2014)) which highlight that financial frictions can *amplify* the effect of fundamental shocks via adverse feedback loops: when productive agents (experts) are levered, a negative fundamental shock decreases their net worth. This leads them to scale back their operations and fire-sale their assets to less productive agents. As consequence, asset prices decline hurting productive agents' net worth further.

A key element of the described mechanism is the bidirectional relationship between wealth distribution and asset prices: an increase in the price of capital disproportionally benefit levered productive agents, and larger wealth for productive agents allows them to increase their capital demand, which translates into a higher capital price. This opens the door for self-fulfilling fluctuations in asset prices as long as there is coordination device available for agents. Fundamental shocks can serve as such devices.

In the amplification mechanism emphasized above, after a negative fundamental shock asset prices decrease triggering an adverse spiral. This is certainly a possible outcome. However, the model has an alternative prediction. The fundamental shock can serve as a coordination device for asset price movements in a counterintuitive direction. If agents believe that asset prices will increase following a negative fundamental shock and that the rise will be sufficiently high to overcome the effect of the fundamental shock over (levered) productive agents, then their expectations become self-fulfilling. A positive spiral is triggered by a negative real shock. I refer to this alternative equilibrium as the *hedging* equilibrium because asset prices hedge experts' net worth against real shocks. The following proposition formalizes this intuition. I use tilde notation to distinguish equilibrium objects associated with the hedging equilibrium.

**Proposition 2.1.** *If there exist a equilibrium capital price $q(\eta)$ with "amplification", i.e. $\sigma+\sigma^q(\eta) > 0$ for $\eta < \eta^\psi \in (0,1)$, then there exist another one $\tilde{q}(\eta)$ with "hedging", i.e. $\sigma + \tilde{\sigma}^q(\eta) < 0$ for $\eta < \tilde{\eta}^\psi \in (0,1)$. Moreover, the crisis region is larger in the amplification equilibrium, i.e. $\tilde{\eta}^\psi < \eta^\psi$. Also, capital price and allocation efficiency are larger in the hedging equilibrium, i.e. $q(\eta) < \tilde{q}(\eta)$ and $\psi(\eta) < \tilde{\psi}(\eta)$ for $\eta \in (0, \eta^\psi)$.*

A key lesson from the existence of the hedging equilibrium is that in an economy with financial friction, agents can use fundamental shocks as coordination devices for asset price movements independently of the direct effect of the fundamental shock. This follows from the possibility of self-fulfilling price movements explained above. To illustrate this further, Appendix B highlights that in the seminal credit cycles model of Kiyotaki Moore (1997) there is an alternative solution for the impulse response functions following a positive productivity shock in which capital price decreases, i.e. *hedges* the effect of the real shock.

### 2.3.1 *Amplification* vs. *hedging*

Given the equilibrium multiplicity, a natural question is how does the two equilibria compare to each other. Proposition 2.1 establishes that allocation efficiency and asset prices are higher in the hedging equilibrium (for any given wealth distribution). In the hedging equilibrium, return on capital is less volatile because fundamental shocks are hedged by asset price movements. This encourages productive agents to hold a larger fraction of the capital stock and the more efficient capital allocation translates into a higher price of capital. Figure 2.1(a) and 2.1(b) illustrate the comparison between both equilibria in terms of experts' wealth share $\psi$ and total volatility of capital return $|\sigma + \sigma^q|$. Recall that capital price $q(\eta)$ is a monotonic transformation of experts capital share $\psi(\eta)$.

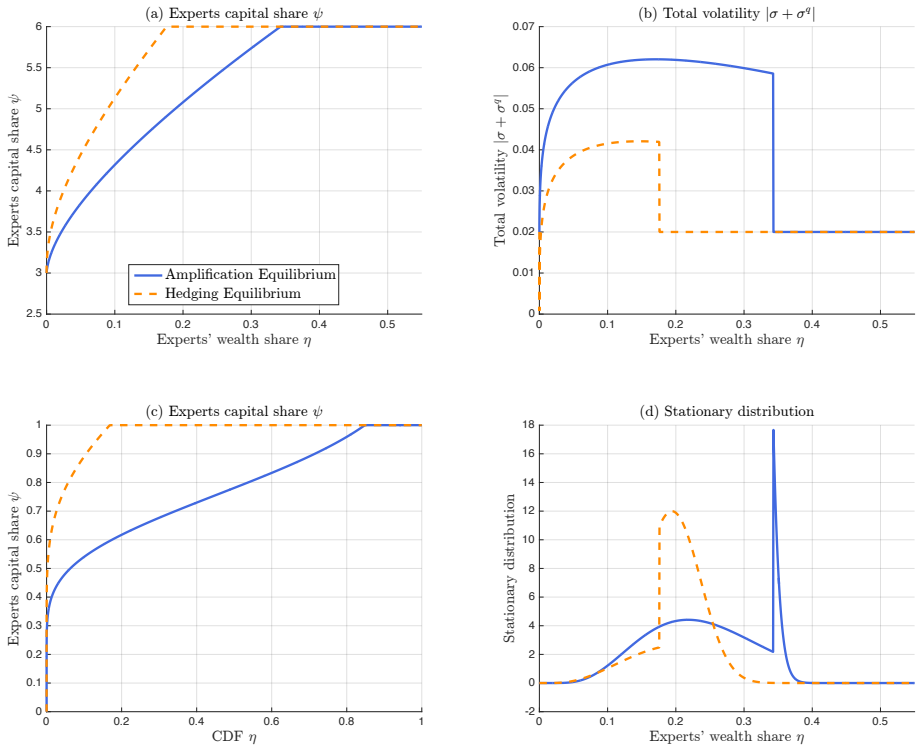
Of course, the comparison is incomplete without acknowledging that the two equilibria present different dynamics, i.e. the frequency with which the economy visits the different states $\eta$ differs across equilibria. Figure 2.1(c) presents a comparison between capital allocation in both equilibria taking into account that stationary distributions are different. It shows the experts capital share $\psi$ in the $y$-axis and the CDF of the state in the $x$-axis. Using this transformation, we can compare both curves as if they had the same (uniform) distribution over the $x$-axis. Again, the experts' capital share is larger in the hedging equilibrium and the difference is even more pronounced.

The economy spends less time in the crisis region, i.e. $\{\eta : \psi(\eta) < 1\}$, in the hedging equilibrium because (i) it is smaller (see Proposition 2.1) and (ii) the transition out of it is faster. There are two opposing forces that influence the comparison between the speed of transition. The larger leverage in the hedging equilibrium helps a faster transition out of the crisis region, however, the lower (expected) excess return of capital over bonds (due to lower total volatility, $|\sigma + \sigma^q|$) slows down the transition. The comparison between stationary distributions is presented in Figure 2.1(d).

An interesting feature is that both equilibria are observationally equivalent when experts are sufficiently capitalized to hold the entire capital stock (in both), i.e. $\eta > \eta^\psi$. It is only after a series of bad fundamental shocks that agents will coordinate in one or the other equilibrium.

**Welfare.** The more efficient allocation capital allocation should reflect into larger welfare for agents. The tractability of the model allows to (numerically) verify this intuition. The following lemma characterizes welfare for agents in the economy.

Figure 2.1: Amplification and Hedging Equilibria




**Lemma 2.3.** *The value function for an expert with individual net worth $n$ can be written as*

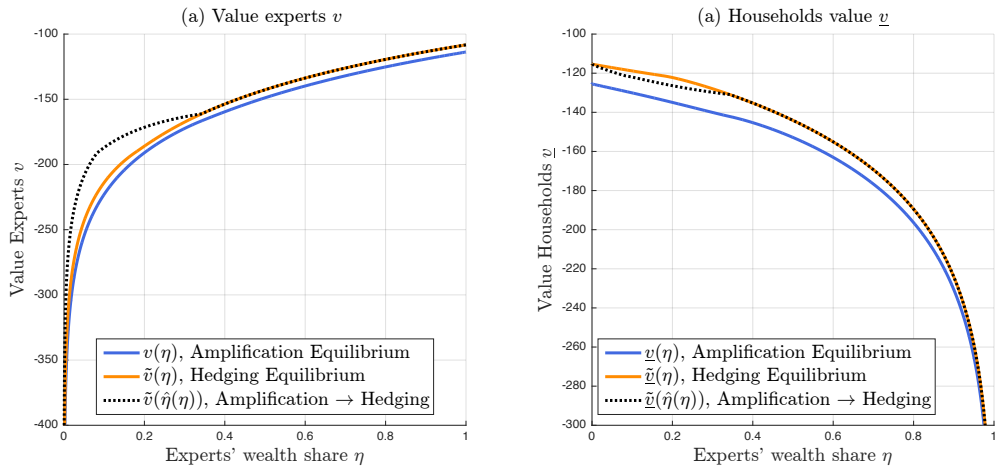
$$V(n;\eta) = v(\eta) + \frac{1}{\rho}\log(K) + \frac{1}{\rho}\log\left(\frac{n}{N}\right)$$

*where $v(\eta)$ satisfies functional equations (2.15) in Appendix A. In the case of households, the expression is analogous with a different first term $\underline{v}(\eta)$ and the corresponding definition of wealth share within sector, i.e. $n/\underline{N}$.*

Numerical exercises consistently show that welfare measures for experts, $v(\eta)$, and households, $\underline{v}(\eta)$, is larger in the hedging equilibrium than in the amplification one, i.e. $\tilde{v}(\eta) \geq v(\eta)$ and $\tilde{\underline{v}}(\eta) \geq \underline{v}(\eta)$ for $\eta \in [0,1]$. The interpretation is that conditional on the wealth distribution both agents prefer the hedging equilibrium. This is illustrated in Figure 2.2(a) and 2.2(b).

An alternative welfare comparison is to examine if agents would benefit from an unexpected switch from the amplification to the hedging equilibrium. In this exercise,

Figure 2.2: Welfare comparison




wealth distribution changes since capital price reacts to the unexpected change but value of bonds is fixed. Denote the ex-post wealth distribution as $\hat{\eta}$. Given that experts are levered, they experience larger capital gains than households so their ex-post wealth share, $\hat{\eta}(\eta)$, is larger, i.e. $\hat{\eta}(\eta) \geq \eta$. The third curve in Figure 2.2(a) shows the new value function level for experts, $\tilde{v}(\hat{\eta}(\eta))$. Recall that the initial situation is the amplification equilibrium, $v(\eta)$. Figure 2.2(b) shows the analogous exercise for households' value function. In summary, an unexpected switch to the hedging equilibrium is a Pareto improvement.[5]

### 2.3.2 *Volatility Paradox* revisited

This subsection discusses how volatility of capital price (growth) $|\sigma^q|$ and total volatility of capital returns $|\sigma + \sigma_t^q|$ vary across economies with different volatility of the fundamental shock $\sigma$.

Consider only the amplification equilibrium, then the exercise replicates the "volatility paradox" result by Brunnermeier and Sannikov (2014): the endogenous volatility of capital price $\sigma_t^q > 0$ increases as the volatility of the fundamental shock decreases. The reason behind is that lower risk induces experts to take larger leverage which increases the sensitivity of their capital demand to capital quality shocks and therefore magnify the spiral between asset prices and wealth distribution. Importantly, the increase in the endogenous component of capital returns $\sigma_t^q$ does not overcome the change in the exogenous component $\sigma$ so total volatility of capital returns decreases.

[5]Integrating over the stationary distribution of $\eta$ would deliver the expected welfare of an agent that draws his wealth from this distribution. I favor the exercises presented in the text over this one.

In case of the hedging equilibrium, the effect over volatility of capital price $|\sigma^q|$ is reversed. In the crisis region, a reduction of fundamental volatility $\sigma$ in fact increases the total volatility of capital returns $|\sigma + \sigma_t^q|$ because real shocks are hedging large capital price movements, i.e. $\sigma + \sigma_t^q < 0$. Therefore, experts choose a smaller leverage which dampens the spiral between asset prices and wealth distribution reducing the endogenous volatility of capital returns $|\sigma_t^q|$, i.e. $\sigma_t^q < 0$ increases. The latter reduces total volatility of returns. Again, the endogenous response of capital price volatility is not sufficiently strong to overcome the change in $\sigma$ and total volatility of capital returns increases.

Note that the increase in volatility of capital returns in the hedging equilibrium only applies for the crisis regime. In the safe regime, capital price is constant and therefore total volatility of returns is equivalent to fundamental volatility $\sigma$. In contrast, for the amplification equilibrium, the volatility of capital returns decreases with $\sigma$ in both regions. Figure 2.3 illustrates how the two equilibria change with the fundamental volatility of the economy.

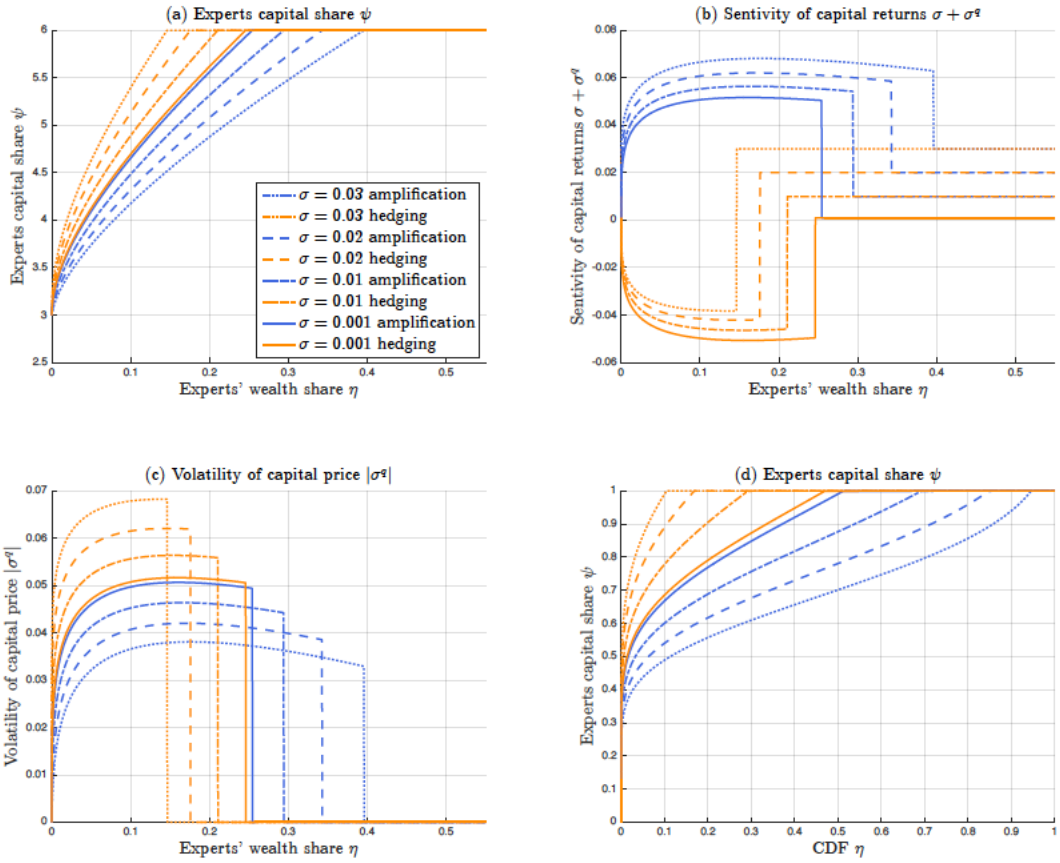
## 2.4 Policy: asset purchase program

In previous sections, we established that: 1) the two equilibrium are observationally equivalent while experts are well capitalized and 2) both agents prefer the hedging equilibrium. Then, it is only after a sequence of adverse fundamental shocks that agents coordinate in one of the two equilibria discussed, and a benevolent policy-maker or planner would induce them to coordinate in the hedging equilibrium if possible.

This section discusses a simple intervention that allows such coordination: the purchase of assets (financed by taxes) to avoid a premature drops in asset prices. This policy can be implemented at no cost even if the government is not able to use the assets as productively as the private sector because asset purchases are never realized on the equilibrium path, i.e., they remain as an off-equilibrium threat. It is key that the government does not intend to support asset prices when productive agents are severely undercapitalized since this would lead to effective asset purchases by the government which are largely inefficient.

**Government.** I extend the model to include a government. The government rises taxes (proportionally to wealth) to finance capital purchases and makes transfers to redistribute proceeds from capital sales. The government is less productive than private agents at managing capital, in fact, I assume that government cannot use

Figure 2.3: Volatility Paradox revisited




capital for production purposes. Capital held by government is also subject to the aggregate shock in the economy.

The policy I consider is the following. The government offers to buy or sell capital at its efficient price $\bar{q} = a/\rho$ whenever experts wealth share is above threshold $\eta^g$. The next proposition characterizes the thresholds that allow agents to coordinate in the hedging equilibrium.

**Proposition 2.2.** *Consider Markov equilibria with a unique aggregate state $\eta$. If the threshold for government asset purchases, $\eta^g$, is sufficiently low to prevent the initial capital price drop in the amplification equilibrium and sufficiently large to allow corresponding drop in the hedging equilibrium, i.e. $\eta^g \in (\tilde{\eta}^\psi, \eta^\psi)$, the economy coordinates in the latter. Moreover, the government does not make any capital purchases in equilibrium.*

**Mechanism.** The policy eliminates endogenous volatility, i.e. $\sigma^q(\eta) = 0$, when experts are sufficiently well capitalized: $\eta > \eta^g$. Then, consider the dynamics of the economy at crisis region threshold for the amplification equilibrium, $\eta^\psi$. In this equilibrium, a negative fundamental shock at this point triggers a drop in asset prices because experts reduce their capital demand as response to the larger volatility of capital return, $|\sigma + \sigma^q|$. The policy implemented supports the efficient capital price $\bar{q}$ ensuring $\sigma^q(\eta) = 0$ at this point. Given that capital returns only face fundamental uncertainty $\sigma$ now, experts do not contract their capital demand, and capital price remains at $\bar{q}$ *without* the government making any capital purchases.

The intuition outlined rests on the fact that the crisis region presents two subregions. The first corresponds to the situations in which experts are not willing to hold the entire capital stock even in the absence of endogenous volatility due to capital price, i.e. $\sigma^q = 0$. This subregion corresponds to the situation in which experts are severely undercapitalized: $\eta < \eta^*$ where

$$\eta^* \equiv \frac{\bar{q}}{(a - \underline{a})}\sigma^2 \ .$$

The second subregion corresponds to range of $\eta$ in which experts do not hold the entire capital stock because of the endogenous uncertainty about capital price $\sigma^q$. In particular, total volatility of capital returns is larger due to endogenous volatility, $|\sigma + \sigma^q| > \sigma$, so experts prefer to fire-sale some capital to households even though they would not do so if capital return was only exposed to fundamental volatility $\sigma$. This subregion corresponds to $\eta \in (\eta^*, \eta^\psi)$ in the case of the amplification equilibrium (similar with $\tilde{\eta}^\psi$ for the hedging equilibrium). It can be shown that $\eta^* < \tilde{\eta}^\psi < \eta^\psi$.

In the first subregion, the crisis (capital misallocation and endogenous volatility) is a fundamental phenomena given the financial friction. The crisis *propagates* to the second subregion where the crisis occurs because agents anticipate low asset prices if experts' net worth deteriorates further. In a sense, asset purchases seek to prevent this propagation.

So far, the argument seems to apply for any threshold $\eta^g > \eta^*$ since those are the situations in which the efficient capital price (and allocation) can be supported absent endogenous volatility. However, this is not the case. The reason is analogous to the one for the existence of the second crisis subregion: there has to be equilibrium dynamics for the crisis region that do not drive the capital price down in the range of bank capitalizations in which policy is applied. This translates to requiring equilib-

rium dynamics for $\eta \leq \eta^g$ such that $q(\eta^g) = \bar{q}$. This is where the hedging equilibrium plays a key role. As long as the policy implemented satisfies $\eta^g > \tilde{\eta}^\psi$, the hedging equilibrium dynamics satisfy this requirement.

**Introduction of asset purchases program.** The unexpected introduction of the asset purchases program would switch the economy from an amplification to the hedging equilibrium. This implies a sudden (weakly) increase in asset prices and a redistribution of wealth towards levered productive agents (debt value does not change). In particular, the new wealth share of experts $\tilde{\eta}$ satisfies

$$\hat{\eta} = \left( \frac{1 + \kappa x(\eta)}{1 + \kappa} \right) \eta$$

where $\kappa$ is the percentage increase in capital price at impact, i.e. $1 + \kappa = \tilde{q}(\hat{\eta})/q(\eta)$. The change in welfare for experts and households, i.e. $\tilde{v}(\hat{\eta}) - v(\eta)$ and $\underline{\tilde{v}}(\eta) - \underline{v}(\eta)$, respectively, depend on the initial wealth distribution. As mentioned before, Figure 2.2 illustrates that for our numerical example the implementation of the asset purchases program (equivalent to a switch towards the hedging equilibrium) is a Pareto improvement independently of the initial wealth distribution of the economy $\eta$.

## 2.5 Extrinsic uncertainty

The hedging equilibrium follow from the possibility self-fulfilling asset price changes in an economy with financial frictions. In particular, in the hedging equilibrium agents use the realization of the fundamental shock to coordinate changes in their demand for capital that translate into movements in its price which in turn justify the initial demand shift. In this section, I highlight that this coordination is possible even when the driving shock does not affect fundamentals, i.e., when it represents extrinsic uncertainty. I illustrate this by considering the model with $\sigma = 0$ and showing that there is a equilibrium in which economic outcomes fluctuate with extrinsic uncertainty. Moreover, I show that both the amplification and the hedging equilibrium converge to this limit at $\sigma \to 0$.

### 2.5.1 Model with non-fundamental shocks

Consider the economy where the Brownian shock does not affect the evolution of capital ($\sigma = 0$), i.e. it is just a sunspot and total capital is fixed $\bar{K}$. In this case, the

economy's fundamentals are deterministic. A *sunspot equilibrium* corresponds to an equilibrium in which allocations change with extrinsic uncertainty, and a *non-sunspot equilibrium* corresponds to the standard equilibrium in which extrinsic uncertainty does not influence allocations.

**Non-sunspot equilibrium**. Intuitively, the economy should admit an equilibrium in which uncertainty not related to fundamentals is disregarded and economic fluctuations are only driven by real shocks (e.g. to preferences or technology). In this economy, such an equilibrium would be deterministic since there are no real shocks. Moreover, in this deterministic set-up, the presence of a risk-free bond is enough to complete markets and obtain a first-best outcome. The following proposition confirms this intuition and characterizes the unique *non-sunspot* equilibrium.

**Proposition 2.3.** *There exists a unique non-sunspot equilibrium, i.e.* $\sigma^q(\eta) = 0$ *for all* $\eta \in [0,1]$*, which corresponds to the first-best outcome. All capital is held by experts, capital price and output are constant, i.e.* $q(\eta) = a/\rho$ *and* $y_t = a\bar{K}$*, and returns are perfectly shared among agents (proportionally to their net worths).*

**Sunspot equilibrium.** The following propositions shows that the economy also admits a sunspot equilibrium and characterizes it.

**Proposition 2.4.** *There exist a Markov sunspot equilibrium that satisfies the following*

- *Capital price,* $q(\eta)$*, is the unique solution to*

$$q' = \frac{q(a-\underline{a})}{\rho q - \underline{a} - \eta(a-\underline{a})}$$

  *with boundary conditions* $q(0) = \underline{a}/\rho$ *and* $q(\eta^\psi) = a/\rho$ *for some threshold* $\eta^\psi \in (0,1]$*. Moreover,* $q(\eta) = a/\rho$ *for* $\eta \in [\eta^\psi, 1]$*.*

- *Capital share owned by bankers,* $\psi(\eta)$*, satisfies*

$$\rho q(\eta) = \psi(\eta)(a-\underline{a}) + \underline{a}$$

- *Capital price sensitivity to the Brownian shock, $\sigma^q(\eta)$, satisfies*

$$[\sigma^q(\eta)]^2 \frac{(\psi(\eta) - \eta)}{\eta(1-\eta)} = \frac{a - \underline{a}}{q(\eta)}$$

*in the interval $\eta \in (0, \eta^\psi]$. For $\eta > \eta^\psi$, $\sigma^q(\eta) = 0$.*

The propositions shows that the economy admits a sunspot equilibrium in which capital is misallocated $\psi < 1$ and asset prices are volatile $|\sigma^q| > 0$ whenever experts are not well capitalized, i.e. $\eta < \eta^\psi$. In this equilibrium, economic fluctuations inherit the stochastic behavior of the Brownian motion. Note that this equilibrium is unique up to the sign of $\sigma^q(\eta)$ which is intuitive since the shock is a sunspot and agents can coordinate on increasing asset prices when Brownian motion moves up or down.

**Role of financial frictions.** The financial friction that allows for a sunspot equilibrium is the absence of a financial asset to trade exposure to the sunspot shocks. This opens the possibility for a bidirectional dependence between capital price and the wealth share of experts. An increase in capital price increases the wealth share of experts because they choose to hold more capital than households in order to exploit their productivity advantage. Also, an increase in the wealth share of experts reduces their effective risk aversion, increasing their capital demand, and ultimately their capital holdings, which translates into a more efficient allocation and a higher capital price. The following proposition formalizes this idea.

**Proposition 2.5.** *If agents were able to write short-term contracts conditional on sunspots $Z_t$, then there would be no sunspot equilibria.*

The argument for the proof relies on the fact that production and risk exposure decisions would be independent. There is no reason for experts not to hold the entire capital stock given their higher productivity since they would be able to achieve any desire risk exposure through financial markets. The result also follows immediately from the first-welfare theorem since markets would be complete.

**Convergence to sunspot equilibrium.** The sunspot equilibrium illustrated in this section corresponds to the limit of the amplification and the hedging equilibrium as $\sigma \to 0$, i.e. the two equilibriums converge to the same limit. The only difference in

of these limits is the sign of $\sigma^q$ which is irrelevant once the shock is a sunspot. There is no discontinuity at $\sigma = 0$. Instead, there is convergence to a sunspot equilibrium because agents can coordinate their believes about asset price movements even in non-fundamental shocks. The underlaying reason is that the economy is vulnerable to self-fulfilling asset price movements due to the financial constraints.

## 2.6 Conclusion

A key lesson about the role of financial frictions in macroeconomic fluctuations is that they amplify the effects of fundamental shocks via feedback loop: a negative fundamental shock hurts the net worth of productive agents because they are levered. As their net worth decreases, they scale back operations and fire sale their assets to less productive agents. This leads to a decline in asset prices that hurts productive agents' net worth again.

This paper dissects this logic in two parts: the fundamental shock and the spiral between asset prices and wealth distribution. Then, it highlights that the main role for the fundamental shock is to provide a coordination device to trigger the spiral and not its direct effect. I illustrate this in two ways. First, I show that it possible for agents to use a positive fundamental shock as a coordination device for asset price drops. This leads to an alternative equilibrium in which real shock and asset price movements hedge each other. Second, I illustrate that it is also possible to trigger the spiral by coordination on a non-fundamental shock.

The hedging equilibrium Pareto dominates the standard amplification one because it exhibits lower overall volatility of economic outcomes. The analysis suggests a policy intervention that helps the economy to coordinate in a hedging equilibrium: an asset purchase program that prevent premature drops in asset prices.

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# Appendix

## 2.A Analytical results

***Proof Lemma 2.1.*** The HJB for the problem of agent with productivity $a^i \in \{a, \underline{a}\}$ is

$$\rho V(n,\eta) = \max_{c,x\geq 0} \log(c) + \mathbb{E}\left[\frac{dV(n,\eta)}{dt}\right]$$

s.t.

$$\frac{dn}{n} = x dr^{k,i} + (1-x) r dt - \frac{c}{n} dt$$

*Solution.* Conjecture $V(n,\eta) = \rho^{-1}\log(n) + \xi(\eta)$ with $d\xi = \mu^{\xi}\xi dt + \sigma^{\xi}\xi dZ_t$. These dynamics will need to satisfy consistency conditions with the aggregate state's law of motion, detailed below. Given the conjecture and using Ito's lemma, the evolution of $V$ can be written as

$$\begin{aligned} dV = V_n\left(\mu^n n dt + \sigma^n n dZ_t\right) + V_{\xi}\left(\mu^{\xi}\xi dt + \sigma^{\xi}\xi dZ_t\right) + \frac{1}{2}V_{nn}\left(\sigma^n n\right)^2 dt ... \\ ... + \frac{1}{2}V_{\xi\xi}\left(\sigma^{\xi}\xi\right)^2 dt + V_{\xi n}\sigma^n\sigma^{\xi} n\xi dt \end{aligned}$$

Then, the HJB reduces to

$$\log(n) + \rho\xi(\eta) = \max_{c,x\geq 0} \log(c) + \frac{1}{\rho}\mu_n - \frac{1}{2\rho}\sigma_n^2 + \mu_{\xi}\xi$$

The first order conditions for the portfolio and consumption decisions are presented in the text. It remains to verify the conjecture.

Replacing these optimal decisions on the HJB equation,

$$\rho\xi = \log(\rho) + \frac{1}{\rho}\left[x\left(\mu^R - r\right) + r - \rho\right] - \frac{1}{2\rho}x^2\left(\sigma + \sigma^q\right)^2 + \mu_{\xi}\xi \tag{2.6}$$

where $x$ is given by equation (2.5) and $\underline{\mu}_{\xi}$ satisfies

$$\mu_{\xi}\xi = \xi_{\eta}\left(\mu^{\eta}\eta\right) + \frac{1}{2}\xi_{\eta\eta}\left(\sigma^{\eta}\eta\right)^{2} \ ,$$

which follows from Ito's lemma. Since equation (2.6) does not depend on individual state $n$, function $\xi(\eta)$ can be chosen to ensure it is always satisfied. This verifies the conjecture. I omit the technical discussion about the verification argument.

***Proof Lemma 2.2.*** Applying Ito's formula to $\eta\left(N, \underline{N}\right) = N/\left(N + \underline{N}\right)$ and using the law of motions of $N$ and $\underline{N}$, we have

$$\eta\sigma_{\eta} = \eta(1-\eta)(x - \underline{x})(\sigma + \sigma^{q}) \tag{2.7}$$

where $x = \psi/\eta$ and $\underline{x} = (1-\psi)/(1-\eta)$.

Similarly, applying Ito's formula to $q(\eta)$

$$q\sigma_{q} = q_{\eta}\sigma_{\eta}\eta \tag{2.8}$$

Equations (2.7) and (2.8) render

$$\sigma + \sigma^{q} = \frac{\sigma}{1 - \frac{q_{\eta}}{q}\eta(1-\eta)(x - \underline{x})} \ . \tag{2.9}$$

Market clearing for goods market can be written

$$\rho q = (a - \underline{a})\psi + \underline{a} \tag{2.10}$$

where the LHS is total consumption and the RHS is total production (all scaled by capital).

First, consider the case in which households hold some capital, $\psi < 1$. In this case, expression (2.5) holds with equality for experts and households. Subtracting them, we have

$$(x - \underline{x})\left(\sigma + \sigma^{q}\right)^{2} = \frac{a - \underline{a}}{q} \tag{2.11}$$

Conditions (2.9)-(2.11) imply the following ODE for $q(\eta)$

$$\frac{\psi(q) - \eta}{\eta(1-\eta)}\left(\frac{\sigma}{1 - \frac{q'}{q}\left(\psi(q) - \eta\right)}\right)^{2} = \frac{a - \underline{a}}{q} \tag{2.12}$$

where $\psi(q) \equiv [\rho q - \underline{a}]/(a - \underline{a})$. Reordering the last expression yields the two ODE showed in the lemma (depending on the root chosen for the square term), i.e.,

$$\begin{aligned} q' &= \frac{q}{\psi - \eta}\left[1 - \frac{\sigma}{\sigma + \sigma^q}\right] \\ &= \frac{q}{\psi - \eta}\left[1 \pm \sigma\sqrt{\frac{(\psi - \eta)}{\eta(1-\eta)}\frac{q}{(a - \underline{a})}}\right] \end{aligned} \tag{2.13}$$

The boundary condition for this first-order ODE follows from goods market clearing condition evaluated at $\eta = 0$, i.e.,

$$q(0) = \frac{\underline{a}}{\rho} .$$

This ODE describes the solution for $q(\eta)$ as long as respects the non-negativity constraint for capital holdings, i.e., $x, \underline{x} \geq 0$. Since $x > \underline{x}$, we only need to check the condition for households. The latter condition is equivalent to check $\psi \in (0, 1)$.

Second, consider the case in which households hold no capital, i.e., $\psi = 1$. In this case, goods market clearing condition implies $q(\eta) = \bar{q}$ which is defined by

$$\bar{q} = \frac{a}{\rho} .$$

Then, $q_\eta = 0$ and equation (2.9) implies $\sigma^q = 0$. This is a solution as long as households do not desire to hold any capital, i.e.,

$$x\sigma^2 < \frac{a - \underline{a}}{\bar{q}}.$$

We are looking for a continuous capital price function so we need boundary condition $q(\eta^\psi) = a/\rho$ for some $\eta^\psi \in (0, 1)$ to hold for ODE (2.13), i.e., at some point experts need to hold the entire capital stock $\psi = 1$. I rule out a boundary condition $\psi = 0$ for $\eta > 0$ because it is not compatible with equilibrium since experts have a productivity advantage.

*Sign of first derivative.* We know $\psi > \eta$ from optimal portfolio decisions. Then, for the negative root (which corresponds to the positive sign in the ODE), $\sigma + \sigma^q < 0$, it is straightforward that capital price is an increasing function, i.e., $q' > 0$. For positive root, $\sigma + \sigma^q > 0$, the capital price function is increasing, if and only if,

$$\sigma^q > 0$$

Recall that in this solution only ensures $\sigma^q > -\sigma$ so we cannot rule out $\sigma^q \in (-\sigma, 0)$.

***Proof Proposition 2.1.*** Given the existence of an amplification solution, there exist a solution for ODE (2.13) with negative sign such that $q(\eta^\psi) = a/\rho$ for some $\eta^\psi \in (0,1)$. Then, to ensure the existence of a different equilibrium associated with the solution of ODE (2.13) with positive sign, it is sufficient to show that $\tilde{q}(\eta) \geq q(\eta)$ with strict inequality for some $\eta$ whenever both satisfy the corresponding ODEs. I show this in two steps. First, I show that $\lim_{\eta\to 0} q'(\eta) < \lim_{\eta\to 0} \tilde{q}(\eta)$. Then, I show that $q'(\hat{q}, \eta) > \tilde{q}'(\hat{q}, \eta)$ for any $\hat{q}$.

The first step follows from directly taking limits to ODE. In the case of the amplification solution (negative sign in ODE)

$$\lim_{\eta\to 0} q'(\eta) = \frac{a - \underline{a}}{\rho}\left(\frac{a - \underline{a}}{\sigma^2 \underline{q}} + 1\right) > 0$$

and in the case of the hedging solution

$$\lim_{\eta\to 0} q'(\eta) = \infty$$

I rule out $\lim_{\eta\to 0} q'(\eta) = -\infty$ because it would imply $\psi < 0$.

The second step follows from the fact that $\psi > \eta$ for any equilibrium.

***Proof Lemma 2.3.*** *Households.* The value function of a households with net worth $n$ can be written as

$$\begin{aligned}
\underline{V}(n,\eta) &= \rho^{-1}\log(\underline{n}) + \underline{\xi}(\eta) \\
&= \rho^{-1}\log\left(\frac{n}{\underline{N}}(1-\eta)qK\right) + \underline{\xi}(\eta) \\
&= \rho^{-1}\left[\log\left(\frac{n}{\underline{N}}\right) + \log(K)\right] + \underbrace{\rho^{-1}\log\left((1-\eta)q\right) + \underline{\xi}(\eta)}_{\equiv \underline{v}(\eta)}
\end{aligned}$$

where the second from $\underline{N} = (1-\eta)qK$, and the third just illustrates how $\underline{v}(\eta)$ is defined in terms of equilibrium objects already derived. Using this definition to replace $\underline{\xi}(\eta)$ with $\underline{v}(\eta)$ and market clearing conditions, HJB becomes

$$\rho\underline{v}(\eta) = \log(\rho(1-\eta)q) - \frac{1}{2\rho}\sigma^2 + \underline{v}\underline{\mu}_v \tag{2.14}$$

where

$$\underline{\mu}_v\underline{v} = \underline{v}_\eta\left(\mu^\eta\eta\right) + \frac{1}{2}\underline{v}_{\eta\eta}\left(\sigma^\eta\eta\right)^2$$

*Experts.* Derivation is symmetric using $N = \eta q K$. In this case, $v(\eta) \equiv \rho^{-1} \log(\eta q) + \xi(\eta)$. Using this definition to replace $\xi(\eta)$ with $v(\eta)$ and market clearing conditions, HJB becomes

$$\rho v(\eta) = \log(\rho \eta q) - \frac{1}{2\rho}\sigma^2 + v\mu_v \tag{2.15}$$

where

$$\mu_v v = v_\eta \left(\mu^\eta \eta\right) + \frac{1}{2} v_{\eta\eta} \left(\sigma^\eta \eta\right)^2$$

The alternative procedure described for value function of households also applies for experts.

***Proof Proposition 2.2.*** The amplification solution is not longer an equilibrium outcome because for $\eta \in (\eta^g, \eta^\psi)$ there is no price uncertainty ($\sigma^q = 0$) so experts are willing to hold the entire capital stock. Condition $\eta^g > \tilde{\eta}^\psi$ ensures the hedging solution is still an equilibrium and that the government does not purchase any capital. I assume the economy starts at a situation where experts are well capitalized $\eta_0 > \eta^\psi$.

***Proof Proposition 2.3.*** Since non-fundamental uncertainty is ignored by agents, non-contingent bonds imply complete markets and the result follows from the first welfare theorem.

***Proof Proposition 2.4.*** This is just the special case of Lemma 2.2 where $\sigma = 0$.

***Proof Proposition 2.5.*** Markets are complete and the result follows from the first welfare theorem.

## 2.B Hedging in Kiyotaki Moore (1997)

One of the seminal contributions to the literature on financial frictions in macroeconomic models is Kiyotaki Moore (1997). This appendix highlights that there is an alternative solution for their impulse response functions following a positive productivity shock. This alternative solution follows the *hedging* logic stressed in this paper.

Kiyotaki and Moore (1997) study the propagation mechanism of an unanticipated temporary productivity shock that affects financially constrained agents, i.e the gatherers. The main insight is that productivity shocks are amplified and exhibit persistence due to financial constraints, a leverage constraint in the model. A temporary negative productivity shock today decreases the net worth of gatherers which, due to

the leverage constraint, impairs their capital (asset) demand today. The lower capital demand implies a lower capital price (and therefore an even larger decrease in capital demand today) and lower net worth tomorrow (and therefore lower capital demand tomorrow as well).

Figure 2.4: Impulse responses to a positive productivity shock (Kiyotaki Moore)

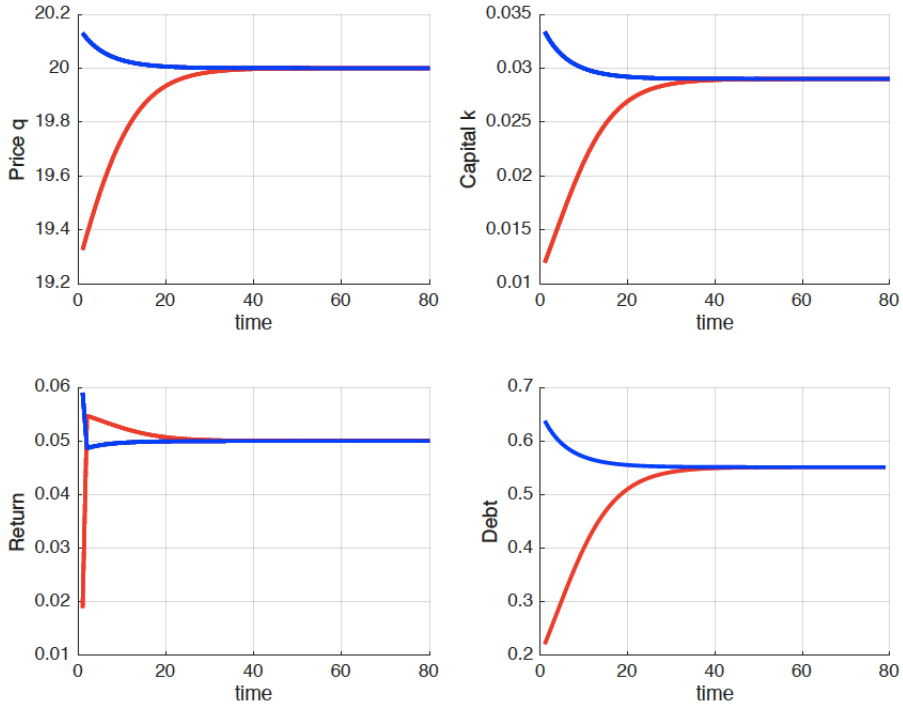



Figure. The red line is the impulse response with hedging property, i.e. with price movement in the opposite direction to the initial productivity impulse.

Using their baseline model, I illustrate that there is a second solution path to the initial productivity shock in which asset price movements hedge instead of amplify the initial impulse. In fact, the capital price movement dominates the initial change in productivity. Consider a temporary positive productivity shock. There is a solution path in which capital price decreases just after the shock and the drop is large enough to decrease net worth despite the original increase in productivity. Then, the amplification mechanism through a tightening of the leverage constraint is triggered which in turn justifies the initial capital increase. The original paper recognize the possibility of multiple solution paths on footnote 16.

**Functions and parameters.** Production function for gatherers

$$G(x) = Ax^{\alpha}$$

The parametrization is the following: $\beta = R^{-1} = 0.95$, $m = 1$, $a = 1$, $\bar{K} = 0.15$, $A = 0.8$, and $\alpha = 0.3$. I use $T = 120$.

**Impulse response.** The following figure shows the possible impulse responses to an unexpected temporary increase in productivity of 5% (just at period 0) when the economy starts at steady state. The solution with a positive impulse on capital price and capital quantity managed by farmers is the one where capital price change reinforces the initial shock (blue line). The solution path with a large drop in capital price and capital managed by farmers is the one where a positive productivity shock translates into a large price drop, large enough to generate a decrease in productive agents' net worth despite productivity gains.

## 2.C Extrinsic Uncertainty: Poisson shocks

This appendix illustrates that a sunspot equilibrium can arise with a different process for extrinsic uncertainty. In particular, I consider the version of the model with no real shocks and a Poisson process $J_t$ (with arrival intensity $p_t$) as sunspot. The following proposition characterizes a sunspot equilibrium in this case.

**Proposition 2.6.** *A (Markov) sunspot equilibrium exists if there are functions* $\{\kappa^q(\eta) \neq 0, \psi(\eta), q(\eta)\}$ *that satisfy the following conditions*

- *Optimal portfolio allocation*

$$p\left[\frac{\kappa^q(\eta)}{1-\kappa^q(\eta)x(\eta)} - \frac{\kappa^q(\eta)}{1-\kappa^q(\eta)x^h(\eta)}\right] \leq \frac{a-\underline{a}}{q(\eta)}$$

  *where* $x = \psi(\eta)/\eta$ *and* $x^h = (1-\psi(\eta))/(1-\eta)$. *This holds with equality, if and only if,* $\psi(\eta) < 1$.

- *Goods market clearing*

$$\rho q(\eta) = \psi(\eta)(a-\underline{a}) - \underline{a}$$

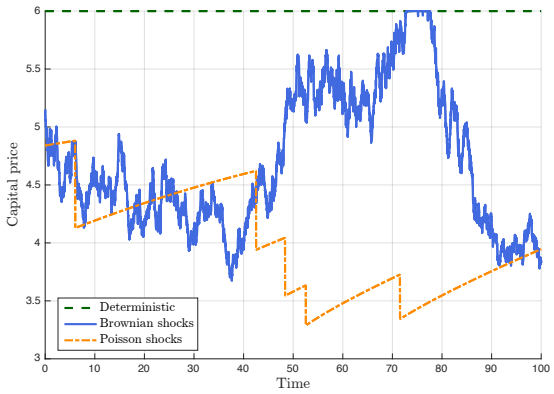
- *Consistency of capital price and wealth distribution changes*

$$\kappa^q(\eta) = 1 - \frac{q(\hat{\eta})}{q(\eta)}$$

  *where*

$$\hat{\eta} = \frac{1-x(\eta)\kappa^q(\eta)}{1-\kappa^q(\eta)}\eta$$

Figure 2.5: Sample paths for capital price




In this case, an analytical proof of the existence of the solution is not provided., instead, I provide a numerical example. Figure 2.5 summarizes the mentioned functions.

**Economic fluctuations.** The equilibria described above translate into drastically different dynamics for economic outcomes, e.g. asset prices and production. Figure 2.5 shows capital price paths for the three different equilibria: the deterministic, the one associated with a Brownian sunspot and the one associated with the Poisson sunspot. Recall that the fundamentals are symmetric and constant across these equilibria. This illustrates how extrinsic uncertainty and financial frictions can generate excess volatility.[6]

**Sunspot equilibria and multiplicity.** In the model presented, sunspots are modeled as particular stochastic processes for tractability, i.e. Brownian motion and Poisson process. Of course, extrinsic uncertainty that manifest as a different stochastic process can potentially generate other sunspot equilibrium that translates into different dynamics.[7]

[6]The simulations presented in Figure 2 come from a stationary environment which is achieved through the introduction of idiosyncratic Poisson shocks that runs experts into households and vice-versa.

[7]Moreover, even for the sunspot processes presented above (one of which has a unique sunspot solution), there can be other sunspot equilibrium if a recursive representation with only intrinsic states is not required.

# Chapter 3

# Safety Traps in a Global Economy

## 3.1 Introduction

The motivation for this paper originates from two stylized macroeconomic facts: First, the large demand for safe stores of value in the global economy compared to a relatively scarce supply. Second, the fact that a large fraction of safe assets in the global economy is supplied by the US, whereas most of the demand for safe assets comes from China, oil producing economies and Japan. These facts have been identified by the literature concerned with "global imbalances" yet this literature remains silent on if (and how) these phenomena impact aggregate economic activity. Recently, Caballero and Farhi (2015) – referred to CF (2015) below – have taken a step into this direction: They analyze how excess demand for safe assets can drag the economy into a liquidity trap in the market for safe assets ("safety trap"), causing a drop in output. However, their paper focuses exclusively on a closed economy environment. This is the gap our work aims to fill: To study the effects of safe asset shortages on output in a *global* economy.

CF(2015) study an OLG model with two kinds of agents: Neutrals (risk-neutral) and Knightians (risk-averse). Neutrals own Lucas trees that are subject to aggregate risk. Neutrals can (and will in equilibrium) securitize the safe part of the tree and sell it to Knightians. However, the securitization process is subject to a financial friction. Neutrals can only pledge a fraction $\rho$ of future payouts. The key market to clear is the market for safe assets. The equilibrating price for this market is the interest rate on safe assets. By introducing price stickiness and money, the safe interest rate is subject to a zero-lower bound. If there is excess demand for safe assets at a positive

interest rate, the interest rate drops, equilibrating supply and demand. However, if there is excess demand at an interest rate of zero, the adjusting margin switches from interest rates to output – since rates cannot drop any further. A drop in output brings about the equilibrating force needed to clear the market. That is, the economy ends up in a safety trap where output is depressed. A key policy implication is that the government can alleviate the safety trap (push up output) by supplying safe public debt backed by taxes.

We build an international version of the model in CF (2015) in order to study the *global* dimension of safety traps. Our model features two countries (or regions), labeled "Home" and "Foreign." It is helpful to think of Home representing the US and, broadly speaking, Foreign representing the rest of the world. We emphasize two dimensions of heterogeneity between Home and Foreign. First, we assume that Home's Neutrals can pledge a larger fraction of future payouts than Foreign's Neutrals. Second, we assume that there is a larger fraction of Knightians in Foreign compared to Home. The first dimension of heterogeneity proxies for a higher development of US financial markets compared to the rest of the world. The second dimension captures the large demand for safe savings vehicles by countries other than the US.

First, we analyze the effects of the highlighted heterogeneities between countries on "natural" safe interest rates (where "natural" refers to the safe rate that equilibrates supply and demand for safe assets without a drop in output). We find that the natural Home autarky safe interest rate lies above the natural Foreign autarky rate and when the economy integrates, the natural world safe interest rate lies in between those autarky rates. That is, opening up the economy depresses safe interest rates from Home's point of view but raises safe rates from Foreign's point of view. The intuition for this results stems from from the scarcity of safe assets. In Home, the scarcity of safe assets is less severe compared to Foreign, which results in a higher market-clearing safe interest rate. When the economy opens up, the scarcity of safe assets is essentially averaged between Home and Foreign, which implies a market-clearing world safe rate in between the autarky safe rates. This result is also mirrored in net foreign asset positions. Home builds up a positive NFA position in safe assets while Foreign has the corresponding negative position. Importantly, when safe assets are sufficiently scarce in Foreign, we find that the natural world safe interest rate is negative although the natural Home autarky rate is positive. When interest rates are constrained by the ZLB and cannot fall below zero, this implies the world economy suffers from a safety trap with depressed output. That is, Foreign's demand for safe assets pushes Home into a recession.

Second, we analyze the role of safe public debt in a global economy by allowing the government to issue safe claims backed by taxes on endowments of future agents. We impose two restrictions on the way taxation operates. Taxes are distortionary and can only be adjusted imperfectly over time. We consider these realistic assumptions capturing that lump-sum taxes are infeasible in practice and changes in tax codes are subject to a sluggish political process. In CF (2015), safe public debt has output expansionary effects without any cost. In contrast, the restrictions we impose on fiscal policy imply that besides the upside of expanding output, safe public debt has the downside of generating wasteful distortions. Under this cost-benefit trade-off for safe public debt, a number of important implications arise. We find that (i) a government may choose to not exhaust its fiscal capacity and allow the global economy to suffer from a safety trap, even though it has the ability to pull the economy out of it. Moreover, (ii) safe public debt issuance in any of the countries has expansionary effects on output in both countries, leading to an externality of public debt: As both countries gain from a public debt issuance but only one country bears the cost, there will be underprovision of safe public debt from a global perspective.

Third, we investigate if Home would find it optimal to close its capital account in order to avoid being pushed into a safety trap by Foreign. For this exercise, it is important to distinguish whether the government's object of interest is output or consumption. This is the case as in the open economy, net exports can drive a wedge between output and consumption. In the case of output, we require Home to maintain debt above a positive threshold. We find that closing down the capital account has the upside of pushing up output (by avoiding a safety trap) but the downside of paying higher debt servicing cost. When Home closes down its capital account, its safe interest rate is positive so that it foregoes a recessionary safety trap but also pays strictly positive debt servicing costs. In contrast, in the open economy, Home suffers some output losses due to the safety trap, yet avoids debt servicing costs altogether as the safe rate is at the zero lower bound. We illustrate that a higher debt threshold and higher severeness of tax distortions tilt the trade-off towards an open capital account. In addition, we show that when the debt threshold approaches zero, only the output expansionary motive remains and a closed capital account is always preferred.

When the government's object of interest is consumption, we do not require debt to be above a threshold. Thus, the debt servicing cost motive is absent. However, there is still a trade-off for the following reason. We find that the upside of closing the capital account is still present as the expansionary effect on output translates into an expansionary effect on consumption. Nonetheless, closing down the capital account

also means that Home foregoes potential net interest income on its asset position. If the world economy is such that Home earns positive net interest income, then it uses this income to permanently finance consumption above net output. By closing the capital account, Home gives up these additional resources.

Finally, we show that our results on the intermediate exhaustion of fiscal capacity and global underprovision of safe assets hold irrespectively of whether the government's object of interest is output or consumption.

**Related Literature.** As outlined above, our paper is most closely related to CF (2015) as we directly build on their framework and extend it to an international model. Importantly, in contrast to CF (2015), we model not only the benefit but also the cost side of safe public debt provision. This leads to a number of interesting implications in the international context, e.g. an externality of safe public debt provision in the global economy.

There is also a recent paper by Caballero et al. (2015) that uses a framework similar to CF (2015) and studies the diffusion of liquidity traps and corresponding policy options in a global economy. Their paper and our paper have been written concurrently. They focus mostly on liquidity traps in general, though also derive some implications for liquidity traps in the market for safe assets. Some of their results overlap with our findings. They also find that countries with severe asset shortages drag other countries into recessions, that public debt provision in any country expands output everywhere and that the distribution of output drops in a liquidity trap is governed by the relative degree of nominal rigidities. Their paper extensively studies the role of nominal exchange rates devaluations, a channel absent in our paper. In contrast to our work, they do not consider costs of safe public debt provision, which is why our results related to this cost side are not present in their work.

Apart from these two papers, our work is related to several strands of the literature. First, the motivation for our paper stems from the literature that identifies the shortage of safe assets in the global economy (Barclays (2012), Caballero (2006, 2010), Caballero and Krishnamurthy (2009)) and the literature rationalizing "global imbalances" by heterogeneities in financial markets (Bernanke (2005), Bernanke et al. (2011), Caballero et al. (2008), Mendoza et al. (2009), Maggiori (2012)).

Second, there is a recent literature that focuses on how the market for safe assets affects the macroeconomy. Barro and Mollerus (2014) study a model with heterogenous degrees of risk-aversion across agents and quantitatively match safe real interest rates, equity premia and crowding-out coefficients of public bonds. Benhima and

Massenot (2013) study a model where agents have decreasing relative risk-aversion and show the existence of multiple equilibria – one of these equilibria is "fear-driven" and features depressed output.

Third, there is a large literature on liquidity traps. See for example, Krugman (1998), Eggertsson and Woodford (2006), Eggertsson and Krugman (2012), Werning (2011) and references therein. A subbranch of the literature on liquidity traps focuses on this phenomenon in an open economy context. Svensson (2001, 2003) discusses policy options involving exchange rate devaluations to escape from a liquidity trap. Jeanne (2009) shows that a demand shock in one country can push the world economy into a liquidity trap and discusses how a coordinated monetary policy response can bring the economy back to first-best. Benigno and Romei (2014) analyze how global liquidity traps can arise due to sovereign deleveraging. Cook and Devereux (2013, 2014) study optimal coordinated monetary and fiscal policy responses in a global liquidity trap and evaluate whether in this context a single-currency area is superior/inferior to a regime with flexible exchange rates. Devereux and Yetman (2014) explore how capital controls can be used once monetary policy loses bite due to a worldwide liquidity trap. We are related to these papers as we study a global liquidity trap in a particular market: the market for safe assets. However, in contrast to the policy options examined in these papers, we explore the role of safe public debt as a tool for stabilization in a global economy.

Furthermore, we are related to the literature that studies the role of the government in providing additional stores of values, see Woodford (1990) and Holmstrom and Tirole (1998). In our model – as in CF (2015) – there is a role for the government to improve market outcomes as it has an advantage over the private sector in providing a particular store of value (safe assets) as it can levy taxes on agents' future income.

There is also a literature that studies the so-called "safe haven" and/or reserve currency status. See Eichengreen (1998, 2012) and Eichengreen and Flandreau (2009) for a historical discussion of this topic. More recently, He et al. (2015) identify a number of economic mechanisms that influence the determination of which country becomes the "safe haven" of the world economy. This is literature is important with respect to our work for the following reason. When there is only a single country in the world economy with the ability to provide safe assets, then our results on the externality and global underprovision of safe public debt are particularly relevant.

Finally, there is also a growing body of literature concerned with "secular stagnation," e.g. Kocherlakota (2014) and Eggertsson and Mehrotra (2014). As will become clear

later on, in our model – as in CF (2015) – safety traps can be permanent so that the economy remains in a recession for an arbitrarily long period of time. Whereas the work mentioned above considers closed economy environments, there is also a recent paper by Eggertsson et al. (2015) that analyzes secular stagnation in an international context. In contrast to our paper, they explore exchange rate devaluations as a policy tool. Similar to our work, they also think about whether economies gain from closing down their capital accounts.

The remainder of this paper is structured as follows. Section 2 lays out our baseline model and derives results on interest rates and net foreign asset positions. Section 3 extends our baseline model to allow for public provision of safe assets under imperfect fiscal policy and derives the results on intermediate exhaustion of fiscal capacity, public debt externality and considers the open vs. closed capital account trade-off. Finally, Section 4 concludes. Proofs and detailed derivations are included in the Appendix.

## 3.2 Baseline model

In this section, we lay out our baseline model. This is an international version of the model studied in CF (2015). First, we present a (real) endowment economy version to illustrate the key forces at play in a simple way, and then incorporate nominal rigidities and production to explore output consequences of these forces.

### 3.2.1 The real endowment economy

We study a single-good economy that consists of two countries (or regions), which we label "Home" and "Foreign." In the following, variables and parameters corresponding to Home will be denoted without superscript, while a *-superscript is used for Foreign. Time is continuous with infinite horizon, $t \in [0, \infty)$.

**Demographics.** Each country is populated by a unit mass of agents. This mass is constant over time. However, individual agents are born and die at rate $\theta$ ($\theta^*$), independent across agents. For analytical simplicity, we work with $\theta = \theta^*$.

**Aggregate Risk.** At each point in time, Home (Foreign) receives endowment $X_t$ ($X_t^*$) of the numeraire good. The world economy is subject to aggregate risk in the fol-

lowing sense. There are 2 Poisson-processes that arrive with intensities $(\lambda^+, \lambda^-)$, and affect endowment in both countries. Their stopping times are denoted by $(\sigma^+, \sigma^-)$. We define the stopping time of the process that hits first by $\sigma \equiv \min\{\sigma^+, \sigma^-\}$. Then, endowments for the Home country are given by

$$X_t = \begin{cases} X & t < \sigma \\ \mu^- X & t \geq \sigma, \sigma = \sigma^- \\ \mu^+ X & t \geq \sigma, \sigma = \sigma^+, \end{cases}$$

where $\mu^- < 1 < \mu^+$ (analogous process for Foreign endowment). Endowment is fixed at $X$ ($X^*$) before the realization of the shock and permanently jumps up/down after the shock has been realized.[1] We focus on the limiting case where $\lambda^+ \to 0, \lambda^- \to 0$.

**Lucas-trees and distribution of endowments.** There is one Lucas-tree in each country. A part $\delta X_t$ ($\delta^* X_t^*$) of endowment accrues as dividends to the Home (Foreign) Lucas tree. The remaining part $(1-\delta)X_t$ ($(1-\delta^*)X_t^*$) is equally distributed as endowment to newborn agents. We work with $\delta = \delta^*$.

**Agents.** There are two kinds of agents in each country: Knightians (infinitely risk-averse), and Neutrals (risk-neutral). A fraction $\alpha$ ($\alpha^*$) are Knightians, and their preferences are given by

$$U_t^K = \mathbb{1}\{t - dt \leq \sigma_\theta < t\} c_t + \mathbb{1}\{t \leq \sigma_\theta\} \min\{U_{t+dt}^K\}$$

The remaining fraction $(1-\alpha)$ $((1-\alpha^*))$ are Neutrals, and their preferences are given by

$$U_t^N = \mathbb{1}\{t - dt \leq \sigma_\theta < t\} c_t + \mathbb{1}\{t \leq \sigma_\theta\} \mathbb{E}_t\{U_{t+dt}^N\}.$$

Importantly, we will assume that agents consume exclusively at time of their death. That is, agents save the endowment they receive at birth during their life time and consume all of their wealth when they get hit by the death-shock.

[1] Note that we have one symmetric shock for the world as a whole. We could generalize this to imperfectly correlated shocks across countries and also to shocks of heterogenous depth. It will become obvious that what matters in the end for the determination of equilibrium is simply the worst possibly outcome.

**Assets.** We assume that the Lucas-trees are in the hands of Neutrals. Neutrals can securitize the Lucas-trees' dividends into assets that they potentially sell to Knightians.

**Financial friction.** We assume the securitization process is subject to a financial friction. Neutrals can only pledge a fraction $\rho$ ($\rho^*$) of the Lucas-tree's returns to other agents.

We emphasize two key dimensions of heterogeneity between Home and Foreign. First of all, the pledgeability of returns is higher in Home than in Foreign, i.e. $\rho > \rho^*$. Second of all, there is a larger fraction of Knightians (risk-averse agents) in Foreign compared to Home, i.e. $\alpha^* > \alpha$. The first source of heterogeneity is supposed to proxy for a higher development of US financial markets compared to the rest of the world. The second source of heterogeneity is supposed to capture the large demand for safe savings vehicles by emerging markets in the global economy.[2]

Our focus is the equilibrium before the Poisson event is realized (ex ante equilibrium), in particular the steady state. However, as will become clear later, even when the Poisson intensities of the shocks go to zero, the equilibrium after the bad Poisson event will feed back into the ex ante equilibrium because of Knightians' preferences. In particular, it will determine the value of safe assets. In the following subsections, we first describe the equilibrium dynamics before the Poisson event, and then move to the (deterministic solution) after the bad Poisson event to retrieve the value of safe assets (and complete the ex ante equilibrium characterization). Finally, we characterize the steady state solution of the ex ante equilibrium.

### Equilibrium before the Poisson event

**Agents' decisions.** Agents receive their endowment at birth, exchange it for assets, and reinvest their wealth until death when they liquidate all of their assets and consume. The only choice is their portfolio composition. Due to their preferences, Knightians will only hold safe assets, i.e. assets whose value is invariant to the state of the world. Neutrals hold trees and will find optimal to issue safe assets to Knightians. These insights are enough to derive the aggregate dynamics.

[2] Apart from these two dimensions, we also allow for heterogeneity in output/size of the countries, i.e. $X \neq X^*$.

**Assets returns.** The value of the Home (Foreign) tree is denote by $V_t$ ($V_t^*$). This value is divided into two parts: the total value of safe assets that can be issued against the tree's dividend payments, $V_t^S$, and the residual, $V_t^R = V_t - V_t^S$, which we label the risky asset. The value of safe assets is the maximum value the owner of the tree can promise to deliver next instant independently of the aggregate state. Let $r_t^S$ be the interest rate on the safe asset, and $r_t$ the one on the risky asset. Moreover, $\delta_t^S$ ($\delta_t^{S*}$) denotes the safe part's dividend which is to be determined endogenously.[3] The pricing equation for the safe part of Home's tree is given by

$$r_t^S V_t^S = \underbrace{\delta_t^S X_t}_{\text{dividend}} + \underbrace{\dot{V}_t^S}_{\text{capital gain}} \tag{3.1}$$

The intuition for this equation is standard. The return on the safe part of the tree is made up of the dividend payment plus the capital gain. Similarly, the remaining asset pricing equations are

$$r_t V_t^R = (\delta - \delta_t^S) X_t + \dot{V}_t^R \tag{3.2}$$

$$r_t^S V_t^{S*} = \delta_t^{S*} X_t^* + \dot{V}_t^{S*} \tag{3.3}$$

$$r_t V_t^{R*} = (\delta - \delta_t^{S*}) X_t^* + \dot{V}_t^{R*} \tag{3.4}$$

**Wealth dynamics.** Let $W_t^K$ and $W_t^N$ denote (aggregate) Home Knightian's and Neutral's wealth, respectively. Then,

$$\dot{W}_t^K = \underbrace{-\theta W_t^K}_{\text{Consumption}} + \underbrace{\alpha(1-\delta) X_t}_{\text{Newborns' wealth}} + \underbrace{r_t^S W_t^K}_{\text{return}} \tag{3.5}$$

This can be understood intuitively when decomposed into 3 components. The change in wealth is given by the sum of a consumption term (which equals a fraction $\theta$ of wealth as a representative subsample dies and consumes), an endowment term capturing wealth of newborn agents and finally a return term which accounts for interest payments surviving agents earn on their wealth (Knightians only hold safe

[3] Once we have solved for the endogenous values of $\delta_t^S$ and $\delta_t^{S*}$, we have to check that $\delta^S \leq \delta\rho$ and $\delta^{S*} \leq \delta^*\rho^*$. These inequalities can be verified from our solution to the equilibrium system.

assets). The remaining wealth evolution equations follow from the same intuition.

$$\dot{W}_t^N = -\theta W_t^N + (1-\alpha)(1-\delta)X_t + r_t W_t^N \tag{3.6}$$
$$\dot{W}_t^{K*} = -\theta W_t^{K*} + \alpha^*(1-\delta^*)X_t^* + r_t^S W_t^{K*} \tag{3.7}$$
$$\dot{W}_t^{N*} = -\theta W_t^{N*} + (1-\alpha^*)(1-\delta^*)X_t^* + r_t W_t^{N*} \tag{3.8}$$

For Neutral's wealth return, note that Neutrals always hold the risky asset, and when they also hold some of the safe asset, they must be incentivized to do so by $r_t = r_t^S$.

**Market Clearing.** Our economy has two markets that clear in equilibrium: The market for the consumption good and the asset market. The goods market clearing condition is given by

$$\underbrace{\theta(W_t^K + W_t^{K*} + W_t^N + W_t^{N*})}_{\text{demand for goods}} = \underbrace{X_t + X_t^*}_{\text{supply of goods}} \tag{3.9}$$

This says that all consumption (given by a fraction $\theta$ of wealth) equals all output. The asset market clearing conditions is

$$\underbrace{W_t^K + W_t^{K*} + W_t^N + W_t^{N*}}_{\text{aggregate asset demand}} = \underbrace{V_t^S + V_t^R + V_t^{S*} + V_t^{R*}}_{\text{aggregate asset supply}}, \tag{3.10}$$

which simply is that the value of all wealth (asset demand) equals the value of asset supply.

**Regimes.** We have to account for the fact that there are two possible regimes in this economy. First, when the supply of safe assets ($V^S + V^{S*}$) is larger than Knightians demand ($W^K + W^{K*}$). In this case, Neutrals are the marginal holders of the safe asset, which requires $r_t = r_t^S$. In the other regime, Knightian's demand exceeds safe asset supply at $r_t = r_t^S$, which causes a drop in the safe rate to equilibrate the market such that $r_t^S < r_t$. In this regime, Knightians are the marginal holder of the safe asset. We can summarize the two regimes by including the following complementary slackness conditions in our system.

$$0 = (V_t^S + V_t^{S*} - W_t^K - W_t^{K*}) \cdot (r_t - r_t^s) \tag{3.11a}$$
$$V_t^S + V_t^{S*} \geq W_t^K + W_t^{K*} \tag{3.11b}$$
$$r_t \geq r_t^s \tag{3.11c}$$

**Value of Safe Assets.** To complete the equilibrium characterization, we need to determine the value of the safe part of each tree. The Home (Foreign) safe asset value is equal to the lowest value the tree could have next instant times the fraction of this value that can be pledged, $\rho$ ($\rho^*$). In this model, the mentioned lower bound is constant and corresponds to the value of the tree if the bad Poisson event hits.[4] Moreover, this value will be independent of the ex ante system we have just described.

Let $V^-$ and $V^{-*}$ denote the values of the Lucas-trees *after* the realization of the bad shock. Then, the value of safe assets that are available *before* the shock is given by

$$V_t^S = V^S = \rho V^- \tag{3.12}$$
$$V_t^{S*} = V^{S*} = \rho^* V^{-*} \tag{3.13}$$

We solve for the values $V^-$ and $V^{-*}$ from the system that is in place *after* the bad shock. This solution is presented in the next subsection.

Note that, even though we focus on the situation where the probability of an aggregate shock is vanishing, i.e. $\lambda^+, \lambda^- \to 0$, equations (3.12)-(3.13) illustrate how the bad aggregate state feeds back into the ex ante equilibrium system. Intuitively, this happens due to Knightians' min-preferences over future outcomes.

### Equilibrium after the bad Poisson shock

Once the aggregate shock has hit, the economy is deterministic. Therefore, there are no longer risky and safe parts of the tree, so we solve for the total value of the trees.[5] Goods and asset market clearing conditions are

$$\theta \left( W_t^{K-} + W_t^{N-} + W_t^{K-*} + W_t^{N-*} \right) = \mu^-(X + X^*)$$
$$W_t^{K-} + W_t^{N-} + W_t^{K-*} + W_t^{N-*} = V_t^- + V_t^{-*}$$

Together, they imply a constant value for the aggregate value of both trees,

$$V_t^- + V_t^{-*} = \frac{\mu^-(X + X^*)}{\theta}$$

[4] This result is intuitive and will be checked for the steady state solution we analyze.

[5] Moreover, we assume $\bar{\rho} > \bar{\alpha}$, so the pedgeability constraint does not influence this ex post equilibrium

Return equations are

$$r_t^- V_t^- = \delta \mu^- X + \dot{V}_t^-$$
$$r_t^- V_t^{-*} = \delta^* \mu^- X^* + \dot{V}_t^{-*}$$

The last three equations imply

$$r_t^- = r^- = \delta\theta \tag{3.14}$$
$$V_t^- = V^- = \frac{\mu^- X}{\theta} \tag{3.15}$$
$$V_t^{-*} = V^{-*} = \frac{\mu^- X^*}{\theta} \tag{3.16}$$

This pins down the values for $V^S$ and $V^{S*}$ and completes the the ex ante equilibrium characterization.

### Steady state before the Poisson event

Our focus is the **steady state** equilibrium before the shock. Recall that $\lambda^+, \lambda^- \to 0$, so this ex ante steady state equilibrium will effectively be in place forever. We begin by providing a formal definition of equilibrium. Let $(\#')$ refer to the static version of equation $(\#)$, i.e. the version where changes are set to zero and time subscripts are dropped.

**Definition 3.1.** [Ex Ante Equilibrium for Endowment Economy] The steady state equilibrium before the Poisson event is a wealth distribution over agents' types $\{W^K, W^N, W^{K*}, W^{N*}\}$, asset values $\{V^S, V^R, V^{S*}, V^{R*}\}$, returns $\{r, r^S\}$, and share of dividends accrued to safe assets $\{\delta^S, \delta^{S*}\}$ such that

- Asset pricing equations hold: $(3.1')$-$(3.4')$.
- Wealth distribution over agents' types is constant: $(3.5')$-$(3.8')$.
- Markets clear: $(3.9')$-$(3.10')$.
- One of the 2 possible regimes holds: $(3.11')$.
- Safe assets' values are constant across aggregate states and respect the financial constraint: $(3.12')$-$(3.13')$.

where $V^-$ and $V^{-*}$ are given by (3.15) and (3.16), respectively.

Note that the equilibrium system is exactly identified as we solve for 12 variables and have 12 independent equations (one of the market clearing equations in redundant by Walras' law).

**Solution.** First, note that safe asset supply is determined by the equilibrium after the Poisson event, and is independent of the effective regime in the ex ante equilibrium. In particular,

$$V^S = \rho\mu^- \frac{X}{\theta} \tag{3.17}$$

$$V^{S*} = \rho^*\mu^- \frac{X^*}{\theta} \tag{3.18}$$

However, wealth distributions, asset values, and interest rates will depend on the effective regime before the shock. As will be clear later, the regime selection will depend on whether the safe asset supply is enough to absorb all of Knightians' wealth. The regime where it is enough is labeled unconstrained regime, while the other is labeled constrained regime. We present both solutions and focus on the latter.

In order to characterize the solution with convenient expressions, we define for any parameter $\varsigma$ the weighted average of this parameter across countries as

$$\bar{\varsigma} \equiv \omega\varsigma + (1-\omega)\varsigma^*,$$

where $\omega \equiv \frac{X}{X+X^*}$ is home country's share of (potential) output .

**Case 1: The unconstrained regime.** In this case, Neutrals are the marginal holder of safe assets, so $r = r^S$. In this regime, we have[6]

$$r = r^S = \delta\theta$$

$$W^K = \alpha\frac{X}{\theta}$$

$$W^{K*} = \alpha^*\frac{X^*}{\theta}$$

[6]The full solution is provided in Appendix 3.A.1.

This solution is valid only if aggregate Knightian's wealth is less than total safe assets value. We find that

$$W^K + W^{K*} \leq V^S + V^{S*} \quad \Leftrightarrow \quad \bar{\rho}\mu^- \geq \bar{\alpha}$$

We assume that the latter condition does *not* hold, and therefore, the global economy is in the constrained regime.

**Assumption 1.** [Safe Asset Shortage Condition] The economy is in the constrained regime where all safe assets are hold by Knightians, i.e.

$$\bar{\rho}\mu^- < \bar{\alpha}.$$

This condition is intuitive. On the one hand, both the average securitization capacity ($\bar{\rho}$) and the depth of the bad shock ($\mu^-$) govern safe asset supply. On the other hand, the average fraction of Knightians in the economy ($\bar{\alpha}$) governs safe asset demand. If the demand factors overweigh the supply factors, safe assets are scarce and the economy ends up in the constrained regime.

**Case 2: The constrained regime** From now on, we focus on the analysis of the constrained regime.[7] It is important to note that in this regime a spread between risky and safe rates opens up.

$$r^S = \delta\theta - (1-\delta)\theta\left(\frac{\bar{\alpha} - \bar{\rho}\mu^-}{\bar{\rho}\mu^-}\right)$$
$$r = \delta\theta + (1-\delta)\theta\left(\frac{\bar{\alpha} - \bar{\rho}\mu^-}{1 - \bar{\rho}\mu^-}\right)$$
$$r - r^S = (1-\delta)\theta\frac{\bar{\alpha} - \bar{\rho}\mu^-}{(1-\bar{\rho}\mu^-)\bar{\rho}\mu^-} > 0$$

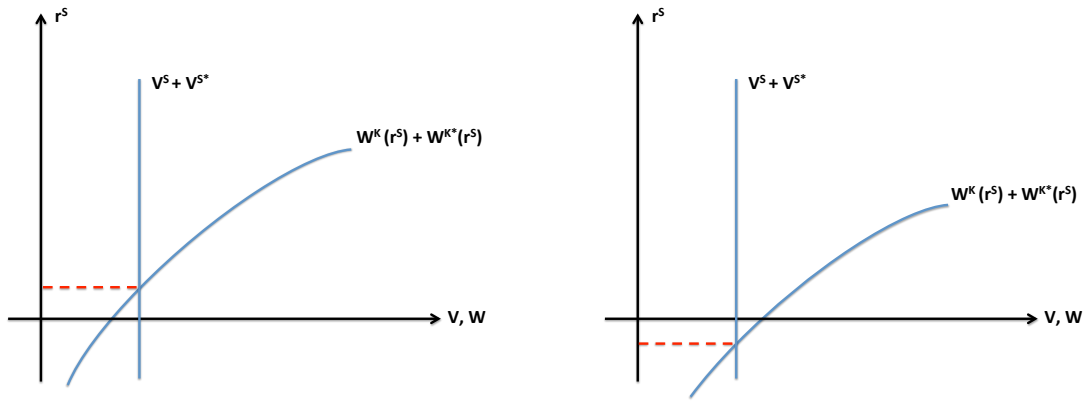
The key element of our analysis is the market for safe assets. As we have seen, the supply of safe assets is fixed by the ex post system,

$$V^S + V^{S*} = \bar{\rho}\mu^-\frac{(X + X^*)}{\theta}$$

[7] The full solution is provided in Appendix 3.A.2.

Figure 3.1: Global safe asset market equilibrium

(a) Positive equilibrium safe real interest rate (b) Negative equilibrium safe real interest rate




However, we can write the demand for safe assets as an increasing function of $r^S$.

$$W^K(r^S) + W^{K*}(r^S) = \frac{(1-\delta)\bar{\alpha}\,(X + X^*)}{\theta - r^S}$$

Figure 3.1 illustrates the market for safe assets graphically.

The safe real interest rate is the price that equilibrates the market for safe assets. If there is an excess demand for safe assets, the safe real rate drops, bringing about lower demand for safe assets such that supply and demand are equalized. Importantly, depending on the parameterization, the equilibrium safe real rate can be either positive or negative. We find that

$$r^S \leq 0 \quad \Leftrightarrow \quad \bar{\rho}\mu^- \leq (1-\delta)\bar{\alpha}. \tag{3.19}$$

This is intuitive. If the safe asset shortage is particularly severe, a negative safe real rate is needed to equilibrate supply and demand. So far, this does not create any issues. Since the model is entirely in real terms and agents have no alternative stores of value available, they will use safe assets to transfer wealth across time even though earning negative returns. However, once we introduce nominal rigidities and money into the model, this changes drastically. Then, negative market-clearing safe real rates become a plague for the economy.

### 3.2.2 Nominal Rigidities and Endogenous Output Determination

In this section, we introduce nominal rigidities and money into the economy. In combination, these two factors lead to a zero-lower bound on interest rates and endogenous output determination.

We now assume that there are 2 kinds of goods: input and consumption goods. Endowment is still given by $X_t$ ($X_t^*$) at every point in time, though this is in terms of input goods. Agents can transform input goods into consumption goods via a 1-1 production technology. Prices for the output good are denoted in nominal terms and are entirely rigid.[8] We assume that the price for the output good in terms of the home currency is $P = 1$ and in terms of the foreign currency is $P^*$. By the law-of-one price, the exchange rate is given by $E = 1/P^*$ and is constant over time. We assume that at the given prices, agents service demand until they run out of resources. Hence, output is demand determined. We define "capacity utilizations" $\xi_t, \xi_t^* \in [0, 1]$ such that output in terms of the consumption good is given by $\xi_t X_t$ and $\xi_t^* X_t^*$.

We introduce money into the economy via a cash-in-advance constraint. When the economy has money, agents have another store-of-value vehicle at hand. If the nominal interest rate is positive (note that $r = i$ due to the price rigidity), agents hold money exclusively for transaction services. However, if rates were to fall below zero, agents would substitute away from other savings vehicles and towards money, using money for both transaction services and as a store of value. Thus, we can rule out equilibria with negative interest rates and obtain a zero-lower-bound: $i_t = r_t \geq 0$. We consider a cashless limit such that we can keep the constraint on interest rates and do not have to carry around any explicit terms referring to money holdings by agent.[9]

The ex ante steady state equilibrium system of the economy with a zero-lower bound and endogenous output is given by the ex ante system from above accounting for two differences. First, endowment in terms of the output good is now given by $\xi_t X$ ($\xi_t^* X^*$). Second, there is a zero lower bound constraint. This can be written as

$$(r_t^S > 0 \ \wedge \ \xi_t = \xi_t^* = 1) \ \text{ or } \ (r_t^S = 0 \ \wedge \ \min\{\xi_t, \xi_t^*\} < 1) \tag{3.20a}$$

$$\xi_t, \xi_t^* \in [0, 1] \tag{3.20b}$$

[8] This is an extreme form of rigidity that enables us to state our insights most clearly. We relax this assumption later when we consider Philipps-curves.

[9] See CF(2015) for a closed economy equilibrium system that explicitly accounts for money holdings.

This is the key friction in our model. As in CF (2015), if there is excess demand for safe assets at a safe real rate of zero, then the equilibrating force switches from a drop in safe rates (which becomes impossible at the ZLB) to a drop in output, captured by capacity utilizations falling below unity.

Next, we formally characterize the ex ante equilibrium in this production economy. Let (#″) refer to the static version of equation (#) where endowment $X$ ($X^*$) is replaced by output $\xi X$ ($\xi^* X^*$). Also, note that the solution after the Poisson event is the same as in the endowment model since the interest rate is strictly positive ex post. The full system and its solution are included in Appendix 3.A.3.

**Definition 3.2.** [Ex Ante Equilibrium for Production Economy] A steady state equilibrium before the Poisson event is a wealth distribution over agents' types $\{W^K, W^N, W^{K*}, W^{N*}\}$, asset values $\{V^S, V^R, V^{S*}, V^{R*}\}$, returns $\{r, r^S\}$, share of dividends accrued to safe assets $\{\delta^S, \delta^{S*}\}$, and utilization capacities $\{\xi, \xi^*\}$ such that

- Asset pricing equations hold: (3.1″)-(3.4″).
- Wealth distribution over agents' types is constant: (3.5″)-(3.8″).
- Markets clear: (3.9″)-(3.10″).
- One of the 2 possible regimes holds: (3.11″).
- Safe assets' values are constant across aggregate states and respect the financial constraint: (3.12″)-(3.13″).
- The ZLB condition holds: (3.20″).

where $V^-$ and $V^{-*}$ are given by (3.15) and (3.16), respectively.

There is a crucial difference to the analysis in the closed economy. In the closed economy, the drop in output that brings about an equilibrium safe real interest rate of zero determines $\xi$. In the open economy, when safe interest rates are against the ZLB, a drop in world output will equilibrate world supply and world demand for safe assets. While we can determine the magnitude of the drop in world output, its distribution across countries is indeterminate.[10] This is illustrated in Figure 3.2. We can determine the amount by which the safe asset demand curve has to shift,

[10]Note that in Definition 3.2 the system has 14 variables and 13 independent equations at the ZLB.

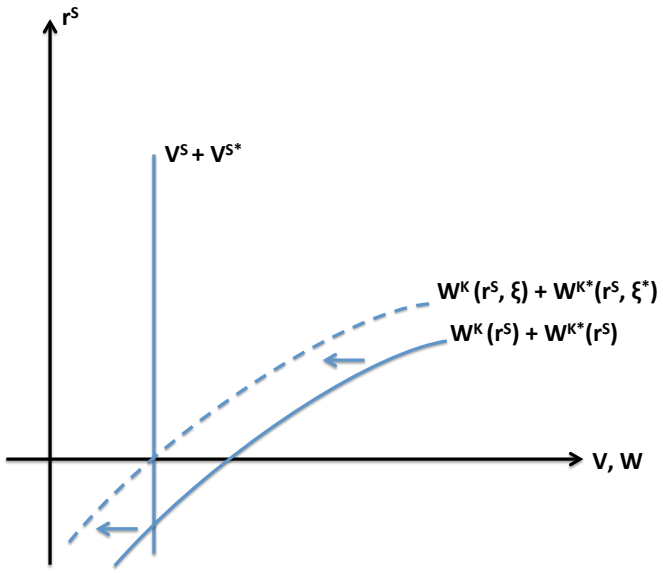



Figure 3.2: Output drop due to excess demand for safe assets

though we cannot pin down how much of that shift comes through a drop in Home vs. Foreign output. That is, we cannot uniquely determine the equilibrium values of $\xi$ and $\xi^*$ at the ZLB. In particular, we have that

$$\omega\alpha\xi + (1-\omega)\alpha^*\xi^* = \frac{\bar{\rho}\mu^-}{(1-\delta)} \tag{3.21}$$

This indeterminacy depends very strongly on the assumptions we have made about price rigidities. Once we move to a model that relaxes the extreme assumption of completely rigid prices, the indeterminacy goes away. In Appendix 3.A.4, we study an extension of the model where movements in prices are governed by Philipps-curves (motivated by sticky wages) which allows us to exactly pin down the distribution of output drops across countries. In such environment, the relative degree of nominal rigidity degree is the key factor: the economy which has more rigid wages bears a larger share of the drop in world output. For the time being, we assume that output drops are equally distributed across countries (which would amount to an equal degree of nominal rigidity) and solve (3.21) as

$$\xi = \xi^* = \frac{\bar{\rho}\mu^-}{\bar{\alpha}(1-\delta)}. \tag{3.22}$$

### 3.2.3 Real Interest Rates and Net Foreign Asset Positions

In order to study our model's implications for world safe interest rates and net foreign asset positions, it is useful to define the following objects. Let $r_{aut}^{S,n}$ and $r_{aut}^{S,n*}$ be the full-capacity autarky safe real interest rates in Home and Foreign. Also, let $r^{S,n}$ be the full-capacity safe real rate of the global economy. These "natural" interest rates are potentially negative.

Recall that we emphasize two dimensions of heterogeneity between countries: Higher fraction of Knightians in Foreign compared to Home and higher securitization capacity in Home compared to Foreign, i.e. $\alpha < \alpha^*$ and $\rho > \rho^*$. We have the following

**Proposition 3.1** (Depression of Safe Interest Rates)**.** *It holds that*

$$r_{aut}^{S,n*} < r^{S,n} < r_{aut}^{S,n} \quad \Leftrightarrow \quad \frac{\bar{\alpha}}{\bar{\rho}} > \frac{\alpha}{\rho}.$$

Under the parameterization we assume, opening up the world economy depresses safe rates from Home's point of view and increases safe rates from Foreign's point of view. Intuitively, the equilibrium safe real interest rate is determined by the severeness of the safe asset shortage. If the financially integrated economy has a more severe scarcity of safe assets compared to the Home autarky economy, the world safe real rate will be lower than the Home autarky rate (and vice versa for Foreign). This relationship of interest rates will be mirrored in net foreign asset positions. Denote Home's net foreign asset position in safe assets by $NFA^S$.

**Proposition 3.2** (Safe Net Foreign Asset Positions)**.** *We have that*

$$NFA^S > 0 \quad \Leftrightarrow \quad r^{S,n} < r_{aut}^{S,n}.$$

A further illustration of these results is given by Figure 3.3. When the Home autarky safe rate is above the Foreign autarky safe rate, then the safe rate in a global economy will be between those two autarky rates. At this interest rate, there is an excess supply for safe assets in Home and an excess demand for safe assets in Foreign. Accordingly, Home has a positive NFA in safe assets and Foreign has a negative NFA in safe assets.

Importantly, when the safe asset shortage in Foreign is sufficiently severe, the economy can end up in the following situation.

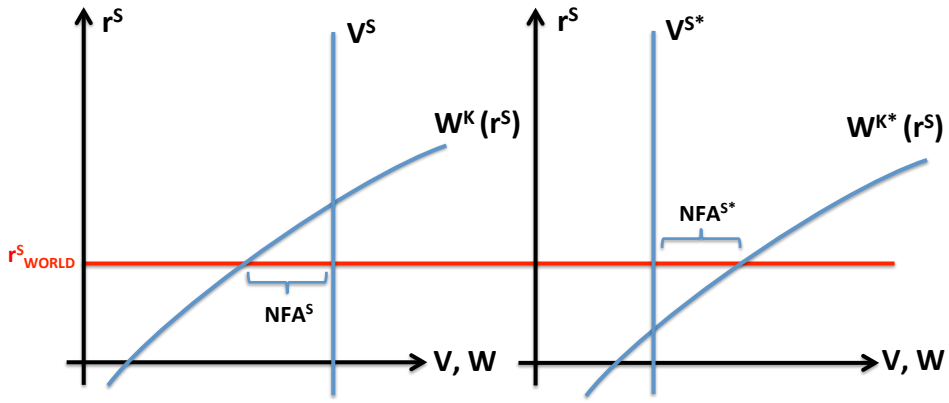



Figure 3.3: Safe NFA positions in a global economy

**Proposition 3.3** (Global Safety Traps)**.** *If* $\bar{\rho}\mu^- < (1-\delta)\bar{\alpha}$ *and* $\rho\mu^- > (1-\delta)\alpha$, *then*

$$r^{S,n} < 0 < r^{S,n}_{aut}.$$

If the safe asset shortage is mild in Home but sufficiently severe in Foreign, then the world economy features a negative natural safe rate even though the Home autarky natural safe rate is above zero. Of course, in our economy with nominal rigidities and the ZLB, negative rates are not attainable. Hence, in this scenario, while the Home autarky economy is at full capacity with a safe rate above zero, the world economy is against the ZLB and experiences a recession. In short, Home gets pushed into a safety trap by Foreign's demand for safe assets.

## 3.3 Public Debt in Global Safety Traps

In this section, we analyze the role of safe public debt in global safety traps. In particular, we allow a government to issue (safe) claims, backed up by taxes on endowments of future agents. We impose two plausible restrictions on the way

taxation operates. Namely, taxes are distortionary and the tax rate can only be adjusted imperfectly over time.

We analyze the situation where the government cares about (net) output, and present the following set of results. First, restricting attention to a global safety trap environment, we arrive at two main insights. (i) *Intermediate Exhaustion of Fiscal Capacity:* a government may optimally choose not to exhaust its fiscal capacity, and may allow the global economy to remain in a safety trap even when it has the ability to eliminate the safe asset shortage. (ii) *Public Debt Externality and Underprovision:* public debt issuance in any country has expansionary effects in both countries. Hence, when only the issuing country bears the costs, the debt level that maximizes domestic (net) output is always below the optimal debt level from a global perspective.

Second, we study an environment where countries are required to maintain a minimum debt level and explore whether a country that could avoid the safety trap by closing its capital account would find it optimal to do so. We show that the following trade-off arises. Compared to a financially integrated economy, closing down one's capital account has the upside of avoiding a recessionary safety trap though implies the downside of higher debt servicing cost. In accordance with this trade-off, we illustrate numerically that larger tax distortions and higher minimum requirements on the debt level tilt the government's decision towards maintaining an open capital account. Moreover, we show analytically that when the minimum debt level becomes arbitrarily small, a closed capital account is always preferred.

Then, we move to an environment where government's focus is consumption as opposed to (net) output. In this setup, we do not require a minimum debt level so that the debt servicing cost motive is absent. We illustrate an extra force at play when considering to close the capital account: net interest payments. In particular, positive net interest payments allow the country to consume beyond its production permanently. Therefore, this translates into an additional cost of closing the capital account. Finally, we show that intermediate exhaustion of fiscal capacity and public debt underprovision are also present in the environment where the government cares about consumption.

### 3.3.1 Model with costly debt provision

**Public debt and taxation restrictions.** We extend the baseline model by introducing public debt backed by taxes. In the following, only the home country can issue

public debt as a safe asset. This asymmetry aims to capture the role of the U.S. as "safe haven" and simplifies the exposition, but results in this subsection do not rely on it. The government taxes the neutral newborns' production,[11] $(1-\alpha)(1-\delta)\xi_t X_t$. We study steady states of the model where the tax rate, $\tau_t$ , takes a constant value for each aggregate state. In particular, let $\tau^-$ ($\tau^+$) be the tax rate after the negative (positive) realization of the Poisson event, and $\tau$, the tax rate before it. The government issues a fixed amount of bonds at time zero,[12] whose value we denote by $D$ (constant over time since it is a safe asset).[13] Now, we introduce restrictions on the way taxation operates that will be key for our results.

**Assumption 2.** [Distortionary Taxes] A fraction $\psi_t$ of tax collection is wasted.

**Assumption 3.** [Imperfect Adjustment]

$$\tau^- \leq \tau(1+\phi) \tag{3.23}$$

Assumption 2 captures the standard idea that there is a cost in raising taxes, and it can be thought of as a shortcut for unmodeled price distortions or administrative costs. We also assume that these distortions are only in place before the Poisson shock to simplify exposition, though this is not necessary for our results to hold. Assumption 3 captures the fact that the government cannot perfectly adjust the tax rate according to the aggregate state of the economy. In particular, the constraint asserts that the tax rate cannot increase more than $\phi$ percent after the bad Poisson event.[14] We consider this a plausible assumption that can be rationalized in several ways. For example, a large increase of taxes may not be politically feasible during recessions. Moreover, it might be hard for the government to exactly fine-tune the adjustment of the tax rate contingent on aggregate shocks, e.g. due to limited observability.

Also, we assume that $\tau^+$ is set to maintain debt value after the good Poisson event, and drop this tax rate from the analysis since there is no feedback from the situation

[11] Knightians are not taxed to avoid taxation to have expansionary effects through wealth reduction of agents that demand safe assets. This allows for a clearer exposition of results, but it is not necessary for our results.

[12] The government rebates lump sum the proceedings from the issuance. To whom these proceedings are rebated is not relevant because we study the steady state where there are no new issuances.

[13] Debt is "short term" or "instantaneous", in the sense that dividend flows, and not the asset value, adjust with market returns.

[14] We could impose the same constraint for taxes after the good Poisson event realization, but this would not influence the ex ante equilibrium.

after the good shock to the ex ante equilibrium. This will be equivalent to allow the government to reduce taxes after the good shock.

Before the Poisson event, effective tax collection can be greater than the interest paid on debt, in which case surpluses ($S_t$) are rebated to taxed agents. After the (bad) Poisson event, the government runs a balance budget, i.e. $S_t = 0$ for $t \geq \sigma$. Then, the flow of funds equation for the government can be written as

$$S_t = \tau_t(1-\psi_t)\nu\xi_t X_t - r_t^S D \tag{3.24a}$$

$$S_t \geq 0 \tag{3.24b}$$

where, for notational convenience, we denote the neutral newborn's share of output as $\nu \equiv (1-\alpha)(1-\delta)$. Accordingly, wealth dynamics for Home Neutrals are now given by

$$\dot{W}_t^N = (r_t^S - \theta)W_t^N + (1-\tau_t)\nu\xi_t X_t + S_t \tag{3.25}$$

Distortions are a fixed share of the tax collection before the Poisson event, $\psi_t = \psi$ for $t < \sigma$, and zero afterwards, $\psi_t = 0$ for $t \geq \sigma$. The latter is not necessary for results, but allows a simple characterization of solutions. It is helpful to think of the government as choosing tax rates $\{\tau, \tau^-\}$ subject to (3.23), letting the debt value and fiscal surpluses (before the Poisson event) be endogenous objects. However, as will become clear below, there will be a one-to-one map between the tax rate set after the bad Poisson event, $\tau^-$, and the value of public debt, $D$, which allows the government to target the latter.

**Equilibrium.** Pricing equations (3.1″)-(3.4″), wealth dynamics for Home Knightians (3.5″), and wealth dynamics for the foreign country (3.7″)-(3.8″) are the same as in the model with production. This is also true the ZLB condition (3.20) and private safe asset equations (3.12)-(3.13). However, the lowest possible values of private assets $(V^-, V^{-*})$ in (3.12)-(3.13) change due to introduction of public debt as we describe in detail below. The regime equation (which was given by (3.11″) beforehand) is now given by

$$0 = (V_t^S + V_t^{S*} + D - W_t^K - W_t^{K*}) \cdot (r_t - r_t^s) \tag{3.26a}$$

$$V_t^S + V_t^{S*} + D \geq W_t^K + W_t^{K*} \tag{3.26b}$$

$$r_t \geq r_t^s \tag{3.26c}$$

The goods market clearing condition is altered to incorporate the waste due to distortionary taxation

$$\theta(W_t^K + W_t^{K*} + W_t^N + W_t^{N*}) = (1 - \psi_t \nu)\xi_t X_t + \xi_t^* X_t^*. \tag{3.27}$$

The asset market clearing accounts for the additional assets supplied by the government, i.e.

$$V_t^S + V_t^{S*} + D + V_t^R + V_t^{R*} = W_t^K + W_t^{K*} + W_t^N + W_t^{N*}. \tag{3.28}$$

where $V_t^S$ is the home country's supply of *private* safe assets. To sum up, the equilibrium is characterized by the set of equation {(3.1″)-(3.5″), (3.7″)-(3.8″), (3.12)-(3.13), (3.24)-(3.28)} taking as given values $(D, V^-, V^{-*})$ derived below and policy instruments $\{\tau, \tau^-\}$ which must satisfy (3.23). As in the previous section, the equilibrium system has an indeterminacy with respect to the distribution of output drops at the ZLB. As above, we will resolve this for the time being by assuming symmetric losses in output.

**Solution after bad Poisson event.** Following the same steps as in subsection 3.2.1, we can derive the lowest possible value of private safe assets $(V^-, V^{-*})$, and the value of public debt, $D$. These value are given by[15]

$$D = \left(\frac{\omega\nu\tau^-}{\omega\nu\tau^- + \delta}\right)\frac{\mu^-(X + X^*)}{\theta} \tag{3.29}$$

$$V^- = \left(\frac{\omega\delta}{\omega\nu\tau^- + \delta}\right)\frac{\mu^-(X + X^*)}{\theta} \tag{3.30}$$

$$V^{-*} = \left(\frac{(1-\omega)\delta}{\omega\nu\tau^- + \delta}\right)\frac{\mu^-(X + X^*)}{\theta} \tag{3.31}$$

The values of private safe assets for the ex ante system are given by $V^S = \rho V^-$, and $V^{S*} = \rho^* V^{-*}$. Hence, the total safe asset supply can be written as

$$D + V^S + V^{S*} = \left(\frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta}\right)\frac{\mu^-(X + X^*)}{\theta} \tag{3.32}$$

[15]Recall that $\omega$ is the share of potential output of the Home country.

Importantly, as is clear from (3.29), the value of debt is a strictly increasing function of government's tax rate set after the bad Poisson event.[16]

**Government securitization advantage and crowding out of private safe assets.** There are two types of market incompletenesses in this model: the non-pledgeability of future newborn's production, and the non-pledgeability of the $(1-\rho)$ fraction of dividends. In our framework, the government alleviates the second market incompleteness by allowing to securitize a part of future newborns' output. However, the relevant measure is *the fraction of these newly pledgeable output flows that is accrued to safe assets in the bad aggregate state* (after the bad Poisson event). Suppose the government uses a fraction $\varphi$ of its taxes revenues to create safe assets (and the rest to create risky assets). Then, the total value of safe assets is given by

$$D + V^S + V^{S*} = \left( \frac{\varphi \omega \nu \tau^- + \bar{\rho}\delta}{\omega \nu \tau^- + \delta} \right) \frac{\mu^-(X + X^*)}{\theta} \tag{3.33}$$

and we can think of our model as the case where $\varphi = 1$. In general, the government is able to increase the supply of safe assets if and only if the fraction of output flows it securitizes into safe assets (as opposed to risky ones) is larger than private sector's pledgeability, i.e. $\varphi > \bar{\rho}$.[17] The latter requirement ensures that the increase in public *safe* assets is larger than the crowding out of private *safe* assets. Even though taxation does not affect the tree dividends of the private asset, there is crowding out of private safe assets through an interest rate channel. After the bad Poisson event,[18] the presence of government policy (public debt backed by taxes) increases dividend payments of total assets (because there are now dividends on debt). At a fixed interest rate, this implies larger value of total assets (or total wealth). Since consumption is a fixed fraction of wealth, this would lead to larger consumption, however total production is fixed after the Poisson event. Therefore, interest rate must adjust. In particular, the interest rate increases to keep total asset (wealth) value constant and equilibrate the goods market. Note that there is a complete crowding out between debt, $D$, and private assets value after the bad shock, $(V^- + V^{-*})$. Since only a $\bar{\rho}$

[16] Also, the smaller the fraction of global output accrued to taxed agents, $\omega\nu \equiv \left(\frac{X}{X+X^*}\right)(1-\alpha)(1-\delta)$, the larger the tax rate increase needed to achieve a given value of debt. This implies that larger economies have an advantage in providing safe assets.

[17] Technically, this follows from the observation that $\frac{d}{d\tau^-}\left[D + VS + VS*\right] > 0 \Leftrightarrow \varphi > \bar{\rho}$.

[18] Recall that after the Poisson event the economy is deterministic so all assets have the same rate of return

fraction of latter become safe assets in the ex ante equilibrium, this fraction represents the crowding out of public debt over safe assets.

The observations discussed provide useful insights. First, as in CF (2015), it is only taxation capacity after the bad Poisson shock that can potentially increase the safe asset supply ex ante. Second, it is not enough to pledge a larger fraction of future output (after the bad shock), this newly pledgeable output flows need to be accrued (in a fraction larger than $\bar{\rho}$) to a safe asset. If a country does not have enough fiscal capacity to ensure the value of debt after the bad Poisson event and effectively issues risky claims (think of $\varphi = 0$), then it ends up decreasing the supply of safe assets (due to crowding out). Third, financial integration allows for a decrease (increase) in the crowding out of private safe assets for the more (less) financially developed country, i.e. the country with larger (smaller) $\rho$.

**Solution before Poisson event.** We begin by providing a formal definition of the steady state equilibrium.

**Definition 3.3.** [Ex Ante Equilibrium for Production Economy with Costly Debt Provision] Given government policy instruments $\{\tau, \tau^-\}$ that satisfy (3.23), a steady state equilibrium before the Poisson event is a wealth distribution over agents' types $\{W^K, W^N, W^{K*}, W^{N*}\}$, asset values $\{V^S, V^R, V^{S*}, V^{R*}, D\}$, returns $\{r, r^S\}$, share of dividends accrued to safe assets $\{\delta^S, \delta^{S*}\}$, utilization capacities $\{\xi, \xi^*\}$, and government surplus $\{S\}$ such that

- Asset pricing equations hold: (3.1″)-(3.4″).
- Wealth distribution over agents' types is constant: (3.5″), (3.25), (3.7″)-(3.8″)
- Markets clear: (3.27), (3.28).
- One of the 2 possible regimes holds: (3.26).
- Safe assets' values are constant across aggregate states and respect the financial constraint: (3.12″)-(3.13″).
- The ZLB condition holds: (3.20).
- Government satisfies its budget flow constraint: (3.24)

where $D$, $V^-$, and $V^{-*}$ are given by (3.29), (3.30), and (3.31), respectively. Recall that (#″) refers to the static version with endogenous production of equation (#).

At the ZLB, the described system has 15 equations and 14 independent variables. As before, we keep our assumption of symmetric utilization capacities, $\xi = \xi^*$. The solution for the natural safe interest rate, i.e. the interest rate at full capacity ($\xi = 1$), is given by

$$r^{S,n}(\tau^-) = \delta\theta - (1-\delta)\theta\left[\frac{\bar{\alpha}}{\mu}\left(\frac{\omega\nu\tau^- + \delta}{\omega\nu\tau^- + \bar{\rho}\delta}\right) - 1\right] \tag{3.34}$$

which is increasing in $\tau^-$ as it expands the supply of total safe assets. If the shortage of safe assets is large enough, i.e. $r^{S,n} < 0$, then the global economy is in a safety trap[19] characterized by $r^S = 0$ and

$$\begin{aligned} \xi(\tau^-) &= \frac{\mu^-}{(1-\delta)\bar{\alpha}}\left[\frac{\theta(V^S + V^{S*} + D)}{X + X^*}\right] \\ &= \frac{\mu^-}{(1-\delta)\bar{\alpha}}\left[\frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta}\right] \end{aligned} \tag{3.35}$$

The latter expression shows that global utilization capacity is proportional to the global supply of safe assets. Therefore, in a global safety trap, the effect of public debt on global safe asset supply (discussed in detail above) translates into effects on global output (utilization capacity).

The remainder of this section considers the government's choice of tax rates $\{\tau, \tau^-\}$. First, we assume the government focuses on output net of the waste associated with distortionary taxation (Subsections 3.3.2 and 3.3.3). Then, we let the government's focus change to consumption (Subsection 3.3.4).

### 3.3.2 Intermediate Exhaustion of Fiscal Capacity and Public Debt Underprovision

We restrict our attention to global safety traps, and illustrate the trade-offs the government faces. The analysis is restricted to (ex ante) steady state comparisons, i.e. long run effects. In particular, we assume that the government maximizes the (ex

[19] See Appendix 3.A.5 for the full steady state solution.

ante) home country's net output. Formally, it solves

$$\begin{aligned} \max_{\{\tau,\tau^-\}} & [1-\tau\psi\nu]\xi(\tau^-)X \\ \text{s.t. } & \tau^- \leq (1+\phi)\tau \\ & \tau,\tau^- \in [0,1] \end{aligned} \tag{3.36}$$

Denote $(\tau_D^-, \tau_D)$ as the solution to this problem.

**Intermediate Exhaustion of Fiscal Capacity.** Larger $\tau^-$ increases the supply of safe assets expanding global utilization capacity. Larger $\tau$ generates more distortionary taxation, decreasing net output. Thus, if the government could perfectly decouple tax rates, it will choose $\tau^- = 1$,[20] and $\tau = 0$. However, this would violate the assumption on imperfect adjustment (3.23). Therefore, the latter will hold with equality. In other words, distortionary taxation and imperfect tax rate adjustments introduce an effective cost of providing safe assets, absent in CF (2015), generating a meaningful trade-off. Increasing the supply of safe assets alleviates the global safety trap, though in order to do this the government needs to increase not only taxes after the bad Poisson event (which is costless), but also before it. The latter generates tax distortions in the (ex ante) steady state, introducing an output cost of debt provision.

This trade-off implies that, when distortion considerations are large enough, the government chooses not to exhaust its fiscal capacity. In this model, the latter can be understood as $\tau^- < 1$. However, large enough distortions can prevent government from exhausting its taxation capacity independently of its limits, e.g. when the tax rate is limited by an upper bound $\bar{\tau}$ strictly lower than 1. The following proposition provides a characterization of the lower threshold on tax distortions needed for intermediate exhaustion of fiscal capacity.

**Proposition 3.4** (Intermediate Exhaustion of Fiscal Capacity)**.** *The domestically optimal (ex post) tax rate $\tau_D^-$ is smaller than 1, if and only if,*

$$\underline{\psi} \equiv \frac{\omega\delta(1-\bar{\rho})(1+\phi)}{(\nu\omega)^2 + (1+\bar{\rho})\nu\omega\delta + \delta^2\bar{\rho}} < \psi \tag{3.37}$$

The lower bound $\underline{\psi}$ decreases with average private pledgeability ($\bar{\rho}$), and increases with allowed tax rate adjustment ($\phi$). First, larger private safe assets reduce the

[20] If some $\hat{\tau}^- < 1$ is enough to escape from the safety trap, then the government would be indifferent to set any $\tau^- \in [\hat{\tau}^-, 1]$

benefits from public debt generating less incentives to exhaust fiscal capacity, so even low tax distortions are enough to prevent the government to issue all the public debt it could. Second, when the government is able to adjust tax rates more flexibly, public debt issuance is less costly, so it requires larger distortions to justify not exhausting public debt capacity. The described intuition is present in a closed economy (set $\omega = 1$), but there are some additional insights from the open economy case. Most importantly, it is less likely for the home country to exhaust fiscal capacity when the rest of the world is less financially developed, i.e. $\rho^* < \rho$.

From now onwards, we restrict our analysis to an environment where tax distortions are high enough to imply intermediate exhaustion of fiscal capacity.

**Public Debt Externality and Underprovision.** The are two key components that deliver an externality of public debt provision. First, in a global safety trap, public (safe) debt issuances from a government generate output expansions in every country. Second, the cost associated with public debt (distortionary taxation) is borne only by the issuing country. The first point is a feature of financial integration, once in the safety trap, a global output drop is needed to equilibrate the safe asset market, however our model is silent about the distribution of the output drops across countries. We have assumed symmetric output drops, yet we only need that output drops are shared to some degree, i.e. $\max\{\xi, \xi^*\} < 1$, for the debt externality to appear. The second point is driven by the fact that the government needs to collect distortionary taxation to meet its debt service.

The formal result follows from comparing the solution to (3.36) to the solution of

$$\begin{aligned} \max_{\{\tau,\tau^-,\}} & [1 - \tau\psi\nu]\xi(\tau^-)X + \xi^*(\tau^-)X^* \\ \text{s.t. } & \tau^- \leq (1+\phi)\tau \\ & \tau, \tau^- \in [0,1] \end{aligned} \tag{3.38}$$

Note that the first (second) term in the objective function is the home (foreign) country's net output. Therefore, as long as $\xi^*(\tau^-)$, is an increasing function, i.e. as long as the foreign country is bearing some of the output drop due to the safety trap, there is a benefit from debt issuance not internalized from the Home's perspective. Let $\tau_G^-$ be the solution to (3.38) and $D(.)$ be defined by (3.29), then we can state the result as follows.

**Proposition 3.5** (Public Debt Underprovision)**.** *The optimal debt provision from a global perspective $D(\tau_G^-)$ is strictly larger than the optimal debt provision $D(\tau_D^-)$ from a domestic perspective, i.e. $D(\tau_G^-) > D(\tau_D^-)$.*

It is insightful to note what kind of market failure generates this externality. It is the fixed debt price due to the ZLB that plays the key role. For example, if the foreign government were able to offer to buy home country's debt at lower interest rate (understanding the expansionary effect it would have on its economy), it would find it optimal to do so. The market would act in a similar way if public debt's price was not fixed.

### 3.3.3 Open vs. closed capital account

In this subsection, we focus on the situation where the financially integrated (global) economy experiences a safety trap, but the home country can escape from it by closing its capital account, i.e.

$$r^{S,n} < 0 < r_{aut}^{S,n}$$

We restrict attention to an environment where each country is required to maintain a certain level of public debt, and explore the cost and benefits for the home country of closing its capital account. The analysis compares the financial autarky outcome against the financial integration outcome, both under the (domestically) optimal public debt policy.

The minimum debt requirement assumption is motivated by factors that are not explicitly included in our model. For example, the government might have issued debt in the past to finance infrastructure investments or the creation of public institutions. Moreover, one can broadly think of the lower bound on debt as a proxy for required ongoing government expenditures. We formally state the assumption regarding the minimum debt level before beginning our analysis.

**Assumption 4.** [Minimum Debt Level]

$$D \geq \underline{D} > 0 \tag{3.39}$$

In the following, we characterize the solutions for the home economy with an open and a closed financial account focusing on the solutions for debt levels and tax rates

since we will compare the (net) output of each solution in steady state. In particular, (net) output will be given by $(1-\psi\tau\nu)\xi X$.

**Open capital account.** Given our focus on a global safety trap situation, the problem the government faces is the one described by (3.36) with the additional restriction (3.39). Recall that (3.29) is an equilibrium condition that always holds, and $\tau_D^-$ denotes the solution to the problem when there is no minimum debt requirement. Then, the optimal (ex post) tax rate in the open economy $\tau_{open}^-$ can be characterized as[21]

$$\tau_{open}^- = \begin{cases} \tau_D^- & \text{if } D(\tau_D^-) > \underline{D} \\ D^{-1}(\underline{D}) & \text{if } D(\tau_D^-) \leq \underline{D}, \end{cases} \tag{3.40}$$

where $D^{-1}(\underline{D})$ is the tax rate corresponding to $\underline{D}$. Moreover, the (ex ante) rate is given by $\tau_{open} = (1+\phi)^{-1}\tau_{open}^-$, and net output, $(1-\psi\tau_{open}\nu)\xi(\tau_{open}^-)X$, is decreasing in $\underline{D}$ when the debt requirement is binding.

The intuition behind the latter characterization is simple. If the debt level requirement is small, it will not be binding since it is optimal to choose a debt level above the minimum requirement due to its expansionary effects on output. If this requirement is large, then it is optimal to set tax rates to match this requirement, which implies that at the margin the additional debt has a negative effect on net output because the larger taxation needed outweighs the expansionary effect of additional safe assets.

**Closed capital account.** We assume that under financial autarky the home country does not experience a safety trap even with no debt provision, i.e. $\rho\mu^- > \alpha(1-\delta)$. Then, output is always at potential, i.e. $\xi = 1$, and there are no output-expansionary benefits from providing public debt, yet there is still a cost due to distortionary taxation. The optimal debt provision is to maintain it as low as possible, i.e. $D = \underline{D}$. Solving the autarky ($X^* = 0$) equilibrium after the Poisson event, we recover a relation between debt value and ex post tax rate, $\tau_{closed}^-$.

$$D_{closed}(\tau^-) = \left(\frac{\nu\tau^-}{\nu\tau^- + \delta}\right)\frac{\mu^- X}{\theta}$$

As before, this is a strictly increasing function of $\tau^-$. The minimum debt level requirement imposes two constraints on the (ex ante) tax rate $\tau_{closed}$. First, the

[21] The proof is provided in Appendix 3.A.6

(ex ante) tax rate needs to ensure that the government effective tax collection is at least enough to cover interests on public debt, i.e.

$$\tau_{closed}(1-\psi)\nu X \geq r^S_{aut}(\underline{D})\underline{D} > 0, \tag{3.41}$$

where $r^S_{aut}(\underline{D})$ is the home country's interest rate under financial autarky and debt level $\underline{D}$, which can be characterized using (3.34) when $X^* = 0$ (so, $\omega = 1$, $\bar{\rho} = \rho$, and $\bar{\alpha} = \alpha$). In particular,

$$r^S_{aut}(\underline{D}) = \delta\theta - (1-\delta)\theta\left[\frac{\alpha}{\mu}\left(\frac{\nu D^{-1}_{closed}(\underline{D}) + \delta}{\nu D^{-1}_{closed}(\underline{D}) + \rho\delta}\right) - 1\right]$$

where $D^{-1}_{closed}(D)$ is the tax rate implied by debt level $D$. Second, the government also has to satisfy the restriction on adjustments of tax rates, i.e.

$$\tau_{closed} \geq (1+\phi)^{-1} D^{-1}_{closed}(\underline{D}) \tag{3.42}$$

The presence of tax distortions ensures that the optimal solution for the ex ante tax rate is the minimum rate that satisfy both requirements, i.e.

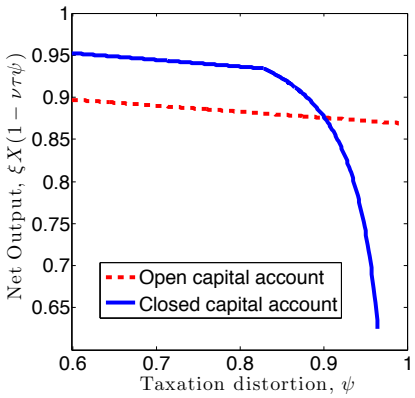
$$\begin{aligned} \tau_{closed} &= \max\left\{\frac{r^S_{aut}(\underline{D})}{(1-\psi)\nu}\frac{\underline{D}}{X}, (1+\phi)^{-1} D^{-1}_{closed}(\underline{D})\right\} \\ &\equiv \max\left\{\tau^1_{closed}, \tau^2_{closed}\right\} \end{aligned} \tag{3.43}$$

where the last line defines the *revenue motive bound*, $\tau^1_{closed}$, and the *adjustment motive bound*, $\tau^2_{closed}$, of the (ex ante) tax rate. Note that the tax rate is an strictly increasing function of $\underline{D}$, which implies that net output, $(1-\psi\tau_{closed}\nu)X$, is a decreasing function of $\underline{D}$.

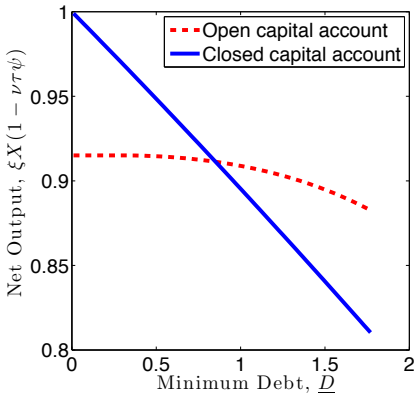
**Costs and benefits of closing the capital account.** The downside of financial integration for Home is that the excess demand for safe assets from the foreign country pushes the global economy into a safety trap that features less than full utilization capacity, i.e. $\xi(\tau^-) < 1$. The upside is that it pays no interest on its debt because the safe rate is zero. The latter implies that it has no *revenue motive* to collect taxes independently of how large its debt is. Large required debt levels only have effects on the (ex ante) tax rate through the *adjustment motive* since at some point large debt

Figure 3.4: Open vs. closed capital account

(a) Tax distortion




(b) Minimum debt requirement




requirements force an increase in the tax rate after the bad Poisson event.[22] Closing the capital account has the advantage of insulating the home economy from the safety trap, $\xi = 1$, but generates a *revenue motive* for tax collection since the safe interest rate is now positive.

Two parameter are key in determining how the trade-off between output losses due to a safety trap and avoiding debt servicing cost plays out. First, the parameter $\psi$ governing the severeness of tax distortions. The larger $\psi$, the more attractive becomes an open capital account. This is due to the fact that larger inefficiencies imply a larger tax rate to collect any given amount of resources. We illustrate this force in Figure 3.4a. For a given minimum debt requirement $\underline{D}$, we vary the severeness of tax distortions. We can see that for low levels of $\psi$, a closed capital account is preferable but once $\psi$ crosses a threshold, an open capital account is desirable. Second, the minimum debt requirement $\underline{D}$ is important. A low (high) debt requirement translates into a low (high) cost of closing the capital account since a positive safe interest rate will generate low (high) interest payments, and therefore tilt the trade-off towards a closed (open) capital account. This is illustrated in Figure 3.4b, where we hold $\psi$ fixed and vary $\underline{D}$. While the forces described above are present in general, how they exactly play out depends on the parameterization of the model.[23]

[22]Moreover, it can be proved that when the *adjustment motive* is binding in the open and closed economy, that a lower tax rate is needed in the open economy to maintain the same debt level, i.e. $D^{-1}(\underline{D}) < D^{-1}_{closed}(\underline{D})$. This follows from the fact that the (ex post) interest rate is lower in the open economy, so the same dividend tax flows generate more value.

[23]This is why we do not prove a formal proposition but rather illustrate them by means of a numerical example. The exact numerical values used to generate Figure 3.4 are provided in Appendix 3.A.7.

Finally, we can show analytically that for for sufficiently small values of $\underline{D}$, a closed capital account is always preferable.

**Proposition 3.6** (Closed Capital Account with Small Debt)**.** $\exists \hat{D} > 0$ *such that for any* $\underline{D} < \hat{D}$*, net output is larger with a closed capital account.*

The intuition for this stems from the fact that there is no safety trap with a closed capital account and when the debt requirement becomes arbitrarily small, the downside of debt servicing payments vanishes.

### 3.3.4 Consumption, Imports and Net Interest Income

So far, we have treated (net) output as the object of interest for the government. However, agents ultimately derive utility from consumption and in the open economy, a country's net output and consumption do not necessarily align perfectly. Thus, we turn to examine the situation where the government cares about consumption. The solution for consumption can be written as

$$C = \underbrace{\xi(\tau^-)X(1-\nu\psi\tau)}_{\text{Net Output}} + \underbrace{\xi(\tau^-)X\delta(1-\omega)\frac{(\nu(1-\tau\psi)-\nu^*)}{\omega\nu(1-\tau\psi)+(1-\omega)\nu^*}}_{\text{Imports/Exports}} \tag{3.44}$$

This implies that the Home country can permanently consume beyond its (net) production, if and only if, its neutral newborns' share of output (net of taxes and rebates) is larger than the one in the Foreign country, i.e. $(1-\tau\psi)\nu > \nu^*$. The intuition behind this follows from interest payments and the risk premium. For Home to be able to consume above its net output permanently, it needs to finance its imports with the return on its asset position, i.e. it needs to generate a net income from interest payments.[24] The presence of the risk premium ensures larger returns for Neutrals' wealth, so a larger fraction of Neutrals is associated with greater interest payments; in turn, greater interest payments allow for financing of permanent imports.

This gap between consumption and (net) output is a key element when considering to close the capital account. In the last subsection, we analyzed the case where $r^{S,n} < 0 < r^{S,n}_{aut}$ which follows from sufficiently large differences in the heterogeneities we emphasize, $\rho > \rho^*$ and $\alpha < \alpha^*$. In such an environment, when the government's

[24]Recall that NFA positions are constant in steady state.

objective is consumption (not output), closing the capital account comes with an extra cost (compared to the analysis focusing on output): it prevents the Home country from enjoying a positive flow of resources due to asset returns. That is, even when there is no debt servicing motive ($\underline{D} = 0$), it might still be desirable to maintain an open capital account as this allows to enjoy additional consumption financed via returns earned on net foreign assets.

Now, we emphasize that the intermediate exhaustion of fiscal capacity and the debt underprovision results are also present when the government targets consumption. The government's problem is given by

$$\begin{aligned} \max_{\{\tau,\tau^-\}} C &\equiv [1-\Gamma(\tau)]\xi(\tau^-)X \\ \text{s.t. } \tau^- &\leq (1+\phi)\tau \\ \tau, \tau^- &\in [0,1] \end{aligned} \tag{3.45}$$

where

$$\Gamma(\tau) = \nu\psi\tau - \delta(1-\omega)\frac{(\nu(1-\tau\psi)-\nu^*)}{\omega\nu(1-\tau\psi)+(1-\omega)\nu^*}$$

which is strictly increasing in $\tau$. Let $(\tau^-, \tau_C^-)$ denote the solution for this problem. The following proposition illustrates the intermediate exhaustion of fiscal capacity result. The intuition is analogous to the one described for output.

**Proposition 3.7** (Intermediate Exhaustion of Fiscal Capacity)**.** *The optimal (ex post) tax rate $\tau_C^-$ is smaller than 1, if and only if, $\psi > \underline{\psi}(\bar{\rho}, \phi)$ where the $\underline{\psi}$ is increasing in the allowed tax rate adjustment $\phi$ and decreasing in $\bar{\rho}$.*

Next, we turn to the debt underprovision result. Note that maximizing global consumption is equivalent to maximizing global (net) output,[25] so $\tau_G^-$ is the tax rate that maximizes global consumption. To study the presence of an externality, we need to analyze the effects of taxes on the part of global net output (or consumption) that is not accrued to the Home country. In previous subsections, this part was Foreign output, and the externality followed from the fact that $\xi^*$ was increasing in $\tau^-$. Now, the relevant part is foreign consumption which can be written as

$$C^* = \xi(\tau^-)(X+X^*)\frac{(1-\omega)\nu}{\omega\nu(1-\tau\psi)+(1-\omega)\nu^*} \tag{3.46}$$

[25]By goods market clearing, it holds that $C + C^* = (1-\psi\nu)\xi X + \xi^* X^*$.

Thus, Foreign consumption is increasing in $\tau^-$ and $\tau$. The first effect is the output effect, larger (ex post) taxes in Home expand global safe assets and therefore output in both countries. The second effect is particular to the consumption analysis. A larger tax rate in Home reduces the Neutrals' share of wealth which decreases the overall interest payments to Home. This translates into less imports for the Home country. The effect is exactly the opposite for the Foreign country: larger interest payments, and larger imports, i.e. more consumption. The following proposition formalizes the latter intuition.

**Proposition 3.8** (Public Debt Underprovision)**.** *The optimal debt provision from a global perspective $D(\tau_G^-)$ is strictly larger than the optimal debt provision $D(\tau_C^-)$ from a domestic perspective when targeting consumption, i.e. $D(\tau_G^-) > D(\tau_C^-)$.*

## 3.4 Conclusion

This paper has explored the output implications of safe asset shortages in a global economy. Safe asset demand from countries with low development of financial markets and large number of risk-averse agents depresses world interest rates. When interest rates are subject to a zero lower bound, then this safe asset demand pushes the world economy into a safety trap in which all countries experience output losses.

Under imperfect fiscal policy (taxes are distortionary and imperfectly adjustable over time), safe public debt has the upside of expanding output though the downside of generating wasteful distortions. Due to this trade-off, an externality arises and the global economy will suffer from underprovision of safe assets. Since safe asset demand is at the root of the problem, countries which do not suffer from safe asset shortages in autarky, have an incentive to close their capital account in order to insulate their economy from safety traps. Yet, this can imply foregoing net interest income on their asset positions.

All in all, our work indicates that safety traps are global phenomena that originate in some countries but affect the world economy as a whole. Due to the externality of safe public debt provision, it seems unlikely that the global undersupply of safe assets will be resolved by the actions of a "safe haven" alone. Moreover, if a world economy has a several countries supplying safe assets on net, these countries have an incentive to move towards a closed capital account, perhaps leading to a "capital controls war" among them. Hence, policy responses towards safety traps cry for coordination on a global scale.

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# Appendix

## 3.A Analytical Results

### 3.A.1 Solution to ex-ante system in unconstrained case

The solution to the ex-ante system in the unconstrained regime is characterized by the following expressions.

$$r = r^S = \delta\theta$$

$$\begin{aligned} \delta^S &= \mu^- \rho \delta & \qquad \delta^{S*} &= \mu^- \rho^* \delta^* \\ V^S &= \rho\mu^- \frac{X}{\theta} & \qquad V^{S*} &= \rho^*\mu^- \frac{X^*}{\theta} \\ V^R &= (1-\rho\mu^-)\frac{X}{\theta} & \qquad V^{R*} &= (1-\rho^*\mu^-)\frac{X^*}{\theta} \\ W^K &= \frac{\alpha X}{\theta} & \qquad W^{K*} &= \frac{\alpha^* X^*}{\theta} \\ W^N &= \frac{(1-\alpha)X}{\theta} & \qquad W^{N*} &= \frac{(1-\alpha)X^*}{\theta} \end{aligned}$$

### 3.A.2 Solution to ex-ante system in constrained case

The solution to the ex-ante system in the constrained regime is characterized by the following expressions.

$$\begin{aligned} r^S &= \delta\theta - \theta(1-\delta)\frac{\tilde{\alpha} - \tilde{\rho}\mu^-}{\tilde{\rho}\mu^-} \\ r &= \delta\theta + \theta(1-\delta)\frac{\tilde{\alpha} - \tilde{\rho}\mu^-}{1-\tilde{\rho}\mu^-} \end{aligned}$$

$$\delta^S = \mu^- \delta\rho - \frac{\rho}{\bar{\rho}}(1-\delta)(\bar{\alpha} - \bar{\rho}\mu^-) \qquad \delta^{S*} = \mu^- \delta\rho^* - \frac{\rho^*}{\bar{\rho}}(1-\delta)(\bar{\alpha} - \bar{\rho}\mu^-)$$

$$V^S = \rho\mu^- \frac{X}{\theta} \qquad V^{S*} = \rho^*\mu^- \frac{X*}{\theta}$$

$$V^R = \frac{(\delta - \delta^S)X}{r} \qquad V^{R*} = \frac{(\delta - \delta^{S*})X^*}{r}$$

$$W^K = \frac{\alpha}{\bar{\alpha}}\frac{X}{\theta}\bar{\rho}\mu^- \qquad W^{K*} = \frac{\alpha^*}{\bar{\alpha}}\frac{X^*}{\theta}\bar{\rho}\mu^-$$

$$W^N = \frac{(1-\alpha)(1-\delta)X}{\theta - r} \qquad W^{N*} = \frac{(1-\alpha^*)(1-\delta)X^*}{\theta - r}$$

### 3.A.3 Ex-ante system and solution with nominal rigidities and ZLB

The ex-ante steady state equilibrium system of the economy with a zero-lower bound and endogenous output can be written as follows.

Wealth dynamics

$$0 = -\theta W^K + r^S W^K + \alpha(1-\delta)\xi X \tag{3.47}$$

$$0 = -\theta W^N + r W^N + (1-\alpha)(1-\delta)\xi X \tag{3.48}$$

$$0 = -\theta W^{K*} + r^S W^{K*} + \alpha^*(1-\delta^*)\xi^* X^* \tag{3.49}$$

$$0 = -\theta W^{N*} + r W^{N*} + (1-\alpha^*)(1-\delta^*)\xi^* X^* \tag{3.50}$$

Pricing equations

$$r^S V^S = \delta^S \xi X \tag{3.51}$$

$$r V^R = (\delta - \delta^S)\xi X \tag{3.52}$$

$$r^S V^{S,*} = \delta^{S*}\xi^* X^* \tag{3.53}$$

$$r V^{R,*} = (\delta^* - \delta^{S*})\xi^* X^* \tag{3.54}$$

Market clearing

$$\theta(W^K + W^{K*} + W^N + W^{N*}) = \xi X + \xi^* X^* \tag{3.55}$$

$$W^K + W^{K*} + W^N + W^{N*} = V^S + V^R + V^{S*} + V^{R*} \tag{3.56}$$

Regime

$$0 = (V^S + V^{S*} - W^K - W^{K*}) \cdot (r - r^S) \tag{3.57}$$
$$V^S + V^{S*} \geq W^K + W^{K*}$$
$$r \geq r^S$$

Safe assets

$$V^S = \rho V^- \tag{3.58}$$
$$V^{S*} = \rho^* V^{-*} \tag{3.59}$$

ZLB

$$(r^S > 0 \ \wedge \ \xi = \xi^* = 1) \ \text{ or } \ (r^S = 0 \ \wedge \ \min\{\xi, \xi^*\} < 1) \tag{3.60}$$

To express the solution of this system, define

$$\tilde{\alpha}(\xi, \xi^*) \equiv \frac{\alpha \xi X + \alpha^* \xi^* X^*}{\xi X + \xi^* X^*}, \qquad \tilde{\rho}(\xi, \xi^*) \equiv \frac{\rho X + \rho^* X^*}{\xi X + \xi^* X^*}$$

Then, the solution for the unconstrained regime can be characterized as

$$r^S = \delta\theta - \theta(1-\delta)\frac{\tilde{\alpha} - \tilde{\rho}\mu^-}{\tilde{\rho}\mu^-}$$
$$r = \delta\theta + \theta(1-\delta)\frac{\tilde{\alpha} - \tilde{\rho}\mu^-}{1 - \tilde{\rho}\mu^-}$$

$$\delta^S = \mu^-\delta\rho - \frac{\rho}{\tilde{\rho}}(1-\delta)(\tilde{\alpha} - \tilde{\rho}\mu^-) \qquad \delta^{S*} = \mu^-\delta\rho^* - \frac{\rho^*}{\tilde{\rho}}(1-\delta)(\tilde{\alpha} - \tilde{\rho}\mu^-)$$
$$V^S = \rho\mu^-\frac{X}{\theta} \qquad V^{S*} = \rho^*\mu^-\frac{X*}{\theta}$$
$$V^R = \frac{(\delta - \delta^S)X}{r} \qquad V^{R*} = \frac{(\delta - \delta^{S*})X^*}{r}$$
$$W^K = \frac{\alpha}{\tilde{\alpha}}\frac{X}{\theta}\tilde{\rho}\mu^- \qquad W^{K*} = \frac{\alpha^*}{\tilde{\alpha}}\frac{X^*}{\theta}\tilde{\rho}\mu^-$$
$$W^N = \frac{(1-\alpha)(1-\delta)X}{\theta - r} \qquad W^{N*} = \frac{(1-\alpha^*)(1-\delta)X^*}{\theta - r}$$

We have basically solve the system treating $\xi, \xi^*$ as parameters. Having obtained the solution laid out above, we check whether $r^S \geq 0$ for $\xi = \xi^* = 1$. If this is the case,

the solution is given by the expressions above with $\xi = \xi^* = 1$. If $r^S < 0$, we have $r^S = 0$ and obtain a relationship between $\xi = \xi^*$ that we display in the text.

### 3.A.4 Relaxing fully rigid prices

This section lays out a version of our international model where prices are sticky, though not completely rigid. This is interesting for the following reason. When we study the 2-country model without inflation, we can only arrive at a linear relationship between $\xi$ and $\xi^*$ – we cannot pin them down separately. However, in the version with *some* price stickiness, we are able to do that. This extension is an international version of the model with price adjustments in CF (2015).

We assume that prices are sticky downward (not upward) and adjustments are governed by Philipps-curves. In particular, we specify

$$\begin{aligned} \pi &\geq -\left(\kappa_0 + \kappa_1(1-\xi)\right) \\ \pi^* &\geq -\left(\kappa_0^* + \kappa_1^*(1-\xi^*)\right), \end{aligned}$$

where the $\kappa$s are parameters. We assume that when the economy is below potential, i.e. $\xi < 1$ ($\xi^* < 1$), prices fall as fast as possible. These restrictions will on the prices in Home and Foreign will be captured by two complementary slackness conditions in the equilibrium system. We assume monetary authorities set nominal interest rates according to Taylor-rules that are specified below.

The steady-state equilibrium is characterized by wealth dynamics (3.47) - (3.50), asset pricing equations (3.51)-(3.54), market clearing conditions (3.55) - (3.56), and the following.

Safe asset regime condition

$$W^K + W^{K*} = V^S + V^{S*} \tag{3.61}$$

Philipps-curves

$$\left[\pi + \left(\kappa_0 + \kappa_1(1-\xi)\right)\right](1-\xi) = 0 \tag{3.62}$$

$$\left[\pi^* + \left(\kappa_0^* + \kappa_1^*(1-\xi^*)\right)\right](1-\xi^*) = 0 \tag{3.63}$$

Taylor-rules and ZLB

$$i = \max\{0, r^{S,nat} + \hat{\pi} + \phi(\pi - \hat{\pi})\} \tag{3.64}$$

$$i^* = \max\{0, r^{S,nat} + \hat{\pi}^* + \phi^*(\pi^* - \hat{\pi}^*)\} \tag{3.65}$$

Fisher equations

$$i = r^S + \pi \tag{3.66}$$

$$i^* = r^S + \pi^* \tag{3.67}$$

Safe assets

$$V^S = \bar{V}^S \tag{3.68}$$

$$V^{S*} = \bar{V}^{S*} \tag{3.69}$$

Law-of-one-price

$$\frac{\dot{E}}{E} = \pi - \pi^*. \tag{3.70}$$

In full rigor, we would use a backward-inductive step to solve for the value of safe assets as functions of parameters (this would include solving the system after the aggregate shock has been realized). This will include judgement about ex-post equilibrium selection (potentially there are multiple ex-post equlibria). We abstract from these issues here by assigning *some* fixed number to the value of safe assets.

We have a set of 19 variables to solve for: real returns $\{r, r^S\}$, wealth distribution over agents' types $\{W^K, W^N, W^{K*}, W^{N*}\}$, asset values $\{V^S, V^R, V^{S*}, V^{R*}\}$, share of dividends accrued to safe assets $\{\delta^S, \delta^{S*}\}$, utilization capacities $\{\xi, \xi^*\}$ , nominal returns $\{i, i^*\}$, inflation rates $\{\pi, \pi^*\}$, and exchange rate depreciation $\frac{\dot{E}}{E}$. And we have 19 independent equations from the system above. We can solve this but the solution is possibly non-unique. Thus, we attempt a solution by means of an AS-AD approach.

For Home, while $\xi < 1$, the AS-curve is given by

$$\pi = -\kappa_0 - \kappa_1 + \kappa_1 \xi.$$

At $\xi = 1$, the AS-curve becomes vertical. The AD-curve has two segments. The first is when $i = 0$, then

$$\pi = \frac{\alpha(1-\delta)\xi X + \alpha^*(1-\delta)\xi^* X^*}{\bar{V}^S + \bar{V}^{S*}} - \theta.$$

The second is when $i > 0$, which gives

$$\pi = \hat{\pi} + \frac{1}{\phi - 1}\frac{\alpha(1-\delta)(1-\xi)X + \alpha^*(1-\delta)(1-\xi^*)X^*}{\bar{V}^S + \bar{V}^{S*}}$$

The kink connecting the two segments is characterized by

$$\pi = \frac{\phi - 1}{\phi}\hat{\pi} - \frac{r^{S,nat}}{\phi}$$

Completely analogous, we have for Foreign that as long as $\xi^* < 1$, the AS-curve is given by

$$\pi^* = -\kappa_0^* - \kappa_1^* + \kappa_1^*\xi^*$$

and turns vertical once $\xi^* = 1$. The upward-sloping part of the AD-curve ($i^* = 0$) is given by

$$\pi^* = \frac{\alpha(1-\delta)\xi X + \alpha^*(1-\delta)\xi^* X^*}{\bar{V}^S + \bar{V}^{S*}} - \theta.$$

The downward-sloping part ($i^* = 0$) is given by

$$\pi^* = \hat{\pi}^* + \frac{1}{\phi^* - 1}\frac{\alpha(1-\delta)(1-\xi)X + \alpha^*(1-\delta)(1-\xi^*)X^*}{\bar{V}^S + \bar{V}^{S*}}$$

The kink for Foreign is characterized by

$$\pi^* = \frac{\phi^* - 1}{\phi^*}\hat{\pi}^* - \frac{r^{S,nat}}{\phi^*}$$

In the following, we characterize the equilibrium where both countries are in a recession. Equalizing the AS- and AD-curve in Home and Foreign respectively yields

$$-\kappa_0 - \kappa_1 + \kappa_1\xi = \frac{\alpha(1-\delta)X}{\bar{V}^S + \bar{V}^{S*}}\xi + \frac{\alpha^*(1-\delta)X^*}{\bar{V}^S + \bar{V}^{S*}}\xi^* - \theta$$
$$-\kappa_0^* - \kappa_1^* + \kappa_1^*\xi^* = \frac{\alpha(1-\delta)X}{\bar{V}^S + \bar{V}^{S*}}\xi + \frac{\alpha^*(1-\delta)X^*}{\bar{V}^S + \bar{V}^{S*}}\xi^* - \theta$$

This gives 2 equations in $\xi$ and $\xi^*$ that we can solve. Some algebra yields

$$\xi^* = \frac{\frac{\alpha(1-\delta)X}{\bar{V}^S+\bar{V}^{S*}}\frac{\theta-\kappa_0^*-\kappa_1^*}{\frac{\alpha(1-\delta)X}{\bar{V}^S+\bar{V}^{S*}}-\kappa_1} + \kappa_1^* + \kappa_0^* - \theta}{\kappa_1^* - \frac{\alpha^*(1-\delta^*)X^*}{\bar{V}^S+\bar{V}^{S*}} + \frac{\alpha(1-\delta)X}{\bar{V}^S+\bar{V}^{S*}}\frac{\frac{\alpha^*(1-\delta^*)X^*}{\bar{V}^S+\bar{V}^{S*}}}{\frac{\alpha(1-\delta)X}{\bar{V}^S+\bar{V}^{S*}}-\kappa_1}}$$

We can exploit the relationship between the two AS-curves to find a simple expression for $\xi$ as a function of $\xi^*$.

$$\xi = \frac{(\kappa_0 - \kappa_0^*) + (\kappa_1 - \kappa_1^*)}{\kappa_1} + \frac{\kappa_1^*}{\kappa_1}\xi^*$$

Finally, note that in the recessionary equilibrium, we have $i = i^* = 0$ which implies $\pi = \pi^* = -r^{S,nat}$. Thus, exchange rate will be constant, i.e. $\frac{\dot{E}}{E} = 0$.

### 3.A.5 Solution for Model with Costly Debt Provision

We study the ex-ante steady state equilibrium in the constrained regime where safe rates are against the ZLB. This is characterized by the following set of equations.

Asset pricing equations

$$r^S V^S = \delta^S \xi X \tag{3.71}$$
$$r^S V^{S*} = \delta^{S*} \xi X^* \tag{3.72}$$
$$rV^R = (\delta - \delta^S)\xi X \tag{3.73}$$
$$rV^{R*} = (\delta - \delta^{S*})\xi X^* \tag{3.74}$$

Wealth dynamics

$$(\theta - r^S)W^K = \alpha(1-\delta)\xi X \tag{3.75}$$
$$(\theta - r^S)W^{K*} = \alpha^*(1-\delta)\xi X^* \tag{3.76}$$
$$(\theta - r)W^N = (1-\tau)\nu\xi X + S \tag{3.77}$$
$$(\theta - r)W^{N*} = \nu^*\xi X^* \tag{3.78}$$

Market Clearing

$$W^K + W^{K*} + W^N + W^{N*} = V^S + V^R + V^{S*} + V^{R*} + D \tag{3.79}$$
$$\theta\left(V^S + V^{S*} + V^R + V^{R*} + D\right) = (1-\psi\nu)\xi X + \xi X^* \tag{3.80}$$

Constrained regime

$$V^S + V^{S*} + D = W^K + W^{K*} \tag{3.81}$$

Safe assets

$$V^S = \rho V^- \tag{3.82}$$
$$V^{S*} = \rho^* V^{*-} \tag{3.83}$$

ZLB

$$r^S = 0 \tag{3.84}$$

Government flow

$$S = \tau(1-\psi)\nu\xi X - r^S D \tag{3.85}$$

These blocks correspond to asset pricing, wealth dynamics, market clearing, the constrained regime equation, the equations for safe assets, the (binding) ZLB constraint and the government flow budget constraint. The full solution to this system is given by

$$\xi = \frac{\mu^-}{\bar{\alpha}(1-\delta)}\frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta}$$
$$r = \theta - \theta\frac{(1-\delta)(1-\bar{\alpha}) - \psi\tau\nu\omega}{(1-\delta)(1-\bar{\alpha}) - \psi\tau\nu\omega + \delta}$$

$$\delta^S = 0 \qquad \delta^{S*} = 0$$
$$V^S = \rho \frac{\omega\delta}{\omega\nu\tau^- + \delta} \frac{\mu^-(X + X^*)}{\theta} \qquad V^{S*} = \rho^* \frac{(1-\omega)\delta}{\omega\nu\tau^- + \delta} \frac{\mu^-(X + X^*)}{\theta}$$
$$W^K = \frac{\alpha}{\bar{\alpha}} \frac{\mu^-}{\theta} \frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta} X \qquad W^{K*} = \frac{\alpha^*}{\bar{\alpha}} \frac{\mu^-}{\theta} \frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta} X^*$$

and with these expressions at hand, we can calculate

$$W^N = \frac{(1-\psi\tau)\nu X}{\theta - r} \frac{\mu^-}{\bar{\alpha}(1-\delta)} \frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta} \qquad W^{N*} = \frac{\nu^* X^*}{\theta - r} \frac{\mu^-}{\bar{\alpha}(1-\delta)} \frac{\omega\nu\tau^- + \bar{\rho}\delta}{\omega\nu\tau^- + \delta}$$
$$V^R = \frac{\delta\xi X}{r} \qquad V^{R*} = \frac{\delta\xi X^*}{r},$$

which completes the full solution.

### 3.A.6 Solution with minimum debt requirement

We consider the same problem as in (3.38) with the additional constraint $D > \underline{D}$. Since (3.29) delivers an strictly increasing relationship between $D$ and $\tau^-$, the constraint is effectively a lower bound on $\tau^-$, denote it as $D^{-1}(\underline{D})$. If the solution to (3.38) satisfies the constraint, i.e. $\tau_G^- < D^{-1}(\underline{D})$, then the extra constraint is not binding and the solution is unchanged. If $\tau_G^-$ does not satisfies debt constraint, the solution is given by $D^{-1}(\underline{D})$ because the function is globally concave, i.e. $Y_G'(\tau^-) > 0$ and $Y_G''(\tau^-) < 0$.

### 3.A.7 Numerical example

The parametrization used for the numerical examples is reported in the following table. The examples aim to illustrate the forces present in the model, and not to provide quantitative statements. When illustrating the effects of debt requirement $\underline{D}$ (tax distortions $\psi$), we fix $\psi = 0.8$ ($\underline{D} = 0.6$).

| Parameter | Value |
|---|---|
| $\delta$ | 0.5 |
| $\theta$ | 0.05 |
| $\mu^-$ | 0.2 |
| $(\rho, \rho^*)$ | (0.8, 0.5) |
| $(\alpha, \alpha^*)$ | (0.2, 0.8) |
| $(X, X^*)$ | (1,0.33) |

To present the effects neatly, we use different values of $\phi$ in each example. For the debt requirement (tax distortions) example, we use $\phi = 1.5$ ($\phi = 0.11$).

## 3.A.8 Proofs

### Proof of Proposition 3.1

Note that the natural safe real interest rate in the world economy, $r^{S,n}$, is given by

$$r^{S,n} = \delta\theta - (1-\delta)\theta\left(\frac{\bar{\alpha} - \bar{\rho}\mu^-}{\bar{\rho}\mu^-}\right).$$

We obtain the natural autarky safe real rate in Home, $r^{S,n}_{aut}$, as the limiting case of the above expression when $X^* \to 0$. Analogously, we obtain $r^{S,n*}_{aut}$ from the above expression when $X \to 0$. We have

$$r^{S,n}_{aut} = \delta\theta - (1-\delta)\theta\left(\frac{\alpha - \rho\mu^-}{\rho\mu^-}\right)$$
$$r^{S,n*}_{aut} = \delta\theta - (1-\delta)\theta\left(\frac{\alpha^* - \rho^*\mu^-}{\rho^*\mu^-}\right).$$

Now, it follows straightforwardly that the respective inequalities hold if and only if the condition given in the Proposition is satisfied.

### Proof of Proposition 3.2

We calculate Home's NFA position in safe assets as

$$NFA^S \equiv V^S - W^K = \rho\mu^-\frac{X}{\theta} - \frac{\alpha}{\bar{\alpha}}\frac{X}{\theta}\bar{\rho}\mu^-.$$

Hence,

$$NFA^S > 0 \Leftrightarrow \frac{\bar{\alpha}}{\bar{\rho}} > \frac{\alpha}{\rho},$$

which completes the proof.

## Proof of Proposition 3.3

We have that

$$r^{S,n} = \delta\theta - (1-\delta)\theta\left(\frac{\bar{\alpha} - \bar{\rho}\mu^{-}}{\bar{\rho}\mu^{-}}\right) < 0 \Leftrightarrow \bar{\rho}\mu^{-} < (1-\delta)\bar{\alpha}$$

and

$$r^{S,n}_{aut} = \delta\theta - (1-\delta)\theta\left(\frac{\alpha - \rho\mu^{-}}{\rho\mu^{-}}\right) > 0 \Leftrightarrow \rho\mu^{-} > (1-\delta)\alpha.$$

## Proof proposition 3.4

We start from problem (3.36). Recall that function $\xi(\tau^{-})$ is given by (3.35), and $\tau_D^{-}$ denotes the solution for $\tau^{-}$. It is straightforward to proof that the objective function is increasing in $\tau^{-}$, and decreasing in $\tau$, so constraint $\tau^{-} \leq (1+\phi)\tau$ will be binding. Then, home net output (the objective function) can be written as

$$Y(\tau^{-}) = (1 - (1+\phi)^{-1}\tau^{-}\psi\nu)\xi(\tau^{-})X$$

It is straightforward to verify that $Y'(\tau^{-}) > 0$ and $Y''(\tau^{-}) < 0$. Let $Y'(1,\psi)$ denote $Y'(1)$ as a function of tax inefficiencies $\psi$. It is straightforward to show $\frac{dY'(1,\psi)}{d\psi} < 0$. Define $\underline{\psi}$ as the value of $\psi$ such that $Y'(1,\underline{\psi}) = 0$. Then, for any $\psi > \underline{\psi}$, we have that $Y'(1) < 0$ holds, i.e. $\tau_D^{-} < 1$. Also, for any $\psi < \underline{\psi}$, we have $Y'(1) > 0$ and $\tau_D^{-} = 1$. The boundary shown in the proposition is equivalent to $Y'(1,\underline{\psi}) = 0$.

## Proof proposition 3.5

Recall that we focus on the situation where there is intermediate exhaustion of fiscal capacity, i.e. $\tau_D^{-} < 1$. Problem (3.38) is symmetric to problem (3.36) except for an extra term in the objective function, $\xi(\tau^{-})X$. It is straightforward to prove that the objective is increasing in $\tau^{-}$ and decreasing in $\tau$ which implies that the adjustment constraint will be binding (as in the domestic problem). Let $Y_G(\tau^{-})$ denote global net output as function of $\tau^{-}$ imposing the adjustment constraint at equality.

$$Y_G(\tau^{-}) = Y(\tau^{)} + \xi(\tau^{-})X^{*}$$

where $Y(\tau^{)}$ the Home net output. It can be shown that $Y'_G(\tau^-) > 0$ and $Y''_G(\tau^-) < 0$. Therefore, for any $a \in [0, 1)$, we have that $\tau_G^- > a$, if and only if, $Y'(a) > 0$. Let $a = \tau_D^-$, then we have that

$$Y'_G(\tau_D^-) = \underbrace{Y'(\tau_D^-)}_{=0} + \xi'(\tau_D^-)X^* > 0$$

where the first term is zero because of the FOC of problem (3.36) and the inequality follows from the fact that $xi(\tau^-)$ is an strictly increasing function. This implies $\tau_G^- < \tau_D^-$.

## Proof proposition 3.6

With no minimum debt requirement,$\underline{D} = 0$, net output in the Home country with a closed capital account is given by $X$, while the corresponding value with an open capital account is given by $\xi(\tau^-)(1 - \nu\psi\tau^-)X$. Recall that we are studying the situation where the financially integrated economy is in a safety trap while the Home country is not in financial autarky. It is clear that $\xi(\tau^-)(1 - \nu\psi\tau^-)X > X$, so the proposition follows from the continuity the solutions on $\underline{D}$.

## Proof proposition 3.7

We start from problem (3.45). Recall that function $\xi(\tau^-)$ is given by (3.35), and $\tau_C^-$ denotes the solution for $\tau^-$. It is straightforward to proof that the objective function is increasing in $\tau^-$, and decreasing in $\tau$, so constraint $\tau^- \leq (1 + \phi)\tau$ will be binding. Then, home net output (the objective function) can be written as

$$C(\tau^-) = (1 - \Gamma((1 + \phi)^{-1}\tau^-))\xi(\tau^-)X$$

It is straightforward to verify that $C'(\tau^-) > 0$ and $C''(\tau^-) < 0$. Therefore, for any $a \in (0, 1]$, we have that $\tau_C^- < a$, if and only if, $C'(a) < 0$. Let $C'(1, \psi, \phi, \bar{\rho})$ denote $C'(1)$ as a function of $\psi$, $\phi$, and $\bar{\rho}$. It can be shown that $\frac{C'(1,\psi,\phi,\bar{\rho})}{d\psi} < 0$, $\frac{C'(1,\psi,\phi,\bar{\rho})}{d\phi} > 0$, and $\frac{C'(1,\psi,\phi,\bar{\rho})}{d\bar{\rho}} < 0$. Define $\underline{\psi}^c(\phi, \bar{\rho})$ as the value of $\psi$ such that $C'(1, \underline{\psi}^c, \phi, \bar{\rho}) = 0$. It follows from the derivatives' signs that $\underline{\psi}^c_\phi > 0$, and $\underline{\psi}^c_{\bar{\rho}} < 0$. Moreover, for any $\psi > \underline{\psi}^c$, we have that $C'(1) < 0$ holds, i.e. $\tau_C^- < 1$. Also, for any $\psi < \underline{\psi}^c$, we have $C'(1) > 0$ and $\tau_C^- = 1$.

## Proof proposition 3.8

By an argument completely symmetric to the one in proof of proposition 3.5, it is enough to prove that Foreign consumption $C^*$ is increasing in in $\tau^-$ when imposing the adjustment constraint at equality. We can write foreign consumption as in equation (3.46), so the proposition follows.